Homework 1
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1. a) $\langle x, y \rangle = x_1 \cdot y_1 + x_2 y_2$

b) $\|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2 = 2(\|x\|^2 + \|y\|^2)$

c) $\|x+y\|^2 + \|x-y\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) + (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) =$

$= \vec{x}^2 + 2\vec{x}\vec{y} + \vec{y}^2 + \vec{x}^2 - 2\vec{x}\vec{y} + \vec{y}^2 = \|x\|^2 + 2\cdot\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2\cdot\langle x, y\rangle + \|y\|^2 =$

$= 2\|x\|^2 + 2\|y\|^2 = 2(\|x\|^2 + \|y\|^2)$

d) $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 - \|x\|^2 + 2\langle x, y\rangle - \|y\|^2) =$

$= \frac{1}{4}\cdot(2\cdot\langle x, y\rangle + 2\cdot\langle x, y\rangle) = \langle x, y\rangle$

2. a) $f(x,y) = x^2 + 2xy + y^2$ $\qquad\qquad$ $x^2 + y = 1$

$g(x,y) = x^2 + y^2$

$$\nabla f(x,y) = \nabla(x^2 + 2xy + y^2) = \begin{bmatrix} 2x+2y \\ 2y+2x \end{bmatrix}$$

$$\nabla g(x,y) = \nabla(x^2+y^2) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix} = \lambda \cdot \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{cases} 2x+2y = \lambda \cdot 2x \\ 2x+2y = \lambda \cdot 2y \\ x^2+y^2 = 1 \end{cases} \Rightarrow x = y \Rightarrow \begin{array}{l} x^2 = 1 - y^2 \\ x^2 = 1 - x^2 \\ 2x^2 = 1 \end{array} \Rightarrow \underline{x = y = \frac{1}{\sqrt{2}}}$$

$\Rightarrow$ $x + y = \lambda \cdot x$

$x + y = \sqrt{2} + \sqrt{2} = \lambda \cdot \sqrt{2}$

$\lambda = \frac{2\sqrt{2}}{\sqrt{2}} = \underline{2}$

b) $f(x,y) = x^2 + 2xy + y^2$ $\qquad x + y = 1$

$g(x,y) = x + y$

$\nabla f(x,y) = \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\nabla g(x,y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{cases} 2x+2y = \lambda \\ 2x+2y = \lambda \\ x+y = 1 \end{cases} \Rightarrow \begin{cases} 2(x+y) = \lambda \\ x+y = 1 \end{cases} \Rightarrow 2 \cdot 1 = \underline{\lambda = 2}$

3. A) $\frac{1}{5}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \right)^T \cdot \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1. & 1 \end{bmatrix} \right) =$

$= \frac{1}{5}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right)^T \cdot \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \right) =$

$= \frac{1}{5} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = R_{A;j}$

```
>>> import numpy as np
>>> x = np.array([1,1,1,1,1,1])
>>> y = np.array([1,1,1,1,1,1])
>>> print (np.cov(x, y))
[[ 0.  0.]
 [ 0.  0.]]
```

B)

$$\frac{1}{5}\left(\begin{bmatrix}1&0\\1&0\\1&0\\0&1\\0&1\\0&1\end{bmatrix}-\begin{bmatrix}1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\end{bmatrix}\right)^T\cdot\left(\begin{bmatrix}1&0\\1&0\\1&0\\0&1\\0&1\\0&1\end{bmatrix}-\begin{bmatrix}1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\end{bmatrix}\right)=$$

$$=\frac{1}{5}\begin{bmatrix}1/2&1/2&1/2&-1/2&-1/2&-1/2\\-1/2&-1/2&-1/2&1/2&1/2&1/2\end{bmatrix}\begin{bmatrix}1/2&-1/2\\1/2&-1/2\\1/2&-1/2\\-1/2&1/2\\-1/2&1/2\\-1/2&1/2\end{bmatrix}=\frac{1}{5}\cdot\begin{bmatrix}(1/4)\cdot 6&-\frac{1}{4}\cdot 6\\-\frac{1}{4}\cdot 6&\frac{1}{4}\cdot 6\end{bmatrix}=$$

$$=\frac{1}{5}\begin{bmatrix}\frac{3}{2}&-\frac{3}{2}\\-\frac{3}{2}&\frac{3}{2}\end{bmatrix}=\begin{bmatrix}3/10&-3/10\\-3/10&3/10\end{bmatrix}=R_{B,ij}$$

```
>>> import numpy as np
>>> x = np.array([1,1,1,0,0,0])
>>> y = np.array([0,0,0,1,1,1])
>>> print (np.cov(x, y))
[[ 0.3 -0.3]
 [-0.3  0.3]]
```

c) $$\frac{1}{5}\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1/2&1/2\end{bmatrix}\right)^T\cdot\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1/2&1/2\end{bmatrix}\right)=$$

$$=\frac{1}{5}\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\end{bmatrix}\right)^T\cdot\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\\1/2&1/2\end{bmatrix}\right)=$$

$$=\frac{1}{5}\cdot\begin{bmatrix}1/2&1/2&1/2&-1/2&-1/2&-1/2\\1/2&1/2&1/2&-1/2&-1/2&-1/2\end{bmatrix}\begin{bmatrix}1/2&1/2\\1/2&1/2\\1/2&1/2\\-1/2&-1/2\\-1/2&-1/2\\-1/2&-1/2\end{bmatrix}=$$

$$=\frac{1}{5}\begin{bmatrix}\frac{1}{4}\cdot 6&\frac{1}{4}\cdot 6\\\frac{1}{4}\cdot 6&\frac{1}{4}\cdot 6\end{bmatrix}=\begin{bmatrix}3/10&3/10\\3/10&3/10\end{bmatrix}=R_{c,ij}$$

```
>>> import numpy as np
>>> x = np.array([1,1,1,0,0,0])
>>> y = np.array([1,1,1,0,0,0])
>>> print (np.cov(x, y))
[[ 0.3  0.3]
 [ 0.3  0.3]]
```

$$D)\quad \frac{1}{4}\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\frac{3}{5}&\frac{3}{5}\end{bmatrix}\right)^{T}\cdot\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}3/5&\frac{3}{5}\end{bmatrix}\right)=$$

$$=\frac{1}{4}\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}3/5&3/5\\3/5&3/5\\3/5&3/5\\3/5&3/5\\3/5&3/5\end{bmatrix}\right)^{T}\cdot\left(\begin{bmatrix}1&1\\1&1\\1&1\\0&0\\0&0\end{bmatrix}-\begin{bmatrix}3/5&3/5\\3/5&3/5\\3/5&3/5\\3/5&3/5\\3/5&3/5\end{bmatrix}\right)=$$

$$=\frac{1}{4}\cdot\begin{bmatrix}2/5&2/5&2/5&-3/5&-3/5\\2/5&2/5&2/5&-3/5&-3/5\end{bmatrix}\begin{bmatrix}2/5&2/5\\2/5&2/5\\2/5&2/5\\-3/5&-3/5\\-3/5&-3/5\end{bmatrix}=$$

$$=\frac{1}{4}\begin{bmatrix}\frac{30}{25}&\frac{30}{25}\\\frac{30}{25}&\frac{30}{25}\end{bmatrix}=\begin{bmatrix}\frac{3}{10}&\frac{3}{10}\\\frac{3}{10}&\frac{3}{10}\end{bmatrix}=R_{D,ij}$$

```
>>> import numpy as np
>>> x = np.array([1,1,1,0,0])
>>> y = np.array([1,1,1,0,0])
>>> print (np.cov(x, y))
[[ 0.3  0.3]
 [ 0.3  0.3]]
```