A HISTORY OF ANALYTICAL ORBIT MODELING IN THE UNITED STATES SPACE SURVEILLANCE SYSTEM

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Abstract

The launch of the Sputnik satellite in 1957 created a need within the United States to track and maintain a catalog of orbital elements for artificial Earth satellites. Since that time, the United States has cataloged over 27,000 satellites with more than 8,000 still in orbit. At the foundation of the cataloging process are the analytical orbit prediction models. The development and implementation of such models for maintenance of a large catalog has been centered at two primary agencies, the Air Force Space Command and the Naval Space Command. Historical highlights of these developments, the algorithms created, and the primary historical characters are traced from those early days in 1957 to the present. Specific equations for the models used at each agency are provided in appendices.

Introduction

Space surveillance became a military mission almost as soon as the first Sputnik satellite was launched on October 4, 1957. In addition to the intense civilian and scientific interest in knowing the locations of space objects, the Air Force needed a practical way to prevent false missile-warning alarms as satellites transited through the coverage of warning systems, while the Navy needed a way to alert fleet units against possible overhead reconnaissance by satellites. Both needs led to the creation of a complete catalog of detectable space objects, with satellite tracking data forwarded continually to a central processing facility and updated orbital data distributed routinely to defense users. Naturally, the catalog also served, and still serves, a variety of civilian and scientific purposes.

To date, cataloged orbits have been represented by some type of mean orbital elements, though the operational models have become more elaborate over time as computers have improved. It has always been known that special perturbations can provide better accuracy than general perturbations, at least in principle. However, the sheer number of satellite orbits to be processed for the catalog has meant that only simplified, analytic orbit models could be used in practice. Only recently, with the advent of multi-processor computer techniques, has it been possible to consider maintaining the satellite catalog with special perturbations[1] and this implementation is in progress. Throughout the history of orbital mechanics, the interaction between the development of orbit models and the development of computational facilities has been often noted but seldom studied. While we cannot offer such a study here, we can note that

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this interaction has been crucial in the development of U.S. space surveillance capabilities in general and in the development of the basic orbit models in particular.

Early Methods: 1957-1963

The first formalized effort to catalog satellites occurred at the National Space Surveillance Control Center (NSSCC) located at Hanscom Field in Bedford, Massachusetts. The procedures used at the NSSCC were first reported in 1959 by Eberhard Wahl[2] who was the technical director of the NSSCC. In 1960, under Project SPACETRACK, Philip Fitzpatrick and George Findley[3] developed detailed documentation of the procedures used at the NSSCC. The following description is based on that historic documentation.

Observation of satellites was performed at more than 150 individual sites. Contributions came from radars, Baker-Nunn cameras, telescopes, radio receivers, and the Moon Watch participants. These dedicated individuals took observations on satellites by visual means. Table 1 describes the various observation types and sources.

Table 1 Space Surveillance Observation Description

Observation Type	Content	Source
Type 1	2 angles and slant range	Radars
Type 2	2 angles	Baker-Nunn cameras, telescopes, binoculars, visual sightings
Type 3	Azimuth	Direction finders
Type 4	Time of closest approach (Doppler)	Radars, radio receivers (for transmitting satellites)

The observations were transferred to the NSSCC by teletype, telephone, mail and personal messenger. There a duty analyst reduced the data, determining corrections that should be made to the orbital elements before they were used for further prediction. After his analysis, these corrections were fed into an IBM-709 computer that computed the updated orbital data. These updated orbital data were then used in another phase of the same computer program to yield the geocentric ephemeris. From the geocentric ephemeris, three different products were computed and sent back to the observing stations for their planning of future observing opportunities.

The first, a bulletin, was a listing of the updated orbital elements and was the forerunner of today's two-line element set. Additionally, it contained a table of pertinent data for a given satellite that provided an observing station with a simple tool for determining a geocentric ephemeris. The essential content of the bulletin was a table of revolution number, time of passage through the ascending node, and longitude of ascending node for 3 to 7 days in advance. Additionally, there was a grid containing latitude, longitude and height for a revolution near the middle of the time period for which the bulletin was valid. This information could be used in conjunction with the other data to determine an ephemeris for a given observing station.

The second product used the ephemeris in the General Look Angles Program (GLAP) to produce a tabulation of all satellite passes observable by a specific ground site. The tabulation contained one or more time points for each pass as well as the azimuth, elevation and slant range at that time point.

The third product, the Fence Look Angles Program (FLAP), is a modified form of the GLAP which was sent to the stations which composed the U. S. Navy and U. S. Army observation fences. Instead of a sequence of look angles for a given pass, the FLAP program gave a time and point of intersection of the satellite orbit with the vertical plane containing the sensing beam.

The ephemeris model employed in the NSSCC made use of the following empirical and theoretical equations to predict the time T of passage through the ascending node, the right ascension Ω of ascending node, and the argument of perigee ω, all at revolution number N. All times are in days and all angles are in degrees.

$$T_N = T_0 + P_0(N - N_0) + c(N - N_0)^2 + d(N - N_0)^3$$

$$\Omega_N = \Omega_0 + \dot{\Omega}_0(T_N - T_0) + \frac{1}{2}\ddot{\Omega}_0(T_N - T_0)^2$$

$$\omega_N = \omega_0 + \dot{\omega}_0(T_N - T_0) + \frac{1}{2}\ddot{\omega}_0(T_N - T_0)^2$$

with

$$\dot{\Omega}_0 = K \frac{\cos i_0}{a_0^{7/2}(1 - e_0^2)^2}$$

$$\ddot{\Omega}_0 / 2 = -\dot{a}_0(7 - e_0)\frac{\dot{\Omega}_0}{2a_0(1 + e_0)}$$

$$\dot{\omega}_0 = -(5\cos^2 i_0 - 1)\frac{\dot{\Omega}_0}{2\cos i_0}$$

$$\frac{\ddot{\omega}_0}{2} = (5\cos^2 i_0 - 1)\frac{\ddot{\Omega}_0}{4\cos i_0}$$

$$K = -9.96^0 / \text{day}$$

where a dot above a term indicates the time rate of change of that term, a subscript 0 indicates the value of that term at the epoch time, a is a unitless semimajor axis normalized by Earth radii, and P is the period of the satellite. The inclination i is assumed to be constant while the eccentricity e is determined based on the assumption that perigee height remains constant throughout the prediction span. The semimajor axis is related to the period by

$$P = 0.058672947a^{3/2}$$

where the period has units of days per revolution. The parameters c and d are the results of a least squares fit to the residual differences between the predicted and observed times of passage through the ascending node. The parameter c may be expressed as

$$c = P_0 \frac{\dot{P}_0}{2}$$

which provides the time rate of change of the period P_0 from which the time rate of change of the semimajor axis a_0 may be computed.

Meanwhile, the Navy developed a largely automatic satellite detection and cataloging system to meet fleet tactical requirements. The Advanced Research Projects Agency (ARPA) began constructing the Naval Space Surveillance System (NAVSPASUR), commonly known as the Fence, in 1958, based on technical ideas and assistance from Naval Research Laboratory (NRL). After the concept was proven, Naval Space Surveillance operations began in June 1960. NAVSPASUR was commissioned as an operational Navy command in February 1961.

The present Fence concept is little changed from the first notions put forward at NRL. A continuous-wave multi-static radar interferometer, ultimately consisting of three transmitters and six receivers, was deployed along a great-circle arc from San Diego, California to Savannah, Georgia. The system transmitted, and still transmits, in real time, to the NAVSPASUR processing facility at Dahlgren, Virginia raw signal phase and amplitude data measured on more than 12 miles of dipole arrays. The raw data is converted by interferometric methods into apparent direction cosines (as seen from the receivers) for use in updating the satellite catalog. Near-Earth satellites pass through the field of view of the Fence four to six times per day, and are, on average, illuminated by two transmitters and detected by four receivers on each pass. The result is an abundant data set produced without any cueing or *a priori* knowledge of the satellite population. More than 98.5% of the orbits visible to the Fence can be updated without human intervention.

The reason for sending all the raw Fence data to Dahlgren is purely historical. In 1958, the only computer in the Navy that was able to handle the surveillance data flow and the orbital updates was the Naval Ordinance Research Calculator (NORC) at the Naval Weapons Laboratory (NWL) in Dahlgren[4,5]. The actual data flow was miniscule by current standards, and the NORC was one of the most powerful computers in the world at the time. Moreover, the catalog processing methods benefited from extensive refinement by C.J. Cohen of NWL and Paul Herget of the University of Cincinnati (Cincinnati Observatory). Nevertheless, the NORC needed about 15 minutes of processing time to update a single satellite orbit, using essentially the same orbit model described above, in addition to the time needed for the Fence data reduction and data association with the catalog. It is interesting that the use of an accurate special-perturbations orbit model had been considered explicitly in the Fence proof-of-concept phase. However, that idea had to be abandoned when it was discovered that the processing time per satellite on the NORC was as much as several hours[6]. Only recently, after several generations of computer technology development, has the special-perturbations approach begun to bear fruit.

The original Fence catalog processing was re-hosted on an IBM 7090 computer in 1961, a step that immediately reduced the processing time to 5 minutes per satellite. New programming techniques and data-handling efficiencies were developed that year, which further reduced the processing time to about 1 minute[5]. This improvement came just in time to avert what would otherwise have been a computational catastrophe created by the breakup of satellite 1961-Omicron. That satellite broke up into several hundred trackable pieces in a short time span, tripling the number of detectable space objects and creating thousands of unassociated Fence observations. Many of the objects were lightweight balloon fragments that exhibited unprecedented decay rates. The NORC would have been hopelessly inadequate for the data association and orbit update tasks, but the IBM 7090 (aided by heroic human efforts) processed the backlogged data in a matter of days. No comparable event has happened since then in the history of space surveillance, but the lessons were well learned. The experience demonstrated, besides the crucial advantage of high computational power, the need for a more accurate orbit model to aid in distinguishing between nearby orbits and in following a satellite through high-decay conditions.

Theoretical Foundations: 1959-1969

In 1959, under Project SPACETRACK, Dirk Brouwer developed a solution[7] for the motion of a near Earth satellite under the influence of the zonal harmonics J_2, J_3, J_4, and J_5. This work was later published in the Astronomical Journal[8]. In that same journal and on adjacent pages, Yoshide Kozai[9] published another solution to the same problem. Most analytical orbit prediction models in the United States Space Surveillance System today still have one of these two methods as their foundation.

In 1961, Brouwer and Gen-Ichiro Hori[10] developed a modification to the 1959 Brouwer solution that included the effects of atmospheric drag. The atmospheric drag model was based on a static exponential representation for atmospheric density with a constant scale height. This choice of density model led to series expansions in the scale height. The complete model was much too extensive to be run on the computers of that time for numerous satellites, given the slow convergence of the series and given that numerous orbits had to be computed.

In the early 1960's the same group that had developed the original NSSCC documentation for Hanscom performed some seminal work in atmospheric density modeling. Starting from the basic equations of hydrostatic equilibrium and assuming that scale height varied linearly with altitude, they derived a density representation using power functions with integral exponents[11]. The importance of this work is that, when applied to an artificial satellite theory, it completely avoids series expansions that occur with exponential representations. This made possible the inclusion of drag in the Brouwer model in a more complete and compact manner. The resulting analytic orbit model was developed by Max Lane[12] in 1965 with further improvements by Lane and Cranford[13] in 1969. Fitzpatrick[14] gives a brief account of Lane's density modeling technique in his textbook.

A very important contribution to analytic satellite theory was made by R. H. Lyddane[15] in 1963. Lyddane showed that the Brouwer solution based on Delaunay variables could be reformulated in terms of Poincaré variables to avoid the small divisors of eccentricity and the sine of inclination while maintaining the first-order character of the theory.

Operational Implementations: 1964-1979

The transition from journal article to operational implementation took two paths. The tracking operation of the Naval Space Surveillance System (NAVSPASUR) adopted the entire 1959 solution of Brouwer with the modifications developed by Lyddane to avoid small divisors of eccentricity or the sine of inclination. This analytic satellite prediction model is now known as PPT3 (Position and Partials as functions of Time). The equations of the original PPT model were implemented on an IBM 7090 computer in 1964 under the guidance of Richard H. Smith who also provided supplemental equations to account for atmospheric drag. At that time, the results of Brouwer and Hori could not be implemented operationally because of computer limitations, and the results of Lane and Cranford were not yet available. Smith adapted ideas from King-Hele[16] in a simple original model that is still in use. His semi-empirical drag model assumes that the effect of atmospheric drag on the mean motion can be represented as a quadratic time function. The linear and quadratic coefficients are treated as solved-for parameters during the orbit determination process. A time rate of change of the eccentricity is represented in terms of the mean motion rate by the following equations.

$$\dot{e}_0 = e_0(1-e_0{}^2)\frac{\dot{a}_0}{a_0} \quad , \quad \dot{a}_0 = -\frac{4}{3}\frac{a_0}{n_0}\left(\frac{\dot{n}_0}{2}\right)$$

The integral of the mean motion equation provides the model for along track drag effect.

PPT retains all long-periodic terms, including the ones with a zero divisor at the critical inclination. However, PPT handles these critical terms in a special way, as described in Appendix C6. A special feature of PPT is that the "mean" mean motion is defined differently from Brouwer's quantity of the same name. Brouwer defined mean motion in terms of mean semimajor axis by essentially the Keplerian formula. However, for PPT, it was decided for computational reasons to define the mean motion as the entire coefficient of time in the linear term of the perturbed mean anomaly. That is, the PPT mean motion includes the zonal secular perturbation rate of mean anomaly that Brouwer derived. As a result, the expression for PPT mean motion explicitly contains perturbation parameters and functions of the other mean elements, similarly to the definition adopted by Kozai[9]. Numerically, the PPT mean motion is closer to Kozai's value than to Brouwer's.

The other path from journal article to operational implementation took place in Colorado Springs. In 1961, the NSSCC was relocated to Colorado Springs, CO and became known as the Space Detection and Tracking System (SPADATS) Center. The NSSCC algorithms were rehosted on a Philco Model 211 computer and the group at Hanscom began to serve as the backup for the SPADATS Center. Following the rehosting in Colorado Springs, Geoff Hilton[17] provided updated documentation of the NSSCC algorithms. In 1960, Aeronutronic had begun developing the astrodynamics basis for a new system. The analytic orbit prediction model was based on the works of Brouwer and Kozai. In order to avoid small divisors of eccentricity or the sine of inclination, Arsenault, Chaffee, and Kuhlman[18] transformed the solution to a series in non-singular parameters keeping only the most important terms. They included from Brouwer only those long and short-period terms in position that do not contain eccentricity as a factor. They also adopted from Kozai the non-Keplerian convention relating mean motion to semimajor axis. The model is known as the Simplified General Perturbations (SGP) model. A complete documentation of SGP is provided by C. G. Hilton and J. R. Kuhlman[19]. Atmospheric drag was included in a similar manner to that of Smith except that the time rate of change of eccentricity was derived based on the assumption that perigee height remains constant as semimajor axis shrinks. In addition to becoming the principal analytic prediction model for centralized processing, SGP was also implemented at many of the satellite tracking sites around the world. In 1964, the SGP model became the primary orbital prediction model for the SPADATS system.

The improvement offered by an analytic rather than an empirical density model led to a decision to implement the development of Lane and Cranford. However, by 1969 the number of satellites in the catalog had grown to a point that computers would not be able to manage the extensive terms in the model. Consequently, a simplified version of the Lane and Cranford work, known as SGP4, was developed and implemented operationally in 1970.

The simplifications leading to SGP4 were accomplished by retaining only the main terms that modeled the secular effect of drag. Similarly, the gravitational modeling was shortened by retaining from Brouwer only those long- and short-periodic terms in position that do not contain eccentricity as a factor. The details of the derivation of SGP4 from the complete development of Lane and Cranford were documented in 1979 by Lane and Hoots[20]. The SGP4 model was used side by side with the SGP model until 1979 when it became the sole model for satellite catalog maintenance.

Deep-Space Modeling: 1965-1997

In 1965, the first highly-eccentric, 12-hour-period satellite was launched. Soon it became apparent that a theory was needed which included terms to account for Lunar and Solar gravitation as well as the resonance effects of Earth tesseral harmonics. A semianalytic treatment of this special class of orbits, which included Lunar and Solar gravity as well as geopotential resonance effects, was developed by Bruce Bowman[21] in 1967. By 1977, Dick Hujsak[22] had incorporated portions of Bowman's work in a new first-order model, which included all perturbations treated by Bowman and an extension to geosynchronous satellites. This new model was fully integrated with the SGP4 model for near Earth satellites. This work completed the SGP4 model in use today. A complete listing of the equations was provided by Hoots and Roehrich[23] and is repeated in Appendix A.

In 1997 the Lunar, Solar, and resonance terms from the SGP4 model were added to the Naval Space Command PPT model to provide improved prediction of higher altitude satellites. This modified model became known as PPT3 and is documented in the work of Paul Schumacher and Bob Glover[24]. A complete listing of the equations is provided in Appendix B.

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Appendix A - SGP4 Model

The United States Space Command two-line element sets can be used for prediction with SGP4. All equations below are taken from SPACETRACK Report #3 by Hoots and Roehrich[23]. The element set consists of

$$
\begin{aligned}
t_0 &= \text{epoch time} \\
n_0 &= \text{mean motion (revolutions/day)} \\
e_0 &= \text{eccentricity} \\
i_0 &= \text{inclination (degrees)} \\
\omega_0 &= \text{argument of perigee (degrees)} \\
\Omega_0 &= \text{right ascension of ascending node (degrees)} \\
M_0 &= \text{mean anomaly (degrees)} \\
B^* &= \text{Atmospheric drag coefficient (units of 1/Earth radii)}
\end{aligned}
$$

where all orbital elements except mean motion are the mean double-prime quantities defined by Brouwer[8] and where the subscript 0 will indicate the value of a quantity at epoch. The mean motion on the two-line element set follows the convention of Kozai[9].

<u>Initialization</u>

Many terms used in the prediction of SGP4 are independent of time. Thus, the algorithm begins with computation of numerous constant terms. The first step in the initialization is the recovery of the Brouwer mean motion from the Kozai mean motion by the equations

$$
\begin{aligned}
a_1 &= \left(\frac{k_e}{n_o}\right)^{2/3} \\
\delta_1 &= \frac{3}{2}\,\frac{k_2}{a_1^{\,2}}\,\frac{(3\cos^2 i_o - 1)}{(1-e_o^{\,2})^{3/2}} \\
a_2 &= a_1\left(1 - \frac{1}{3}\delta_1 - \delta_1^{\,2} - \frac{134}{81}\delta_1^{\,3}\right) \\
\delta_o &= \frac{3}{2}\,\frac{k_2}{a_2^{\,2}}\,\frac{(3\cos^2 i_o - 1)}{(1-e_o^{\,2})^{3/2}} \\
n_o'' &= \frac{n_o}{1+\delta_o} \\
a_o'' &= \left(\frac{k_e}{n_0''}\right)^{2/3} .
\end{aligned}
$$

where

$$k_2 = \frac{1}{2} J_2 a_E{}^2 \text{[units of (Earth radii)}^2\text{]}$$
$$J_2 = 1.082616\, x 10^{-3}$$
$$k_e = \sqrt{GM} = .0743669161 \text{ [units of (Earth radii)}^{1.5} \text{/ minute]}$$
$$G = \text{universal gravitational constant}$$
$$M = \text{mass of the Earth}$$
$$a_E = \text{Equatorial radius of the Earth}$$

The SGP4 model is set in the FK4 and WGS72 reference standards, referred to the J2000.0 epoch.
From this point on, the mean motion $n_0^{''}$ and the semimajor axis $a_0^{''}$ follow the Brouwer convention. Also, all quantities on the right hand side of equations are understood to be double-prime mean elements.

Initialization for Secular Effects of Atmospheric Drag

Atmospheric drag modeling is based on a power-law density function[11] given by

$$\rho = \rho_0 (q_0 - s)^4 / (r - s)^4$$

where r is the radial distance of the satellite from the center of the Earth with q_0 and s being altitude parameters of the power law density function. The parameter q_0 is a constant equal to 120 km plus one Earth radius whereas the parameter s is determined based of epoch perigee height above a spherical Earth. If perigee height is greater than or equal 156 km, the value of s is fixed to be 78 km plus one Earth radius. For altitudes greater than or equal to 98 km but less than 156 km, s is defined to be perigee height minus 78 km plus one Earth radius. For altitudes below 98 km, s is 20 km plus one Earth radius. In all equations that follow, the parameters q_0 and s should be in units of Earth radii.

$$\theta = \cos i_o$$
$$\xi = \frac{1}{a_0 - s}$$
$$\beta_o = (1 - e_o^2)^{1/2}$$
$$\eta = a_0 e_o \xi$$
$$C_2 = (q_o - s)^4 \xi^4 n_0 (1 - \eta^2)^{-7/2} \left[a_0 \left(1 + \frac{3}{2}\eta^2 + 4 e_o \eta + e_o \eta^3\right) + \frac{3}{2} \frac{k_2 \xi}{(1 - \eta^2)} \left(-\frac{1}{2} + \frac{3}{2}\theta^2\right)(8 + 24\eta^2 + 3\eta^4) \right]$$

$$C_1 = B^* C_2$$

$$C_3 = \frac{(q_o - s)^4 \xi^5 A_{3,0} n_0 a_E \sin i_o}{k_2 e_o}$$

$$C_4 = 2n_0 (q_o - s)^4 \xi^4 \; a_0 \beta_o^2 \; (1-\eta^2)^{-7/2} \{[2\eta(1+e_o\eta) + \frac{1}{2} e_o + \frac{1}{2}\eta^3] - \frac{2k_2\xi}{a_0(1-\eta^2)} [3\,(1-3\theta^2)(1+ \frac{3}{2}\eta^2 - 2e_o\eta - \frac{1}{2}e_o\eta^3) + \frac{3}{4}(1-\theta^2)(2\eta^2 - e_o\eta - e_o\eta^3)\cos 2\omega_o]\}$$

$$C_5 = 2\left(q_o - s\right)^4 \xi^4 a_0 \beta_o^2 (1-\eta^2)^{-7/2} \left[1 + \frac{11}{4}\eta\,(\eta + e_o) + e_o\eta^3\right]$$

$$D_2 = 4 a_0 \xi \, C_1^2$$

$$D_3 = \frac{4}{3}\, a_0 \xi^2 (17 a_0 + s)\, C_1^3$$

$$D_4 = \frac{2}{3}\, a_0^2 \xi^3 (221 a_0 + 31 s) C_1^4 .$$

where

$$A_{3,0} = -J_3 a_E^{\,3}$$

$$J_3 = -0.253881 \times 10^{-5}$$

Initialization for Secular Effects of Earth Zonal Harmonics

The secular effects of gravitation are included through the equations

$$\dot{M} = \left[\frac{3k_2(-1+3\theta^2)}{2a_0^2\beta_o^3} + \frac{3k_2^2(13 - 78\theta^2 + 137\theta^4)}{16a_0^4\beta_o^7}\right] n_0$$

$$\dot{\omega} = \left[-\frac{3k_2(1-5\theta^2)}{2a_0^2\beta_o^4} + \frac{3k_2^2(7 - 114\theta^2 + 395\theta^4)}{16a_0^4\beta_o^8} + \frac{5k_4(3 - 36\theta^2 + 49\theta^4)}{4a_0^4\beta_o^8}\right] n_0$$

$$\dot{\Omega} = \left[-\frac{3k_2\theta}{a_0^2\beta_o^4} + \frac{3k_2^2(4\theta - 19\theta^3)}{2a_0^4\beta_o^8} + \frac{5k_4\theta(3 - 7\theta^2)}{2a_0^4\beta_o^8}\right] n_0$$

where

$$k_4 = -\frac{3}{8} J_4 a_{\mathrm{E}}^4$$

$$J_4 = -1.65597 \times 10^{-6}$$

Initialization for Secular and Long-Period Coefficients of Lunar and Solar Gravity

For satellites with periods greater than or equal to 225 minutes, additional terms are included to model the effect of Lunar and Solar gravitation on the satellite. Such satellites are referred to as "deep space" satellites. The equations for calculation of the orbital element secular rates and long-period coefficients due to the Moon and Sun gravitation are provided in Appendix C1.

Initialization for Resonance Effects of Earth Gravity

For orbits with periods that result in repeating satellite position in relation to the Earth's figure, the effects of the non-zonal harmonics can be significant. This resonance condition is treated in the SGP4 model for orbits with ½-day (semisynchronous and highly eccentric) and 1-day (geosynchronous) periods. The equations for initialization of the resonance effects of Earth gravity are provided in Appendix C2.

Update

Predictions of satellite motion are performed using the constants computed in the initialization.

Secular Update for Earth Zonal Gravity and Partial Atmospheric Drag Effects

The angles $M, \omega,$ and Ω are first updated to include the effects of the Earth zonal harmonics and atmospheric drag effects.

$$
\begin{aligned}
& M_{DF} = M_o + n_0(t-t_0) + \dot{M}(t-t_0) \\
& \omega_{DF} = \omega_o + \dot{\omega}(t-t_o) \\
& \Omega_{DF} = \Omega_o + \dot{\Omega}(t-t_0) \\
& \delta\omega = B^* C_3 (\cos\omega_o)(t-t_o) \\
& \delta\mathrm{M} = -\frac{2}{3}(q_o - s)^4 B^* \xi^4 \frac{a_E}{e_o\eta}\left[(1+\eta\cos M_{DF})^3 - (1+\eta\cos M_o)^3\right] \\
& M = M_{DF} + \delta\omega + \delta M \\
& \omega = \omega_{DF} - \delta\omega - \delta M \\
& \Omega = \Omega_{DF} - \frac{21}{2}\frac{n_0 k_2 \theta}{a_0{}^2 \beta_o{}^2} C_1 (t-t_o)^2
\end{aligned}
$$

where $(t-t_o)$ is time since epoch in minutes. It should be noted that when epoch perigee height is less than 220 kilometers or for "deep space" satellites, the terms $\delta\omega$, and $\delta\mathrm{M}$ are dropped.

Secular Updates for Effects of Lunar and Solar Gravity.

For satellites with periods greater than or equal to 225 minutes, the secular effects of Lunar and Solar gravity are included as detailed in Appendix C3.

Secular Update for Resonance Effects of Earth Gravity

The resonance effects are applied to mean anomaly, mean motion, and semimajor axis using a numerical integration scheme as detailed in Appendix C4.

Secular Update for Remaining Atmospheric Drag Effects

$$e = e_o - B^* C_4 (t - t_o) - B^* C_5 (\sin M - \sin M_o)$$
$$a = \left(\frac{k_e}{n}\right)^{2/3} \left[1 - C_1 (t - t_o) - D_2 (t - t_o)^2 - D_3 (t - t_o)^3 - D_4 (t - t_o)^4\right]^2$$
$$IL = M + \omega + \Omega + n_0 [\frac{3}{2} C_1 (t - t_o)^2 + (D_2 + 2C_1^{\ 2})(t - t_o)^3 + \frac{1}{4}(3D_3 + 12C_1 D_2 + 10C_1^{\ 3})(t - t_o)^4$$
$$+ \frac{1}{5}(3D_4 + 12C_1 D_3 + 6D_2^{\ 2} + 30C_1^{\ 2} D_2 + 15C_1^{\ 4})(t - t_o)^5]$$
$$\beta = \sqrt{1 - e^2}$$
$$n = k_e / a^{3/2}$$

where $(t - t_o)$ is time since epoch in minutes. It should be noted that when epoch perigee height is less than 220 kilometers or for “deep space” satellites, the equations for a and IL are truncated after the linear and quadratic terms, respectively, and the term involving C_5 is dropped.

Update for Long-Period Periodic Effects of Lunar and Solar Gravity

The long-period effects due to the third-body perturbation depend on the position of the Sun or Moon in its orbit. The mean anomaly of the perturbing body at the prediction time is

$$M_X = M_{O_X} + n_X \Delta t$$

where Δt is the time since the Lunar/Solar ephemeris epoch. The remaining equations for computation of the long-period periodic effects of Lunar and Solar gravity are provided in Appendix C5.

The contributions of the Sun and Moon are combined for each term computed above and applied as follows.

$$e = e + \delta e_{LS}$$
$$i = i + \delta i_{LS}$$

For $i > 0.2$ radians

$$\Omega = \Omega + \delta\Omega_{LS} / \sin i$$
$$\omega = \omega + (\delta\omega_{LS} + \cos i \, \delta\Omega_{LS}) - \delta\Omega_{LS} \cos i / \sin i$$
$$M = M + \delta M_{LS}$$

For $i \le 0.2$ radians

$$\alpha = \sin i \sin\Omega + \sin i \cos\Omega\delta\Omega_{LS} + \cos i \sin\Omega\delta i_{LS}$$
$$\beta = \sin i \cos\Omega - \sin i \sin\Omega\delta\Omega_{LS} + \cos i \cos\Omega\delta i_{LS}$$
$$\Omega = \tan^{-1}\left(a / \beta\right)$$
$$M = M + \delta M_{LS}$$
$$\omega = \omega + (\delta\omega_{LS} + \cos i\delta\Omega_{LS}) - \Omega \sin i\delta i_{LS}$$

Update for Long-Period Periodic Effects of Earth Gravity

Add the long-period periodic terms

$$a_{xN} = e \cos\omega$$
$$IL_L = \frac{A_{3,0} \sin i}{8k_2 a\beta^2}(e \cos \omega)\left(\frac{3+5\cos i}{1+\cos i}\right)$$
$$a_{yNL} = \frac{A_{3,0} \sin i}{4k_2 a\beta^2}$$
$$IL_T = IL + IL_L$$
$$a_{yN} = e \sin \omega + a_{yNL}.$$

Update for Short-Period Periodic Effects of Earth Gravity

Solve Kepler's equation for $(E+\omega)$ by defining

$$U = IL_T - \Omega$$

and using the iteration equation

$$(\mathrm{E}+\omega)_{i+1} = (\mathrm{E}+\omega)_i + \Delta(\mathrm{E}+\omega)_i$$

with

$$\Delta(\mathrm{E}+\omega)_1 = \frac{U - a_{yN}\cos(\mathrm{E}+\omega)_i + a_{xN}\sin(\mathrm{E}+\omega)_i - (\mathrm{E}+\omega)_i}{1 - a_{yN}\sin(\mathrm{E}+\omega)_i - a_{xN}\cos(\mathrm{E}+\omega)_i}$$

and

$$(\mathrm{E}+\omega)_1 = U.$$

The following equations are used to calculate preliminary quantities needed for short-period periodics.

$$e\cos E = a_{xN}\cos(\mathrm{E}+\omega) + a_{yN}\sin(\mathrm{E}+\omega)$$
$$e\sin E = a_{xN}\sin(\mathrm{E}+\omega) - a_{yN}\cos(\mathrm{E}+\omega)$$
$$e = ({a_{xN}}^2 + {a_{yN}}^2)^{1/2}$$
$$p_L = a(1-e^2)$$
$$r = a(1-e\cos\mathrm{E})$$
$$\dot{r} = k_e\frac{\sqrt{a}}{r}e\sin\mathrm{E}$$
$$r\dot{f} = k_e\frac{\sqrt{p_L}}{r}$$
$$\cos u = \frac{a}{r}\left[\cos(\mathrm{E}+\omega) - a_{xN} + \frac{a_{yN}(e\sin\mathrm{E})}{1+\sqrt{1-e^2}}\right]$$
$$\sin u = \frac{a}{r}\left[\sin(\mathrm{E}+\omega) - a_{yN} - \frac{a_{xN}(e\sin\mathrm{E})}{1+\sqrt{1-e^2}}\right]$$
$$u = \tan^{-1}\left(\frac{\sin\ u}{\cos\ u}\right)$$
$$\Delta r = \frac{k_2}{2p_L}(1-\cos^2 i)\cos 2u$$
$$\Delta u = -\frac{k_2}{4{p_L}^2}(7\cos^2 i - 1)\sin 2u$$
$$\Delta\Omega = \frac{3k_2\cos\ i}{2{p_L}^2}\sin 2u$$
$$\Delta i = \frac{3k_2\cos\ i}{2{p_L}^2}\sin\ i\cos\ 2u$$
$$\Delta\dot{r} = -\frac{k_2 n}{p_L}(1-\cos^2 i)\sin 2u$$
$$\Delta r\dot{f} = \frac{k_2 n}{p_L}\left[(1-\cos^2 i)\cos 2u - \frac{3}{2}(1-3\cos^2 i)\right]$$

The short-period periodics are added to give the osculating quantities

$$r_k = r\left[1 - \frac{3}{2} k_2 \frac{\sqrt{1-e^2}}{p_L^{\;2}} (3\cos^2 i - 1)\right] + \Delta r$$
$$u_k = u + \Delta u$$
$$\Omega_k = \Omega + \Delta\Omega$$
$$i_k = i + \Delta i$$
$$\dot{r}_k = \dot{r} + \Delta\dot{r}$$
$$r\dot{f}_k = r\dot{f} + \Delta r\dot{f}.$$

Then unit orientation vectors (denoted by boldface) are calculated by

$$\boldsymbol{U} = \boldsymbol{M}\sin u_k + \boldsymbol{N}\cos u_k$$
$$\boldsymbol{V} = \boldsymbol{M}\cos u_k - \boldsymbol{N}\sin u_k$$

where

$$\boldsymbol{M} = \begin{Bmatrix} M_x = -\sin\Omega_k \cos i_k \\ M_y = \cos\Omega_k \cos i_k \\ M_z = \sin i_k \end{Bmatrix}$$
$$\boldsymbol{N} = \begin{Bmatrix} N_x = \cos\Omega_k \\ N_y = \sin\Omega_k \\ N_z = 0 \end{Bmatrix} \quad .$$

Then position and velocity are given by

$$\boldsymbol{r} = r_k \boldsymbol{U}$$

and

$$\dot{\boldsymbol{r}} = \dot{r}_k \boldsymbol{U} + r\dot{f}_k \boldsymbol{V}.$$

Appendix B - PPT3 Model

The Naval Space Command two-line element sets can be used for prediction with PPT3. All equations below are adapted from Schumacher and Glover[24]. The element set consists of

$$
\begin{aligned}
&t_0 = \text{epoch time} \\
&n_0 = \text{mean motion (revolutions/day)} \\
&e_0 = \text{eccentricity} \\
&I_0 = \text{inclination (degrees)} \\
&\omega_0 = \text{argument of perigee (degrees)} \\
&\Omega_0 = \text{right ascension of ascending node (degrees)} \\
&M_0 = \text{mean anomaly (degrees)} \\
&\mathit{Decay1} = \dot{n}/2 \ \ (\text{revolutions/day}^2) \\
&\mathit{Decay2} = \ddot{n}/6 \ \ (\text{revolutions/day}^3)
\end{aligned}
$$

where all orbital elements except mean motion are the mean double-prime quantities defined by Brouwer[8] and where the subscript 0 will indicate the value of a quantity at epoch. The mean motion on the two-line element set follows the convention of Kozai[9], although PPT3 uses its own convention for mean motion which is slightly different from Kozai's, as explained in the main text and shown explicitly below. (This mathematical incompatibility has long been noted in space surveillance operations, and is periodically rediscovered by newcomers. Because the two types of mean motion are numerically close together, the potential incompatibility is easily overcome by special processing at Naval Space Command before two-line elements are transmitted. However, the processing details are beyond the scope of this paper.) The two drag parameters, Decay1 and Decay2, are empirically determined during the orbit correction process.

The PPT3 orbit theory as implemented at Naval Space Command uses a specific value of the gravitational constant that defines the canonical units of the system.

$$k_e = \sqrt{GM} = .0743669161 \text{ [units of (Earth radii)}^{1.5} \text{ / minute]}$$

$$G = \text{universal gravitational constant}, \ M = \text{mass of the Earth}$$

The PPT3 model is set in the FK4 and WGS72 reference standards, referred to the J2000.0 epoch.

We define the variables that will be used throughout the mathematical development. These definitions also provide the values used in PPT3 for the zonal coefficients. The notation closely follows that used by Brouwer[7,8].

$$\gamma_2 = \frac{k_2}{a''^2} \qquad \gamma_3 = \frac{A_{3.0}}{a''^3} \qquad \gamma_4 = \frac{k_4}{a''^4} \qquad \gamma_5 = \frac{A_{5.0}}{a''^5}$$

$$\gamma_2' = \frac{\gamma_2}{\eta^4} \qquad \gamma_3' = \frac{\gamma_3}{\eta^6} \qquad \gamma_4' = \frac{\gamma_4}{\eta^8} \qquad \gamma_5' = \frac{\gamma_5}{\eta^{10}}.$$

where

$$k_2 = \frac{1}{2} J_2 R_\oplus^2 \; = 0.54130789\text{x}10^{-3}$$

$$\text{A}_{3.0} = -J_3 R_\oplus^3 \; = 0.25388100\text{x}10^{-5}$$

$$k_4 = -\frac{3}{8} J_4 R_\oplus^4 \; = 0.62098875\text{x}10^{-6}$$

$$\text{A}_{5.0} = -J_5 R_\oplus^5 \quad = 0.21848266\text{x}10^{-6}$$

$$\eta = \sqrt{1 - e''^2} \qquad \theta = \cos I''$$

<u>Initialization</u>

Many terms used in the prediction of PPT3 are independent of time. Thus, the algorithm begins with computation of numerous constant terms. The first step in the initialization is the recovery of the Brouwer semimajor axis from the Kozai-type PPT3 mean motion. Form the initial semimajor axis from the Kozai-type PPT3 mean motion

$$a_{\mathrm{i}} = n_0''^{-2/3}$$

The semimajor axis is transformed by iterating the following sequence five times. The second and fourth Brouwer gamma prime variables are formed from the current semimajor axis value. Then the semimajor axis is recomputed, using the zonal secular variation of the mean anomaly M defined directly below.
For i = 1,5

$$\gamma_2' = \frac{k_2}{a_{i-1}^{\,2} \eta^4} \qquad \gamma_4' = \frac{k_4}{a_{i-1}^4 \eta^8}$$

$$a_i = \left[\frac{1 + \delta_s M}{n_0''} \right]^{2/3}$$

After the fifth iteration, the result is defined as the Brouwer semimajor axis a_0''. From this point on, the semimajor axis a_0'' follows the Brouwer convention. Also, all quantities on the right hand side of equations are understood to be double-prime mean elements.

Initialization for Secular Effects of Earth Zonal Harmonics

The secular effects of gravitation are included through the equations

$$\delta_s M = \frac{3}{2}\gamma_2^{'}\eta(-1+3\theta^2)$$

$$+\ \frac{3}{32}\gamma_2^{'\,2}\eta[-15+16\eta+25\eta^2+(30-96\eta-90\eta^2)\theta^2+(105+144\eta+25\eta^2)\theta^4]$$

$$+\ \frac{15}{16}\gamma_4^{'}\eta e''^2(3-30\theta^2+35\theta^4)$$

$$\delta_s\omega = \frac{3}{2}\gamma_2^{'}(-1+5\theta^2)$$

$$+\ \frac{3}{32}\gamma_2^{'\,2}[-35+24\eta+25\eta^2+(90-192\eta-126\eta^2)\theta^2+(385+360\eta+45\eta^2)\theta^4]$$

$$+\ \frac{5}{16}\gamma_4^{'}[21-9\eta^2+(-270+126\eta^2)\theta^2+(385-189\eta^2)\theta^4]$$

$$\delta_s\Omega = -3\gamma_2^{'}\theta + \frac{3}{8}\gamma_2^{'\,2}[(-5+12\eta+9\eta^2)\theta+(-35-36\eta-5\eta^2)\theta^3]$$

$$+\ \frac{5}{4}\gamma_4^{'}(5-3\eta^2)(3-7\theta^2)\theta$$

Initialization for Secular and Long-Period Coefficients of Lunar and Solar Gravity

Additional terms are included to model the effect of Lunar and Solar gravitation on the satellite. The equations for calculation of the orbital element secular rates and long-period coefficients due to the Moon and Sun gravitation are provided in Appendix C1.

Initialization for Resonance Effects of Earth Gravity

For orbits with periods that result in repeating satellite position in relation to the Earth's figure, the effects of the non-zonal harmonics can be significant. This resonance condition is treated in the PPT3 model for orbits with ½-day (semisynchronous and highly eccentric) and 1-day (geosynchronous) periods. The equations for initialization of the resonance effects of Earth gravity are provided in Appendix C2.

<u>Orbit Propagation</u>

Predictions of satellite motion are performed using the constants computed in the initialization. The secular effects are applied first. These effects are produced by the zonal gravity field, the third-body perturbations, and resonance and drag when these two forces are applied. Next, the long-period periodics arising from the zonal gravity field and the third-body forces are added. Finally, the zonal short-period periodics are applied to produce an osculating element set. Position and velocity vectors are then calculated.

Secular Update

A unique feature of PPT3 is that the secular rates for ω and Ω combine both the zonal and third-body effects and are posed in terms of the change in mean anomaly since epoch rather than the change in time. The changes in mean anomaly produced by all the perturbations are carefully accounted for in the PPT3 secular update processing so that ω and Ω can be correctly propagated.

The angles ω, Ω, and I are first updated to the time of interest t include the effects of the Earth zonal harmonics and the third-body perturbations. The Brouwer zonal secular update is shown in the initialization section above while the secular effects of Lunar and Solar gravity are detailed in Appendix C1.

$$\Delta t = (t - t_0)$$

$$\Delta M_Z = n''_0 \Delta t$$

where the Kozai-type PPT3 mean motion implicitly includes the zonal secular rate $\delta_s M$.

$$\Omega'' = \Omega''_0 + (\delta_s \Omega + \delta_{TBS} \Omega) \Delta M_Z$$

$$\omega'' = \omega''_0 + (\delta_s \omega + \delta_{TBS} \omega) \Delta M_Z$$

$$I'' = I''_0 + (\delta_{TBS} I) \Delta t$$

where δ_s is the zonal secular and δ_{TBS} is the combined third-body secular for each element.
If the satellite is a non-resonant case, the mean motion is updated for the zonal secular effects:

$$M'' = M''_0 + \Delta M_Z$$

If the satellite is resonant, the mean motion is directly integrated as described below.

Secular Update for Resonance

The resonance effects are applied to mean anomaly and semimajor axis using a numerical integration scheme as detailed in Appendix C4. The integrations produce an updated value for the mean anomaly M'' (which is used directly) and an updated mean motion n''. The change in mean anomaly and mean motion due to resonance are formed, and the change in mean motion is used to update the semimajor axis.

$$\Delta M_R = M'' - M''_0$$

$$\Delta n_R = n'' - n''_0$$

$$a'' = a''_0 - \frac{2}{3} \frac{a''_0 \Delta n_R}{n''_0}$$

Secular Update for Atmospheric Drag Effects

$$\Delta M_D = \frac{\dot{n}}{2}(t - t_0)^2 + \frac{\ddot{n}}{6}(t - t_0)^3$$

$$\Delta M_{TOT} = \Delta M_Z + \Delta M_R + \Delta M_D$$

$$M'' = M'' + \Delta M_D$$

$$\dot{a}_D = -\frac{2}{3}\frac{a_0'' n}{n_0''}$$

$$a'' = a'' + \dot{a}_D \, \Delta t$$

$$e'' = e_0'' + \left[\frac{\dot{a}_D e_0'' (1 - e_o''^2)}{a_0''} + \delta_{TBS} e\right]\Delta t$$

The values at the time of interest for Ω and ω are recomputed using the total change in the mean anomaly produced by all of the secular effects.

$$\Omega'' = \Omega_0'' + (\delta_s \Omega + \delta_{TBS} \Omega)\, \Delta M_{TOT}$$

$$\omega'' = \omega_0'' + (\delta_s \omega + \delta_{TBS} \omega)\, \Delta M_{TOT}$$

Kepler's equation is solved by iteration (Aitken's delta-squared method) at this point to obtain the true anomaly. The secularly updated mean anomaly and eccentricity are used. The initial value for the eccentric anomaly is set to the mean anomaly

$$E_a = M''$$

The iteration is performed in a twenty-step DO loop

$$E_1 = E_a$$
$$E_a = M'' + e \sin E_a$$
$$If \; \left| E_a - E_1 \right| \; < 10^{-8} \quad \text{Compute true anomaly}$$
$$E_2 = E_a$$
$$E_a = M'' + e \sin E_a$$
$$If \; \left| E_a - E_2 \right| \; < 10^{-8} \quad \text{Compute true anomaly}$$
$$E_a = E_a + \frac{(E_a - E_2)^2}{(2E_2 - E_1 - E_a)}$$

If the tolerance criteria are met or twenty iteration steps have been performed, the sine and cosine of the true anomaly are formed

$$\sin f = \frac{\eta \sin E_a}{1 - e'' \cos E_a}$$

$$\cos f = \frac{\cos E_a - e''}{1 - e'' \cos E_a}$$

Update for Long-Period Periodic Effects of Lunar and Solar Gravity

The long-period effects due to the third-body perturbation depend on the position of the Sun or Moon in its orbit.

$$\Delta t_{LS} = t - t_{LS} \text{ where } t_{LS} \text{ is the epoch of third-body representation}$$

$$M_{LS} = M_{LS_0} + \dot{M}_{LS} \Delta t_{LS} + \ddot{M}_{LS} \Delta t_{LS}^2 + \dddot{M}_{LS} \Delta t_{LS}^3$$

The remaining equations for computation of the long-period periodic effects of Lunar and Solar gravity are provided in Appendix C5. The contributions of the Sun and Moon are combined for each term computed above and applied at the same time as the other periodics. The coefficients in this polynomial for third-body mean anomaly are given in the *Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac*, Her Majesty's Stationery Office, London, 1961, page 98 for the Sun and page 107 for the Moon.

Update for Long-Period Periodic Effects of Earth Gravity

The Brouwer long-period corrections are given by:

$$\delta_1 e = e''\eta^2\left(\frac{1}{8}\gamma_2'\left[1-11\theta^2-\frac{40\theta^4}{1-5\theta^2}\right]-\frac{5\gamma_4'}{12\gamma_2'}\left[1-3\theta^2-\frac{8\theta^4}{1-5\theta^2}\right]\right)\cos 2\omega''$$

$$+\frac{\eta^2\sin I''}{4\gamma_2'}\left(\gamma_3'+\frac{5\gamma_5'}{16}\left(4+3e''^2\right)\left[1-9\theta^2-\frac{24\theta^4}{1-5\theta^2}\right]\right)\sin\omega''$$

$$-\frac{35\gamma_5'}{384\gamma_2'}e''^2\eta^2\sin I''\left[1-5\theta^2-\frac{16\theta^4}{1-5\theta^2}\right]\sin 3\omega''$$

$$\delta_1 I = -\frac{e''\delta_1 e}{\eta^2\tan I''}$$

$$\delta_1 M = \eta^3\left(\frac{1}{8}\gamma_2'\left[1-11\theta^2-\frac{40\theta^4}{1-5\theta^2}\right]-\frac{5\gamma_4'}{12\gamma_2'}\left[1-3\theta^2-\frac{8\theta^4}{1-5\theta^2}\right]\right)\sin 2\omega''$$

$$-\frac{\eta^3\sin I''}{4\gamma_2'e''}\left(\gamma_3'+\frac{5\gamma_5'}{16}\left(4+9e''^2\right)\left[1-9\theta^2-\frac{24\theta^4}{1-5\theta^2}\right]\right)\cos\omega''$$

$$+\frac{35\gamma_5'}{384\gamma_2'}e''\eta^3\sin I''\left[1-5\theta^2-\frac{16\theta^4}{1-5\theta^2}\right]\cos 3\omega''$$

$$\delta_1\omega = \left\{-\frac{1}{16}\gamma_2'\left[(2+e''^2)-11(2+3e''^2)\theta^2-\frac{40(2+5e''^2)\theta^4}{-5\theta^2}-\frac{400e''^2\theta^6}{(1-5\theta^2)^2}\right]\right.$$

$$\left.+\frac{5\gamma_4'}{24\gamma_2'}\left[(2+e''^2)-3(2+3e''^2)\theta^2-\frac{8(2+5e''^2)\theta^4}{1-5\theta^2}-\frac{80e''^2\theta^6}{(1-5\theta^2)^2}\right]\right\}\sin 2\omega''$$

$$+\frac{1}{4\gamma_2'}\left\{\gamma_3'\left(\frac{\sin I''}{e''}-\frac{e''\theta^2}{\sin I''}\right)\right.$$

$$+\frac{5\gamma_5'}{16}\left[\left(\frac{\eta^2\sin I''}{e''}-\frac{e''\theta^2}{\sin I''}\right)(4+3e''^2)+e''\sin I''(26+9e''^2)\right]\left[1-9\theta^2-\frac{24\theta^4}{1-5\theta^2}\right]$$

$$\left.-\frac{15\gamma_5'}{8}e''\theta^2\sin I''(4+3e''^2)\left[3+\frac{16\theta^2}{1-5\theta^2}+\frac{40\theta^4}{(1-5\theta^2)^2}\right]\right\}\cos\omega''$$

$$+\frac{35\gamma_5'}{576\gamma_2'}\left\{-\frac{1}{2}\left(e''\sin I''(3+2e''^2)-\frac{e''^3\theta^2}{\sin I''}\right)\left[1-5\theta^2-\frac{16\theta^4}{1-5\theta^2}\right]\right.$$

$$\left.+e''^3\theta^2\sin I''\left[5+\frac{32\theta^2}{1-5\theta^2}+\frac{80\theta^4}{(1-5\theta^2)^2}\right]\right\}\cos 3\omega''$$

$$\delta_1\Omega = e''^2\theta\left(-\frac{\gamma_2'}{8}\left[11 + \frac{80\theta^2}{1-5\theta^2} + \frac{200\theta^4}{(1-5\theta^2)^2}\right] + \frac{5\gamma_4'}{12\gamma_2'}\left[3 + \frac{16\theta^2}{1-5\theta^2} + \frac{40\theta^4}{(1-5\theta^2)^2}\right]\right)\sin 2\omega''$$

$$+ \frac{e''\theta}{4\gamma_2'}\left\{\frac{\gamma_3'}{\sin I''} + \frac{5\gamma_5'}{16\sin I''}\left(4 + 3e''^2\right)\left[1 - 9\theta^2 - \frac{24\theta^4}{1-5\theta^2}\right]\right.$$

$$\left. + \frac{15\gamma_5'}{8}\sin I''\left(4 + 3e''^2\right)\left[3 + \frac{16\theta^2}{1-5\theta^2} + \frac{40\theta^4}{(1-5\theta^2)^2}\right]\right\}\cos\omega''$$

$$- \frac{35\gamma_5'}{576\gamma_2'}e''^3\theta\left\{\frac{1}{2\sin I''}\left[1 - 5\theta^2 - \frac{16\theta^4}{1-5\theta^2}\right]\right.$$

$$\left. + \sin I''\left[5 + \frac{32\theta^2}{1-5\theta^2} + \frac{80\theta^4}{(1-5\theta^2)^2}\right]\right\}\cos 3\omega''$$

Update for Short-Period Periodic Effects of Earth Gravity

To simplify the following formulas, let W21 and W22 be defined as follows:

$$\mathrm{W21} = 3\sin(2\omega' + 2f') + 3e''\sin(2\omega' + f') + e''\sin(2\omega' + 3f')$$

$$\mathrm{W22} = \frac{a''^2}{r'^2}\eta^2 + \frac{a''}{r'}.$$

Then the Brouwer short-period corrections are:

$$\delta_2 a = a''\gamma_2\left[\left(-1 + 3\theta^2\right)\left(\frac{a''^3}{r'^3} - \frac{1}{\eta^3}\right) + 3(1 - \theta^2)\frac{a''^3}{r'^3}\cos(2\omega' + 2f')\right]$$

$$\delta_2 e = \frac{\eta^2\gamma_2}{2e''}\left[(-1 + 3\theta^2)\left(\frac{a''^3}{r'^3} - \frac{1}{\eta^3}\right) + 3(1 - \theta^2)\left(\frac{a''^3}{r'^3} - \frac{1}{\eta^4}\right)\cos(2\omega' + 2f')\right]$$

$$- \frac{\eta^2\gamma_2'}{2e''}(1 - \theta^2)\,[3e''\cos(2\omega' + f') + e''\cos(2\omega' + 3f')\,]$$

$$\delta_2 I = \frac{\gamma_2'}{2}\theta\sin I''[3\cos(2\omega' + 2f') + 3e''\cos(2\omega' + f') + e''\cos(2\omega' + 3f')]$$

$$\delta_2 M = -\frac{\eta^3\gamma_2'}{4e''}\{2(-1 + 3\theta^2)(\mathrm{W22} + 1)\sin f'$$

$$+ 3(1 - \theta^2)[(1 - \mathrm{W22})\sin(2\omega' + f') + (\mathrm{W22} + \tfrac{1}{3})\sin(2\omega' + 3f')]\}$$

$$\delta_2\omega = -\frac{\delta_2 M}{\eta} + \frac{\gamma_2'}{4}[6(-1+5\theta^2)(f' - M' + e''\sin f') + (3-5\theta^2)\mathrm{W21}]$$

$$\delta_2\Omega = -\frac{\gamma_2'}{2}\theta[6(f' - M' + e''\sin f') - \mathrm{W21}]$$

Add all periodics at one time using the Lyddane modification

$$e\cos M = (e'' + \delta e)\cos M'' - (e''\delta M)\sin M''$$

$$e\sin M = (e'' + \delta e)\sin M'' + (e''\delta M)\cos M''$$

$$\sin\frac{I}{2}\cos\,\Omega = \left[\sin\frac{I''}{2} + \left(\frac{\delta I}{2}\right)\cos\frac{I''}{2}\right]\cos\,\Omega'' - \left(\sin\frac{I''}{2}\delta\Omega\right)\sin\,\Omega''$$

$$\sin\frac{I}{2}\sin\,\Omega = \left[\sin\frac{I''}{2} + \left(\frac{\delta I}{2}\right)\cos\frac{I''}{2}\right]\sin\,\Omega'' + \left(\sin\frac{I''}{2}\delta\Omega\right)\cos\,\Omega''$$

$$z = (M'' + \omega'' + \Omega'') + (\delta z)$$

Get the osculating classical elements from the osculating Lyddane variables

$$e = \sqrt{(e\cos M)^2 + (e\sin M)^2}$$

$$M = \tan^{-1}\left(\frac{e\sin M}{e\cos M}\right)$$

$$\cos I = 1 - 2\sin^2\frac{I}{2} = 1 - 2\left[\left(\sin\frac{I}{2}\cos\,\Omega\right)^2 + \left(\sin\frac{I}{2}\sin\,\Omega\right)^2\right]$$

$$\Omega = \tan^{-1}\left(\frac{\sin(I/2)\sin\,\Omega}{\sin(I/2)\cos\,\Omega}\right)$$

$$\omega = z - M - \Omega$$

Kepler's equation is solved again using the method described above with the osculated mean motion and eccentricity. The position and velocity are computed by standard formula. First, the orientation unit vectors are formed

$$\boldsymbol{U} = \begin{pmatrix} \cos\Omega\cos(f+\omega) - \sin\Omega\sin(f+\omega)\cos I \\ \sin\Omega\cos(f+\omega) + \cos\Omega\sin(f+\omega)\cos I \\ \sin(f+\omega)\sin I \end{pmatrix}$$

$$\boldsymbol{V} = \begin{pmatrix} -\cos\Omega\sin(f+\omega) - \sin\Omega\cos(f+\omega)\cos I \\ -\sin\Omega\sin(f+\omega) + \cos\Omega\cos(f+\omega)\cos I \\ \cos(f+\omega)\sin I \end{pmatrix}$$

$$\boldsymbol{W} = \begin{pmatrix} \sin\Omega\sin I \\ -\cos\Omega\sin I \\ \cos I \end{pmatrix}$$

Then the position and velocity are

$$\boldsymbol{r} = \frac{a(1-e^2)}{1+e\cos f}\boldsymbol{U}$$

$$\dot{\mathbf{r}} = \frac{\sqrt{\mu}\left(e\sin f\right)}{\sqrt{a(1-e^2)}}\mathbf{U} + \frac{\sqrt{\mu}\left(1+e\cos f\right)}{\sqrt{a(1-e^2)}}\mathbf{V}$$

Appendix C - Deep Space Equations

C1 - Initialization for Secular and Long-Period Coefficients of Lunar and Solar Gravity

The first step in the initialization process is to compute the position of the Moon and Sun at the epoch time of the satellite element set using the following equations:

$$\Omega_{m_\varepsilon} = [\Omega_{m_{\varepsilon 0}} + \dot{\Omega}_{m_\varepsilon}\,\Delta t + \ddot{\Omega}_{m_\varepsilon}\,\Delta t^2 + \dddot{\Omega}_{m_\varepsilon}\,\Delta t^3]_{\mathrm{mod}\,2\pi}$$

$$\cos I_m = \cos\varepsilon\cos I_{m_\varepsilon} - \sin\varepsilon\sin I_{m_\varepsilon}\cos\Omega_{m_\varepsilon}$$

The Lunar longitude of perigee referred to the ecliptic is:

$$\gamma = u_{0_\varepsilon} + \dot{u}_\varepsilon\,\Delta t + \ddot{u}_\varepsilon\,\Delta t^2 + \dddot{u}_\varepsilon\,\Delta t^3$$

where u_{0_ε} is the epoch longitude of perigee (with respect to the ecliptic).

The Lunar right ascension of the ascending node referred to the equator is:

$$\sin\Omega_m = \sin I_{m_\varepsilon}\sin\Omega_{m_\varepsilon} / \sin I_m$$

$$\cos\Omega_m = \sqrt{1-\sin^2\Omega_m}$$

Then

$$\sin\Delta = \sin\varepsilon\sin\Omega_{m_\varepsilon} / \sin I_m$$

$$\cos\Delta = \cos\Omega_m\cos\Omega_{m_\varepsilon} + \sin\Omega_m\sin\Omega_{m_\varepsilon}\cos\varepsilon$$

$$\Delta = \tan^{-1}\left(\frac{\sin\Delta}{\cos\Delta}\right)$$

$$\omega_m = \gamma - \Omega_{m_\varepsilon} + \Delta = G_{o_m}$$

$$M_s = M_o + \dot{M}\Delta t + \ddot{M}\,\Delta t^2 + \dddot{M}\Delta t^3$$

where Δt is the time since the Lunar/Solar ephemeris epoch and where the elements of the Moon and Sun are obtained from equations supplied in the *Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac* (pages 107 and 98 for the Moon and Sun, respectively). The constants for calculating Lunar and Solar positions are defined as:

$I_{m_\varepsilon} = 5.^\circ 145396374$ (the moon's inclination with respect to the ecliptic)

$\varepsilon = 23.^\circ 4441$ (the obliquity of the ecliptic)

$e_m = .05490$ (lunar eccentricity)

$e_s = .01675$ (solar eccentricity)

$n_m = 1.583521770 \times 10^{-4}$ radians/minute (lunar mean motion)

$n_s = 1.19459 \times 10^{-5}$ radians/minute (solar mean motion)

$I_s = \varepsilon = 23.^\circ 4441$ (solar inclination)

$\Omega_s = 0$

$\omega_s = 281.^\circ 2208 = G_{o_s}$

$C_m = 4.796806521 \times 10^{-7}$ radians/minute (lunar perturbation coefficient)

$C_s = 2.98647972 \times 10^{-6}$ radians/minute (solar perturbation coefficient)

The Lunar and Solar elements are epoched at 0.5 January 1900 (Julian date 2415020.0).

For each body X, either the Sun or the Moon, terms are calculated which depend solely upon the epoch satellite orbital elements $(\Omega_0, \omega_0, I_0)$ and the orbital elements of the Moon and Sun. In the calculations of these terms, the following conventions apply:

(a) Quantities on the right side of the equation with subscript 0 refer to mean elements of the satellite orbit.

(b) Quantities on the right side of the equation with subscript X refer to the orbit of body X.

(c) Quantities on the left side of the equation refer to the satellite's orbit as affected exclusively by body X.

(d) n_X = mean motion of perturbing body X.

(e) All orbital elements of the Moon and Sun, except mean anomaly, are treated as constant at the epoch of the satellite

Calculate the constants

$$a_1 = \cos\omega_x \cos(\Omega_o - \Omega_x) + \sin\omega_x \cos I_x \sin(\Omega_o - \Omega_x)$$

$$a_3 = -\sin\omega_x \cos(\Omega_o - \Omega_x) + \cos\omega_x \cos I_x \sin(\Omega_o - \Omega_x)$$

$$a_7 = -\cos\omega_x \sin(\Omega_o - \Omega_x) + \sin\omega_x \cos I_x \cos(\Omega_o - \Omega_x)$$

$$a_8 = \sin\omega_x \sin I_x$$

$$a_9 = \sin\omega_x \sin(\Omega_o - \Omega_x) + \cos\omega_x \cos I_x \cos(\Omega_o - \Omega_x)$$

$$a_{10} = \cos\omega_x \sin I_x$$

$$a_2 = a_7 \cos i_o + a_8 \sin i_o$$

$$a_4 = a_9 \cos i_o + a_{10} \sin i_o$$

$$a_5 = -a_7 \sin i_o + a_8 \cos i_o$$

$$a_6 = -a_9 \sin i_o + a_{10} \cos i_o$$

$$X_1 = a_1\cos\omega_o + a_2\sin\omega_o$$
$$X_2 = a_3\cos\omega_o + a_4\sin\omega_o$$
$$X_3 = -a_1\sin\omega_o + a_2\cos\omega_o$$
$$X_4 = -a_3\sin\omega_o + a_4\cos\omega_o$$
$$X_5 = a_5\sin\omega_o$$
$$X_6 = a_6\sin\omega_o$$
$$X_7 = a_5\cos\omega_o$$
$$X_8 = a_6\cos\omega_o$$
$$Z_{31} = 12X_1^2 - 3X_3^2$$
$$Z_{32} = 24X_1X_2 - 6X_3X_4$$
$$Z_{33} = 12X_2^2 - 3X_4^2$$
$$Z_1 = 6(a_1^2 + a_2^2) + (1+e_0^2)Z_{31}$$
$$Z_2 = 12(a_1a_3 + a_2a_4) + (1+e_0^2)Z_{32}$$
$$Z_3 = 6(a_3^2 + a_4^2) + (1+e_0^2)Z_{33}$$
$$Z_{11} = -6a_1a_5 + e_0^2(-24X_1X_7 - 6X_3X_5)$$
$$Z_{13} = -6a_3a_6 + e_0^2(-24X_2X_8 - 6X_4X_6)$$
$$Z_{21} = 6a_2a_5 + e_0^2(24X_1X_5 - 6X_3X_7)$$
$$Z_{23} = 6a_4a_6 + e_0^2(24X_2X_6 - 6X_4X_8)$$
$$Z_{22} = 6a_4a_5 + 6a_2a_6 + e_0^2(24X_2X_5 + 24X_1X_6 - 6X_4X_7 - 6X_3X_8)$$
$$Z_{12} = -6a_1a_6 - 6a_3a_5 - e_0^2(24X_2X_7 + 24X_1X_8 + 6X_3X_6 + 6X_4X_5)$$

The secular rates are computed separately for both the Sun and Moon, and then combined into a single third-body secular rate. The secular rates due to the third-body perturbation are:

$$\dot{a}_x = 0$$
$$\dot{e}_x = -15C_xn_x\frac{e_0\eta_0}{n_0}(X_1X_3 + X_2X_4)$$
$$\dot{I}_x = \frac{-C_xn_x}{2n_0\eta_0}(Z_{11} + Z_{13})$$
$$\dot{M}_x = \frac{-C_xn_x}{n_0}(Z_1 + Z_3 - 14 - 6e_0^2)$$
$$\dot{\Omega}_x = \begin{cases} \dfrac{C_xn_x}{2n_0\eta_0\sin I_0}(Z_{21} + Z_{23}) & \text{if } I_0 \ge 3^\circ \\ 0 & \text{if } I_0 < 3^\circ \end{cases}$$

$$\dot{\omega}_x = \begin{cases} \dfrac{C_x n_x \eta_0}{n_0''}(Z_{31} + Z_{33} - 6) - \dot{\Omega}_x \cos I_0'' & \text{if } I_0'' \geq 3^\circ \\ \dfrac{C_x n_x \eta_0}{n_0''}(Z_{31} + Z_{33} - 6) & \text{if } I_0'' < 3^\circ \end{cases}$$

C2 - Initialization for Resonance Effects of Earth Gravity

If the satellite period in minutes is in the closed interval [1200, 1800], then it is assumed to be in a one-day resonance condition. The following constants are satellite-independent for 1-day period satellites:

$$Q_{22} = \sqrt{C_{22}^2 + S_{22}^2},\ \ Q_{31} = \sqrt{C_{31}^2 + S_{31}^2},\ \ Q_{33} = \sqrt{C_{33}^2 + S_{33}^2}$$

where

$$Q_{31} = 2.1460748x10^{-6}$$
$$Q_{22} = 1.7891679x10^{-6}$$
$$Q_{33} = 2.2123015x10^{-7}$$

The three phase angles are:

$$\lambda_{22} = \frac{1}{2}\tan^{-1}\frac{S_{22}}{C_{22}},\ \ \lambda_{31} = \tan^{-1}\frac{S_{31}}{C_{31}},\ \ \lambda_{33} = \frac{1}{3}\tan^{-1}\frac{S_{33}}{C_{33}}$$

where

$$\lambda_{31} = 0.13130908$$
$$\lambda_{22} = 2.88431980$$
$$\lambda_{33} = 0.37448087$$

The functions of inclination (F) and eccentricity (G) for 1-day period satellites (which are dependent solely upon epoch quantities) are calculated as follows:

$$F_{220} = \frac{3}{4}(1 + \cos I_o'')^2$$
$$F_{311} = \frac{15}{16}\sin^2 i_o(1 + 3\cos I_o'') - \frac{3}{4}(1 + \cos I_o'')$$
$$F_{330} = \frac{15}{8}(1 + \cos I_o'')^3$$

$$
\begin{aligned}
G_{200} &= 1 - \frac{5}{2} e_o''^2 + \frac{13}{16} e_o''^4 \\
G_{310} &= 1 + 2 e_o''^2 \\
G_{300} &= 1 - 6 e_o''^2 + \frac{423}{64} e_o''^4
\end{aligned}
$$

Compute the following coefficients of the resonance terms.

$$\delta_1 = \frac{3n_o^2}{a_o^3} F_{311} G_{310} Q_{31}$$

$$\delta_2 = \frac{6n_o^2}{a_o^2} F_{220} G_{200} Q_{22}$$

$$\delta_3 = \frac{9n_o^2}{a_o^3} F_{330} G_{300} Q_{33}$$

If the satellite period in minutes is in the closed interval [680, 760] and the eccentricity is greater than or equal to 0.5, then it is assumed to be in a ½-day resonance condition. The following constants are satellite-independent for ½-day period satellites:

$$\sqrt{C_{22}^2 + S_{22}^2} = 1.7891679x10^{-6}$$

$$\sqrt{C_{32}^2 + S_{32}^2} = 3.7393792x10^{-7}$$

$$\sqrt{C_{44}^2 + S_{44}^2} = 7.3636953x10^{-9}$$

$$\sqrt{C_{52}^2 + S_{52}^2} = 1.1428639x10^{-7}$$

$$\sqrt{C_{54}^2 + S_{54}^2} = 2.1765803x10^{-9}$$

$$D_{lmpq} = \frac{3n_o^2}{a_o^l} \sqrt{C_{lm}^2 + S_{lm}^2} F_{lmp} G_{lpq}$$

for the following (l,m,p,q) quadruples: (2,2,0,1), (2,2,1,1), (3,2,1,0), (3,2,2,2), (5,2,2,0), (5,2,3,2), (4,4,2,2), (5,4,2,1), (5,4,2,3), (4,4,1,0). The functions of inclination (dependent upon epoch quantities) are as follows:

$$F_{220} = \frac{3}{4}(1+\cos\ I''_o)^2$$

$$F_{221} = \frac{3}{2}(\sin\ I''_o)^2$$

$$F_{321} = \frac{15}{8}\sin\ I''_o(1-2\cos\ I''_o - 3\cos^2\ I''_o)$$

$$F_{322} = \frac{-15}{8}\sin\ I''_o(1+2\cos\ I''_o - 3\cos^2\ I''_o)$$

$$F_{441} = \frac{105}{4}\sin^2\ I''_o(1+\cos\ I''_o)^2$$

$$F_{442} = \frac{315}{8}\sin^4 I''_o$$

$$F_{522} = \frac{315}{32}\left[\sin^3 I''_o - 2\sin^3 I''_o \cos\ I''_o - 5\sin^3 I''_o \cos^2 I''_o + \sin I''_o(\frac{-2}{3} + \frac{4}{3}\cos\ I''_o + 2\cos^2 I''_o)\right]$$

$$F_{523} = \frac{105}{16}\sin\ I''_0\{1+2\cos\ I''_o - 3\cos^2 I''_o - \frac{3}{2}\sin^2 I''_o[1+2\cos\ I''_o - 5\cos^2 I''_o]\}$$

$$F_{542} = \frac{945}{32}\sin\ I''_o\{2-8\cos\ I''_o + \cos^2 I''_o[-12+8\cos\ I''_o + 10\cos^2 I''_o]\}$$

$$F_{543} = \frac{945}{32}\sin\ I''_o\{\cos^2 I''_o[12+8\cos\ I''_o - 10\cos^2 I''_o] - 2 - 8\cos\ I''_o\}$$

and the functions of eccentricity are:

$$G_{211}=\begin{cases} 3.616-13.247e''_o+16.29e''^2_o & e''_o \le 0.65 \\ -72.099+331.819e''_o-508.738e''^2_o+266.724e''^3_o & e''_o > 0.65 \end{cases}$$

$$G_{201}= -0.306 - 0.44\,(e''_o - 0.64)$$

$$G_{310}=\begin{cases} -19.302+117.39e''_o-228.419e''^2_o+156.591e''^3_o & e''_o \le 0.65 \\ -346.844+1582.851e''_o-2415.925e''^2_o+1246.113e''^3_o & e''_o > 0.65 \end{cases}$$

$$G_{322}=\begin{cases} -18.9068+109.7927e''_o-214.6334e''^2_o+146.5816e''^2_o & e''_o \le 0.65 \\ -342.585+1554.908e''_o-2366.899e''^2_o+1215.972e''^3_o & e''_o > 0.65 \end{cases}$$

$$G_{410}=\begin{cases} -41.122+242.694e''_o-471.094e''^2_o+313.953e''^3_o & e''_o \le 0.65 \\ -1052.797+4758.686e''_o-7193.992e''^2_o+3651.957e''^3_o & e''_o > 0.65 \end{cases}$$

$$G_{422}=\begin{cases} -146.407+841.88e''_o-1629.014e''^2_o+1083.435e''^3_o & e''_o \le 0.65 \\ -3581.69+16178.11e''_o-24462.77e''^2_o+12422.52e''^3_o & e''_o > 0.65 \end{cases}$$

$$G_{520}=\begin{cases} -532.114+3017.977e''_o-5740.032e''^2_o+3708.276e''^3_o & e''_o \le 0.65 \\ 1464.74-4664.75e''_o+3763.64e''^2_o & 0.65 < e''_o < 0.715 \\ -5149.66+29936.92e''_o-54087.36e''^2_o+31324.56e''^3_o & e''_o \ge 0.715 \end{cases}$$

$$G_{521}=\begin{cases} -822.71072+4568.6173e''_o-8491.4146e''^2_o+5337.524e''^3_o & e''_o < 0.70 \\ -51752.104+218913.95e''_o-309468.16e''^2_o+146349.42e''^3_o & e''_o \ge 0.70 \end{cases}$$

$$G_{532}=\begin{cases} -853.666+4690.25e''_o-8624.77e''^2_o+5341.4e''^3_o & e''_o < 0.70 \\ -40023.88+170470.89e''_o-242699.48e''^2_o+115605.82e''^3_o & e''_o \ge 0.70 \end{cases}$$

$$G_{533}=\begin{cases} -919.2277+4988.61e''_o-9064.77e''^2_o+5542.21e''^3_o & e''_o < 0.70 \\ -37995.78+161616.52e''_o-229838.2e''^2_o+109377.94e''^3_o & e''_o \ge 0.70 \end{cases}$$

C3 - Secular Updates for Effects of Lunar and Solar Gravity

The secular effects of Lunar and Solar gravity are included by the following equations:

$$M = M + \dot{M}_{LS}(t-t_0)$$
$$\omega = \omega + \dot{\omega}_{LS}(t-t_0)$$
$$\Omega = \Omega + \dot{\Omega}_{LS}(t-t_0)$$
$$e = e_0 + \dot{e}_{LS}(t-t_0)$$
$$I = I_0 + \dot{I}_{LS}(t-t_0)$$

where the rates with subscript LS are the sum of the effects of Lunar and Solar perturbations.

C4 - Secular Update for Resonance Effects of Earth Gravity

Define an auxiliary variable λ for the resonance treatment as

$$\lambda = M + \Omega + \omega - \theta_G$$

for orbits in the one-day-period band and

$$\lambda = M + 2\Omega - 2\theta_G$$

for orbits in the ½-day-period band where θ_G is the longitude of Greenwich. Simultaneously, numerically integrate the resonance equations for mean motion and the resonance variable λ. The numerical integration scheme is the Euler-Maclaurin method with a step size of 12 hours (720 minutes).

At epoch $\lambda_i = \lambda_o, \quad n_i = n_o.$

and the Euler-Maclaurin equations are

$$\lambda_i = \lambda_{i-1} + \dot{\lambda}_i(\Delta t) + \frac{\ddot{\lambda}_i}{2}(\Delta t)^2$$

$$n_i = n_{i-1} + \dot{n}_i(\Delta t) + \frac{\ddot{n}_i}{2}(\Delta t)^2$$

The derivatives are computed as follows:

For 1-day-period orbits:

$$\begin{aligned}
\dot{\lambda}_1 &= n_i + \dot{\lambda}_0 \\
\dot{n}_i &= \delta_1 \sin(\lambda_i - \lambda_{31}) + \delta_2 \sin(2\lambda_i - 2\lambda_{22}) + \delta_3 \sin(3\lambda_i - 3\lambda_{33}) \\
\frac{\ddot{\lambda}_i}{2} &= \frac{\dot{n}_i}{2} \\
\frac{\ddot{n}_i}{2} &= \frac{\dot{\lambda}_i}{2}[\delta_1 \cos(\lambda_i - \lambda_{31}) + 2\delta_2 \cos(2\lambda_i - \lambda_{22}) + 3\delta_3 \cos(3\lambda_i - \lambda_{33})]
\end{aligned}$$

For the ½-day-period orbits, (using the ½-day resonance (*l*, *m*, *p*, *q*) quadruplets):

$$\begin{aligned}
\dot{\lambda}_i &= n_i + \dot{\lambda}_0 \\
\dot{n}_i &= \sum_{(l,m,p,q)} D_{lmpq} \sin[(l-2p)\omega_i + \frac{m}{2}\lambda_i - G_{lm}] \\
\frac{\ddot{\lambda}_i}{2} &= \frac{\dot{n}_i}{2} \\
\frac{\ddot{n}_i}{2} &= \frac{\dot{\lambda}_i}{2}\left[\sum_{(l,m,p,q)} \frac{m}{2} D_{lmpq} \cos[(l-2p)\omega_i + \frac{m}{2}\lambda_i - G_{lm}]\right]
\end{aligned}$$

where

$$
\begin{aligned}
G_{22} &= 5.7686396 \\
G_{32} &= 0.95240898 \\
G_{44} &= 1.8014998 \\
G_{52} &= 1.0508330 \\
G_{54} &= 4.4108898
\end{aligned}
$$

and $\omega_i = \omega_o + \dot{\omega}_o \Delta t$ is the secularly-updated argument of perigee.
The 1-day-period initial conditions are

$$
\begin{aligned}
\lambda_0 &= M_0 + \omega_0 + \Omega_0 - \theta_0 \\
\dot{\lambda}_0 &= \dot{M}_0 + \dot{M}_{LS} + \dot{\Omega}_0 + \dot{\Omega}_{LS} + \dot{\omega}_0 + \dot{\omega}_{LS} - \dot{\theta}
\end{aligned}
$$

where θ is the Greenwich hour angle.
The ½-day initial conditions are:

$$
\begin{aligned}
\lambda_o &= M_0 + 2\Omega_o - 2\theta_0 \\
\dot{\lambda}_0 &= \dot{M}_o + \dot{M}_{LS} + 2\dot{\Omega}_o + 2\dot{\Omega}_{LS} - 2\dot{\theta}
\end{aligned}
$$

When λ_i, n_i are obtained at the time of interest, compute:

$$
\begin{aligned}
n &= n_i \\
M &= \begin{cases} \lambda_i - \Omega_s - \omega_s + \theta_t & \text{for 1-day period} \\ \lambda_i - 2\Omega_s + 2\theta_t & \text{for 1/2-day period} \end{cases}
\end{aligned}
$$

and Ω_s and ω_s are the mean elements updated with the secular rates of the other perturbations.

C 5 - Update for Long-Period Periodic Effects of Lunar and Solar Gravity

The true anomaly of the perturbing body is approximated by

$$
f_X = M_X + 2e_X \sin M_X
$$

Defining:

$$
\begin{aligned}
F_2 &= \tfrac{1}{2} sin^2 f_X - \tfrac{1}{4} \\
F_3 &= -\tfrac{1}{2} sin\ f_x cos\ f_X
\end{aligned}
$$

We have, for each perturbing body:

$$\delta e_x = -\frac{30\eta_0 C_x e_0}{n_0}[F_2(X_2X_3 + X_1X_4) + F_3(X_2X_4 - X_1X_3)]$$

$$\delta I_x = -\frac{C_x}{n_0\eta_0}[F_2Z_{12} + F_3(Z_{13} - Z_{11})]$$

$$\delta M_x = -\frac{2C_x}{n_0}[F_2Z_2 + F_3(Z_3 - Z_1) - 3e_x \sin f_x(7 + 3e_0{}^2)]$$

$$(\delta\omega_x + \cos I_x \delta\Omega_x) = \frac{2\eta_0 C_x}{n_0}[F_2Z_{32} + F_3(Z_{33} - Z_{31}) - 9e_x \sin f_x]$$

$$\sin I_x \delta\Omega_x = \frac{C_x}{n_0\eta_0}[F_2Z_{22} + F_3(Z_{23} - Z_{21})]$$

The long-period periodics are computed separately for both the Sun and Moon, and then combined into a single third-body long-period term.

C6 - Critical Inclination in PPT3

Brouwer[8] showed that the perturbation theory should remain valid for all inclinations except for an interval of about 1.5° on either side of the critical inclination. Within this narrow range, special procedures are required in any implementation of a Brouwer-type theory. In PPT3, the procedure is as follows. First, compute the critical factor

$$x = 1 - 5\cos^2 I''$$

This factor vanishes at about I'' = 63.43°. Then all occurrences of $1/x$ are replaced by the approximation

$$\frac{1}{x} \approx \frac{1 - \exp\left(-100x^2\right)}{x} \equiv C(x)$$

Away from the critical inclination, $C(x)$ tends rapidly to $1/x$. But in the neighborhood of the critical inclination, $C(x)$ is bounded, and in fact vanishes at $x = 0$. It can be shown that $C(x)$ has a maximum amplitude of about 6.382, and that there are two extrema having this amplitude, a minimum near I'' = 61.86° and a maximum near I'' = 65.08°.

The value of $C(x)$ is not computed directly from the above expression because of numerical ill conditioning. Even the direct power-series expansion of the exponential function exhibits poor convergence because of the factor of 100. Both problems are avoided by repeatedly factoring the numerator of $C(x)$, expanding one factor in series and formally canceling x from the denominator. In particular, repeatedly factor the difference of squares to obtain the exact expression

$$C(x) = \frac{1}{x}\left(1 - \exp\left(-\beta x^2\right)\right)\prod_{m=0}^{10}\left(1 + \exp\left(-2^m \beta x^2\right)\right)$$

where $\beta = 100/2^{11}$. Then the first factor is computed by a series expansion truncated to a practical number of terms, which is feasible because of the smallness of β. PPT3 currently uses a twelve-term expansion:

$$\frac{\left(1 - \exp\left(-\beta x^2\right)\right)}{x} \cong \beta x \sum_{n=0}^{12} (-1)^n \frac{\beta^n x^{2n}}{(n+1)!}$$