# Linear Independence and Series Expansions in Function Spaces

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**1. INTRODUCTION.** The concept of *linear independence* plays a central role in the theory of vector spaces. Indeed, its history is as old as vector spaces themselves. Linear independence is one of the two conditions defining a *basis* $\{v_k\}_{k=1}^n$ for a vector space $V$: while one requirement is that $\{v_k\}_{k=1}^n$ span $V$ (i.e., that each $v$ in $V$ have a representation $v = \sum_{k=1}^n c_k v_k$ for some coefficients $c_k$), linear independence guarantees that the representation of $v$ in terms of the basis vectors is *unique*.

In this article we consider complex vector spaces generated by certain special functions and examine whether their linear combinations have unique representations. We start with the most elementary case, namely, trigonometric functions. After that we consider complex exponential functions and other more complicated systems of functions (Gabor systems and wavelet systems) that have recently attracted much attention both in pure mathematics and in applied science. We present some open problems related to those systems, problems that are easy to formulate but apparently very difficult to solve. Finally, we introduce frames, which generalize the concept of an orthonormal basis. The motivation for this generalization comes from Gabor analysis, where we show that certain desirable properties are incompatible with the orthonormal basis requirement. We show how the concept of linear dependence for wavelet systems plays a key role in modern constructions of frames having wavelet structure.

**2. SINE AND COSINE FUNCTIONS.** In this section we study the question of linear independence for a set of cosine functions and sine functions. The most general case is to consider cosine functions $\cos(\lambda_k x)$ with some real parameters $\lambda_k$ ($k = 1, 2, \ldots, n$) and sine functions $\sin(\mu_\ell x)$ with parameters $\mu_\ell$ ($\ell = 1, \ldots, m$) for some positive integers $n$ and $m$. That is, we consider the functions in the set

$$\{\cos(\lambda_k x)\}_{k=1}^n \cup \{\sin(\mu_\ell x)\}_{\ell=1}^m. \tag{1}$$

A moment's thought reveals that one must impose conditions on the numbers $\lambda_k$ and $\mu_\ell$ in order to guarantee that the functions in (1) are linearly independent. An example illustrates this point.

**Example 1.** Consider numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ for which $\lambda_1 = -\lambda_2$. Since $\cos x = \cos(-x)$ for all real $x$, it follows that

$$1 \cdot \cos(\lambda_1 x) - 1 \cdot \cos(\lambda_2 x) + 0 \cdot \cos(\lambda_3 x) + \cdots + 0 \cdot \cos(\lambda_n x) = 0$$

for all $x$ in $\mathbb{R}$. This shows that the functions $\cos(\lambda_k x)$ are linearly dependent on any interval.

Example 1 demonstrates that we have to assume that $|\lambda_k| \neq |\lambda_j|$ when $k \neq j$ if we want the functions $\cos(\lambda_k x)$ to be linearly independent. Similarly, we must have $|\mu_\ell| \neq |\mu_j|$ when $\ell \neq j$ if we want the set $\{\sin(\mu_\ell x)\}_{\ell=1}^m$ to be linearly independent. Moreover, since $\sin(0) = 0$, in the latter case we also need $\mu_\ell \neq 0$ for all $\ell$. Interestingly, these assumptions turn out to be sufficient for the combined system in (1) to be linearly independent on an *arbitrary interval*:

**Theorem 1.** *If $\{\lambda_k\}_{k=1}^n$ and $\{\mu_\ell\}_{\ell=1}^m$ are sets of real numbers such that $\mu_l \neq 0$ for each $l$ and such that $|\lambda_l| \neq |\lambda_j|$ and $|\mu_l| \neq |\mu_j|$ when $l \neq j$, then the set of functions*

$$\{\cos(\lambda_k x)\}_{k=1}^n \cup \{\sin(\mu_\ell x)\}_{\ell=1}^m$$

*is linearly independent on any interval $I$.*

*Proof.* We may assume that the $\lambda_k$ and $\mu_\ell$ are nonnegative and satisfy

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_m.$$

Now suppose that for coefficients $c_k$ and $d_\ell$ it is true that

$$\sum_{k=1}^n c_k \cos(\lambda_k x) + \sum_{\ell=1}^m d_\ell \sin(\mu_\ell x) = 0 \tag{2}$$

for all $x$ in $I$. Our objective is to prove that $c_k = d_\ell = 0$ for all $k$ and $\ell$, to which we proceed as follows. First assume that $\lambda_n > \mu_m$ (i.e., the largest $\lambda$-value is strictly greater than the largest $\mu$-value). Differentiating both sides of equation (2) $4N$ times for arbitrary $N$ in $\mathbb{N}$ leads to

$$\sum_{k=1}^n c_k \lambda_k^{4N} \cos(\lambda_k x) + \sum_{\ell=1}^m d_\ell \mu_\ell^{4N} \sin(\mu_\ell x) = 0$$

on $I$, which can be rewritten as

$$c_n \cos(\lambda_n x) = -\sum_{k=1}^{n-1} c_k \left(\frac{\lambda_k}{\lambda_n}\right)^{4N} \cos(\lambda_k x) - \sum_{\ell=1}^m d_\ell \left(\frac{\mu_\ell}{\lambda_n}\right)^{4N} \sin(\mu_\ell x). \tag{3}$$

Choose $x$ to be a point in $I$ for which $\cos(\lambda_n x) \neq 0$. Since the right-hand side of (3) tends to 0 as $N \to \infty$, this forces $c_n = 0$.

A completely analogous argument establishes that $d_m = 0$ if $\mu_m > \lambda_n$. If all $\lambda$-values are different from all $\mu$-values, repeated application of the foregoing argument shows that $c_k = d_\ell = 0$ for all $k$ and $\ell$. Handling the case of coincidence between certain $\lambda$-values and $\mu$-values demands only a slight modification of the proof. Suppose, for example, that $\lambda_n = \mu_m$. Then the argument shows that

$$c_n \cos(\lambda_n x) + d_m \sin(\mu_m x) = -\sum_{k=1}^{n-1} c_k \left(\frac{\lambda_k}{\lambda_n}\right)^{4N} \cos(\lambda_k x) - \sum_{\ell=1}^{m-1} d_\ell \left(\frac{\mu_\ell}{\mu_m}\right)^{4N} \sin(\mu_\ell x)$$

for all $x$ in $I$. Letting $N \to \infty$, we conclude that

$$c_n \cos(\lambda_n x) + d_m \sin(\lambda_n x) = 0 \tag{4}$$

on $I$. Differentiating (4) leads to

$$\lambda_n (d_m \cos(\lambda_n x) - c_n \sin(\lambda_n x)) = 0. \tag{5}$$

Since $\lambda_n = \mu_m \neq 0$, equations (4) and (5) imply that $c_n = d_m = 0$. ∎

Theorem 1 is related to classical Fourier analysis. A Fourier series is an expansion of a $2\pi$-periodic function $f : \mathbb{R} \to \mathbb{C}$ in terms of the functions

$$1, \cos x, \cos 2x, \ldots, \cos nx, \ldots, \sin x, \sin 2x, \ldots, \sin nx, \ldots. \tag{6}$$

As a special case of Theorem 1 we see that any finite subfamily of the functions in (6) is linearly independent.

Fourier series are often used in *complex form.* Let $i$ signify the complex unit and recall that the *complex exponential function* $e^{i\lambda x}$ ($\lambda \in \mathbb{R}$) is defined by

$$e^{i\lambda x} = \cos(\lambda x) + i \sin(\lambda x).$$

With this notation, the Fourier series of a periodic function $f : \mathbb{R} \to \mathbb{C}$ of period $2\pi$ that is (Lebesgue) integrable on $(-\pi, \pi)$ can be written

$$f(x) \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx} \qquad \left( c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx \right). \tag{7}$$

The methods discussed hitherto can be used to prove that a family of complex exponentials is linearly independent if no $\lambda$-value is repeated:

**Proposition 1.** *If $\{\lambda_k\}_{k=1}^n$ is a set of real numbers for which $\lambda_l \neq \lambda_j$ when $l \neq j$, then the set of complex exponentials $\{e^{i\lambda_k x}\}_{k=1}^n$ is linearly independent on an arbitrary interval.*

**3. BASES IN HILBERT SPACES.** In this section we give a precise interpretation of the Fourier series (7) in terms of orthonormal bases in Hilbert spaces. We first provide a short introduction to Hilbert spaces. A more detailed treatment can be found in most textbooks on analysis, for example, in [**22**].

From linear algebra we know many pleasant properties of a finite-dimensional vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. For example, such a space $V$ has orthonormal bases. Denoting an orthonormal basis by $\{e_k\}_{k=1}^n$, we can easily express each $f$ in $V$ in terms of the vectors $e_k$:

$$f = \sum_{k=1}^{n} \langle f, e_k \rangle e_k.$$

On a more theoretical level, we know that each Cauchy sequence in a finite-dimensional normed vector space, such as an inner product space with its associated norm $|| \cdot ||$ ($||f|| = \sqrt{\langle f, f \rangle}$), is convergent.

The nice properties of finite-dimensional vector spaces make it natural to ask whether similar properties hold for infinite-dimensional vector spaces with inner products. It turns out that if we restrict our attention to the inner product spaces with the property that each Cauchy sequence (relative to the norm arising from the inner product) is convergent, then we are able to extend most of the results from linear algebra to the infinite-dimensional setting. This leads to the formal definition of a Hilbert space: a *Hilbert space* $\mathcal{H}$ is a vector space equipped with an inner product and having the property that each Cauchy sequence is convergent. For technical reasons, we assume in addition that our Hilbert spaces $\mathcal{H}$ are separable in the metric space topology associated with $|| \cdot ||$, meaning that there exists a countable dense subset of $\mathcal{H}$.

In the following we restrict our attention to complex separable Hilbert spaces $\mathcal{H}$, with the inner product $\langle \cdot, \cdot \rangle$ chosen to be linear in the first entry. A family $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is called a *basis* for $\mathcal{H}$ if for each $f$ in $\mathcal{H}$ there exist unique scalar coefficients $c_k(f)$ such that

$$f = \sum_{k=1}^{\infty} c_k(f) f_k. \tag{8}$$

The convergence in (8) is in the norm of $\mathcal{H}$; that is, (8) means that

$$\left\| f - \sum_{k=1}^{N} c_k(f) f_k \right\| \to 0$$

as $N \to \infty$. In practice, it is important to know how to calculate the coefficients $c_k(f)$. For this reason, it is often an advantage to consider an *orthonormal basis* (i.e., a basis $\{f_k\}_{k=1}^{\infty}$ for which $\langle f_j, f_k \rangle$ equals one if $k = j$ and zero if $k \neq j$). In fact, if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis, the reader can check that

$$c_k(f) = \langle f, f_k \rangle;$$

thus, the expansion (8) takes the convenient form

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k. \tag{9}$$

In the context of Fourier series, the relevant vector space is

$$L^2(-\pi, \pi) = \left\{ f : (-\pi, \pi) \to \mathbb{C} \quad \text{with} \quad \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty \right\}.$$

If we identify functions that are equal almost everywhere and outfit the resulting space with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx,$$

then $L^2(-\pi, \pi)$ becomes a Hilbert space.

One of the main results for Fourier series is that the functions $\{1/\sqrt{2\pi} e^{ikx}\}_{k \in \mathbb{Z}}$ form an orthonormal basis for $L^2(-\pi, \pi)$. The expansion (9) corresponds to the Fourier series in (7). Thus, the exact meaning of (7) is that for $f$ in $L^2(-\pi, \pi)$

$$\left\| f - \sum_{|k| \leq N} c_k e^{ik\cdot} \right\|_{L^2(-\pi,\pi)} \to 0$$

as $N \to \infty$, which translates to

$$\int_{-\pi}^{\pi} \left| f(x) - \sum_{|k| \leq N} c_k e^{ikx} \right|^2 dx \to 0$$

as $N \to \infty$. The complex exponentials $e^{ikx}$ $(k \in \mathbb{Z})$ all have period $2\pi$. In contrast, for arbitrary real $\lambda_k$ the system of complex exponentials $\{e^{i\lambda_k x}\}_{k \in \mathbb{Z}}$ might not even have a common period. The branch of mathematics dealing with properties of $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ for such $\lambda_k$ is called *nonharmonic Fourier analysis*. Very often it requires a deep understanding of complex analysis, and we will not be able to go further into this fascinating subject here (see the book [**27**] for an outstanding presentation). There are, however, a few cases where interesting results can be obtained by means of relatively easy techniques. Frequently this is the case if the numbers $\lambda_k$ can be considered as small perturbations of $k$ (i.e., if $|k-\lambda_k|$ is uniformly small for $k$ in $\mathbb{Z}$). An important example of this is the famous "Kadec 1/4-Theorem": it asserts that if

$$\sup_{k\in\mathbb{Z}} |k-\lambda_k| < 1/4, \tag{10}$$

then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is a basis for $L^2(-\pi,\pi)$. Kadec proved this result in 1963, and an elementary proof can be found in [**27**]. We note that the hypothesis in Kadec's 1/4-theorem is just a sufficient condition, not a necessary one. For example, if the condition (10) is satisfied when the supremum over $\mathbb{Z}$ is replaced with the supremum over $\mathbb{Z}\setminus\{k_0\}$ for some $k_0$ in $\mathbb{Z}$ and if $\lambda_{k_0}\neq\lambda_k$ for all $k$ in $\mathbb{Z}\setminus\{k_0\}$, then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ is still a basis for $L^2(-\pi,\pi)$. In words, this result says that the parameter in *one* of the exponential functions can be perturbed by an arbitrary amount.

**4. GABOR SYSTEMS AND WAVELETS.** In this section we consider some systems of functions that do not appear in classical analysis but have attracted much attention in recent years. The functions in these systems belong to the function space

$$L^2(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{C} \quad \text{with} \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty \right\}.$$

Like $L^2(-\pi,\pi)$, the vector space $L^2(\mathbb{R})$ becomes a Hilbert space if we identify functions that are equal almost everywhere and endow the space with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

The complex exponential functions do not belong to $L^2(\mathbb{R})$. In fact,

$$\int_{-\infty}^{\infty} |e^{i\lambda x}|^2\,dx = \int_{-\infty}^{\infty} dx = \infty$$

for any real $\lambda$. On the other hand, if $g$ belongs to $L^2(\mathbb{R})$, then the function $x\mapsto e^{i\lambda x}g(x-\mu)$ is also in $L^2(\mathbb{R})$ for all $\lambda$ and $\mu$ in $\mathbb{R}$, because

$$\int_{-\infty}^{\infty} |e^{i\lambda x}g(x-\mu)|^2\,dx = \int_{-\infty}^{\infty} |g(x-\mu)|^2\,dx = \int_{-\infty}^{\infty} |g(x)|^2\,dx < \infty.$$

Note that the graph of the function $x\mapsto g(x-\mu)$ is obtained simply by translating the graph of $g$ $\mu$ units to the right. One can check that the operation of multiplying $g$ by $e^{i\lambda x}$ corresponds to a translation of the *Fourier transform* of $g$. To be more precise, if we define the Fourier transform $\hat{g}:\mathbb{R}\to\mathbb{C}$ of the function $g$ in $L^2(\mathbb{R})$ by

$$\hat{g}(\gamma) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i\gamma x}\,dx,$$

then the function $h(x) = e^{i\lambda x}g(x)$ has

$$\hat{h}(\gamma) = \hat{g}\left(\gamma - \frac{\lambda}{2\pi}\right).$$

Now let $a$ and $b$ be given positive numbers, and let $g$ in $L^2(\mathbb{R})$ be a given function. We consider collections of functions of the form

$$\{e^{2\pi imbx}g(x - na)\}_{m,n\in\mathbb{Z}}. \tag{11}$$

A system of functions of the type (11) is called a *regular Gabor system.* If $I$ is a countable (or finite) index set and $\{(\lambda_n, \mu_n)\}_{n\in I}$ is an arbitrary family of points in $\mathbb{R}^2$, the collection of functions given by

$$\{e^{2\pi i\lambda_n x}g(x - \mu_n)\}_{n\in I}$$

is called an *irregular Gabor system.*

Note that a regular Gabor system (and certain irregular Gabor systems) consists of *infinitely many* functions. By definition, we say that such a system is linearly independent if each *finite subfamily* is linearly independent.

The question of linear independence of Gabor systems is very complicated. Heil, Ramanathan, and Topiwala considered the question in 1994 and were able to prove that an irregular Gabor system is linearly independent under certain conditions on the function $g$ and the points $\{(\lambda_n, \mu_n)\}_{n\in I}$ [**15**]. Based on their results they formulated the following conjecture:

**Conjecture.** *A Gabor system $\{e^{2\pi i\lambda_n x}g(x - \mu_n)\}_{n\in I}$ with $g$ in $L^2(\mathbb{R}) \setminus \{0\}$ is linearly independent provided that the points $(\lambda_n, \mu_n)$ for $n$ in $I$ are distinct.*

In 1996 Linnell was able to prove the conjecture for regular Gabor systems (see [**19**]), but his methods do not apply to irregular Gabor systems. Considerable effort has been invested in the conjecture, but till now nobody has been able to prove (or disprove) it. A detailed description of the conjecture and the results concerning it obtained so far is given by Heil in [**16**]; in particular, that paper exhibits a Gabor system consisting of four functions for which the conjecture remains unresolved.

Gabor systems are widely used in signal analysis. Another, and even more popular, system of functions in signal analysis is a wavelet system. For a given function $\psi$ in $L^2(\mathbb{R})$ the associated *wavelet system* consists of the functions

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) \quad (j, k \in \mathbb{Z}). \tag{12}$$

We say that the wavelet system (12) is *generated* by the function $\psi$. For reasons that will be clear soon, we sometimes denote the generating function by $\psi$ and at other instances denote it by $\phi$.

Linearly dependent wavelet systems exist. For example, if $\phi = \chi_{[0,1)}$ (i.e., $\phi$ is the characteristic function of the interval $[0, 1)$), then one easily verifies that

$$\phi_{0,0} = \frac{1}{\sqrt{2}}(\phi_{1,0} + \phi_{1,1}).$$

Wavelet systems with (a particular type of) linear dependence play an important role in wavelet theory. In fact, most constructions of wavelet systems of practical use are based on so-called multiresolution analysis: this consists of a set of conditions implying that a certain function $\phi$, called the *scaling function*, satisfies an equation of the type

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(2x - k). \tag{13}$$

In general, we want this equation to be satisfied for a sequence $\{c_k\}_{k\in\mathbb{Z}}$ belonging to the vector space

$$\ell^2 := \left\{ \{a_k\}_{k\in\mathbb{Z}} \ : \ \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty \right\}.$$

The technical reason for this is that $\ell^2$ can be given an inner product that makes it a Hilbert space. We note that in the case where only finitely many coefficients $c_k$ in (13) are nonzero, this equation implies that the wavelet system $\{\phi_{j,k}\}_{j,k\in\mathbb{Z}}$ is linearly dependent; in fact, already the subsystem $\{\phi_{j,k}\}_{j\in\{0,1\},k\in\mathbb{Z}}$ is linearly dependent. We return to a discussion of the implications of this in section 5.

Multiresolution analysis aims at the construction of a function $\psi$ that generates an orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. The function $\psi$ is usually defined in terms of the function $\phi$ satisfying (13), namely, via

$$\psi(x) = \sum_{k\in\mathbb{Z}} d_k \phi(2x - k)$$

for certain coefficients $d_k$. This explains the sudden change of notation earlier: we need *two* wavelet systems $\{\phi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ in this construction!

In practice, (13) is usually verified in the Fourier domain: (13) holds for some coefficient sequence $\{c_k\}_{k\in\mathbb{Z}}$ in $\ell^2$ if and only if

$$\hat{\phi}(2\gamma) = \sum_{k\in\mathbb{Z}} \tilde{c}_k e^{2\pi i k\gamma} \hat{\phi}(\gamma) \tag{14}$$

holds on $\mathbb{R}$ for some sequence $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ from $\ell^2$. A condition of this type is satisfied, for example, by even-order B-splines.

**Example 2.** The first-order B-spline is defined by $B_1 = \chi_{[-1/2,1/2]}$; the higher-order B-splines are defined inductively by convolution:

$$B_{n+1}(x) = B_n * B_1(x) = \int_{-\infty}^{\infty} B_n(x - t) B_1(t) dt.$$

A direct calculation shows that

$$\widehat{B_1}(\gamma) = \frac{\sin(\pi\gamma)}{\pi\gamma},$$

from which it follows by a standard property of convolution that

$$\widehat{B_n}(\gamma) = \left(\widehat{B_1}(\gamma)\right)^n = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^n.$$

Thus

$$\widehat{B_n}(2\gamma) = \left(\frac{\sin(2\pi\gamma)}{2\pi\gamma}\right)^n = \left(\frac{2\cos(\pi\gamma)\sin(\pi\gamma)}{2\pi\gamma}\right)^n = (\cos(\pi\gamma))^n\,\widehat{B_n}(\gamma).$$

When the order $n$ is even, we can write

$$(\cos(\pi\gamma))^n = \left(\frac{e^{i\pi\gamma} + e^{-i\pi\gamma}}{2}\right)^n = \sum_{k=-n/2}^{n/2} c_k e^{2\pi i k\gamma}$$

for a certain set of coefficients $\{c_k\}_{k=-n/2}^{n/2}$. This demonstrates that (14) is satisfied (i.e., the wavelet system generated by a B-spline of even order is linearly dependent). We note that the B-splines can be written explicitly in terms of functions that are piecewise polynomial. When $n > 1$, the B-spline $B_n$ is a continuous, piecewise polynomial function, where the highest order of the polynomials involved is $n - 1$. For example,

$$B_2(x) = \begin{cases} 1 + x & \text{if } x \in [-1, 0], \\ 1 - x & \text{if } x \in [0, 1], \\ 0 & \text{otherwise;} \end{cases}$$

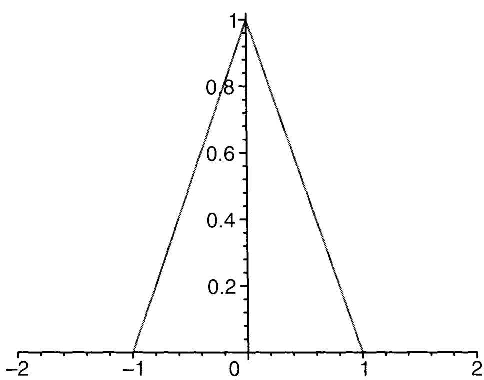


**Figure 1.** The B-spline $B_2$.

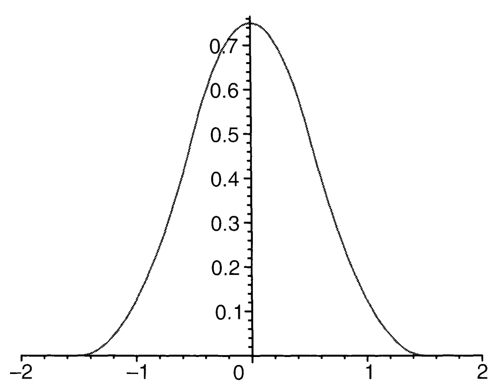


**Figure 2.** The B-spline $B_3$ in (15), which does not satisfy an equation of the type (13).

$$B_3(x) = \begin{cases} \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8} & \text{if } x \in [-\frac{3}{2}, -\frac{1}{2}], \\ -x^2 + \frac{3}{4} & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8} & \text{if } x \in [\frac{1}{2}, \frac{3}{2}], \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

(see Figures 1 and 2). The concrete version of (13) for $B_2$ is

$$B_2(x) = \frac{1}{2}B_2(2x-1) + B_2(2x) + \frac{1}{2}B_2(2x+1).$$

Since $B_3$ is an odd-order spline, it does not satisfy (13). However, it satisfies a similar equation in which the translation-step in the wavelet system is $1/2$ instead of 1:

$$B_4(x) = \frac{1}{4}B_2(2x-3/2) + \frac{3}{4}B_2(2x-1/2) + \frac{3}{4}B_2(2x+1/2) + \frac{1}{4}B_2(2x+3/2).$$

**5. FRAMES IN $L^2(\mathbb{R})$.** This section is more advanced than the previous sections. Its purpose is to explain the role of Gabor systems and wavelet systems in the context of series expansions of functions in $L^2(\mathbb{R})$.

If the function $f$ belongs to $L^2(\mathbb{R})$ and if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$, then the expansion (8) is valid in the $L^2(\mathbb{R})$-sense with

$$c_k(f) = \int_{-\infty}^{\infty} f(x)\overline{f_k(x)}\,dx.$$

An orthonormal basis is automatically linearly independent, but (8) might very well hold for families $\{f_k\}_{k=1}^{\infty}$ that are *not* linearly independent. A quite general way to obtain representations of the type (8) is to consider "frames."

**Definition.** A family of functions $\{f_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist positive constants $A$ and $B$ such that

$$A\int_{-\infty}^{\infty} |f(x)|^2\,dx \le \sum_{k=1}^{\infty}\left|\int_{-\infty}^{\infty} f(x)\overline{f_k(x)}\,dx\right|^2 \le B\int_{-\infty}^{\infty} |f(x)|^2\,dx \tag{16}$$

holds for all $f$ in $L^2(\mathbb{R})$.

An orthonormal basis is automatically a frame; on the other hand, a frame $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis if (16) holds with $A = B = 1$ and if, in addition,

$$\int_{-\infty}^{\infty} |f_k(x)|^2\,dx = 1$$

for each $k$. More generally, a frame $\{f_k\}_{k=1}^{\infty}$ is a basis if

$$\sum_{k=1}^{\infty} c_k f_k = 0 \Rightarrow c_k = 0 \quad (k \in \mathbb{N}). \tag{17}$$

Note that (17) can be viewed as an infinite-dimensional version of the usual condition for linear independence.

One can prove that a frame $\{f_k\}_{k=1}^{\infty}$ leads to a representation of the type (8), where the coefficients $c_k(f)$ in the expansion of $f$ have the form

$$c_k(f) = \int_{-\infty}^{\infty} f(x)\overline{h_k(x)}\,dx \tag{18}$$

for a certain family of functions $\{h_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{R})$ (see any of the references [**27**], [**8**], [**13**], or [**4**]). However, in contrast to the situation for a basis, there might exist other choices for the coefficient sequence $\{c_k(f)\}_{k=1}^{\infty}$. For example, if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis, then the family

$$\{f_1, f_1, f_2, f_3, f_4, f_5, \dots\}$$

is a frame, and each $f$ in $L^2(\mathbb{R})$ has several representations of the type (8).

For general frames it is quite involved to find the coefficients $c_k(f)$ in (18): we must first identify suitable functions $h_k$. A significant simplification appears if the frame condition (16) holds with $A = B = 1$: in that case, we can take $h_k = f_k$ (see [**8**] or [**4**]). In general, a frame for which we can take $A = B$ is said to be *tight*. Note that the condition of $\{f_k\}_{k=1}^{\infty}$ being a tight frame with $A = B = 1$ does not imply that the $f_k$ are pairwise orthogonal: for example, if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$, then the family

$$\left\{f_1, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \dots\right\}$$

constitutes a tight frame, but not all pairs of vectors are orthogonal.

We are now ready to connect Gabor systems with frames. If an infinite Gabor system $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ is a frame, we simply call it a *Gabor frame*. The simplest Gabor frame is quite easy to construct:

**Example 3.** We have seen that the functions $\{\frac{1}{\sqrt{2\pi}}e^{imx}\}_{m\in\mathbb{Z}}$ form an orthonormal basis for $L^2(-\pi,\pi)$. In fact, since they are periodic with period $2\pi$, they actually form an orthonormal basis for $L^2(I)$ for any interval $I$ of length $2\pi$. If we want to put emphasis on the fact that we are looking at the exponential functions on the interval $[-\pi+2n\pi, \pi+2n\pi)$ we can do so by asserting that

$$\left\{\frac{1}{\sqrt{2\pi}}e^{imx}\chi_{[-\pi+2n\pi,\pi+2n\pi)}(x)\right\}_{m\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(-\pi+2n\pi, \pi+2n\pi)$. Now observe that the intervals $[-\pi+2n\pi, \pi+2n\pi)$ are disjoint and cover the entire real line. This ensures that the union of all these bases, the family

$$\left\{\frac{1}{\sqrt{2\pi}}e^{imx}\chi_{[-\pi+2n\pi,\pi+2n\pi)}(x)\right\}_{m,n\in\mathbb{Z}},$$

is an orthonormal basis for $L^2(\mathbb{R})$.

The simplicity of this example is misleading: in general, it might be difficult to check that a given Gabor system forms a frame. Even if $g$ is the characteristic function of an interval, there are cases where it is unknown whether the Gabor system generated by $g$ for given parameters $a$ and $b$ constitutes a frame or not. A detailed analysis was performed by Janssen and published in [**17**]: it specifies at least eight types of conditions under which the conclusion is known. We state a few of them in what follows.

**Example 4.** It can be shown that

$$\left\{e^{2\pi imbx}\chi_{[0,1)}(x-na)\right\}_{m,n\in\mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ if $b = 1$ and $0 < a \le 1$ .

We now replace the characteristic function $\chi_{[0,1)}$ with $\chi_{[0,c)}$ for some positive number $c$, and consider the Gabor system $\{e^{2\pi imbx}\chi_{[0,c)}(x-na)\}_{m,n\in\mathbb{Z}}$ with parameters $a$ and $b$. Via a change of variable, the analysis of such systems can be reduced to the case $b = 1$. The following is known for $G = \{e^{2\pi imx}\chi_{[0,c)}(x-na)\}_{m,n\in\mathbb{Z}}$:

(i) $G$ is a frame if $1 \ge c \ge a$;

(ii) $G$ is a frame if $1 < c < 2$, $0 < a < 1$, and $a$ is irrational;

(iii) $G$ is not a frame if $a > 1$;

(iv) $G$ is not a frame if $a = p/q$ for relatively prime integers $p$ and $q$ such that $2 - 1/q < c < 2$.

These results indicate that rationality of the parameter $a$ plays a key role.

For an arbitrary Gabor frame $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ general frame theory tells us that every function $f$ in $L^2(\mathbb{R})$ has an expansion of type (8) in terms of the functions in the frame. Even more holds: there exists a function $h$ in $L^2(\mathbb{R})$ such that each $f$ in $L^2(\mathbb{R})$ has a series expansion

$$f(x) = \sum_{m,n\in\mathbb{Z}} c_{m,n}(f)e^{2\pi imbx}g(x-na), \tag{19}$$

where

$$c_{m,n}(f) = \int_{-\infty}^{\infty} f(x)e^{-2\pi imbx}\overline{h(x-na)}\,dx \tag{20}$$

and convergence is in the $L^2$-norm. These coefficients should be compared with the general coefficients in (18): whereas for general frames we have to find an infinite family of functions $\{h_k\}_{k=1}^{\infty}$ in order to calculate the coefficients, it suffices to find one function $h$ in the Gabor case. There exist iterative procedures for finding this function $h$, but we will not go into this subject here (see [**25**]).

Case (iii) in Example 4 was based on the assumption that $b = 1$, so the condition can also be written as $ab > 1$. The result turns out to be typical for Gabor systems: a Gabor system can never form a frame if $ab > 1$. It is illustrative to consider the case for $g = \chi_{[0,1)}$ and $b = 1$, where this can be seen directly. Indeed, when $a > 1$ all functions in the Gabor system $\{e^{2\pi imx}\chi_{[0,1)}(x-na)\}_{m,n\in\mathbb{Z}}$ are zero on the interval $(1, a)$ and therefore cannot be used to expand arbitrary functions in $L^2(\mathbb{R})$. An amazing fact about Gabor frames is that if $ab < 1$, then the coefficients $c_{m,n}(f)$ in (19) are *never* unique (i.e., we can always choose the coefficients as in (20), but there exist other options). Compare this with the aforementioned result by Linnell stating that a finite regular Gabor system is linearly independent. The latter result implies that any function $f$ *has at most one representation as a finite sum*

$$f(x) = \sum_{|m|,|n|\le N} c_{m,n}(f)e^{2\pi imbx}g(x-na).$$

Thus, there is a fundamental difference between finite and infinite Gabor systems. In particular, the linear independence of finite Gabor systems does *not* imply that the representation in (19) is unique. The explanation is that the correct concept of linear independence for the elements in a frame $\{f_k\}_{k=1}^{\infty}$ for $L^2(\mathbb{R})$ is given by (17), which is a stronger condition than the mere linear independence of each *finite* subset $\{f_k\}_{k=1}^{n}$. It is known, however, how the two concepts are related: (17) holds if and only if $\{f_k\}_{k=1}^{n}$ is linearly independent for each $n$ and the quantity

$$\inf_n \min_f \left\{ \sum_{k=1}^{n} \left| \int_{-\infty}^{\infty} f(x)\overline{f_k(x)}\, dx \right|^2 \; : \; f \in \operatorname{span}\{f_k\}_{k=1}^{n}, \int_{-\infty}^{\infty} |f(x)|^2 dx = 1 \right\}$$

is positive. The nontrivial proof of this fact can be found in [**6**].

At this point it is natural to think about the relevance of general Gabor frames. Example 3 shows that we can construct Gabor systems that form orthonormal bases, so why bother with the more complicated frame concept? The explanation is that frames provide a more flexible tool than orthonormal bases: whereas Gabor-type orthonormal bases exist, it might be that we can not find one that satisfies certain *additional constraints*. A concrete case of this situation is given by the "Balian-Low Theorem," which asserts that if a function $g$ generates an orthonormal basis (or, more generally, a so-called Riesz basis, meaning a basis that is equivalent to an orthornormal basis) with Gabor structure, then

$$\left( \int_{-\infty}^{\infty} |xg(x)|^2 \, dx \right) \left( \int_{-\infty}^{\infty} |\gamma \hat{g}(\gamma)|^2 \, d\gamma \right) = \infty. \tag{21}$$

Described in words, this result tells us that it is impossible in the orthonormal setting for both $g$ and $\hat{g}$ to decay rapidly at infinity. This is very inconvenient in signal analysis, where the time-behavior and frequency-behavior of functions are considered simultaneously. The good news is that Gabor frames do not suffer from this shortcoming: it is possible to construct functions $g$ that generate frames for which the product (21) is finite. A simple example is the *Gaussian* $g(x) = e^{-x^2}$, which generates a frame if $ab < 1$. For this function, $\hat{g}(\gamma) = \sqrt{\pi} e^{-\pi^2 \gamma^2}$, so the product (21) is indeed finite.

We note in passing that the proof that the Gaussian generates a frame whenever $ab < 1$ has an interesting history. Daubechies and Grossmann [**9**] proved around 1987 that $\{e^{2\pi imbx} g(x - na)\}_{m,n\in\mathbb{Z}}$ is a frame whenever $ab < 0.994$ and conjectured the general result. In 1991, the conjecture was proved by Lyubarski [**20**] and independently by Seip and Wallsten [**23**], [**24**]. The original proofs are complicated and use advanced complex analysis. Janssen later gave a shorter proof [**18**].

We also remark that the Gaussian does not generate a frame if $ab = 1$. This is easier. Namely, it is known that *if* a Gabor system with $ab = 1$ is a frame, then it is actually a basis [**4**]. For the case where $g$ is the Gaussian this would contradict the Balian-Low theorem. For more information about Gabor systems we refer the reader to the book by Gröchenig [**13**].

Certain infinite wavelet systems also yield representations of all functions in $L^2(\mathbb{R})$. In other words, there exist functions $\psi$ in $L^2(\mathbb{R})$ such that each $f$ in $L^2(\mathbb{R})$ has a representation (convergent in $L^2$-norm)

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{j,k} 2^{j/2} \psi(2^j x - k). \tag{22}$$

Wavelet analysis is currently one of the most active research areas in mathematics, mainly because the subject is not only mathematically fascinating but also important for several areas of applied science. A large part of wavelet theory aims at the construction of functions $\psi$ such that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. This was initiated by Mallat and Meyer in 1989, who developed the aforementioned multiresolution analysis. A further analysis by Daubechies demonstrated how to construct orthonormal bases $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for which $\psi$ has compact support (see Figure 3).

It is interesting to notice that the first frame constructions with wavelet structure appeared even earlier, namely, in a 1985 paper by Daubechies, Grossman, and Meyer [**10**]. Also in the context of wavelets, frames are attractive, for wavelet frames can enjoy properties that orthonormal bases rule out. For example, one can produce a frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for which the function $\psi$ is infinitely differentiable and decays exponentially, whereas no basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ with these properties exists. For further reading we recommend the book [**8**] by Daubechies (a classic) or [**26**] by Walnut (which requires fewer prerequisites). More intuitive presentations aimed at a general audience can be found in [**2**] and [**5**].

The fact that linearly dependent wavelet systems exist is central to recent developments of wavelet theory. As discussed earlier, most constructions of functions $\psi$ for which representations of the type (22) are possible are based on scaling functions $\phi$ that satisfy equations of the sort

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(2x - k). \tag{23}$$

However, in classical wavelet analysis, it is possible that infinitely many coefficients $c_k$ are nonzero, so (23) is not yet a statement about linear dependence. But as a follow-up to a series of papers by Ron and Shen, Daubechies et al. [**11**] have recently constructed tight frames of the form $\{\psi^1_{j,k}\}_{j,k\in\mathbb{Z}} \cup \{\psi^2_{j,k}\}_{j,k\in\mathbb{Z}}$ (i.e., systems obtained by combining the wavelet systems associated with *two* functions $\psi^1$ and $\psi^2$). A similar construction was discovered simultaneously by Chui et al. [**7**]. The functions $\psi^1$ and $\psi^2$ are usually constructed by means of expressions of the form

$$\psi^i(x) = \sum d^i_k \phi(2x - k),$$

where $\phi$ satisfies an equation of the type (23) for a *finite* sequence $\{c_k\}$.

The sequences $\{d^i_k\}$ used to obtain $\psi^1$ and $\psi^2$ are also typically finite. In the case where $\phi$ is chosen to be a B-spline of even order, this approach leads to explicitly given functions $\psi^i$ (see Figure 4). Comparing the functions in Figure 4 with the "classical" wavelets of Figure 3, we see immediately that we are dealing with much simpler functions. In fact, the constructions in [**7**] and [**11**] can be used to exhibit tight frames such that the generators $\psi^1$ and $\psi^2$ are splines of any desired order and with compact support.

Frames themselves were introduced long before the interest in wavelets surged, specifically, in a 1952 paper by Duffin and Schaeffer [**12**]. Apparently the paper was far ahead of its time, for it took almost thirty years before the topic appeared in print again. The original Duffin and Schaeffer paper contained a general definition of frames in Hilbert spaces and used them in the context of nonharmonic Fourier series. We refer the reader to [**4**] for more information on frames.

We conclude this paper with a few words concerning the decomposition of frames into linearly independent subsets. In fact, we have already seen that the elements in a frame might be linearly dependent. Thus it is natural to ask whether we can at least decompose a frame $\{f_k\}_{k=1}^{\infty}$ for $L^2(\mathbb{R})$ into a finite number of subsets each of which is linearly independent. The answer turns out to be yes (see [**3**]) provided that the

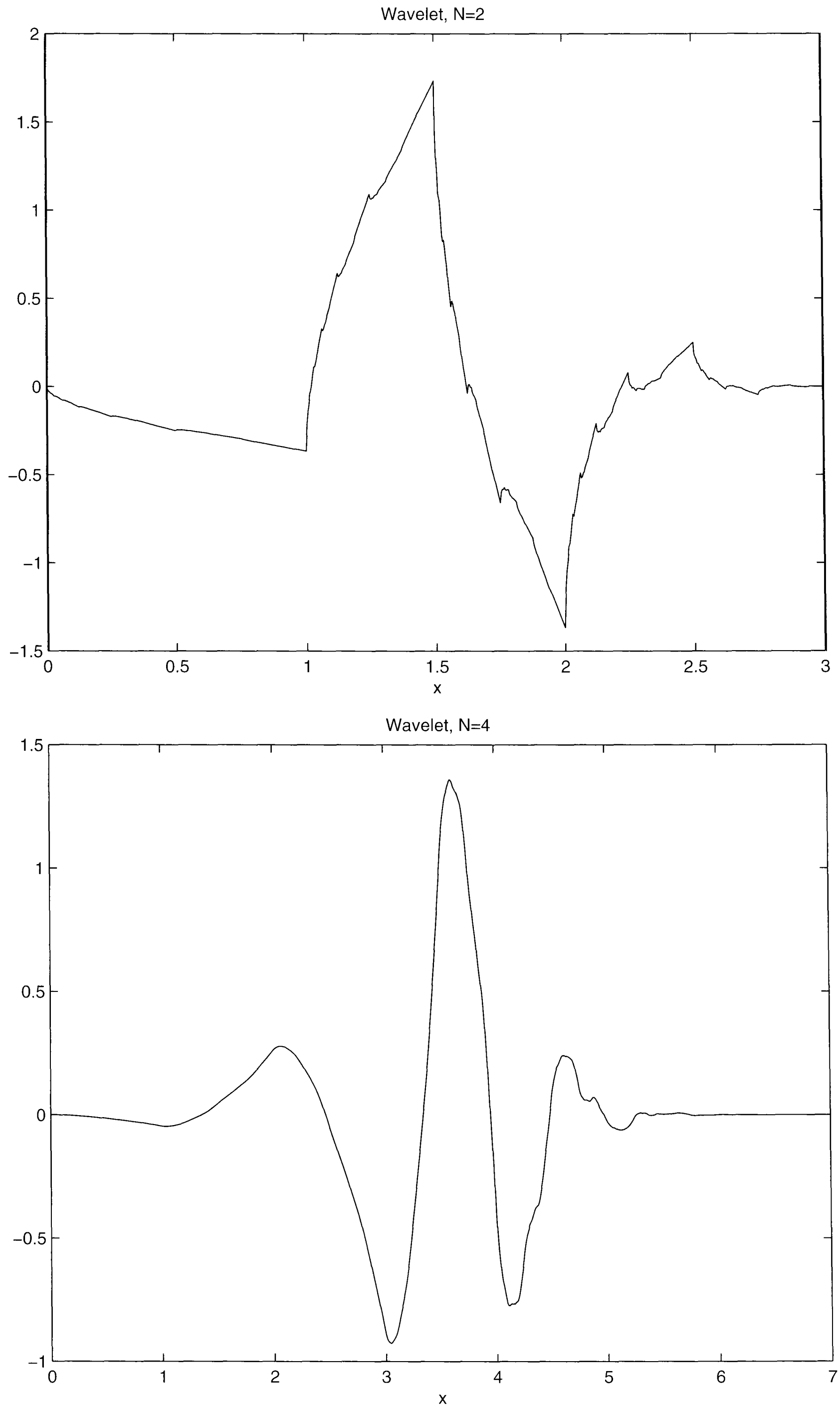


**Figure 3.** Some of Daubechies's wavelets. The number $N$ is a parameter that appears in the numbering of these wavelets.

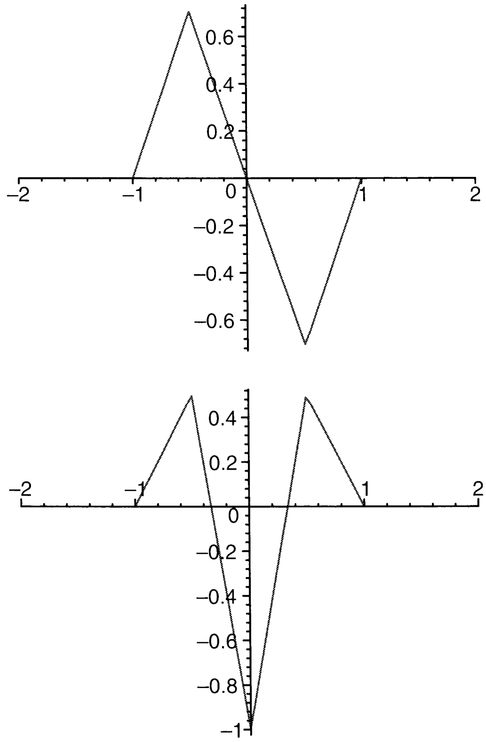


**Figure 4.** Functions $\psi^1$ and $\psi^2$ for which $\{\psi^1_{j,k}\}_{j,k\in\mathbb{Z}} \cup \{\psi^2_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$.

following condition holds:

$$\inf_k \int_{-\infty}^{\infty} |f_k(x)|^2\, dx > 0. \tag{24}$$

**Theorem 2.** *If $\{f_k\}_{k=1}^{\infty}$ is a frame for which* (24) *is satisfied, then there exists a finite partition*

$$\{f_k\}_{k=1}^{\infty} = \bigcup_{j=1}^{N} \{f_k\}_{k\in I_j} \tag{25}$$

*such that each set $\{f_k\}_{k\in I_j}$ is linearly independent* (*in the sense that each finite subset of $\{f_k\}_{k\in I_j}$ is linearly independent*).

The problem of decomposing a frame into a finite set of families satisfying (17) is more subtle. In 2001 Feichtinger formulated the following conjecture in conversations with several colleagues:

**Conjecture.** *Every frame $\{f_k\}_{k=1}^{\infty}$ satisfying* (24) *can be partitioned into a finite union as in* (25), *where each sequence $\{f_k\}_{k\in I_j}$ satisfies* (17).

Feichtinger even conjectured that the sequences $\{f_k\}_{k\in I_j}$ can be chosen to be frames.

It is known that Feichtinger's conjecture is true under slightly stronger assumptions (see [**3**] and [**14**]). However, the conjecture itself remains open. We point out that the technical condition (24) is necessary for Theorem 2 as well as for the conjecture. For example, if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$, then

$$\left\{f_1, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \dots\right\}$$

is a frame that can not be decomposed into a finite union of linearly independent sets. Condition (24) is automatically satisfied in the important wavelet case.

**6. APPLICATIONS OF WAVELETS AND FRAMES.** So far we have related the classical concept of linear independence to the modern theory of wavelets and frames without going into much detail about the practical use of these systems. In this final section we mention a few concrete cases where the use of wavelets and frames is an advantage when compared with the use of classical methods.

One of the most important properties of wavelet systems is that they lead to efficient compression methods. For a large class of signals (for example, images) a wavelet representation contains a large number of small coefficients. By thresholding (i.e., replacing small coefficients with zeroes) we obtain a very good approximation of the signal that requires much less capacity to store or to transmit. For this reason, wavelets are used in many instances where efficient representations of signals is important. For example, the FBI employs wavelets to store fingerprints electronically: through wavelet compression, a fingerprint can be stored using about 1Mb capacity, whereas a fingerprint without compression would require about 13Mb. This compression also explains the preference for wavelets over classical methods like Fourier analysis. In fact, a fingerprint can also be stored using Fourier methods, but they do not furnish an efficient way of compressing information.

We have already seen some benefits of frames compared with orthonormal bases (see Figure 3 and Figure 4). From the signal processing perspective, frames have yet another advantage. In short, frames suppress noise. By transmitting a signal from one place to another, a signal will always be corrupted by some noise. While a representation of the signal via orthonormal bases will let all the noise pass through to the receiver of the signal, the redundancy of frames has the pleasant property of removing parts of the noise without affecting the signal itself. A more detailed explanation of this would take us too far away from the central object of this article, so we refer the reader to the books [**8**] by Daubechies and [**21**] by Mallat for more precise information.

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