FINAL Study Guide

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Discrete Final

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Contents

1 The Foundations: Logic and Proofs

1.1 Propositional Logic

Examples of a proposition:

$p(x) = x$ is a cat.
$q(x) = x$ has fur.

1. Negation, $\neg p(x)$, changing the statement to x is not a cat

2. Conjunction, $p(x) \wedge q(x)$, changing the statement to say x is a cat and it has fur.

3. Disjunction, $p(x) \vee q(x)$, changing the statement to say x is a cat and it does not have fur.

4. Exclusive Or, $p(x) \oplus q(x)$, where the statement is true only when exactly one of $p(x)$ or $q(x)$ is true, otherwise the statement is false.

5. Conditional Statement, $p(x) \rightarrow q(x)$, where the statement is read "If p, then q" the statement is false when p is true and q is false, otherwise the statement is true.

6. Bi-Conditional Statement, $p(x) \leftrightarrow q(x)$, where the statement is true if the values of p and q match. The statement is read as any of the following;

   (a) "p if and only if q"
   (b) "p is necessary and sufficient for q"
   (c) "if p then q, and conversely"
   (d) "p iff q"

Definitions

1. Hypothesis - p is considered the hypothesis in the statement $p(x) \rightarrow q(x)$.

2. Conclusion- q is considered the conclusion in the statement $p(x) \rightarrow q(x)$.

3. Bi-Implications - Another way to express "Bi-Conditional statements"

1.2 Propositional Equivalences

1. Logically Equivalent, $p(x) \equiv q(x)$, when two statements share the same truth values then the statement is said to be logically equivalent.

2. Identity Laws
$$p(x) \wedge T \equiv p(x)$$
$$p(x) \vee F \equiv p(x)$$

3. Domination Laws
$$p(x) \vee T \equiv T$$
$$p(x) \wedge F \equiv F$$

4. Idempotent Laws
$$p(x) \vee p(x) \equiv p(x)$$
$$p(x) \wedge p(x) \equiv p(x)$$

5. Double Negation Law
$$\neg(\neg p(x)) \equiv p(x)$$

6. Commutative Laws
$$p(x) \vee q(x) \equiv q(x) \vee p(x)$$
$$p(x) \wedge q(x) \equiv q(x) \wedge p(x)$$

7. Associative Laws
$$(p(x) \vee q(x)) \vee r(x) \equiv p(x) \vee (q(x) \vee r(x))$$
$$(p(x) \wedge q(x)) \wedge r(x) \equiv p(x) \wedge (q(x) \wedge r(x))$$

8. Distributive Laws
$$p(x) \vee (q(x) \wedge r(x)) \equiv (p(x) \vee q(x)) \wedge (p(x) \vee r(x))$$
$$p(x) \wedge (q(x) \vee r(x)) \equiv (p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$$

9. De Morgan's Laws
$$\neg\,(p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x)$$
$$\neg\,(p(x) \vee q(x)) \equiv \neg p(x) \wedge \neg q(x)$$

10. Absorption Laws
$$p(x) \vee (p(x) \wedge q(x)) \equiv p(x)$$
$$p(x) \wedge (p(x) \vee q(x)) \equiv p(x)$$

11. Negation Laws
$$p(x) \vee \neg p(x) \equiv T$$
$$p(x) \wedge \neg p(x) \equiv F$$

Definitions

1. <u>Tautology</u> - When all cases in the statement are determined to be true, it is said to be a tautology.

2. <u>Contradiction</u>- When all cases in the statement are determined to be false, it is said to be a contradiction.

3. <u>Contingency</u> - When the statement is neither a tautology, nor a contradiction then it is said to be a contingency.

1.3   Predicates and Quantifiers

1. <u>Universal Quantification</u>, $\forall p(x)$ read as, "For all x, $p(x)$."

2. <u>Existential Quantification</u>, $\exists p(x)$ read as, "There exists an element x, that $p(x)$."

3. <u>De Morgan's Laws for Quantifiers</u>:

$$\neg \forall p(x) \equiv \exists \neg p(x)$$

$$\neg \exists p(x) \equiv \forall \neg p(x)$$

Definitions

1. <u>Quantification</u> - Used to express the extent that a predicate is true. In English, the words; all, some, many, none, and few are used in quantifications.

2. <u>Counterexample</u> - When there is a value, x in which $\forall p(x)$ is false, then that value of x is called a counterexample of $\forall p(x)$.

1.4   Rules of Inference (ROI)

Modus Ponens

$$p$$
$$\underline{p \rightarrow q}$$
$$\therefore q$$

Modus Tollens

$$\neg q$$
$$\underline{p \rightarrow q}$$
$$\therefore \neg p$$

Hypothetical Syllogism

$$p \rightarrow q$$
$$\underline{q \rightarrow r}$$
$$\therefore p \rightarrow r$$

Disjunctive Syllogism

$$p \vee q$$
$$\underline{\neg p}$$
$$\therefore q$$

Addition

$$\underline{p}$$
$$\therefore p \vee q$$

Simplification

$$\underline{p \wedge q}$$
$$\therefore p$$

Conjunction

$$p$$
$$\underline{q}$$
$$\therefore p \wedge q$$

Universal Instantiation

$$\underline{\forall p(x)}$$
$$\therefore p(c)$$

Universal Generalization

$$p(c) \text{ for an arbitrary } c$$
$$\therefore \forall p(x)$$

Existential Instantiation

$$\underline{\exists p(x)}$$
$$\therefore p(c) \text{ for some element } c$$

Existential Generalization

$$p(c) \text{ for some element } c$$
$$\therefore \exists p(x)$$

Definitions

1. <u>Argument</u> - Sequence of propositions.

2. <u>Premises</u> - All propositions in the argument with the exclusion of the conclusion

3. <u>Conclusion</u> - The final proposition in the argument.

4. <u>Argument Form</u> - A sequence of compound propositions involving propositional variables.

5. <u>Valid</u> - A form that makes it impossible for the premises to be true and the conclusion nevertheless to be false.

1.5 Proofs

1. <u>direct proof</u>:

   Directly prove that if n is an odd integer then n^2 is also an odd integer.

   PROOF ($\rightarrow$):

   An odd number is denoted by the equation, $2k + 1$

   $$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1$$

   $2(2k^2 + 2) + 1$, is another form of 2k + 1.

   Therefore, if n is an odd integer then n^2 is also an odd integer.

2. <u>Proofs by Contradiction</u>:

   Definitions

1. <u>Theorem</u> - A statement that can be shown to be true.

2. <u>Proof</u> - A demonstration that a theorem is true.

3. <u>Axioms</u> (or postulates) - Statements we assume to be true.

4. <u>Lemma</u> - A less important theorem that is helpful in the proof of other results.

5. <u>Corollary</u> - A theorem that can be established directly from a theorem that has been proved.

6. <u>Conjecture</u> - A statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert. When a proof of a conjecture is found, the conjecture becomes a theorem.

2 Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets

A set is an unordered collection of objects, called elements or members of the set.

   We write $a \in A$ to denote that a is an element of the set A.

   We write $a \notin A$ denotes that a is not an element of the set A