Steady State Solution

Sand Processing Problem

October 2025

To model the sand processing problem we were given we came up with the following advection-diffusion equation:

$$
\begin{aligned}
\partial_t h(x,t) + v(x,t)\partial h_x(x,t) &= D\partial_{xx} h(x,t) + S(x,t), \\
h(0,t) &= 0, \\
h(l,t) &= 0.
\end{aligned}
\tag{1}
$$

Where,

- $h(x,t)$ is the height of the sand in metres at a given time, t, and point in space, x,
- $l > 0$ is the length of the conveyor belt,
- $v(x,t)$ is the velocity of the conveyor belt,
- $D > 0$ is the constant of diffusivity of sand,
- $S(x,t)$ is the rate at which sand is added.

Steady State Equation

In steady state, we assume that the height does not changes with time:

$$\partial_t h(x,t) = 0$$

We also assume that $\partial_t S(x,t) = 0$ and $v(x,t) = v$ for some constant $v > 0$. So, we can take $S(x,t)$ as some function of x such that $S(x,t) = s(x)$. Hence, the differential equation follows as

$$v\frac{dh}{dx} = D\frac{d^2h}{dx^2} + s(x) \quad \Longrightarrow \quad Dh''(x) - vh'(x) = -s(x). \tag{2}$$

Let

$$\lambda = \frac{v}{D}, \qquad w(x) = h'(x),$$

then (2) becomes a first-order linear ODE

$$w'(x) - \lambda w(x) = -\frac{s(x)}{D} \tag{3}$$

Multiplying both side of (3) by $e^{-\lambda x}$, we get

$$e^{-\lambda x} w'(x) - \lambda e^{-\lambda x} w(x) = -e^{-\lambda x} \frac{s(x)}{D}$$

$$e^{-\lambda x} w'(x) + \frac{d}{dx}(e^{-\lambda x}) w(x) = -e^{-\lambda x} \frac{s(x)}{D}$$

$$\frac{d}{dx}\left(e^{-\lambda x} w(x)\right) = -\frac{s(x)}{D} e^{-\lambda x}.$$

Integrating from 0 to x,

$$e^{-\lambda x} w(x) - w(0) = -\frac{1}{D} \int_0^x s(z)\, e^{-\lambda z}\, dz.$$

Hence

$$h'(x) = w(x) = e^{\lambda x} \left[w(0) - \frac{1}{D} \int_0^x s(z)\, e^{-\lambda z}\, dz \right]. \tag{4}$$

Integrating again from from 0 to x,

$$h(x) - h(0) = \int_0^x e^{\lambda t} \left[w(0) - \frac{1}{D} \int_0^t s(z)\, e^{-\lambda z}\, dz \right] dt \tag{5}$$

Since, $h(0) = 0$,

$$h(x) = \int_0^x e^{\lambda t} w(0)\, dt - \frac{1}{D} \int_0^x e^{\lambda t} \left(\int_0^t s(z) e^{-\lambda z}\, dz \right) dt$$

$$\int_0^x e^{\lambda t} w(0)\, dt = w(0) \int_0^x e^{\lambda t}\, dt = w(0) \left[\frac{e^{\lambda t}}{\lambda} \right]_0^x = w(0) \frac{e^{\lambda x} - 1}{\lambda}$$

Switching the order of double integration,

$$\int_0^x e^{\lambda t} \left(\int_0^t s(z) e^{-\lambda z}\, dz \right) dt = \int_0^x s(z) e^{-\lambda z} \left(\int_z^x e^{\lambda t} dt \right) dz$$

$$\int_z^x e^{\lambda t} dt = \left[\frac{e^{\lambda t}}{\lambda} \right]_z^x = \frac{e^{\lambda x} - e^{\lambda z}}{\lambda}$$

Hence,

$$\int_0^x e^{\lambda t} \left(\int_0^t s(z) e^{-\lambda z}\, dz \right) dt = \int_0^x s(z) e^{-\lambda z} \cdot \frac{e^{\lambda x} - e^{\lambda z}}{\lambda}\, dz = \frac{1}{\lambda} \int_0^x s(z) \left(e^{\lambda (x-z)} - 1 \right) dz$$

So, the final solution is,

$$\boxed{h(x) = \frac{w(0)}{\lambda} \left(e^{\lambda x} - 1 \right) - \frac{1}{D\lambda} \int_0^x s(z) \left(e^{\lambda (x-z)} - 1 \right) dz} \tag{6}$$

where, $\lambda = \frac{v}{D}$ and as $h(L) = 0$, hence $w(0) = \frac{1}{D\left(e^{\lambda L} - 1\right)} \int_0^L s(z) \left(e^{\lambda (L-z)} - 1 \right) dz$