COMPLEX VARIABLES: STUDY GUIDE FOR EXAM 2

1 Improper integrals via theory of residues – basic skills

Use residue theory to compute the following integrals. Note: you'll get two of these kinds of problems on the upcoming exam. As you practice the problems below do put mental energy into justifying every step involved in the (proofs of the) relevant theorems (see below). On the exam you will be allowed to use results / theorems from class without justification.

1. $\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^3}\, dx$
2. $\int_0^{\infty} \frac{x^2}{x^4+4}\, dx$ (Note: this is an even function.)
3. $\int_0^{\infty} \frac{1}{x^4+5x^2+4}\, dx$
4. $\int_0^{+\infty} \frac{\cos(x)}{(x^2+1)^2}\, dx$
5. $\int_{-\infty}^{\infty} \frac{x\sin(sx)}{x^2+1}\, dx$
6. $\int_0^{\infty} \frac{\cos(sx)}{(x^4+4)^2}\, dx$
7. $\int_{-\infty}^{\infty} \frac{\sin(x)\, dx}{x(x^2+1)}$
8. $\int_0^{\infty} \frac{\sqrt{x}}{(x+1)(x+2)}\, dx$
9. $\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+1}\, dx.$

2 Integration in the complex plane: thinking required

You will have one problem of this sort (as well as some extra credit opportunities) on the exam. You will be expected to show all the details.

1. Use the substitution $z = \exp(i\theta)$ to convert the following into contour integrals in the complex plane. Don't forget that $\cos(\theta) = \frac{z+z^{-1}}{2}$ and $\sin(\theta) = \frac{z-z^{-1}}{2i}$ for $z = \exp(i\theta)$. Use residue theory to evaluate the resulting contour integrals:
   (a) $\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)}$;
   (b) $\int_0^{2\pi} \frac{d\theta}{5+3\sin(\theta)}$.

2. Use the contour of the rectangle with vertices at $-R$, R, $R + i\pi$ and $-R + i\pi$ to compute the integrals

   (a) $\int_{-\infty}^{\infty} \frac{\cos(x)}{e^x + e^{-x}} \, dx$

   (b) $\int_{0}^{\infty} \frac{\sin(x)}{e^x - e^{-x}} \, dx$

3. One of the classic integrals one does in multivariable calculus is that of $\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}$. Use this result, and use the contour in the shape of a pizza wedge with vertex at the origin and the circular edge going from R to $R \exp(i\frac{\pi}{4})$ to argue that

$$\int_0^\infty \cos(x^2) \, dx = \int_0^\infty \sin(x^2) \, dx = \frac{\sqrt{2\pi}}{4}.$$

4. Compute the following integral

$$\int_0^\infty \frac{dx}{x^3 + 1}$$

   by using residue theory and the domain in the shape of the pizza wedge with the vertex at the origin and the angle of $2\pi/3$.

5. Let $f(z)$ be a holomoprhic function of complex variable defined everywhere in the complex plane. According to the extended Cauchy Integral Formula, the derivatives of $f(z)$ satisfy

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} \, ds,$$

   where C is a small counterclockwise circle centered at z. Use this version of the Cauchy Integral Formula to show that if $|f(z)| \leq A|z|^m + B$ for some integer m and some positive numbers A and B, then $f(z)$ is a polynomial of degree at most m.