Basic concepts in probability

Terminologies

- Random Experiment: Experiment whose outcome cannot predict.
- Event: outcome of a random experiment.
- Sample space: The set of all possible outcomes of a random experiment.
- Favourable event: The trials which entail the happening of an event.
- Equally likely event: Each of the events has an equal chance of happening.
- Mutually exclusive events: The occurrence of one event prevents the occurrence of another; $P(A \cap B)=0$

Terminologies

- Independent events: The occurrence of one event does not affect the occurrence of another. Both can occur simultaneously; $P(A \cap B) = P(A)P(B)$

- Dependent events: If the occurrence or non-occurrence of event A affects the occurrence of event B

- Exhaustive events: A set of events are called exhaustive events if at least one of them necessarily occurs whenever the experiment is performed.

Definition of probability

- If a trial results in **n** exhaustive mutually exclusive and equally likely events and **m** of them are favourable to the happening of an event A,

Probability of A p(A)= $\frac{Number\ of\ favourable\ cases}{Exhaustive\ number\ of\ cases} = \frac{n(A)}{n(S)} = \frac{m}{n}$

- E.g. Throwing a die

Possible cases are S= { 1,2,3,4,5,6}; n(S)= 6

Event A → Event of getting number 5

n(A) = 1

p(A=5) = $\frac{1}{6}$

p(A) = 1 , A is a certain event

P(A)=0, A is an impossible event

Axiomatic definition of probability

- $0 \leq p(A) \leq 1$
- $p(S) = 1, p(\emptyset) = 0$
- *If A and B are mutually exclusive events*

$$p(A \cup B) = p(A) + p(B)$$
$$p(A \cap B) = 0$$

If $A_1, A_2, \ldots A_n$ are mutually exclusive events

$$p(A_1 \cup A_2 \cup \ \ldots \cup A_n) = \sum_{n=1}^{\infty} p(A_n)$$

- *If A and B are independent events*

$$p(A \cap B) = p(A)p(B)$$

Probability - Questions

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that 2 balls drawn are white and blue?
2. If you twice flip a balanced coin, what is the probability of getting at least one head?
3. What is the chance that a leap year selected at random will contain 53 sundays?
4. Two unbiased dice are thrown and the difference between the number of spots turned up is noted. Find the probability that the difference between the number is 4.
5. A room has 3 electric lamps. From a collection of 10 electric bulbs of which 6 are good, 3 are selected at random and put in the lamps. Find the probability that the room is lighted.

Probability - Questions

6. A sample space contains three sample points with associated probabilities given by $2p$, p^2, and $4p-1$. Find value of p.
7. What is the chance of getting two sixes in two rollings of a single die?
8. A coin is tossed thrice. What is the chance of getting all heads.
9. Find the probability of a card drawn at random from an ordinary pack is a diamond.
10. From a pack of 52 cards, 1 card is drawn at random. Find the probability of getting a queen.

Conditional Probability

- The probability of event A provided the event B has already occurred is called the is called conditional probability $p(A/B)$

$$p(A/B) = \frac{p(A\cap B)}{p(B)} \text{ provided } p(B) \neq 0$$

$$p(B/A) = \frac{p(A\cap B)}{p(A)} \text{ provided } p(A) \neq 0$$

(How likely event A has occurred is subspace of B)

- If A and B are independent events

$$p(A/B) = \frac{p(A)p(B)}{p(B)} = p(A)$$

$$p(B/A) = \frac{p(A)p(B)}{p(A)} = p(B)$$

Conditional Probability

- If A and B are mutually exclusive events,

$$p(A/B) = 0$$

$$p(B/A) = 0$$

$$p(A \cap B) = 0$$

- If A and B are dependent events:

$$p(A \cap B) = p(A/B) * p(B) = p(B/A) * p(A)$$

(Multiplication law of probability)

Conditional Probability - Questions

1. A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?
2. A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both the balls drawn are black.
3. If the probability that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that is will have both is 0.18. Find the probability that a system with high fidelity will have high selectivity.
4. Find the probability of drawing a queen, a king and a knave in that order from a pack of cards in three consecutive draws, the cards drawn not being replaced.

Conditional Probability – H. W. Questions

1. A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one and examined. The ones examined are not put back. What is the probability that the ninth one examined is the last defective?
2. A manufacturer of airplane parts knows that the probability is 0.8 that an order will be ready for shipment on time and it is 0.7 that an order will be ready for shipment and will be delivered on time. What is the probability that such an order will be delivered on time given that it was also ready for shipment on time?

Total Probability

- If B_1, B_2, …, B_n are mutually exclusive and exhaustive set of events of a sample space S and A is any event associated with events B_1, B_2, …, B_n then

$$p(A) = p(A/B_1) * p(B_1) + p(A/B_2) * p(B_2) + \ldots + p(A/Bn) * p(B_n)$$

$$p(A) = \sum_{i=1}^{n} p(A/B_\mathrm{i}) * p(B_i)$$

Bayes' Theorem

- If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive set of events of a sample space S and A is any event associated with events $B_1, B_2, \ldots, B_n$ then

$$p(B_i/A) = \frac{p(A/Bi) * p(Bi)}{\sum_{i=1}^{n} p(A/B_i) * p(B_i)}$$

we know that

$$p(A) = \sum_{i=1}^{n} p(A/B_i) * p(B_i)$$

$$p(B_i/A) = \frac{p(A/Bi) * p(Bi)}{p(A)}$$

Questions

1. The contents of urns I, II, III are as follows:
    i. 1 white, 2 black, and 3 red balls
    ii. 2 white, 1 black, 1 red balls
    iii. 4 white, 5 black, and 3 red balls

    One is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urns I, II and III?

2. A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colours being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue.

Questions

4. A factory produces its entire output with three machines. Machines I, II, and III produce 50%, 30%, and 20% of the output, but 4%, 2% and 4% of their outputs are defective respectively. What fraction of the total output is defective?
5. A box contains 5 red and 4 white balls. Two balls are drawn successively from the box without replacement and it is noted that the second one is white. What is the probability that the first one is also white?
6. A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters and plant II the rest. At plant I, 85 out of 100 scooters are rated higher quality and at plant II, only 65 out of 100 scooters are rated higher quality. A scooter is chosen at random. What is the probability that the scooter came from plant II, if it is known that the scooter is of high quality.

Questions – H.W

1. In a coin-tossing experiment if the coin shows a head, one dice is thrown, and the number is recorded. If the coin shows a tail, two dice are thrown, and their sum is recorded. What is the probability that the recorded number will be 2?
2. A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?
3. Two shipments of parts are received. The first shipment contains 1000 parts with 10% defective and the second shipment contains 2000 parts with 5% defective. One shipment is selected at random. Two parts are tested and found good. Find the probability (a posterior) that the tested parts were selected from first shipment.

Random Variables

- A real number X connected with the outcome of a random experiment.

(Assigns a real value to each outcome of a random experiment)

- A function or rule that assigns a time function to every outcome of a random experiment is called a random process.

E.g. Let E be the random experiment consisting of two tosses of a coin

$$S = \{HH, HT, TH, TT\}$$

Random variable X denotes the number of heads

$$X = \{2, 1, 0\}$$

Discrete Random Variables

- A random variable X takes at most a countable number of values.

E.g. A coin is tossed

The number of defective bulbs in a box of 10 bulbs

Continuous Random Variables

- A random variable X takes at most an infinite number of possible values between certain limits.

E.g. Amount of sugar in orange

Possible values of the temperature outside

Lifetime of a CF Lamp

Height & weight

Probability Mass Function

- Suppose X is a one-dimensional discrete random variable taking at most a countably infinite number of values x_1, x_2, …

  With each possible outcome x_i, we associate a p_i ,

$$P(X = x_i) = p(x_i) = p_i$$

- The function $p(x_i)$, i= 1, 2, … satisfying the following conditions is called probability mass function (Probability distribution of random variable X)

  1) $p(x_i) \geq 0 \ \forall \ i$

  2) $\sum_{i=1}^{\infty} p(x_i) = 1$

- The collection of pairs $\{x_i, p_i\}$, i= 1, 2, 3 … is called the probability distribution of the random variable X.
- The set of values X takes is called spectrum of the random variable.

Distribution Function

- Let X be a random variable, then the $F_x(x) = F(x) = P(X \le x), -\infty < x < \infty$ function is called distribution function of X.
- Properties:
  1. If $F(x)$ is a distribution function of the random variable X and if a< b, then $P(a < X \le b) = F(b) - F(a)$.
  2. If $F(x)$ is the distribution function of one-dimensional random variable X, then
     i. $0 \le F(x) \le 1$
     ii. $\mathrm{F}(x) \le F(y), if\ x < y$

  3. If $F(x)$ is the distribution function of one-dimensional random variable X,

$\mathrm{F}(\infty) = \lim_{x\to\infty} F(x) = 1$, and $\mathrm{F}(-\infty) = \lim_{x\to-\infty} F(x) = 0$

$$\mathrm{F}(\infty) = 1, \mathrm{F}(-\infty) = 0$$

Discrete Distribution Function

- The distribution function of the random variable X with PMF $p(x_i), i = 1, 2, 3, \ldots$ is defined as

$$F(x) = \sum_{i:\, x_i \leq x} p(x_i)$$

Note:

1. $p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$
2. Mean of the random variable X = $\mathrm{E(X)} = \sum_x x\, p(x)$
3. Variance of the random variable X

$$Var(X) = \sum x^2 p(x) - \left[\sum x\, p(x)\right]^2$$

Questions

1. If X is a discrete random variable having the probability distribution

$X = x$	1	2	3
$P(X = x)$	k	$2k$	k

Find $P(X \leq 2)$.

2. If X is a discrete random variable having the PMF

$X = x$	-1	0	1
$P(x)$	k	$2k$	$3k$

Find $P(X \geq 0)$

3. If X is a discrete random variable having the probability distribution

$X = x$	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$4k$

Find $P(2 < X < 4)$

Questions

4. If X is a discrete random variable having the probability distribution

$X = x$	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

Find $P\left(\frac{1}{2} < X < \frac{7}{2} \, /X > 1\right)$.

5. A random variable X has the probability function

$X = x$	-2	-1	0	1
$P(X = x)$	0.4	k	0.2	0.3

Find K and mean value of X.

6. A coin is tossed two times, if X is the number of heads. Find the probability distribution of X.

Questions

7. If $P(X = x) = \begin{cases} kx & x=1,2,3,4,5 \\ 0 & Otherwise \end{cases}$ represents a probability, find
   - K
   - P(X being a prime number)
   - $P(\frac{1}{2} < X < \frac{5}{2} / X > 1)$
   - The distribution function
8. The CDF of a random variable is given by

$$F(x) = \begin{cases} 0 & for\ x < 0 \\ \frac{x^2}{16} & for\ 0 \leq x \leq 4 \\ 1 & for\ 4 < x \end{cases}$$

Find $P(X > 1/X < 3)$

Questions

8. If the probability mass function of a random variable is given by $P(X=r) = kr^3$, r= 1,2,3,4 Find

- Value of k
- $P\left[\frac{1}{2} < X < \frac{5}{2} / X > 1\right]$
- The mean and variance of X
- The distribution function of X

Continuous Random Variable

Probability Density Function (PDF)

- Consider the small interval $\left(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}\right)$of length Δx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that x falls in the interval $\left(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}\right)$.

$$P\left(x - \frac{\Delta x}{2} \leq x \leq \ x + \frac{\Delta x}{2}\right) = f(x)dx$$

- Properties
- $f(x) \geq 0 \ \ \forall x \in R$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a < X < b) = \int_{a}^{b} f(x)dx$

Cumulative Distribution function (CDF)

Probability Density Function (PDF)

- The cumulative distribution function of a random variable X with PDF $f(x)$ is
- $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx \quad -\infty < x < \infty$
- Relation between PDF and CDF

$$f(x) = \frac{d}{dx}F(x)$$

- Mean= E(x) = $\int_{-\infty}^{\infty} xf(x)dx$
- Variance= $E(x^2) - [E(x)]^2$

Questions

1. A random variable X has the PDF $f(x)$ given by

$$f(x) = \begin{cases} cxe^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

   Find the value of c and CDF of x

2. A continuous random variable X follows the probability law

$f(x) = ax^2$, $0 \leq x \leq 1$. Determine a and find the probability that x lies between $\frac{1}{4}$ and $\frac{1}{2}$.

3. If the PDF of a random variable X is f(x) $= \frac{x}{2}$ $in\ 0 \leq x \leq 2$, find $p(X > 1.5/X$>1).

4. If $f(x) = kx^2, 0 < x < 3$ is to be the density function, Find the value of k.

Questions

5. If the CDF of a random variable X is given by $F(x) = 0 \; for \; x < 0; = \frac{x^2}{16} \; for \; 0 \le x < 4, and = 1 \; for \; x \ge 4$. Find P(X>1/X<3).

6. The diameter of an electric cable , say X is assumed to be a continuous random variable with PDF given by

   $f(x) = kx(1-x), 0 \le x \le 1$. Determine K and $p\left(x \le \frac{1}{3}\right)$

7. The length of time (in minutes) that a certain lady speaks on the phone is found to be random phenomenon with probability function specified by the PDF

$$f(x) = \begin{cases} Ae^{\frac{x}{5}}, & x \ge 0 \\ 0, & Otherwise \end{cases}$$

Find the value of A that makes $f(x)$ a PDF.