FINAL Study Guide

Andrew Reed
Discrete Final

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Contents

1 The Foundations: Logic and Proofs

1.1 Propositional Logic

Examples of a proposition:
$p(x) = x$ is a cat.
$q(x) = x$ has fur.

1. Negation, $\neg p(x)$, changing the statement to x is not a cat
2. Conjunction, $p(x) \wedge q(x)$, changing the statement to say x is a cat and it has fur.
3. Disjunction, $p(x) \vee q(x)$, changing the statement to say x is a cat and it does not have fur.
4. Exclusive Or, $p(x) \oplus q(x)$, where the statement is true only when exactly one of $p(x)$ or $q(x)$ is true, otherwise the statement is false.
5. Conditional Statement, $p(x) \rightarrow q(x)$, where the statement is read "If p, then q" the statement is false when p is true and q is false, otherwise the statement is true.
6. Bi-Conditional Statement, $p(x) \leftrightarrow q(x)$, where the statement is true if the values of p and q match. The statement is read as any of the following;
   (a) "p if and only if q"
   (b) "p is necessary and sufficient for q"
   (c) "if p then q, and conversely"
   (d) "p iff q"

Definitions

1. Hypothesis - p is considered the hypothesis in the statement $p(x) \rightarrow q(x)$.
2. Conclusion- q is considered the conclusion in the statement $p(x) \rightarrow q(x)$.
3. Bi-Implications - Another way to express "Bi-Conditional statements"

1.2 Propositional Equivalences

1. Logically Equivalent, $p(x) \equiv q(x)$, when two statements share the same truth values then the statement is said to be logically equivalent.

2. Identity Laws
$$p(x) \wedge T \equiv p(x)$$
$$p(x) \vee F \equiv p(x)$$

3. Domination Laws
$$p(x) \vee T \equiv T$$
$$p(x) \wedge F \equiv F$$

4. Idempotent Laws
$$p(x) \vee p(x) \equiv p(x)$$
$$p(x) \wedge p(x) \equiv p(x)$$

5. Double Negation Law
$$\neg(\neg p(x)) \equiv p(x)$$

6. Commutative Laws
$$p(x) \vee q(x) \equiv q(x) \vee p(x)$$
$$p(x) \wedge q(x) \equiv q(x) \wedge p(x)$$

7. Associative Laws
$$(p(x) \vee q(x)) \vee r(x) \equiv p(x) \vee (q(x) \vee r(x))$$
$$(p(x) \wedge q(x)) \wedge r(x) \equiv p(x) \wedge (q(x) \wedge r(x))$$

8. Distributive Laws
$$p(x) \vee (q(x) \wedge r(x)) \equiv (p(x) \vee q(x)) \wedge (p(x) \vee r(x))$$
$$p(x) \wedge (q(x) \vee r(x)) \equiv (p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$$

9. De Morgan's Laws
$$\neg\left(p(x) \wedge q(x)\right) \equiv \neg p(x) \vee \neg q(x)$$
$$\neg\left(p(x) \vee q(x)\right) \equiv \neg p(x) \wedge \neg q(x)$$

10. Absorption Laws
$$p(x) \vee (p(x) \wedge q(x)) \equiv p(x)$$
$$p(x) \wedge (p(x) \vee q(x)) \equiv p(x)$$

11. Negation Laws
$$p(x) \vee \neg p(x) \equiv T$$
$$p(x) \wedge \neg p(x) \equiv F$$

Definitions

1. Tautology - When all cases in the statement are determined to be true, it is said to be a tautology.
2. Contradiction- When all cases in the statement are determined to be false, it is said to be a contradiction.
3. Contingency - When the statement is neither a tautology, nor a contradiction then it is said to be a contingency.

1.3 Predicates and Quantifiers

1. Universal Quantification, $\forall p(x)$ read as, "For all x, $p(x)$."
2. Existential Quantification, $\exists p(x)$ read as, "There exists an element x, that $p(x)$."
3. De Morgan's Laws for Quantifiers:

$$\neg \forall p(x) \equiv \exists \neg p(x)$$

$$\neg \exists p(x) \equiv \forall \neg p(x)$$

Definitions

1. Quantification - Used to express the extent that a predicate is true. In English, the words; all, some, many, none, and few are used in quantifications.
2. Counterexample - When there is a value, x in which $\forall p(x)$ is false, then that value of x is called a counterexample of $\forall p(x)$.

1.4 Rules of Inference (ROI)

<u>Modus Ponens</u>

$$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$$

<u>Modus Tollens</u>

$$\begin{array}{c} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

<u>Hypothetical Syllogism</u>

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

<u>Disjunctive Syllogism</u>

$$\begin{array}{c} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

<u>Addition</u>

$$\begin{array}{c} p \\ \hline \therefore p \lor q \end{array}$$

<u>Simplification</u>

$$\begin{array}{c} p \land q \\ \hline \therefore p \end{array}$$

<u>Conjunction</u>

$$\begin{array}{c} p \\ q \\ \hline \therefore p \land q \end{array}$$

<u>Universal Instantiation</u>

$$\begin{array}{c} \forall p(x) \\ \hline \therefore p(c) \end{array}$$

<u>Universal Generalization</u>

$$\begin{array}{c} p(c) \text{ for an arbitrary c} \\ \hline \therefore \forall p(x) \end{array}$$

<u>Existential Instantiation</u>

$$\begin{array}{c} \exists p(x) \\ \hline \therefore p(c) \text{ for some element c} \end{array}$$

<u>Existential Generalization</u>

$$\begin{array}{c} p(c) \text{ for some element c} \\ \hline \therefore \exists p(x) \end{array}$$

Definitions

1. <u>Argument</u> - Sequence of propositions.
2. <u>Premises</u> - All propositions in the argument with the exclusion of the conclusion
3. <u>Conclusion</u> - The final proposition in the argument.
4. <u>Argument Form</u> - A sequence of compound propositions involving propositional variables.
5. <u>Valid</u> - A form that makes it impossible for the premises to be true and the conclusion nevertheless to be false.

1.5 Proofs

1. direct proof:

   Directly prove that if n is an odd integer then n^2 is also an odd integer.

   PROOF ($\rightarrow$):

   An odd number is denoted by the equation, $2k + 1$

   $$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1$$

   $2(2k^2 + 2) + 1$, is another form of 2k + 1.

   Therefore, if n is an odd integer then n^2 is also an odd integer.

2. Proofs by Contradiction:

Definitions

1. Theorem - A statement that can be shown to be true.
2. Proof - A demonstration that a theorem is true.
3. Axioms (or postulates) - Statements we assume to be true.
4. Lemma - A less important theorem that is helpful in the proof of other results.
5. Corollary - A theorem that can be established directly from a theorem that has been proved.
6. Conjecture - A statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem.

2 Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets

A set is an unordered collection of objects, called elements or members of the set.
We write $a \in A$ to denote that a is an element of the set A.
We write $a \notin A$ denotes that a is not an element of the set A

Well known sets include;
$\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of natural numbers
$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, the set of integers
$\mathbb{Z}+ = \{1, 2, 3, ...\}$, the set of positive integers
$Q = \{p/q|p \in \mathbb{Z},\ q \in \mathbb{Z},\ and\ q \neq 0\}$, the set of rational numbers R, the set of real numbers
$\mathbb{R}+$, the set of positive real numbers
$\mathbb{C}$, the set of complex numbers.
Ω, the universal set

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \nsubseteq B$, find a single x $\in$ A such that x $\notin$ B.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Every set (S) includes the following two sets;

$$S \subseteq S \qquad AND \qquad \emptyset \subseteq S$$

A set (S) is said to be a finite set when it has exactly n distinct elements and n is said to be the cardinality of S, denoted as $|S|$

The Power Set of a set (S) is the set of all subsets of S. It is denoted as $\wp(S)$

2.2 Set Operations

Let A and B be sets. The Cartesian product of A and B, denoted as $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

Let A and B be sets. The intersection of the sets A and B, denoted by A $\cap$ B, is the set containing those elements in both A and B.

Two sets are called disjoint if their intersection is the empty set.

Let A and B be sets. The difference (complement) of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

The complement of the set A, denoted by $\overline{\mathrm{A}}$, is the complement of A with respect to Ω . Therefore, the complement of the set A is Ω - A.

2.3 Set Identities

1. Identity Laws
$$A \cap \Omega = A$$
$$A \cup \emptyset = A$$

2. Domination Laws
$$A \cup \Omega = \Omega$$
$$A \cap \emptyset = \emptyset$$

3. Idempotent Laws
$$A \cup A = A$$
$$A \cap A = A$$

4. Complementation Laws
$$\overline{(\overline{\mathrm{A}})} = A$$

5. Commutative Laws
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

6. Associative Laws
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

7. Distributive Laws
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

8. De Morgan's Laws
$$\overline{\mathrm{A} \cup \mathrm{B}} = \overline{\mathrm{A}} \cap \overline{\mathrm{B}}$$
$$\overline{\mathrm{A} \cap \mathrm{B}} = \overline{\mathrm{A}} \cup \overline{\mathrm{B}}$$

9. Absorption Laws
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

10. Complement Laws
$$A \cup \overline{\mathrm{A}} = \Omega$$
$$A \cap \overline{\mathrm{A}} = \emptyset$$