CLIQUES

Refer Chapter 7

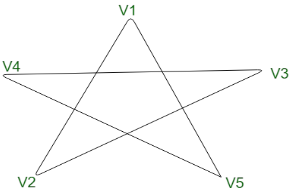
$G : (V, E)$

$G(S) : V', E'$

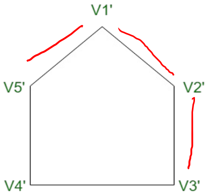
CLIQUE

- A clique in a (an undirected) graph G is a subgraph of G isomorphic to a complete graph
- A Clique is a subset of vertices such that each vertex is connected to every other one. Vertices and edges in a clique makes that subgraph a complete graph.

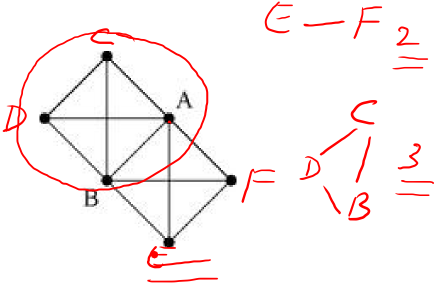
Isomorphism



G



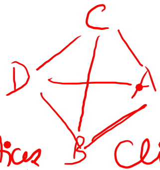
G'



NP - Complete

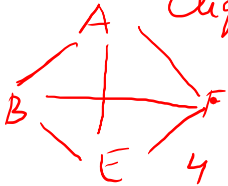
A graph w' n vertices

2^n subsets possible



Clique

K_4



Clique

CLIQUES

CLIQUES ..

- Cliques of order 1 are just single vertices.
- Cliques of order 2 are just pairs of vertices connected by an edge, so for simple undirected graphs the number of cliques of order 2 is just the number of edges.

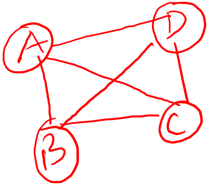
CLIQUES ..

- Cliques of order 3 are triangles in the graph

Example: K_4

- This has one clique of order 4 and 4 of order 3, 6 of order 2 and 4 of order 1

- The clique number of G is the maximum order of a clique in G.
  - For K_n , it is n.
  - For C_n it is 2 unless n = 3 (for $C_3 = 3$).



$\underline{K_4}$ → Complete graph w/ 4 vertices

Cliques of order 1 → A D C B

A subgraph w/ 1 vertex

How many cliques of order 1 = 4

$\underline{G}$

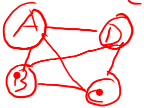
G is subgraph of G

Cliques of order $\underline{2}$ → A — D

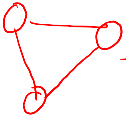
= 6 cliques

• Cliques of order 3 → Triangles = 4 cliques

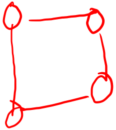
Cliques of order 4 →



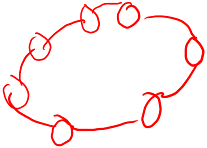
C_n – Circular graph



$\rightarrow 3$ Maximal clique order



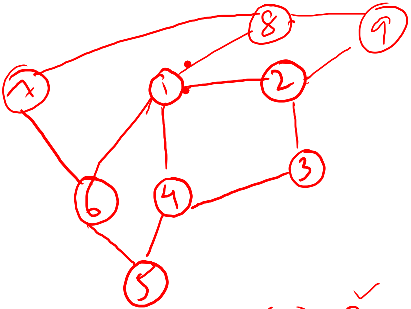
$\rightarrow 2$



$\rightarrow 2$

$C_n = 2$; if $n = 3$, 3

$n = 9$



Algorithm to find clique

1. Find a vertex v of the smallest possible degree in G.
2. If the degree of v is $n-1$, stop; G is a clique, so the largest clique in G has size n.
3. Otherwise, remove v and all of its edges from G. Find the largest clique in the smaller graph. Report that as the largest clique in G.

$Deg(1) = 4$
$(2) = 3$
$(3) = 2$ ✓
$(4) = 2$ ✓

$Deg(5) = 2$ ✓
$(6) = 3$
$(7) = 2$
$(8) = 3 \to 2$

$Deg(9) = 2$ ✓ — X

n = 8

n = 7

n = 6

n = 5

n = 4

n : 3

n = 2

Clique of order 2

Deg(1): 4 − 3 / 2 − 1

(2) = 3 − 2 − 1 X

(3) : 2 ✓ − X

(4) = 3 − 2

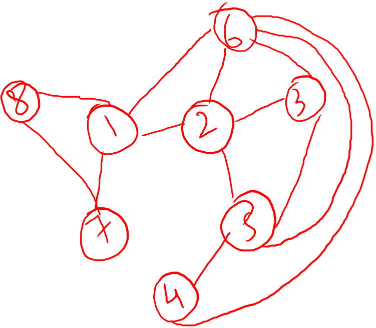
Deg(5): 2 − 1

(6) = 3 → 2 − X

(7) = 2 − X

(8) = 3 → 2 → 1 X

~~Deg(9): 2~~ − X



Deg(1) = 4 → 3 → 2 — X

Deg(2) = 4 → 3 ✓

Deg(3) = 3 ✓

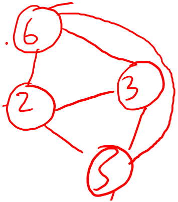
(4) = 2 → X

(5) = 4 → 3 ✓

(6) = 5 → 4 → 3 ✓

(7) = 2 → 1 → X

(8) = 2 → X



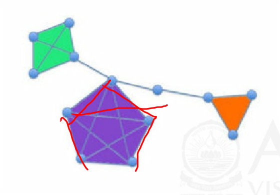
Clique of order 4

Find maximal clique.

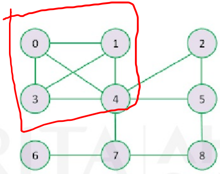
(1,2), (2,3), (3,1), (4,3), (4,1), (4,2) ✓

(1,2), (2,3), (3,1), (4,3), (4,5), (5,3) ✓

Cliques ..

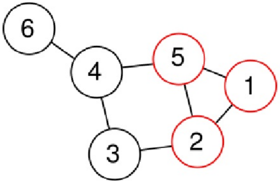


- Cliques → 2, 3, 4 and ~~6~~ 5
- Clique Number → ~~6~~ 5



- Cliques → 2, 3 and 4
- Clique Number → 4

Maximal order



- Cliques → 2 and 3
- Clique Number → 3

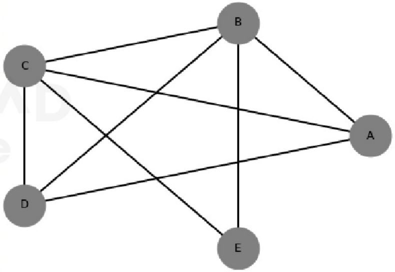
Finding Cliques ..

"Cliques" in NetworkX

- `find_cliques(G[, nodes])`: Returns all maximal cliques in an undirected graph
- `graph_clique_number(G[, cliques])`: Returns the clique number of the graph
- `cliques_containing_node(G[, nodes, cliques])`: Returns a list of cliques containing the given node
- `node_clique_number(G[, nodes, cliques, ...])`: Returns the size of the largest maximal clique containing each given node

CLIQUES USING NETWORKX: EXAMPLES

```
list(nx.find_cliques(G))
[['B', 'C', 'E'], ['B', 'C', 'A'],
 ['B', 'D', 'E']]
nx.graph_clique_number(G)
4
nx.cliques_containing_node(G)
{'A': [['C', 'B', 'A', 'D']],
 'B': [['C', 'B', 'E'],
       ['C', 'B', 'A', 'D']],
 'C': [['C', 'B', 'E'],
       ['C', 'B', 'A', 'D']],
 'D': [['C', 'B', 'A', 'D']],
 'E': [['C', 'B', 'E']]}
 nx.node_clique_number(G)
 {'A': 4, 'B': 4, 'C': 4, 'D': 4, 'E': 3}
```



K-plex

A k-plex of size n is a maximal subset of n vertices within a
network such that each vertex is connected to at least
n - k of the others.

If k = 1, -> ordinary clique

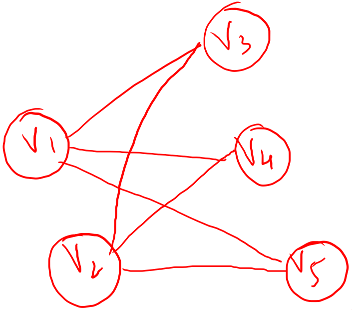
n-k = n-1 of the vertices

n-3

Useful concept for discovering groups within networks: in real life many groups
in social and other networks form k-plexes. There is no solid rule about what value k should take.

K-core

atleast k vertices

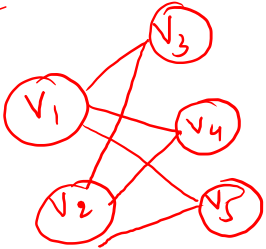


1-plex

X

k-plex ?

k = ?



3-plex.

5 vertices

3 →

$n - k$

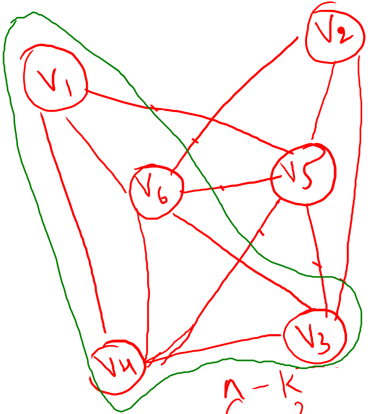
$k = 2$

3 vertices.

k-plex

$n - k = 2$

3-plex



n - K
6 - 3

k-plex?

Min deg(G) = ?

deg(v_1) = 3
(v_2) = 3
(v_3) = 4
(v_4) = 4
v_5 = 5
v_6 = 5

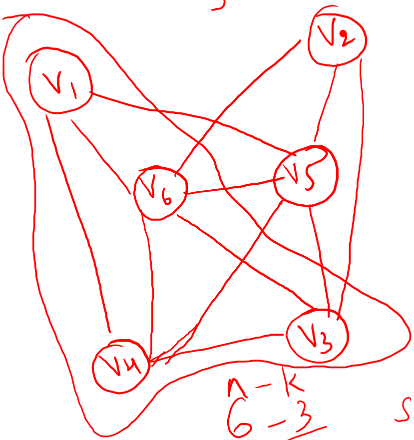
G'

→ 2-plex.

n = 5

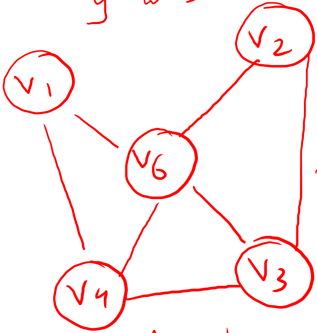
For G' to be 2-plex
deg(v') = n - 2 = 3

G



$n-k$
$6-3$

G' w/ 5 vertices.



X
2-plex

Is it 2-plex?

$n = 5$. Each vertex should be $n-K \Rightarrow n-2$ vertices connected to $\rightarrow$ 3 vertices

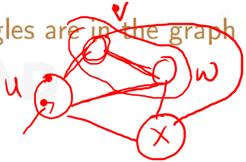
Objectives

Clustering in Networks

- Clustering Coefficient
- Transitivity

CLUSTERING (1/3)

- Cliques are unstable under minor changes, and hence this measure is not considered as robust.
  - *cohesiveness*: tightly-knit-groups
  - identifying cohesive subgraphs: Clustering
    - ∗ Visually this is the notion of how many triangles are in the graph and how much they bunch together.



DEFINITION (CLUSTERING COEFFICIENT)

Let $G = (V, E)$ be a simple undirected graph. The (individual) clustering coefficient of a node $u \in V$ is the probability $C\ell(u)$ that two neighbors of u are themselves neighbors

Clustering Coefficient

$$C\ell(u) = \frac{|\{(v,w) : v,w \in V; v \neq w; (u,v),(u,w),(v,w) \in E\}|}{|\{(v,w) : v,w \in V; v \neq w; (u,v),(u,w) \in E\}|}$$

Clustering (2/3)

Clustering Coefficient ..

$$C\ell(u) = \frac{|\{(v,w) : v,w \in V; v \neq w; (u,v),(u,w),(v,w) \in E\}|}{deg(u)\,(deg(u)-1)}$$

Clustering Coefficient: Simplified Form

$$C\ell(i) = \frac{2\,L_i}{k_i\,(k_i-1)}$$

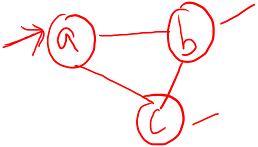
where, L_i is the number of edges between k_i neighbors, and k_i is the degree of node i.

Example

Friendship networks: measures how many of your friends are friends with each other

- The **average clustering coefficient** is the average of $Cl(u)$ over all $u \in V$

Clustering (3/3)



Overall Clustering Coefficient: Transitivity

$$C\ell(G) = \frac{\sum_{u \in V} |\{(v,w) : v,w \in V; v \neq w; (u,v),(u,w),(v,w) \in E\}|}{\sum_{u \in V} |\{(v,w) : v,w \in V; v \neq w; (u,v),(u,w) \in E\}|}$$

Transitivity Simplified

$$C\ell(G) = \frac{\textit{Number of Triangles X 3}}{\textit{Number of Connected Triplets}}$$

CLUSTERING COEFFICIENT: EXAMPLES (1/2)

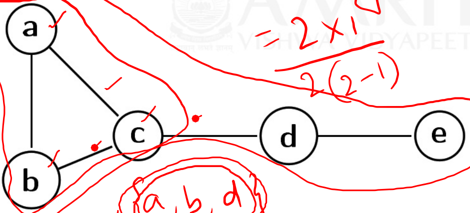
$$C\ell(i) = \frac{2\,L_i}{k_i\,(k_i-1)}$$

deg(i)

$$C\ell(G) = \frac{\textit{Number of Triangles X 3}}{\textit{Number of Connected Triplets}}$$

Cl(a) = 2 + no. of edges bet^n nbrs of a

$= \frac{2 \times 1}{2(2-1)}$



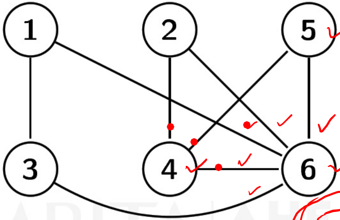
{a, b, d}

a-b

$C\ell(i)$ & $C\ell(G)$

- $C\ell(a) = \frac{2\times 1}{2(2-1)} = 1$
- $C\ell(b) = \frac{2\times 1}{2(2-1)} = 1$
- $C\ell(c) = \frac{2\times 1}{3(3-1)} = 1/3$
- $C\ell(d) = \frac{2\times 0}{1(1-1)} = 0$
- $C\ell(e) = \frac{2\times 0}{2(2-1)} = 0$
- $C\ell(G) = \frac{1\times 3}{6} = 1/2$

Clustering Coefficient: Examples (2/2)



nbrs C_2

$cl(6) = \frac{2L_i}{k_i(k_i-1)} = \frac{2*3}{5*4}$

5, 2, 1, 4, 3

(1,3), (5,4), (2,4),

$C\ell(G)$

- No. of Triangles: 3, No. of Triplets: 17
  - Node 1: $\binom{2}{2} = 1$
  - Node 2: $\binom{2}{2} = 1$
  - Node 3: $\binom{2}{2} = 1$
  - Node 4: $\binom{3}{2} = 3$
  - Node 5: $\binom{2}{2} = 1$
  - Node 6: $\binom{5}{2} = 10$
- $C\ell(G) = \frac{3\times 3}{17} = 9/17$

OBJECTIVES

Clustering using NetworkX

- Clustering Coefficient
- Transitivity
- Triangles

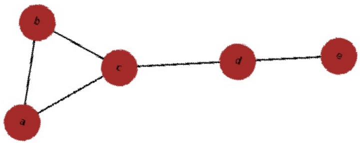
Functions for Clustering

In NetworkX

- clustering(G[, nodes): Compute the clustering coefficient for nodes
- average_clustering(G[, nodes]): Compute the average clustering coefficient for the graph G
- transitivity(G): Compute graph transitivity, the fraction of all possible triangles present in G.
- triangles(G[, nodes]): Compute the number of triangles.

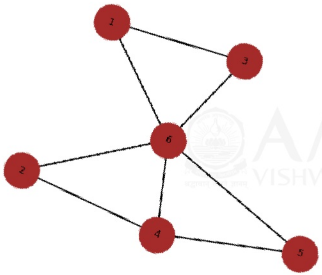
EXAMPLE 1

```python
import networkx as nx
import matplotlib . pyplot as plt
edge_widths = [2]
G2 = nx.Graph([('a', 'b'),('a', 'c'),('b', 'c'), ('c', 'd'), ('d', 'e')
nx . draw (G2 , with_labels = True, node_size=2000, node_color="brown",
width=edge_widths)
```




```python
print(nx.clustering(G2))
{'a': 1.0, 'b': 1.0,
 'c': 0.33, 'd': 0, 'e': 0}
nx.transitivity(G2)
0.5
```

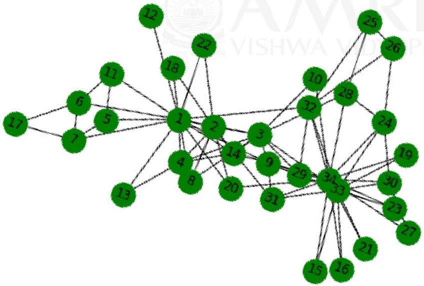
EXAMPLE 2




```
nx.transitivity(G2)
0.5294
print(nx.clustering(G2))
{1: 1.0, 3: 1.0, 6: 0.3,
 2: 1.0, 4: 0.66, 5: 1.0}
nx.average_clustering(G2)
0.8277
nx.triangles(G2)
{1: 1, 3: 1, 6: 3,
 2: 1, 4: 2, 5: 1}
```

Example 3

```
G2 = nx.karate_club_graph()
G2 = nx.convert_node_labels_to_integers(G,first_label=1)
```



```
nx.transitivity(G2)
0.2556818181818182
print(nx.clustering(G2))
{1: 0.15, 2: 0.33, 3: 0.24,..,
17: 1.0,.. 33: 0.196, 34: 0.110}
nx.average_clustering(G2)
0.5706384782076823
nx.triangles(G2,1)
18
```

Summary

NetworkX

- Clustering Coefficient
- Transitivity
- Triangles

References

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