A Tiny Bit of Fluid Dynamics

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Abstract

We consider the vector field $\vec{U}(\mathrm{y})$, where $\mathrm{y} = (x, y, z)$. From the fundamental theorem of ODE'S for each given initial condition $\mathrm{x} = (a, b, c)$ there is a unique solution of the IVP

$$\begin{cases} \frac{\partial \mathrm{y}}{\partial t} = \vec{U}(\mathrm{y}(t)) \\ \mathrm{y}(0) = \mathrm{x} \end{cases}$$

Problem 1

We first consider the fluid modeled by the velocity vector field

$$\vec{U}(\mathrm{y}) = -2x\vec{\imath} + (-2y + z)\vec{\jmath} + (y - 2z)\vec{k}$$

From this equation we aim to determine the trajectory of a molecule located at an arbitrary initial location.

First we convert the equation for $\vec{U}$ into a matrix form.

$$\vec{U} = \begin{pmatrix} -2x \\ -2y + z \\ y - 2z \end{pmatrix}$$

$$\vec{U} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Using the matrix representation of U, we can now attempt to find a general solution to the equation. We do this by first determining the eigenvalues of the matrix.

$$\det \begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix} = 0$$

Solving the equation we find that the eigenvalues are as follows.

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

Using the eigenvalues and the original matrix representation of U, we can find the eigenvectors using $U \cdot V = \lambda \cdot V$. Thus we can obtain the respective eigenvectors for each eigenvalue.

$$V_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Using these eigenvalues and eigenvectors, we can determine a general formula for y to be:

$$\mathrm{y}(t) = C_1 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{-3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Using the initial condition of $\mathrm{y}(0) = \mathrm{x}$, we can solve for the values of $C_1,\ C_2,\ C_3$.

$$\begin{aligned} \mathrm{y}(0) &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \\ \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned}$$

Using the values of $C_1,\ C_2,\ C_3$ we can determine the equation for y based off of the initial conditions (a, b, c).

$$\begin{aligned} \mathrm{y}(t) = &\left(\frac{b}{2} + \frac{c}{2}\right) e^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &+ (a) e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &+ \left(\frac{b}{2} - \frac{c}{2}\right) e^{-3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

We plot this equation using matplotlib, to demonstrate that the phase portrait diagram for the equation consists of a three dimensional sink, where all lines converge to the origin.

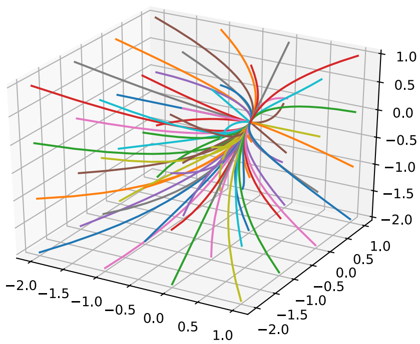

Figure 1: This image represents the phase portrait diagram for the equation, with a set of different initial conditions.

Problem 2

First we find the formula for $F_t : \mathrm{x} \mapsto \mathrm{y}\,(t)$

$$F_t\,(a,b,c) = \begin{pmatrix} ae^{-2t} \\ \left(\frac{b}{2}+\frac{c}{2}\right)e^{-t} + \left(\frac{b}{2}-\frac{c}{2}\right)e^{-3t} \\ \left(\frac{b}{2}+\frac{c}{2}\right)e^{-t} - \left(\frac{b}{2}-\frac{c}{2}\right)e^{-3t} \end{pmatrix}$$

Using this equation we find the Jacobi matrix DF_t.

$$DF_t = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix}$$

$$DF_t = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & \frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} \\ 0 & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t} \end{pmatrix}$$

$$\det\left(DF_t\right) = e^{-6t}$$

$$S\,(t) = e^{-6t}$$

This result demonstrates that as $t \to \infty$ then $S(t) \to 0$. This shows that as time goes on, the solution curve compresses closer and closer to zero.

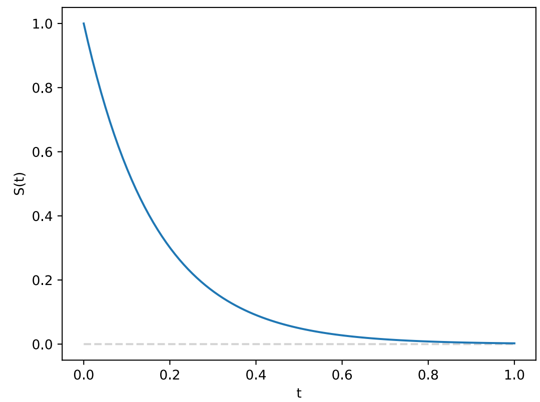

Figure 2: This image shows the plot of the stretch factor $S(t)$. It can be seen that as t grows, the value of $S(t)$ nears zero, indicating the compression of the solution curves at larger times.

Problem e

Using a simplified generalization of the matrix for $S(t)$ we assume that $\mathrm{x} \mapsto \mathrm{y}$ is only two dimensional, so $(a,b) \mapsto (x,y)$. Thus the Jacobian matrix becomes a 2×2 matrix instead of a 3×3 matrix.

$$DF_t = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{pmatrix}$$

Now we utilize this generalized matrix to determine a general formula for $S(t)$.

$$S(t) = \frac{\partial x}{\partial a}\cdot\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\cdot\frac{\partial x}{\partial b}$$

Now we can determine the derivative of S in terms of t like so.

$$\frac{\partial S}{\partial t} = \frac{\partial^2 x}{\partial a \partial t}\cdot\frac{\partial y}{\partial b} + \frac{\partial^2 y}{\partial b \partial t}\cdot\frac{\partial x}{\partial a} - \frac{\partial^2 y}{\partial a \partial t}\cdot\frac{\partial x}{\partial b} - \frac{\partial^2 y}{\partial b \partial t}\cdot\frac{\partial y}{\partial a}$$

We note that $\frac{\partial x}{\partial t}$ appears in this equation, and thus we can use the equation from a velocity vector U like so.

$$\frac{\partial x}{\partial t} = U_1\left(x\,(t)\,, y\,(t)\right)$$

$$\frac{\partial y}{\partial t} = U_2\left(x\,(t)\,, y\,(t)\right)$$

Using these equations we can apply chain rule to the equation for $\frac{\partial S}{\partial t}$.

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= \left(\frac{\partial U_1}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial U_1}{\partial y}\cdot\frac{\partial y}{\partial a}\right)\cdot\frac{\partial y}{\partial b}\\
&+\left(\frac{\partial U_2}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial U_2}{\partial y}\cdot\frac{\partial y}{\partial b}\right)\cdot\frac{\partial x}{\partial a}\\
&-\left(\frac{\partial U_2}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial U_2}{\partial y}\cdot\frac{\partial y}{\partial a}\right)\cdot\frac{\partial x}{\partial b}\\
&-\left(\frac{\partial U_1}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial U_1}{\partial y}\cdot\frac{\partial y}{\partial b}\right)\cdot\frac{\partial y}{\partial a}
\end{aligned}
$$

After simplification we find

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= \frac{\partial U_1}{\partial x}\cdot\left(\frac{\partial x}{\partial a}\cdot\frac{\partial y}{\partial b}-\frac{\partial x}{\partial b}\cdot\frac{\partial y}{\partial a}\right)\\
&+\frac{\partial U_2}{\partial y}\cdot\left(\frac{\partial y}{\partial b}\cdot\frac{\partial x}{\partial a}-\frac{\partial y}{\partial a}\cdot\frac{\partial x}{\partial b}\right)
\end{aligned}
$$

We can notice that the value of $S(t)$ appears, and so we can then factor it out.

$$
\frac{\partial S}{\partial t} = S(t)\cdot\left(\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}\right)
$$

We note that this represents the divergence of the vector U.

Problem 3

Using our solutions for Problem 2 and Problem e, we can verify our results, as the solution to Problem 2 should satisfy the differential equation found in Problem e.

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= \left(\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}+\frac{\partial U_3}{\partial z}\right)\cdot e^{-6t}\\
\frac{\partial U_1}{\partial x} &= -2 \quad \frac{\partial U_2}{\partial y} = -2 \quad \frac{\partial U_3}{\partial z} = -2\\
\frac{\partial S}{\partial t} &= -6\cdot e^{-6t}
\end{aligned}
$$

This verifies the solution that we expect.

Problem π

Now we consider a *incompressible fluid* which is modeled by the velocity vector field of the form

$$
\vec{U}(\mathrm{y}) = \alpha x\vec{\imath} + (-2y+z)\vec{\jmath} + (y-2z)\vec{k}
$$

We first must determine the value of α such that the fluid this equation models is incompressible. To do so, we use the solution in Problem e. As the value of S is the *stretch factor* of the fluid, and if the fluid needs to be incompressible, then this value must be constant. Thus we can determine that $\frac{\partial S}{\partial t} = 0$. Using this knowledge we are able to find the value of α.

$$
\begin{aligned}
0 &= (\alpha + (-2) + (-2))\cdot S\\
0 &= (\alpha - 4)\\
\alpha &= 4
\end{aligned}
$$

Using this value of α we can see that the equation that models an incompressible fluid is

$$
\vec{U}(\mathrm{y}) = 4x\vec{\imath} + (-2y+z)\vec{\jmath} + (y-2z)\vec{k}
$$

Problem π.1

Using the same methods that we utilized in Problem 1 we solve for and equation for $\mathrm{y}(t)$.

$$
\vec{U} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}
$$

$$
\det\begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix}
$$

Solving this for λ's we find:

$$
\lambda_1 = 4 \quad \lambda_2 = -3 \quad \lambda_3 = -1
$$

Using these λ's we find the corresponding eigenvectors to be.

$$
V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
$$

Using these eigenvalues and eigenvectors we are able to create an equation for y for the incompressible fluid:

$$
\mathrm{y}\,(t) = C_1 e^{4t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
$$

Using the same methods from Problem 1 we can solve for the values of C_1, C_2, C_3. We find these to be

$$
C_1 = a \quad C_2 = \frac{b}{2} - \frac{c}{2} \quad C_3 = \frac{b}{2} + \frac{c}{2}
$$

Thus our equation for y becomes

$$\mathrm{y}\,(t) = ae^{4t}\begin{pmatrix}1\\0\\0\end{pmatrix} + \left(\frac{b}{2}-\frac{c}{2}\right)e^{-3t}\begin{pmatrix}0\\1\\-1\end{pmatrix} + \left(\frac{b}{2}+\frac{c}{2}\right)e^{-t}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Using matplotlib with the equation for the incompressible fluid equation, we can generate a phase portrait for the fluid. Not the saddle shape, as either a sink or source would indicate that the fluid is being compressed (or *negatively compressed*).

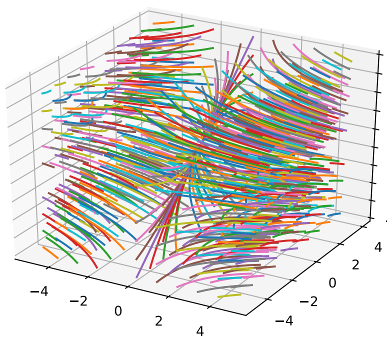

Figure 3: This image represents the phase portrait for the incompressible fluid, showing that there is a 3D saddle shape present in the space.

Problem π.2

Now we determine the stretch factor for the incompressible fluid. As mentioned previously the stretch factor should be constant. First is to find the Jacobi matrix DF_t for the equation we found in Problem π.1.

$$DF_t = \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & \frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} \\ 0 & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t} \end{pmatrix}$$

Thus finding the det DF_t we can find the equation for $S(t)$.

$$\det\left(DF_t\right) = 1$$

This mirrors our initial predictions that the stretch factor would be constant, and shows that the fluid is not being stretched at all, as that would contradict the incompressibility of it.