Objectives

Spectral Graph Theory

- Adjacency Matrix Spectra
- D-regular Graph

Spectral Graph Theory

- Analyze the spectrum of matrix representing G

Spectrum

- Spectrum: Eigenvectors of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues:

$$\Lambda = \{\lambda_1, \lambda_2, \ldots \lambda_n\}$$

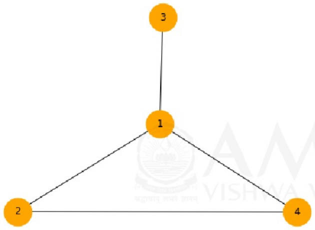
eigen values

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

  - Relate properties of graphs with the eigenvalues and eigenvectors (spectral properties)
  - How eigenvalues and vectors related to graph structure
  - Components in the graph
  - How denser the graph is
  - Denser parts in the graph

GRAPH REPRESENATIONS

ADJACENCY MATRIX



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The adjacency matrix captures the entire network structure and the matrix properties tell us about various useful things about the network.

- Matrix of size $|V| \times |V|$
- $A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$

$$\begin{pmatrix} a_{11} & \dots & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ a_{31} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

ADJACENCY MATRIX SPECTRA

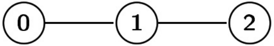
- Let λ is eigenvalue and υ is eigenvector

$$A\upsilon = \lambda\upsilon$$

- $\upsilon \in \mathbb{R}^V$ or $\upsilon : V \to \mathbb{R}$
- Symmetric - n real eigen values and n real eigen vectors

Example

```python
import networkx as nx
G= nx.path_graph(3)
```



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix Spectra

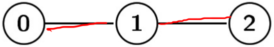
- Let λ is eigenvalue and υ is eigenvector

$$A\upsilon = \lambda\upsilon$$

- $\upsilon \in \mathbb{R}^V$ or $\upsilon : V \to \mathbb{R}$
- Symmetric - n real eigen values and n real eigen vectors

Example

```
import networkx as nx
G= nx.path_graph(3)
```



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\lambda_1 \leq \lambda_2 \leq \lambda_3$

λ_1= -1.414, λ_2= 0, λ_3= 1.414

$$\upsilon_1 = \begin{pmatrix} 0.5 \\ -0.707 \\ 0.5 \end{pmatrix}, \upsilon_2 = \begin{pmatrix} 0.707 \\ 0 \\ -0.707 \end{pmatrix}, \upsilon_3 = \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix}$$

n0
n1
n2

Adjacency Matrix Spectra

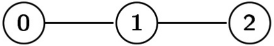
- Let λ is eigenvalue and υ is eigenvector

$$A\upsilon = \lambda \upsilon$$

- $\upsilon \in \mathbb{R}^V$ or $\upsilon : V \to \mathbb{R}$
- Symmetric - n real eigen values and n real eigen vectors

Example

```python
import networkx as nx
G= nx.path_graph(3)
```



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

λ_1= -1.414, λ_2= 0, λ_3= 1.414

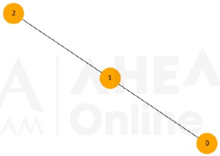
$$\upsilon_1 = \begin{pmatrix} 0.5 \\ -0.707 \\ 0.5 \end{pmatrix}, \upsilon_2 = \begin{pmatrix} 0.707 \\ 0 \\ -0.707 \end{pmatrix}, \upsilon_3 = \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix}$$

- Eigenvectors forms orthonormal basis for the space

Computing using Python

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
import numpy as np
from numpy import linalg as LA
G7=nx.path_graph(3)
S= nx.to_numpy_matrix(G7)
print(S)
evals, evecs = LA.eig(S)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
nx.draw(G7, node_size = 1500,
        node_color="orange",
        with_labels = True)
plt.show()
```

```
   [[0.  1.  0.]
A=  [1.  0.  1.]
    [0.  1.  0.]]
      lambda_1         lambda_2         lambda_3
[-1.41421356e+00  9.77950360e-17  1.41421356e+00]
         v1               v2               v3
[[ 5.00000000e-01  7.07106781e-01  5.00000000e-01]
 [-7.07106781e-01  9.06753788e-17  7.07106781e-01]
 [ 5.00000000e-01 -7.07106781e-01  5.00000000e-01]]
```



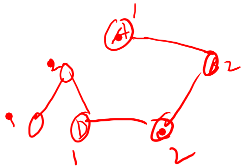
- Eigenvector centrality: eigenvector corresponding to the largest eigen value
  - principal eigenvector

```
centrality = nx.eigenvector_centrality(G7)
print(['%s %0.2f'%(node,centrality[node]) for node in centrality])
```

```
['0 0.50', '1 0.71', '2 0.50']
```

More on Adjacency Matrix Spectra

$$A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$



- Consider famous $\boxed{Ax = b}$ equation,
  - What is Ax here,
  - think x as a set of labels or values assigned to nodes
- Note Ax is a n-dimensional column vector

$$\begin{array}{ccc} A & x & \\ \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ a_{31} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} & = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix} \end{array}$$

y_3

$Av_3 = \lambda_3 v_3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix} = 1.414 \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 1.0 \\ 0.707 \end{pmatrix}$$

- What is y_i
  - Say $y_2 = \overset{0}{a_{21}}.x_1 + \overset{0}{a_{22}}.x_2 + \dots \overset{1}{a_{23}} \dots a_{2n}.x_n$
- $y_i = \sum_{j:(i,j) \in E} x_j$
  - Ax is a vector whose i_{th} coordinate contains the sum of the x_j who are in-neighbors of i

$$y_3 = \sum_{(3,j)} x_j$$

D-REGULAR GRAPH

K-regular

Each node has d nbrs.

- A d-regular graph G has all nodes with $\overset{2}{d}$ degree

SPECTRA OF D-REGULAR GRAPH

- Consider a connected d-regular graph

$$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

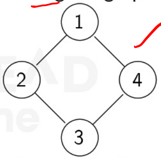
- Since all nodes having equal degree of d, summation of a row: $\sum_{j=1}^{n} a_{kj} = d$

- $$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$$

- Hence $\lambda = d$, and corresponding eigenvector $\upsilon = (1, 1, \ldots 1)^T$

$A \cdot \begin{bmatrix} \vdots \end{bmatrix} = 2 \begin{bmatrix} \vdots \end{bmatrix}$

- 2-regular graph



$$\underset{A}{\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}} \overset{x}{\underset{x}{\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}}} = \underset{y}{\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}}$$

no. of n_1 nbrs

n_2 nbrs

n_3 nbrs

n_4 nbrs

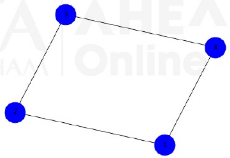
$A x = \overset{d}{\lambda} x \rightarrow \begin{bmatrix} \vdots \end{bmatrix}$

Computing using Python

- 2-regular, 4 node graph

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
import numpy as np
from numpy import linalg as LA
G8=nx.Graph([(1,2),(2,3),(3,4),(4,1)])
S= nx.to_numpy_matrix(G8)
print(S)
evals, evecs = LA.eig(S)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
nx.draw(G8, node_size = 1500,
        node_color="orange",
        with_labels = True)
plt.show()
```

```
 [[0.  1.  0.  1.]
 [1.  0.  1.  0.]
 [0.  1.  0.  1.]
 [1.  0.  1.  0.]]

 [ 2.     -2.      0.        0.]
   v1     v2      v3        v4
[[-0.5   -0.5     0.        0.    ]
 [-0.5    0.5    -0.707     0.707]
 [-0.5   -0.5     0.        0.    ]
 [-0.5    0.5     0.707    -0.707]]
```



```
centrality = nx.eigenvector_centrality(G8)
print(['%s %0.2f'%(node,centrality[node]) for node in centrality])
```

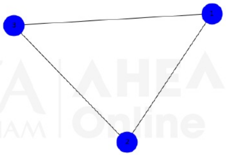
```
['1 0.50', '2 0.50', '3 0.50', '4 0.50']
```

Computing using Python

- 2-regular, 3 node graph

```python
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
import numpy as np
from numpy import linalg as LA
G8=nx.Graph([(1,2),(2,3),(3,1)])
S= nx.to_numpy_matrix(G9)
print(S)
evals, evecs = LA.eig(S)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
nx.draw(G9, node_size = 1500,
        node_color="orange",
        with_labels = True)
plt.show()
```

```
[[0. 1. 1.]
 [1. 0. 1.]
 [1. 1. 0.]]
 [ -1.    2.    -1.]
[[-0.816  0.577  0.192]
 [ 0.408  0.577 -0.783 ]
 [ 0.408  0.577  0.591]]
```



```python
centrality = nx.eigenvector_centrality(G9)
print(['%s %0.2f'%(node,centrality[node]) for node in centrality])
```

```
['1 0.58', '2 0.58', '3 0.58']
```

Objectives

Spectral Graph Theory

- Disconnected D-regular Graph
- Laplacian Matrix

NOT-CONNECTED D-REGULAR GRAPH

- A d-regular disconnected G has all nodes with d degree

SPECTRA OF D-REGULAR DISCONNECTED GRAPH

- Consider a a not-connected d-regular graph

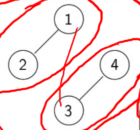
$$\begin{pmatrix} a_{11} & a_{12} & . & 0 & 0 \\ a_{21} & a_{22} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & . & a_{n(n-1)} & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & . & 0 & 0 \\ a_{21} & a_{22} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & . & a_{n(n-1)} & a_{nn} \end{pmatrix} \begin{pmatrix} d \\ d \\ . \\ 0 \\ 0 \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ . \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} a_{11} & a_{12} & . & 0 & 0 \\ a_{21} & a_{22} & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & . & a_{n(n-1)} & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ . \\ d \\ d \end{pmatrix} = d \begin{pmatrix} 0 \\ 0 \\ . \\ 1 \\ 1 \end{pmatrix}$$

- Not-connected 1-regular graph



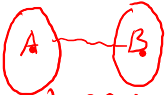
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- $\boxed{\lambda_1 = \lambda_2 = \text{-}1,}\ \boxed{\lambda_3 = \lambda_4 = 1}$

$$v_1 = \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 0.707 \\ 0.707 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -0.707 \\ 0.707 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ -0.707 \\ 0.707 \end{pmatrix}$$

d



0 $\lambda_3 \approx \lambda_4$ 0.01

2 eigen values equal

$\lambda_3 = 1$

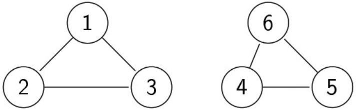
λ_4

2 disconnected Components.

Not-Connected D-regular Graph

Matrix has a block structure

- $A = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$.
- $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ d \\ \cdot \\ \cdot \\ d \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

6 vertices
so 6 eigen values

- Eigenvalues

$-1,\ 2,\ -1,\ -1,\ 2,\ -1$

Adj. matrix is
→ symmetric
→ Eigen values are real
Corr. eigenvectors will be orthogonal $\lambda_i = \lambda_j$

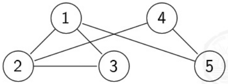
Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



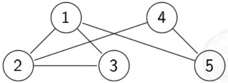
Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

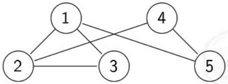
Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, & \text{if } i == j \\ -1, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad [d_{ii}] = D$$

$n \times n$ $\qquad$ $n \times n$

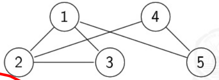
Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, & \text{if } i == j \\ -1, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$D - A$$

$$L = \underbrace{\begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}}_{0} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = b$

$Ax = \lambda x$

$x = [1 \ldots\ldots 1]$

$L \cdot x = [0 \ldots 0].$

$L \cdot x = 0 \cdot x$

$\lambda = 0 \rightarrow$ Trivial eigen pair.

$L = L^T.$

$L \cdot x = 0 \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

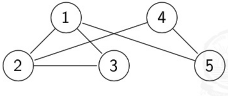
Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- L is symmetric, observe that the row sum, $\sum_j L_{ij} = 0$.
- Since row sum is zero, $L1 = 0$ where $1 = (1,1,1...1)^T$
- or $LC = 0$, where C is a constant vector $(c,c,c...c)^T$
- Hence $LC = 0\ C$, one of the eigenvalue $\lambda = 0$

$$\begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= 0$$

OBJECTIVES

SPECTRAL GRAPH THEORY

- Laplacian Spectra
- Components in a Graph
- Properties of Laplacian Spectra

Revisiting Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i

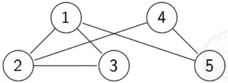
Revisiting Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



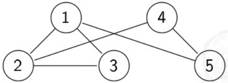
Revisiting Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

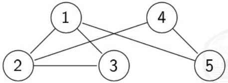
Revisiting Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

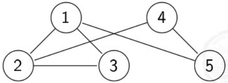
REVISITING GRAPH LAPLACIAN

DEFINITION

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, \text{ if } i == j \\ -1, \text{ if } i \neq j \text{ and } (i,j) \in E \\ 0, \text{ otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

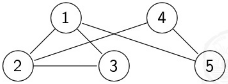
Revisiting Graph Laplacian

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$A_{ij} = \begin{cases} k_i, & \text{if } i == j \\ -1, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

where k_i denotes the degree of node i



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

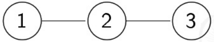
- L is symmetric, observe that the row sum, $\sum_j L_{ij} = 0$.
- Since row sum is zero, $L1$ =0 where 1= $(1,1,1...1)^T$
- or LC =0, where C is a constant vector $(c,c,c...c)^T$
- Hence $LC = 0\ C$, one of the eigenvalue $\lambda = 0$

$$\begin{pmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Laplacian Spectra

Properties

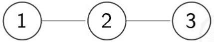
- Eigenvalues are real and non negative
- Eigenvectors are real and orthogonal
- First Eigenvalue $\lambda_1 = 0$ and corresponding eigenvector is a constant vector (same value in all direction)
    - Trivial eigenvalue and eigenvector



Laplacian Spectra

Properties

- Eigenvalues are real and non negative
- Eigenvectors are real and orthogonal
- First Eigenvalue $\lambda_1 = 0$ and corresponding eigenvector is a constant vector (same value in all direction)
    - Trivial eigenvalue and eigenvector

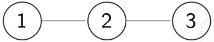


$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Laplacian Spectra

Properties

- Eigenvalues are real and non negative
- Eigenvectors are real and orthogonal
- First Eigenvalue $\lambda_1 = 0$ and corresponding eigenvector is a constant vector (same value in all direction)
    - Trivial eigenvalue and eigenvector



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

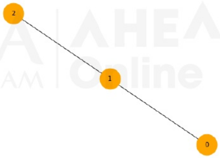
- $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -0.707 \\ 0 \\ 0.707 \end{pmatrix}, v_3 = \begin{pmatrix} -0.408 \\ 0.816 \\ -0.408 \end{pmatrix}$$

Graph Laplacian using Python

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
from numpy import linalg as LA
GL1=nx.Graph([(1,2),(2,3)])
S=nx.laplacian_matrix(GL1)
T=S.todense()
print(T)
nx.draw(GL1, node_size = 1500, node_color="green", with_labels = True)
plt.show()
evals, evecs = LA.eig(T)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
```

```
    [[ 1 -1  0]
L=   [-1  2 -1]
     [ 0 -1  1]]

lambda_1          lambda_2          lambda_3
[ 3.00000000e+00  1.00000000e+00  -3.36770206e-17]

         v1               v2               v3
[[-4.08248290e-01  -7.07106781e-01  5.77350269e-01]
 [ 8.16496581e-01   2.61239546e-16  5.77350269e-01]
 [-4.08248290e-01   7.07106781e-01  5.77350269e-01]]
```

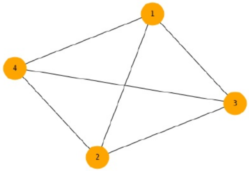


DIRECTLY FROM THE GRAPH

```
nx.laplacian_spectrum(GL2)
```

```
array([1.11022302e-16, 1.00000000e+00, 3.00000000e+00, 4.00000000e+00])
```

GRAPH LAPLACIAN: COMPLETE GRAPH




```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
from numpy import linalg as LA
GL3=nx.Graph([(1,2),(1,3),(1,4),(2,3),(2,3),(2,4),(3
S=nx.laplacian_matrix(GL3)
T=S.todense()
print(T)
nx.draw(GL3, node_size = 1500, node_color="green", wit
plt.show()
evals, evecs = LA.eig(T)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
```

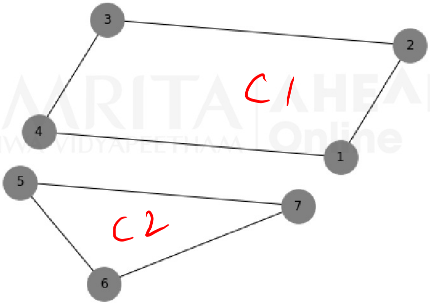
```
   [[ 3 -1 -1 -1]
L=  [-1  3 -1 -1]
    [-1 -1  3 -1]
    [-1 -1 -1  3]]
[4.00000000e+00 -1.11022302e-16  4.00000000e+00
4.00000000e+00]

[[ 0.8660254  -0.5 -0.18959995  0.07751764]
 [-0.28867513 -0.5  0.85988861  0.1713954 ]
 [-0.28867513 -0.5 -0.33514433 -0.80769725]
 [-0.28867513 -0.5 -0.33514433  0.55878421]]
```

Components in a Graph

- If there are two connected components, the similar argument as for the adjacency matrix applies, and $\lambda_1 = \lambda_2 = 0$
- In general, the multiplicity of eigenvalue 0 is the number of connected components

2.5^- 2.5^-
$\lambda_1 = \lambda_2$

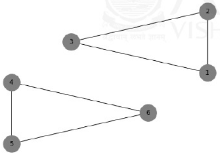
Eigenvalues λ_i



[0 0 2 2 3 3 4]

Two Components Graph

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
from numpy import linalg as LA
GL5=nx.Graph([(1,2),(2,3),(3,1),(4,5),(5,6),(6,4)])
nx.draw_kamada_kawai(GL5, node_size = 1000, node_color="g
plt.show()
S= nx.laplacian_matrix(GL5)
T=S.todense()
print(T)
evals, evecs = LA.eig(T)
evals=evals.real
print(evals)
evecs=evecs.real
print(evecs)
```




```
[[ 2 -1 -1  0  0  0]
 [-1  2 -1  0  0  0]
 [-1 -1  2  0  0  0]
 [ 0  0  0  2 -1 -1]
 [ 0  0  0 -1  2 -1]
 [ 0  0  0 -1 -1  2]]

[ 3      0      3      3      0      3     ]

[[0.816 -0.577  0.309  0.     0.     0.   ]
 [-0.408 -0.577 -0.809  0.     0.     0.   ]
 [-0.408 -0.577  0.499  0.     0.     0.   ]
 [ 0.     0.     0.     0.816 -0.577  0.309]
 [ 0.     0.     0.    -0.408 -0.577 -0.809]
 [ 0.     0.     0.    -0.408 -0.577  0.499]]
```

Properties of the Laplacian

Simple Properties

Edge Union If G and H are two graphs on the same vertex set with disjoint edge sets, $L_{G\cup H} = L_G + L_H$

Isolaited Vertices If a vertex $i \in G$ is isolated, then the corresponding row and column of the Laplacian are zero, i.e. $[LG]_{i,j} = [LG]_{j,i} = 0$ for all j.

Disjoint Union These together imply that the Laplacian of the disjoint union of G and H is direct sum of L_G and L_H , i.e.: $L_{G\cup H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$

Theorem (Disjoint Union Spectrum)

If L_G has eigenvectors $v_1, v_2, \ldots, v_n$ with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and L_H has eigenvectors $\omega_1, \omega_2, \ldots, \omega_n$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ then $L_{G\cup H}$ has eigenvectors

$$v_1 \oplus 0, v_2 \oplus 0, \ldots, v_n \oplus 0, 0 \oplus \omega_1, 0 \oplus \omega_2, \ldots, 0 \oplus \omega_n$$

with eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n$$

Proof: $L_{G\cup H} * (v_1 \oplus 0) = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ 0 \end{pmatrix}$

Thus $v_1 \oplus 0$ is an eigenvector of $L_{G\cup H}$ with eigenvalue λ_1 . The rest follow by symmetry. ■

$\omega_1 \cap \omega_2 = \{0, 0, 0\}$

$\omega_1 \oplus \omega_2$

L_G has v_1 $L_{G\cup H} \rightarrow v_1$ & λ_1

Direct Sum.

W_1, W_2 $[1, 2, 3], [5, 3, 15] \dots$

$\rightarrow W_1 : (x, y, 3x)$ $[1, 2, 5], [5, 10, 3] \dots [1, 2, 3]$

$\rightarrow W_2 : (x, 2x, z)$

$W_1 \cap W_2 \neq \{0\}$!!

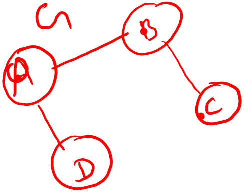
Not Direct Sum

$W_1 \cap W_2 = \{0\} \rightarrow$ Direct Sum

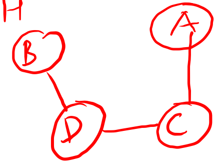
$W_1 \oplus W_2$

(A, B, C, D)

G



H



Edge union : $L_{\underline{G} \cup \underline{H}} = \underline{L_G} + \underline{L_H}$

Edges are disjoint

Edges disjoint. $\{AB, AD, BC\}$

$\{BD, AC, CD\}$

$L_G = D_G - A_G$

$$L_H = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$L_G = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$$