Pythagorean Hodograph curves

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Introduction

Arc length of polynomial plane curve $r : [0,1] \to \mathbb{R}^2, r(t) = (x(t), y(t))$:

$$length(r) = \int_0^1 ||r'(t)||^2 dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)} dt$$

In general the integral doesn't have a closed form solution.

Definition (Pythagorean hodograph property)

Polynomial plane curve $r(t) = (x(t), y(t))$ has a pythagorean hodograph if there exists a non-negative polynomial $\sigma(t)$ such that

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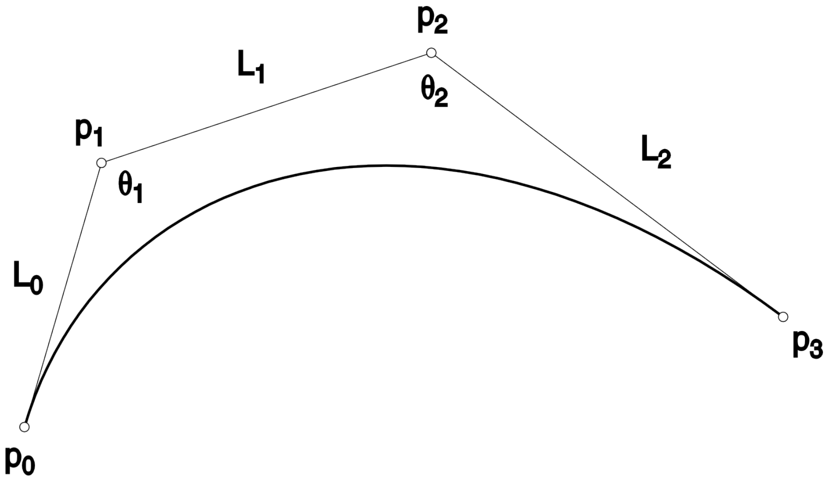
PH curves are exactly the curves for which the arc length function is a polynomial:

$$length(r) = \int_0^1 \sqrt{x'^2(t) + y'^2(t)}dt = \int_0^1 \sigma(t)dt$$

Cubic PH curves



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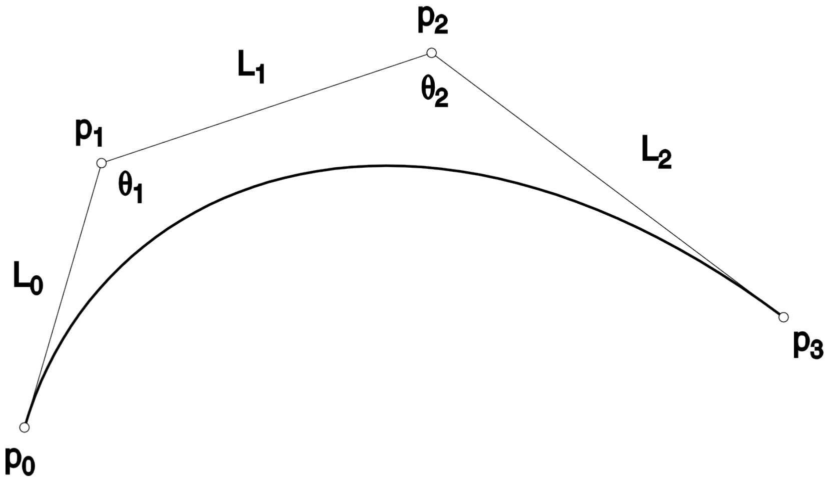

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- $L_1 = \sqrt{L_2 L_0}$
- $\theta_0 = \theta_1$

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- Offset curves $r_d(t) = r(t) + dn(t)$ are rational curves (of degree $2n - 1$)
- Degree is always odd
- Everything is easier

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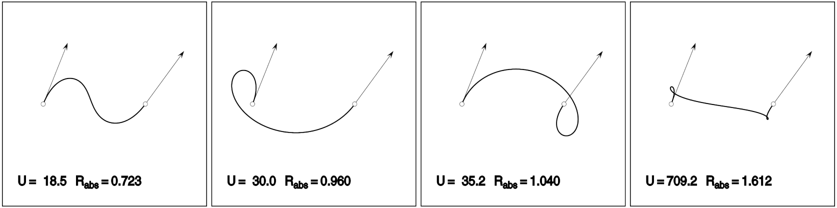
- For quintic PH curves, there are four solutions:



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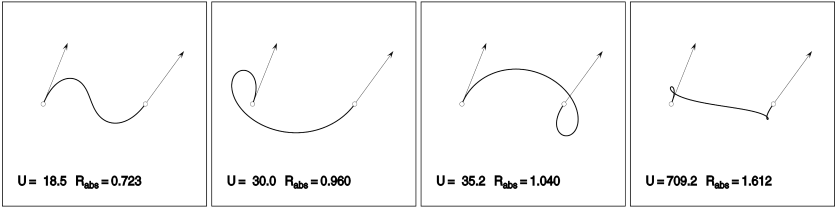

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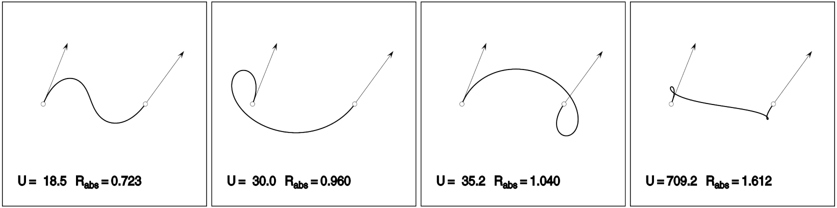
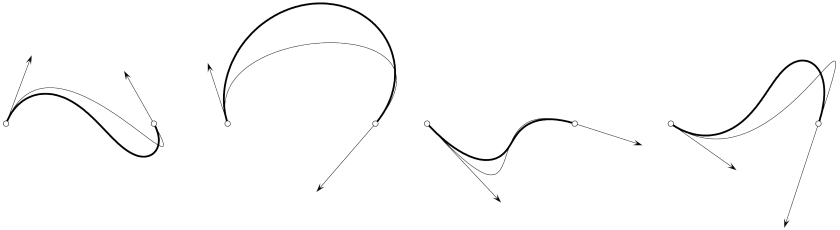


Alternative ways:

- A pair of cubic PH curves
- A cubic PH composed with a möbius transformation

Comparison with cubic interpolants



Defined in Minkowski space $\mathbb{R}^{2,1}$ which has indefinite inner product:

$$\langle (u_0, u_1, u_2), (v_0, v_1, v_2) \rangle = u_0 v_0 + u_1 v_1 - u_2 v_2$$

Minkowski PH curves

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PH property for the curve $m(t) = (x(t), y(t), r(t)) : [0, 1] \to \mathbb{R}^{2,1}$:

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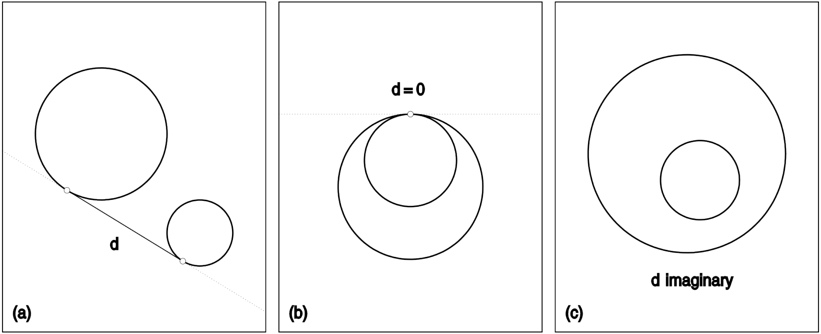
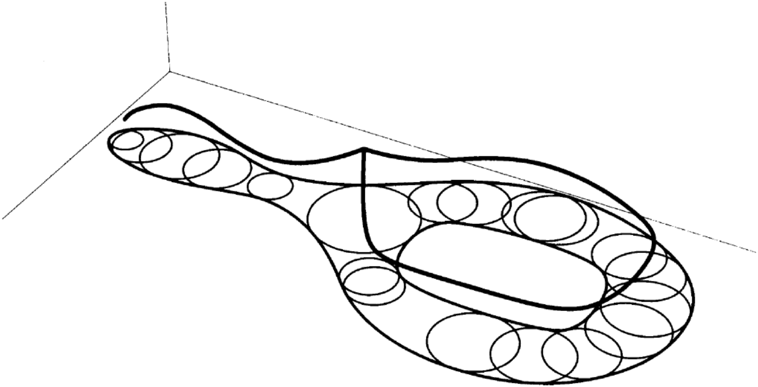
Motivation: interpret $(x(t), y(t), r(t)$ as a circle of radius $r(t)$ at $(x(t), y(t))$

Minkowski distance defined by: $d(u, v)^2 = \langle u - v, u - v \rangle$

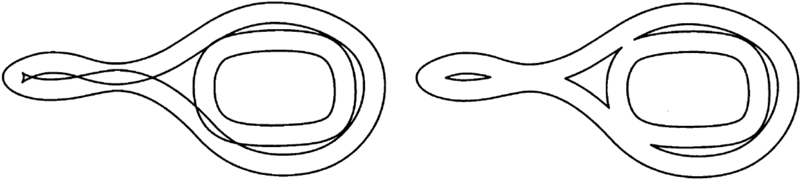


Medial Axis Transform



MAT as a graph of MPH curves

MAT offset curve trimming



Untrimmed / trimmed offset curves