$$\partial_t h(x,t) + v(x,t)\,\partial_x h(x,t) = D\partial_{xx} h(x,t) + S \quad , \quad h(0,t)=0,\ h(\ell,t)=0$$

$$v>0$$

let $x_m = m\Delta x, \quad t_n = n\Delta t$

$$h''(x_m) = \frac{h(x_{m+1}) - 2h(x_m) + h(x_{m-1})}{(\Delta x)^2} + O((\Delta x)^2)$$

$$h'(x_m) = \frac{h(x_m) - h(x_{m-1})}{\Delta x}$$

let $H_m = H(x_m, t)$

$$\frac{dH_m}{dt} = D\,\frac{H_{m+1} - 2H_m + H_{m-1}}{(\Delta x)^2} - v\,\frac{H_m - H_{m-1}}{\Delta x} + S$$

let $H_m^n = H(x_m, t_n)$

$$\frac{\partial h}{\partial t}(x,t) \approx \frac{H_m^{n+1} - H_m^n}{\Delta t} = D\,\frac{H_{m+1}^n - 2H_m^n + H_{m-1}^n}{(\Delta x)^2} - v\,\frac{H_m^n - H_{m-1}^n}{\Delta x} + S$$

$$H_m^n = H_m^n + D\Delta t\,\frac{H_{m+1}^n - 2H_m^n + H_{m-1}^n}{(\Delta x)^2} - v\Delta t\,\frac{H_m^n - H_{m-1}^n}{\Delta x} + S$$

Discretize using a backward difference (backward Euler is unconditionally stable)

$$h'(t_n) = \frac{h(t_n) - h(t_{n-1})}{\Delta t} + O(\Delta t)$$

$$\frac{H_m^n - H_m^{n-1}}{\Delta t} = D\,\frac{H_{m+1}^n - 2H_m^n + H_{m-1}^n}{(\Delta x)^2} - v\,\frac{H_m^n - H_{m-1}^n}{\Delta x} + S$$

<u>time</u> shift by 1 forward

$$\frac{H_m^{n+1} - H_m^n}{\Delta t} = D\,\frac{H_{m+1}^{n+1} - 2H_m^{n+1} + H_{m-1}^{n+1}}{(\Delta x)^2} - v\,\frac{H_m^{n+1} - H_{m-1}^{n+1}}{\Delta x} + S$$

$$H_m^n = -D\Delta t\,\frac{H_{m+1}^{n+1} - 2H_m^{n+1} + H_{m-1}^{n+1}}{(\Delta x)^2} + v\Delta t\,\frac{H_m^{n+1} - H_{m-1}^{n+1}}{\Delta x} - S\Delta t + H_m^{n+1}$$

let $C = \frac{D\Delta t}{(\Delta x)^2}$ and $U = \frac{v\Delta t}{\Delta x}$

$$H_m^n = -C\left[H_{m+1}^{n+1} - 2H_m^{n+1} + H_{m-1}^{n+1}\right] + U\left[H_m^{n+1} - H_{m-1}^{n+1}\right] - S\Delta t + H_m^{n+1}$$

$$H_m^n = -C\left[H_{m+1}^{n+1} - 2H_m^{n+1} + H_{m+1}^{n+1}\right] + U\left[H_m^{n+1} - H_{m-1}^{n+1}\right] - \underbrace{S\Delta t}_{\text{constant}} + H_m^{n+1}$$

$$\therefore \underbrace{-C}_{\text{upper}} H_{m+1}^{n+1} + \underbrace{(1+2C+U)}_{\text{diag}} H_m^{n+1} - \underbrace{(C+U)}_{\text{lower}} H_{m-1}^{n+1} = H_m^n + S\Delta t$$

$AH^{n+1} = b$ where $b = H_m^n + S\Delta t$ $\quad b \in \mathbb{R}^M$, $\quad 0 \le m \le M$

Dirichlet B.C. $\quad U_0(t) = 0$ corresponds to $h(0,t) = 0$

$\quad U_M(t) = 0$ corresponds to $h(L,t) = 0$