Wave-equation

• Waves on a string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

$x \in [0, L)$, $t \in (0, T)$

$x \in [0, L]$

$u_{tt} = c^2 \cdot u_{xx}$

IC

$\rightarrow u(x, 0) = I(x)$

$\rightarrow \frac{\partial}{\partial t} u(x, 0) = 0$

BC: $u(0, t) = 0$, $u(L, t) = 0$

Discretizing time domain: $0 = t_0 < t_1 < \dots < t_{N_t} = T$

space $0 = x_0 < x_1 \quad < x_{N_x} = L$

$u_i^n = u(x_i, t_n) \leftarrow$ discrete solution

$\frac{\partial^2}{\partial t^2} u(x_i, t_n) = c^2 \cdot \frac{\partial^2}{\partial x^2} u(x_i, t_n)$ at mesh points replacing derivatives by finite differences

$$\frac{\partial^2}{\partial t^2} u(x_i,t_n) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}$$

same, but difference operator notation introduced

$$[D_t D_t u]_i^n = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2}$$

$$\frac{\partial^2}{\partial x^2} u(x_i,t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = [D_x D_x u]_i^n$$

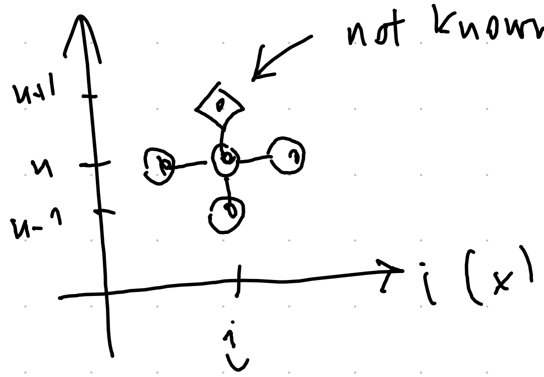
<u>Algebraic version of the PDE</u>

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n+1}}{(\Delta t)^2} = c^2 \cdot \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

<u>Algebraic version of the initial conditions</u>

$$\frac{\partial}{\partial t} u(x_i,0) \approx \frac{u_i^1 - u_{i-1}^1}{2\Delta t} = [D_{2t} u]_i^0 \rightarrow u_i^{-1} = u_i^1 \quad i=0..N_x$$

$$u(x_i,0) \longrightarrow u_i^0 = I(x_i) \quad i=0..N_x$$

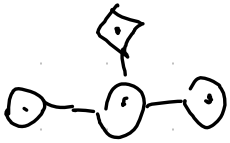


$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C_r^2 \cdot (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

where $C_r = c \cdot \frac{\Delta t}{\Delta x}$

(Courant number)

initial condition



$$u_i^1 = u_i^0 - \frac{1}{2} C_r^2 \cdot (u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

Slightly generalized model problem

$u_{tt} = c^2 \cdot u_{xx} + f(x,t)$; $u(x,0) = I(x)$; $u_t(x,0) = V(x)$; $u(0,t) = 0$; $u(L,t) = 0$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + C_r^2 \cdot (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \Delta t^2 \cdot f_i^n \qquad (*)$$

IC: $u_t(x,0) = V(x) \Rightarrow [D_{2t} u = V]_i^0$ $\quad u_i^{-1} = u_i^1 - 2 \cdot \Delta t \cdot V_i \Rightarrow$ using (*) for $n=0$

$$\Rightarrow \boxed{u_i^1 = u_i^0 - \Delta t \cdot V_i + \frac{1}{2} C_r^2 \cdot (u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \frac{1}{2} (\Delta t)^2 \cdot f_i^0}$$

Stability of scheme:

$$\Delta t \le \frac{\Delta x}{c}$$

Working with a scaled PDE model

$$\bar{x} = \frac{x}{L} \; ; \; \bar{t} = \frac{c}{L} \cdot t \; ; \; \bar{u} = \frac{u}{a}$$

where $a = \max_x |I(x)|$

(typical size of u)

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial \bar{t}}(a\bar{u}) \cdot \frac{d\bar{t}}{dt} = a \cdot \frac{c}{L} \cdot \frac{\partial \bar{u}}{\partial \bar{t}} \\ \frac{\partial u}{\partial x} &= \frac{\partial}{\partial \bar{x}} \cdot (a\bar{u}) \cdot \frac{d\bar{x}}{dx} = \frac{a}{L} \cdot \frac{\partial \bar{u}}{\partial \bar{x}} \end{aligned}\right\}$$

$$\partial_{tt} u = c^2 \cdot \partial_{xx} u$$

$$\frac{a^2 c^2}{L^2} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{a^2}{L^2} \cdot c^2 \cdot \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad x \in (0,1) \; ; \; t \in \left(0, \frac{cT}{L}\right)$$

ICs: $\left.\begin{aligned} \frac{a}{L/c} \cdot \frac{\partial \bar{u}}{\partial t}(\bar{x}, 0) &= V(L\bar{x}) \\ a\,\bar{u}(\bar{x}, 0) &= I(L\bar{x}) \end{aligned}\right\} \Rightarrow \bar{u}(\bar{x}, 0) = \frac{I(L\bar{x})}{\max_x |I(x)|} \; , \; \frac{\partial \bar{u}}{\partial \bar{t}}(\bar{x}, 0) = \frac{L}{a \cdot c} V(L\bar{x})$

Reflecting boundary

- $u=0$ reflects the wave, but u changes sign at the boundary
- $\boxed{u_x=0}$ reflects the wave as a mirror (and preserves sign)

$$u_x=0 \Rightarrow \frac{\partial u}{\partial n} = \vec{n}\cdot\nabla u = 0$$

in 1D: $\left.\frac{\partial}{\partial n}\right|_{x=L} = \left.\frac{\partial}{\partial x}\right|_{x=L}$ and $\left.\frac{\partial}{\partial n}\right|_{x=0} = -\left.\frac{\partial}{\partial x}\right|_{x=0}$

- $u_i^{n+1} = -u_i^{n-1} + 2u_i^n + 2C^2(u_{i+1}^n - u_i^n)$ $\quad i=0$ (left side)
- $u_i^{n+1} = -u_i^{n-1} + 2u_i^n + 2C^2\cdot(u_{i-1}^n - u_i^n)$ $\quad i=N_x$ (right side)

Other method - using ghost cells

→ extra points, so the u_{-1}^n and $u_{N_x+1}^n$ fictitious values are represented

$\Rightarrow u_{-1}^n = u_1^n$ and $u_{N_x+1}^n = u_{N_x-1}^n$ for homogeneous N-BC

Generalization: variable wave velocity

$c = c(x)$ sampling at mesh points

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(q(x)\cdot\frac{\partial u}{\partial x}\right) + f(x,t)$$

$$\phi = q(x)\cdot\frac{\partial u}{\partial x} \longrightarrow \left.\frac{\partial \phi}{\partial x}\right|_i^n \simeq \frac{\phi_{i+1/2}-\phi_{i-1/2}}{\Delta x} = [D_x\phi]_i^n$$

$$\phi_{i+1/2} = q_{i+1/2}\cdot\left.\frac{\partial u}{\partial x}\right|_{i+1/2}^n \simeq q_{i+1/2}\cdot\frac{u_{i+1}^n - u_i^n}{\Delta x} = [q D_x u]_{i+1/2}^n$$

$$\phi_{i-1/2} = q_{i-1/2}\cdot\left.\frac{\partial u}{\partial x}\right|_{i-1/2}^n \simeq q_{i-1/2}\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = [q D_x u]_{i-1/2}^n$$

$$\left[\frac{\partial}{\partial x}\left(q\,\frac{\partial u}{\partial x}\right)\right]_i^n \simeq \frac{1}{\Delta x}\left(\frac{1}{\Delta x}\left(q_{i+1/2}\cdot(u_{i+1}^n - u_i^n) - q_{i-1/2}\,(u_i^n - u_{i-1}^n)\right)\right)$$

$$= \frac{1}{(\Delta x)^2}\left(q_{i-1/2}\,u_{i-1}^n - \left(q_{i+1/2} + q_{i-1/2}\right)u_i^n + q_{i+1/2}\cdot u_{i+1}^n\right)$$

Coefficient between mesh points

$q_{i+1/2} \simeq \frac{1}{2}(q_i + q_{i+1})$ arithmetic mean

$q_{i+1/2} \simeq 2\cdot\left(\frac{1}{q_i} + \frac{1}{q_{i+1}}\right)^{-1}$ harmonic mean $\rightarrow$ large jumps (geological media)

$q_{i+1/2} \simeq \sqrt{q_i\cdot q_{i+1}}$ geometric mean (to linearize quadratic nonlinearities)

$$[D_t D_t u = D_x\cdot \bar{q}^x \cdot D_x u + f]_i^n$$

$$\boxed{\begin{aligned} u_i^{n+1} &= -u_i^{n-1} + 2u_i^n + \\ &+ \left(\frac{\Delta t}{\Delta x}\right)^2\cdot\left(\frac{1}{2}(q_i + q_{i+1})(u_{i+1}^n - u_i^n) - \frac{1}{2}(q_i + q_{i-1})(u_i^n - u_{i-1}^n)\right) + \\ &+ \Delta t^2\cdot f_i^n \end{aligned}}$$

for internal points

stability in case of variable coefficient

$$\Delta t \leq \beta\cdot\frac{\Delta x}{\max_{x\in[0,L]} c(x)}$$

$\uparrow$ safety factor (0,9 acts as an all-round value)

Neumann - condition

$$[D_{2x}u]_i^n := \frac{u_{i+1}^n - u_{i-1}^n}{2\cdot\Delta x} = 0 \Rightarrow u_{i+1}^n = u_{i-1}^n \qquad i = N_x$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{at } x = L$$

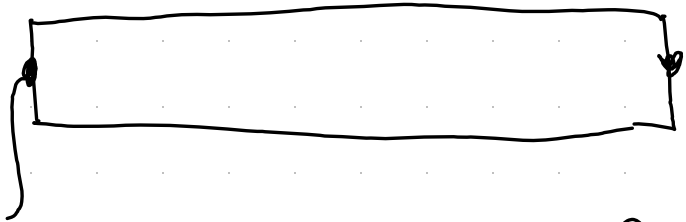
(using $u_{i+1}^n = u_{i-1}^n$)

$$\Rightarrow \quad u_i^{n+1} \simeq -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \cdot \underbrace{2q_i\left(u_{i-1}^n - u_i^n\right)}_{\text{alternative way yields}} + \Delta t^2 \cdot f_i^n$$

(57)

$$\underbrace{\left(q_i + \frac{1}{2}\left(q_{i+1} + q_{i-1}\right)\right)}_{\simeq 2q_i}\left(u_{i-1}^n - u_i^n\right)$$

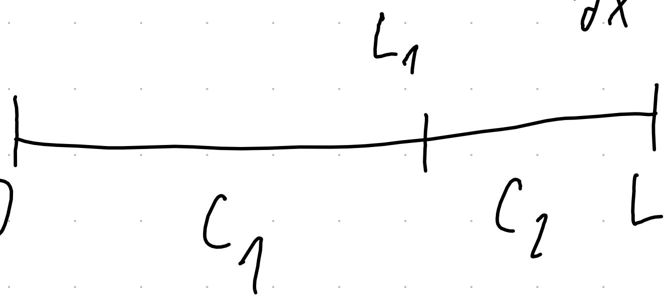
$$\boxed{u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \cdot \left(2q_{i-1/2}\right) \cdot \left(u_{i-1}^n - u_i^n\right) + \Delta t^2 \cdot f_i^n}$$



$u = 0$

$\frac{\partial u}{\partial x} = 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



$$c = \begin{cases} c_1, & x < L_1 \\ c_2, & x > L_2 \end{cases}$$

$$\boxed{c_1 = 1}$$

$$\boxed{c_2 = 3}$$