Z325FU04 - Modèles Linéaires de la Recherche Opérationnelle

Two-Phase Simplex Method

Hervé Kerivin, PhD

Université Clermont Auvergne
Institut d'Informatique
LIMOS UMR 6158 CNRS
Clermont-Ferrand

Office: ISIMA B133

kerivin@isima.fr

Three Kinds of Pitfalls

The simplex method starts from a feasible dictionaries

Initialization: how do we get hold of a feasible dictionary?

The simplex method moves from one dictionary to another one in its search for an optimal solution

Iteration: can we always choose an entering variable, find the leaving variable, and construct the next feasible dictionary by pivoting?

The simplex method stops when no nonbasic variable can enter the basis without making the value of the objective function decrease

Termination: Can the simplex method construct an endless sequence of dictionaries without ever reaching an optimal solution?

Example

Consider the following feasible dictionary

$$\begin{array}{rcrcrcrcr} x_4 & = & 1 & & & & & - & 2x_3 \\ x_5 & = & 3 & - & 2x_1 & + & 4x_2 & - & 6x_3 \\ x_6 & = & 2 & + & x_1 & - & 3x_2 & - & 4x_3 \\ \hline z & = & & & 2x_1 & - & x_2 & + & 8x_3 \end{array}$$

Choose x_3 as the entering variable

Three candidates (i.e., x_4, x_5, x_6) for being the leaving variable; choose x_4 with $x_3 \leq \frac{1}{2}$

After pivoting, the new feasible dictionary is

$$\begin{array}{rcrcrcrcr} x_3 & = & \frac{1}{2} & & & & & - & \frac{1}{2}x_4 \\ x_5 & = & & - & 2x_1 & + & 4x_2 & + & 3x_4 \\ x_6 & = & & & x_1 & - & 3x_2 & + & 2x_4 \\ \hline z & = & 4 & + & 2x_1 & - & x_2 & - & 4x_4 \end{array}$$

Two basic variables (i.e., x_5, x_6) take the value zero

Example (cont'd)

Current feasible dictionary

$$
\begin{array}{rcrcrcrcr}
x_3 & = & \frac{1}{2} & & & & & - & \frac{1}{2}x_4 \\
x_5 & = & & - & 2x_1 & + & 4x_2 & + & 3x_4 \\
x_6 & = & & & x_1 & - & 3x_2 & + & 2x_4 \\
\hline
z & = & 4 & + & 2x_1 & - & x_2 & - & 4x_4
\end{array}
$$

Current feasible solution: $x_3 = \frac{1}{2}, x_1 = x_2 = x_4 = x_5 = x_6 = 0$

Choose x_1 as the entering variable

Three candidates (i.e., x_5, x_6) for being the leaving variable; choose x_5 with $x_1 \leq 0$

After pivoting, the new feasible dictionary is

$$
\begin{array}{rcrcrcrcr}
x_1 & = & & & 2x_2 & + & \frac{3}{2}x_4 & - & \frac{1}{2}x_5 \\
x_3 & = & \frac{1}{2} & & & - & \frac{1}{2}x_4 & & \\
x_6 & = & & - & x_2 & + & \frac{7}{2}x_4 & - & \frac{1}{2}x_5 \\
\hline
z & = & 4 & + & 3x_2 & - & x_4 & - & x_5
\end{array}
$$

Current feasible solution: $x_3 = \frac{1}{2}, x_1 = x_2 = x_4 = x_5 = x_6 = 0$

Degenerate Solution and Iteration

Degenerate solution

Basic solutions with one or more basic variables at zero are called degenerate

Degenerate iteration

Simplex iterations that do not change the basic solution are called degenerate

Degeneracy is annoying since the motivation behind the simplex method is to increase the value of the objective function in each iteration

Degeneracy abounds in LP problems arising from practical applications

Whenever that happens, the simplex method may stall by going through a few (and sometime quite a few) degenerate iterations in a row; such a block of degenerate iterations ends with a breakthrough represented by a nondegenerate iteration

Another Example

Consider the following feasible dictionary

$$
\begin{array}{rclrrrrrrrr}
x_5 & = & & - & \frac{1}{2}x_1 & + & \frac{11}{2}x_2 & + & \frac{5}{2}x_3 & - & 9x_4 \\
x_6 & = & & - & \frac{1}{2}x_1 & + & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
\hline
z & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4
\end{array}
$$

Apply the following rules

- the entering variable will always be the nonbasic variable that has the largest coefficient in the last row of the dictionary
- if two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave

Another Example - After Six Iterations

1. entering variable: x_1; leaving variable: x_5; $(0,0,0,0,0,0,1)$; $z = 0$
2. entering variable: x_2; leaving variable: x_6; $(0,0,0,0,0,0,1)$; $z = 0$
3. entering variable: x_3; leaving variable: x_1; $(0,0,0,0,0,0,1)$; $z = 0$
4. entering variable: x_4; leaving variable: x_2; $(0,0,0,0,0,0,1)$; $z = 0$
5. entering variable: x_5; leaving variable: x_3; $(0,0,0,0,0,0,1)$; $z = 0$
6. entering variable: x_6; leaving variable: x_4; $(0,0,0,0,0,0,1)$; $z = 0$

Resulting feasible dictionary

$$\begin{array}{rcrcrcrcrcr}
x_5 & = & & - & \frac{1}{2}x_1 & + & \frac{11}{2}x_2 & + & \frac{5}{2}x_3 & - & 9x_4 \\
x_6 & = & & - & \frac{1}{2}x_1 & + & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
\hline
z & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4
\end{array}$$

Cycling

Cycling

The simplex method cycles if one dictionary appears in two different iterations

Cycling can occur only in the presence of degeneracy

Theorem 1

If the simplex method fails to terminate, then it must cycle

Smallest-subscript rule

The smallest-subscript rule refers to breaking ties in the choice of the entering and leaving variables by always choosing the candidate x_k that has the smallest subscript k

Theorem 2 (Bland's rule)

The simplex method terminates as long as the entering and leaving variables are selected by the smallest-subscript rule in each iteration

Back to Another Example

After the fifth iteration, the dictionary is

$$
\begin{array}{rclrrrrrrr}
x_5 & = & & & 9x_6 & + & 4x_1 & - & 8x_2 & - & 2x_3 \\
x_4 & = & & - & x_6 & - & \frac{1}{2}x_1 & + & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 \\
x_7 & = & 1 & & & - & x_1 & & & & \\
\hline
z & = & & & 24x_6 & + & 22x_1 & - & 93x_2 & - & 21x_3
\end{array}
$$

x_1 is chosen as the entering variable (smallest-subscript rule)

x_4 is the leaving variable ($x_1 \leq 1$)

After pivoting, the new feasible dictionary is

$$
\begin{array}{rclrrrrrrrr}
x_1 & = & & - & 2x_6 & + & 3x_2 & + & x_3 & - & 2x_4 \\
x_5 & = & & & x_6 & + & 4x_2 & + & 2x_3 & - & 8x_4 \\
x_7 & = & 1 & + & 2x_6 & - & 3x_2 & - & x_3 & + & 2x_4 \\
\hline
z & = & & - & 20x_6 & - & 27x_2 & + & x_3 & - & 44x_4
\end{array}
$$

Next iteration

- entering variable: x_3
- leaving variable: x_7 with $x_3 \leq 1$
- the value of z increases

Perturbation Method

Idea: add a small positive value to each right-hand side values so that those small values do not operate on the same scale

1. Choose m positive scalar $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ so that

$$0 < \epsilon_m \ll \epsilon_{m-1} \ll \ldots \ll \epsilon_2 \ll \epsilon_1 \ll 1 \tag{1}$$

2. Apply the simplex method to

$$\begin{array}{lll} \text{maximize } z & = & \displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{subject to} & & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i + \epsilon_i \quad \text{for } i = 1, 2 \ldots, m \\ & & x_j \geq 0 \quad\quad\quad\quad\quad\ \text{for } j = 1, 2 \ldots, n \end{array}$$

No linear combinations of the ϵ_i using coefficients arising in the course of the simplex method

- can produce a number whose size is of the same order as the problem's data
- can make an ϵ_i reach a larger ϵ_j

Back Again to Another Example

Initial perturbated dictionay

$$\begin{array}{rcrcrcrcrcrcrcr}
x_5 & = & \epsilon_1 & & & - & \frac{1}{2}x_1 & + & \frac{11}{2}x_2 & + & \frac{5}{2}x_3 & - & 9x_4 \\
x_6 & = & \epsilon_2 & & & - & \frac{1}{2}x_1 & + & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & - & x_4 \\
x_7 & = & 1 & + & \epsilon_3 & - & x_1 & & & & & & \\
\hline
z & = & & & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4
\end{array}$$

Entering variable: x_1

Leaving variable: x_6 $(2\epsilon_2 < 2\epsilon_1 < 1 + \epsilon_3)$

New dictionary

$$\begin{array}{rcrcrcrcrcrcrcrcr}
x_1 & = & 2\epsilon_2 & & & & & + & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_6 \\
x_5 & = & \epsilon_1 & - & \epsilon_2 & & & + & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_6 \\
x_7 & = & 1 & - & 2\epsilon_2 & + & \epsilon_3 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_6 \\
\hline
z & = & & & 20\epsilon_2 & & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_6
\end{array}$$

Back Again to Another Example (cont'd)

$$
\begin{array}{rclcrcrcrcrcrcrcr}
x_1 & = & 2\epsilon_2 & & & & & + & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_6 \\
x_5 & = & \epsilon_1 & - & \epsilon_2 & & & + & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_6 \\
x_7 & = & 1 & - & 2\epsilon_2 & + & \epsilon_3 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_6 \\
\hline
z & = & & & 20\epsilon_2 & & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_6
\end{array}
$$

Entering variable: x_3

Leaving variable: x_7

New dictionary

$$
\begin{array}{rclcrcrcrcrcrcrcr}
x_3 & = & 1 & - & 2\epsilon_2 & + & \epsilon_3 & & & - & 3x_2 & + & 2x_4 & + & 2x_6 & - & x_7 \\
x_1 & = & 1 & + & \epsilon_3 & & & & & & & & & & & - & x_7 \\
x_5 & = & 2 & + & \epsilon_1 & - & 5\epsilon_3 & + & 2\epsilon_3 & - & 2x_2 & - & 4x_4 & + & 5x_6 & - & 2x_7 \\
\hline
z & = & 1 & + & 18\epsilon_2 & + & \epsilon_3 & & & - & 30x_2 & - & 42x_4 & - & 18x_6 & - & x_7
\end{array}
$$

This is an optimal dictionay for the perturbated problem

Disregarding all the terms involving ϵ_1, ϵ_2, and ϵ_3 gives an optimal dictionary of the original problem

How to Choose the ϵ-Values?

Finding the entering variables: the ϵ values do not affect the choice

Finding the leaving variables: each basic variable x_i leads to a row

$$x_i = (r_{i0} + r_{i1}\epsilon_1 + \ldots + r_{in}\epsilon_m) + \sum_{j \notin B} \bar{a}_{ij} x_j$$

Perturbation method

In the perturbation method, the right-hand side values (i.e., b_i for $i = 1, 2, \ldots, m$) are slightly changed by adding extremely small values (i.e., ϵ_i for $i = 1, 2, \ldots, m$) satisfying (1)

Remarks: if the ϵ_i's were randomly chosen independently to each other (yet still small enough), the probability of an exact cancellation would be zero

Lexicographically smaller

An expression $r = r_0 + r_1\epsilon_1 + \ldots + r_m\epsilon_m$ is lexicographically smaller than an expression $s = s_o + s_1\epsilon_1 + \ldots + s_m\epsilon_m$ if there is a smaller subscript k so that $r_k \neq s_k$ and $r_k < s_k$.

r is lexicographically smaller than s if and only if r is numerically smaller than s for all values of $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ that satisfy (1)

Lexicographic Method

Theorem 3

The simplex method terminates as long as the leaving variable is selected by the lexicographic rule in each iteration

Lexicographic method

The lexicographic method is an implementation of the perturbation method where $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ are symbols representing indefinite quantities which satisifies (1)

The symbols (or numbers) $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$

- are only needed when a tie between candidates for leaving the basis needs to be broken
- are only added as a temporary perturbation when needed

Preventing Occurrences of Cycling

First method:

- smallest subscript (Bland, 1977)
- may be computationally inefficient with respect to the length of the simplex path generated

Second method:

- the perturbation method (Charnes, 1952)
- the lexicographic method (Dantzig, Orden, and Wolfe, 1955)
- may be computationally expensive to implement

Preventing Occurrences of Cycling

First method:

- smallest subscript (Bland, 1977)
- may be computationally inefficient with respect to the length of the simplex path generated

Second method:

- the perturbation method (Charnes, 1952)
- the lexicographic method (Dantzig, Orden, and Wolfe, 1955)
- may be computationally expensive to implement

Coming across practical problems which cycle is very rare

Computer round-off errors: updated right-hand sides are rarely exact zero values

Computer implementations of the simplex methods tend to disregard the possibility of cycling

Example

$$
\begin{array}{llrrrrrl}
\text{maximize } z = & x_1 & - & x_2 & + & x_3 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\
 & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\
 & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Infeasible origin

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\
x_5 & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 \\
x_6 & = & -1 & + & x_1 & - & x_2 & + & 2x_3 \\
\hline
z & = & & & x_1 & - & x_2 & + & x_3
\end{array}
$$

Feasible dictionary

$$
\begin{array}{rcrcrcrcr}
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 \\
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 \\
\hline
z & = & -0.6 & + & 0.2x_1 & - & 0.2x_5 & + & 0.4x_6
\end{array}
$$

Initialization: how do we get hold of a feasible dictionary?

Example of an Auxiliary Problem

Consider the following auxiliary LP problem

$$
\begin{array}{lrcrcrcrcl}
\text{minimize} & & & & & & & & & x_0 \\
\text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 & + & x_0 \\
& 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 & + & x_0 \\
& -x_1 & + & x_2 & - & 2x_3 & \leq & -1 & + & x_0 \\
& & & & x_1, x_2, & x_3, x_0 & \geq & 0 & &
\end{array}
$$

It is feasible: take $x_1 = x_2 = x_4 = 0$ and $x_0 \geq 5$

If a feasible solution $\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix}^T$ of the auxiliary problem is so that $x_0 = 0$, then the original LP problem also has a feasible solution $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$

If the original LP has a feasible solution $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \end{bmatrix}^T$, then the auxiliary problem has a feasible solution $\begin{bmatrix} 0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \end{bmatrix}^T$

The original LP problem has a feasible solution if and only if the auxiliary LP problem has a feasible solution with $x_0 = 0$

Auxiliary LP Problem

Auxiliary problem

Consider an LP problem

$$\begin{array}{lcll} \text{maximize } z & = & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2 \ldots, m \\ & & x_j \geq 0 & \text{for } j = 1, 2 \ldots, n \end{array}$$

with an infeasible origin. Its auxiliary LP problem is

$$\begin{array}{lcll} \text{maximize } w & = & -x_0 & \\ \text{subject to} & & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i & \text{for } i = 1, 2 \ldots, m \\ & & x_j \geq 0 & \text{for } j = 0, 1, 2 \ldots, n \end{array}$$

Theorem 4

The original LP problem has a feasible solution if and only if the optimum value of the auxiliary LP problem is zero

Solving the Auxiliary LP Problem

Initial dictionary for the auxiliary problem

$$
\begin{array}{rcrcrcrcrcr}
x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 & + & x_0 \\
x_5 & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 & + & x_0 \\
x_6 & = & -1 & + & x_1 & - & x_2 & + & 2x_3 & + & x_0 \\
\hline
w & = & & & & & & & & - & x_0
\end{array}
$$

Infeasible!

Solving the Auxiliary LP Problem

Initial dictionary for the auxiliary problem

$$\begin{array}{rcrcrcrcrcr} x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 & + & x_0 \\ x_5 & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 & + & x_0 \\ x_6 & = & -1 & + & x_1 & - & x_2 & + & 2x_3 & + & x_0 \\ \hline w & = & & & & & & & & - & x_0 \end{array}$$

Infeasible!

A feasible dictionary can be obtained by a single pivot: x_0 enters, x_5 leaves

Feasible initial dictionary for the auxiliary problem

$$\begin{array}{rcrcrcrcrcr} x_0 & = & 5 & + & 2x_1 & - & 3x_2 & + & x_3 & + & x_5 \\ x_4 & = & 9 & & & - & 2x_2 & - & x_3 & + & x_5 \\ x_6 & = & 4 & + & 3x_1 & - & 4x_2 & + & 3x_3 & + & x_5 \\ \hline w & = & -5 & - & 2x_1 & + & 3x_2 & - & x_3 & - & x_5 \end{array}$$

Feasible initial dictionary for the auxiliary LP problem

The infeasible initial dictionary can be transformed into a feasible one by a single pivot, with x_0 entering and the "most infeasible" slack variable x_{n+i} leaving the basis

Solving the Auxiliary LP Problem (cont'd)

Construct a feasible initial dictionary by a single pivot, with x_0 entering and the "most infeasible" slack variable x_{n+i} leaving the basis (this is not an iteration of the simplex method1)

Solving the Auxiliary LP Problem (cont'd)

Construct a feasible initial dictionary by a single pivot, with x_0 entering and the "most infeasible" slack variable x_{n+i} leaving the basis (this is not an iteration of the simplex method1)

Use the simplex method to solve the auxiliary LP problem, starting from the feasible initial dictionary, following the rule: if x_0 competes with other variables for leaving the basis, always choose x_0 as the leaving variable

At the end of the simplex method, x_0 cannot be basic and the value of w cannot be zero

Solving the Auxiliary LP Problem (cont'd)

Construct a feasible initial dictionary by a single pivot, with x_0 entering and the "most infeasible" slack variable x_{n+i} leaving the basis (this is not an iteration of the simplex method1)

Use the simplex method to solve the auxiliary LP problem, starting from the feasible initial dictionary, following the rule: if x_0 competes with other variables for leaving the basis, always choose x_0 as the leaving variable

At the end of the simplex method, x_0 cannot be basic and the value of w cannot be zero

- in the last iteration (where w was negative), the value of x_0 must have dropped from some positive level to zero
- x_0 was then a candidate for leaving the basis

Two possible types of optimal dictionaries for the auxiliary LP problem

Solving the Auxiliary LP Problem (cont'd)

Construct a feasible initial dictionary by a single pivot, with x_0 entering and the "most infeasible" slack variable x_{n+i} leaving the basis (this is not an iteration of the simplex method1)

Use the simplex method to solve the auxiliary LP problem, starting from the feasible initial dictionary, following the rule: if x_0 competes with other variables for leaving the basis, always choose x_0 as the leaving variable

At the end of the simplex method, x_0 cannot be basic and the value of w cannot be zero

- in the last iteration (where w was negative), the value of x_0 must have dropped from some positive level to zero
- x_0 was then a candidate for leaving the basis

Two possible types of optimal dictionaries for the auxiliary LP problem

- a dictionary where x_0 is nonbasic, and so the value of w is zero
- a dictionary where x_0 is basic, and the value of w is nonzero

Going Back to the Original LP Problem

After two iterations, the optimal dictionary for the auxiliary problem is

$$\begin{array}{rcrcrcrcrcr}
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 & - & 0.8x_0 \\
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 & - & 0.6x_0 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 & + & 2x_0 \\
\hline
w & = & & & & & & & & - & x_0
\end{array}$$

Feasible solution to the original LP problem:
$x_1 = 0$, $x_2 = 2.2$ and $x_3 = 1.6$

Feasible dictionary to the original LP problem

Going Back to the Original LP Problem

After two iterations, the optimal dictionary for the auxiliary problem is

$$
\begin{array}{rcrcrcrcrcr}
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 & - & 0.8x_0 \\
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 & - & 0.6x_0 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 & + & 2x_0 \\
\hline
w & = & & & & & & & & - & x_0
\end{array}
$$

Feasible solution to the original LP problem:
$x_1 = 0$, $x_2 = 2.2$ and $x_3 = 1.6$

Feasible dictionary to the original LP problem

$$
\begin{array}{rcrcrcrcr}
x_3 & = & 1.6 & - & 0.2x_1 & + & 0.2x_5 & + & 0.6x_6 \\
x_2 & = & 2.2 & + & 0.6x_1 & + & 0.4x_5 & + & 0.2x_6 \\
x_4 & = & 3 & - & x_1 & & & - & x_6 \\
\hline
z & = & -0.6 & + & 0.2x_1 & - & 0.2x_5 & + & 0.4x_6
\end{array}
$$

obtained by expressing the original objective function using only the nonbasic variables

Two-Phase Simplex Method

Two-phase simplex method

1. In the first phase, set up and solve the auxiliary problem
   - if x_0 competes with other variables for leaving the basis, choose x_0 as the leaving variable
   - optimal dictionary has one of the two properties
     - x_0 is nonbasic, and the value w^* is zero
     - x_0 is basic, and the value w^* is nonzero
2. In the second phase,
   - if $w^* = 0$, solve the original problem itself
   - if $w^* > 0$, conclude that the original problem is infeasible

Fundamental Theorem of Linear Programming

The first phase of the two-phase simplex method discovers

Fundamental Theorem of Linear Programming

The first phase of the two-phase simplex method discovers whether an LP problem is infeasible

If the LP problem is infeasible, the first phase delivers a basic feasible solution

The second phase of the two-phase simplex method discovers

Fundamental Theorem of Linear Programming

The first phase of the two-phase simplex method discovers whether an LP problem is infeasible

If the LP problem is infeasible, the first phase delivers a basic feasible solution

The second phase of the two-phase simplex method discovers whether an LP problem is unbounded

If the LP problem is not unbounded, the second phase delivers a basic optimal solution

Theorem 5 (Fundamental theorem of linear programming)

Every LP problem in the standard form has the following three properties

1. *if it has no optimal solution, then it is either infeasible or unbounded*
2. *if it has a feasible solution, then it has a basic feasible solution*
3. *if it has an optimal solution, then it has a basic optimal solution*