Complex Variables (Exam #1)

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Contents

1 Complex Plane

1.1 Conjugate

Definitions

$$\bar{z} = \overline{a + ib} = a - ib$$

Properties

$$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$$
$$\overline{z_1 \cdot z_2} - \bar{z_1} \cdot \bar{z_2}$$

1.2 Modulus

Definition

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{z \cdot \bar{z}}$$

Properties

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

1.3 Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

1.4 Exponential Form

$$z = x + iy$$
$$z = |z| e^{i \mathrm{Arg}(z)}$$

product

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 \cdot r_2)\, e^{i(\theta_1 + \theta_2)}$$

Quotient

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

n-th power

$$\left(r e^{i\theta}\right)^n$$

2 Basic Elementary Functions

2.1 Exponential Function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\exp(x+iy) = e^x \cos y + ie^x \sin y$$

2.2 Logarithmic Function

$$\log(z) = \ln(|z|) + i\mathrm{arg}(z) \quad \text{Multi-valued}$$
$$\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z) \quad \text{Single-Valued}$$

2.3 Power Function

$$z^w = \exp(w \log(z)) \quad \text{Multi-valued}$$
$$P.V.z^w = \exp(w\mathrm{Log}(z)) \quad \text{Single-valued}$$

2.4 Trigonometic Functions

sin

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$
$$\sin(z) = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$
$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\sin(z) = -i\sinh(iz)$$

cos

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2i}$$
$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
$$\cos(z) - \cosh(iz)$$

sinh

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}$$
$$\sinh(z) = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$
$$i\sinh(z) = \sin(iz)$$

cosh

$$\cosh(s) = \frac{\exp(z) + \exp(-z)}{2}$$
$$\cosh(z) = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \cdots$$
$$\cosh(z) = \cos(iz)$$

3 Holomorphic functions and Cauchy-Riemann equations

3.1 Terminology

Differentiable If a function is differentiable, then the Jacobi matrix exists.

Holomorphic A function $f(z)$ is holomorphic if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

If a function is holomorphic, it implies that it is differentiable and the Cauchy-Remann equations hold for it.

Analytic A function is analytic if it is expressable as a sum of power series.

$$\sum a_n (z - z_0)^n$$

If a function is analytic then it is also holomorphic.

3.2 Jacobian

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

3.3 Cauchy-Riemann Equations

$f(z)$ is differentiable exactly when

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Derivation

$$
\begin{aligned}
z &= x + iy \\
f(z) &= u(x,y) + iv(x,y) \\
&= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\
f'(z) &= \lim_{a+ib \to 0} \left[\frac{y(x+a, y+b) + iv(x+a, y+b) - u(x,y) - iv(x,y)}{a+ib} \right] \\
&= \lim_{a \to 0} \left[\lim_{b \to 0} \left[\frac{y(x+a, y+b) + iv(x+a, y+b) - u(x,y) - iv(x,y)}{a+ib} \right] \right] \\
&= \lim_{a \to 0} \left[\frac{u(x+a, y) - u(x,y)}{a} + i \frac{v(x+a, y) - v(x,y)}{a} \right] \\
&= \boxed{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}} \\
&= \lim_{b \to 0} \left[\lim_{a \to 0} \left[\frac{y(x+a, y+b) + iv(x+a, y+b) - u(x,y) - iv(x,y)}{a+ib} \right] \right] \\
&= \lim_{b \to 0} \left[\frac{u(x, y+b) - u(x,y)}{ib} + i \frac{v(x, y+b) - v(x,y)}{ib} \right] \\
&= \boxed{-i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \\
\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
&= \begin{cases} \frac{\partial y}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}
\end{aligned}
$$