

FINAL Study Guide

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1 The Foundations: Logic and Proofs

1.1 Propositional Logic

Examples of a proposition:

$p(x) = x$ is a cat.

$q(x) = x$ has fur.

1. Negation, $\neg p(x)$, changing the statement to x is not a cat
2. Conjunction, $p(x) \wedge q(x)$, changing the statement to say x is a cat and it has fur.
3. Disjunction, $p(x) \vee q(x)$, changing the statement to say x is a cat and it does not have fur.
4. Exclusive Or, $p(x) \oplus q(x)$, where the statement is true only when exactly one of $p(x)$ or $q(x)$ is true, otherwise the statement is false.
5. Conditional Statement, $p(x) \rightarrow q(x)$, where the statement is read "If p, then q" the statement is false when p is true and q is false, otherwise the statement is true.
6. Bi-Conditional Statement, $p(x) \leftrightarrow q(x)$, where the statement is true if the values of p and q match. The statement is read as any of the following;
 - (a) "p if and only if q"
 - (b) "p is necessary and sufficient for q"
 - (c) "if p then q, and conversely"
 - (d) "p iff q"
7. Converse, Contrapositive, and Inverse

Given $p \rightarrow q$;

 - (a) Converse: $q \rightarrow p$
 - (b) Contrapositive: $\neg q \rightarrow \neg p$
 - (c) Inverse: $\neg p \rightarrow \neg q$

Definitions

1. Hypothesis - p is considered the hypothesis in the statement $p(x) \rightarrow q(x)$.
2. Conclusion- q is considered the conclusion in the statement $p(x) \rightarrow q(x)$.
3. Bi-Implications - Another way to express "Bi-Conditional statements"

1.2 Propositional Equivalences

1. Logically Equivalent, $p(x) \equiv q(x)$, when two statements share the same truth values then the statement is said to be logically equivalent.

2. Identity Laws

$$p(x) \wedge T \equiv p(x)$$

$$p(x) \vee F \equiv p(x)$$

3. Domination Laws

$$p(x) \vee T \equiv T$$

$$p(x) \wedge F \equiv F$$

4. Idempotent Laws

$$p(x) \vee p(x) \equiv p(x)$$

$$p(x) \wedge p(x) \equiv p(x)$$

5. Double Negation Law

$$\neg(\neg p(x)) \equiv p(x)$$

6. Commutative Laws

$$p(x) \vee q(x) \equiv q(x) \vee p(x)$$

$$p(x) \wedge q(x) \equiv q(x) \wedge p(x)$$

7. Associative Laws

$$(p(x) \vee q(x)) \vee r(x) \equiv p(x) \vee (q(x) \vee r(x))$$

$$(p(x) \wedge q(x)) \wedge r(x) \equiv p(x) \wedge (q(x) \wedge r(x))$$

8. Distributive Laws

$$p(x) \vee (q(x) \wedge r(x)) \equiv (p(x) \vee q(x)) \wedge (p(x) \vee r(x))$$

$$p(x) \wedge (q(x) \vee r(x)) \equiv (p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$$

9. De Morgan's Laws

$$\neg(p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x)$$

$$\neg(p(x) \vee q(x)) \equiv \neg p(x) \wedge \neg q(x)$$

10. Absorption Laws

$$p(x) \vee (p(x) \wedge q(x)) \equiv p(x)$$

$$p(x) \wedge (p(x) \vee q(x)) \equiv p(x)$$

11. Negation Laws

$$p(x) \vee \neg p(x) \equiv T$$

$$p(x) \wedge \neg p(x) \equiv F$$

Definitions

1. Tautology - When all cases in the statement are determined to be true, it is said to be a tautology.
2. Contradiction- When all cases in the statement are determined to be false, it is said to be a contradiction.
3. Contingency - When the statement is neither a tautology, nor a contradiction then it is said to be a contingency.

1.3 Predicates and Quantifiers

1. Universal Quantification, $\forall p(x)$ read as, "For all x , $p(x)$."
2. Existential Quantification, $\exists p(x)$ read as, "There exists an element x , that $p(x)$."
3. De Morgan's Laws for Quantifiers:

$$\neg \forall p(x) \equiv \exists \neg p(x)$$

$$\neg \exists p(x) \equiv \forall \neg p(x)$$

Definitions

1. Quantification - Used to express the extent that a predicate is true. In English, the words; all, some, many, none, and few are used in quantifications.
2. Counterexample - When there is a value, x in which $\forall p(x)$ is false, then that value of x is called a counterexample of $\forall p(x)$.

1.4 Rules of Inference (ROI)

Modus Ponens

$$\frac{p \quad p \rightarrow q}{\therefore q}$$

Modus Tollens

$$\frac{\neg q \quad p \rightarrow q}{\therefore \neg p}$$

Hypothetical Syllogism

$$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$$

Disjunctive Syllogism

$$\frac{p \vee q \quad \neg p}{\therefore q}$$

Addition

$$\frac{p}{\therefore p \vee q}$$

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Conjunction

$$\frac{p \quad q}{\therefore p \wedge q}$$

Universal Instantiation

$$\frac{\forall p(x)}{\therefore p(c)}$$

Universal Generalization

$$\frac{p(c) \text{ for an arbitrary } c}{\therefore \forall p(x)}$$

Existential Instantiation

$$\frac{\exists p(x)}{\therefore p(c) \text{ for some element } c}$$

Existential Generalization

$$\frac{p(c) \text{ for some element } c}{\therefore \exists p(x)}$$

Definitions

1. Argument - Sequence of propositions.
2. Premises - All propositions in the argument with the exclusion of the conclusion
3. Conclusion - The final proposition in the argument.
4. Argument Form - A sequence of compound propositions involving propositional variables.
5. Valid - A form that makes it impossible for the premises to be true and the conclusion nevertheless to be false.

1.5 Proofs

1. direct proof:

Directly prove that if n is an odd integer then n^2 is also an odd integer.

PROOF (\rightarrow):

An odd number is denoted by the equation, $2k + 1$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

$2(2k^2 + 2k) + 1$, is another form of $2k + 1$.

Therefore, if n is an odd integer then n^2 is also an odd integer.

2. Proofs by Contradiction:

Definitions

1. Theorem - A statement that can be shown to be true.
2. Proof - A demonstration that a theorem is true.
3. Axioms (or postulates) - Statements we assume to be true.
4. Lemma - A less important theorem that is helpful in the proof of other results.
5. Corollary - A theorem that can be established directly from a theorem that has been proved.
6. Conjecture - A statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem.

2 Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets

A set is an unordered collection of objects, called elements or members of the set.

We write $a \in A$ to denote that a is an element of the set A .

We write $a \notin A$ denotes that a is not an element of the set A

Well known sets include;

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of integers

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, the set of positive integers

$Q = \{p/q | p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$, the set of rational numbers \mathbb{R} , the set of real numbers

\mathbb{R}^+ , the set of positive real numbers

\mathbb{C} , the set of complex numbers.

Ω , the universal set

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B .

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Every set (S) includes the following two sets;

$$S \subseteq S \quad \text{AND} \quad \emptyset \subseteq S$$

A set (S) is said to be a finite set when it has exactly n distinct elements and n is said to be the cardinality of S , denoted as $|S|$

The Power Set of a set (S) is the set of all subsets of S . It is denoted as $\mathcal{P}(S)$

2.2 Set Operations

Let A and B be sets. The Cartesian product of A and B, denoted as $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

Two sets are called disjoint if their intersection is the empty set.

Let A and B be sets. The difference (complement) of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

The complement of the set A, denoted by \bar{A} , is the complement of A with respect to Ω .
Therefore, the complement of the set A is $\Omega - A$.

2.3 Set Identities

1. Identity Laws

$$A \cap \Omega = A$$

$$A \cup \emptyset = A$$

2. Domination Laws

$$A \cup \Omega = \Omega$$

$$A \cap \emptyset = \emptyset$$

3. Idempotent Laws

$$A \cup A = A$$

$$A \cap A = A$$

4. Complementation Laws

$$\overline{(\overline{A})} = A$$

5. Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

6. Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

7. Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

8. De Morgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

9. Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

10. Complement Laws

$$A \cup \overline{A} = \Omega$$

$$A \cap \overline{A} = \emptyset$$

2.4 Functions

Given A function f from A to B

1. One To One: For a function to be one to one or every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
2. Onto: For a function to be onto $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f .

Definitions:

1. Function - Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \rightarrow B$.
2. Domain - The set of possible values of the variables of a function.
3. Range - The set of all output values of a function.
4. Surjective - Another way to say onto.
5. Injective - Another way to say one to one.
6. One-to-one Correspondence (Bijection) - Both onto and one to one.

2.5 Sequences and Summations

2.6 Matrices

3 Induction and Recursion

3.1 Divisibility and Modular Arithmetic

3.2 Mathematical Induction

3.3 Strong Induction

3.4 Recursive Algorithms

4 Counting

4.1 The Pigeonhole Principle

1. The Pigeonhole Principle -If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

4.2 Generalized Permutations and Combinations

1. Permutation Without Repetition - $P(n, r) = \frac{n!}{(n-r)!}$
2. Combination Without Repetition - $\binom{n}{k} = \frac{n!}{(n-k)! \times k!}$
3. Permutation With Repetition - $P(n, r) = n^r$
4. Combination With Repetition - $\binom{n}{k} = \frac{(n+r-1)!}{r!(n-1)!}$

5 Relation

5.1 Relations

5.2 n-ary Relations

5.3 Closures of Relations