REAL II REVIEW

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1. Differentiability

1.1. Definition.

Definition 1.1. *Suppose* $F : \mathcal{U} \to \mathbb{R}^m$ *is differentiable at* P*. Then*

$$F(P + \vec{h}) = F(P) + [DF(P)]\vec{h} + \vec{E}, \quad \textit{with} \quad \lim_{\|\vec{h}\| \to 0} \frac{\left\|\vec{E}\right\|}{\left\|\vec{h}\right\|} = 0.$$

1.2. **Differentiable / Linearizable.** Clearly when $\vec{h} = \vec{0}$ then the linearization nust equal the function at that point, thus

$$L(P + \vec{h}) = F(P) + A\vec{h} + \vec{E}$$

for some matrix A. Thus when we take the i-th component of F, we get

$$F^i(p^1, \ldots, p^j + h, \ldots, p^n) = F^i(p^1, \ldots, p^n) + A^i_j h + E^i$$

Rearanging to solve for A^i_j we find that $A^i_j = \frac{\partial F^i}{\partial x^j}$. Thus if a function is differentiable/linearizable then the matrix of the linearization *has to be* the Jacobi matrix.

1.3. **Error term.** We want the error term to die faster than $\vec{h}$ because we get the error term from Taylors Theorem, so if this is not the case, then as $\vec{h} \to 0$ we would have the error exploding, and that would be bad. So part of differentiability is that the error term must die.

1.4. **Existence is not differentiable.** A partial derivative can exists at all points, but by some nature of the function, it may not be a continuous partial derivative. Thus one cannot assume that if the partial derivative exists, that the function is differentiable. However, it is safe to say that if the partial derivative exists and is continuous then the function is differentiable.

1.5. **Mean Value Theorem / Taylor Theorem.**

Theorem 1.1 (Mean Value Theorem)**.** *Suppose $\mathcal{U} \subseteq \mathbb{R}^n$ is convex and suppose $F : \mathcal{U} \to \mathbb{R}$ has continuous partial derivatives. Then for each $P_1, P_2 \in \mathcal{U}$ there exists Q such that*

$$F(P_2) - F(P_1) = [DF(Q)] \cdot \overrightarrow{P_1P_2}$$

Proof. Let F be continuously differentiable. Take $f : \mathbb{R} \to \mathbb{R}$ as

$$f(t) = F(P_1 + t\overrightarrow{P_1P_2})$$

where $P_1, P_2 \in \mathcal{U}$. with $t \in [0, 1]$. Then applying one dimension mean value theorem $\exists z \in R$ such that

$$f(1) - f(0) = f'(z) \cdot 1$$

we can find $f'(z)$ to be

$$f'(z) = DF|_{P_1 + z\overrightarrow{P_1P_2}} \cdot \overrightarrow{P_1P_2}$$

thus

$$\begin{aligned} f(1) - f(0) &= DF|_{P_1 + z\overrightarrow{P_1P_2}} \cdot \overrightarrow{P_1P_2} \\ F(P_1) - F(P_2) &= DF|_{P_1 + z\overrightarrow{P_1P_2}} \cdot \overrightarrow{P_1P_2} \\ F(P_2) - F(P_1) &= [DF(Q)] \cdot \overrightarrow{P_1P_2} \quad Q = P_1 + z\overrightarrow{P_1P_2} \end{aligned}$$

□

Theorem 1.2 (Taylor Theorem)**.** *Suppose $\mathcal{U} \subseteq \mathbb{R}^n$ is convex. Suppose further that $F : \mathcal{U} \to \mathbb{R}$ is smooth.*

- *Suppose F has bounded second order partial derivatives. Then there is a constant M_2 depending only on the bounds on the second partial derivatives such that for each $P \in \mathcal{U}$ and each $\vec{h}$ with $P + \vec{h} \in \mathcal{U}$ we have*

$$F(P + \vec{h}) = F(P) + [DF(P)]\vec{h} + E_1$$

  with $|E_1| \le \frac{1}{2!} M_2 \left\|\vec{h}\right\|^2$.
- *Suppose F has bounded third order partial derivatives. Then there is a constant M_3 depending only on the bounds on the third partial derivatives such that for each $P \in \mathcal{U}$ and each $\vec{h}$ with $P + \vec{h} \in \mathcal{U}$ we have*

$$F(P + \vec{h}) = F(P) + [DF(P)]\vec{h} + \frac{1}{2!}\vec{h}^T[HessF(P)]h\vec{+}E_2$$

  with $|E_2| \le \frac{1}{3!} M_3 \left\|\vec{h}\right\|^3$.

Proof.

$$f(t) = F(P + t\vec{h})$$

$$\begin{aligned}
f(1) &= f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(z) \cdot \beta \\
f'(t) &= \frac{\partial F}{\partial x^1}(P + t\vec{h})h^1 + \ldots + \frac{\partial F}{\partial x^n}(P + t\vec{h})h^n \\
f''(t) &= \frac{\partial^2 F}{\partial {x^1}^2}(P + t\vec{h})(h^1)^2 + \frac{\partial^2 F}{\partial x^1 \partial x^2}(P + t\vec{h})h^1 h^2 \cdot 2 + \ldots
\end{aligned}$$

$$f^{(k)}(t) = \left(h^1 \frac{\partial}{\partial x^1} + \ldots + h^n \frac{\partial}{\partial x^n}\right)^k F(P + t\vec{h})$$

Where $\|E\| \leq C \left\|\vec{h}\right\|^3$ where C is some constant which depends on

$$\sup_{i,j,k} \left| \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} \right|$$

□

2. Measuring sizes of stuff

2.1. **k-forms.**

2.1.1. *Motivation.* k-forms set a "standard" for measuring the size and the orientation of a k-parallelotope.

2.1.2. *Determinant.* For a n-form on n-dimensional vector space leads to the concept of determinant.

$$\begin{aligned}
&\omega(\vec{V}_1, \ldots, \vec{V}_n) \\
&= \underbrace{\sum \operatorname{sign}(\sigma) V_1^{\sigma(1)} \cdots V_n^{\sigma(n)}}_{\text{determinant}} \omega(\partial_1, \ldots, \partial_n) \\
&= \omega(\partial_1, \ldots, \partial_n) \det [\ \ ]
\end{aligned}$$

2.2. **Measure Theory.**

2.2.1. *Jordan Measure.* Jordan measure is a method to measure the size of an "arbitrary" shape, Because a k-form can only measure the size of a parallelotope, and can't handle arbitrary shapes.

2.2.2. *What is not Jordan measurable.* $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable, because $\mu_{int} = 0$, and $\mu_{out} = 1$, this is also because $\partial\mathbb{Q} = \bar{\mathbb{Q}} \setminus \mathrm{Int}\mathbb{Q} = \mathbb{R}$. Then the measure of the boundary is not zero, this is an issue.

2.2.3. *Jordan measure of zero.* We know that a set has Jordan measure of zero if we can completely cover it in boxes, where the sum of the area of the boxes is less than some ϵ. Then we are able to make $\epsilon \to 0$, and thus the measure must also be zero. This entails ensuring that the boxes completely cover the set, and that we can make them as small as we want. For our homework we used the angle to change the size of the boxes.

2.2.4. *Measurable by boundary.* This is an IFF. If the set is measurable, then we can construct two coverings, B_{in} and B_{out}, where $B_{in} \subset \mathrm{Int}(\mathcal{U})$, and B_{out} intersects $\bar{\mathcal{U}}$ non-trivially, i.e. also covers the boundary or it over estimates. As we refine these coverings, they should both approach $\mu(\mathcal{U})$. Thus the difference between them will become less than ϵ. The difference between the two sets will be exactly boxes that contain $\partial\mathcal{U}$, thus $\mu(\partial\mathcal{U}) < \epsilon$.

If the measure of the boundary is zero, then we extend the grid by the hyper planes which define the sides of the boxes, thus effectively cutting the boxes up. There are sets in this global grid, that are entirely within $\mathcal{U}$, and then there are sets that cover $\bar{\mathcal{U}}$, the difference of these sets will be a covering of $\partial\mathcal{U}$, not that it may not be the same covering that we started with, but it will also be less than ϵ, thus since the difference of $\mathcal{B}_{out}$, and $\mathcal{B}_{in}$ must be less than ϵ, since this is true for all ϵ, than $\mathcal{U}$ must me measurable.

3. Riemann Integration

3.1. **Definition.**

Definition 3.1. *We say that the function f is Riemann-integrable over $\mathcal{U}$ if there exists $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all decompositions $\mathcal{U}_*$ with $s(\mathcal{U}_*) < \delta$ and all sample points ξ_* we have*

$$|RS(f, \mathcal{U}_*, \xi_*)| < \varepsilon$$

Definition 3.2. *We say that the function f is Riemann-integrable over $\mathcal{U}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such tat for all decompositions $\mathcal{U}_*$ and $\widetilde{\mathcal{U}_*}$ with $s(\mathcal{U}_*), s(\widetilde{\mathcal{U}_*}) < \delta$ and all sample points $\xi_*, \widetilde{\xi_*}$ we have*

$$\left| RS(f, \mathcal{U}_*, \xi_*) - RS(f, \widetilde{\mathcal{U}_*}, \widetilde{\xi_*}) \right| < \varepsilon$$

Theorem 3.1. *If f is uniformly continuous over $\mathcal{U}$ then f is Riemann integrable over $\mathcal{U}$.*

Theorem 3.2. *f is Riemann integrable iff the set of discontinuities is of measure zero.*

3.2. **Show not integrable.** I would show this by contradiction, Assume that it is integrable, then we set $\varepsilon < 0.5$, then using the Cauchy definition, we select two partitions, and we select the sample points ξ_* to be rational values, and $\widetilde{\xi_*}$ to be irrational values. Thus when we use the Cauchy definition we see that we will have a contradiction, thus the function is not Riemann integrable. This only works for sure because we are able to pick any selection of sample points.

3.3. Fundamental Theorem of Calculus.

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Then*

$$\int_{x\in[a,b]} \frac{\partial f}{\partial x} dx = f(b) - f(a)$$

Proof. Since $\frac{\partial f}{\partial x}$ is continuous we know it is Riemann-integrable over $[a, b]$; let I be the value of its Riemann integral over $[a, b]$. This means that for each ε there is $\delta > 0$ such that for all decompositions $\mathcal{U}_*$ of $[a, b]$ with $s(\mathcal{U}_*) < \delta$ and all sample points ξ_* we have

$$|RS(f', \mathcal{U}_*, \xi_*) - I| < \varepsilon$$

Consider any subdivision points

$$a = a_0 < a_1 < \ldots < a - p = b,$$

with $a_{i+1} - a_i < \delta$ for all i. Note that this gives us a decomposition $\mathcal{U}_*$ of $[a, b]$ with $s(\mathcal{U}_*) < \delta$. By the Mean Value Theorem there exists $\xi_i \in (a_i, a_{i+1})$ such that

$$\frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} = \frac{\partial f}{\partial x}(\xi_i)$$

By design we have

$$RS(f', \mathcal{U}_*, \xi_*) = f(b) - f(a)$$

The overall conclusion is that for all $\varepsilon > 0$ we have

$$|(f(b) - f(a)) - I| < \varepsilon$$

In other words, $I = f(b) - f(a)$. □

4. Universal Properties

4.1. **Exterior Product.**

$$\begin{tikzcd}
V \times \ldots \times V \arrow[rr, "\omega"] \arrow[d, "a"'] & & \mathcal{U} \\
V \wedge \ldots \wedge V \arrow[urr, "\exists! L"'] & &
\end{tikzcd}$$

Let V be a vector space. By the *k-th exterior power of V* we mean a vector space E^k together with an alternating multi linear map $a : V \times \ldots \times V \to E^k$ (on k copies of V) such that for each alternating multi linear map $\omega : V \times \ldots \times V \to \mathcal{U}$ there exists a unique linear map $L : E^k \to \mathcal{U}$ for which $\omega = L \circ a$.

Proof. The mapping a is alternating and multilinear. We start our proof by addressing the mapping a in more detail.

Once a basis for V is fixed, we have the decomposition

$$\vec{V}_1 \wedge \ldots \wedge V_k = a(\vec{V}_1, \ldots, \vec{V}_k) = \sum a^{i_1 \ldots i_k}(\vec{V}_1, \ldots, \vec{V}_k) \partial_{i_1} \wedge \ldots \wedge \partial_{i_k}$$

in which

$$a^{i_1 \ldots i_k}(\vec{V}_1, \ldots, \vec{V}_k) = \vec{V}_1 \wedge \ldots \wedge \vec{V}_k(dx^{i_1}, \ldots, dx^{i_k})$$

However, each individual coordinate mapping $a^{i_1 \ldots i_k}$ is itself a k-form and can be expressed as

$$a^{i_1 \ldots i_k} = \sum a^{i_1 \ldots i_k}(\partial_{j_1}, \ldots, \partial_{j_k}) dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

It now follows that

$$a^{i_1 \ldots i_k}(\vec{V}_1, \ldots, \vec{V}_k) = \vec{V}_1 \wedge \ldots \wedge \vec{V}_k(dx^{i_1}, \ldots, dx^{i_k}) = dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\vec{V}_1, \ldots, \vec{V}_k)$$

i.e. that $a^{i_1 \ldots i_k} = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. We are now ready to proceed with the main part of the proof.

To see that there is at most one mapping L corresponding to a given ω it suffices to observe that any such L must satisfy

$$L(\partial_{i_1} \wedge \ldots \wedge \partial_{i_k}) = \omega(\partial_{i_1}, \ldots, \partial_{i_k})$$

The latter in fact, defines a unique linear map $L : \Lambda^k(V) \to W$. It remains to argue that $\omega = L \circ a$.

To do so it suffices to focus on the coordinate mappings $\omega^i := dy^i \circ \omega$ and $L^i := dy^i \circ L$. As above, coordinate mappings are k-forms and thus we may write

$$\omega^i = \sum \omega^i(\partial_{i_1}, \dots, \partial_{i_k}) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Upon evaluation we obtain

$$\begin{aligned}\omega^i(\vec{V}_1, \dots, \vec{V}_k) &= \sum L^i(\partial_{i_1} \wedge \dots \wedge \partial_{i_k}) a^{i_1 \dots i_k}(\vec{V}_1, \vec{V}_k) \\ &= L^i \left(\sum a^{i_1 \dots i_k}(\vec{V}_1, \dots, \vec{V}_k) \partial_{i_1} \wedge \dots \wedge \partial_{i_k} \right) \\ &= L^i(a(\vec{V}_1, \dots, \vec{V}_k))\end{aligned}$$

that is, $\omega^i = L^i \circ a$ and $\omega = L \circ a$.

To See that the exterior power is unique up to isomorphism, suppose you had two: $(\Lambda^k(V)_1, a_1)$ and $(\Lambda^k(V)_2, a_2)$. By the Universal Property there would exists unique linear mapping L_1 and L_2 such that

$$a_2 = L_1 \circ a_1 \quad a_1 = L_2 \circ a_2$$

Since then

$$a_1 = (L_2 \circ L_1) \circ a_1 \quad a_2 = (L_1 \circ L_2) \circ a_2$$

the uniqueness portion of the Universal Property implies that both $L_2 \circ L_1$ and $L_1 \circ L_2$ are identity mappings. In particular, one concludes that L_1 and L_2 are isomorphisms. $\square$

4.2. **Tensor Product.**

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[r, "\omega"] \arrow[d, "b"'] & \mathcal{U} \\ V_1 \otimes \dots \otimes V_k \arrow[ur, "\exists ! L"'] & \end{tikzcd}$$

Let $V_1, \dots, V_k$ be vector spaces. By the tensor product of $V_1, \dots, V_k$ we man a vector space denoted $V_1 \otimes \dots \otimes V_k$ together with a bilinear map $b : V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$ such that for each multilinear map $\omega : V_1 \times \dots \times V_k \to \mathcal{U}$ there exists a unique linear map $L : V_1 \otimes \dots \otimes V_k \to \mathcal{U}$ for which $\omega = L \circ b$.

5. Random bits of linear / tensor algebra

5.1. **Dual Space.**

Definition 5.1. *Let V be a vector space. The set of all covectors of $\omega : V \to \mathbb{R}$ is called the dual space to V and is denoted by V^*.*

The dual space and the vector space have the same dimension, were we define the basis to be of the form

$$dx_i(\partial_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We verified that this acts as a basis for the dual space. Because V and V^* have the same dimension then there is a correspondence between ∂_i and dx_i.

5.2. **Tensor terminology.**

$$T \in \left[\bigotimes^p V^*\right] \otimes \left[\bigotimes^q V\right]$$

$$T_{i_1 \ldots i_p}^{j_1 \ldots j_q} \equiv T(\partial_{i_1}, \ldots, \partial_{i_p}, dx^{j_1}, \ldots, dx^{j_q})$$

5.3. **Raising and Lowering indices.**

6. Volume Forms

6.1. Orientation of a Space. The orientation of a space is based on a set of basis vectors and a volume form, The orientation of space can be thought of as how much a parallelotope is in the same direction of the basis vectors, or agrees with how the volume form measures the space. This can be show with vectors, were a vector in the x, y, z direction would be positive but a vector in the $-x, -y, -z$ direction would be considered as negatively oriented.

6.2. **Corollary 2 in Part II.**

Corollary 6.1. *Let $\{\partial_1, \ldots, \partial_n\}$ be an orthonormal basis of an n-dimensional innerproduct space, with innerproduct g. If ω is a volume form with $\omega(\partial_1, \ldots, \partial_n) = 1$ then*

$$\left|\omega(\vec{V}_1, \ldots, \vec{V}_n)\right| = \sqrt{\det\left[g(\vec{V}_i, \vec{V}_j)\right]}$$

6.3. **dvol.**

$$\text{dvol}_g = \sqrt{\det\left[g(\partial_i, \partial_j)\right]} dx^1 \wedge \ldots \wedge dx^n$$

This is able to determine the orientation because of the alternating nature of $\wedge$ products, and the size is due to the linearity of wedge products, and the determinant. The determinant represents the measure of one "unit" cell based on the basis vectors, and the dot product, then the size of each vector multiplied together then scaled by this amount provide the area.

6.4. **Hodge start.** Given some alternating multilinear mapping

$$(\vec{V_1}, \dots, \vec{V_k}) \to \omega(\vec{V_1}, \dots, \vec{V_k}, \_, \dots, \_)$$

then by the Universal Property we have a linear mapping

$$\star_\omega : \Lambda^k(V) \to \Lambda^{n-k}(V^*)$$

It basically it places in the first k values into the function of ω and returns a new function that takes $n - k$ values.

$$\star_\omega(\vec{V_1}, \dots, \vec{V_k}) = \omega(\vec{V_1}, \dots, \vec{V_k}, \_, \dots \_)$$

There is also an equivalent representation that works for co-vectors. It is pretty much the same thing.

6.5. **Cross product.**

6.5.1. *What is it?*

6.5.2. *Does it have the properties we want?*

6.5.3. *5.10.4.*

7. Category Theory

7.1. **What did you learn?** A category is a construct of *objects* and *morphisms* between the objects. Objects could be for example vector spaces, and then the associated morphisms would be linear transformations. It seems pretty straightforward, and very useful. Most of the proofs are done through diagram chasing.

7.2. **What is a functor?** A *functor* is a method for converting from one category to another. Functors must take isomorphisms to isomorphisms, and take compositions to compositions. This is useful if examining a problem in one category is hard, then it can be converted to an easier category. Note that if a functor maps an isomorphism, it does not mean that the morphism in category 1 is an isomorphism. Thus this can only be used as a contrapositive tool.