Z325FU04 - Modèles Linéaires de la Recherche Oérationnelle

Additional Facts

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Problem #1

Solve the following LP problem using the simplex method

$$
\begin{array}{lrcrcl}
\text{maximize } z = & x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & 2x_2 & \leq & 14 \\
 & -x_1 & + & 2x_2 & \geq & 5 \\
 & 2x_1 & + & 4x_2 & = & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Two possibilities

1. write the LP problem in standard form, and solve it using the two-phase simplex method
2. use of an artificial LP problem

Artificial LP Problem

Original LP problem into equality form

$$
\begin{array}{lrcrcrcrcl}
\text{maximize } z = & x_1 & + & 4x_2 & & & & & & \\
\text{subject to} & x_1 & + & 2x_2 & + & x_3 & & & = & 14 \\
& -x_1 & + & 2x_2 & & & - & x_4 & = & 5 \\
& 2x_1 & + & 4x_2 & & & & & = & 18 \\
& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

Problem: The last two equations have no basic variables

Artificial LP problem

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{minimize } w = & & & & & & & & & y_1 & + & y_2 & & \\
\text{subject to} & x_1 & + & 2x_2 & + & x_3 & & & & & & & = & 14 \\
& -x_1 & + & 2x_2 & & & - & x_4 & + & y_1 & & & = & 5 \\
& 2x_1 & + & 4x_2 & & & & & & & + & y_2 & = & 18 \\
& & & & & & & & & & & x_1, x_2, x_3, x_4, y_1, y_2 & \geq & 0
\end{array}
$$

Feasible basis $\{x_3, y_1, y_2\}$

Another Type of Two-Phase Method

Solve the artificial LP problem using the simplex method

Two cases need to be considered

- if the optimal value of the artificial LP problem is positive (i.e., $w^* > 0$), then the original LP problem is infeasible
- if the optimal value of the artificial LP problem is zero (i.e., $w^* = 0$), then the original LP problem is feasible

The second case implies that all the artificial variables (i.e., y_1 and y_2) are equal to zero, yet it does not mean that they are all nonbasic

If all the artificial variables are nonbasic, then solve the original LP problem starting from the optimal basis found in the first phase

If some of the artificial variables are basic and $w^* = 0$ (i.e., the dictionary is degenerate), try to make them leave the basis by a series of pivots; it may happen that some artificial variables have to remain basic!

Problem #2

Solve the following LP problem using the simplex method

$$\begin{array}{lrcrcl} \text{maximize } z = & x_1 & + & 2x_2 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & \geq & 6 \\ & 3x_1 & + & 2x_2 & \leq & 12 \\ & x_1 & & & \geq & 0 \end{array}$$

Two possibilities

1. write the LP problem in standard form, and solve it using the two-phase simplex method
2. use of the big M method

Simplex Big-M Method

Solve a combined Phase I, Phase II LP problem

1. write the LP problem in the standard form (i.e., maximization problem with all the constraints $\leq$)
2. if constraint i has a negative right-hand side b_i, subtract an artificial variable $y_i \geq 0$
3. let M be a very large positive number; add $-My_i$ to the objective function for each artificial variable
4. eliminate the artificial variables from the last row of the dictionary; perform the simplex method

If any artificial variable appears as a nonzero basic variable in the optimal solution, the original LP problem is infeasible

Problem #3

Solve the following LP problem using the simplex method (with the largest-coefficient rule)

$$\begin{array}{lrcrcrcl}
\text{maximize } z = & 100x_1 & + & 10x_2 & + & x_3 & & \\
\text{subject to} & x_1 & & & & & \leq & 1 \\
 & 20x_1 & + & x_2 & & & \leq & 100 \\
 & 200x_1 & + & 20x_2 & + & x_3 & \leq & 10,000 \\
 & x_1 & & & & & \geq & 0 \\
 & & & x_2 & \geq & & & 0 \\
 & & & & & x_3 & \geq & 0
\end{array}$$

Solution

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 1 & - & x_1 & & & & \\
x_5 & = & 100 & - & 20x_1 & - & x_2 & & \\
x_6 & = & 10,000 & - & 200x_1 & - & 20x_2 & - & x_3 \\
\hline
z & = & & & 100x_1 & + & 10x_2 & + & x_3
\end{array}
$$

Largest-coefficient rule

Always choose as the entering variable that candidate having the largest coefficient in the objective-function row

With the largest-coefficient rule, the simplex method requires 8 dictionaries (i.e, 7 iterations)

Largest-increase rule

Always choose as the entering variable that candidate whose entrance into the basis brings about the largest increase in the objective function

With the largest-increase rule, the simplex method requires 2 dictionaries (i.e., 1 iteration)

Klee-Minty Problem

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j & \\ \text{subject to} & \displaystyle 2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \leq 100^{i-1} & \text{for } i = 1, 2, \ldots, n \\ & x_j \geq 0 & \text{for } i = 1, 2, \ldots, n \end{array}$$

A Klee-Minty problem has as a feasible region a distortion of the *n*-dimensional cube; the simplex method with the largest-coefficient rule passes through all 2^n vertices

Klee and Minty, 1972

Applying the simplex method with the largest-coefficient rule to this LP problem requires 2^n dictionaries

With the largest-increase rule, the simplex method requires 2 dictionaries

Polynomial-Time Algorithm?

Polynomial-time algorithm

A polynomial-time algorithm is an algorithm which terminates after a number of steps bounded by a polynomial in the input size

Corollaries

- The simplex method with the largest-coefficient rule is not a polynomial-time algorithm
- The simplex method with the largest-increase rule is not a polynomial-time algorithm (Jeroslow, 1973)
- The simplex method with Bland's rule is not a polynomial-time algorithm (Avis and Chvátal, 1978)

Polynomial-time algorithms for linear programming

- Khachiyan (1979) - Ellipsoid method
- Karmarkar (1984) - Interior-point (projective) method