

Practice Homework 17 – Assigned November 13th, due November 16th

**Note:** Remember that you must show your work to get full credit for a problem.

1. Consider the following false proof that  $\text{Card}(\mathbb{N}) \neq \text{Card}(\mathbb{Q})$ . (This is a revised version of the first problem on Practice HW #16.)

*Proof.* [FALSE]

We proceed by contradiction. Suppose there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{Q}$ .

Then we create a number  $z$  as follows. We will define  $z = .z_1z_2z_3\cdots$  where  $z_n$  is a single digit number and represents the  $n^{\text{th}}$  number after the decimal point.

We define  $z_n$  to be 1 if the  $n^{\text{th}}$  decimal place of  $f(n)$  is not 1, and define  $z_n$  to be 0 if the  $n^{\text{th}}$  decimal place of  $f(n)$  is 1.

Then by construction  $z$  is not in the image of  $f$  and thus  $f$  is not a bijection.

Thus no bijection exists from  $\mathbb{N}$  to  $\mathbb{Q}$ .

Thus  $\text{Card}(\mathbb{N}) \neq \text{Card}(\mathbb{Q})$ .

□

State the logical error(s) in the prove and explain why the proof fails.

2. Let  $f : A \rightarrow B$  be a function.

- (a) Prove that for all  $C \subseteq B$ , we have  $f(f^{-1}(C)) \subseteq C$ .
- (b) Find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a set  $C \subseteq \mathbb{R}$  such that  $f(f^{-1}(C)) \neq C$ .
- (c) Prove that for all  $C \subseteq A$ , we have  $C \subseteq f^{-1}(f(C))$ .
- (d) Find a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a set  $C \subseteq \mathbb{R}$  such that  $C \neq f^{-1}(f(C))$ .