Lab Report 1

RLC-Circuits - Transient Response

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Contents

1 Introduction

In this experiment, behavior of RLC-Circuit with a square wave signal was observed. After conducting the experiment, the theoretically and experimentally obtained numbers were compared and will be discussed in the evaluation section. Matlab was used to compare expected signal's behavior with experimentally seen behavior.
Also, in the evaluation section, a way of solving second order differential equations will be described. Transient responses and steady-state responses will be studied to have a complete solution of second order equations.

2 Execution Transient response of RLC-Circuits

2.1 Experimental Preparation and Execution

First, the following RLC-Circuit was constructed.

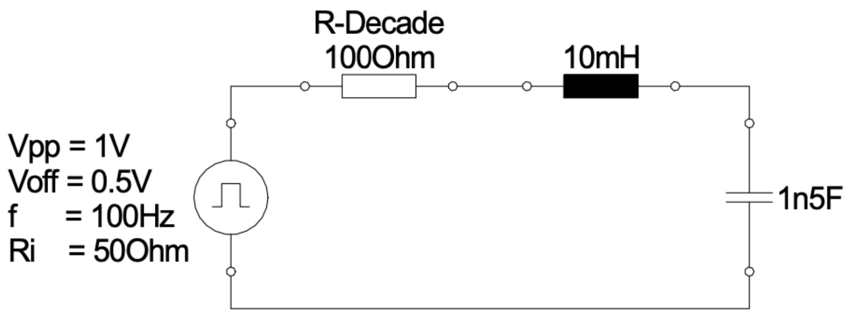

Figure 1: RLC-Circuit

The function generator was set to produce a 100Hz square wave with an amplitude of 0.5V and an offset of 0.5V. Also, the oscilloscope was connected in parallel to the capacitor. After assembling the circuit, the damped frequency was recorded by using the cursors in the oscilloscope. Figure 3 shows the damped frequency. Then, damped radian frequency can be determined as follows:

$$f_d = 41.66kHz \quad \Rightarrow \quad \omega_d = 2\pi f_d = 261.757 \times 10^3 rad/s$$

Also, we can calculate the damped radian frequency theoretically. Considering that the function generator has internal resistance of 50Ω, it can be written as:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{(R+50)^2C}{4L}} =$$
$$= \frac{1}{\sqrt{10^{-2}\times 1.5\times 10^{-9}}}\sqrt{1-\frac{(100+50)^2\times 1.5\times 10^{-9}}{4\times 10^{-2}}} = 258.090\times 10^3 rad/s$$

The theoretical value and experimentally measured value confirm each other.

Knowing that the circuit is critically damped when $\zeta = 1$, the resistance can be determined as:

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 1 \quad \Rightarrow \quad R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{10^{-2}}{1.5 \times 10^{-9}}} = 5164\Omega$$

The R-decade was set to 5164Ω so that the circuit was critically damped. Figure 4 shows the behavior of the voltage of the generator and of the capacitor.
Then, the resistance of the R-decade was changed to $R = 3000\Omega$ and to $R = 7000\Omega$. Displayed signals on the oscilloscope were used to compare with the critically damped case. At $R = 3000\Omega$, which is shown in first picture of Figure 5, it can be observed that voltage of the capacitor goes a little bit higher than voltage of the generator in the beginning. Meanwhile, at $R = 7000\Omega$, which is shown in the second pictore of Figure 5, voltage of the capacitor gets the same value as voltage of the generator a little bit later.
Finally, the R-decade was set to $R = 30k\Omega$, so that the circuit was over-damped. The first picture of Figure 6 shows the signals in the oscilloscope at the same resolution. It is much easier to compare several signals at the same resolution. Also, the second picture of Figure 6 shows the same signal, but at different resolution to have a full view.

2.2 Results

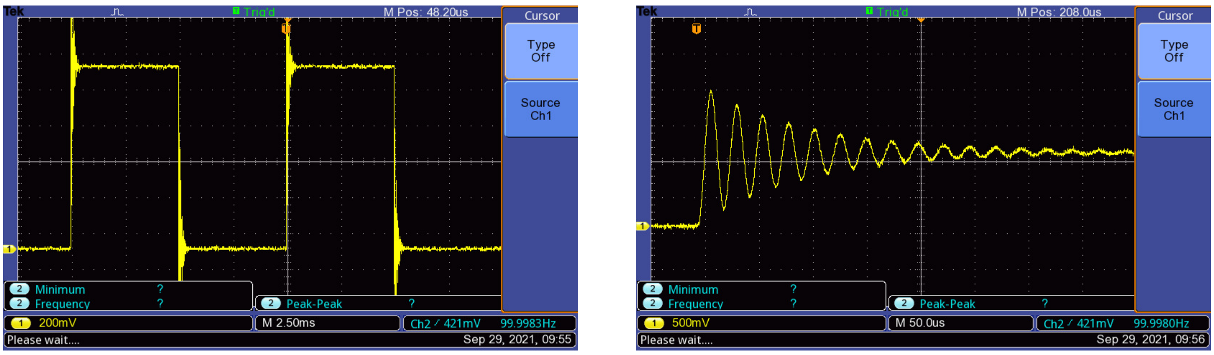

Figure 2: Ringing phenomenon.

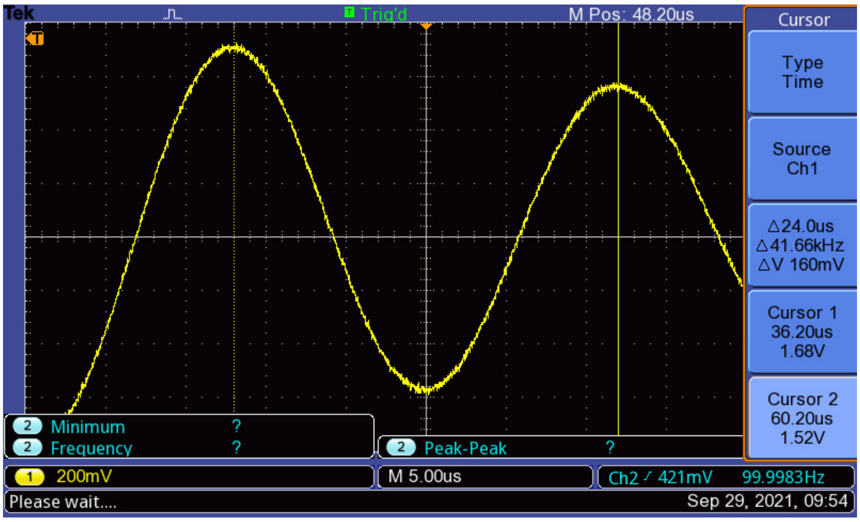

Figure 3: One period of the damped signal.

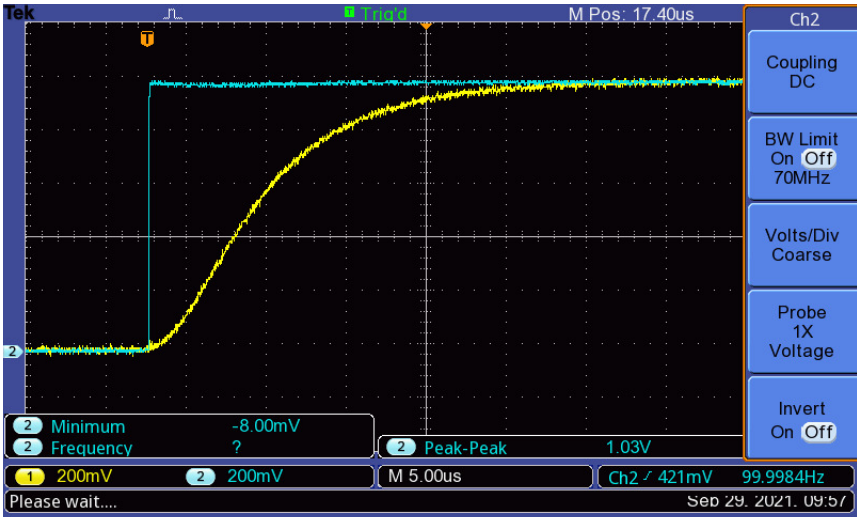

Figure 4: Voltage of the generator and of the capacitor at $R = 5164\Omega$.

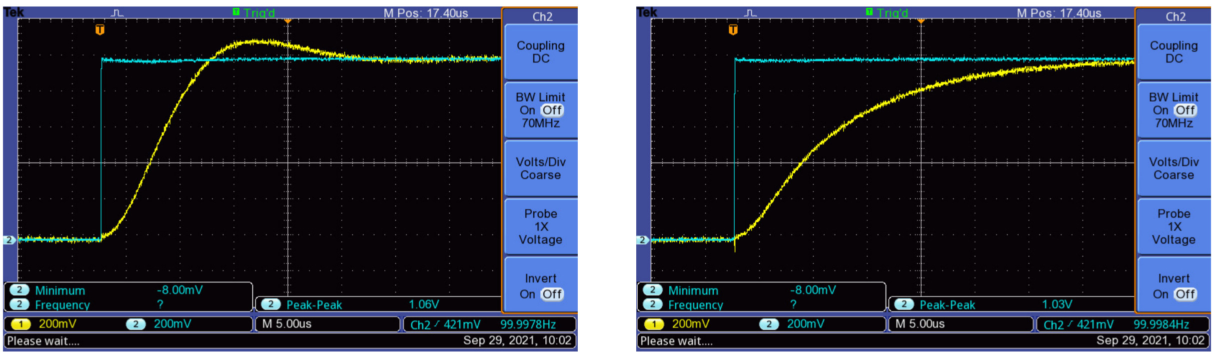

Figure 5: Voltage of the generator and of the capacitor at $R = 3000\Omega$ and at $R = 7000\Omega$.

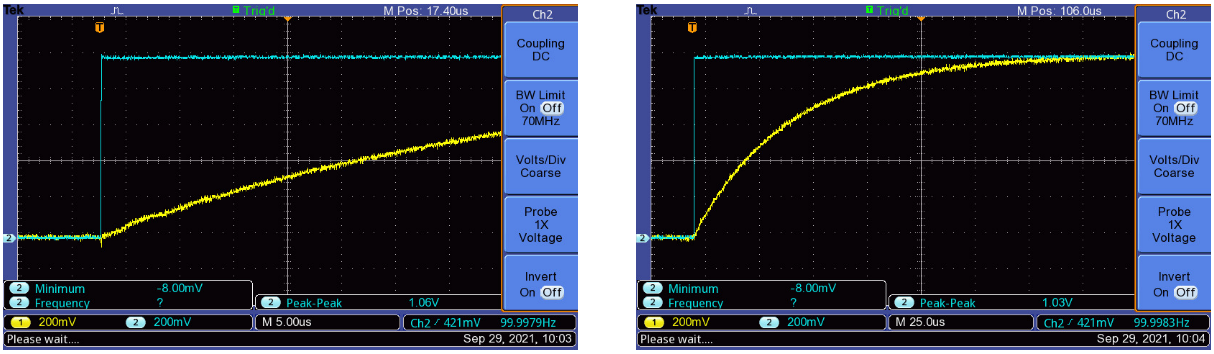

Figure 6: Voltage of the generator and of the capacitor at $R = 30k\Omega$ with the same and with different resolution.

3 Evaluation

3.1 Question 1

Since all elements are located in series, the current through every element remains the same. So, current through the capacitor and voltage of the inductor can be described as:

$$i = C\frac{dv_c}{t} \tag{1}$$

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left[C\frac{dv_c}{dt}\right] = LC\frac{d^2v_c}{dt^2} \tag{2}$$

Applying KVL to the circuit,

$$v_L + v_R + v_c = v_s \tag{3}$$

where v_s is the voltage from the generator. Substituting (2) into (3) and using Ohm's law with (1) gives:

$$\begin{gathered} LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = v_s \Rightarrow \\ 1.5 \times 10^{-11}\frac{d^2v_c}{dt^2} + 2.25 \times 10^{-7}\frac{dv_c}{dt} + v_c = v_s \end{gathered} \tag{4}$$

Now the damping nature of the circuit when $R = 100\Omega$ can be found

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{150}{2}\sqrt{\frac{1.5 \times 10^{-9}}{10^{-2}}} = 29.047 \times 10^{-3} \tag{5}$$

which is much smaller than 1 meaning that the circuit is under-damped.
Since the circuit is under-damped, v_c can be written as:

$$v_c(t) = exp(-\zeta\omega_n t) \times [C_1 cos(w_d t) + C_2 sin(w_d t)] + K \tag{6}$$

where $\zeta = 29.047 \times 10^{-3}$, K is the forced solution, and

$$w_n = \frac{1}{\sqrt{LC}} = \frac{1}{10^{-2} \times 1.5 \times 10^{-9}} = 258.200 \times 10^3 rad/s \tag{7}$$

$$w_d = w_n\sqrt{1-\zeta^2} = 258.2 \times 10^3 \times \sqrt{1-(29.047 \times 10^{-3})^2} = 258.090 \times 10^{-3} \tag{8}$$

Initial conditions should be used to find C_1 and C_2. Figure 7 shows the circuit at $t = 0$. The function generator has $V_{pp} = 1$, so it is possible to change the AC voltage source with DC voltage source of $1V$.

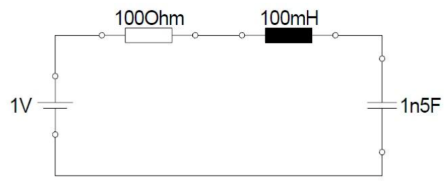

Figure 7: RLC-circuit at t=0

Since before $t = 0$, there was no source, the capacitor acts as a short circuit and the inductor acts as an open circuit. It means that there was no current and no voltage across any element.

$$v_c(t=0) = C_1 + K = 0 \quad \Rightarrow \quad C_1 = -K = -1 \tag{9}$$

Also, from the current across the capacitor at $t = 0$:

$$i_c(t=0) = C\frac{dv_c(0)}{dt} = 0 \quad \Rightarrow \quad \frac{dv_c(0)}{dt} = 0 \tag{10}$$

Taking derivative of (6):

$$\begin{aligned} \frac{dv_c(t)}{dt} &= -\zeta\omega_n exp(-\zeta\omega_n t) \times [C_1 cos(\omega_d t) + C_2 sin(\omega_d t)] + \\ &exp(-\zeta\omega_n t) \times [C_1\omega_d sin(\omega_d t) + C_2\omega_d cos(\omega\omega_d t)] \end{aligned} \tag{11}$$

Thus, at $t = 0$, this equation gives the following:

$$\frac{dv_c(0)}{dt} = -\zeta\omega_n C_1 + C_2\omega_d = 0 \quad \Rightarrow \tag{12}$$

$$C_2 = \frac{\zeta\omega_n C_1}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}} C_1 = -\frac{29.047 \times 10^-3}{\sqrt{1-(29.047 \times 10^-3)^2}} = -29.060 \times 10^-3 \tag{13}$$

3.2 Question 2

The voltage equation over the capacitor was written in Matlab with corresponding variables. Figure 8 shows the results obtained from Matlab.

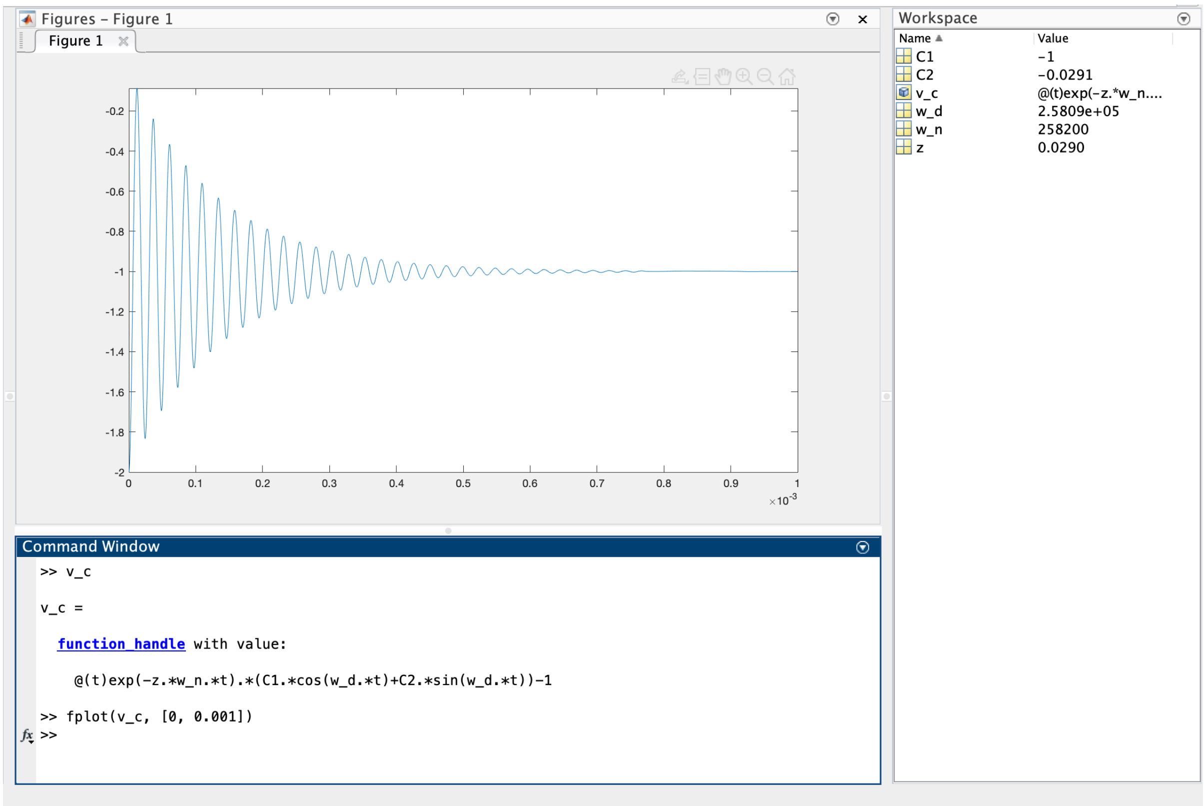

Figure 8: Matlab code

3.3 Question 3

Knowing that the circuit is critically damped when $\zeta = 1$, the resistance can be determined as:

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = 1 \quad \Rightarrow \quad R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{10^{-2}}{1.5 \times 10^{-9}}} = 5164\Omega \tag{14}$$

Therefore, the equation for describing the voltage $v_c(t)$ can be written as:

$$v_c(t) = C_1 exp(-\zeta\omega_n t) + C_2 t exp(-\zeta\omega_n t) + K \tag{15}$$

where $\zeta = 1$, K is the forced solution, and

$$w_n = \frac{1}{\sqrt{LC}} = \frac{1}{10^{-2} \times 1.5 \times 10^{-9}} = 258.200 \times 10^3 rad/s \tag{16}$$

Initial conditions should be used to find C_1 and C_2. Figure 7 shows the circuit at $t = 0$. The function generator has $V_{pp} = 1$, so it is possible to change the AC voltage source with DC voltage source of $1V$.

Since before $t = 0$, there was no source, the capacitor acts as a short circuit and the inductor acts as an open circuit. It means that there was no current and no voltage across any element.

$$v_c(t = 0) = C_1 + K = 0 \quad \Rightarrow \quad C_1 = -K = -1 \tag{17}$$

Also, from the current across the capacitor at $t = 0$:

$$i_c(t=0) = C\frac{dv_c(0)}{dt} = 0 \quad \Rightarrow \quad \frac{dv_c(0)}{dt} = 0 \tag{18}$$

Taking derivative of (15):

$$\frac{dv_c(t)}{dt} = -\zeta\omega_n C_1 exp(-\zeta\omega_n t) - \zeta\omega_n C_2 texp(-\zeta\omega_n t) + C_2 exp(-\zeta\omega_n t) \tag{19}$$

Thus, at $t = 0$, this equation gives the following:

$$\frac{dv_c(0)}{dt} = -\zeta\omega_n C_1 + C_2 = 0 \quad \Rightarrow \tag{20}$$

$$C_2 = \zeta\omega_n C_1 = -258.200 \times 10^3 \tag{21}$$

The voltage equation over the capacitor was written in Matlab with corresponding variables. Figure 9 shows the results obtained from Matlab.

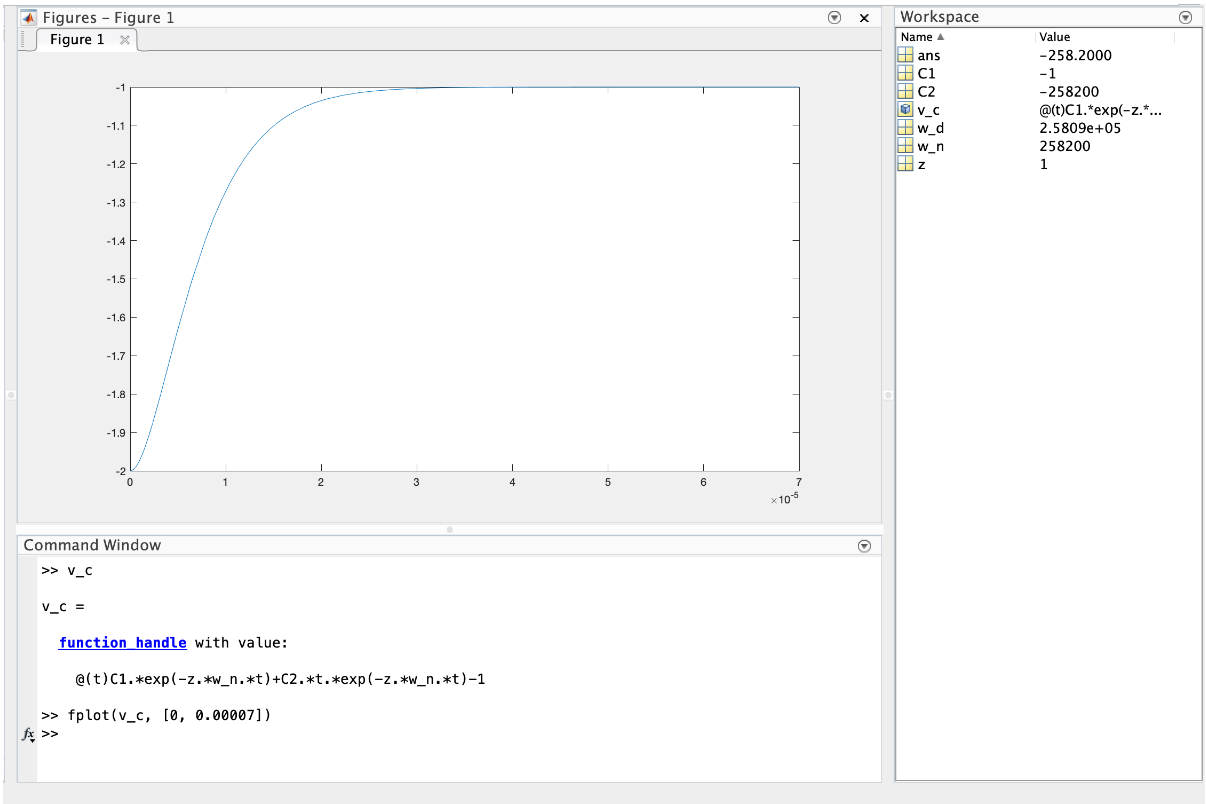

Figure 9: Matlab code

3.4 Question 4

The damped radian frequency, which is obtained during the experiment, was $\omega_d = 261.757 \times 10^3 rad/s$ and theoretical value was $\omega_d = 258.090 \times 10^3 rad/s$. Thus, the experimental value is confirmed by expected value. It can easily be calculate that experimental number has relative error of 1.42% which is small enough. However, there are still some other error such as inaccurate values of capacitor, inductor, resistors, and inaccuracy of the oscilloscope. To calculate the damped radian frequency, the cursors from oscilloscope were used. Also, the plots from Matlab and signals from oscilloscope are very similar, meaning that signal behaved as expected.

3.5 Question 5

The circuit shown in Figure 10 was used to calculate the following numbers and equations.

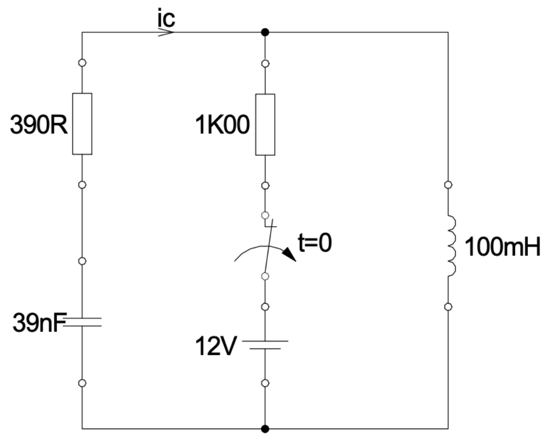

Figure 10: RLC-circuit

1. After opening the switch, all elements will be located in series. Therefore, the current through all elements will be the same. So, the current through the inductor and the voltage of the capacitor would be

$$v_c = \frac{1}{C}\int i(t)dt \tag{22}$$

$$v_L = L\frac{di(t)}{dt} \tag{23}$$

Applying KVL to the circuit,

$$v_L + v_R + v_c = 0 \tag{24}$$

Substituting (23) and (22) into (24) and using Ohm's law gives:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt = 0 \tag{25}$$

Multiplying everything by C and taking derivative of everything gives the following:

$$
\begin{aligned}
LC\frac{d^2i(t)}{dt^2} + RC\frac{di(t)}{dt} + i(t) = 0 \Rightarrow \\
3.9{\times}10^{-9}\frac{d^2i(t)}{dt^2} + 1.521 \times 10^{-5}\frac{di(t)}{dt} + i(t) = 0
\end{aligned} \tag{26}
$$

Now the damping nature can be determined as:

$$
\zeta = \frac{RC}{2\sqrt{LC}} = 121.777 \times 10^{-3} \tag{27}
$$

which is much smaller than 1 meaning that the circuit is under-damped. Thus, the equation of $i(t)$ can be written as:

$$
i(t) = exp(-\zeta\omega_n t) \times [C_1 cos(\omega_d t) + C_2 sin(\omega_d t)] \tag{28}
$$

where $\zeta = 121.777 \times 10^-3$ and

$$
\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{0.1 \times 39 \times 10^{-9}} = 15.894 \times 10^3 rad/s \tag{29}
$$

$$
\omega_d = \omega_n\sqrt{1-\zeta^2} = 15.894 \times 10^3 \times \sqrt{1-(121.777 \times 10^-3)^2} = 258.090 \times 10^{-3} \tag{30}
$$

Before $t = 0$, the DC voltage source was connected long enough for capacitor to acquire enough charge to have almost full voltage across it. It means that no more current flows though capacitor. Also, since the current through the inductor is constant and $v_L = Ldi(t)/dt$, it is clear that the voltage across the inductor will be zero, meaning the inductor works as a short circuit. These give the following circuit:

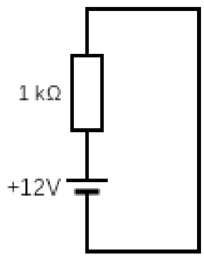

Figure 11: Circuit before $t = 0$

Now, the current through the resistor can be found by using Ohm's law:

$$
I = \frac{V}{R} = \frac{12}{1000} = 12mA \tag{31}
$$

At $t = 0$, the switch is opened. It means that there is no more current through the resistor $R = 1K\Omega$ and that the capacitor is no longer open-circuit and the inductor is no longer short-circuit. It gives the circuit shown in Figure 12.

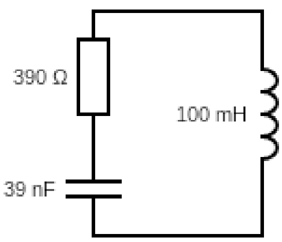

Figure 12: Circuit at $t = 0$

Note that the current can not change instantaneously, meaning that the current through the inductor will be $I = 12mA$. Thus, the current through the capacitor will be $I = 12mA$ at $t = 0$.
At $t = 0$, (28) gives the following form:

$$i(0) = exp(o) \times [C_1 \times 1 + C_2 \times 0] = C_1 = I = 12mA \quad \Rightarrow \quad C_1 = 12mA \tag{32}$$

Also, note that the voltage across the inductor was 0 initially. Since the voltage can not change instantaneously as well, at $t = 0$, the voltage across the inductor still will be 0.

$$v_L = 0 = L\frac{di(0)}{dt} \quad \Rightarrow \quad \frac{di(0)}{dt} = 0 \tag{33}$$

Let's take a derivative of $i(t)$ from (28):

$$\begin{aligned} \frac{di(t)}{dt} &= -\zeta\omega_n exp(-\zeta\omega_n t) \times [C_1 cos(\omega_d t) + C_2 sin(\omega_d t)]+ \\ &exp(-\zeta\omega_n t) \times [C_1\omega_d sin(\omega_d t) + C_2\omega_d cos(\omega\omega_d t)] \end{aligned} \tag{34}$$

At $t = 0$, this equation becomes:

$$\frac{di(0)}{dt} = -\zeta\omega_n C_1 + C_2\omega_d = 0 \quad \Rightarrow \tag{35}$$

$$C_2 = \frac{\zeta\omega_n C_1}{\omega_d} = C_1\frac{\zeta}{\sqrt{1-\zeta^2}} = 12\frac{121.777 \times 10^-3}{\sqrt{1 - (121.777 \times 10^-3)^2}} = 1.472mA \tag{36}$$

Finally, the following equation for the current is obtained:

$$i(t) = exp(-1935.524t) \times [12cos(258.090 \times 10^-3t) + 1.472sin(258.090 \times 10^-3t)] \tag{37}$$

2. The current equation was written in Matlab with corresponding variables. Figure 13 shows the results obtained from Matlab.

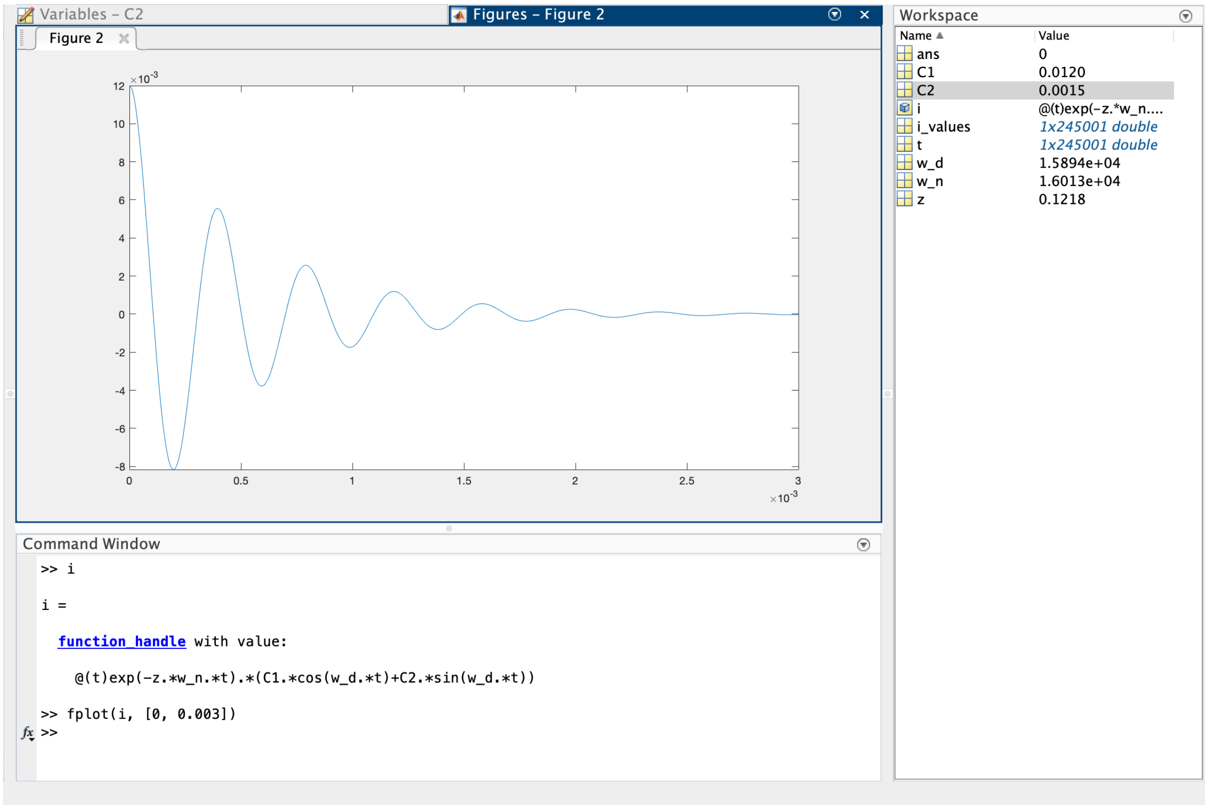

Figure 13: Matlab code

4 Conclusion

During the experiment, the transient response of an RLC-circuit with square wave input was analyzed. All 3 types of transient response (under damped, critically damped, and over damped) were visualized with oscilloscope experimentally. Then, in the evaluation part, these graphs were confirmed with Matlab. In the evaluation part, the voltage across the capacitor of the first circuit and current across the capacitor of the second circuit were described. Also, in the evaluation part, difference between theoretically and experimentally obtained numbers were described.

Overall, experiment was successfully conducted. All numbers and graphs were as expected. Two homogeneous equations were solved by using the initial conditions.

5 Prelab RLC Circuits - Frequency response

5.1 Question 1

Figure 14 shows the code used in this part and Figure 15 shows the graph obtained from Matlab.

Prelab_1.m

```
%% Variables
s = tf('s');
R = 390;
L = 10 * 10^-3;
C = 270 * 10^-9;
Z_L = L*s;
Z_C = 1/(s*C);
%% Bode magnitude
%defining denominator
D = R + Z_L + Z_C;

%Bode magnitude across the resistor
H_R = R/D;
bodemag(H_R, {100, 10^5});
grid on;
hold on;

%Bode magnitude across the capacitor
H_L = Z_L/D;
bodemag(H_L, {100, 10^5});
hold on;

%Bode magnitude across the inductor
H_C = Z_C/D;
bodemag(H_C, {100, 10^5});
hold on;

%Bode magnitude across the inductor
H_LC = (Z_C+Z_L)/D;
bodemag(H_LC, {100, 10^5});
```

Figure 14: Matlab code

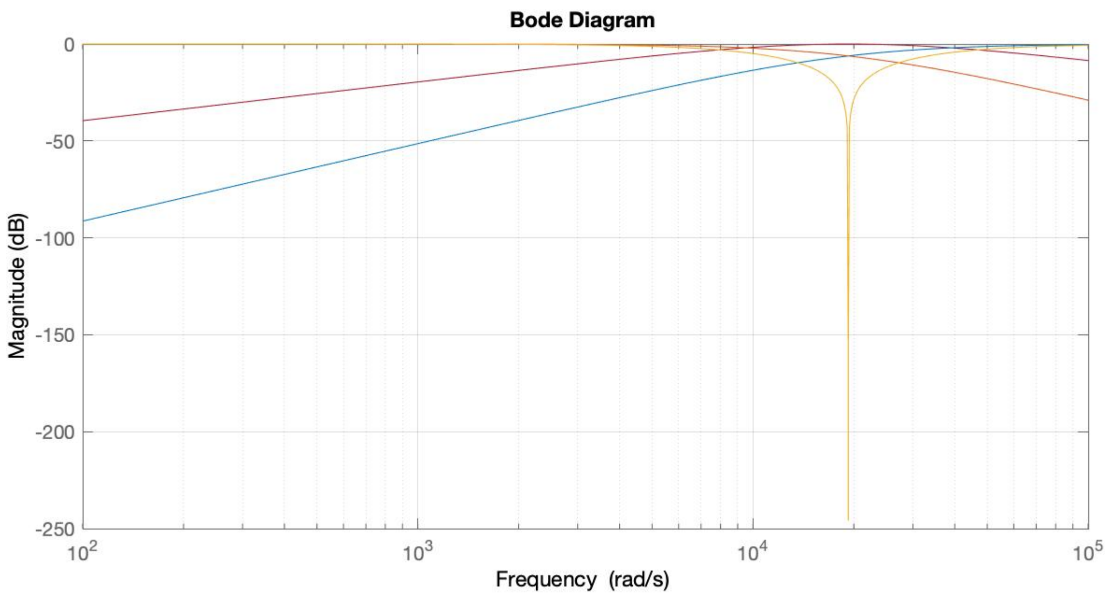

Figure 15: Matlab Graph

5.2 Question 2

To extract the bandwidth, the graph was zoomed to get more accurate numbers as shown in Figure 16.

$$f_1 = 7.91 \times 10^3 \qquad f_2 = 4.68 \times 10^4 \quad \Rightarrow \quad Bandwidth = f_2 - f_1 = 3.998 \times 10^4 \tag{38}$$

$$f_0 = \frac{f_1 + f_2}{2} \quad \Rightarrow \quad Q = \frac{f_0}{f_2 - f_1} = \frac{f_1 + f_2}{2(f_2 - f_1)} = 0.7034 \tag{39}$$

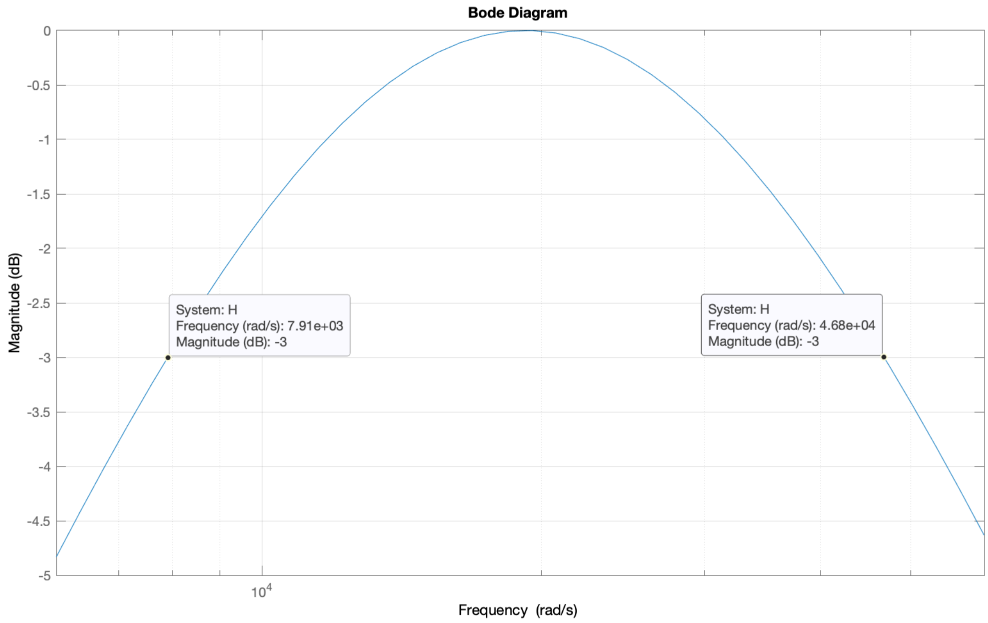

Figure 16: Matlab Graph

5.3 Question 3

Filter characteristic:

- Resistor: Band-pass filter
- Capacitor: Low-pass filter
- Inductor: High-pass filter
- Capacitor and Inductor: Notch filter

6 References

References

[1] CO-520-B Signals and Systems Lab
Instructors: Uwe Pagel and Mojtaba Joodaki.
Fall Semester, 2021

[2] Inductor in a DC circuit
`http://www.cmm.gov.mo/eng/exhibition/secondfloor/MoreInfo/2_3_6_ResistanceInductance.html`
Link

[3] How capacitor works with DC
`https://binaryupdates.com/how-capacitor-works-with-dc/`
Link