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ESTIMATION OF UNSATURATED FLOW IN LAYERED SOILS WITH THE FINITE CONTROL VOLUME METHOD[1]

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ABSTRACT

Water movement in unsaturated soil is described by Richards' equation, which is strongly nonlinear and cannot be solved analytically. For this reason numerical methods such as finite difference and finite element methods have been used to solve it. This paper presents another numerical solution of Richards' equation, based on the finite control volume method. This method has important advantages over other numerical methods, such as conservativeness of the system and flexibility of the grid intervals. To validate the numerical model a series of experiments were carried out in the laboratory in a vertical column of unsaturated two-layered soil (coarse and fine sand). The upper boundary condition was a second kind or Newman one and the lower boundary condition was a third kind or Newton's law condition. The soil water content was measured using the γ-ray absorption method, while the water pressure in the pore media was measured using a tensiometer system with ceramic cups and pressure transducers. The numerical results of the new computational scheme are in good agreement with the experimental points. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: finite control volume; water mass balance; drainage; infiltration

RÉSUMÉ

Le mouvement de l'eau dans un sol non saturé est décrit par une équation aux dérivées partielles, appelée l'équation de Richards, qui n'accepte pas de solutions analytiques, mais seulement numériques: différences finies et éléments finis. Ici une nouvelle méthode numérique est presentée, basée sur les éléments de volume de contrôle. La nouvelle méthode présente certains avantages par rapport aux autres en ce qui concerne la conservation de la masse et la flexibilité du maillage. Le problème physique étudié dans le laboratoire sur une colonne de deux couches de sol, présente deux conditions aux limites différentes: à la surface de la colonne une condition de Neuman, tandis' que en bas de la colonne une condition d'aération ou de la loi de Newton. La mesure de la teneur en eau s'est faite par gammametrie et le dispositif des mesures tensiométriques comprenait des tensiomètres et des capteurs de pression. Les résultats numériques de la nouvelle méthode sont en bon accord avec les points expérimentaux. Copyright © 2001 John Wiley & Sons, Ltd.

MOTS CLÉS: éléments de volume de contrôle; bilan de masse d'eau; infiltration; drainage

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INTRODUCTION

Prediction of water movement in unsaturated soils is a major problem in many branches of science and engineering, such as soil science, agricultural engineering, environmental engineering and groundwater hydrology. In virtually all studies of the unsaturated zone, the fluid motion is assumed to obey the classical Richards' equation (Hillel, 1980). This equation may be written in several forms, with either pressure head or water content as the dependent variable. The constitutive relationship (characteristic curve) between pressure head and water content allows conversion of one form of the equation to the other. Three standard forms of the unsaturated flow equation may be identified: the "h-based" form, the "θ-based" form and the "mixed" form.

As Richards' equation is strongly nonlinear, an analytical solution is not possible except for special cases. Numerical approximations are therefore typically used to solve it. The standard approximations applied to the spatial domain are the finite difference and the finite element methods. Tzimopoulos (1978) solved it using the finite element method. Hills *et al*. (1989) presented the advantages and disadvantages of Richards' equation between the "h-based" form and the "θ-based" form. Moldrup *et al*. (1989) studied vertical water movement from fine to medium soils numerically. Celia *et al*. (1990) presented a general mass-conservative numerical solution for the unsaturated flow equation. Kirkland *et al*. (1992) presented two new solutions of Richards' equation with the finite difference method. Sakellariou-Makrantonaki (1997) experimented in layered soil and simulated water movement using the Laasonen computational scheme. Aldama and Aparicio (1998) studied the coefficient stability of numerical solutions on Richards' equation. There are also many other researchers who simulated water movement by numerical methods.

This paper presents the numerical solution of a new computational scheme based on Richards' equation, which was solved by the finite control volume method. In nature more layered soil appears than homogeneous soil. For validation of the numerical model a series of experiments were carry out at the laboratory in a vertical column of unsaturated two-layered soil (coarse and fine sand). The experiments took place at the Laboratory of Agricultural Hydraulics in the Department of Rural Engineering in the Aristotle University of Thessaloniki (A.U.Th.). The numerical results, which were obtained by the finite control volume method, agree well with experimental data.

MATERIAL AND METHODS

The physical problem was studied in the laboratory using a plexiglass cylindrical vertical column, 100 cm long and with a 6 cm inside diameter (ID), with two layered soils. The coarse sand was graded from 2 to 0.85 mm and the fine sand from 0.85 to 0.212 mm. The experiment took place in two cycles. In the first cycle, the coarse sand was the upper layer with a mean bulk density of 1.435 ± 0.02 g cm^{-3} and was 30 cm high. The fine sand was the lower layer with a mean bulk density of 1.4 ± 0.016 g cm^{-3} and was 40 cm high. In the second cycle of the experiment, the coarse sand was the lower layer with a mean bulk density of 1.435 ± 0.015 g cm^{-3} and 40 cm height, while the fine sand was the upper layer with a mean bulk density of 1.4 ± 0.018 g cm^{-3} and 30 cm height. The column was packed using a soil-raising method with free-failing soil passing through a sequence of sieves. With this method, a good homogeneity of sand packing can be achieved. All experiments were run at a constant temperature of 21 ± 1°C.

The bulk densities and the water content were measured by the γ-ray absorption method (Davidson *et al*., 1963; Reginato and van Bavel, 1964; Vachaud and Thony, 1971). The γ-ray device contained a 300 mCi americium-241 source. The americium source and the photomultiplier detector (including a NaI crystal and preamplifier) were set on a horizontal platform connected to a step-by-step motor. In this way one can follow the development of water profiles in the column over time.

The water pressure was measured at the same time at six different depths in the column using tensiometers with ceramic cups and pressure transducers. Each tensiometer was connected to its own pressure transducer. Pressure transducers were connected by a multichannel data acquisition system to a digital voltmeter and then to a printer. The accuracy of the pressure transducers was within 1.5% and the response time was 1 ms. The position of the tensiometers corresponded respectively to depths of 6, 16, 26, 36, 46 and 56 cm from the soil surface. The assumption that the tensiometers measure the average pressure at the point of insertion was made in order to process the measurements.

The soil column was wetted and drained from the bottom by means of a Mariotte burette. The drainage was achieved by lowering the burette step by step and the infiltration done by raising it in the same way.

The unsaturated hydraulic conductivity was estimated using a dosimetric pump that fed the soil column with distilled water from the top. In order to determine the hydraulic conductivity at the appropriate soil moisture, the moisture must be stabilized at the same value at an upper and a lower level of each soil layer. This experimental procedure is based on Darcy's law:

$$q(\theta) = -K(\theta)\left[\frac{\mathrm{d}\Psi(\theta)}{\mathrm{d}z} - 1\right] \tag{1}$$

where q is Darcy's velocity [L T^{-1}], K is the unsaturated hydraulic conductivity [L T^{-1}], θ is the soil moisture [L^3 L^{-3}], Ψ is the suction [L] and z is the distance from an arbitrary control surface [L]. In Equation (1) the vertical axis z has a positive direction downwards. Assuming that the soil moisture θ is constant along the soil column at each layer, then $\mathrm{d}\Psi/\mathrm{d}z = (\mathrm{d}\Psi/\mathrm{d}\theta)(\mathrm{d}\theta/\mathrm{d}z) = 0$ and Equation (1) becomes

$$q(\theta) = |K(\theta)| \tag{2}$$

During the experiment for the measurement of hydraulic conductivity, a problem appeared because it was not possible to measure $K(\theta)$ near the saturation point at the same time in the two layers. When the coarse sand was at the saturation point, the fine sand was not and it was impossible for the soil column to be fed with a higher volume of distilled water. For this reason hydraulic conductivity of the fine sand was measured for the points near saturation, before the second cycle of the experiment and then the coarse sand was packed as the upper layer. This was found to be the only solution to the problem.

The saturated hydraulic conductivity was approximated by fitting the experiment points, using the Brooks and Corey's model (1964), because it was very difficult to obtain it experimentally (Touma, 1984).

The results of Brooks and Corey's model (1964) were also tested with van Genuchten's prediction model (1978) and they gave extremely close results by comparing the mean square error and coefficient correlation between them (Table I). So van Genuchten's prediction model (1978) was used in the application for computational reasons only.

Table I. Mean square error (MSE) and coefficient correlation between Brooks and Corey's model (1964) and van Genuchten's prediction model (1978)

	First cycle		Second cycle	
	Coarse sand	Fine sand	Coarse sand	Fine sand
MSE	1.1×10^{-3}	8.6×10^{-4}	8.4×10^{-4}	1.2×10^{-3}
Coefficient correlation	0.998	0.999	0.998	0.998

NUMERICAL MODEL

The Richards' equation describing the one-dimensional unsaturated flow in porous media is

$$C\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial z}\left[K\left(\frac{\partial \Psi}{\partial z} + 1\right)\right] \tag{3}$$

where C is the specific water capacity equal to $d\theta/d\Psi$ [L^{-1}], Ψ is the suction [L], t is the time coordinate [T], z is the space coordinate pointing downwards [L], K is the unsaturated hydraulic conductivity [$L\ T^{-1}$] taken as a function of θ, because this form does not exhibit significant hysterisis.

In order to solve Richards' equation (3) with the finite control volume method, the domain of numerical integration (z, t) is divided into rectangular elements in space and time (Figure 1) with dimensions $(\delta z)_i$ and Δt respectively. The axis z has a positive direction downwards and we examine the grid point P, surrounded by the points E and W on the right- and the left-hand side respectively. The shaded domain shows the faces of the finite control volume. The letters e and w denote these faces. The distances $(\delta z)_e$ and $(\delta z)_w$ are not necessarily equal. Indeed, the use of non-uniform grid spacing is often desirable to deploy computing power effectively. In general, we shall obtain an accurate solution only when the grid is sufficiently fine, but there is no need to employ a fine grid in regions where the dependent variable changes rather slowly with z. On the other hand, a fine grid is required where the variation is steep. Moreover, there are no universal rules defining maximum (or minimum) ratio of the adjacent grid intervals.

Equation (3) was solved by the finite control volume numerical method. This method was presented by Patankar (1980) in order to solve the equation of heat transfer and was considered as a special case of the method of weighted residuals with weighting function $W = 1$. Using the method of weighted residuals:

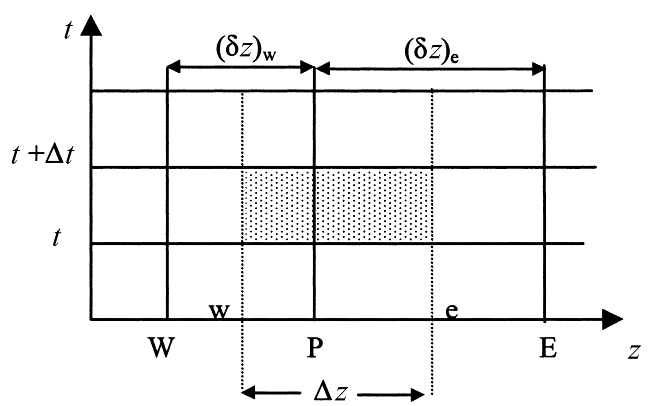

Figure 1. Grid-point cluster for the numerical integration

$$\iint WR \, \mathrm{d}\forall = 0 \Rightarrow$$

$$\Rightarrow \int_{\mathrm{w}}^{\mathrm{e}} \int_{t}^{t+\Delta t} \left(C \frac{\partial \Psi}{\partial z} \right) \mathrm{d}t \, \mathrm{d}z \int_{t}^{t+\Delta t} \int_{\mathrm{w}}^{\mathrm{e}} \frac{\partial}{\partial z} \left[K \left(\frac{\partial \Psi}{\partial z} + 1 \right) \right] \mathrm{d}z \, \mathrm{d}t \tag{4}$$

Equation (4) is discretized as follows (Figure 1):

$$\bar{C}_{\mathrm{P}}(\Psi_{\mathrm{P}}^{t+\Delta t} - \Psi_{\mathrm{P}}^{t})\Delta z = \left(K_{\mathrm{e}} \frac{\Psi_{\mathrm{E}} - \Psi_{\mathrm{P}}}{(\delta z)_{\mathrm{e}}} - K_{\mathrm{w}} \frac{\Psi_{\mathrm{P}} - \Psi_{\mathrm{W}}}{(\delta z)_{\mathrm{w}}} + K_{\mathrm{e}} - K_{\mathrm{w}} \right)_{t+\Delta t} \Delta t \tag{5}$$

The discretization equation (5) takes the following form:

$$A_{\mathrm{P}}\Psi_{\mathrm{W}}^{t+\Delta t} + B_{\mathrm{P}}\Psi_{\mathrm{P}}^{t+\Delta t} + C_{\mathrm{P}}\Psi_{\mathrm{E}}^{t+\Delta t} = D_{\mathrm{P}} \tag{6}$$

where

$$A_{\mathrm{P}} = -\left(K_{\mathrm{w}} \frac{\Delta t}{\Delta z (\delta z)_{\mathrm{w}}} \right)_{t+\Delta t} \tag{6a}$$

$$B_{\mathrm{P}} = \left(\bar{C}_{\mathrm{P}} + K_{\mathrm{e}} \frac{\Delta t}{\Delta z (\delta z)_{\mathrm{e}}} + K_{\mathrm{w}} \frac{\Delta t}{\Delta z (\delta z)_{\mathrm{w}}} \right)_{t+\Delta t} \tag{6b}$$

$$C_{\mathrm{P}} = -\left(K_{\mathrm{e}} \frac{\Delta t}{\Delta z (\delta z)_{\mathrm{e}}} \right)_{t+\Delta t} \tag{6c}$$

$$D_{\mathrm{P}} = \bar{C}_{\mathrm{P}}\Psi_{\mathrm{P}}^{t} + (K_{\mathrm{e}} - K_{\mathrm{w}})_{t+\Delta t} \frac{\Delta t}{\Delta z} \tag{6d}$$

$$\bar{C}_{\mathrm{P}} = \bar{C}_{\mathrm{P}}\big|_{t} \tag{6e}$$

This three-diagonal system of equations allows the estimation of vertical water movement in layered soils.

APPLICATION

The new computational scheme was tested with experimental data, which were obtained as described above. The experiment took place using two-layered soils. In the first cycle the coarse sand was packed as the upper layer and the fine sand as the lower layer. In the second cycle the two layers were reversed. The main purpose of the experiment was the measurement of unsaturated hydraulic conductivity and water mass balance in layered soils. The infiltration of water, for the case of characteristic curve measurement, was controlled from the button of the column for two reasons:

To prevent the air from being caged in the soil pores.
To have a slow flow as the gravity was negative.

The new computational scheme (5) was used to estimate the water mass balance. In order to describe the infiltration and the drainage, the following conditions were used:

Initial condition

As initial condition the equilibrium profile $\Psi = z$ was considered for both cases, infiltration and drainage.

Boundary conditions

In the top of the column it is considered that at the surface of the upper layer the flux is zero (second type or Newman condition). So for $z = 0$ and $t > 0$, $q = 0$.

In the bottom of the column there is a ceramic cup, which has thickness l and hydraulic conductivity K_c. The Darcy's velocity at the ceramic cup between an inside point P and an outside point M (Figure 2) is

$$q = -K_c\left(\frac{\Phi_M - \Phi_P}{l}\right) \tag{7}$$

The Darcy's velocity in the soil sample at a point P near the ceramic cup is

$$q = -K\frac{d\Phi}{dz} \tag{8}$$

So the lower boundary condition is the third kind or Newton's law:

$$-K\frac{\partial \Phi}{\partial z} + K_c\left(\frac{\Phi_M - \Phi_P}{l}\right) = 0 \tag{9}$$

For the simulation of water movement in layered soils we assumed that at the interface between layers the following conditions hold (Touma, 1975; Elmaloglou, 1980):

- The pressure profiles of soil water are continuous at the interface between two layers and pressure values are the same in both layers. On the other hand the moisture profiles are not continuous because the layers have different characteristic curves.
- The water supply at the interface between two layers, which pass from one layer to the other, is the same. However, the gradient of the pressure profile is not the same from one layer to the other, because the hydraulic conductivity for each layer is different.
- At the interface of two layers the hydraulic conductivity is taken as the average.

Numerical stability of the new computational scheme required a time step Δt, very short, equal to 0.0001 h, and a depth step Δz equal to 1 cm. Table II summarizes all the hydrodynamic parameters of the two layers. The soil moisture characteristic curve and the unsaturated hydraulic conductivity, which were required for the execution of the computer program, were obtained using van Genuchten's models (Tzimopoulos and Arampatzis, 1999).

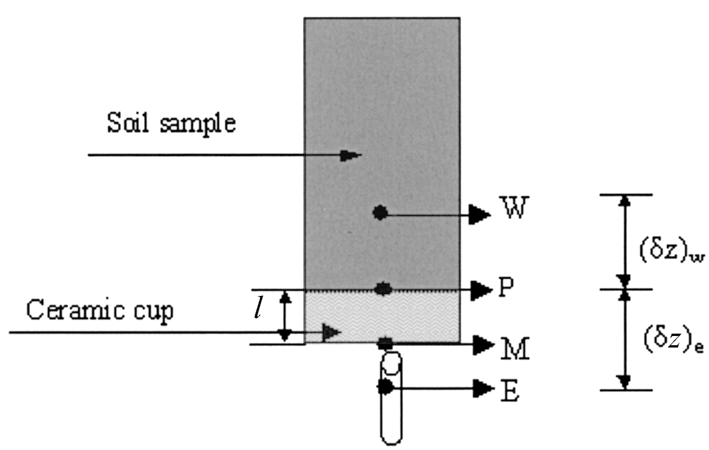

Figure 2. Lower boundary condition

Table II. Coefficients of van Genuchten's (1978) equations $\theta(\Psi)$ and $K(\theta)$ and mean square error (MSE) and coefficient correlation between them and experimental points

	First cycle		Second cycle	
	Coarse sand	Fine sand	Coarse sand	Fine sand
K_s (cm min^{-1})	19.21	2.78	19.77	2.97
θ_r	0.07	0.08	0.07	0.06
θ_s	0.40	0.36	0.40	0.36
Drainage				
a	0.13	0.056	0.095	0.063
n	12.5	13.64	12.84	10.81
$m = 1 - 1/n$	0.92	0.93	0.92	0.91
MSE	1.5×10^{-3}	3.6×10^{-4}	2.7×10^{-3}	2.5×10^{-4}
Coefficient correlation	0.965	0.985	0.921	0.992
Infiltration				
a	0.24	0.088	0.205	0.777
n	4.93	25.69	5.25	7.09
$m = 1 - 1/n$	0.80	0.96	0.81	0.86
MSE	5.4×10^{-4}	1.6×10^{-3}	4.5×10^{-4}	3.6×10^{-5}
Coefficient correlation	0.986	0.92	0.989	0.999

Figures 3–6 show cumulative water volumes during infiltration and drainage for both cycles of the experiment. The experimental points were taken or by weighting, during the experiment, or by integration of experimental moisture profiles. The computational points were obtained by integration of computational moisture profiles. In both cases a good agreement was observed between experimental and computed values for the cumulative water volumes. In Figure 6 the

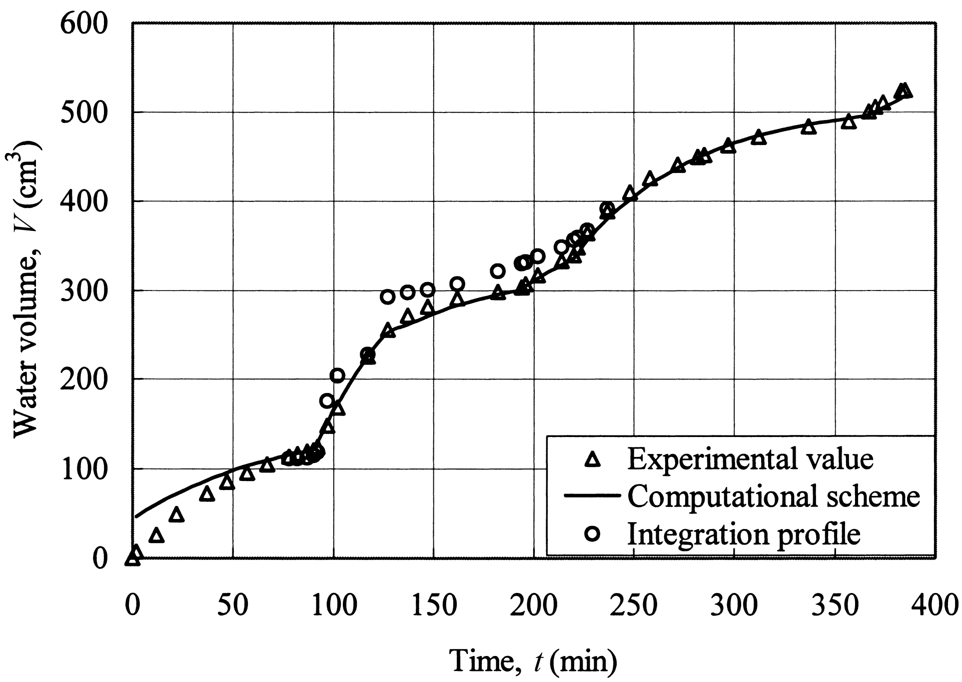

Figure 3. Cumulative water volumes during drainage for the first cycle of the experiment

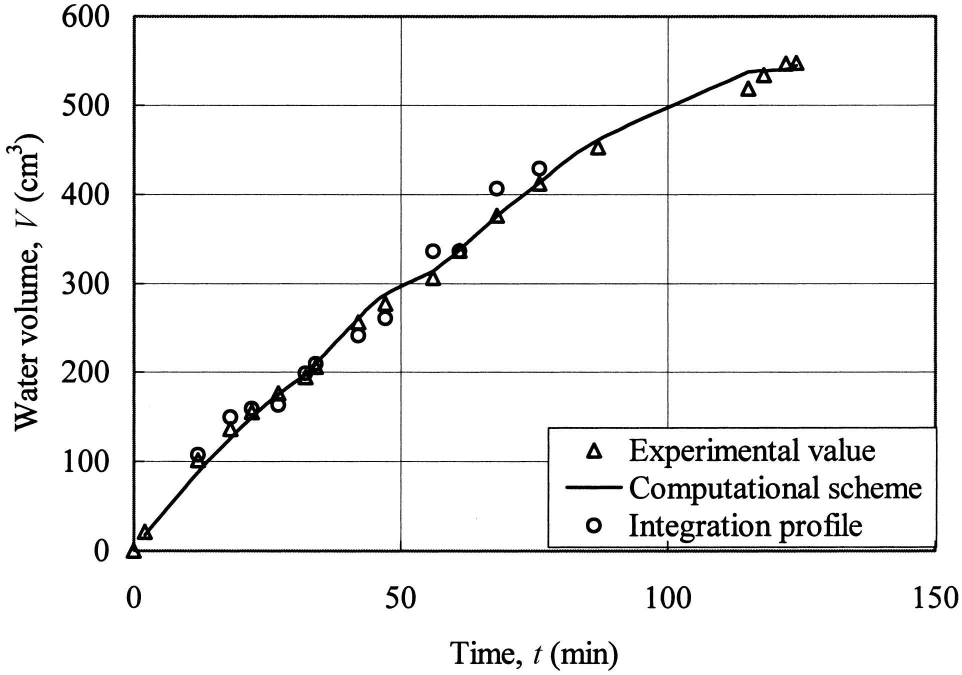

Figure 4. Cumulative water volumes during infiltration for the first cycle of the experiment

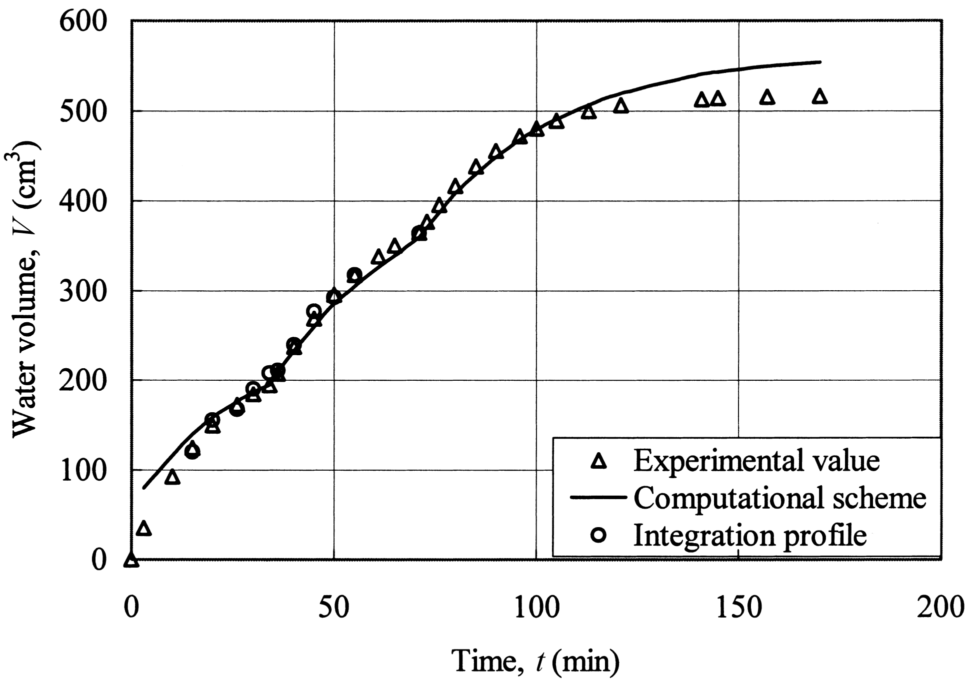

Figure 5. Cumulative water volumes during drainage for the second cycle of the experiment

form of computational curve looks like that for drainage, but it is not. This happened because in the last 10 min of the infiltration the pressure head was low and the flow in the sample was very slow.

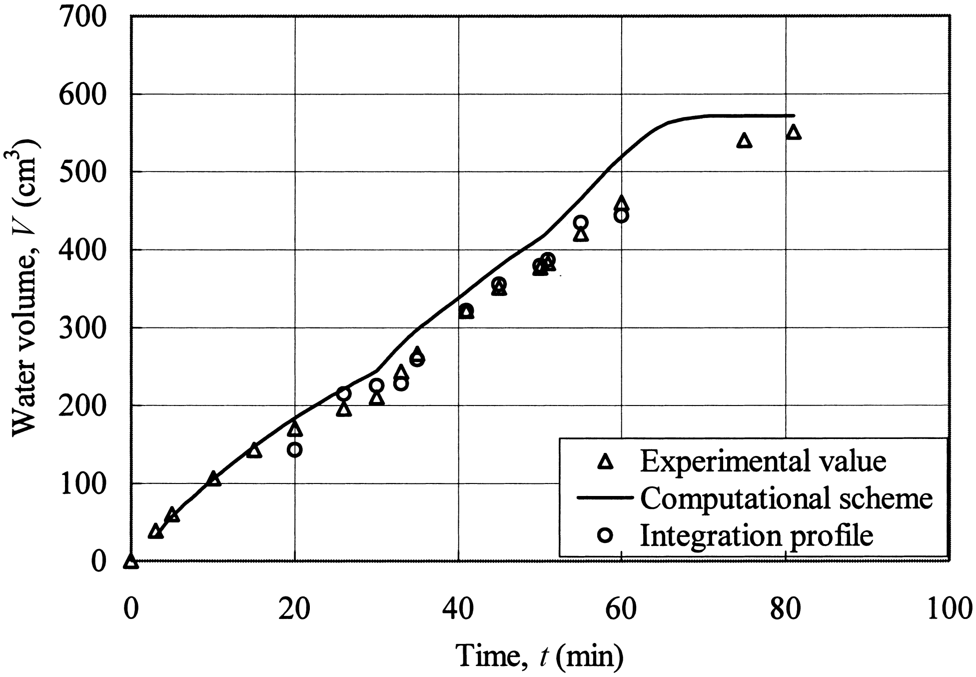

Figure 6. Cumulative water volumes during infiltration for the second cycle of the experiment

CONCLUSIONS

- The most attractive feature of the finite control volume formulation is that the resulting solution would imply that the integral conservation of quantities such as mass is exactly satisfied over any group of finite volumes and, of course, over the whole calculation domain. This characteristic exists for any number of grid points and not just in a limiting sense when the number of grid points becomes large. Thus, even the coarse grid solution exhibits exact integral balances (Patankar, 1980).
- Good agreement was observed in the cumulative water volumes, taken by experiment and by numerical calculations. So the new numerical scheme can be considered mass-conservative and no instabilities were observed during calculations.
- The estimation of unsaturated hydraulic conductivity in layered soils presents some difficulties especially for points near saturation, as the two layers have different water content saturation. For this reason a special technique is applied for the measurement of hydraulic conductivity near saturation.

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