ABSTRACT ALGEBRA – FIRST HOMEWORK ASSIGNMENT ON HOMOMORPHISMS

(1) Let R denote the set of 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with real entries, i.e let

$$R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

(a) Show that $(R, +, \cdot)$ is a subring of the ring of 2×2 real matrices. In other words, show the following.

(i) Show that R is closed under matrix addition and multiplication.

(ii) Show that R contains both the additive and the multiplicative identities of matrix algebra.

(iii) Show that R is closed under additive inverses. That is, show that for all elements of R their additive inverses are also in R.

(b) Consider the mapping

$$F : R \to \mathbb{C}, \text{ given by } F : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + bi.$$

Show this mapping is an isomorphism of rings with identity.

(c) Is $(R, +, \cdot)$ a field? Why / why not?

(2) Examine each of the following mappings – determine whether or not they are homomorphisms of rings with identity. In cases when they are: compute the kernel and the image. In cases when they are not: make it clear which homomorphism properties hold and which do not.

(a) $F : \mathbb{Z}[i] \to \mathbb{Z}[2i]$ given by $F(a + bi) = a + 2bi$.

(b) $F : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ given by $F(a + b\sqrt{2}) = a - b\sqrt{2}$.

(c) $F : \mathbb{Z} \to \mathbb{Z}$ given by $F(n) = 7n$.

(d) $F : \mathbb{Z} \to \mathbb{Z}$ given by $F(n) = n^2$.

(e) $F : \mathbb{Z}/(2) \to \mathbb{Z}/(2)$ given by $F : [n] \mapsto [n]^2$.

(f) $F : \mathbb{R}[X] \to \mathbb{R}$ given by $F : P(X) \mapsto P(1)$.

(g) $F : \mathbb{C}[X] \to \mathbb{C}$ given by $F : P(X) \mapsto P(i)$;

(h) $F : \mathbb{R}[X] \to \mathbb{C}$ given by $F : P(X) \mapsto P(i)$;

(i) $F : \mathbb{R}[X] \to \mathbb{R}[X]$ given by $F : P(X) \mapsto P(X^2)$.

(3) The concepts of *kernel and image* appears in other parts of (linear) algebra.

(a) What are homomorphisms between two vector spaces? What do we mean under *kernel and image* in this context? Show that both *kernel and image* are subspaces of relevant vector spaces.

(b) What are homomorphisms in the context of groups? What do we mean under *kernel and image* in this context? Show that both *kernel and image* are subgroups of relevant groups.

(c) What are homomorphisms in the context of rings with identity? What do we mean under *kernel and image* in this context? Show that *kernel* is an ideal and that the *image* is a subring of the relevant rings.

(d) Apart from groups, rings and fields in abstract algebra we study .... *algebras*! This is not a joke. By *algebra* we mean rings which also happen to be vector spaces (over some field of scalars). For example, when we say *matrix algebra* the word *algebra* is loaded! It is a technical term meaning that we are considering the ring of matrices as well as the fact that matrices form a vector space over real or complex numbers. What are homomorphisms in the context of algebras? What do we mean under *kernel and image* in this context? Can you say anything more about *kernel and image* in this context – are they sub-anything? What are they? Do you have examples of homomorphisms of algebras on this very homework? If yes, tell me which ones they are!