

## REAL II REVIEW

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### 1. DIFFERENTIABILITY

#### 1.1. Definition.

**Definition 1.1.** Suppose  $F : \mathcal{U} \rightarrow \mathbb{R}^m$  is differentiable at  $P$ . Then

$$F(P + \vec{h}) = F(P) + [DF(P)]\vec{h} + \vec{E}, \quad \text{with} \quad \lim_{\|\vec{h}\| \rightarrow 0} \frac{\|\vec{E}\|}{\|\vec{h}\|} = 0.$$

**1.2. Differentiable / Linearizable.** Clearly when  $\vec{h} = \vec{0}$  then the linearization must equal the function at that point, thus

$$L(P + \vec{h}) = F(P) + A\vec{h} + \vec{E}$$

for some matrix  $A$ . Thus when we take the  $i$ -th component of  $F$ , we get

$$F^i(p^1, \dots, p^j + h, \dots, p^n) = F^i(p^1, \dots, p^n) + A_j^i h + E^i$$

Rearranging to solve for  $A_j^i$  we find that  $A_j^i = \frac{\partial F^i}{\partial x^j}$ . Thus if a function is differentiable/linearizable then the matrix of the linearization *has to be* the Jacobi matrix.

**1.3. Error term.** We want the error term to die faster than  $\vec{h}$  because we get the error term from Taylors Theorem, so if this is not the case, then as  $\vec{h} \rightarrow 0$  we would have the error exploding, and that would be bad. So part of differentiability is that the error term must die.

**1.4. Existence is not differentiable.** A partial derivative can exist at all points, but by some nature of the function, it may not be a continuous partial derivative. Thus one cannot assume that if the partial derivative exists, that the function is differentiable. However, it is safe to say that if the partial derivative exists and is continuous then the function is differentiable.

#### 1.5. Mean Value Theorem / Taylor Theorem.

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*Date:* March 4, 2019.

## 2. MEASURING SIZES OF STUFF

### 2.1. k-forms.

#### 2.1.1. *Motivation.*

#### 2.1.2. *Determinant.*

### 2.2. Measure Theory.

2.2.1. *Jordan Measure.* Jordan measure is a method to measure the size of an "arbitrary" shape, Because a k-form can only measure the size of a parallelotope, and can't handle arbitrary shapes.

2.2.2. *What is not Jordan measurable.*  $\mathbb{Q} \cap [0, 1]$  is not Jordan measurable, because  $\mu_{int} = 0$ , and  $\mu_{out} = 1$ , this is also because  $\partial\mathbb{Q} = \bar{\mathbb{Q}} \setminus \text{Int}\mathbb{Q} = \mathbb{R}$ . Then the measure of the boundary is not zero, this is an issue.

2.2.3. *Jordan measure of zero.* We know that a set has Jordan measure of zero if we can completely cover it in boxes, where the sum of the area of the boxes is less than some  $\epsilon$ . Then we are able to make  $\epsilon \rightarrow 0$ , and thus the measure must also be zero. This entails ensuring that the boxes completely cover the set, and that we can make them as small as we want. For our homework we used the angle to change the size of the boxes.

2.2.4. *Measurable by boundary.* This is an IFF. If the set is measurable, then we can construct two coverings,  $B_{in}$  and  $B_{out}$ , where  $B_{in} \subset \text{Int}(\mathcal{U})$ , and  $B_{out}$  intersects  $\bar{\mathcal{U}}$  non-trivially, i.e. also covers the boundary or it over estimates. As we refine these coverings, they should both approach  $\mu(\mathcal{U})$ . Thus the difference between them will become less than  $\epsilon$ . The difference between the two sets will be exactly boxes that contain  $\partial\mathcal{U}$ , thus  $\mu(\partial\mathcal{U}) < \epsilon$ .

If the measure of the boundary is zero, then we extend the grid by the hyper planes which define the sides of the boxes, thus effectively cutting the boxes up. There are sets in this global grid, that are entirely within  $\mathcal{U}$ , and then there are sets that cover  $\bar{\mathcal{U}}$ , the difference of these sets will be a covering of  $\partial\mathcal{U}$ , not that it may not be the same covering that we started with, but it will also be less than  $\epsilon$ , thus since the difference of  $\mathcal{B}_{out}$ , and  $\mathcal{B}_{in}$  must be less than  $\epsilon$ , since this is true for all  $\epsilon$ , than  $\mathcal{U}$  must be measurable.

## 3. RIEMANN INTEGRATION

## 3.1. Definition.

**Definition 3.1.** We say that the function  $f$  is Riemann-integrable over  $\mathcal{U}$  if there exists  $I \in \mathbb{R}$  such that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all decompositions  $\mathcal{U}_*$  with  $s(\mathcal{U}_*) < \delta$  and all sample points  $\xi_*$  we have

$$\|RS(f, \mathcal{U}_*, \xi_*)\| < \varepsilon$$

**Definition 3.2.** We say that the function  $f$  is Riemann-integrable over  $\mathcal{U}$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all decompositions  $\mathcal{U}_*$  and  $\widetilde{\mathcal{U}}_*$  with  $s(\mathcal{U}_*), s(\widetilde{\mathcal{U}}_*) < \delta$  and all sample points  $\xi_*, \widetilde{\xi}_*$  we have

$$\left\| RS(f, \mathcal{U}_*, \xi_*) - RS(f, \widetilde{\mathcal{U}}_*, \widetilde{\xi}_*) \right\| < \varepsilon$$

**Theorem 3.1.** If  $f$  is uniformly continuous over  $\mathcal{U}$  then  $f$  is Riemann integrable over  $\mathcal{U}$ .

**Theorem 3.2.**  $f$  is Riemann integrable iff the set of discontinuities is of measure zero.

## 3.2. Show not integrable.

## 3.3. Fundamental Theorem of Calculus.

## 4. UNIVERSAL PROPERTIES

## 4.1. Exterior Product.

## 4.2. Tensor Product.

## 5. RANDOM BITS OF LINEAR / TENSOR ALGEBRA

## 5.1. Dual Space.

## 5.2. Tensor terminology.

## 5.3. Raising and Lowering indices.

## 6. VOLUME FORMS

## 6.1. Orientation of a Space.

## 6.2. Corollary 2 in Part II.

6.3.  $d\text{vol}_g$ .

## 6.4. Hodge star.

## 6.5. Cross product.

6.5.1. *What is it?*

6.5.2. *Does it have the properties we want?*

6.5.3. *5.10.4.*

## 7. CATEGORY THEORY

**7.1. What did you learn?** A category is a construct of *objects* and *morphisms* between the objects. Objects could be for example vector spaces, and then the associated morphisms would be linear transformations. It seems pretty straightforward, and very useful. Most of the proofs are done through diagram chasing.

**7.2. What is a functor?** A *functor* is a method for converting from one category to another. Functors must take isomorphisms to isomorphisms, and take compositions to compositions. This is useful if examining a problem in one category is hard, then it can be converted to an easier category. Note that if a functor maps an isomorphism, it does not mean that the morphism in category 1 is an isomorphism. Thus this can only be used as a contrapositive tool.