Robin boundary condition on steady state analysis

$Dh'' - vh' + S = 0$

$h(0) = 0$

$Dh'(L) = \alpha h(L) \longrightarrow$ Robin BC at $x = L$

$v > 0$, $D > 0$, S const. $\alpha \geq 0$ (now absorbing outlet is)

<u>finding sol.</u>

$\hat{h}(x) = ax + b$

$\hat{h}'(x) = a$

$\hat{h}''(x) = 0$

$D \cdot 0 - va + S = 0 \quad a = \frac{S}{V}$

b arbitrary $\therefore \quad \hat{h}(x) = \frac{S}{V} x$

(homogenous eq.)

$Dh'' - vh' = 0$

$Dr^2 - vr = 0$

$r(Dr - v) = 0 \Rightarrow \quad r = 0, \; r = \frac{v}{D}$

$\dot{h}(x) = A + Be^{\left(\frac{v}{D}\right)x}$

<u>General sol.</u>

$h(x) = \tilde{h}(x) + \dot{h}(x)$

$h(x) = A + Be^{\left(\frac{v}{D}\right)x} + \frac{S}{V}x$

at $x = 0$ $\quad (h(0) = 0)$

$$0 = A + B + 0 \Rightarrow A = -B$$

$$h(x) = B\left(e^{\left(\frac{v}{D}\right)x} - 1\right) + \frac{S}{v}x$$

$$h'(x) = B\left(\frac{v}{D}\right)e^{\left(\frac{v}{D}\right)x} + \frac{S}{v}$$

at $x = L$ $\quad \left(Dh'(L) = \alpha h(L)\right)$

LHS:

$$Dh'(L) = D\left[B\left(\frac{v}{D}\right)e^{\left(\frac{v}{D}\right)L} + \frac{S}{v}\right] = vBe^{\left(\frac{v}{D}\right)L} + \frac{DS}{v}$$

RHS:

$$\alpha h(L) = \alpha\left[B\left(e^{\left(\frac{v}{D}\right)L} - 1\right) + \frac{S}{v}(L)\right] \cancel{+\alpha B}$$

$$= \alpha B\left(e^{\left(\frac{v}{D}\right)L} - 1\right) + \alpha\frac{S}{v}L$$

$\therefore$

$$vBe^{\left(\frac{v}{D}\right)L} + \frac{DS}{v} = \alpha B\left(e^{\left(\frac{v}{D}\right)L} - 1\right) + \alpha\frac{S}{v}(L)$$

$$B\left(ve^{\left(\frac{v}{D}\right)L} - \alpha\left(e^{\left(\frac{v}{D}\right)L} - 1\right)\right) = \cancel{\alpha}\frac{S}{v}\left(\alpha L - D\right)$$

$$B = \frac{\frac{S}{v}\left(\alpha L - D\right)}{ve^{\left(\frac{v}{D}\right)L} - \alpha\left(e^{\left(\frac{v}{D}\right)L} - 1\right)}$$

$$\therefore \; h(x) = \left(\frac{\frac{S}{V}(\alpha L - D)}{V e^{\left(\frac{V}{D}\right)L} - \alpha \left(e^{\left(\frac{V}{D}\right)L} - 1\right)} \right) \left(e^{\left(\frac{V}{D}\right)x} - 1 \right) + \frac{S}{V} x$$