Review Suggestions for Final Exam
Math 215, Fall 2017

Please use this study guide to identify what to study for the Math 215 Final Exam and to practice some new problems. Once you feel well-prepared use the practice test to simulate the exam experience. (This means working at a desk in a quiet space without the aid of notes or the textbook, and without any distracting technology.) General suggestions for taking timed math tests are posted on Liz's website: `https://sites.google.com/a/lclark.edu/stanhope/study-strategies`.

The final exam will be in the usual classroom on Monday, December 18 from 8:30-11:30.

You may bring a 3×5 inch notecard of notes to the exam.

Here are office hours between now and Monday: Liz: Thursday & Friday 2:30-4:00, Saturday 11:00-12:30
Everett: Thursday & Friday 10:00-4:00

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1 Overall suggestions:

Please first practice problems from Practice Homework and the three Midterm Exams, and practice similar problems from the text. This will confirm that your foundational knowledge of course topics is strong. After that review Theory Homework problems to refresh problem solving skills for more challenging problems and to recall proof techniques. The problems listed below provide another opportunity to sharpen skills in course topics.

If you get stuck on a problem please maximize the partial credit you can earn by doing the following:

- Neatly list definitions or theorems that you believe are important in the proof.
- In a sentence or two explain a proof method that you tried, and why it failed. If you've tried more than one proof method, please include a sentence or two for each.

An incorrect proof that is recognized and analyzed effectively is likely to earn more credit than an incorrect proof that is not recognized as incorrect.

2 Topics that will not appear on the exam.

Converting a logical statement to disjunctive normal form, truth tables, and cardinality will not be tested on this exam.

3 Counting Practice Problems

1. How many ways are there to select an 11-member soccer team and a 5-member basketball team from a class of 30 students if
   a. nobody can be on both teams?
   b. any number of students can be on both teams?
   c. at most one student can be on both teams?
2. a. How many poker hands are there with exactly two clubs or exactly two diamonds or exactly two spades?
   b. How many poker hands are there with at least one king and at least one queen?

c. How many poker hands are there with at least one king, at least one queen and no more than three diamonds?

3. In how many ways can the following poker hands occur?

   a. Straight flush - 5 consecutive ranks, all in the same suit, ace ranks high, i.e. ace, 2, 3, 4, 5 isn?t valid.

   b. Four of a kind - all 4 cards of a single rank plus one other card.

   c. Full house - 3 cards of one rank and 2 cards of another rank.

   d. Flush - 5 cards of the same suit but don?t count the straight flushes.

   e. Straight - 5 consecutive ranks, suits don?t matter, but don?t count the straight flushes.

   f. Three of a kind - 3 cards of a single rank, and two other cards of distinct ranks.

   g. Two pair - 2 cards of one rank, 2 cards of another rank, and 1 card of a distinct rank.

4 Number Theory

1. Is it true that for every natural number n, 3 never divides $n^2 + 1$? Please prove your answer.

2. Let $m, n \in \mathbb{N}$. If $d|mn$ where $gcd(m, n) = 1$, show that d can be written as $d = rs$ where $r|m$, $s|n$ and $gcd(r, s) = 1$. (Hint: Consider $r = gcd(d, m)$.)

3. Please prove that $8^n - 3^n$ is divisible by 5 for all natural numbers n.

4. Prove that the cube of a positive integer can always be written as the difference of two squares. (Hint: How can you use sums of consecutive cubes?)

5 Logic & Quantified Statements

1. Consider the following statement: $\forall n \in \mathbf{N}, \frac{n+1}{n} > 1$. Please write the negation of this statement. Please then determine if the original or the negation is true and give a proof of the one that is true.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x + 1$. Consider the following statement:

$$\forall \epsilon > 0, \exists \delta > 0, (|x - 2| < \delta \longrightarrow |f(x) - f(2)| < \epsilon)$$

   Please write the negation of this statement. Please then determine if the original or the negation is true and give a proof of the one that is true.

3. Consider the following statement: $\forall n \in \mathbf{Z}, \exists a \in \mathbf{N}, \exists b \in \mathbf{N}, (a + b)^n > a^n + b^n$. Please write the negation of this statement. Please then determine if the original or the negation is true and give a proof of the one that is true.

6 Set Theory

1. Please state and prove both of DeMorgan's Laws for sets.

2. Consider the following two sets

$$A \times (A \backslash B) \quad \boxed{} \quad (A \backslash B) \times A.$$

   Choose one of the four symbols below to put in the box above to make the statement true.

$$= \quad \neq \quad \subseteq \quad \supseteq .$$

   For each direction prove either that the given set is a subset of the other by using an element casing argument, or prove that is is not necessarily a subset of the other by providing an explicit example.

3. Let S be the relation on $\mathbb{R}$ given by xSy if and only if $x - y \in \mathbb{Z}$. Is this relation an equivalence relation? Please either give a proof, or examples of where it fails.

7 Functions

1. Define $t : \mathbb{R}^2 \to \mathbb{R}$ by $t(x, y) = x + y$.

   i. What is the range of t?

   ii. Is t onto? Why/Why not?

   iii. What is the pre-image of zero?

   iv. What is the pre-image of 1?

   v. Is t one-to-one? Why/Why not?

   vi. If t is not one-to-one, please change the domain of t to a subset of $\mathbb{R}^2$ so that t restricted to this new subset is one-to-one AND has the same range as t.

2. Suppose $f : A \to B$ is a function. Please show that f is injective if and only if $\forall S \subseteq A$, $f^{-1}(f(S)) = S$.

3. For a set of integers A, define $A^{+1} = \{x \ : \ x - 1 \in A\}$. Then define the function $f : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ which sends a set $S \in \mathcal{P}(\mathbb{Z})$ to S^{+1}, so $f(S) = S^{+1}$. Also define the function $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which sends a set $S \in \mathcal{P}(\mathbb{N})$ to S^{+1}, so $g(S) = S^{+1}$.

   1. Is f onto? If so, prove your answer, otherwise prove that f is not onto.
   2. Is f one-to-one? If so, prove your answer, otherwise prove that f is not one-to-one.
   3. Is g onto? If so, prove your answer, otherwise prove that f is not onto.
   4. Is g one-to-one? If so, prove your answer, otherwise prove that f is not one-to-one.

8 Graph Theory

1. Find a simple graph with each of the following degree sequences, or prove that one cannot exist.

   i. 1,2,3,3,4,5

   ii. 3,5,7,9

   iii. 3,3,3,3,3,3

   iv. 1,1,3,3,5,5,7

2. Let G be a simple, bipartite, and planar graph. If each vertex of G has degree at least d, at most how large can d be?

3. Let T be a tree with n vertices that has no vertex of degree 2. Show that T has more than $n/2$ vertices of degree 1.