HW.6

Problem 4

> Suppose F is a field, and R is an integral domain that contains F as a subring. If R, considered as a vector space over F, is finite dimensional then show R is a field.

Considering R as a finite dimensional vector space over F, with dimension n, then we known R is spanned by a set of linearly independent basis vectors, which we will denote $\{v_1, \ldots, v_n\}$. Now we define a mapping $f : V \mapsto F^n$. defined as

$$f(v_i) = (\overbrace{0, \ldots, 0}^{i-1}, 1, \overbrace{0, \ldots, 0}^{n-i-2})$$

Now we demonstrate that f is an isomorphism between these two vector spaces.

First consider $a, b \in R$, we can write these as

$$\begin{aligned} a &= a_0 v_0 + \ldots + a_n v_n \\ b &= b_0 v_0 + \ldots + b_n v_n \end{aligned}$$

Let us compute $f(a+b)$

$$\begin{aligned} f(a+b) &= f(a_0 v_0 + \ldots + a_n v_n + b_0 v_0 + \ldots + b_n v_n) \\ &= f((a_0 + b_0) v_0 + \ldots + (a_n + b_n) v_n) \\ &= (a_0 + b_0) f(v_0) + \ldots + (a_n + b_n) f(v_n) \\ &= a_0 f(v_0) + \ldots + a_n f(v_n) + b_0 f(v_0) + \ldots + b_n f(v_n) \\ &= f(a) + f(b) \end{aligned}$$

Thus f is a homomorphism.

Consider some element $a \in F^n$, we can write this as $a = (a_0, a_1, \ldots, a_n)$, and thus this is equivalent to $a_0 e_0 + a_1 e_1 + \ldots + a_n e_n$, and now we can take $a_0 v_0 + \ldots + a_n v_n \in R$, then

$$\begin{aligned} f(a_0 v_0 + \ldots + a_n v_n) &= a_0 f(v_0) + \ldots + a_n f(v_n) \\ &= a_0 e_0 + \ldots + a_n e_n = a \end{aligned}$$

Thus f is injective.

Consider some $a, b \in R$ such that $f(a) = f(b)$. This means that

$$\begin{aligned} f(a_0 v_0 + \ldots + a_n v_n) &= f(b_0 v_0 + \ldots + b_n v_n) \\ a_0 f(v_0) + \ldots + a_n f(v_n) &= b_0 f(v_n) + \ldots + b_n f(v_n) \end{aligned}$$

Thus we can conclude that $a_i = b_i$ for $i = 1, \ldots, n$, and so $a = b$, and thus f is surjective.

Thus f is an isomorphism between R and F^n, so R and F^n are isomorphic. We have previously shown that F^n is a field, and so since R is isomorphic to F^n then we conclude that R must also be a field.

Problem 5

> Let θ be a complex root of the irreducible polynomial $x^3 - 3x + 4 \in \mathbb{Q}[x]$. Find the inverse of $\theta^2 + \theta + 1$ in $\mathbb{Q}(\theta)$ explicitly in the form $a + b\theta + c\theta^2$ with $a, b, c \in \mathbb{Q}$.

First we compute

$$\begin{aligned}(\theta^2 + \theta + 1)(c\theta^2 + b\theta + a) &= 1 \\ (a - 4c - 4b) + (4b + a - c)\theta + (4c + b + a)\theta^2 &= 1\end{aligned}$$

Then we can construct a system of linear equations

$$\begin{aligned}4c + b + a &= 0 \\ 4b + a - c &= 0 \\ a - 4c - 4b &= 1\end{aligned}$$

We then solve this system for c to find $c = -\frac{3}{49}$, $b = -\frac{5}{49}$, and $a = \frac{17}{49}$. So we conclude that the inverse if given by

$$\frac{17}{49} - \frac{5}{49}\theta - \frac{3}{49}\theta^2$$