Objectives

Random Graph Models

- Basics
- Erdos-Renyl Model

Social Network Models

Random Graph

- A random graph is a graph in which properties such as the number of graph vertices, graph edges, and connections between them are determined in some random way.

Random Graph

- Choose m pairs of vertices uniformly at random from all possible pairs and connect them with an edge.
- Strictly, in fact, the random graph model is not defined in terms of a single randomly generated network, but as an ensemble of networks, i.e., a probability distribution over possible networks.

Erdos-Renyi Model

$n \rightarrow \frac{n(n-1)}{2}$

$2m$

$V_1 \rightarrow 4$ (V_1, V_2) (V_1, V_3) (V_1, V_4) V_1, V_5

$V_2 \rightarrow 4$ (V_2, V_1) (V_2, V_3) V_4 V_5

Basics

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$
Erdos and Renyi, 1959.
Random graph models

- GNM — $G(n, m)$, a randomly selected graph from the set of C_N^m graphs, $N = \frac{n(n-1)}{2}$, with n nodes and m edges
- GNP — $G(n, p)$, each pair out of $N = \frac{n(n-1)}{2}$ pairs of nodes is connected with probability p, m-random number

n vertices w/ prob p, an edge exist bet a pair of vertices

Erdos-Renyi Model ..

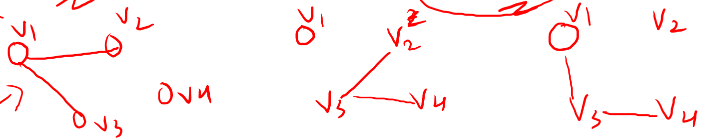
N - No. of vertices.
M - No of edges.

C^M_N

GNM Model

- Random Graph Models: GNM
  Let $n \geq 1$ and $0 \leq M \leq n(n-1)/2$. The $G(n,M)$ model for random graphs is the probability space (S,P), where S denotes the set of simple undirected graphs G on $\{1,2,\ldots,n\}$ with M edges (or equivalently, the set of all subgraphs G of K_n with n nodes and M edges) and each graph $G \in S$ is assigned equal probability $P(G) = \frac{1}{|S|}$.

vertices 4
2 edges



$\frac{1}{10}$

GNM Model ..

- Binomial Coefficient: $\binom{n}{m}$ (read as **n choose m**)

Computing $P(G)$

- The number of edges in K_n: $\binom{n}{2}$ $= \frac{n(n-1)}{2}$

  n_2 $(n-2)!\,2!$

- The number of traingle in K_n: $\binom{n}{3}$
- Note that $\binom{n}{m} = \binom{n}{n-m}$

Computing $P(G)$

- Thus the number of graphs in S in the $G(n, M)$ model is simply the number of edges in K_n choose M.
- ie., $\binom{n(n-1)/2}{M}$ as we just need to choose which M edges to include in our graph.

Choose M edges from $\frac{n(n-1)}{2}$

GNM MODEL ..

$P(G)$

- Hence $P(G) = \frac{1}{\binom{n(n-1)/2}{M}}$, $G \in S$

$G(N, M)$

$5, 3$

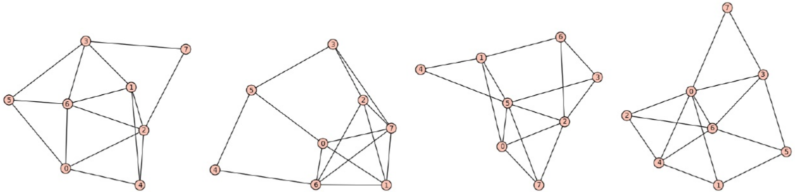
$5(4)/2 \rightarrow$ edges

$\frac{1}{\binom{10}{3}}$.

$\binom{10}{3}$ edges

IN SAGE

```
from sage.graphs.graph_generators_pyx import RandomGNP
D = graphs.RandomGNM(8, 15)
```



OBJECTIVES

RANDOM GRAPH MODELS

- Erdos-Renyl Model
- GNP Model

Erdos Renyi Model

G NM

$p(G) \to \dfrac{1}{\binom{n(n-1)/2}{m}}$

(n, p)

GNP Model

- Second Model: GNP Model
  Let $n \geq 1$ and $0 \leq p \leq 1$. The $G(n,p)$ model for random graphs is the probability space (S, P) where S is the set of all (simple undirected) graphs on $\{1, 2, \ldots, n\}$, where for $G \in S$ with m edges, we define the probability function to be $P(G) = p^m(1-p)^{\frac{n(n-1)}{2}-m}$

no of vertices

prob. of an edge bet a pair of vertices.

$\frac{n(n-1)}{2} - m$

$p^m (1-p)^{\frac{n(n-1)}{2} - m}$

$p^x (1-p)^{n-x}$ → Bernoulli trials.

COMPUTING $P(G)$

- Let G be a random $G(n,p)$ graph.
  - For $1 \leq i < j \leq n$, let $A_{i,j}$ denote the event that the edge {i, j} is included G
  - $B_{i,j}$ the event that the edge {i, j} is not included in G.
  - Say G has m edges: $\{i_1, j_1\}, .., \{i_m, j_m\}$.
  - $\{i_{m+1}, j_{m+1}\}, .., \{i_N, j_N\}$ denote the remaining pairs of non-edges
    - $*$ where $N = \frac{n(n-1)}{2}$
- $A_{i,j}$ events and $B_{i,j}$ are pairwise independent events

edge is present | edge is not present

A_{ij} | B_{ij}

Erdos Ranyi Model: GNP ..

$p(A \cap B)$

$p(A) \cdot p(B)$

Computing $P(G)$

$\frac{n(n-1)}{2}$

- $P(A_{i_1,j_1} \cap A_{i_2,j_2} \cap \ldots \cap A_{i_m,j_m}) \cap B_{i_{m+1},j_{m+1}} \cap \ldots \cap B_{i_N,j_N}) =$
- $P(A_{i_1,j_1}) \cdot P(A_{i_2,j_2}) \cdot \ldots \cdot P(A_{i_m,j_m}) \cdot P(B_{i_{m+1},j_{m+1}}) \ldots P(B_{i_N,j_N}) = p^m(1-p)^{N-m}$

$P(G)$

- $P(G) = p^m(1-p)^{N-m}$
- $= p^m(1-p)^{\frac{n(n-1)}{2} - m}$

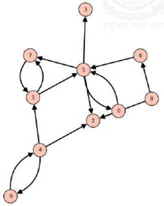
m $\quad N-m$

Erdos Ranyi Model: GNP ..

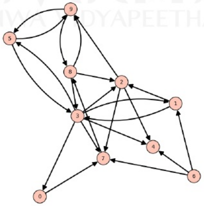
In Sage..

```
from sage.graphs.graph_generators_pyx import RandomGNP
D = RandomGNP(10, .2, directed = True)
```

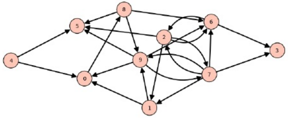
RandomDirectedGNP(10,0.200000000000000): Digraph on 10 vertices

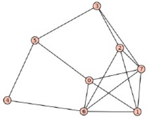


RandomDirectedGNP(10,0.200000000000000): Digraph on 10 vertices



RandomDirectedGNP(10,0.200000000000000): Digraph on 10 vertices



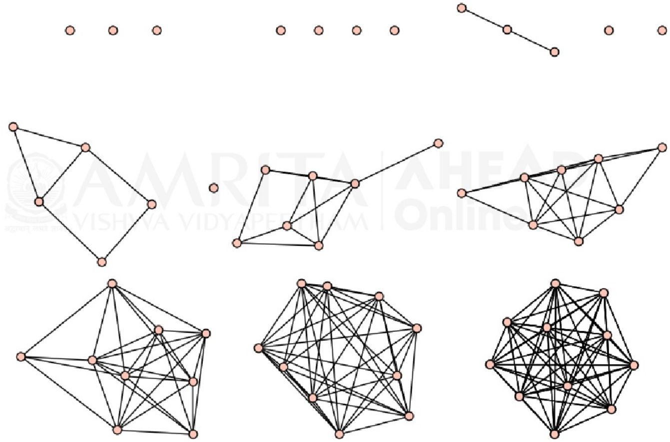


Erdos Ranyi Model: GNM & GNP .. (1/4)

In Sage..

```
g = []
j = []
for i in range(9):
    k = graphs.RandomGNM(i+3, i^2-i)
    g.append(k)
for i in range(3):
        n = []
        for m in range(3):
            n.append(g[3*i + m].plot(vertex_size=50,
                                    vertex_labels=False))
        j.append(n)
G = sage.plot.graphics.GraphicsArray(j)
G.show()  # long time
```

ERDOS RANYI MODEL: GNM & GNP .. (2/4)

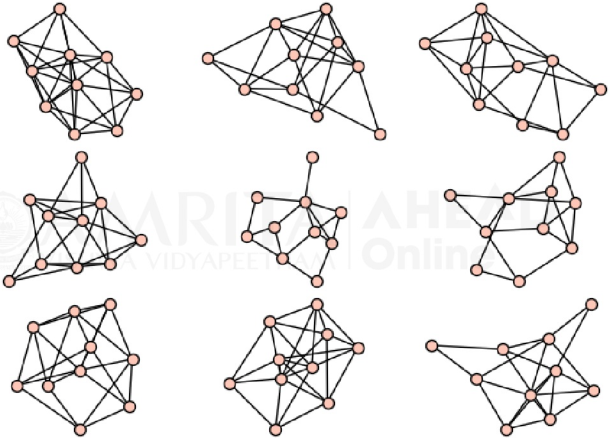


Erdos Ranyi Model: GNM & GNP .. (3/4)

In Sage..

```
g = []
j = []
for i in range(9):
    k = graphs.RandomGNP(10, 0.5)
    g.append(k)
for i in range(3):
        n = []
        for m in range(3):
            n.append(g[3*i + m].plot(vertex_size=50,
                     vertex_labels=False))
        j.append(n)
G = sage.plot.graphics.GraphicsArray(j)
G.show()  # long time
```

Erdos Ranyi Model: GNM & GNP .. (4/4)



Objectives

Random Graph Models

- Properties of GNP Model

Properties of GNP Model

Mean Number of Edges

GNP

- Consider random graph $G(n, p)$. The number of edges in the model is not fixed, but we can calculate its mean or expectation value as follows
  - The number of graphs with exactly n vertices and m edges is equal to the number of ways of picking the positions of the edges from the $\binom{n}{2}$ distinct vertex pairs.

$\binom{n}{2} \rightarrow$

2

MEAN NUMBER OF EDGES ..

- Each of these graphs appears with the same probability P(G) and hence the total probability of drawing a graph with m edges from our ensemble is

$$P(m) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\frac{n(n-1)}{2}-m}$$

- Binomial Distribution: mean value of m is

distinct set of vertices.

$$\sum x \cdot P(x)$$

$$\langle m \rangle = \sum_{m=0}^{\binom{n}{2}} mP(m) = \binom{n}{2} p$$

For each pair (i,j) the prob. of an edge $= p$

$\binom{n}{2} * p$

Properties of GNP Model..

A — B

Mean Degree

- Let k be the mean degree for a n-vertex graph
- then $k.n = 2m$, or $k = \frac{2m}{n}$
- hence mean degree in $G(n, p)$

$\sum x \cdot p(x)$

$$\langle k \rangle = \sum_{m=0}^{\binom{n}{2}} \frac{2m}{n} P(m) = \frac{2}{n}\binom{n}{2} p = (n-1)p$$

$\sum k \cdot p(k)$

- The mean degree of a random graph is often denoted **c**

m

$2m \rightarrow$ undirected

n vertices

$n.k = 2m$

$k = 2m/n$

$\langle k \rangle = \frac{2}{n} \sum m \, p(m) = \frac{2}{n}\binom{n}{2} p = \frac{2}{n} \frac{n(n-1)!}{(n-2)!\,2!} p$

PROPERTIES OF GNP MODEL..

Deg. dist

DEGREE DISTRIBUTION

how many node have deg 0 = 2

- A given vertex in the graph is connected with independent probability p to each of the $n-1$ other vertices.

deg 1 = 3

- Thus the probability of being connected to a particular k other vertices and not to any of the others is $p^k(1-p)^{n-1-k}$

2 =

- There are $\binom{n-1}{k}$ ways to choose those k other vertices, and hence the total probability of being connected to exactly k others is

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Properties of GNP Model...

Directed graph

$$\frac{2 \cdot L_i}{k_i (k_i - 1)}$$

Clustering Coefficient

- Local clustering coefficient C_i measures the density of links in node $i's$ immediate neighborhood: $C_i = 0$ means that there are no links between i's neighbors; $C_i = 1$ implies that each of the i's neighbors link to each other
- To calculate C_i for a node in a random network we need to estimate the expected number of links L_i between the node's k_i neighbors.
- In a random network the probability that two of i's neighbors link to each other is say p_c

PROPERTIES OF GNP MODEL...

$\frac{n(n-1)}{2}$

CLUSTERING COEFFICIENT ..

- As there are $k_i(k_i-1)/2$ possible links between the k_i neighbors of node i, the expected value of L_i is

$$\langle L_i \rangle = p_c \frac{k_i(k_i-1)}{2}$$

- The clustering coefficient is

$$p_c = C_i = \frac{2\langle L_i \rangle}{k_i(k_i-1)}$$

Assume undirected

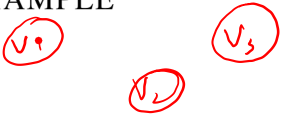
Since mean node degree is 2m/n:

- $C_i = p_c = \frac{\langle k \rangle}{n-1}$

$k = \frac{(n-1)}{2} p$

- Which is nothing but simply **p**

GNP Model: Example



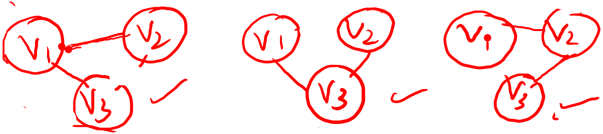
Example

Let $n = 3$ and $p = 0.5$. On $V = \{1, 2, 3\}$.

- A_m denote the subset of graphs on V with m edges.
- Hence, in the $G(n,p) = G(3, 0.5)$ model.

$\binom{3}{0}$ 1 graph with 0 edges	$P(A_0) = P(0 edges) = 1.(0.5)^3 = 0.125$
$\binom{3}{1}$ 3 graph with 1 edge	$P(A_1) = P(1 edges) = 3.(0.5)^3 = 0.375$
$\binom{3}{2}$ 3 graph with 2 edges	$P(A_2) = P(2 edges) = 3.(0.5)^3 = 0.375$
$\binom{3}{3}$ 1 graph with 3 edges	$P(A_3) = P(3 edges) = 1.(0.5)^3 = 0.125$

3 vertices and no edges



OBJECTIVES

ERDOS-RENYI GRAPHS

- Problems and Solutions

Problem 1 (1/2)

GNP Model

- Consider random graph $G(n, p)$ with mean degree c
  - Show that in the limit of large n, number of triangles in the network is $\frac{1}{6}c^3$.
  - Show that expected number of connected triplets is $\frac{1}{2}nc^2$ and hence calculate $C\ell(G)$.

Edges $\binom{n}{3}p$.

Number of triangles

- Let p be the probability of being connected

Number of triangles $= \binom{n}{3} \times p$

Given that mean degree is $c \approxeq (n-1)p$

Hence, $p = \frac{c}{n-1}$

$\therefore$ Number of triangles $= \binom{n}{3} \times \frac{c^3}{(n-1)^3} = \frac{1}{6}c^3$

Problem 1 (2/2)

Number of Connected Triplets

- Let p be the probability of being connected
- Consider a single vertex, there are $\binom{n-1}{2}$ possible pair of neighbors.

Total connected triplets
$= \binom{n}{2} \times p^2 \times n$
For large n, $\approxeq \binom{n-1}{2} \times p^2 \times n$
$= \frac{n(n-1)}{2} \times \frac{c^2}{(n-1)^2} \times n$
$= \frac{1}{2}nc^2$

Clustering Coefficient

- $C\ell(G) = \frac{Number\ \ of\ \ triangles \times 3}{Number\ \ of\ \ connected\ \ triplets}$
  $= \frac{\frac{1}{6}c^3 \times 3}{\frac{1}{2}nc^2}$
  $= \frac{c}{n}$

Problem 2 (1/4)

$F_1 \to F_2 \to F_3 \to F_4$

Transaction Network

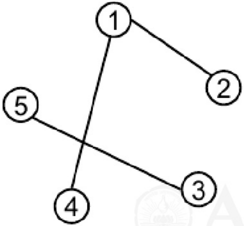
N firms produce goods by buying and selling intermediate products from and to each other. The resulting inter-firm transaction network is modeled as an (undirected) Erdos-Renyi random graph. Based on this model, the expected average nearest-neighbor degree is estimated as $\langle k^{nn} \rangle \approx 2$ and the expected average clustering coefficient as $\langle C \rangle \approx 0.01$

- Determine the total number N of firms and the probability p that any two firms are connected.
  - $N = \frac{\langle k^{nn} \rangle}{\langle C \rangle} = \frac{2}{0.01} = 200$
  - $p = \langle C \rangle = 0.01$

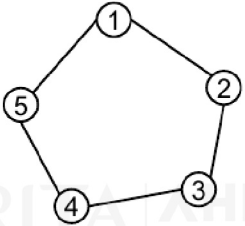
PROBLEM 2 (2/4)

- Determine the average number $\langle k \rangle$ of firms each firm is connected to, the total expected number $\langle L \rangle$ of links, and the average firm-firm distance $\langle D \rangle$ in the network.
  - $\langle k \rangle = (N-1)p \approxeq 2$
  - $\langle L \rangle = \frac{N(N-1)}{2} p = \frac{200 \times 199}{2} \times 0.01 = 199$
  - $\langle D \rangle = \frac{logN}{logk} = 7.64$
- Consider the following three possible configurations for the subgraph involving the first 5 firms only. Calculate the probability of occurrence for each of the three possibilities in the inter-firm network considered above.

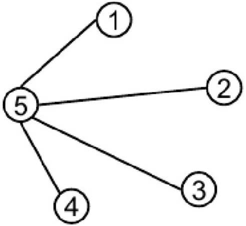
PROBLEM 2 (3/4)



Subgraph 1



Subgraph 2



Subgraph 3

- Total possible edges $= \binom{5}{2} = 10$
- Subgraph 1 - $p^3(1-p)^7 = (0.01)^3(0.99)^7$
- Subgraph 2 - $p^5(1-p)^5 = (0.01)^5(0.99)^5$
- Subgraph 3 - $p^4(1-p)^6 = (0.01)^4(0.99)^6$

Problem 2 (4/4)

Discussion

- Critical condition probability $p_c \approxeq {}^1/_N = {}^1/_{200}$
  - if $p < p_c$ - Network is divided into many small connected components
  - if $p > p_c$ - Network has a gaint component and have some smaller components
- In this example $p = {}^1/_{100} > p_c$, hence, there must be a gaint component.

References

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