CAT FLIPPING

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1. Moment of Inertia of the Cat

We model a falling cat with two cylinders A and B connected by a spherical joint (which we assume to be massless) with A and B separated by angle θ.

Claim 1. About the center of mass, the y axis of the cat is principle. Further, we can write $\vec{L} = (2I_{yy})\omega_y\hat{y}$. For an I_{yy} dependent on the geometry of the cylinders and θ.

Proof. Let the moment of inertia tensors of cylinders A and B about the center of mass of the cat as a whole be denoted I_A and I_B. Recall the definition of the moment of inertia tensor:

$$I = \begin{pmatrix} \sum(y^2+z^2) & \sum xy & \sum xz \\ \sum yx & \sum(x^2+z^2) & \sum yz \\ \sum zx & \sum zy & \sum(x^2+y^2) \end{pmatrix}$$

Then, by the reflectional symmetry of bodies A and B, the calculations for I_A and I_B are identical except for the replacement of y by $-y$. So, we can safely write

$$I_A = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \text{ and } I_B = \begin{pmatrix} I_{xx} & -I_{xy} & I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ I_{zx} & -I_{zy} & I_{zz} \end{pmatrix},$$

And so the moment of inertia tensor I of the cat as a whole is of the form

$$I = I_A + I_B = \begin{pmatrix} 2I_{xx} & 0 & 2I_{xz} \\ 0 & 2I_{yy} & 0 \\ 2I_{zx} & 0 & 2I_{zz} \end{pmatrix}$$

Then $\vec{y}$ is a clear eigenvector of I confirming that the y axis is a principle axis. □

Note that the above argument can be generalized to any two bodies with reflectional symmetry through the x, y plane.

Claim 2. The I_{yy} in claim 1 can be written $I_{y'y'}\cos^2(\theta/2) + I_{z'z'}\sin^2(\theta/2)$ where $I_{y'y'}$ and $I_{z'z'}$ are the moment of inertia elements corresponding to the principle axes.

Proof. Let x', y', z' denote the principle axes of cylinder A about the center of mass with z' going down the length of the cylinder, x' parallel to x, and y' in the corresponding location for a right-handed orthogonal coordinate system. Then, with this coordinate system corresponding to the principle axes, we can write the moment of inertia tenor I'_A of cylinder A about the center of mass of A.

$$I_A = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix}$$

Now, we adjust to the x, y, z coordinate system established above and solve for I_{yy}. Note $\hat{y}$ is given by $(0, \cos(\theta/2), \sin(\theta/2))$ in x', y', z' coordinates. Thus by adjusting coordinates we have

$$\begin{aligned} I_{yy} &= \begin{pmatrix} 0 & \cos(\theta/2) & \sin(\theta/2) \end{pmatrix} \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix} \begin{pmatrix} 0 \\ \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \\ &= I_{y'y'} \cos^2(\theta/2) + I_{z'z'} \sin^2(\theta/2) \end{aligned}$$

The above gives the value of I_{yy} about the center of mass of A, but the center of mass of the cat as a whole shares identical x and z values. So, by $I_{yy} = \sum x^2 + z^2$, the value of I_{yy} holds about the center of mass the cat.

$\square$

Again, note the above argument is generalizable given the principle axes.

Then overall we have a simple relationship between the angular momentum of the cat and an angular velocity ω of the cat in the $\vec{y}$ direction.

$$\vec{L} = (I_{y'y'} \cos^2(\theta/2) + I_{z'z'} \sin^2(\theta/2))\omega\hat{y} \tag{1}$$

In particular, if A and B are two solid cylinders of equal density with length L and radius R, we have

$$\vec{L} = (/ * */) \tag{2}$$

2. ????