QUIZ STUDY GUIDE

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1. Terminology

(1) State the commutative and the associative rules for an operation. State the distributive rule.

Commutative Rule:

$$a \star b = b \star a$$

Associative Rule:

$$(a \star b) \star c = a \star (b \star c)$$

Distributive Rule:

$$(a \star b) \circ c = (a \circ c) \star (b \circ c)$$
$$a \circ (b \star c) = (a \circ b) \star (a \circ c)$$

(2) What do we mean when we say that...

(a) Something is an "identity" with respect to an operation? E.g. what do we mean by an "additive identity"? Or by a "multiplicative identity"?

Something is an identity if $a \star \mathrm{Id} = a$, and $\mathrm{Id} \star a = a$. That operation applied to the identity and any other value is equal to that other value.

(b) An element is "invertible with respect to an operation"? What do we mean by a "multiplicative inverse"?

An element is invertible with respect to an operation if there exists an "identity" for that operation, and there exists a value such that $a \star a^{-1} = \mathrm{Id}$, and $a^{-1} \star a = \mathrm{Id}$.

(c) $(G, \star)$ constitutes a group?

- $\star$ is associative.
- $\star$ has an identity.
- Every element of G has an inverse with respect to $\star$.
- $\star$ May or may not be commutative.

(d) $(R, \oplus, \otimes)$ constitutes a ring? A commutative ring? How about a field?

Ring:
- $(R, \oplus)$ is a commutative group.
- $\otimes$ is associative.
- $\otimes$ distributes over $\oplus$.
- $\otimes$ may or may not be commutative.
- $\otimes$ may or may not have identity.
- Elements of R may or may not have an inverse.

Field:
- Every element of R has a multiplicative inverse.

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(e) an element is a "unit" in a ring?

Any element which is invertible in a ring.

(f) an element is a "zero divisor" in a ring?

$a \in R, a \neq 0$ is a zero divisor if there is $b \in R, b \neq 0$ such that $ab = 0$.

(g) something is an "integral domain"?

An integral domain is a ring with no zero divisors.

2. Examples

(1) Consider a collection $\mathcal{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$. Composition of functions is an operation in the set $\mathcal{F}$.

$$f, g \in \mathcal{F} \to f \circ g \in \mathcal{F}.$$

(a) Which rules does this operation satisfy?

$\circ$ satisfies the Associative rule.

(b) Does it have an identity?

Yes $\mathrm{Id}(x) = x$.

(c) Does every element of $\mathcal{F}$ have its inverse?

No, consider $f(x) = e^x$, the inverse would be $f^{-1}(x) = \ln(x)$. But $\ln : \mathbb{R} > 0 \mapsto \mathbb{R}$, thus $f^{-1}(x) \notin \mathcal{F}$.

(d) Does $\mathcal{F}$ constitute a group? A ring? A field?

Only one operation is defined, so it is not a ring or a field, and since not every element has an inverse, then it is not a group.

(2) Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices whose entries are real numbers.

(a) Which rules do matrix addition and multiplication satisfy?

Matrix addition satisfies commutative rule, and the Associative rule. Matrix multiplication satisfies Associative rule, and the distributive rule.

(b) Does matrix addition have an identity? If so, what is it? Likewise is there a multiplicative identity? If so, what is it?

The identity of matrix addition is the zero matrix, the multiplicative identity is the identity matrix.

(c) Does every matrix in $M_n(\mathbb{R})$ have its additive inverse? For matrices with additive inverse explain how one would go about finding the supposed inverse. How about multiplicative inverse? Explain how one would go about finding it.

Yes, every matrix has its additive inverse, to find it, multiply every element of the matrix by -1. Not all matrices have a multiplicative inverse.

(d) Is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ a unit in $M_2(\mathbb{R})$? What if I changed the framework of the problem and considered $M_2(\mathbb{Z})$, the set of 2×2 matrices whose entries are integers, in in place of $M_2(\mathbb{R})$?

Yes this matrix is invertible in $M_2(\mathbb{R})$, so it is a unit. In the framework of $M_2(\mathbb{Z})$, then it is not invertible, so it is not a unit.

(e) Is $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ a zero divisor in $M_2(\mathbb{R})$? How about in $M_2(\mathbb{Z})$? How about $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is it a zero division in $M_2(\mathbb{R})$? $M_2(\mathbb{Z})$?

No this is not a zero divisor, there is no matrix such that when multiplied with this one equals the zero matrix. Yes $\begin{pmatrix}0 & 1//0 & 0\end{pmatrix}$ is a zero divisor in $M_2(\mathbb{R})$, and $M_2(\mathbb{Z})$.

(f) Does $M_n(\mathbb{R})$ constitute a group? A ring? A field? An integral domain?

It does not constitute a integral domain, as there are zero divisors, and it does not constitute a field, as not all elements have a multiplicative inverse, thus it is a field.

(3) Let $S \neq \emptyset$. Union $\cup$ and intersection $\cap$ are operation on the power set $\mathcal{P}(S)$.

(a) Which rules do $\cup$ and $\cap$ satisfy?

$\cup$ satisfies commutative and associative. $\cap$ also satisfies commutative and associative. And $\cap$ distributes over $\cup$, so the distributive rule is satisfied.

(b) Does $\cup$ have an identity? If so, what is it? Does $\cap$ have an identity? If so, what is it?

The identity for $\cup = \emptyset$, and the identity for $\cap = S$.

(c) Does every $X \in \mathcal{P}(S)$ have an inverse with respect to $\cup$? If so, what is it? Does every $X \in \mathcal{P}(S)$ have an inverse with respect to $\cap$? If so, what is it?

No, there is no inverse with respect to $\cup$, there is no $Y \in \mathcal{P}(S)$ such that $X \cup Y = \emptyset$, only $X = \emptyset$ has an inverse. Similarly the only element of $\mathcal{P}(S)$ with an inverse with respect to $\cap$ is S.

(d) Does $\mathcal{P}(S)$ constitute a group? A ring? A field? An integral domain?

Since $(\mathcal{P}(S), \cup)$ is not a group, because not all elements have an inverse. Since this is not a group, then $(\mathcal{P}(S), \cup, \cap)$ is not a ring, field, or integral domain.

(4) Consider the set

$$\mathbb{Z}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} | a, b \in \mathbb{Z}\right\}$$

(a) Is the regular addition of numbers an operation on $\mathbb{Z}\left[\sqrt{2}\right]$? I.e. is it true that

$$x, y \in \mathbb{Z}\left[\sqrt{2}\right] \to x + y \in \mathbb{Z}\left[\sqrt{2}\right]?$$

If so, address the situation with the identity and the inverses with respect to this operation on $\mathbb{Z}\left[\sqrt{2}\right]$.

Yes it is an operation. The identity is $a = b = 0$, or $0 + b\sqrt{2}$. Thus inverses should exists for all elements. Inverses would be found as $x = a + b\sqrt{2}$, then $x^{-1} = (-a) + (-b)\sqrt{2}$.

(b) Is the regular multiplication of numbers an operation on $\mathbb{Z}\left[\sqrt{2}\right]$? I.e. is it true that

$$x, y \in \mathbb{Z}\left[\sqrt{2}\right] \to x \cdot y \in \mathbb{Z}\left[\sqrt{2}\right]?$$

If so, address the situation with the identity and the inverses with respect to this operation on $\mathbb{Z}\left[\sqrt{2}\right]$. Specifically, describe the units in $\mathbb{Z}\left[\sqrt{2}\right]$.

Yes it is also an operation. The identity is $a = 1, b = 0$, or $1+0\sqrt{2}$. Inverses may not always exists though. For example consider $2+0\sqrt{2}$, the inverse would have to be $\frac{1}{2}$, but this is not in the set. Thus the units in $\mathbb{Z}\left[\sqrt{2}\right]$ are $\pm 1 + 0\sqrt{2}$.

(c) Is $\mathbb{Z}\left[\sqrt{2}\right]$ a ring? A field? An integral domain?

This is a ring, and an integral domain, but it is not a field.

(d) What if instead we were to consider

$$\mathbb{Q}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} | a, b \in \mathbb{Q}\right\}?$$

Using this definition, then more elements would have inverses, but not all of them. Consider $0 + 1\sqrt{2}$. The multiplicative inverse would have to be $\frac{1}{\sqrt{2}}$, but this is not in the domain. So this would still only be a ring, and an integral domain.