

1. THE WAVE FUNCTION

1.1. The Schrödinger Equation.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

1.2. Operators.

$$\begin{aligned}\langle \hat{x} \rangle &= \langle \Psi | \hat{x} | \Psi \rangle \\ \langle \hat{p} \rangle &= m \frac{d\langle \hat{x} \rangle}{dt} = \langle \Psi | -i\hbar \frac{\partial}{\partial x} | \Psi \rangle \\ \langle \hat{H} \rangle &= \langle \Psi | -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) | \Psi \rangle \\ [\hat{A}, \hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A}\end{aligned}$$

1.3. The Uncertainty Principle.

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

2. TIME-INDEPENDENT SCHRÖDINGER EQUATION

2.1. Stationary States.

$$\begin{aligned}-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V\psi &= E\psi \\ \Psi(x, t) &= \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-i\frac{E_n t}{\hbar}}\end{aligned}$$

2.2. The Infinite Square Well.

$$\begin{aligned}V(x) &= \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{otherwise} \end{cases} \\ E_n &= \frac{n^2 \pi^2 \hbar^2}{2mL^2} \\ \psi_n(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \\ \langle \psi_i | \psi_j \rangle &= \delta_{ij} \\ c_n &= \langle \psi_n | \Psi(x, t=0) \rangle\end{aligned}$$

The probability of being in state n is equal to $|c_n|^2$.

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

2.3. The Harmonic Oscillator.

2.3.1. Base State.

$$\begin{aligned}\Psi_0 &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar} x^2} \\ E_0 &= \frac{\hbar\omega}{2}\end{aligned}$$

2.3.2. Ladder Operators.

$$\begin{aligned}a_+ &= \frac{1}{\sqrt{2m\omega\hbar}} [-i\hat{p} + m\omega x] \\ a_- &= \frac{1}{\sqrt{2m\omega\hbar}} [i\hat{p} + m\omega x] \\ [a_-, a_+] &= 1\end{aligned}$$

2.3.3. Solving.

$$\begin{aligned}a_+ \psi_n &= \sqrt{n+1} \psi_{n+1} \\ a_- \psi_n &= \sqrt{n} \psi_{n-1} \\ \psi_n &= \frac{1}{\sqrt{n!}} (a_+)^n \psi_0 \\ E_n &= \left(n + \frac{1}{2}\right) \hbar\omega\end{aligned}$$

2.3.4. Operators.

$$\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) \\ \hat{p} &= i\sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-)\end{aligned}$$

2.4. The Free Particle.

$$\begin{aligned}k^2 &= \frac{2mE}{\hbar^2} \\ k &\in \mathbb{R} \\ E &= \frac{\hbar^2 k^2}{2m} \\ p &= \hbar k \\ \Psi(x, t) &= \underbrace{Ae^{ikx - i\frac{E}{\hbar}t}}_{\text{right mover}} + \underbrace{Be^{-ikx - i\frac{E}{\hbar}t}}_{\text{left mover}} \\ \Phi(k) &= \frac{1}{\sqrt{a\pi}} \frac{\sin(ka)}{k}\end{aligned}$$

2.5. Step Potential.

$$\begin{aligned}V(x) &= \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases} \\ J_x &= -\frac{\hbar i}{2m} \left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right] \\ T &\equiv \left| \frac{J_{trans}}{J_{inc}} \right| \quad 0 \leq T \leq 1 \\ R &\equiv \left| \frac{J_{refl}}{J_{inc}} \right| \quad 0 \leq R \leq 1\end{aligned}$$

2.5.1. Case 1. $E > V_0$

$$\begin{aligned}k_1^2 &= \frac{2mE}{\hbar^2} \\ \Psi_I &= \underbrace{Ae^{ik_1 x}}_{\text{Incident}} + \underbrace{Be^{-ik_1 x}}_{\text{Reflected}} \\ k_2^2 &= \frac{2m(E - V_0)}{\hbar^2} \\ \Psi_{II} &= \underbrace{Ce^{ik_2 x}}_{\text{Transmitted}} + \underbrace{De^{-ik_2 x}}_{\text{DNE}} \\ T &= \frac{4}{\left[1 + \frac{k_2}{k_1}\right]^2} \frac{k_2}{k_1} \\ R &= \frac{\left[1 - \frac{k_2}{k_1}\right]^2}{\left[1 + \frac{k_2}{k_1}\right]^2}\end{aligned}$$

2.5.2. Case 2. $E < V_0$

$$\begin{aligned}k_1^2 &= \frac{2mE}{\hbar^2} \\ \Psi_I &= \underbrace{Ae^{ik_1 x}}_{\text{Incident}} + \underbrace{Be^{-ik_1 x}}_{\text{Reflected}} \\ k_2^2 &= \frac{2m(E - V_0)}{\hbar^2} \\ \Psi_{II} &= \underbrace{Ce^{-k_2 x}}_{\text{Transmitted}} + \underbrace{De^{k_2 x}}_{\text{DNE}} \\ T &= 0 \\ R &= 1\end{aligned}$$

3. FORMALISM

3.1. **Observables.** Observances are represented by hermitian operators, and $\langle O \rangle = \mathbb{R}$.

$$\hat{O}|\psi\rangle = o|\psi\rangle$$

3.2. The Uncertainty Principle.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Observables whose operators do not commute are called incompatible observables, meaning that measuring one changes the other.

3.3. **Dirac Notation.** We can express functions as a basis vector in Hilbert space.

$$|\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \langle \alpha| = (\alpha_1^* \alpha_2^* \cdots \alpha_n^*)$$

The identity operator is

$$\hat{I} = \sum |\alpha\rangle \langle \alpha|$$

For some operator \hat{O} and eigenvalues o_n

$$\hat{O}|n\rangle = o_n |n\rangle$$

3.3.1. Matrix Operators.

$$O_{nm} \equiv \langle n | \hat{O} | m \rangle$$

Hermitian if

$$O_{nm} = O_{mn}^*$$

3.3.2. Harmonic Oscillator.

$$\begin{aligned}a_+ |n\rangle &= \sqrt{n+1} |n+1\rangle \\ a_- |n\rangle &= \sqrt{n} |n-1\rangle\end{aligned}$$

3.3.3. Wave Function.

$$\begin{aligned}|\Psi\rangle &= \sum_n A_n |n\rangle \\ A_n &= \langle n | \Psi \rangle \\ |\Psi\rangle &= \sum_n \langle n | \Psi \rangle |n\rangle\end{aligned}$$

4. QUANTUM MECHANICS IN THREE DIMENSIONS

4.1. Schrödinger Equation in Spherical Coordinates.

4.1.1. Legendre.

$$P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l$$

$$P_l^m(x) \equiv (1 - x^2)^{|m|/2} \left(\frac{d}{dx} \right)^{|m|} P_l(x)$$

$$\begin{aligned} P_0 &= 1 & P_0^0 &= 1 \\ P_1 &= x & P_1^1 &= \sin \theta \\ P_2 &= \frac{1}{2}(3x^2 - 1) & P_1^0 &= \cos \theta \\ & & P_2^2 &= 3 \sin^2 \theta \\ P_3 &= \frac{1}{2}(5x^3 - 3x) & P_2^1 &= 3 \sin \theta \cos \theta \\ P_4 &= \frac{1}{8}(35x^4 - 30x^2 - 3) & P_2^0 &= \frac{1}{2}(3 \cos^2 \theta - 1) \end{aligned}$$

4.1.2. Spherical Harmonics.

$$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos \theta)$$

$$\epsilon = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m \leq 0 \end{cases}$$

$$\langle Y_l^m | Y_{l'}^{m'} \rangle = \delta_{ll'} \delta_{mm'}$$

$$Y_0^0 = \left(\frac{1}{4\pi} \right)^{1/2}$$

$$Y_1^0 = \left(\frac{3}{4\pi} \right)^{1/2} \cos \theta$$

$$Y_1^{\pm 1} = \mp \left(\frac{3}{8\pi} \right)^{1/2} \sin \theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi} \right)^{1/2} (3 \cos^2 \theta - 1)$$

$$Y_2^{\pm 1} = \mp \left(\frac{15}{8\pi} \right)^{1/2} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_2^{\pm 2} = \left(\frac{15}{32\pi} \right)^{1/2} \sin^2 \theta e^{\pm 2i\phi}$$

4.2. The Hydrogen Atom.

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$

$$E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} \quad n \in \mathbb{N}$$

$$a \equiv \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 0.529 \times 10^{-10} m$$

$$E_1 = -13.6 \text{ eV}$$

4.2.1. Laguerre.

$$L_q(x) \equiv e^x \left(\frac{d}{dx} \right)^q (e^{-x} x^q)$$

$$L_{q-p}^p(x) \equiv (-q)^p \left(\frac{d}{dx} \right)^p L_q(x)$$

$$L_0 = 1$$

$$L_1 = -x + 1$$

$$L_2 = x^2 - 4x + 2$$

$$L_3 = -x^3 + 9x^2 - 18x + 6$$

$$L_0^0 = 1$$

$$L_1^0 = -x + 1$$

$$L_2^0 = x^2 - 4x + 2$$

$$L_0^1 = 1$$

$$L_1^1 = -2 + 4$$

$$L_0^2 = 2$$

$$L_1^2 = -6x + 18$$

$$L_2^2 = 12x^2 - 96x + 144$$

$$L_0^3 = 6$$

$$L_1^3 = -24 + 96$$

4.2.2. Radial.

$$R_{nl}(r) = \sqrt{\left(\frac{2}{na} \right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \cdot e^{-r/na} \left(\frac{2r}{na} \right)^l \left[L_{n-l-1}^{2l+1} \left(\frac{2r}{na} \right) \right]$$

$$R_{10} = 2a^{-3/2} \exp(-r/a)$$

$$R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a} \right) \exp(-r/2a)$$

$$R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp(-r/2a)$$

4.2.3. Wave Function.

$$n \in \mathbb{N}$$

$$0 \leq l < n$$

$$-l \leq m \leq l$$

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na} \right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} \cdot e^{-r/na} \left(\frac{2r}{na} \right)^l \left[L_{n-l-1}^{2l+1} \left(\frac{2r}{na} \right) \right] Y_l^m(\theta, \phi)$$

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$\langle \psi_{nlm} | \psi_{n'l'm'} \rangle = \delta_{nn'} \delta_{ll'} \delta_{mm'}$$

4.3. Angular Momentum.

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

$$[L_x, L_y] = i\hbar L_z, [L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y$$

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2$$

$$[L^2, L_x] = 0, [L^2, L_y] = 0, [L^2, L_z] = 0$$

$$L_{\pm} \equiv L_x \pm iL_y$$

$$[L_z, L_{\pm}] = \pm \hbar L_{\pm}$$

$$[L^2, L_{\pm}] = 0$$

4.4. Spin.

$$S_i = \frac{\hbar}{2} \sigma_i$$

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$