MŰFI25 – LECTURE 7.

SOME PROBLEMS THAT INVOLVE SOLUTION OF ORDINARY DIFFERENTIAL EQUATION(S)

(ONLY PROBLEMS ARE SHOWN)

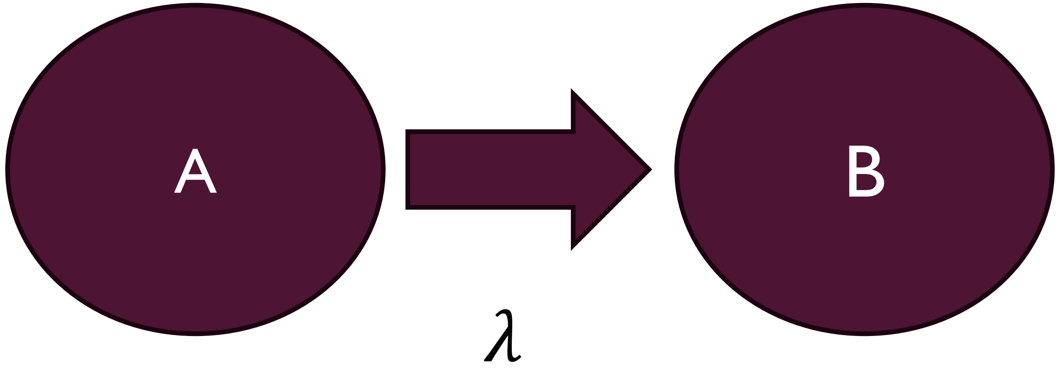
NEWTON'S LAW OF COOLING

- A hot cup of coffee cooling in the room. Temperature change is proportional to temperature difference between coffee and room's air.
- Equation that describes cooling

$$\frac{dT}{dt} = -k \cdot (T - T_{air})$$

where T is temperature of coffee, T_air is air temperature and k is heat convection constant.

RADIOACTIVE DECAY



- Number of radioactive atoms decreases exponentially over time.

$$\frac{dN}{dt} = -\lambda \cdot N$$

- where N is number of atoms that not decayed so far, λ is inverse of half-life time

CIRCUIT WITH CAPACITOR AND RESISTOR IN SERIES

- Let's use a simple circuit of capacitor, resistor and battery connected serialy. Than we can write for voltage of capacitor if battery is turned on (from 0 V to V_0 when $t = 0$)

$$\frac{dV}{dt} + \frac{V}{R \cdot C} = \frac{V_0}{R \cdot C}$$

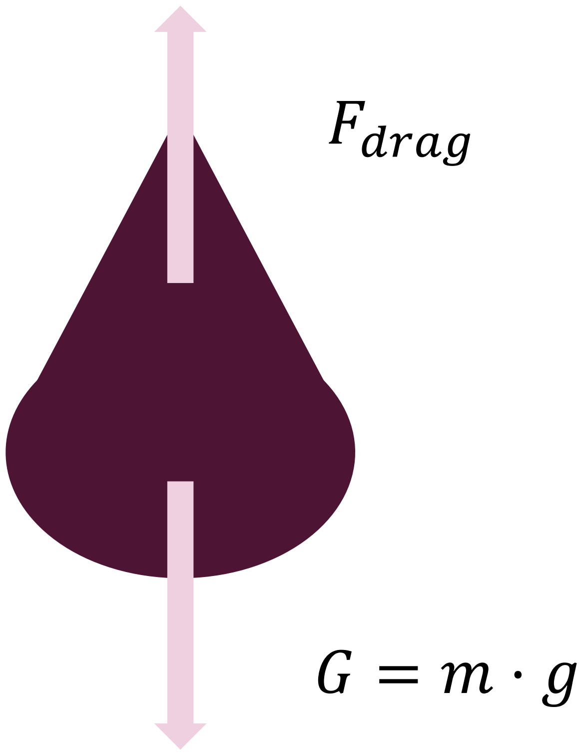
- where V is voltage of capacitor, R and C are resistance and capacitance, V_0 is battery's voltage
- Analytical solution :

DRAG FORCE OF A FALLING OBJECT

- Let us use falling problem. An object falling through air with velocity-dependent drag.
- If obkect's mass is m, and gravitational constant is m and drag force coefficient is b :

$$m \cdot \left(\frac{dv}{dt}\right) + b \cdot v = m \cdot g$$

- We can calculate solution of this problem :



BERNOULLI EQUATIONS - POPULATION WITH LIMITED RESOURCES

- Population with limited resources (like rabbits and grass)
- Population of rabbits is denoted by P, grass (what rabbits eat) is limited and rabbits can not eat as many as they want.

$$\frac{dP}{dt} = r \cdot P \; - a \cdot P^2$$

rabbit (P)

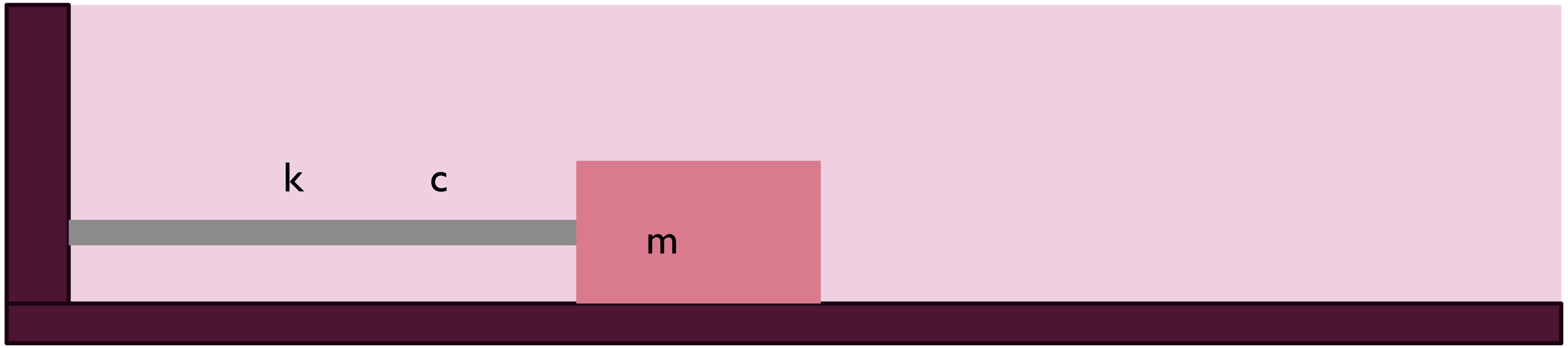
grass

OSCILLATORS' OF DIFFERENT KINDS / MECHANICAL SYSTEM

- Simplest form of a harmonic oscillator (spring-mass system) :

$$m \cdot \frac{dx^2}{dt^2} + c \cdot \frac{dx}{dt} + k \cdot x = 0$$

- is ODE of problem, where x is placement, k is spring constant, c is damping constant.
- Oscillations are not forced.



OSCILLATORS’ OF DIFFERENT KINDS / ELECTRICAL SYSTEM

- A sinusoidal voltage is set to source, while capacitor, resistor, battery is connected in serial way.

$$L \cdot \left(\frac{d^2Q}{dt^2}\right) + R\left(\frac{dQ}{dt}\right) + \frac{Q}{C} = V_0 \cdot \backslash\mathrm{cos}(\omega t)$$

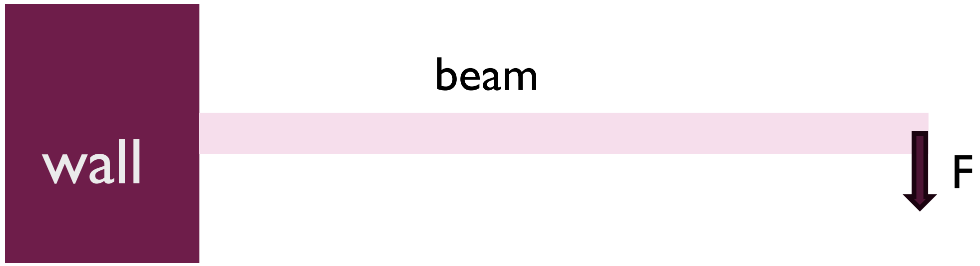
- where R, C means resistance and capacitance. Unknown is charge of capacitor

HIGHER ORDER LINEAR ODES

- Higher order ODEs are like harmonic motion (2nd derivative of space is velocity) that can be converted to set of linear equations of lower degree.
- Beam deflection : A beam is bending under load

$$E \cdot I \cdot \left(\frac{d^4 y}{dx^4}\right) = w(x)$$

- where E is Young's modulus, I moment of inertia, w is function of distributed load
- Don't forget that we need 4 boundary conditions to start.

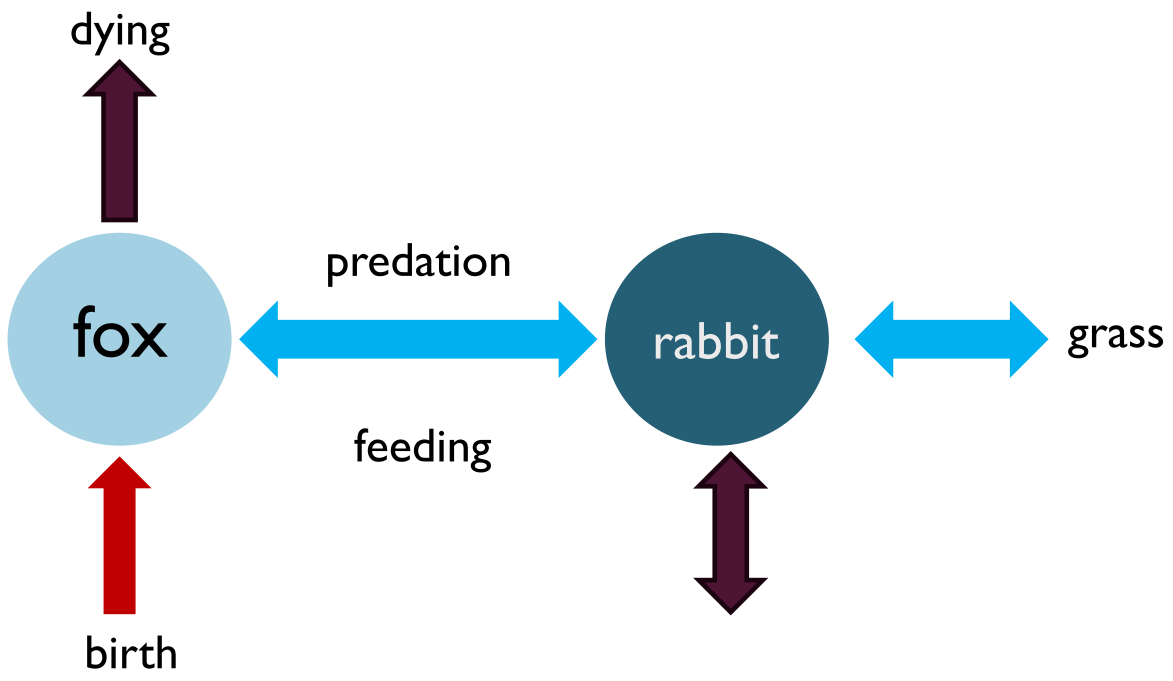


PREDATOR-PREY (LOTKA-VOLTERRA) MODE

- Basic equation of population dynamics
- Given a set of rabbits (R) and foxes (F) that eat rabbits.
  Expectation that population has effect on each other.

$$\begin{cases} \dfrac{dR}{dt} = a \cdot R - b \cdot R \cdot P \\ \dfrac{dF}{dt} = -c \cdot F + d \cdot R \end{cases}$$

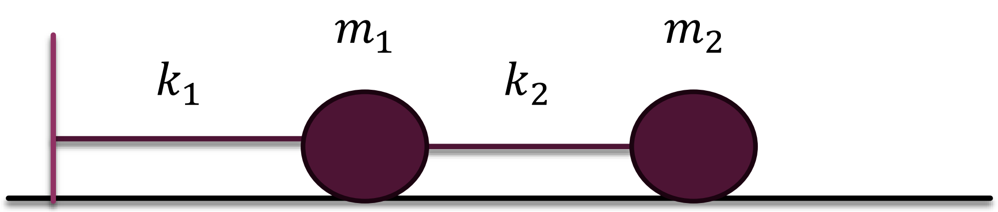
- If rabbit population grows, foxes eat rabbits, population oscillate.



COUPLED OSCILLATIONS

- Let’s connect two oscillatory systems.

$$m_1\left(\frac{d^2x_1}{dt^2}\right) = -k_1 \cdot x_1 + k_2(x_2 - x_1)))$$



- Twos masses(m_2 and m_1) are connected together, oscillations will travel across masses

$$m_2\left(\frac{d^2x_2}{dt^2}\right) = -k_2 \cdot (x_2 - x_1)$$

LOGISTIC GROWTH

- Population with carrying capacity K :

$$\frac{dP}{dt} = r \cdot P \cdot \left(1 - \frac{P}{K}\right)$$

- Bacteria in petri dish
- Fish population in lake
- Viral spread with limited susceptible population

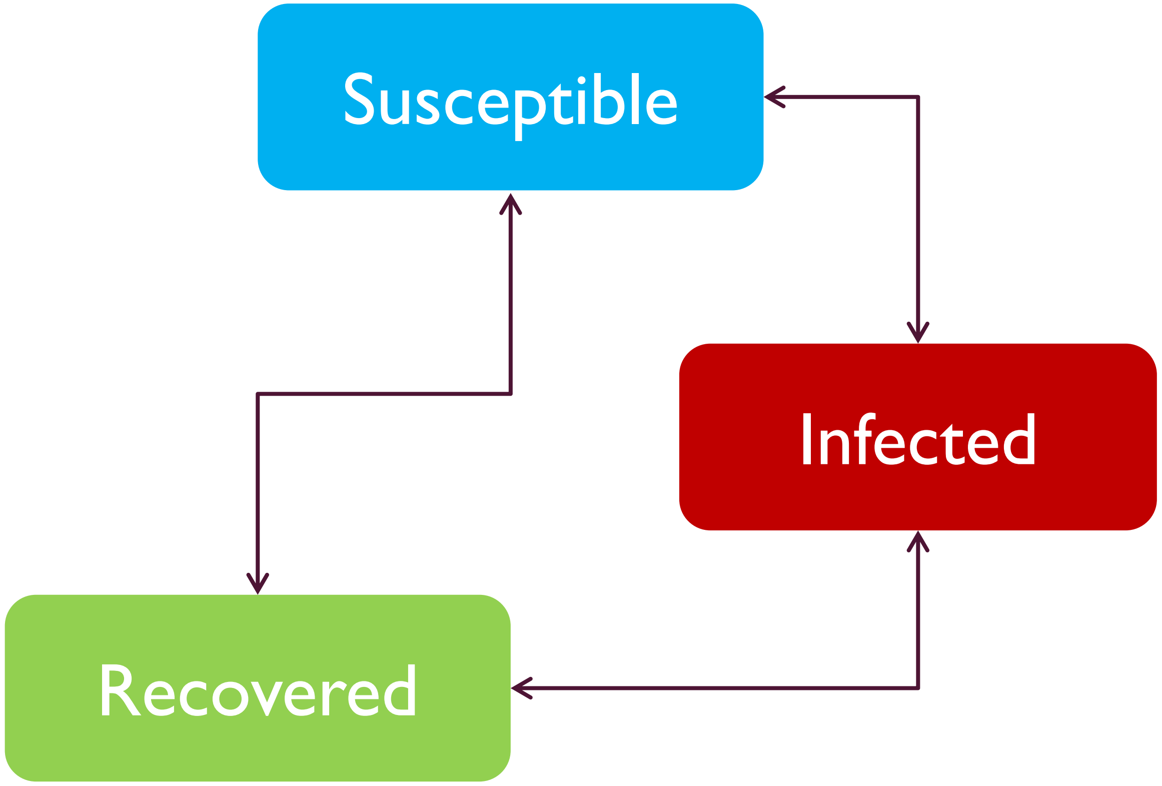
SIR DISEASE MODEL (NON-LINEAR PROBLEM)

- Epidemic spread (COVID-19, flu, measles)
- S = susceptible, I = infected, R = recovered

$$\frac{dS}{dt} = -\beta \cdot S \cdot I$$

$$\frac{dI}{dt} = \beta \cdot S \cdot I - \gamma \cdot I$$

$$\frac{dR}{dt} = \gamma \cdot I$$



AUTONOMOUS EQUATIONS

- Autonomous equations describe systems where the rules don't change over time – the system's behaviour depends only on its current state, not on what time it is.
- Form : $\frac{dy}{dt} = f(y)$ or $\frac{dx}{dt} = f(x)$

The independent variable doesn't appear explicitly on the right side. So, if y(t) is a solution, then y(t+c) is also a solution for any constant c. *("time-translation invariance")* shifting when you start doesn't change the pattern of behavior.

SOLVING STRATEGY OF ODES

- using ode45, ode89 and for stiff probles ode15s
- ode45 : explicit Runge-Kutta (4,5) formula, the first one to be used
- function of DE (differential equation) : `dydt = function odefun(t,y)`
- using default values, but set relative tolerance to 10^{-6} :
  `opts = odeset('RelTol',1e-6)`
- set timespan (start time to end time) : `tspan = [0 7000];`
- set initial state : `x0 = [1000];`
- solve ODE : `[T,Y] = ode45(@odefun, Tspan, x0, opts);`
  T : time of solutions (Nx1), Y : solution of variable(s) (NxM)

EXAMPLE – SOLUTION OF DECAY EQUATION

- Decay equation : $\frac{dN}{dt} = -\lambda \cdot N$
- set intial state : `N0 = 1000;`
- setting parameters of ODE :

```
tau_half = 5730; % in years
la = log(2)/tau_half;
```

- Solving equation using Runge-Kutta 4.5 order

```
[T,Y] = ode45(@(t,x) decayode(x,la),[0 
7000],[N0]);
```