ABSTRACT ALGEBRA – ASSIGNMENT ON THE FIRST ISOMORPHISM THEOREM (AND SUCH)

(1) This is a long problem. Bear with me. The problem has many valuable points, all of its pieces connect, and they also connect with stuff we did earlier in the class.

  (a) Consider the polynomial $X^3 - 2$. Viewed as an element of $\mathbb{R}[X]$ it factors into a product of irreducibles. (It better – $\mathbb{R}[X]$ is a UFD!) Please factor it into irreducibles! (Yes, verify that the factors you come up with indeed are irreducible.) (And if you are really stuck, think that $X^3 - 2 = X^3 - (\sqrt[3]{2})^3$ and go see what The Internet says about factoring $a^3 - b^3$.)

  (b) Consider the same polynomial $X^3 - 2$ but now as an element of $\mathbb{Q}[X]$. Show that $X^3 - 2$ viewed as an element of $\mathbb{Q}[X]$ is in fact irreducible.

  (c) Is the ideal $(X^3 - 2)$ inside of $\mathbb{R}[X]$ prime? Is it maximal? (Did I mention the word PID recently?)

  (d) Is the ideal $(X^3 - 2)$ inside of $\mathbb{Q}[X]$ prime? Is it maximal?

  (e) Consider the mapping

$$F : \mathbb{R}[X] \to \mathbb{R} \text{ given by } \quad F(P) = P(\sqrt[3]{2}).$$

  It can easily be verified this is a homomorphism. What are its kernel and image? What does the First Isomorphism Theorem have to say about this situation?

  (f) Consider the mapping

$$F : \mathbb{Q}[X] \to \mathbb{R} \text{ given by } \quad F(P) = P(\sqrt[3]{2}).$$

  It can easily be verified this is a homomorphism. What are its kernel and image? What does the First Isomorphism Theorem have to say about this situation? (If you have hard time writing what the image is, I recommend thinking about $\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}] = \{a\sqrt[3]{4} + b\sqrt[3]{2} + c \mid a, b, c \in \mathbb{Q}\}$.)

  (g) Use the First Isomorphism Theorem to show that

$$\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}] = \{a\sqrt[3]{4} + b\sqrt[3]{2} + c \mid a, b, c \in \mathbb{Q}\}$$

  is a field.

  (h) Actually, in the technical sense of the word $\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}]$ is an algebra over $\mathbb{Q}$. What is its dimension?

  (i) So, supposedly $\sqrt[3]{4} - \sqrt[3]{2} - 1$, being a non-zero element of $\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}]$, has a multiplicative inverse of the form $a\sqrt[3]{4} + b\sqrt[3]{2} + c$. To find it, follow the steps given below:

  - Consider the polynomial $X^2 - X - 1$. What is the GCD of $X^2 - X - 1$ and $X^3 - 2$ as elements of $\mathbb{Q}[X]$?

- What does the GCD theorem tell you about this situation? Do you know of any polynomials $A(X)$ and $B(X)$ with
$$A(X) \cdot (X^2 - X - 1) + B(X) \cdot (X^3 - 2) = 1?$$
(Note: the solution has a small amount of fractions here for a reason. I want you to feel / see that there is an abstract algebra based algorithm involved and not just some ad hoc smarty-pants move that one kind of knows because one somehow / maybe had an ambitious algebra teacher in high school.)
- What is the multiplicative inverse of $[X^2 - X - 1]$ as an element of the quotient ring $\mathbb{Q}[X]/(X^3 - 2)$?
- What is the multiplicative inverse of $\sqrt[3]{4} - \sqrt[3]{2} - 1$ in $\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}]$?

(j) Explain in words how to go about finding inverses of non-zero elements of $\mathbb{Q}[\sqrt[3]{4}, \sqrt[3]{2}]$.

(2) The set of all continuous real-valued functions of real variable, equipped with the standard operations of addition and multiplication, forms a commutative ring with identity. In fact, it forms an algebra over real numbers, in the technical sense of the word. Let us denote this algebra by $\mathscr{F}$:
$$\mathscr{F} = \{\varphi : \mathbb{R} \to \mathbb{R} \,\big|\, \varphi \text{ is continuous}\}.$$
(a) What are the additive and the multiplicative identities here? What is the situation with additive and multiplicative inverses? Is $\mathscr{F}$ an integral domain?

(b) What is the dimension of $\mathscr{F}$ as a vector space over real numbers?

(c) Consider the mapping $F : \mathscr{F} \to \mathbb{R}$ given by $F(\varphi) = \varphi(1)$. Is this map a homomorphism? If so, what are its kernel and image? If so, what does the First Isomorphism Theorem have to say about this situation?

(d) Consider the set
$$\mathfrak{M} := \{\varphi \in \mathscr{F} \,\big|\, \varphi(1) = 0\}.$$
Show that:
- Show that $\mathfrak{M}$ is a *prime* ideal of $\mathscr{F}$.
- Show that $\mathfrak{M}$ is a *maximal* ideal of $\mathscr{F}$.

(3) Let R be a commutative ring with identity, and let $I \subseteq J$ be two of its ideals. Let
$$J/I := \{[r] \in R/I \,\big|\, r \in J\}.$$
(a) Show that J/I is an ideal of R/I.

(b) Use the First Isomorphism Theorem to show that the quotient ring $(R/I)/(J/I)$ is isomorphic to the quotient ring R/J:
$$(R/I)/(J/I) \cong R/J.$$