Practice Exam 2

March 25, 2018

QUESTION 1 (80 POINTS)

Consider a two-level system of $N_\uparrow$ (spin up) and $N_\downarrow$ (spin down) particles adding up to N total particles. The multiplicity of the system is given by

$$\Omega = \frac{N!}{N_\downarrow! N_\uparrow!}\,. \tag{0.1}$$

1. **U**se Stirling's formula to **f**ind the entropy S of the system as a function of N and $N_\downarrow$ **o**nly. (Hint:) Stirling's formula is $x! \cong x^x e^{-x}$.

2. In the limit $N_\downarrow \ll N$ **s**how that the entropy of the system is given by

$$S = kN_\downarrow \left[1 + \ln\left(\frac{N}{N_\downarrow}\right)\right]\,. \tag{0.2}$$

   (Hint:) You may need to use the Taylor expansion $\ln(1+x) \cong x$ for $|x| \ll 1$.

3. **F**ind the multiplicity of the system in the limit $N_\downarrow \ll N$.

4. Now, we turn on a magnetic field in the up direction, so most of the spins are alined in the up direction at low temperature. The total energy of the system is

$$U = \mu B (N_\downarrow - N_\uparrow)\,, \tag{0.3}$$

   where μ is a constant. **U**se the above expression of entropy and thermodynamic identities to **f**ind the average number of the down-spin particles as a function of temperature $N_\downarrow(T)$ at low temperature.

5. **F**ind the average thermal energy of the system $U(T)$ at low temperature.

6. **F**ind the heat capacity of the system at constant volume and low temperature. **W**hat is the value of the heat capacity at $T = 0$, **i**s this expected?

QUESTION 2 (20 POINTS)

Two monoatomic gases, A and B, are held in storage tanks of Volumes $V_A = V$ and $V_B = 3V$. The two tanks have identical number of molecules $N_A = N_B$ and both gases are held at the same temperature. There is a valve that separates the two tanks. If the valve is opened, **c**alculate the change of the total entropy of the system (assume that the molecular weight is the same for both gases).

MISCELLANEOUS FORMULA

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\begin{aligned}
PV &= NkT\,, \\
Nk &= nR\,, \\
\Delta U &= W + Q\,, \\
dW &= -PdV\,, \\
U &= \frac{f}{2}NkT\,, \\
C_V \text{ for ideal gas} &= \frac{f}{2}Nk\,, \\
C_P \text{ for ideal gas} &= \frac{2+f}{2}Nk\,, \\
k &= 1.38 \times 10^{-23} \text{ Joule/K}\,, \\
R &= 8.31 \text{ Joule/(Mole.K)}\,, \\
1 \text{ m}^3 &= 10^3 \text{ liter}\,, \\
1 \text{ atm} &= 10^5 \text{ Pa}\,, \\
0 \text{ C} &= 273.15^\circ\text{K}\,, \\
\frac{1}{T} &= \left(\frac{\partial S}{\partial U}\right)_{V,N}\,, \\
C_V &= \left(\frac{\partial U}{\partial T}\right)_{V,N}\,, \\
P &= T\left(\frac{\partial S}{\partial V}\right)_{U,N}\,, \\
\Delta S &= \frac{Q}{T}\,, \\
\Delta S &= \int_{T_i}^{T_f} \frac{C_P}{T}\,dT\,.
\end{aligned}
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