Math 215
Practice Final Exam

December 15, 2017

Name (in block capital letters):

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Instructor (tick one box): ☐ **Section 1: L. Stanhope (12:50)**

☐ **Section 2: E. Sullivan (12:50)**

Instructions:

1. To answer these questions, use only the knowledge in your head, and on your $3\times$ 5 card.
2. You have 3 hours in which to take the exam.
3. Be sure to carefully read the directions for each problem. Please show your work – correct answers to problems with no justification of where they come from will earn little credit.
4. Finally, do your best to think logically and clearly, and then trust your judgment. There are no "trick" questions here. You can do this!

Problem	Score
1	
2	
3	
4	
Total	

1. Assume the standard deck of cards: 52 cards, 4 different suits and 13 different ranks.

   (a) How many 5-card hands have exactly two aces and exactly two distinct suits? Show your work.

   (b) How many hands have exactly two aces and at least one queen?

   (c) The number 5-card hands which have exactly two distinct ranks.

2. For the statement below please do the following.

   (a) Translate the statement into words.

   (b) Write the negation of the statement such that it contains no implications and a negation sign does not appear before an extensional quantifier.

   (c) If the statement is true, provide a proof.

   (d) If the statement is false, negate the statement and then provide a proof.

$$\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, \forall z \in \mathbb{N}, \quad (((x \mid z) \wedge (y \mid z)) \rightarrow ((x - y) \mid z))$$

3. Use induction to prove $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \binom{n+1}{2}^2$.

4. Suppose $a, b, c \in \mathbb{Z}$. Without using the Fundamental Theorem of Arithmetic, please show that if $GCD(a, c) = 1$ and $a|bc$ then $a|b$.

5. Show that a natural number and its cube have the same remainder after division by 6.

6. Consider the following two sets

$$A \times (B \backslash C) \quad \boxed{} \quad (A \times B) \setminus (A \times C)\,.$$

   Choose one of the four symbols below to put in the box above to make the statement true.

$$= \quad \neq \quad \subseteq \quad \supseteq\,.$$

   For each direction prove either that the given set is a subset of the other by using an element casing argument, or prove that is is not necessarily a subset of the other by providing an explicit example.

7. Let $\mathbb{P}$ denote the set of all prime numbers. So $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$. Define a function $f : \mathbb{N} \to \mathcal{P}(\mathbb{P})$ by $f(n)$ equals the set of prime factors of n. So for example, $f(12) = \{2, 3\}$.

   a. Please determine whether or not f is injective. Then prove injectivity or give a specific counterexample that it does not hold.

   b. Please determine whether or not f is surjective. Then prove surjectivity or give a specific counterexample that it does not hold.

8. Consider a sphere of radius 1 inch. Prove that for any 9 points on the surface of the sphere that there is at least one pair of points for which the distance along the sphere between them is no more than $\pi/2$ inches.

9. Suppose $f : A \to B$ is a function. Please prove the following statement. "Function f is surjective if and only if for all subsets S of A, $B \backslash f(S) \subseteq f(A \backslash S)$."

10. Consider the set of real numbers $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$. (Here $\mathbb{Q}$ is the set of all rational numbers. Its complement is the set of irrational numbers such as π and e.) Define the relation R on $\mathbb{R}$ by

$$xRy \iff \{x, y\} \subseteq \mathbb{Q} \text{ or } \{x, y\} \subseteq \overline{\mathbb{Q}}$$

Is R an equivalence relation? If so, how many distinct equivalence classes does R have? If not, which of the three criteria for equivalence relations does R fail?