QUIZ STUDY GUIDE

ARDEN RASMUSSEN

1. Terminology

(1) State the commutative and the associative rules for an operation. State the distributive rule.

Commutative Rule:

$$a \star b = b \star a$$

Associative Rule:

$$(a \star b) \star c = a \star (b \star c)$$

Distributive Rule:

$$(a \star b) \circ c = (a \circ c) \star (b \circ c)$$
$$a \circ (b \star c) = (a \circ b) \star (a \circ c)$$

(2) What do we mean when we say that...

(a) Something is an "identity" with respect to an operation? E.g. what do we mean by an "additive identity"? Or by a "multiplicative identity"?

Something is an identity if $a \star \mathrm{Id} = a$, and $\mathrm{Id} \star a = a$. That operation applied to the identity and any other value is equal to that other value.

(b) An element is "invertible with respect to an operation"? What do we mean by a "multiplicative inverse"?

An element is invertible with respect to an operation if there exists an "identity" for that operation, and there exists a value such that $a \star a^{-1} = \mathrm{Id}$, and $a^{-1} \star a = \mathrm{Id}$.

(c) $(G, \star)$ constitutes a group?

- $\star$ is associative.
- $\star$ has an identity.
- Every element of G has an inverse with respect to $\star$.
- $\star$ May or may not be commutative.

(d) $(R, \oplus, \otimes)$ constitutes a ring? A commutative ring? How about a field?

- $(R, \oplus)$ is a commutative group.
- $\otimes$ is associative.
- $\otimes$ distributes over $\oplus$.
- $\otimes$ may or may not be commutative.
- $\otimes$ may or may not have identity.
- Elements of R may or may not have an inverse.

(e) an element is a "unit" in a ring?

(f) an element is a "zero divisor" in a ring?

Date: September 11, 2019.

(g) something is an "integral domain"?

2. Examples

(1) Consider a collection $\mathcal{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$. Composition of functions is an operation in the set $\mathcal{F}$.

$$f, g \in \mathcal{F} \to f \circ g \in \mathcal{F}.$$

  (a) Which rules does this operation satisfy?
     $\circ$ satisfies the Associative rule.
  (b) Does it have an identity?
     Yes $\mathrm{Id}(x) = x$.
  (c) Does every element of $\mathcal{F}$ have its inverse?
     No, consider $f(x) = e^x$, the inverse would be $f^{-1}(x) = \ln(x)$. But $\ln : \mathbb{R} > 0 \mapsto \mathbb{R}$, thus $f^{-1}(x) \notin \mathcal{F}$.
  (d) Does $\mathcal{F}$ constitute a group? A ring? A field?

(2) Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices whose entries are real numbers.
  (a) Which rules do matrix addition and multiplication satisfy?
     Matrix addition satisfies commutative rule, and the Associative rule. Matrix multiplication satisfies Associative rule, and the distributive rule.
  (b) Does matrix addition have an identity? If so, what is it? Likewise is there a multiplicative identity? If so, what is it?
     The identity of matrix addition is the zero matrix, the multiplicative identity is the identity matrix.
  (c) Does every matrix in $M_n(\mathbb{R})$ have its additive inverse? For matrices with additive inverse explain how one would go about finding the supposed inverse. How about multiplicative inverse? Explain how one would go about finding it.
     Yes, every matrix has its additive inverse, to find it, multiply every element of the matrix by -1. Not all matrices have a multiplicative inverse.
  (d) Is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ a unit in $M_2(\mathbb{R})$? What if I changed the framework of the problem and considered $M_2(\mathbb{Z})$, the set of 2×2 matrices whose entries are integers, in in place of $M_2(\mathbb{R})$?
  (e) Is $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ a zero divisor in $M_2(\mathbb{R})$? How about in $M_2(\mathbb{Z})$? How about $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is it a zero division in $M_2(\mathbb{R})$? $M_2(\mathbb{Z})$?
  (f) Does $M_n(\mathbb{R})$ constitute a group? A ring? A field? An integral domain?

(3) Let $S \neq \emptyset$. Union $\cup$ and intersection $\cap$ are operation on the power set $\mathcal{P}(S)$.
  (a) Which rules do $\cup$ and $\cap$ satisfy?
  (b) Does $\cup$ have an identity? If so, what is it? Does $\cap$ have an identity? If so, what is it?
  (c) Does every $X \in \mathcal{P}(S)$ have an inverse with respect to $\cup$? If so, what is it? Does every $X \in \mathcal{P}(S)$ have an inverse with respect to $\cap$? If so, what is it?
  (d) Does $\mathcal{P}(S)$ constitute a group? A ring? A field? An integral domain?

(4) Consider the set

$$\mathbb{Z}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} | a, b \in \mathbb{Z}\right\}$$

(a) Is the regular addition of numbers an operation on $\mathbb{Z}\left[\sqrt{2}\right]$? I.e. is it true that

$$x, y \in \mathbb{Z}\left[\sqrt{2}\right] \to x + y \in \mathbb{Z}\left[\sqrt{2}\right]?$$

If so, address the situation with the identity and the inverses with respect to this operation on $\mathbb{Z}\left[\sqrt{2}\right]$. Specifically, describe the units in $\mathbb{Z}\left[\sqrt{2}\right]$.

(b) Is $\mathbb{Z}\left[\sqrt{2}\right]$ a ring? A field? An integral domain?

(c) What if instead we were to consider

$$\mathbb{Q}\left[\sqrt{2}\right] = \left\{a + b\sqrt{2} | a, b \in \mathbb{Q}\right\}?$$