ABSTRACT ALGEBRA – FIRST HOMEWORK ON EUCLIDEAN DOMAINS

The point of the first problem is to help you process the notion of an integral domain. The point behind the remaining problems is in getting used to working in non-integer, non-polynomial Euclidean domains.

(1) Consider $\mathbb{R}^2$, the set of all pairs of real numbers, equipped with the following operations.
- $(a, b) + (c, d) = (a + c, b + d)$.
- $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$.

  (a) Show that $(\mathbb{R}^2, +, \cdot)$ is a commutative ring with identity. (You are expected to provide a verification for everything. Be clear about additive and multiplicative identities and inverse(s).)

  (b) Consider the element $(0, 1)$ and denote it by $\mathbf{i}$. What is $\mathbf{i}^2$?

  (c) Using the notation introduced above we could write $a + b\mathbf{i} = (a, b)$. Schematically, this looks like complex numbers. How do the addition and the multiplication rules for complex numbers differ from the ones in this problem?

  (d) Address the situation with the multiplicative units of $(\mathbb{R}^2, +, \cdot)$. Which elements $a + bi$ have the multiplicative inverse, and what is it? Is $(\mathbb{R}^2, +, \cdot)$ a field? Is it an integral domain?

  (e) Find all solutions w of the following equations

    (i) $(1 + 2\mathbf{i}) \cdot w = 0$.

    (ii) $(1 + 2\mathbf{i}) \cdot w = 2 + \mathbf{i}$.

    (iii) $(1 + \mathbf{i}) \cdot w = 0$.

    (iv) $(1 + \mathbf{i}) \cdot w = 1 + \mathbf{i}$.

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(2) Implement the argument from class / previous homework which shows that
- The units in $\mathbb{Z}[i]$ are ± 1 and $\pm i$, only.
- The units in $\mathbb{Z}[\sqrt{5}i]$ are ± 1, only.

(3) Provide a factorization of 2, 3, 5, 7, 11 and 13 into irreducibles of $\mathbb{Z}[i]$.

(4) Provide two different factorizations of 9 into products of irreducibles of $\mathbb{Z}[\sqrt{5}i]$.

(5) Find <u>*a*</u> quotient and the corresponding remainder upon division of the following within $\mathbb{Z}[i]$. (For some nerdy entertainment find several quotients and a remainder.)

  (a) $7 + 11i$ divided by $4 + i$.

  (b) $6 + i$ divided by $1 - i$.

  (c) 10 divided by $-9 + 7i$.

(6) Find <u>*a*</u> GCD of the above: $\mathrm{GCD}(7 + 11i, 4 + i)$, $\mathrm{GCD}(6 + i, 1 - i)$ and $\mathrm{GCD}(10, -9 + 7i)$. (For some nerdy entertainment find several.) (To check your answer assemble a suitable $\mathbb{Z}[i]$-linear combination which equals the GCD.)