Linear Algebra & Probability

3 blue 1 brown
linear algebra

$$2x - y = 0$$
$$-x + 2y = 0$$

$$AX = B$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$ → Row perspective

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$ → clm perspective (it allow to understand dimensn) – ie, 2D...

→ column approach :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 5 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

3×3
3×1

→ Row approach :

$$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + 3 \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} + 5 \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$
$$= [49, 61, 90]$$

1×3
3×3

Linear System :

A Linear Combination of $x_1, \dots x_n$ has the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 \dots a_n x_n$$

where the no: $a_1, a_2 \dots a_n \in \mathbb{R}$ are combinatn of coeffients

$x_1, x_2 \dots x_n$ are unknowns.

Eg:- This is a linear Combinatn of x, y, z

$$(1/4)x + y - z = 0$$

Q

$x + 2y + z = 2$

$3x + 8y + z = 12$

$0x + 4y + z = 2$

$$\underset{A}{\begin{bmatrix}1 & 2 & 1\\3 & 8 & 1\\0 & 4 & 1\end{bmatrix}} \xrightarrow{R_2 - 3R_1} \underset{A_1}{\begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 4 & 1\end{bmatrix}} \xrightarrow{R_3 - 2R_2} \underset{A_2}{\begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 0 & 5\end{bmatrix}}$$

OR

$R_2 - 3R_1$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\3 & 8 & 1\\0 & 4 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 0 & 5\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 4 & 1\end{bmatrix}$$

$$\underset{A_{21}}{\begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\0 & 0 & 1\end{bmatrix}}\underset{A}{\begin{bmatrix}1 & 2 & 1\\3 & 8 & 1\\0 & 4 & 1\end{bmatrix}} = \begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 4 & 1\end{bmatrix} \qquad \underset{A_{32}}{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}}\underset{A_1}{\begin{bmatrix}1 & 2 & 1\\0 & 2 & -2\\0 & 4 & 1\end{bmatrix}}$$

~~A32~~

Q

$$\begin{bmatrix} & \end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}b & a\\d & c\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}b & a\\d & c\end{bmatrix}$$

but $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}c & d\\a & b\end{bmatrix}$

We cannot put a matrix on left of main matrix for clm exchange.. If we x by on left we are doing row operatn for clm operatn, xleatn should be on righ

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}b & a\\d & c\end{bmatrix} \quad //$$

In General transformation of A to U can write as: ~~A=LU~~

$\underbrace{(E_{32}, E_{31}\, E_{21})}_{E} A = U$

Column Approach:

Combination of clms of A, scaled by clms of C
(Clms of C are combi' of clm of A)

$$\overset{A}{\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}}\begin{bmatrix}4\\5\end{bmatrix} = 4\begin{bmatrix}2\\4\end{bmatrix} + 5\begin{bmatrix}3\\5\end{bmatrix}$$

$$\overset{A}{[\quad]}\,\overset{B}{[\quad]} = \overset{C}{[\quad]}$$

Row approach:

Rows of C are Combination of rows of B.

Q.

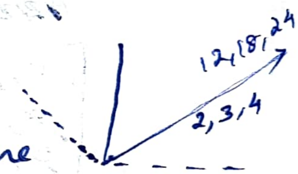
$$\overset{A}{\begin{bmatrix}2\\3\\4\end{bmatrix}}\,\overset{B}{\begin{bmatrix}1 & 6\end{bmatrix}} = \begin{bmatrix}2\\3\\4\end{bmatrix}1 + \begin{bmatrix}2\\3\\4\end{bmatrix}6$$

3 × 1
1 × 2

u/

$$\overset{C}{\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}}$$

Peculiarity: 2, 3, 4
12, 18, 24 } if in 3D plane the both the vectors are lying on same plane/line
{They can't span the entire space}



Another Approach [clm of A × Row of B]

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix} \quad \begin{bmatrix}1 & 6\\0 & 0\end{bmatrix}$$

Clms of A × Row of B

3×1
1×2

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

Inverse:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad \det = 0$$

2D perspective: 2 vectors are lying on same line.
(1,2) & (3,6)

Gauss Jordan Method

Q $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

$\boxed{AA^{-1} = I = A^{-1}A}$

Note: Inverse of AB is $B^{-1}A^{-1}$
{We remove shoe then socks, to get back to original state, we put back socks first}

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

← {Gauss Jordan Method}

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

A, I → I, A^{-1} (REF) → I, A^{-1} (RREF)

Checking:

$$\overset{A^{-1}}{\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$7 - 6$

$E\,[A\ I] = I$
elimination matrix

$\boxed{EA = I}$

$\therefore \boxed{E = A^{-1}}$

* $(A^T)^{-1} = (A^{-1})^T$

$A = LU$

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \xrightarrow[R_2 = R_2 - 4R_1]{} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 0 \\ 8 & 7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 6 & 0 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$R_2 - 4R_1$

$(0\ 3) \quad (8, 7)$

$R_2 = R_2 - 4R_1$

$\downarrow$

$R_2 = R_2 + 4R_1$

$(8, 7)\ (0, 3)$

Then

$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

Lower Δ Upper Δ

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$R_2 + 4R_1$

No Row Change Assumptn.

$$\boxed{E_{32} E_{31} E_{21} A = U}$$

$$\boxed{A = (E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}) U}$$

$\downarrow$

I

$\Rightarrow$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} E_{32} \times E_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +5 & 0 \end{bmatrix} \quad \{\text{remove -ve}\}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- * $E = E_{32} \times E_{21}$
- * find $E_{21}^{-1}, E_{32}^{-1}, E_{31}^{-1}$ {remove -ve}
- * Then $E = E_{21}^{-1} \times E_{32}^{-1}$

$$\underset{\{E_{21}^{-1} \times E_{32}^{-1}\}}{E} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

(L) -> Lower digonal matrix//

$$n^2+(n-1)^2+(n-2)^2+\dots 3^2+2^2+1^2$$
$$= \frac{1}{3}n^3 \quad (\text{or}) \quad \int^n n^2\,dx$$

Q. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Row Exchange

~~Matrix~~[x] × matrix = matrix{with row 1 & 2 swapped}
How to get?

$$\underset{\{\text{Swapped mat}\}}{P_{12}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\text{Swap row 1 \& 2}\} \quad — 1$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \{\text{Swap 3 \& 1}\} \quad — 2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \{\text{Swap 2 \& 3}\} \quad — 3$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ Swap {3 & 1}, Swap {2 & 3} — 4

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ {swap 2 & 1, 2 & 3} — 5

$\boxed{3!\ \text{number of Swaps possibilities}}$

Suppose for taking Inverse of matrixing

for 1, 2 & 3 Inverse will ·

for 4, 5 Inverse of 4 = 5

$$\underset{A}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

for taking transpose.

for 1 → transpose will be 1, 2 → transp will be 2 ....

Q $AX = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$AX = 0$

$A^{-1}AX = 0 \times A^{-1}$

$IX = 0$

$\underline{X = 0}$

* $PA = LU$
* $P^T = P^{-1}$
* $PP^{-1} = I$
* $PP^T = I$

* $A = LU$ can represent as $PA = LU$, where $P = I$, where there is no row exchange.

* $\cancel{R \times R^T \neq I/R}$

$$\underset{R}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}} \underset{R^T}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 7 \\ 7 & 11 & 17 \end{bmatrix} \longrightarrow \text{Symmetric matrix } \{P^T = P\}$$

$$\boxed{R \times R^T = \text{Symmetric matrix}}$$

VECTOR SPACES - SUB SPACES

→ o/p need tobe within that space..

→ line meets criteria of subspace

→ origin (0,0) / $\vec{0}$ vector meet criteria of subspace

→ In R^3, if any plane ~~falls~~ not pass throughorigin then it will not satisfy $c\vec{v} + d\vec{w}$

∴ Plane should pass through (0,0)

→ Will Union of 2 Subspace make a subspace → No $[P \cup L]$

→ Will Intersectn of 2 subspace make a subspace → Yes $[P \cap L]$

$Ax = b$

Q.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

4×3 3×1 4×1

4 diam plane with 3 vectors

Can it does $Ax=b$, hold good for any b?

No we cant span R^4 with just 3 vectors ie, we wont able to Cover all b's //

How to know which b will be suitable?

all possible b's will be all the possible linear Combinatn of 3 vectors In A.

* Colum space (C(A)): The linear Combinatn of the colvectors In A

* col 3 of A is dependent of Col 1 & 2 [1^{st} clm + 2^{nd} clm = 3^{rd} clm].

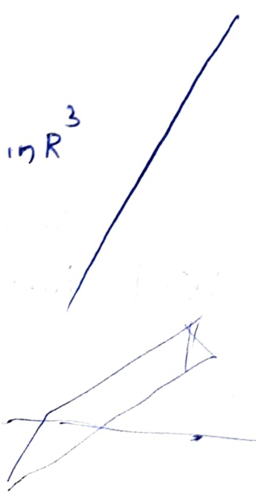
↳ C(A) is a 2 diam subspace of R^4

<u>Null Space</u>

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \text{line in } R^3$$



Both Solutn does not pass through origin //
∴ it cant Consider as a subspace //

Nullspace

Q. $Ax = 0$

Special Solutions & Pivot Value

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

Pivot Position

Staircase

* We find that, Row 3 is dependent of Row 1 & 2

ie, $R_3 = R_1 + R_2$ //

Reduced Row Echelon Form [rref]

it has 2 things: above & below pivot positions, we will have 0.

~~$R_2 = R_2/2$~~

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~$R_1 - 2R_2$~~

$x_1 \quad x_2 \quad x_3 \quad x_4$

pivot variable

free variables

Solving $Ux = 0$, $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

Backsubstitution:

$$\begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \rightarrow x_2 \\ \\ \rightarrow x_4 \end{matrix} \quad \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \begin{matrix} \rightarrow x_2 \\ \\ \\ \rightarrow x_4 \end{matrix}$$

assume $x_4 = 0$

$2x_3 + 4x_4 = 0$

$2x_3 = \cancel{4x}$ $x_3 = 0$ //

$\therefore x_1 + 2 + 0 + 0 = 0$

$x_1 = -2$

$x_4 = 1$

$2x_3 + 4x_4 = 0$

$2x_3 + 4 = 0$

$x_3 = -2$ //

$x_2 = 0$

$x_1 + 2(0) + -4 + 2$

$x_1 - 2 = 0$

$x_1 = 2$

ie, In back substitution.

Consider $\begin{bmatrix} \square \\ 1 \\ \square \\ 0 \end{bmatrix}$ & Consider $\begin{bmatrix} \square \\ 0 \\ \square \\ 1 \end{bmatrix}$

x_4 as 0 & find x_1, x_2, x_3 ; x_4 as 1 & find x_1, x_2, x_3

ie, $c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + $ ie, $d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Longrightarrow (*)$

Reduce row echelon form

$\xrightarrow{\substack{R_1 - 2R_2 \\ R_2 = R_2/2}}$

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array}$$

$x_1 + 2x_2 + 0x_3 - 2x_4$

$x_3 + 2x_4$

Looking at RREF, we find

Pivot

$$\begin{array}{cc} x_1 & x_3 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{cc} x_2 & x_4 \end{array} \begin{bmatrix} 2 & -2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

I F

ie, $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (x_1, x_3)

$R = [I \quad F]$

ie, $\begin{array}{cc} x_1 & x_3 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{array}{cc} x_2 & x_4 \end{array} \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ {Ignoring rows of 0s}

Rank = # pivot clms

free variable $= n - r$
$= 4 - 2$
$= 2$

n = no: clms {$m \times n$ matrix}

Solve $R x = 0$
$\downarrow$
nullspace //

$$RN = 0$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} N = 0$$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = -FI + FI$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = 0$$

Q. $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \xrightarrow[\substack{R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 2R_1 \\ R_4 = R_4 - 2R_1}]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\substack{R_2 \leftrightarrow R_3 \\ \text{Interchange} \\ R_4 = R_4 - 2R_2}]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{array}{ccc} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 \quad x_2 \quad x_3 \end{array}$$

Pivot variable, Free variable
(x_1, x_2) // (x_3)

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 2x_3 = 0$

$r = 2$
$n = 3$

$\#$ free $= 3 - 2 = 1$ //

x_3 is free variable

$\therefore$ we can give any value //

Let $x_3 = 1$

$$\begin{bmatrix} \boxed{-1} \\ \boxed{-1} \\ 1 \end{bmatrix} \quad \therefore \begin{aligned} 2x_2 + 2(1) &= 0 \\ 2x_2 &= -2 \\ x_2 &= -1 \end{aligned}$$

$\therefore x_1 + 2(-1) + 3(1) = 0$

$x_1 - 2 + 3 = 0$

$x_1 = -1$

$\therefore$ when $x_3 = 1$, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ //

$c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$Ax = b$$

Q)

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

$$\underbrace{\begin{matrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{matrix}}_{\{\text{Augmented Matrix} \rightarrow [A\,b]\}}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - [b_2 - 2b_1] \\ x_1 & x_2 & x_3 & x_4 & \underline{\underline{b_3 - b_2 - b_1}} \end{bmatrix}$$

Solvability Criteria for $Ax = b$

b has to be in the clm space of A

~~C(A)~~ $\{b \text{ is in } C(A)\}$

let $b_1, b_2, b_3 \Rightarrow 1, 5, 6$

$\therefore$ $b_1 = 1$ //
$b_2 - 2b_1 = 3$ //
$b_3 - b_2 - b_1 = 0$ //

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 3$$
~~x_3~~

Free variable $\Rightarrow$ $\underline{\underline{x_4, x_2}}$ $\therefore$ $\underline{\underline{= 0}}$

$\therefore x_4 = 0$

$2x_3 + 4x_4 = 3$

$\cancel{2(0) + 4(x_4) = 3}$

$2x_3 = 3$

$x_3 = 3/2$

$x_2 = 0$

$x_1 + 2(0) + 2(3/2) + 2(0) = 1$

$x_1 + 3 = 1$

$x_1 = -2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

$$\boxed{\underbrace{X}_{\text{Complete soln}} = \underbrace{X_p}_{\text{particular soln}} + \underbrace{X_N}_{\text{Nullspace soln}}}$$

$$A X_p = b$$
$$A X_N = 0$$

* {before a page}//

$$\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Particular soln

Null sp Soln {2D}

Rank → no: pivots {clm}

RANK

$m \times n$
$m = 4$
$n = 2$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 - 2R_1 \\ R_3 - 6R_1 \\ R_4 - 5R_1}]{} \begin{array}{cc} x_1 & x_2 \\ \hline \end{array} \begin{bmatrix} 1 & 1 \\ 0 & \boxed{-1} \\ 0 & \boxed{-6} \\ 0 & \boxed{-4} \end{bmatrix} \xrightarrow[\substack{R_3 + 5R_3 \\ R_4 + 4R_4}]{} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow[\substack{R_1 = R_1 \\ -R_2}]{} \begin{bmatrix} 1 & 0 \\ 0 & \boxed{1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$-5 - 5(-1)$

Pivot variables ...
1 pivot clm//

→ No dependency b/w 2 clm
→ Rank = 2
→ what will be Null space, when we have Independent clm?

$N(A) = $ Zero Vector

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

→ I Pivot rows tranforms to identity matri

→ F

0 or 1 soln

Case 1 ⤴ $r = n < m$//

II case $\Rightarrow$ $r = \overset{\text{row}}{m} < n$ //

$$\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{cccc} \multicolumn{2}{c}{I} & \multicolumn{2}{c}{F} \end{array} \begin{bmatrix} 1 & 0 & \underbrace{- \;\; -} \\ 0 & 1 & - \;\; - \end{bmatrix}$$

These value can be anything

~~Free variables~~

∞ solutions

III case $\Rightarrow$ $r = m = n$ [Square matrix]

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Unique soln
[Single soln]

$$R = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Nullspace = Zero Vector

unique soln

IV Case

$r < m$, $r < n$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

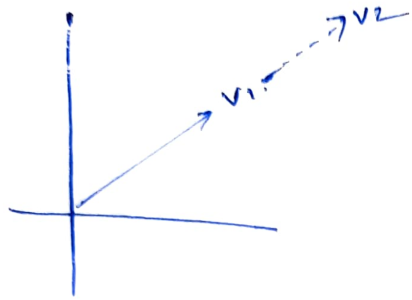
0 or ∞ soln

Rank of matrix tells u abt the no: soln's

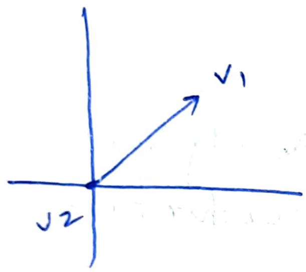
Let there be vectors $v_1, v_2, v_3 \cdots v_n$

If there exist some combinatn $c_1v_1 + c_2v_2 + c_3v_3 + \dots c_nv_n = 0$, where

- * If the only soln is $v_1 = 0, v_2 = 0, \dots v_n = 0 \Rightarrow$ Independent //
- * If non zero soln exist. (Eg:- $c_1 = 1, c_2 = 2 \dots$) $\Rightarrow$ dependent
  ie, $c_n \neq 0$



dep



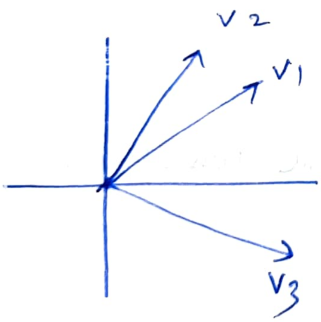
v_2 = zero vector.

Moment the 0 vector comes in play <u>Dependence</u>

$0v_1 + \overset{c}{\cancel{0}}v_2 = 0$

↓

can be any constant //



$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Rightarrow AC = 0$$

2×3

Scope for free variables --

$\therefore$ There will be some dependency.

→ If there is Independence, $N(A) = \{0\}$

[if rank = n, each clm is a pivot col, no free variables]

SPAN

Linear Combination of vectors.

BASIS

2 properties:

1) They are Independent -
2) They span the space / subspace.

Eg:- $\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$ Is this the only basis? Are there other bases aswell?

Consider $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$

Put them as a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$ and use row elimination to do?

* for a space any # basis is possible

Dimension of a space

(# of vectors in a basis?)

3×4
$m \times n$ 1×4

Q. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$ 3×4 //

3×4
1×4

There will be some free variables
Since there is dependency & $m < n$

$-1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + -1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Nullspace $= \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$? {assign $x_3 = 1, x_4 = 0$}

$\boxed{\text{Rank}(A) = \#\text{pivot col} = \text{Diamension of basis} = \text{diamension of clm space of A } C(A)}$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot var (x_1, x_2) freevar (x_3, x_4)

$\therefore$ Rank $= 2$ //

$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

pivot pivot free $= 2$

$\therefore$ Rank $= 2$

$\boxed{\begin{aligned} \text{Dim } N(A) = \#\text{free Variable} &= n - \text{rank} \\ &= 4 - 2 \\ &= 2 \text{ //} \end{aligned}}$

FOUR FUNDAMENTAL SUBSPACES

Other subspaces

→ Clm space $(C(A))$ → $Ax = b$ → Clm space is in R^m → all Combinatn of clms of A

→ Null space $(N(A))$ → $Ax = 0$ → Null space is in R^n

4 Subspaces:

→ Row space → $R(A)$ → all Combinatn of rows of A

Row space of A $(R(A))$ = Clm space (A^T)

$$\boxed{R(A) = C(A^T)}$$

→ Row space is in R^n

→ Null space of A^T (Left Null space) $N(A^T)$:→ Null space of A^T is in R^m

1) $C(A) \to R^m \to$ diamensn (r)
2) $N(A) \to R^n \to$ diamen $(n-r)$
3) $R(A) \to R^n \to$ diamensn (r)
4) $N(A^T) \to R^m \to$ diamenn $(n-r)$

diam $C(A) \to$ diam $R(A)$
diam $N(A) \to$ diam $N(A^T)$

Q.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 = -(R_2)} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{array}{c} \begin{array}{cccc} I & & F & \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$\underbrace{x_1 \; x_2}_{\text{pivot}}$ $\underbrace{x_3 \; x_4}_{\text{free}}$

$\boxed{R}$

Diamensn(R) = # freevar. = 2 //

R is the row reduced for of A.

∴ Combinatn of rows of R & A are equal {rowspace} //

$C(A) \neq C(A)$
but
$R(A) = R(A)$ as we perform row operatn

Left Null Space $N(A^T)$

$A^Ty = 0$ or $y^TA = 0^T$ (taking transpose on both side)

elimination matrix

$$\text{Rref}\left[A_{m\times n}\quad I_{m\times m}\right] = R_{m\times n}\quad E_{m\times m}$$

$$E\left[A_{m\times n}\quad I_{m\times m}\right] = R_{m\times n}\quad E_{m\times m}$$

$$\Rightarrow \underline{\underline{EA = R}}$$

For Square Matrix

$$EA = I$$

$$\boxed{E = A^{-1}}$$

Taking identity matrix & do all operation

$$\underset{I}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix}1&0&0\\-1&1&0\\-1&0&1\end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix}1&0&0\\1&-1&0\\-1&0&1\end{bmatrix} \xrightarrow{R_1 - 2R_2} \underbrace{\begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}}_{\text{Elimination Matrix } (E)}$$

Rank:
$$\begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix}-1&2&0\\1&-1&0\\0&-2&\end{bmatrix}$$

Rank (I) = 2 //

$$\begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\end{bmatrix}$$

free Pivot

-1 +

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_3} \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$A^T \to$ rank $= 2$

dim $N(A^T) = 3 - 2 = 1$ // {basis has only 1 vector}

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$\boxed{\text{Symetric matrix + Sym matrix = Symetric matric}}$

$\{AA^T = I\}$

5/11/22

Symmetric $\cap$ Upper Δ ar matrix = diganal matrix of diam 3

$\boxed{S \cap U = \text{Diagonal Matrix of diamension 3}}$

* $S \cup U \to$ not an useful Concept.
* $S + U \to$

diam $S = 6$

Diam $U = 6$ $\therefore$ dim $(S+U) = 9$ //

$\boxed{\text{Dim } S + \text{Dim } U = \text{Dim}(S \cap U) + \text{Dim}(S + U)}$

$6 + 6 = 3 + 9$

$12 = 12$

Rank : Maximum number of linearly independent coloumns (vectors) → (formal defntn)

For $m \times n$ matrix
$\text{rank} \le m$ or $\text{rank} \le n$

Rank 1 Matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ \square & \square & \square \end{bmatrix}$$ what will be 2nd row ?

2×3, $m < n$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$ {Scalar x le}

$2 \times 3, m < n$ //

Basis for Row space = $[1 \ 4 \ 5]$
Dimension of Row space = 1
∴ Dim. of clm space = 1

{Since R_2 is linearly dep of R_1, R_2 will cut off after eliminatn}
Since R_1 is Indep //

We can represent A as (every Rank 1 matrix)
$$\underset{2 \times 1}{\overset{(U)}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}} \underset{1 \times 3}{\overset{\{V^T\}}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}}$$

Every Rank 1 matrix can represent as $\boxed{A = UV^T}$, U & V are clm vectors

★ Consider 4 vectors in R^4

(5) $\{v_1 + v_2 + v_3 + v_4 = 0$

What is diamension of vectors in subspace ?

$AX = 0$ ⟶ Null Space.

$m \times n$
1×4

$$\underbrace{\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \end{bmatrix}}_{(A)} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$$

Rank(A) = 1 //

Diamension of null space $= n - R$ {Since Rank = 1}
$= 4 - 1 = 3$ //

Row space, $R(S) = 1$
$N(S) = 3$ ⟶ $[\boxed{1}\ 1\ 1\ 1]$

free variable = 3 //
Pivot variable = 1 //

Give any values to x_2, x_3, x_4

$$\begin{bmatrix} \square \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \square \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \square \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 + x_2 = 0 \Rightarrow x_1 + 1 = 0 \Rightarrow x_1 = -1$
$x_1 + x_3 = 0 \Rightarrow x_1 + 1 = 0 \Rightarrow x_1 = -1$
$x_1 + x_4 = 0 \Rightarrow x_1 + 1 = 0 \Rightarrow x_1 = -1$

$$\begin{bmatrix} \boxed{-1} \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \boxed{-1} \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \boxed{-1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ 3 ?

Since we have 3 free variables

Sum of diam of
$R(S) + N(S) = N$
$1 + 3 = 4$ //

$$\boxed{R(S) + N(S) = n}$$

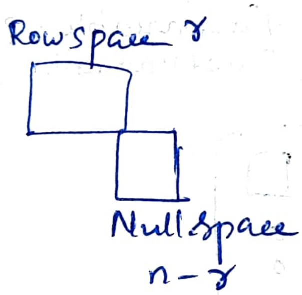
$$1 + 3 = 4$$

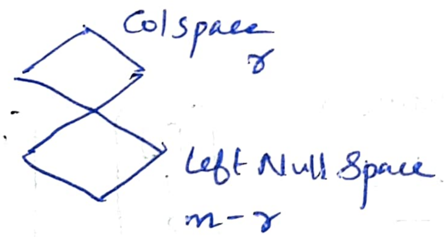
$$\boxed{C(S) + N(S^T) = m}$$

$$1 + 0 = 1$$

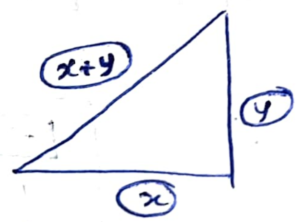
Orthogonal Vectors of Subspaces

dimension of Rowspace = r
dimension of Colm space = r





Two Vectors are orthogonal, if dot product is 0



$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$x^T x \qquad y^T y$$

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $Y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\quad X + Y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$1^2 + 2^2 + 3^2 = \underline{\underline{14}}$ $\quad 2^2 + (-1)^2 + 0^2 = \underline{\underline{5}}$ $\quad 3^2 + 1^2 + 3^2 = 19$

$$\underbrace{14 + 5 = 19}$$

This is true since X & Y are orthogonal

$$X^T X + Y^T Y = (X + Y)^T (X + Y)$$

$$= X^T X + Y^T Y + X^T Y + Y^T X$$

$$\{X^T Y = Y^T X\}$$

$$\cancel{X^T X} + \cancel{Y^T Y} = \cancel{X^T X} + \cancel{Y^T Y} + X^T Y + Y^T X$$

$$0 = 2X^T Y$$

$$\therefore \boxed{X^T Y = 0}$$

~~Vectors are orthogonal~~.

Q. If we have 2 vectors, and if 1 is 0 vector are they still orthogonal...?

Yes, 0 vector is orthogonal to all vectors.

SubSpaces.

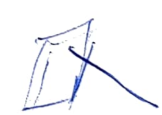
Suppose we have 1 subspace S & another subspace T.

1. When will subspace S orthogonal to subspace T ?

Every subspace of S & T are orthogonal only if all the vectors satisfy the condition from the basis.

$$AX = 0$$

$$\begin{bmatrix} \text{Row 1 of A} \dots \\ \text{Row 2 of A} \dots \\ \\ \text{Row m of A} \dots \end{bmatrix} \begin{bmatrix} \\ X \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

All the line & plane cannot be orthogonal. 

Zero Subspace is always orthogonal

$$c_1 \, Row_1^T x + c_2 \, Row_2^T x + \dots$$

Orthogonal Subspace in 3 dimension . R_3

Row space → line

Null space → line .

Dim (Row Space) = 1 → its a line //

Dim (Null Space) = $n - r$

$= 3 - 1$

$= 2$ ∴ its a plane //

$$\boxed{Ax = b}$$

× by both side by A^T

$\Rightarrow \quad A^T A \hat{x} = A^T b$

$A = m \times n$.

$A^T = n \times m$

$A = m \times n$

∴ $A^T A = n \times n$

$$\Rightarrow \boxed{A A^T \hat{x} = A^T b}$$

→ $A^T A = A A^T$

Invertability of AA^T

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$$

clms are Independent

& no: clms = 2

$\therefore$ Rank = 2

$$\underset{3\times 2}{A}\;\; x \;\; b$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This will have a Soln if

b_1, b_2, b_3 is in clm space of A or

~~b_1, b_2, b_3 Could be express~~

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

Let

let

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$$

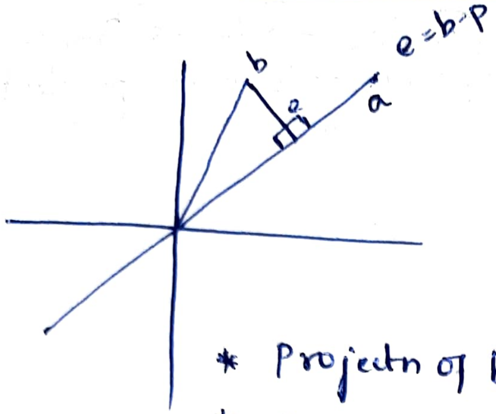
$$= \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

$|A| = 0$

NonInvertable

$1+4+15$

PROJECTIONS



* Projectn of b on to a → ⊥ar.
* error = $e = b - p$

* $P = xa = ax$
* a is ⊥ to e means a is ⊥ar to $b - p$ ie, a is ⊥ar to $b - xa$.
* $a^T(b - xa) = 0$ [∵ dotproduct is equal to 0]

$xa^Ta = a^Tb$

$x = \dfrac{a^Tb}{a^Ta} = \dfrac{b}{a}$ → will be a matrix / will be a number

∴ $P = xa = ax$

$= a \cdot \dfrac{a^Tb}{a^Ta}$

→ If b, become twice. P will also get twice

&

→ if a become twice there will be no change

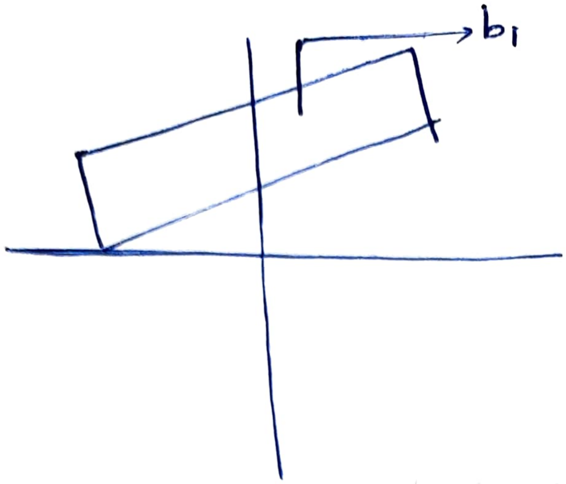
① Symmetric {projectn matrix must be symmetric}

② $P^2 = P$ {projectn of b on a is on P, whereas projectn of P on a is P itself}

① x

② $P = ax$

③ P



$a_1, a_2 \rightarrow$ basis and has to be independent.

$P = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$P = A\hat{x}$

Goal is to find set of $\hat{x}$

$e = b - A\hat{x}$, is going to be orthogonal to plane.

$a_1^T(b - A\hat{x}) = 0$
$a_2^T(b - A\hat{x}) = 0$

Convert it into matrix form

$$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A^T(b - A\hat{x}) = 0$$

$$\Rightarrow A^T e = 0$$

$\hookrightarrow$ e is in the left null space of A
or
e is in null space of A^T

e is orthogonal to clm space of A

$$A^T A\hat{x} = A^T b$$

$$\boxed{\hat{x} = [A^T A]^{-1} A^T b}$$

$\therefore P = A\hat{x}$

$= \underline{\underline{A(A^TA)^{-1}A^Tb}}$

Suppose A was a Square matrix

$\therefore P = A\,[A^TA]^{-1}A^Tb$

$= A\,A^{-1}\;A^{T^{-1}}\;A^T$

$= \underline{\underline{I}}$

→ we can't apply this since we have a Rectangular matrix.

$P = \underbrace{A(A^TA)^{-1}A^T}\,b$

↳ $A(A^TA)^{-1}A^T \Rightarrow$

$A(A^TA)^{-1}\,(A^TA)\,(A^TA)^{-1}\,A^T$

~~Suppose A, we~~

$P = A\,(A^TA)^{-1}A^T$

[no: equatn > no: unknown
$m > n$.]

$Pb = b$

$\underbrace{Pb = 0}$ (happens when $\perp$)

~~$m>n$~~
~~$m<n$~~

$\boxed{\text{Col space} \perp N(A^T)}$

$Pb = A\,(A^TA)^{-1}\,(A^Tb) \to 0$ when $\perp$

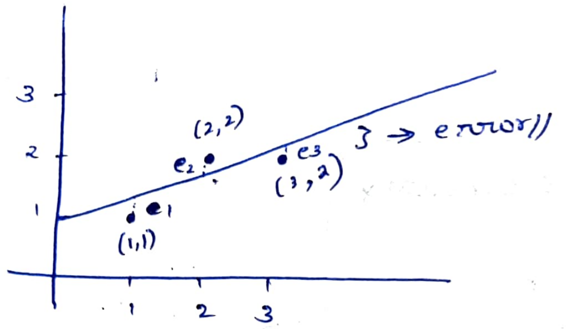
$\Rightarrow Pb = 0$

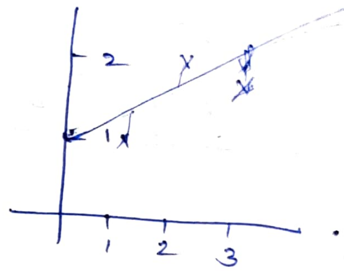
$Pb = A\,(A^TA^{-1})\,A^T\,Ax$ [as $Ax = b$, when b is in $C(A)$]

$= \underbrace{A(A^TA^{-1})\,(A^TA)}\,x$

$= Ax$ → $(\underline{\underline{A^{-1}A = I}})$

$= b$ //





1,05

Eg.,

$$C + D = 1$$
$$C + 2D = 2$$
$$C + 3D = 2$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} c \\ d \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

We want the best representatn of these pts to minimize error...

∴ We use minimize the sum of squares of error

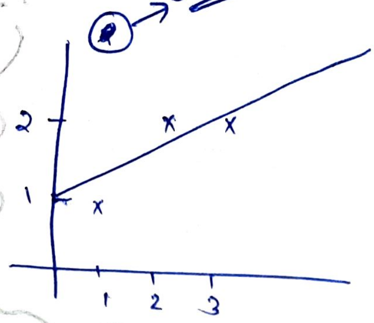
$$Ax = b$$

$$e = \|Ax - b\|^2$$

If b ~~are~~ is in the $C(A)$ then $\|Ax - b\|^2 = 0$

↳ clmspace

Note.



outliers → this have to treated seperately

Find $\hat{x}$ and P

$$x = \begin{bmatrix} C \\ D \end{bmatrix} \quad P$$

$$\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} \quad \hat{P}$$

$$A^T A \hat{x} = A^T b$$

2×3
3×1

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}}_{\text{Symetric}} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

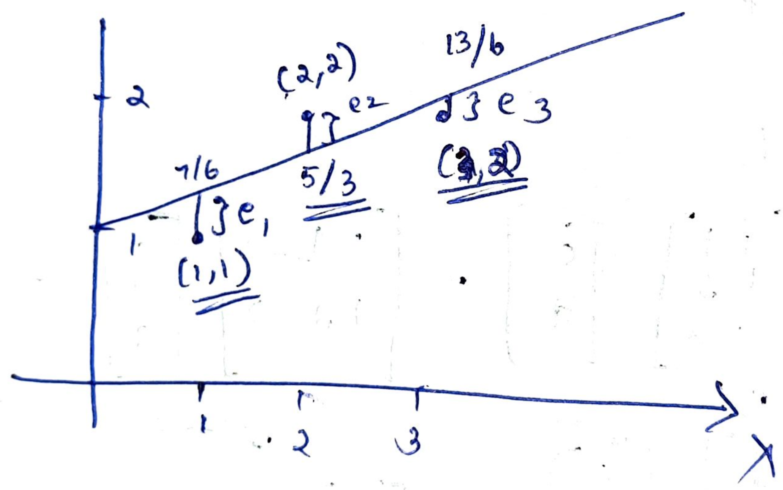
Normal Equations
$$3C + 6D = 5 \quad \text{—— (1)}$$
$$6C + 14D = 11 \quad \text{—— (2)}$$

$$(1) \times 2 \rightarrow 6C + 12D = 10$$
$$6C + 14D = 11$$
$$2D = 1$$
$$D = 1/2$$

$$\therefore C = 2/3$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$$

* $C + D = 1 \longrightarrow 2/3 + 1/2 = \frac{4+3}{6} = 7/6$ — P_1
* $C + 2D = 2 \longrightarrow 2/3 + 2/2 = \frac{4+6}{6} = 10/6 = 5/3$ — P_2
* $C + 3D = 2 \longrightarrow 2/3 + 3/2 = \frac{4+9}{6} = 13/6$ — P_3



$\cancel{e_1 = 7/6 - 1/1 = 7/6 - 6/6 = 1/6}$

$\cancel{e_2 = 5/3 - 2 = \frac{5-12}{6} = -7/6}$

Taking $P - y$

$e_1 = 1 - 7/6 = -1/6$

$e_2 = 2 - 5/3 = \frac{12-10}{6} = 2/6$

$e_3 = 2 - 13/6 = \frac{12-13}{6} = -1/6$

$\frac{2}{1} - \frac{5}{3} = \frac{6-5}{3}$

$$\underset{P}{\begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}} + \underset{E}{\begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \cancel{\begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}} = \begin{bmatrix} 21/6 & 42/6 \\ 15/3 & 30/3 \\ 39/6 & 76/6 \end{bmatrix}$$

$$7/6 + 10/6 + 13/6$$
$$30/6 = 5$$

$$\cancel{3 \times 1}$$
$$3 \times 2$$

$$-\frac{1}{6} - \frac{2}{6} - \frac{3}{6} =$$

$$\begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -3/6 & -8/6 \\ 6/6 & 12/6 \\ -3/6 & -6/6 \end{bmatrix}$$

$$7/6 + \frac{4}{6} + \frac{6}{6}$$

$$\frac{5}{3} +$$

$$3/6 + 4/6$$

$$\frac{5}{3} + \frac{10}{3} + \frac{15}{3}$$

- A has Independent coloums.

$A^T A$ is Invertible

$$Ax = 0$$

$$A^T A x = 0$$

Multiply x^T

$$x^T A^T A x = 0$$

$$(Ax)^T (Ax) = \|Ax\|^2$$

$$\Rightarrow Ax = 0$$

$$\frac{13}{6} + 26 + 39$$

26
13
78

$$7/6 + 14/6 + 21$$

21
14
35

Orthonormal Vectors

Orthogonal Matrix and Gram Schmidt

Take 2 orthogonal vectors and $x \perp y \rightarrow 0$

$$q_i^T q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$Q = \begin{bmatrix} \vdots & & \vdots \\ Q_1 & \cdots & Q_n \\ \vdots & & \vdots \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} Q^T & \cdots \\ \vdots & \\ Q_n^T & \cdots \end{bmatrix} \begin{bmatrix} \vdots & & \vdots \\ q_1 & \cdots & q_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Property

$$Q^T = Q^{-1}$$

$$Q^T Q = I$$

for orthogonal matrices

[Square]

$Q^TQ = I$

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Orthonormal

Orthonomalmatrix =
Eg:

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}}_{\text{dotproduct} = 0}$$

$\longrightarrow 1 \times -2 = -2$

$\longrightarrow 2 \times -1 = -2$

$\longrightarrow 2 \times 2 = 4$

$\longrightarrow -4 + 4 = 0$

Normalyzing $\longrightarrow$ $Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

Formula for projection matrix P

$$P = A(A^TA)^{-1}A^T$$

$A = Q$

$\Rightarrow$ $Q(Q^TQ)^{-1}Q^T = P$

ie, $\boxed{P = QQ^T}$

is this square / rectangular matrix //

If it is square matrix

$\boxed{P = I}$

- Q is symmetric
- square of projects brings back to same pt

$$P^2 = QQ^TQQ^T$$

$$\boxed{P^2 = QQ^T}$$

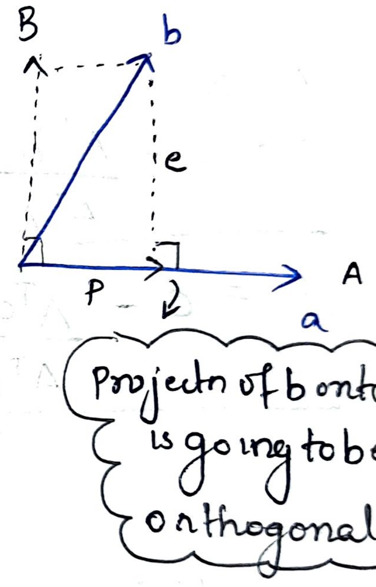
$A = Q$.

$$A^T A \hat{x} = A^T b$$

$$A = Q \Rightarrow Q^T Q \hat{x} = Q^T b$$

$$\therefore \boxed{\hat{x} = Q^T b}$$

Gram Schmidt



given : vectors a, b which are independent

step1 : Convert to orthogonal vector A, B.

step2 : Convert to orthonormal vector q_1, q_2

where $q_1 = \frac{A}{\|A\|}$ and $q_2 = \frac{B}{\|B\|}$

ie, $a, b \Rightarrow A, B \Rightarrow q_1, q_2$

$$B = b - \square A$$

$$B = b - \frac{A^T b}{A^T A} A$$

Step1:

Let $A = a$

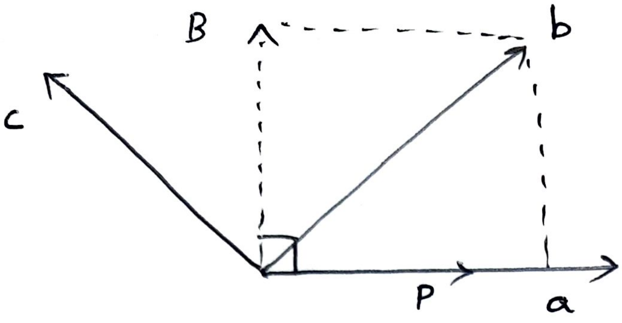
Then : $\boxed{B = b - \frac{A^T b}{A^T A} A}$

$A^T B = 0$

$\Rightarrow$ A and B are orthogonal.

Verifying : $A^T B = A^T b - \frac{A^T A^T b A}{A^T A} = 0$

Now Consider a 3rd vector c



Then a, b, c $\Rightarrow$ A, B, C $\Rightarrow$ q_1, q_2, q_3

$A = a$

$B = b - \frac{A^T b}{A^T A} A$

$$\boxed{C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B} \Rightarrow \begin{matrix} C \perp A \\ C \perp B \end{matrix}$$

$q_1 = \frac{A}{\|A\|}$

$q_2 = \frac{B}{\|B\|}$

$q_3 = \frac{C}{\|C\|}$

Q $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ are they orthogonal.

$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \square \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$B = b - \square A$

$\frac{A^T b}{A^T A} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{3}{3} = \underline{\underline{\frac{3}{3}}}$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A^T \cdot B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$\|A\| = \sqrt{1^2 +$

$\|B\| = \sqrt{\ }$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$Q = [q_1\ q_2]$$

$$= \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

dot pr = 0

length of vec

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \xrightarrow[\text{Convert into}]{} Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

Orthonormal

→ Is there any diff b/w clm space of A & Q ?

No

Q check if $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$ is orthonormal

→ check if it is orthogonal

ie, dot product = 0 . ie, $\left.\begin{array}{l} 1\times1 = 1 \\ 1\times0 = 0 \\ 1\times3 = 3 \end{array}\right\}$ Sum = 4 $\neq 0$

→ check length of each vector is 1

$1^2+1^2+1^2 = 3 \neq 1$

$1^2+0^2+3^2 = 10 \neq 1$

∴ not orthonormal.

★ There is no diff. b/w $C(A)$ & $C(Q)$

★ In elimination, $A = LU$

Similarly for Gram-Schmidt $\boxed{A = QR}$

Q → othonormal

R → Something

Determinants.

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Singular matrix, ~~Ann~~ $|A| = 0$ //

∗/. (determinant change sign, if a row/clm are exchanged.)

$$|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 //$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$$

∗/.
$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

∗/.
$$A = \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$4 - 0 + 0 - 1$ $\quad$ $8 - 8 + 1$

$= 3$

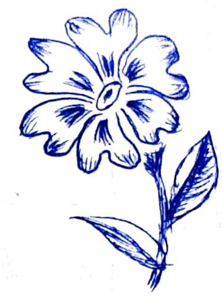
∗/.

If 2 rows of A are equal, then $|A| = 0$. $\quad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$

∗/. Subtracting a xle from 1 row from another row

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

A matrix with a row of 0 has $|A| = 0$



$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$$

If A is Δar, $|A| = a_{11} a_{22} \cdots a_{nn}$
= Prod. of diag. entries.

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \qquad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$$

Pivot of 2x2 matrix if $a \neq 0$ are a & $d - (c/a)b$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d-(c/a)b \end{vmatrix} = \underline{\underline{ad - bc}}.$$

Det of AB = |A| times |B| ie, $|AB| = |A|\,|B|$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

The transpose A^T has the same det as A

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \underline{\underline{ad - bc}}$$

$\underline{\underline{|AB| = |A|\,|B|}}$

Eigen Vector

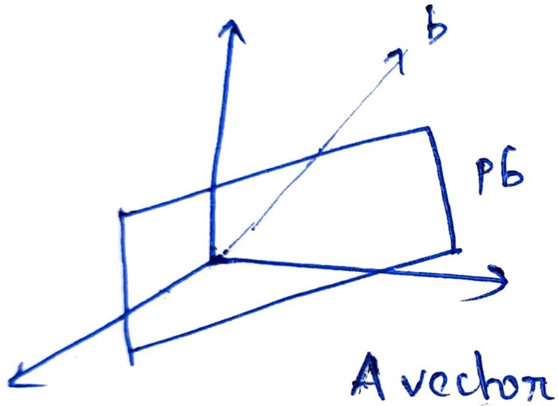
Any dimension

$$\boxed{Ax = \lambda x}$$

matrix

eigen vector

Eigen value.



What vector remains in same plane even after projection?

A vector that lies in the plane to be projected on to. [ie, the projection will give back the same vector]

$Px = \lambda x$ $(\lambda = 1)$ ⟶ other than this special case, Project will take the vector away from the plane it was originally in

0 $(\lambda = 0)$

$A = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$

Q) $Ax = \lambda x$ $(\lambda = 1)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \Rightarrow$$

$$\boxed{\underset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \underset{\lambda x_1}{\overset{x_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}}$$

Since $\lambda = 1$ $\therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Q) $Ax = \lambda x$ $(\lambda = -1)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x = -1\, x$$

$$\boxed{\underset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} -1 \\ +1 \end{bmatrix}} = \underset{\lambda x_2}{\overset{x_2}{\begin{bmatrix} -1 \\ +1 \end{bmatrix}}}}$$

$\therefore$ Since $\lambda = -1$ $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

[Eigenvectors are orthogonal]

We find that x_1 & x_2 are orthogonal.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$1 \times -1 + 1 \times 1$

$= -1 + 1 = 0$

&

$\lambda_1 + \lambda_2 = 1 + -1 = 0$

$\lambda_1 \lambda_2 = 1 \times -1 = -1$

$\lambda_1 \lambda_2 = |A|$

x_1, x_2 are orthogonal

$\lambda_1 + \lambda_2 = 0$

$\lambda_1 \lambda_2 = -1$

$\lambda_1 \lambda_2 = |A|$

$$Ax = \lambda x$$

scalar

$$Ax - \lambda x = 0$$

$$(A - \lambda I)\, x = 0$$

det $A - \lambda I$

trivial case: $x=0$, above eq. will always true.

Other case : $(A - \lambda I) = 0$

must be non-invertible

[singular]

Symetric

Q. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$(A - \lambda I)\, x = 0$ same as $|A - \lambda I|$

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(3-\lambda) - 1 = 0$$

$$9 - 3\lambda - 3\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

8 → determinant

6λ → sum of diagonal

$$\lambda = 4, 2$$

when $\lambda = 4$

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot X = 0$$

when $\lambda = 2$

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot X = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0 \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ +1 \end{bmatrix} = 0$$

(*) $$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}}_{\text{eigen vectors Same, but EignValue changed...}} \Rightarrow (A + 3I)x = (\lambda + 3)x$$

$\lambda_1 = 1$ $\quad$ $\lambda_1 = 4$ $\quad$ $1 + 3 = 4$

$\lambda_2 = -1$ $\quad$ $\lambda_2 = 2$ $\quad$ $-1 + 3 = 2$

$*$ $Ax = \lambda x$

$Bx = \alpha x$

$(A+B)x = (\lambda + \alpha) x$

Non Symetric

$*$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix} = \underset{Q}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$

rotatnmatrix

$\mathrm{Det}(Q - \lambda I) = 0$

$x = \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \sqrt{-1} = \underline{\underline{i, -i}}$ $\rightarrow$ Complex soln [for NonSymetric Matrix]

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ i.e, $\underline{\underline{Q^T = -Q}}$ here

Q) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

$\text{Dec}(Q - \lambda I) = 0 \rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}$

$x = \begin{vmatrix} 3-\lambda & 0 \\ -1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(3-\lambda) = 0$

$9 - 3\lambda - 3\lambda + \lambda^2 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

S:- -6
P:- 9

$\lambda^2 - 3\lambda - 3\lambda + 9 = 0$

$-3, -3$

$\lambda(\lambda - 3) - 3(\lambda - 3) = 0$

$\underline{\underline{\lambda = 3, 3}}$

$(A - \lambda I)\, x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \Rightarrow x = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

When $\lambda_1 = 3$ $\lambda_2 = 3$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

basis for x

x_1 & x_2 are same.
x_1 & x_2 are not indep (dep)
x_1 & x_2 are not orthogonal ... $\{1 \times 1 - 0 \times 0\} \neq 0\}$

$\therefore$ This forms a special soln $\rightarrow$ Degenerate Solution