4.4 Polynomial Regression

So far we have been discussing a Multiple Linear Regression model

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_p X_{ip} + \varepsilon_i$$

The aforementioned model is **linear** both with respect to the predictors (X_{ij}) and the coefficients (β_j).

Consider now the following models:

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i1}^2 + \beta_3 X_{i2} + \beta_4 X_{i2}^2 + \beta_5 X_{i1} X_{i2} + \varepsilon_i$$

or

$$\log_{10} Y_i = \beta_1 X_{i1} + \beta_2 \sqrt{X_{i1}} + \beta_3 e^{X_{i3}} + \varepsilon_i$$

These two models are both *non-linear* with respect to the predictors but **linear** with respect to the coefficients β_j.

On the other side, the model below

$$Y_i = \frac{\gamma_0}{1 + \gamma_1 e^{\gamma_2 X_i}} + \varepsilon_i$$

is non-linear both with respect to the parameters *and* the predictors. More generally, we denote a **non-linear model** as

$$y_i = f(X_i, \gamma) + \varepsilon_i$$

where we assume that f is a smooth function.

If we want to estimate the model parameters in the model above, we can use the *Least-Squares* approach as usual. We write the *RSS* as

$$RSS = \sum_{i=1}^{n} \left(y_i - f(X_i, \gamma) \right)^2$$

and we *minimize* it with respect to γ. The **normal equations** are

$$\frac{\partial RSS}{\partial \gamma_k} = -2 \sum_{i=1}^{n} \left(y_i - f(X_i, \gamma) \right) \left[\frac{\partial f(X_i, \gamma)}{\partial \gamma_k} \right], \ \forall k = 1, \ldots, p$$

Setting these equations equal to 0 yields the *least-squares* estimators for γ_k. However, solving this non-linear system of equations is highly complex, and it typically requires either *direct numerical search methods* (such as Gauss-Newton) or a method that searches for the minimum in *RSS* (such as a method similar to steepest descent).

The approach above only works when we know function f. When this is not known, then we need to *approximate* it using the data. For simplicity in the presentation of the results, we consider only one predictor.

Consider the following model

$$y_i = f(x_i) + \varepsilon_i,$$

where y_i is the response variable, x_i is a predictor, f is a *smooth* function, and the error terms ε_i are *IID* $\mathrm{N}(0, \sigma^2)$ random variables.

Our goal is to estimate f using methods that we have already discussed. This implies that we need to represent f in such a way that the model above becomes a *linear* model. This can be done by choosing a *basis*, i.e. defining the space of functions of which f is an element. Once we choose the basis functions, they will be treated as **completely known**.

4.4.1 Polynomial Basis Functions

If $b_j(x)$ is the jth basis function, then f has the following representation

$$f(x) = \sum_{j=0}^{d} b_j(x)\beta_j$$

for some values β_j. Therefore, we write the nonlinear model $y_i = f(x_i) + \varepsilon_i$ as a linear model

$$y_i = \beta_0 + \sum_{j=1}^{d} b_j(x_i)\beta_j + \varepsilon_i$$

Suppose that f is believed to be a 4th order polynomial, so *the space of polynomials of order 4 and below contains* f.

A basis for this space is

$$\begin{aligned} b_0(x) &= 1 \\ b_1(x) &= x \\ b_2(x) &= x^2 \\ b_3(x) &= x^3 \\ b_4(x) &= x^4 \end{aligned}$$

so that the model becomes

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 \underbrace{x_i^2}_{=\beta_0 + \sum_{j=1}^{d} b_j(x_i)\beta_j} + \beta_3 x_i^3 + \beta_4 x_i^4 + \varepsilon_i$$

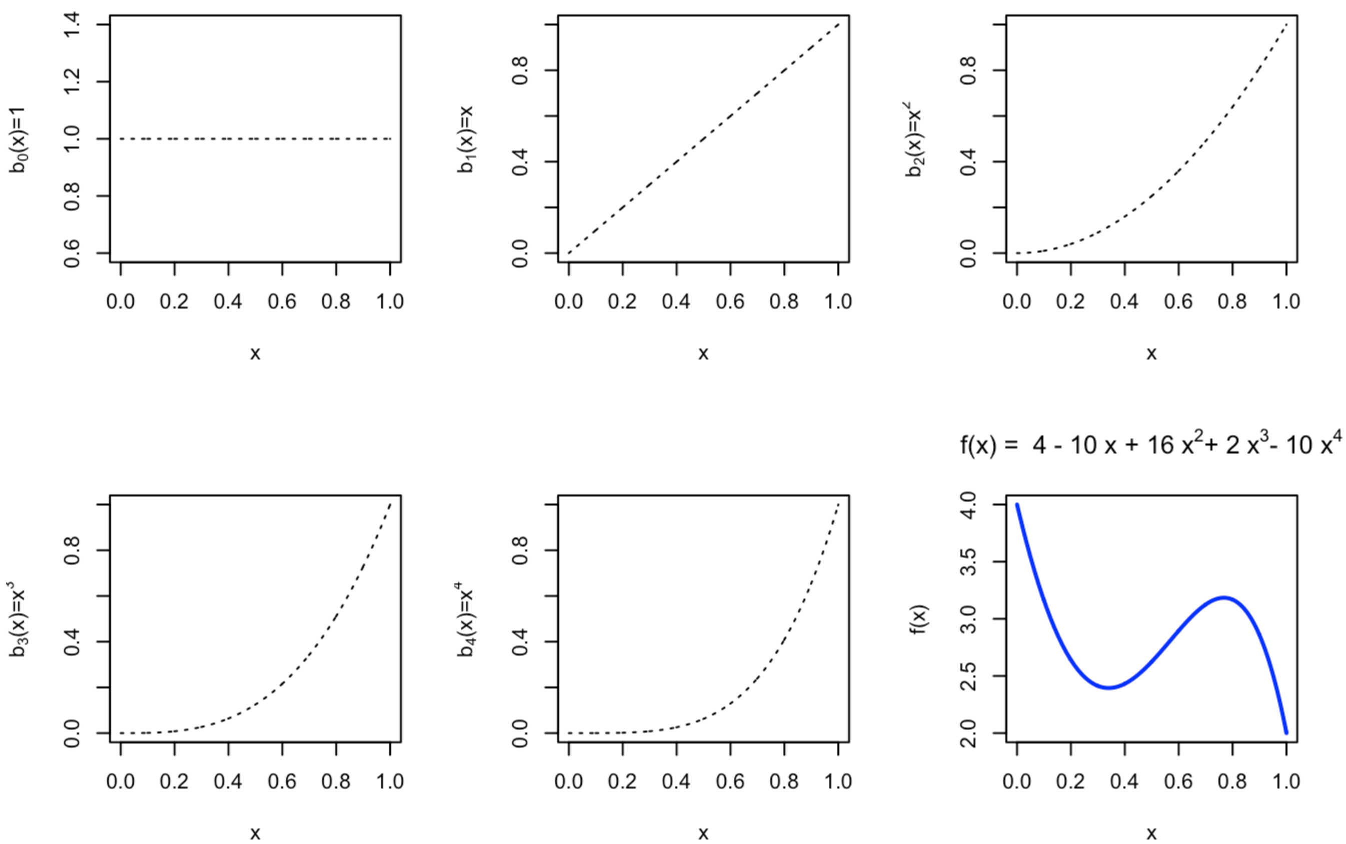
Illustration of the Polynomial Basis Functions

Representation of a function in terms of basis functions using a polynomial basis. The following code creates the plots the polynomial basis function up to order 4

```r
x=seq(0, 1, by=0.001)
b0 = rep(1, length(x))
b1 = x
b2 = x^2
b3 = x^3
b4 = x^4

fun1 = 4*b0 -10* b1 + 16*b2 + 2*b3 -10*b4

par(mfrow = c(2,3))
plot(x, b0, type='l', lty=3, ylab=expression("b"[0]*"(x)=1"))
plot(x, b1, type='l',lty=3, ylab=expression("b"[1]*"(x)=x"))
plot(x, b2, type='l',lty=3, ylab=expression("b"[2]*"(x)=x"^2))
plot(x, b3, type='l',lty=3, ylab=expression("b"[3]*"(x)=x"^3))
plot(x, b4, type='l',lty=3, ylab=expression("b"[4]*"(x)=x"^4))
plot(x, fun1, type='l', ylab="f(x)", main=expression("f(x) =  4 - 10 x + 16 x"^2*
```



Polynomial Regression Model

A *non-linear* model can be represented using a basis of polynomial functions as follows:

$$y_i = f(x_i) + \varepsilon_i \quad \longrightarrow \quad y_i = \beta_0 + \sum_{j=1}^{d} b_j(x_i)\beta_j + \varepsilon_i$$

where d is the degree of the polynomial component.

How do we choose d?

1. *Forward Approach*: Keep *adding* terms until the *last* added term is not significant.

2. *Backward Approach*: Start with a large d, and keep *eliminating* the terms that are not statistically significant, starting with the highest order term.

Once we pick a value of d, then we usually do **not** test the significance of the lower-order terms. Therefore, when we decide to use a polynomial of degree d, by default, we include *all the lower-order terms in our model*.

Reasoning In regression analysis, we do not want our results to be affected by a change of location/scale of the data. Consider the following example:
Suppose the data $\{y_i, x_i\}_{i=1}^n$ are generated by the model:

$$y_i = x_i^2 + \varepsilon_i, \quad \varepsilon_i \sim \mathrm{N}(0, \sigma^2)$$

But, they are instead recorded as $\{z_i, x_i\}_{i=1}^n$, where $z_i = x_i + 2$, that is,

$$y_i = (z_i - 2)^2 + \varepsilon_i = 4 - 4z_i + z_i^2 + \varepsilon_i$$

The linear term could become significant, if we shift the x values.

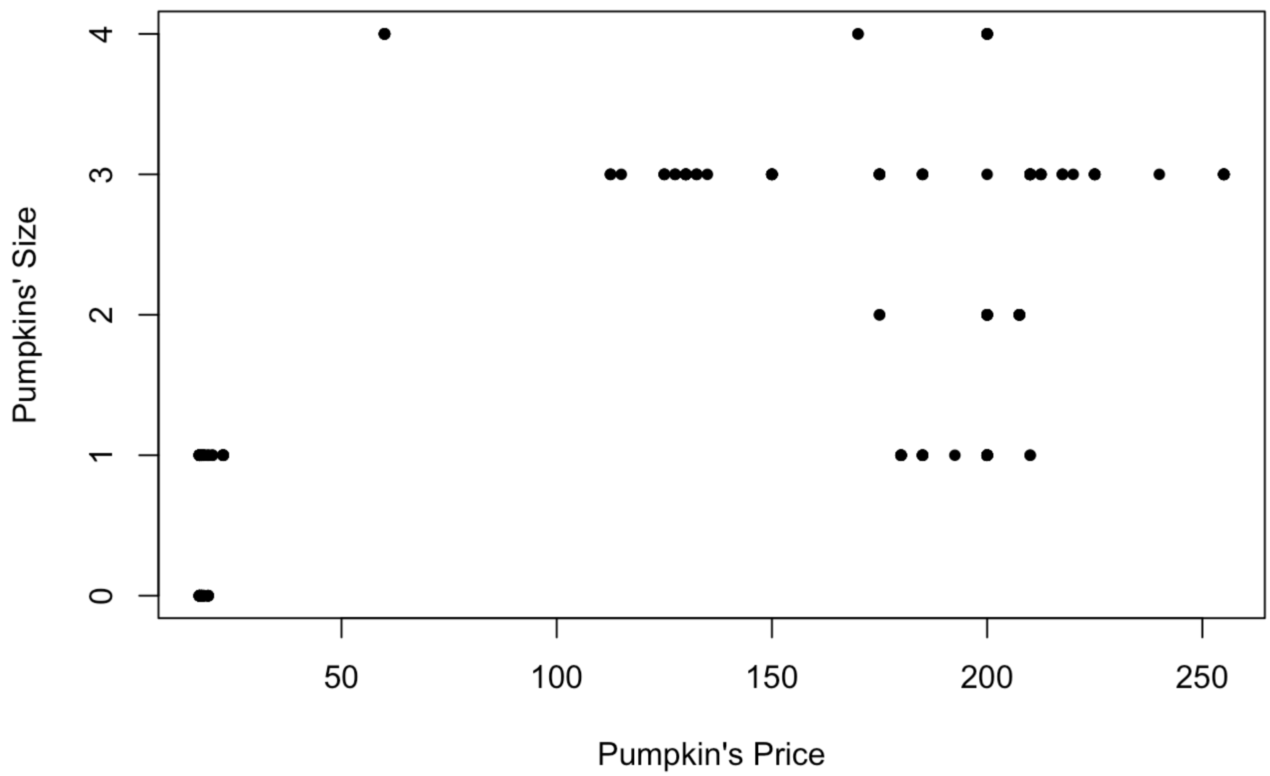
Exception: When we have a particular polynomial function in mind, e.g. the data are collected to test a particular physics formula $Y \approx X^2 + constant$, then you should test whether you can drop the linear term.

The Chicago Pumpkins Example

The `pumpkins.csv` data set contains information regarding the `size` and `price` of pumpkins sold in the Chicago area. Our goal in this example is to *predict* the `size of the pumpkin` (response) based on its `price` (predictor).

The scatter plot of the data is shown below:

```
pumpkins = read.csv("data/ch4/chicagopumpkins.csv",header=TRUE)
plot(pumpkins$price, pumpkins$size, pch=20, xlab="Pumpkin's Price", ylab="Pumpkir
```



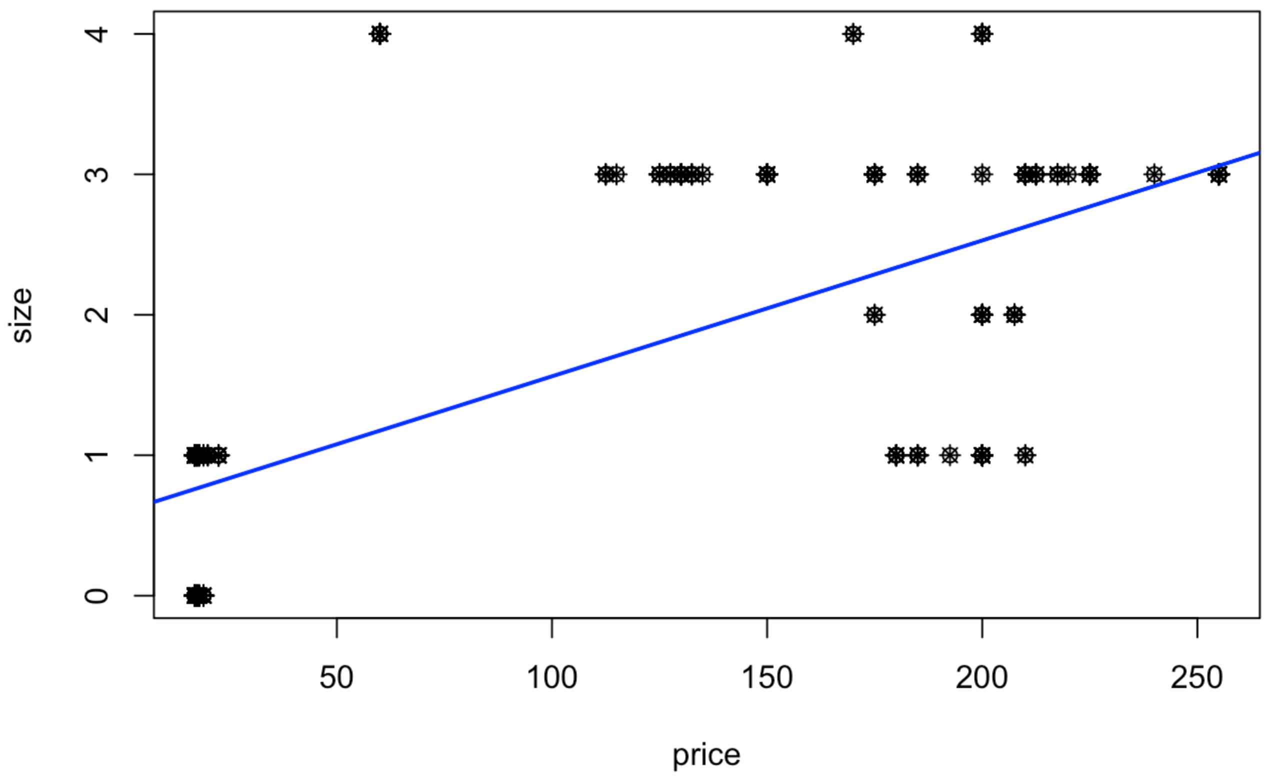
We see that a linear fit is probably *not* a good idea. Indeed,

```
lm.pumpkins = lm(size ~ price, data=pumpkins)
summary(lm.pumpkins)
```

```
##
## Call:
## lm(formula = size ~ price, data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.6272 -0.7594  0.2244  0.3728  2.8245
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.5948315  0.0988339   6.018 6.35e-09 ***
## price       0.0096781  0.0006856  14.117  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9477 on 246 degrees of freedom
## Multiple R-squared:  0.4475, Adjusted R-squared:  0.4453
## F-statistic: 199.3 on 1 and 246 DF,  p-value: < 2.2e-16
```

Although the predictor is significant at explaining the response, the R^2 is on the lower end and the scatter plot does not support a straight line as a good fit.

```
plot(size ~ price, data=pumpkins)
points(size ~ price, data=pumpkins, pch=8)
abline(lm.pumpkins, col="blue", lwd=2)
```



We want to select a *higher order* model and we will do so following a Forward Selection approach first and a Backward Selection method second:

1. We start with a **Forward Selection** approach, i.e. we start by a linear model, and we keep **adding** higher order terms until the added term becomes *statistically insignificant*.

```
# Forward Selection
lm.pumpkins =  lm(size ~ price , data=pumpkins)
summary(lm.pumpkins)
```

```
##
## Call:
## lm(formula = size ~ price, data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.6272 -0.7594  0.2244  0.3728  2.8245
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.5948315  0.0988339   6.018 6.35e-09 ***
## price       0.0096781  0.0006856  14.117  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9477 on 246 degrees of freedom
## Multiple R-squared:  0.4475, Adjusted R-squared:  0.4453
## F-statistic: 199.3 on 1 and 246 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins2 =  lm(size ~ price + I(price^2), data=pumpkins)
summary(lm.pumpkins2)
```

```
##
## Call:
## lm(formula = size ~ price + I(price^2), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.6438 -0.6029  0.3695  0.4895  2.3941
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.093e-01  1.174e-01   0.932    0.353
## price        3.037e-02  3.208e-03   9.469  < 2e-16 ***
## I(price^2)  -9.051e-05  1.375e-05  -6.582  2.8e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8754 on 245 degrees of freedom
## Multiple R-squared:  0.5305, Adjusted R-squared:  0.5267
## F-statistic: 138.4 on 2 and 245 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins3 =  lm(size ~ price + I(price^2) + I(price^3), data=pumpkins)
summary(lm.pumpkins3)
```

```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.2786 -0.4937 -0.1368  0.5531  1.8633
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.414e+00  1.691e-01   -8.36 4.83e-15 ***
## price        1.256e-01  9.079e-03   13.83  < 2e-16 ***
## I(price^2)  -9.845e-04  8.239e-05  -11.95  < 2e-16 ***
## I(price^3)   2.228e-06  2.034e-07   10.95  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7182 on 244 degrees of freedom
## Multiple R-squared:  0.6853, Adjusted R-squared:  0.6814
## F-statistic: 177.1 on 3 and 244 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins4 =  lm(size ~ price + I(price^2) + I(price^3) + I(price^4), data=pump
summary(lm.pumpkins4)
```

```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3) + I(price^4),
##     data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.3200 -0.4497 -0.1241  0.5539  1.7925
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -2.871e+00  3.782e-01  -7.590 6.83e-13 ***
## price        2.314e-01  2.630e-02   8.800 2.60e-16 ***
## I(price^2)  -2.565e-03  3.785e-04  -6.776 9.25e-11 ***
## I(price^3)   1.061e-05  1.973e-06   5.378 1.76e-07 ***
## I(price^4)  -1.470e-08  3.443e-09  -4.271 2.80e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6941 on 243 degrees of freedom
## Multiple R-squared:  0.7073, Adjusted R-squared:  0.7025
## F-statistic: 146.8 on 4 and 243 DF,  p-value: < 2.2e-16
```

```r
lm.pumpkins5 =  lm(size ~ price + I(price^2) + I(price^3) + I(price^4)+ I(price^5
summary(lm.pumpkins5)
```

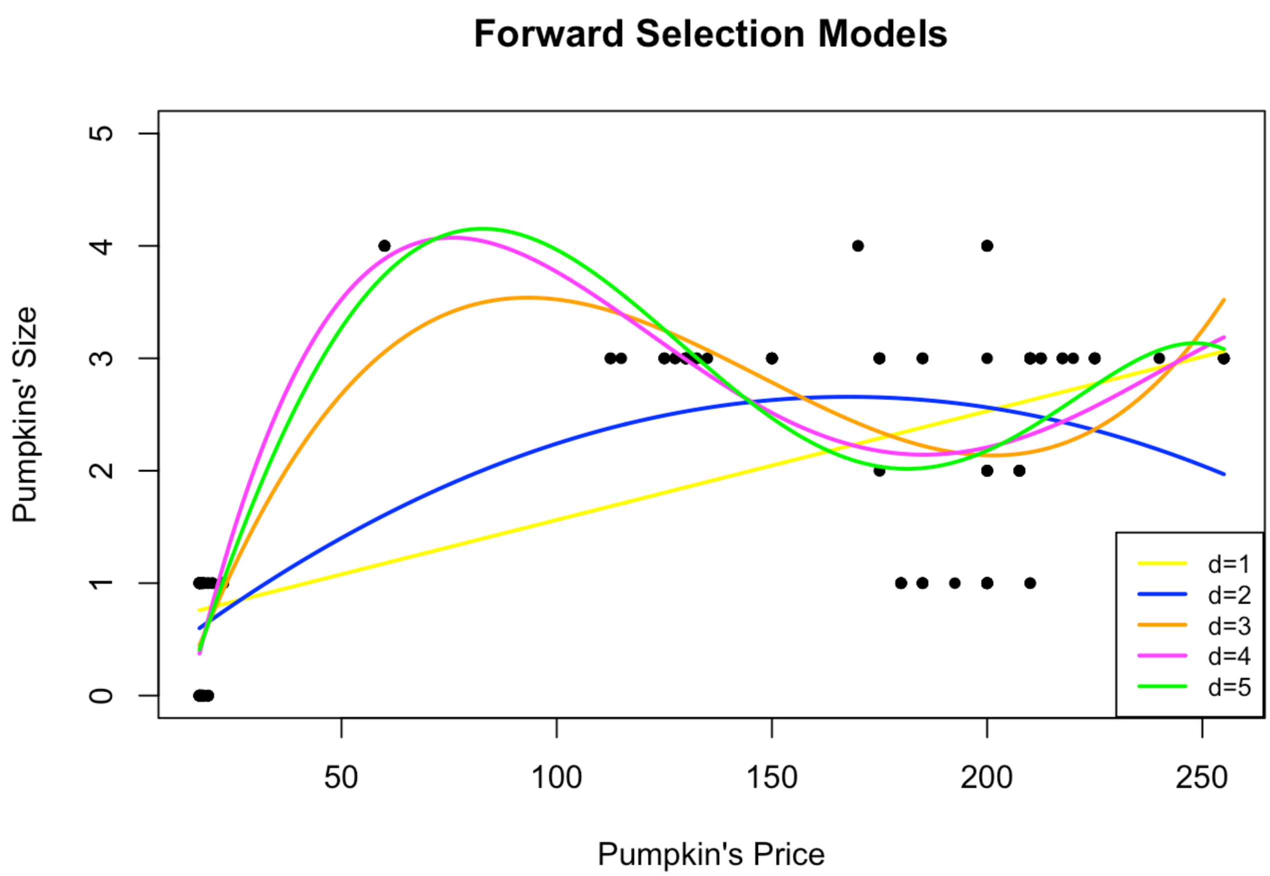
```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3) + I(price^4) +
##     I(price^5), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.38082 -0.46537 -0.08211  0.53463  1.91954
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.691e+00  7.112e-01  -2.378   0.0182 *
## price        1.305e-01  5.791e-02   2.253   0.0251 *
## I(price^2)  -2.479e-04  1.244e-03  -0.199   0.8422
## I(price^3)  -1.078e-05  1.113e-05  -0.969   0.3333
## I(price^4)   7.118e-08  4.409e-08   1.615   0.1077
## I(price^5)  -1.248e-10  6.386e-11  -1.954   0.0519 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6901 on 242 degrees of freedom
## Multiple R-squared:  0.7118, Adjusted R-squared:  0.7059
## F-statistic: 119.6 on 5 and 242 DF,  p-value: < 2.2e-16
```

We see that the *5th order model* has a 5th order term with *p-value equal to 0.0519*, which is **higher than 5%**, so we conclude that the optimal order for the polynomial, according to the forward selection method is $d = 4$. So, the fitted model is:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 x^2 + \hat{\beta}_3 x^4 + \hat{\beta}_4 x^4$$

If we plot all the fitted models, we have:

```
newprice = data.frame(price=seq(17, 255, 1))
plot(pumpkins$price, pumpkins$size, pch=20, ylim=c(0,5), xlab="Pumpkin's Price",
lines(newprice$price, predict(lm.pumpkins, newprice), col="yellow", lty=1, lwd=2)
lines(newprice$price, predict(lm.pumpkins2, newprice), col="blue", lty=1, lwd=2);
lines(newprice$price, predict(lm.pumpkins3, newprice), col="orange", lty=1, lwd=2
lines(newprice$price, predict(lm.pumpkins4, newprice), col="magenta", lty=1, lwd=
lines(newprice$price, predict(lm.pumpkins5, newprice), col="green", lty=1, lwd=2)
legend(230, 1.45, legend=c("d=1", "d=2", "d=3", "d=4", "d=5"),
       col=c("yellow", "blue", "orange", "magenta", "green"), lty=c(1,1,1,1,1), c
```



The magenta line is the one that corresponds to the 4th order model.

2. We can also select d using the **Backward Elimination** approach, that is we start with a large value for d and we eliminate terms *until the highest order term in the model is statistically significant*:

```
lm.pumpkins10 =  lm(size ~ price + I(price^2) + I(price^3) + I(price^4)+ I(price^
summary(lm.pumpkins10)
```

```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3) + I(price^4) +
##     I(price^5) + I(price^6) + I(price^7) + I(price^8) + I(price^9) +
##     I(price^10), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.44082 -0.45530 -0.01853  0.53535  2.12546
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.331e+01  2.671e+01   0.498    0.619
## price       -2.191e+00  4.274e+00  -0.513    0.609
## I(price^2)   1.396e-01  2.625e-01   0.532    0.595
## I(price^3)  -4.389e-03  8.140e-03  -0.539    0.590
## I(price^4)   8.198e-05  1.458e-04   0.562    0.575
## I(price^5)  -9.818e-07  1.624e-06  -0.605    0.546
## I(price^6)   7.731e-09  1.163e-08   0.664    0.507
## I(price^7)  -3.970e-11  5.374e-11  -0.739    0.461
## I(price^8)   1.276e-13  1.549e-13   0.824    0.411
## I(price^9)  -2.321e-16  2.535e-16  -0.916    0.361
## I(price^10)  1.821e-19  1.800e-19   1.012    0.313
##
## Residual standard error: 0.6483 on 237 degrees of freedom
## Multiple R-squared:  0.7509, Adjusted R-squared:  0.7404
## F-statistic: 71.45 on 10 and 237 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins9 =  lm(size ~ price + I(price^2) + I(price^3) + I(price^4)+ I(price^5
summary(lm.pumpkins9)
```

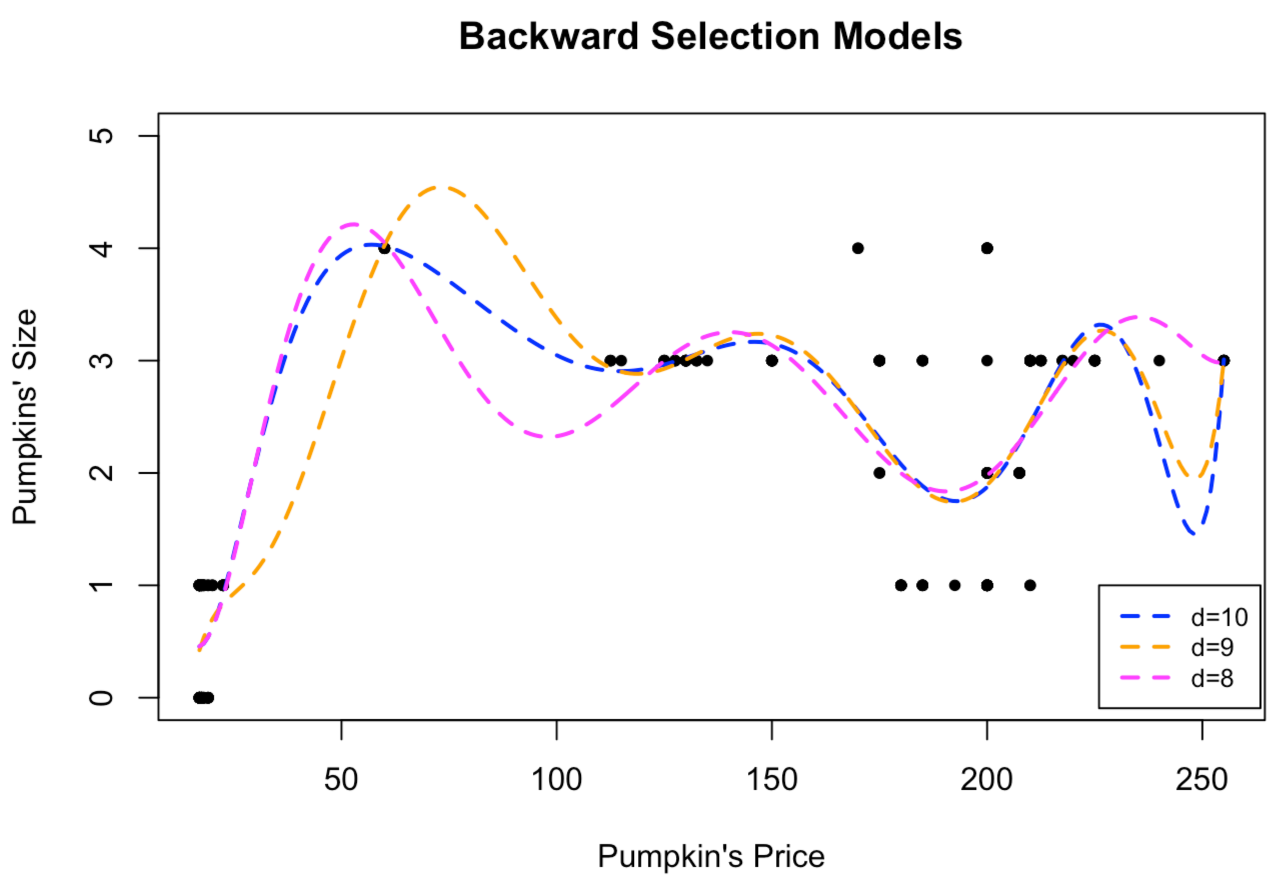
```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3) + I(price^4) +
##     I(price^5) + I(price^6) + I(price^7) + I(price^8) + I(price^9),
##     data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.4514 -0.4230 -0.0091  0.5177  2.1072
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.191e+01  9.589e+00  -1.242   0.2154
## price        1.877e+00  1.450e+00   1.295   0.1967
## I(price^2)  -1.125e-01  8.271e-02  -1.360   0.1751
## I(price^3)   3.506e-03  2.316e-03   1.514   0.1314
## I(price^4)  -6.094e-05  3.623e-05  -1.682   0.0939 .
## I(price^5)   6.252e-07  3.391e-07   1.844   0.0664 .
## I(price^6)  -3.875e-09  1.945e-09  -1.993   0.0475 *
## I(price^7)   1.425e-11  6.704e-12   2.125   0.0346 *
## I(price^8)  -2.859e-14  1.276e-14  -2.241   0.0259 *
## I(price^9)   2.411e-17  1.030e-17   2.340   0.0201 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6483 on 238 degrees of freedom
## Multiple R-squared:  0.7499, Adjusted R-squared:  0.7404
## F-statistic: 79.27 on 9 and 238 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins8 =  lm(size ~ price + I(price^2) + I(price^3)+ I(price^4)+ I(price^5)
summary(lm.pumpkins8)
```

```
##
## Call:
## lm(formula = size ~ price + I(price^2) + I(price^3) + I(price^4) +
##     I(price^5) + I(price^6) + I(price^7) + I(price^8), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.39988 -0.45678 -0.05827  0.53309  2.02119
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  9.140e+00  3.347e+00   2.731 0.006791 **
## price       -1.368e+00  4.257e-01  -3.215 0.001486 **
## I(price^2)   7.522e-02  2.032e-02   3.702 0.000266 ***
## I(price^3)  -1.798e-03  4.783e-04  -3.759 0.000214 ***
## I(price^4)   2.262e-05  6.154e-06   3.675 0.000293 ***
## I(price^5)  -1.611e-07  4.541e-08  -3.549 0.000466 ***
## I(price^6)   6.535e-10  1.918e-10   3.407 0.000771 ***
## I(price^7)  -1.406e-12  4.316e-13  -3.259 0.001282 **
## I(price^8)   1.246e-15  4.008e-16   3.108 0.002112 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6544 on 239 degrees of freedom
## Multiple R-squared:  0.7441, Adjusted R-squared:  0.7355
## F-statistic: 86.87 on 8 and 239 DF,  p-value: < 2.2e-16
```

Starting with an order *10* model, we identify that an *9th* order model is *optimal* according to the backward elimination criterion. If we plot all the fitted models, we have:

```
newprice = data.frame(price=seq(17, 255, 1))
plot(pumpkins$price, pumpkins$size, ylim=c(0,5), pch=20, xlab="Pumpkin's Price",
lines(newprice$price, predict(lm.pumpkins10, newprice), col="blue", lty=2, lwd=2)
lines(newprice$price, predict(lm.pumpkins9, newprice), col="orange", lty=2, lwd=2
lines(newprice$price, predict(lm.pumpkins8, newprice), col="magenta", lty=2, lwd=
legend(226, 1, legend=c("d=10", "d=9", "d=8"),
       col=c("blue", "orange", "magenta"), lty=c(2,2,2), cex=0.8, lwd=2)
```



The magenta line is the one that corresponds to the *9th* order model.

For the fitted model we finally choose, we should (as always) perform diagnostic tests.

4.4.2 Orthogonal Polynomials

Fitting high order polynomials is generally **not recommended**, since they are very *unstable* and *difficult to interpret*. In addition, successive predictors x^j are *highly correlated* introducing multicollinearity problems. One way around this is to fit orthogonal polynomials of the form:

$$y_i = \beta_0 + \beta_1 z_1 + \ldots + \beta_d z_d + \varepsilon_i$$

where each $z_j = a_1 + b_2 x + \ldots + \kappa_j x^j$ is a polynomial of order j with coefficients chosen such that $z_i^\top z_j = 0$ (i.e. the inner product of any two polynomials is zero).

The Chicago Pumpkins Example

In `R`, we can fit orthogonal polynomials using the `poly` function. In the code below, we repeat the same process as before (for choosing d) using the orthogonal polynomials.

```
# Forward Selection
lm.pumpkins02 =  lm(size ~ poly(price,2), data=pumpkins)
summary(lm.pumpkins02)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 2), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.6438 -0.6029  0.3695  0.4895  2.3941
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)      1.70161    0.05559  30.612  < 2e-16 ***
## poly(price, 2)1 13.37846    0.87538  15.283  < 2e-16 ***
## poly(price, 2)2 -5.76139    0.87538  -6.582  2.8e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.8754 on 245 degrees of freedom
## Multiple R-squared:  0.5305, Adjusted R-squared:  0.5267
## F-statistic: 138.4 on 2 and 245 DF,  p-value: < 2.2e-16
```

```
lm.pumpkinsO3 =  lm(size ~ poly(price,3), data=pumpkins)
summary(lm.pumpkinsO3)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 3), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.2786 -0.4937 -0.1368  0.5531  1.8633
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       1.7016     0.0456  37.313  < 2e-16 ***
## poly(price, 3)1  13.3785     0.7182  18.628  < 2e-16 ***
## poly(price, 3)2  -5.7614     0.7182  -8.022 4.35e-14 ***
## poly(price, 3)3   7.8672     0.7182  10.954  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7182 on 244 degrees of freedom
## Multiple R-squared:  0.6853, Adjusted R-squared:  0.6814
## F-statistic: 177.1 on 3 and 244 DF,  p-value: < 2.2e-16
```

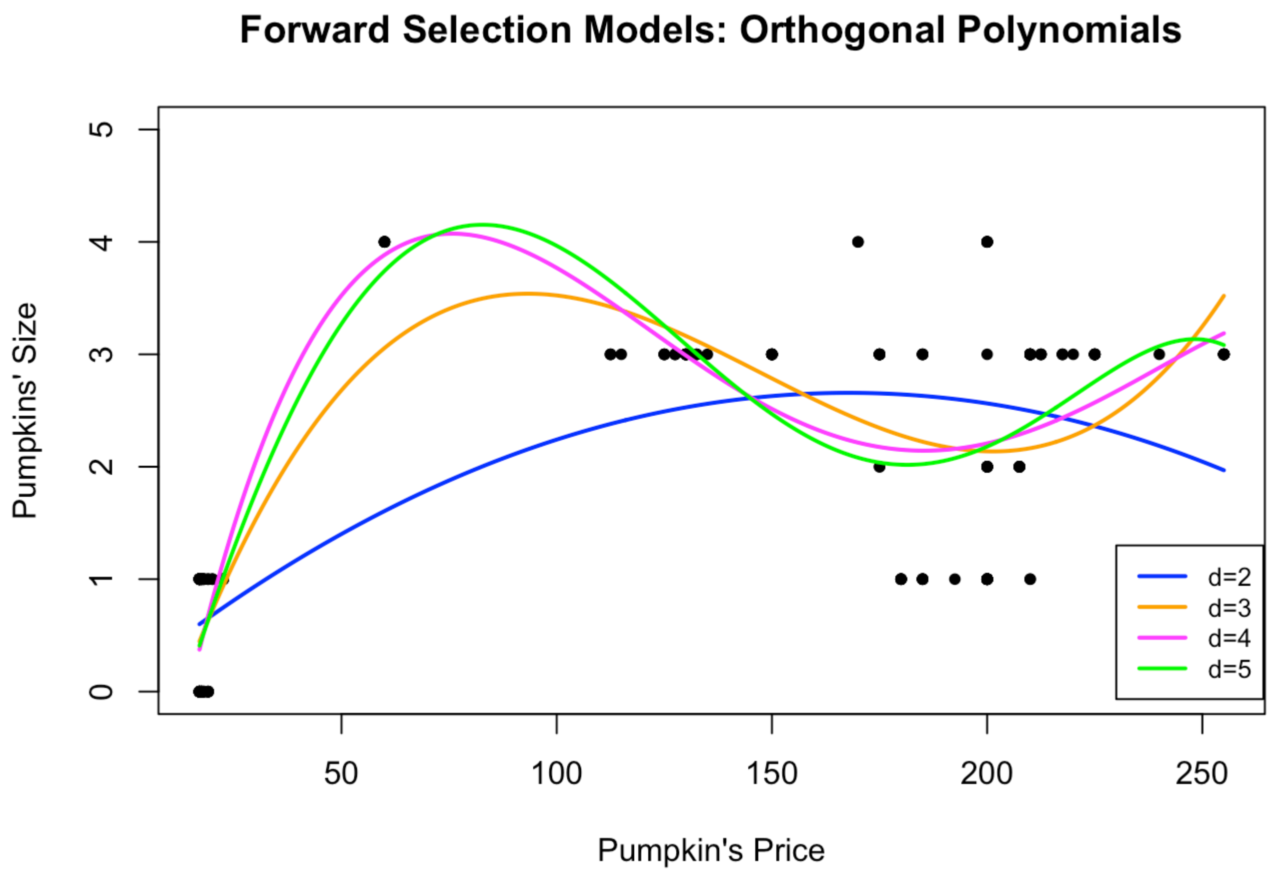
```
lm.pumpkins04 =  lm(size ~ poly(price,4), data=pumpkins)
summary(lm.pumpkins04)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 4), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.3200 -0.4497 -0.1241  0.5539  1.7925
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)      1.70161    0.04407  38.608  < 2e-16 ***
## poly(price, 4)1 13.37846    0.69408  19.275  < 2e-16 ***
## poly(price, 4)2 -5.76139    0.69408  -8.301 7.22e-15 ***
## poly(price, 4)3  7.86718    0.69408  11.335  < 2e-16 ***
## poly(price, 4)4 -2.96423    0.69408  -4.271 2.80e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6941 on 243 degrees of freedom
## Multiple R-squared:  0.7073, Adjusted R-squared:  0.7025
## F-statistic: 146.8 on 4 and 243 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins05 =  lm(size ~ poly(price,5), data=pumpkins)
summary(lm.pumpkins05)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 5), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.38082 -0.46537 -0.08211  0.53463  1.91954
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)       1.70161    0.04382  38.831  < 2e-16 ***
## poly(price, 5)1 13.37846    0.69009  19.387  < 2e-16 ***
## poly(price, 5)2 -5.76139    0.69009  -8.349 5.35e-15 ***
## poly(price, 5)3  7.86718    0.69009  11.400  < 2e-16 ***
## poly(price, 5)4 -2.96423    0.69009  -4.295 2.53e-05 ***
## poly(price, 5)5 -1.34841    0.69009  -1.954   0.0519 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6901 on 242 degrees of freedom
## Multiple R-squared:  0.7118, Adjusted R-squared:  0.7059
## F-statistic: 119.6 on 5 and 242 DF,  p-value: < 2.2e-16
```

```
newprice = data.frame(price=seq(17, 255, 1))
plot(pumpkins$price, pumpkins$size, pch=20,  ylim=c(0,5),xlab="Pumpkin's Price",
lines(newprice$price, predict(lm.pumpkinsO2, newprice), col="blue", lty=1, lwd=2)
lines(newprice$price, predict(lm.pumpkinsO3, newprice), col="orange", lty=1, lwd=
lines(newprice$price, predict(lm.pumpkinsO4, newprice), col="magenta", lty=1, lwc
lines(newprice$price, predict(lm.pumpkinsO5, newprice), col="green", lty=1, lwd=2
legend(230, 1.3, legend=c("d=2", "d=3", "d=4", "d=5"),
       col=c("blue", "orange", "magenta", "green"), lty=c(1,1,1,1), cex=0.8, lwd=
```




```
# Backward Selection
lm.pumpkinsO10 =  lm(size ~ poly(price,10), data=pumpkins)
summary(lm.pumpkinsO10)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 10), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.44082 -0.45530 -0.01853  0.53535  2.12546
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)        1.70161    0.04117  41.335  < 2e-16 ***
## poly(price, 10)1  13.37846    0.64829  20.636  < 2e-16 ***
## poly(price, 10)2  -5.76139    0.64829  -8.887  < 2e-16 ***
## poly(price, 10)3   7.86718    0.64829  12.135  < 2e-16 ***
## poly(price, 10)4  -2.96423    0.64829  -4.572 7.77e-06 ***
## poly(price, 10)5  -1.34841    0.64829  -2.080  0.03861 *
## poly(price, 10)6  -1.45456    0.64829  -2.244  0.02578 *
## poly(price, 10)7  -2.57955    0.64829  -3.979 9.20e-05 ***
## poly(price, 10)8   2.03378    0.64829   3.137  0.00192 **
## poly(price, 10)9   1.51698    0.64829   2.340  0.02012 *
## poly(price, 10)10  0.65591    0.64829   1.012  0.31269
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6483 on 237 degrees of freedom
## Multiple R-squared:  0.7509, Adjusted R-squared:  0.7404
## F-statistic: 71.45 on 10 and 237 DF,  p-value: < 2.2e-16
```

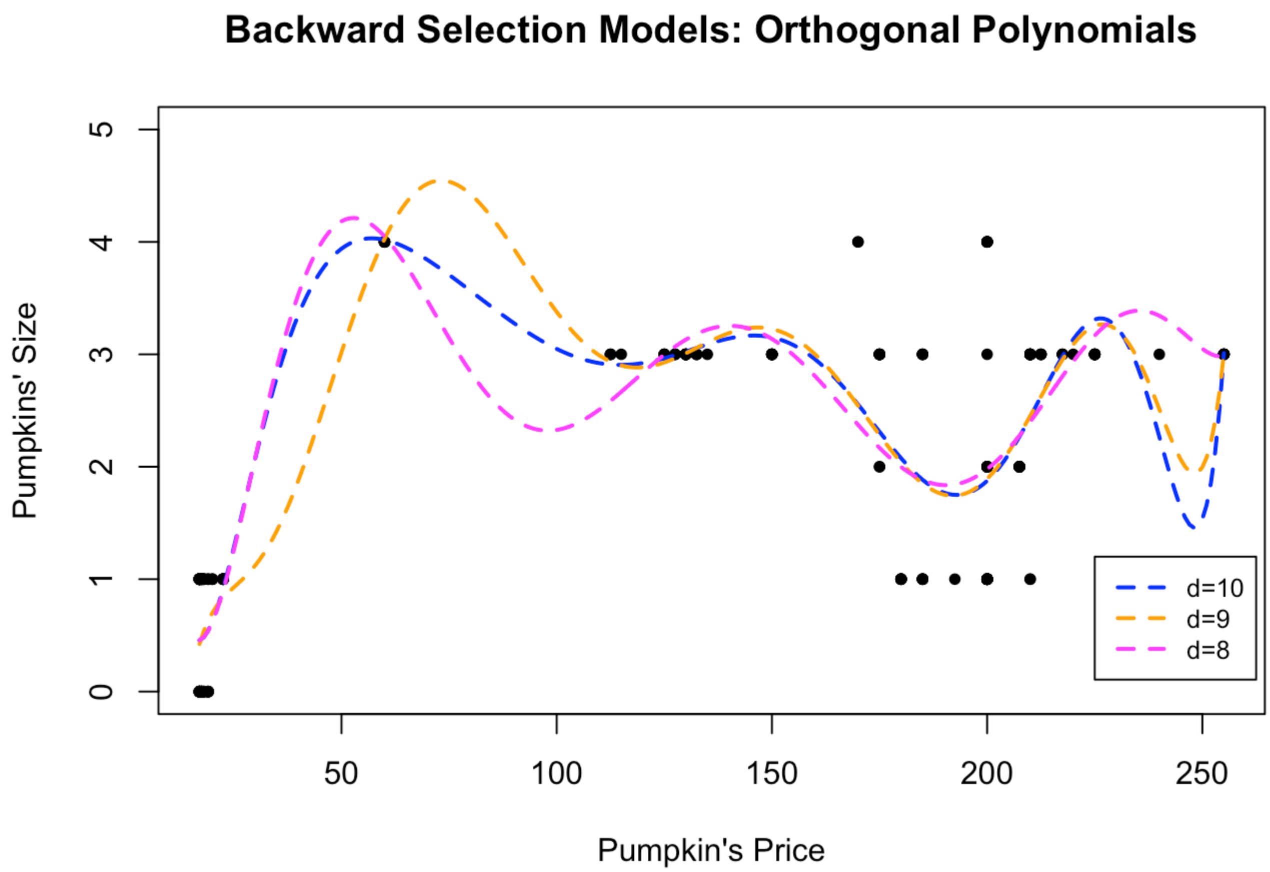
```r
lm.pumpkins09 =  lm(size ~ poly(price,9), data=pumpkins)
summary(lm.pumpkins09)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 9), data = pumpkins)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.4514 -0.4230 -0.0091  0.5177  2.1072
##
## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)       1.70161    0.04117  41.333  < 2e-16 ***
## poly(price, 9)1 13.37846    0.64833  20.635  < 2e-16 ***
## poly(price, 9)2 -5.76139    0.64833  -8.887  < 2e-16 ***
## poly(price, 9)3  7.86718    0.64833  12.135  < 2e-16 ***
## poly(price, 9)4 -2.96423    0.64833  -4.572 7.76e-06 ***
## poly(price, 9)5 -1.34841    0.64833  -2.080  0.03861 *
## poly(price, 9)6 -1.45456    0.64833  -2.244  0.02578 *
## poly(price, 9)7 -2.57955    0.64833  -3.979 9.20e-05 ***
## poly(price, 9)8  2.03378    0.64833   3.137  0.00192 **
## poly(price, 9)9  1.51698    0.64833   2.340  0.02012 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6483 on 238 degrees of freedom
## Multiple R-squared:  0.7499, Adjusted R-squared:  0.7404
## F-statistic: 79.27 on 9 and 238 DF,  p-value: < 2.2e-16
```

```
lm.pumpkins08 =  lm(size ~ poly(price,8), data=pumpkins)
summary(lm.pumpkins08)
```

```
##
## Call:
## lm(formula = size ~ poly(price, 8), data = pumpkins)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.39988 -0.45678 -0.05827  0.53309  2.02119
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)      1.70161    0.04155  40.951  < 2e-16 ***
## poly(price, 8)1 13.37846    0.65437  20.445  < 2e-16 ***
## poly(price, 8)2 -5.76139    0.65437  -8.805 2.71e-16 ***
## poly(price, 8)3  7.86718    0.65437  12.023  < 2e-16 ***
## poly(price, 8)4 -2.96423    0.65437  -4.530 9.32e-06 ***
## poly(price, 8)5 -1.34841    0.65437  -2.061 0.040421 *
## poly(price, 8)6 -1.45456    0.65437  -2.223 0.027162 *
## poly(price, 8)7 -2.57955    0.65437  -3.942 0.000106 ***
## poly(price, 8)8  2.03378    0.65437   3.108 0.002112 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6544 on 239 degrees of freedom
## Multiple R-squared:  0.7441, Adjusted R-squared:  0.7355
## F-statistic: 86.87 on 8 and 239 DF,  p-value: < 2.2e-16
```

```
plot(pumpkins$price, pumpkins$size, pch=20, ylim=c(0,5), xlab="Pumpkin's Price",
lines(newprice$price, predict(lm.pumpkinsO10, newprice), col="blue", lty=2, lwd=2
lines(newprice$price, predict(lm.pumpkinsO9, newprice), col="orange", lty=2, lwd=
lines(newprice$price, predict(lm.pumpkinsO8, newprice), col="magenta", lty=2, lwc
legend(225, 1.2, legend=c("d=10", "d=9", "d=8"),  col=c("blue", "orange", "magent
```



4.4.3 Piece-wise Polynomials

If the true mean of $\mathrm{E}(Y|X=x)=f(x)$ is *too wiggly*, we might need to fit a higher order polynomial, which is not always a good idea. Instead we consider **piece-wise polynomials**:

1. we divide the range of x into several intervals, and
2. within each interval, $f(x)$ is a low-order polynomial, e.g., cubic or quadratic, but the polynomial coefficients will be different from interval to interval
3. we require the overall $f(x)$ to be continuous up to certain derivatives.

This method is also called *"broken-stick regression"*. Its benefit is that it localizes the influence of each data point to a particular segment, but overall it is not a very smooth line as the one we obtain by fitting a single polynomial for the whole data set.