

# CAT FLIPPING

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## 1. MOMENT OF INERTIA OF THE CAT

We model a falling cat with two cylinders  $A$  and  $B$  connected by a spherical joint (which we assume to be massless) with  $A$  and  $B$  separated by angle  $\theta$ .

**Claim 1.** About the center of mass, the  $y$  axis of the cat is principle. Further, we can write  $\vec{L} = (2I_{yy})\omega_y\hat{y}$ . For an  $I_{yy}$  dependent on the geometry of the cylinders and  $\theta$ .

*Proof.* Let the moment of inertia tensors of cylinders  $A$  and  $B$  about the center of mass of the cat as a whole be denoted  $I_A$  and  $I_B$ . Recall the definition of the moment of inertia tensor:

$$I = \begin{pmatrix} \sum(y^2 + z^2) & \sum xy & \sum xz \\ \sum yx & \sum(x^2 + z^2) & \sum yz \\ \sum zx & \sum zy & \sum(x^2 + y^2) \end{pmatrix}$$

Then, by the reflectional symmetry of bodies  $A$  and  $B$ , the calculations for  $I_A$  and  $I_B$  are identical except for the replacement of  $y$  by  $-y$ . So, we can safely write

$$I_A = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \text{ and } I_B = \begin{pmatrix} I_{xx} & -I_{xy} & I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ I_{zx} & -I_{zy} & I_{zz} \end{pmatrix},$$

And so the moment of inertia tensor  $I$  of the cat as a whole is of the form

$$I = I_A + I_B = \begin{pmatrix} 2I_{xx} & 0 & 2I_{xz} \\ 0 & 2I_{yy} & 0 \\ 2I_{zx} & 0 & 2I_{zz} \end{pmatrix}$$

Then  $\vec{y}$  is a clear eigenvector of  $I$  confirming that the  $y$  axis is a principle axis.  $\square$

Note that the above argument can be generalized to any two bodies with reflectional symmetry through the  $x, y$  plane.

**Claim 2.** The  $I_{yy}$  in claim 1 can be written  $I_{y'y'}\cos^2(\theta/2) + I_{z'z'}\sin^2(\theta/2)$  where  $I_{y'y'}$  and  $I_{z'z'}$  are the moment of inertia elements corresponding to the principle axes.

*Proof.* Let  $x', y', z'$  denote the principle axes of cylinder  $A$  about the center of mass with  $z'$  going down the length of the cylinder,  $x'$  parallel to  $x$ , and  $y'$  in the corresponding location for a right-handed orthogonal coordinate system. Then, with this coordinate system corresponding to the principle axes, we can write the moment of inertia tensor  $I'_A$  of cylinder  $A$  about the center of mass of  $A$ .

$$I_A = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix}$$

Now, we adjust to the  $x, y, z$  coordinate system established above and solve for  $I_{yy}$ . Note  $\hat{y}$  is given by  $(0, \cos(\theta/2), \sin(\theta/2))$  in  $x', y', z'$  coordinates. Thus by adjusting coordinates we have

$$\begin{aligned} I_{yy} &= \begin{pmatrix} 0 & \cos(\theta/2) & \sin(\theta/2) \end{pmatrix} \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix} \begin{pmatrix} 0 \\ \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \\ &= I_{y'y'} \cos^2(\theta/2) + I_{z'z'} \sin^2(\theta/2) \end{aligned}$$

The above gives the value of  $I_{yy}$  about the center of mass of  $A$ , but the center of mass of the cat as a whole shares identical  $x$  and  $z$  values. So, by  $I_{yy} = \sum x^2 + z^2$ , the value of  $I_{yy}$  holds about the center of mass the cat.

□

Again, note the above argument is generalizable given the principle axes.