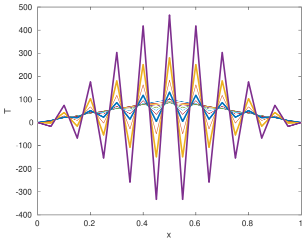
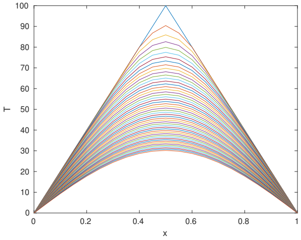
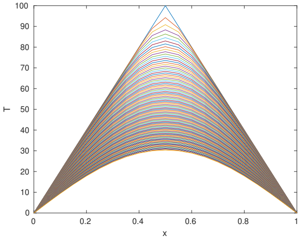
Szélessávú Hírközlés és Villamosságtan Tanszék
Antennák és EMC kutatócsoport
NES Laboratórium

MűFi23

Hyperbolic PDEs - Finite Difference Solution







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The Lax Method

- solution of

$$f_t + u \cdot f_x = 0$$

- blablabal
- final scheme :

$$\boxed{f_i^{n+1} = \frac{1}{2} \cdot \left(f_{i+1}^n + f_{i-1}^n\right) - \frac{c}{2}\left(f_{i+1}^n - f_{i-1}^n\right)}$$

- where

$$c = \frac{u \cdot \Delta t}{\Delta x}$$

  is the convection constant
- conditionally stabile : $c \leq 1$

- using some back substitutions for f_t and f_{tt}
- final scheme :

$$\boxed{f_i^{n+1} = f_i^n - \frac{c}{2}\left(f_{i+1}^n - f_{i-1}^n\right) + \frac{c^2}{2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n\right)}$$

- consistent, conditionally stable

$$c = \frac{u \cdot \Delta t}{\Delta x} \quad \leq 1$$

- quite complicated for nonlinear PDE or set of PDEs

- multiples step method shown for

$$f_t + u \cdot f_x = 0$$

- 1st step : Lax method used as provisional values at $\underline{n+1}$
- 2nd step : leapfrog used to find values at $\underline{n+2}$
- provisional value (at an internal point) :

$$f_i^{n+1} = \frac{1}{2}\left(f_{i+1}^n + f_{i-1}^n\right) - \frac{c}{2}\left(f_{i+1}^n - f_{i-1}^n\right)$$

- final value :

$$f_i^{n+2} = f_i^n - c \cdot \left(f_{i+1}^{n+1} - f_{i-1}^{n+1}\right)$$

- predictor-corrector method, 1969
- predicted value $(\overline{f_i^{n+1}})$:

$$\overline{f_i^{n+1}} = f_i^n - c \cdot (f_{i+1}^n - f_i^n)$$

- corrected value (f_i^{n+1}) :

$$f_i^{n+1} = \frac{1}{2}\left[f_i^n + \overline{f_i^{n+1}} - c \cdot \left(\overline{f_i^{n+1}} - \overline{f_{i-1}^{n+1}}\right)\right]$$

- first-order upwind approximation of the convection equation

$$f_t + u \cdot f_x = 0$$

- $$f_i^{n+1} = f_i^n - c \cdot \left(f_i^n - f_{i-1}^n\right)$$

- used a first-order, single-sided approximation for f_x

- use a second-order approximation for f_x
- second-order upwind approximation :

$$f_i^{n+1} = f_i^n - \cdot (f_i^n - f_{i-1}^n) - \frac{c(1-c)}{2} \left(f_i^n - 2f_{i-1}^n + f_{i-2}^n\right)$$

- also called fully implicit method
- starting from

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + u \cdot \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x} = 0$$

- ith line of equation set to solve for to step to $n+1$

$$-\frac{c}{2} f_{i-1}^{n+1} + f_i^{n+1} + \frac{c}{2} f_{i+1}^{n+1} = f_i^n$$

- (non)linear set of equations has a form of

$$\mathbf{A} \cdot \mathbf{f}^{n+1} = \mathbf{b}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{f}^{n+1}$ is the solution column vector and $\mathbf{b}$ is the vector of nonhomogeneous terms

- in general

$$f_{tt} = c^2 \cdot f_{xx}$$

- equivalent to set of two coupled first-order convection equations :

$$\left.\begin{aligned} f_t + c g_x &= 0 \\ g_t + c f_x &= \end{aligned}\right\}$$

- exact solution is

$$f(x,t) = F(x - ct) + G(x + ct)$$

- initial conditions are $f(x,0) = \phi(x)$ and $f_t(x,0) = \Theta(x,0)$
- after some calculations we get *D'Alembert solution*

$$f(x,t) = \frac{1}{2}\left(\phi(x - ct) + \phi(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \Theta(\xi) d\xi\right)$$

Lax-Wendroff one-step method

-
- starting from

Flux-splitting method