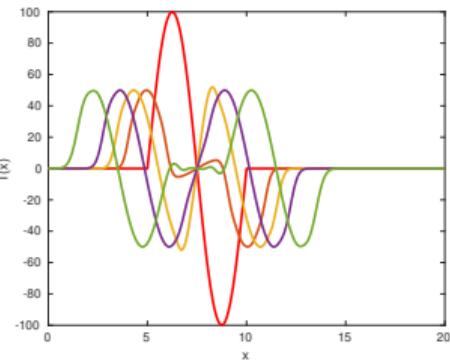
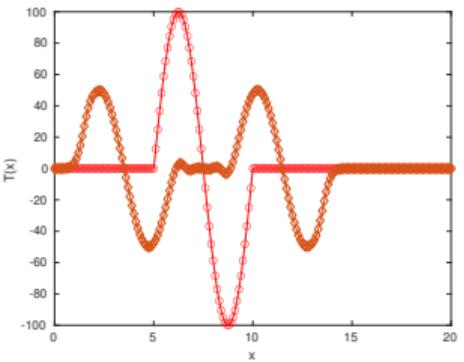
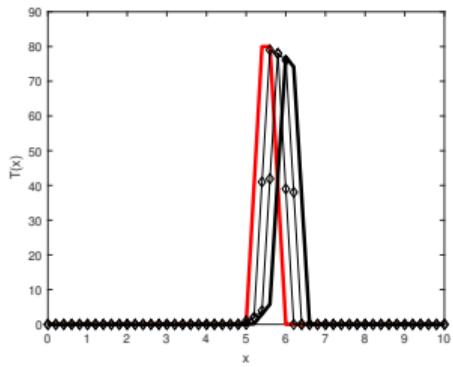


MűFi23

Hyperbolic PDEs - Finite Difference Solution



1 Solution of convection equations

2 Upwind Method(s)

3 Implicit Methods

4 Wave Equation

- ▶ solution of

$$f_t + u \cdot f_x = 0$$

- ▶ blablabal
- ▶ final scheme :

$$f_i^{n+1} = \frac{1}{2} \cdot (f_{i+1}^n + f_{i-1}^n) - \frac{c}{2} (f_{i+1}^n - f_{i-1}^n)$$

- ▶ where

$$c = \frac{u \cdot \Delta t}{\Delta x}$$

is the convection constant

- ▶ conditionally stable : $c \leq 1$



- ▶ using some back substitutions for f_t and f_{tt}
- ▶ final scheme :

$$f_i^{n+1} = f_i^n - \frac{c}{2} (f_{i+1}^n - f_{i-1}^n) + \frac{c^2}{2} (f_{i+1}^n - 2f_i^n + f_{i-1}^n)$$

- ▶ consistent, conditionally stable

$$c = \frac{u \cdot \Delta t}{\Delta x} \leq 1$$

- ▶ quite complicated for nonlinear PDE or set of PDEs



- multiples step method shown for

$$f_t + u \cdot f_x = 0$$

- 1st step : Lax method used as provisional values at $n+1$
- 2nd step : leapfrog used to find values at $n+2$
- provisional value (at an internal point) :

$$f_i^{n+1} = \frac{1}{2} (f_{i+1}^n + f_{i-1}^n) - \frac{c}{2} (f_{i+1}^n - f_{i-1}^n)$$

- final value :

$$f_i^{n+2} = f_i^n - c \cdot (f_{i+1}^{n+1} - f_{i-1}^{n+1})$$



- ▶ predictor-corrector method, 1969

- ▶ predicted value ($\overline{f_i^{n+1}}$) :

$$\overline{f_i^{n+1}} = f_i^n - c \cdot (f_{i+1}^n - f_i^n)$$

- ▶ corrected value (f_i^{n+1}) :

$$f_i^{n+1} = \frac{1}{2} \left[f_i^n + \overline{f_i^{n+1}} - c \cdot \left(\overline{f_i^{n+1}} - \overline{f_{i-1}^{n+1}} \right) \right]$$



1 Solution of convection equations

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- ▶ first-order upwind approximation of the convection equation

$$f_t + u \cdot f_x = 0$$

- ▶
$$f_i^{n+1} = f_i^n - c \cdot (f_i^n - f_{i-1}^n)$$
- ▶ used a first-order, single-sided approximation for f_x



- ▶ use a second-order approximation for f_x
- ▶ second-order upwind approximation :

$$f_i^{n+1} = f_i^n - \cdot(f_i^n - f_{i-1}^n) - \frac{c(1-c)}{2} (f_i^n - 2f_{i-1}^n + f_{i-2}^n)$$



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- ▶ also called fully implicit method

- ▶ starting from

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + u \cdot \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x} = 0$$

- ▶ i th line of equation set to solve for to step to $n + 1$

$$-\frac{c}{2}f_{i-1}^{n+1} + f_i^{n+1} + \frac{c}{2}f_{i+1}^{n+1} = f_i^n$$

- ▶ (non)linear set of equations has a form of

$$\mathbf{A} \cdot \mathbf{f}^{n+1} = \mathbf{b}$$

where \mathbf{A} is the coefficient matrix, \mathbf{f}^{n+1} is the solution column vector and \mathbf{b} is the vector of nonhomogeneous terms



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- ▶ in general

$$f_{tt} = c^2 \cdot f_{xx}$$

- ▶ equivalent to set of two coupled first-order convection equations :

$$\begin{cases} f_t + cg_x = 0 \\ g_t + cf_x = \end{cases}$$

- ▶ exact solution is

$$f(x, t) = F(x - ct) + G(x + ct)$$

- ▶ initial conditions are $f(x, 0) = \phi(x)$ and $f_t(x, 0) = \Theta(x, 0)$
- ▶ after some calculations we get *D'Alembert solution*

$$f(x, t) = \frac{1}{2} \left(\phi(x - ct) + \phi(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \Theta(\xi) d\xi \right)$$



- ▶
- ▶ starting from



