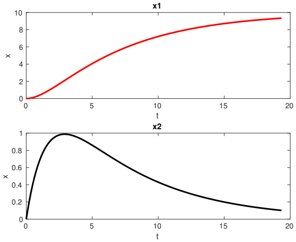
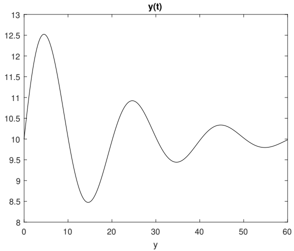
MŰEGYETEM 1782

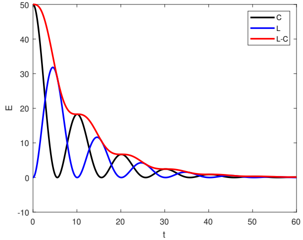
Műszaki és fizikai problémák megoldása számítógéppel
Solution of selected problems using computer from physics and engineering

MűFi24 - 2024

Matrix manipulation and linear equations







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ra | (September 17, 2024)

- Consider a metropolitan area with a constant total population of 1 million individuals.
- This area consists of a city and its suburbs, and we want to analyze the changing urban and suburban populations.
- Let C_k denote the city population and S_k the suburban population after k years. Suppose that each year 15% of the people in the city move to the suburbs, whereas 10% of the people in the suburbs move to the city.
- Then it follows that

$$
\begin{aligned}
C_{k+1} &= 0.85C_k + 0.10S_k \\
S_{k+1} &= 0.15C_k + 0.90S_k
\end{aligned}
$$

- We envision a physical system—such as a population with n specified subpopulations—that evolve through a sequence of successive states described by the vectors

$$\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_k}$$

- **transition matrix** – gives probability of jump from one state to other $\mathbf{A}_{i,k}$ defines probability that state changes from k to i, i.e. $i \to k$ transition's probability
- stochastic matrix if the sum of elements in each column is 1.

- $$\mathbf{x[k]} = \begin{bmatrix} u[k] \\ s[k] \end{bmatrix}$$
  means u - urban population, s - suburban population at time k
- $$\mathbf{A} = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}$$
- `A = [0.85 0.1;0.15 0.9]`
- `x0= [900000;100000];`
- `k = 1:10;`

1. Use different initial values! Look for the populations after a few years!
2. Change rates in transition matrix! What is the consequence of it?
3. Extend the model with another suburban region! What is the transition matrix in this case?

- Consider a predator-prey population consisting of foxes and rabbits.
- Initially there are F_0 foxes and R_0 rabbits. After k months there are F_k foxes and R_k rabbits.
- It is assumed transition is given by

$$
\begin{aligned}
F_{k+1} &= 0.4F_k + 0.3R_k \\
R_{k+1} &= -r \cdot F_k + 1.23R_k
\end{aligned}
$$

  where r called *capture rate*, that represents the average number of rabbits consumed monthly by each fox.
- evolution in time is given by

$$\mathbf{x}_{k+1} = \mathbf{A} \cdot \mathbf{x}_k \text{ hence } \mathbf{x}_k = \mathbf{A}^k \cdot \mathbf{x}_0$$

  where $\mathbf{x}_k = \begin{bmatrix} F_k \\ R_k \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 0.4 & 0.3 \\ -r & 1.2 \end{bmatrix}$

- $0.4F_k$ means that without rabbits only 40% of the foxes would survive each month
- $0.3R_k$ represents the growth in the fox population due to the available food supply of the rabbits
- The term $1.2R_k$ in the second equation indicates that, in the absence of any foxes, the rabbit population would increase by 20% each month.
- Long time behaviour of model :
  - stable limiting population
  - mutual extinction
  - population explosion

Notes : Write out characteristic equation of the transition matrix ... which gives eigenvalues of A in terms of the capture rate r. There are three possibilities :

1. F_k and R_k may approach constant nonzero values. (*stable limiting populations* that coexist in equilibrium with one another)
2. F_k and R_k may both approach zero. (*mutual extinction of the two species*)
3. F_k and R_k may both increase without bound (*population explosion*)

Find corresponding capture rate, r for the three possible outrun.

- Write a simple simulator that shows how the population changes in time! capture rate should be changeable! Plot results!
- Write a function that finds limiting parameters for stable limiting populations case!
- Change model to get seasonal effects! (Give meaning of time as different seasons are coming!) What should be changed?

- $x = \begin{bmatrix} F \\ R \end{bmatrix}$, F number of foxes, R number of rabbits
- `function x = PPmodel(r, F0, R0, k)`
  A function that calculates number of foxes and rabbits starting from F0 and R0, at a given capture rate (r) for times k.

- In a certain town, 30 percent of the married women get divorced each year and 20 percent of the single women get married each year.
- There are 8000 married women and 2000 single women.
- Assuming that the total population of women remains constant, how many married women and how many single women will there be after one year? After two years?

In his **Liber abaci** (Book of Calculation) published in 1202, Leonardo Fibonacci ("Leonardo, son of Bonacci", 1175-1250?, also known as Leonardo of Pisa (Leonardo Pisano)) asked the following question :
How many pairs of rabbits are produced from a single original pair in one year, if every month each pair begets a new pair, which is similarly productive beginning in the second succeding month?

The answer

The answer is provided by the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... in which each term is the sum of the two immediate predecessors. That is, the sequence is defined recurcively as follows :

$$s_0 = s_1 = 1; s_{n+1} = s_n + s_{n-1}; \text{ for } n \geq 1$$

The s_n is the number of rabbit pairs present after n months. Note that if

$$x_n = [s_{n+1}; s_n] \text{ and } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$x_n = Ax_{n-1} \text{ and so } x_n = A^n x_0$$

where $x_0 = (1, 1)$

1. Show that A has eigenvalues
2. Compute

$$[s_{n+1}; s_n]A^n x_0 = PD^nP^{-1}[1; 1]$$

to derive formula of Fibonacci expression ...

- Sex-linked genes are genes that are located on the X chromosome. For example, the gene for blue-green color blindness is a recessive sex-linked gene. To devise a mathematical model to describe color blindness in a given population, it is necessary to divide the population into two classes: males and females.
- Let $x_1(0)$ be the proportion of genes for color blindness in the male population, and let $x_2(0)$ be the proportion in the female population. [Since color blindness is recessive, the actual proportion of color-blind females will be less than $x_2(0)$.]
- Because the male receives one X chromosome from the mother and none from the father, the proportion $x_1(1)$ of color-blind males in the next generation will be the same as the proportion of recessive genes in the present generation of females.
- Because the female receives an X chromosome from each parent, the proportion $x_2(1)$ of recessive genes in the next generation of females will be the average of $x_1(0)$ and $x_2(0)$.

- Thus
$$x_2(0) = x_1(1)$$
$$\frac{1}{2}x_1(0) + \frac{1}{2}x_2(0) = x_2(1)$$
- If $x_1(0) = x_2(0)$ then the proportion will not change in future generations.
- If $x_1(0) \neq x_2(0)$ then we should write as a matrix equation

$$\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}$$

- The proportions of genes for color blindness in the male and female populations will tend to the same value as the number of generations increases.

- Suppose that 10,000 men and 10,000 women settle on an island in the Pacific that has been opened to development.
- Suppose also that a medical study of the settlers finds that 200 of the men are color blind and only 9 of the women are color blind.
- Let x(1) denote the proportion of genes for color blindness in the male population and let x(2) be the proportion for the female population.
- Assume that x(1) is equal to the proportion of color-blind males and that x(2)2 is equal to the proportion of color- blind females.

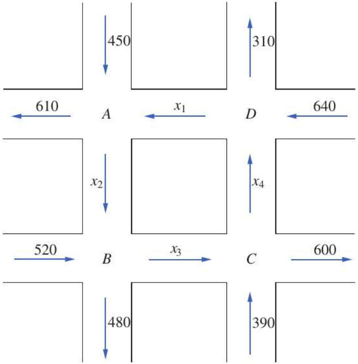
Determine x(1) and x(2) and enter them in MATLAB as a column vector x. Enter also the matrix A from Application 3 of Section 6.3. Set MATLAB to format long, and use the matrix A to compute the proportions of genes for color blindness for each sex in generations 5, 10, 20, and 40. What are the limiting percentages of genes for color blindness for this population? In the long run, what percentage of males and what percentage of females will be color blind?

- The management and preservation of many wildlife species depend on our ability to model population dynamics. A standard modeling technique is to divide the life cycle of a species into a number of stages.
- The models assume that the population sizes for each stage depend only on the female population and that the probability of survival of an individual female from one year to the next depends only on the stage of the life cycle and not on the actual age of an individual.
- Four-stage model : eggs, hatchlings $\rightarrow$ Juveniles and subadults $\rightarrow$ Novice breeders $\rightarrow$ Mature breeders

- At each stage, we estimate the probability of survival over a one-year period. We also estimate the ability to reproduce in terms of the expected number of eggs laid in a given year.
- The approximate ages for each stage are listed in parentheses next to the stage description.

Stage number	Description	age	Annual Survivalship	Eggs layed
1	Eggs, hatchlings	< 1	0.67	0
2	Juveniles, subadults	$1 - 21$	0.74	0
3	Young breeders	22	0.81	127
4	Mature breeders	23-54	0.81	79

In the downtown section of a certain city, two sets of one-way streets intersect as shown in Figure 1.2.2. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram. Determine the amount of traffic between each of the four intersections.



- At each intersection, the number of automobiles entering must be the same as the number leaving.
- Equations can be written :

$$
\begin{aligned}
x_1 + 450 &= x_2 + 160 \\
x_2 + 520 &= x_3 + 480 \\
x_3 + 390 &= x_4 + 600 \\
x_4 + 640 &= x_1 + 310
\end{aligned}
$$

- `A = [1 -1 0 0;0 1 -1 0;0 0 1 -1;-1 0 0 1];`
- `B = [160;-40;210;-330];`
- `x = inv(A)*B;`

Polynomial interpolation

- data points are given (x_0, y_0), (x_1, y_1), $\ldots$ (x_n, y_n) that are to be presented by a specific type of function $y = f(x)$
- Let's find a polynomial of degree n solution !

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

  where $a_0, a_1, a_2, \ldots, a_n$ are constants.
- data points (x_i, y_i) lies on the curve $y = f(x)$, so $f(x_i) = y_i$ yields for $i = 0, 1, 2, \ldots, n$ the $n + 1$ equations

$$\begin{aligned}
a_0 + a_1 x_0 + a_2 (x_0)^2 + \cdots + a_n (x_0)^2 &= y_0 \\
a_0 + a_1 x_1 + a_2 (x_1)^2 + \cdots + a_n (x_1)^2 &= y_1 \\
a_0 + a_1 x_2 + a_2 (x_2)^2 + \cdots + a_n (x_2)^2 &= y_2 \\
&\vdots \\
a_0 + a_1 x_n + a_2 (x_n)^2 + \cdots + a_n (x_n)^2 &= y_n
\end{aligned}$$

- $$\mathbf{A} \cdot \mathbf{x_i} = \mathbf{y}_i$$

  where $\mathbf{A}$ an $(n+1) \times (n+1)$ coefficient matrix is the Vandermonde-matrix

  $$\mathbf{A} = \begin{bmatrix} 1 & x_0 & (x_0)^2 & \cdots & (x_0)^n \\ 1 & x_1 & (x_1)^2 & \cdots & (x_1)^n \\ & & & & \\ 1 & x_n & (x_n)^2 & \cdots & (x_n)^2 \end{bmatrix}$$

- if the x-coordinates x_0, x_1, x_2, ..., x_n are distinct, then the matrix A is nonsingular, hence the system has a unique solution for the coefficients $a_0, a_1, a_2, \ldots, a_n$.

- Find polynomial of the form

$$y = A + B \cdot x + C \cdot x^2 + D \cdot x^3$$

that interpolates $(-1, 4)$, $(1, 2)$, $(2, 1)$ and $(3, 16)$!

- Use population data given in `population.csv`!
- Find a linear fit on datas to the 1995 and 2005 population values!
- Fit a quadratic polynomial $P_2(t) = a + bt + ct^2$ to the 1995, 2000, 2005 datas!
- Fit a cubic polynomial $P_3(t) = a + bt + ct^2 + dt^3$ to the (1995, 2000,2005, 2010) population values!
- Fit a fourth-degree population model of the form

$$P_4(t) = a + bt + ct^2 + dt^3 + et^4$$

  to the population values of 1990-1995-2000-2005-2010!
- Calculate error of fits, calculate predictions for year 2030! Plot known datas and all the fitted polynomials!

Better approximation

- Use updated data (find in website) and compare it with the extrapolation made in previous example!

- Let's have three points! $P(-1,5)$, $Q(5,-3)$ and $R(6,4)$!
- Find the one and only circle that passes these points!

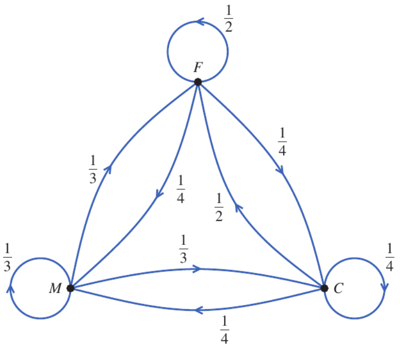
$$(x-h)^2+(y-k)^2=r^2$$

- Find the rotated conic section (an ellipse, parabola or hyperbola) centered at the origin of the xy-coordinate system!

$$Ax^2+Bxy+Cy^2=1$$

Suppose that in a primitive society the members of a tribe are engaged in three occupations: farming, manufacturing of tools and utensils, and weaving and sewing of clothing. Assume that initially the tribe has no monetary system and that all goods and services are bartered. Let us denote the three groups by F, M, and C, and suppose that the directed graph in Figure 1.2.4 indicates how the bartering system works in practice. The figure indicates that the farmers keep half of their produce and give one-fourth of their produce to the manufacturers and one-fourth to the clothing producers. The manufacturers divide the goods evenly among the three groups, one-third going to each group. The group producing clothes gives half of the clothes to the farmers and divides the other half evenly between the manufacturers and themselves. The result is summarized in the following table:

Distribution as a directed graph shown



	F	M	C
F	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
M	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
C	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$

- The first column of the table indicates the distribution of the goods produced by the farmers, the second column indicates the distribution of the manufactured goods, and the third column indicates the distribution of the clothing.

- As the size of the tribe grows, the system of bartering becomes too cumbersome and, consequently, the tribe decides to institute a monetary system of exchange. For this simple economic system, we assume that there will be no accumulation of capital or debt and that the prices for each of the three types of goods will reflect the values of the existing bartering system. The question is how to assign values to the three types of goods that fairly represent the current bartering system.
- Nobel Prize–winning economist Wassily Leontief

- it leads to a homogeneous system, we have a free parameter
- $$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 = x_1$$
- $$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_2$$
- $$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_3$$
- Make it non-homogeneous system and solve it for integers!

Problem statement

- sinusoidal excitation in linear systems
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