COMPLEX VARIABLES: HOMEWORK DUE MONDAY, 2/19/2018

1. Plug and chug problems based on Sections 5.1, 5.2 and 5.3

(1) Express $\exp(z)$ in the $u+iv$-form for the following values of z:
  (a) $z = \frac{1}{2} - i\frac{\pi}{4}$;
  (b) $z = -1 + i\frac{3\pi}{2}$;
  (c) $z = \frac{\pi}{3} - 2i$.

(2) Find all values of
  (a) $\mathrm{Log}(ie^2)$;
  (b) $\mathrm{Log}(\sqrt{3} - i)$;
  (c) $\mathrm{Log}(-\sqrt{2} + \sqrt{2}i)$;
  (d) $\mathrm{Log}((1+i)^4)$;
  (e) $\log(-3)$;
  (f) $\log(8)$;
  (g) $\log(4i)$;
  (h) $\log(-\sqrt{3} - i)$.

(3) Find **the principal value** of
  (a) 4^i;
  (b) $(1+i)^{i\pi}$;
  (c) $(-1)^{\frac{1}{\pi}}$;
  (d) $(1+i\sqrt{3})^{i/2}$.

(4) Find all the values of the following:
  (a) i^i;
  (b) $(-1)^{\sqrt{2}}$;
  (c) $i^{2/\pi}$;
  (d) $(1+i)^{2-i}$;
  (e) $(-1)^{3/4}$;
  (f) $i^{2/3}$.

(5) Find all values of z for which:
  (a) $\exp(z) = -4$;
  (b) $\exp(z) = 2 + 2i$;
  (c) $\exp(z) = \sqrt{3} - i$;
  (d) $\exp(z) = -1 + i\sqrt{3}$;
  (e) $\mathrm{Log}(z) = 1 - i\frac{\pi}{4}$;
  (f) $\mathrm{Log}(\mathrm{z} - 1) = i\frac{\pi}{2}$;
  (g) $\exp(z) = -ie$;
  (h) $\exp(z+1) = i$.

2. Other problems

(1) Provide justifications for the following:
   (a) $\exp(z)^n = \exp(nz)$ for integer values of n;
   (b) $|\exp(z)| \leq \exp(|z|)$. (When does the equality hold?)
   (c) $|\exp(z)| \leq 1$ if $\text{Re}(z) < 0$.

(2) Find the domain of the function $f(z) = \sum_{n=0}^{\infty} \exp(inz)$.

(3) Discuss circumstances under which the following are true:
   (a) $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(\text{z}_2)$;
   (b) $\text{Log}(\frac{z_1}{z_2}) = \text{Log}(z_1) - \text{Log}(\text{z}_2)$;
   (c) $\text{Log}(z^n) = n\text{Log}(z)$. Assume integer values of n.

(4) Cook up examples which show the following no longer hold.
   (a) $\text{P.V}(z_1 z_2)^{1/3} = \text{P.V}(z_1)^{1/3} \cdot \text{P.V}(z_2)^{1/3}$;
   (b) $\text{P.V}z^{2/3} = (\text{P.V}z^2)^{1/3}$.

(5) Find branches of the following logarithmic functions satisfying the following conditions; give an explicit (piece-wise) formula relating the branches to $\text{Log}(z)$, and interpret in terms of the Riemann surface. (Note: This is a modification of Problem 9 from your textbook.)
   (a) Find a branch of $f(z) = \log(z)$ which is continuous at $z = -1$ and attains the value of πi there.
   (b) Find a branch of $f(z) = \log(z)$ which is continuous at $z = 1$ and attains the value of $4\pi i$ there.
   (c) Find a branch of $f(z) = \log(z)$ which is continuous at $z = -i$ and attains the value of $-5\frac{\pi}{2}i$ there.

(6) Find the images of the following regions under the following maps. Provide an illustration and color code some of the corresponding edges. (Note: This is a modification of Problem 12 from your textbook.)
   (a) Find the image of the left half-plane ($\text{Re}(z) \leq 0$) under the exponential map $f(z) = \exp(z)$.
   (b) Find the image of the rectangle given by $-1 \leq \text{Re}(z) \leq 1$, $-\frac{\pi}{2} \leq \text{Im}(z) \leq \frac{\pi}{2}$ under the exponential map $f(z) = \exp(z)$.
   (c) Find the image of the rectangle given by $0 \leq \text{Re}(z) \leq 1$, $-\frac{\pi}{4} \leq \text{Im}(z) \leq \frac{\pi}{4}$ under the map $f(z) = (\exp(z))^2$.
   (d) Find the image of the second quadrant under the logarithmic map $f(z) = \text{Log}(z)$.
   (e) Find the image of the third quadrant under the map $f(z) = -i \cdot \text{Log}(z) + 1$.
   (f) Find the image of the unit disk centered at the origin under the map $f(z) = \text{Log}(2z)$.

(7) Consider complex logarithm as a function defined on its Riemann surface. Find the pre-images, on the Riemann surface, of the following subsets of the complex plane. Please provide accompanying illustrations!
   (a) The pre-image of the imaginary axis.
   (b) The pre-image of the vertical strip $-1 \leq \text{Re}(w) \leq 1$.
   (c) The pre-image of the real axis.
   (d) The pre-image of the vertical strip $0 \leq \text{Im}(w) \leq 4\pi$.