Industrial Mathematics Analytic Solution

Sand Processing Problem

October 2025

To model the sand processing problem we were given we came up with the following advection-diffusion equation:

$$\begin{aligned} \partial_t h(x,t) + v(x,t)\partial h_x(x,t) = D\partial_{xx}h(x,t) + S(x,t), \\ h(0,t) = 0, \\ h(l,t) = 0. \end{aligned} \tag{1}$$

Where,

- $h(x,t)$ is the height of the sand in metres at a given time, t, and point in space, x,
- $l > 0$ is the length of the conveyor belt,
- $v(x,t)$ is the velocity of the conveyor belt,
- $D > 0$ is the constant of diffusivity of sand,
- $S(x,t)$ is the rate at which sand is added.

However, this is difficult to solve analytically. To simplify things we'll start by assuming that $h(x,t) = \alpha$ for some constant α. This implies that $\partial_t h(x,t) = 0$. We'll also assume that $S(x,t) = S$ and $v(x,t) = v$ for some constants $S > 0$ and $v > 0$. This gives us the one dimensional steady state of the advection-diffusion equation that can be defined as follows,

$$\begin{aligned} D\partial_{xx}h(x,t) - v\partial_x h(x,t) + S = 0, \\ h(0,t) = 0, \\ h(l,t) = 0. \end{aligned} \tag{2}$$

We can now regard $h(x,t)$ as a function of x such that $h(x,t) = g(x)$. This reduces the problem to solving the ordinary differential equation below,

$$\begin{aligned} Dg''(x) - vg'(x) + S = 0, \\ g(0) = 0, \\ g(l) = 0. \end{aligned} \tag{3}$$

The complementary equation is $Dg''(x) - vg'(x) = 0$, which has the characteristic polynomial $Dm^2 - vm = 0$. This implies that the homogeneous solution is $g_h(x) = C_1 e^{\frac{v}{d}x} +$

C_2. Since S is constant and a constant term appears in the homogeneous solution, we try a linear function, $g_p(x) = Ax$, for the particular solution. Substituting into Equation 3 gives $-Av + S = 0 \Rightarrow A = \frac{S}{v}$. So the general solution is,

$$g(x) = C_1 e^{\frac{v}{d}x} + C_2 + \frac{S}{v}x. \tag{4}$$

Using the initial conditions to solve for C_1 and C_2, we see that,

$$\begin{aligned} C_1 &= -C_2 \text{ and} \\ C_1 &= \frac{Sl}{v(1 - e^{\frac{vl}{d}})}. \end{aligned} \tag{5}$$

Therefore the solution is,

$$g(x) = \frac{Sl}{v(1 - e^{\frac{vl}{d}})} \left(e^{\frac{v}{d}x} - 1 \right) + \frac{S}{v}x. \tag{6}$$

We wish to solve the one dimensional steady state of the advection-diffusion equation

$$\begin{aligned} D\partial_{xx}h(x,t) - v\partial_x h(x,t) + S &= 0, && (7)\\ h(0,t) &= 0, && (8)\\ h(l,t) &= 0. && (9) \end{aligned}$$

Where

- $h(x,t)$ is the height of the sand in metres at a given time t and point in space x
- $l > 0$ is the length of the conveyor
- $v > 0$ is the velocity of the conveyor belt
- $D > 0$ is the constant of diffusivity of sand
- $S > 0$ is amount of sand added, constant with respect to space and time

Since $\partial_t h(x,t) = 0$, we can regard h as a function of x, say $h(x,t) = g(x)$. So the problem reduces to solving the ODE

$$\begin{aligned} Dg''(x) - vg'(x) + S &= 0,\\ g(0) &= 0, \\ g(l) &= 0. \end{aligned} \tag{10}$$

The complementary equation is $Dg''(x) - vg'(x) = 0$, which has the characteristic polynomial $Dm^2 - vm = 0$. This implies that the homogeneous solution is $g_h(x) = C_1 e^{\frac{v}{d}x} + C_2$. For the particular solution, since S is constant and a constant term appears in the homogeneous solution, we try a linear function, $g_p(x) = Ax$. Substituting into the ODE gives $-Av + S = 0 \Rightarrow A = \frac{S}{v}$. So the general solution is $g(x) = C_1 e^{\frac{v}{d}x} + C_2 + \frac{S}{v}x$. Using the initial conditions, we see

$$\begin{aligned} C_1 &= -C_2\\ C_1 &= \frac{Sl}{v(1 - e^{\frac{vl}{d}})}. \end{aligned}$$

Therefore the solution is

$$g(x) = \frac{Sl}{v(1 - e^{\frac{vl}{d}})}\left(e^{\frac{v}{d}x} - 1\right) + \frac{S}{v}x$$