Linear Algebra Review

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1 Determinant

1.1 Solving

1.1.1 Cofactor Expansion

$$det(A) = det \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \tag{1}$$

$$det(A) = \alpha_{1,1}[\alpha_{2,2}\alpha_{3,3}] - \alpha_{1,2}[\alpha_{2,1}\alpha_{3,3}] - \alpha_{3,1}\alpha_{2,3}] \tag{2}$$

$$+ \alpha_{1,3}[\alpha_{3,2}\alpha_{3,2} - \alpha_{3,1}\alpha_{2,2}] \tag{3}$$

1.1.2 Gaussian Operations

using row and column operations create a row echelon matrix from A, by folowing the rules;

1. Swapping two rows multiplies the determinant by -1.
2. Multipling a row by a nonzero scalar multiplies the determinant by the same scalar.
3. Adding to one row a scalar multiple of another does not change the determinant.

$$B = ref(A) \tag{4}$$

$$d = \text{The product of scalars by which the determinant has been multiplied} \tag{5}$$

$$det(A) = \frac{\prod diagonal(B)}{d} \tag{6}$$

1.1.3 Permutations

$sign(p)$ is the number of swaps necesary to achive that permutation.

$$det(A) = \sum_p sign(p)\alpha_{1,p(1)}\alpha_{2,p(2)} \cdots \alpha_{n,p(n)} \tag{7}$$

1.2 Properties

$det(A) = 0$ Then A is singular and A^{-1} does not exist.

$det(A) \neq 0$ Then A is invertible and A^{-1} exists.

2 Inverse

2.1 Gaussian Elimination

Using row operations only, convert the matrix on the left form A to I, and the resulting matrix on the right will be A^{-1}.

$$A|II \qquad\qquad |A^{-1} \tag{8}$$

2.2 Cofactor

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} \widehat{A_{1,1}} & \widehat{A_{1,2}} & \widehat{A_{1,3}} \\ \widehat{A_{2,1}} & \widehat{A_{2,2}} & \widehat{A_{2,3}} \\ \widehat{A_{3,1}} & \widehat{A_{3,2}} & \widehat{A_{3,3}} \end{bmatrix}^T \tag{9}$$

$$\widehat{A_{1,1}} = det \begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \tag{10}$$

2.3 Properties

$$(AB)^{-1} = B^{-1}A^{-1}A^{-1}A \qquad = IAA^{-1} = I \tag{11}$$

3 Transposse

Transpose is found by swapping the rows and columns of a matrix.

$$A^T = \begin{bmatrix} \alpha_{1,1} & \alpha_{2,1} & \alpha_{3,1} \\ \alpha_{1,2} & \alpha_{2,2} & \alpha_{3,2} \\ \alpha_{1,3} & \alpha_{2,3} & \alpha_{3,3} \end{bmatrix} \tag{12}$$

3.1 symmetric and Anti-Symmetric

If a matrix is symmetric then:

$$A^T = A \tag{13}$$

If a matrix is anti-symmetric then:

$$A^T = -A \tag{14}$$

3.2 Properties

$$(AB)^T = B^T A^T \tag{15}$$

4 Rank and Nullity

Rank is the dimension of the image, nullity is the dimension of the kernel of A. Kernel is the vectors that are collaped to a point.

$$\text{For } A_{nxn} \tag{16}$$

$$Rank(A) = dim(Im(A)) \tag{17}$$

$$Nullity(A) = dim(Ker(A)) \tag{18}$$

$$det(A) \neq 0 \Rightarrow \text{Columns are linearindependent} \Rightarrow Rank(A) = n \tag{19}$$

$$Rank(A) + Nullity(A) = n \tag{20}$$

4.1 Solve for Rank

$$\text{Use Gaussian Elimination on } A \tag{21}$$
$$Rank(A) = \text{The number of pivots} \tag{22}$$

4.2 Image and Pre-Image

$$A\vec{v} = \vec{w} \tag{23}$$
$$\vec{v} \text{ is the pre-image of } \vec{w} \tag{24}$$
$$\vec{w} \text{ is the image of } \vec{w} \tag{25}$$

5 Eigenvalues and Eigenvectors

$$A\vec{v} = \lambda\vec{v} \tag{26}$$

5.1 Eigenvalues

λ is an eigenvalue of A if $A\vec{v} = \lambda\vec{v}$ and $\vec{v}$ is non zero. To find eigen values solve the folowing equation fo λ.

$$det(A - \lambda I) = 0 \tag{27}$$

5.2 Eigenvectors

$\vec{v}$ is an eigenvector of A if $A\vec{v} = \lambda\vec{v}$ and $\vec{v}$ is non zero. To find eigenvectors find the rref of the augmented matrix.

$$\vec{v} = rref(A - \lambda I) \tag{28}$$

6 Linear Independance

Vectors are linear independant if the nullity of the matrix formed by the vectors is zero.

$$\vec{v_1}\ \vec{v_2}\ \vec{v_3} \tag{29}$$
$$Nullity\begin{pmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{pmatrix} = 0 \tag{30}$$
$$\Rightarrow \vec{v_1}, \vec{v_2}, \text{ and } \vec{v_3} \text{ are linear independent.} \tag{31}$$

7 Basis

A basis is a span of linear independent vectors which can express everything (within the space) as a linear comination of the vectors.

8 Matrix Properties

8.1 Algebra

8.1.1 Addition

Matrix addition is done term by term.

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} + \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{bmatrix} = \begin{bmatrix} \alpha_{1,1}+\beta_{1,1} & \alpha_{1,2}+\beta_{1,2} & \alpha_{1,3}+\beta_{1,3} \\ \alpha_{2,1}+\beta_{2,1} & \alpha_{2,2}+\beta_{2,2} & \alpha_{2,3}+\beta_{2,3} \\ \alpha_{3,1}+\beta_{3,1} & \alpha_{3,2}+\beta_{3,2} & \alpha_{3,3}+\beta_{3,3} \end{bmatrix} \tag{32}$$

8.1.2 Subtraction

Matrix subtraciton is done term by term.

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{bmatrix} - \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{bmatrix} = \begin{bmatrix} \alpha_{1,1}-\beta_{1,1} & \alpha_{1,2}-\beta_{1,2} & \alpha_{1,3}-\beta_{1,3} \\ \alpha_{2,1}-\beta_{2,1} & \alpha_{2,2}-\beta_{2,2} & \alpha_{2,3}-\beta_{2,3} \\ \alpha_{3,1}-\beta_{3,1} & \alpha_{3,2}-\beta_{3,2} & \alpha_{3,3}-\beta_{3,3} \end{bmatrix} \tag{33}$$

8.1.3 Multiplication

For matrix multiplicaiton to work, the columns of the first matrix must match the rows of the second matrix.

$$A \text{ is a nxm matrix} \tag{34}$$

$$B \text{ is a mxp matrix} \tag{35}$$

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,m} \end{bmatrix} \tag{36}$$

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,p} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,p} \end{bmatrix} \tag{37}$$

$$AB = \begin{bmatrix} (AB)_{1,1} & (AB)_{1,2} & \cdots & (AB)_{1,p} \\ (AB)_{2,1} & (AB)_{2,2} & \cdots & (AB)_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ (AB)_{n,1} & (AB)_{n,2} & \cdots & (AB)_{n,p} \end{bmatrix} \tag{38}$$

$$(AB)_{i,j} = \sum_{k=1} A_{i,k} B_{k,j} \tag{39}$$

8.1.4 Division

Matrix division is just multiplicaiton by the inverse matrix.

$$\frac{B}{A} = B * \frac{1}{A} = B * A^{-1} \tag{40}$$

8.2 Commutative and Associative

$$AB \neq BA \tag{41}$$

$$ABC = (AB)C = A(BC) \tag{42}$$