Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

How Does the Simplex Method Work?

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History

- Linear programming is quite young
- It started in 1947 when G.B. Dantzig designed the “simplex method” for solving linear programming formulations of U.S. Air Force planning problems

Example

Consider the following LP

$$\begin{array}{lrcrcr} \text{maximize } z = & -x_1 & - & 2x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 3 \\ & 3x_1 & - & x_2 & \leq & 5 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

Obvious feasible solution: $x_1 = 0$, $x_2 = 0$

This solution is optimal: increasing either x_1 or x_2 would decrease the value of z

Slack

Consider the inequality $x_1 + x_2 \leq 3$

For every feasible solution x_1, x_2, the left-hand side of this constraint is at most the value of the right-hand side; often, there may be a slack between the two values

Let x_3 denote the slack, that is, $x_3 = 3 - x_1 - x_2$

With this notation, inequality $x_1 + x_2 \leq 3$ may now be written as $x_3 \geq 0$

Slack Variables

Slack variables and system of linear equations

Transform the constraints of an LP problem (written in a standard form) into a system of linear equations by adding slack variables

$$\left\{\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & +\ldots+ & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\ldots+ & a_{2n}x_n & \leq & b_2 \\ \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & +\ldots+ & a_{mn}x_n & \leq & b_m \\ & & & & x & \geq & 0 \end{array}\right. \tag{1}$$

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & +\ldots+ & a_{1n}x_n & + & x_{n+1} & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\ldots+ & a_{2n}x_n & + & x_{n+2} & = & b_2 \\ \vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & +\ldots+ & a_{mn}x_n & + & x_{n+m} & = & b_m \\ & & & & & & x & \geq & 0 \end{array}\right. \tag{2}$$

Remark: the slack variables are nonnegative (i.e., $x_{n+1} \geq 0$, $x_{n+2} \geq 0$, ..., $x_{n+m} \geq 0$)

A First Example

$$
\left\{\begin{array}{lrcrcrcl}
\text{maximize } z = & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
& x_1 & & & & & \geq & 0 \\
& & & x_2 & & & \geq & 0 \\
& & & & & x_3 & \geq & 0
\end{array}\right. \tag{3}
$$

$x_1 = 0, x_2 = 0, x_3 = 0$ is feasible for (3)
$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0$ is feasible for (3)

A First Example

$$\left\{\begin{array}{lrcrcrcl} \text{maximize } z = & 5x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\ & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\ & x_1 & & & & & \geq & 0 \\ & & & x_2 & & & \geq & 0 \\ & & & & & x_3 & \geq & 0 \end{array}\right. \tag{3}$$

$x_1 = 0, x_2 = 0, x_3 = 0$ is feasible for (3)
$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0$ is feasible for (3)

$$\left\{\begin{array}{lrcrcrcrcr} \text{maximize} & z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 \\ \text{subject to} & x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ & x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ & x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ & & & & & \multicolumn{5}{r}{x_1, x_2, x_3, x_3, x_4, x_5, x_6 \geq 0} \end{array}\right. \tag{4}$$

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8$ is feasible for (4)
$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2}$ is feasible for (4)

Equivalence Result - Extension

Extension

Every feasible solution $x_1, x_2, \ldots, x_n$ of the LP problem

$$\max\{z = cx : x \text{ satisfies } (1)\}$$

can be extended in the unique way determined by

$$\left\{\begin{array}{rclclclcl} x_{n+1} & = & b_1 & - & a_{11}x_1 & - & a_{12}x_2 & - \ldots - & a_{1n}x_n \\ x_{n+2} & = & b_2 & - & a_{21}x_1 & - & a_{22}x_2 & - \ldots - & a_{2n}x_n \\ \vdots & & & & & & & & \\ x_{n+m} & = & b_m & - & a_{m1}x_1 & - & a_{m2}x_2 & - \ldots - & a_{mn}x_n \\ z & = & & & c_1x_1 & + & c_2x_2 & + \ldots + & c_nx_n \end{array}\right. \tag{5}$$

into a feasible solution $x_1, x_2, \ldots, x_{n+m}$ of

$$\max\{z : x_1, x_2, \ldots, x_{n+m} \geq 0\}$$

Equivalence Result - Restriction

$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$ is feasible for (4)

$x_1 = 2, x_2 = 0, x_3 = 1$ is feasible for (3)

Restriction

Every feasible solution $x_1, x_2, \ldots, x_{n+m}$ of

$$\max\{cx : x_1, x_2, \ldots, x_{n+m} \geq 0\}$$

can be restricted, simply by deleting the slack variables, into a feasible solution $x_1, x_2, \ldots, x_n$ of the LP problem

$$\max\{z = cx : x \text{ satisfies } (1)\}$$

Back to our example

(4) is equivalent to

$$\left\{\begin{array}{lcccccccc} \text{maximize} & z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 \\ \text{subject to} & x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ & x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ & x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\ & & & & & & & & & x_1, x_2, x_3, x_3, x_4, x_5, x_6 \geq 0 \end{array}\right. \qquad (6)$$

Back to our example

(4) is equivalent to

$$\left\{\begin{array}{lrcrcrcrcr} \text{maximize} & z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 \\ \text{subject to} & x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ & x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ & x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\ & & & & & \multicolumn{5}{r}{x_1, x_2, x_3, x_3, x_4, x_5, x_6 \geq 0} \end{array}\right. \qquad (6)$$

$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$ is optimal for (4); optimal value $z^* = 13$

$x_1 = 2, x_2 = 0, x_3 = 1$ is optimal for (3); optimal value $z^* = 13$

Equivalence Result - Optimality

Optimality

Every optimal solution to the LP problem

$$\max\{z = cx : x \text{ satisfies } (1)\}$$

determines an optimal solution to

$$\max\{cx : x_1, x_2, \ldots, x_{n+m} \geq 0\}$$

and conversely; the optimal values are equal

Back to our First Example Again

We are given the following LP problem

$$\left\{\begin{array}{lrcrcrcl}
\text{maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
& x_1 & & & & & \geq & 0 \\
& & & x_2 & & & \geq & 0 \\
& & & & & x_3 & \geq & 0
\end{array}\right.$$

Define

$$\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3
\end{array}$$

where x_4, x_5, and x_6 are the slack variables for the first, second, and third inequalities, respectively

Initial Dictionary

Given a LP written in standard form

$$
\begin{aligned}
\text{maximize } z &= \sum_{j=1}^{n} c_j x_j \\
\text{subject to } & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2 \ldots, m \\
& x_j \ge 0 \quad \text{for } j = 1, 2 \ldots, n
\end{aligned}
\tag{7}
$$

Introduce the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m} \ge 0$

Initial dictionary

$$
\begin{array}{rcccl}
x_{n+i} & = & b_i & - & \sum_{j=1}^{n} a_{ij} x_j \quad \text{for } i = 1, 2 \ldots, m \\
\hline
z & = & & & \sum_{j=1}^{n} c_j x_j
\end{array}
$$

Basic/Nonbasic variables

Basic/Nonbasic Variables

The equations of every dictionary must express m of the variables $x_1, x_2, \ldots, x_n$ (called the basic variables) and the objective function z in terms of the remaining n variables (called the nonbasic variables)

Example: In the previous dictionary

- x_4, x_5, x_6 are the basic variables
- x_1, x_2, x_3 are the nonbasic variables

Basis

The basic variables constitute a basis

Example: In the previous dictionary, the basis is $\{x_4, x_5, x_6\}$

Dictionaries

$$
\left\{
\begin{array}{lrcrcrcrcr}
\text{maximize} & z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 \\
\text{subject to} & x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
 & x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
 & x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
 & & & & & \multicolumn{5}{r}{x_1, x_2, x_3, x_3, x_4, x_5, x_6 \geq 0}
\end{array}
\right.
$$

Dictionary

Let B be the subscripts of the m basic variables of LP (7). The dictionary associated with B is

$$
\begin{array}{rclcll}
x_i & = & \bar{b}_i & - & \sum_{j \notin B} \bar{a}_{ij} x_j & \text{for all } i \in B \\
\hline
z & = & \bar{z} & + & \sum_{j \notin B} \bar{c}_j x_j &
\end{array}
$$

Example: $B = \{1, 3, 5\}$

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6
\end{array}
$$

Feasible Dictionary

In the previous dictionaries

- set the nonbasic variables at zero, that is,
  - $x_1 = x_2 = x_3 = 0$
  - $x_2 = x_4 = x_6 = 0$
- evaluate the basic variables, that is,
  - $x_4 = 5, x_5 = 11, x_6 = 8$
  - $x_1 = 2, x_3 = 1, x_5 = 1$
- the basic variables are nonnegative, the dictionaries are feasible

Feasible dictionary

If, when

1. setting the n right-hand side variables at zero and
2. evaluating the m left-hand side variables,

we arrive at a feasible solution, then the dictionary is called a feasible dictionary

Basic Solutions

Proposition 1

Every feasible dictionary describes a feasible solution

Proposition 2

Not every feasible solution is described by a feasible dictionary

Example:

- feasible solution to (3): $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$, $x_5 = 5$, $x_6 = 3$
- three equations in the dictionary and five nonzero variables; in a feasible dictionary, one has at most three nonzero variables (the basic ones)

Basic solutions

Feasible solutions that can be described by dictionaries are called basic

Feasible Origin

The slack variables $(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$ are basic, the decision variables $(x_1, x_2, \ldots, x_n)$ are nonbasic

Solution associated with the initial dictionary:

- $x_1 = x_2 = \cdots = x_n = 0$
- $x_{n+i} = b_i$ for $i = 1, 2, \ldots, m$

Proposition 3

The initial dictionary is feasible if and only if each right-hand side b_i in (7) is nonnegative

The set of zero values for the decision variables is sometimes called the origin

Feasible origin

LP problems (7) with each right-hand side, b_i, nonnegative are referred to as problems with feasible origin

Main Idea

The simplex method works exclusively with basic feasible solutions (i.e., feasible dictionaries) and ignores all other feasible solutions

The grand strategy of the simplex method is of successive improvements:

1. having found some basic feasible solution $x_1, x_2, \ldots, x_{n+m}$
2. try to proceed to another basic feasible solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+m}$ which is better in the sense that
$$\sum_{j=1}^{n} c_j \bar{x}_j > \sum_{j=1}^{n} c_j x_j$$

A Second Example

LP in standard form

$$
\begin{array}{lrcrcrcl}
\text{maximize } z = & 5x_1 & + & 5x_2 & + & 3x_3 & & \\
\text{subject to} & x_1 & + & 3x_2 & + & x_3 & \leq & 3 \\
 & -x_1 & & & + & 3x_3 & \leq & 2 \\
 & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & + & 3x_2 & - & x_3 & \leq & 2 \\
 & x_1 & & & & & \geq & 0 \\
 & & & x_2 & & & \geq & 0 \\
 & & & & & x_3 & \geq & 0
\end{array}
$$

Feasible initial dictionary

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

The basic variables are the slack variables (i.e., $B = \{4, 5, 6, 7\}$)

Better Feasible Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

Improving the feasible solution

Look for a feasible solution that yields a higher value of z

Keep $x_2 = 0$ and $x_3 = 0$

Increase the value of x_1

Better Feasible Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

Improving the feasible solution

Look for a feasible solution that yields a higher value of z

Keep $x_2 = 0$ and $x_3 = 0$

Increase the value of x_1

Obtain $z = 5x_1 > 0$

Better Feasible Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

Improving the feasible solution

Look for a feasible solution that yields a higher value of z

Keep $x_2 = 0$ and $x_3 = 0$

Increase the value of x_1

Obtain $z = 5x_1 > 0$

If $x_1 = 1$, then

$$x_4 = 2, \quad x_5 = 3, \quad x_6 = 2, \quad x_7 = 0, \quad z = 5, \quad \text{feasible}$$

Better Feasible Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

Improving the feasible solution

Look for a feasible solution that yields a higher value of z

Keep $x_2 = 0$ and $x_3 = 0$

Increase the value of x_1

Obtain $z = 5x_1 > 0$

If $x_1 = 1$, then

$$x_4 = 2, \quad x_5 = 3, \quad x_6 = 2, \quad x_7 = 0, \quad z = 5, \quad \text{feasible}$$

If $x_1 = 2$, then

$$x_4 = 1, \quad x_5 = 4, \quad x_6 = 0, \quad x_7 = -2, \quad z = 10, \quad \text{not feasible}$$

Better Feasible Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & - & x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 2 & + & x_1 & & & - & 3x_3 \\
x_6 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_7 & = & 2 & - & 2x_1 & - & 3x_2 & + & x_3 \\
\hline
z & = & 0 & + & 5x_1 & + & 5x_2 & + & 3x_3
\end{array}
$$

Improving the feasible solution

Look for a feasible solution that yields a higher value of z

Keep $x_2 = 0$ and $x_3 = 0$

Increase the value of x_1

Obtain $z = 5x_1 > 0$

If $x_1 = 1$, then

$$x_4 = 2, \quad x_5 = 3, \quad x_6 = 2, \quad x_7 = 0, \quad z = 5, \quad \text{feasible}$$

If $x_1 = 2$, then

$$x_4 = 1, \quad x_5 = 4, \quad x_6 = 0, \quad x_7 = -2, \quad z = 10, \quad \text{not feasible}$$

Cannot increase x_1 too much

How much can we increase x_1 (keeping $x_2 = x_3 = 0$ at the same time) and still maintaining the feasibility (i.e., $x_4, x_4, x_5, x_7 \geq 0$)?

Better Feasible Solution (cont'd)

The condition $x_4 = 3 - x_1 - 3x_2 - x_3 \geq 0$ implies $x_1 \leq 3$

The condition $x_5 = 2 + x_1 - 3x_3 \geq 0$ implies $x_1 \geq -2$

The condition $x_6 = 4 - 2x_1 + x_2 - 2x_3 \geq 0$ implies $x_1 \leq 2$

The condition $x_7 = 2 - 2x_1 - 3x_2 + x_3 \geq 0$ implies $x_1 \leq 1$

Better Feasible Solution (cont'd)

The condition $x_4 = 3 - x_1 - 3x_2 - x_3 \geq 0$ implies $x_1 \leq 3$

The condition $x_5 = 2 + x_1 - 3x_3 \geq 0$ implies $x_1 \geq -2$

The condition $x_6 = 4 - 2x_1 + x_2 - 2x_3 \geq 0$ implies $x_1 \leq 2$

The condition $x_7 = 2 - 2x_1 - 3x_2 + x_3 \geq 0$ implies $x_1 \leq 1$

$x_1 \leq 1$ is the most stringent

Better Feasible Solution (cont'd)

The condition $x_4 = 3 - x_1 - 3x_2 - x_3 \geq 0$ implies $x_1 \leq 3$

The condition $x_5 = 2 + x_1 - 3x_3 \geq 0$ implies $x_1 \geq -2$

The condition $x_6 = 4 - 2x_1 + x_2 - 2x_3 \geq 0$ implies $x_1 \leq 2$

The condition $x_7 = 2 - 2x_1 - 3x_2 + x_3 \geq 0$ implies $x_1 \leq 1$

$x_1 \leq 1$ is the most stringent

Increasing x_1 up to 1, we obtain the following feasible solution

- $x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$
- $x_4 = 2, \quad x_5 = 3, \quad x_6 = 2, \quad x_7 = 0$
- $z = 5$

New Dictionary

$x_1 = 1 > 0$ means x_1 has become a basic variable

$x_7 = 0$ means x_7 has become a nonbasic variable

We want to move x_1 to the left-hand side and x_7 to the right-hand side in the dictionary

Fourth equation of the initial dictionary gives

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7 \tag{8}$$

Substituting from (8) into the first three equations and the last equation of the initial dictionary, we arrive at the new dictionary

$$\begin{array}{ccccccccc}
x_1 & = & 1 & - & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_7 \\
x_4 & = & 2 & - & \frac{3}{2}x_2 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_7 \\
x_5 & = & 3 & - & \frac{3}{2}x_2 & - & \frac{5}{2}x_3 & - & \frac{1}{2}x_7 \\
x_6 & = & 2 & + & 4x_2 & - & 3x_3 & + & x_7 \\
\hline
z & = & 5 & - & \frac{5}{2}x_2 & + & \frac{11}{2}x_3 & - & \frac{5}{2}x_7
\end{array}$$

Iteration of the Simplex Method

Iteration of the simplex method

At each iteration, we shall attempt to increase the value of z by making

- one of the right-hand side variables positive
- one of the left-hand side variables zero (keeping the others nonnegative)

In other words:

- Given a feasible dictionary
- Select an entering variable
- Find a leaving variable
- Construct the next feasible dictionary by pivoting

Entering variable

Entering variable

A nonbasic variable, that will increase the value of z when made positive, chosen to enter the basis is called the entering variable

Back to our second example:

- any of the three nonbasic variables (i.e., x_1, x_2, x_3) can be chosen
- common practice to choose the variable having the largest coefficient in the formula for z
- choose (arbitrarily) x_1 to be the entering variable

Choosing the Entering Variable

Entering variable

The entering variable is a nonbasic variable x_j with a positive coefficient $\bar{c}_j$ in the last row of the current dictionary

$$z = z^* + \sum_{j \in N} \bar{c}_j x_j$$

where N is the set of subscripts of nonbasic variables

If there is more than one candidate for entering the basis, then any of these candidates may serve

If $\bar{c}_j \leq 0$ whenever $j \in N$, then the current solution is optimal to LP problem (7)

- every feasible solution $x_1, x_2, \ldots, x_{n+m}$ satisfies $x_1 \geq 0, x_2 \geq 0, \ldots, x_{n+m} \geq 0$
- the nonbasic variables are set at zero
- since $\bar{c}_j \leq 0$ whenever $j \in N$, increasing the value of a nonbasic variable would decrease the value of z

Leaving Variable

Leaving variable

The basic variable whose nonnegativity imposes the most stringent upper bound on the increment of the entering variable is called the leaving variable

Back to our second example:

- as the value of x_1 increases, so does the value of x_5; the values of x_4, x_6, x_7 decrease
- $x_4 \geq 0$ implies $x_1 \leq 3$
- $x_6 \geq 0$ implies $x_1 \leq 2$
- $x_7 \geq 0$ implies $x_1 \leq 1$
- x_7 is the leaving variable

Finding the Leaving Variable

Leaving variable

The leaving variable is that basic variable whose nonnegativity imposes the most stringent upper bound on the increase of the entering variable

If there is more than one candidate for entering the basis, then any of these candidates may serve

If there is no candidate for leaving the basis, then the LP problem (7) is unbounded

- there is no candidate for leaving the basis
- we can make the entering variable as large as we want
- the obtained solution still is feasible
- none of values of the basic variable has decreased
- the value of the objective function has increased
- the more the value of the entering variable increases, the more the value of z increases
- since the entering variable has no upper bound, z has no upper bound, and the LP problem is unbounded

Pivoting

Pivoting

The new dictionary is obtained by

- making the entering variable go from the right-hand side to the left, whereas the leaving variable goes in the opposite direction (on the pivot row)
- substituting the entering variable from the pivot row into the remaining rows of the dictionary

The computational process of constructing the new dictionary is referred to as pivoting

Once you are done with pivoting,

- the m basic variables must have nonnegative values
- the value of the objective function cannot have decreased

Second Iteration

Improved dictionary:

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 1 & - & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_7 \\
x_4 & = & 2 & - & \frac{3}{2}x_2 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_7 \\
x_5 & = & 3 & - & \frac{3}{2}x_2 & - & \frac{5}{2}x_3 & - & \frac{1}{2}x_7 \\
x_6 & = & 2 & + & 4x_2 & - & 3x_3 & + & x_7 \\
\hline
z & = & 5 & - & \frac{5}{2}x_2 & + & \frac{11}{2}x_3 & - & \frac{5}{2}x_7
\end{array}
$$

Improved feasible solution:
$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 3, x_6 = 2, x_7 = 0$

Improve objective function value:
$z = 5$

Second Iteration

Improved dictionary:

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 1 & - & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_7 \\
x_4 & = & 2 & - & \frac{3}{2}x_2 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_7 \\
x_5 & = & 3 & - & \frac{3}{2}x_2 & - & \frac{5}{2}x_3 & - & \frac{1}{2}x_7 \\
x_6 & = & 2 & + & 4x_2 & - & 3x_3 & + & x_7 \\
\hline
z & = & 5 & - & \frac{5}{2}x_2 & + & \frac{11}{2}x_3 & - & \frac{5}{2}x_7
\end{array}
$$

Improved feasible solution:
$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 3, x_6 = 2, x_7 = 0$

Improve objective function value:
$z = 5$

Entering variable: x_3 (only nonbasic variable with a positive coefficient in the last row of the dictionary)

Second Iteration

Improved dictionary:

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 1 & - & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_7 \\
x_4 & = & 2 & - & \frac{3}{2}x_2 & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_7 \\
x_5 & = & 3 & - & \frac{3}{2}x_2 & - & \frac{5}{2}x_3 & - & \frac{1}{2}x_7 \\
x_6 & = & 2 & + & 4x_2 & - & 3x_3 & + & x_7 \\
\hline
z & = & 5 & - & \frac{5}{2}x_2 & + & \frac{11}{2}x_3 & - & \frac{5}{2}x_7
\end{array}
$$

Improved feasible solution:
$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 2, x_5 = 3, x_6 = 2, x_7 = 0$

Improve objective function value:
$z = 5$

Entering variable: x_3 (only nonbasic variable with a positive coefficient in the last row of the dictionary)

Leaving variable: x_6 (basic variable imposing the most stringent upper bound on the increase of x_3)

Third Iteration

Pivoting, we arrive at our third dictionary

$$
\begin{array}{rcrcrcrcr}
x_3 & = & \frac{2}{3} & + & \frac{4}{3}x_2 & + & \frac{1}{3}x_7 & - & \frac{1}{3}x_6 \\
x_1 & = & \frac{4}{3} & - & \frac{5}{6}x_2 & - & \frac{1}{3}x_7 & - & \frac{1}{6}x_6 \\
x_4 & = & 1 & - & \frac{7}{2}x_2 & & & + & \frac{1}{2}x_6 \\
x_5 & = & \frac{4}{3} & - & \frac{29}{6}x_2 & - & \frac{4}{3}x_7 & + & \frac{5}{6}x_6 \\
\hline
z & = & \frac{26}{3} & + & \frac{29}{6}x_2 & - & \frac{2}{3}x_7 & - & \frac{11}{6}x_6
\end{array}
$$

Improved feasible solution:
$x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{2}{3}, x_4 = 1, x_5 = \frac{4}{3}, x_6 = 0, x_7 = 0$

Improve objective function value:
$z = \frac{26}{3}$

Third Iteration

Pivoting, we arrive at our third dictionary

$$
\begin{array}{rcrcrcrcr}
x_3 & = & \frac{2}{3} & + & \frac{4}{3}x_2 & + & \frac{1}{3}x_7 & - & \frac{1}{3}x_6 \\
x_1 & = & \frac{4}{3} & - & \frac{5}{6}x_2 & - & \frac{1}{3}x_7 & - & \frac{1}{6}x_6 \\
x_4 & = & 1 & - & \frac{7}{2}x_2 & & & + & \frac{1}{2}x_6 \\
x_5 & = & \frac{4}{3} & - & \frac{29}{6}x_2 & - & \frac{4}{3}x_7 & + & \frac{5}{6}x_6 \\
\hline
z & = & \frac{26}{3} & + & \frac{29}{6}x_2 & - & \frac{2}{3}x_7 & - & \frac{11}{6}x_6
\end{array}
$$

Improved feasible solution:
$x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{2}{3}, x_4 = 1, x_5 = \frac{4}{3}, x_6 = 0, x_7 = 0$

Improve objective function value:
$z = \frac{26}{3}$

Entering variable: x_2 (only nonbasic variable with a positive coefficient in the last row of the dictionary)

Third Iteration

Pivoting, we arrive at our third dictionary

$$\begin{array}{ccccccccc}
x_3 & = & \frac{2}{3} & + & \frac{4}{3}x_2 & + & \frac{1}{3}x_7 & - & \frac{1}{3}x_6 \\
x_1 & = & \frac{4}{3} & - & \frac{5}{6}x_2 & - & \frac{1}{3}x_7 & - & \frac{1}{6}x_6 \\
x_4 & = & 1 & - & \frac{7}{2}x_2 & & & + & \frac{1}{2}x_6 \\
x_5 & = & \frac{4}{3} & - & \frac{29}{6}x_2 & - & \frac{4}{3}x_7 & + & \frac{5}{6}x_6 \\
\hline
z & = & \frac{26}{3} & + & \frac{29}{6}x_2 & - & \frac{2}{3}x_7 & - & \frac{11}{6}x_6
\end{array}$$

Improved feasible solution:
$x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{2}{3}, x_4 = 1, x_5 = \frac{4}{3}, x_6 = 0, x_7 = 0$

Improve objective function value:
$z = \frac{26}{3}$

Entering variable: x_2 (only nonbasic variable with a positive coefficient in the last row of the dictionary)

Leaving variable: x_5 (basic variable imposing the most stringent upper bound on the increase of x_3)

Forth Iteration

Pivoting, we arrive at our fourth dictionary

$$
\begin{array}{rcrcrcrcr}
x_2 & = & \frac{8}{29} & - & \frac{8}{29}x_7 & + & \frac{5}{29}x_6 & - & \frac{6}{29}x_5 \\
x_3 & = & \frac{30}{29} & - & \frac{1}{29}x_7 & - & \frac{3}{29}x_6 & - & \frac{8}{29}x_5 \\
x_1 & = & \frac{32}{29} & - & \frac{3}{29}x_7 & - & \frac{9}{29}x_6 & + & \frac{5}{29}x_5 \\
x_4 & = & \frac{1}{29} & + & \frac{28}{29}x_7 & - & \frac{3}{29}x_6 & + & \frac{21}{29}x_5 \\
\hline
z & = & 10 & - & 2x_7 & - & x_6 & - & x_5
\end{array}
$$

Improved feasible solution:
$x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29}, x_4 = \frac{1}{29}, x_5 = 0, x_6 = 0, x_7 = 0$

Improve objective function value:
$z = 10$

Forth Iteration

Pivoting, we arrive at our fourth dictionary

$$
\begin{array}{rcccccccc}
x_2 & = & \frac{8}{29} & - & \frac{8}{29}x_7 & + & \frac{5}{29}x_6 & - & \frac{6}{29}x_5 \\
x_3 & = & \frac{30}{29} & - & \frac{1}{29}x_7 & - & \frac{3}{29}x_6 & - & \frac{8}{29}x_5 \\
x_1 & = & \frac{32}{29} & - & \frac{3}{29}x_7 & - & \frac{9}{29}x_6 & + & \frac{5}{29}x_5 \\
x_4 & = & \frac{1}{29} & + & \frac{28}{29}x_7 & - & \frac{3}{29}x_6 & + & \frac{21}{29}x_5 \\
\hline
z & = & 10 & - & 2x_7 & - & x_6 & - & x_5
\end{array}
$$

Improved feasible solution:
$x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29}, x_4 = \frac{1}{29}, x_5 = 0, x_6 = 0, x_7 = 0$

Improve objective function value:
$z = 10$

No nonbasic variable can enter the basis without making the value of z decrease

The last dictionary describes an optimal solution of our second example

Stopping Criterion

Stopping criterion

Repeat the process until no nonbasic variable can enter the basis without making the value of z decrease. The feasible solution associated with the current dictionary is an optimal solution

The optimal solution of our second example is

- $x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29}$
- $z = 10$

Stopping Criterion

Stopping criterion

Repeat the process until no nonbasic variable can enter the basis without making the value of z decrease. The feasible solution associated with the current dictionary is an optimal solution

The optimal solution of our second example is

- $x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29}$
- $z = 10$

The final dictionary for our first example reads

$$\begin{array}{ccccccccc}
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6
\end{array}$$

- The last row shows that every feasible solution with $z = 13$ satisfies $x_2 = x_4 = x_6 = 0$
- The rest of the dictionary shows that every such solution satisfies $x_1 = 2$, $x_3 = 1$, and $x_5 = 1$
- Therefore, there is just one optimal solution

Multiple Optimal Solutions

Consider the following dictionary

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & + & x_2 & - & 2x_5 & + & 7x_3 \\
x_1 & = & 1 & - & 5x_2 & + & 6x_5 & - & 8x_3 \\
x_6 & = & 4 & + & 9x_2 & + & 2x_5 & - & x_3 \\
\hline
z & = & 8 & & & & & - & x_3
\end{array}
$$

Multiple Optimal Solutions

Consider the following dictionary

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & + & x_2 & - & 2x_5 & + & 7x_3 \\
x_1 & = & 1 & - & 5x_2 & + & 6x_5 & - & 8x_3 \\
x_6 & = & 4 & + & 9x_2 & + & 2x_5 & - & x_3 \\
\hline
z & = & 8 & & & & & - & x_3
\end{array}
$$

The last row shows that every optimal solution satisfies $x_3 = 0$, but not necessarily $x_2 = 0$ and $x_5 = 0$

Multiple Optimal Solutions

Consider the following dictionary

$$\begin{array}{rcrcrcrcr} x_4 & = & 3 & + & x_2 & - & 2x_5 & + & 7x_3 \\ x_1 & = & 1 & - & 5x_2 & + & 6x_5 & - & 8x_3 \\ x_6 & = & 4 & + & 9x_2 & + & 2x_5 & - & x_3 \\ \hline z & = & 8 & & & & & - & x_3 \end{array}$$

The last row shows that every optimal solution satisfies $x_3 = 0$, but not necessarily $x_2 = 0$ and $x_5 = 0$

For such solutions, the rest of the dictionary implies

$$\begin{array}{rcrcrcr} x_4 & = & 3 & + & x_2 & - & 2x_5 \\ x_1 & = & 1 & - & 5x_2 & + & 6x_5 \\ x_6 & = & 4 & + & 9x_2 & + & 2x_5 \end{array}$$

Multiple Optimal Solutions

Consider the following dictionary

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 3 & + & x_2 & - & 2x_5 & + & 7x_3 \\
x_1 & = & 1 & - & 5x_2 & + & 6x_5 & - & 8x_3 \\
x_6 & = & 4 & + & 9x_2 & + & 2x_5 & - & x_3 \\
\hline
z & = & 8 & & & & & - & x_3
\end{array}
$$

The last row shows that every optimal solution satisfies $x_3 = 0$, but not necessarily $x_2 = 0$ and $x_5 = 0$

For such solutions, the rest of the dictionary implies

$$
\begin{array}{rcrcrcr}
x_4 & = & 3 & + & x_2 & - & 2x_5 \\
x_1 & = & 1 & - & 5x_2 & + & 6x_5 \\
x_6 & = & 4 & + & 9x_2 & + & 2x_5
\end{array}
$$

Every optimal solution arises from some x_2 and x_5 such that

$$
\begin{array}{rcrcr}
-x_2 & + & 2x_5 & \leq & 3 \\
5x_2 & - & 6x_5 & \leq & 1 \\
-9x_2 & - & 2x_5 & \leq & 4 \\
 & & x_2, x_5 & \geq & 0
\end{array}
$$

Tableau Format

Another way to present the simplex method (probably a more popular way, yet a less straightforward one)

9Modified form) dictionary for our first example

$$\begin{array}{rrrrrrrrrrrrrrrrr}
 & & 2x_1 & + & 3x_2 & + & x_3 & + & x_4 & & & & & & & = & 5 \\
 & & 4x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & & & & & = & 11 \\
 & & 3x_1 & + & 4x_2 & + & 2x_3 & & & & & & + & x_6 & & = & 8 \\
\hline
-z & + & 5x_1 & + & 4x_2 & + & 3x_3 & & & & & & & & & = & 0
\end{array}$$

Tableau for our first example : recording just the coefficients of the x_j's, together with the right-hand sides

$$\begin{array}{cccccccc}
2 & 3 & 1 & 1 & 0 & 0 & & 5 \\
4 & 1 & 2 & 0 & 1 & 0 & & 11 \\
3 & 4 & 2 & 0 & 0 & 1 & & 8 \\
\hline
5 & 4 & 3 & 0 & 0 & 0 & & 0
\end{array}$$

Entering Variable

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0

Examine all the numbers (but the farthest right) in the last row

- they all are nonpositive: the tableau describes an optimal solution
- choose one column with a positive number in the last row (pivot column $\rightarrow$ entering variable)

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0

Leaving Variable

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0

For each row whose entry r in the pivot column is positive, look up the entry s in the rightmost column

- if all the entries in the pivot column are nonpositive, the LP problem is unbounded
- the row with the smallest ration $\frac{s}{r}$ is the pivot row ($\rightarrow$ leaving variable)

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0

The entry at the intersection of the pivot column with the row column is called the pivot number

Pivoting

$$
\begin{array}{cccccc|c}
2 & 3 & 1 & 1 & 0 & 0 & 5 \\
4 & 1 & 2 & 0 & 1 & 0 & 11 \\
3 & 4 & 2 & 0 & 0 & 1 & 8 \\
\hline
5 & 4 & 3 & 0 & 0 & 0 & 0
\end{array}
$$

Divide every entry of the pivot row by the pivot number

$$
\begin{array}{cccccc|c}
1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\
4 & 1 & 2 & 0 & 1 & 0 & 11 \\
3 & 4 & 2 & 0 & 0 & 1 & 8 \\
\hline
5 & 4 & 3 & 0 & 0 & 0 & 0
\end{array}
$$

For every remaining row, subtract a suitable multiple of the new pivot row (goal: make all the entries in the pivot column but the pivot number be zero)

$$
\begin{array}{cccccc|c}
1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\
0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \\
\hline
0 & -\frac{7}{2} & \frac{1}{2} & -\frac{5}{2} & 0 & 0 & -\frac{25}{2}
\end{array}
$$