Differential Equations (Exam #1)

Arden Rasmussen

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1 Summary

- Mixing Problems
  - Setup
  - Analyze
- Find equilibrium solutions and preform qualitative analysis.
- Find solution using analytic methods
- Fundamental Theorem
  - Solution curves don't cross unless RHS turns *ugly* on you.
- Euler method
  - Within reason
- Slope field
  - Not much of number crunching
- Some discussion about a parameter

2 Reading & Writing Differential Equations

The form of a differential equation must be the rate equals some equation that incorporates the equation.

$$\frac{dy}{dt} = f(t, y(t)) \tag{1}$$

Where $f(t, y(t))$ is some function that involves $y(t)$ and optional involves t.

2.1 Exponential Model

Idea: Relative growth rate is some constant: k.

$$\frac{dy}{dt} = k \cdot y(t) \tag{2}$$

2.2 Logistic Model

Idea: The relative growth rate may not be constant, and may depend on circumstances. It may depend on $y(t)$ itself. Where M is the maximum limit for $y(t)$.

$$\frac{dy}{dt} = ky(t)\left(1 - \frac{y(t)}{M}\right) \tag{3}$$

3 The Fundamental Theorem of ODE'S

Given a differential equation of the form:

$$\frac{dy}{dt} = f(t, y) \tag{4}$$

There are many solutions to a single differential equation. The solution is pinned to a single equation when the initial condition y_0 is set to a value.

3.1 Initial Value Problems

Initial value problems (IVP) come in the form:

$$\begin{cases} y\left(0\right) = y_0 \\ \frac{dy}{dt} = f(t, y) \end{cases} \tag{5}$$

Expect to have exactly one solution. Suppose $f(t, y)$ is a function which:

- Is continuous in a neighborhood of (t_0, y_0).
- Has continuous $\frac{\partial f}{\partial y}$ in a neighborhood of (t_0, y_0)

Then the IVP as exactly one solution. If either of these conditions are not met, then the fundamental theorem of ODE'S tells us nothing about the solution to the IVP. A General rule of thumb is:

> If $f(t, y)$ is nice then there is exactly one solution, if solution is messy, then $f(t, y)$ is not nice.

Note that this is only in the neighborhood of (t_0, y_0), so this says nothing about the long term existence or uniqueness. This only shows short term existence and uniqueness.

3.2 Method of Barriers

If the solutions for $f(t, y)$ are always nice, then no solution curves can cross or intersect, so any solution is stuck between other solution curves. Using equilibrium solutions (6.2) causes solutions to be sandwiched between these easer to find solutions.

4 Analytic Methods

Analytic methods are used to find the exact equation for the differential equation.

4.1 Separable Equations

Separable equations works by separating the terms for y and t in the differential equation to different sides of the equation. Given an equation of the form:

$$\frac{dy}{dt} = A(y) \cdot B(t) \tag{6}$$

This method can be applied in the following format.

$$\frac{1}{A(y)} \cdot \frac{dy}{dt} = B(t) \tag{7}$$

$$\int_{s=t_0}^{s=t} \frac{1}{A(y)} \cdot \frac{dy}{ds} ds = \int_{s=t_0}^{s=t} B(s) ds \tag{8}$$

$$\int_{y=y_0}^{y=y(t)} \frac{1}{A(y)} dy = \int_{s=t_0}^{s=t} B(s) ds \tag{9}$$

$$f\left(y(t), y_0\right) = g\left(t, t_0\right) \tag{10}$$

$$y(t) = g\left(t, t_0, y_0\right) \tag{11}$$

Note that due to [7] the case when $A(y) = 0$ must be checked to see if it is a solution. And f and g are arbitrary functions that are the results of the integration, they merely are involving int parameters.

4.2 Propagators

Propagators uses the method of propagating the initial value, and any additional values added. This method works when given an equation of the form:

$$\frac{dy}{dt} = r(t)s(t) + f(t) \tag{12}$$

The first step is to identify the propagator function.

$$P(t, t_0) = \exp\left(\int_{s=t_0}^{s=t} r(s) ds\right) \tag{13}$$

Using the propagator, the solution for $y(t)$ is of the form:

$$y(t) = P(t, t_0) \cdot y_0 + \int_{s=t_0}^{s=t} P(t, s) \cdot f(s) ds \tag{14}$$

Some notes on the Propagator

$$P(a, b) \cdot P(b, a) = 1 \tag{15}$$
$$P(a, b) \cdot P(b, c) = P(a, c) \tag{16}$$

5 Numeric Methods

Numeric methods are focused on getting approximate numeric values, without actually solving for the equation like analytic methods.

5.1 Euler's Method

Euler's method uses crating a table to approximate the value of $y(t)$ for different values of t. By using a step size of Δt, and calculating the slope for each step, the solution can be approximated through the following equations.

$$y_{n+1} = y_n + \Delta t \cdot \left.\frac{dy}{dt}\right|_{(t_n, y_n)} \tag{17}$$

Some notes on Euler's Method Because this is a method of approximation, it becomes more accurate with the smaller the value for Δt. With a value that is too large, the approximation could miss asymptotes, or could cross equilibrium solutions that cannot be crossed. (3.2)

6 Qualitative Analysis

Qualitative analysis is viewing the general trends of the solution of the differential equation, without finding values at all.

6.1 Stream Plots/Slope Fields

Basic Idea: Do careful accounting of slopes and connect the points. Creating a stream plot from a slope field (vector field).

Create a slope field, utilizing the Fundamental Theorem of ODE'S (3). Then connect the points of the slope field, creating an approximation of the solution plots.

6.2 Equilibrium Solutions

Equilibrium solutions are solutions that are easy to estimate, are are where $y(t) \equiv C$, where C is some constant. This can also be viewed as when $\frac{dy}{dt} = 0$.