HW.8

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Problem 1:

Prove that for all positive integers n that

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. We will prove by induction. We will prove the statement

$$P(n): \quad \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Base Case: First we will prove for $P(1)$

$$\begin{aligned} 1^2 &= \frac{1(1+1)(2(1)+1)}{6} \\ 1 &= \frac{1(2)(3)}{6} \\ 1 &= \frac{6}{6} \\ 1 &= 1 \end{aligned}$$

Thus we see that the case for $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some $k \geq 1$.

$$\sum_{i=1}^{k} i^2 = 1 + 4 + 9 + \cdots + (k-1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step: Prove $P(k+1)$ using the *inductive hypothesis*. So

$$\begin{aligned}
1+4+9+\cdots+k^2+(k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
\frac{k(k+1)(2k+1)}{6}+(k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
\frac{2k^3+3k^2+k}{6}+\left(k^2+2k+1\right) &= \frac{2k^3+9k^2+13k+6}{6} \\
\frac{2k^3+3k^2+k+6k^2+12k+6}{6} &= \frac{2k^3+9k^2+13k+6}{6} \\
\frac{2k^2+9k^2+13k+6}{6} &= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}$$

Thus we see that the case of $P(k+1)$ is true if the case of $P(k)$ is true. Thus we have shown that

$$\sum_{i=1}^{n} i^2 = 1+4+9+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for any positive interger n. □

Problem 2:

Prove that for all non-negative integers n that

$$\sum_{i=0}^{n} 3^i = 1+3+9+\cdots+3^{n-1}+3^n = \frac{3^{n+1}-1}{2}$$

Proof. We will prove by induction. We will prove the statement

$$P(n): \quad \sum_{i=0}^{n} 3^i = 1+3+9+\cdots+3^{n-1}+3^n = \frac{3^{n+1}-1}{2}$$

Base Case: First we will prove for $P(0)$.

$$\begin{aligned}
3^0 &= \frac{3^{0+1}-1}{2} \\
1 &= \frac{3^1-1}{2} \\
1 &= \frac{3-1}{2} \\
1 &= \frac{2}{2} \\
1 &= 1
\end{aligned}$$

We see that the equality is true for $P(0)$.

Inductive Hypothesis: Assume $P(k)$ is true for some $k \geq 0$.

$$\sum_{i=0}^{k} 3^i = 1 + 3 + 9 + \cdots + 3^{k-1} + 3^k = \frac{3^{k+1} - 1}{2}$$

Inductive Step: Prove $P(k+1)$ using the *inductive hypothesis*.

$$\begin{aligned}
1 + 3 + 9 + \cdots + 3^k + 3^{k+1} &= \frac{3^{(k+1)+1} - 1}{2} \\
\frac{3^{k+1} - 1}{2} + 3^{k+1} &= \frac{3^{k+2} - 1}{2} \\
\frac{3^{k+1} - 1 + 2 \cdot 3^{k+1}}{2} &= \frac{3^{k+2} - 1}{2} \\
\frac{3 \cdot 3^{k+1} - 1}{2} &= \frac{3^{k+2} - 1}{2} \\
\frac{3^{k+1+1} - 1}{2} &= \frac{3^{k+2} - 1}{2} \\
\frac{3^{k+2} - 1}{2} &= \frac{3^{k+2} - 1}{2}
\end{aligned}$$

Thus we see that the case of $P(k+1)$ is true. So we conclude that

$$\sum_{i=0}^{n} 3^i = 1 + 3 + 9 + \cdots + 3^{n-1} + 3^n = \frac{3^{n+1} - 1}{2}$$

is true for any non-negative integer n. □

Problem 3:

Prove that the sum of the first n odd numbers is n^2. That is, prove

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

Proof. We will prove by induction. We will prove the statement

$$P(n): \quad \sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

Base Case: First we will prove for $P(1)$.

$$\begin{aligned}
1 &= 1^2 \\
1 &= 1
\end{aligned}$$

We see that the case for $P(1)$ is true.

Inductive Hypothesis: Assume $P(k)$ is true for some $k \geq 1$.

$$\sum_{i=1}^{k}(2i-1) = 1+3+5+\cdots+2k-1 = k^2$$

Inductive Step: Prove $P(k+1)$ using the *inductive hypothesis.*

$$\begin{aligned} 1+3+5+\cdots+2(k+1)-1 &= (k+1)^2 \\ k^2+2k+2-1 &= k^2+2k+1 \\ k^2+2k+1 &= k^2+2k+1 \end{aligned}$$

Thus we see that the case of $P(k+1)$ is true. So we can conclude that

$$\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+2n-1 = n^2$$

is true for all intergers n. □

Problem 4:

Let $f_1, f_2, f_3, \ldots, f_n$ be functions. Let $g_0(x) = x, g_1 = f_1$, and $g_i = f_i \circ g_{i-1}$ for all integers $i \geq 2$. Prove that for all integers $n \geq 1$,

$$g_n'(x) = \prod_{i=1}^{n} f_i'(g_{i-1}(x))$$

Proof. We will prove by induction. We will prove the statement

$$P(n): \quad g_n'(x) = \prod_{i=1}^{n} f_i'(g_{i-1}(x))$$

Base Case: First we will prove for $P(1)$

$$g_2'(x) = \prod_{i=1}^{2} f_i'(g_{i-1}(x))$$

□