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Quantum Cognition

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Abstract

Uncertainty is an intrinsic part of life; most events, affairs, and questions are uncertain. A key problem in behavioral sciences is how the mind copes with uncertain information. Quantum probability theory offers a set of principles for inference, which align well with intuition about psychological processes in certain cases: cases when it appears that inference is contextual, the mental state changes as a result of previous judgments, or there is interference between different possibilities. We motivate the use of quantum theory in cognition and its key characteristics. For each of these characteristics, we review relevant quantum cognitive models and empirical support. The scope of quantum cognitive models encompasses fallacies in decision-making (such as the conjunction fallacy or the disjunction effect), question order effects, conceptual combination, evidence accumulation, perception, over-/underdistribution effects in memory, and more. Quantum models often formalize psychological ideas previously expressed in heuristic terms, allow unified explanations of previously disparate findings, and have led to several surprising, novel predictions. We also cast a critical eye on quantum models and consider some of their shortcomings and issues regarding their further development.

Contents

INTRODUCTION

Information in our environment is mostly uncertain, and much of cognition is about managing this uncertainty to generate useful conclusions. For example, we worry about things like whether it will rain tonight (a natural worry of Londoners); whether next year will be free from dangerous viruses; and the cause of some funny spots suddenly appearing on our child's face. What are the foundations for our capacity for probabilistic inference? This is a question which encompasses research related to three Nobel Prizes in Economics (awarded to Herbert Simon in 1978, Daniel Kahneman in 2002, and Richard Thaler in 2017), a philosophical debate that goes back to antiquity, and surprising implications for norms for correct reasoning.

Probabilities are used to quantify uncertainty and make inferences from uncertain premises. A probability theory is a set of mathematical axioms for how to combine and update probabilities. This review concerns three overarching traditions regarding the relevance of probability theory to cognition. The first is Bayesian/classical probability theory (CPT). CPT axioms embody some of our basic intuitions regarding how to deal with probabilities. In a famous quote by Laplace (cited in Perfors et al. 2011, p. 313), CPT is described as "nothing but common sense reduced to calculation." The axioms of CPT number only four, yet they are the foundation of a mathematical edifice which encompasses any kind of probabilistic reasoning. CPT cognitive models have clearly attracted great interest in the last few decades (Griffiths et al. 2010, Oaksford & Chater 1994, Tenenbaum et al. 2011). Second, there are heuristics and biases, a toolbox of rules which offer fast and frugal accounts that describe numerous behavioral findings. Heuristics and biases have also attracted significant interest (Gigerenzer & Todd 1999, Kahneman et al. 1982). Third, there is quantum probability theory (QPT), which is a newer direction and is the focus of the present review. Like CPT, QPT is a general probability theory, that is, a set of rules for how to combine and update probabilities. QPT and CPT axioms are different, so we often reach different conclusions when we employ QPT versus CPT. We can consider any of the questions above (e.g., "Will it rain tonight?") and compute the corresponding probabilities with either CPT or QPT.

Some readers may have come across quantum mechanics, which is a theory of physics. The pioneering physicists who developed quantum mechanics soon realized that CPT was not suitable for this new physical theory—it seemed that uncertain information for microscopic particles obeyed probability rules different from the familiar ones from CPT. So, together with a new physics theory, they developed a new theory of probability as well—what we call QPT. QPT is the theory of probability from quantum mechanics, without any of the physics. In fact, Bohr (1958), one of the founding fathers of quantum theory, was one of the earliest to propose that principles of quantum physics, such as complementarity, could be applied outside of physics to human knowledge (for a recent example, see Lu & Busemeyer 2014). An important qualification is that the use of QPT in cognitive science makes no assumptions regarding the nature of brain neurophysiology; all current quantum cognitive models do not rely on a quantum brain hypothesis, which has been heatedly contested (Hameroff 2007, Litt et al. 2006).

As with the physicists who developed quantum mechanics, some pioneering researchers in psychology have asked whether there are cognitive phenomena for which CPT or heuristic explanations are not sufficiently satisfactory. These researchers initiated the quantum cognition research program (Aerts & Aerts 1995, Atmanspacher et al. 2002, Bordley 1998, Khrennikov 1999). However, the application of these ideas to empirical data started becoming more widespread after the publication of a special issue of the *Journal of Mathematical Psychology* about 10 years ago (Bruza & Gabora 2009). Overviews of the key ideas and advances are presented by Ashtiani & Azgomi (2015), Bruza et al. (2015b), Busemeyer & Bruza (2011), and Pothos & Busemeyer (2013). Haven & Khrennikov (2013) and Wendt (2015) describe applications beyond psychology in social sciences. These references also serve as tutorials for the QPT formalism (along with Yearsley 2017 and Yearsley & Busemeyer 2016). Note that describing these contributions as the quantum cognition research program has two purposes. First, it brings together models which employ broadly similar mathematical tools and concepts, those from QPT. Second, it implies a commitment to the specific way in which probabilities are computed, from quantum mechanics, and the associated mathematical theorems (the Born rule, Gleason's theorem, the Kochen–Specker and Bell theorems, the Lüders postulate).

The general questions which guide this review include the following: How general is the applicability of probabilistic reasoning in cognitive processing? If we see a part of cognition as a probabilistic engine, when is the mind better described by CPT versus QPT principles? What is the importance of context in cognition, and how can we formalize contextual influences? How can prior decisions shape subsequent thought?

MOTIVATION FOR QUANTUM PROBABILITY THEORY IN COGNITION

For the average psychologist, the proposal that quantum theory might have something to do with cognition initially shades between ambitious and implausible. Early QPT models have been met with skepticism. These early models therefore had to focus on the behavioral findings that have proved the most persistently challenging for classical (CPT or otherwise) formalisms. In this section, we outline some of the findings which initially motivated QPT models, reserving additional empirical coverage for subsequent sections.

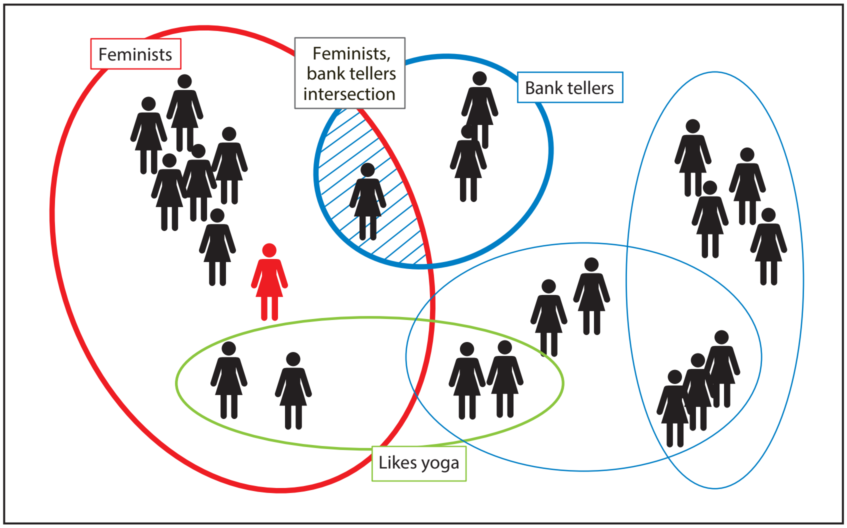

Figure 1

A classical probability theory representation for the conjunction fallacy from Tversky & Kahneman (1983). We consider a large set of all possible realizations for Linda that we can imagine. The red and blue ellipses correspond to the subsets of possible Lindas consistent with the bank teller and feminist properties (other subsets correspond to other properties). The conjunction is the intersection between these two subsets. The "real" Linda is shown in red, representing all possible information about Linda.

CPT inference is considered to be rational (de Finetti et al. 1993). Additionally, employing CPT forces decisions to be consistent with the basic CPT axioms. These and other powerful arguments have been made for adopting CPT in inference (Griffiths et al. 2010). So why are there apparent discrepancies between CPT prescription and human behavior? The key problem is that full CPT inference can be intractably complex for many real problems; even limited situations can require effortful computation. Limiting CPT to make processes manageable by realistic agents is the fundamental problem of bounded rationality (Simon 1955)—and of course there is no single answer. Moreover, there is abundant evidence that human inference is characterized by a mix of more analytic and more intuitive inference, where the former can sometimes be associated with more accurate approximations of CPT processes (Elqayam & Evans 2013, Fernbach & Sloman 2009, Kahneman 2001, Sloman 1996).

Despite the many successes of CPT, researchers have accumulated a large body of empirical findings that are hard (though not impossible) to reconcile with CPT cognitive models. These findings challenge the cognitive ubiquity of some of the most basic CPT intuitions. Consider the conjunction fallacy (Tversky & Kahneman 1983) (see **Figure 1**). In one example, participants were told of a hypothetical person, Linda, who was described as looking like a feminist but not a bank teller. Participants were asked to rank-order the likelihood of several statements about Linda. The critical statements were that Linda is a feminist, Linda is a bank teller, and Linda is a feminist and a bank teller. The results indicated that

$$P(\text{feminist}) > P(\text{feminist \& bank teller}) > P(\text{bank teller}),$$

where $P(X)$ indicates the probability of event X. The conjunction fallacy refers to the finding that the conjunction is judged more probable than the bank teller possibility alone.

Why is this result problematic for CPT? The formal foundations of CPT are essentially set theory. To compute CPT probabilities, we calculate fractions for the outcomes of interest relative to all possible outcomes. This is best explained with the conjunction fallacy variant presented by Tentori et al. (2004), which concerned Scandinavian individuals and the probabilities that they might have blue eyes and blond hair. The key finding was

$$P(\text{blond hair \& blue eyes}) > P(\text{blond hair}).$$

P(blond hair) would be computed as the fraction of blond Scandinavians relative to all Scandinavians. We can imagine enumerating Scandinavian individuals, selecting out the ones with blond hair, and then from this subset selecting out the ones with blue eyes (and blond hair). It seems clearly incorrect that there will be more Scandinavian individuals in the conjunctive set than in the set corresponding to the individual premise, yet this is what people do—and we, erudite readers, may also find it hard to avoid the intuition that the conjunction seems more probable than the individual statements (Gilboa 2000). Tentori et al.'s (2004) formulation makes the set-theoretic structure of the problem obvious, but the CPT situation is identical whether we employ frequentist probabilities (as in Tentori et al. 2004) or probabilities as subjective degrees of belief (Tversky & Kahneman 1983). The conjunction fallacy is an extensively replicated finding and has resisted all kinds of disambiguation manipulations to ensure that participants correctly understand the conjunction and individual statements as intended (as opposed to, for example, understanding 'bank teller' as 'bank teller & not feminist'; Dulany & Hilton 1991, Moro 2009).

There is also a disjunction fallacy—the probability that Linda is a bank teller or a feminist is judged to be less likely than the probability that she is a feminist alone (Bar-Hillel & Neter 1993, Carlson & Yates 1989)—and other, related fallacies including unpacking effects (Rottenstreich & Tversky 1997, Sloman et al. 2004), as well as more complex conjunctions (Gronchi & Strambini 2017, Winman et al. 2010). Conjunction and disjunction types of fallacies also occur in conceptual combinations called overextensions and underextensions, respectively (Aerts et al. 2016; Hampton 1988a,b), and in memory processes (Brainerd & Reyna 2008; Brainerd et al. 1999, 2015).

Next, consider a Prisoner's Dilemma game in which two players make a choice to defect or cooperate and receive payoffs depending on their combined choices. Shafir & Tversky (1992) designed a study using this game to test a rational axiom of decision-making called the sure-thing principle (Savage 1954). According to this principle, if under each possible state of the world one always prefers action A over B, then one should prefer action A over B even when the state of the world is unknown. Shafir & Tversky (1992) tested this principle by examining three conditions: one in which a player was informed that their opponent had already defected, another in which the player was informed that the opponent cooperated, and a third in which the opponent's play remained unknown. Many players chose to defect when the opponent defected and when the opponent cooperated, but then switched and decided to cooperate in the unknown case, violating the sure-thing principle. This result, called the disjunction effect, shows a violation of the law of total probability (**Figure 2**). As for conjunction fallacies, violations of the law of total probability challenge CPT intuition at a basic level. The number of times a participant cooperates splits cleanly into the number of times the participant cooperates and the opponent cooperates or defects:

$$\begin{aligned} P(\mathrm{C}_{\text{participant}}) &= P(\mathrm{C}_{\text{participant}} \,\&\, \mathrm{C}_{\text{opponent}}) + P(\mathrm{C}_{\text{participant}} \,\&\, \mathrm{D}_{\text{opponent}}) \\ &= P(\mathrm{C}_{\text{participant}}|\mathrm{C}_{\text{opponent}}) \cdot P(\mathrm{C}_{\text{opponent}}) + P(\mathrm{C}_{\text{participant}}|\mathrm{D}_{\text{opponent}}) \cdot P(\mathrm{D}_{\text{opponent}}). \end{aligned}$$

This expression involving conditional probabilities corresponds to what is measured and shows that $P(\mathrm{C}_{\text{participant}})$ is bounded between $P(\mathrm{C}_{\text{participant}}|\mathrm{C}_{\text{opponent}})$ and $P(\mathrm{C}_{\text{participant}}|\mathrm{D}_{\text{opponent}})$.

Contrary to this prediction, the probability to cooperate in the unknown condition was 37%, which is higher than the probabilities to cooperate both when the opponent was known to defect

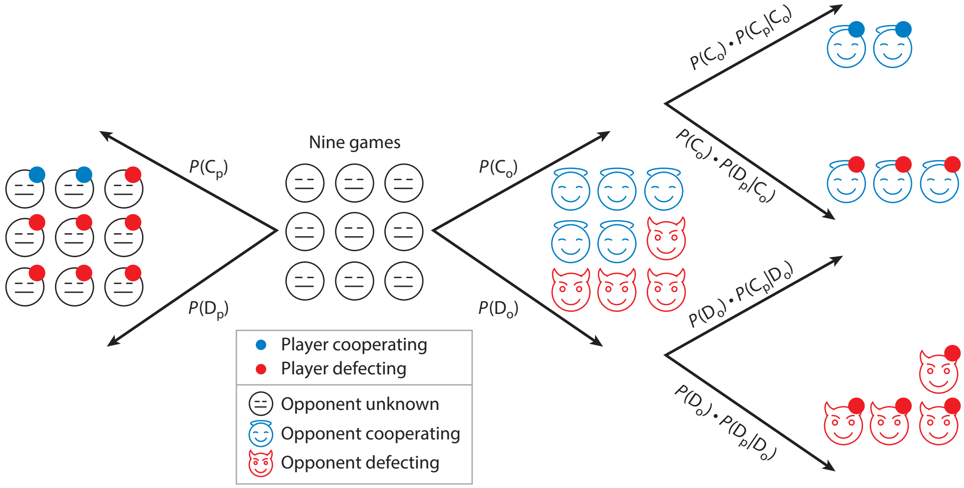

Figure 2

On the left versus right, the player decides to defect or cooperate without versus with knowledge of the opponent's action, respectively. The law of total probability means that the number of times the player cooperates is cleanly split into the number of times the player cooperates and the opponent cooperates versus defects (there is the same number of blue/red dots in both panels). C_p and D_p refer to the player cooperating or defecting, and C_o and D_o refer to the opponent cooperating or defecting.

(3%) and when the opponent was known to cooperate (16%) (Shafir & Tversky 1992; see also Busemeyer et al. 2006, Croson 1999, Kvam et al. 2014, Tesař 2020). Similar results were obtained in other tasks, notably a two-stage gambling task (Broekaert et al. 2020, Tversky & Shafir 1992) and a categorization–decision paradigm (Townsend et al. 2000, Wang & Busemeyer 2016).

Another example of a puzzling result lacking a natural classical explanation concerns order effects in judgment. A popular example is the following. Using a Gallup opinion poll, Moore (2002) examined the probability of "yes" responses to the pair of questions "Is Clinton honest?" and "Is Gore honest?" presented in both possible orders. When the Clinton question was first, Moore (2002) observed $P(\text{Clinton}_{\text{yes}}) = 0.50$ and $P(\text{Gore}_{\text{yes}}) = 0.68$. When the order was reversed, $P(\text{Gore}_{\text{yes}}) = 0.60$ and $P(\text{Clinton}_{\text{yes}}) = 0.57$. This is a surprising difference, both because the putative cause, the change in ordering, appears innocuous and because of the practical importance of the result: For many public debates, Gallup polls can have a substantial influence on opinion, especially for undecided individuals. Similar order effects have been reported in assessing evidence, for example, concerning the probability of a disease given two pieces of evidence presented in different orders, with participants who were medical professionals (Bergus et al. 1998) and in mock jury decision situations (McKenzie et al. 2002, Trueblood & Busemeyer 2011). Order effects also occur with similarity judgments. For example, Tversky (1977) found that judging the similarity of Korea to China produced higher ratings than judging these two countries the other way around. Explanations for question order effects invariably invoke ideas related to the constructive nature of judgment; for example, answers to earlier questions activate thoughts which affect our perspective on later ones (Hogarth & Einhorn 1992, Schwarz 2007). These order effects can naturally be explained by the noncommutative nature of QPT. However, they are more challenging for CPT, because conjunction in CPT is commutative, which means that order does not matter: $P(A \& B) = P(B \& A)$. Commutativity sometimes makes intuitive sense, but again we are confronted with empirical situations which diverge from this CPT intuition.

People are sensitive to measurements, such as asking an opinion, and taking a measurement can change their future behavior. This is difficult for CPT theories, because when one is asked a question (e.g., "Is Linda a feminist?"), one's answer is based on existing (albeit not accessed) knowledge and the answer does not create any new information. No Bayesian updating is required, so the answer should not affect responses to future questions. Surprisingly, human cognition sometimes does not work like this. Asking for an opinion or making a judgment can apparently change the relevant mental state, as evidenced by effects on subsequent behavior (Ariely & Norton 2008, Brehm 1956, Lichtenstein & Slovic 2006, Schwarz 2007, Sharot et al. 2010). For example, in a hypothetical legal case, Holyoak & Simon (1999) showed that the evaluation of arguments changed to become more consistent with the produced verdict. Such effects can be called constructive influences, because the act of, for instance, making a decision helps construct a particular mental state. These constructive influences naturally agree with QPT because it was originally formulated by physicists to account for the fact that measurement can change and disturb the state of a system (a cognitive system in our case).

The tension between such so-called fallacies and (especially) CPT is not insurmountable, so these findings do not disprove the applicability of CPT. Below, we review some principled attempts to reconcile fallacies, such as the conjunction fallacy, with CPT (Costello & Watts 2014, Tentori et al. 2013, Zhu et al. 2020). However, for many researchers, fallacies do reveal a persistent tension between fundamental CPT principles and human intuition. Below, we also consider psychological accounts of the fallacies outside CPT, spanning ideas across areas as diverse as cognitive and social psychology. These ideas are often expressed in the form of simple heuristics, which are principles intended to explain specific aspects of cognition (Gigerenzer & Goldstein 1996, Hertwig et al. 2013, Tversky & Kahneman 1983).

There is little doubt that at least some of cognition relies on probabilistic reasoning captured by CPT reasoning, while other parts of cognition rely on a toolbox of simple heuristics. However, these two opposing views cannot easily be integrated in a formal way. A key idea motivating the use of quantum theory in cognition is that it lies between CPT models (which are constrained by strict axiomatic rules) and simple heuristics (which are free from any axiomatic constraints) by employing axioms that are less constraining than CPT but more formal and systematic than simple heuristics.

The consideration of QPT in behavioral modeling is appealing for three reasons. First, it provides a coherent explanation for the wide range of puzzling findings summarized above, using a common set of principles. For example, the same QPT principles that account for probability judgment fallacies, such as the conjunction fallacy, can also account for the disjunction effect in decision-making. Such common explanations reveal links between findings that had not previously been considered together. Second, QPT enriches psychological theory with several novel concepts, such as incompatibility, superposition, collapse, and entanglement, which are the foundation for new hypotheses and have led to the discovery of new phenomena. Finally, in a certain formal way, QPT represents the next step away from CPT (Sorkin 1994), if computations are required to have a more local focus. As discussed in the next section, in QPT there are compatible and incompatible questions. For compatible questions, everything is classical; for incompatible ones, apparent classical errors and inconsistencies arise. QPT can be considered a more local version of CPT, where, instead of having a large space of classical questions, classical inference exists only within smaller subsets of questions (which are compatible with one another); across subsets, questions are incompatible (cf. Fernbach & Sloman 2009, Lewandowsky et al. 2002, Pothos et al. 2021, Trueblood et al. 2017).

BASIC DESCRIPTION OF QUANTUM PROBABILITY THEORY

The predominance of CPT intuition has been so complete that it is hard to imagine alternatives. To understand how QPT provides an alternative form of probabilistic intuition, we briefly consider the picture of the world according to each probability theory, using Linda from the conjunction fallacy example. In each case, we have to represent our uncertain information about Linda, formulate different questions about possible characteristics she might have, and then estimate the probabilities of question outcomes (see also Busemeyer & Bruza 2011, Khrennikov 2014).

CPT begins with a sample space such as that shown in **Figure 1**, which contains all the various possible realizations for Linda, represented by the female characters. A possible outcome of a question, such as whether or not Linda is a feminist, is represented by a subset of the sample space, such as the red bounded ellipse for yes to feminism. The outcome from a pair of questions, such as whether Linda is a feminist and bank teller, is represented by the intersection of subsets, as in the shaded region of overlap between the red and blue ellipses. The beliefs a person has about these questions are represented by a probability function that assigns a probability to each subset. For example, the probability that Linda is a feminist is the probability assigned to the red ellipse. The probabilities assigned to the union of mutually exclusive events must add. The larger the subset for a question outcome is, the more possible Lindas we can imagine consistent with this question outcome, and the more likely this question outcome will be—that is, the probability of a question outcome depends on the size of the corresponding subset. Crucially, we could resolve all possible questions about Linda and identify the "real" Linda in the sample space.

Note that the subset for 'bank teller' is formed by the union of the subset for 'bank teller & feminist' with the subset for 'bank teller & not feminist.' Consequently, the probability of 'bank teller' equals the sum of the probabilities assigned to these two events and thus must exceed the probability of the single 'bank teller & feminist' event. This illustrates why the conjunction fallacy is so puzzling according to CPT.

QPT begins with what is essentially a vector space, such as the two-dimensional space shown in **Figure 3**. A vector can be thought of as a line with a specific direction and length (e.g., arrows in **Figure 3**). Such a vector space contains all possible outcomes for questions about Linda. For example, for the question "Is Linda a bank teller?" there are two question outcomes (yes or no), represented by two unit-length vectors at a 90° angle to each other, basically forming the x and y axes. Specifically, in **Figure 3**, the outcome that Linda is a bank teller is represented by the vertical axis and that she is not a bank teller by the horizontal axis. The answers to a different question, like the feminist question, can be represented by a different pair of orthogonal vectors rotated by some angle. That is, the x and y axes for different questions are related to each other through simple rotations. A vector representing a question outcome spans a one-dimensional subspace, called a ray. Subspaces are very important in QPT; QPT is a way to assign probabilities to subspaces, and the key difference between QPT and CPT is that the latter involves assigning probabilities to subsets.

The set of beliefs a person has about these questions is represented by a (unit length) state vector (**Figure 3**). The probability of a question outcome is obtained by projecting (i.e., laying down) the state vector onto the subspace representing the answer, and then computing the squared length. To compute the conjunction of two question outcomes, we typically have to employ a sequential projection, which corresponds to resolving one question after the other. For example, suppose we are interested in P(F & BT). **Figure 3** shows that there is no single ray corresponding to both feminism and being a bank teller. Therefore, we have to compute P(F & then BT), which involves two steps. First, we project the state vector onto the F ray. Second, we project this previous projection onto the BT ray. P(F & then BT) is the squared length of the last projection. Alternatively, suppose we want to compute P(BT & then F). We first project the state vector to the BT ray and then project the resulting projection to the F ray.

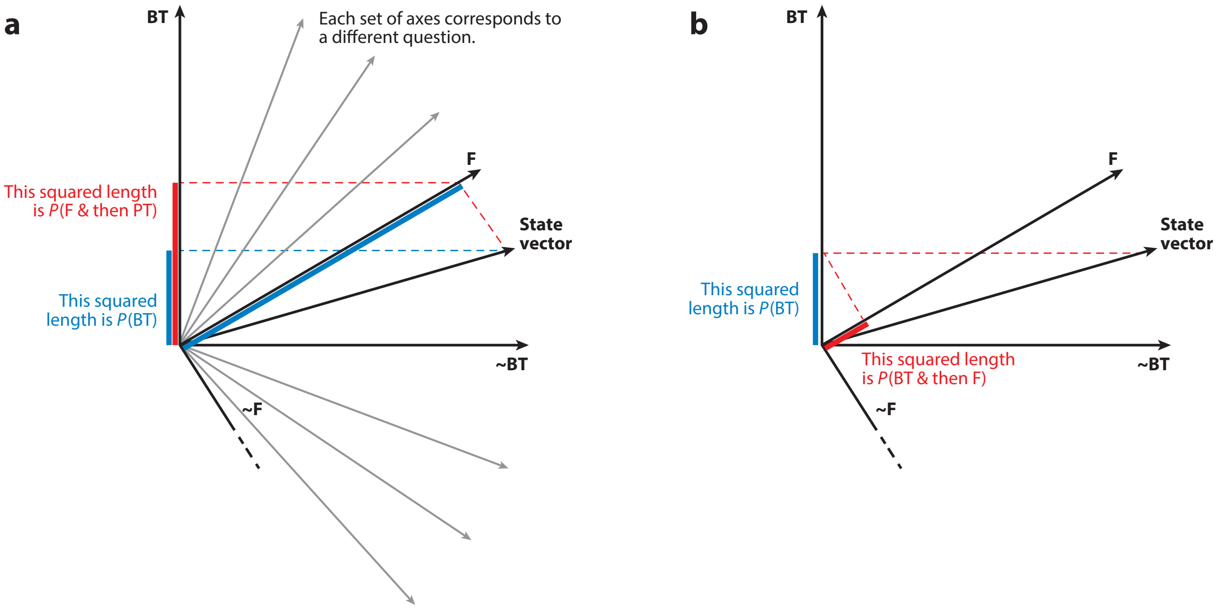

Figure 3

A quantum probability theory representation of the Linda problem in two dimensions. Particular axes (one-dimensional subspaces, also called rays) correspond to question outcomes. Probabilities are computed as the squared length of the projection of the state vector onto the corresponding axis. In panel *a*, the projection onto feminism is the blue bar on the F line, and the probability equals the squared length of this bar. The red bar along the BT ray illustrates *P*(F & then BT). To compute *P*(F & then BT), first project the state vector onto the F ray, then project this previous projection onto the BT ray (follow the two red perforated lines). In panel *b*, the red bar along the F ray illustrates *P*(BT & then F). To compute *P*(BT & then F), first project the state vector onto the BT ray, then project the resulting projection to the F ray (follow the red perforated lines). Abbreviations: BT, bank teller; F, feminist; tilde (~), not.

These simple computations illustrate two key properties of QPT. First,

$$P(\text{F \& then BT}) \neq P(\text{BT \& then F});$$

that is, a sequence of projections can be noncommutative. Unlike in CPT, order matters, and such noncommutativity leads to interference effects (discussed in detail in the section titled Quantum Interference, below). Second, every time we resolve a question, the state vector has to change in a specific way. This is the QPT property of vector collapse (also considered below).

As for CPT, we might wonder what happens if we try to resolve as much uncertainty about Linda as possible, by considering all the possible questions about Linda. First, suppose we determine that Linda is a feminist. We would then have to place the state vector along the 'feminist' subspace, since in this way we have maximum overlap between state vector and subspace. If we then decide that Linda is not a bank teller, we would align the state vector with the subspace for 'not-bank-teller.' But a state vector aligned with the 'not-bank-teller' subspace has partial overlap with both the 'feminist' and 'nonfeminist' subspaces. That is, resolving the bank teller question made us uncertain about the feminist one!

In general, in QPT, it is often impossible to resolve all questions concurrently, and certainty about one question introduces uncertainty about others; that is, there are uncertainty relations. The impossibility of complete knowledge contrasts sharply with CPT intuition. Psychologically, at a broad level, these ideas resonate with a view of cognition as strongly context and perspective dependent (Fodor 1983). Answers to questions like whether Linda is a bank teller and a feminist in **Figure 3**, for which there are uncertainty relations, are called incompatible and are unique to QPT. Two questions are incompatible if the presence of one question alters our perspective for the other (that is, if the questions are contextual) or if the questions are such that answering one question disturbs the relevant system for the other one. For example, we might want to ask a teenager the pair of questions "How well is the revision of your paper going?" and "What will you do this Friday night?" We can imagine that the first question makes the teenager anxious, so that he understands the second question differently than if we had asked the second question individually. Other pairs of answers can be compatible, such as whether Linda is a feminist or not and whether she has black hair or not (i.e., it seems unlikely that, for these questions, answering one changes our perspective for the other). For compatible questions, the probabilities add up just like in CPT. Therefore, when we contrast QPT with CPT below, we focus mostly on incompatible questions.

In **Figure 3**, the sequential probability P(F & then BT) is greater than the direct probability P(BT); that is, even with this simple representation, we have an example where P(F & then BT) > P(BT). Such an approach, suitably generalized, could therefore be a cognitive model for the conjunction fallacy (Busemeyer et al. 2011). An interesting question is: Why would someone ever think to use this approach to model the conjunction fallacy? There are three parts to the answer. First, CPT does not appear to offer a natural approach to understanding the conjunction fallacy (but note that there are more elaborate CPT models, which we consider below). Second, heuristic explanations developed for the original conjunction fallacy, based on, for instance, the similarity between Linda and her various possible characteristics (Tversky & Kahneman 1983), are intuitive, but vague and with limited predictive value. Third, since the conjunction fallacy problem is about probabilistic judgment, we might be biased to develop a model for the conjunction fallacy based on some kind of probability theory. What is a probability theory, not constrained in the same way as CPT, which is consistent with the intuitions expressed in the heuristic, similarity-based models for the conjunction fallacy? The most immediate answer is QPT.

One of the important points of this review is that QPT provides a notion of probabilistic correctness that is an alternative to CPT. Classically, the conjunction fallacy appears blatantly incorrect, in the same way that $1 + 1 = 3$ appears incorrect (**Figure 1**). Yet if we accept the QPT axioms, a conjunction fallacy can seem reasonable. Moreover, QPT inference can be said to be rational on exactly the same basis as CPT. A hallmark of rationality in probabilistic inference is the Dutch book theorem. Suppose you have to assign probabilities to a combination of bets. The Dutch book theorem states that, if probabilities obey a small set of requirements, then you are guaranteed to be protected from a certain loss (otherwise, there may be combinations of bets such that, whatever happens, you will suffer a net loss). CPT is consistent with the Dutch book theorem requirements (Oaksford & Chater 2007), but, interestingly, so is QPT (Pothos et al. 2017). How can a decision like the conjunction fallacy be both correct and incorrect, both rational and irrational? This boils down to whether the corresponding questions can be assumed to be compatible or incompatible. If the latter, then QPT is the appropriate theory of inference; if the former, then CPT is (in which case QPT agrees with CPT). So, QPT will be the correct and rational way to approach certain questions.

In the case of an imaginary Linda, a person's understanding of what is meant by 'bank teller' in the context of questions about feminism might be different from the understanding emerging without that context, in which case the conjunction fallacy is not unreasonable. However, sometimes naïve observers represent questions as incompatible, when it seems clear that they are not. In the case of a Scandinavian person, we can easily check whether or not a person has blue eyes and blond hair, so these questions should be treated as compatible. We may conclude that the conjunction fallacy in this case is a mental error of representing as incompatible questions which are not so. Why might this occur? There is evidence that lack of familiarity or reduced effort increases the likelihood of incompatible representations (Nilsson et al. 2013, Trueblood et al. 2017). So, we can provisionally assume that QPT representations are less demanding than CPT ones and offer a more default option under reflexive versus reflective modes of reasoning (Kahneman 2001). These issues are not without practical importance. For example, in medical or legal decision-making, the rational status of an inference can be paramount (Wojciechowski & Pothos 2018).

In the following sections, we consider in more detail key QPT features that distinguish QPT from CPT and offer novel directions for behavioral research. In doing so, we describe some of the main QPT cognitive models and how they account for the puzzling findings reviewed above.

QUANTUM INTERFERENCE

Quantum interference is a unique property which concerns a breakdown of the law of total probability. Suppose we give the decision maker some preliminary information and then compare two conditions. In the first condition, we simply measure event B alone; in the second condition, we first measure event A and then measure event B. What we expect from the CPT law of total probability is the following:

$$P(\mathrm{B}) = P(\mathrm{A}\ \&\ \mathrm{B}) + P(\sim\mathrm{A}\ \&\ \mathrm{B}). \qquad 1.$$

Now let us see how this works with QPT. After being given the preliminary information, the decision maker is in a state represented by state vector ψ. A positive outcome from the measurement of B is represented by a projector P_{B} (a matrix). It projects the state vector onto the subspace representing B by the matrix product $P_{\mathrm{B}} \cdot \psi$ (above, we have considered one-dimensional subspaces, but a subspace can have any dimensionality). The probability of a positive outcome to question B then equals the squared length $\|P_{\mathrm{B}} \cdot \psi\|^2$ (where the double vertical bars indicate length). Consider another projector, P_{A}, which is used to project onto a positive outcome for question A, and define $I - P_{\mathrm{A}}$ as the orthogonal projector that projects onto the negative outcome for A. Then we can rewrite the projector for B as the sum of two products: $P_{\mathrm{B}} = P_{\mathrm{B}}\, P_{\mathrm{A}} + P_{\mathrm{B}}(I - P_{\mathrm{A}})$. The probability of B can be broken down as follows:

$$\|P_{\mathrm{B}} \cdot \psi\|^2 = \left\|P_{\mathrm{B}}P_{\mathrm{A}} \cdot \psi + P_{\mathrm{B}}(I - P_{\mathrm{A}}) \cdot \psi\right\|^2 = \|P_{\mathrm{B}}P_{\mathrm{A}} \cdot \psi\|^2 + \left\|P_{\mathrm{B}}(I - P_{\mathrm{A}}) \cdot \psi\right\|^2 + \Delta, \qquad 2.$$

where Δ is the sum of cross products produced by squaring the sum. The first term on the right of Equation 2, $\|P_{\mathrm{B}}P_{\mathrm{A}} \cdot \psi\|^2$, is $P(\mathrm{A}\ \&\ \mathrm{B})$, and the second term, $\|P_{\mathrm{B}}(I - P_{\mathrm{A}}) \cdot \psi\|^2$, is $P(\sim \mathrm{A}\ \&\ \mathrm{B})$. The last term, Δ, is called the interference term, which can be positive, negative, or zero. If the measures are compatible, then Δ is zero and QPT satisfies the CPT law of total probability; but if the measures are incompatible, then Δ can be negative or positive and the CPT law of total probability is violated. To emphasize the importance of the order, we usually write P(A & then B) for P(A & B), when A is measured first. Next, we show how this simple property of interference accounts for a variety of the puzzling findings mentioned above.

Probability Judgments

CPT assumes that we can resolve any combination of questions concurrently, so that we can assign a probability for any combination of question outcomes. In fact, this is required because of a property called closure—this is why CPT calculations are intractable for bounded rational agents (CPT theorists are aware of this problem and incorporate it in their modeling; e.g., Lake et al. 2015, Tenenbaum et al. 2011). By contrast, in QPT, uncertainty relations preclude the concurrent resolution of incompatible questions. Pairs of questions have to be evaluated in a sequential way, one question outcome at a time. For example, for the conjunction fallacy, we cannot evaluate P(F & BT); instead, we have to commit to an order and evaluate, for instance, P(F & then BT).

Busemeyer et al. (2011, 2015) and Franco (2009) (see also Aerts 2009, Gronchi & Strambini 2017, Miyadera & Phillips 2012) showed how QPT can account for not only the conjunction fallacy but also the disjunction fallacy, unpacking effects, and more complex conjunctions. **Figure 3** illustrates a toy version of their model; however, Busemeyer et al. (2011) used a more general model, based on subspaces of arbitrary dimensionality. In Equation 2, we assume that vector ψ is determined by the Linda story; we use a projector P_{BT} to represent a positive answer to the bank teller question and a projector P_{F} to represent a positive answer to the feminism question. We assume that the bank teller and feminist questions are incompatible so that the interference term $\Delta \neq 0$. A conjunction fallacy in the observed direction is produced by having $\Delta < -\|P_{\mathrm{BT}}(I - P_{\mathrm{F}}) \cdot \psi\|^2$, a requirement which can be justified from the way different question outcomes about Linda are expected to correlate. A similar argument can be used for the disjunction fallacy. This is a simple illustration of how, in QPT, it can be entirely correct to have a (sequential) conjunction as more probable than an individual statement. Psychologically, participants may find it hard to imagine Linda as a bank teller from the initial perspective of the story. But once they accept that Linda is a feminist, they might think that bank teller may not be such an unlikely profession for feminists. QPT sequential processes are like successive abstractions; with each question evaluated, some of the original information is lost, but new insights and perspectives may be acquired.

The same principles, and interference effects, have been used to account for over- and underextensions of membership judgments that occur with conceptual combinations (e.g., Hampton 1988a, Osherson & Smith 1981). An overextension effect occurs when the membership of an item is stronger for a conjunction of two concepts than for either concept individually. A classic example is a goldfish, which is rated as a better example of the combined concept 'pet fish' than either concept 'pet' or 'fish' individually. An underextension effect occurs when the strength of membership of an item is weaker for a disjunction of two concepts in comparison to the individual concepts. For example, an ashtray is considered a better example of 'home furnishings' in comparison to 'home furnishings or furniture.' These and related findings were modeled with QPT models by Aerts and colleagues (Aerts 2009, Aerts & Gabora 2005, Aerts et al. 2016). Such models have been intended to cover overextension and underextension effects in conceptual combination (Hampton 1988a,b), as well as noncompositionality.

Question Order Effects

According to QPT, both the conjunction fallacy and question order effects arise from the same principle: interference, which results from incompatibility. The co-occurrence of the two effects is an important a priori prediction from QPT models, which was confirmed by Yearsley & Trueblood (2018) (see also Gavanski & Roskos-Ewoldsen 1991; Stolarz-Fantino et al. 2003, experiment 2). Yearsley & Trueblood (2018) additionally showed that the extent of conjunction fallacies was within the bounds predicted by QPT.

For order effects, QPT models go a step further. QPT makes an a priori, quantitative prediction regarding order effects, called the QQ equality (Wang et al. 2014). Consider a typical question order experiment in which two binary-valued questions are asked back to back, but in different orders. The basic model in this case is very simple: The probability for a pair of answers to questions A and then B is computed from the product of projectors $P_{\mathrm{B}}P_{\mathrm{A}}\psi^2$, and the probability for the opposite order of questions is based on the reverse product $P_{\mathrm{A}}P_{\mathrm{B}}\psi^2$. Despite its simplicity, this model makes the following prediction for any dimensionality and any pair of projectors:

$$\begin{aligned} \mathrm{QQ} = &\left[P\left(\mathrm{A_{yes}} \text{ \& then } \mathrm{B_{no}}\right) + P\left(\mathrm{A_{no}} \text{ \& then } \mathrm{B_{yes}}\right)\right] \\ &- \left[P\left(\mathrm{B_{yes}} \text{ \& then } \mathrm{A_{no}}\right) + P\left(\mathrm{B_{no}} \text{ \& then } \mathrm{A_{yes}}\right)\right] = 0. \end{aligned} \tag{3}$$

The QQ equality has been considered one of the most important a priori predictions from QPT. Wang et al. (2014) examined question order effects across 70 national surveys in the USA, with the number of participants varying between 651 and 3,006, and reported good consistency with the QQ equality.

A related experimental direction concerns the ABA paradigm, whereby participants see one question followed by a different one followed, finally, by the original one. With such a paradigm, a QPT approach predicts that if the A and B questions are incompatible, then the second instance of A (measure A, then B, then A again) might produce responses different from the first instance because of interference, but not (or less so) if the questions are compatible (Khrennikov et al. 2014). Examining corresponding predictions is complicated by response biases from simply remembering the first question. Nevertheless, Busemeyer & Wang (2017) obtained results supporting the QPT predictions.

Decision-Making

We now turn to the disjunction effect, as observed in, for instance, the Prisoner's Dilemma (Shafir & Tversky 1992; see also Busemeyer et al. 2006, Croson 1999, Kvam et al. 2014, Tesař 2020). Recall that the main finding is that the probability to cooperate in the unknown condition is much higher than in both known conditions, violating the CPT law of total probability (Equation 1). Pothos & Busemeyer (2009) applied Equation 2 to account for the disjunction effect, using a projector P_{PD} to project the state vector onto the subspace for the player deciding to defect and using another incompatible projector P_{OD} for the opponent's decision to defect. Pothos & Busemeyer (2009) went a step further, using the payoffs from the game and a cognitive dissonance principle from social psychology (Festinger 1957) to build the projectors P_{PD} and P_{OD}; they then quantitatively predicted the sign and magnitude of the interference term Δ required to account for the disjunction effect. More recent research by Broekaert et al. (2020) used a model very similar to the one used by Pothos & Busemeyer (2009) to account for the disjunction effect obtained with a two-stage gambling task (Tversky & Shafir 1992). Other QPT accounts of the disjunction effect for the Prisoner's Dilemma are presented by Asano et al. (2011), Denolf et al. (2017), and Martínez-Martínez & Sánchez-Burillo (2016); other QPT accounts of the disjunction effect for the two-stage gambling task include those by Khrennikov & Haven (2009) and Yukalov & Sornette (2011).

QPT interference was also the basis for Wang & Busemeyer's (2016) model for the disjunction effect (see also Busemeyer et al. 2009) in the categorization–decision paradigm (Townsend et al. 2000). Wang & Busemeyer (2016) built the corresponding projectors, employing payoffs in the task and the same cognitive dissonance principles as above to predict the sign and magnitude of the interference term Δ. An interesting alternative approach for the computation of interference terms in this paradigm is that presented by Moreira & Wichert (2017), who used a quantum-like network (Tucci 1995) to associate interference terms with image similarities.

Memory Recognition

In one paradigm, Brainerd, Reyna, and colleagues (Brainerd & Reyna 2008, Brainerd et al. 2015) asked participants to encode a set of memory targets, for example, a word list. Participants were presented with the targets, related distractors that were semantically related to the targets, and unrelated distractors. The test probes included "Is it is a target?", "Is it a related distractor?", and "Is it a target or a related distractor?", which produced probabilities $P(\mathrm{T})$, $P(\mathrm{R})$, and $P(\mathrm{T\ or\ R})$, respectively. Classically, because the target and related distractor categories are mutually exclusive, we expect that $P(\mathrm{T}) + P(\mathrm{R}) = P(\mathrm{T\ or\ R})$. The major finding is overdistribution, such that $P(\mathrm{T}) + P(\mathrm{R}) > P(\mathrm{T\ or\ R})$. It appears that some items were being remembered as both presented and not presented (Brainerd & Reyna 2008, Brainerd et al. 2015). An analogous subadditivity effect was also observed: $P(\mathrm{T}) + P(\mathrm{R}) + P(\mathrm{NR}) > 1$, where NR is a nonrelated distractor and T, R, and NR constitute the set of mutually exclusive and exhaustive cues. Brainerd et al. (2015) developed a quantum recognition memory model that was viewed as a formalization of fuzzy trace theory (Reyna 2008, Reyna & Brainerd 1995). However, it relied on compatible measurements rather than using interference produced by incompatible measurements. Later, Denolf & Lambert-Mogiliansky (2016) proposed a quantum memory model in which the verbatim information and gist information were represented by incompatible measurements, and used interference to account for the results. The latter approach was extended and more rigorously tested by Trueblood & Hemmer (2017) (see also Broekaert & Busemeyer 2017).

Comparison with Alternative Models

Overall, QPT models have provided fairly straightforward accounts of a wide range of puzzling findings that have resisted compelling classical descriptions for several decades. They have provided a unified account of these various findings, such as the conjunction fallacy for probability judgments and the disjunction effect in decision-making, as arising from the common explanatory idea of interference produced by incompatible projectors. Moreover, they have led to surprising predictions, such as the QQ equality, that have advanced our understanding of constraints in behavior. However, some of these findings have already been intensely studied, at least separately. How do QPT models compare with these alternative explanations? We evaluate some of these alternative explanations, proceeding from more heuristic to more formal accounts.

First, let us examine some heuristic alternative accounts. Tversky & Kahneman's (1983) explanation of the conjunction fallacy is that it is generated by a representativeness heuristic, according to which probabilistic judgments depend on the similarity between an instance (e.g., Linda) and a category (e.g., feminists). Such an approach is very intuitive but lacks precision. QPT can be viewed as a way to formalize representativeness, because probabilities are computed as the overlap between the state vector and the subspace representing a question outcome (Sloman 1993). The advantage is that QPT can then encompass related findings beyond the conjunction fallacy and reveal commonalities between seemingly disparate phenomena. Shafir & Tversky's (1992) explanation for the disjunction effect was based on the idea of failure of consequential reasoning (for the Prisoner's Dilemma finding): When the opponent's play is known, the player can readily evaluate the consequences of each action, but when the opponent's play is unknown, these consequences become unclear. Again, this explanation is intuitive but imprecise, and QPT provides a way to make these ideas quantitatively rigorous: The effect of the two lines of reasoning that are clear when the opponent's play is known is canceled out by interference, Δ, when the opponent's play is unknown. The advantage of the quantum model is that it can lead to new predictions, notably order effects that have been shown to moderate the disjunction effect (Broekaert et al. 2020). Concerning question order effects, a well-known idea from social psychology is that these effects arise because earlier questions create a unique context or perspective for evaluating subsequent ones (Schwarz 2007). For example, an earlier judgment can activate thoughts or perspectives that alter the perception of subsequent ones. Alternatively, it is possible that a choice biases a reinterpretation of preferences to avoid cognitive dissonance (Festinger 1957). Again, the QPT model is completely consistent with these intuitive ideas from social psychology, but the advantage of QPT is the capability of making precise a priori predictions, such as the QQ equality.

Now let us turn to more formal alternative accounts. A fairly natural hypothesis regarding the conjunction fallacy is that it might arise from an erroneous cognitive rule for combining probabilities for a conjunction. Averaging models assume that both conjunctions and disjunctions are based on averages of the probabilities of the conjuncts (Abelson et al. 1987, Fantino et al. 1997, Nilsson et al. 2009), but with different weights for conjunction and disjunction. One problem with these models is that they provide no account of conditional probabilities. Additionally, unless weights change in a post hoc way, such models predict conjunction fallacies regardless of causal links between the conjuncts (high rate of conjunction fallacy) or not (lower rate). However, we expect a conjunction fallacy in cases such as "Mr. F has had one or more heart attacks and Mr. F is older than 55" but not in "Mr. F has had one or more heart attacks and Mr. G is older than 55" (Tversky & Kahneman 1983). Causal strength between the individual premises is captured particularly well by models such as inductive confirmation, which purports that probabilistic assessment follows the extent to which particular pieces of information strengthen or weaken our beliefs in an initial hypothesis (Tentori et al. 2013). A weakness of inductive confirmation is that it does not provide any explanation for disjunction errors.

Another influential approach for the conjunction fallacy is that probabilistic computation is classical, but in an error-prone way. Costello & Watts (2014) (see also Costello et al. 2018, Zhu et al. 2020, and earlier research by Dougherty et al. 1999) proposed that the mind computes probabilities via a mental sampling process, either retrieving relevant instances from one's experience (e.g., "How many women like Linda have I come across?") or generating relevant instances through a mental simulation. Importantly, the mental sampling processes for different questions are independent from one another, so that it is perfectly possible to have an estimate for a conjunction higher than an estimate for an individual premise. With an additional assumption that more complex probability terms, such as conjunctions and disjunctions, are more error prone than simpler terms, it is possible to produce conjunction and disjunction fallacies, as well as a series of probabilistic identities, that appear to be satisfied by human judgments (Costello & Watts 2018). The main concern with this research is that it has been difficult to provide a consistent approach to conditional probabilities, with differences across publications, so there is limited coverage of question order effects (Zhu et al. 2020). Also, some of the empirical results appear to go against predictions from CPT noise models (Tentori et al. 2013, Yearsley & Trueblood 2018). However, there is a key idea consistent with QPT models, namely the requirement of independent sampling for the conjunctions and the individual probabilities. QPT essentially formalizes this idea in the way that incompatibility breaks up a sample space into smaller ones. For the Linda problem, instead of having a single sample space as in **Figure 1**, we have a separate sample space each time a question is assessed that is incompatible with the previous one (Hughes 1989). Note that QPT models can also embody a distinction between errorless and error-prone measurement, but in Busemeyer et al.'s (2011) model this was not needed to cover the conjunction fallacy.

The conjunction fallacy and question order effects are the areas for which QPT models have been most closely scrutinized. The probabilistic identities produced by Costello et al. (2018) and Zhu et al. (2020) are not obeyed by the QPT model, in contrast to empirical results. This challenge has yet to be addressed by QPT. Boyer-Kassem et al. (2016) examined a QPT property called the law of reciprocity and reported violations of this property. However, this property is restricted to single-dimensional subspaces (like those shown in **Figure 3**) and does not apply to higher-dimensional subspaces, such as those used by Busemeyer et al. (2011). Both Kellen et al. (2018) and Costello & Watts (2018) questioned the necessity of QPT to account for the QQ equality, and they constructed alternative models to account for this finding. Clearly, it is always easier to make up post hoc heuristics to reproduce a known result than to predict a result a priori. QPT models have also been compared with other models of order effects on inference, such as the anchoring and adjustment model (Hogarth & Einhorn 1992, McKenzie et al. 2002). However, the QPT models have been shown to provide a better account for these order effects than anchoring and adjustment models (Trueblood & Busemeyer 2011).

SUPERPOSITION AND STATE VECTOR COLLAPSE

Recall from the QPT description of the Linda questions (**Figure 3**) that the beliefs of a person judging Linda are represented by a state vector lying in the vector space. This state vector is not aligned with either the vector spanning the 'definitely feminist' ray or the vector spanning the 'definitely-not-feminist' ray. Instead, the state vector is a superposition state with respect to feminism, because it is formed by a linear combination of these two feminism vectors. At the same time, this state vector can also be considered a superposition with respect to 'bank teller' formed by a linear combination of the 'bank teller' vector and the 'not-bank-teller' vector. More generally, the same state vector can represent beliefs for all kinds of questions about Linda. Finally, a superposition state implies that beliefs about Linda cannot be described by a probability mixture of some conjunction of answers to the feminism and bank teller questions, because this conjunction does not even exist in this picture. In short, superposition states are not the same as probability mixtures. A superposition state contains ontic uncertainty (lack of existence of a feature) about Linda, whereas a probability mixture represents epistemic uncertainty (lack of knowledge) about an existing feature of Linda. In other words, ontic uncertainty reflects the nature of the system itself, not our lack of knowledge about it (Atmanspacher & Primas 2003, Griffiths 2013, Spekkens 2007). This way of interpreting uncertainty as superpositions is a mathematical requirement from a fundamental theorem in QPT (the Kochen–Specker theorem).

A person reaching a decision about Linda must resolve the ontic uncertainty and (probabilistically) create an answer from the superposition state. The resolution of ontic uncertainty is called the collapse of the state vector, and the state vector changes in a precise way, so that it aligns with the question outcome (related to another fundamental principle in QPT, the Lüders projection postulate). As noted above, for example, if the person decides that Linda is a feminist, then the new state will be aligned with the 'feminist' ray. That is, resolving a question changes the person's mental state.

There have been several examinations of QPT predictions regarding constructive influences and the collapse principle. White et al. (2014, 2020) employed pairs of stimuli of strong positive or negative valence, such that the second would always be rated but the first would be rated only half the time. With a simple QPT model, they predicted that the impact of rating the first stimulus would be to intensify the judgment for the second one (e.g., make it more positive if the stimulus was already positive). This prediction is called an evaluation bias and has been repeatedly confirmed. Yearsley & Pothos (2016) examined a prediction from the collapse principle called the quantum Zeno effect. In their experiment, participants were asked to judge the likelihood that a defendant was guilty of a crime after being provided with a sequence of evidence; the defendant was initially assumed to be innocent. In different conditions, participants were asked to make judgments after each of n pieces of information, with n changing across conditions. According to the collapse principle, the more intermediate judgments there were (lower n), the more likely it would be that the judgments would keep collapsing the state back to the initial one, thereby resetting the person's view that, say, Smith is innocent. This is the quantum Zeno effect, which in a decision setting is the idea that the frequency of intermediate judgments slows down opinion change (colloquially, a watched pot never boils). Zeno effects have also been investigated using quantum models with bistable perception tasks (Atmanspacher & Filk 2010).

Another interesting application of the collapse principle concerns the so-called zero priors paradox. A requirement from Bayes's rule in CPT is that if a person assigns a probability of zero or one to any hypothesis, then no new evidence can change the probability from these extremes. This rule turns out to conflict with human inferences. Basieva et al. (2017) conducted two experiments in which participants judged the likelihood of guilt for various possible candidates of a crime. The initial evidence caused participants to place zero (or very low) probability on one candidate, but later evidence moved these zero priors to a high probability of guilt. In Basieva et al.'s (2017) quantum model, it was possible to move from zero priors to nonzero probability of guilt by taking advantage of the way an earlier decision changes the mental state.

Comparison with Alternative Models

As in the case for question order effects, social psychologists have long explained constructive influences on judgment by assuming that an earlier judgment can increase memory or attention to some information, which affects later judgments (e.g., Ariely & Norton 2008, Schwarz 2007). QPT is a way to formalize such ideas. Yearsley & Pothos (2016) compared their quantum model of the Zeno effect with a Bayesian updating model, but only the former was able to reproduce the Zeno effect. Basieva et al. (2017) compared their quantum model with a Bayesian model that assumed a very small prior instead of an exactly zero prior, but the latter was still unable to account for the large change in new evidence.

REPRESENTING SIMILARITY STRUCTURE IN VECTOR SPACES

The dimensionality of a vector space is determined by the number of orthogonal basis vectors (informally, the number of axes) required to span the space, that is, to reach every point by a linear combination. The vector space in **Figure 3** is two dimensional, because only two vectors are needed to span the plane, and all subspaces are rays in the plane. However, it is not necessary, and often not even psychologically reasonable, to assume that each answer to a question is represented by a ray. Instead, subspaces can be of arbitrary dimensionality, and this dimensionality can be associated with the complexity of question outcomes. This flexibility offers a way to capture differences in extent of knowledge in similarity and categorization. For example, suppose we want to represent the concepts of '(Red) China' and '(North) Korea,' when there would be more knowledge about the former than the latter (Tversky 1977). The higher dimensionality of the China subspace would correspond to the more-available information about China, for example, about language, culture, industry, or the political system. Note that each basis vector in that subspace is not necessarily a unique feature but rather is better understood as a summary of related features, in a way analogous to the principal components in factor analysis. Differences in subspace dimensionality can be employed to model the internal structure of concepts and have allowed coverage of empirical results in the similarity literature.

The predominant approach to similarity and representation has been that concepts are points in some vector space and similarity is a simple function of the distance between points (Nosofsky 1992, Shepard 1987). However, if similarities are modeled in this way, then they must be consistent with the metric axioms, three simple properties that all distances must obey: minimality (i.e., the distance between a point to itself is zero), symmetry (i.e., the distance between two points is the same regardless of starting point), and the triangle inequality (i.e., one side of a triangle will always be less than the sum of the other two sides). Tversky (1977) reported similarity findings which sharply contrasted with all three metric axioms. For example, regarding symmetry, he reported that the similarity of China to Korea was judged to be lower than the similarity of Korea to China (Aguilar & Medin 1999, Tversky & Gati 1982). There are simple parametric ways to cover such findings within distance-based models. For example, for symmetry, distances can be multiplied by a directionality parameter that allows asymmetries (Nosofsky 1992). However, such approaches build in the asymmetry by adding parameters rather than having it follow from more basic principles.

Pothos & Trueblood (2015) and Yearsley et al. (2017) proposed that similarity can be modeled using QPT by representing the two compared objects using incompatible subspaces, so that the

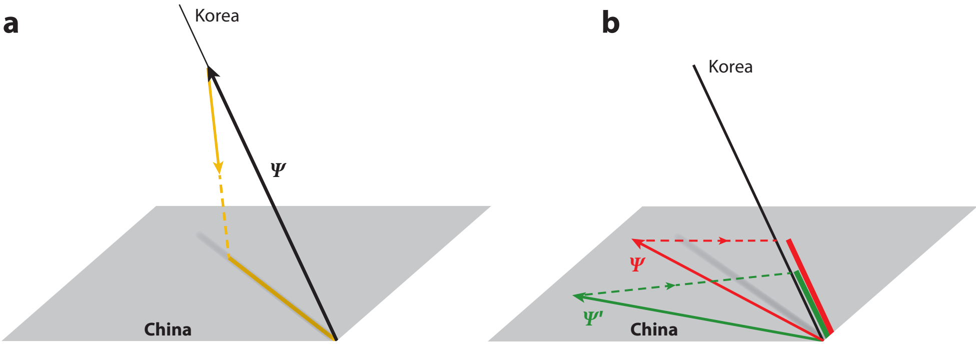

Figure 4

(*a*) We are initially thinking about Korea (ψ is a normalized vector along the Korea ray), and the similarity to China is a projection from Korea to China. Since the Korea subspace is a ray, there is only one way to have a projection of a state from Korea to China. (*b*) We are initially thinking about China, but in this case there are infinite possibilities for a normalized state within the China subspace (such as ψ or ψ'). Therefore, the projection from China to Korea can likewise vary, with the maximum attained only for a specific state.

order of evaluation is important. For example, the similarity of China to Korea involves thinking about China and then about Korea, with asymmetries arising from incompatibility of subspaces, and the direction of the effect resulting from the differences in subspace dimensionality (**Figure 4**). This calculation is identical to that used to compute the probability of a pair of answers, so QPT assumes that the same fundamental processes are used to make similarity and probability judgments (see Shafir et al. 1990 for supporting evidence). Similarity structure in the QPT model arises from the way subspaces relate to one another; Busemeyer & Wang (2017) offered a proposal for how such representations can be computed. The QPT similarity model covers Tversky's (1977) and others' main findings in the similarity literature, including structural constraints in similarity judgment. For example, when comparing two individuals, one would compare the hair color of the first person with the hair color of the second person, not with the color of her shoes (Gentner 1983, Goldstone 1994). The QPT similarity model can be extended to capture structural constraints by employing a different space for each relevant object part and combining these individual representations into an overall one (using an operation called tensor product; Pothos & Trueblood 2015).

Comparison with Alternative Models

Similarity has been intensively studied. Tversky (1977) proposed that similarity depends on common and distinctive features, but with weights determined by which concept is the target and which the referent in a similarity comparison. Ashby & Perrin (1988) considered representations as distributions of perceptual effects from perceiving the same object. Then, the overlap between these distributions would determine similarity; this is their influential general recognition theory. Krumhansl (1978) adopted a modified distance metric, such that the local density of a point would compress or expand distance. Hahn et al. (2003) equated similarity with algorithmic transformational complexity. Pothos et al. (2013) pointed out some difficult assumptions in all these models. For example, regarding the triangle inequality, Ashby & Perrin (1988) assumed unequal perceptual distributions, and Krumhansl (1978) postulated similarity computations on subspaces of the overall space. Criticizing the QPT model, Kintsch (2014) argued that differences in extent of knowledge can be modeled without the (QPT) subspaces, in a standard representation, by employing vectors differing in length. Even though Kintsch's (2014) proposal captures differences in the extent of knowledge, the internal structure of his representations is uninformative about what this knowledge is. In the QPT model, higher-dimensionality subspaces would be associated with more basis vectors, with the latter being interpretable as features (or feature bundles).

ENTANGLEMENT

Although we have focused our attention on incompatible measurements, QPT also incorporates compatible questions. With compatible questions, QPT and CPT generally converge—with an important exception. With compatible questions, QPT, like CPT, can form events that are conjunctions of outcomes from a pair of measurements. The state vector in this case is a superposition over all possible conjunctions. A special type of superposition state for a pair of compatible questions is an entangled state, which is a state that cannot be decomposed into the product of states for each question separately. This is analogous to a dependent joint probability distribution in CPT. However, entangled quantum states can produce dependencies that are called contextual, which cannot be produced by dependent joint distributions in CPT. The term contextual refers to the idea that the meaning of one question changes depending on the other questions occurring at the time. This leads us to the notion of supercorrelation.

Any psychology student learns that the highest possible correlation between two variables is when the variables perfectly align with each other (or are perfectly opposite each other). How can it be otherwise? Suppose we have four different binary questions denoted $a1$, $a2$, $b1$, and $b2$. We can form four 2×2 joint frequency tables produced by all question combinations, $(a1 \times b1)$, $(a1 \times b2)$, $(a2 \times b1)$, and $(a2 \times b2)$. From each of these tables we can compute a correlation, denoted $C(a1,b2)$ for the table for $(a1 \times b2)$, and so forth (strictly speaking, these are expectation values). Then, we define the quantity CHSH as equal to $C(a1,b1) + C(a1,b2) + C(a2,b1) - C(a2,b2)$, where CHSH stands for the scientists Clauser, Horne, Shimony, and Holt, who invented this quantity to test CPT versus QPT theories (Clauser et al. 1969). Suppose that $a1$, $b1$, and $a2$ all maximally correlate with one another, so we find that CHSH $= 3 - C(a2,b2)$. Classically, if $a1$, $b1$, and $a2$ correlate maximally, then we must also find that $C(a2,b2) = 1$ and, thus, CHSH $= 2$. This value of CHSH is called Bell's bound (Bell 2004).

One of the most subtle and powerful aspects of QPT is that Bell's bound can be exceeded, in which case we say the variables are supercorrelated. For example, we could have $C(a2,b2) = -1$, yielding a CHSH value of 4 (but note that QPT allows S values only up to approximately 2.8, which is Tsirelson's bound). If $C(a1,b2) = 1$ and $C(a2,b2) = -1$, then it is as if question $b2$ is treated differently, depending on whether we consider it with question $a1$ or $a2$. We can say that question $b2$ depends contextually on which other question we have. Therefore, supercorrelation means perfect coordination between the a and the b questions, such that the b questions contextually depend on which a question we consider. Supercorrelations can arise in QPT only when the state vector for the a and b questions is entangled.

Supercorrelations indicate that questions $a1$ and $a2$ cannot be understood independently of questions $b1$ and $b2$. This lack of independence is relevant for theories of conceptual combination. For the example of a pet fish (which, as described above, relates to a conjunction fallacy), the meaning of the combination cannot be determined by independently (i.e., compositionally; Fodor 1994) combining the meaning from the individual concepts. Bruza et al. (2015a) demonstrated supercorrelations for novel, ambiguous conceptual combinations composed of two words, such that each word had two (fairly) distinct meanings. For example, in 'boxer bat,' boxer can refer to a person or a dog and bat to a sporting equipment or an animal (for a similar proposal in memory, see Bruza et al. 2009). Other researchers have focused on decisions across two pairs of questions. For example, Aerts et al. (2018) employed questions about pairs of wind directions, Cervantes & Dzhafarov (2018) posed questions about the Snow Queen fairy tale, and

Basieva et al. (2019) asked questions about meal choices. The latter two studies employed a generalized test of contextuality that takes into consideration violations of marginal consistency, which reduces to the CHSH in the special case when the marginals are consistent (as required by the CHSH inequality to test contextuality; Dzhafarov et al. 2016). Some researchers have also explored the possibility of supercorrelations in time, though direct empirical confirmations have been limited (Atmanspacher & Filk 2010, Yearsley & Pothos 2014). Aerts et al. (2018) and Busemeyer & Wang (2018) proposed models to account for measurement context effects that use both concepts of incompatible measurements and entangled states.

Comparison with Alternative Models

Busemeyer & Wang (2018) conducted two large experiments designed to investigate measurement context effects. In one study, participants were presented with public health service announcements and made binary decisions regarding six pairs of questions concerning the effectiveness of the announcements. In another study, participants viewed computer-generated avatars and made binary decisions regarding six pairs of questions formed by four measurements concerning attitudes toward the avatars. In both studies, Busemeyer & Wang (2018) showed that QPT models were superior to Bayesian causal network models for predicting the joint frequencies in the six two-way table measures. QPT models for conceptual combinations have also employed entanglement and supercorrelations (Aerts et al. 2018).

QUANTUM DYNAMICS

In psychology, probabilistic dynamic systems are most frequently modeled using Markov models, such as random walk/diffusion models of decision-making (Ratcliff et al. 2016). Quantum theory provides another general system for modeling probabilistic dynamic systems. Markov systems describe the evolution of probability distributions across time using the Kolmogorov equation. Quantum systems describe the evolution of the state vector across time using the Schrödinger equation.

One important application of both approaches has been to signal detection tasks, which require accumulation of evidence across time. Participants are presented with information across time, so that the information is sampled from one of two possible states of nature. The information is noisy, which makes the decision difficult and time-consuming. The decision maker's task is to determine which of the two states of nature is generating the information across time. At various points, the decision maker can report his or her decision.

In such tasks, the Markov and QPT models are similar, but there is a critical difference when successive measurements are taken across time. The Markov model obeys a type of law of total probability across time, called the Chapman–Kolmogorov equation, but the QPT model allows violations of this law. Kvam et al. (2015) tested the two types of dynamic models using a perceptual decision paradigm, in which participants had to judge the predominant motion direction in a dynamic dot display. In a choice–confidence condition, participants studied the dots, made a decision regarding motion, studied the dots again, and finally provided a confidence rating. In a confidence-only condition, participants simply studied the dots and then provided the confidence rating. According to the Chapman–Kolmogorov equation, the distribution of confidence ratings should be identical across the two conditions, but this was not the case. The difference between the conditions revealed an interference effect predicted by the quantum model. Additionally, better fits were provided by Kvam et al.'s (2015) QPT model as compared with a matching Markov one.

A second critical difference between Markov and QPT models concerns the way dynamics evolve across time. Traditional Markov models of evidence accumulation predict a monotonic

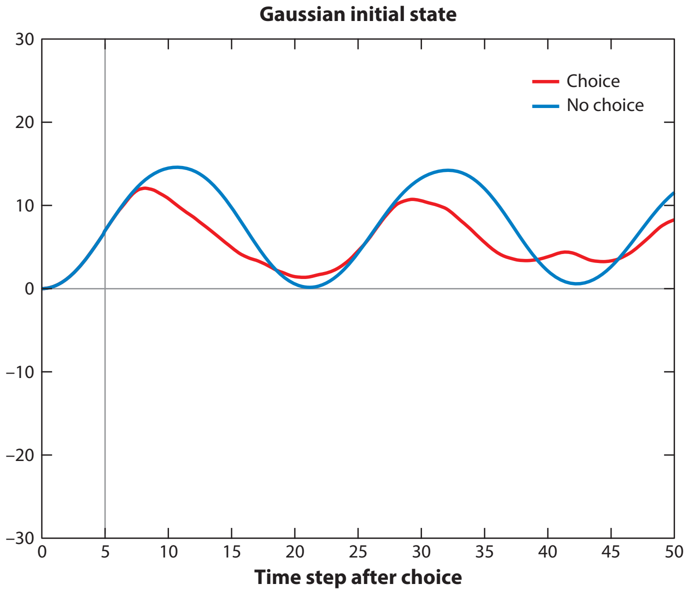

Figure 5

Oscillation and interference predicted by quantum dynamics. The vertical line indicates the time point when a choice was made for the condition with choice, followed by preference ratings across time, and the red curve represents the predictions for this condition. The blue curve represents the predictions when no choice was made at the vertical line. The choice attenuates the oscillation.

growth in the mean of the accumulation across time, making multiple preference reversals plausible only in multiattribute choice and even, in such cases, dependent on the particular way the available options are compared (Diederich 2003, Johnson & Busemeyer 2005, Usher & McClelland 2001). By contrast, the typical dynamics in QPT predict systematic oscillation effects across time. Thus, QPT dynamical models are suitable when we expect ambivalence and rumination in the decision process.

Kvam et al. (2020) tested these predictions using a preference task. Participants were given a choice between two coupons for restaurants that differed with respect to quality, cost, and distance. Participants were asked for preference ratings at several time points, with and without a prior choice. As for Kvam et al. (2015), the prior choice produced interference effects. Moreover, the authors found that preferences systematically oscillated across time, as predicted by a QPT model (**Figure 5**).

Comparisons with Alternative Models

Kvam et al. (2015) compared the predictions for choice and confidence from a quantum walk model with those from a Markov random walk model using a Bayes factor, and the Bayes factor favored the quantum model for the majority of participants. Busemeyer et al. (2019) compared the two models using a generalization test: The model parameters were estimated from two timing intervals for measuring confidence, and these same parameters were then used to predict the confidence ratings for a third time interval. The quantum walk model produced more accurate generalization predictions. Markov and quantum walk models have also been compared using choice and response time distributions. Fuss & Navarro (2013) found a small advantage for a quantum dynamic model over a Markov random walk model for predicting choice and response time in a signal detection type of task. The oscillatory feature of QPT dynamics offers a precise expression of ideas in social psychology that, sometimes, we go through several cycles of back-and-forth before making up our mind (Brehm & Wicklund 1970, Walster 1964).

INTEGRATING CLASSICAL AND NONCLASSICAL PROBABILITIES

The final feature of QPT we discuss concerns the way QPT models can bring together both QPT and CPT probabilistic inference. The results discussed above indicate that QPT thinking emerges when there is less familiarity with a problem and participants are responding in a more reflexive way, and vice versa concerning CPT thinking (Trueblood et al. 2017). QPT thinking can be considered less complex than CPT thinking. In the former case, because of incompatibility, we can embed the representations for several questions in the same low-dimensionality space, but for the latter, the amount of probabilistic information increases rapidly (exponentially) with increasing questions (cf. Atmanspacher & Romer 2012, Pothos et al. 2021). Therefore, QPT might serve as a default thinking mode, gradually replaced by CPT as experience with a task accumulates, and depending on the importance of the task in the first place.

Trueblood et al. (2017) studied the relations between CPT and QPT in the case of causal reasoning, using Rehder's (2014) paradigm. Causal reasoning concerns how information about causes affects conclusions about possible effects. For example, a fallacy in causal reasoning is to confuse effects and causes by considering the probability of an effect given a cause as the same as the probability of a cause given an effect. It appears that there is a CPT component to human causal reasoning, as well as influences from several biases. Trueblood et al. (2017) and Mistry et al. (2018) proposed a model which would reflect varying degrees of quantumness. In one extreme case, all representations would be quantum (incompatible); in another, classical (compatible); and intermediate cases were included, too, such that some representations would be quantum and some classical. With this framework, these authors were able to offer a unified description of results in causal reasoning, for which Rehder (2014) had employed four separate components (a CPT one and three heuristic ones).

SUMMARY EVALUATION OF QUANTUM PROBABILITY THEORY MODELS

QPT models have worked best when it appears that there are interference effects, the mental state is altered by earlier choices or measurements, and questions are contextual (i.e., the meaning of any question depends on which other questions are present). There have been several compelling heuristic ideas and models designed individually for each separate result, across very different theoretical traditions (e.g., Brainerd et al. 1999, Hogarth & Einhorn 1992, Schwarz 2007, Tversky & Kahneman 1983). We suggest that a main advantage of quantum cognitive models is that they provide a single and common set of coherent, formal principles to accommodate the role of measurement and contextuality in cognition, across disparate cognitive areas. Several findings that have been considered separately, such as conjunction and disjunction fallacies, question order effects, and disjunction effects in decision-making, have been shown using QPT to have the same theoretical origin—interference in this case—and have been explained with similar models. In other cases, separate models can be subsumed into a single QPT one (e.g., as done in Trueblood et al. 2017 for the separate model components presented in Rehder 2014).

We have reviewed several examples of successful QPT cognitive models, but not all (e.g., Al Nowaihi & Dhami 2017, Basieva et al. 2017, Blutner et al. 2013, Favre et al. 2016, LaMura 2009). We expect that there will be cases where QPT models fail, for example, when the circumstances are appropriate for CPT reasoning (as Trueblood et al. 2017 considered) or when neither QPT nor CPT can offer satisfactory descriptions. Indeed, there are examples of QPT papers reporting problems with QPT models (Busemeyer et al. 2009). However, existing critical research for QPT models appears to rely on concepts already embodied in QPT (notably, the independent sampling assumption of Costello & Watts 2014); offers redescriptions of results in heuristic terms, which can always happen (Kellen et al. 2018); or focuses on limited QPT models (Boyer-Kassem et al. 2016). Overall, most (but not all) comparisons between QPT and matched classical models (CPT or otherwise) have favored the QPT models. There is a simple explanation: Most QPT modeling has focused on empirical situations that appear to embody key properties of QPT, such as interference and/or collapse. QPT models suit such situations particularly well, so it is unsurprising that they offer good descriptions of empirical results.

If QPT models embody both incompatible and compatible representations, but CPT ones only the latter, is it not the case that QPT models will always be more complex than CPT ones? This is not necessarily true. QPT models typically employ incompatible questions so that a complexity consideration involves a QPT model with incompatible questions and a broadly matched CPT one. Several model comparisons have incorporated complexity and still favored QPT models over matched CPT ones (Broekaert et al. 2020, Busemeyer et al. 2015, Mistry et al. 2018, Trueblood & Hemmer 2017, Trueblood et al. 2017). Another issue concerns whether QPT models have implications for the neural substrate supporting cognition. The explanatory objective of QPT models is best stated using Griffiths et al.'s (2010) approach, as top-down or function-first models which focus on the mathematical principles that characterize behavior (see also Marr 1982). As such, implications for neuronal implementation are limited, and, in particular, QPT cognitive models do not require quantum processes at the neurophysiological level (cf. Litt et al. 2006). In any case, there have been proposals for how the interference terms required for QPT models can naturally arise from neuronal processes (Khrennikov et al. 2018, Suppes et al. 2012).

Can the novel theoretical concepts offered by QPT models translate to generative value? We have discussed several surprising predictions, some entirely unexpected on the basis of predominant theory. For example, the QQ equality shows that response probabilities for several different question pairs combine in a very precise way, as predicted by QPT theory (Wang et al. 2014). The co-occurrence of conjunction fallacy and question order effects is another unique prediction from QPT, and that connection was never made by the earlier heuristic accounts (Yearsley & Trueblood 2018). Even though it has been well established that decisions can alter a mental state, inertia in opinion change from multiple decisions (quantum Zeno effect) offers new support for the specific account of constructive changes in QPT (Yearsley & Pothos 2016). Oscillations during evidence accumulation and interference effects of choice on later confidence are also surprising predictions generated by QPT (Kvam et al. 2020).

The key weakness of QPT models is that we are still unsure which conditions are necessary or sufficient to produce incompatible representations. It is straightforward to conclude that, in some cases, CPT complexity will be intractably complex for bounded agents. But why should a drive to simplify from CPT translate to QPT, as opposed to, for example, heuristics or bounded rational approximations to CPT? There are a few possible answers. QPT might be the next best thing to CPT, in that, as noted above, representations are locally classical—that is, there are subsets of compatible questions, but across subsets we have incompatibility and thus nonclassical effects (Fernbach & Sloman 2009, Lewandowsky et al. 2002, Pothos et al. 2021). Beim Graben & Atmanspacher (2006) considered how incompatibility between classical questions can sometimes emerge if the description of the answers to some question are coarse, so that questions have some irreducible fuzziness regarding possible outcomes. Finally, Aerts & Sassoli de Bianchi (2015) proposed that the quantum probability rule in cognition arises from averaging across experimental conditions. Consider a typical decision experiment, in which participants are asked to assign probabilities to possibilities. Suppose that different participants employ different probability rules. According to Aerts & Sassoli de Bianchi (2015), averaging across participants can be shown to reduce to the QPT rule for probabilistic assignment, which is why quantum theory appears successful in cognitive applications.

These ideas have yet to be fleshed out in a satisfactory way. However, the situation is analogous to the application of QPT in physics—successful empirical applications by far predated the developmental of foundational arguments (Hardy 2001). Moreover, incompatibility can always be empirically determined, for example, with question order effects, constraining any subsequent predictions. We finish with three key points. First, if one thinks that probabilistic inference is an important part of cognition, then he or she should not be restricted to just CPT. Second, QPT is the most established alternative to CPT. Third, the progress of the QPT cognitive program has so far been very promising.

CONCLUDING COMMENTS

QPT offers to psychology several new concepts with potential explanatory value, a sophisticated framework for modeling, and the potential to formalize powerful intuitions which have so far been heuristically explained. QPT cognitive models appear to work particularly well in certain empirical cases, such as when it appears that there is interference, constructive influences, or contextuality, and it is in this area that researchers in this community have focused their efforts. At the same time, we believe that we have only scratched the surface of QPT models in terms of their potential to revolutionize psychological theory. This is a challenge for the next 10 years.

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