Math 215 – Fall 2017

Theory Homework 9 – Assigned November 13th, due November 20th

**Note:** Remember that you must show your work to get full credit for a problem.

1. In this problem you will complete a proof that a set and its power set must have different cardinalities. Here is the claim:

   *Theorem:* Let $A$ be a set. Then $Card(A) \neq Card(\mathcal{P}(A))$.

   To prove this claim argue two cases. In Case 1 suppose $A$ is empty and show that the claim is true. In Case 2 suppose that $A$ is not empty. Now argue by contradiction. Carefully write down what the contradiction assumption $Card(A) = Card(\mathcal{P}(A))$ implies. We won't tell you the whole proof, but we'll give you a few major pieces. You will need to consider the set

   $$D = \{a \in A \ : \ a \notin f(a)\}$$

   for a particular function $f : A \to \mathcal{P}(A)$. (Do you see where this function comes from?) You will also need to apply surjectivity to get a special element $d \in A$. Where does $d$ fall in $A$ relative to $D$?

2. Prove that $\mathcal{P}(\mathbb{N})$ is uncountably infinite. (Use the previous problem, this should be a short write-up.)

3. Are $Card(\mathbb{N})$ and $Card(\mathbb{R})$ the only possibilities for the cardinality of an infinite set? If yes, why is this so? If no, what other possibilities are there? A well-reasoned paragraph is sufficient for this problem.