Schedule for the Linear Algebra class

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Contents

What is this class about?
(Introduction to vector spaces)

The main goal for the first two weeks of class is to get a solid understanding of what the course is about. The secondary goal is to master the technique called Gaussian Elimination.

Week 1

Tuesday, 1/17: SNOW DAY!!!

Thursday, 1/19: In brief: what is Linear Algebra about? Some of the key words to pay attention to in this class are *coordinate system, vector, linear combination, basis, coordinates, dimension.* The examples from class are not included in your textbook, so make sure you take good notes.

Friday, 1/20: Gaussian Elimination method for solving systems of linear equations. Note: This is *the method* for solving linear equations that you will be required to use throughout the course.

Week 2

Monday, 1/23: Geometry behind solving linear systems of equations, primarily in three unknowns. This lecture explores the connection between the previous two lectures. It also serves as an introduction to the concepts of *subspace, basis, dimension.*

Tuesday, 1/24: Geometry behind solving linear systems of equations, primarily in four unknowns. This lecture is a follow up to Friday's lecture, but it raises the level of abstraction to *four dimensional spaces and its subspaces.*

Thursday, 1/26: Advanced Gaussian Elimination. Emphasis in this class will be on discussion of possible outcomes of Gaussian Elimination.

Friday, 1/27: Geometry behind solving linear systems of equations with an arbitrary number of unknowns. This lecture continues the overall theme of the Monday's class, but deals with yet higher dimensional spaces. A new concept which may be discussed is that of *linear independence.*

Homework due Monday 1/23/17

The purpose of the following assignment is to

- Practice Gaussian elimination.

Note: not using Gaussian elimination is defeating the purpose of this assignment.

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1. Solve the following systems.

   (a) $\begin{cases} x + 2y - 3z = 0 \\ 2x - 2y + 3z = 0 \\ x + y + z = 5 \end{cases}$

   (b) $\begin{cases} y - 2z = 1 \\ 3x + z = 4 \\ 2x + y - 5z = 0 \end{cases}$

2. Solve the following system and interpret its solution geometrically.

$$\begin{cases} 3x - y + z = 0 \\ 5x - y + 2z = 0 \end{cases}$$

3. For what values of λ does the set of solutions of the system

$$\begin{cases} x + y - 2z = 0 \\ x - y + z = 0 \\ 3x + y + \lambda z = 0 \end{cases}$$

   correspond to a line? Which line or lines are those?

Homework due Thursday 1/26/17

The purpose of the following assignment is to:

- Practice the use of words basis, coordinates, dimension and subspace.
- Think about the phrase "four dimensional space";
- Practice the linear algebra content covered so far in the context of four dimensional vector space $\mathbb{R}^4$.

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1. Consider the equation $x + 3y - 2z = 0$.

   (a) Solve this equation.

   (b) The set of solutions of this particular equation forms a subspace of our space. What kind of a subspace? Find its basis and dimension.

   (c) By inspection I can see that $x = 1$, $y = 3$ and $z = 5$ solve the equation
   $$x + 3y - 2z = 0.$$
   In other words, I can see that the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ lies in the subspace which you addressed above. What are the coordinates of the vector $\mathbf{v}$ with respect to the basis you found above?

2. Consider the equation $x - y + 3z = 0$. Geometrically, it describes a plane.

   (a) Solve this equation in two different ways. The first time keep y and z as free variables. The second time keep x and z as free variables.

   (b) Based on the work you have done, find two different bases of the plane
   $$x - y + 3z = 0.$$
   For future purposes call these two bases F and G.

   (c) By inspection I can see that $\mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ lies in this plane. Find the coordinates $[\mathbf{w}]_F$ of $\mathbf{w}$ with respect to F and find the coordinates $[\mathbf{w}]_G$ of $\mathbf{w}$ with respect to G.

   (d) The solutions of the system
   $$\begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases}$$
   correspond to a subspace of our space. What kind of subspace? Find its basis and dimension.

3. Consider the system
   $$\begin{cases} x - y + 2z + w = 0 \\ x - z - w = 0 \\ 3x - y - z - w = 0. \end{cases}$$
   The solutions of this system correspond to a subspace of the four dimensional space $\mathbb{R}^4$. What kind of subspace? What is its dimension? What is it spanned by?

Homework due Monday 1/30/17

The purpose of the following assignment is to:

- Practice the use of words "consistent", "homogeneous", "reduced row echelon form", "un- derdetermined" and "overdetermined" in the context of systems of equations.
- Think about the concepts of linear dependence and independence.

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1. (a) Give an example, other than the one done in class, of a system of equations which is not consistent.

   (b) What does the phrase "homogeneous linear system" refer to? Why is it that every homogeneous linear system of equations is consistent?

   (c) What does the terminology "underdetermined" and "overdetermined" linear system stand for? Cook up an example of a consistent underdetermined system and then solve the system. Cook up an example of a consistent overdetermined system, and then solve the system. Cook up an example of an inconsistent overdetermined system.

   (d) If the reduced row echelon form of a $m \times n$ homogeneous system of equations has r pivots, how many free variables does it have? How many dimensions does the space of solutions have?

2. (a) When somebody talks about an n-dimensional vector space, what are they referring to?

   (b) Give me an example of a line in a six-dimensional space.

   (c) Give me an example of a plane (two-dimensional subspace) of a five-dimensional space.

3. (a) Give an example of three vectors in $\mathbb{R}^3$ which are linearly dependent. Justify your claim that the vectors are linearly dependent.

   (b) Give an example of three vectors in $\mathbb{R}^3$ which are linearly independent. Justify your claim that the vectors are linearly independent.

4. Consider the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}.$$

   Is the vector $\vec{v}_3$ a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_2$? If so, find one non-trivial solution of the 4×3 system of equations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}.$$

5. Repeat the above with

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

6. Is the following set of vectors of $\mathbb{R}^4$ linearly dependent or independent. Justify your claim.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 8 \\ -8 \end{pmatrix}.$$

7. Find a basis and find the dimension of $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ inside of $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Review Questions: Weeks 1 and 2

1. What is a linear system of equations? Give several very different examples, or articulate the schematic form of a linear system in full generality. Also write down the meanings for the following phrases: consistent linear system, homogeneous linear system, underdetermined linear system, overdetermined linear system.

2. You learned a method for solving linear systems: Gaussian elimination. Outline – in your own words – what the steps involved in this method are and what the "final product" of Gaussian elimination is.

3. Include four examples of linear systems: a consistent system with only one solution, a consistent underdetermined system, a consistent overdetermined system, and an inconsistent system. For each of the 2×2 systems you choose to include also provide some pictorial context.

4. Depending on the number of pivots in the (reduced) row echelon form of a consistent linear system, discuss the number of free variables in the set of solutions.

5. What exactly is meant under *vector space* $\mathbb{R}^n$? What are vectors here? Can you make an analogy with our "every-day" space?

6. We constantly do certain operations on vectors. What operations are those? Write a general formula describing these operations, or at the very least show some examples of vector operations.

7. When we say that a vector v is a linear combination of vectors $f_1, f_2, ..., f_n$, what does that mean?

8. Finish the following sentences: "When one of the vectors $v_1, v_2, ..., v_n$ can be expressed as a linear combination of the remaining ones we say that the vectors $v_1, v_2, ..., v_n$ are _________. Otherwise, we say that they are _________."

9. In practice, how do we check if vectors $v_1, v_2, ..., v_n$ are linearly independent? Work out an explicit example where the vectors end up being linearly independent and an example where the vectors end up being linearly dependent.

10. Describe informally the concept of a subspace.

11. What does the phrase "span of vectors $v_1, v_2, ..., v_n$" mean? Can you exhibit an example of this concept?

12. What does the phrase "basis" refer to in the field of Linear Algebra? Illustrate this concept on several examples: standard basis for $\mathbb{R}^2$, standard basis for $\mathbb{R}^3$, standard basis for $\mathbb{R}^n$, a basis for a "slanted" plane of your choice in $\mathbb{R}^3$, a basis for a subspace of $\mathbb{R}^5$ coming from solutions of a homogeneous 2×5 linear system.

13. What does the word "dimension" mean in Linear Algebra? For each example you just brought up also emphasize the corresponding dimension.

14. What do we mean when we say "coordinates of a vector with respect to a basis"? What notation do we use in this context? Show the meaning of this concept on an example.

15. Planes passing through the origin of the standard three-dimensional space have equations of the form $ax + by + cz = 0$. Work out an example in which you are computationally determining the intersection of two such planes in space.

16. Work out an example in which you are computing the intersection of three four-dimensional subspaces of a five dimensional space.

17. Based on what you saw in this class during the first two weeks, what do you think is the subject of the field of Linear Algebra? What is the point?

Matrix Algebra and Determinants

By this stage of the course you will without a doubt become aware of the overwhelming extent to which rectangular arrays of numbers, also known as *matrices* appear in linear algebra. A number of techniques of linear algebra, including those pertaining to systems of linear equations, are best expressed in terms of matrices, a particular kind of algebra we do with them, and a particular numerical value (called the *determinant*) that can be associated to each square matrix.

Week 3

Monday, 1/30: Algebraic operations with matrices.

Tuesday, 1/31: Properties of matrix operations.

Thursday, 2/2: Practice. Comments will be made in regards to using "technology" for the purposes of matrix computation.

Friday, 2/3: The definition of the determinant. Take good notes!

Week 4

Monday, 2/6: Properties of determinants, with emphasis on row and column operations.

Tuesday, 2/7: Properties of determinants, with emphasis on invertibility of a matrix.

Thursday, 2/9: Summary: solvability of a linear system $A\vec{x} = \vec{b}$. By nature this will be a synthesis section. Take good notes!

Friday, 2/10: Another application of determinants to solving linear systems: Cramer's Rule.

Homework due Thursday 2/2/17

The purpose of the following assignment is to:

- Practice algebraic operations with matrices.
- Develop an understanding on the concept of the inverse of a matrix.

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1. Let
$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
   Compute the following:
   - $2A - 3B$, if possible.
   - $A + B + C$, if possible.
   - AB and BA, if possible.
   - AC and CA, if possible.

2. Express the following systems in the matrix form. Do not solve the systems.

   (a) $\begin{cases} x + y = 3 \\ 2x - 3y = 0 \end{cases}$

   (b) $\begin{cases} x + y + z = 1 \\ x - z = 1 \end{cases}$

   (c) $\begin{cases} x + y + z + w = 0 \\ x + z = 1 \\ y + w = 1 \end{cases}$

3. Let
$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
   Compute A^2, A^3, A^4 and A^n for $n \geq 5$.

4. Cook up an example of a 2×2 non-zero matrix A for which $A^2 = 0$.

5. In class we discussed a method for finding inverses of matrices. Explain in your own words why this method works.

6. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute A^{-1} and B^{-1}.

Homework due Monday 2/6/17

The purpose of the following assignment is to

- Wrap up some (terminological) details regarding matrix inverses, transposes etc.
- Become familiar with the concept of matrix determinant.

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1. (a) What does the phrase singular matrix refer to? What does the phrase invertible matrix refer to?

   (b) Complete the following sentence: A square matrix is if its reduced row echelon form is . Otherwise, we say that it is .

   (c) What property did you learn in class regarding the inverse of a product of two invertible matrices?

2. (a) What are the properties of transposes in regards to matrix addition, multiplication and inverse?

   (b) What does the phrase symmetric matrix refer to? What does the phrase anti-symmetric matrix refer to? Illustrate on examples.

   (c) Let A be a symmetric invertible matrix. Use the properties of transposes to argue that both A^2 and A^{-1} have to be symmetric.

   (d) Let A be any square matrix. Are the matrices $A + A^T$ and $A - A^T$ symmetric? Anti-symmetric? Justify your claim(s).

3. Complete the following sentence: A square matrix is if its determinant is . On the other hand, if then the matrix is .

4. Based on the expressions discussed in class find the determinants of the following matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

   Make sure you show the steps involved in this process.

Homework due Thursday 2/9/17

The purpose of the following assignment is to:

- Think about properties of matrix determinants.
- Practice evaluating determinants using row and column operations.
- Use determinants to determine (pun intended!) if a matrix is invertible or not.

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1. (a) Let A be a matrix in which one row is a multiple of another row. What do you know about $\det(A)$ and why?

   (b) Let A be a matrix in which one column is a multiple of another column. What do you know about $\det(A)$ and why?

2. Let α be a scalar and A an $n \times n$ matrix. How does $\det(\alpha A)$ relate to $\det(A)$?

3. Suppose A is some anti-symmetric matrix. Argue that $\det(A)$ must be zero.

4. Compute the following determinants. You are expected to use row or column operations.

   (a) $\begin{vmatrix} 7 & 1 & 10 \\ 17 & 1 & 20 \\ 27 & 1 & 30 \end{vmatrix}$

   (b) $\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{vmatrix}$

   (c) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{vmatrix}$

   (d) $\begin{vmatrix} 2t+p & 2u+q & 2v+r \\ p-a & q-b & r-c \\ a/4 & b/4 & c/4 \end{vmatrix}$ if it is known that $\begin{vmatrix} a & b & c \\ p & q & r \\ t & u & v \end{vmatrix} = 2$.

5. Which of the matrices from the previous problem are invertible? Explain how you made your decision.

6. Compute the following determinant:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Random tips: Your answer should be $(b-a)(c-a)(c-b)$. Also, don't forget that $b^2 - a^2 = (b-a)(b+a)$ etc.

Homework due Monday 2/13/17

The purpose of the following assignment is to:

- Practice using determinants in the context of linear systems.
- Use cofactors (in the context of matrices and determinants).

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1. Use the theorem from class regarding the solvability of an $n \times n$ linear system $A\vec{x} = \vec{b}$ to determine (mind the math pun!) the number of solutions of the following systems. You should not, however, find the solutions.

   (a) $\begin{cases} 4x + y - 5z = 0 \\ x + 4y - 5z = 0 \\ x + y + z = 0 \end{cases}$

   (b) $\begin{cases} x - 2y = 0 \\ y - 2z = 0 \\ -x + 4z = 0 \end{cases}$

2. Use the same methodology to answer the following questions. For what value of λ does the system

$$\begin{cases} x - 2y + 3z = 0 \\ 3x - y + z = 0 \\ 4x - 3y + \lambda z = 0 \end{cases}$$

   (a) have only one (trivial) solution?

   (b) have infinitely many solutions?

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 9 & 9 \end{pmatrix}$. Compute $\det(A)$ by using:

   (a) The cofactor expansion along the third row;

(b) The cofactor expansion along the second column.

4. Compute the following determinants relying primarily on cofactor expansions. Remember that it is most convenient to use expansions along rows or columns which contain lots of zeros.

$$\text{a)}\ \begin{vmatrix} 4 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 3 & 0 \end{vmatrix} \qquad \text{b)}\ \begin{vmatrix} 0 & 1 & 0 & 0 \\ 4 & 3 & 2 & 5 \\ 2 & 7 & 0 & 1 \\ 5 & 9 & 0 & 3 \end{vmatrix} \qquad \text{c)}\ \begin{vmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 7 & 0 & 1 & 1 \\ 0 & 9 & 0 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 \end{vmatrix}$$

5. Use the method of cofactors to find A^{-1} for the matrix A used in problem 3. Check your answer with a calculator.

6. Use Cramer's Rule to solve the following systems.

(a) $\begin{cases} 2x + y = -1 \\ 3x + 2y = -3 \end{cases}$

(b) $\begin{cases} x + y + z = 0 \\ x + 3y - 3z = 1 \\ x + 9y + 9z = 0 \end{cases}$

Review Questions: Weeks 3 and 4

1. What is each of the following pieces of terminology referring to? Illustrate on an example.

   - Matrix of format $m \times n$.
   - The ij-entry of a matrix.
   - Square matrix.
   - Zero matrix; Identity matrix.
   - Diagonal matrix.
   - Upper-triangular/ lower-triangular matrices.
   - Symmetric matrix.

2. There are four basic matrix operations:

   - Multiplication of a matrix by a scalar;
   - Addition of (two) matrices;
   - Multiplication of (two) matrices;
   - Taking the transpose of a matrix.

   Illustrate each of the operations listed above on an example, and discuss when we are and when we are *not* able to perform the indicated operations.

3. Matrix multiplication is not commutative. Illustrate this on an example. Bring up at least one more "unusual" thing about matrix algebra.

4. What, by definition, is inverse of a matrix? In practice, how do we find inverses of matrices? Give general directions and work out an explicit 3×3 example.

5. Not every square matrix has an inverse. Show an example of a 2×2 non-invertible matrix, along with a justification why the matrix is noninvertible.

6. Write down all conditions under which you know a matrix is invertible. Write down all conditions under which you know a matrix is noninvertible.

7. Write down all properties of matrix algebra having to do with inverses and transposes. More specifically, how do the inverses and transposes behave with respect to the four basic matrix operations?

8. What, by definition, is the determinant of a 2×2 matrix? A 3×3 matrix? A $n \times n$ matrix? Give general directions and work out a 2×2 and a 3×3 example.

9. Write down all properties of the determinant having to do with row and column operations. Then, work out a 4×4 determinant using the row and column operations.

10. Write down the properties of the determinants having to with matrix multiplication and inverses.

11. Determinants can be computed using cofactors. Describe the method in general terms, and then illustrate on a 3×3 example.

12. There is a method of computing the inverse of a matrix using determinants and cofactors. Describe the method in general terms, and then illustrate on a 3×3 example.

13. Matrices are useful when dealing with linear systems of equations. Show on an example how a linear system of equations can be expressed in matrix a form.

14. State the big theorem we talked about in class regarding the solvability of an $n \times n$ linear system $A\vec{x} = \vec{b}$.

15. There is a method for solving linear equations which involves determinants; it is called Cramer's Rule. Include a description of the rule in general terms, and an explicit example which shows how a system can solved using Cramer's Rule.

16. You will need to use your calculator in order to perform some of the matrix operations. Write down a set of instructions for how your calculator can be used for:
    - Row reducing matrices;
    - Multiplying matrices;
    - Finding matrix inverse;
    - Finding the determinant of the matrix.

Introduction to linear transformations, eigenvalues and eigenvectors

At this stage we re-visit the material covered during the first two weeks of our class. We deepen our understanding of the geometry of linear systems of equations by using matrices and determinants. We do so by first developing a way of looking at matrices as geometric transformations.

Week 5

Monday, 2/13: Interpreting matrices as geometric linear transformations. Take detailed notes!

Tuesday, 2/14: The concept of linear transformations – practice.

Thursday, 2/16: Nullspace of a matrix / kernel of a linear transformation. In other words, we re-cast the problems addressed in the first two weeks of class in the language of matrices.

Friday, 2/17: Column space of a matrix / image of a linear transformation.

Week 6

Monday, 2/20: On row space, column space and the rank of a matrix. Note that there is a big result here that everyone should learn; it is called *The Rank+Nullity Theorem.*

Tuesday, 2/21: Practice.

Thursday, 2/23: The concepts of eigenvalues and eigenvectors, with emphasis on their geometric interpretation.

Friday, 2/24: Day two of eigenvalues and eigenvectors. You should think of this lecture as an algebraic practice for the concepts developed on Thursday.

Homework due Thursday 2/16/17

The purpose of the following assignment is to:

- Practice thinking about matrices and linear transformations geometrically.

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1. Illustrate linear transformations corresponding to following matrices. Pictures, like those presented in class, and explanations in words are expected.

   (a) $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

   (e) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$

   (f) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Homework due Monday 2/20/17

The purpose of the following assignment is to:

- Practice computing the effect of a linear transformation through the computation null- and column-spaces / kernel and image.

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1. Compute the nullspace and column space of the following matrices. In other words, compute the kernel and the image of the linear transformations associated to the following matrices. Then comment in words (or pictures) on the effect of these linear transformations.

   (a) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 5 & 1 & -3 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

Homework due Thursday 2/23/17

The purpose of the following assignment is to:

- Work with the concept of matrix rank.
- Work with the Rank+Nullity Theorem.
- Re-visit the concept of linear independence, in the context of matrices, determinants etc.
- Give you an opportunity to synthesize all the concepts we talked about in this class thus far.

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1. Answer the following conceptual questions:

   (a) Complete the following sentence: If A is an $m \times n$ matrix, then the row space is a subspace of   , while the column space is a subspace of   .

   (b) How does the dimension of the row space relate to the dimension of the column space?

   (c) If an $m \times n$ matrix is interpreted as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, what does rank signify geometrically?

   (d) If an $m \times n$ matrix is interpreted as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, what does nullity signify?

   (e) What is the relationship between the rank and the nullity of an $m \times n$ matrix?

   (f) If the rank of an $m \times n$ matrix A is n, what can you say about its nullity?

   (g) If A is an invertible $n \times n$ matrix, what can you say about its rank and nullity?

2. Let $A\vec{x} = \vec{b}$ be a consistent system. How many free parameters does it have if A is a 5×4 matrix of rank 3?

3. Let $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

   (a) Compute the rank of the matrix A, without computing its row or the column space per se. To show me that you learned something new use both row and column operations.

   (b) Based on your result above, what is the nullity of this matrix?

4. Use the concept of matrix rank to find the dimension of the span of vectors

$$\begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 9 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 27 \\ 1 \end{pmatrix}$$

   in $\mathbb{R}^4$. Are these vectors linearly independent?

5. Answer the following conceptual questions. To get the most out of this exercise you should address the questions in the order in which they are listed.

   (a) If columns of an $m \times n$ matrix are linearly independent, what can you say about the rank and the nullity of the matrix?

   (b) If rows of an $m \times n$ matrix are linearly independent, what can you say about the rank and the nullity of the matrix?

   (c) If columns of a square $n \times n$ matrix A are linearly dependent, what can you say about the determinant of A?

   (d) If columns of a square $n \times n$ matrix A are linearly independent, what can you say about the determinant of A?

   (e) Let $v_1, ..., v_n$ be n vectors in $\mathbb{R}^n$ and let A be the matrix whose columns are the vectors $v_1, ..., v_n$. Explain how determinant of A can be used to decide whether the vectors $v_1, ..., v_n$ are linearly independent.

   (f) Let $n > m$ and let the rows of an $m \times n$ matrix be linearly independent. Is it true that one could erase several columns of A and obtain an invertible $m \times m$ matrix? Explain.

   (g) Let r be an integer smaller than both m and n. Let A be a matrix of rank r. Is it true that one could erase several columns of and several rows of A, and obtain an invertible $r \times r$ matrix?

(h) Let r be an integer smaller than both m and n. Let A be a matrix of rank r and let B be any $(r+1)\times(r+1)$ matrix which can be made from A by erasing some of its rows or columns. What can you say about the determinant of B?

6. Are vectors
$$\begin{pmatrix}1\\2\\1\end{pmatrix},\quad \begin{pmatrix}0\\1\\1\end{pmatrix},\quad \begin{pmatrix}3\\2\\1\end{pmatrix}$$
linearly dependent or independent? What is the span of these vectors? Try to do this problem in the most efficient way possible.

Homework due Monday 2/27/17

The purpose of the following assignment is to:

- Think about eigenvalues and eigenvectors.
- Make you practice computing eigenvalues and eigenvectors.
- Give you exposure to some complications that might arise.

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1. Let us "visualize" an $n \times n$ matrix A as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Describe in words the visual meaning of its eigenvalues and eigenvectors.

2. Explain the logic that leads to the characteristic equation $\det(A-\lambda I)=0$.

3. Compute the eigenvalues and the eigenvectors of the following matrices. Comment on the geometric implications of your computations.
$$\text{a)}\ \begin{pmatrix}3 & -2\\-2 & 6\end{pmatrix} \qquad \text{b)}\ \begin{pmatrix}7 & 2\\-15 & -4\end{pmatrix} \qquad \text{c)}\ \begin{pmatrix}1 & 0 & 0\\1 & 2 & 0\\1 & 2 & 3\end{pmatrix}$$

4. For each of the following matrices find its eigenvalues and the corresponding eigenvectors (or in certain cases, eigenspaces).

(a) $\begin{pmatrix}0 & 0 & -2\\1 & 2 & 1\\1 & 0 & 3\end{pmatrix}$

(b) $\begin{pmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

5. Let A be an invertible $n \times n$ matrix with n distinct eigenvalues. What, if anything, can you say about the eigenvalues and eigenvectors of A^{-1} and A^2?

Review Questions: Weeks 5 and 6

1. Let A be an $m \times n$ matrix. Rule-wise, what is meant under "linear transformation corresponding to the matrix A"? What are the domain and the codomain here?

2. We mentioned two very important properties of linear transformations. Assuming L_A denotes the linear transformation corresponding to the matrix A, what do we get to say about $L_A(\vec{X} + \vec{Y})$ and $L_A(k \cdot \vec{X})$? Can you justify that?

3. Explain the concept of linear transformations on pictures. Show two very different examples.

4. When do we say that a linear transformation is one-to-one? When do we say that it is onto? Explain on examples.

5. What is the nullspace of a matrix, what is its dimension called and what does it geometrically represent? Explain on an easy example which is *not* computationally intensive.

6. Suppose two vectors are taken by the linear transformation L_A to the same element of the codomain. What can you say about the difference of these two vectors? What visual consequence does this observation have? Explain on an easy example which is *not* computationally intensive.

7. What is

   - the column space,
   - the row space,

   of an $m \times n$ matrix? Explain on an easy example which is *not* computationally intensive.

8. What is the rank of a matrix, and what does it have to do with the row and column spaces?

9. What is the geometrical significance of a column space of a matrix? What is the geometrical significance of the rank?

10. What is the relationship between the rank and the nullity of a $m \times n$ matrix?

11. How is the rank of a matrix computed? What kind of operations are permissible here? Work out an explicit example.

12. There is an efficient way of simultaneously finding the nullspace and the column space of a matrix. Explain it on a not-too-easy example.

13. Finding matrix rank can help with:

    (a) determining if a set of vectors is linearly dependent or independent;

    (b) determining the dimension of the span of a bunch of vectors.

    Explain each of these on an easy example which is *not* computationally intensive.

14. Determinants can help with finding out whether a collection of n vectors in $\mathbb{R}^n$ forms a basis. Explain how, and include an example.

15. What is the technical definition of eigenvalues and eigenvectors of a square $n \times n$ matrix? What is their geometrical significance? Explain on a super-easy example which is *not* computationally intensive.

16. What is the characteristic polynomial equation of a matrix, and why does it determine the eigenvalues?

17. Execute three non-trivial examples of eigenstuff computations. To show what all could happen, show one example which features an eigenspace of dimension at least 2, and one example in which the dimensions of the eigenspaces do not add up to the dimension of the domain.

The Exam

The material covered in the course thus far makes up the very basics of Linear Algebra. At this point an exam is in order.

Week 7

Monday, 2/27: Summary of the first part of the course. Think of this as a review day.

Tuesday, 2/28: Evening exam – no class!

Thursday, 3/2: New stuff, see the following pages.

Friday, 3/3: New stuff, see the following pages.

Abstract vector spaces and their linear transformations

In the next two to three weeks of the class you need to get to a point where you can deal with the abstract versions of the concepts discussed in the first part of the course.

Week 7 (i.e. the exam week)

Thursday, 3/2: Abstract vector spaces. Expect examples such as $C[a, b]$ or vector spaces over complex numbers. Take good lecture notes!

Friday, 3/3: Abstract vector spaces. Think of this as the follow-up to Thursday's lecture, with emphasis shifted from examples to abstract properties of vector spaces.

Week 8

Monday, 3/6: Subspaces.

Tuesday, 3/7: The concept of linear independence.

Thursday, 3/9: Basis, coordinates and dimension.

Friday, 3/10: More on basis, coordinates and dimension, i.e a follow-up to the last lecture.

Week 9

Monday, 3/13: The concept of linear transformations, kernel and image.

Tuesday, 3/14: Linear transformations practice.

Thursday, 3/16: Matrix representations of linear transformations.

Friday, 3/17: More on matrices of linear transformations / practice.

Homework due Monday 3/6/17

The purpose of the following assignment is to:

- Think about abstract vector spaces.

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1. List all the requirements that a certain set V needs to fulfill in order to be a *vector space*.

2. Let $(f_1, f_2, f_3, ...)$ denote an infinite sequence of functions $f_n(t)$ of real variable t; you should assume that the domain of each f_n consists of all real numbers. Let V denote the set of all such sequences of functions.

   - Describe formulaically what $+$ operation in the set V is.
   - Describe formulaically what the scalar multiplication operation in V is.
   - Go through the list of all eight defining properties of vector spaces and verify that <u>each one</u> is fulfilled.

3. Let V denote the set of all positive real numbers, let $\oplus$ be an operation on elements of V defined by
$$x \oplus y = xy,$$
and let $\odot$ be an operation defined on scalars α and elements x of V by
$$\alpha \odot x = x^{\alpha}.$$
Go through the list of all eight defining properties of vector spaces and verify that <u>each one</u> is fulfilled. In other words, prove that V is a vector space under the indicated operations $\oplus$ and $\odot$.

4. Replace the set V of the previous problem by, say, the closed interval $[1, +\infty)$ and keep the rules of operations $\oplus$ and $\odot$ the same. Would we still have a vector space? Explain in a sentence why / why not?

5. Let V denote the set of all *positive* numbers. Assume that $+$ just means the common addition operation, but that scalar multiplication means the following:
$$\alpha \odot x = x^{\alpha}.$$
Would V together with $+$ and $\odot$ constitute a vector space? If yes, prove that each one of the eight properties is fulfilled. If not, discuss which <u>all</u> of the eight properties fail and why.

Homework due Thursday 3/9/17

The purpose of the following assignment is to:

- Practice the concept of "being closed under an operation".
- Think about the concept of a vector subspace.
- Re-visit the concept of linear independence as understood in $\mathbb{R}^n$.
- Practice the concept of linear independence in abstract context.

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1. Recall that $\mathbb{R}^{3\times 3}$ denotes the vector space of all 3×3 matrices. Which of the following subsets of $\mathbb{R}^{3\times 3}$ are closed under addition $+$? Which are closed under the scalar multiplication operation $\cdot$? I want you to justify every one of your claims!
   (a) The set S_1 of all symmetric 3×3 matrices, that is, the set of all matrices A such that $A^T = A$.
   (b) The set S_2 of all anti-symmetric 3×3 matrices, that is, the set of all matrices A such that $A^T = -A$.
   (c) The set S_3 of all invertible 3×3 matrices.
   (d) The set S_4 of all non-invertible matrices.
2. Which of S_1, S_2, S_3, S_4 from the previous problem is a vector subspace of $\mathbb{R}^{3\times 3}$?
3. Recall the vector space P_n of all polynomials of degree up to but not including n. Which of the following subsets of P_n are closed under addition $+$? Which are closed under the scalar multiplication operation $\cdot$? I want you to justify every one of your claims!
   (a) The set S_1 of all polynomials $P(x)$ of degree less than n such that $P(1) = 0$.
   (b) The set S_2 of all polynomials $P(x)$ of degree less than n such that $P(1) \geq 0$.
   (c) The set S_3 of all polynomials $P(x)$ of degree less than n such that $P(1)+P'(2) = 0$.
   (d) The set S_4 of all polynomials $P(x)$ of degree less than n such that $\int_0^1 P(x)\,dx = 0$.
   (e) Which of S_1, S_2, S_3, S_4 is a vector subspace of P_n?
4. **Linear Independence in $\mathbb{R}^n$.**
   (a) Explain why / how verification of linear independence of several vectors in $\mathbb{R}^n$ boils down to computing the rank or a determinant of a matrix.

(b) Are vectors $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$, $\begin{pmatrix}1\\1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\2\\1\end{pmatrix}$ in $\mathbb{R}^4$ linearly independent?

(c) What about vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}4\\9\\16\end{pmatrix}$ in $\mathbb{R}^3$?

5. **Linear Independence in $\mathbb{R}^{m\times n}$.** Are the matrices

$$A_1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \quad A_2 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad A_3 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}, \quad A_4 = \begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix}$$

linearly independent?

6. **Linear Independence in Function Spaces.**

(a) What is the Wronskian $W[f_1, f_2, ..., f_k](x)$ of functions $f_1, f_2, ..., f_k$ at x?

(b) Explain why / how verification of linear independence of several functions / polynomials boils down to the Wronskian computation. What does Wronskian have to satisfy in order to have linear independence?

(c) Are polynomials $P_1(x) = 1 - x + x^2$, $P_2(x) = 1 + x + x^2$ and $P_3(x) = 1 + 2x + x^2$ linearly independent?

(d) Are polynomials $P_1(x) = 1 - x + x^2$, $P_2(x) = 1 + x + x^2$ and $P_3(x) = x^2$ linearly independent?

Homework due Monday, 3/13/17

The purpose of the following assignment is to:

- Re-visit the concept of span as understood in $\mathbb{R}^n$.
- Practice the concept of span in abstract context.
- Practice working with bases and coordinates.

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1. **Span of vectors in $\mathbb{R}^n$.**

(a) Consider the vectors $\vec{v}_1 = \begin{pmatrix}1\\3\end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix}2\\1\end{pmatrix}$ of $\mathbb{R}^2$. Can $\begin{pmatrix}1\\1\end{pmatrix}$ be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$? If so, what are the coefficients of the linear combination?

(b) Consider again the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ of $\mathbb{R}^2$. Can any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ of $\mathbb{R}^2$ be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ or are there some limitations? If yes, what are the relevant coefficients of the linear combination?

(c) Based on the above, what can you conclude about $\mathrm{Span}(\vec{v_1}, \vec{v_2})$?

(d) Employ strategies similar to those above to decide whether the set of vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ spans $\mathbb{R}^3$.

2. **Span in P_n.**

(a) Consider the polynomials $v_1(x) = 1 + 2x$ and $v_2(x) = 3 + 5x$ of P^2. Can the polynomial $f(x) = 7 + x$ be expressed as a linear combination of v_1 and v_2? If so, what are the coefficients of the linear combination?

(b) Consider again the polynomials $v_1(x) = 1 + 2x$ and $v_2(x) = 3 + 5x$ of P_2. Can any polynomial $v(x) = a + bx$ of P_2 be expressed as a linear combination of v_1 and v_2? If so, what are the coefficients of the relevant linear combination?

(c) Is the collection v_1, v_2 a spanning set for P_2?

(d) Employ strategies similar to those above to decide whether 1, $1 + x$, $1 + x + x^2$ span P_3.

(e) Employ strategies similar to those above to decide whether $1+x+x^2$, $1+x+2x^2$ and $1 + x + 3x^2$ span P_3.

3. **Bases in $\mathbb{R}^n$.**

(a) Verify that $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a basis for $\mathbb{R}^2$ by verifying explicitly that

- $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$;
- are linearly independent.

(b) Using the basis from the previous problem, what are the coordinates of the following vectors? Illustrate your computation by a drawing in $\mathbb{R}^2$.

i. $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ii. $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(c) Employ the strategies from the above to explicitly show that the collection of vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

forms a basis of $\mathbb{R}^3$. What are the coordinates of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in this basis?

4. **Bases in abstract vector spaces.**

   (a) Verify that $v_1(x) = 4$, $v_2(x) = 4 - 2x$ and $v_3(x) = 4 - 2x + x^2$ is a basis for P_3 by explicitly verifying that

   - v_1, v_2 and v_3 span P_3;
   - are linearly independent.

   (b) Using the basis from the previous problem, what are the coordinates of the following?

   i. $v = 1$

   ii. $v = x$

   iii. $v = 1 + x$

   iv. $v = x^2$

   v. $v = 1 + x + x^2$

   vi. $v = 7 + 3x + x^2$

Homework due Thursday 3/16/17

The purpose of the following assignment is to:

- Practice working with the concept of dimension.
- Become familiar with the concept of (abstract) linear transformations.
- Develop an understanding of kernel, image, eigenvalues and eigenvectors of abstract linear transformations.

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1. (a) What is the maximum number of linearly independent vectors in an n-dimensional vector space?

   (b) What can you say about n linearly independent vectors of an n-dimensional vector space?

2. Verify – in the most efficient (and yet mathematically complete) way possible – that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   form a basis of $\mathbb{R}^4$.

3. What is the dimension of the following vector spaces? For each vector space write down what would be its (standard) basis. (You do not need to "officially" prove that what you have is a basis.)

   (a) $\mathbb{R}^{m\times n}$

   (b) P_n

   (c) The vector space V of all pairs (P, Q) of polynomials P and Q each of which is of degree at most 2. (e.g. $(1+2x, 3+x^2)$ is in V.)

   (d) The vector space V of all 2×2 matrices whose entries are polynomials of degree at most 3. (e.g. $\begin{pmatrix} 0 & x^3 \\ 1+x+x^3 & 2-x^2 \end{pmatrix}$ is in V.)

4. What do you think is the dimension of the vector space of *all* polynomials? What do you think is the dimension of the vector space $C[a, b]$? Some (perhaps not very rigorous) explanation is expected.

5. Which of the following are linear transformations?

   (a) $L : C[0, 1] \to C[0, 1]$ with the rule $L(f) = f + 2$ (so that, for instance, $L(x^3) = x^3 + 2$);

   (b) $L : C[0, 1] \to C[0, 1]$ with the rule $L(f) = e^x f$ (so that, for instance, $L(x^3) = e^x x^3$);

   (c) $L : C[0, 1] \to C[0, 1]$ with the rule $L(f) = e^f$ (so that, for instance, $L(x^3) = e^{x^3}$);

   (d) $L : C[0, 1] \to \mathbb{R}^3$ with the rule $L(f) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$ (so that, for instance, $L(x^3) = \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$);

   (e) $L : C^\infty[0, 1] \to C^\infty[0, 1]$ with the rule $L(f) = f'' + 5f' + 4f$ (so that, for instance, $L(x^3) = 6x + 15x^2 + 4x^3$).

6. Let $V = C^\infty[0, 1]$ and let $L : V \to V$ be given by the rule $L(f) = -2f'$.

   - Find $L(e^x)$ and $L(e^{\alpha x})$;
   - Verify that L is a linear transformation;
   - Find $\ker(L)$. What is $\dim \ker(L)$? Is L one-to-one?
   - Find $\mathrm{Im}(L)$. Is L onto?
   - Find eigenvalues and eigenfunctions of L.

Homework due Monday 3/20/17

The purpose of the following assignment is to:

- Practice working with abstract linear transformations on a theoretical level.
- Re-visit examples of linear transformations between function spaces, and ponder their relationship to differential equations.
- Become familiar with the concept of a matrix representation of a linear transformation.
- Utilize the matrix representation to address linear transformations between finite dimensional vector spaces.

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1. Let $L : V \to W$ be a linear transformation.
   (a) If $L(x_1) = L(x_2) = 0$, is it necessarily true that $L(x_1 + x_2) = 0$?
   (b) If $L(x) = 0$ and if α is a scalar, is it necessarily true that $L(\alpha x) = 0$?
   (c) Is $\ker(L)$ closed under addition and scalar multiplication?
   (d) Is $\ker(L)$ a subspace of V? Is $\ker(L)$ a subspace of W?
   (e) If $x_1 = L(v_1)$ and $x_2 = L(v_2)$, is it necessarily true that $x_1 + x_2$ is an image of some vector in V? If so, which one?
   (f) If $x = L(v)$ and if α is a scalar, is it necessarily true that αx is an image of some vector in V? If so, which one?
   (g) Is $\mathrm{Im}(L)$ closed under addition and scalar multiplication?
   (h) Is $\mathrm{Im}(L)$ a subspace of V? Is $\mathrm{Im}(L)$ a subspace of W?

2. Let $L : C^\infty[0, 1] \to C^\infty[0, 1]$ be defined by $L(f) = f''$. It can be easily seen that this is a linear transformation. (Compare to the following homework problem.)
   (a) Find the kernel and the image of L.
   (b) If the function $g(x) = xe^x$ is in the image, find *all* of its pre-images.
   (c) Find $L(e^{2x})$, $L(e^{-2x})$, $L(\cos(3x))$, $L(\sin(3x))$. Based on your answers, make an educated guess regarding the eigenvalues and the eigenfunctions of L.

3. In differential equations one learns to solve (for the unknown function f) equations such as $mf''(x) + bf'(x) + kf(x) = 0$ where m, b and k are some constants. For instance, one learns that all solutions of $f''(x) + 2f'(x) + 2f(x) = 0$ can be expressed in the form of $f(x) = \alpha e^{-x} \sin(x) + \beta e^{-x} \cos(x)$ for some constants α and β.
   (a) Verify that $L : C^\infty[a, b] \to C^\infty[a, b]$ given by $L(f) = f'' + 2f' + 2f$ is a linear transformation.

(b) Find $\ker(L)$ and its dimension.

(c) What is $L(e^x)$? Based on this knowledge, find *all* f with $f''(x)+2f'(x)+2f(x) = 5e^x$.

(d) Linear transformations between function spaces are tightly related to differential equations. Address very briefly.

4. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain why it necessarily takes the form of $L(\vec{x}) = A\vec{x}$ for some matrix A. Also explain what columns of A represent in this case.

5. Find matrix representations of the following linear transformations. Use standard bases.

   (a) $L : P_3 \to P_3$ given by $L(P) = P(0) + P(1)x + P(2)x^2$. (So, $L(1 + 2x) = 1 + 3x + 5x^2$.)

   (b) $L : P_3 \to P_3$ given by $L(P) = (x \cdot P(x))'$. (So, $L(1 + 2x) = 1 + 4x$.)

6. For each of the two linear transformations in Problem 5 find:

   (a) The nullspace of the matrix representation. Based on that computation find the kernel of the linear transformation itself.

   (b) The column space of the matrix representation. Based on that computation find the image of the linear transformation itself.

Review Questions: Weeks 7, 8 and 9

1. What is the technical definition of a vector space? Give four completely different examples of vector spaces, but explicitly work out only one example in which you will verify that something is a vector space.

2. Technically, what is a subspace of a vector space? (Which two properties does a subset of a vector space need to obey in order to be a vector subspace?) Mention an intuitive example of a vector subspace (no proofs needed there), and an explicit abstract example (proof expected).

3. What is the technical definition of a span of several vectors (e.g. Span$(v_1, ..., v_n)$)?

4. How does one in practice verify if a vector is an element of Span$(v_1, ..., v_n)$? Work out an explicit example in which you verify that a vector is in the span, and work our an explicit example in which the vector does not belong to the span.

5. When do we say that vectors $v_1, ..., v_n$ of a vector space V are linearly independent? Give an intuitive (and not particularly rigorous) interpretation of linear independence and linear dependence.

6. How does one verify linear independence in $\mathbb{R}^n$? Address in general words, and offer an example. How does one verify linear independence in function spaces? Address in general words, and offer an example.

7. What is/are....

   - ... a "basis of a vector space"? Illustrate on an example (drawing!) and then technically show that what you have indeed is a basis.

   - ... "dimension" of a vector space? For several different vector spaces we commonly deal with list the dimension and standard basis. (No proofs expected.)

   - ... the maximum number of linearly independent vectors in an n-dimensional vector space?

   - ... the most efficient way of verifying that n vectors of an n-dimensional vector space V forms a basis? Explain on an example.

   - ... coordinates of a vector with respect to a basis? Explain on an explicit example.

8. Do you know any infinite dimensional vector space? Justify your claim.

9. What is meant under "linear transformation" between V and W? Provide two very different examples of linear transformations, and verify that they indeed are linear transformations. At least one of your examples needs to feature an infinite dimensional vector space.

10. Assuming L is a linear transformation, describe the following concepts in general terms and explain them on an example other than those coming from $\mathbb{R}^n$.

    - the kernel of L.
    - the image of L.
    - an eigenvalue of L.
    - an eigenfunction of L.

11. If $L : V \to W$, then show that $\ker(L)$ and $\operatorname{Im}(L)$ are subspaces of V and W respectively.

12. - What do the words "one-to-one" and "onto" mean?
    - What do dimensions of the kernel and the image have to do with being "one-to-one" and "onto"?
    - Linear "one-to-one" and "onto" transformations permit inverse transformations. Explain the concept of the inverse of a linear transformation on an example.

13. Explain why knowing what a linear transformation does to basis vectors is enough for knowing what the transformation does to any vector.

14. Given a basis E of a n-dimensional vector space V there is a canonical linear, "one-to-one" and "onto" linear transformation mapping $\mathbb{R}^n$ and V. What is this canonical transformation, and can you explain it on a example which does not involve standard bases?

15. Given bases E and F of finite dimensional vector spaces V and W respectively, what is the matrix representation of a linear transformation $L : V \to W$ with respect to E and F? Explain the concept in general and on an example.

16. Matrix representations of linear transformations (between finite dimensional vector spaces) are useful for finding the kernel, the image, the eigenvalues and the eigenvectors of the linear transformations. Explain on two examples.

17. Linear transformations between function spaces are tightly related to differential equations. Address very briefly.

Changing bases

One often needs to be able to change coordinate systems so that they suit the geometry of the situation at hand the best. For instance, rotations around an axis are most easily studied when the axis of the rotation lines up with one of the axis of our coordinate system (i.e. the span of one of our basis vectors). During week 10 of the course you will learn how to change bases or, in layman's terms, switch from one coordinate system to another.

Week 10

Monday, 3/20: How do coordinates of a vector change when we change bases?

Tuesday, 3/21: How does the matrix of a linear transformation change when we change bases? Take good notes!

Thursday, 3/22: Practice. Do note that some examples covered on this day will come from "eigen-bases". This is a prelude to the material we will do on Friday.

Friday, 3/23: Diagonalization.

SPRING BREAK

Homework due Thursday 3/23/17

The purpose of the following assignment is to:

- Become familiar with how change of basis and coordinates work.
- Think about changing bases; think about the effects changing bases has on coordinates and matrix representations of linear transformations.

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1. Let E denote the standard basis of $\mathbb{R}^2$ and let F denote the basis obtained from E by means of the 45^o counterclockwise rotation.

   - Express the basis vectors of F in terms of the basis vectors of E and assemble the corresponding transition matrix S.
   - How do coordinates $[\vec{x}]_F$ relate to coordinates $[\vec{x}]_E$ and vice versa?
   - Use this relationship to find the coordinates of the vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with respect to the basis F, i.e. $[\vec{x}]_F$.
   - Apply your latest formula to the vector $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and provide an illustration for what is going on.

2. Let E denote the standard basis of P_3 and let F denote the basis $[1, 1+x, (1+x)^2]$.

   - Express the basis vectors of F in terms of the basis vectors of E and assemble the corresponding transition matrix S.
   - How do coordinates $[\vec{v}]_F$ relate to coordinates $[\vec{v}]_E$ and vice versa?
   - Use this relationship to find the coordinates of the vector $\vec{v} = a + bx + cx^2$ with respect to the basis F, i.e. $[\vec{v}]_F$.
   - Apply your latest formula to the vector $\vec{v} = 1+x+x^2$ and compute $[1+x+x^2]_F$.

3. Let E and F be two different bases for an n-dimensional vector space V. One can express the relationship between E and F through a formula of the form $F = E \cdot S$.

   (a) What is S and how do we in practice find it?

   (b) How do coordinates $[\vec{x}]_F$ relate to coordinates $[\vec{x}]_E$ and vice versa? Prove your claim in full generality, and work out the relevant formulas in the context of the example you brought up above.

(c) Let E and F be as above, and let $L : V \to V$ be a linear transformation from an n-dimensional vector space V to itself. One can find the matrix representation of L in two different ways. Using the basis E we get one matrix representation, call it A. Using the basis F we get another matrix representation of L, call it B. How are A and B related to one another? Why does this relationship hold? (Actually prove the relevant formula.)

4. The above can be used to find matrix representations of linear transformations with respect to non-standard bases. Demonstrate this procedure on the following examples.

   (a) Find the matrix representation of the linear transformation $L : P_3 \to P_3$ given by $L(P) = x \cdot P'(x)$ with respect to the basis $F = [2, 2 + x, (2 + x)^2]$.

   (b) Find the matrix representation of the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y+z \\ x-y+z \\ x-y-z \end{pmatrix}.$$

   with respect to the basis

$$F = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right].$$

Homework due right after you come back from Spring Break

The purpose of the following assignment is to:

- Work with the concept of diagonalisation.

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1. Outline the procedure for the diagonalization of matrices, and execute the procedure on the examples of following matrices.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

   (b) $B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

2. For each of the three matrices A, B and C from the previous problem find A^n, B^n and C^n.

3. If A is an $n \times n$ matrix, what does e^A refer to? Relate your answer to what you learned in Calculus II.

4. For the 2×2 matrix A from the above, find e^A.

Dot product in $\mathbb{R}^n$

Up to now our study of vector spaces avoided anything that had to do with measurements of length or angles. To be entirely honest, linear algebra up to this point is a study of coordinatization, rather than the study of geoMETRY. (Look up the origin of the word geometry.) As those of you who went through multivariable calculus or basic physics know, dot product (or inner product as it is sometimes called) is one way to getting measurements involved. The goal for the first two weeks after the break is to master linear algebra in $\mathbb{R}^n$ equipped with the standard, Euclidean dot product.

Week 11

Monday, 4/3: Dot (scalar, inner) product in $\mathbb{R}^n$.

Tuesday, 4/4: Orthonormal bases.

Thursday, 4/6: Orthogonal complement.

Friday, 4/7: Projections and approximations.

Week 12

Monday, 4/10: Adjoint of a linear transformation and *the Fundamental Subspaces Theorem.* Take good notes!

Tuesday, 4/11: Application: least squares problem and the linear regression.

Thursday, 4/13: Gram-Schmidt procedure.

Friday, 4/14: FESTIVAL OF SCHOLARS – no class!

Homework due Thursday 4/6/17

The purpose of the following assignment is to:

- Become familiar with the concept of dot product in $\mathbb{R}^n$, and how it relates to geometric concepts such as length and angle measurement.
- Practice working with orthogonal sets and orthonormal bases in $\mathbb{R}^n$.

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1. What is the angle between the following vectors in $\mathbb{R}^4$:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ -3 \end{pmatrix}$$

2. Let $\vec{x}$ and $\vec{y}$ be two unit vectors in $\mathbb{R}^n$, i.e. vectors such that $\|\vec{x}\| = \|\vec{y}\| = 1$.
   (a) Compute $< \vec{x} + \vec{y}, \vec{x} - \vec{y} >$.
   (b) What can you say about the angle between $\vec{x} + \vec{y}$ and $\vec{x} - \vec{y}$?

3. Let x and y be two orthogonal vectors is $\mathbb{R}^n$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

4. If $F = [f_1, f_2, ..., f_n]$ is an orthogonal basis of $\mathbb{R}^n$, then the coordinates $[v]_F$ of a vector v in $\mathbb{R}^n$ with respect to F can be easily found through dot product. Explain how and why. What can you say about $[v]_F$ if in addition F is orthonormal?

5. Consider the collection of vectors

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; f_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; f_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

   (a) Verify that this is an orthogonal collection of vectors.
   (b) Verify explicitly that $[f_1, f_2, f_3]$ form a basis of $\mathbb{R}^3$.
   (c) Find the coordinates of the vector $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with respect to this basis.
   (d) Adjust the basis from the above so that it be orthonormal.
   (e) What are the coordinates of v with respect to this new, orthonormal basis?

6. Repeat the above with

$$f_1 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \; f_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}, \; f_3 = \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix}$$

and $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Homework due Monday 4/10/17

The purpose of the following assignment is to:

- Get further practice with orthogonal and orthonormal sets of vectors.
- Become familiar with the concept of *orthogonal matrices*.
- Become familiar with the concepts of
  - orthogonal complement;
  - projection onto a subspace.

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1. Let $F = [f_1, f_2, ..., f_n]$ be **any** orthonormal basis for $\mathbb{R}^n$. Let

$$[x]_F = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{pmatrix} \quad \text{and} \quad [y]_F = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_n \end{pmatrix}$$

be the coordinates of vectors x and y with respect to the basis F (not necessarily the standard basis). Use the fact that F is orthonormal to argue that

(a) $< x, y >= \alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n$;

(b) $\|x\|^2 = \alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2$.

2. Argue, like we did in class, that orthogonal sets of vectors are linearly independent.

3. In class we discussed a method, based on orthogonal matrices, of verifying that a certain set of vectors forms an orthonormal basis for $\mathbb{R}^n$. Describe that method in general terms and apply it to the following example:

$$F = \left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right]$$

4. Answer the following questions about orthogonal matrices.

   (a) Is identity matrix I an orthogonal matrix?

   (b) Assume A and B are orthogonal matrices. Argue that AB is also an orthogonal matrix.

   (c) Assume A is an orthogonal matrix. Argue that A^{-1} is an orthogonal matrix.

5. Let S be the line in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

   (a) Would it be fair to say that $S^\perp$ is the plane described by $x + 2y - 3z = 0$? Why or why not?

   (b) Find a basis for the plane $S^\perp$.

6. Let S be the plane in $\mathbb{R}^5$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Find $S^\perp$.

7. Let S be the line spanned by $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$ in $\mathbb{R}^3$. Find the projection of the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ onto the line S. What would be the projection of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ onto the plane $3x + 4y + 12z = 0$?

8. Find the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the plane $x - y - z = 0$.

Homework due Thursday 4/13/17

The purpose of the following assignment is to:

- Practice projection-themed computations.

- Think about least squares solutions.

1. Consider the plane S in $\mathbb{R}^4$ spanned by

$$\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

   (a) Find $S^\perp$.

   (b) Find an orthonormal basis for S, as well as an orthonormal basis for $S^\perp$.

   (c) Find the projections of the vector

$$\vec{v} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

   onto S and onto $S^\perp$.

   (d) Find the closest point in the plane S to the point $(1, 1, 1, 1)$, i.e. the endpoint of the vector $\vec{v}$.

   (e) Find the distance between (the endpoint of) the vector $\vec{v}$ and the plane S.

   (f) A theorem discussed in class expresses the projection onto S as a linear transformation

$$\vec{x} \mapsto UU^T\vec{x}.$$

   What is the matrix representation $A = UU^T$ of the projection onto S discussed above?

   (g) Double-check your work by computing the projection of $\vec{v}$ from the above by means of the matrix representation $A = UU^T$ of the previous problem.

   (h) Combine the orthonormal bases for S and $S^\perp$ to get an orthonormal basis F of $\mathbb{R}^4$. Find the coordinates of the vector $\vec{v}$ with respect to F.

2. Explain why a linear system $Ax = b$ is consistent when and only when one of the following holds:

   - b is in the column space of A;
   - b is orthogonal to $N(A^T)$.

3. Explain what we do when we need to find at least an approximate solution to $Ax = b$ in the situation when b is not in the column space of A. In other words, explain the logic behind the formula

$$x = (A^TA)^{-1}A^Tb.$$

4. Suppose you need to fit a line $y = \alpha + \beta x$ through the data points $(0, 2)$, $(3, 10)$, $(5, 16)$, $(8, 26)$ and $(10, 33)$. Go through the steps behind the linear regression discussed in class on Friday, while using these explicit data points, in order to find this line of best fit.

5. Suppose you need to fit a parabola $y = \alpha + \beta x + \gamma x^2$ through the data points

$$(0, 0), (1, 2), (2, 10) \quad \text{and} \quad (3, 16).$$

Go through the steps similar to those taken in the discussion of linear regression on Friday in order to find this supposed parabola of best fit.

Homework due Monday 4/17/17

The purpose of the following assignment is to:

- Become familiar with the Gram-Schmidt orthogonalization process.

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1. Apply the Gram-Schmidt orthogonalization process to the following basis of $\mathbb{R}^3$:

$$\left[\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right].$$

2. Find an orthonormal basis of:

   (a) The 2-dimensional plane S of $\mathbb{R}^3$ spanned by $u_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

   (b) The 3-dimensional subspace S of $\mathbb{R}^4$ spanned by

$$u_1 = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \ u_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}.$$

Review Questions: Weeks 11 and 12

1. What is the definition of the dot product in $\mathbb{R}^n$? Address how this dot product in $\mathbb{R}^n$ determines

   - Length of a vector? Show on an example.
   - Angle between two vectors. Show on an example.

2. The following is a list of several very important properties of the dot product in $\mathbb{R}^n$. Can you provide a justification / explanation for these properties?

   (a) $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0$ only if $x = \vec{0}$.

   (b) $\langle x, y\rangle = \langle y, x\rangle$.

   (c) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$.

3. What does it mean for two vectors to be orthogonal? Show on an example.

4. Does Pythagorean Theorem hold in n-dimensional space? If so, state it and justify it.

5. (a) What are orthogonal bases? What are orthonormal bases? What is the main example of such bases?

   (b) If $F = [f_1, f_2, ..., f_n]$ is an orthogonal basis of $\mathbb{R}^n$, then the coordinates $[v]_F$ of a vector v in $\mathbb{R}^n$ with respect to F can be easily found through dot product. Explain how and why. What can you say about $[v]_F$ if in addition F is orthonormal?

6. When do we say that a matrix is orthogonal? Comment on the fact that columns of an orthogonal $n \times n$ matrix form an orthonormal basis of $\mathbb{R}^n$, and vice versa.

7. What is meant by *the orthogonal complement* of a subspace S of $\mathbb{R}^n$? Also address the following:

   (a) Show an easy informal example of this idea.

   (b) Discuss how one finds orthogonal complements in general. Then work out a specific example.

   (c) Discuss the dimension of the orthogonal complement.

   (d) Make a connection with the Fundamental Subspaces Theorem.

8. In relation to the Fundamental Subspaces Theorem we discussed the identity

$$\langle Ax, y\rangle = \langle x, A^T y\rangle.$$

   Justify this identity, and show that you can use it in abstract situations. For example, show that an orthogonal matrix A satisfies:

$$\langle Ax, Ay\rangle = \langle x, y\rangle \quad \text{and} \quad \|Ax\| = \|x\|.$$

9. What is the definition of the projection $\mathrm{pr}_S\,\vec{v}$ of a vector $\vec{v}$ onto a subspace S? Can you provide a picture illustrating this concept?

10. (a) How do we find a projection of a vector onto a line? Discuss any relevant formulas. An example may be helpful.

    (b) How do we find a projection of a vector onto a subspace whose orthogonal basis $[u_1, u_2, ..., u_k]$ in known? Discuss any relevant formulas. An example may be helpful.

11. (a) Projection of a vector $\vec{v}$ onto a subspace S finds the closest vector $\vec{x}$ to $\vec{v}$ within S. Justify this claim.

    (b) The above idea is used in finding "the best" approximate solutions to inconsistent (usually overdetermined) systems $Ax = b$. Discuss the logic which leads to the formula $x = (A^T A)^{-1} A^T b$. Then work out a specific example.

12. There is a procedure which allows us to modify a basis into an orthogonal or orthonormal basis. How is this procedure called, what does it entail and what is the logic behind it? Implement the procedure on an example of a three-dimensional (sub)space of your choice.

Abstract inner-product spaces

The task now is to generalize what we learned about dot product to more abstract situations.

Week 13

Monday, 4/17: Abstract inner-product spaces. The emphasis here is on examples such as $C[a, b]$.

Tuesday, 4/18: Orthogonality (orthonormal bases, orthogonal complements, projections, approximations, Gram-Schmidt procedure).

Thursday, 4/20: Orthogonality practice, with emphasis on approximation examples

Friday, 4/21: Matrix representation of inner-products. Do take good notes!

Week 14

Monday, 4/24: Self-adjoint operators and the Spectral Theorem.

Tuesday, 4/25: (Course evaluations and) Spectral Theorem practice.

Thursday, 4/27: Summary of the course.

Homework due Thursday 4/20/17

The purpose of the following assignment is to:

- Think about dot-products of functions and abstract dot-products in general.

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1. Compute the following:

   (a) The distance between functions $f(x) = x$ and $g(x) = x^2$ in $C[0,1]$;

   (b) The angle between functions $f(x) = e^x$ and $g(x) = e^{-x}$ in $C[0,1]$;

   (c) The angle between the functions $f(x) = \cos(x)$ and $g(x) = x$ in $C[-\pi, \pi]$.

2. Find the projection of the function $f(x) = x^2$ onto the line in $C[-1,1]$ spanned by the constant function $g(x) = 1$.

3. Verify that

   (a) $\langle x, y\rangle = 9x_1y_1 + 16x_2y_2$ defines a dot product on $\mathbb{R}^2$;

   (b) $\langle P, Q\rangle = P(0)Q(0) + P(1)Q(1)$ defines a dot product on P_2;

   (c) $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)e^{-x}\,dx$ defines a dot product on $C[-1,1]$.

4. None of the following is a dot-product on the indicated vector space. Explain why.

   (a) $\langle x, y\rangle = (x_1y_1)^2 + (x_2y_2)^2$ on $\mathbb{R}^2$;

   (b) $\langle P, Q\rangle = P(0)Q(0) + P(1)Q(1)$ on P_3.

5. Equip P_3 with the dot-product

$$\langle P, Q\rangle = \int_0^1 P(x)Q(x)\,dx.$$

   Use the Gram-Schmidt process to modify the basis $F = [1, x, x^2]$ of P_3 into

   (a) an orthogonal basis with respect to the above dot-product;

   (b) an orthonormal basis with respect to the above dot-product.

6. Equip $\mathbb{R}^2$ with the weighted dot-product $\langle x, y\rangle = 9x_1y_1 + 16x_2y_2$. Working in this nonstandard inner-product space, find

   (a) The angle between the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$;

   (b) The projection of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ onto the line spanned by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Homework due Monday 4/24/17

The purpose of the following assignment is to:

- Think about (linear) algebra with complex numbers.
- Think about complex inner-product spaces.
- Make a comparison between real and complex inner-product spaces.

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1. (a) What is a complex number?

   (b) How do we perform arithmetic operations with complex numbers? Explain addition, subtraction and multiplication of complex numbers on an example.

   (c) What is a conjugate of a complex number? How is the conjugate of a complex number z denoted?

   (d) If $z = a + bi$, what is $z\bar{z}$?

   (e) To divide complex numbers one can "unreduce" the fraction with the conjugate of the divisor (denominator). For instance,

   $$\frac{4+i}{1+2i} = \frac{(4+i)(1-2i)}{(1+2i)(1-2i)} = ... = \frac{6-7i}{5} = \frac{6}{5} - \frac{7}{5}i.$$

   Use this idea to divide the following:

   i. $\frac{2-i}{i}$

   ii. $\frac{2-4i}{1+i}$

   iii. $\frac{1}{2+i}$

2. Solve the following systems of linear equations with complex numbers. Use Gaussian elimination.

   $$\text{a)} \begin{cases} ix + y = 2i \\ x + iy = 0 \end{cases} \qquad \text{b)} \begin{cases} (1+i)x + y = 0 \\ 2x + (1-i)y = 0 \end{cases}.$$

3. The rules of matrix operations apply to matrices whose entries are in complex numbers. Let

   $$A = \begin{pmatrix} 1-i & 1-i \\ i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

   Find the following.

   (a) $A + B$

   (b) $A \cdot B$

   (c) $\det(A)$

(d) A^{-1}, if it exists. What about B^{-1}?

(e) Eigenvalues and eigenvectors of B.

4. Matrices A for which $A = A^\dagger$ are called Hermitian; they are a complex linear algebra analogue of symmetric matrices. Provide an example of a Hermitian matrix.

5. Explain why, both in the cases of real and complex inner-product spaces, knowing what the dot-product does to basis vectors is sufficient for knowing what the dot-product does to any two vectors. In your response you may use general formulas, or you may use examples.

6. Virtually every abstract linear algebra object has its representation in coordinates. Given a basis $E = [e_1, ..., e_n]$ of a (real or complex) inner-product space V, what is meant under the matrix representation of the dot-product with respect to E? How does the knowledge of this matrix representation (and the coordinates $[\vec{v}]_E$ and $[\vec{w}]_E$ of vectors $\vec{v}$ and $\vec{w}$ with respect to E) help us find the dot-product $\langle \vec{v}, \vec{w} \rangle$ of $\vec{v}$ and $\vec{w}$?

7. Find the matrix representation of the standard dot-product in $\mathbb{R}^2$ with respect to the basis
$$E = [e_1, e_2] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].$$
Based on this matrix representation find $\langle 2e_1 - e_2, -3e_1 + e_2 \rangle$.

8. Explain why the matrix representation of a real inner-product is necessarily symmetric. What kind of a matrix does the matrix representation of a complex inner-product have to be?

Homework due Thursday 4/27/17

The purpose of the following assignment is to:

- Think about self-adjoint operators.
- Think about the Spectral Theorem.

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1. What does the phrase *self-adjoint matrix* by definition mean? Prove that symmetric matrices whose entries are real numbers, as well as Hermitian matrices whose entries are complex numbers, are self-adjoint.

2. What does the Spectral Theorem state? Try to prove it!

3. Find an orthonormal basis of eigenvectors for the following matrices.

   (a) $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$;

   (b) $B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$;

   (c) $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$;

   (d) $D = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

4. Diagonalize matrices A, B, C and D from the previous problem. Use these diagonalizations to find A^n, B^n, C^n and D^n.

Review Questions: Weeks 13 and 14

1. - What is the technical definition of a real inner-product space?
   - What examples of real inner-product spaces do you know? (Make sure you mention function and weighted spaces.)
   - Use an example of your choice to explain the mechanics of computing in function inner-product spaces. For instance, you may want to show how one can find an orthonormal basis for the subspace $S = \text{Span}\{1, x, x^2\}$ in $C[-1, 1]$.

2. - What is the technical definition of a complex inner-product space?
   - What is the main example of a complex inner-product space?

3. What is Hermitian transpose of a matrix? What is a Hermitian matrix? Can you give an example?

4. What does the phrase *self-adjoint matrix* by definition mean? Prove that symmetric matrices whose entries are real numbers, as well as Hermitian matrices whose entries are complex numbers, are self-adjoint.

5. What does the Spectral Theorem state? Can you give some arguments in favor of it?

6. Find an orthonormal basis of eigenvectors for

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

   Then, diagonalize A and use the diagonalizations to find A^n.