ABSTRACT ALGEBRA - FIRST MIDTERM EXAM

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Problem 1

Let $\omega \in \mathbb{C}$ be a solution of the equation

$$\omega^2 + \omega + 1 = 0.$$

Consider the set $\mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$. Show that the set $\mathbb{Z}[\omega]$ is closed under the ordinary addition and under the ordinary multiplication. Conclude that $\mathbb{Z}[\omega]$ is a ring which is a subring of the field of complex numbers.

Proof. Consider the set $\mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$. Let $a_1 + b_1\omega, a_2 + b_2\omega \in \mathbb{Z}[\omega]$, then we compute

$$(a_1 + b_1\omega) + (a_2 + b_2\omega) = (a_1 + a_2) + (b_1 + b_2)\omega.$$

It is clear that $a_1 + a_2 \in \mathbb{Z}$ and $b_1 + b_2 \in \mathbb{Z}$, so we can conclude that $\mathbb{Z}[\omega]$ is closed under ordinary addition. Now we compute

$$(a_1 + b_1\omega) \cdot (a_2 + b_2\omega) = a_1a_2 + a_1b_2\omega + a_2b_1\omega + b_1b_2\omega^2.$$

Since $\omega^2 + \omega + 1 = 0$, then we know that $\omega^2 = -\omega - 1$, so we can rewrite this to be

$$\begin{aligned} &a_1a_2 + a_1b_2\omega + a_2b_1\omega - b_1b_2(\omega + 1) \\ &\qquad = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1 - b_1b_2)\,\omega. \end{aligned}$$

Again, we can see that $a_1a_2 - b_1b_2 \in \mathbb{Z}$ and $a_1b_2 + a_2b_1 - b_1b_2 \in \mathbb{Z}$, thus we conclude that $\mathbb{Z}[\omega]$ is closed under ordinary addition.

TODO

We are able to conclude that $(\mathbb{Z}[\omega], +, \cdot)$ is a ring, and is a subring of $\mathbb{C}$. $\square$

Problem 2

Consider the set $\mathbb{Z}[2i] = \{a + 2bi | a, b \in \mathbb{Z}\}$. Standard number addition and multiplication turn $\mathbb{Z}[2i]$ into a commutative integral domain with identity.

(a). Prove that 2 is irreducible in this ring.

(b). Prove that $2i$ is irreducible in this ring.

(c). Is it true that $2|2i$ in this ring?

(d). Are 2 and $2i$ associates in this ring?

(e). Can you provide two factorizations of 4 into irreducible?

(f). Is 2 prime in this ring? Justify your claim.

(g). Is $2i$ prime in this ring?

(g). Is $\mathbb{Z}[2i]$ a Euclidean domain? Is it a PID?

Problem 3

Let I be an ideal of a commutative ring R with identity. Define the following set:

$$\operatorname{rad}(I) = \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Note: $\mathbb{N}$ is the set of positive integers only. In particular, $0 \notin \mathbb{N}$.

(a). Suppose temporarily that $R = \mathbb{Z}$. Find $\operatorname{rad}(I)$ for the following choices of I:

(i). $I = (9)$

(i). $I = (43)$

(iii). $I = (72)$

(b). Going back to the general situation, show $\operatorname{rad}(I)$ is an ideal. Hint: Look at your very first homework assignment.