Advection–Diffusion Equation with a Source Term: Forward Euler Numerical Solution and Analytical Validation

Toy Sand Conveyor Model

Here we use the one-dimensional diffusion equation with advection and a source term. This model represents sand being transported along a moving conveyor belt while being continuously deposited at a uniform rate.

1 Model Description

Advection–diffusion–source equation, with a uniform source term:

$$\partial_t h + v\,\partial_x h \;=\; D\,\partial_{xx} h + s_0, \qquad x \in [0, L],\; t > 0. \tag{1}$$

The boundaries are fixed-height (Dirichlet) to match the analytic setup:

$$h(0,t) = 0, \qquad h(L,t) = 0, \tag{2}$$

with initial condition $h(x, 0) = 0$.

2 Analytical Steady-State Solution

At steady state, $Dh'' - vh' + s_0 = 0$ with $h(0) = h(L) = 0$. The known closed form is

$$h(x) = \frac{s_0 L}{v\big(1 - e^{vL/D}\big)}\Big(e^{(v/D)x} - 1\Big) \;+\; \frac{s_0}{v}\,x, \tag{3}$$

which we use solely as a benchmark for the numerics.

3 Numerical Scheme: Forward Euler Implementation

The PDE is discretised on a uniform grid $x_i = i\Delta x$, $i = 0, \ldots, N_x - 1$ with time step Δt.

3.1 Spatial Derivatives

The advection term describes how sand is carried along the belt at a constant speed v. Since the sand at each point depends mainly on material arriving from the upstream side, the spatial derivative is taken using a one–sided difference in the direction of motion:

$$(\partial_x h)_i \approx \frac{h_i - h_{i-1}}{\Delta x}, \quad \text{for } v > 0.$$

If the belt were moving in the opposite direction, the difference would simply reverse.

The diffusion term represents small random spreading of sand, which acts equally in both directions. It is therefore computed using the standard central difference:

$$(\partial_{xx} h)_i \approx \frac{h_{i+1} - 2h_i + h_{i-1}}{\Delta x^2}.$$

3.2 Time Stepping

We use the Forward Euler method:

$$h_i^{n+1} = h_i^n + \Delta t\Big[- v\,(\partial_x h)_i^n + D\,(\partial_{xx} h)_i^n + s_0 \Big].$$

Stability requires

$$\Delta t \;<\; \frac{1}{|v|/\Delta x + 2D/\Delta x^2}.$$

The Dirichlet–Dirichlet boundaries are imposed after each update:

$$h_0^{n+1} = 0, \qquad h_{N_x-1}^{n+1} = 0.$$

3.3 Parameter Selection

The numerical parameters were selected to represent a small conveyor system while ensuring a stable and clearly interpretable simulation. A belt length of $L = 10\,\mathrm{m}$ and velocity $v = 0.5\,\mathrm{m/s}$ correspond to a slow, small scale setup where material takes about 20 s to travel from source to outlet. The diffusivity $D = 0.02\,\mathrm{m^2/s}$ produces a short diffusive boundary layer ($D/v \approx 0.04\,\mathrm{m}$) near the end of the belt, and the uniform source rate $s_0 = 0.02$ results in a steady-state height of approximately 0.4 m. These values yield a Péclet number $\mathrm{Pe} = vL/D \approx 250$, so advection dominates but diffusion remains visible, providing a physically reasonable and numerically stable test case.

4 Results and Discussion

Figure 1 shows the evolution of $h(x, t)$ at successive times, with the analytic steady-state profile.

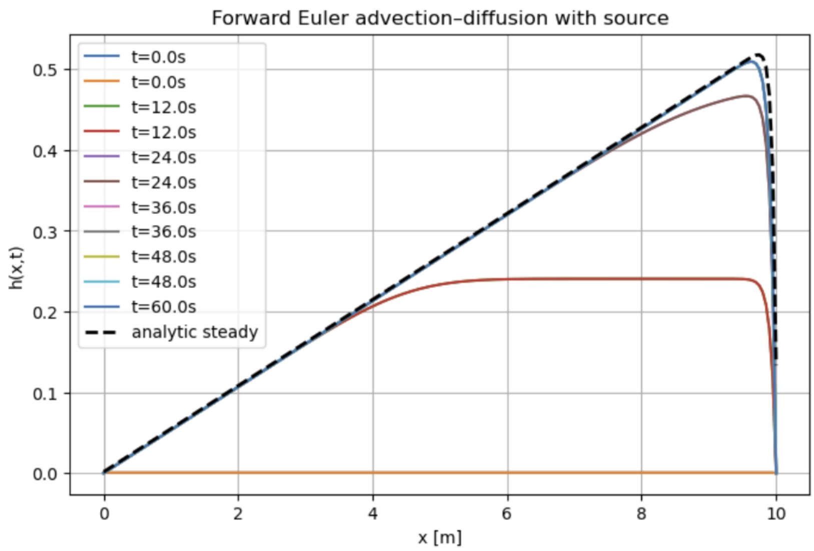

Figure 1: Forward Euler simulation of the advection–diffusion–source equation. Snapshots at $t = 12, 24, 36, 48$ s converge to the analytic steady state (black dashed line).

The numerical solution evolves smoothly from the empty initial condition toward the steady-state ramp. After roughly $t \approx L/u$, the system reaches equilibrium, and the computed profile overlays the analytic curve almost exactly.

This confirms both the correctness of the finite-difference implementation and the validity of the analytical derivation.

5 Conclusion

We have derived and validated an explicit numerical solver for the one dimensional advection diffusion equation with a uniform source and physically meaningful boundary conditions. The forward Euler method accurately reproduces the analytic steady-state solution when the stability condition is respected. This framework provides a baseline for more advanced models.