Orbital Mechanics

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Abstract

Orbital mechanics or astrodynamics is the applicaiton of pallistics and celestial mechanics to the practicale problems concerning the motion of rockets and other spacecraft. The motion of these objects is usually calculated from Newton's laws of motion and Newton's law of universal gravitation. It is a core disciplin within space mission design and control. Celestial mechanics trats more broadly the orbital dynamics of systems under the influence of gravity, including both spacecraft and natural astronomical bodies such as star systems, planets, moons, and comets. Orbital mechanics focuses on spacecraft trajectories, including orbital maneuvers, orbit plane changes, and interplanetary transfers, and is used by mission planners to predict the result of propulsive maneuvers. General relativity is a more exact theory than Newton's laws for calculating orbits, and is sometimes necessary for greater accuracy or in high-gravity situations (such as orbits close to the Sun).

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1 Laws of astrodynamics

The fundamental laws of astrodynamics are Newton's law of universal gravitation and Newton's laws of motion, while the fundamental mathematical tool is his differential calculus.

Every orbit and trajectory outside atmospheres is in principle reversible, i.e., in the space-time function the time is reversed. The velocities are reversed and the accelerations are the same, including those due to rocket bursts. Thus if a rocket burst is in the direction of the velocity, in the reversed case it is opposite to the velocity. Of course in the case of rocket bursts there is no full reversal of events, both ways the same delta-v is used and the same mass ratio applies.

Standard assumptions in astrodynamics include non-interference form outside bodies, negligible mass for one of the bodies, and negligible other forces (such as from the solar wind, atmospheric drag, etc.). More accurate calculations can be made without these simplifing assumptions, but they are more complicated. The increased accuracy often does not make enough of a difference in the calculation to be worthwhile.

Kepler's laws of planetary motion may be derived from Newton's laws, when it is assumed that the orbiting body is subject only to the gravitational force of the central attractor. When an engine thrust or propulsive force is present, Newton's laws still apply, but Kepler's laws are invalidated. When the thrust stops, the resulting orbit will be different but will one again be described by Kepler's laws. The three laws are:

1. The orbit of every planet is an ellipse with the sun at one of the foci.
2. A line joining a planet and the sun sweeps out equal areas durring equal intervals of time.
3. The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits.

1.1 Escape velocity

The formula for an escape velocity is easily derived as follows. The specific energy (energy per unit mass) of any space vehicle is composed of two components, the specific potential energy and the specific kinetic energy. The specific potential energy associatied with a planet of mass M is given by

$$\epsilon_p = -\frac{GM}{r}$$

while the specific kinetic energy of an object is given by

$$\epsilon_k = \frac{v^2}{2}$$

Since energy is conserved,

$$\epsilon = \frac{v^2}{2} - \frac{GM}{r}$$

does not depend on the distance, r, from the center of the central body to the space vehicle in question. Therefor, the object can reach infinite r only if this quantity is nonnegative, which implies

$$v \geq \sqrt{\frac{2GM}{r}}$$

The escape velocity from Earth's surface is about $11\frac{km}{s}$, but that is insufficient to send the body an infinite distance because of the gravitational pull of the Sun. To escape the Solar System from a location at a distance from the Sun equal to the distance Sun-Earth, but not close to the Earth, requires around $42\frac{km}{s}$ velocity, but there will be "part credit" for the Earth's orbital velocity for spacecraft launced from Earth, if their further acceleration (due to the propulsion system) carries them in the same direction as Earth travels in its orbit.

1.2 Formulae for free orbits

Orbits are conic sections, so the formula for the distance of a body for a given angle corresponds to the formula for that curve in polar corrdinates, which is:

$$r = \frac{p}{1 + e\cos\theta}$$
$$\mu = G(m_1 + m_2)$$
$$p = \frac{h^2}{\mu}$$

μ is called the gravitational parameter. m_1 and m_2 are the masses of objects 1 and 2, and h is the specific angular momentum of object 2 with respect to object 1. The parameter θ is known as the true anomaly, p is the smi-latus rectum, while e is the orbital eccentricity, all obtainable form the various forms of the six independent orbital elements.

1.3 Circular orbits

All bound orbits where the gravity of a central body dominates are elliptical in nature. A special case of this is the circular orbit, which is an ellipse of zero eccentricity. The formula for the velocity of a body in a circular orbit at distance r from the center of gravity of mass M can be derived as follows -

Centribetal acceleration matches the acceleration due to gravity. So,

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

Therefore,

$$v = \sqrt{\frac{GM}{r}}$$

where G is the gravitational constant, equal to

$$6.67384x10^{-11}\frac{m^3}{kg \cdot s^2}$$

To properly use this formula, the units must be consistent; for example M must be in kilograms, and r must be in meters. The answer will be in meters per second.

The quantity GM is often termed the standard gravitational parameter, which has a different value for every planet or moon in the solar system.

Once the circular orbital velocity is known, the escape velocity is easily found by multiplying by the square root of 2:

$$v = \sqrt{2}\sqrt{\frac{GM}{r}} = \sqrt{\frac{2GM}{r}}$$

To escape from gravity, the kinetic energy must at least match the negative potential energy. So,

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

and therefore,

$$v = \sqrt{\frac{2GM}{r}}$$

1.4 Elliptical orbits

If $0 < e < 1$, then the denominator of the equation of free orbits varies with the true anomaly θ, but remains positive, never becoming zero. Therefore, the relative posiiton vector remains bound, having its smallest magnitude at periapsis r_p, which is given by:

$$r_p = \frac{p}{1+e}$$

The maximum value r is reached when $\theta = 180°$. This point is called the apoapsis, and its radial coordinate, denoted r_a, is

$$r_a = \frac{p}{1-e}$$

Let $2a$ be the distance measured along the apse line from periapsis P to apoapsis A, as illustrated in the equation below:

$$2a = r_p + r_a$$

Substituting the equations above, we get:

$$a = \frac{p}{1-e^2}$$

a is the semimajor axis of the sllipse. Solving for p, and substituting the result in the conicc section curve formula above, we get:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

1.4.1 Orbital period

Under standard assumptions the orbital period (T) of a body traveling along an elliptic orbit can be computed as:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

where:

- μ is standard gravitational parameter,
- a is length of semi-major axis.

Conclusions:

- The orbital period is equal to that for a circular orbit with the orbit radius equal to the semi-major axis (a),
- For a given semi-major axis the orbital period does not depend on the eccentricity.

1.4.2 Velocity

Under standard assumptions the orbital speed (v) of a body traveling along an elliptic orbit can be computed from the vis-viva equation as:

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

where:

- μ is the standard gravitational parameter,
- r is the distance between the orbiting bodies.
- a is the length of the semi-major axis.

The velocity equaiton for a hyperbolic trajectory has either $+\frac{1}{a}$, or it is the same with the convention that in that case a is negative.

1.4.3 Energy

Under standard assumptions, specific orbital energ (ϵ) of elliptic orbit is negative and the orbital energy conservation equaiton (the vis-viva equaiton) for this orbit can take the form:

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = \epsilon < 0$$

where:

- v, is the speed of the orbiting body,
- r, is the distance of the orbiting body from the center of mass of the central body,
- a, is the semi-major axis,
- μ, is the standard gravitational parameter.

Conclusions:

- For a given semi-major axis the specific orbital energy is independent of the eccentricity.

Using the viral theorem we find:

- the time-average of the specific potential energy is equal to 2ϵ
  - the time-average of v^{-1} is a^{-1}
- the time-average of the specific kinetic energy is equal to $-\epsilon$

1.5 Parabolic orbits

If the eccentricity equals 1, then the orbit equaiton becomes:

$$r = \frac{h^2}{\mu} \frac{1}{1 + \cos\theta}$$

where:

- r, is the radial distance of the orbiting body from the mass center of the central body,
- h, is specific angular momentum of the orbiting body,
- θ, is the true anomaly of the orbiting body,
- μ, is the standard ravitational parameter.

As the true anomaly θ approaches $180°$, the denominator approaches zero, so that r tends towards infinity. Hence, the energy of the trajectory for which $e = 1$ is zero, and is given by:

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r} = 0$$

where:

- v, is the speed of the orbiting body.

In other words, the speed anywhere on a parabolic path is:

$$v = \sqrt{\frac{2\mu}{r}}$$

1.6 Hyperbolic orbits

if $e > 1$, the orbit formula,

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}$$

describes the geometry of the hyperbolic orbit. The system consits of two symmetric curves. The orbiting body occupies one of them. The other one is its empty mathmatical image. Clearly, the denominator of the equation above goes to zero when $\cos\theta = \frac{-1}{e}$. We denote this value of true anomaly

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right)$$

since the radial distance approaches infinity as the true anomaly approaches θ_∞, known as the true anomaly of the asymptote. Observe that θ_∞

lies between 90° and 180°. From the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ it follows that:

$$\sin\theta_\infty = \frac{1}{e}\sqrt{e^2 - 1}$$

1.6.1 Energy

Under standard assumptions, specific orbital energy ϵ of a hyperbolic trajectory is greater than zero and the orbital energy conservation equaiton for this kind of trajectory takes form:

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{-2a}$$

where:

- v, is the orbital velocity of orbiting body,
- r, is the radial distance of orbiting body from central body,
- a, is the negative semi-major axis,
- μ, is standard gravitational parameter.

1.6.2 Hyperbolic excess velocity

Under standard assumptions the body traveling along hyperbolic trajectory will attain in infinity an orbital velocity called hyperbolic excess velocity (v_∞) that can be computed as:

$$v_\infty = \sqrt{\frac{\mu}{-a}}$$

where:

- μ is standard gravitational parameter,
- a is the negative semi-major axis of orbit's hyperbola.

The hyperbolic excess velocity is related to the specific orbital energy or characteristic energy by

$$2\epsilon = C_3 = v_\infty^2$$

2 Calculating trajectories

2.1 Kepler's equation

One approach to calculaiting orbits (mainly used historically) is to use Kepler's equation:

$$M = E - \epsilon \cdot \sin E$$

where M is the mean anomaly, E is the eccentric anomaly, and ϵ is the eccentricity.

With Kepler's formuala, finding the time-of-flight to reach an angle (true anomaly) of θ from periapsis is broken into two steps:

1. Compute the eccentric anomaly E from true anomaly θ
2. Compute the time-of-flight t from the eccentric anomaly E

Finding the eccentric anomaly at a given tim e(the inverse problem) is more difficlut. Kepler's equation is transcendental in E, meaning it cannot be solved for E algebraically. Kepler's equaiton can be solved for E analytically by inversion.

A solution of kepler's equation, valid for all real values of ϵ is:

$$E = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{M^{\frac{n}{3}}}{n!} \lim_{\theta \to 0} \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}\theta^{n-1}} \left(\left(\frac{\theta}{\sqrt[3]{\theta - \sin\theta}} \right)^n \right) \right), & \epsilon = 1 \\ \displaystyle\sum_{n=1}^{\infty} \frac{M^{n}}{n!} \lim_{\theta \to 0} \left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}\theta^{n-1}} \left(\left(\frac{\theta}{\theta - \epsilon \cdot \sin\theta} \right)^n \right) \right), & \epsilon \neq 1 \end{cases}$$

Evaluating this yields:

$$E = \begin{cases} x + \frac{1}{60}x^3 + \frac{1}{1400}x^5 + \frac{1}{25200}x^7 + \frac{43}{17248000}x^9 + \frac{1213}{7207200000}x^{11} + \frac{151439}{12713500800000}x^{13} \cdots \mid x = (6M)^{\frac{1}{3}}, & \epsilon = 1 \\ \frac{1}{1-\epsilon}M - \frac{\epsilon}{(1-\epsilon)^4}\frac{M^3}{3!} + \frac{(9\epsilon^2+\epsilon)}{(1-\epsilon)^7}\frac{M^5}{5!} - \frac{(225\epsilon^3+54\epsilon^2+\epsilon)}{(1-\epsilon)^{10}}\frac{M^7}{7!} = \frac{(11025\epsilon^4+4131\epsilon^3+243\epsilon^2+\epsilon)}{(1-\epsilon)^{13}}\frac{M^9}{9!}\cdots, & \epsilon \neq 1 \end{cases}$$

Alternatively, Kepler's Equation can be solved numerically. First one must guess a value of E and solve for time-of-flight; then adjust E as necessary to bring the computed time-of-flight closer to the desired value until the required precision is achieved. Usually, Newton's method is used to achieve relatively fast convergentce.

The main difficulty with this approach is that is=t can take prohibitively long to converge for the extream elliptical orbits. FOr near-parabolic orbits, eccentricity ϵ is nearly 1, and plugging $e = 1$ into the formula mean anomaly, $E - \sin E$, we find ourselves subtracting two nearly-equal values, and accuracy suffers. For near-circular orbits, it is hard to find the periapsis in the first place (and truly circular orbits have no periapsis at all). Furthermore, the equation was derived on the assumption of an elliptical orbit, and so does not hold for parabolic or hyperbolic orbits. These difficulties are what led to the development of the universal variable formulation, described below.

2.2 Conic orbits

For simple procedures, such as computing the delta-v for coplanar transfer ellipses, traditional approaches are fairl effective. Others, such as time-of-flight are far more complicated, especially for near-circular and hyperbolic orbits.

2.3 The patched conic approximation

2.4 The universal variable formulation

2.5 Perturbation

3 Orbital maneuver

3.1 Orbital transfer

3.2 Gravity assist and the Oberth effect

3.3 Interplanetary Transport Network and fuzzy orbits