Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

Duality

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Motivation

Consider the following LP problem

$$\begin{array}{lrcrcrcrcl} \text{maximize } z = & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Can we prove that a given feasible solution is optimal without using the simplex method

Every feasible solution gives a lower bound on the optimal value z^* (e.g., $x_1 = 3, x_2 = 0, x_3 = 2, x_4 = 0$ yields $z^* \geq 22$)

Need an upper bound on z^* to prove that a given feasible solution which hits z^* is optimal (e.g., $x_1 = 0, x_2 = 14, x_3 = 0, x_4 = 5$ yields $z^* \geq 29$)

Upper Bounds

Multiply the second inequality by $\frac{5}{3}$

$$\frac{5}{3}(5x_1 + x_2 + 3x_3 + 8x_4 \leq 55)$$

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Every feasible solution (x_1, x_2, x_3, x_4) satisfies

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

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Multiply the second inequality by $\frac{5}{3}$

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$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Since

$$4x_1 \leq \frac{25}{3}x_1,\ x_2 \leq \frac{5}{3}x_2,\ 5x_3 \leq 5x_3,\ 3x_4 \leq \frac{40}{3}x_4$$

Upper Bounds

Multiply the second inequality by $\frac{5}{3}$

$$\frac{5}{3}(5x_1 + x_2 + 3x_3 + 8x_4 \leq 55)$$

Every feasible solution (x_1, x_2, x_3, x_4) satisfies

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Since

$$4x_1 \leq \frac{25}{3}x_1,\ x_2 \leq \frac{5}{3}x_2,\ 5x_3 \leq 5x_3,\ 3x_4 \leq \frac{40}{3}x_4$$

we obtain

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

that is,

$$z^* \leq \frac{275}{3}$$

Upper Bounds (cont'd)

Add the second and third constraints

Every feasible solution (x_1, x_2, x_3, x_4) satisfies

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

Since

$$4x_1 \leq 4x_1,\ x_2 \leq 3x_2,\ 5x_3 \leq 6x_3,\ 3x_4 \leq 3x_4$$

we obtain

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$$

that is,

$$z^* \leq 58$$

Can we do better?

Upper Bounds (cont'd)

Construct a linear combination of the constraints

$$\begin{array}{ll} y_1(x_1 - x_2 - x_3 + 3x_4 \leq 1) & \text{with } y_1 \geq 0 \\ y_2(5x_1 + x_2 + 3x_3 + 8x_4 \leq 55) & \text{with } y_2 \geq 0 \\ y_3(-x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3) & \text{with } y_3 \geq 0 \end{array}$$

Every feasible solution (x_1, x_2, x_3, x_4) satisfies

$$\begin{aligned}(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 \\ + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3\end{aligned}$$

Objective function: $z = 4x_1 + x_2 + 5x_3 + 3x_4$

To conclude that $z^* \leq y_1 + 55y_2 + 3y_3$ we want

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3 \end{aligned}$$

Upper Bounds (cont'd)

To get the smallest upper bound on z^*, we need to solve the following LP problem

$$
\begin{array}{lrcrcrcr}
\text{minimize} & y_1 & + & 55y_2 & + & 3y_3 & & \\
\text{subject to} & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
& -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
& -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
& 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
& & & & & y_1, y_2, y_3 & \geq & 0
\end{array}
$$

This LP problem is called the dual problem of our original LP problem (called the primal problem)

Dual Problem

Primal problem

Primal problem

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\ & x_j \geq 0 & \text{for } j = 1, 2 \ldots, n \end{array}$$

Dual problem

Dual problem

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} a_{ij} y_i \geq c_j & \text{for } j = 1, 2 \ldots, n \\ & y_i \geq 0 & \text{for } i = 1, 2 \ldots, m \end{array}$$

Dual Problem (cont'd)

Constraint-Variable correspondence

- m Primal constraints $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ are in a one-to-one correspondence with the m dual variables y_i $(i = 1, 2, \ldots, m)$
- n Primal variables x_j are in a one-to-one correspondence with the n dual constraints $\sum_{i=1}^{m} a_{ij}y_i \geq c_j$ $(j = 1, 2, \ldots, n)$

Objective-Rhs correspondence

- Coefficient c_j (in the primal objective function) appears as the right-hand side of the dual constraint corresponding to variable x_j $(j = 1, 2, \ldots, n)$
- Right-hand side b_i in the primal appears as the coefficient of dual variable y_i in the dual objective function $(i = 1, 2, \ldots, m)$

Problem #1

Write the dual problem of the following LP problem

$$
\begin{array}{lrcrcrcrcrl}
\max & 4x_1 & + & 3x_2 & + & 7x_3 & + & 9x_4 & & & \\
\text{s.t.} & 2x_1 & - & 4x_2 & + & x_3 & & & \leq & -1 & (y_1 \geq 0) \\
 & x_1 & + & 5x_2 & + & x_3 & + & x_4 & \leq & 16 & (y_2 \geq 0) \\
 & x_1 & & & + & x_3 & & & \leq & 5 & (y_3 \geq 0) \\
 & 2x_1 & + & 4x_2 & & & - & x_4 & \leq & 8 & (y_4 \geq 0) \\
 & & & x_2 & - & 3x_3 & + & x_4 & \leq & 0 & (y_5 \geq 0) \\
 & & & -4x_2 & & & + & 3x_4 & \leq & 4 & (y_6 \geq 0) \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & \leq & 19 & (y_7 \geq 0) \\
 & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \geq & 0 &
\end{array}
$$

The primal variables are non-negative and it is a maximization primal problem: the dual constraints are $\geq$ constraints

Problem #2

Write the dual problem of the following LP problem

$$
\begin{array}{lrcrcrcll}
\min & 5x_1 & + & 2x_2 & + & 6x_3 & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & \geq & 5 & (y_1 \geq 0) \\
 & & & x_2 & + & 2x_3 & \geq & 4 & (y_2 \geq 0) \\
 & x_1 & & & + & x_3 & \geq & 4 & (y_3 \geq 0) \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

The primal variables are non-negative and it is a minimization primal problem: the dual constraints are $\leq$ constraints

Problem #3

Write the dual problem of the following LP problem

$$
\begin{array}{lrcrcrcll}
\max & 3x_1 & + & 2x_2 & + & 5x_3 & & & \\
\text{s.t.} & 5x_1 & + & 3x_2 & + & x_3 & = & -8 & (y_1 \text{ free}) \\
 & 4x_1 & + & 2x_2 & + & 8x_3 & \leq & 1 & (y_2 \geq 0) \\
 & 6x_1 & + & 7x_2 & + & 3x_3 & \geq & 1 & (y_3 \leq 0) \\
 & x_1 & & & & & \leq & 4 & (y_4 \geq 0) \\
 & & & & & x_3 & \geq & 0 &
\end{array}
$$

One primal variable is non-negative and the other ones are free; it is a maximization primal problem: one dual constraint is an $\geq$ constraint, the two other ones are equations

Dual Problem (cont'd)

PRIMAL	**DUAL**
maximization	minimization
$\leq$ constraint	variable ≥ 0
$\geq$ constraint	variable ≤ 0
$=$ constraint	unconstrained variable
variable ≥ 0	$\geq$ constraint
variable ≤ 0	$\leq$ constraint
unconstrained variable	$=$ constraint
right-hand side	objective function
objective function	right-hand side

Proposition 1

The dual of the dual is the primal

Primal/Dual solution

- A feasible solution of the primal problem is called a primal solution
- A feasible solution of the dual problem is called a dual solution

Primal and Dual Problems

- Primal problem

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\
& x_j \geq 0 & \text{for } j = 1, 2 \ldots, n
\end{array}
$$

- Dual problem

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\
\text{subject to} & \displaystyle\sum_{i=1}^{m} a_{ij} y_i \geq c_j & \text{for } j = 1, 2 \ldots, n \\
& y_i \geq 0 & \text{for } i = 1, 2 \ldots, m
\end{array}
$$

Weak Duality Theorem

Theorem 2 (Weak duality)

Suppose that x is a primal solution and y is a dual solution. Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Weak Duality Theorem

Theorem 2 (Weak duality)

Suppose that x is a primal solution and y is a dual solution. Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Corollary 3

Suppose that x^* is a primal solution, y^* is a dual solution and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Then x^* is an optimal primal solution, and y^* is an optimal dual solution

Back to our example:

- primal solution $x_1^* = 0, x_2^* = 14, x_3^* = 0, x_4^* = 5$
- dual solution $y_1^* = 11, y_2^* = 0, y_3^* = 6$

Optimal Dual Solution and Primal Dictionary

Optimal dictionary for the primal

$$
\begin{array}{rcrcrcrcrcr}
x_2 & = & 14 & - & 2x_1 & - & 4x_3 & - & 5x_5 & - & 3x_7 \\
x_4 & = & 5 & - & x_1 & - & x_3 & - & 2x_5 & - & x_7 \\
x_6 & = & 1 & + & 5x_1 & + & 9x_3 & + & 21x_5 & + & 11x_7 \\
\hline
z & = & 29 & - & x_1 & - & 2x_3 & - & 11x_5 & - & 6x_7
\end{array}
$$

Optimal Dual Solution and Primal Dictionary

Optimal dictionary for the primal

$$
\begin{array}{rcrcrcrcrcr}
x_2 & = & 14 & - & 2x_1 & - & 4x_3 & - & 5x_5 & - & 3x_7 \\
x_4 & = & 5 & - & x_1 & - & x_3 & - & 2x_5 & - & x_7 \\
x_6 & = & 1 & + & 5x_1 & + & 9x_3 & + & 21x_5 & + & 11x_7 \\
\hline
z & = & 29 & - & x_1 & - & 2x_3 & - & 11x_5 & - & 6x_7
\end{array}
$$

x_5 is the slack variable for the first primal constraint; y_1 is the dual variable associated with the first primal constraint

The slack variables x_5, x_6, x_7 can be matched up with the dual variables y_1, y_2, y_3 as follows

$$x_5 \leftrightarrow y_1,\ x_6 \leftrightarrow y_2,\ x_7 \leftrightarrow y_3$$

Assign, with reversed signed, the reduced-cost coefficients of the slack variables to the corresponding dual variables, that is,

$$y_1 = 11,\ y_2 = 0,\ y_3 = 6$$

Duality Theorem

Theorem 4 (Strong duality)

If the primal problem has an optimal solution $(x_1^*, x_2^*, \ldots, x_n^*)$, then the dual problem has an optimal solution $(y_1^*, y_2^*, \ldots, y_m^*)$ so that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Corollary 5

(i) *The primal problem has an optimal solution if and only if the dual solution has an optimal solution*

(ii) *If the primal is unbounded, then the dual is infeasible*

(iii) *If the dual is unbounded, then the primal is infeasible*

Self-Duality

Consider the following LP problem

$$
\begin{array}{lrcrcrcr}
\min & 4x_1 & + & 7x_2 & + & 9x_3 & & \\
\text{s.t.} & & & x_2 & + & 2x_3 & \geq & -4 \\
 & -x_1 & & & + & 6x_3 & \geq & -7 \\
 & 2x_1 & + & 6x_2 & & & \leq & 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Write its dual

Both primal and dual problems are equivalent

Both primal and dual problems may be infeasible at the same time

Skew-symmetric matrix : $A^T = -A$

Primal/Dual Combinations

		DUAL		
		Optimal	**Infeasible**	**Unbounded**
	Optimal	Possible	Impossible	Impossible
PRIMAL	**Infeasible**	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

Problem #4

$$
\begin{array}{llrcrcr}
\max & z = & -x_1 & - & 2x_2 & & \\
\text{s.t.} & & -3x_1 & + & x_2 & \leq & -1 \\
 & & x_1 & - & x_2 & \leq & 1 \\
 & & -2x_1 & + & 7x_2 & \leq & 6 \\
 & & 9x_1 & - & 4x_2 & \leq & 6 \\
 & & -5x_1 & + & 2x_2 & \leq & -3 \\
 & & 7x_1 & - & 3x_2 & \leq & 6 \\
 & & & & x_1, x_2 & \geq & 0
\end{array}
$$

Solve the dual problem using the simplex method

Read the optimal primal solution off the optimal dictionary

Problem #4 (cont'd)

Dual problem

$$\begin{array}{lrcrcrcrcrcrcl}
\min & -y_1 & + & y_2 & + & 6y_3 & + & 6y_4 & - & 3y_5 & + & 6y_6 & & \\
\text{s.t.} & -3y_1 & + & y_2 & - & 2y_3 & + & 9y_4 & - & 5y_5 & + & 7y_6 & \geq & -1 \\
 & y_1 & - & y_2 & + & 7y_3 & - & 4y_4 & + & 2y_5 & - & 3y_6 & \geq & -2 \\
 & & & & & & & & & \multicolumn{3}{r}{y_1, y_2, y_3, y_4, y_5, y_6} & \geq & 0
\end{array}$$

Optimal dictionary

$$\begin{array}{rclcrcrcrcrcrcr}
y_5 & = & \frac{1}{5} & - & \frac{3}{5}y_1 & + & \frac{1}{5}y_2 & - & \frac{2}{5}y_3 & + & \frac{9}{5}y_4 & + & \frac{7}{5}y_6 & - & \frac{1}{5}y_7 \\
y_8 & = & \frac{12}{5} & - & \frac{1}{5}y_1 & - & \frac{3}{5}y_2 & + & \frac{31}{5}y_3 & - & \frac{2}{5}y_4 & - & \frac{1}{5}y_6 & - & \frac{2}{5}y_7 \\
\hline
-z & = & \frac{3}{5} & - & \frac{4}{5}y_1 & - & \frac{2}{5}y_2 & - & \frac{36}{5}y_3 & - & \frac{3}{5}y_4 & - & \frac{9}{5}y_6 & - & \frac{3}{5}y_7
\end{array}$$

Optimal primal solution $x_1^* = \frac{3}{5}$ $x_2^* = 0$

Problem #4 (cont'd)

Suppose somebody claims that

- $x_1 = \frac{3}{5}\ x_2 = 0$
- $y_1 = 0\ y_2 = 0\ y_3 = 0\ y_4 = 0\ y_5 = \frac{1}{5}\ y_6 = 0$

are optimal solutions for the primal and dual problems, respectively
Can you check it without using the simplex method?

Problem #4 (cont'd)

Suppose somebody claims that

- $x_1 = \frac{3}{5}$ $x_2 = 0$
- $y_1 = 0$ $y_2 = 0$ $y_3 = 0$ $y_4 = 0$ $y_5 = \frac{1}{5}$ $y_6 = 0$

are optimal solutions for the primal and dual problems, respectively
Can you check it without using the simplex method?

Same question with the following claim

- $x_1 = \frac{3}{5}$ $x_2 = 0$
- $y_1 = \frac{1}{5}$ $y_2 = 0$ $y_3 = 0$ $y_4 = 0$ $y_5 = \frac{2}{15}$ $y_6 = 0$

are optimal solutions for the primal and dual problems, respectively

Same question with the following claim

- $x_1 = \frac{1}{5}$ $x_2 = \frac{1}{5}$
- $y_1 = 0$ $y_2 = 0$ $y_3 = 0$ $y_4 = 0$ $y_5 = \frac{1}{5}$ $y_6 = 0$

are optimal solutions for the primal and dual problems, respectively

First Claim

PRIMAL slack variables	DUAL decision variables
$x_3 = \frac{4}{5}$	$y_1 = 0$
$x_4 = \frac{2}{5}$	$y_2 = 0$
$x_5 = \frac{36}{5}$	$y_3 = 0$
$x_6 = \frac{3}{5}$	$y_4 = 0$
$x_7 = 0$	$y_5 = \frac{1}{5}$
$x_8 = \frac{9}{5}$	$y_6 = 0$

PRIMAL decision variables	DUAL slack variables
$x_1 = \frac{3}{5}$	$y_7 = 0$
$x_2 = 0$	$y_8 = \frac{12}{5}$

PRIMAL objective function	DUAL objective function
$-\frac{3}{5}$	$-\frac{3}{5}$

Second Claim

PRIMAL slack variables	DUAL decision variables
$x_3 = \frac{4}{5}$	$y_1 = \frac{1}{5}$
$x_4 = \frac{2}{5}$	$y_2 = 0$
$x_5 = \frac{36}{5}$	$y_3 = 0$
$x_6 = \frac{3}{5}$	$y_4 = 0$
$x_7 = 0$	$y_5 = \frac{2}{15}$
$x_8 = \frac{9}{5}$	$y_6 = 0$

PRIMAL decision variables	DUAL slack variables
$x_1 = \frac{3}{5}$	$y_7 = -\frac{4}{15} < 0$
$x_2 = 0$	$y_8 = 3$

PRIMAL objective function	DUAL objective function
$-\frac{3}{5}$	$-\frac{3}{5}$

Third Claim

PRIMAL slack variables	DUAL decision variables
$x_3 = -\frac{3}{5} < 0$	$y_1 = 0$
$x_4 = 1$	$y_2 = 0$
$x_5 = 5$	$y_3 = 0$
$x_6 = 5$	$y_4 = 0$
$x_7 = -\frac{12}{5} < 0$	$y_5 = \frac{1}{5}$
$x_8 = \frac{26}{5}$	$y_6 = 0$

PRIMAL decision variables	DUAL slack variables
$x_1 = \frac{1}{5}$	$y_7 = 0$
$x_2 = \frac{1}{5}$	$y_8 = \frac{12}{5}$

PRIMAL objective function	DUAL objective function
$-\frac{3}{5}$	$-\frac{3}{5}$

Primal and Dual problems

Primal problem

$$
\begin{array}{lll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\
& x_j \geq 0 & \text{for } j = 1, 2 \ldots, n
\end{array}
$$

Dual problem

$$
\begin{array}{lll}
\text{minimize} & \sum_{i=1}^{m} b_i y_i & \\
\text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & \text{for } j = 1, 2 \ldots, n \\
& y_i \geq 0 & \text{for } i = 1, 2 \ldots, m
\end{array}
$$

Complementary Slackness Conditions

Complementary slackness conditions

Let x^* and y^* be feasible solutions for the primal and dual problems, respectively. We say that they satisfy the complementary slackness conditions if

$$x_j^* = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ (or both) for } j = 1, 2, \ldots, n \tag{1}$$

and

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \text{ or } y_i^* = 0 \text{ (or both) for } i = 1, 2, \ldots, m \tag{2}$$

(1) for every primal variable, either it is equal to zero or the slack variable of the associated dual constraint is zero

(2) for every primal constraint, either its slack variable is zero or the associated dual variable is zero

Complementary Slackness Theorem

Theorem 6 (Complementary slackness theorem)

Let x^* and y^* be feasible solutions for the primal and dual problems, respectively. The complementary slackness conditions are necessary and sufficient conditions for simultaneous optimality of x^* and y^*.

Proof. The following expression

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*$$

holds with equality throughout if and only if

$$c_j x_j^* = (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* \quad \text{for } j = 1, 2, \dots, n$$

$$(\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* = b_i y_i^* \quad \text{for } i = 1, 2, \dots, m$$

Proof of the Theorem

The first n equations gives Condition (1)

The last m equations gives Condition (2)

Therefore, Conditions (1) and (2) are necessary and sufficient for

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

to hold

On the other hand, the Duality theorem shows that this equality is necessary and sufficient for simultaneous optimality of x^* and y^* $\square$

A More Applicable Form

Theorem 7

A feasible solution x^* to the primal problem is optimal if and only if there are numbers $y_1^*, y_2^*, \ldots, y_m^*$ so that (complementary slackness conditions)

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \qquad \text{whenever } x_j^* > 0$$

$$y_i^* = 0 \qquad \text{whenever } \sum_{j=1}^{n} a_{ij} x_j^* < b_j$$

and so that (y^* is a dual solution)

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \qquad \text{for } j = 1, 2, \ldots, n$$

$$y_i^* \geq 0 \qquad \text{for } i = 1, 2, \ldots, m$$

Uniqueness

This strategy for verifying optimality of allegedly optimal solutions is applicable only if the system of equations of Theorem 7 has a unique solution

Theorem 8

If x^* is a nondegenerate basic primal solution, then the system of equations of Theorem 7 has a unique solution