Computational Methods - III

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Lecture Objectives

In this lecture we will discuss:

- Numerical schemes for linear Hyperbolic PDEs
- Numerical methods for parabolic equations
- Numerical methods for elliptic equations

Classification of PDEs

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x \partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff + G = 0$$

- Hyperbolic if $B^2 - 4AC > 0$
- Parabolic if $B^2 - 4AC = 0$
- Elliptic if $B^2 - 4AC < 0$

Fundamental PDEs

- Hyperbolic: Wave Equation 2nd order, 1st order (inviscid flows):

$$\frac{\partial^2 f}{\partial t^2} - \omega^2 \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0$$

- Parabolic: Heat Transfer Equation (boundary layers, heat conduction)

$$\frac{\partial f}{\partial t} = \nabla^2 f, \quad \frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$

- Elliptic: Laplace Equation, Poisson Equation (low-speed flows):

$$\nabla^2 f = 0, \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = g\left(\vec{r}, t\right),$$

Hyperbolic PDEs

- Model Problem:

$$\frac{\partial f}{\partial t}+\frac{\partial F}{\partial x}=\frac{\partial f}{\partial t}+\frac{\partial uf}{\partial x}=0$$

- Finite-volume conservative formulation:

$$\frac{\partial f}{\partial t}+\frac{F_{i+\frac{1}{2}}-F_{i-\frac{1}{2}}}{\Delta x}=0$$

- Problem - find the flux through the cell faces $F_{i+\frac{1}{2}}$

Methods for Hyperbolic PDEs

There is a wealth of methods available, however we will go only through the following methods:

- 1st order:
  - Upwind Explicit/Implicit
  - Lax Explicit
- 2nd order:
  - Lax-Wendroff Explicit
  - Warming & Beam Explicit
- 3rd Order: Rusanov Explicit

Explicit/Implicit 1st Order Upwind

- Explicit:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + u\frac{f_{i+1}^n - f_i^n}{\Delta x} = 0, \quad \frac{f_i^{n+1} - f_i^n}{\Delta t} + u\frac{f_i^n - f_{i-1}^n}{\Delta x} = 0$$

- Implicit:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + u\frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} = 0, \quad \frac{f_i^{n+1} - f_i^n}{\Delta t} + u\frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x} = 0$$

- Or in the flux form...

Generalised form of upwind

Upwind is stable/unstable depending on the value of u. For the stability:

$$f_i^{n+1} = f_i^n - u\frac{\Delta t}{\Delta x}\left(f_i^n - f_{i-1}^n\right), \quad u > 0$$

$$f_i^{n+1} = f_i^n - u\frac{\Delta t}{\Delta x}\left(f_{i+1}^n - f_i^n\right), \quad u < 0$$

Let us introduce the following definition:

$$u^+ = \frac{1}{2}(u + |u|) = \begin{cases} u & u > 0 \\ 0 & u < 0 \end{cases}$$

$$u^- = \frac{1}{2}(u - |u|) = \begin{cases} 0 & u > 0 \\ u & u < 0 \end{cases}$$

Generalised form of upwind

Generalised form of the upwind method:

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{\Delta x}\left(u^+\left(f_i^n - f_{i-1}^n\right) + u^-\left(f_{i+1}^n - f_i^n\right)\right)$$

Or using the definitions of $u^\pm$:

$$f_i^{n+1} = f_i^n - u\frac{\Delta t}{\Delta x}\frac{f_{i+1}^n - f_{i-1}^n}{2} + \frac{|u|\,\Delta t}{2\Delta x}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n\right)$$

Flux form?

Same as central differences + dissipation term

Lax-Friedrichs Method

The central differences which are unstable unconditionally can be stabilised via the amended approximation of time derivative:

$$\frac{f_i^{n+1} - \frac{f_{i+1}^n + f_{i-1}^n}{2}}{\Delta t} + u\frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x} = 0$$

The resulting method was suggested by Lax in 1954:

$$f_i^{n+1} = \frac{f_{i+1}^n + f_{i-1}^n}{2} - u\frac{\Delta t}{2\Delta x}\left(f_{i+1}^n - f_{i-1}^n\right)$$

Stability & Accuracy -?

Lax-Wendroff Method

Lax-Wendroff Method utilises second order discretisation in order to compensate for the negative artificial viscosity of the central-difference scheme:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

From Taylor series:

$$f_{n+1} = f_n + \frac{\partial f}{\partial t}\Delta t + \frac{\partial^2 f}{\partial t^2}\frac{\Delta t^2}{2} + O\left(\Delta t^3\right)$$

Or in the difference approximation form:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + u\frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x} = \frac{\Delta t u^2}{2}\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{\Delta x^2} + O\left(\Delta t^2, \Delta x^2\right)$$

Accuracy, Stability - ?

MacCormack Method

A two-step scheme similar to Lax-Wendroff which is widely used for fluid flow equations:

- Predictor:

$$\frac{\overline{f_i^{n+1}} - f_i^n}{\Delta t} + u\frac{f_{i+1}^n - f_i^n}{\Delta x} = 0$$

- Corrector:

$$\frac{f_i^{n+1} - \frac{f_i^n + \overline{f_i^{n+1}}}{2}}{\Delta t} + u\frac{\overline{f_i^{n+1}} - \overline{f_{i-1}^{n+1}}}{2\Delta x} = 0$$

For a linear equation - equivalent to Lax-Wendroff.

Warming-Beam Method

Two-step:

- predictor (u>0):

$$\frac{\overline{f_i^{n+1}} - f_i^n}{\Delta t} + u\frac{f_i^n - f_{i-1}^n}{\Delta x} = 0$$

- corrector:

$$f_i^{n+1} = \frac{f_i^n + \overline{f_i^{n+1}}}{2} - \frac{u\Delta t}{\Delta x}\frac{\overline{f_i^{n+1}} - \overline{f_{i-1}^{n+1}}}{2} - \frac{u\Delta t}{\Delta x}\frac{f_i^n - 2f_{i-1}^n + f_{i-2}^n}{2}$$

Rusanov Method

Three-step $c = \frac{u\Delta t}{\Delta x}$:

- Step 1:

$$f^{(1)}_{i+\frac{1}{2}} = \frac{f^n_{i+1} + f^n_i}{2} - \frac{1}{3}c\left(f^n_{i+1} - f^n_i\right)$$

- Step 2:

$$f^{(2)}_i = f^n_i - \frac{2}{3}c\left(f^{(1)}_{i-\frac{1}{2}} - f^{(1)}_{i-\frac{1}{2}}\right)$$

- Step 3:

$$\begin{aligned} f^{n+1}_i = f^n_i &- \frac{c}{24}\left(-2f^n_{i+2} + 7f^n_{i+1} - 7f^n_{i-1} + 2f^n_{i-2}\right) \\ &- \frac{3}{8}c\left(f^{(2)}_{i+1} - f^{(2)}_{i-1}\right) \\ &- \frac{\omega}{24}\left(f^n_{i+2} - 4f^n_{i+1} + 6f^n_i - 4f^n_{i-1} + f^n_{i-2}\right) - \end{aligned}$$

Rusanov Method

- Modified equation:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = -u\frac{(\Delta x)^3}{24}\left(\frac{\omega}{c} - 4c + c^3\right)\frac{\partial^4 f}{\partial x^4} + O\left((\Delta x)^4\right)$$

- Stability:

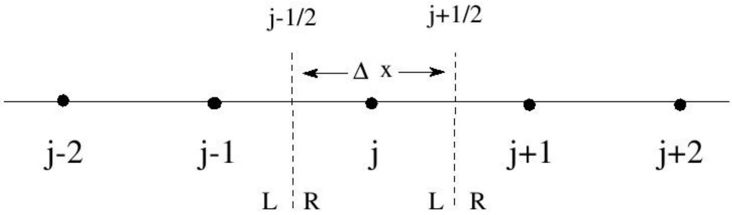
$$|c| < 1$$

$$4c^2 - c^4 \leq \omega \leq 3$$

Increasing the order of finite volume schemes

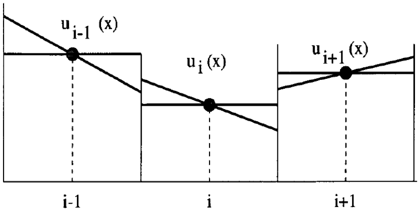
Lax-Friedrichs is too dissipative when applied in the canonic form. We can reduce the dissipation through variable interpolation.

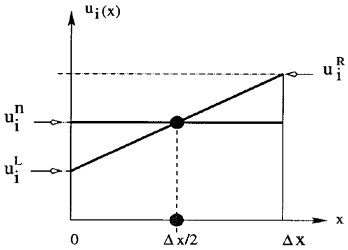
$$\mathbf{F}^{LF}_{i+\frac{1}{2}} = \frac{\mathbf{F}^n\left(\mathbf{U}^n_{i+\frac{1}{2},L}\right) + \mathbf{F}^n\left(\mathbf{U}^n_{i+\frac{1}{2},R}\right)}{2} + \frac{1}{2}\frac{\Delta x}{\Delta t}\left(\mathbf{U}^n_{i+\frac{1}{2},L} - \mathbf{U}^n_{i+\frac{1}{2},R}\right)$$



Linear interpolation

For a function $u(x)$





Linear interpolation

Interpolate function f inside the cell:

$$f(x) = f_j^n + (x - x_j)\frac{\Delta_j}{\Delta x}, \quad x \in [0, \Delta x]$$

he slope Δ_i can be chosen in a number of ways. The general expression:

$$\Delta_j = \frac{1+\omega}{2}\Delta f_{j-\frac{1}{2}} + \frac{1-\omega}{2}\Delta f_{j+\frac{1}{2}}$$

$$\Delta f_{j-\frac{1}{2}} = f_j^n - f_{j-1}^n, \quad \Delta f_{j+\frac{1}{2}} = f_{j+1}^n - f_j^n$$

Where $\omega \in [-1, 1]$ (for $\omega = 0$ - represents central differences). Note that for piecewise-linear form the conservation is satisfied by default.

Quadratic interpolation

Interpolate the function inside the cell using 2nd order polynomial:

$$f(\xi) = a\xi^2 + b\xi + c$$

$$\frac{1}{\Delta x}\int_{-\frac{3\Delta x}{2}}^{-\frac{\Delta x}{2}} f(\xi)\,dx = f_{j-1}, \quad \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} f(\xi)\,dx = f_j$$

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{3\Delta x}{2}} f(\xi)\,dx = f_{j+1}$$

Quadratic interpolation

Resulting coefficients:

$$a = \frac{f_{j+1} - 2f_j + f_{j-1}}{2\Delta x^2}$$

$$b = \frac{f_{j+1} - f_{j-1}}{2\Delta x}, \quad c = \frac{-f_{j-1} + 26f_j - f_{j+1}}{24}$$

$$f_{j+\frac{1}{2},L} = \frac{1}{6}\left(2f_{j+1} + 5f_j - f_{j-1}\right)$$

$$f_{j+\frac{1}{2},R} = \frac{1}{6}\left(-f_{j+2} + 5f_{j+1} + 2f_j\right)$$

(*) H. Lomax, "Fundamentals of computational fluid dynamics", Springer, 2001
(**) E.F. Toro, "Riemann solvers and numerical methods for fluid dynamics", Springer, 1997

Quadratic interpolation

For a linear equation, with $u = 1$ for simplicity, this choice of flux leads to:

$$F_{j+\frac{1}{2}} = \frac{1}{2}\left(F\left(f^L_{j+\frac{1}{2}}\right) + F\left(f^R_{j+\frac{1}{2}}\right)\right) = \frac{1}{12}\left(-f_{j+2} + 7f_{j+1} + 7f_j - f_{j-1}\right)$$

$$F_{j-\frac{1}{2}} = \frac{1}{2}\left(F\left(f^L_{j-\frac{1}{2}}\right) + F\left(f^R_{j-\frac{1}{2}}\right)\right) = \frac{1}{12}\left(-f_{j+1} + 7f_j + 7f_{j-1} - f_{j-2}\right)$$

Resulting scheme:

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{12\Delta x}\left(-f_{j+2} + 8f_{j+1} - 8f_{j-1} + f_{j-2}\right)$$

Equivalent to 4th order central difference approximation.

Parabolic PDEs

1D parabolic PDE (heat transfer):

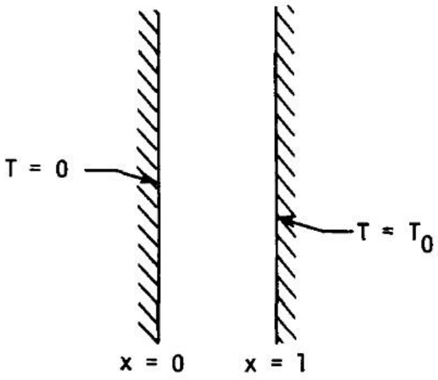
$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} + S$$

Schemes:

- Simple Explicit
- Simple Implicit
- Crank-Nicholson

Analytic Solution

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \quad T(0,t) = 0, T(1,t) = T_0, \quad T(x,0) = 0$$



Simple Explicit

1D parabolic PDE:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = D \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

The scheme is:

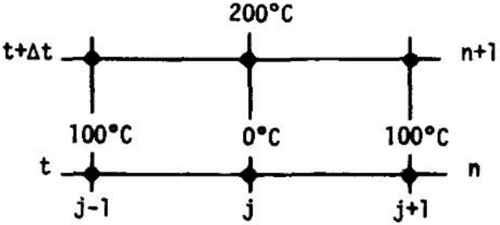
- Stable if $\Delta t \leq \frac{\Delta x^2}{2D} \rightarrow r = \frac{D\Delta t}{\Delta x^2} < \frac{1}{2}$
- Accurate to $O\left(\Delta t, \Delta x^2\right)$

Simple Explicit

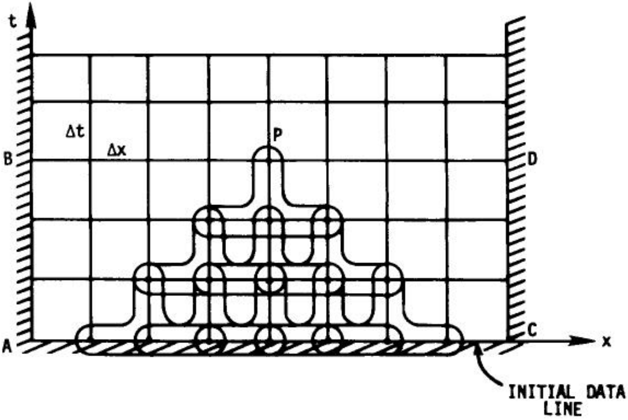
Rewrite in the form:

$$T_i^{n+1} = r\left(T_{i+1}^n + T_{i-1}^n\right) + (1 - 2r)\, T_i^n$$

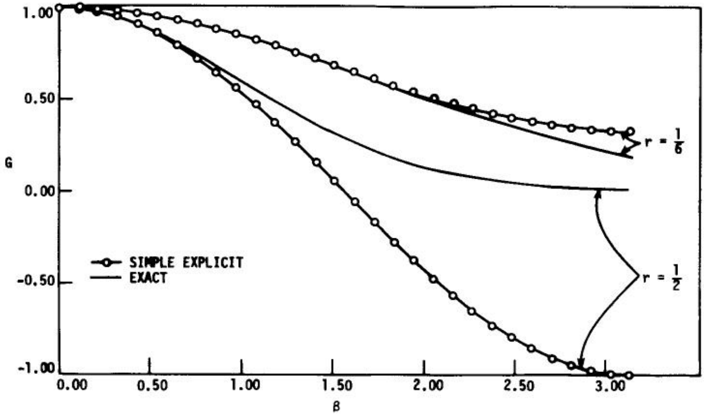
Look at a simple case and set $r = 1$



Simple Explicit



Simple Explicit



Example

Consider transient heat transfer in a rod $x \in [0, 1]$ with the thermal diffusivity coefficient $D = 0.05$, same temperature on both sides and initial condition:

$$T = \sin(2\pi x)$$

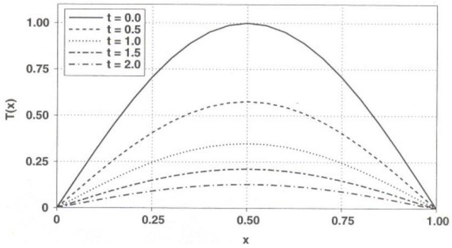
What will the amplitude error be after 10 time steps with $\Delta t = 0.1$ on a grid of 10 cells.

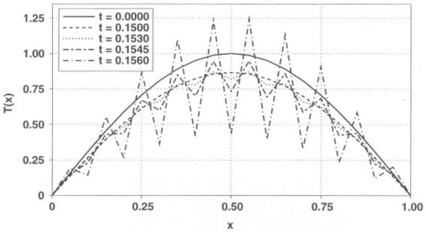
Simple Explicit

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} + \left(\pi^2 - 1\right) e^{-t} \sin\left(\pi x\right)$$

$$T\left(0,t\right) = T\left(1,t\right) = 0, \quad T\left(x,0\right) = \sin\left(\pi x\right)$$

$$h_1 = 0.001, h_2 = 0.0015, \Delta x = 0.05, D = 1$$





Simple Implicit (Laasonen)

1D parabolic PDE:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = D\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2}$$

The scheme is:

- Unconditionally stable
- Accurate to $O\left(\Delta t, (\Delta x)^2\right)$
- Leads to a system of linear equations

Crank-Nicholson (Trapezoidal)

1D parabolic PDE:

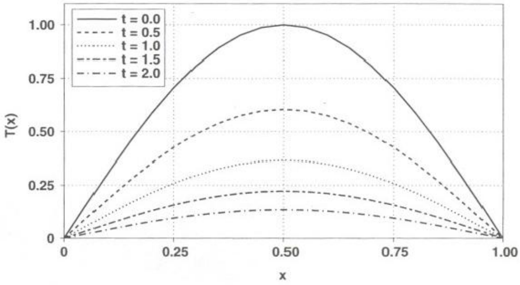
$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = D\frac{1}{2}\left(\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2} + \frac{T_{i+1}^{n} - 2T_i^{n} + T_{i-1}^{n}}{\Delta x^2}\right)$$

The scheme is:

- Unconditionally stable
- Accurate to $O\left((\Delta t)^2, (\Delta x)^2\right)$
- Leads to a system of linear equations

Crank-Nicholson (Trapezoidal)

$$h_1 = 0.05, \Delta x = 0.05, D = 1$$



DuFort-Frankel

1D parabolic PDE:

$$\frac{T_i^{n+1} - T_i^{n-1}}{2\Delta t} = D\frac{T_{i+1}^n - \left(T_i^{n+1} + T_i^{n-1}\right) + T_{i-1}^n}{\Delta x^2}$$

The scheme is:

- Unconditionally stable
- Accurate to $O\left(\left(\frac{\Delta t}{\Delta x}\right)^2, (\Delta t)^2, (\Delta x)^2\right)$
- Leads to an explicit form:

$$T_i^{n+1}(1+2r) = T_i^{n-1} + 2r\left(T_{i+1}^n - T_i^{n-1} + T_{i-1}^n\right), \quad r = \frac{D\Delta t}{\Delta x^2}$$

DuFort-Frankel

Truncation error:

$$TE = \left(\frac{1}{12} D (\Delta x)^2 - D^3 \frac{(\Delta t)^2}{(\Delta x)^2} \right) \frac{\partial^4 T}{\partial x^4} +$$
$$\left(\frac{1}{360} D (\Delta x)^4 - \frac{1}{3} D^3 (\Delta t)^2 + 2 D^5 \frac{(\Delta t)^4}{(\Delta x)^4} \right) \frac{\partial^6 T}{\partial x^6} +$$
$$\ldots$$

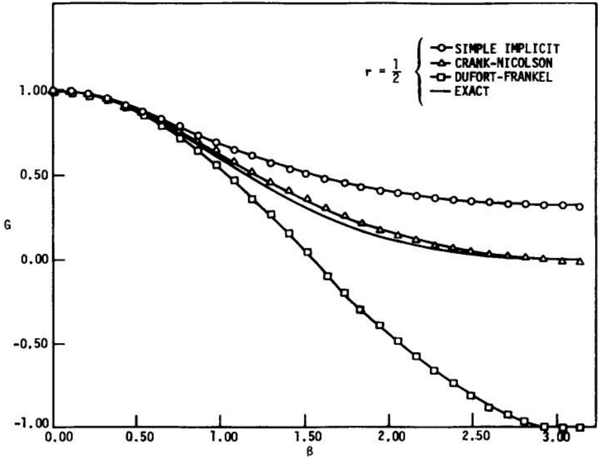
Amplification factor:

$$G = \frac{2r\cos\beta \pm \sqrt{1 - 4r^2 \sin^2 \beta}}{1 + 2r`}$$

In order for the scheme to be consistent, it is necessary to have

$$\frac{\Delta t}{\Delta x} = r \frac{\Delta x}{D} \to 0$$

Amplification factors



Combined Explicit/Implicit

Explicit, implicit and Crank-Nicholson schemes can be combined:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{D}{\Delta x^2}\left(\theta\left(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}\right) + (1-\theta)\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)\right)$$

cases:

- $\theta = 0$ - explicit, TE = $O\left(\Delta t, (\Delta x)^2\right)$
- $\theta = 1$ - implicit, TE = $O\left(\Delta t, (\Delta x)^2\right)$
- $\theta = \frac{1}{2}$ - trapezoidal, TE = $O\left((\Delta t)^2, (\Delta x)^2\right)$

Combined Explicit/Implicit

Combined method can yield higher orders for specific values of θ:

- $\theta = \frac{1}{2} - \frac{1}{12r}$, TE = $O\left((\Delta t)^2, (\Delta x)^4\right)$
- $\theta = \frac{1}{2} - \frac{1}{12r}$, and $r = \frac{1}{\sqrt{20}}$ TE = $O\left((\Delta t)^2, (\Delta x)^6\right)$

Stability:

- Unconditionally stable if $\frac{1}{2} \leq \theta \leq 1$
- If $0 \leq \theta < \frac{1}{2}$, stable only if $0 \leq r \leq \frac{1}{2-4\theta}$

2D: Try Crank-Nicholson

1D parabolic PDE:

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = D\frac{1}{2}\left(\frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{\Delta x^2} + \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2}\right) +$$

$$D\frac{1}{2}\left(\frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{\Delta y^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2}\right)$$

What would the system matrix look like?

2D: Approximate Factorisation

Write Crank-Nicholson for a 2D heat equation, for example:

$$\frac{T^{n+1}-T^n}{\Delta t}=\frac{D}{2}A_x\left(T^{n+1}+T^n\right)+\frac{D}{2}A_y\left(T^{n+1}+T^n\right)+O\left(\Delta t^2+\Delta x^2+\Delta y^2\right)$$

With operators:

$$A_x(T)=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2},\quad i=1..M-1,\quad j=1...N-1$$

$$A_y(T)=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2},\quad i=1..M-1,\quad j=1...N-1$$

...

Rearrange:

$$\left(I - \frac{D\Delta t}{2}A_x - \frac{D\Delta t}{2}A_y\right)T^{n+1} = \left(I + \frac{D\Delta t}{2}A_x + \frac{D\Delta t}{2}A_y\right)T^n + \Delta t O\left(\Delta t^2 + \Delta x^2 + \Delta y^2\right)$$

We can form a full product with by adding and subtracting $A_x A_y$ term:

$$\left(I - \frac{D\Delta t}{2}A_x - \frac{D\Delta t}{2}A_y\right)T^{n+1} = \left(I - \frac{D\Delta t}{2}A_x\right)\left(I - \frac{D\Delta t}{2}A_y\right)T^{n+1} - \frac{D^2\Delta t^2}{4}A_x A_y T^{n+1}$$

Final system:

$$\left(I - \frac{D\Delta t}{2}A_x\right)\left(I - \frac{D\Delta t}{2}A_y\right)T^{n+1} = \left(I + \frac{D\Delta t}{2}A_x\right)\left(I + \frac{D\Delta t}{2}A_y\right)T^n$$

1D-split:

$$\left(I - \frac{D\Delta t}{2}A_x\right)\zeta^{n+1} = T$$

Unconditionally stable.

2D: Alternating Direction Implicit

For a 2D heat equation, for example:

$$\frac{T_{i,j}^{n+\frac{1}{2}} - T_{i,j}^{n}}{\Delta t} = \frac{D}{2}\left(\frac{\partial^2 T^{n+\frac{1}{2}}}{\partial x^2} + \frac{\partial^2 T^{n}}{\partial y^2}\right)$$

$$\frac{T^{n+1} - T^{n+\frac{1}{2}}}{\Delta t} = \frac{D}{2}\left(\frac{\partial^2 T^{n+\frac{1}{2}}}{\partial x^2} + \frac{\partial^2 T^{n+1}}{\partial y^2}\right)$$

Alternate directions to make the scheme symmetric. Analogous to approximate factorisation.

Alternating Direction Explicit

A multistep method by Saul'yev (1957):
Step 1, from left to right, i.e. T_{i-1}^{n+1} is known:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{D}{\Delta x^2}\left(T_{i-1}^{n+1} - T_i^{n+1} - T_i^n + T_{i+1}^n\right)$$

Step 2, from right to left, i.e. T_{i+1}^{n+2} is known::

$$\frac{T_i^{n+2} - T_i^{n+1}}{\Delta t} = \frac{D}{\Delta x^2}\left(T_{i-1}^{n+1} - T_i^{n+1} - T_i^{n+2} + T_{i+1}^{n+2}\right)$$

Accuracy: $O\left((\Delta x)^2, (\Delta t)^2, (\Delta t/\Delta x)^2\right)$. Unconditionally stable.

Elliptic PDEs

2D Laplace equation:

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

Simple!

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} = 0$$

Accuracy - ?

Elliptic PDEs: Higher order 9-point

2D Laplace equation on a uniform grid with Δx and Δy

$$\frac{-p_{i-2,j}+16p_{i-1,j}-30p_{i,j}+16p_{i+1,j}-p_{i+2,j}}{12\Delta x^2}+\frac{-p_{i,j-2}+16p_{i,j-1}-30p_{i,j}+16p_{i,j+1}-p_{i,j+2}}{12\Delta y^2}=0$$

Accuracy

$$h=k: \quad ??$$

$$\Delta x \neq \Delta y: \quad O\left(\Delta x^4, \Delta y^4\right)$$

Elliptic PDEs: Higher order

2D Laplace equation on a uniform grid with $\Delta x = h$ and $\Delta y = k$

$$p_{i+1,j+1} + p_{i-1,j+1} + p_{i+1,j-1} + p_{i-1,j-1} - 2\frac{h^2 - 5k^2}{h^2 + k^2}\left(p_{i+1,j} + p_{i-1,j}\right) + 2\frac{5h^2 - k^2}{h^2 + k^2}\left(p_{i,j+1} + p_{i,j-1}\right) - 20p_{i,j} = 0$$

Accuracy:

$$h = k : \quad O\left(h^6\right)$$

$$h \neq k : \quad O\left(h^2, k^2\right)$$

Summary

- Stability and accuracy of methods for hyperbolic equations can be determined via modified equation analysis
- Accuracy of methods can be raised through sub-grid interpolation
- Increased accuracy can bring oscillatory behaviour
- We reviewed a number of methods for parabolic equations.
- Implicit schemes, as a rule lead to a system of linear equations to be solved. This system can be changed to a tri-diagonal one by approximate factorisation methods.
- Elliptic PDEs lead to a system of linear equations (potentially sparse).