~~$a = $~~ A

$$C = C - \frac{A^T C A}{A^T A} - \frac{B^T C B}{B^T B}$$

$=$

15. 12. 2022

Diagonalization of a Matrix

capital λ

Similar to $A = LU$, we have $\boxed{A = S \Lambda S^{-1}}$

We put our eigen vectors into matrix S.

$$S = \begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}$$ $\Rightarrow$ n independent eigen vectors form the columns of S

$$AS = A[x_1 \;\; x_2 \;\cdots\; x_n]$$ $\rightarrow$ eigen vectors

$$= [\lambda_1 x_1 \;\; \lambda_2 x_2 \;\cdots\; \lambda_n x_n] \qquad [\because Ax = \lambda x]$$

$$= \underbrace{[x_1 \;\; x_2 \;\cdots\; x_n]}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$\hookrightarrow$ eigen values

Then we get:

$$AS = S\Lambda$$

we can derive 2 eqns from the base eqn

$$\boxed{S^{-1} A S = \Lambda}$$

OR

$$\boxed{A = S \Lambda S^{-1}}$$

based on the assumption that: but this may not always be the case

$$A^2 = S\Lambda \underbrace{S^{-1} . S}_{I} \Lambda S^{-1} = \underline{S \Lambda^2 S^{-1}}$$ (much more profound)

Amazing thing: $A^{100} = S \Lambda^{100} S^{-1}$ $\Rightarrow$ an easy computation compared to $A^{100} = LU \times \cdots \times LU$ or QR and so on.

★ Caveat: may not hold / work for repeated eigen values

(please note) or watch out — only instance where it will work: I i.e. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & \end{bmatrix}_{n \times n} = I$

Eg

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)^2 - 0 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = 2, 2$$

why? because of this

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ In such a case, diagonalization will not work.

★ Consider the following system:

$$U_{k+1} = AU_k$$

$$U_1 = AU_0$$

$$U_2 = AU_1 = A \cdot AU_0 = A^2 U_0$$

The base case U_0 should be expressed as a combination of eigen vectors or decomposed into

$$U_0 = c_1 x_1 + \dots + c_n x_n = Sc$$

Multiplying both sides by A:

$$AU_0 = A(c_1 x_1 + \dots + c_n x_n)$$

$$= c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n \qquad [\because Ax = \lambda x]$$

16.12.2022

$u_{k+1} = A u_k$

$\boxed{u_0 = c_1 x_1 + \dots + c_n x_n = Sc}$

$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + \dots + c_n \dots$

$= \lambda^{100} Sc$

fibonacci

$\boxed{F_{k+2} = F_{k+1} + F_k}$ ⟶ main eq, 2nd order

$F_{k+1} = F_{k+1}$ ⟶ add this eq to make it first order

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ and $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{2\times2} u_k$

$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{k+1} \\ f_k \end{bmatrix} = A u_k$

↳ symmetric

Now, find eigen values and vectors:

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - \lambda - 1 = 0$

$\lambda = \dfrac{1 \pm \sqrt{5}}{2}$

$\dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$\dfrac{+1 \pm \sqrt{1 + 4 \times 1 \times +1}}{2}$

$\dfrac{+1 \pm \sqrt{5}}{2}$

i.e. $\lambda_1 = \dfrac{1+\sqrt{5}}{2} = 1.618$ (golden ratio)

$\lambda_2 = \dfrac{1-\sqrt{5}}{2} = -0.618$

☆ $F_{100} = \underbrace{(1.618)^{100} c_1 x_1}_{\text{leading term}} + \underbrace{(-0.618)^{100} c_2 x_2}_{\text{tends to } 0}$, i.e. $F_{100} = (1.618)^{100} c_1 x_1$

⟹ eigen value controls the growth rate

$1 - \lambda(1-\lambda)$

$1 - \lambda + \lambda^2$

$\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda+\lambda^2 \end{bmatrix}$

To find e. vectors:

$(A - \lambda I)x = 0$

$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now: $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

i.e. $c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

To find 'n' term:

- rep u_0 as $c_1 x_1 + \cdots + c_n x_n$
- find e. values and e. vectors
- find how λ contributes to growth rate

17.12.2022

Markov Models

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix} \rightarrow \text{Markove Matrix}$$

Properties of MM:

- All entries ≥ 0 ①
- All columns add upto 1 (all entries in a col) ② → if a matrix is given s.t its row elements add upto 1, transpose and proceed as normal

★ $\lambda = 0 \Rightarrow$ leads to a steady state [in the case of diff eq]

★ In the case of MM, $\lambda = 1 \Rightarrow$ Steady state ③

↓

the 1st eigen value.

In almost all cases, the $\underbrace{\text{remaining}}$ e. values < 1 ④

↓

$$\Rightarrow u_k = A^k u_0 = c_1 \lambda_1^k x_1 + \underbrace{c_2 \lambda_2^k x_2 + \cdots + c_n \lambda_n^k x_n}_{\lambda \to 0 \Rightarrow \text{these terms vanish}}$$

$\therefore \boxed{u_k = c_1 x_1} \Rightarrow$ steady state [$\because \lambda_1 = 1$]

$A - 1 \cdot I = 0$

$$\begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & 0.6 \end{bmatrix} = 0$$

has to be singular to get x, i.e. the nullspace

$\longrightarrow$ we find that the rows are dependent

$row_1 + row_2 + row_3 = 0$

$\downarrow$

$$A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

as $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq 0$

$\Rightarrow$ ~~x is in $N(A^T)$~~

NOTE: E-values for A and A^T are the same.

E-vectors aren't same, as A and A^T, and thus $N(A)$ and $N(A^T)$ are guarenteed to be the same.

$$\left.\begin{array}{c} |A - \lambda I| = 0 \\ \Updownarrow \\ |A^T - \lambda I| = 0 \end{array}\right\}$$

property 10 of det.

$\det(A) = \det(A^T)$

finding E-vectors, we get $x_1 = \begin{bmatrix} 0.6 \\ 0.33 \\ 0.7 \end{bmatrix}$ when $\underbrace{\lambda_1 = 1}$

$\downarrow$

thus, $\lambda = 1$ works.

Q. People migrate from Cal to Ma. Capture this in a Markove Model.

$u_{k+1} = A^k u_0$

$$\begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_k$$

$$\begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

Easy way to solve: find e. values and vectors.

$$\begin{bmatrix} u_{CA} \\ u_{MA} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$\underline{\underline{\lambda = 1}}$, $A - \lambda I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}$

Find x_1: $\begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

To find other λ:

$tr(A) = 0.9 + 0.8 = \cancel{0.17}\ 1.7$

$\lambda_1 + \lambda_2 = \cancel{0.17}\ 1.7$

$\lambda_2 = \cancel{0.17}\ 1.7 - 1$

$= \underline{\underline{0.7}}$

$\underline{\underline{\lambda = 0.7}}$ $A - \lambda I = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now: $u_k = c_1 (1)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \underbrace{c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{Vanish}} = \underline{\underline{c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}}} = c_1 x_1 + 0$

20.12.2022

N-dim space $q_1, q_2, \ldots, q_n$ | q refers to orthonormal

$V = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$ We were in vector space

how do we find them?

$q_1^T V = x_1 q_1 q_1^T + \underbrace{x_2 q_2 q_1^T + \cdots + x_n q_n q_1^T}_{\text{Vanishes}}$

$= x_1$

$Qx = V \Rightarrow \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\boxed{x = Q^{-1} V = Q^T V}$ $[\because Q^{-1} = Q^T]$

Fourier Series (works in function spaces)

$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$ Infinite series

★ orthonormal Basis: $\sin x, \cos x, \sin 2x, \cos 2x$

★ Periodicity: $\boxed{f(x) = f(x+2\pi)}$

inner dot product is used here

If there are two functions f and g,

$$f^T g = \int_0^{2\pi} f(x) \cdot g(x)$$

Eg: $$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} (\sin x)^2 = 0$$

If we want to find a_1:

$$\int_0^{2\pi} f(x) \cos x \, dx = a_1 \int_0^{2\pi} (\cos x)^2 \, dx = \pi$$

i.e. $a_1 = \frac{1}{\pi} \int f(x) \cos x \, dx$ ⇒ Euler's formula

Symmetric Matrices

- Eigen values are real.
- Eigen vectors are orthogonal / orthonormal (i.e. they are $\perp^r$) } vip properties

We saw that $A = S \Lambda S^{-1}$ powerful

if A is symmetric, then S (which contains e. vectors) is orthonormal

i.e. $\boxed{A = Q \Lambda Q^{-1} = Q \Lambda Q^T}$

$$= \boxed{\lambda_1 \underbrace{q_1 q_1^T}_{\text{projection matrix}} + \lambda_2 q_2 q_2^T + \cdots}$$ ⇒ Principal Axis Theorem OR Spectral Theorem

★ $(q_1 q_1^T)^2 = q_1 q_1^T$

$A = LU$
$A = QR$
$A = Q \Lambda Q^T$
major decompositions

The idea is that a signal can be decomposed into various components

22.12.2022

Positive Definite Matrices (PD)

- PD
- Test for PD
- Test for Min ($x^T A x > 0$)
- Some geometry (Ellipsoids)

★ The ~~[illegible]~~ signs (+ve, -ve) of the pivots will be the same as the signs of e.values.

★ det = prod of pivots (no row exchanges)
= prod of e.values

★ A PD is one which adheres to the following properties:

- All λ are +ve ($\lambda > 0$)
- All pivots are +ve
- All sub-det are +ve
- given: always symmetric (PD is subset of symmetric matrix)

Eg:
$$\begin{array}{c|cc} & a & b \\ 1\times1 & 5 & 2 \\ 2\times2 & 2 & 3 \\ & b & c \end{array}$$

pivots = 5, $11/5$ (a, $\frac{(ac-b^2)}{a}$)

det = 15 - 4 = 11 = $ac - b^2$

$|A - \lambda I| = 0$

$\lambda^2 - \underset{\text{trace}}{8}\lambda + \underset{\text{det}}{11} = 0 \Rightarrow \lambda = 4 \pm \sqrt{5}$

pivots = a, $\frac{ac-b^2}{a}$

det = $ac - b^2$

for $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

★ Sub determinants: for a n×n matrix, look at the det for 1×1, 2×2, ... all the way upto n×n.

Eg:
$$\begin{array}{c|ccc} 1\times1 & 5 & 2 & 1 \\ 2\times2 & 2 & 3 & 1 \\ 3\times3 & 1 & 1 & 1 \end{array}$$

$$\begin{matrix} 2 & 6 \\ 6 & ? \end{matrix}$$

if ? = 17 then det is -ve

Positive Semi-Definite Matrix

Det = 0

Eg: $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$

pivots = 2 (only 1 pivot)

rank 1 matrix

★ $x^T A x$ (an easy alternative test for PD or PSD)

when $c = 7$, det is -ve

when $c = 20$, its +ve

$$[x_1 \;\; x_2] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \;\; x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2 = f(x_1, x_2)$$

this is of the form: $ax_1^2 + 2bx_1x_2 + cx_2^2$

where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

★

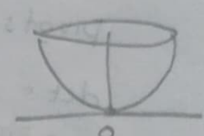
Saddle point: in some directions, it's the max point and in others it's the min point

★ $x^T A x = [x_1 \;\; x_2] \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Happy Christmas Sah!

$$= 2x_1^2 + 12x_1x_2 + 20x_2^2 = f(x_1, x_2) = 2(x_1 + 3x_2)^2 + 2x_2^2$$

which can never be 0



In some cases, we might express $f(x_1, x_2)$ as $2(x_1 + 3x_2)^2 - \underline{11}\, x_2^2$ (for eg) and it could be +ve or -ve depending on x_2. (Saddle pt)

★ $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \xrightarrow{3R_1 - R_2} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

pivots = 2, 2

multipliers = 3

multiplier

We find that: $\underbrace{2(x_1 + 3x_2)^2 + 2x_2^2}_{\text{pivots}}$

23.12.2022

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

($a = f_{xx}$, $b = f_{xy}$, $c = f_{xy}$, $d = f_{yy}$)

Calculus $\Updownarrow$ matrix

$ac - b^2$

Q. Is it PD?

$$\begin{array}{ccc} +ve & +ve & +ve \end{array}$$

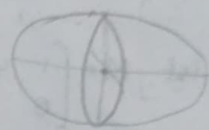
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Det $= 2, 3, 4 > 0$

pivots $= 2, \frac{3}{2}, \frac{4}{3} > 0$

$\lambda = 2-\sqrt{2}, \ 2, \ 2+\sqrt{2} > 0$

3 axis, 3 diff evalues



$f(x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0 \quad (= x^TAx)$

$= 1$

★ Is the inverse P.D?

Yes, λ values for inverse will be $\frac{1}{\lambda}$ which are still +ve.

★ A+B P.D?

$x^TAx > 0$

$x^TBx > 0$

$\Rightarrow \quad x^T(A+B)x > 0$

★ For a rectangular matrix $A_{m \times n}$, A^TA is symmetric. Is A^TA P.D? or PSD?

$(Ax)^T(Ax)$

If $x^TA^TAx > 0$, then it's PD.

$(Ax)^2 > 0$

Now: $x^TA^TAx = (Ax)^T(Ax) = ||Ax||^2 \geq 0$

not equal to 0 $\Rightarrow$ no nullspace

Thus proved or Q.E.D.

row exchanges aren't needed

Similar Matrices

If A and $B^{n\times n}$ are similar, then:

$$\boxed{B = M^{-1} A M}$$

Similarity is in terms of e. values

$\Lambda = S^{-1} A S$

M can be anything?

Eg $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\lambda = 3, 1$ det $= 4-1 = 3$, trace $= 4$

$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

Now let $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Then: $B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix}$

$= \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$ det $= 3$, trace $= 4$

$$\begin{matrix} 2-4 & 1-8 \\ 1 & 2 \\ = -2 & -7 \\ 1 & 2 \\ = -2 & -8-7 \\ 1 & \end{matrix}$$

$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$

★ For every matrix, ~~th~~ there is a family of matrices (which are similar to it in terms of λ)

★ $Ax = \lambda x$

$AIx = \lambda x$

$A M M^{-1} x = \lambda x$

$(M^{-1} A M) M^{-1} x = \lambda M^{-1} x$

$\boxed{B(M^{-1}x) = \lambda(M^{-1}x)}$ $= M^{-1}(Ax)$

Only λ remains the same; A and B are different

$\lambda_1 = \lambda_2 = 4$ (BAD case)

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ Nearest equivalent eq to the diagonal. $\Rightarrow$ Jordan Form

$$M^{-1}AM = M^{-1}4IM = 4M^{-1}M = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Jordan Blocks

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \not\approx \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

e. values: 0, 0, 0, 0

Rank = 2

2 e. vectors, 2 missing

idea is to divide the matrix into blocks

** Every square matrix A is similar to Jordan Matrix J.

$$\begin{bmatrix} \boxed{J_1} & \cdots & \cdots \\ \cdots & \boxed{J_2} & \cdots \\ \cdots & \cdots & \boxed{J_d} \end{bmatrix}$$

10. 1 2023

Probability

- Random Experiment: experiment whose outcome we cannot predict
- Event: outcome of a random experiment
- Sample space: the set of all possible outcome of random experiments
- Favourable Event: the trials which entail the happiness of an event
- Equally-Likely Event: each event has an equal chance of happening
- Mutually Exclusive Event: the occurance of 1 event prevents the occurance of another. $\boxed{P(A \cap B) = 0}$
- Independent Event: Occurance of 1 event doesn't affect the occurance of another event. $\boxed{P(A \cap B) = P(A) \cdot P(B)}$ Both can occur simultaneously. Eg: tossing 2 coins
- Dependent Event: occurance ~~&~~ or non-occurance of event A affects the occurance of event B.
- Exhaustive Event: a set of events is called exhaustive event if at least one of them

$$P(A) = \frac{\#\text{ favourable events}}{\text{total }\#\text{ events}} = \frac{n(A)}{n(S)}$$

Dice

$S = \{1, 2, 3, 4, 5, 6\}$

$n(S) = 6$

$P(A) = \frac{1}{6}$

$P(\text{getting even number}) = \frac{3}{6} = \frac{1}{2}$

Axiomatic Definition of Probability

$0 \le P(A) \le 1$

$P(S) = 1, \quad P(\phi) = 0$

If A and B are mutually exclusive, then:

$$\boxed{\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ P(A \cap B) &= 0 \end{aligned}}$$

Q. A bag contains 3 red, 6 white, 7 blue balls. What is the probability that 2 balls drawn are white and blue?

Total possible events = $16C_2$

Favourable events = # occurance of W and B = $6C_1$, $7C_1$

$$P(A) = \frac{6C_1 \times 7C_1}{16C_2} = \frac{6 \times 7}{\left(\frac{16 \times 15}{2}\right)} = \frac{7}{20}$$

Q. If you ~~had~~ twice flip a balanced coin, what is the prob of getting atleast one head?

$S = \{HH, HT, TH, TT\}$

$P(\text{getting atleast 1 head}) = \frac{3}{4}$

Q. What is the chance that a leap year selected at random has 53 Sundays?

366 days → 52 weeks and 2 days

$\downarrow$

SM, MT, TW, WT, TF, FS, SS

$$\begin{array}{r} 52 \\ 7\overline{)366} \\ \underline{35} \\ 16 \\ \underline{14} \\ 2 \end{array}$$

$n = 7$, $m = 2$

$\therefore P(A) = \frac{2}{7}$

Q. Two unbiased dice are thrown and the difference b/w the # spots turned up is noted. Find the prob. that diff b/w the numbers is 4.

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1) \ldots\ldots\}$

$m = \{(1,5), (5,1), (2,6), (6,2)\}$

$P(A) = \frac{4}{36} = \frac{1}{9}$

Q. A room has 3 electric lamps. From a collection of 10 electric bulbs, ~~which~~ where 6 are good, 3 are selected at random and put in the lamps. Find the prob that the room is lighted.

Select atleast 1 good bulb :

$= 1 - P$ (taking all 3 defective)

$= 1 - \frac{{}^4C_3}{{}^{10}C_3}$ $\left[\because \begin{array}{l} m = {}^4C_3 \\ n = {}^{10}C_3 \end{array} \right]$

$= 1 - \frac{4}{120}$

$= \frac{116}{120}$

Q. A sample space contains 3 sample points with associated prob: $2P$, P^2, $4P-1$. Find P.

$2P = 1$

$P^2 = 1$

$4P - 1 = 1$

Q. What is the chance of getting 2 sixes in 2 rollings of a single dice?

Q. A coin is tossed thrice. What is the chance of getting all heads?

Q. Find the probability of a card drawn at random?

Conditional Probability

The probability of event A provided event B has already occured is called conditional probability $P(A/B)$

$$P(A \cap B) = P(A/B) \cdot P(B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

$$P(A \cap B) = P(B/A) \cdot P(A)$$

[How likely event A has occured in subspace of B]

$$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$$

★ If A and B are independent events:

$$P(A/B) = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

$$P(B/A) = \frac{P(A) \cdot P(B)}{P(A)} = P(B)$$

$$\boxed{P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)}}$$

Bayes Theorem

★ If A and B are mutually exclusive,

$$P(A/B) = 0$$
$$P(B/A) = 0$$
$$P(A \cap B) = 0$$

★ If A and B are dependent,

$$P(A \cap B) = P(A/B) \cdot P(B)$$
$$= P(B/A) \cdot P(A)$$

[∵ multiplicative law]

12.1.2023

Q. A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is good?

Let A = event that tube 1 is good
B = event that tube 2 is good.

~~P(A) = ~~
~~P(B) = 5/9~~

$P(A) = \frac{6C_1}{10C_1}$ $P(A \cap B) = \frac{6C_2}{10C_2}$

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{6C_2}{10C_2} \times \frac{10C_1}{6C_1}$

OR $P(B/A) = \frac{5}{9}$

Q2. A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the prob that both the balls ~~are~~ drawn are black.

A: ball 1 is black
B: ball 2 is black.

$P(A) = \frac{3C_1}{8C_1}$ $P(B/A) = \frac{2C_1}{7C_1}$

$A = 3/8$ $= 2/7$

$P(A \cap B) = P(B/A) \cdot P(A)$

$= \frac{\cancel{2}}{7} \times \frac{3}{\cancel{8}4} = \frac{3}{28}$

Q. If the prob that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that it will have both is 0.18. Find the probability that a system with high fidelity will have high selectivity.

A: high fidelity
B: high selectivity

$P(B/A) = \frac{P(A \cap B)}{P(A)} =$

Q. Find the prob of drawing a queen, a king and a knave in the order from a pack of cards in 3 ~~consequ~~ consecutive draws, the cards drawn not being replaced.

A: queen in 1^{st} draw $\quad P(A) = \frac{4}{52}$

B: king in 2^{nd} draw $\quad P(B) = \frac{4}{51}$

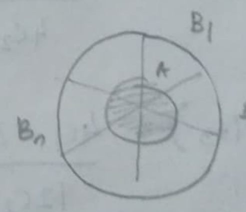
C: knave in 3^{rd} draw $\quad P(C) = \frac{4}{50}$

$$P(\text{total}) = \frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}$$

HW
~~Q. A consignment of 15 record players~~ contains 4 defectives. The record players

<u>Total Probability</u>

If $B_1, B_2, \cdots B_n$ are mutually exclusive and exhaustive set of events of a Sample space S and A is any event associated with events $B_1, B_2 \cdots B_n$, then:



$$P(A) = P(A/B_1) \cdot P(B_1) + \cdots + P(A/B_n) \cdot P(B_n)$$

$$= \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)$$

$A = (B_1 \cap A) \cup \cdots \cup (B_n \cap A)$

$$P(B_i/A) = \frac{P(A/B_i)\ P(B_i)}{\Sigma\ P(A/B_i)\ P(B_i)} \quad \text{Bayes Theorem}$$

$P(A) = P[\quad]$

$= P(B_1 \cap A) + \cdots$

$= P(A/B_1) \cdot P(B_1) + \cdots$

$= \sum_{i=1}^{n} P(A/B_i) \cdot P(B_i)$

Why? $P(A/B_i) =$

The contents of urns I, II and III are as follows:

i) 1 W, 2 B, 3 R balls

ii) 2 W, 1 B, 1 R balls.

iii) 4 W, 5 B, 3 R balls

One is chosen at random and 2 balls are drawn. They happen to be W and R. What is the probability that they come from urns I, II and III?

A: getting 1 W and 1 R

B_1: from urn I

B_2: from urn II

B_3: from urn III

To find:

$P(B_1/A) = \dfrac{P(A/B_1) \cdot P(B_1)}{\sum_i P(A/B_i) \cdot P(B_i)}$ 0.27

$P(B_2/A)$ = 0.46

$P(B_3/A)$ 0.25

$P(B_1) = P(B_2) = P(B_3) = \dfrac{1}{3}$

$P(A/B_1) = \dfrac{1C_1 \times 3C_1}{6C_2} = \dfrac{1}{5}$

$\dfrac{1 \times \cancel{3}}{\frac{\cancel{6} \times 5}{\cancel{2}}} = \dfrac{1}{5}$

$P(A/B_2) = \dfrac{2C_1 \times 1C_1}{4C_2} = \dfrac{1}{3}$

$\dfrac{\cancel{2} \times 1}{\frac{4 \times 3}{\cancel{2}}}$

$P(A/B_3) = \dfrac{4C_1 \times 3C_1}{12C_2} = \dfrac{2}{11}$

$\dfrac{\cancel{4} \times \cancel{3}}{\frac{\cancel{12} \times 11}{2}}$

$P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + P(A/B_3) \cdot P(B_3)$

$= \dfrac{1}{5} \times \dfrac{1}{3} + \dfrac{1}{3} \times \dfrac{1}{3} + \dfrac{2}{11} \times \dfrac{1}{3}$

$= \dfrac{1}{15} + \dfrac{1}{9} + \dfrac{2}{33} =$

$P(B_1/A) = \dfrac{1/5 \times \cancel{1/3}}{1/5 \times \cancel{1/3} + \frac{1}{3} \times \cancel{1/3} + \frac{2}{11} \times \cancel{1/3}} = \dfrac{1/5}{1/5 + 1/3 + 2/11} =$

13.1.2023

Q. A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colours being seen. If a 3rd ball is drawn randomly, and observed to be red, what is the probability that both of the discarded balls were blue?

A: 3rd ball is ~~blue~~ red

~~P(A) =~~

B_3: 1st two balls are blue.

B_2: red, blue

B_1: red, red

possible events

To find: $P(B_3/A) = \dfrac{P(A|B_3)\cdot P(B_3)}{P(A)}$

$P(A) \leftarrow \{ P(A|B_3)\cdot P(B_3) + P(A|B_2)\cdot P(B_2) + P(A|B_1)\cdot P(B_1)$

$P(B_1) = \frac{1}{3}$

$P(B_2) = \frac{1}{3}$

$P(B_3) = 1/3$

$P(A|B_1) = 5/18$

$P(A|B_2) = 1/3$

$P(A|B_3) = 7/18$

$\frac{5C_1}{18C_1} = \frac{5}{18}$

$\frac{6C_1}{18C_1} = \frac{6}{18}$

$\frac{7C_1}{18C_2} = \frac{7}{18}$

$$P(B_3/A) = \frac{\frac{7}{18}\times\cancel{\frac{1}{3}}}{\frac{7}{18}\times\cancel{\frac{1}{3}} + \frac{1}{3}\times\cancel{\frac{1}{3}} + \frac{5}{18}\times\cancel{\frac{1}{3}}}$$

$$= \frac{\frac{7}{18}}{\frac{7}{18}+\frac{1}{3}+\frac{5}{18}}$$

$$= \frac{7}{18}$$

$$\frac{\frac{7}{\cancel{18}}}{\frac{7+6+5}{\cancel{18}}} = \frac{7}{18}$$

Q. A factory produces its entire o/p with 3 machines. Machines I, II, III produce 50%, 30%, 20% respectively of the o/p, but 4%, 2% and 4% of their o/p are defective respectively. What fraction of the total o/p is defective?

A: o/p is defective.
B_1: Machine I o/p
B_2: Machine II o/p
B_3: Machine III o/p

$P(B_1) = \frac{50}{100}$ $\quad P(A/B_1) = 4/100$

$P(B_2) = \frac{30}{100}$ $\quad P(A/B_2) = 2/100$

$P(B_3) = \frac{20}{100}$ $\quad P(A/B_3) = 4/100$

To find: $P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + P(A/B_3) \cdot P(B_3)$

$$= \frac{4}{100} \cdot \frac{50}{100} + \frac{2}{100} \cdot \frac{30}{100} + \frac{4}{100} \cdot \frac{20}{100}$$

$$= \frac{200 + 60 + 80}{10000}$$

$$= \frac{340}{10000} = 0.034$$

200
140
340

Q. A box contains 5 R and 4 W balls. Two balls are drawn successively from the box without replacement and it is noted that the 2nd one is white. What is the probability that the 1st one is also white?

A: 2nd ball is white.
B_1: 1st is red
B_2: 1st is white

$P(B_1) = \frac{5C_1}{9C_1} = 5/9$ $\quad P(A/B_1) = \frac{4C_1}{8C_1} = \frac{4}{8}$

$P(B_2) = \frac{4C_1}{9C_1} = 4/9$ $\quad P(A/B_2) = \frac{3C_1}{8C_1} = \frac{3}{8}$

$$P(B_2/A) = \frac{P(A/B_2) \cdot P(B_2)}{P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2)}$$