REAL ANALYSIS II

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1. Category Theory

1.1. Category. Category is a pair of objects and morphisms

$$\mathcal{C} = (\mathcal{O}, \mathcal{M})$$

1.1.1. *Examples.*

- (Vector Spaces, Linear Transformations)
- (Metric Spaces, Continuous Maps)
- (Manifolds, Differentiable Maps)

1.1.2. *Homeomorphisms.*

$$\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$$

This is all okay functions between $\mathcal{O}_1$, and $\mathcal{O}_2$.

$$\begin{aligned} \mathrm{Id}(\mathcal{O}) &= \mathcal{O} \\ f \circ \mathrm{Id} = &f = \mathrm{Id} \circ f \\ (f \circ g) \circ h &= f \circ (g \circ h) \end{aligned}$$

Definition 1.1. *$f \in Hom(Obj_1, Obj_2)$ is an isomorphism if $\exists f^{-1} \in Hom(Obj_2, Obj_1)$ with $f \circ f^{-1} = Id = f^{-1} \circ f$.*

1.2. Functors.

Definition 1.2. *$F : \mathcal{C}_1 \to \mathcal{C}_2 \quad \forall\ Obj \in \mathcal{C}_1\ \exists\ Obj \in \mathcal{C}_2$. With $F(Id) = Id$, and $F(T \circ L) = F(T) \circ F(L)$ (or $F(T \circ L) = F(L) \circ F(T)$). This means that identity and compositions are preserved. Functors can either be covariant or contravairant.*

Covariant functors push morphisms forwards, contravariant pull morphisms back.

$$\begin{tikzcd} \mathrm{Obj}_1 \arrow[r, "f"] \arrow[d, "F"'] & \mathrm{Obj}_2 \arrow[d, "F"] \\ F(\mathrm{Obj}_1) \arrow[r, "F(f)"] & F(\mathrm{Obj}_2) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Obj}_1 \arrow[r, "f"] \arrow[d, "F"'] & \mathrm{Obj}_2 \arrow[d, "F"] \\ F(\mathrm{Obj}_1) & F(\mathrm{Obj}_2) \arrow[l, "F(f)"'] \end{tikzcd}$$

Figure 1. covariant and contravariant functors

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1.2.1. *Examples.* Exterior powers: $V \mapsto \Lambda^k(V)$ is a functor.

1.3. Cohomology Functor.

If $F : \mathcal{M}_1 \to \mathcal{M}_2$ then $F^* : \Gamma(\Lambda^k(M_2)) \to \Gamma(\Lambda^k(\mathcal{M}_1))$. We use this functor to determine if manifolds are the same. However it is only one directional. If $\mathcal{M}_1 \cong \mathcal{M}_2 \implies H^k(\mathcal{M}_1) \cong H^k(\mathcal{M}_2)$.

So since $H^0([0,1]) \ncong H^0(S^1)$, then we know that $[0,1] \ncong S^1$. The same applies for T^2 and S^2 just using H^1.

To show that there is no smooth functions, assume there is, and then show that the dimensions don't match up of H^1, and so there is nonsense, so that becomes a contradiction.

1.4. Algebraic Topology.

Functor from category of manifolds to vector spaces, "measuring" the manifold by associating vector spaces to it. Then work can be done on the vector spaces, which is much simpler than to do it on the manifolds.

2. Manifolds

2.1. Definition of Manifold.

Definition 2.1. *A metric space $(\mathcal{M}, d)$ is said to be a n-dimensional manifold if there exists a collection of mappings $\{\varphi : U_\varphi \to \mathcal{M}\}$. Such that if $\varphi(U_\varphi) = V\varphi$ then*

- $U_\varphi \subseteq \mathbb{R}^n$
- $\varphi : U_\varphi \to \varphi(U_\varphi) \subseteq \mathcal{M}$ *is a homeomorphism. (Continuous mapping with continuous inverse).*
- $\bigcup_\varphi V_\varphi = \mathcal{M}$
- *For all $\varphi_1, \varphi_2 \in \{\varphi\}$ we have*
$$\varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(V_{\varphi_1} \cap V_{\varphi_2}) \to \varphi_2^{-1}(V_{\varphi_1} \cap V_{\varphi_2})$$
is a diffeomorphism. $\varphi_2^{-1} \circ \varphi_1$ are called transition maps.

These are all manifolds, because they can be paramatrixes with some number of bijective maps. For S^1 we use the angle, but need to to cover the point of where θ is discontinuous. So we get

$$\begin{aligned} &\varphi_1 : (-\pi, \pi) \to S^1 && \varphi_0(t) = (\cos(t), \sin(t)) \\ &\varphi_2 : (0, 2\pi) \to S^1 && \varphi_0(t) = (\cos(t), \sin(t)) \end{aligned}$$

2.2. Orientability.

Definition 2.2. *A manifold is called orientable if there is an atlas of charts $\{\varphi : U_\varphi \to \mathcal{M}\}$ covering $\mathcal{M}$ such that each function $\Phi_x^y := \varphi_2^{-1} \circ \varphi_1$ has a positive Jacobian* $\det[D\Phi_x^y]$. *In these notes we will use the phrase orientable atlas when we refer to an atlas in which transitions functions always have positive Jacobian.*

Definition 2.3. *A manifold is called orientable if it permits a nowhere vanishing n-form.*

These two definitions are equivalent. Suppose ω is a nowhere vanishing volume form on $\mathcal{M}$. Then for any parametrization $\varphi(x^1, \ldots, x^n)$ we have

$$\omega(\partial_{x^1}, \ldots, \partial_{x^n}) dx^1 \wedge \cdots \wedge dx^n \neq 0.$$

By swapping x^1 and x^2 variables in the parametrization (if necessary!) we can in fact ensure that

$$\omega(\partial_{x^1}, \dots, \partial_{x^n}) > 0$$

for all φ. Given two overlapping parametrizations we have the relationship

$$\omega(\partial_{x^1}, \dots, \partial_{x^n}) = \det [D\Phi_x^y]\, \omega(\partial_{y^1}, \dots, \partial_{y^n}),$$

which then further implies

$$\det [D\Phi_x^y] > 0$$

for all transition function Φ_x^y. In other worlds, if a manifold is orientable according to the second definitions it is also orientable according to the first definition.

To prove the converse we use dvol_g, we note that $g_{kl} = \sum \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} g_{ij}$. By taking the determinant, we can then get rid of the absolute values, by our assumption. We then have

$$dy^1 \wedge \cdots \wedge dy^n = \det [D\Phi_x^y]\, dx^1 \wedge \cdots \wedge dx^n.$$

Then using chain rule we get dvol_g on both y and x. Since this is coordinate invariant, and will always be positive, then it is defined on $\mathcal{M}$, and is non vanishing.

2.2.1. *Examples.* Most other manifolds, take the sphere for example. You can construct a nowhere vanishing n-form to the sphere, it would be something like ∂r, or something of that sort.

2.2.2. *Non-Examples.* Möbuis band. If you follow a normal vector all the way around the manifold, at some point you will wrap back around to your self, but be pointing the other way. This means that at some point the normal vector field must have passed through zero, and so it vanishes. So this is not orientable.

2.3. **Tangent Vector and Tangent Space.** $\gamma(t)$ on $\mathcal{M}$ is smooth mapping $\gamma : I \to \mathcal{M}$. Let $f : \mathcal{M} \to \mathbb{R}$ be some observable. $f \circ \gamma : I \to \mathbb{R}$ is f as observed by γ, so γs going in different directions should observe different rates of change of f with $\frac{\partial}{\partial t}\Big|_{t=0}(f \circ \gamma)$ being perceived rate of change.

We then say

$$\gamma_1 \sim \gamma_2 \iff \frac{\partial}{\partial t}\Big|_{t=0}(f \circ \gamma_1) = \frac{\partial}{\partial t}\Big|_{t=0}(f \circ \gamma_2)$$

The tangent space of M at P is the set of equivalence classes of curves at p. All of the tangent vectors to M at P.

2.4. **Cotangent Vector Field.** Cotangent vector field is a section of the cotangent bundle. A section is a mapping such that each point of a subset of the manifold stays within the associated vector space based at that point on the manifold. For the circle and cylinder, a section s can move point up and down, but not around the cylinder. This way π will bring the points back to the starting point. We can write this as $\pi \circ s = \mathrm{Id}$.

2.5. **Tangent Bundle.** The tangent bundle is the disjoint union of the tangent space at all points on the manifold. $\dim(TM) = 2n$. Think of this like the cylinder to the circle(Although this is not tangent).

There exists a projection map, that takes any point in a tangent space and projects it to the base point P that the tangent space is tangent to. This is called π.

2.6. $M \to TM$**.** Yes, $M \to TM$ will result in getting something like $[\gamma] \to [F \circ \gamma]$. Where F is the "functor". This proves that this is a covariant functor?????

3. Universal Property

Definition 3.1 (Exterior Powers)**.** *Let V be a vector space. By the k-th exterior power of V we mean the vector space E^k together with an alternating multilinear map $a : V \times V \times \ldots \times V \to E^k$ (on k copies of V) such that for each alternating multilinear map $\omega : V \times V \times \ldots \times V \to W$ there exists a* ***unique*** *linear map $L : E^k \to W$ for which $w = L \circ a$.*

$$\begin{tikzcd}
V_1 \times V_2 \times \ldots \times V_n \arrow[r, "\omega"] \arrow[d, "b"'] & W \\
V_1 \otimes V_2 \otimes \ldots \otimes V_n \arrow[ur, dashed, "\exists! L"'] &
\end{tikzcd}$$

FIGURE 2. Universal property for tensor products of vector spaces $V_1, \ldots, V_n$.

The universal property can be applied to different functors, so it can be used for $\otimes$, and for $\wedge$.

4. Tensor Algebra

4.1. **Using Universal Property.**

$$\begin{tikzcd}
V^* \times V \arrow[r, "\omega(\eta, v) = \eta(v)"] \arrow[d, "b"'] & \mathbb{R} \\
V^* \otimes V \arrow[ur, dashed, "\exists! L"'] &
\end{tikzcd}$$

This proves that there must be some linear mapping L that satisfies this. This mapping is called the contraction operation.

4.2. **Basis and Dimension.** The basis for $V_1 \otimes \cdots \otimes V_n$ is going to be

$$\{\partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_n}}\}$$

where the $\partial_{x^{i_j}}$ basis vector is a basis for the V_j vector space.

The dimension of $V_1 \otimes \cdots \otimes V_n$ is going to be the product of the dimensions of each individual vector spaces.

$$\dim(V_1 \otimes \cdots \otimes V_n) = \dim(V_1)\dim(V_2)\cdots\dim(V_n)$$

4.3. **Change of Basis.**

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \partial_{x^{i_1}} \otimes \ldots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}$$

$$T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \sum T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{j_q}}{\partial y^{\beta_q}}$$

4.4. **Tensor Bundles and Sections.** A Tensor bundle is gotten, by doing tensor products and such of each individual $T_P M$ and by then taking the union over all P gives us a tensor bundle of sorts. Sections of it are tensor fields.

Tensor bundles are

$$((\otimes^p)T^*M) \otimes ((\otimes^q)TM)$$

Tensor fields can be expressed as

$$T = \sum T^{j_1 \ldots j_q}_{i_1 \ldots i_p} dx^{i_1} \otimes \ldots \otimes dx^{i_p} \otimes \partial_{x^{j_1}} \otimes \ldots \otimes \partial_{x^{j_q}}$$

4.5. **Mapping between Tensor Bundles.**

$$\begin{tikzcd}
M_1 \arrow[r, "F"] \arrow[d] & M_2 \arrow[d] \\
TM_1 \arrow[r, "DF"] \arrow[d] & TM_2 \arrow[d] \\
\otimes^p TM_1 \arrow[r, "\otimes DF"] & \otimes^p TM_2
\end{tikzcd}$$

5. Exterior Calculus

5.1. **Exterior Product on the Level of Linear Algebra.** $\wedge$ can be though of as giving us the parallelotope spanned by the wedges vectors. It is defined by determinant. The basis of $V \wedge \cdots \wedge V$ is

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad \text{for } 1 \leq i_1 < i_1 < \cdots < i_k \leq n.$$

This means that the dimension of the exterior product will be n choose k. Things change under diffeomorphism, by the determinant of the map. For change of basis, this is the determinant of the transition map.

5.2. **Bundles and k-forms.**

5.3. **dvol$_g$.**

Definition 5.1.

$$dvol_g = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$$

dvol$_g$ can be used to find the "volume" of a parallelotope. It only exists if a manifold is orientable, and if we have a definition of g.

5.4. **Integration of n-forms on n-dimensional Manifolds.**

5.5. **Flux Integration.**

5.5.1. *Circulation Integrals.* 1-form to circulation integrals

$$\int_\gamma \omega = \int_{t_i}^{t_f} \omega(\partial_t)dt$$

raising an index:

$$\omega^\sharp = \vec{v} \implies \omega(\partial_t) = g(\vec{v}, \partial_t)dt$$

so we have:

$$\int_\gamma \omega = \int_{t_i}^{t_f} g(\vec{v}, \partial_t)dt = \int_\gamma g(\vec{v}, \vec{T})ds$$

therefore, we set

$$\int_\gamma \omega \leftrightarrow \int_\gamma \vec{v}\vec{T}ds$$

5.5.2. *Flux Integrals.* 1-form in 2d to flux Suppose $\Sigma \subseteq M($ a curve on manifold $)$, M is 2d. Assume counter-clockwise Euclidean $\mathbb{R}^2$. Suppose ω is 1-form.

First, note that

$$\omega(\partial_t)dt = \omega(\vec{T})ds$$
$$\vec{N} = (\star\vec{T})^\sharp$$

Since $\star$ is isomorphism, we have

$$-\omega = \star(\vec{v})$$

for some vector field $\vec{v}$. Hence we set

$$\omega(\vec{T}) = -\text{dvol}_g(\vec{v}, \vec{T}) = \text{dvol}_g(\vec{T}, \vec{v}) = g(\vec{N}, \vec{V})$$

This is the flux integral we learned in calc III.

2-form in 3d. Suppose $\Sigma \subseteq M$, an oriented surface, M is 3d, ω is a 2-form. We know that

$$\omega(\vec{V_1}, \vec{V_2}) = g(\vec{V_1}, \vec{V_1} \times \vec{V_2})$$

where $\vec{V} = \star\omega$

$$\begin{aligned}\omega(\partial_{u^1}, \partial_{u^2}) &= \star(\vec{v})(\partial_{u^1}, \partial_{u^2}) \\ &= \mathrm{dvol}_g(\vec{v}, \partial_{u^1}, \partial_{u^2}) \\ &= \mathrm{dvol}_g(\partial_{u^1}, \partial_{u^2}, \vec{v}) \\ &= g(\partial_{u^1} \times \partial_{u^2}, \vec{v}) \\ &= g(\vec{N}, \vec{v})\sqrt{\det(\sim)} \\ &= g(\vec{N}, \vec{V})dA\end{aligned}$$

Hence we set

$$\int_\Sigma \omega = \int_\Sigma g(\vec{v}, \vec{N})dA$$

Flux of $\vec{V}$ across Σ

5.6. Exterior Derivative.

$$d\omega = \sum d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

d is only thing that has the following properties

(1) df is $d : \Gamma(\Lambda^0) \to \Gamma(\Lambda^1)$ where df is 1-form

$$df = \sum df(\partial_{x^i})dx^i = \sum \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} dx^i$$

(2) d distributes over $+$ and scalar.
(3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d\eta$
(4) $d^2\omega = 0$

$$\begin{aligned}\vec{grad} f &= (df)^\sharp \\ df(\vec{v}) &= \sum \frac{\partial f}{\partial x^i} dx^i(\vec{v}) = \sum \frac{\partial f}{\partial x^i} v^i \\ d\omega(\vec{v}) &= \star \circ d \circ \star(\vec{v}) \\ \mathrm{curl}(\vec{v}) &= \star \circ d \circ \flat(\vec{v})\end{aligned}$$

5.7. Stokes' Theorem.

5.8. **Green-Gauss-Stokes' Theorems.**

$$\begin{aligned}
\int_\Sigma \operatorname{div}(\vec{v}) dA &= \int_\Sigma d\omega \quad \text{where } d\omega = \star\vec{v} \\
&= \int_\gamma \omega \\
&= \int_\gamma \star\vec{v}(\partial_t) dt \\
&= \int_\gamma \star\vec{v}(\vec{T}) ds \\
&= \int_\gamma \mathrm{dvol}_g(\vec{v}, \vec{T}) \\
&= -\int_\gamma \mathrm{dvol}_g(\vec{T}, \vec{v}) \\
&= -\int_\gamma \star\vec{T}(\vec{v}) ds \\
&= -\int_\gamma g(\star\vec{T}_\flat, \vec{v}) ds \\
&= \int_\gamma g(\vec{v}, \vec{N} ds) \quad \text{Since } \star\vec{T}_\flat = -\vec{N}
\end{aligned}$$

Greens

$$\begin{aligned}
\int_M d\omega &= \int_{\partial M} \omega \\
\int_{\mathcal{M}} g(\star d\omega, \vec{N}) dA &= \int_{\partial M} \omega(\partial_t) dt \\
\int_M g(\star d\flat\vec{v}, \vec{N}) dA &= \int_{\partial M} g(\vec{v}, \partial_t) dt \\
\int_M g(\mathrm{curl}\vec{v}, \vec{N}) dA &= \int_{\partial M} g(\vec{v}, \vec{T}) ds
\end{aligned}$$

Stokes

$$\begin{aligned}
d\omega &= (-1)^{i-1} \partial f x^i dx^1 \wedge \ldots \wedge dx^n \\
\int_M d\omega &= \int_{U_\varphi} (-1)^{i-1} \partial f x^i dx^1 \ldots dx^n \\
&= 0 \quad \text{unless } U_\varphi \text{ hist the boundary} \\
&= \int_{\partial M} \omega
\end{aligned}$$

6. Cohomology

6.1. **Cochain Complex.**

$$0 \xrightarrow{d} \Gamma(\Lambda^0(T^*M)) \xrightarrow{d} \Gamma(\Lambda^1(T^*m)) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Lambda^n(T^*M)) \xrightarrow{d} 0.$$

Definition 6.1. *Forms in Ker(d) are called closed forms, and forms in Im(d) are called exact forms.*

Definition 6.2. *The k-th cohomology of $\mathcal{M}$, denoted $H^k(\mathcal{M})$, is the quotient vector space*

$$H^k(\mathcal{M}) = Ker(d^k)/Im(d^{k-1}).$$

In other words $H^k(\mathcal{M})$ consists of equivalent classes of k-forms ω for which

$$d\omega = 0 \quad \text{under} \quad \omega_1 \sim \omega_2 \leftrightarrow \exists \eta, \quad \omega_1 - \omega_2 = d\eta.$$

6.2. **Functorial Nature.** Closed forms pull back to closed forms, and exact forms pull back from exact forms, allows H^k to be a functor. This is reliant on

$$d(F^*\omega) = F^* d\omega.$$

It is a contravariant functor.

6.3. $H^0(\mathcal{M})$**.** This "counts" the number of path connected components in the manifold $\mathcal{M}$. We did this proof as homework. Generally it goes as we know that $df = 0$. So f must be constant locally. Then because of path conceitedness it must be constant on the entirety of the connected component. But for multiple components, then each component can be its own constant.

Let $\mathcal{M}$ be a manifold, let d be the number of its connected components. Then $H^0(\mathcal{M}) \cong \mathbb{R}^d$.

6.4. $d\theta$ **on** S^1**.** $d\theta$ is a special function, such that there does not exist any function f such that $df = d\theta$. This will result in $H^1(S^1) \cong \mathbb{R}$.

6.5. $H^1(\mathcal{M})$**.** $H^1(\mathcal{M})$ gives us the "number of ways that $\mathcal{M}$ wraps in on itself".

6.6. Scalar and Vector Potentials.