Automata groups

Andrzej Zuk
Paris 7

Abstract. We give an introduction to the theory of automata groups and present most important recent developments.

INTRODUCTION

The class of automata groups contains several remarkable countable groups. Their study has led to the solution of a number of important problems in group theory. Its recent applications have extended to the fields of algebra, geometry, analysis and probability.

Together with arithmetic [23] and hyperbolic groups [19], automata groups dominate the modern landscape of theory of infinite groups.

Many groups from this paper could be presented under another common name like branched or self-similar groups (the definitions will be given later). We have chosen the name *automata groups* to stress the importance of this construction, which leads to groups with interesting properties.

As examples of this construction we have chosen the following problems:

- Burnside problem. Infinite finitely generated torsion groups.
- Milnor problem. Constructions of groups of intermediate growth.
- Atiyah problem. Computation of L^2 Betti numbers.
- Day problem. New examples of amenable groups.
- Gromov problem. Groups without uniform exponential growth.

For each of them we have chosen historically first examples of automata groups considered to solve these problems. Namely the Aleshin group, the lamplighter group (which can be generated by a two state automaton), the Wilson group and a group generated by a three state automaton.

There are other examples of automata groups which were important for the development of the theory (see [9], [30] and [20]). Let us mention the Fabrykowski-Gupta group [9], the Sushchanskii group [30] and the Gupta-Sidki group [20].

1. Automata groups

1.1. Definition of groups generated by automata

The automata which we consider are finite, reversible and have the same input and output alphabets, say $D = \{0, 1, \ldots, d-1\}$ for a certain integer $d > 1$. To such an automaton A are associated a finite set of states Q, a transition function $\phi : Q \times D \to Q$ and the exit function $\psi : Q \times D \to D$. The automaton A is caracterized by a quadruple (D, Q, ϕ, ψ).

The automaton A is inversible if, for every $q \in Q$, the function $\psi(q, \cdot) : D \to D$ is a bijection.

In this case, $\psi(q, \cdot)$ can be identified with an element σ_q of the symmetric group S_d on $d = |D|$ symbols.

There is a convenient way to represent a finite automaton by a marked graph $\Gamma(A)$ which vertices correspond to elements of Q.

Two states $q, s \in Q$ are connected by an arrow labelled by $i \in D$ if $\phi(q, i) = s$; each vertex $q \in Q$ is labelled by a corresponding element σ_q of the symmetric group.

The automata we just defined are non-initial. To make them initial we need to mark some state $q \in Q$ as the initial state. The initial automaton $A_q = (D, Q, \phi, \psi, q)$ acts on the right on the finite and infinite sequences over D in the following way. For every symbol $x \in D$ the automaton immediately gives $y = \psi(q, x)$ and changes its initial state to $\phi(q, x)$.

By joning the exit of A_q to the input of another automaton $B_s = (S, \alpha, \beta, s)$, we get an application which corresponds to the automaton called the composition of A_q and B_s and is denoted by $A_q \star B_s$.

This automaton is formally described as the automaton with the set of the states $Q \times S$ and the transition and exit functions Φ, Ψ defined by

$$\Phi((x, y), i) = (\phi(x, i), \alpha(y, \psi(x, i))),$$

$$\Psi((x, y), i) = \beta(y, \psi(x, i))$$

and the initial state (q, s).

The composition $A \star B$ of two non-initial automata is defined by the same formulas for input and output functions but without indicating the initial state.

Two initial automata are equivalent if they define the same application. There is an algorithm to minimize the number of states.

The automaton which produces the identity map on the set of sequences is called trivial. If A is an invertible then for every state q the automaton A_q admits an inverse automaton A_q^{-1} such that $A_q \star A_q^{-1}$, $A_q^{-1} \star A_q$ are equivalent to the trivial one. The inverse automaton can be formally described as the automaton $(Q, \widetilde{\phi}, \widetilde{\psi}, q)$ were $\widetilde{\phi}(s, i) = \phi(s, \sigma_s(i))$, $\widetilde{\psi}(s, i) = \sigma_s^{-1}(i)$ for $s \in Q$. The equivalence classes of finite invertible automata over the alphabet D constitute a group called a group of finite automata which depends on D. Every set of finite automata generates a subgroup of this group.

Now let A be an invertible automaton. Let $Q = \{q_1, \dots, q_t\}$ be the set of states of A and let $A_{q_1}, \dots, A_{q_t}$ be the set of initial automata which can be obtained from A. The group $G(A) = \langle A_{q_1}, \dots, A_{q_t} \rangle$ is called the group genrated or determined by A.

1.2. Automata groups and wreath products

There is a relation between automata groups and wreath products. For a group of the form $G(A)$ one has has the following interpretation.

Let $q \in Q$ be a state of A and let $\sigma_q \in S_d$ be the permutation associated to this state. For every symbol $i \in D$ we denote $A_{q,i}$ the initial automaton having as the initial state $\phi(q,i)$ (then $A_{q,i}$ for $i = 0, 1, \dots, d-1$ runs over the set of initial automata which are neighbors of A_q, i.e. such that the graph $\Gamma(A)$ has an arrow from A_q to $A_{q,i}$).

Let G and F be the groups of finite type such that F be a group of permutation of the set X (we are interested in the case were F is the symmetric group S_d and X is the set $\{0, 1, \dots, d-1\}$). We define the wreath product $G \wr F$ of these groups as follows. The elements of $G \wr F$ are the couples (g, γ) where $g : X \to G$ is a function such that $g(x)$ is different from the identity element of G, denoted Id, only for a finite number of elements x of X, and where γ is an element of F. The multiplication in $G \wr F$ is defined by:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(x) = g_1(x)g_2(\gamma_1^{-1}(x)) \quad \text{for } x \in X.$$

We write the elements of the group $G \wr S_d$ as $(a_0, \dots, a_{d-1})\sigma$, where $a_0, \dots, a_{d-1} \in G$ and $\sigma \in S_d$.

The group $G = G(A)$ admits the embedding into a wreath product $G \wr S_d$ via the application

$$A_q \to (A_{q,0}, \dots, A_{q,d-1})\sigma_q,$$

where $q \in Q$. The right expression is called a wreath decomposition of A. We write $A_q = (A_{q,0}, \dots, A_{q,d-1})\sigma_q$.

For simplicity we denote a the generator of A_a of the group generated by the automaton A.

1.3. Action on the tree

The finite sequences over the alphabet $D = \{0, \ldots, d-1\}$ are in bijection with the vertices of a rooted tree T_d of degree d (whose root corresponds to an empty sequence).

An initial automaton A_q acts on the sequences over D and thus acts on T_d by automorphisms. Therefore for each group generated by an automaton, in particular for a group of the form $G(A)$, there exists a canonical action on a tree (for a theory of actions on non-rooted trees, see [29]).

Let now G be a group acting on a rooted tree T. The boundary ∂T, consisting of infinite geodesic rays starting at the root vertex is endowed with a natural topology which makes it homeomorphic with the Cantor set.

The action of G on T induces an action on ∂T by homeomorphisms and admits a canonical invariant measure μ on ∂T which is the Bernoulli measure on $D^{\mathbb{N}}$ given by the distribution $\left\{\frac{1}{d}, \ldots, \frac{1}{d}\right\}$.

There is a canonical way to associate a unitary representation to a dynamical system endowed with an invariant measure. In our case we obtain a regular representation π sur $L^2(\partial T, \mu)$, defined by $(\pi(g)f)(x) = f(g^{-1}x)$.

1.4. Projections of stabilisers

For a group $G = G(A)$ acting by automorphisms on T, we denote $\mathrm{St}_G(n)$ the subgroup of G made of elements of G which act trivially on the level n of the tree T. In a similar way, for a vertex $u \in T$ we denote $\mathrm{St}_G(u)$ the subgroup of G composed of the elements fixing u. The embedding of G into the wreath product $G \wr S_d$ induces $\phi : \mathrm{St}_G(1) \to G^d$ into the base group of the wreath product. This defines the canonical projections $\psi_i : \mathrm{St}_G(1) \to G$ $(i = 1, \ldots, d)$ defined by $\psi_i(g) = \phi(g)|_i$ for $g \in \mathrm{St}_G(1)$.

1.5. Branch and fractal groups

The stabilizer $\mathrm{St}_G(n)$ of the n-th level is the intersection of the stabilizers of all vertices on this level. For a vertex $u \in T$ we can define the projection $\psi_u : \mathrm{St}_G(u) \to G$.

Definition 1.1. A group G is fractal if for every vertex u, we have $\psi_u(\mathrm{St}_G(u)) = G$ after the identification of the tree T with the subtree T_u issued from the vertex u.

The rigid stabilizer of the vertex u is a subgroup $\mathrm{Rist}_G(u)$ of the automorphisms of G which act trivially on $T \setminus T_u$. The rigid stabilizer of the n-th level $\mathrm{Rist}_G(n)$ is the subgroup generated by the rigid stabilizers on this level.

A group G acting on a rooted tree T is called spherically transitive if it acts transitively on each level. A spherically transitive group $G \subseteq \mathrm{Aut}(T)$ is branched if $\mathrm{Rist}_G(n)$ is finite index subgroup for each $n \in \mathbb{N}$. A spherically transitive group $G \subseteq \mathrm{Aut}(T)$ is weakly branched if $|\mathrm{Rist}_G(n)| = \infty$ for all $n \in \mathbb{N}$.

If there is no risk of confusion we omit the index G in $\mathrm{St}_G(u)$, $\mathrm{Rist}_G(u)$, etc.

The embedding $G \to G \wr S_d$, $g \to (g_0, \ldots, g_{d-1})\sigma$ defines a restriction g_i of g at the vertex i of the first level. The iteration of this procedure leads to a notion of restriction g_u of g at the vertex u.

Definition 1.2. We say that the group G is regularly weakly branched over a subgroup $K \neq \{1\}$ if $K \supseteq K \times \ldots \times K$ (direct product of d factors, each of them acting on the corresponding subtree T_u, $|u| = 1$).

We use the notations $x^y = y^{-1}xy, [x, y] = x^{-1}y^{-1}xy$ and denote $\langle X\rangle^Y$ the normal closure of X in Y. The length of a word w and an element g is denoted $|w|$ and $|g|$ respectively.

1.6. The word problem

The word problem has a solution for every group generated by a finite automaton due to the following algorithm:

Proposition 1.3. *The word problem is solvable for automata groups.*

PROOF. Let w be a word over the alphabet composed of the labelings of the states of the automaton and their inverses.

1. Verify if $w \in \mathrm{St}_G(1)$ (otherwise $w \neq 1$ in G).
2. Compute $w = (w_0, \ldots, w_{d-1})$. Then

$$w = 1$$

in G iff $w_i = 1$ in G for $i = 0, \ldots, d-1$. Go to 1. by replacing w by w_i and proceeding with every w_i as with w.

If in some step we obtain a word which is not in $\mathrm{St}_G(1)$ then $w \neq 1$ in G. If in every step all the words $w_{i_1}, \ldots, w_{i_n}$ already appeared in the algorithm then $w = 1$ in G.

This algorithm converges because the lengths of w_0, ..., w_{d-1} are at most the length of w and after sufficiently many steps there is repetition of the words.

1.7. Classification of the automata groups on two states with the alphabet $\{0, 1\}$

For the alphabet on two letters the automata with just one state produce only the trivial group or the group of oder two.

We are going to analyze all groups generated by the automata on two states with the alphabet on two letters.

In addition to the lmplighter group from Figure 2, there are five other groups generated by the automata from Figure 1 (we denote 1 or Id the identity of S_2 and e the nontrivial element of S_2).

The first two automata generate the trivial group and the group of order two. The group given by the third automaton is isomorphic to the Klein group $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. The fourth automaton defines the dihedral group $\mathbb{D}_\infty$. The last automaton defines the infinite cyclic group.

These are the only possiblities [14].

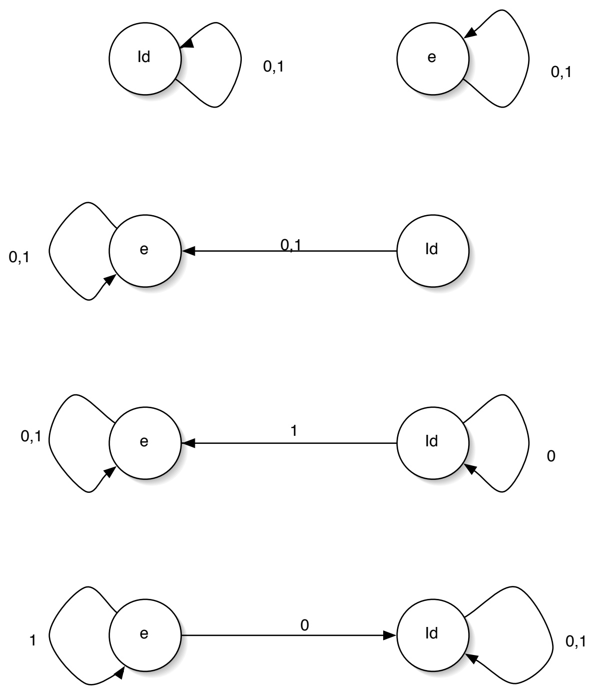

Figure 1: The automata generating the trivial group, the group of order two, the Klein group, the dihedral group and the infinite cyclic group.

Theorem 1.4. *The only groups generated by the automata on two states over the alphabet on two letters are:*

- *the trivial group;*
- *the group of order two* $\mathbb{Z}/2\mathbb{Z}$*;*
- *the Klein group* $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$*;*
- *the infinite cyclic group* $\mathbb{Z}$*;*

- *the infinite dihedral group* $\mathbb{D}_\infty$*;*
- *the lamplighter group* $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$.

Proof. We denote a and b two states of the automaton. If both states are labelled by the identity or by both by e, then the group generated by the automaton is either trivial or $\mathbb{Z}/2\mathbb{Z}$.

Thus we can suppose that one state, say a, is labelled by the identity and the other by e. By exchanging if necessary 0 with 1, we can suppose that $a = (a, a)$ or $a = (b, b)$ or $a = (a, b)$.

(i) Case $a = (a, a)$.

In this case a corresponds to the identity in the group. The exchange of 0 and 1 (this does not change a) reduces b to three possibilities: $b = (b, b)\mathsf{e}$, $b = (a, b)\mathsf{e}$ or $b = (a, a)\mathsf{e}$.

The first case corresponds to $\mathbb{Z}/2\mathbb{Z}$, the second to $\mathbb{Z}$ and the third to $\mathbb{Z}/2\mathbb{Z}$.

(ii) Case $a = (b, b)$.

The exchange of 0 and 1 (this does not change a) reduces b to three possibilities: $b = (b, b)\mathsf{e}$, $b = (a, a)\mathsf{e}$ or $b = (a, b)\mathsf{e}$.

The first two possibilities correspond to the Klein group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. indeed a and b are of order two and commute.

The third case corresponds to the infinite cyclic group. Indeed

$$ab = (ba, b^2)\mathsf{e},$$

$$ba = (ab, b^2)\mathsf{e},$$

so a and b commute. Secondly

$$b^2a = (b^2a, b^2a),$$

which implies the triviality of b^2a.

Therefore the group is cyclic. The preceeding relation ensures that the order of a is twice the order of b. But a and b have the same order according to the relation $a = (b, b)$. As a and b are non-trivial it implies that the group is $\mathbb{Z}$.

(iii) Case $a = (a, b)$.

By considering if necessary the inverse automaton (which generates the same group and does not change a) we can suppose that b satisfies one of the three possibilities: $b = (b, b)\mathsf{e}$, $b = (a, b)\mathsf{e}$ or else $b = (a, a)\mathsf{e}$.

In the first case $b^2 = (b^2, b^2)$ so b is of order 2. As $a^2 = (a^2, b^2)$, a is also of order two. The relations $a^{-1}b = (a^{-1}b, 1)\mathsf{e}$ and $(a^{-1}b)^2 = (a^{-1}b, a^{-1}b)$ imply that $a^{-1}b$ is of infinite order. Therefore it is the infinite dihedral group $\mathbb{D}_\infty$.

The second case corresponds to the lamplighter group (see the next section).

The third case can be analyzed in a similar way.

1.8. Important examples

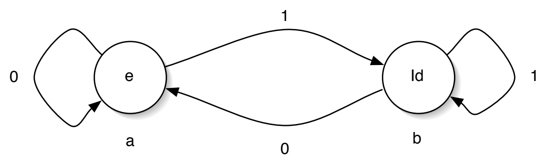

Figure 2: The automaton which generates the lamplighter group

In the following sections we present important examples of automata groups. Namely the lamplighter group, the Aleshin group, a group generated by a three state automaton and the Wilson group.

There are several other groups which played an important role in the devellopment of the theory. Let us mention the Fabrykowski-Gupta group [9], the group of Sushchansky [30] and the group of Gupta-Sidki [20].

For a general theory of automata groups one can refer to [2], [11] and [33].

2. The lamplighter group as an automaton group

The automaton group from Figure 2 generates the lamplighter group [14]. This group can be defined as the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ or as a semi-direct product $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ with the action of $\mathbb{Z}$ on $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ by translation.

Let a and b be the generators of the lamplighter group $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ such that $a = (f_a, g_a)$, $b = (f_b, g_b)$, where $g_a = g_b \in \mathbb{Z}$ is a generator of $\mathbb{Z}$, $f_a \in \oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is the identity and $f_b = (\ldots, 0, 0, 1, 0, 0, \ldots) \in \oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is such that 1 is in a position 1. There is an isomorphism between this group and the group generated by the automaton from Figure 2, where a and b correspond to the initial states of the automaton.

The study of this group and its action on a rooted tree of degree two enabled one to answer a question of Atiyah.

2.1. Operator reccurence

Let now G be the group generated by the automaton from Figure 2. We denote $\partial T = E_0 \sqcup E_1$ the partition of the boundary ∂T associated to the subtrees T_0 et T_1 issused from two vertices on the first level. We have the isomorphism $L^2(\partial T, \mu) \simeq L^2(E_0, \mu_0) \oplus L^2(E_1, \mu_1)$ where μ_i is the restriction of μ à E_i, as well as the isomorphism $L^2(\partial T, \mu) \simeq L^2(E_i, \mu_i)$, for $i = 0, 1$, coming from $T \simeq T_i$.

In this way we get an isomorphism between $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of infinite dimension. Thanks to this isomorphism, the operaters $\pi(a)$, $\pi(b)$ (also denoted by a et b, respectively), where π is a representation as in Section 1.3, satisfy the following operator relations:

$$a = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

which correspond to the wreath product relations: $a = (a, b)\mathsf{e}$ and $b = (a, b)$.

Let π_n a permutation representation of the group G induced by the action of G on the level n of the associated tree and let $\mathcal{H}_n$ be the space of functions on the n-th level. Let a_n and b_n be the matrices corresponding to generators for the representation π_n. Then $a_0 = b_0 = 1$ and

$$a_n = \begin{pmatrix} 0 & a_{n-1} \\ b_{n-1} & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix} \tag{2.1}$$

keeping in mind the natural isomorphism $\mathcal{H}_n \simeq \mathcal{H}_{n-1} \oplus \mathcal{H}_{n-1}$.

2.2. The lamplighter group and spectral measure

We are interested in the spectrum and the spectral measure of the Markov operator for the lamplighter group.

For a finite generating subset S which is symmetric ($S = S^{-1}$) we consider the simple random walk on the Cayley graph $\mathrm{Cay}(G, S)$. Then the random walk operator $A : \ell^2(G) \to \ell^2(G)$ is defined by

$$Af(g) = \frac{1}{|S|} \sum_{s \in S} f(sg),$$

where $f \in \ell^2(G)$ and $g \in G$.

As the operator A is bounded (we have $\|A\| \leq 1$) and self-adjoint, it admits a spectral decomposition

$$A = \int_{-1}^{1} \lambda \mathrm{d}E(\lambda),$$

where E is a spectral measure. This measure is defined on the Borel subsets of the interval $[-1, 1]$ and takes its values in the space of projectors of the Hilbert space $\ell^2(G)$. The Kesten spectral measure μ on the interval $[-1, 1]$ is defined by

$$\mu(B) = \langle E(B)\delta_{\mathrm{Id}}, \delta_{\mathrm{Id}} \rangle,$$

where B is a Borel subset of $[-1, 1]$ and $\delta_{\mathrm{Id}} \in \ell^2(G)$ is a function equal to 1 for the identity element and 0 elsewhere.

For a closed and G-invariant subspace H of $\ell^2(G)$ we define its von Neumann dimension $\dim(H)$ as

$$\dim(H) = \langle \Pr(H)\delta_{\mathrm{Id}}, \delta_{\mathrm{Id}} \rangle,$$

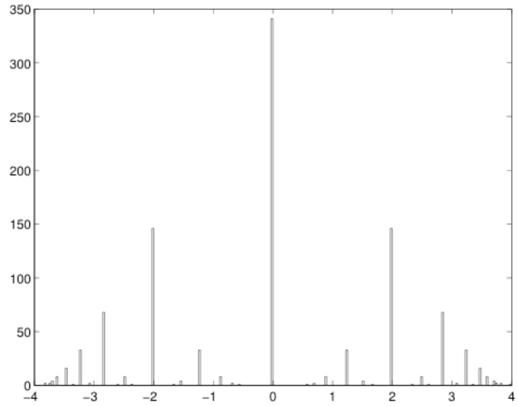

Figure 3: The histograme of the spectrum of $a_n + a_n^{-1} + b_n + b_n^{-1}$ for $n = 10$

where $\Pr(H)$ is a projection of $\ell^2(G)$ on H.

For the lamplighter group we can compute this measure [14]:

Theorem 2.1. *Let G be the group defined be the automaton from Figure 2, with generators a et b. The random walk operator A on $\ell^2(G)$ has the following eigenvalues:*

$$\cos\left(\frac{l}{q}\pi\right)$$

where $q = 2, 3, 4, \ldots$ and $l = 1, \ldots, q-1$.

The von Neumann dimension of the corresponding eigenspace is

$$\dim\left(\ker\big(A - \cos(\frac{l}{q}\pi)\big)\right) = \frac{1}{2^q - 1}$$

where $(l, q) = 1$.

In order to prove this theorem we use finite dimensional approximations π_n described above.

The computation of the spectral measure has several applications to random walks. In the following section we present an application of this computation to a problem of Atiyah about L^2 Betti numbers of closed manifolds.

2.3. A question of Atiyah

In 1976 Atiyah [3] defined L^2 Betti numbers for closed manifolds. He ended his paper with a question about the values of these numbers. Later this question gave rise to so called Atiyah conjecture.

For a group Γ we denote $\mathrm{fin}^{-1}(\Gamma)$ the subgroup of $\mathbb{Q}$ generated by the inverses of the orders of finite subgroups of Γ. For a closed manifold M we denote by $b_i^{(2)}(M)$ its i-th L^2 Betti number.

Conjecture. — *Let M be a closed manifold whose fundamental group $\pi_1(M)$ is isomorphic to Γ. Then*

$$b_i^{(2)}(M) \in \mathrm{fin}^{-1}(\Gamma)$$

for every integer i.

There are several texts presenting results about this conjecture, the most recent one is the book by Lück [21]. Many results confirm different forms of the Atiyah conjecture. However we show that the strong version mentioned above is false [17].

Theorem 2.2. *Let G be the group given by the presentation*

$$G = \langle a, t, s \mid a^2 = 1, [t, s] = 1, [t^{-1}at, a] = 1, s^{-1}as = at^{-1}at \rangle.$$

Every finite subgroup of G is an abelian 2-group, in particular the order of any finite subgroup of G is a power of 2. There exists a closed Riemannian manifold (M, g) of dimension 7 such that $\pi_1(M) = G$ for which the third L^2 Betti number is equal to

$$b_3^{(2)}(M, g) = \frac{1}{3}.$$

The proof of Theorem 2.2 relies on the results explained earlier concerning the spectral mesure of the random walk operator A on the lamplighter group, for which G is an HNN extension. The results imply

$$\dim\bigl(\ker(A)\bigr) = \frac{1}{3},$$

but the denominator 3 does not divide the powers of 2, which are the orders of finite subgroups of the lamplighter group.

The Atiyah conjecture can be equivalently formulated in terms of the dimension of the proper subspaces of the operators in $\mathbb{Z}[G]$ acting on $\ell^2(G)$ where $G = \pi_1(M)$. If G is a finitely presented group and A a random walk operator on G, there is a construction of a closed manifold M with a fundamental group G and such that the third L^2 Betti number of M is equal to the von Neumann dimension of the kernel of the operator A.

The lamplighter group is not finitely presented. However its admits a recursive presentation and therefore is a subgroup of a finitely presented group.

3. THE ALESHIN GROUP

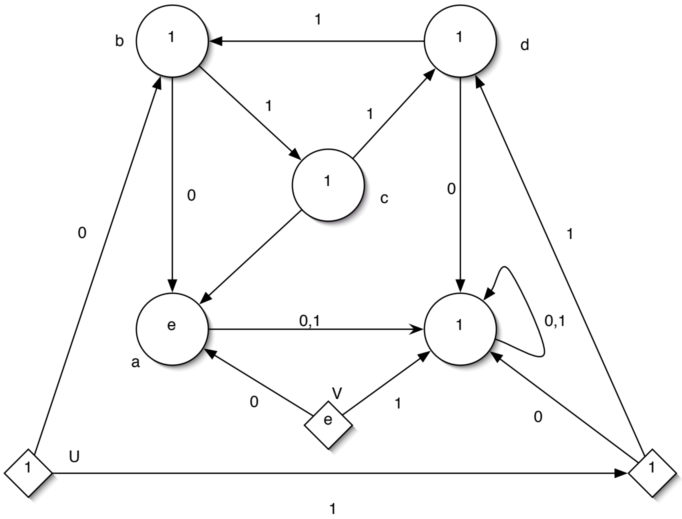

Figure 4: The Aleshin automaton from 1972

Let us consider the finite inversible automaton from Figure 4. The Aleshin group [1] is the group G generated by U and V.

Its study enabled one to give a particularly simple answer to the Burnside problem and to solve a problem of Milnor.

3.1. An answer to the Burnside problem

In 1902 Burnside asked if there is an infinite finitely generated group such that all elements are of finite order. The most important result concerning the existence of such groups is the theorem of Adyan-Novikov [27]. The Aleshin group gives a very simple answer even if, unlike the groups of Adyan-Novikov, the orders of its elements are not uniformely bounded. Aleshin [1] proved:

Theorem 3.1. *The group generated by U and V is torsion and infinite.*

The original proof enables one to construct a non-countable quantity of infinite p-groups for every prime number p.

The Aleshin group is by definition of finite type.

Let G be the group generated by the states a, b, c and d of the automaton from Figure 4. It is easy to see that this group is commensurable to the group generated by the states U and V and to the group generated by all states of the automaton (i.e. these groups have finite index subgroups which are isomorphic).

Proposition 3.2. *The group G is infinite.*

Proof. Consider $\mathrm{St}_G(1)$ which is of index 2 in G. Its projection on the first coordinates contains a, b, c and d. Thus $\mathrm{St}_G(1)$ surjects onto G which show that G is infinite.

Lemma 3.3. *The group generated by b, c and d is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. It is a simple verfication.

For $g \in G$ we have $g^8 \in \mathrm{St}_G(3)$.

Proposition 3.4. *For g^8 consider its image in G^8. Denote $g^8 = (g_1, \dots, g_8)$. Then*

$$|g_i| < |g|, \tag{3.2}$$

for the length with respect to generators a, b, c and d.

Proof. This follows from the same estimates as in the proof about the growth (see proof of Lemma 3.6).

One shows by induction on the length that every element is of finite order.

3.2. Growth

For a group G generated by a finite set S which we suppose to be symmetric (i.e. such that $S = S^{-1}$), we denote $|g|_S$ the minimal number number of generates needed to represent g. The growth of the group G describes the asymptotic behavior of the function

$$b_G(n) = |\{g \in G : |g|_S \leq n\}|.$$

This type of growth is independent of the generating set. For instance for nilpotent groups it is polynomial and for a group which contains a subgroup or even semigroup which free the growth is exponential. For the history of this notion see [18]. In [24] it was asked it there are other types of growth. We present a solution given in [13].

Proposition 3.5. *The group G is not of polynomial growth.*

Proof. A polynomial growth group contains a finite index subgroup which is nilpotent (Gromov) and contains a finite index subgroup which is torsion free (Malcev). However Aleshin's theorem show that G is infinite and torsion. One can also prove this proposition by a simple calculation.

Let us show that the Aleshin group has sub-exponential growth.
We have relations:

$$a = (1, 1)\mathsf{e}$$

$$\begin{array}{rcl} b & = & (a,c) \\ c & = & (a,d) \\ d & = & (1,b) \end{array} \tag{3.3}$$

and also

$$\begin{array}{rcl} aba & = & (c,a) \\ aca & = & (d,a) \\ ada & = & (b,1). \end{array} \tag{3.4}$$

Let $\Gamma = \mathrm{St}_G(3)$. Then $[G:\Gamma] < \infty$.

Lemma 3.6. *For $g \in \Gamma$ consider its image in G^8. Denote $g = (g_1, \ldots, g_8)$. Then*

$$\sum_{i=1}^{8} |g_i| \leq \frac{3}{4}|g| + 8, \tag{3.5}$$

for the length with respect to generators a, b, c and d.

Proof. As a is of order 2 and b, c, d are elements of the Klein group, every element $g \in G$ can be written as:

$$g = \overline{a}k_1ak_2a \ldots ak_n\overline{a},$$

where $k_i \in \{b, c, d\}$ and the first and last a, denoted by $\overline{a}$, does not appear necessarly.

Consider the bloc $\gamma = k_iak_{i+1}a$. Thanks to relations (3.3) et (3.4) its image $\gamma = (\gamma_1, \gamma_2)$ in $G \times G$ verifies

$$|\gamma_1| + |\gamma_2| \leq |\gamma|.$$

If k_i or k_{i+1} is equal to d, this inequality becomes

$$|\gamma_1| + |\gamma_2| \leq \frac{3}{4}|\gamma|. \tag{3.6}$$

The relations (3.3) and (3.4) show that the image of k_i or ak_ia in $G \times G$ gives c if $k_i = b$ and gives d if $k_i = c$. Thus if we iterate this procedure 3 times we are sure to be in the situation (3.6). Therefore we have an inequality (3.5) (the term 8 is due to the fact that $|g|$ is not necessarily divisible by 8).

Proposition 3.7. *The Aleshin group has sub-exponential growth.*

Proof. The inequality (3.5) shows that

$$|b_\Gamma(k)| \leq \sum_{k_1+\ldots+k_8 \leq \frac{3}{4}k+8} |b_G(k_1)| \times \ldots \times |b_G(k_8)|. \tag{3.7}$$

It is important to compute the length with respect to a, b, c and d even if a does not belong to Γ. As Γ is of finite index in G, we have

$$\lim_{n\to\infty} \sqrt[n]{|b_G(n)|} = \lim_{n\to\infty} \sqrt[n]{|b_\Gamma(n)|} = \alpha. \tag{3.8}$$

For every $\varepsilon > 0$ there exists $c > 0$ such that for n sufficiently large, we get

$$|b_G(n)| \leq c(\alpha + \varepsilon)^n.$$

The majoration (3.7) ensures there is c' such that

$$|b_\Gamma(n)| \leq c' n^8 (\alpha + \varepsilon)^{\frac{3}{4}n+8}.$$

Thus $\lim_{n\to\infty} \sqrt[n]{|b_\Gamma(n)|} \leq \alpha^{\frac{3}{4}}$ which together with (3.8), implies

$$\lim_{n\to\infty} \sqrt[n]{|b_G(n)|} = 1.$$

Therefor the Aleshin group has subexponential growth.

The precise behavior of the growth function for the Aleshin group is unknown. For the best estimates see [5].

4. GROUP GENERATED BY AN AUTOMATON ON THREE STATES

We are interested in the group generated by the automaton from Figure 5 introduced in [15]. This group appears also as the Galois group of iteration of the polynomial $x^2 - 1$ over finite fields (Pink) and as a monodromy group of the ramified covering of the Riemann sphere given by the polynomial $z^2 - 1$ (see [25]).

One of the remarkable properties of this group is related to the amenability.

4.1. Algebraic properties of G

Theorem 4.1 ([15]). *Let G be the group generated by the automaton from Figure 5.*

The group G has following properties:

a) *it is fractal;*

b) *it is regularly weakly branch over G';*

c) *it has no torsion;*

d) *the semi-group generated by a and b is free;*

e) *it admits a presentation:*

$$G = \langle a, b | \sigma^\varepsilon(\theta^m([a, a^b])) = 1, m = 0, 1, \ldots, \varepsilon = 0, 1\rangle,$$

where

$$\sigma : \begin{cases} a & \mapsto & b^2 \\ b & \mapsto & a \end{cases} \qquad \theta : \begin{cases} a & \mapsto & a^{b^2+1} \\ b & \mapsto & b. \end{cases}$$

We present here proofs of some algebraic properties of G mentioned in Theorem 4.1.

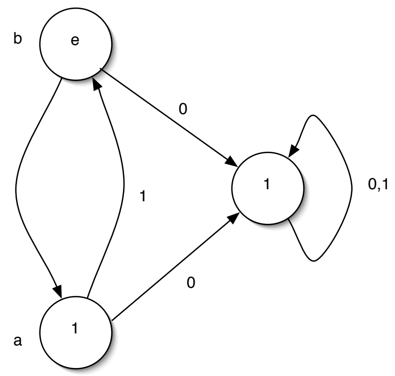

Figure 5: An automaton on three states

For $G = \langle a, b \rangle$, we have relations $a = (1, b)$ and $b = (1, a)\mathsf{e}$.

Proposition 4.2. *The groupe G is fractal.*

Proof. We have

$$\mathrm{St}_G(1) = \langle a, a^b, b^2 \rangle.$$

But

$$\begin{array}{rcl} a & = & (1, b) \\ a^b & = & \mathsf{e}(1, a^{-1})(1, b)(1, a)\mathsf{e} = (b^a, 1) \\ b^2 & = & (a, a), \end{array} \tag{4.9}$$

and any of the images of two projections of $\mathrm{St}_G(1)$ is G, i.e. G is fractal.

Proposition 4.3. *The group G is regularly weakly branch over G', i.e.*

$$G' \geq G' \times G'.$$

Proof. Indeed as

$$[a, b^2] = (1, [b, a]),$$

using fractalness of G, we get $G' \geq \langle [a, b^2] \rangle^G \geq 1 \times \langle [b, a] \rangle^G = 1 \times G'$ et $(1 \times G')^b = G' \times 1$. Thus G' contains $G' \times G'$ and as $G' \neq 1$ the group G is regularly weakly branch over G'.

Lemma 4.4. *The semi-group generated by a et b is free.*

Proof. Consider two different words $U(a, b)$ et $V(a, b)$ which represent the same element and such that $\rho = \max\{|U|, |V|\}$ is minimal. A direct verification shows that ρ cannot be neither 0 nor 1.

Suppose that $|U|_b$, the number of occurrences of b in U, is even (and thus $|V|_b$ as well). If this is not the case we can consider the words bU and bV thus increasing ρ by 1.

Now U and V are products of

$$a^m = (1, b^m)$$

and

$$ba^m b = (1, a)\mathsf{e}(1, b^m)(1, a)\mathsf{e} = (b^m a, a).$$

If one of these words has no b, say $U = a^m$, after projecting U and V on the forst coordinate we get $1 = V_0$ where the projection V_0 is a non-empty word verifying $|V_0| < |V| \leq \rho$. This contradicts the minimality of U and V.

We consider now the situtation where b appears in both words at least two times. If the number of ocurences of b in U and V where one, then by minimality they have to be equal to ba^n and $a^m b$. But $ba^n = (b^n, a)\mathsf{e}$ et $a^m b = (1, b^m a)\mathsf{e}$ which shows that these words are different and represent different elements.

Thus two words contain two b and $|U|, |V| \leq \rho$, where one of them contains at least four b and $|U|, |V| \leq \rho + 1$.

If we consider projections of U and V on the second coordinate, we get two different words (because of the minimality of U and V, they have to end with different letters) and of lengths which are shorter. This contradicts the minimality of ρ.

Lemma 4.5. *We have the following relation:*

$$\gamma_3(G) = (\gamma_3(G) \times \gamma_3(G)) \rtimes \langle [[a, b], b] \rangle$$

where $\gamma_3(G) = [[G, G], G]$.

Proof. We start with the relations

$$\gamma_3(G) = \langle [[a, b], a], [[a, b], b] \rangle^G,$$

$$[[a, b], a] = [(b^a, b^{-1}), (1, b)] = 1,$$

$$[[a, b], b] = (b^{-a}, b)\mathsf{e}(1, a^{-1})(b^a, b^{-1})(1, a)\mathsf{e} = (b^{-a}, b)(b^{-a}, b^a) = (b^{-2a}, bb^a). \tag{4.10}$$

The first two imply

$$\gamma_3(G) = \langle [[a, b], b] \rangle^G.$$

Thanks to the relation

$$[a, b^2] = (1, [a, b])$$

we have

$$[[a, b^2], a] = [(1, [a, b]), (1, b)] = (1, [[a, b], b]).$$

Let $\xi = [[a, b], b]$. Direct computations show that $\xi^a, \xi^{a^{-1}}, \xi^b, \xi^{b^{-1}} \in \langle \xi \rangle$ mod $\gamma_3(G) \times \gamma_3(G)$ and $\langle \xi \rangle \cap (\gamma_3(G) \times \gamma_3(G)) = 1$ because of (4.10) and $(bb^a)^n \in G'$ if and only if $n = 0$ et $\gamma_3(G) \leq G'$.

Lemma 4.6. *We have the following relation:*

$$G'' = \gamma_3(G) \times \gamma_3(G).$$

PROOF. Let $f = (1, c) \in G$ where $c = [a, b]$. For $d = (b, b^{-1}) \in G'$ we get

$$[f, d^{-1}] = [(1, [a, b]), (b^{-1}, b)] = (1, [[a, b], b]) \in G''.$$

This implies that $G'' \supseteq 1 \times \gamma_3(G)$and thus $G'' \supseteq \gamma_3(G) \times \gamma_3(G)$. As $G'' \subseteq \gamma_3(G)$ according to Lemma 4.5 it is enough to show $\langle \xi \rangle \cap G'' = 1$.

One can easily show

$$G' = (G' \times G') \rtimes \langle c \rangle. \tag{4.11}$$

Using the relation (4.10) and the relation (4.11) we have

$$G'' = \langle [c, f] \rangle^G.$$

But

$$[c, f] = [(b^a, b^{-1}), (1, c)] = [1, [b^{-1}, [a, b]]] \in 1 \times \gamma_3(G).$$

This ends the proof.

Here is another general property of regular weakly branched groups which is easy to prove.

Proposition 4.7. *Let G be regular weakly branched group over K. Then for every normal subgroup $N \lhd G$ there exists n such that*

$$K'_n < N$$

where $K_n = K \times \ldots \times K$ (direct product of d^n factors, each one acting on the corresonding subtree).

4.2. Amenability

In 1929 von Neumann [26] defined the notion of amenability which became fundamental.

Definition 4.8. The group G is amenable if there is a measure μ defined on all subsets of G such that

- $\mu(G) = 1$;
- $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subset G$;
- $\mu(gA) = \mu(A)$ for every $g \in G$ and every $A \subset G$.

It follows form the work of von Neumann [26] that the groups of subexponential growth are amenable and that this class is closed under following elementary operations: extensions, quotients, subgroups and direct limits.

Before the construction of the group generated by an automaton from Figure 5, all amenable groups could be obtained from groups of subexponential growth using elementary operations described above. For the history of different conjectures concerning the class of amenable groups see [16], and first reference is the paper of Day [6].

Let SG_0 be the class of groups such that all finitely generated subgroups are of subexponential growth. Suppose that $\alpha > 0$ is an ordinal and we have defined SG_β for any ordinal $\beta < \alpha$. Now if α is a limit ordinal let

$$SG_\alpha = \bigcup_{\beta<\alpha} SG_\beta.$$

If α is not a limit ordinal let SG_α be the classe of groups which can be obtained from groups in $SG_{\alpha-1}$ using either extenstions or direct limits. Let

$$SG = \bigcup_\alpha SG_\alpha.$$

The groups in this class are called subexponentially amenable.

SG is the smallest class of groups which contains groups of subexponential growth and is closed under elementary operations. The classes SG_α are closed under taking subgroups and quotients.

Proposition 4.9 ([15]). *The group G is not subexponentially amenable, i.e. $G \notin SG$.*

Proof. We start with the following lemmas:

Lemma 4.10. *We have the relation*

$$\psi_1(\gamma_3(G)) = \langle \gamma_3(G), b^{2a} \rangle.$$

Proof. It is a consequence of Lemma 4.5 and the relation (4.10).

Lemma 4.11. *We have*

$$\psi_1(\langle \gamma_3(G), b^{2a} \rangle) = \langle \gamma_3(G), b^{2a}, a \rangle.$$

Proof. It is a consequence of the preceding lemma and the relation $b^{2a} = (a, a^b)$.

Lemma 4.12. *For the projection on the second coordinate we have:*

$$\psi_2(\langle \gamma_3(G), b^{2a}, a \rangle) = G.$$

Proof. It follows from Lemma 4.5 and the relations $b^{2a} = (a, a^b)$ et $a = (1, b)$.

We can now prove Proposition 4.9. Suppose that $G \in SG_\alpha$ for α minimal. Then α cannot be 0 as G has exponential growth (the semi-group generated by a et b is free according to Lemma 4.4). Moreover α is not a limit ordinal as if $G \in SG_\alpha$ for a limit ordinal then $G \in SG_\beta$ for an ordinal $\beta < \alpha$. Also G is not a direct limit (of an increasing sequence of groups) as it is finitely generated. Thus there exist N, $H \in SG_{\alpha-1}$ such that the following sequence is exact:

$$1 \to N \to G \to H \to 1.$$

Thanks to Proposition 4.7 there exists n such that $N > (\mathrm{Rist}_G(n))' \geq G'' \times \ldots \times G''$ (2^n fois). So $G'' \in SG_{\alpha-1}$ and then $\gamma_3(G) \in SG_{\alpha-1}$ according to Lemma 4.6. Each class SG_α is closed with respect to quotients and subgroups. From Lemmas 4.10, 4.11, 4.12 we deduce that $G \in SG_{\alpha-1}$. Contradiction.

To show amenablity of G one uses a criterion of Kesten [22] concerning random walks on G.

Let μ be a symmetric probability measure supported on the generating set S of G, i.e. $G = \langle S \rangle$, $\mu(s) = \mu(s^{-1})$ for every $s \in S$ and $\mu(S) = 1$.

Let p_n be the probability of return to the identity after n steps of the random walk given by μ, i.e.

$$p_n(\mathrm{Id}, \mathrm{Id}) = \mu^{*n}(\mathrm{Id})$$

whereμ^{*n} is the n-th power of the convolution of μ on G.

Theorem 4.13 (Kesten [22]). *The group G is amenable if and only if*

$$\lim_{n\to\infty} \sqrt[2n]{p_{2n}(\mathrm{Id}, \mathrm{Id})} = 1.$$

Amenablity of G was proven by Virag [31]. This proof was published in [4].

On G we consider the random walk Z_n given by a symmetric measure μ on $S = \{a, a^{-1}, b, b^{-1}\}$ with weights $\{1, 1, r, r\}$, i.e. $\mu(a^{-1}) = \mu(a) = \frac{1}{2r+2}$, $\mu(b^{-1}) = \mu(b) = \frac{r}{2r+2}$.

The image Z_n by the embedding of G in $G \wr S_2$ is denoted:

$$Z_n = (X_n, Y_n)\varepsilon_n$$

where $X_n, Y_n \in G$ and $\varepsilon_n \in S_2$.

We define the stopping times σ and τ:

$$\begin{aligned}
\sigma(0) &= 0 \\
\sigma(m+1) &= \min\{n > \sigma(m) : \varepsilon_n = 1, X_n \neq X_{\sigma(m)}\} \\
\tau(0) &= \min\{n > 0 : \varepsilon_n = \mathsf{e}\} \\
\tau(m+1) &= \min\{n > \tau(m) : \varepsilon_n = \mathsf{e}, Y_n \neq Y_{\tau(m)}\}
\end{aligned}$$

A simple computation shows:

Lemma 4.14. *$X_{\sigma(m)}$ and $Y_{\tau(m)}$ are simple random walks on G according to the distribution $\mu'(a^{-1}) = \mu'(a) = \frac{r}{2r+4}$, $\mu'(b^{-1}) = \mu'(b) = \frac{1}{r+2}$.*

We remark that for $r = \sqrt{2}$ we get the same distribution on Z_n, $X_{\sigma(n)}$ and $Y_{\tau(n)}$.

One can also verify

Lemma 4.15. *Almost surely*

$$\lim_{m\to\infty} \frac{m}{\sigma(m)} = \lim_{m\to\infty} \frac{m}{\tau(m)} = \frac{2+r}{4+4r} < \frac{1}{2}.$$

To conclude we need to modify the distance on G, in order to control the norm of Z_n by the norms of X_n and Y_n.

Let T_n be a finite subtree of n level of T on which acts G. For $g \in G$ we define $||| \cdot |||_{T_n}$ by:

$$||| \, g \, |||_{T_n} = \sum_{\gamma \in \partial T_n} \big(|g_{|_\gamma}| + 1\big) - 1.$$

Finally we define the distance $||| \ \ |||$ on G :

$$||| \, g \, ||| = \min_n ||| \, g \, |||_{T_n}.$$

One checks that for $g = (g_0, g_1)\mathsf{e}^{0,1}$

$$|||g_0||| + |||g_1||| \le |||g||| \le |||g_0||| + |||g_1||| + 1$$

and that the growth with respect to the metric $||| \cdot |||$ is at most exponential, i.e. there exists $a > 1$ such that

$$|\{g : |||g||| \le n\}| \le a^n. \tag{4.12}$$

We have

Proposition 4.16. *Almost surely*

$$\lim_{n\to\infty} \frac{|||Z_n|||}{n} = 0.$$

Proof. The existance of the limit which we denote by s is a consequence of Kingman's ergodic theorem. Now

$$\frac{|||Z_n|||}{n} \le \frac{|||X_n|||}{n} + \frac{|||Y_n|||}{n} + \frac{1}{n}.$$

But

$$\lim_{n\to\infty} \frac{|||X_n|||}{n} = \lim_{n\to\infty} \frac{|||X_{\sigma(n)}|||}{\sigma(n)} = \lim_{n\to\infty} \frac{|||X_{\sigma(n)}|||}{n} \lim_{n\to\infty} \frac{n}{\sigma(n)}$$

and similarly for Y_n. So for $r = \sqrt{2}$ if $s > 0$, by Lemma (4.15), $s < s\frac{1}{2} + s\frac{1}{2} = s$. This contradiction implies that $s = 0$.

Proposition 4.17. *The probability $p(Z_{2n} = \mathrm{Id})$ does not decay exponentially.*

Proof. For every $\varepsilon > 0$, we have

$$p(|||Z_{2n}||| \le \varepsilon n) = \sum_{g \in G, |||g||| \le \varepsilon n} p(Z_{2n} = g) \le p(Z_{2n} = \mathrm{Id}) \times |\{g \in G; |||g||| \le \varepsilon n\}|.$$

Thus according to (4.12)

$$p(Z_{2n} = \mathrm{Id}) \ge p\left(\frac{|||Z_n|||}{n} < \varepsilon\right) \cdot a^{-\varepsilon n}.$$

Following Proposition 4.16 and Kesten's criterion the group generated by the automaton from Figure 5 is amennable.

Using HNN extentions of the group G, one can construct amenable finitely presented groups which are subexponentially amenable. In [16] we show that the group

$$\widetilde{G} = \langle b, t | [b^{tb}, b^t] = 1, b^{t^2} = b^2 \rangle$$

has these properties.

5. WILSON GROUP

We prensent a group constructed by Wilson to solve a problem of Gromov. To define it we use the language of wreath products (see Section 1.2).

5.1. Problem of Gromov

For groups of exponential growth, the growth function depends on the generating set. It is natural to ask if one can define an invariant independent of the generating set.

More precisely for a group G generated by a finite set S one defines

$$h(G, S) = \lim_{n \to \infty} \sqrt[n]{|\{g \in G : |g|_S \leq n\}|}.$$

The entropy of the group G is then

$$h(G) = \inf_{S; \langle S \rangle = G} h(G, S).$$

In 1981, Gromov [18] asked if for every G of exponential growth

$$h(G) > 1,$$

i.e. wether it has uniform exponential growth which means that there exists $a > 1$ such that for every generating set

$$|\{g \in G : |g|_S \leq n\}| \geq a^n.$$

The answer is positive for several classes of groups like hyperbolic or finitely generated linear [8].

The first group without uniform exponential growth was constructed by Wilson in 2003 [32].

5.2. Construction of Wilson

Let us consider A_{31} the alternating subgroup of the symmetric group on 31 elements.

Theorem 5.1. *Let H be a perfect group of finite type which satisfies the property $H \simeq H \wr A_{31}$. Then there exists a sequence (x_n) of elements of order 2 and a sequence (y_n) of elements of order 3 such that*

1. $\langle x_n, y_n \rangle = H$ *for every* n ;
2. $\lim_{n\to\infty} h(H, \{x_n, y_n\}) = 1$.

Construction of H

Let T_{31} be a rooted tree of degree 31. Let $x \in \mathrm{Aut}(T_{31})$ be such that it acts nontrivially only on the first level. We define $\overline{x} \in Aut(T_{31})$ by its image in the wreath product

$$\overline{x} = (x, \overline{x}, \mathrm{Id}, \ldots, \mathrm{Id}).$$

Finally let

$$H = \langle x, \overline{x} | x \in A_{31} \rangle.$$

The group H is of finite type and H is perfect as A_{31} is.

Proposition 5.2. *We have*

$$H \simeq H \wr A_{31}.$$

Proof. Let $\sigma = (2,3,4)$, $\rho = (1,3,2) \in A_{31}$ and consider $x, y \in A_{31}$. Then $[\overline{x}, \sigma\overline{y}] = ([x,y], \mathrm{Id}, \ldots, \mathrm{Id})$. As H is perfect this shows that for every $x \in A_{31}$ we have $(x, \mathrm{Id}, \ldots, \mathrm{Id}) \in H$. Then $\rho(x, \mathrm{Id}, \ldots, \mathrm{Id})^{-1}\overline{x} = (\overline{x}, \mathrm{Id}, \ldots, \mathrm{Id})$. Thus H contains $\{(h, \mathrm{Id}, \ldots, \mathrm{Id}) | h \in H\}$ and using $x \in A_{31}$ we get $H \wr A_{31} \subseteq H$.

Now we explain what are the properties of the group A_{31} which we need.

Proposition 5.3. *The group A_{31} can be generated by an element of order 2 and an element of order 3.*

As $H \simeq H \wr A_{31}$ and H is perfect this implies that there exists $u, v \in H$ such that $u^2 = v^3 = \mathrm{Id}$ and $H = \langle u, v \rangle$.

Proposition 5.4. *Let $H \simeq H \wr A_{31}$ be a perfect group generated by u and v such that $u^2 = v^3 = id$. Then there exist $x, y \in A_{31}$ such that*

- *there exists* $\alpha, \beta \in \{1, \ldots, 31\}$, $\alpha \neq \beta$

$$\begin{aligned} x(\alpha) &= x^y(\alpha) = \alpha \\ y(\beta) &= \beta \end{aligned}$$

- *the elements*

$$\begin{aligned} \widehat{x} &= (\ldots, u, \ldots)x \\ \widehat{y} &= (\ldots, v, \ldots)y, \end{aligned}$$

where u is in position α and v in position β, satisfy $\widehat{x}^2 = \widehat{y}^3 = id$ and $\langle \widehat{x}, \widehat{y} \rangle = H$.

Proof. We easily verify this proposition with explicit x, y, α and β [32].

Now let

$$\begin{aligned}\gamma'(n) &= |\{w \in H : |w|_{\langle \widehat{x},\widehat{y}\rangle} \leq n\}|,\\ \gamma(n) &= |\{w \in H : |w|_{\langle u,v\rangle} \leq n\}|.\end{aligned}$$

Proposition 5.5. *If we denote* $\lim_{n\to\infty} \sqrt[n]{\gamma(n)} = c$ *and* $\lim_{n\to\infty} \sqrt[n]{\gamma'(n)} = c'$*, then for* $s \geq 3$ *we have*

$$c' \leq \max\left(c^{1-\frac{1}{2c}}, (1+2/s)(s+2)^{2/s}\right).$$

Proof. We start by explaining the second term. Consider $\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Let

$$\rho_n = \{w \in \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}; |w|_{\langle x,y\rangle} \leq n \text{ et } \mid \{xy^{-1}xy \in w\} \mid \leq [n/s]\}.$$

Then $\lim_{n\to\infty} \sqrt[n]{\rho_n} \leq (1+2/s)(s+2)^{2/s}$.

Now let

$$\begin{aligned}B(n) &= \{w \in \langle \widehat{x},\widehat{y}\rangle; |w| \leq n\}\\ B_+(n) &= \{w \in B(n); \mid \{\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y} \in w\} \mid \leq [n/s]\}\\ B_-(n) &= B(n) \setminus B_+(n).\end{aligned}$$

We get

$$\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y}\widehat{x} = (1,\ldots,1,v^{-1},\ldots,u,\ldots,v)xy^{-1}xyx$$

where v^{-1} is in position $xyx(\beta)$, u is in $yx(\alpha)$ and v in $x(\beta)$.

If $w \in B_+(n)$ then $\mid \{\widehat{x}\widehat{y}^{-1}\widehat{x}\widehat{y}\widehat{x}\} \mid$ is at least $\frac{1}{2}[n/s] = r$. Thus

$$|B_+(n)| \leq |A_{31}| \sum_{n_1+\ldots+n_{31}\leq n-2r} \prod_{j=1}^{31} \gamma(n_i) \leq K(n)(c+\varepsilon)^{n-2r} = K(n)(c+\varepsilon)^{n(1-1/2s)}$$

where $K(n)$ is a polynomial in n. We get an estimate from proposition.

The proof of the theorem is thus reduced to the following elementary lemma:

Lemma 5.6. *There exists a sequence* $s_n \to \infty$ *such that*

$$c_n \to 1$$

where $c_1 = 2$ *and* $c_n = \max\left(c_{n-1}^{1-\frac{1}{2s_n}}, (1+2/s_n)(s_n+2)^{2/s_n}\right)$ *for* $n \geq 2$.

Finally to show that H has exponential growth, one shows that it contains a free semi-group.

References

[1] S. V. ALESHIN, *Finite automata and the Burnside problem for periodic groups*, Mat. Zametki 11 (1972), 319–328.

[2] S. V. ALESHIN, V. B. KUDRYAVTSEV, A. S. PODKOLZIN, *Introduction to automata theory*, Nauka, Moscow, 1985.

[3] M. F. ATIYAH, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque Analyse et Topologie en l'honneur d'Henri Cartan (Orsay, 1974), pp. 43–72, Astérisque, No. 32-33, Soc. Math. France, Paris, 1976.

[4] L. BARTHOLDI, B. VIRAG, *Amenability via random walks*, Duke Math. J. 130 (2005), no. 1, 39–56.

[5] J. BRIEUSSEL, *On groups of intermediate growth*, to appear.

[6] M. M. DAY, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544.

[7] W. DICKS, T. SCHICK, *The spectral measure of certain elements of the complex group ring of a wreath product*, Geom. Dedicata 93 (2002), 121–137.

[8] A. ESKIN, S. MOZES, H. OH, *On uniform exponential growth for linear groups*, Invent. Math. Vol 160 (1), (2005), 1–30.

[9] J. FABRYKOWSKI, N. GUPTA, *On groups with sub-exponential growth functions*, J. Indian Math. Soc. (N.S.) 49 (1985), no. 3-4, 249–256.

[10] J. FABRYKOWSKI, N. GUPTA, *On groups with sub-exponential growth functions II*, J. Indian Math. Soc. (N.S.) 56 (1991), no. 1-4, 217–228.

[11] F. GECSEG, I. PEAK, *Algebraic theory of automata*, Disquisitiones Mathematicae Hungaricae, 2, Akadémiai Kiadó, Budapest, 1972.

[12] Y. GLASNER S. MOZES, *Automata and square complexes*, Geom. Dedicata 111 (2005), 43–64.

[13] R. I. GRIGORCHUK, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR-Izv. 25 (1985), no. 2, 259–300.

[14] R. I. GRIGORCHUK, A. ZUK, *The lamplighter group as a group generated by a 2-state automaton and its spectrum*, Geometriae Dedicata 87 (2001), 209–244.

[15] R. I. GRIGORCHUK, A. ZUK, *On a torsion-free weakly branch group defined by a three state automaton*, Internat. J. Algebra Comput., Vol. 12, No. 1 (2002), 1-24.

[16] R. I. GRIGORCHUK, A. ZUK, *On the spectrum of a torsion-free weakly branch group defined by a three state automaton*, in Computational and Statistical Group Theory (Ed. by R. Gilman, A. Myasnikov, V. Shpilrain), Contemp. Math., Amer. Math. Soc. 2002, vol. 298, 57–83.

[17] R. I. GRIGORCHUK, P. LINNELL, T. SCHICK, A. ZUK, *On a question of Atiyah*, C. R. Acad. Sci. Paris, t. 331, Série I, p. 663–668, 2000.

[18] M. GROMOV, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques, CEDIC, Paris, 1981.

[19] M. GROMOV, *Hyperbolic groups*, Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[20] N. GUPTA, S. SIDKI, *Some infinite p-groups*, Algebra i Logika 22 (1983), no. 5, 584–589.

[21] W. LÜCK, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 44, Springer-Verlag, Berlin, 2002.

[22] H. KESTEN, *Full Banach mean on countable groups*, Math. Scand. 7 (1959), 146–156.

[23] G. A. MARGULIS, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17, Springer-Verlag, Berlin, 1991.

[24] J. W. MILNOR, Amer. Math. Monthly 75 (1968), 685–686.

[25] V. NEKRASHEVYCH, *Self-similar groups*, Mathematical Surveys and Monographs, 117, American Mathematical Society, Providence, RI, 2005.

[26] J. von NEUMANN, *Zur allgemeinen Theorie des Masses*, Fund. Math., 13, 73–116 (1929).

[27] P. S. NOVIKOV, S. I. ADJAN, *Infinite periodic groups. I, II, III*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 212–244, 251–524, 709–731.

[28] J.-F. PLANCHAT, *Groupe d'Aleshin*, mémoire de DEA, Paris 7, 2005.

[29] J.-P. SERRE, *Arbres, amalgames,* SL_2, Astérisque, No. 46, Société Mathématique de France, Paris, 1977.

[30] V. I. SUSHCHANSKY, *Periodic p-groups of permutations and the unrestricted Burnside problem*, Dokl. Akad. Nauk SSSR 247 (1979), no. 3, 557–561.

[31] B. VIRAG, *Self-similar walk on a self-similar group*, preprint 2003.

[32] J. S. WILSON, *On exponential growth and uniformly exponential growth for groups*, Invent. Math. 155 (2004), no. 2, 287–303.

[33] A. ZUK, *Automata groups*, book to appear.