7.4 Special Cases

7.4.1 Unbalanced ANOVA

When the treatment sample sizes are unequal, the analysis of variance for two-factor studies becomes more complex. The least-squares equations are no longer of a simple structure and the regular analysis of variance formulas are now inappropriate. Furthermore, the factor effect components sums of squares are no longer orthogonal; which means that **they do not sum up to TSS**.

The Rats Experiment: Unbalanced Scenario

Consider the `rats` example from the previous lecture.Remove the first observation to make the data **unbalanced**.

```
newrats=rats
newrats=newrats[-1,]
```

Use the `anova()` command for *each* of the following models: (1) `poison*treat` - i.e. `poison` first, `treat` second (2) `treat*poison` - i.e. `treat` first, `poison` second

```
anova(lm(1/time ~ poison*treat, newrats))
```

```
## Analysis of Variance Table
##
## Response: 1/time
##              Df Sum Sq Mean Sq F value    Pr(>F)
## poison        2 36.672 18.3358 81.0799 7.276e-14 ***
## treat         3 18.567  6.1889 27.3670 2.706e-09 ***
## poison:treat  6  1.980  0.3300  1.4592    0.2207
## Residuals    35  7.915  0.2261
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
anova(lm(1/time ~ treat*poison, newrats))
```

```
## Analysis of Variance Table
##
## Response: 1/time
##              Df Sum Sq Mean Sq F value    Pr(>F)
## treat         3 20.136  6.7121 29.6807 9.986e-10 ***
## poison        2 35.102 17.5510 77.6094 1.362e-13 ***
## treat:poison  6  1.980  0.3300  1.4592    0.2207
## Residuals    35  7.915  0.2261
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Observe that the results change depending on the order the factors are introduced in the model.

Although the ANOVA output changes with the order of the two factors, the information stays the same: the interaction is not significant; the two factors are significant no matter whether the other factor is included or not included in the model. So the data support the additive model.

Due to the lack of orthogonality, the ANOVA table F-tests are not applicable. Therefore, we choose to work with the regression approach to ANOVA models. This means that we will introduce indicator (dummy) variables. We need $a-1$ indicator variables for factor A main effects and $b-1$ indicator variables for factor B main effects. The interactions correspond to the cross products of the indicator variables for A and B.

We then perform *partial F tests* to test the various interactions and main effects in the model as follows:

$$\begin{cases} H_0 : \text{Smaller model} \\ H_\alpha : \text{Larger model} \end{cases}$$

The partial F-test is:

$$F = \frac{(RSS_0 - RSS_\alpha)/(df_0 - df_\alpha)}{RSS_\alpha/df_\alpha} \sim F_{df_0 - df_\alpha, df_\alpha}$$

The numerator captures the variation in the data *not* explained by the reduced model, *but* explained by the full model, while the denominator quantifies the variation in the data not explained by the full model (i.e., not explained by either model), which is used to estimate the error variance.

As usual we *reject* H_0, if F test statistic is large, that is, the variation missed by the reduced model, when being compared with the error variance, is significantly large.

The Rats Experiment: Unbalanced Scenario

```
full.rats.model = lm(1/time ~ poison*treat, newrats)
additive.rats.model = lm(1/time ~ poison+treat, newrats)
anova(additive.rats.model, full.rats.model)
```

```
## Analysis of Variance Table
##
## Model 1: 1/time ~ poison + treat
## Model 2: 1/time ~ poison * treat
##   Res.Df    RSS Df Sum of Sq      F Pr(>F)
## 1     41 9.8951
## 2     35 7.9151  6      1.98 1.4592 0.2207
```

The interaction term is still not statistically significant, so we remove it from the model. We can also test for the main effects:

```
treat.rats.model = lm(1/time ~ treat, newrats)
poison.rats.model = lm(1/time ~ poison, newrats)
anova(treat.rats.model, additive.rats.model)
```

```
## Analysis of Variance Table
##
## Model 1: 1/time ~ treat
## Model 2: 1/time ~ poison + treat
##   Res.Df    RSS Df Sum of Sq      F   Pr(>F)
## 1     43 44.997
## 2     41  9.895  2    35.102 72.722 3.28e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
anova(poison.rats.model, additive.rats.model)
```

```
## Analysis of Variance Table
##
## Model 1: 1/time ~ poison
## Model 2: 1/time ~ poison + treat
##   Res.Df     RSS Df Sum of Sq      F    Pr(>F)
## 1     44 28.4618
## 2     41  9.8951  3    18.567 25.644 1.669e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We see that both main effects are statistically significant so we conclude that the additive model is the “best” model here.

Type III Sums of Squares

There is an alternative way to construct an ANOVA Table and compute the SS so that they add up to the total SS. These are the so-called **Type III** Sums of Squares. These are obtained by running the `Anova()` function in nthe `car` package in `R`. This type of SS tests for the presence of an effect given that all the other effects are in the model.

The Rats Experiment: Unbalanced Scenario

```
library(car)
Anova(lm(1/time ~ treat*poison, newrats), type="III")
```

```
## Anova Table (Type III tests)
##
## Response: 1/time
##                Sum Sq Df   F value    Pr(>F)
## (Intercept)  316.160  1 1398.0416 < 2.2e-16 ***
## treat         17.849  3   26.3085 4.354e-09 ***
## poison        35.596  2   78.7029 1.115e-13 ***
## treat:poison   1.980  6    1.4592    0.2207
## Residuals      7.915 35
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Anova(lm(1/time ~ treat+poison, newrats), type="III")
```

```
## Anova Table (Type III tests)
##
## Response: 1/time
##              Sum Sq Df  F value    Pr(>F)
## (Intercept) 319.33  1 1323.129 < 2.2e-16 ***
## treat        18.57  3   25.644 1.669e-09 ***
## poison       35.10  2   72.722 3.280e-14 ***
## Residuals     9.90 41
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

7.4.2 Sample Size of 1

Another scenario that arises quite often when running experiments is when we have only one observation per factor level combination. If there are no missing observations, then we have a balanced ANOVA with sample size $n = 1$.

When we have $n = 1$ we do not have enough degrees of freedom to estimate the error and the $RSS = 0$ when the model includes main effects and interaction term, which means that we cannot perform any hypothesis tests when fitting the full model with interactions.

For example,

Strength of a Thermoplastic Composite Experiment

The composite data frame has 9 rows and 3 columns. Data comes from an experiment to test the **strength** of a thermoplastic composite depending on the `power` of a laser and `speed` of a tape.

```
library(faraway)
?composite
```

As we can see we have 9 factor level combinations (i.e. treatments) and only 9 observations.

```
composite.full = lm(strength ~ laser*tape, composite)
anova(composite.full)
```

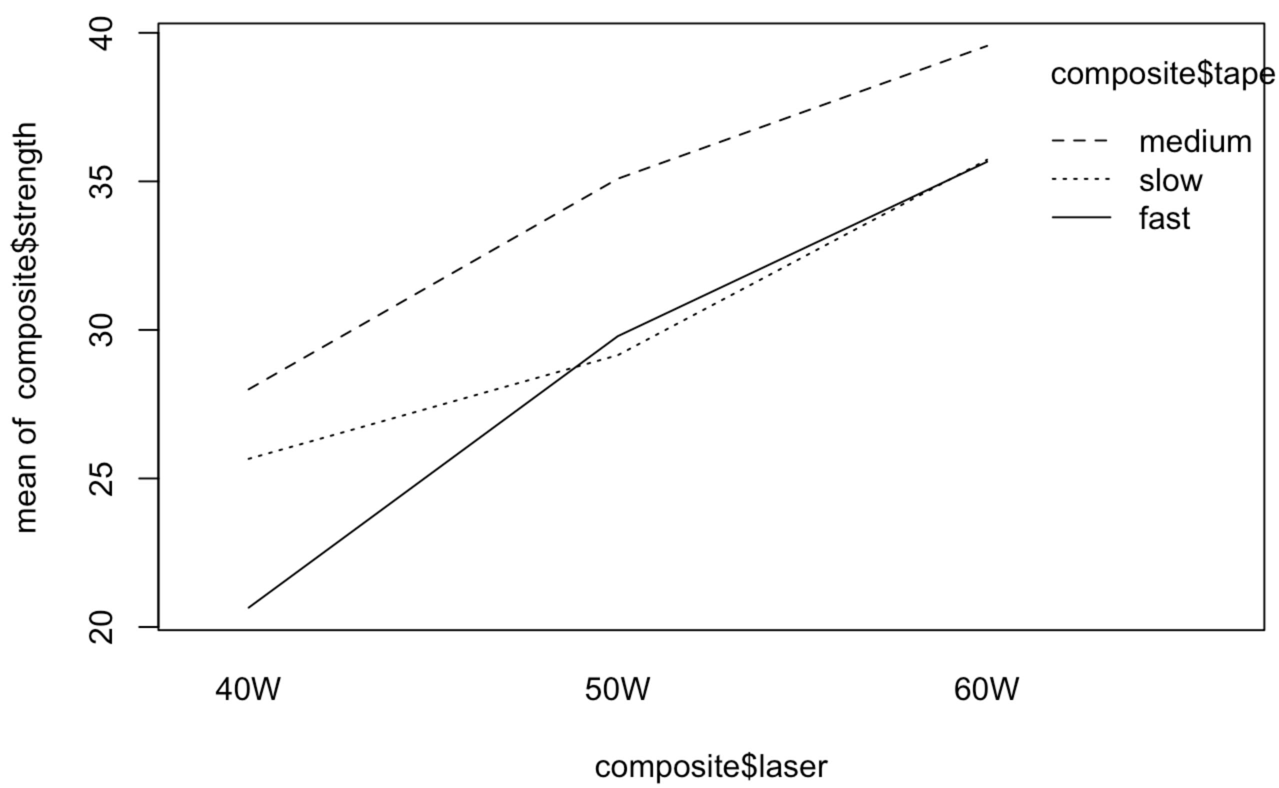
```
## Warning in anova.lm(composite.full): ANOVA F-tests on an essentially perfect f
## are unreliable
```

```
## Analysis of Variance Table
##
## Response: strength
##            Df  Sum Sq Mean Sq F value Pr(>F)
## laser       2 224.184 112.092
## tape        2  48.919  24.459
## laser:tape  4  10.503   2.626
## Residuals   0   0.000
```

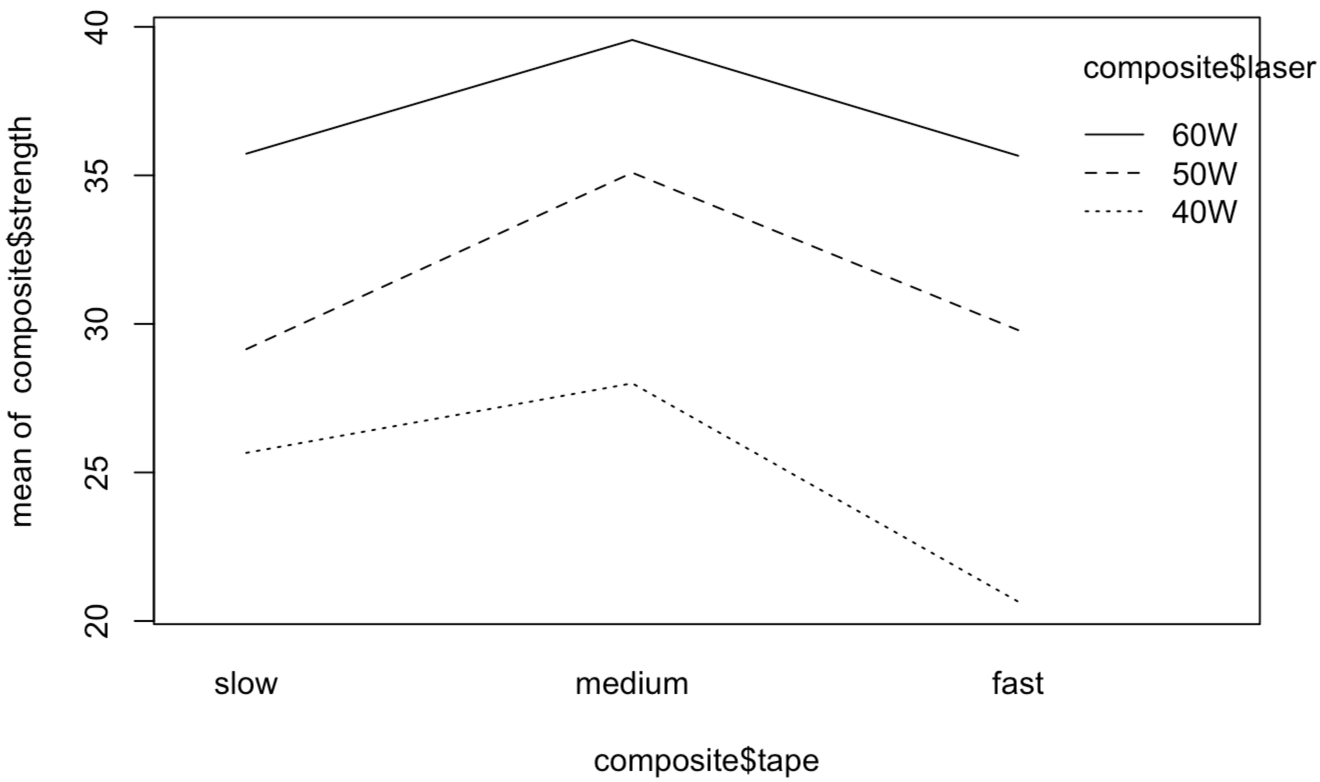
In this output, we observe the 0 degrees of freedom for the error and the 0 RSS. So, the first idea onn how to proceed is to remove the interaction term and work with the additive model. But, what if interactions are present? We might miss some valuable information.

So, let's check the interaction plots to decide whether interactions are present:

```
interaction.plot(composite$laser, composite$tape, composite$strength)
```



```
interaction.plot(composite$tape, composite$laser, composite$strength)
```



Based on these interaction plots we conclude that interactions are indeed present.

Tukey's Test for Additivity

One way to test for interactions, when we do not have enough degrees of freedom is to specify the type of interactions. Specifically, instead of adding a term $(\alpha\beta)_{ij}$ in the model, we **assume** that the interactionns are the product of the main effects α_i and β_j, i.e.

$$(\alpha\beta)_{ij} = \theta\alpha_i\ \beta_j$$

Therefore, the model becomes

$$y_{ij} = \mu + \alpha_i + \beta_j + \theta\ \alpha_i\ \beta_j + \epsilon_{ij}$$

This means that in order to check for the significance of the interactions, we only need to test the significance of the θ coefficient. To construct an appropriate test statistic, let's look at the updated Sums of Squares Decomposition for this model. Consider the SSA, SSB as before and :

$$SSAB^* = \frac{\left(\sum_i \sum_j (\bar{y}_{i.} - \bar{y}_{..})(\bar{y}_{.j} - \bar{y}_{..}) y_{ij}\right)^2}{\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 \ \sum_j (\bar{y}_{.j} - \bar{y}_{..})^2}$$

The TSS is computed as usual and is decomposed as

$$TSS = SSA + SSB + SSAB^* + SSRem^*$$

where the remainder is

$$SSRem^* = TSS - SSA - SSB - SSAB^*$$

Then, if we want to test the following hypothesis, i.e. connduct a test for model additivity,

$$\begin{cases} H_0 : \ \theta = 0 \quad \text{(no interactions)} \\ H_\alpha : \ \theta \neq 0 \quad \text{(interactions)} \end{cases}$$

the test statistic becomes

$$F^* = \frac{SSAB^*/1}{SSRem^*/(ab - a - b)} \sim F_{1, ab-a-b}$$

Implementation of Tukey's Additivity Test in R

Strength of a Thermoplastic Composite Experiment

```
meffectmodel = lm(strength ~ laser+tape, composite)
lasercoefs = rep(c(0, 6.5733, 12.2133), 3)
tapecoefs = rep(c(0, 4.0367, -1.48), each=3)

newmodel = update(meffectmodel, .~. + I(lasercoefs*tapecoefs))
anova(newmodel)
```

```
## Analysis of Variance Table
##
## Response: strength
##                            Df  Sum Sq Mean Sq F value   Pr(>F)
## laser                       2 224.184 112.092 36.8201 0.007745 **
## tape                        2  48.919  24.459  8.0344 0.062401 .
## I(lasercoefs * tapecoefs)  1   1.370   1.370  0.4501 0.550340
## Residuals                   3   9.133   3.044
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Based on Tukey's Additivity test we conclude that the interactions are not statistically significant, therefore the additive model is adequate.

1. Of course, there are also *meta-analysis* studies, but we will "ignore" this type of studies here.↩
2. If you are interested in designed experiments, you should check STAT 424 in SP 2024!↩
3. More on this in the second half of the semester.↩
4. More details can be found here. A video of the Galton example here.↩
5. $IID(0, \sigma^2)$ means independent and identically distributed with mean 0 and variance σ^2 .↩
6. Unit-free and location/scale invariant variable is a standardized variable, i.e. the one that is centered and scaled by subtracting its mean and dividing by its variance.↩
7. Kimber, H. (1995). The `golden egg'. Teaching Statistics, 17(2), 34-7↩
8. Estimation looks to get information from the data about a fixed but parameter, while prediction looks to get information about a random variable↩
9. It is a linear combination of the n data points $y_1, y_2, \ldots, y_n$↩

10. Note that if two random variables are uncorrelated, they are not necessarily independent, unless they have a joint Normal distribution.↩

11. Note here that the F test we perform is one-sided. This has to do with the decision rule we define: we say that we reject the null when the F statistic value exceeds the corresponding critical value.↩

12. Note that no matter how large the sample size becomes, the width of a PI, unlike a CI, will never approach 0.↩

13. There are also Lack-of-Fit tests when replicates are available, but those will be discussed later in the course.↩

14. The transformation for $\lambda = 0$ is justified because $\lim_{\lambda \to 0} \frac{y^\lambda - 1}{\lambda} = \log(y)$↩

15. Recall how we do transformation of random variables in higher dimensions - this is something taught in STAT 410.↩

16. You can find more details here: Cholesky↩

17. $n = 4$ is used for simplicity, since technically that should be a 42$\times$ 4 matrix↩

18. this is a randomized experiment↩

19. Formulas are given for factor A↩