Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

Sensitivity Analysis

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An Example: Primal LP Problem

$$
\begin{array}{lrcrcrcl}
\max z = & 4x_3 & - & 13x_4 & + & 7x_5 & & \\
\text{s.t.} & 3x_3 & + & 2x_4 & + & 5x_5 & \leq & 5 \\
 & x_3 & - & 3x_4 & + & 2x_5 & \leq & 3 \\
 & & & & & x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- Initial dictionary

$$
\begin{array}{rclcrcrcr}
x_1 & = & 5 & - & 3x_3 & - & 2x_4 & - & 5x_5 \\
x_2 & = & 3 & - & x_3 & + & 3x_4 & - & 2x_5 \\
\hline
z & = & & & 4x_3 & - & 13x_4 & + & 7x_5
\end{array}
$$

- First-iteration dictionary (x_3 enters, x_1 leaves)

$$
\begin{array}{rclcrcrcr}
x_3 & = & \frac{5}{3} & - & \frac{2}{3}x_4 & - & \frac{5}{3}x_5 & - & \frac{1}{3}x_1 \\
x_2 & = & \frac{4}{3} & + & \frac{11}{3}x_4 & - & \frac{1}{3}x_5 & + & \frac{1}{3}x_1 \\
\hline
z & = & \frac{20}{3} & - & \frac{47}{3}x_4 & + & \frac{1}{3}x_5 & - & \frac{4}{3}x_1
\end{array}
$$

An Example: Dual LP Problem

$$
\begin{array}{lrcrcr}
\min w = & 5y_1 & + & 3y_2 & & \\
\text{s.t.} & 3y_1 & + & y_2 & \geq & 4 \\
 & 2y_1 & - & 3y_2 & \geq & -13 \\
 & 5y_1 & + & 2y_2 & \geq & 7 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

- Associated dual original dictionary

$$
\begin{array}{rcrcrcr}
y_3 & = & -4 & + & 3y_1 & + & y_2 \\
y_4 & = & 13 & + & 2y_1 & - & 3y_2 \\
y_5 & = & -7 & + & 5y_1 & + & 2y_2 \\
\hline
-w & = & & & -5y_1 & - & 3y_2
\end{array}
$$

- Associated dual first-iteration dictionary

$$
\begin{array}{rcrcrcr}
y_4 & = & \frac{47}{3} & + & \frac{2}{3}y_3 & - & \frac{11}{3}y_2 \\
y_5 & = & -\frac{1}{3} & + & \frac{5}{3}y_3 & + & \frac{1}{3}y_2 \\
y_1 & = & \frac{4}{3} & + & \frac{1}{3}y_3 & - & \frac{1}{3}y_2 \\
\hline
-w & = & -\frac{20}{3} & - & \frac{5}{3}y_3 & - & \frac{4}{3}y_2
\end{array}
$$

Mirror Images: Initial Dictionaries

- Primal

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 5 & - & 3x_3 & - & 2x_4 & - & 5x_5 \\
x_2 & = & 3 & - & x_3 & + & 3x_4 & - & 2x_5 \\
\hline
z & = & & & 4x_3 & - & 13x_4 & + & 7x_5
\end{array}
$$

- Dual

$$
\begin{array}{rcrcrcr}
y_3 & = & -4 & + & 3y_1 & + & y_2 \\
y_4 & = & 13 & + & 2y_1 & - & 3y_2 \\
y_5 & = & -7 & + & 5y_1 & + & 2y_2 \\
\hline
-w & = & & - & 5y_1 & - & 3y_2
\end{array}
$$

Mirror Images: First-Iteration Dictionaries

- Primal

$$
\begin{array}{rcrcrcrcr}
x_3 & = & \frac{5}{3} & - & \frac{2}{3}x_4 & - & \frac{5}{3}x_5 & - & \frac{1}{3}x_1 \\
x_2 & = & \frac{4}{3} & + & \frac{11}{3}x_4 & - & \frac{1}{3}x_5 & + & \frac{1}{3}x_1 \\
\hline
z & = & \frac{20}{3} & - & \frac{47}{3}x_4 & + & \frac{1}{3}x_5 & - & \frac{4}{3}x_1
\end{array}
$$

- Dual

$$
\begin{array}{rcrcrcr}
y_4 & = & \frac{47}{3} & + & \frac{2}{3}y_3 & - & \frac{11}{3}y_2 \\
y_5 & = & -\frac{1}{3} & + & \frac{5}{3}y_3 & + & \frac{1}{3}y_2 \\
y_1 & = & \frac{4}{3} & + & \frac{1}{3}y_3 & - & \frac{1}{3}y_2 \\
\hline
-w & = & -\frac{20}{3} & - & \frac{5}{3}y_3 & - & \frac{4}{3}y_2
\end{array}
$$

Mirror Images

- Primal dictionary

$$\begin{array}{rclcll} x_r & = & \bar{b}_r & - & \sum_{s \in N} \bar{a}_{rs} x_s & (r \in B) \\ \hline z & = & \bar{d} & + & \sum_{s \in N} \bar{c}_s x_x & \end{array}$$

with $x_r (r \in B)$ basic and $x_s (s \in N)$ nonbasic

- Dual dictionary

$$\begin{array}{rclcll} y_s & = & -\bar{c}_s & + & \sum_{r \in B} \bar{a}_{rs} y_r & (s \in N) \\ \hline -w & = & -\bar{d} & - & \sum_{r \in B} \bar{b}_r x_r & \end{array}$$

with $y_s (s \in N)$ basic and $y_r (r \in B)$ nonbasic

Dual-Feasibility

Dual-feasible dictionary

The primal dictionary

$$\begin{array}{rcrcll} x_r & = & \bar{b}_r & - & \sum_{s \in N} \bar{a}_{rs} x_s & \quad (r \in B) \\ \hline z & = & \bar{d} & + & \sum_{s \in N} \bar{c}_s x_x & \end{array} \tag{1}$$

is dual-feasible if the corresponding dual dictionary

$$\begin{array}{rcrcll} y_s & = & -\bar{c}_s & + & \sum_{r \in B} \bar{a}_{rs} y_r & \quad (s \in N) \\ \hline -w & = & -\bar{d} & - & \sum_{r \in B} \bar{b}_r x_r & \end{array} \tag{2}$$

is feasible

The primal dictionary is dual-feasible if and only if $\bar{c}_s \leq 0$ for all $s \in N$

Main Idea

- A dual variable y_k is basic in (2) if and only if the corresponding primal variable x_k is nonbasic in (1)
- A dual dictionary arising from (2) by a single pivot, with y_i entering and y_j leaving the basis, will correspond to the primal dictionary arising from (1) by a single pivot, with x_i leaving and x_j entering the basis

Main idea

- Begin with a dual-feasible dictionary (1)
- Iteration
  - choose a subscript $i \in B$ so that
  $$\overline{b}_i < 0$$
  - find a subscript $j \in N$ so that
  $$\overline{a}_{ij} < 0 \text{ and } \frac{\overline{c}_j}{\overline{a}_{ij}} \leq \frac{\overline{c}_s}{\overline{a}_{is}} \text{ for all } s \in N \text{ with } \overline{a}_{is} < 0$$
  - pivot with x_i leaving and x_j entering the basis

An Example

- Primal LP problem

$$
\begin{array}{lrcrcrcl}
\max z = & 4x_3 & - & 13x_4 & + & 7x_5 & & \\
\text{s.t.} & 3x_3 & + & 2x_4 & + & 5x_5 & \leq & 5 \\
 & x_3 & - & 3x_4 & + & 2x_5 & \leq & 3 \\
 & & & & & x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- Primal dictionary (dual-feasible)

$$
\begin{array}{rcrcrcrcr}
x_1 & = & -4 & + & 3x_2 & - & 11x_4 & + & x_5 \\
x_3 & = & 3 & - & x_2 & + & 3x_4 & - & 2x_5 \\
\hline
z & = & 12 & - & 4x_2 & - & x_4 & - & x_5
\end{array}
$$

- Correspondent dual dictionary

$$
\begin{array}{rcrcrcr}
y_2 & = & 4 & - & 3y_1 & + & y_3 \\
y_4 & = & 1 & + & 11y_1 & - & 3y_3 \\
y_5 & = & 1 & - & y_1 & + & 2y_3 \\
\hline
-w & = & -12 & + & 4y_1 & - & 3y_3
\end{array}
$$

An Iteration

- Leaving variable
  - basic variable x_1: $\bar{b}_1 = -4$
  - basic variable x_3: $\bar{b}_3 = 3$
- Entering variable
  - nonbasic variable x_2: $\frac{\bar{c}_2}{\bar{a}_{12}} = \frac{4}{3}$
  - nonbasic variable x_4 is ignored since $\bar{a}_{14} = 11 \geq 0$
  - nonbasic variable x_5: $\frac{\bar{c}_5}{\bar{a}_{15}} = \frac{1}{1} = 1$
- Pivot with x_5 entering and x_1 leaving the basis yields the dual-feasible dictionary

$$\begin{array}{rcrcrcrcr}
x_3 & = & -5 & - & 2x_1 & + & 5x_2 & - & 19x_4 \\
x_5 & = & 4 & + & x_1 & - & 3x_2 & + & 11x_4 \\
\hline
z & = & 8 & - & x_1 & - & x_2 & - & 12x_4
\end{array}$$

Termination

- If no $i \in B$ satisfies

$$\overline{b}_i < 0$$

  then, dictionary (1) is not only dual-feasible but also feasible; it describes an optimal solution $x_1^*, x_2^*, \ldots, x_{n+m}^*$ by
    - $x_s^* = 0$ for all $s \in N$
    - $x_r^* = \overline{b}_r$ for all $r \in B$
- If no $j \in N$ satisfies

$$\overline{a}_{ij} < 0 \text{ and } \frac{\overline{c}_j}{\overline{a}_{ij}} \leq \frac{\overline{c}_s}{\overline{a}_{is}} \text{ for all } s \in N \text{ with } \overline{a}_{is} < 0$$

  then the dual problem is unbounded, and the primal problem is infeasible

Four Questions

1. How is the leaving variable found?
2. How is the entering variable found?
3. How are the numbers $\overline{b}_r$ updated?
4. How are the numbers $\overline{c}_s$ updated?

Dual-feasible dictionary in matrix terms

$$\begin{array}{lclcl} x_B & = & B^{-1}b & - & B^{-1}A_N x_N \\ \hline z & = & c_B B^{-1}b & + & (c_N - c_B B^{-1}A_N)x_N \end{array}$$

We write

- $x_B^* = B^{-1}$
- $\overline{c}_N = c_N - c_B B^{-1} A_N$

Original LP Problem

$$
\begin{array}{lrcrcrcrcr}
\max z = & -5x_1 & - & 3x_2 & - & 3x_3 & - & 6x_4 & & \\
\text{s.t.} & -6x_1 & + & x_2 & + & 2x_3 & + & 4x_4 & \leq & 14 \\
 & 3x_1 & - & 2x_2 & - & x_3 & - & 5x_4 & \leq & -25 \\
 & -2x_1 & + & x_2 & & & + & 2x_4 & \leq & 14 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

$$
A = \begin{pmatrix} -6 & 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & -2 & -1 & -5 & 0 & 1 & 0 \\ -2 & 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 14 \\ -25 \\ 14 \end{pmatrix}
$$

$$
c = \begin{pmatrix} -5 & -3 & -3 & -6 & 0 & 0 & 0 \end{pmatrix}
$$

Initialization

We may initialize by

$$x_B^* = \begin{pmatrix} x_5^* \\ x_6^* \\ x_7^* \end{pmatrix} = \begin{pmatrix} 14 \\ -25 \\ 14 \end{pmatrix}$$

$$\overline{c}_N = \begin{pmatrix} \overline{c}_1 & \overline{c}_2 & \overline{c}_3 & \overline{c}_4 \end{pmatrix} = \begin{pmatrix} -5 & -3 & -3 & -6 \end{pmatrix}$$

Step 1

How is the leaving variable found?

- Is there a subscript $i \in B$ so that
$$\overline{b}_i < 0?$$
- The numbers $\overline{b}_r (r \in B)$ are readily available

STEP 1

- If $x_B^* \geq 0$,then stop: x^* is an optimal solution
- Otherwise, choose the leaving variable: it may be any basic variable x_i with $x_i^* < 0$

- Our example: x_6 is the leaving variable

Step 2

How is the entering variable found? (part 1)

- Is there a subscript $j \in N$ so that

$$\overline{a}_{ij} < 0 \text{ and } \frac{\overline{c}_j}{\overline{a}_{ij}} \leq \frac{\overline{c}_s}{\overline{a}_{is}} \text{ for all } s \in N \text{ with } \overline{a}_{is} < 0$$

- First m rows of the dual-feasible dictionary

$$x_B = B^{-1}b - B^{-1}A_N x_N$$

- If the leaving variable x_i appears in the pth position in the basis heading, then the desired numbers $\overline{a}_{is}$ composed the pth row of $B^{-1}A_N$
- Let v be the pth row of B^{-1}; the pth row of $B^{-1}A_N$ equals vA_N
- Vector v may be found by solving the system

$$vB = e$$

with e standing for the pth row of the $m \times m$ identity matrix

Step 2 (cont'd)

Step 2

Let p be so that x_i appears in the pth position in the basis heading. Let e stand for the pth row of the $m \times m$ identity matrix.

- Solve the system $vB = e$
- Compute $w_N = vA_N$

Each $\overline{a}_{is}$ is a component w_s of this vector w_N

Step 2 (Our Example)

- $B = I$
- $p = 2$
- The system $vB = e$ is

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

- $v = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$
- The vector $w_N = vA_N$ is

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 1 & 2 & 4 \\ 3 & -2 & -1 & -5 \\ -2 & 1 & 0 & 2 \end{pmatrix}$$

- $w_N = \begin{pmatrix} 3 & -2 & -1 & -5 \end{pmatrix}$

Step 3

How is the entering variable found? (part 2)

- The numbers $\bar{a}_{is} (s \in N)$ are known (i.e., $\bar{a}_{is} = w_s$)
- Find the nonbasic variable x_j so that
  - $w_j < 0$, and
  - $\min\{\frac{\bar{c}_s}{w_s} : s \in N \text{ with } w_s < 0\} = \frac{\bar{c}_j}{w_j}$

STEP 3

Let J be the set of those nonbasic variables x_j for which $w_j < 0$.

- If $J = \emptyset$, then stop: the problem is infeasible
- Otherwise, find the x_j in J that minimizes $\frac{\bar{c}_j}{w_j}$, and let it be the entering variable

Step 3 (Our Example)

- $J = \{x_2, x_3, x_4\}$
- nonbasic variable x_2: $\frac{\bar{c}_2}{w_2} = \frac{-3}{-2} = \frac{3}{2}$
- nonbasic variable x_3: $\frac{\bar{c}_3}{w_4} = \frac{-3}{-1} = 3$
- nonbasic variable x_4: $\frac{\bar{c}_3}{w_4} = \frac{-6}{-5} = \frac{6}{5}$
- x_4 is the entering variable

Step 4

How are the numbers $\overline{b}_r (r \in B)$ updated?

- First m rows of the dual-feasible dictionary

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}A_N x_N \\ &= x_B^* - B^{-1}A_N x_N \\ &= \overline{b}_B - B^{-1}A_N x_N \end{aligned}$$

- The numbers $\overline{b}_r (r \in B)$ can be updated as for the revised simplex method
- Having determined the entering variable x_j and therefore the entering column a
  - solve the system $Bd = a$
  - replace x_B^* by $x_B^* - td$ with $t = \frac{\overline{b}_i}{\overline{a}_{ij}} = \frac{x_i^*}{w_j}$

Step 4

Solve the system $Bd = a$ with a standing for the entering column

Step 4 (Our Example)

- Entering column

$$a = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$$

- The system $Bd = a$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$$

- $d = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$

Step 5

How are the numbers $\overline{c}_s (s \in N)$ updated?

- The z-row can be recorded as

$$z = \overline{d} + \overline{c}_j x_j + \sum_{s \in N \setminus \{s\}} \overline{c}_s x_s$$

- Pivoting: substituting for x_j from

$$x_j = \frac{1}{\overline{a}_{ij}} (\overline{b}_i - x_i - \sum_{s \in N \setminus \{s\}} \overline{a}_{is} x_s)$$

- The formula for z get updated into

$$z = \overline{d} + \overline{c}_j \frac{1}{\overline{a}_{ij}} (\overline{b}_i - x_i - \sum_{s \in N \setminus \{s\}} \overline{a}_{is} x_s) + \sum_{s \in N \setminus \{s\}} \overline{c}_s x_s$$

- Writing w_s for each $\overline{a}_{is} (s \in N)$, we obtain

$$z = (\overline{d} + \overline{b}_i \frac{\overline{c}_j}{w_j}) - \frac{\overline{c}_j}{w_j} x_i + \sum_{s \in N \setminus \{s\}} (\overline{c}_s - w_s \frac{\overline{c}_j}{w_j}) x_s$$

Step 5 (cont'd)

- New coefficient $\overline{c}_i = -\frac{\overline{c}_j}{w_j}$
- for each $s \in N = \setminus\{j\}$, new coefficient is $\overline{c}_s + \overline{c}_i w_s$

Step 5

- Set the value x_j^* at $t = \frac{x_j^*}{w_j}$
- Replace the values x_B^* by $x_B^* - td$
- Replace the leaving column of B by the entering column
- In the basis heading, replace the leaving variable by the entering variable
- Set $\overline{c}_i = -\frac{\overline{c}_j}{w_j}$; add $\overline{c}_i w_s$ to each $\overline{c}_s$ with $s \neq i$

Step 5 (Our Example)

- Leaving variable: x_6
- Entering variable: x_4
- $t = \frac{x_6^*}{w_4} = \frac{-25}{-5} = 5$
- $x_B^* = \begin{pmatrix} x_5^* \\ x_4^* \\ x_7^* \end{pmatrix} = \begin{pmatrix} 14 - 4t \\ t \\ 14 - 2t \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \\ 4 \end{pmatrix}$
- New basis matrix $B = E_1$ with the eta matrix

$$E_1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -5 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

- New reduced-cost vector; $\overline{c}_6 = -\frac{\overline{c}_4}{w_6} = -\frac{6}{5}$ and

$$\begin{aligned} \overline{c}_N &= \begin{pmatrix} \overline{c}_1 & \overline{c}_2 & \overline{c}_3 & \overline{c}_6 \end{pmatrix} \\ &= \begin{pmatrix} -5 + 3\overline{c}_6 & -3 - 2\overline{c}_6 & -3 - \overline{c}_6 & \overline{c}_6 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{43}{5} & -\frac{3}{5} & -\frac{9}{5} & -\frac{6}{5} \end{pmatrix} \end{aligned}$$

Sensitivity Analysis

- Numerical data often represent only rough estimates of quantities
- Replacing the original data by more pessimistic or optimistic estimates of the quantities; adding new data; removing some data
- How does the optimal policy changes as the data change?
- Solving the initial LP problem is only the starting point for further analysis of the situation

Sensitivity analysis

Sensitivity analysis is a method for making changes in the original problem data and computing a new optimal solution without re-solving the problem

Variations

- One may create a number a variations on the original LP problem
  - changes to the objective function
  - changes to the right-hand side
  - adding a new variable
  - adding a new constraint
  - changes to the left-hand side
- Idea: If the problem changes are small, one expects the optimal solution to change in a predictable fashion
- Similar problems are likely to have similar solutions, and so it is often easier to exploit the results obtained in solving the original problem; no need to solve each of the modified problems from scratch

Original LP Problem

- Consider the following LP in the standard form

$$
\begin{array}{lll}
\text{maximize} & z = \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\
& x_j \geq 0 & \text{for } j = 1, 2 \ldots, n
\end{array}
\tag{3}
$$

- How the results found by the simplex method in solving (3) can be exploited in solving modified version

- After adding slack variables, (3) becomes

$$
\begin{array}{ll}
\text{maximize} & cx \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
\tag{4}
$$

- Let x^* be the optimal solution to (4)

Changes to the Objective Function

- Changes are restricted to the objective function cx

$$\begin{array}{ll} \text{maximize} & \tilde{c}x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \tag{5}$$

- x^* remains feasible in (5); it can be used to initialize the simplex method on (5)
- If x^* is dual-feasible, then it is optimal to (5)
- If $\tilde{c}$ differs from c only slightly, then an optimal solution to (5) is likely to differ from the optimal solution x^* only slightly, and the number of iteration leading from x^* to an optimal solution is likely to be small

Changes to the Right-Hand Side

- Changes are restricted to the right-hand side b

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = \tilde{b} \\ & x \geq 0 \end{array} \tag{6}$$

- x^* defines a basic solution $\tilde{x}$ of (6) by $\tilde{x}_N = x^*_N$ and $B\tilde{x}_B = \tilde{b} - A_N\tilde{x}_N$
- $\tilde{x}$ is dual-feasible; it can be used to initialize the dual-simplex method on (6)
- If $\tilde{x}$ is feasible in (6), then it is optimal to (6)
- If $\tilde{b}$ differs from b only slightly, then an optimal solution to (6) is likely to differ from the optimal solution x^* only slightly, and the number of iteration leading from x^* to an optimal solution is likely to be small

Adding a New Variable

- New variable x_{n+1} is introduced

$$\begin{array}{ll} \text{maximize} & cx + \tilde{c}_{n+1} x_{n+1} \\ \text{subject to} & Ax + \tilde{a}_{n+1} x_{n+1} = b \\ & x \geq 0 \\ & x_{n+1} \geq 0 \end{array} \tag{7}$$

- $\begin{pmatrix} x_1^* & x_2^* & \dots & x_n^* & 0 \end{pmatrix}^T$ is a feasible basic solution; it can be used to initialize the simplex method on (7) (x_{n+1} is a nonbasic variable)

- If the reduced cost of x_{n+1} has the right sign (and $\begin{pmatrix} x_1^* & x_2^* & \dots & x_n^* & 0 \end{pmatrix}^T$ is feasible), then it is optimal to (7)

Adding a New Constraint

- New constraint $\sum_{j=1}^{n} \tilde{a}_j x_j \leq \tilde{b}_0$ is introduced

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & \sum_{j=1}^{n} \tilde{a}_j x_j + x_{n+1} = \tilde{b}_0 \\ & x \geq 0 \\ & x_{n+1} \geq 0 \end{array} \tag{8}$$

- $\begin{pmatrix} x_1^* & x_2^* & \dots & x_n^* & x_{n+1}^* \end{pmatrix}^T$ with

$$x_{n+1}^* = \tilde{b}_0 - \sum_{j=1}^{n} \tilde{a}_j x_j$$

is a dual-feasible basic solution; it can be used to initialize the dual-simplex method on (8) (x_{n+1} is a basic variable)

Changes to the Left-Hand Side

- Changes to the left-hand side of problem (4) into

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & \tilde{A}x = b \\ & x \geq 0 \end{array} \tag{9}$$

- If the changes leading from (4) to (9) are only slight, then an optimal solution to (9) is likely to differ from the optimal solution x^* only slightly
- Changes of a_{ij} at basic variable x_j might make B into a singular matrix $\tilde{B}$
- As long as the entries a_{ij} of A remain unchanged for all variables x_j basic in x^*, the vector $\tilde{x}$ may be obtained by
  - first, converting x^*_N into $\tilde{x}_N$
  - then, solving the system $B\tilde{x}_B = \tilde{b} - \tilde{A}_N\tilde{x}_N$
  - $\tilde{x}$ is a feasible basic solution; it can be used to initialize the simplex method on (9)

Changes to a Basic Column

- Changes in column a_j with x_j basic variable can be thought as
  - introduction of a new variable $x_{n+1} \geq 0$ with
    - column $\tilde{a}_j$
    - objective function coefficient $\tilde{c}_{n+1} = c_j$
  - combined with setting $0 \leq x_j \leq 0$
- The resulting LP problem may almost be handled as a problem of type (9)