OBJECTIVES

REVISITING LINEAR ALGEBRA

- Vectors
- Linear Functions

What is Linear Algebra

Basics

- Many difficult problems can be handled easily once relevant information is organized in a certain way.
- Certain mathematical structures are present in the information.
- Linear algebra is, in general, the study of those structures.

Linear algebra is the study of vectors and linear functions.

Three Notions

- Vectors
- Linear functions
- Matrices

VECTORS

DEFINITION

Vectors are stack of numbers

Stacks of numbers are not only the vectors

PROPERTY OF VECTORS

Vectors are closed under addition and scalar multiplication

Vectors (1/2)

Example

- Numbers: Both 3 and 5 are numbers and so is $3+5$.
- 3-vectors: $\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$.
- Polynomials: If $p(\mathrm{x}) = 1+x-2x^2+3x^3$ and $q(\mathrm{x}) = x+3x^2-3x^3+x^4$ then their sum $p(\mathrm{x})+q(\mathrm{x})$ is the new polynomial $1+2x+x^2+x^4$.
- Power series: If $\mathrm{f(x)=1+x+}\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ and $\mathrm{g(x)=1-x+}\frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots$ then $\mathrm{f(x)+g(x)=2+}\ x^2 + \frac{2}{4!}x^4 + \cdots$ is also a power series.
- Functions: If $\mathrm{f(x)}=e^x$ and $\mathrm{g(x)}=e^{-x}$ then their sum $\mathrm{f(x)+g(x)}$ is the new function $2\cosh x$.

VECTORS (2/2)

EXAMPLE

- 2(3)=6 (is a number)
- $0\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ (The zero 3-vector)
- $2(1+x-2x^2+3x^3)= 2+2x-4x^2+6x^3$ (another polynomial)
- $0(e^x)$ =0 (The zero function)

Vectors: Real world example

Example

(Of organizing and reorganizing information)
You own stock in 3 companies: *Google*, *Netflix*, and *Apple*. The value V of your stock portfolio as a function of the number of shares you own s_N, s_G, s_A of these companies is

$24s_G + 80s_A + 35s_N$. what could be $V\begin{pmatrix}1\\2\\3\end{pmatrix}$?

- 1 share of G, 2 shares of N and 3 shares of A?

we can denote V itself as an ordered triple of numbers

$$\text{Denote } V \text{ by } (24 \;\; 80 \;\; 35) \text{ and thus write } V\begin{pmatrix}1\\2\\3\end{pmatrix}_B = (24 \quad 80 \quad 35)\begin{pmatrix}1\\2\\3\end{pmatrix}$$

to remind us to calculate $24(1)+80(2)+35(3)=334$

VECTORS: REAL WORLD EXAMPLE..

ORDERED LIST

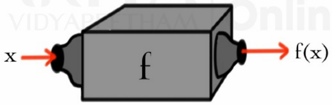
EXAMPLE

In vectors, order does matter!!!

Denote V by $(35 \;\; 80 \;\; 24)$ and thus write $V \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{B'} = (35 \;\; 80 \;\; 24) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to remind us to calculate $35(1) + 80(2) + 24(3) = 264.$ because we chose the order $\begin{pmatrix} N & A & G \end{pmatrix}$ and named that order B' so that inputs are interpreted as $\begin{pmatrix} s_N \\ s_A \\ s_G \end{pmatrix}$.

Linear Functions (1/1)

- In general, function as a machine f into which one may feed a real number.
- For each input x this machine outputs a single real number $f(x)$.
- In linear algebra, functions will have vectors (of some type) as both inputs and outputs.



- What vector X satisfies $f(X) = B$?

Linear combination

Concept

A *linear combination* of x_1, ..., x_n has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the combination's *coefficients*.

This is a linear combination of x, y, and z.

$$(1/4)x + y - z$$

More on Liner Functions

Examples

- What number x satisfies $10x = 3$?
- What 3-vector u satisfies $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times u = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?
- What polynomial p satisfies $\int_{-1}^{1} p(y)dy = 0$ and $\int_{-1}^{1} yp(y)dy = 1$?
- What power series $f(x)$ satisfies $x\frac{d}{dx}f(x) - 2f(x) = 0$?
- What number x satisfies $4x^2 = 1$?
- All of these are of the form
  - What vector X satisfies $f(X) = B$?
- with a function, f known, a vector B known, and a vector X unknown.

Objectives

Revisiting Linear Algebra

- Linear Functions
- Matrices

Linear Functions: Charactristics

Linearity

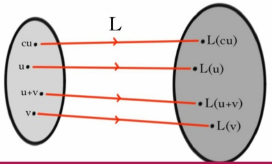
- L to denote an arbitrary linear function
- v and u are vectors and c is a number
  - $L(u)$ and $L(v)$ are Vectors
    - ∗ Additivity:

$$L(u+v) = L(u) + L(v).$$

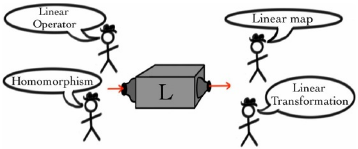
    - ∗ Homogeneity:

$$L(cu) = cL(u).$$

- Linearity:- Additivity + Homogeneity



Linear Operator



$$\boxed{\text{Function} = \text{Transformation} = \text{Operator}}$$

$$\boxed{Lv = w}$$

- where v is an unknown, w a known vector, and L is a known linear transformation.
- Solving the equation $Lv = w$ often amounts to solving systems of linear equations,

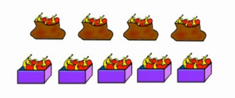
Matrices (1/3)

What is a matrix

Matrices are the result of organizing information related to linear functions.

Example

A room contains x bags and y boxes of fruit.



Each bag contains 2 apples and 4 bananas and each box contains 6 apples and 8 bananas. There are 20 apples and 28 bananas in the room. Find x and y.

Matrices (2/3)

Example

.. The values are the numbers x and y that simultaneously make both of the following equations true:

$$\begin{aligned} 2x + 6y &= 20 \\ 4x + 8y &= 28\,. \end{aligned}$$

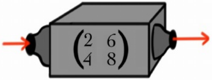
$$\left.\begin{aligned} 2x + 6y = 20 \\ 4x + 8y = 28 \end{aligned}\right\} \iff \begin{pmatrix} 2x+6y \\ 4x+8y \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix} \iff x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}.$$

The function $\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$ is defined by $\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

Matrices (3/3)

Viewed as a machine that inputs and outputs 2-vectors, our 2×2 matrix does the following:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} 2x + 6y \\ 4x + 8y \end{pmatrix}.$$

Our fruity problem is now rather concise.

Example

(This time in purely mathematical language):

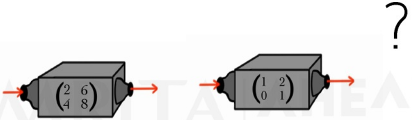
What vector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$?

This is of the same $Lv = w$ form as our opening examples.

Linear Functions and Matrices (1/2)

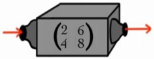
Notion of Matrix multiplication

What would happen if we placed two of our expensive machines end to end?

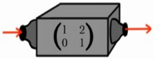


The output of the first machine would be fed into the second.

$$\begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} 2x+6y \\ 4x+8y \end{pmatrix}$$



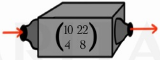
$$\begin{pmatrix} 1.(2x+6y)+2.(4x+8y) \\ 0.(2x+6y)+1.(4x+8y) \end{pmatrix} = \begin{pmatrix} 10x+22y \\ 4x+8y \end{pmatrix}$$

Linear Functions and Matrices (2/2)

Notion of Matrix multiplication..

Notice that the same final result could be achieved with a single machine:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} 10x + 22y \\ 4x + 8y \end{pmatrix}.$$

There is a simple matrix notation for this called *matrix multiplication*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 10 & 22 \\ 4 & 8 \end{pmatrix}.$$

Objectives

Revisiting Linear Algebra

- Eigenvalues & Eigenvectors
- Eigenvector Centrality

Eigen Vectors

Basics

- $x \rightarrow \boxed{\text{A}} \rightarrow Ax$
  - Matrix A could be seen as a function in Linear Algebra
- Eigenvectors
  - Ax is parallel to x
  - $Ax = \lambda x$; λ is a scalar value
- When $\lambda = 0$ Ax=0, x is in $N(A)$
- In $Ax = \lambda x$, both λ and x are unknowns

Eigen Vectors (1/1)

A Simple Example

- $Ax = \lambda x$, both λ and x are unknowns
- We find both λ and x through observation

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = 1$

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
  - $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\lambda = -1$

Solving for Eigenvectors (1/2)

Solution to $Ax = \lambda x$

- $Ax = \lambda x$
    - $(A - \lambda \mathrm{I})x = 0$
        * Matrix A shifted by λ
        * Singular $\rightarrow |A - \lambda \mathrm{I}| = 0$
    - $|A - \lambda \mathrm{I}| = 0$: characteristics equation
    - Solve characteristics equation for λs
    - Find the null space for each λ; $Ax = \lambda_i x = 0$

Solving for Eigenvectors (2/2)

Example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$|A - \lambda \mathrm{I}| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow$$

$$(\lambda - 4)(\lambda - 2) = 0 \Rightarrow \lambda = 4,\ 2$$

For $\lambda = 4$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,\ \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,\ \hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvector Centrality

- Eigenvector centrality is an extension of degree centrality.
- Eigenvector centrality (also called eigencentrality) is a measure of the influence of a node in a network.
- A vertex's importance in a network is increased by having connections to other vertices that are themselves important.
- All the entries in the eigenvector must be non-negative.
- In general, eigen centrality is applicable to undirected networks
- Pagerank is a varient of eigenvector centrality

Eigenvector Centrality

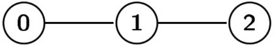
- Let λ is eigenvalue and υ is eigenvector

$$A\upsilon = \lambda\upsilon$$

- $\upsilon \in \mathbb{R}^V$ or $\upsilon : V \to \mathbb{R}$
- Symmetric - n real eigen values and n real eigen vectors

Example

```
import networkx as nx
G= nx.path_graph(3)
```



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvector Centrality

- Let λ is eigenvalue and υ is eigenvector

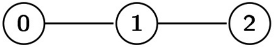
$$A\upsilon = \lambda\upsilon$$

- $\upsilon \in \mathbb{R}^V$ or $\upsilon : V \to \mathbb{R}$
- Symmetric - n real eigen values and n real eigen vectors

Example

```python
import networkx as nx
G= nx.path_graph(3)
```

λ_1= -1.414, λ_2= 0, λ_3= 1.414

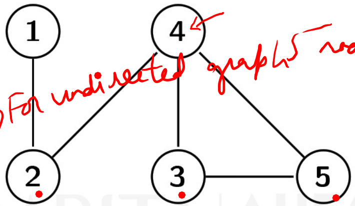


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\upsilon_1 = \begin{pmatrix} 0.5 \\ -0.707 \\ 0.5 \end{pmatrix}, \upsilon_2 = \begin{pmatrix} 0.707 \\ 0 \\ -0.707 \end{pmatrix}, \upsilon_3 = \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix}$$

- Eigenvectors forms orthonormal basis for the space

Computing Eigenvector Centrality (1/2)



5 m edges

2m → For undirected graphs nodes

$Av = \lambda v$

$|A - \lambda I|$

$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix}$ → ev centrality

- The Power iteration method can be used to solve the eigenvector centrality problem.

→ none be

$x_1 \quad \sqrt{1+4+4+9+4}$

1/4.69 2/4.69 1/3 2/3

Iteration 1

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{x_0} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 2 \end{pmatrix}}_{x_1} = \begin{pmatrix} 0.213 \\ 0.426 \\ 0.426 \\ 0.639 \\ 0.426 \end{pmatrix}$$

4.69

Iteration 2

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0.213 \\ 0.426 \\ 0.426 \\ 0.639 \\ 0.426 \end{pmatrix}}_{x_1} = \underbrace{\begin{pmatrix} 0.426 \\ 0.852 \\ 1.065 \\ 1.278 \\ 1.065 \end{pmatrix}}_{x_2} = \underbrace{\begin{pmatrix} 0.195 \\ 0.389 \\ 0.486 \\ 0.584 \\ 0.486 \end{pmatrix}}_{\text{normalized } x_2}$$

2.19

Computing Eigenvector Centrality (2/2)

Iteration 3

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.195 \\ 0.389 \\ 0.486 \\ 0.584 \\ 0.486 \end{pmatrix} = \begin{pmatrix} 0.389 \\ 0.779 \\ 1.07 \\ 1.361 \\ 1.07 \end{pmatrix}_{2.21} = \begin{pmatrix} 0.176 \\ 0.352 \\ 0.484 \\ 0.616 \\ 0.484 \end{pmatrix}$$

$A \quad x_2 \quad x_3$ normalized

Iteration 4

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.176 \\ 0.352 \\ 0.484 \\ 0.616 \\ 0.484 \end{pmatrix} = \begin{pmatrix} 0.352 \\ 0.792 \\ 1.100 \\ 1.320 \\ 1.100 \end{pmatrix}_{2.21} = \begin{pmatrix} 0.159 \\ 0.358 \\ 0.497 \\ 0.597 \\ 0.497 \end{pmatrix}$$

$A \quad x_3 \quad x_4$

Eigenvector Centrality

$n \to O(n)$

n

2^n

node 1

node 2

node 5

$$x = \begin{pmatrix} 0.159 \\ 0.358 \\ 0.497 \\ 0.597 \\ 0.497 \end{pmatrix}$$

leading eV

x_4

$x_5 - x_8$

Caveats in Power method

1. Will not work if the initial vector x(0) happens to be orthogonal to the leading eigenvector.
2. The elements of the vector have a tendency to grow on each iteration—they get multiplied by approximately a factor of the leading eigenvalue each time, which is usually greater than 1.
3. How long do we need to go on multiplying by the adjacency matrix before the result converges to the leading eigenvalue?

$A \rightarrow v_1 \; v_2 \; v_3$

$v_1 \perp v_2$

$v_1 \perp v_3$

x_0 x_0'

$x_1 \rightarrow A x_0$

$x_2 \rightarrow A x_1 = A(Ax_0) = A^2 x_0$

$x_3 \rightarrow A x_2 = A(AAx_0) = A^3 x_0$

$x_k \rightarrow A^k x_0$

$x_0 \rightarrow$ non-ve values

approx original eV

if $v_1 \cdot v_2 = 0$ then $v_1 \perp v_2$

Computational Complexity.

$G(n, m)$ — $n \to$ nodes, $m \to$ m edges

$A \cdot x_0$

A will have n^2 values.

$x_1 = A \cdot x_0 = n^2$ multiplications.

2nd iteration $\to A \cdot x_1 = n^2$ multiplications.

For k iterations, we have

$(K \cdot n^2)$ multiplications.

$O(n \cdot n^2) = O(n^3)$

Computational Complexity

If G is represented as Adj. list.

1 → 2
2 → 1, 4
3 → 4, 5
4 → 2, 3, 5
5 → 3, 4

10 values

$2m$

$2m$ multiplications

for 1 iteration.

If there are n iterations.

$n \cdot 2m$

$\Rightarrow O(nm) \Rightarrow$ worst case scenario $O(n \cdot n^2) = O(n^3)$

For complete graph → K_5

1 → 4
2 → 4
3 → 4

$\frac{n(n-1)}{2}$

$O(n^2)$

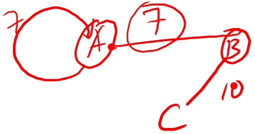
Leading ev $\Rightarrow A^k x_0$.

$$\frac{Ax = \lambda x}{x\cdot}$$

What is λ

$$\lambda = \frac{Ax\cdot x}{x\cdot x} = \frac{\lambda x\cdot x}{x\cdot x} = \lambda$$

$A = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix}$



$x_0 : \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓ 2

$$\begin{array}{ccc} & & x_1 \\ \to A x_0 = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} & = & \begin{bmatrix} 7 \\ 9 \end{bmatrix} \end{array} \xrightarrow{\text{abs max value}} \begin{array}{c} x_1 \\ \begin{bmatrix} 7/9 \\ 9/9 \end{bmatrix} = \begin{bmatrix} 0.77 \\ 1 \end{bmatrix} \end{array}$$

$$A x_1 = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix} \begin{bmatrix} 0.77 \\ 1 \end{bmatrix} = \begin{bmatrix} 14.39 \\ 13.93 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0.97 \end{bmatrix}$$

$A x_2 =$

$\vdots$

$x_n = \begin{bmatrix} 1 \\ 0.999 \end{bmatrix} \to$ approximation for leading ev. $\to \begin{array}{c} ev \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$

$x_0 \cdot lev = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= 1$

$$\rightarrow X_0 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} \qquad \text{leev} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_0 \cdot \text{leev} \neq 0$$

So x_0 is acceptable.

$$Ax_2 = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

No convergence

$$x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow$$ wrongly chosen random initial vector

$$Ax_0 = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2/2 \\ -2/2 \end{bmatrix} = \overset{x_1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$Ax_1 = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \rightarrow \overset{x_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$