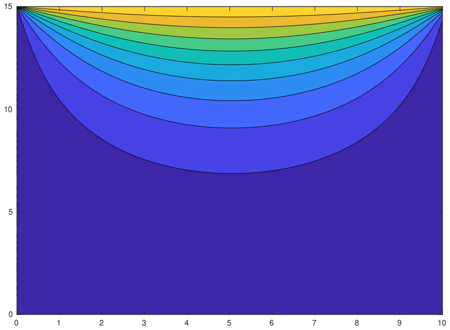
Szélessávú Hírközlés és Villamosságtan Tanszék
Antennák és EMC kutatócsoport
NES Laboratórium

Műszaki és fizikai problémák számítógépes megoldása - 2023

Elliptic PDEs - Problems to solve in many dimensions



- Laplace-equation
$$\nabla^2 f = 0$$
- applies to ideal fluid flow, mass diffusion, heat diffusion, electrostatics, etc.
- sample case – Electrostatics

$$\nabla \times \vec{\mathbf{E}} = 0 \qquad \text{rotation-less electric field}$$

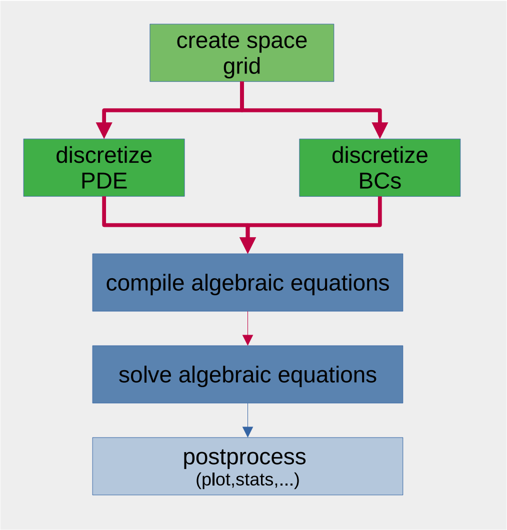
$$\vec{\mathbf{E}} = -\text{grad } \Phi$$

$$\nabla \vec{\mathbf{D}} = \varrho \quad \Rightarrow \nabla \left(\varepsilon \nabla \Phi \right) = \varrho$$

$$\text{charge-less medium : } \nabla \nabla \Phi = \Delta \Phi 0$$

- in presence of matter : $\varepsilon \neq$ const. everywhere

$$\nabla \left(\varepsilon \nabla \Phi \right) = 0$$



- creating space mesh : x_i
- discretize PDE :

$$\frac{\mathrm{d}^2\Phi}{\mathrm{dx}^2} \rightarrow \Phi_i, \Phi_{i-1}, \ldots$$

- discretize BCs : Dirichlet : $\Phi_N = U_1$, Neumann : $\frac{\partial \Phi}{\partial n} = E$
- algebraic equation -

$$\mathbf{A}\vec{\mathbf{\Phi}} = \vec{\mathbf{B}}$$

- one-dimensional Laplace-equation with boundary conditions

$$T(0) = T_L = 100 \quad \text{and} \quad T(L) = T_R = 10$$

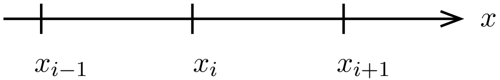
- PDE :

$$\frac{\mathrm{d}^2 T}{\mathrm{dx}^{\,2}} = \frac{d^2 T}{dx^2} = 0$$

- coordinate in x-direction : (equidistant $h_i = \Delta x$)

$$x_i = (i-1) \cdot \Delta x, \qquad i = 1, 2, \ldots, N$$

- environment of the point x_i :

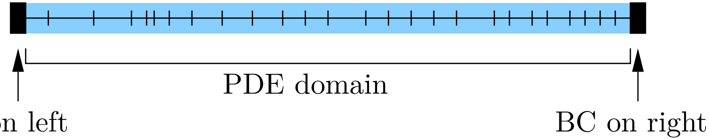


One-dimensional elliptic problem - equidistant grid

- $$\text{discretization for inside points : } \frac{1}{h^2} \cdot T_{i-1} - \frac{2}{h^2} \cdot T_i + \frac{1}{h^2} \cdot T_{i+1} = 0$$
- Dirichlet boundary on left $T_1 = T_L$ (on right $T_N = T_R$)
- using Neumann-BC on right :

$$\frac{dT}{dx} = Q \qquad \Rightarrow \qquad \frac{T_N - T_{N-1}}{x_N - x_{N-1}} = Q$$

- domain of simulation in one-dimension :

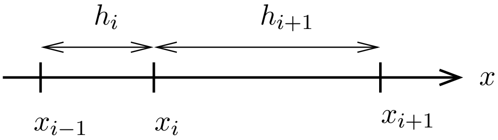


$$\text{Laplace-equation in one-dimension : } \Delta\Phi = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2\Phi}{\mathrm{dx}^2} = 0$$

space grid is non-equidistant :

$$x = \{x_1, x_2, \ldots, x_N\}$$

difference in space : $h_i = x_i - x_{i-1}$ and $h_{i+1} = x_{i+1} - x_i$
environment of the point x_i with non-equal distances :



- $$\text{PDE :} \quad \nabla \left(\varepsilon \nabla \Phi \right) = 0$$
- ε is permittivity (dielectric constant) of medium $\varepsilon = \varepsilon(x)$
- 1D form :
$$\frac{d}{dx} \left(\varepsilon \frac{d}{dx} \Phi \right) = 0$$
- boundary conditions : $\Phi = 100$ at $x = 0$ and $\Phi = 10$ at $x = L$
- permittivity :
$$\varepsilon = \begin{cases} \varepsilon_1 & \text{if } x < L_1 \\ \varepsilon_2 & \text{if } x \geq L_1 \end{cases}$$

- discretizing 2nd derivative

$$\frac{d}{dx}\Phi\Big|_i \simeq \frac{\left(\varepsilon\frac{d}{dx}\Phi\right)\Big|_{i+1/2} - \left(\varepsilon\frac{d}{dx}\Phi\right)\Big|_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} =$$

- using 1st derivative approximation

$$\frac{d}{dx}\Phi\Big|_i \simeq \frac{\Phi_{i+1/2} - \Phi_{i+1/2}}{x_{i+1/2} - x_{i-1/2}} =$$

- discretization :

$$\ldots = \frac{2}{h_i + h_{i+1}}\left\{\frac{\varepsilon_{i-1/2}}{h_i}\Phi_{i-1} - \left(\frac{\varepsilon_{i-1/2}}{h_i} + \frac{\varepsilon_{i+1/2}}{h_{i+1}}\right)\Phi_i + \frac{\varepsilon_{i+1/2}}{h_{i+1}}\Phi_{i+1}\right\}$$

- Laplace-equation for T in two-dimensional domain

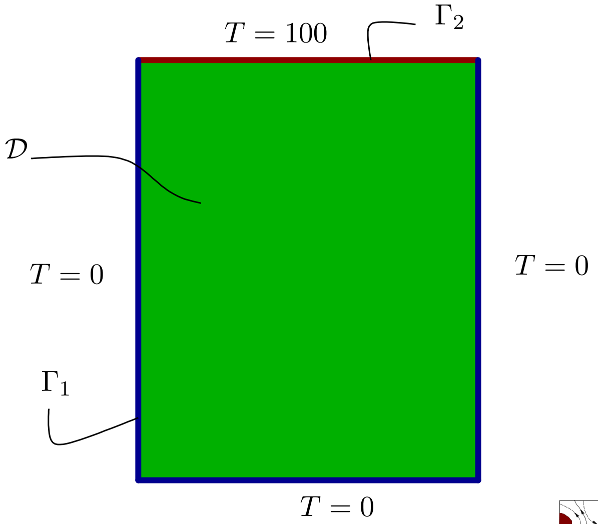
$$\Delta T = 0 \text{ on } \mathcal{D}(x, y);$$

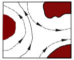
$$\mathcal{D}(x, y) = (0, 10) \times (0, 15)$$

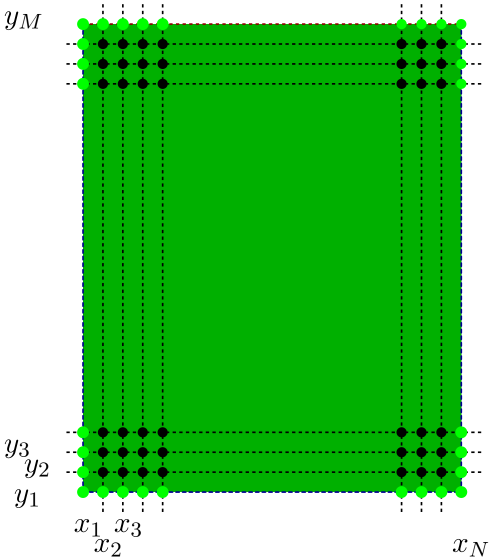
- Boundary conditions :

$$\Gamma_1 : T(x, 0) = T(0, y) = T(10, y) = 0$$

$$\Gamma_2 : T(x, 10) = 100$$







- equidistant in x-direction :
  $h_i = x_i - x_{i-1} = \Delta x = \text{const.}$
  N coordinate lines
- equidistant in y-direcion :
  $k_j = y_j - y_{j-1} = \Delta y = \text{const.}$
  M coordinate lines
- number of variables : $N \times M$
- green points - boundary points (apply BC)
- black points - internal points (apply PDE)

Discretization

- discretizing PDE using equidistant propery in each directions :

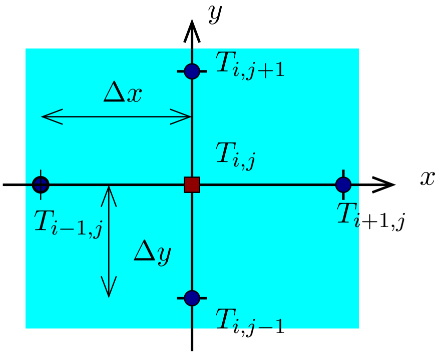
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

$$\beta^2 \cdot T_{i,j-1} + T_{i-1,j} - 2(1+\beta^2)T_{i,j} + T_{i+1,j} + \beta^2 T_{i,j+1} = 0$$

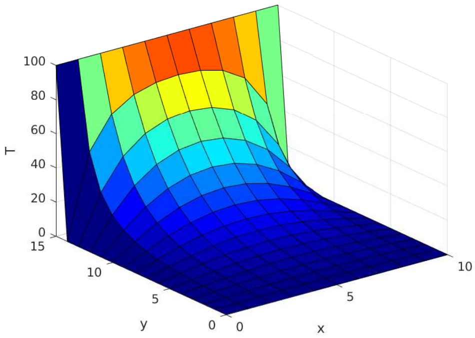
$$\text{where} \qquad \beta^2 = \left(\frac{\Delta x}{\Delta y}\right)^2$$

- Dirichlet Boundary condition :

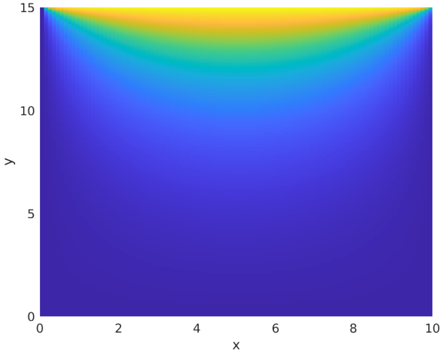
$$T_{i,j} = T_1$$



Elliptic problem in 2D - discretized – algebraic equations



solution on a coarser grid ($\Delta x = 1$, $\Delta y = 1$, 21×31 points, solution time : 0.54s)



solution on a finer grid ($\Delta x = 0.1$, $\Delta y = 0.1$, 101×151 points, solution time : 12.32s)