REAL ANALYSIS FALL 2018

ARDEN RASMUSSEN

Problem 1

Show that the exponential function is differentiable, and that $\exp' = \exp$.

Proof. Consider the definition of $\exp(x)$ from Defn. 4.21. We consider for some $n \in \mathbb{N}$

$$\begin{aligned} E_n(x) &= \sum_{k=0}^{n} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \end{aligned}$$

Consider the derivative of $E_n(x)$

$$\begin{aligned} E_n'(x) &= 0 + 1 + x + \frac{x^2}{2!} + \cdots \\ &= \sum_{k=0}^{n-1} \frac{x^k}{k!}. \end{aligned}$$

as $n \to \infty$, from Defn. 4.21, clearly

$$\begin{aligned} E_n(x) &\to \exp(x) \\ E_n'(x) &\to \exp(x). \end{aligned}$$

Thus by Prop. 4.16 we can conclude that $\exp'(x) = \exp(x)$. Thus since the derivative exists we conclude that $\exp(x)$ is differentiable. ∎

Problem 2

Show that $\exp(x) > 0$ for all $x \in \mathbb{R}$.

Proof. Consider $\exp(x)$, and $x \in \mathbb{R}$.

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Case 1. If $x > 0$ then

$$\begin{aligned}\exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(+)^k}{(+)} \\ &= \sum_{k=0}^{\infty} (+) \\ &> 0\end{aligned}$$

since this is a sum of only positive values, then the sum must also be positive. Thus $\exp(x) > 0$ if $x > 0$.
Case 2. If $x = 0$ then

$$\begin{aligned}\exp(0) &= \sum_{k=0}^{\infty} \frac{0^k}{k!} \\ &= 1 + 0 + 0 + 0 + \cdots \\ &= 1 \\ &> 0\end{aligned}$$

Since 1 is greater than 0 then $\exp(0) > 0$.
Case 3. If $x < 0$ then we utilize a later proof. We notice that in our proof from Prob. 4, we never use the result of this proof. Thus we are able to use the result of Prob. 4 in this proof. Thus we consider

$$\begin{aligned}\frac{\exp(x)}{\exp(x)} &= 1 \\ \exp(x)\exp(-x) &= 1.\end{aligned}$$

Since $-x > 0$ and we have shown that $\exp((+)) > 0$ then we know

$$\exp(x)(+) = (+).$$

Thus the only way this is true is if $\exp(x) > 0$.
We conclude that $\exp(x) > 0$ for all $x \in \mathbb{R}$. □

Problem 3

Use the fact that $\exp(x) > 0$ for all $x \in \mathbb{R}$ to conclude that the exponential function is strictly increasing, and thus a bijection $\mathbb{R} \to (0, \infty)$.

Proof. Consider $\exp(x)$. From Prob. 2 $\exp(x) > 0$ for all $x \in \mathbb{R}$. Since $\exp(x)$ is continuous, from $[-L, L]$ for any $L > 0$, we consider some sub interval defined by $[\alpha, \beta]$, such that $[\alpha, \beta] \subset [-L, L]$. Thus we know that $\exp(x)$ is continuous on the interval $[\alpha, \beta]$, and from Prob. 1, it is also differentiable on the interval (α, β). Thus by the Mean Value Theorem there exists some $z \in [\alpha, \beta]$ such that

$$\exp'(x) = \frac{\exp(\beta) - \exp(\alpha)}{\beta - \alpha}.$$

From Prob. 1 we substitute $\exp'(x) = \exp(x)$, then due to the assumption that $\exp(x) > 0$ for all $x \in \mathbb{R}$, we know that $\exp(z) > 0$. From the construction of the interval $[\alpha, \beta]$, it is clear that

$$\begin{aligned} \beta &> \alpha \\ \beta - \alpha &> 0. \end{aligned}$$

Thus

$$\begin{aligned} (+) &= \frac{\exp(\beta) - \exp(\alpha)}{(+)} \\ (+) &= \exp(\beta) - \exp(\alpha) \\ \implies 0 &< \exp(\beta) - \exp(\alpha) \\ \exp(\alpha) &< \exp(\beta) \end{aligned}$$

Thus we can clearly see that

$$\beta > \alpha \implies \exp(\beta) > \exp(\alpha)$$

Thus we conclude that $\exp(x)$ must be strictly increasing. $\square$

Proof. We now prove that exp is a bijection from $\mathbb{R} \to (0, \infty)$.
Injective. First we prove that exp is injective. This means that $\exp(x) = \exp(y) \implies x = y$. Take $x, y \in \mathbb{R}$ and assume that $\exp(x) = \exp(y)$. We proceed by contradiction. Assume that $x > y$ (without loss of generality, this can be applied to $y > x$). Thus by the previous proof

$$x > y \implies \exp(x) > \exp(y).$$

But this is a contradiction of our assumption that $\exp(x) = \exp(y)$. Using the same argument for if $y > x$, we find that it must be true that $x = y$, because $x \not> y$ and $y \not> x$. Thus exp is injective
Surjective. Now we prove that that for all $y \in (0, \infty)$ there is some $x \in \mathbb{R}$ such that $\exp(x) = y$. Take $y \in (0, \infty)$. We construct two sequences of numbers defined as

$$\begin{aligned} x_n \to y \quad \alpha_n &= \exp(x_n) \\ z_n \to y \quad \gamma_n &= \exp(z_n) \end{aligned}$$

such that $x_n < y < z_n$. By the proof above, we can see that

$$\begin{aligned} \exp(x_n) &< \exp(y) < \exp(z_n) \\ \alpha_n &< \exp(y) < \beta_n. \end{aligned}$$

Then as $n \to \infty$ by squeeze theorem, we can clearly see that α_n converges to some value γ, and β_n also converges to this gamma. We call this $\gamma = \exp(y)$. Thus for any $y \in (0, \infty)$ $\exists \gamma \in \mathbb{R}$ such that $\gamma = \exp(y)$. Thus exp is surjective.

Since exp is both injective and surjective, then exp must be a bijection from $\mathbb{R} \to (0, \infty)$. $\square$

Problem 4

In this problem you show that $\exp(a+b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{R}$. To do this, fix $a \in \mathbb{R}$ and define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \exp(a+x) - \exp(a)\exp(x)$$

The plan of the proof is to show that $g(x) = 0$ for all $x \in \mathbb{R}$.

(a). Show that g is differentiable with $g' = g$.

Proof. Let $a \in \mathbb{R}$, and consider $g(x) = \exp(a+x) - \exp(a)\exp(x)$, where $x \in \mathbb{R}$. Consider the derivative of $g(x)$

$$\begin{aligned} g'(x) &= (\exp(a+x) - \exp(a)\exp(x))' \\ &\text{by Thm. 4.6.a, and } \exp(a) \text{ is a constant} \\ &= (\exp(a+x))' - \exp(a)(\exp(x))' \\ &\text{by Thm. 4.7} \\ &= \exp'(a+x)(a+x)' - \exp(a)\exp'(x) \\ &\text{by Prob. 1} \\ &= \exp(a+x) - \exp(a)\exp(x) \\ g'(x) &= g(x) \end{aligned}$$

Thus g is differentiable with $g' = g$. □

(b). Show that $g(0) = 0$.

Proof. Take $g(x)$,

$$\begin{aligned} g(0) &= \exp(a+0) - \exp(a)\exp(0) \\ &= \exp(a) - \exp(a)\exp(0) \\ &\text{from Prob. 2.2, } \exp(0) = 1 \\ &= \exp(a) - \exp(a) \\ &= 0 \end{aligned}$$

Thus $g(0) = 0$. □

(c). Let $M = \sup\left\{|g(x)| : x \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right\}$. Use the Mean Value Theorem to show that $|g(x)| \leq \frac{1}{4}M$ for all $x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. Deduce that $M \leq \frac{1}{4}M$ and thus $M = 0$. Conclude $g(x) = 0$ for all $x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$.

Proof. We consider the function $g(x)$ as defined above. Let

$$M = \sup\left\{|g(x)| : x \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right\}.$$

Since g is continuous because exp is continuous, and we have shown that g is differentiable in Prob. 4.a, we can conclude that $|g|$ will also be continuous and differentiable.

Consider the bounds $[0, x]$ for $x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. Clearly $|g|$ is continuous over the interval $[0, x]$, and $|g|$ is differentiable on the interval $(0, x)$. Thus by the Mean Value Theorem we know that there exists some z such that

$$\begin{aligned} |g'(z)| &= \frac{|g(x)| - |g(0)|}{|x - 0|} \\ &\text{by Prob. 4.b} \\ &= \frac{|g(x)|}{|x|} \\ |x||g'(z)| &= |g(x)| \\ &= \text{by Prob. 4.a} \\ |x||g(z)| &= |g(x)|. \end{aligned}$$

Then by the constraints on x and the assumption of M we find

$$\begin{aligned} \frac{1}{4}M &\geq |x||g(z)| = |g(x)| \\ \frac{1}{4}M &\geq |g(x)| \end{aligned}$$

This expression holds for all $x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. We note that $|g|$ must be bounded, as exp is bounded on the interval $\left[-\frac{1}{4}, \frac{1}{4}\right]$, since $|g|$ is bounded, then by the definition of supremum of $|g|$ Defn. 3.42, there must exist some value in this interval called x_* such that $|g(x_*)| = M$.

We now use this value in the previous expression

$$\begin{aligned} \frac{1}{4}M &\geq |g(x_*)| \\ \frac{1}{4}M &\geq M \end{aligned}$$

clearly this can only be the case if $M = 0$. By the definition of supremum, it is clear that $|g(x)| = 0 = g(x)$ for all $x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. $\square$

(d). Adapt the idea of the previous step to show that

$$g(x_0) = 0 \implies g(x) = 0 \text{ for all } x \in \left[x_0 - \frac{1}{4}, x_0 + \frac{1}{4}\right].$$

Proof. We use the same method from the previous proof here. Consider $g(x)$. Assume

$$M = \sup\left\{|g(x)| : x \in \left[x_0 - \frac{1}{4}, x_0 + \frac{1}{4}\right]\right\}.$$

for some $x_0 \in \mathbb{R}$ such that $g(x_0) = 0$.

Now consider the interval between 0 and x for $x \in [x_0 - \frac{1}{4}, x_0 + \frac{1}{4}]$. Again by the Mean Value Theorem, there exists some z such that

$$\begin{aligned} |g'(z)| &= \frac{|g(x)| - |g(x_0)|}{|x - x_0|} \\ |g(z)| &= \frac{|g(x)|}{|x - x_0|)} \\ |x - x_0||g(z)| &= |g(x)| \end{aligned}$$

Because of out limitations on x we know that $|x - x_0| \leq \frac{1}{4}$. And by our assumption of M we know

$$\frac{1}{4}M \geq |x - x_0||g(z)| = |g(x)|$$
$$\frac{1}{4}M \geq |g(x)|$$

By the same reasoning as the previous proof, there must be some x_* in the interval such that $|g(x_*)| = M$, thus we can use this value of x_* and find

$$\frac{1}{4}M \geq M$$

Clearly this can only be true if $M = 0$, and from this we know that $g(x) = 0$ for all $x \in \left[x_0 - \frac{1}{4}, x_0 + \frac{1}{4}\right]$. □

(e). Conclude that $g(x) = 0$ for all $x \in \mathbb{R}$.

Proof. We begin by using the result of Prob. 4.c, stating that $g(x) = 0\ \forall x \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. We know select x_1 to be $\frac{1}{4}$. From Prob. 4.c we know that $g(x_1) = 0$, and from Prob. 4.d, we know that this implies that $g(x) = 0\ \forall x \in \left[x_1 - \frac{1}{4}, x_1 + \frac{1}{4}\right]$. We can repeat this process for all increments, by constructing a general expression for x_n

$$x_n = \frac{n}{4} \quad \forall n \in \mathbb{Z}$$

Thus for any $x_* \in \mathbb{R}$, there is some interval centered around some x_n that covers x_* and thus $g(x_*) = 0$. Thus we conclude that $g(x) = 0$ for all $x \in \mathbb{R}$. □

Explain. Explain how this completes the proof that $\exp(a + b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{R}$. Show also that $\exp(-a) = 1/\exp(a)$ for all $a \in \mathbb{R}$.

Since $g(x) = 0\ \forall x \in \mathbb{R}$, then we can see that

$$0 = \exp(a + x) - \exp(a)\exp(x)$$
$$\exp(a + x) = \exp(a)\exp(x)$$

since we made no assumptions on a or x this can be generalized to

$$\exp(a + b) = \exp(a)\exp(b)$$

Consider

$$\begin{aligned}&\exp(0)\\ &= \exp(-a + a)\\ &= \exp(-a)\exp(a).\end{aligned}$$

From Prob. 2 we know that $\exp(0) = 1$, thus this expression must equal 1.

$$1 = \exp(-a)\exp(a)$$
$$\frac{1}{\exp(a)} = \exp(-a)$$

Thus $\exp(-a) = 1/\exp(a)$.

Problem 5

In this problem you show that $\exp(q) = e^q$ for all $q \in \mathbb{Q}$.

(a). Explain why it suffices to show this only for $q > 0$.

It is sufficient to show this only for $q > 0$, because from Prob. 4, we showed that $\exp(-a) = \frac{1}{\exp(a)}$, thus is we show this for $q > 0$ we know that

$$\exp(-q) = \frac{1}{\exp q} = \frac{1}{e^q} = e^{-q}.$$

Thus we are able to neglect the cases when $q < 0$, and only prove this for when $q > 0$.

(b). Show that $\exp(n) = e^n$ for $n \in \mathbb{N}$.

Proof. First we prove that $\exp(1) = e$. From the definition of exp, we use the value of $x = 1$ and expand into the series definition.

$$\begin{aligned} \exp(1) &= \sum_{k=0}^{\infty} \frac{1^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \end{aligned}$$

Notice that this is exactly the series definition of e, thus we know that $\exp(1) = e$. $\square$

Proof. Let $n \in \mathbb{N}$. Consider $\exp(n)$, we know that since $n \in \mathbb{N}$ that it can be represented as a sum of 1's, thus

$$\begin{aligned} \exp(n) &= \exp(\underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}}) \\ &\text{then by induction and Prob. 4} \\ &= \underbrace{\exp(1)\exp(1)\cdots\exp(1)}_{n \text{ times}} \\ &= \exp(1)^n \end{aligned}$$

From the previous proof, we know that $\exp(1) = e$ thus we can clearly see that $\exp(n) = e^n$ for all $n \in \mathbb{N}$. $\square$

(c). Show that $\exp(1/m) = e^{1/m}$ for all $m \in \mathbb{N}$.

Proof. Consider $m \in \mathbb{N}$. We consider $\exp\left(\frac{1}{m}\right)$. Consider rasing this to the power of m to obtain

$$\begin{aligned}
\exp\left(\frac{1}{m}\right)^m &= \underbrace{\exp\left(\frac{1}{m}\right)\exp\left(\frac{1}{m}\right)\cdots}_{m \text{ times}} \\
&\text{by induction and Prob. 4} \\
&= \exp\left(\underbrace{\frac{1}{m}+\frac{1}{m}+\cdots}_{m \text{ times}}\right) \\
&= \exp\left(\frac{m}{m}\right) \\
&= \exp(1) \\
&\text{from Prob. 5.b} \\
&= e \\
&= e^{\frac{m}{m}} \\
&= \left(e^{\frac{1}{m}}\right)^m
\end{aligned}$$

Thus we can see that

$$\exp\left(\frac{1}{m}\right)^m = \left(e^{\frac{1}{m}}\right)^m$$

Then we take the square root of each side and we obtain that

$$\exp\left(\frac{1}{m}\right) = e^{\frac{1}{m}}$$

Thus we can conclude that $\exp(1/m) = e^{1/m}$ for all $m \in \mathbb{N}$. $\square$

(d). Conclude that $\exp(q) = e^q$ for $q \in \mathbb{Q}$.

Proof. Let $q \in \mathbb{Q}$, such that $q = n/m$ for $n, m \in \mathbb{N}$. Consider

$$\begin{aligned}
\exp(q) &= \exp\left(\frac{n}{m}\right) \\
&= \exp\underbrace{\left(\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}\right)}_{n \text{ times}} \\
&\text{by induction and Prob. 4} \\
&= \underbrace{\exp\left(\frac{1}{m}\right)\exp\left(\frac{1}{m}\right)\cdots\exp\left(\frac{1}{m}\right)}_{n \text{ times}} \\
&= \exp\left(\frac{1}{m}\right)^n \\
&\text{by Prob. 5.c} \\
&= e^{1/m^n} \\
&= e^{n/m} \\
&= e^q
\end{aligned}$$

Thus $\exp(q) = e^q$ for all $q \in \mathbb{Q}$. ∎

Problem 6

We define the **logarithm function** to be the inverse of the exponential function, so $\log(x) = \exp^{-1}(x)$.

(a). Explain how we know that the logarithm function is differentiable and compute its derivative.

Proof. We know that the logarithm is differentiable, because $\exp(x)$ is a bijection. And if a function is a bijection, and differentiable, then that functions inverse must also be a bijections, and differentiable. Thus log must be differentiable.

We prove by the use of the Inverse Function Theorem (Thm. 4.11). Since exp is differentiable, and since $\exp' = \exp$ clearly, $\exp'$ is continuous. From Prob. 2, we know that $\exp'(x_*) \neq 0$ for all $x_* \in \mathbb{R}$, since $\exp(x) > 0$ for all $x \in \mathbb{R}$. Then, by the Inverse Function Theorem, there exists $\delta > 0$ such that within $B_\delta(x_*)$ is inevitable, and that $\exp^{-1}$ is differentiable and satisfying

$$\begin{aligned}
(\exp^{-1})'(y) &= \frac{1}{\exp'(\exp^{-1}(y))} \\
\log'(y) &= \frac{1}{\exp(\log(y))}.
\end{aligned}$$

By the definition of an inverse function, we know that $\exp(\log(x)) = x$, thus

$$\log'(y) = \frac{1}{y}.$$

Since we made no assumptions on x_* or y, then this is satisfied for all $y \in (0, \infty)$. ∎

(b). Show that the "usual log rules" hold:

$$\log(xy) = \log(x) + \log(y), \quad \log(x/y) = \log(x) - \log(y), \quad \log(x^y) = y\log(x).$$

$\log(xy) = \log(x) + \log(y)$*:*

Proof. Consider $\log(xy)$ for any $x, y \in (0, \infty)$. Define some α and β such that

$$\begin{aligned} \log(x) &= \alpha, & \log(y) &= \beta \\ \exp(\alpha) &= x, & \exp(\beta) &= y. \end{aligned}$$

Thus we rewrite the expression as

$$\begin{aligned} \log(xy) &= \log(\exp(\alpha)\exp(\beta)) \\ &\text{by Prob. 4} \\ &= \log(\exp(\alpha+\beta)) \\ &\text{by inverse} \\ &= \alpha + \beta \\ &= \log(x) + \log(y) \end{aligned}$$

Thus $\log(xy) = \log(x) + \log(y)$ is true. □

$\log(x/y) = \log(x) - \log(y)$*:*

Proof. Consider $\log\left(\frac{x}{y}\right)$ for any $x, y \in (0, \infty)$. We again define $\alpha, \beta \in \mathbb{R}$, in the same way as before. Thus our expression becomes

$$\begin{aligned} \log\left(\frac{x}{y}\right) &= \log\left(\frac{\exp(\alpha)}{\exp(\beta)}\right) \\ &\text{by Prob. 4} \\ &= \log(\exp(\alpha)\exp(-\beta)) \\ &= \log(\exp(\alpha-\beta)) \\ &= \alpha - \beta \\ &= \log(x) - \log(y). \end{aligned}$$

Thus $\log(x/y) = \log(x) - \log(y)$ holds. □

$\log(x^y) = y\log(x)$*:*

Proof. Consider $\log(x^y)$ for any $x, y \in (0, \infty)$. We again define α in the same was as before. Thus we can write

$$\begin{aligned} \log(x^y) &= \log(\exp(\alpha)^y) \\ &\text{by Prob. 5} \\ &= \log(e^{\alpha^y}) \\ &= \log(e^{y\alpha}) \\ &= y\alpha \\ &= y\log(x) \end{aligned}$$

Thus $\log(x^y) = y\log(x)$ holds. □

Problem 7

The real numbers

(a). What are the axioms/properties that we are assuming about the real number. What purpose do each of these axioms serve? (That is, what motivates each of these axioms?)

The axioms of the real numbers are:

- Field axioms
  - Addition axioms
    * Commutative $x + y = y + x$ $\forall x, y \in \mathbb{F}$.
    * Associative $(x + y) + z = x + (y + z)$ $\forall x, y \in \mathbb{F}$.
    * There exists an Additive Identity $0 \in \mathbb{F}$ such that $0 + x = x$ $\forall x \in \mathbb{F}$.
    * For all $x \in \mathbb{F}$ there exists an additive inverse $-x \in \mathbb{F}$ such that $x + (-x) = 0$.
  - Multiplication axioms
    * Commutative $x \cdot y = y \cdot x$ $\forall x, y \in \mathbb{F}$.
    * Associative $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ $\forall x, y, z \in \mathbb{F}$.
    * There exists a multiplicative identity $1 \in \mathbb{F} \setminus 0$ such that $1 \cdot x = x$ $\forall x \in \mathbb{F}$.
    * For all $x \in \mathbb{F} \setminus \{0\}$ there exists a multiplicative inverse $x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$.
  - Distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$ $\forall x, y, z \in \mathbb{F}$.
- Ordered fields
  - For each $x, y \in \mathbb{F}$ precisely one of the following holds: $x < y$, $x = y$, or $y < x$.
  - if $x, y, z \in \mathbb{F}$ with $x < y$ and $y < z$, then $x < z$.
  - if $x, y, z \in \mathbb{F}$ with $x < y$ then $x + z < y + z$.
  - if $x, y \in \mathbb{F}$ with $0 < x$ and $0 < y$ then $0 < x \cdot y$.
- Archimedian property. For each $x \in \mathbb{F}$ with $x > 0$ there exists $n \in \mathbb{N}$ with $n > x$.
- Bisection Search is successful. For every sequence of closed intervals I_n with
  (1) $I_n \subset I_{n+1}$ $\forall n \in \mathbb{N}$.
  (2) $|I_n| \to 0$.
  we have $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

The field axioms provide us with a system where our expressions for addition, subtraction, multiplication and division work as we expect them to. This means that we want the operations to be commutative and associative. We also want some construction of additive identity, additive inverse, multiplicative identity, and the multiplicative inverse. These field axioms provide the basis of natural numbers that are needed to construct the reals. The ordered field axiom allows us to construct sequences of natural numbers, which is frequently used to converge to a real number. Then the archimedian property says that there is some number that is greater than a given x, e.g. there is always a number larger. Bisection search says that there will always be some value between two other numbers, as the interval goes to zero, but will always contain some elements.

(b). Define what it means for a sequence $\{x_n\} \subset \mathbb{R}$ to converge to $x_* \in \mathbb{R}$.

A sequence $\{x_n\} \subset \mathbb{R}$ converges to $x_* \in \mathbb{R}$ if for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$k > N \implies |x_k - x_*| < \varepsilon.$$

In this case we write $x_n \to x_*$.

(c). Give an example of a theorem (or other fun fact) that makes use of sequences. What makes this result noteworthy?

A theorem that makes uses of sequences is Squeeze theory (Prop. 1.17). This theorem is very noteworthy, because it is necessary for other axioms, such at bisection search. It allows us to say that two things converging toward one another, will squeeze some third sequence and they will all converge to the same value. This is incredibly useful and very intuitive.

Problem 8

Metric spaces

(a). Give the definition of a metric space.

Let X be a set. A function $d : X \times X \to [0, \infty)$ is a metric on X if d is

(1) definite, meaning that for all $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$.
(2) symmetric meaning that $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(3) satisfies the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y)$$

for all $x, y, z \in X$.

if d is a metric on X then the pair (X, d) is called a metric space.

(b). Give the definition of what it means for a set to be open in a metric space.

Let $Y \subset X$.

(1) An open ball is defined as $B_r(p) = \{q \in X | d(p, q) < r\}$.
(2) A point $p \in Y$ is an interior point if there exists an open ball $B_r(q)$ such that $p \in B_r(q) \subset Y$.
(3) Y is open in X if each element of Y is an interior point.

(c). Give three examples of metric spaces – one example where the elements are number, one example where the elements are pairs of numbers, and one example where the elements are functions. Give an example of an open ball in each of your three examples.

(1) $(\mathbb{R}, |x - y|)$.

$$B_{0.5}(0) = \left\{ y \in \mathbb{R} \;\middle|\; |x - y| < 0.5 \right\} = (-0.5, 0.5).$$

(2) $\left(\mathbb{R}^2, \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}\right)$.

$$B_{0.5}(0) = \left\{ y \in \mathbb{R}^2 \;\middle|\; \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} < 0.5 \right\} = \left\{ y \in \mathbb{R} \;\middle|\; \sqrt{y_1^2 + y_2^2} < 0.5 \right\}$$

(3) $(C_b(X), 0 \text{ if } f = g \text{ else } 1) \equiv$ All continuous bounded functions $f : X \to \mathbb{R}$, with the norm $\|f\| = \sup_X |f|$ is finite. With the discrete metric.

$$B_{0.5}(f) = \left\{ g \in C_b(X) \;\middle|\; 0 \text{ if } f = g \right\} = \{f\}$$

(d). Give an example of a theorem (or other fun fact) that makes use of open sets. What makes this result noteworthy?

An interesting theorem is the open set definition of continuous functions (Thm. 3.23). This theorem states that a function is continuous if and only if every open set in the co-domain the preimage of that set is also open in the domain.

This is a very noteworthy result, as we con construct a notion of continuous functions without every really having a function, we just need to know how open sets relate over some mapping. Because we only need to think of the open sets of a metric space, it is very cool that we can then uses these open sets to determine weather a function is continuous or not.

Problem 9

Continuous functions

(a). State the four equivalent notions of continuity for functions. Illustrate each notion for the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$.

Suppose (X, dx) and (Y, dy) are metric spaces and $f : X \to Y$. Then the following are equivalent.

(1) f is continuous.
(2) For each $x_* \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$dx(x, x_*) < \delta \implies dy(f(x), f(x_*)) < \varepsilon.$$

(3) For each open set $U \subset Y$ the preimage $f^{-1}(U)$ is open in X.
(4) For each closed set $V \subset Y$ the preimage $f^{-1}(V)$ is closed in X.

(b). Give an example of a theorem (or other fun fact) that makes use of continuous functions. What makes this result noteworthy?

The basic properties of continuous functions are very fun (Prob. 1 Exam 1). I think that these properties are very cool, because they prove that combinations of continuous function are then continuous. Because of this all that needs to be shown is that a very limited number of functions must be proven to be continuous, then by these properties, combinations, products, compositions and any combination of these is then also continuous.

This can be especially useful, when showing a function is continuous, if all of the individual components are continuous, then the function as a whole is also continuous. This greatly simplifies continuity proofs, and that is very convenient and useful.

Problem 10

Synthesis

Write a few words indicating how the ideas about numbers, metric spaces, and continuous functions discussed in this course help lay the foundation for a more rigorous theory of calculus.

Using these concepts for numbers, metrics, and functions, we are able to construct most aspects of the theory of calculus, while only taking the axioms that we listed in Prob. 9.a. Because these are the only assumptions that need to be made, the definitions that we construct for calculus are more rigorous. As they do not require any more assumptions. Normally we calculus requires many more assumptions in order to construct some of the concepts, but with this process, we are able to minimize the number of axioms required for the theory of calculus. This allows for larger generalizability, as anything that satisfies the short list of assumptions can be applied to use in the theory of calculus. Thus this method lays the foundation for a more rigorous theory of calculus.