ABSTRACT ALGEBRA – FIRST MIDTERM EXAM

(1) Let $\omega \in \mathbb{C}$ be a solution of the equation

$$\omega^2 + \omega + 1 = 0.$$

Consider the set $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$. Show that the set $\mathbb{Z}[\omega]$ is closed under the ordinary addition and under the ordinary multiplication. Conclude that $\mathbb{Z}[\omega]$ is a ring which is a subring of the field of complex numbers.

(2) Consider the set $\mathbb{Z}[2i] = \{a+2bi \mid a, b \in \mathbb{Z}\}$. Standard number addition and multiplication turn $\mathbb{Z}[2i]$ into a commutative integral domain with identity.

(a) Prove that 2 is irreducible in this ring.

(b) Prove that $2i$ is irreducible in this ring.

(c) Is it true that $2 \mid 2i$ in this ring?

(d) Are 2 and $2i$ associates in this ring?

(e) Can you provide two factorizations of 4 into irreducibles?

(f) Is 2 prime in this ring? Justify your claim.

(g) Is $2i$ prime in this ring? Justify your claim.

(h) Is $\mathbb{Z}[2i]$ a Euclidean domain? Is it a PID?

(3) Let I be an ideal of a commutative ring R with identity. Define the following set:

$$\mathrm{rad}(I) = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Note: $\mathbb{N}$ is the set of positive integers only. In particular, $0 \notin \mathbb{N}$.

(a) Suppose temporarily that $R = \mathbb{Z}$. Find $\mathrm{rad}(I)$ for the following choices of I:

(i) $I = (9)$;

(ii) $I = (43)$;

(iii) $I = (72)$.

(b) Going back to the general situation, show $\mathrm{rad}(I)$ is an ideal. Hint: Look at your very first homework assignment.