CONTINUOUS REAL-VALUED FUNCTIONS

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1. Basic Properties

Suppose that $f, g : \Omega \to \mathbb{R}$ are continuous. Show that the following functions are also continuous:

(a) $f + g$ and $f - g$
(b) $f \cdot g$
(c) $c \cdot f$, where c is some constant
(d) f/g, provided $g \neq 0$

1.a.

Proof. Let $\varepsilon > 0$. We assume $f, g : \Omega \to \mathbb{R}$ are continuous. Assmue that the sequence $x_n \to x_*$. Let $N_f, N_g \in \mathbb{N}$ such that

$$
\begin{aligned}
n > N_f &\implies |f(x_n) - f(x_*)| < \frac{\varepsilon}{2} \\
n > N_g &\implies |g(x_n) - g(x_*)| < \frac{\varepsilon}{2}
\end{aligned}
$$

Take $N = max(N_f, N_g)$. Thus when $n > N$ both of the previous statements are true. Consider

$$
\begin{aligned}
&|(f+g)(x_n) - (f+g)(x_*)| \\
&= |f(x_n) + g(x_n) - f(x_*) - g(x_*)| \\
&= |(f(x_n) - f(x_*)) + (g(x_n) - g(x_*))| \\
&\leq |f(x_n) - f(x_*)| + |g(x_n) - g(x_*)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{when } n > N \\
&= \varepsilon
\end{aligned}
$$

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Now we can conclude that $(f+g)(x_n) \to (f+g)(x_*)$ and thus $f+g$ must be continuous. Now consider

$$\begin{aligned}
&|(f-g)(x_n) - (f-g)(x_*)| \\
&= |f(x_n) - g(x_n) - f(x_*) + g(x_*)| \\
&= |(f(x_n) - f(x_*)) + (g(x_*) - g(x_n))| \\
&\leq |f(x_n) - f(x_*)| + |g(x_*) - g(x_n)| \\
&= |f(x_n) - f(x_*)| + |g(x_n) - g(x_*)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{when } n > N \\
&= \varepsilon
\end{aligned}$$

We can see that $(f-g)(x_n) \to (f-g)(x_*)$ and thus $f-g$ must be continuous. □

1.b.

Proof. Let $\varepsilon > 0$. We define M such that

$$M = max\left(\{|f(x_*)|, |g(x_*)|, 1\}\right)$$

Let $N \in \mathbb{N}$ such that

$$n > N \implies \begin{cases} |f(x_n) - f(x_*)| < \frac{\varepsilon}{3M} \\ |g(x_n) - g(x_*)| < \frac{\varepsilon}{3M} \end{cases}$$

Consider

$$\begin{aligned}
&|(f \cdot g)(x_n) - (f \cdot g)(x_*)| \\
&= |f(x_n)g(x_n) - f(x_*)g(x_*)| \\
&= |f(x_n)g(x_n) - f(x_n)g(x_*) + f(x_n)g(x_*) - f(x_*)g(x_*)| \\
&= |f(x_n)(g(x_n) - g(x_*)) + g(x_*)(f(x_n) - f(x_*))| \\
&\leq |f(x_n)(g(x_n) - g(x_*))| + |g(x_*)(f(x_n) - f(x_*))| \\
&= |f(x_n)||g(x_n) - g(x_*)| + |g(x_*)||f(x_n) - f(x_*)| \\
&\text{Since } |g(x_*)| \leq M \text{ and } |f(x_n)| < M + \frac{\varepsilon}{3M} \\
&< \left(M + \frac{\varepsilon}{3M}\right)|g(x_n) - g(x_*)| + M|f(x_n) - f(x_*)| \\
&< \left(M + \frac{\varepsilon}{3M}\right)\frac{\varepsilon}{3M} + M\frac{\varepsilon}{3M} \quad \text{when } n > N \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9M^2} + \frac{varepsilon}{3} \\
&\text{Since } M \geq 1 \text{ then we know} \qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

□

2. Composition

Suppose $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are continuous, and that $f(U) \subset V$. Show that $g \circ f : U \to \mathbb{R}$ is continuous.

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3. Examples

(a) Show that all polynomial functions are continuous
(b) Show that $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous.

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4. Non-Example

Let $\sigma : \mathbb{R} \to \mathbb{R}$ is given by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that σ is not continuous.

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5. The Square Root Function

Consider the square root function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$.

(a) Show that the square root function is *strictly increasing*, meaning that

$$a < b \implies \sqrt{a} < \sqrt{b}$$

(b) Show that the square root function is continuous.

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6. Intermediate Value Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that y_* is between $f(a)$ and $f(b)$. Prove that there exists $x_* \in [a, b]$ such that $f(x_*) = y_*$.

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7. $\varepsilon - \delta$ Criterion for Continuity

Show that the following are equivalent:

(a) $f : \Omega \to \mathbb{R}$ is continuous at $x_* \in \Omega$.
(b) For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \Omega$ we have

$$|x - x_*| < \delta \implies |f(x) - f(x_*)| < \varepsilon$$

Then illustrate the second condition with a picture.

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8. Example

Show directly that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ satisfies condition (b) above at $x_* = 2$.

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9. Examples

(a) Show that $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$ is uniformly continuous.
(b) Show that $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is uniformly continuous.
(c) Show that $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.