Linear Algebra Final Study Guide

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In the given material that follows in this study guide these definitions will be used multiple times throughout. As such let...

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

1 Vectors

1.1 The Geometry and Algebra of Vectors

Definition: Vector addiction is defined as such,

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_k + v_k]$$

Definition: Vector scalar Multiplication is defined as such,

$$c\mathbf{u} = [c \times u_1, c \times u_2, \ldots, c \times u_k]$$

Algebraic Properties of Vectors in $\mathcal{R}^n$;

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + 0 = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = 0$
5. $c(\mathbf{u} + \mathbf{v}) = c \times \mathbf{u} + c \times \mathbf{v}$
6. $(c + d)\mathbf{u} = c \times \mathbf{u} + d \times \mathbf{u}$
7. $c(d \times \mathbf{u}) = cd \times \mathbf{u}$
8. $1 \times \mathbf{u} = \mathbf{u}$

Definition: A vector $\mathbf{v}$ is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there are scalars, $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1 \times \mathbf{v}_1 + c_2 \times \mathbf{v}_2 + \cdots + c_k + \mathbf{v}_k$.

1.2 Length and Angle: The Dot Product

Definition: The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \cdot \mathbf{v}$, and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 \times v_1 + u_2 \times v_2 + \cdots + u_k \times v_k$$

Theorem: Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathcal{R}^n$ and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w})$
3. $(c \cdot \mathbf{u}) \cdot \mathbf{v} = c \cdot (\mathbf{u} \cdot \mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$
5. $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Definition: The **length** (or **norm**) of a vector $\mathbf{v}$ in $\mathcal{R}^k$ is the non-negative scalar $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}_1^2 + \mathbf{v}_2^2 + \cdots + \mathbf{v}_k^2}$$

Theorem: let $\mathbf{v}$ be a vector in $\mathcal{R}^n$ and let c be a scalar. Then;

1. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
2. $||c \times \mathbf{v}|| = |c| \times ||\mathbf{v}||$

Definition: The distance $d(\mathbf{u}, \mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$ is defined

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

Definition: For non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}|| \times ||\mathbf{v}||}$$

Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$ are considered **orthogonal** to eachother if their dot product is equal to 0. Thus if the following holds true then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal;

$$\mathbf{u}\cdot\mathbf{v} = 0$$

Definition: if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathcal{R}^n$ and $\mathbf{u} \neq 0$, then the **projection of v onto u** is the vector $proj_u(v)$ defined by;

$$proj_u(v) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

1.3 Lines and Planes

2 Linear Systems

2.1 Introduction to Systems of Linear Equations

Definition: A **Linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is a equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

A system of linear equations with real coefficients has either

1. A unique solution (a consistent system)
2. Infinitely many solutions (a consistent system)
3. No solution (an inconsistent system)

2.2 Direct Methods for Solving Linear Systems

Definition: A matrix is in **row echelon form** if it satisfies the following properties:

1. Any row consisting entirely of zeros are at the bottom.
2. In each non-zero row, the first non-zero entry(called the **leading entry**) is in a column to the left of any leading entries below.

Definition: The following **elementary row operations** can be preformed on a matrix:

1. Interchange two rows
2. Multiply a row by a non-zero constant
3. Add a multiple of a row to another

Definition: Matricies A and B are **row equivalent** if there is a sequence of elementary row operations that converts A into B.

Theorem: Given two matrices A and B, A and B are row equivalent if and only if they can be reduced to te same row echelon form.

Gaussian Elimination

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to a row echelon form.
3. Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

Definition: The **rank** of a matrix is the number of non-zero rows in its row echelon form.

The Rank Theorem
Let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then
number of free variables $= n - rank(A)$

Definition: A matrix is in **reduced row echelon form** if it satisfies the following properties:

1. It is in row echelon form.
2. The leading entry in each non-zero row is a 1 (called the **leading 1**)
3. Each column containing a leading 1 has zeros everywhere else

Gauss-Jordan Elimination

1. Write the augmented matrix of the system of linear equations
2. Use elementary row operations to reduce the augmented matrix to reduced row echelon form.
3. If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

Definition: A system of linear equations is called **homogeneous** if the constant in each term in each equation is zero.

Theorem:
if $[A|\mathbf{0}]$ is a homogeneous system of m linear equations with n variables, where $M < n$, then the system has infinitely many solutions.

2.3 Spanning Sets and Linear Independence

Theorem:

A system of linear equations with augmented matrix $[A|textbfb]$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Definition:

If $S = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, is a set of vectors in $\mathcal{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called a **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and denoted by $span(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $span(S)$.

If $span(S) = \mathcal{R}^n$, then S is called the **spanning set** for $\mathcal{R}^n$.

Definition:

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is **linear dependent** if the scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that

$$c_1 \times \mathbf{v}_1 + c_2 \times \mathbf{v}_2 + \cdots + c_k \times \mathbf{v}_k$$

A set of vectors that is not linearly dependent is called **linearly independent**

Theorem: Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathcal{R}^n$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Theorem: let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathcal{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A|textbf0]$ has a nontrivial solution.

Theorem: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (row) vectors in $\mathcal{R}^n$ and let A be the $m \times n$ matrix $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ with these vectors as its rows. then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $rank(A) < m$.

Theorem: Any set of vectors in $\mathcal{R}^n$ is linearly dependent if $m > n$

3 Matrices

3.1 Matrix Operations

Definition: A **matrix** is a rectangular array of numbers called the **entries**, or **elements**, of the matrix.

Definition: If A is a $m \times n$ matrix and B is an $n \times r$ matrix, then the **product** $C = AB$ is an $m \times r$ matrix. The (i, j) entry of the product is computed as follows:

$$c_{i,j} = a_{i,j} \times b_{1,j} + a_{i,2} \times b_{2,j} + \cdots + a_{i,n} \times b_{n,j}$$

Theorem:

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ a $1 \times m$ standard unit vector. Then

1. $\mathbf{e}_i A$ is the ith row of A.
2. $A\mathbf{e}_j$ is the jth columns of A.

Matrix Powers If A is a square matrix and r and s are non-negative integers, then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{r \times s}$

Definition:
The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A.that is, the ith column of A^T is the ith row of A for all i.

Definition: A square matrix A is **symmetric** if $A^T = A$ that is, if A is equal to its own transpose.

3.2 Matrix Algebra

Theorem:

Algebraic Properties of Matrix Addition and Scalar Multiplication
The A, B and C be matrices of the same size and let c and d be scalars. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + O = A$
4. $A + (-A) = O$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$
7. $c(dA) = (cd)A$
8. $1 \times A = A$

Theorem:

Let A, B, and C be matrices (whose sizes are such that the indicated operations can be preformed) and let k be a scalar. Then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $k(AB) = (kA)B = A(kB)$
5. $I_m A = A = AI_n$ if A is $m \times n$

Theorem:

Properties of the Transpose let A and B be matrices (whose sizes are such that the indicated operations can be preformed) and let k be a scalar. Then

1. $(A^T)^T = A$
2. $(kA)^T = k(A^T)$
3. $(A^r)^T = (A^T)^r$ for all non-negative integers r
4. $(A + B)^T = T^T + B^T$
5. $(AB)^T = B^T A^T$

Theorem:

1. If A is a square matrix, then $A + A^T$ is a symmetric matrix.
2. For any matrix A, AA^T and $A^T A$ are symmetric matrices.

3.3 The Inverse of a Matrix

Definition: If A is an $n \times n$ matrix, an **inverse** of A is an $n \times n$ matrix $A\prime$ with the property that

$$AA\prime = I$$

and

$$A\prime A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an $A\prime$ exists, then A is called invertible.

Theorem: If A is an invertible matrix, then its inverse is unique.

Theorem: If A is an invertible $n \times n$ matrix, then the system of linear equations given $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\mathcal{R}^n$

Theorem: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Theorem:

1. If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1}$$
2. If A is an invertible matrix and c is a non-zero scalar, then cA is an invertable matrix and
$$(cA)^{-1} = \frac{1}{c}A^{-1}$$
3. If A and B are invertible matricies of the same size, then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$
4. If A is an invertible matrix, then A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T$$
5. if A is an invertible matrix, then A^n is invertible for all non-negative integers n and
$$(A^n)^{-1} = (A^{-1})^n$$

Definition: If A is an invertible matrix and n is a positive integer, then A^{-n} is defined by
$$A^{-n} = (A^{-1})^n = (A^n)^{-1}$$

Definition: An **elementary matrix** is any that can be obtained by performing an elementary row operation on an identity matrix.

Theorem: Let E be the elementary matrix obtained bt preforming an elementary row operation on I_n. if the same elementary row operation is preformed on an $n \times r$ matrix A, the result is the same as the matrix EA.

Theorem: Each elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

Theorem:

The Fundamental THeorem of Invertible matrices: Version 1 Let A be an $n \times n$ matrix. The folowing statements are equivalent:

1. A is invertible
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathcal{R}^n$
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is a product of elementary matrices.

Theorem: Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$

Theorem: Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations that transforms I into A^{-1}.

3.4 The LU Factorization

This section was ignored in class

3.5 Subspace, Basis, Dimension, Rank

Definition: A **subspace** of $\mathcal{R}^n$ is any collection S of vectors in $\mathcal{R}^n$ such that

1. The zero vector $\mathbf{0}$ is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S then $\mathbf{u}+\mathbf{v}$ is in S. (S is **closed under addition**)
3. if $\mathbf{u}$ is in S and c is a scalar then c is a scalar, then $c\mathbf{u}$ is in S. (S is **closed under scalar multiplication**)

Theorem: Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathcal{R}^n$. then $span(v_1, v_2, \ldots, v_k)$ is a subspace of $\mathcal{R}^n$.

Definition: Let A be an $m \times n$ matrix

1. The **row space** of A is the subspace $row(A)$ of $\mathcal{R}^n$ spanned by the rows of A
2. The **column space** of A is the subspace $col(A)$ of $\mathcal{R}^n$ spanned by the columns of A.

Theorem: Let B be any matrix that is row equivalent to a matrix A then $row(B) = row(A)$.

Theorem: Let A be a $m \times n$ matrix and let N be the set of solutions of the momogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then N is a subspace of $\mathcal{R}$.

Definition: Let A be an $m \times n$ matrix. The **null space** of A is the subspace of $\mathcal{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. It is denoted by $null(A)$

Theorem: Let A be a matrix whose entries are real numbers, for any system of linear equations $A\mathbf{x} = \mathbf{b}$, exactly one of the following is true

1. There is no solution
2. There is a unique solution
3. There are infinitely many solution

Definition: A **basis** for a subspace S of $\mathcal{R}^n$ is a set of vectors in S that

1. spans S
2. Is linear independent.

Theorem:

The basis Theorem Let S be a subspace of $\mathcal{R}^n$. Then any two bases for S have the same number of vectors.

Definition: If S is a subspace of $\mathcal{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted $dim(S)$

Theorem: The row and column space of a matrix A have the same dimension.

Definition: The **rank** of a matrix A is the dimension of its row and column spaces and is denoted by $rank(A)$

Theorem: For any matrix A,

$$rank(A^T) = rank(A)$$

Definition: The **nullity** of a matrix A is the dimension of its null space and is denoted by $nullity(A)$

Theorem:

The Rank Theorem If A is an $m \times n$ matrix then

$$rank(A) + nullity(A) = n$$

Theorem:

The Fundimental Theorem of Invertible matricies: Version 2

Theorem: Let A be a $m \times n$ matrix. Then:

1. $rank(A^T A) = rank(A)$
2. The $n \times n$ matrix $A^T A$ is invertible if and only if $rank(A) = n$

Theorem: Let S be a subspace of $\mathcal{R}^n$ and let $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$ be the basis for S. For every vector $\mathbf{v}$ in S, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis in $\mathcal{B}$:

$$\mathbf{v} = c_1 \times \mathbf{v}_1 + c_2 \times \mathbf{v}_2 + \cdots + c_k + \mathbf{v}_k$$

Definition: Let S be a subspace of $\mathcal{R}^n$ and let $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$ be the basis for S. Let $\mathbf{v}$ be a vector in S, and write $\mathbf{v} = c_1 \times \mathbf{v}_1 + c_2 \times \mathbf{v}_2 + \cdots + c_k + \mathbf{v}_k$. Then $c_1, c_2, \ldots, c_k$ are called the **coordinates of v with respect to** $\lfloor$ and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the **coordinate vector of v with respect to** $\mathcal{B}$

3.6 Introduction to Linear Transformations

Definition: A linear transformation between two vector spaces v and w is a map $T : v \to w$ such that the following holds;

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$$

$$T(c \times \mathbf{v}_1) = c \times T(\mathbf{v}_1)$$

Definition: Let $T : \mathcal{R}^m \to \mathcal{R}^n$ and $S : \mathcal{R}^n \to \mathcal{R}^p$ be linear transformations. Then $S \circ T : \mathcal{R}^m \to \mathcal{R}^p$ is a linear transformation. Moreover, their standard matricies are related by,

$$[S \circ T] = [S]\,[T]$$

Definition: Let S and T be linear transformations from $\mathcal{R}^n$ to $\mathcal{R}^n$. Then S and T are inverse transformations if $S \circ T = I_n$ and $T \circ S = I_n$

Definition: Let $T : \mathcal{R}^n \to \mathcal{R}^n$ be an invertible linear transformations.Then the standard matrix $[T]$ is an invertible matrix. Thus $[T^{-1}] = [T]^{-1}$

3.7 Markov Chains

4 Eigen

4.1 Introduction to Eigenvalues and Eigenvectors

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called a **eigenvector** of A corresponding to λ.

Definition: Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ and is denoted E_λ.

4.2 Determinants

Definition: Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$. Then the **determinatnt** of A is the scalar

$$det(A) = |A| = a_{1,1} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

4.3 Eigenvalues and Eigenvectors of n $\times$ n Matrix

4.4 Similarity and Diagonalization

5 Orthogonality

This section was not covered in class

6 Vector Spaces

6.1 Vector Spaces and Subspaces