Discrete Math Review

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1 Counting

The addition principle $M_1 + M_2 + M_3 = \#ways$, where M_i are different methods for the same task.

The multiplication principle $t_1 \cdot t_2 \cdot t_3 = \#ways$, where t_i are different tasks to be completed.

Indirect Method Count the total number of ways, and subtract the unwanted ways that were counted.

Combinations A collection of k objects chosen from n distinct objects *without regard to the order*.

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{1}$$

Permutations An *arrangement* of k objects out of a collection of n distinct objects *where order matters*.

2 Basic Logic

Negation $\neg P$, not P.

Conjunction $P \wedge Q$, P and Q.

Disjunction $P \vee Q$, P or Q.

Conditional $P \to Q$, if P then Q.

Biconditional $P \leftrightarrow Q$, P if and only if Q.

Logically Equivalent $P \iff Q$.

Implies $P \implies Q$.

Note that the *Conditional* and *Implies* are often mixed, as well as the *Biconditional* and *logically equivalent*.

Distributive Laws

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R) \tag{2}$$
$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R) \tag{3}$$

Conditional

$$P \to Q \iff \neg P \vee Q \tag{4}$$

De Morgan's Laws

$$\neg(P \wedge Q) \iff \neg P \vee \neg Q \tag{5}$$
$$\neg(P \vee Q) \iff \neg P \wedge \neg Q \tag{6}$$

Parenthesies

$$(P \vee Q) \vee R \iff P \vee Q \vee R \tag{7}$$
$$P \wedge (Q \wedge R) \iff P \wedge Q \wedge R \tag{8}$$

Converse $P \to Q : Q \to P$.

Inverse $P \to Q : \neg P \to \neg Q$.

Contrapositive $P \to Q : \neg Q \to \neg P$.

3 Direct and Indirect Proofs in Number Theory

Contrapositive of $P \to Q$ is $\neg Q \to \neg P$ (Switch order and negate both).

Definition of divides If there is an integer q such that $n = kq$, then $k|n$.

Division Algorithm For $m, n \in \mathbb{Z}$ there is a unique pair $r, q \in \mathbb{Z}$ such that $n = mq + r$ where $m > 0$ and $0 \leq r < m$.

Odd and Even Odd if there is a $q \in \mathbb{Z}$ such that $n = 2q + 1$, it is odd. Event if there is a $q \in \mathbb{Z}$ such that $2q = n$, it is even.

Prime and Ireducible p is prime if got $a, b \in \mathbb{Z}$, $p|ab \to p|a$ or $p|b$. p is irreducible if $p = ab \to a = 1$ or $b = 1$.If p is prime,it is irreducible.

4 Induction

Use induction when dealing with natural numbers, often in a series or repetitive numerical situation. Always include *Base case*, *Inductive hypothesis* (for $k \geq BC\#$), *Inductive Step* (use IH to prove $k + 1$). Start with claim $P(n)$. No backward proofs.

5 Greatest Common Divisors

Definition Let $m, k \in \mathbb{Z}$. d is the $GCD(m, k)$ if

(i) $d|m$

(ii) $d|k$

(iii) if $a \in \mathbb{Z}$ and $a|m$ and $a|k$, then $a \leq d$.

Euclidean Algorithm Use *Division algorithm* until you get $r = 0$. Go backwards to express GCD s linear combination of the two numbers.

GCD pre-hammer Given $a, b \in \mathbb{Z}$, $\exists x, y \in \mathbb{Z} \to GCD(a, b) = xa + yb$.

GCD hammer a and b are relatively prime if and only if $\exists x, y \in \mathbb{Z}$ such that $xa + yb = 1$.

If $d|ab$ and $GCD(a, d) = 1$ then $d|b$.

6 The Fundamental Theorem of Arithmetic

Let $n > 1$, $n \in \mathbb{Z}$. Then n has a prime factorization, $n = p_1 \cdot p_2 \cdots p_k$ where each p_i is prime. This factorization is *unique up to reordering the factors.* The $LCM = \frac{ab}{GCD}$ where a, b are two integers we are concerned with.

7 Set Theory Basics

Set A set is a collection of elements $\{a, b, c\}$.

Union $A \cup B$, the combination of all the elements from both A and B, without repeating elements.

Intersection $A \cap B$, The set of elements that both A and B have in common.

Cardinality $|A|$ or $card(A)$, The number of elements in a set.

Empty Set The set with no elements is the empty set $\emptyset$ or $\{\}$.

Power Set $\mathcal{P}(S)$, The set of all possible subsets of S.

Compliment The set of elements in the universe U that are not elements of the original set.

8 Equivalence Relations

Relation Given sets A and B, a relation is a subset of $A \times B$.

Equivalence Relation A relation is an equivalence relation if it is

- **Reflexive** $\forall a \in A, aRa$.
- **Symmetric** $aRb \to bRa$.
- **Transitive** aRb and $bRc \to aRc$.

Partition Let A be a nonempty set. Define $P = \{A_1, A_2, \ldots, A_n\}$ where

(i) For i from $i = 1, 2, \ldots, n$ $A_i \subseteq A$.

(ii) For $i = 1, 2, \ldots, n$, $A_i \neq \emptyset$.

(iii) For pairs i, j with $i \neq j$, $A_i \cap A_j = \emptyset$ (Each subset is unique).

(iv) $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

Such a P is a partition. $A_1, A_2, \ldots, A_n$ are the partition elements. Each is an equivalence class $[a]_R$ under a relation R. Each partition gives an equivalence relation.

9 Functions

Function A function f is a relation from $A \mapsto B$ for which each element of A appears once and only once in the first coordinates of an ordered pair in f.

Injective f is injective (one-to-one) if $\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$. If every element in the domain maps to a unique value in the co domain.

Surjective f is surjective (onto) if $\forall y \in B, \exists x \in A, f(x) = y$. If every element in co domain is mapped to by at least one element in the domain.

Bijection A function is a bijection is it is both injective and surjective.

Composition Let f, g be functions, the composition of these function is defined as $f \circ g \equiv g(f(x))$.

10 The Pigeonhole Principle

Suppose $|A| = n$ and $|B| = m$ and $n > m$. Then the function $f : A \mapsto B$ cannot be injective. There are too many "pigeons" and not enough "holes".

11 The Inclusion/Exclusion Principle

If $S_1, S_2, \ldots, S_n$ are all finite sets and $n \geq 2$, then

$$\begin{aligned} |S_1 \cup S_2 \cup \cdots \cup S_n| &= |S_1| + |S_2| + \ldots + |S_n| && (9) \\ &- |S_1 \cap S_2| - |S_1 \cap S_3| - \ldots - |S_{n-1} \cap S_n| && (10) \\ &+ \bullet\bullet\bullet && (11) \\ &+ (-1)^{n-1} |S_1 \cup S_2 \cup \cdots \cup S_n| && (12) \end{aligned}$$

$$\left| \bigcup_{k=1}^{n} S_k \right| = \sum_{k=1}^{n} |S_k| - \sum_{1 \leq k < j \leq n} |S_i \cap S_j| + \ldots + (-1)^{n-1} \left| \bigcap_{k=1}^{n} S_k \right| \tag{13}$$

12 Graph Theory

Graph A graph is a set V of objects along with a set E of unordered two element subsets of V. Elements of V are *vertices* and elements of E are *edges*.

Degree The degree of a vertex is the number of elements in E that contain that vertex. The sum of degrees must always be even, as each edge connects exactly two vertexes.

Degree Sequence The degree sequence of a graph is the list of degrees of vertexes of a graph in increasing order.

Walk Walks are routes through a graph that connect two vertices and can repeat edges and vertexes.

Trail Trails are walks that cannot repeat edges.

Path Paths are walks that cannot repeat edges or vertexes.

Circuit A circuit is a closed trail, (one which ends at the beginning vertex).

Connected A graph is connected if any 2 vertices have a path in between them.

Component A component of a graph is the largest connected piece of a graph where you include all connected vertices.

Tree A tree is a connected graph with no circuit's.

Forest A forest is a disconnected graph with no circuit's, or a group of disconnected trees.

Bridge A bridge is an edge that connects two otherwise disconnected components of a graph. If a graph G has a circuit, then there is an edge in G that is not a bridge. If you remove an edge from a connected graph, you split it into at most two components.

Equivilant Definition for Finite Tree If T is a graph with n vertices, then the following are equivalent:

(1) T is a tree.

(2) T has no circuits and $n - 1$ edges.

(3) T is connected and has $n - 1$ edges.

(4) T is connected and removal of an edge disconnects T.

(5) There exists a unique path between any vertices in T.

(6) T has no circuits, and adding any edge creates one.

Minimal Spanning Tree A minimal spanning tree, is a tree that connects a graph using as minimal weight as possible. To create a minimal spanning tree

1. Chose the edge with minimum weight.
2. Look at the remaining edges. Chose edge with minimum weight that does not create a circuit.
3. Stop when you have $n-1$ edges.

Planar A graph is planar if a diagram can be drawn of it where no edges cross.

- To prove a graph is planar, can produce a planar drawing of it.
- Hard to prove it isn't, best to argue by contradiction with *Euler characteristic*.

Euler Characteristic $v - e + f = 2$ for a connected planar graph.