Review Suggestions for Midterm 3
Math 215, Fall 2017

Please use this study guide to identify what to study for Midterm 3 next week, and to practice some new problems. Once you feel well-prepared use the practice test to simulate the exam experience. (This means working at a desk in a quiet space without the aid of notes or the textbook, and without any distracting technology.) I don't yet have solutions to study guide problems written up. You are welcome to come by office hours to discuss solutions. General suggestions for taking timed math tests are posted on Professor Stanhope's website: `https://sites.google.com/a/lclark.edu/stanhope/study-strategies`.

In order to allow more time for Midterm 3 it will be given as a timed, outside of class exam. To do this first identify a 90 minute period of time during your day on Tuesday. Note that our class will be canceled that day. At the beginning of this 90 minute period please stop by Professor Stanhope or Professor Sullivan's office (students from both sections can go to either professor) to pick up the exam. After signing the rules and honor statement for the exam, you will take the exam to a quiet place (Watzek works well, for example) and take the exam in 90 minutes. When you are done you will drop the exam off at Professor Stanhope or Professor Sullivan's office (again, students from both sections can go to either professor). A sample of the rules and honor statement is posted on our Moodle page.

---

1 Introductory Set Theory

Before we can use set theory to define things like equivalence relations, functions, or combinatorial graphs, we need to first set up the language and basic tools of set theory. To be prepared for this section please know how to work with sets. This includes knowing set builder notation, what $\emptyset$ is, what it means for a set to be a subset of another set, how we show sets are equal to one another, and what it means to 'element chase.' Please be familiar with the use of Venn diagrams. In addition please be fluent with set intersections (both finite and countably infinite), set unions (both finite and countably infinite), set difference, set products, the complement of a set, and the power set of a set.

To review for Introductory Set Theory please:

- Be able to explain in a short paragraph why the study of set theory is important to mathematics.
- Answer this: What set is a subset of every set? What set is an element of the power set of any set?
- Be able to use element chasing to show that one set is a subset of another set.
- Be able to use element chasing in both directions to show set equality.
- Be able to move comfortably between statements about a set and statements about its power set. (So "if $A \subseteq B$, then $A \in \mathcal{P}(B)$", or, "$x \in A$ implies $\{x\} \in \mathcal{P}(A)$," for example.)

Practice Problems:

1.1 Decide whether each statement is TRUE or FALSE.

a. $x \in \{x\}$
b. $\emptyset \in \{x\}$
c. $\{2, \{2\}\} = \{\{2\}\}$
d. $\emptyset \subseteq \{x, \emptyset\}$
e. $2 \in \{x \in \mathbb{R} : x = k^2, k \in \mathbb{Z}\}$
f. $\{\{\emptyset\}\} \subseteq \mathcal{P}(\emptyset)$
g. $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

1.2 What is the set $\{n \in \mathbb{Z} \mid n = 2m \text{ for some } m \in \mathbb{Z}\}$?

1.3 What is the set $\{k \in \mathbb{N} \ : \ \exists p, q \in \mathbb{N}, (k = pq, 1 < p < k, 1 < q < k)\}$

1.4 Use set builder notation to write the set of all positive real numbers.

1.5 Use set builder notation to write the set of all rational numbers that have a factor of 5 in their denominators.

1.6 Prove or disprove the following statements. For proofs please use element chasing. Assume that A, B, C, D are sets.

a. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
b. $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (B \cap D)$
c. $(A \backslash B) \backslash C = A \backslash (B \cup C)$
d. $\overline{(A \cap C)} = \overline{A} \cup \overline{C}$
e. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
f. $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

1.7 Is the following statement true? Please prove or give a counterexample.

$$\text{“}A \subseteq C \text{ if and only if } A \cup (B \cap C) = (A \cup B) \cap C\text{”}$$

1.8 Let X be a finite set. Which of the two sets $\mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$ and $\mathcal{P}(\mathcal{P}(X))$ has the most elements?

1.9 Let $A \subseteq B$ and $D \subseteq C$. Prove that

$$(A \times C) \cup (B \times D) \subseteq B \times C$$

1.10 Suppose $A_1, A_2, A_3, \ldots$ is a countably infinite family of sets. Prove that

$$\bigcup_{n=1}^{\infty} \mathcal{P}(A_n) \subseteq \mathcal{P}(\bigcup_{n=1}^{\infty} A_n)$$

1.11 Find the following subsets of the real numbers.

$$\bigcup_{n=1}^{\infty}\left(\frac{1}{n^2+1},+\infty\right) \qquad\qquad \bigcap_{k=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

---

2 Equivalence Relations

For starters, please know the definition of a relation. Then be sure you know the definition of an equivalence relation. Given a relation on a set, be able to determine whether or not it is an equivalence relation. If it is be able to give a proof, if it is not be able to give an example showing how it fails to be an equivalence relation. Given an equivalence relation on a set A, be able to describe the equivalence class of an element of A. Also be able to describe the partition on A determined by an equivalence relation. What is the connection between partitions on A and equivalence relations on A?

Practice problems: Problems #1 – #8, #10 – #13 from Section 6.2 of the text are great practice problems. In Section 6.6 Problems #5 – #8 are good practice problems. Problem #2 is and excellent problem to get another perspective on your understanding of general relations and functions. Practice the notions of equivalence classes and partitions with the problems below.

There will be no questions on Midterm 3 about equivalence relations on products of sets. We will review this topic a bit more later this semester and so save such problems for the final exam.

2.1 For each of the following equivalence relations on $\mathbb{R}$, find $[0]$ (the equivalence class of zero) and $[3]$ (the equivalence class of three). Describe the partition on $\mathbb{R}$ determined by each of these equivalence relation.

- Let R be the relation on $\mathbb{R}$ given by xRy if and only if $|x| = |y|$.
- Let S be the relation on $\mathbb{R}$ given by xSy if and only if $\sin x = \sin y$.

2.2 Let $\sim$ be the equivalence relation on $\mathbb{R}\backslash\{0\}$ given by $x \sim y$ if and only if $xy > 0$. Please describe the partition of $\mathbb{R}$ corresponding to $\sim$.

2.3 Let S be the relation on $\mathbb{R}$ given by xSy if and only if $x - y \in \mathbb{Z}$. Is this relation an equivalence relation? Please either give a proof, or examples of where it fails.

2.4 Let A be a nonempty set and let $B \subseteq A$. We define a relation R on the power set of A by XRY when $B \cap X = B \cap Y$.

1. Prove that R is an equivalence relation.
2. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$, what is the equivalence class of $X = \{1, 3, 5\}$?
3. How many distinct equivalence classes are there in this example (where $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$)?

---

3 Functions

Please know the formal definition of a function. (This is the definition that describes a function as a special type of relation.) Be able to recognize a relation that fails to be a function. Know the vocabulary we use with functions: domain, codomain, image, inverse image, and range. Have both an informal and formal (that is, the precise definition) understanding of what it means for a function to be one-to-one, onto, and/or bijective. Be fluent in proving that a particular function is/is not one-to-one, onto and/or bijective. Given a function, be able to compute the image and inverse image of given points or given sets. Be comfortable using saggital diagrams to specify a function. Understand and be able to use the definition of the composition of two functions.

Also try some of the practice problems below.

3.1 Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \frac{x^2}{1+x^2}$. Please determine the following subsets of $\mathbb{R}$.

  a. The inverse image of $[0, 1]$ under f.
  b. The inverse image of $[0, .5]$ under f.
  c. The domain, codomain and range of f.

3.2 Is each of the following functions one-to-one, onto, both or neither? Please prove your answers.

  a. Let $g : (1, \infty) \to \mathbb{R}$ be defined by $g(x) = \ln x$.
  b. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^4 - 5$.
  c. Let $k : \mathbb{R}^2 \to \mathbb{R}$ be defined by $h((x, y)) = x^2 + y^2$.
  d. Let $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be defined by $F(n) = \{1, 2, \ldots, n\}$.

3.3 Function/set operation questions. Please assume $f : X \to Y$ for each part.

  a. Let $A, B \subseteq X$. Prove or disprove that $f(A \cup B) = f(A) \cup f(B)$.
  b. Let $A, B \subseteq X$. Prove or disprove that $f(A \cap B) = f(A) \cap f(B)$.
  c. Let $C, D \subseteq Y$. Prove or disprove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
  d. Let $C, D \subseteq Y$. Prove or disprove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

3.4 Please prove that the composition of two bijections is a bijection.

3.5. Let A be a nonempty set. Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be defined by $F(X) = A \backslash X$ for all $X \in \mathcal{P}(A)$. Show that F is bijective.