Bayes’ Rule

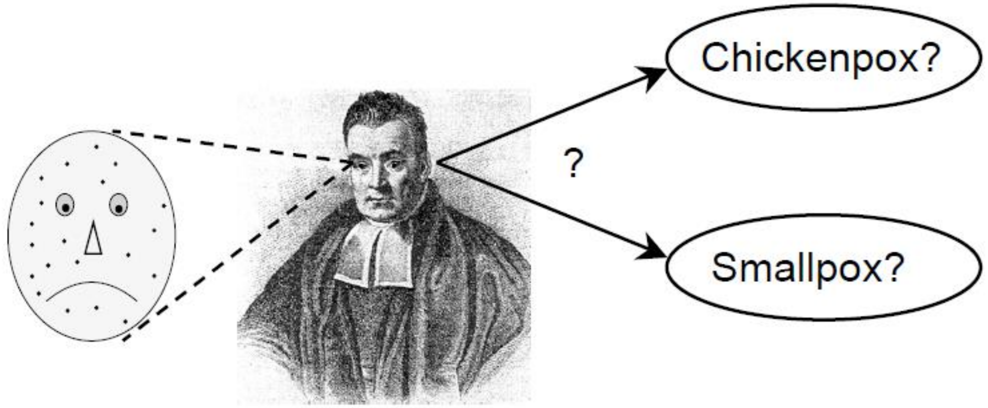
Content sourced from

Bayes’ Rule : A Tutorial Introduction to Bayesian Analysis by James V. Stone

Bayes' Rule

The Patient's Perspective

Suppose that you wake up one day with spots all over your face, as in Figure 1.2. The doctor tells you that 90% of people who have smallpox have the same symptoms as you have. In other words, the probability of having these symptoms given that you have smallpox is 0.9 (ie 90%). As smallpox is often fatal, you are naturally terrified.



$$p(\text{symptoms are spots} \mid \text{disease is smallpox}) = 0.9, \tag{1.1}$$

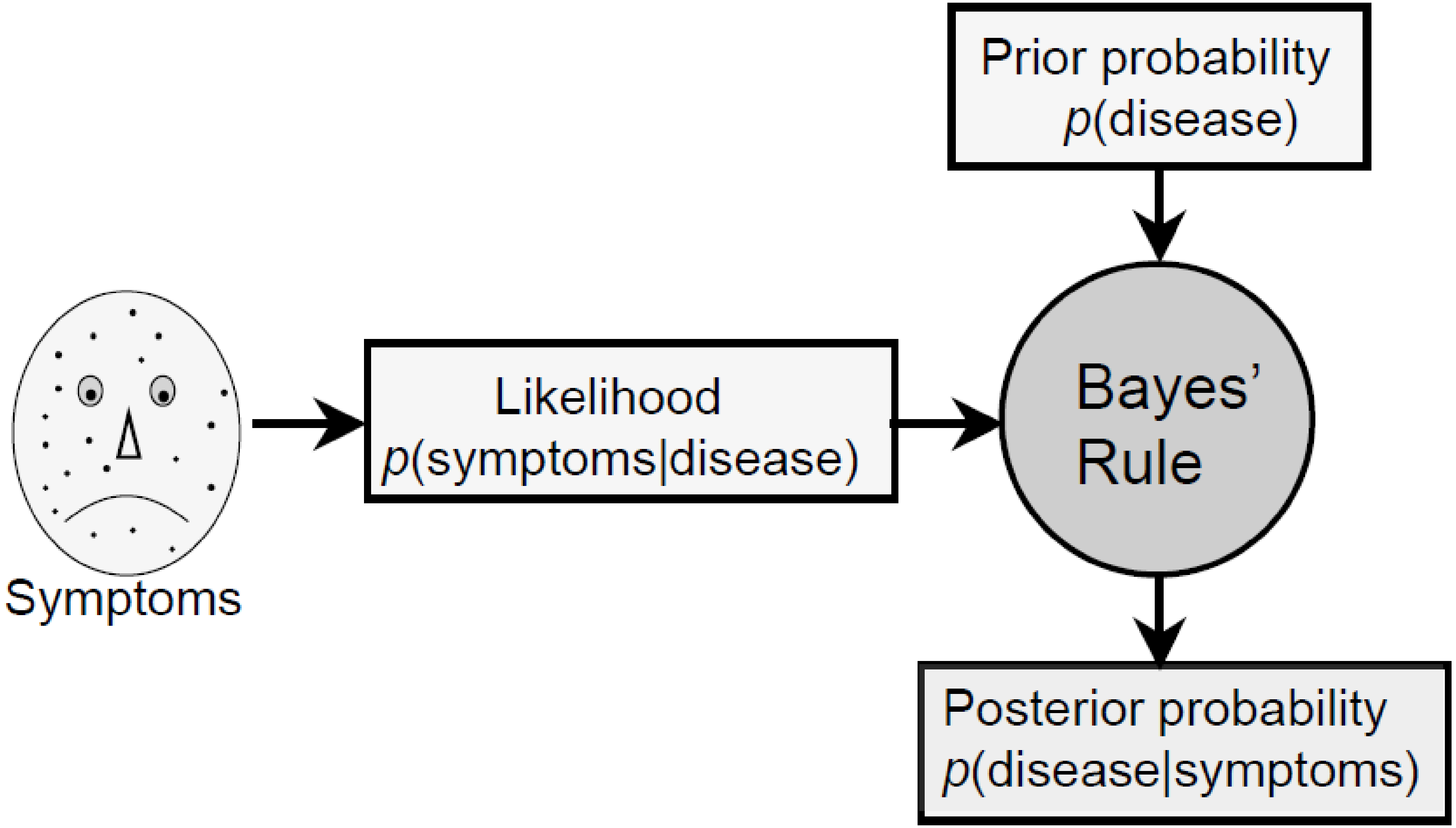
where the letter p stands for probability, and the vertical bar $|$ stands for "given that". So, this short-hand statement should be read as

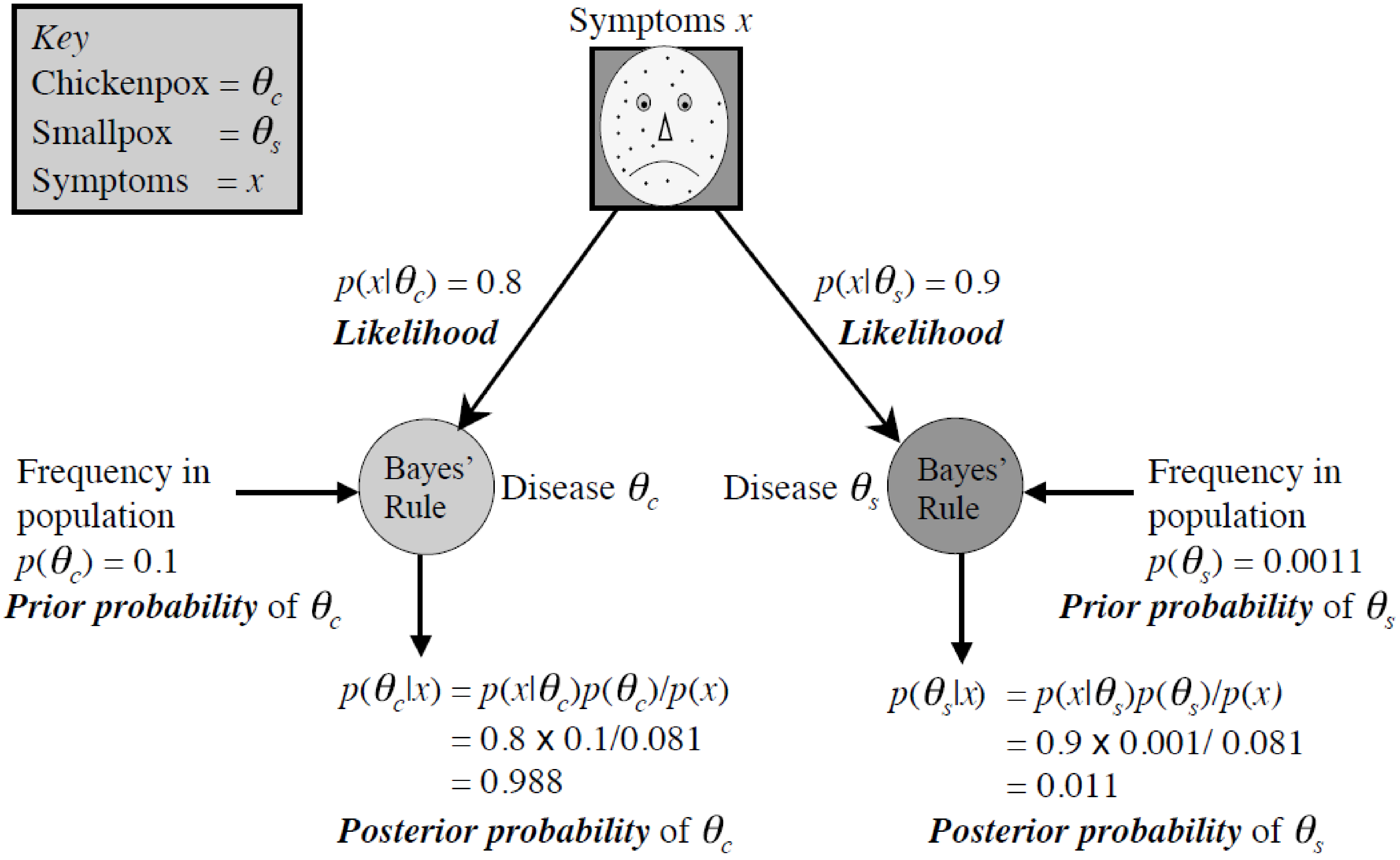
> "the probability that the patients symptoms are spots given that he has smallpox is 90% or 0.9".

Similarly, we might find that spots are observed in 80% of patients who have chickenpox, which is written as

$$p(\text{spots}|\text{chickenpox}) = 0.8. \tag{1.3}$$

Bayes' Rule


Prior probability
p(disease)
Likelihood
p(symptoms|disease)
Bayes' Rule
Symptoms
Posterior probability
p(disease|symptoms)



Key
Chickenpox = θ_c
Smallpox = θ_s
Symptoms = x
Symptoms x
$p(x|\theta_c) = 0.8$
Likelihood
$p(x|\theta_s) = 0.9$
Likelihood
Frequency in population
$p(\theta_c) = 0.1$
Prior probability of θ_c
Bayes' Rule
Disease θ_c
Disease θ_s
Bayes' Rule
Frequency in population
$p(\theta_s) = 0.0011$
Prior probability of θ_s
$p(\theta_c|x) = p(x|\theta_c)p(\theta_c)/p(x)$
$= 0.8 \times 0.1/0.081$
$= 0.988$
Posterior probability of θ_c
$p(\theta_s|x) = p(x|\theta_s)p(\theta_s)/p(x)$
$= 0.9 \times 0.001/ 0.081$
$= 0.011$
Posterior probability of θ_s

Bayes' Rule

The equation used to perform Bayesian inference is called Bayes' rule, and in the context of diagnosis is

$$p(\text{disease}|\text{symptoms}) = \frac{p(\text{symptoms}|\text{disease})p(\text{disease})}{p(\text{symptoms})}, \tag{1.12}$$

which is easier to remember as

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{marginal likelihood}}. \tag{1.13}$$

Bayes' Rule

$$p(\text{hypothesis}|\text{data}) = \frac{p(\text{data}|\text{hypothesis}) \times p(\text{hypothesis})}{p(\text{data})},$$

Sample Problem

1. *You go to see the doctor about an ingrowing toenail. The doctor selects you at random to have a blood test for swine flu, which for the purposes of this exercise we will say is currently suspected to affect 1 in 10,000 people in Australia. The test is 99% accurate, in the sense that the probability of a false positive is 1%. The probability of a false negative is zero. You test positive. What is the new probability that you have swine flu?*

Now imagine that you went to a friend's wedding in Mexico recently, and (for the purposes of this exercise) it is know that 1 in 200 people who visited Mexico recently come back with swine flu. Given the same test result as above, what should your revised estimate be for the probability you have the disease?

Source: http://stony.me/statistics-for-beginners-bayes-rule-4/

Let $P(D)$ be the probability you have swine flu.

Let $P(T)$ be the probability of a positive test.

We wish to know $P(D|T)$.

Bayes theorem says

$$P(D|T) = P(T|D)P(D) / P(T)$$

which in this case can be rewritten as

$$P(D|T) = P(T|D)P(D) / (P(T|D)P(D) + P(T|ND)P(ND))$$

where $P(ND)$ means the probability of not having swine flu.

We have

$P(D) = 0.0001$ (the a priori probability you have swine flu).

$P(ND) = 0.9999$

$P(T|D) = 1$ (if you have swine flu the test is always positive).

$P(T|ND) = 0.01$ (1% chance of a false positive).

Plugging these numbers in we get

$$P(D|T) = 1 \times 0.0001 / (1 \times 0.0001 + 0.01 \times 0.9999) = 0.01$$

That is, even though the test was positive your chance of having swine flu is only 1%.

Source: http://stony.me/statistics-for-beginners-bayes-rule-4/

However, if you went to Mexico recently then your starting $P(D)$ is 0.005. In this case

$$P(D|T) = 1 \times 0.005 / (1 \times 0.005 + 0.01 \times 0.995) = 0.33$$

and you should be a lot more worried.