MIDTERM2

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Problem 1

Festival of finite abelian groups.

a.

Please prove that $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/49\mathbb{Z}$.

b.

Let A be an abelian group of order 392. List all possible isomorphism classes of A.

c.

Assume further that A contains an element of order 196. List the possible isomorphism classes of A.

d.

Let $G = \mathbb{Z}/49\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Find subgroup H, K of G both of order 2 so that G/H and G/K are not isomorphic.

Problem 2

Suppose a group of prime order p acs on a finite set. What are the possible sizes of the orbits of this action?

Problem 3

Here is a nice fact: "If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p". There are many proofs of this fact. For this problem, please follow the steps below to prove this result.

a.

Let S denote the set of ordered p-tuples of element sof G the product of whose coordinates is 1. So

$$X = \{(x_1, x_2, \dots, x_p) : x_i \in G \text{ and } x_1 x_2 \cdots x_p = 1\}.$$

Show that S contains $|G|^{p-1}$ elements.

Date: 2020-04-15.

b.

We would like to define an action of the cyclic group of order p, C_p, on S. Do this by letting a permutation act on the indices of an element of S. Please prove that this is a group action.

c.

Using your work above, including Problem 2, prove the nice fact.

Problem 4

The Class Equation expresses the order of a finite group as the sum of a list of natural numbers $n_1 + n_2 + \cdots + n_k$. Consider the following sums. Please rule out those that could not appear on the right hand side of the Class Equation. Please explain your reasoning.

a.

$3 + 2 + 5$

b.

$1 + 2 + 2 + 5$

c.

$1 + 2 + 3 + 4$

d.

$2 + 2 + 2 + 2 + 2$

Problem 5

Let E/F be an extension of files. Suppose $f(x), g(x) \in F[x]$ are not both zero. Let $d_F(x)$ be the gcd of $f(x)$ and $g(x)$ in $F[x]$. Now view $f(x), g(x)$ as elements of $E[x]$, and let $d_E(x)$ be the gcd of $f(x)$ and $g(x)$ in $E[x]$. Show $d_F(x) = d_E(x)$. (This is a bit surprising since various questions involving divisibility such as irreducibility depend on the field be used.)

Problem 6

The algebraic numbers $\mathcal{A}$ are all numbers in $\mathbb{C}$ that are algebraic over $\mathbb{Q}$. They are a subfield of $\mathbb{C}$; you can assume this without proof. (Its not a bad proof, feel free to enjoy it in a non-test setting.) Please prove that $\mathcal{A}$ is not a finite extension of $\mathbb{Q}$. Git lots of details.