ABSTRACT ALGEBRA – THE MAIN HOMEWORK ASSIGNMENT ON EUCLIDEAN DOMAINS

The point of the first problem is to help you digest the idea of Euclidean Domains in general and to help you appreciate the extent to which the classic introductory results of number theory (and polynomial algebra) have very little to do with integers and polynomials per se. If you finish this assignment and still don't feel The Meta in all of this then ... I guess we need to talk!

The remaining problems are there so you can start playing with the concept of ideals in ring theory. Fuller story to come next week!

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(1) Suppose $(R, +, \cdot)$ is a Euclidean domain, as defined in class.
  (a) Define GCD of two (non-zero) elements of R, and prove the GCD Theorem.
  (b) Prove that irreducible elements of R are also prime.
  (c) Prove that factorizations into irreducibles of R, should they exist, are unique up to permutations and associates.

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(2) Let $(R, +, \cdot)$ be a commutative ring with identity.
  (a) Let $k \in R$ and let
  $$(k) = \{r \cdot k \,\big|\, r \in R\}.$$
  Show that (k) is an ideal of R.
  (b) Let $k_1, k_2 \in R$ and let
  $$(k_1, k_2) = \{r_1 \cdot k_1 + r_2 \cdot k_2 \,\big|\, r_1, r_2 \in R\}.$$
  Show that (k_1, k_2) is an ideal of R.

(3) Go over and present the argument from class which shows that every ideal in a Euclidean domain is principal.

(4) Show that an intersection of ideals in a commutative ring with identity is again an ideal of the ring.

(5) Provide a definition of GCD and LCM in terms of ideals.