THE WORD PROBLEM FOR AUTOMATA GROUPS

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1. Automata Groups

We will first begin with some definitions that are used in the description of automatic groups. Firstly Σ is the finite set of *letters*, this is commonly called the *alphabet*. Then the free monoid generated by Σ will be denoted Σ^*, and the elements of Σ^* are commonly called *words* or *strings*, and it possesses an identety as the *empty word* which will be denoted as ϵ. The free monoid Σ^* can be though of as the set of all possible combinations of letters in Σ. For example consider the alphabet

$$\Sigma = \{a, b\} \Rightarrow \Sigma^* = \{a, b, aa, ab, ba, bb, \ldots\} .$$

The product of two strings $u, v \in \Sigma^*$ is denoted simply as uv and represents the concatination of the strings. We will also make use of the notation $|w|$ to mean the length of a string for some string $w \in \Sigma^*$. A subset of Σ^* is called a *language*, and a subset of $\Sigma^* \times \Sigma^*$ is called a *relation*.

In our case we can commonly view the alphabet Σ to be the set of generators for some group G.

Definition 1.1 (Automata)**.** An Automata is a construct that consists of an *alphabet* $\Sigma = \{0, 1, \ldots, d-1\}$, a finite *set of states* Q, a *transition function* $\phi : Q \times D \to Q$, and an *set of accepting states* $F \subseteq Q$. Automata are commonly denoted by the quadruple $A = (D, Q, \phi, F)$.

Definition 1.2 (Initial Automaton)**.** An Initial automata is an automata with the addition of an initial state $q \in Q$. This type of automata is denoted as $A_q = (D, Q, \phi, \psi, q)$.

Consider the automota represented in Figure 1. Applying this automata to the *word* of `11-111-1-1`, we can see how this automata converts this sequence of generator operations into an element of the group.

State	Word
0	`11-111-1-1`
1	`1-111-1-1`
2	`-111-1-1`
1	`11-1-1`
2	`1-1-1`
3	`-1-1`
2	`-1`
1	

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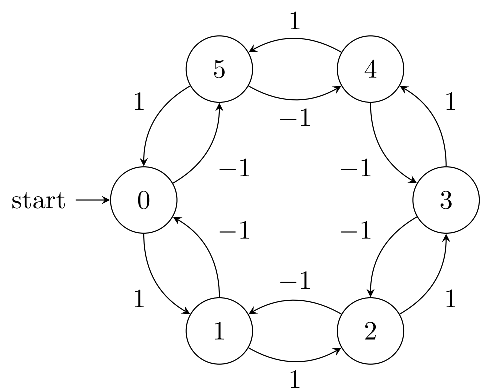

FIGURE 1. Automata representing the group $\mathbb{Z}/6\mathbb{Z} = \langle 1 \rangle$, this is denoted as $A_0 = (\{1\}, \{0, 1, 2, 3, 4, 5\}, \phi, \psi, 0)$, where $\phi(q, d) = q + d$, and $\psi(q, d) = d$.

Thus this word represents the element 1 in the group. This same process can be done for any sequence of generators, and as long as the sequence of generators is finite, then using an automota group can be used to algorithmicly determine when two different representations represent the same element of the group.

2. The Word Problem

The word problem in relation to groups asks if there is some algorithmic method to decide whether an element given in terms of a product of generators is equivalent to another product of generators. It is not possible to solve the word problem in general, this means that given two products of generators one cannot tell if they represent the same element.