Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

The Revised Simplex Method

Hervé Kerivin, PhD

Université Clermont Auvergne
Institut d'Informatique
LIMOS UMR 6158 CNRS
Clermont-Ferrand

Office: ISIMA B133

kerivin@isima.fr

Motivation

Standard Simplex Method

Implementation of the simplex method that updates a dictionary in each iteration

When a basic solution is represented by a dictionary, the new basic solution, found by an iteration of the simplex method, is easy to find and only a small part of the dictionary is used.

Motivation

Standard Simplex Method

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When a basic solution is represented by a dictionary, the new basic solution, found by an iteration of the simplex method, is easy to find and only a small part of the dictionary is used.

The used part of the dictionary may be reconstructed from the original data

Revised Simplex Method

Implementation of the simplex method that works with the original data, not the transformed data (i.e., not with dictionaries or tableaux)

Example

$$
\begin{array}{lrcrcrcr}
\text{maximize } z = & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Slack variables: $x_4, x_5, x_6 \geq 0$

$$
\begin{array}{lllllll}
c_1 = 5 & c_2 = 4 & c_3 = 3 & c_4 = 0 & c_5 = 0 & c_6 = 0 & \\
a_{11} = 2 & a_{12} = 3 & a_{13} = 1 & a_{14} = 1 & a_{15} = 0 & a_{16} = 0 & b_1 = 5 \\
a_{21} = 4 & a_{22} = 1 & a_{23} = 2 & a_{24} = 0 & a_{25} = 1 & a_{26} = 0 & b_2 = 11 \\
a_{31} = 3 & a_{22} = 4 & a_{33} = 2 & a_{34} = 0 & a_{35} = 0 & a_{36} = 1 & b_3 = 8
\end{array}
$$

Motivation (cont'd)

Each iteration of the revised simplex method

- requires solving two systems of linear inequalities (not from scratch)
- may or may not take less time than the corresponding iteration of the standard simplex method

On the typical large and sparse LP problems solved in applications, the revised simplex method works faster than the standard simplex method

Modern computer programs for solving LP problems always use some form of the revised simplex method

Back to Our Example

We record the first three rows of the original dictionary as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 11 \\ 8 \end{bmatrix}$$

We record the last row (i.e., the z-row) of the original dictionary as

$$z = \mathbf{c}^T\mathbf{x}$$

where

$$\mathbf{c} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Second Dictionary

$$
\begin{array}{rcrcrcrcr}
x_1 & = & \frac{5}{2} & - & \frac{3}{2}x_2 & - & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 \\
x_5 & = & 1 & + & 5x_2 & & & + & 2x_4 \\
x_6 & = & \frac{1}{2} & + & \frac{1}{2}x_2 & - & \frac{1}{2}x_3 & + & \frac{3}{2}x_4 \\
\hline
z & = & \frac{25}{2} & - & \frac{7}{2}x_2 & + & \frac{1}{2}x_3 & - & \frac{5}{2}x_4
\end{array}
$$

Since x_1, x_5, x_6 are the basic variables, we write $A\mathbf{x} = \mathbf{b}$ as

$$A_B\mathbf{x}_B = \mathbf{b} - A_N\mathbf{x}_N \tag{1}$$

where

$$
A_B = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_5 \\ x_6 \end{bmatrix}
$$

$$
A_N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

Nonsingular Matrix

Nonsingular matrix

A square matrix A is called nonsingular if for every right-hand side $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has precisely one solution

We know that

- A matrix has an inverse if and only if it is nonsingular
- The inverse of a matrix, if it exists, is unique

The square matrix A_B is nonsingular, that is, A_B has an inverse A_B^{-1}

Multiplying both sides of (1) by A_B^{-1} on the left gives

$$\mathbf{x}_B = A_B^{-1}\mathbf{b} - A_B^{-1}A_N\mathbf{x}_N \quad (2)$$

Second Dictionary (cont'd)

Writing $z = \mathbf{c}^T\mathbf{x}$ as

$$z = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N$$

where

$$\mathbf{c}_B = \begin{bmatrix}5\\0\\0\end{bmatrix} \quad \mathbf{c}_N = \begin{bmatrix}4\\3\\0\end{bmatrix}$$

and substituting $\mathbf{x}_B$ from (2), we obtain

$$z = \mathbf{c}_B^T A_B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N)\mathbf{x}_N$$

The second dictionary can be recorded in matrix term as

$$\begin{array}{rcl} \mathbf{x}_B = A_B^{-1}\mathbf{b} & - & A_B^{-1}A_N\mathbf{x}_N \\ \hline z = \mathbf{c}_B^T A_B^{-1}\mathbf{b} & + & (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N)\mathbf{x}_N \end{array}$$

Third Dictionary

$$
\begin{array}{rcrcrcrcr}
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6
\end{array}
$$

Since x_1, x_3, x_5 are the basic variables, we have

$$
A_B = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} \quad \mathbf{c}_B = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}
$$

$$
A_N = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} \quad \mathbf{c}_N = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}
$$

The third dictionary can be recorded in matrix term as

$$
\begin{array}{rcl}
\mathbf{x}_B = A_B^{-1}\mathbf{b} & - & A_B^{-1}A_N\mathbf{x}_N \\
\hline
z = \mathbf{c}_B^T A_B^{-1}\mathbf{b} & + & (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N)\mathbf{x}_N
\end{array}
$$

General Case

LP problem in the standard form

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & \text{for } i = 1, 2 \ldots, m \\ & x_j \ge 0 & \text{for } j = 1, 2 \ldots, n \end{array}$$

After the introduction of the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m} \ge 0$, this problem can be recorded as

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

General Case (cont'd)

Matrix A has m rows and $n + m$ columns; the last m columns form the identity matrix

The column vector $\mathbf{x}$ has $n + m$ components

The column vector $\mathbf{b}$ has m components

The column vector $\mathbf{c}$ has $n + m$ components; the last m components are zeros

Each basic feasible solution partitions

- $\mathbf{x}$ into m basic variables (i.e., $\mathbf{x}_B$) and n nonbasic variables (i.e., $\mathbf{x}_N$)
- A into A_B and A_N
- $\mathbf{c}$ into $\mathbf{c}_B$ and $\mathbf{c}_N$

Basis Matrix

Basis matrix

A basis matrix B of A is a square submatrix of m linearly independent columns of A

A basis matrix is nonsingular

A_B is a basis matrix; it is customary to denote the basis matrix by B rather than A_B

Dictionary in matrix term

$$\begin{array}{rcl} \mathbf{x}_B = B^{-1}\mathbf{b} & - & B^{-1}A_N\mathbf{x}_N \\ \hline z = \mathbf{c}_B^T B^{-1}\mathbf{b} & + & (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N)\mathbf{x}_N \end{array}$$

First Example

LP problem

$$\begin{array}{lrcrcrcrcl} \text{maximize } z = & 19x_1 & + & 13x_2 & + & 12x_3 & + & 17x_4 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & \leq & 225 \\ & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 117 \\ & 4x_1 & + & 3x_2 & + & 3x_3 & + & 4x_4 & \leq & 420 \\ & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Feasible solution

$$x_1 = 54, \quad x_2 = 0, \quad x_3 = 63, \quad x_4 = 0$$

Matrix Notation

Slack variables: $x_5, x_6, x_7 \geq 0$

$$A = \begin{bmatrix} 3 & 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 225 \\ 117 \\ 420 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 19 \\ 13 \\ 12 \\ 17 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Dictionary

Dictionary associated with the given solution

$$\mathbf{x}_B = B^{-1}\mathbf{b} \quad - \quad B^{-1}A_N\mathbf{x}_N$$

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} \quad + \quad (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}A_N)\mathbf{x}_N$$

where

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_7 \end{bmatrix} \quad \mathbf{c}_B = \begin{bmatrix} 19 \\ 12 \\ 0 \end{bmatrix}$$

$$A_N = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 4 & 0 & 0 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \mathbf{c}_N = \begin{bmatrix} 13 \\ 17 \\ 0 \\ 0 \end{bmatrix}$$

Let

$$\bar{\mathbf{x}}_B = B^{-1}\mathbf{b} = \begin{bmatrix} 54 \\ 63 \\ 15 \end{bmatrix}$$

Inverse of a Matrix?

One might be misled into the belief that the inverse B^{-1} is a handy device for solving systems $B\mathbf{x} = \mathbf{b}$

Gaussian elimination

- P_i permutation matrix ($i = 1, 2, \ldots, m$)
- L_i lower triangular eta matrix ($i = 1, 2, \ldots, m$)
- $U = L_n P_n L_{n-1} P_{n-1} \ldots L_1 P_1 B$ upper triangular matrix with all-one diagonal

The matrices $P_1, L_1, \ldots, P_n, L_n$ and U are referred to as a triangular factorisation of B

There are several good reasons for using a triangular factorization of B instead

- computing B^{-1} takes considerably longer than computing the triangular factorization
- since extra calculations are involved in computing B^{-1}, the results are more likely to suffer from rounding errors
- even very sparse matrices tend to have dense inverses

Entering Variable

We need to compute

$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N$$

to decide

- if the basic feasible solution is optimal
- if not, to find an entering variable

Entering Variable

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$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N$$

to decide

- if the basic feasible solution is optimal
- if not, to find an entering variable

It is done in two steps

1. first, we find $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ by solving the system

$$\mathbf{y}^T B = \mathbf{c}_B^T$$

2. then, calculate

$$\mathbf{c}_N^T - \mathbf{y}^T A_N$$

Our Example

1. Solve the system

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 12 & 0 \end{bmatrix}$$

we find

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{17}{2} & 0 \end{bmatrix}$$

Our Example

1. Solve the system

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 12 & 0 \end{bmatrix}$$

we find

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{17}{2} & 0 \end{bmatrix}$$

2. Calculate

$$\begin{bmatrix} 13 & 17 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{7}{2} & \frac{17}{2} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 4 & 0 & 0 \end{bmatrix}$$

we find

$$\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & -\frac{7}{2} & -\frac{17}{2} \end{bmatrix}$$

Our Example

1. Solve the system

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 12 & 0 \end{bmatrix}$$

   we find

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2. Calculate

$$\begin{bmatrix} 13 & 17 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{7}{2} & \frac{17}{2} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 4 & 0 & 0 \end{bmatrix}$$

   we find

$$\begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & -\frac{7}{2} & -\frac{17}{2} \end{bmatrix}$$

Second component of $\mathbf{x}_N$ (i.e., x_4) enters the basis

Entering Variable (cont'd)

The components of $\mathbf{c}_N^T - \mathbf{y}^T A_N$ may be calculated individually

- a nonbasic variable x_j corresponds to
  - component c_j of $\mathbf{c}_N$
  - column $\mathbf{a}_j$ of A_N
- the corresponding component of $\mathbf{c}_N^T - \mathbf{y}^T A_N$ equals $c_j - \mathbf{y}^T \mathbf{a}_j$

If all the components of $\mathbf{c}_N^T - \mathbf{y}^T A_N$ are nonpositive, then the basic feasible solution is optimal

The entering variable may be any nonbasic variable x_j for which $\mathbf{y}^T \mathbf{a}_j < c_j$

Entering column

The column of A_N corresponding to the entering variable is called the *entering column*

Leaving Variable

Top m rows of a dictionary read

$$\mathbf{x}_B = \bar{\mathbf{x}}_B - B^{-1} A_N \mathbf{x}_N$$

Since all the nonbasic variables but the entering variable x_j remain at zero, $\mathbf{x}_B$ changes from

$$\bar{\mathbf{x}}_B \quad \text{to} \quad \bar{\mathbf{x}}_B - t(B^{-1}\mathbf{a}_j)$$

where

- t is the value of the entering variable
- $\mathbf{a}_j$ is the entering column

Two steps

1. first, we find $\mathbf{d} = B^{-1}\mathbf{a}_j$ by solving the system
$$B\mathbf{d} = \mathbf{a}_j$$
2. then, calculate the largest t so that
$$\bar{\mathbf{x}}_B - t\mathbf{d} \geq \mathbf{0}$$

Our Example

1. Solve the system

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

we find

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Our Example

1. Solve the system

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

we find

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

2. Calculate for basic variable
   - x_1: $54 - \frac{1}{2}t \geq 0 \rightarrow t \leq 108$
   - x_3: $63 - \frac{1}{2}t \geq 0 \rightarrow t \leq 126$
   - x_7: $15 - \frac{1}{2}t \geq 0 \rightarrow t \leq 30$

Our Example

1. Solve the system

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

   we find

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

2. Calculate for basic variable
   - x_1: $54 - \frac{1}{2}t \geq 0 \rightarrow t \leq 108$
   - x_3: $63 - \frac{1}{2}t \geq 0 \rightarrow t \leq 126$
   - x_7: $15 - \frac{1}{2}t \geq 0 \rightarrow t \leq 30$

x_7 leaves the basis; $t = 30$

Leaving Variable (cont'd)

If all the components of **d** are nonpositive, then the LP problem is unbounded

The leaving variable is the basic variable which imposes the smallest upper bound on the value t taken by the entering variable

The component of $\bar{\mathbf{x}}_B - t\mathbf{d}$ corresponding to the entering variable equals zero

Leaving column

The column of B corresponding to the leaving variable is called the leaving column

Update the Basic Feasible Solution

Two steps

1. set the value of the entering variable at t
   our example: $x_4 = 30$
2. replace the values of the basic variables by

$$\overline{\mathbf{x}}_B - t\mathbf{d}$$

our example: $x_1 = 39$, $x_3 = 48$, $x_7 = 0$

New basis matrix

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \quad \overline{\mathbf{x}}_B = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix}$$

Basis Heading

The order of the columns of B is unimportant as long as it matches the order of the components of $\mathbf{x}_B$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \end{bmatrix} \quad \bar{\mathbf{x}}_B = \begin{bmatrix} 48 \\ 30 \\ 39 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 4 \end{bmatrix}$$

Basis heading

A ordered list of the basic variables that specifies the actual order of the m columns of B is called the basis heading

Convenient to replace the leaving variable by the entering variable in each update of the basis heading

Iteration of the Revised Simplex Method

Optimality? Entering variable?

1. Solve the system $\mathbf{y}^T B = \mathbf{c}_B^T$
2. Choose an entering column
   - any column $\mathbf{a}^T$ of A_N such that $\mathbf{y}^T\mathbf{a}$ is less than the corresponding component of $\mathbf{c}_N$
   - if there is no such column, then the current solution is optimal

Unbounded? Leaving variable?

3. Solve the system $B\mathbf{d} = \mathbf{a}$
4. Find the largest t such that $\bar{\mathbf{x}}_B - t\mathbf{d} \geq \mathbf{0}$
   - if there is no such t, then the LP problem is unbounded
   - otherwise, at least one component of $\bar{\mathbf{x}}_B - t\mathbf{d}$ equals zero and the corresponding variable is leaving the basis

Update of the basis

5.
   - Set the value of the entering variable at t
   - Replace the values of the basic variables by $\bar{\mathbf{x}}_B - t\mathbf{d}$
   - Replace the leaving column of B by the entering column
   - In the basis heading, replace the leaving variable by the entering variable

Example

A workshop manufactures four kinds of furniture: bookcases, desks, chairs and bedframes.

- A bookcase takes three hours of labor, one unit of metal and four units of wood.
- A desk takes two hours of labor, one unit of metal and three units of wood.
- A chair takes one hour of labor, one unit of metal and three units of wood.
- A bedframe takes two hours of labor, one unit of metal and four units of wood.

There are 420 units of wood, 117 units of metal and 225 hours of labor available per day. Knowing that a bookcase brings a net profit of \$19, a desk brings a net profit of \$13, a chair brings a net profit of \$12 and a bedframe brings a net profit of \$17, how is the workshop going to maximize its daily net profit?

LP Formulation

LP problem

$$
\begin{array}{lrcrcrcrcl}
\text{maximize } z = & 19x_1 & + & 13x_2 & + & 12x_3 & + & 17x_4 & & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & \leq & 225 \\
 & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 117 \\
 & 4x_1 & + & 3x_2 & + & 3x_3 & + & 4x_4 & \leq & 420 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

where

- x_1 is the number of bookcases manufactured during the day
- x_2 is the number of desks manufactured during the day
- x_3 is the number of chairs manufactured during the day
- x_4 is the number of bedframes manufactured during the day

Economic Significance

$$
\begin{array}{lll}
\text{maximize} & z = \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\
& x_j \geq 0 & \text{for } j = 1, 2 \ldots, n
\end{array}
$$

where

- x_j measures the level of the output of the jth product
- b_i specifies the available amount of the ith resource

$$\sum_{j=1}^{n} a_{ij} x_{j}\text{(units of product } j\text{)} \leq b_{i}\text{(units of resource } i\text{)}$$

- a_{ij} expressed in units of resource i per unit of product j
- c_j expressed in dollars per unit of product j

Economic Significance of the Dual Variables

Dual problem

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & \text{for } j = 1, 2 \ldots, n \\ & y_i \geq 0 & \text{for } i = 1, 2 \ldots, m \end{array}$$

Variables $y_1, y_2, \ldots, y_m$ can be given a meaningful interpretation

Let $x_1^*, x_2^*, \ldots, x_n^*$ and $y_1^*, y_2^*, \ldots, y_m^*$ be optimal primal and dual solutions. At optimum

$$\sum_{j=1}^{n} c_j (\text{dollars per unit of product } j)\; x_j^* (\text{units of product } j)$$

$$=$$

$$\sum_{i=1}^{m} b_i (\text{units of resource } i)\; y_i^*$$

y_i^* represents a value in dollars per unit of resource i

Shadow Price

Shadow price

For $i = 1, 2, \ldots, m$, dual variable y_i^* is called the shadow price of the ith resource; it is also called the marginal value of the ith resource

Change in profit (i.e., z) per unit of change in resource i

$$\frac{\partial z}{\partial b_i} = y_i^*$$

If the optimal solution is not degenerate, with each extra unit of resource i, the profit would increase by y_i^* which specifies the maximum amount the firm should be willing to pay, over and above the trading price, for each extra unit of resource i

Back to Our Example

The program of making 54 bookcases and 63 chairs per day has been proposed to the company. Is this program optimal?

Dual problem

$$
\begin{array}{lrcrcrcl}
\min & 225y_1 & + & 117y_2 & + & 420y_3 & & \\
\text{subject to} & 3y_1 & + & y_2 & + & 4y_3 & \geq & 19 \\
 & 2y_1 & + & y_2 & + & 3y_3 & \geq & 13 \\
 & y_1 & + & y_2 & + & 3y_3 & \geq & 12 \\
 & 2y_1 & + & y_2 & + & 4y_3 & \geq & 17 \\
 & & & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

Complementary Slackness Theorem

Feasible solution **x** is optimal if and only if there are numbers $y_1, y_2, \ldots, y_m$ that satisfy

1. the system of equations

$$\begin{array}{rcrcrcl} 3y_1 & + & y_2 & + & 4y_3 & = & 19 \\ y_1 & + & y_2 & + & 3y_3 & = & 12 \\ & & & & y_3 & = & 0 \end{array}$$

2. the system of inequalities

$$\begin{array}{rcrcrcl} 2y_1 & + & y_2 & + & 3y_3 & \geq & 13 \\ 2y_1 & + & y_2 & + & 4y_3 & \geq & 17 \\ y_1 & & & & & \geq & 0 \\ & & y_2 & & & \geq & 0 \end{array}$$

Complementary Slackness Theorem (cont'd)

Nondegenerate case

If the given primal solution $\mathbf{x}$ is nondegenerate, then

1. the system of equations featured in the complementary slackness theorem is nothing but the system $\mathbf{y}^T B = \mathbf{c}_B^T$ considered in Step 1 of an iteration of the revised simplex method
2. the system of inequalities featured in the complementary slackness theorem is nothing but the system $\mathbf{c}_N^T - \mathbf{y}^T A_N \leq \mathbf{0}$ considered in Step 2 of an iteration of the revised simplex method

Degenerate case

If the given primal solution $\mathbf{x}$ is degenerate, then the system of equations featured in the complementary slackness theorem forms a proper subsystem of $\mathbf{y}^T B = \mathbf{c}_B^T$ considered in Step 1 of an iteration of the revised simplex method

Steps 1 and 2 check if $\mathbf{y}$ corresponds to a dual solution

Economic Interpretation

1. Step 1 (for our example)

$$y_1 = \frac{7}{2} \quad y_2 = \frac{17}{2} \quad y_3 = 0$$

Temporary shadow prices

Solving $\mathbf{y}^T B = \mathbf{c}_B^T$ may be interpreted as assigning temporary shadow prices to the resources so that the total value of the resources consumed by each of the basic activities matches the profit returned by this activity

- time: $\$\frac{7}{2}$/hour
- metal: $\$\frac{17}{2}$/unit
- wood: \$0/unit
- basic activities
  - making bookcases (i.e., decision variable x_1)
  - making chairs (i.e., decision variable x_3)
  - leaving wood unused (i.e., slack variable x_7 associated with the primal constraint related to wood)

Economic Interpretation (cont'd)

2. Step 2 (for our example)

$$c_N - yA_N = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & -\frac{7}{2} & -\frac{17}{2} \end{bmatrix}$$

Pricing out

Evaluating $\mathbf{y}^T A_N$ may be interpreted as finding the total value of the resources consumed by each of the nonbasic activities; this operation is sometimes referred to as pricing out the nonbasic activities

Nonbasic activities

- making desks (i.e., decision variable x_2)
- making bedframes (i.e., decision variable x_4)
- leaving working time unused (i.e., slack variable x_5 associated with the primal constraint related to labor)
- leaving metal unused (i.e., slack variable x_6 associated with the primal constraint related to metal)

Entering variable: nonbasic activity which does pay back more than it consumes

Optimal Primal Solution

If none of the nonbasic activities pays back more than it consumes (i.e., $\mathbf{y}^T A_N \geq \mathbf{c}_N^T$), then the current primal solution is optimal

Remark:

- If the primal solution is nondegenerate, then the converse of this implication is guaranteed by the complementary slackness theorem
- If the primal solution is degenerate, then it may be optimal even if some of the inequalities in $\mathbf{y}^T A_N \geq \mathbf{c}_N^T$ are violated

Mix (Step 3)

In Steps 3, 4 and 5, the revised simplex method attempts to construct an improved program by substituting making bedframes (i.e., entering variable) for a suitable mix of

- making bookcases (i.e., x_1)
- making chairs (i.e., x_3)
- leaving wood unused (i.e., x_7)

(i.e., basic variables)

Mix

The mix, which finds the number d_i units of each basic activity i per unit of the entering activity, must consume resources at the same rate as the entering activity itself; the mix is exactly the solution of the system $B\mathbf{d} = \mathbf{a}$, where $\mathbf{a}$ is the entering column, in Step 3 of an iteration of the revised simplex method

Our Example

Let

- d_1: concentration of making bookcases (i.e., x_1)
- d_2: concentration of making chairs (i.e., x_3)
- d_7: concentration of leaving wood unused (i.e., x_7)

in the mix

$$\begin{array}{rcrcrcl} 3d_1 & + & d_3 & & & = & 2 \\ d_1 & + & d_3 & & & = & 1 \\ 4d_1 & + & 3d_3 & + & d_7 & = & 4 \end{array}$$

Solution $d_1 = \frac{1}{2}$, $d_3 = \frac{1}{2}$, $d_7 = \frac{1}{2}$

Each bedframe will be substituted for

- half a bookcase, plus
- half a chair, plus
- half a unit of unused wood

Substitution

Substitution

The operation consisting of substituting the profitable entering activity for a suitable mix of the basic activities is called a substitution

A substitution raises the objective function value by

$$c_j - \mathbf{y}^T \mathbf{a} > 0$$

per unit of entering activity (a is the entering column)

The largest value t the entering activity can take is given by a substitution which does not consume more resources than available, that is,

$$\max\{t : \overline{\mathbf{x}}_B - t\mathbf{d} \geq \mathbf{0}\}$$

(Step 4 of an iteration of the revised simplex method)

Interpretation of Reduced Costs

Let x_j be a nonbasic variables

Increasing x_j by one unit would make $\mathbf{x}_B$ change by $-B^{-1}\mathbf{a}_j$

Increasing x_j by one unit would make z

- decrease by $\mathbf{c}_B^T B^{-1}\mathbf{a}_j$ for the basic variables
- increase by c_j

The net change of z is

$$c_j - \mathbf{c}_B^T B^{-1}\mathbf{a}_j$$

that is, the reduced cost of x_j

Changing the Right-Hand Sides

Consider the following LP in the standard form

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad && \text{for } i = 1, 2 \ldots, m \\ & x_j \geq 0 && \text{for } j = 1, 2 \ldots, n \end{aligned} \tag{3}$$

Impact of (small) changes of the right-hand sides on the optimal solution z^*?

Theorem

Theorem

If this LP problem has at least one nondegenerate basic optimal solution, then there exists a positive ϵ with the following property: If $|t_i| \leq \epsilon$ for all $i = 1, 2, \ldots, m$, then the problem

$$\begin{array}{lll} \text{maximize} & z = \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i & \text{for } i = 1, 2 \ldots, m \\ & x_j \geq 0 & \text{for } j = 1, 2 \ldots, n \end{array}$$

has an optimal solution and its optimal value equals

$$z^* + \sum_{i=1}^{m} y_i^* t_i$$

with z^* standing for the optimal value of (**??**) and with $y_1^*, y_2^*, \ldots, y_m^*$ standing for the optimal solution of its dual

An Example

A forester has 100 acres of hardwood timber. Felling the hardwood and letting the area regenerate would cost \$10 per acre in immediate resources and bring a subsequent return of \$50 per acre. An alternative course of action is to fell the hardwood and plant the area with pine; that would cost \$50 per acre with a subsequent return of \$120 per acre. Only \$4,000 is available to meet the immediate costs.

1. Should the forester borrow \$100 now and pay back \$180 later?
2. Should the forester invest \$100 now and collect \$180 later?

LP Formulation

Let

- x_1 = number of acres for option 1 (fell and let area regenerate)
- x_2 = number of acres for option 2 (fell and plant with pine)

$$
\begin{array}{lrcrcl}
\text{maximize } z = & 40x_1 & + & 70x_2 & & \\
\text{subject to} & x_1 & + & x_2 & \leq & 100 \\
 & 10x_1 & + & 50x_2 & \leq & 4,000 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Its optimal solution is $x_1^* = 25$ and $x_2^* = 75$

Efficicent Implementation

Efficiency of the revised simplex method hinges on the ease of implementing steps 1 and 3

The systems $\mathbf{y}^T B = \mathbf{c}_B^T$ and $B\mathbf{d} = \mathbf{a}$ are not solved from scratch: some device is used to facilitate their solutions and is updated at the end of each iteration

- eta factorization of the basis (the simplest)
- product form of the inverse (popular)
- more complicated devices

Eta Matrix

Eta matrix

An eta matrix differs from the identity matrix in only one column, referred to as its eta column

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \\ 0 & 7 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \end{bmatrix}$$

From one Basis to the Next One

Let B_{k-1} denote the basis matrix obtained after $k-1$ iterations of the simplex method

$$B_{k-1} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \quad \bar{\mathbf{x}}_{B_{k-1}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_3 \\ \bar{x}_7 \end{bmatrix} = \begin{bmatrix} 54 \\ 63 \\ 15 \end{bmatrix}$$

Let B_k denote the basis matrix obtained after k iterations of the simplex method

$$B_k = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \quad \bar{\mathbf{x}}_{B_k} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix}$$

Each B_k differs from the preceding B_{k-1} in only one column

Eta Column

Suppose that B_{k-1} and B_k differs in the pth column;
our example: $p = 3$

The pth column of B_k corresponds to the entering column **a** selected in step 2;
our example: $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

The eta matrix

$$E_k = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

is so that $B_k = B_{k-1}E_k$

The eta column of E_k (that is, its pth column) is nothing but the vector **d** obtained in step 3 (i.e., $B_{k-1}\mathbf{d} = \mathbf{a}$)

Eta Factorization

Assume that the initial basis consists of the slack variables

Initial basis matrix: $B_0 = I_m$

Using eta matrices as previously described, we have

- $B_1 = E_1$
- $B_2 = E_1 E_2$
- $B_3 = E_1 E_2 E_3$
- $\vdots$
- $B_k = E_1 E_2 \dots E_k$ is the eta factorization of B_k

Steps 1 and 3

Step 1: the system $\mathbf{y}^T B_k = \mathbf{c}_B^T$ can be seen as

$$(((\mathbf{y}^T E_1)E_2)\dots)E_k = \mathbf{c}_B^T$$

and solved by solving the sequence of systems

$$\mathbf{u}_{k-1}^T E_k = \mathbf{c}_B^T \qquad \mathbf{u}_{k-2}^T E_{k-1} = \mathbf{u}_{k-1}^T \quad \dots \quad \mathbf{u}_1^T E_2 = \mathbf{u}_2^T \qquad \mathbf{y}^T E_1 = \mathbf{u}_1^T$$

Steps 1 and 3

Step 1: the system $\mathbf{y}^T B_k = \mathbf{c}_B^T$ can be seen as

$$(((\mathbf{y}^T E_1)E_2)\ldots)E_k = \mathbf{c}_B^T$$

and solved by solving the sequence of systems

$$\mathbf{u}_{k-1}^T E_k = \mathbf{c}_B^T \qquad \mathbf{u}_{k-2}^T E_{k-1} = \mathbf{u}_{k-1}^T \quad \ldots \quad \mathbf{u}_1^T E_2 = \mathbf{u}_2^T \qquad \mathbf{y}^T E_1 = \mathbf{u}_1^T$$

Step 3: the system $B_k\mathbf{d} = \mathbf{a}$ can be seen as

$$E_1(E_2(\ldots(E_3\mathbf{d}))) = \mathbf{a}$$

and solved by solving the sequence of systems

$$E_1\mathbf{v}_1 = \mathbf{a} \qquad E_2\mathbf{v}_2 = \mathbf{v}_1 \quad \ldots \quad E_{k-1}\mathbf{v}_{k-1} = \mathbf{v}_{k-2} \qquad E_k\mathbf{d} = \mathbf{v}_{k-1}$$

Since the E_j's are eta matrices, the $2k$ systems are very simple to solve

Example

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

with

$$A = \begin{bmatrix} 3 & 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 225 \\ 117 \\ 420 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 19 \\ 13 \\ 12 \\ 17 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

First Iteration

Basis matrix: $B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Steps 1 and 2: Entering variable: x_3; entering column: $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Steps 3: Eta column: $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Step 4: $t = 117$; leaving variable: x_6

Second Iteration

Basis matrix: $B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ $(= E_1)$

Steps ! and 2: Entering variable: x_1; entering column: $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Step 3: Eta column: $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Step 4: $t = 54$; leaving variable: x_5

Third Iteration

Basis matrix: $B_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ $(= E_1 E_2)$

Steps 1 and 2: Entering variable: x_4; entering column: $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

Step 3: Eta column: $\mathbf{d} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

Step 4: $t = 30$; leaving variable: x_7

Fourth Iteration

Basis matrix: $B_3 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$ $(= E_1 E_2 E_3)$

Steps 1 and 2: No entering variables: the current solution is optimal