# COMPLEX VARIABLES: FIRST HOMEWORK ASSIGNMENT ON CONTOUR INTGRATION

## 1. Meditation

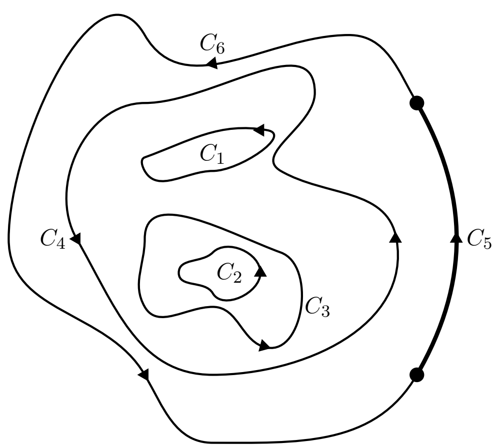
Write down at least three different "proofs" of why

$$\int_C \frac{1}{z}\, dz = 2\pi i$$

for counterclockwise closed contours $C$ which go around the origin exactly once.

## 2. Contour Integration based on Green's / Cauchy Theorem

Consider the configuration on the following diagram. Note that $C_5$ is the bold path with distinct starting and ending points, and that $C_6$ denotes the "left-over".




Suppose $f(z)$ is a holomorphic function of complex variable everywhere except at a couple of points inside the contours $C_1$ and $C_2$. Furthermore, suppose the following:

$$\int_{C_1} f(z)\, dz = -1, \quad \int_{C_2} f(z)\, dz = 1 + i, \quad \int_{C_5} f(z)\, dz = -3 - i.$$

Find the values of $\int_{C_3} f(z)\, dz$, $\int_{C_4} f(z)\, dz$ and $\int_{C_6} f(z)\, dz$.

## 3. Contour Integration via Antiderivatives

(1) Compute the following contour integrals using antiderivatives.
   (a) $\int_C \exp(z)\,dz$ where $C$ is a contour going from $-i$ to $1+i$;
   (b) $\int_C z\exp(z)\,dz$ where $C$ is a contour going from 2 to $i\frac{\pi}{2}$;
   (c) $\int_C \frac{1}{z-2}\,dz$ where $C$ is the counterclockwise contour of a circle of radius 3 centered at the origin.
   (d) $\int_C \frac{1}{z-2}\,dz$ where $C$ is the counterclockwise contour of a circle of radius 1 centered at the origin.
   (e) $\int_C \frac{1}{z^2}\,dz$ where $C$ is the counterclockwise contour going around the origin exactly once. (What happens if the contour goes around the origin multiple times?)
   (f) $\int_C \frac{1}{(2z-1)^2}\,dz$ where $C$ is the counterclockwise circle of very large radius going around the origin exactly once.

## 4. Practicing the Triangle and ML Inequalities

(1) Use the Triangle Inequality to find (upper) bounds for the following. Detailed and legible explanations are expected.
   (a) $|\exp(z)|$ over the disk of radius 3 centered at the origin;
   (b) $|\exp(z)|$ over the disk of radius 3 centered at $i$;
   (c) $\left|\frac{\exp(z)-1}{z}\right|$ over the unit disk centered at the origin;
   (d) $\left|\frac{1}{z+2}\right|$ over the contour of the circle of radius 10 centered at the origin;
   (e) $\left|\frac{1}{z+2}\right|$ over the contour of the circle of radius 1 centered at the origin;
   (f) $\left|\frac{z-3}{z+2}\right|$ over the contour of the circle of radius 1 centered at the origin.

(2) Let $C$ denote the vertical line segment starting at $1-2i$ and ending at $1+2i$. Find, with an explanation, an upper bound for

$$\left|\int_C (\bar{z})^{2018}\,dz\right| \quad \text{and} \quad \left|\int_C (\bar{z})^{-2018}\,dz\right|.$$

(3) Let $C_2$ denote the counterclockwise circle of radius 2 centered at 0. Find, with an explanation, an upper bound for

$$\left|\int_C (\bar{z}-1)^{2018}\,dz\right| \quad \text{and} \quad \left|\int_C (\bar{z}-1)^{-2018}\,dz\right|.$$

(4) Let $C_R$ denote the upper semi-circle of radius $R$ centered at $2-i$, oriented counterclockwise. Find, with an explanation, the following:

$$\lim_{R\to+\infty} \int_{C_R} \frac{dz}{z^3+1}.$$

(5) Let $C_R$ denote the counterclockwise circle of radius $R$ centered at 0. Find, with an explanation, the following:

$$\lim_{R\to+\infty}\int_{C_R}\frac{z^2}{z^4+3z^2+2}\,dz.$$

## 5. Cauchy Integral Formula

(1) Let $C_1$ denote the counterclockwise circle of radius 1 centered at 0, and let $C_\varepsilon$ denote the counterclockwise circle of radius $\varepsilon$ centered at 0; assume that $\varepsilon < 1$.

(a) Use the Cauchy Theorem to conclude the following. Detailed and legible explanation is expected.

$$\int_{C_1}\frac{\exp(z)}{z}dz = 2\pi i + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\exp(z)-1}{z}\,dz.$$

(b) Use the first problem on this homework and the ML inequality to find $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\exp(z)-1}{z}\,dz$. Detailed and legible explanation is expected.

(c) Combine the conclusions from the above to find the value of $\int_{C_1}\frac{\exp(z)}{z}dz$.

**Note:** I understand that one can apply CIF to this problem. That's not the point. I want you to process and internalize the logic which brings us to the CIF in the first place.

(2) Let $C_1$ denote the counterclockwise circle of radius 1 centered at 0, and let $C_\varepsilon$ denote the counterclockwise circle of radius $\varepsilon$ centered at 0; assume that $\varepsilon < 1$.

(a) Use the Cauchy Theorem to conclude the following. Detailed and legible explanation is expected.

$$\int_{C_1}\frac{\cosh(2z)}{z}dz = 2\pi i + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\cosh(2z)-1}{z}\,dz.$$

(b) Use the first problem on this homework and the ML inequality to find $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{\cosh(2z)-1}{z}\,dz$. Detailed and legible explanation is expected.

(c) Combine the conclusions from the above to find the value of $\int_{C_1}\frac{\cosh(2z)}{z}dz$.

**Note:** I understand that one can apply CIF to this problem. That's not the point. I want you to process and internalize the logic which brings us to the CIF in the first place.