Limit Theorems with Multiple Small World Clusters/Networks [*]

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Abstract

We refine the conditions needed to obtain central limit theorems for data with multiple clusters or networks allowing for different patterns of growth rates of the size and the number of clusters/networks. Next, we specialise the results to the setup where we have sparse, small world networks, i.e. networks with limited maximal degree and a relatively small (but growing) diameter, properties encountered in many social and economic networks. In this setup the conditions translate into restrictions on the constant of growth of the network diameter relative to the maximal degree. We consider both means of node- and edge-specific characteristics and show that for the latter imposing within-networks weak dependence may be necessary. Our Monte Carlo simulations confirm that the wild cluster bootstrap provides accurate inference.

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1 Introduction

In many cases economic data come from several groups or individuals interconnected through their social networks and it is important to adjust inference for the presence of related dependence between

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the observations. In particular, it is important to know conditions (if any) under which asymptotic theory can be used to approximate distributions of the sample statistics. When the number of clusters stays the same once we increase the sample size, such conditions have been provided by Bester et al. (2011) and then extended by Djogbenou et al. (2019) (DMN, henceforth) and Hansen & Lee (2019) to a growing number of clusters.

As our first contribution, we refine the latter result by distinguishing cases when we may have various proportions of growing and fixed-size clusters or network components and show that the general restrictions on the size of the largest cluster in DMN can be relaxed in some cases. This refinement is motivated by empirical studies that highlight the relevance of settings where network evolution involves components growing at different rates alongside an increasing number of fixed-size components. For example, Tomassini & Luthi (2007) demonstrate that the coauthorship network in genetic programming evolved in that fashion, with increasing share of nodes in the giant component, decreasing share of nodes (but increasing number) in the second largest component, and the number of components growing proportionally to the number of nodes.

The conditions needed to obtain a CLT vary between different network structures, in particular on the number of growing components and the variation of their sizes. On one end, when the network consist of many components growing at the same rate it is enough that the largest component grows at a rate only marginally slower than N, on the other, when we have both large components growing in size and many fixed-size components, the largest component may need to grow at a rate smaller than $\sqrt{N}$.

As a second contribution, we apply these results to provide low level conditions for the case of sparse, small world networks. Many economic and social networks on top of being sparse exhibit a small-world property (Watts & Strogatz (1998)), namely that the network distance between each pair of connected nodes is small compared to the number of nodes in the network.[1] Formally, a small-world network has a diameter proportional to $\log N$, where N is the number of nodes. We combine this restriction and boundedness of the maximal degree of a node (i.e. sparsity) and show that CLT applies to data coming from multiple small world networks under additional assumptions restricting the size of the diameter relative to the maximal degree for any given N, or, put differently,

[1] The "small-world" property often also includes the characteristic that the network graph is much more clustered than a random graph (see e.g. Definition 4.1.3 in Watts (1999)). However, the latter property is not useful for the purpose of providing CLT in our context so we do not discuss it.

restricting the constant of proportionality relating the diameter to $\log N$.

As a third contribution, in addition to node-specific means we also consider CLTs for edge-specific characteristics, where we distinguish between flows, i.e. purely characteristics of edges, and contrasts, i.e. functions of characteristics of nodes involved in an edge. We show that a CLT for the means of flows holds under relatively mild and natural strengthening of conditions needed for node-specific means. However, we require strong mixing conditions (with respect to network distance) to justify a CLT for the means of contrasts, which is considerably stronger than other conditions we impose. As the clustering literature rarely imposes network structure and, hence, does not define edge-specific means, our results seem novel here.

We confirm our findings in Monte Carlo simulations showing that the CLT applies to t statistics coming from sparse, small world networks. Simulations also show that the wild cluster bootstrap suggested by Cameron et al. (2008) performs much better than CCE errors with small number of large clusters/networks, adding to the evidence supporting it for inference (cf. MacKinnon et al. (2023)).

When viewed from the network literature perspective, our analysis is conditional on network evolution, thus we do not include uncertainty coming from network formation. Hence, our results apply to stable networks with network-mediated dependence as the main source of dependence. An example is a long-term friendship network where we are interested in labour market outcomes. These outcomes are likely to have been affected by network interactions (e.g. referrals) and our results suggest how to proceed with inference on means of such outcomes.

Kojevnikov et al. (2021) provide a CLT for node-specific means for data coming from one large network assuming weak dependence between the nodes in the form of ψ-dependence (see also Leung & Moon (2023)). They do not condition on the observed network in their analysis and provide some primitive conditions and examples of network formation processes consistent with ψ-dependence. They propose a HAC-type variance estimator. Leung (2023) shows, however, that for many networks a cluster-robust inference may perform better. Similarly to our paper Ogburn et al. (2024) provide a CLT conditional on the network formation process but only allow dependence up to friends-of-friends, whereas we allow for any connected nodes (via any path) to be dependent.

There is a large literature on obtaining limit theorems with spatial networks (see Jenish & Prucha (2012), Kuersteiner & Prucha (2013), Kuersteiner (2019) among others). Although many social and economic phenomena could be modelled using these networks, most social and economic networks feature presence of cliques (see Jackson (2008)). But Kojevnikov et al. (2021) demonstrate that spatial networks have limitations in terms of accommodating nontrivial presence of cliques, in particular when the maximal clique size grows with N.

Technically, our results rely on verification of Lyapunov's condition, just as DMN. They can also be seen as related to the results on limit theorems with m-dependence, where m can diverge to infinity (Romano & Wolf (2000)), and the literature on normal approximations under local dependence (Baldi & Rinott (1989), Chen & Shao (2004)) with a difference that we allow the dependence neighbourhoods to grow with the sample size. Other recent articles on inference using network data include Bickel et al. (2011), Bickel et al. (2013), Matsushita & Otsu (2023) among others.

As we aim to apply our results to small world networks, we refer to different dependency groups in the data as network components, rather than clusters. Nevertheless, for the main CLTs the two terms can be used interchangeably.

2 Main idea

Let $\{Y_1, \ldots, Y_N\}$ denote mean zero random variables corresponding to nodes in a network G_N. We are interested in the central limit theorem for the sample mean:

$$\overline{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

assuming that only nodes not connected through any network path have statistically independent Y_i's and dependence between the remaining observations is not restricted. For example, inference on the network sample average is of interest in the studies of economic connectedness (Chetty et al. (2022)): we may want to test if we have perfectly economically connected communities by testing $H_0 : E[Y_i] = 0.5$, where Y_i is the individual economic connectedness for a person i in the low SES group, defined as the fraction of friends in the high SES group.

Let the network consist of c_N separated components of size $N_c, c = 1, \ldots, c_N$ and the number of non-zero pairwise correlations between the Y_i's in a component is of order $N_c^{1+\gamma_c}$, $\gamma_c \in [0, 1]$. Also, let

$c = 1$ correspond to the component for which $N_c^{1+\gamma_c}$ grows the fastest. Then, if $0 < Var(Y_i) < \infty$, we have:

$$Var\left(\sum_{i=1}^{N} Y_i\right) \sim \sum_{c=1}^{c_N} N_c^{1+\gamma_c}$$

Using this structure we provide sufficient conditions to verify the Lyapunov condition, which involve bounds on the rates of growth of N_c's. Next the bound on the maximal number of nodes in a sparse network with a given maximal degree and diameter (Pineda-Villavicencio & Wood (2015)) is used to translate these conditions to the parameters of a small world network.

The refinement compared to DMN comes from the fact that they consider a general case with all clusters possibly growing with N. Thus, in order to verify the Lyapunov's condition, they bound:

$$N^{-2-\delta} \sum_{c=1}^{c_N} N_c^{(1+\gamma_c)(2+\delta)} \leq N^{-1-\delta} N_1^{(1+\gamma_1)(1+\delta)}$$

and the right-hand side converges to zero if $N_1^{1+\gamma_1}/N = \eta_N^{-1/(1+\delta)}$ where $\eta_N \to \infty$. However, if the network evolves in such a way that the number of growing components (c_k) stays the same and the number of components of fixed size $(c_N - c_k)$ grows with N (case (c) below), we can bound:

$$N^{-2-\delta} \sum_{c=1}^{c_N} N_c^{(1+\gamma_c)(2+\delta)} \leq c_k N^{-2-\delta} N_1^{(1+\gamma_c)(2+\delta)} + O(N^{-1-\delta})$$

and now the right-hand side will converge to zero if $N_1^{1+\gamma_1}/N = \eta_N^{-1/(2+\delta)}$. This accommodates "larger" components N_1 than the previous condition, thus, allows us to relax assumptions in DMN in this case. Similar refinements follow under other network evolution assumptions.

Since our conditions imply N_c is of order smaller than N, the results do not apply to networks with a giant component involving almost all nodes or, alternatively, networks with a giant component growing at rate N. This happens, for example, in the Facebook network where 99.91% of individuals belong to the largest connected component (Ugander et al. (2011)). However, if one is interested in actual dependence between individuals on Facebook, instead of using formal links, one may only keep connections that generate some cross-traffic (views, shares, likes etc.), thus (potentially) splitting the giant component into multiple groups that can be viewed as independent, which would fit into our framework.

Recently, for a related problem, Kojevnikov & Song (2023) showed that consistent estimation of the mean in clustered samples, without intra-cluster dependence restrictions, requires presence of at least two large clusters, which implies that the largest cluster has to be of order smaller than $O(N)$. Thus, our findings are in line with that result.

3 Central limit theorems for small-world networks

Recall that $d_{max} = \max_{i \in \mathcal{N}_N} d_i$ is the maximal degree in network G_N, where $\mathcal{N}_N = \{1, 2, \ldots, N\}$. Define $l(i,j)$ to be the network distance on the shortest path between i and j and set $l(i,j) = \infty$ if i and j are not connected by any network path. The diameter of network G_N is now formally defined as $\Delta_N = \max_{i,j \in \mathcal{N}_N : l(i,j) < \infty} l(i,j)$. All our results hold conditionally on network evolution $\{G_N\}_{N=1}^{\infty}$ and, thus, take network formation as given.

3.1 Node specific means

Consider the sample mean $\overline{Y}$ defined above, let $C_c \subset \mathcal{N}_N$ enumerate nodes in a network component c and make the following assumptions:

Assumption CVAR. *(a) $l(i,j) = \infty$ implies $Y_i \perp Y_j$.*

(b) There exist $\{\gamma_c\}_{c=1}^{c_N} : 0 \leq \gamma_c \leq 1, \underline{\sigma}^2 > 0, \bar{\sigma}^2 > 0, \delta > 0$ such that:

$$\underline{\sigma}^2 N_c^{1+\gamma_c} \leq Var\left(\sum_{i \in C_c} Y_i\right) \leq \bar{\sigma}^2 N_c^{1+\gamma_c},$$

$$E\left(\sum_{i \in C_c} Y_i\right)^{2+\delta} \leq K_N^{\frac{2+\delta}{2}} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}.$$

for all c and N, where $K_N = O(1)$.

Part (a) states that unconnected nodes are statistically independent. The second condition has two parts. The first part assumes that the number of non-zero correlations in a component of size N_c is a power function of N_c. Though, certainly imposing some structure on the number of non-zero correlations, it accommodates a variety of cases, in particular, all correlations being non-zero

($\gamma_c = 1, \underline{\sigma}^2 < 1, \bar{\sigma}^2 > 1$), the fraction of non-zero correlations being constant ($\gamma_c = 1, \underline{\sigma}^2 < 1, \bar{\sigma}^2 < 1$) and decreasing toward zero with N_c ($\gamma_c < 1$). We note that the power form is chosen for convenience and clarity of the CLT conditions and one may impose other correlation structures and follow similar arguments as in our proofs to obtain alternative conditions.[2] Also, one does not need to know γ_c's in practice and can assume $\gamma_c = 1$, for all c, for the purpose of the conditions given below. Similarly, Romano & Wolf (2000) impose conditions that imply $Var(\sum_{i=1}^{N} Y_i) \sim N^{1+\gamma}$ for some $-1 \leq \gamma < 1$. The second part of (b) assumes existence of moments for the component-wise averages. For example, with $\delta = 2$, this condition holds if Y_i's have finite fourth moments and the number of nonzero within-component pairwise correlations is proportional to $N_c^{1+\gamma_c}$.

Further conditions needed to obtain a central limit theorem depend on the network evolution scheme, in particular, how many growing network components there are and what their relative growth rates are. Before we proceed, we need to clarify some notation. As the network evolves both the existing components grow in size and new components arise. Thus, we will make the dependence of the size of a component on the number of nodes explicit, i.e. write $N_c(N)$, and keep in mind that c_N is a function of N. Without loss of generality, let the first component ($c = 1$) be a component for which the number of non-zero correlations, i.e. $N_1^{1+\gamma_1}$, grows at the fastest rate.

Theorem 1. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of mean zero random variables and define $B_N^2 = Var(\sqrt{N}\bar{Y})$. Under Assumption CVAR:*

$$\frac{\sqrt{N}\bar{Y}}{B_N} \to^{D} N(0,1)$$

as $N \to \infty$ (conditionally on network evolution) if either of the following holds:

(a) all components grow at the same rate, γ_c's are all equal and $N_1/N \to 0$

(b) all components grow (possibly at different rates) and we have for all c:

(i) the component with the fastest growing number of intra-component correlations is also the fastest growing component, i.e $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0$ implies $\frac{N_c(N)}{N_1(N)} \to 0$,

(ii) $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ is weakly decreasing in N,

(iii) there exists $M < \infty$ such that $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ for all $(N, \tilde{N})$ such that $c_{\tilde{N}} = c_N + 1$ and $N > M, \tilde{N} > M$,

[2] As $N_c^{1+\gamma_c}$ does not need to be an integer, the bounds $\underline{\sigma}^2, \bar{\sigma}^2$ also accommodate rounding.

(iv) $N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}/N \to 0$,

(c) c_k *components grow with* N *and remaining* $c_N - c_k$ *components have fixed size,* c_k *is fixed and* $N_1^{1+\gamma_1}/N \to 0$,

(d) c_k *components grow with* N *and remaining* $c_N - c_k$ *components have fixed size,* $\frac{c_N}{N} \to s > 0$, $c_k \to \infty$ *and one of the following conditions is satisfied:*

(i) $N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}/N \to 0$,

(ii) $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0$ *implies* $\frac{N_c(N)}{N_1(N)} \to 0$, $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ *is weakly decreasing in* N*, there exists* $M < \infty$ *such that* $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ *for all* $(N, \tilde{N})$ *such that* $c_{\tilde{N}} = c_N + 1$ *and* $N > M, \tilde{N} > M$*, and* $N_c^{1+\gamma_c}/N \to 0, \forall c$,

(iii) components $\{1, \ldots, c_k\}$ *grow at the same rate,* $\{\gamma_c\}_{c=1}^{c_k}$ *are all equal and* $N_1^{1+\gamma_1\frac{2+\delta}{\delta}}/N \to 0$,

(e) c_k *components grow with* N *and remaining* $c_N - c_k$ *components have fixed size,* $\frac{c_N}{N} \to 0$ *and either all components* $\{1, \ldots, c_k\}$ *grow at the same rate (with* $\{\gamma_c\}_{c=1}^{c_k}$ *all equal) and* $c_N^{(2+\delta)/2}/c_k^{1+\delta} \to 0$*, or we have for all c:*

(i) $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0$ *implies* $\frac{N_c(N)}{N_1(N)} \to 0$,

(ii) $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ *is weakly decreasing in* N,

(iii) there exists $M < \infty$ *such that* $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ *for all* $(N, \tilde{N})$ *such that* $c_{\tilde{N}} = c_N + 1$ *and* $N > M, \tilde{N} > M$,

(iv) $N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}/N \to 0$ *when* $\frac{c_k}{c_N} \to 1$ *or* $N_1^{\frac{2+\delta}{\delta}}/N \to 0$ *otherwise.*

The proof of Theorem 1 is given in the Appendix and, practically, amounts to the verification of the Lyapunov's condition for different cases above. Note that the presence of the factor $\sqrt{N}$ in the statement of the theorem does not imply that we obtain a square root rate of convergence as in general B_N will not be $O(1)$. The result can be restated as a result conditional on common shocks affecting all the nodes in the network just as in Kojevnikov et al. (2021), and would then apply to networks where there is some dependence between unconnected nodes and the dependence can be modelled through observables. Theorem 1 provides only sufficient conditions for the CLT to hold. Still, it covers a wide range of network evolution setups, including networks with a number of dominant components and a number of small-sized components, a structure commonly encountered in social networks. In particular, case (c) or (d) corresponds to the network evolution in genetic programming coauthorship network discussed in the Introduction (cf. Tomassini & Luthi (2007)).

The result refines the results in DMN by distinguishing different network evolution schemes. This often produces relaxed conditions on the permissible size of the largest cluster/network component. To see that, adapting from their regression setup to a simple sample mean setup, if η_N denotes the rate of divergence of $Var(\sum_{i=1}^{N} Y_i)$, they require (see MacKinnon et al. (2023)): $\left(\frac{\sqrt{\eta_N}}{N}\right)^{-(2+\delta)/(1+\delta)} \frac{N_1}{N} \to 0$. Let us compare this condition to the ones in parts (a) and (c). For the former case, it is easy to derive that both our and their conditions require $c_N \to \infty$, in other words $N_1/N \to 0$. For the latter case, we need $N_1^2/N \to 0$ (assuming $\gamma_1 = 1$), whereas their condition implies that we need:[3]

$$\left(\frac{N^2}{\sum_{c=1}^{c_k} N_c^2 + (c_N - c_k)O(1)}\right)^{\frac{2+\delta}{2(1+\delta)}} \frac{N_1}{N} = \left(\frac{N^{-\frac{\delta}{2+\delta}} N_1^{\frac{2(1+\delta)}{2+\delta}}}{N^{-1}\sum_{c=1}^{c_k} N_c^2 + O(1)}\right)^{\frac{2+\delta}{2(1+\delta)}}$$

to converge to zero, which is satisfied if $N_1^{2(1+\delta)/\delta}/N \to 0$. Note that this is a stronger condition (e.g. requires $N_1^3/N \to 0$ when $\delta = 2$) than the one we impose and both are equivalent only if all the moments exist (i.e. $\delta = \infty$).

Remark 1. *The first condition in (b) requires that the component(s) with the fastest growing number of non-zero between-nodes correlations is(are) also the fastest growing component(s). It is trivially satisfied if all γ_c's are equal. Additionally, note that together conditions (b)(i)-(ii) imply $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0 \Leftrightarrow \frac{N_c(N)}{N_1(N)} \to 0, \forall c$.*

Remark 2. *The conditions (b)(ii) and (b)(iii) require presence of a dominant component(s) in which the number of non-zero correlations grows at the fastest rate, as well as that the new components cannot grow at a faster rate (relative to last previously created component) that this dominant component.*

Remark 3. *The condition on the largest component in part (a) is equivalent to the one in part (e)(iv) if all moments of Y exist, i.e. $\delta = \infty$. Finally, note that the condition on N_1 in (d)(iii) is weaker than the one in (d)(i), but both become equivalent to conditions in (b)(iv) and (c) when $\delta \to \infty$.*

Remark 4. *DMN use Lyapunov's theorem, so serve as a natural reference point for our results.*

[3] Note that the second part of our Assumption CVAR(b) (with $\gamma_c = 1$) is implied by an equivalent of their Assumption 1 (see e.g. their Lemma A.2).

However, we may also compare our conditions to Hansen & Lee (2019) who apply Lindeberg-Feller's theorem. For case (a) they require $N_1^2/(N\lambda_N) \to 0$, where $\lambda_N \leq O(N_1)$, which amounts to our condition $N_1/N \to 0$ if λ_N achieves the upper bound. Additionally, for the case (d)(ii) their condition on the cluster sizes requires the following quantity to be bounded:

$$\frac{\left(\sum_{c=1}^{c_N} N_c^{2+\delta}\right)^{\frac{2}{2+\delta}}}{N} = \frac{\left(\sum_{c=1}^{c_k} N_c^{2+\delta} + (c_N - c_k)O(1)\right)^{\frac{2}{2+\delta}}}{N} \leq c_k^{\frac{2}{2+\delta}} \frac{N_1^2}{N} + o(1),$$

whereas we require $N_1^2/N \to 0$ (with $\gamma_1 = 1$). Thus, our condition is weaker if c_K diverges faster than $(N_1^2/N)^{\frac{2+\delta}{2}}$. Of course, in condition (d)(ii) we also require that at some point along the network evolution the newly created clusters do not grow faster than the existing ones and Hansen & Lee (2019) assume uniform integrability so our conditions are not directly comparable. However, these examples show that our rate conditions do not directly follow from theirs and that there is a tradeoff between approaching the problem from their and our perspective.

Next, we specialise Theorem 1 to the small-world setup by providing sufficient conditions for the restrictions on the largest component.

Assumption SW. *(a) $d_{max} \geq 2, d_{\max} = O(1)$.*

(b) $\Delta_N \leq \log_a(bN)$ for some constants $a > 1, b > 0$.

Assumption SW(a) imposes sparsity of the network. The lower bound on the maximal degree has a technical nature and rules out the case of networks formed only of connected pairs of nodes. Note that sparsity is often imposed as a condition on average degree, namely $1/N \sum_{i=1}^N d_i = O(1)$, which is implied by our condition. Assumption SW(b) imposes the small-world property, namely that the diameter of the network is (at most) proportional to $\log N$.

Proposition 1. *Under Assumptions CVAR and SW the conditions in (a), (b)(iv), (c), (d), (e)(iv) in Theorem 1 of the form $N_c^r/N \to 0$ can be replaced by:*

$$\log_{d_{max}-1} a > r.$$

Combined with Theorem 1, Proposition 1 gives sufficient conditions for a network comprising of small world components to satisfy the central limit theorem. For example, with all components growing at the same rate with equal γ's, we need $a > d_{max} - 1$. This means CLT is consistent with having diameter of order $\log N$, but the conditions imposed on $\log_{d_{max}-1} a$ restrict the scaling factor for the network diameter relative to the maximal degree and require that, for given N, the diameter is not too large relative to the maximal degree (in other words, $1/\log(a)$ cannot be too large).

Remark 5. *To the best of our knowledge, there is no precise formula linking the parameters of a small world model like the Watts-Strogatz small world (SW) model (Watts & Strogatz (1998)) and the Barabási-Albert (BA) preferential attachment model (Barabási & Albert (1999)) to the constants of proportionality in the diameter so it is difficult to translate our conditions on the constant* a *to the parameters of these models.*[4] *For a random graph $G(N,p)$, the diameter is bounded when $Np \to \infty$ and $Np \to 0$ so our conditions on the size of* a *will be satisfied as long as N is large enough. When $Np \to \lambda > 1$ we have $\Delta_N = \log N/\log\lambda$. In such random graph let us truncate the degree at some d_{max} and, on top of that, artificially split the largest component, which is of order N in this case, such that it grows at some slower rate. This way the generated network dynamics will fit our setup in part (b) and condition (b)(iv) becomes $\log\lambda > \log(d_{max}-1)\frac{2+\delta}{\delta}\sup_{c\geq 1}(1+\gamma_c)$ which gives $\lambda > 9$ when $d_{max} = 4, \gamma_c = 1, \delta = \infty$.*

A natural corollary, providing sufficient conditions for a CLT for multiple small-world networks irrespective of the network structure and values of γ_c, follows:

Corollary 1. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of mean zero random variables and define $B_N^2 = Var(\sqrt{N}\overline{Y})$. Under Assumptions CVAR and SW:*

$$\frac{\sqrt{N}\overline{Y}}{B_N} \to^{D} N(0,1)$$

as $N \to \infty$ (conditionally on network evolution) if either of the following holds:

(a) all c_k growing components (where $c_k \leq c_N$) satisfy:

(i) $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0$ implies $\frac{N_c(N)}{N_1(N)} \to 0$,

(ii) $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ is weakly decreasing in N,

(iii) there exists $M < \infty$ such that $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ for all $(N,\tilde{N})$ such that $c_{\tilde{N}} = c_N + 1$ and $N > M, \tilde{N} > M$,

[4] Even if these models are precisely defined. See Bollobás & Riordan (2002) for a discussion of the mathematical definitions of small-world networks.

and $\log_{d_{max}-1} a > \frac{2(2+\delta)}{\delta}$,

(b) *all* c_k *growing components (where* $c_k \leq c_N$*) grow at the same rate and either* $\log_{d_{max}-1} a > 1 + \frac{2+\delta}{\delta}$. *or* $c_N^{(2+\delta)/2}/c_k^{1+\delta} \to 0$.

4 Variance estimation

In this section we suggest estimators of the variance B_N^2 that can be used for inference together with Theorem 1. Since Y_i's have zero mean:

$$B_N^2 = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} E(Y_iY_j)\mathbb{1}\{l(i,j) < \infty\}$$

thus a natural estimator arises:

$$\hat{B}_N^2 = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} Y_iY_j\mathbb{1}\{l(i,j) < \infty\}.$$

Note that this is the cluster covariance estimator (CCE). Bester et al. (2011) provide conditions under which it is valid for spatial networks.

Theorem 2. *Let* $\{Y_i\}_{i=1}^{\infty}$ *be a sequence of mean zero random variables, Assumptions CVAR and SW hold and let* $E|Y_i|^4$ *be bounded for all* i. *Then:*

$$Var(\hat{B}_N - B_N) \to 0.$$

as $N \to \infty$ *(conditionally on network evolution) if:*

(a) *all components grow at the same rate and* $\log_{d_{max}-1} a > 3$,

(b) *all components grow (possibly at different rates) and* $\log_{d_{max}-1} a > 4$,

(c) c_k *components grow with* N *and remaining* $c_N - c_k$ *components have fixed size,* c_k *is fixed and* $\log_{d_{max}-1} a > 2$,

(d) c_k *components grow with* N *and remaining* $c_N - c_k$ *components have fixed size,* $c_k \to \infty$ *and either* $\log_{d_{max}-1} a > 4$ *or all* c_k *components grow at the same rate and* $\log_{d_{max}-1} a > 3$.

The theorem implies consistency of the proposed estimator in our setup. Recall that, when all components grow at the same rate with N, Theorem 1 allows the size of the largest connected component to grow at a rate arbitrarily close to N but here the allowed rate is not higher than $N^{1/3}$. Again, this is in line with findings in Kojevnikov & Song (2023) for clustered samples. They show that one requires much stricter conditions for variance estimation then for consistent discrimination of the mean.

Although consistent, this estimator does not work well in practice if there are a few large components, as shown by Cameron et al. (2008). Our Monte Carlo simulations confirm that – a confidence interval using the CCE estimator severely undercovers even for the sample size $N = 10000$ when there are only seven components. Similarly, a related HAC estimator in Kojevnikov et al. (2021) also undercovers when there is a lot of dependence between Y_i's (i.e. the "autoregressive" parameter is close to 0.5) in a setup with one component.

Thus, following Cameron et al. (2008), as an alternative to the CCE estimator we consider the wild clustered bootstrap and find that it performs much better in our Monte Carlo simulations. Let $c = 1, \dots, c_N$ enumerate separate components of network G_N. The bootstrap procedure for testing the hypothesis $H_0 : E[Y] = 0$ at the α level is as follows:

1. For each connected component draw $v_c = -1$ or 1 with probability 1/2.

2. Calculate $\overline{Y}^* = \frac{1}{N}\sum_{i=1}^{N} Y_i v_{c(i)}$ where $c(i)$ denotes the component that i belongs to.

3. Reject the null hypothesis if $\bar{Y}$ is below the $\alpha/2$ or above the $1 - \alpha/2$ quantile of $\overline{Y}^*$ across the bootstrap samples.[5]

As an alternative one may consider randomisation tests of Canay et al. (2017).

5 Means of edge-specific characteristics

In this section we provide limit theorems for means of characteristics of edges between nodes. Applications include means of input-output flows in production networks (see e.g. Acemoglu et al. (2012)) or mean difference in socio-economic status between individuals belonging to the same local

[5] Estimating B_N^2 by variance of $\sqrt{N}\overline{Y}^*$ across the bootstrap samples is often not theoretically justified. In fact, additional MC simulations not reported in this article suggest that the t test based on such estimate can often be very conservative in small samples.

community (see e.g. Chetty et al. (2022)). Note that the edge characteristics in these two examples have a different structure – in the former they are nonparametric functions of a node pair (i, j) ("flows") whereas in the latter they are known functions of characteristics of a node (i, j) involved in an edge ("contrasts"). These differences lead to distinct analysis, in particular a CLT for contrasts requires stronger conditions.

5.1 Flows

Let Y_{ij} denote the characteristic of an edge between nodes i and j and assume that there are no flows between separate components, i.e. $Y_{ij} = 0$ if nodes i and j are not connected (by any path). With this structure we effectively have c_N components with $N_{c,f} = N_c(N_c - 1)$ outcome pairs Y_{ij} and the analysis resembles the one for node-specific means, but now the effective sample size is $N_f = \sum_{c=1}^{c_N} N_{c,f}$. We can define the edge-specific mean by:

$$\overline{Y}_f = \frac{1}{N_f} \sum_{i=1}^{N} \sum_{j \neq i} Y_{ij} \mathbb{1}\{l(i, j) < \infty\}.$$

Similarly to node-specific means we will assume that flows in separate network components are statistically independent and modify Assumption CVAR to the present context:

Assumption CVAR'. *(a) $l(i, k) = \infty$ implies $Y_{ij} \perp Y_{kl}$.*

(b) There exist $\{\gamma_c\}_{c=1}^{c_N} : 0 \leq \gamma_c \leq 1, \underline{\sigma}^2 > 0, \bar{\sigma}^2 > 0, \delta > 0$ such that:

$$\underline{\sigma}^2 N_{c,f}^{1+\gamma_c} \leq Var\left(\sum_{i \in C_c} \sum_{j \in C_c : j \neq i} Y_{ij}\right) \leq \bar{\sigma}^2 N_{c,f}^{1+\gamma_c},$$

$$E\left(\sum_{i \in C_c} \sum_{j \in C_c : j \neq i} Y_{ij}\right)^{2+\delta} \leq K_N^{\frac{2+\delta}{2}} N_{c,f}^{(1+\gamma_c)\frac{2+\delta}{2}}.$$

for all c and N, where $K_N = O(1)$.

We have the following result:

Theorem 3. *Let $\{Y_{ij}\}_{i,j=1}^{\infty}$ be a sequence of mean zero random variables and define $B_{N,f}^2 =$*

$Var(\sqrt{N_f} Y_f)$. Under Assumptions CVAR' and SW:

$$\frac{\sqrt{N_f} Y_f}{B_{N,f}} \rightarrow^D N(0,1)$$

as $N \to \infty$ (conditionally on network evolution) if

(a) condition (a) in Theorem 1 holds with $\log_{d_{max}-1} a > 1$,

(b) conditions (b)(i)-(iii) in Theorem 1 hold and $\log_{d_{max}-1} a > 2(1+\gamma_c)\frac{2+\delta}{\delta}, \forall c$,

(c) condition (c) in Theorem 1 holds with $\log_{d_{max}-1} a > 2(1+\gamma_c), \forall c$,

(d) condition (d) in Theorem 1 holds with (i)-(iii) replaced by:

(i)' $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0$ implies $\frac{N_c(N)}{N_1(N)} \to 0$, $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ is weakly decreasing in N, there exists $M < \infty$ such that $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ for all $(N, \tilde{N})$ such that $c_{\tilde{N}} = c_N + 1$ and $N > M, \tilde{N} > M$, and $\log_{d_{max}-1} a > 2(1+\gamma_c), \forall c$,

(ii)' components $\{1, \ldots, c_k\}$ grow at the same rate, $\{\gamma_c\}_{c=1}^{c_k}$ are all equal and $\log_{d_{max}-1} a > 2(\frac{1+\delta}{\delta} + \gamma_c \frac{2+\delta}{\delta}), \forall c$,

(e) condition (e) in Theorem 1 holds with part (iv) replaced by: $\log_{d_{max}-1} a > 2(1+\gamma_c)\frac{2+\delta}{\delta}, \forall c$, when $\frac{c_k}{c_N} \to 1$ or $\log_{d_{max}-1} a > \frac{2(2+\delta)}{\delta}$.

Theorem 3 can be used for inference once an estimator of $B_{N,f}$ is available. One would expect that an analogous estimator to the CCE estimator or a wild cluster bootstrap described in Section 4 would work by the same reasoning as for node-specific means. Note that Theorem 3 strengthens conditions of Theorem 1 due to the fact that in the current setup each network "component" contains (up to) $N_c(N_c - 1)$ correlated elements compared to N_c before.

5.2 Contrasts

Let h be a symmetric function and define the edge-specific mean as:[6]

$$\overline{Y}_c = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} h(Y_i, Y_j) \mathbb{1}\{l(i,j) < \infty\}.$$

A leading example would be $h(Y_i, Y_j) = |Y_j - Y_i|$ with Y_i denoting a measure of socio-economic status like income (Chetty et al. (2022)), in which case the statistic would measure average absolute differences in income among neighbourhoods ("economic connectedness") and our results would provide a starting point for conducting inference which takes into account network-dependence between connected units. We point out that, when there is non-negligible dependence between connected individuals, even large sample sizes may not guarantee statistical significance of the findings as in such case the "effective" sample size may be small.

Under stationarity of Y_i we get the following Hoeffding decomposition:

$$\frac{\sqrt{N}\overline{Y}_c}{B_{N,c}} = B_{N,c}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} h_1(Y_i) \frac{N_c(i) - 1}{N-1} + B_{N,c}^{-1} \frac{\sqrt{N}}{N(N-1)} \sum_{i<j} h_2(Y_i, Y_j) \mathbb{1}\{l(i,j) < \infty\}, \tag{1}$$

where $h_1(y) = E_Y[h(y, Y)], h_2(y_1, y_2) = h(y_1, y_2) - h_1(y_1) - h_1(y_2)$, $N_c(i)$ denotes the number of nodes in the component to which i belongs and $B_{N,c}^2 = Var\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} h_1(Y_i) \frac{N_c(i)-1}{N-1}\right)$. In order to obtain a CLT we need to show that the second term in the decomposition converges to zero in probability and show that a CLT holds for the triangular array $\left\{h_1(Y_i) \frac{N_c(i)-1}{N-1}\right\}_{i,N}$.

The variance of the second term in (1) (up to a scaling factor) can be written as:

$$\sum_{c=1}^{c_N} \sum_{i \in C_c} \sum_{j \in C_c, j \neq i} \sum_{k \in C_c} \sum_{l \in C_c, l \neq j} E[h_2(Y_i, Y_j) h_2(Y_k, Y_l)]$$

and under Assumption CVAR(b) this term is of order $\sum_{c=1}^{c_N} N_c^{3+\gamma_c}$, which is the same as the order of $B_{N,c}^2$. This shows a difficulty in obtaining a central limit theorem for contrasts without some further restrictions on dependence between Y_i's. Thus, we impose a strong mixing condition with respect to the network distance $l(\cdot, \cdot)$ following Kojevnikov et al. (2021).

[6] The definition could be extended to functions of characteristics of triples, quadruples etc. of nodes, which can be used to study clique characteristics. The treatment of such statistics would follow similar lines. Hence, for the sake of exposition, we do not analyse them in detail.

For σ-fields $\mathcal{F}, \mathcal{G}$, let $\alpha(\mathcal{F}, \mathcal{G}) = \sup_{F \in \mathcal{F}, G \in \mathcal{G}} |P(F \cap G) - P(F)P(G)|$ and define the component-specific mixing coefficients by:

$$\alpha_{c,N}(s) = \sup\{\alpha(\sigma(Y_A), \sigma(Y_B)) : A, B \subset C_c, l(A, B) \geq s\}$$

where $Y_A = \{Y_i\}_{i \in A}$ and $l(A, B) = \min_{i \in A} \min_{i' \in B} l(i, i')$. Further, note that the data $\{Y_i\}_{i=1}^N$ is α-mixing with $\alpha_N(s) = \max_{c \in \{1,\ldots,c_N\}} \alpha_c(s)$. Let $c_{N_c}(s, m; k)$ be the quantity capturing the network's denseness defined on p. 891 in Kojevnikov et al. (2021). We impose the following assumption.

Assumption WDEP. *For all* $c \in \mathbb{N}$, $\{Y_i\}_{i \in C_c}$ *is a stationary strong mixing process with* $E[h(Y_i, Y_j)] = 0$ *and we have for* $p > 4$*:*

(a) h *is a bounded Lipchitz function satisfying:* $E|h(Y_i, Y_j)|^p < \infty$.

(b) $\frac{1}{N^3 B_{N,c}^2} \sum_{c=1}^{c_N} N_c \sum_{s \geq 0} c_{N_c}(s, N_c; 2) \alpha_c(s) \to 0$.

(c) There exists a positive sequence m_N *such that for* $k = 1, 2$*:*

$$\frac{1}{N^{\frac{k}{2}} B_{N,c}^{2+k}} \sum_{s \geq 0} c_N(s, m_N; k) \alpha_N(s)^{1 - \frac{2+k}{p}} \to 0,$$

$$\frac{N^{3/2} \alpha_N(m_N)^{1 - \frac{1}{p}}}{B_{N,c}} \to 0.$$

Remark 6. *Assumptions (a) and (c) are needed for the asymptotic normality of the first term in* (1)*. Assumptions (a) and (b) are used to show that the second term vanishes. Note that if* $p \to \infty$ *and* $c_{N_c}(s, N_c, 2) \leq c_N(s, m_N, 2)$*, then (c) implies (b), however, in general,* $c_N(s, m, 2)$ *is not monotone in the second argument so this does not follow.*

Remark 7. *Lipschitz continuity in part (a), though relatively strong, is satisfied trivially for our leading example of* $h(y_1, y_2) = |y_2 - y_1|$*.*

We are now ready to state the CLT theorem for contrasts.

Theorem 4. *Under Assumptions CVAR(a) and WDEP we have:*

$$\frac{\sqrt{N} Y_c}{B_{N,c}} \to^D N(0, 1)$$

as $N \to \infty$ (conditionally on network evolution).

Compared to the previous results, this theorem imposes significantly stronger assumptions since the restrictions on the structure of a network and the largest component do not suffice here, as mentioned above. The small-world property and sparsity conditions will affect the network denseness function, $c_N(s, m; k)$. However, establishing a precise link between the diameter, maximal degree and this function is not straightforward.

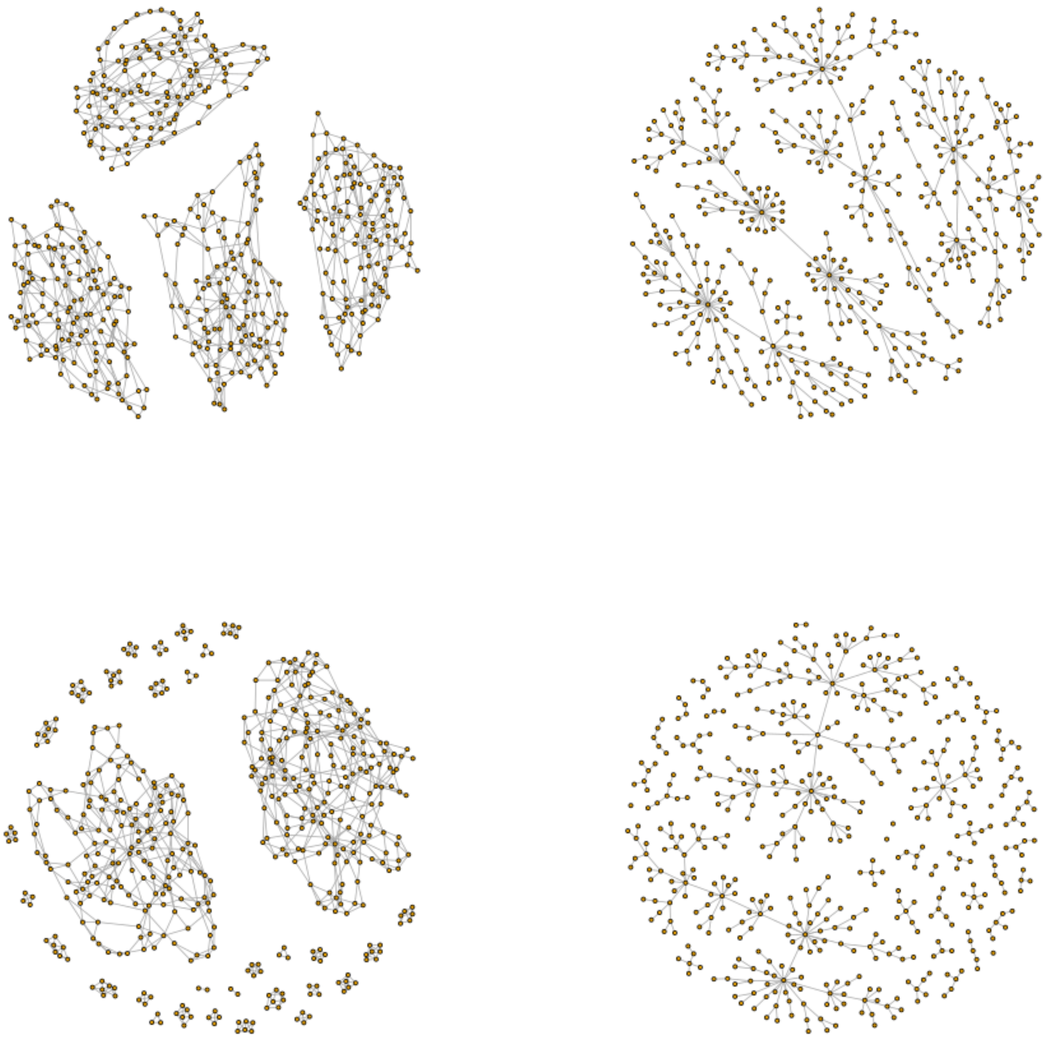
6 Monte Carlo simulations

We conduct a Monte Carlo study to verify our central limit theorems and assess the finite-sample performance of the variance estimators proposed in Section 4. The finite-sample properties of the CCE and the wild cluster bootstrap have been extensively studied in the clustering literature (e.g. Cameron et al. (2008), Bester et al. (2011), MacKinnon & Webb (2017), DMN, Canay et al. (2021)). However, these studies primarily focus on settings with multiple growing clusters, corresponding to cases (a) and (b) in Theorem 1. In contrast, we introduce a design that incorporates a mixture of growing and fixed-size components, with the number of both increasing as the network expands.

We consider two network generating algorithms: the Watts-Strogatz small world (SW) model (Watts & Strogatz (1998)) and the Barabási-Albert (BA) preferential attachment model (Barabási & Albert (1999)). The first model generates networks with diameters proportional to $\log N$ whereas the second model produces diameters proportional to $\log N$ or $\log N / \log \log N$ depending on the parameters (Bollobás & Riordan (2004)). For most parameter values the BA model implies that the maximal degree of a node grows with N, thus we further "prune" the graph to make sure that the maximal degree is stable: (1) we start with a node with the highest degree and randomly erase superfluous edges, (2) check if the maximal degree satisfies the imposed bound, (3) if not, we go back to step (1) and repeat the procedure.

In terms of the architecture of the network, we consider both (approx.) equal-sized components and growing + fixed components. For the former case we start with four connected components for $N = 500$ and add one component for each increase in the sample size above that, hence ending up with seven components for $N = 10000$. For the latter case, we allocate 30% of all nodes to the fixed components and we draw fixed component sizes from the binomial distribution with mean size

Figure 1: Examples of Monte Carlo designs, SW model (left) and BA model (right), $N = 500$.



of 5 nodes and maximal size of 10. We start with two growing components when $N = 500$ and add one more for each increase in the sample size, such that these components grow (approx.) at rates $N^{\{0.45,0.25,0.15,0.1,0.1\}}$, respectively. We perform 1000 MC repetitions and use 1000 replications for the bootstrap procedures. Figure 1 shows four examples of networks generated by SW and BA models (top panel: equal components, bottom panel: growing + fixed).

6.1 Node-specific means

Let $C(i)$ denote the network component containing node i and $N_c(i)$, as before, denote the number of connected nodes in this component. The data is generated from the following process

$$Y_i = \frac{1}{\sqrt{N_c(i) - 1}} \sum_{j \neq i, j \in C_{c(i)}} \varepsilon_j$$

where ε_j's are i.i.d., drawn from a standardised uniform distribution. In other words, node i's outcome is equal to the average of ε's of all the nodes that i is connected to, which implies strong dependence between outcomes belonging to the same network component. We consider coverage of confidence intervals built using known variance ("oracle") , CCE estimator $\hat{B}_N$ ("estim.") and wild cluster bootstrap ("boot.") introduced in Section 4.

Table 1 contains the simulation result for means of node-specific characteristics. The BA model is parametrised by: m – the number of edges added in each step of building the graph, z_a - appeal of nodes that do not have any connections. The algorithm for building an SW network starts with a circle (or, more generally, lattice) graph and "rewires" some of the connections between neighbouring nodes to some more distant nodes, thus is parametrised by: p - probability of rewiring an edge, k - the number of edges per vertex in the initial circle graph. Different values of these parameters produce graphs with different maximal degrees and diameters.

When we use the known variance the coverage is close to the nominal 95% level across the designs and parameter values, thus confirming that the CLT holds for small world networks. However, once we use the estimated variance $\hat{B}_N^2$ the coverage deteriorates substantially, with values somehow close to the nominal values only in the three top left panels of Table 1 for which the networks are pretty sparse with a small degree and a large diameter. This is in line with other MC studies in the literature. As expected, the wild cluster bootstrap works well, besides the small sample size

Table 1: Simulated coverage, node-specific means, 95% level

	BA model							SW model						
						Coverage							Coverage	
N	m	z_a	d_{max}	Δ_N	oracle	estim.	boot.	p	k	d_{max}	Δ_N	oracle	estim.	boot.
	Equal Components													
500	1	0	10	12	0.954	0.844	0.952	0.05	2	6	12	0.958	0.814	0.852
1000	1	0	10	11	0.964	0.930	0.945	0.05	2	6	15	0.950	0.854	0.927
5000	1	0	10	14	0.938	0.941	0.947	0.05	2	7	20	0.949	0.860	0.923
10000	1	0	10	16	0.952	0.950	0.949	0.05	2	7	20	0.960	0.897	0.928
500	1	1	10	16	0.954	0.869	0.944	0.05	5	13	5	0.949	0.836	0.891
1000	1	1	10	16	0.947	0.870	0.962	0.05	5	13	6	0.932	0.862	0.951
5000	1	1	10	18	0.957	0.928	0.947	0.05	5	14	7	0.954	0.876	0.928
10000	1	1	10	20	0.956	0.937	0.953	0.05	5	15	8	0.954	0.893	0.948
500	1	2	10	15	0.942	0.850	0.941	0.05	10	24	3	0.943	0.811	0.880
1000	1	2	10	17	0.955	0.869	0.964	0.05	10	24	4	0.946	0.828	0.941
5000	1	2	10	25	0.950	0.923	0.954	0.05	10	26	5	0.945	0.858	0.931
10000	1	2	10	21	0.941	0.920	0.945	0.05	10	26	5	0.941	0.904	0.943
500	2	0	20	6	0.959	0.818	0.885	0.10	2	8	9	0.947	0.815	0.888
1000	2	0	20	7	0.948	0.847	0.945	0.10	2	7	10	0.954	0.851	0.917
5000	2	0	20	10	0.956	0.876	0.931	0.10	2	8	14	0.951	0.862	0.939
10000	2	0	20	10	0.951	0.860	0.946	0.10	2	9	15	0.957	0.886	0.940
500	2	1	20	7	0.953	0.801	0.877	0.10	5	14	4	0.951	0.803	0.883
1000	2	1	20	7	0.950	0.831	0.927	0.10	5	14	5	0.943	0.836	0.929
5000	2	1	20	10	0.959	0.879	0.935	0.10	5	16	6	0.954	0.857	0.940
10000	2	1	20	10	0.948	0.875	0.944	0.10	5	16	7	0.961	0.883	0.941
500	2	2	20	6	0.951	0.809	0.868	0.10	10	25	3	0.953	0.800	0.864
1000	2	2	20	7	0.952	0.854	0.940	0.10	10	25	4	0.942	0.855	0.940
5000	2	2	20	9	0.955	0.884	0.934	0.10	10	28	4	0.946	0.874	0.927
10000	2	2	20	10	0.955	0.888	0.944	0.10	10	30	5	0.951	0.873	0.937
	Growing + Fixed													
500	1	0	10	10	0.942	0.895	0.956	0.05	2	6	13	0.941	0.673	0.942
1000	1	0	10	13	0.945	0.902	0.946	0.05	2	7	15	0.950	0.782	0.947
5000	1	0	10	12	0.955	0.938	0.949	0.05	2	8	19	0.949	0.768	0.942
10000	1	0	10	13	0.958	0.953	0.947	0.05	2	7	21	0.949	0.774	0.939
500	1	1	10	11	0.952	0.908	0.940	0.05	5	13	6	0.952	0.670	0.949
1000	1	1	10	14	0.953	0.917	0.954	0.05	5	14	6	0.947	0.768	0.951
5000	1	1	10	16	0.945	0.938	0.945	0.05	5	14	8	0.949	0.789	0.934
10000	1	1	10	17	0.953	0.932	0.946	0.05	5	15	9	0.951	0.796	0.950
500	1	2	10	16	0.956	0.712	0.956	0.05	10	24	4	0.940	0.650	0.938
1000	1	2	10	17	0.949	0.818	0.941	0.05	10	24	4	0.948	0.772	0.943
5000	1	2	10	23	0.943	0.918	0.939	0.05	10	27	5	0.941	0.786	0.952
10000	1	2	10	22	0.941	0.934	0.956	0.05	10	26	6	0.964	0.796	0.959
500	2	0	20	7	0.953	0.661	0.944	0.10	2	7	10	0.953	0.658	0.956
1000	2	0	20	8	0.956	0.804	0.942	0.10	2	7	10	0.949	0.792	0.954
5000	2	0	20	9	0.954	0.788	0.956	0.10	2	8	14	0.948	0.801	0.952
10000	2	0	20	10	0.950	0.790	0.954	0.10	2	7	17	0.956	0.816	0.943
500	2	1	20	7	0.958	0.670	0.947	0.10	5	15	5	0.945	0.668	0.937
1000	2	1	20	7	0.936	0.762	0.944	0.10	5	15	5	0.951	0.761	0.944
5000	2	1	20	10	0.953	0.785	0.943	0.10	5	15	7	0.959	0.798	0.943
10000	2	1	20	10	0.951	0.799	0.952	0.10	5	16	7	0.963	0.820	0.951
500	2	2	20	6	0.947	0.658	0.941	0.10	10	26	3	0.956	0.677	0.943
1000	2	2	20	7	0.961	0.791	0.948	0.10	10	26	4	0.954	0.761	0.941
5000	2	2	20	9	0.956	0.786	0.948	0.10	10	28	5	0.941	0.763	0.944
10000	2	2	20	9	0.963	0.810	0.958	0.10	10	29	4	0.960	0.957	0.940

Note: 1000 Monte Carlo simulations, 1000 bootstrap replications. “Oracle” – known variance, “estim.” – variance estimator $\hat{B}_N$, “boot.” – wild cluster bootstrap.

$N = 500$, with coverage values only slightly below the nominal 95% for most designs.

6.2 Edge-specific means

As the case of means of flows discussed in Section 5.1 is very similar to the case of node-specific means, we only run simulations for the means of contrasts. As Theorem 4 requires weak dependence within components, we follow the design in Kojevnikov et al. (2021) and generate outcomes as:

$$Y_i = \sum_{s \geq 0} \frac{\rho^s}{|L_i(s)|} \sum_{j \in L_i(s)} \varepsilon_j$$

where $L_i(s)$ denotes the set of nodes at distance s from i, we set $\rho = 0.5$ and again ε follows the standardised uniform distribution.[7] Further, following our main example, we take:

$$h(Y_i, Y_j) = |Y_j - Y_i|.$$

As variance estimation in the U-statistic setup is more involved than with simple means and we do not provide variance estimators above, we only provide coverage values with known variance.

Table 2 shows that for all specifications of the network formation model the coverage probabilities are close to 95% even for $N = 500$, which is in line with our CLT in Theorem 4.

7 Conclusion

Many social and economic networks are sparse and are small-world. We show that data coming from (multiple) such networks satisfies a central limit theorem under the additional assumption restricting the constant of proportionality of the diameter to $\log N$, even without imposing weak dependence between connected nodes.

We consider a simple setup of undirected unweighted networks but the results should extend naturally to directed networks and networks in which we can assign (bounded) weights to covariances of characteristics between two connected nodes. If these weights would vanish or decrease sufficiently fast between large connected components, then one could potentially be able to extend our results to networks with a giant component of size $O(N)$, i.e. larger than allowed in our current setup.

[7] We have also ran simulations with normal errors and the results are very similar. See Appendix H.

Table 2: Simulated coverage, edge-specific means (contrasts), 95% level

	BA model					SW model				
N	m	z_a	d_{max}	Δ_N	Coverage oracle	p	k	d_{max}	Δ_N	Coverage oracle
	Equal Components									
500	1	0	10	10	0.946	0.05	2	6	12	0.952
1000	1	0	10	11	0.939	0.05	2	6	15	0.939
5000	1	0	10	13	0.951	0.05	2	7	20	0.958
10000	1	0	10	14	0.933	0.05	2	7	20	0.950
500	1	1	10	16	0.953	0.05	5	13	5	0.956
1000	1	1	10	16	0.944	0.05	5	13	6	0.935
5000	1	1	10	23	0.945	0.05	5	14	7	0.934
10000	1	1	10	19	0.937	0.05	5	15	8	0.937
500	1	2	10	14	0.951	0.05	10	24	3	0.957
1000	1	2	10	15	0.937	0.05	10	24	4	0.950
5000	1	2	10	18	0.934	0.05	10	26	5	0.948
10000	1	2	10	21	0.95	0.05	10	26	5	0.937
500	2	0	20	7	0.946	0.10	2	8	9	0.945
1000	2	0	20	8	0.941	0.10	2	7	10	0.944
5000	2	0	20	11	0.953	0.10	2	8	14	0.945
10000	2	0	20	11	0.947	0.10	2	9	15	0.944
500	2	1	20	7	0.962	0.10	5	14	4	0.961
1000	2	1	20	7	0.944	0.10	5	14	5	0.930
5000	2	1	20	9	0.929	0.10	5	16	6	0.952
10000	2	1	20	11	0.957	0.10	5	16	7	0.948
500	2	2	20	6	0.954	0.10	10	25	3	0.954
1000	2	2	20	7	0.950	0.10	10	25	4	0.954
5000	2	2	20	9	0.948	0.10	10	28	4	0.940
10000	2	2	20	9	0.956	0.10	10	30	5	0.948
	Growing + Fixed									
500	1	0	10	10	0.958	0.05	2	6	12	0.961
1000	1	0	10	13	0.957	0.05	2	6	15	0.952
5000	1	0	10	12	0.950	0.05	2	7	20	0.957
10000	1	0	10	13	0.951	0.05	2	7	20	0.957
500	1	1	10	11	0.947	0.05	5	13	5	0.952
1000	1	1	10	14	0.965	0.05	5	13	6	0.950
5000	1	1	10	16	0.954	0.05	5	14	7	0.957
10000	1	1	10	17	0.946	0.05	5	15	8	0.943
500	1	2	10	16	0.947	0.05	10	24	3	0.952
1000	1	2	10	17	0.946	0.05	10	24	4	0.948
5000	1	2	10	23	0.942	0.05	10	26	5	0.944
10000	1	2	10	22	0.936	0.05	10	26	5	0.940
500	2	0	20	7	0.943	0.10	2	8	9	0.947
1000	2	0	20	8	0.942	0.10	2	7	10	0.934
5000	2	0	20	9	0.953	0.10	2	8	14	0.954
10000	2	0	20	10	0.944	0.10	2	9	15	0.954
500	2	1	20	7	0.951	0.10	5	14	4	0.959
1000	2	1	20	7	0.946	0.10	5	14	5	0.947
5000	2	1	20	10	0.962	0.10	5	16	6	0.958
10000	2	1	20	10	0.952	0.10	5	16	7	0.955
500	2	2	20	6	0.947	0.10	10	25	3	0.943
1000	2	2	20	7	0.965	0.10	10	25	4	0.959
5000	2	2	20	9	0.948	0.10	10	28	4	0.963
10000	2	2	20	9	0.941	0.10	10	30	5	0.939

Note: 1000 Monte Carlo simulations, 1000 bootstrap replications. “Oracle” – known variance.

Similarly, a natural extension would be to allow all network components to be "weakly" connected, for example by having low conductance (see Leung (2023)) and weak dependence between the clusters. If a CLT would obtain in such setting with unrestricted intra-component dependence remains an open question.

Appendix

A Proofs

A.1 Useful lemmas

Lemma 1. *(Pineda-Villavicencio, Wood (2015)) Every graph with minimum degree d_{min}, maximum degree d_{max} and diameter Δ_N has at most $2d_{min}(d_{max}-1)^{\Delta_N-1}+1$ vertices.*

Lemma 2. *(Kojevnikov et al. (2021), Prop. 2.2) Let f and g belong to a collection of bounded Lipschitz real functions and sets (A,B) of nodes be such that $l(A,B)\geq s$. If $\{Y_{i,N}\}_{i=1}^{\infty}$ is a strong mixing triangular array (w.r.t. network distance $l(\cdot,\cdot)$) with mixing coefficients $\{\alpha_N(s): s\geq 0\}$, then we have:*

$$|cov(Y_A,Y_B)|\leq 4\|f\|_\infty\|g\|_\infty\alpha_N(s).$$

B General CLT for networked data

The following theorem gives high level conditions for the networked data to satisfy CLT. Our main theorem, Theorem 1, will follow from verifying these conditions for different network evolution structures.

Theorem 5. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of mean zero random variables and define $B_N^2 = Var(\sqrt{N}\overline{Y})$. Let Assumption CVAR hold and:*

(a) $B_N^2\geq L_N\frac{1}{c_N}\sum_{c=1}^{c_N}N_c^{\gamma_c}$,

(b) $\frac{K_N}{L_N}=O(1)$,

(c) $\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}}\frac{\sum_{c=1}^{c_N}N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N}N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}}\to 0.$

Then we have:

$$\frac{\sqrt{N}\overline{Y}}{B_N}\to^D N(0,1)$$

as $N\to\infty$ (conditionally on network evolution).

Proof. Firstly, note that under Assumption CVAR(a) partial sums from different network components are independent and we can write $B_N^2 = 1/N \sum_{c=1}^{c_N} Var\left(\sum_{i\in C_c} Y_i\right)$. Thus, in order to obtain a CLT it will suffice to verify Lyapunov's condition for the partial sums, i.e. show that

$$\frac{\sum_{c=1}^{c_N} E\left|\sum_{i\in C_c} Y_i\right|^{2+\delta}}{(NB_N)^{2+\delta}} \to 0.$$

But this follows from Assumption CVAR(b) and the conditions of the theorem by:

$$\frac{\sum_{c=1}^{c_N} E\left|\sum_{i\in C_c} Y_i\right|^{2+\delta}}{(NB_N)^{2+\delta}} \le \frac{K_N^{\frac{2+\delta}{2}} \sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(L_N \frac{N}{c_N} \sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} = \left(\frac{K_N}{L_N}\right)^{\frac{2+\delta}{2}} \left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} = o(1)$$

□

C Proof of Theorem 1

As mentioned in the text N_c's are really a function of N but, to economise on notation, we will only make this explicit when necessary.

Part (a). We have $N_c = N_1, \forall c$, and $c_N N_1/N = 1$. Using that and Assumption CVAR(b) we obtain:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \ge \frac{\underline{\sigma}^2 N_1^{1+\gamma_1} c_N}{N N_1^{\gamma_1}} = \frac{\underline{\sigma}^2 N_1 c_N}{N} = O(1) \tag{2}$$

which verifies conditions (a) and (b) of Theorem 5 with $L_N = O(1)$.

With equal rates we have:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} = \frac{c_N N_1^{\frac{2+\delta}{2}}}{N^{\frac{2+\delta}{2}}} = c_N^{-\frac{\delta}{2}}$$

which converges to zero if the number of components grows, which is implied by $N_1/N \to 0$.

Part (b). Recall that N_1 denotes a component for which the number of non-zero correlations grows at the fastest rate, i.e. $N_c^{1+\gamma_c}/N_1^{1+\gamma_1} \to 0$ or $\to 1$ for all $c > 1$. For sufficiently large N, we get $N/c_N \leq N_1$ under condition (b)(i). Thus, we have:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O(1)\frac{c_N N_1}{N}\frac{\sum_{c=1}^{c_N} N_c^{1+\gamma_c}}{N_1\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O(1)\frac{\sum_{c=1}^{c_N}\frac{N_c^{1+\gamma_c}}{N_1^{1+\gamma_1}}}{\sum_{c=1}^{c_N}\frac{N_c^{\gamma_c}}{N_1^{\gamma_1}}}$$

Note that both the numerator and the denominator in the last expression are positive so the ratio will be bounded away from zero if the sum in the denominator converges.

In order to properly analyse the convergence of the series, let us define the "jump points" in c_N by $N_c^J = \{N : c_N = c_{N-1} + 1, c_N = c\}$. Now by condition (b)(ii) we have:

$$\sum_{c=1}^{c_N}\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}} \leq \sum_{c=1}^{c_N}\frac{N_c(N_c^J)^{\gamma_c}}{N_1(N_c^J)^{\gamma_1}}$$

which gives an infinite sum indexed by c. For this sum to be finite we apply the ratio test (see e.g. Theorem 3.34 in Rudin (1976)), which requires:

$$\lim_{c_N\to\infty}\frac{N_{c_N+1}(N_{c_N+1}^J)^{\gamma_{c_N+1}}}{N_1(N_{c_N+1}^J)^{\gamma_1}}\Big/\frac{N_{c_N}(N_{c_N}^J)^{\gamma_{c_N}}}{N_1(N_{c_N}^J)^{\gamma_1}} < 1.$$

This latter condition is satisfied (for sufficiently large N) by condition (b)(iii). This verifies assumptions (a) and (b) of Theorem 5 with $L_N = O(1)$.

In order to verify condition (c) of Theorem 5 note that:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}}\frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{(\sum_{c=1}^{c_N} N_c^{\gamma_c})^{\frac{2+\delta}{2}}} \leq \left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}}\frac{c_N N_1^{(1+\gamma_1)\frac{2+\delta}{2}}}{c_N^{\frac{2+\delta}{2}} N^{\epsilon\frac{2+\delta}{2}}} \leq \frac{c_N}{N}\left(\frac{N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2}}$$

for some $\epsilon > 0$ by the fact that all components are growing at some positive rate. The last expression converges to zero since $c_N/N \to 0$ and condition (b)(iv) implies that $N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}/N \to 0$.

Part (c). Here we have the number of fixed size components equal $c_N - c_k \leq N - \sum_{c=1}^{c_k} N_c$ which together with $N_c/N \to 0$ implies that $c_N/N = O(1)$. Furthermore:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O(1)\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c} + c_N - c_k}{\sum_{c=1}^{c_k} N_c^{\gamma_c} + (c_N - c_k)\bar{N}^u} = O(1)\frac{\sum_{c=1}^{c_k}\frac{N_c^{1+\gamma_c}}{N} + O(1)}{\sum_{c=1}^{c_k}\frac{N_c^{\gamma_c}}{N} + O(1)} \geq O(1)$$

where $\bar{N}^u$ denotes an upper bound on the number of nodes in a fixed size component. The last inequality follows from $N_c^{1+\gamma_c}/N \to 0$ in condition (c) of the theorem.

Similarly:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq O(1) \frac{\sum_{c=1}^{c_k} \left(\frac{N_c^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} + O(N^{-\frac{\delta}{2}})}{\left(\sum_{c=1}^{c_k} \frac{N_c^{\gamma_c}}{N} + O(1)\right)^{\frac{2+\delta}{2}}} \to 0$$

which completes the proof of this part.

Part (d). As in the previous part:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O(1) \frac{\sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N} + O(1)}{\sum_{c=1}^{c_k} \frac{N_c^{\gamma_c}}{N} + O(1)} \geq O(1)$$

Note that $c_k/N \to 0$ and we have $(c_N - c_k)/N \leq (1 - \sum_{c=1}^{c_k} N_c/N) \leq 1$. This together with $(c_N - c_k)/N \to s > 0$ implies that $\sum_{c=1}^{c_k} N_c/N = O(1)$. Since $\sum_{c=1}^{c_k} N_c^{1+\gamma_c}/N \geq \sum_{c=1}^{c_k} N_c/N \geq \sum_{c=1}^{c_k} N_c^{\gamma_c}/N$, we obtain $\frac{B_N^2}{\frac{N}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O(1)$.

Moreover:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq O(1) \frac{\sum_{c=1}^{c_k} \left(\frac{N_c^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} + O(N^{-\frac{\delta}{2}})}{\left(\sum_{c=1}^{c_k} \frac{N_c^{\gamma_c}}{N} + O(1)\right)^{\frac{2+\delta}{2}}} \tag{3}$$

as in the previous part, however now the expression involves infinite sums. Using the fact that the slowest growing component grows at a rate N^{ϵ} for some $\epsilon > 0$ we can bound $c_k \leq N^{1-\epsilon}$:

$$\sum_{c=1}^{c_k} \left(\frac{N_c^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} \leq N^{1-\epsilon} \left(\frac{N_1^{1+\gamma_1}}{N}\right)^{\frac{2+\delta}{2}} \leq \left(\frac{N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2}}$$

and the last expression converges to zero by condition (d)(i), which finalises verification of (c) in Theorem 5.

Alternatively, condition (c) in Theorem 5 will be satisfied if $\sum_{c=1}^{c_k} \left(\frac{N_c^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} = o(1)$. First, note that using $N_1^{1+\gamma_1}/N \to 0$ we can write:

$$\sum_{c=1}^{c_k} \left(\frac{N_c^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} \leq \left(\frac{N_1^{1+\gamma_1}}{N}\right)^{\frac{\delta}{2}} \sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N} = o(1) \sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N}$$

so it is enough to show that the latter sum converges using the ratio test. To show that first note that for N large enough we have:

$$\sum_{c=1}^{c_k}\frac{N_c^{1+\gamma_c}}{N}=\sum_{c=1}^{c_k}\frac{N_c^{1+\gamma_c}}{N_1^{1+\gamma_1}}\frac{N_1^{1+\gamma_1}}{N}\leq\sum_{c=1}^{c_k}\frac{N_c^{1+\gamma_c}}{N_1^{1+\gamma_1}}=\sum_{c=1}^{c_k}\frac{N_c}{N_1}\frac{N_c^{\gamma_c}}{N_1^{\gamma_1}}\leq\sum_{c=1}^{c_k}\frac{N_c^{\gamma_c}}{N_1^{\gamma_1}}$$

Now $\sum_{c=1}^{c_k}\frac{N_c^{\gamma_c}}{N_1^{\gamma_1}}=O(1)$ follows from the same argument as the one in part (b).

Finally, if all c_k components grow at the same rate N_1, the expression in (3) simplifies to:

$$O(1)\frac{c_k\left(\frac{N_1^{1+\gamma_1}}{N}\right)^{\frac{2+\delta}{2}}+O(N^{-\frac{\delta}{2}})}{\left(c_k\frac{N_1^{\gamma_1}}{N}+O(1)\right)^{\frac{2+\delta}{2}}}$$

and now $c_k\sim N/N_1$ which gives:

$$c_k\left(\frac{N_1^{1+\gamma_1}}{N}\right)^{\frac{2+\delta}{2}}\simeq\left(\frac{N_1^{1+\gamma_1\left(\frac{2+\delta}{\delta}\right)}}{N}\right)^{\frac{\delta}{2}}$$

and the last expression converges to zero under condition (d)(iii).

Part (e). Here we have $c_N/N\to 0$. We will distinguish three cases: 1) c_k components grow at the same rate, 2) $c_k/c_N\to 1$, 3) $c_k/c_N\to 0$.

1) Consider the case when all N_c's are equal. Note that we have $N/c_N\leq N_1$ and $c_kN_1/N\to 1$, which implies $c_kN_1^{1+\gamma_1}/N\geq O(1)$. Thus:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N}N_c^{\gamma_c}}\geq O(1)\frac{c_kN_1^{1+\gamma_1}+(c_N-c_k)O(1)}{c_kN_1^{1+\gamma_1}+\frac{N}{c_N}(c_N-c_k)O(1)}=O(1)\frac{1+\left(\frac{c_N}{c_k}-1\right)O\left(N_1^{-(1+\gamma_1)}\right)}{1+\left(1-\frac{c_k}{c_N}\right)O\left(Nc_k^{-1}N_1^{-(1+\gamma_1)}\right)}=O(1)$$

since $(\frac{c_N}{c_k}-1)N_1^{-(1+\gamma_1)}=\frac{c_N-c_k}{N}\frac{N}{c_kN_1^{1+\gamma_1}}=o(1)$.

It remains to verify the condition (c) of Theorem 5, which follows by:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq \left(\frac{c_N N_1}{N}\right)^{\frac{2+\delta}{2}} \frac{c_k N_1^{(1+\gamma_1)\frac{2+\delta}{2}} + (c_N - c_k)O(1)}{\left(c_k N_1^{1+\gamma_1} + (c_N - c_k)O(N_1)\right)^{\frac{2+\delta}{2}}}$$

$$= \left(\frac{c_N N_1}{N}\right)^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \frac{1 + \left(\frac{c_N}{c_k} - 1\right) O\left(N_1^{-(1+\gamma_1)\frac{2+\delta}{2}}\right)}{\left(1 + \left(\frac{c_N}{c_k} - 1\right) O\left(N_1^{-\gamma_1}\right)\right)^{\frac{2+\delta}{2}}} = \frac{c_N^{\frac{2+\delta}{2}}}{c_k^{1+\delta}} o(1) = o(1)$$

where the second equality follows from $N \geq c_k N_1$ and the last one is due to the condition stated in the Theorem. Note that $c_N^{\frac{2+\delta}{2}}/c_k^{1+\delta} \to 0$ is implied by $c_k/c_N \to 1$.

2) We have:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq \frac{\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_c}} + \frac{c_N}{N}\frac{c_N/c_k - 1}{1/c_k \sum_{c=1}^{c_k} N_c^{\gamma_c}} O(1)}{1 + \frac{c_N/c_k - 1}{1/c_k \sum_{c=1}^{c_k} N_c^{\gamma_c}} O(1)} = \frac{\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_c}} + o(1)}{1 + o(1)}$$

since $1/c_k \sum_{c=1}^{c_k} N_c^{\gamma_c} > 0$. Assumptions (e)(i)-(ii) imply the first component is growing the fastest. Thus, using $c_N N_1/N \geq 1$:

$$\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_c}} \geq \frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_1 N_c^{\gamma_c}} = \frac{\sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N_1^{1+\gamma_1}}}{\sum_{c=1}^{c_k} \frac{N_c^{\gamma_c}}{N_1^{\gamma_1}}} \geq O(1)$$

where the last inequality follows from conditions (e)(ii)-(iii) following the same arguments as in the proof of part (b).

Furthermore, we obtain for some $\epsilon > 0$:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_c^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_c^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq \left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} N_1^{\frac{2+\delta}{2}} \frac{c_k N_1^{(1+\gamma_1)\frac{2+\delta}{2}} + (c_N - c_k)O(1)}{\left(c_k N_1 N^{\epsilon} + N_1(c_N - c_k)O(1)\right)^{\frac{2+\delta}{2}}}$$

$$= \left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} N_1^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \frac{\left(\frac{N_1^{\gamma_1}}{N^{\epsilon}}\right)^{\frac{2+\delta}{2}} + o(1)}{1 + o(1)}$$

which will converge to zero if $\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} N_1^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \left(\frac{N_1^{\gamma_1}}{N^{\epsilon}}\right)^{\frac{2+\delta}{2}} \to 0$. But:

$$\left(\frac{c_N}{N}\right)^{\frac{2+\delta}{2}} N_1^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \left(\frac{N_1^{\gamma_1}}{N^{\epsilon}}\right)^{\frac{2+\delta}{2}} = \left(\frac{c_N}{c_k}\right)^{\frac{\delta}{2}} c_N \left(\frac{N_1^{1+\gamma_1}}{N^{1+\epsilon}}\right)^{\frac{2+\delta}{2}} \leq \left(\frac{c_N}{c_k}\right)^{\frac{\delta}{2}} \frac{c_N}{N} \left(\frac{N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2}}$$

which converges to zero if $N_1^{(1+\gamma_1)\frac{2+\delta}{\delta}}/N \to 0$, which is implied by condition (e)(iv) (recall that $c_N/N \to 0$ and $c_k/c_N \to 1$).

3) Consider the case $c_k/c_N \to 0$ now. We have:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_c^{\gamma_c}} \geq O\left(\frac{c_N}{N}\right) \frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c} + (c_N - c_k)O(1)}{\sum_{c=1}^{c_k} N_c^{\gamma_c} + (c_N - c_k)O(1)} \tag{4}$$

and we can either have the first or the second term in the denominator diverging faster.

Consider first the case when $1/c_N \sum_{c=1}^{c_k} N_c^{\gamma_c} \leq O(1)$. Now we can rewrite (4) as:

$$O(1)\frac{\sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N} + \left(\frac{c_N}{N}\right)\left(1 - \frac{c_k}{c_N}\right)O(1)}{\frac{1}{c_N}\sum_{c=1}^{c_k} N_c^{\gamma_c} + \left(1 - \frac{c_k}{c_N}\right)O(1)} \geq O(1)$$

using $\sum_{c=1}^{c_k} \frac{N_c^{1+\gamma_c}}{N} \geq \sum_{c=1}^{c_k} \frac{N_c}{N} \to 1$, where the last limit follows from $c_N/N \to 0$ (note that $\sum_{c=1}^{c_k} N_c = N - (c_N - c_k)O(1)$). Furthermore, for the case $1/c_N \sum_{c=1}^{c_k} N_c^{\gamma_c} \to \infty$ we write (4) as:

$$O(1)\frac{\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_c}} + \frac{c_N(c_N - c_k)}{N\sum_{c=1}^{c_k} N_c^{\gamma_c}}O(1)}{1 + \frac{c_N - c_k}{\sum_{c=1}^{c_k} N_c^{\gamma_c}}O(1)} = O(1)\frac{\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_c}} + o(1)}{1 + o(1)} \geq O(1)$$

where the last inequality follows by the same argument as in the proof of part 2) above, with the help of conditions (e)(ii)-(iii).

Finally, we can write:

$$\begin{aligned}\frac{c_N}{N}\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_c^{\gamma_k}} &\le \left(\frac{c_N N_1}{N}\right)^{\frac{2+\delta}{2}} c_N^{-\frac{\delta}{2}} \frac{\left(\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{N_1\sum_{c=1}^{c_k} N_c^{\gamma_c}}\right)^{\frac{2+\delta}{2}} + \frac{c_N-c_k}{N_1\sum_{c=1}^{c_k} N_c^{\gamma_c}}O(1)}{\left(1+\frac{c_N-c_k}{\sum_{c=1}^{c_k} N_c^{\gamma_c}}O(1)\right)^{\frac{2+\delta}{2}}} \\ &\le \frac{c_N}{N}\left(\frac{N_1^{\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2}} \frac{1+o(1)}{\left(1+\frac{c_N-c_k}{\sum_{c=1}^{c_k} N_c^{\gamma_c}}O(1)\right)^{\frac{2+\delta}{2}}} = o(1)\end{aligned}$$

where we have used:

$$\frac{\sum_{c=1}^{c_k} N_c^{1+\gamma_c}}{N_1\sum_{c=1}^{c_k} N_c^{\gamma_c}} \le 1$$
$$\frac{1}{c_N}N_1\sum_{c=1}^{c_k} N_c^{\gamma_c} \ge \frac{1}{c_N}\sum_{c=1}^{c_k} N_c^{1+\gamma_c} \ge \frac{N}{c_N}\frac{\sum_{c=1}^{c_k} N_c}{N} \to \infty$$

and $N_1^{\frac{2+\delta}{\delta}}/N \to 0$ follows from condition (e)(iv). Hence, this part of Theorem 1 follows from Theorem 5.

D Proof of Proposition 1

First, assume that $d_{max} > 2$. Using the bound in Lemma 1 and taking $\log_{d_{max}-1}$, we get that $N_1^r/N \to 0$ is implied by

$$r \log_a N - \log_{d_{max}-1} N \to -\infty$$

which can be rewritten as:

$$\log N \frac{r\log(d_{max}-1) - \log a}{\log a \log(d_{max}-1)} \to -\infty.$$

This is implied by $r\log(d_{max}-1) - \log a < 0$, which gives the condition in the proposition.

Finally, note that if $d_{max} = 2$, taking $d_{min} = 1$, Lemma 1 implies that the largest component cannot contain more than 3 nodes. This implies that $N_1^r/N \to 0$ is satisfied for any r.

E Proof of Theorem 2

We have:

$$
\begin{aligned}
N^2 Var(\hat{B}_N - B_N) &= E\left(\sum_{i=1}^{N}\sum_{j}(Y_iY_j - E(Y_iY_j))\mathbb{1}\{l(i,j)<\infty\}\right)^2 \\
&= \sum_{i=1}^{N}\sum_{j}\sum_{k}\sum_{l} E[(Y_iY_j - E(Y_iY_j))(Y_kY_l - E(Y_kY_l))]\mathbb{1}\{l(i,j)<\infty\}\mathbb{1}\{l(k,l)<\infty\} \\
&= \sum_{i=1}^{N}\sum_{j}\sum_{k}\sum_{l} Cov(Y_iY_j, Y_kY_l)\mathbb{1}\{l(i,j)<\infty\}\mathbb{1}\{l(k,l)<\infty\}. \qquad (5)
\end{aligned}
$$

But $\mathbb{1}\{l(i,j)<\infty\}\mathbb{1}\{l(k,l)<\infty\} = 0$ unless i and j belong to the same network component and same happens for k and l. But for such pairs of (i,j) and (k,l) we have $Cov(Y_iY_j, Y_kY_l) \neq 0$ only when (i,j) and (k,l) belong to the same network component (see Assumption CVAR'). Thus, we can rewrite (5) as:

$$
(5) = \sum_{c=1}^{c_N}\sum_{i\in C_c}\sum_{j\in C_c}\sum_{k\in C_c}\sum_{l\in C_c} Cov(Y_iY_j, Y_kY_l) \leq M\sum_{c=1}^{c_N} N_c^4
$$

because $Cov(Y_iY_j, Y_kY_l) \leq M$ for bounded M by the assumption that $E\,|Y_i|^4$ is bounded (in the statement of the theorem) and Cauchy-Schwartz inequality.

Now consider the three cases in the theorem:

(a) $\sum_{c=1}^{c_N} N_1^4 = c_N N_1^4 = N N_1^3 = o(N^2)$ by the condition $\log_{d_{max}-1} a > 3$,

(b) $\sum_{c=1}^{c_N} N_c^4 \leq c_N N_1^4 = \frac{c_N}{N} N N_1^4 = o(N^2)$ by $c_N/N = o(1)$ and the condition $\log_{d_{max}-1} a > 4$,

(c) $\sum_{c=1}^{c_N} N_c^4 \leq c_k N_1^4 + (c_N - c_k)O(1) = o(N^2)$ by $c_N = o(N^2)$ and the condition $\log_{d_{max}-1} a > 2$,

(d) $\sum_{c=1}^{c_N} N_c^4 \leq c_k N_1^4 + o(N^2)$ by the same argument as above. Now, with equal components, $c_k N_1 = O(N)$, and $\log_{d_{max}-1} a > 3$ implies that the final expression is $o(N^2)$. Otherwise, $c_k N_1^4 = o(N) N_1^4 = o(N^2)$ where the last equality follows from $\log_{d_{max}-1} a > 4$.

F Proof of Theorem 3

First note that the statements (i) $\frac{N_c(N)^{1+\gamma_c}}{N_1(N)^{1+\gamma_1}} \to 0 \Rightarrow \frac{N_c(N)}{N_1(N)} \to 0$, (ii) $\frac{N_c(N)^{\gamma_c}}{N_1(N)^{\gamma_1}}$ is weakly decreasing in N, (iii) there exists $M < \infty$ such that $\frac{N_{c_{\tilde{N}}}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N}(N)^{\gamma_N}} < \left(\frac{N_1(\tilde{N})}{N_1(N)}\right)^{\gamma_1}$ for all $(N, \tilde{N})$ such that $c_{\tilde{N}} = c_N + 1$

and $N > M$, are equivalent (respectively) to statements:

(i)' $\frac{N_{c,f}(N)^{1+\gamma_c}}{N_{1,f}(N)^{1+\gamma_1}} \to 0 \Rightarrow \frac{N_{c,f}(N)}{N_{1,f}(N)} \to 0$,

(ii)' $\frac{N_{c,f}(N)^{\gamma_c}}{N_{1,f}(N)^{\gamma_1}}$ is weakly decreasing in N,

(iii)' there exists $M < \infty$ such that $\frac{N_{c_{\tilde{N}},f}(\tilde{N})^{\gamma_{\tilde{N}}}}{N_{c_N,f}(N)^{\gamma_N}} < \left(\frac{N_{1,f}(\tilde{N})}{N_{1,f}(N)}\right)^{\gamma_1}$ for all $(N, \tilde{N})$ such that $c_{\tilde{N}} = c_N + 1$ and $N > M$,

For most parts the proof follows by the same argument as in the proof of Theorem 1 above, with N_f replacing N, $N_{c,f}$ replacing N_c and $B_{N,f}$ replacing B_N. Thus, we present only arguments that differ. W.l.o.g. we often write N_c^2 instead of $N_c(N_c - 1)$ as these are of the same order.

Part (b). Condition (a). of Theorem 5 is satisfied by the same argument as in the proof of Theorem 1. Note that $N_f = \sum_{c=1}^{c_N} N_c(N_c - 1)$ is minimised subject to $\sum_{c=1}^{c_N} N_c = N$ by setting $N_c = N/c_N$, which implies $N_f \geq N(N-1)/c_N$. For condition (c), by the same reasoning as above, we have:

$$\left(\frac{c_N}{N_f}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_{c,f}^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq \frac{c_N N_{1,f}^{(1+\gamma_1)\frac{2+\delta}{2}}}{N_f^{\frac{2+\delta}{2}}}$$

and the last expression is bounded by:

$$\frac{c_N^{2+\frac{\delta}{2}} N_1^{(1+\gamma_1)(2+\delta)}}{N^{2+\delta}} = \left(\frac{c_N}{N}\right)^{2+\frac{\delta}{2}} \left(\frac{N_1^{2(1+\gamma_1)\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2}} = o(1)$$

where the last equality is implied by $c_N/N \to 0$ and condition (b) in the statement of the theorem.

Part (c). Here the proof follows the same lines as in Theorem 1 with a difference that now we require $N_{1,f}^{1+\gamma_1}/N \to 0$ which is implied by the condition in the statement of the theorem.

Part (d). Consider unequal components case first. Note that we have $\sum_{c=1}^{c_k} N_{c,f}^{\gamma_1}/N_f \leq \sum_{c=1}^{c_k} N_{c,f}/N_f = O(1) \leq \sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_1}/N_f$, which implies:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}} \geq O\left(\frac{c_N}{N}\right) \frac{\sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_c}/N_f + o(1)}{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}/N + O(1)} \geq \frac{O(1)}{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}/N + O(1)}$$

so it remains to show that $\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}/N = O(1)$. Since:

$$\frac{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}}{N} = \frac{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}}{N_{1,f}^{\gamma_1}} \frac{N_{1,f}^{\gamma_1}}{N} \leq \frac{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}}{N_{1,f}^{\gamma_1}}$$

for N large enough, this follows from assumption (d)(i') in the theorem, noting that we also have $N_1^{2\gamma_1}/N \to 0$. Now to verify condition (c) of Theorem 5, by the same reasoning as in the proof of Theorem 1 we need $\sum_{c=1}^{c_k} \left(N_{c,f}^{1+\gamma_c}/N\right)^{\frac{2+\delta}{2}} \to 0$. Noting that:

$$\sum_{c=1}^{c_k} \left(\frac{N_{c,f}^{1+\gamma_c}}{N}\right)^{\frac{2+\delta}{2}} \leq \left(\frac{N_{1,f}^{1+\gamma_1}}{N}\right)^{\frac{\delta}{2}} \sum_{c=1}^{c_k} \frac{N_{c,f}^{1+\gamma_c}}{N}$$

condition (d)(i') implies both that $N_{1,f}^{1+\gamma_1}/N \to 0$ and $\sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_c}/N$ converges, which proves the needed claim.

Consider now the case where c_k components grow at the same rate. Firstly, using $c_N/N_f \geq 1/N_{1,f}$:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}} \geq \frac{c_k N_{1,f}^{1+\gamma_1} + (c_N - c_k)O(1)}{c_k N_{1,f}^{1+\gamma_1} + (c_N - c_k)O(N_f/c_N)} = \frac{1 + \frac{c_N - c_k}{c_k N_{1,f}^{1+\gamma_1}} O(1)}{1 + \frac{(1 - c_k/c_N)N_f}{c_k N_{1,f}^{1+\gamma_1}} O(1)}$$

and the last expression is bounded away from zero since $c_k N_{1,f}^{1+\gamma_1}/N_f \geq O(1)$ (see above) and:

$$\frac{c_N - c_k}{c_k N_{1,f}^{1+\gamma_1}} = \frac{c_N - c_k}{N_f} \frac{N_f}{c_k N_{1,f}^{1+\gamma_1}} = o(1).$$

Now to verify condition (c) of Theorem 5, just as before we need $c_k(N_{1,f}^{1+\gamma_1}/N)^{(2+\delta)/2} \to 0$ and using $c_k \sim N/N_1$ this expression becomes $N_1^{1+\delta+\gamma_1(2+\delta)}/N^{\delta/2} = (N_1^{2(1+\delta)/\delta + 2\gamma_1(2+\delta)/\delta}/N)^{\delta/2}$ and converges to zero by assumption (d)(ii)' of the theorem.

Part (e). First consider the case with c_k components growing at the same rate:

1) Using $c_N/N_f \geq 1/N_{1,f}$ and rearranging we have:

$$\frac{B_N^2}{\frac{1}{c_N}\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}} \geq O(1) \frac{1 + \frac{c_N - c_k}{c_k N_{1,f}^{1+\gamma_1}} O(1)}{1 + \left(1 - \frac{c_k}{c_N}\right) \frac{N_f}{c_k N_{1,f}^{1+\gamma_1}} O(1)}$$

Now $N_f = c_k N_{1,f} + (c_N - c_k)O(1)$ and $(c_N - c_k)/N_f \to 0$ imply $c_k N_{1,f}/N_f \to 1$, which gives $c_k N_{1,f}^{1+\gamma_1}/N_f \geq O(1)$ and $\frac{c_N - c_k}{c_k N_{1,f}^{1+\gamma_1}} = \frac{c_N - c_k}{N_f} \frac{N_f}{c_k N_{1,f}^{1+\gamma_1}} = o(1)$. This verifies condition (a) of Theorem 5.

Now proceeding as in the proof of Theorem 1 we get:

$$\left(\frac{c_N}{N_f}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_{c,f}^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq \left(\frac{c_N N_{1,f}}{N_f}\right)^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \frac{1 + O(1)(c_N - c_k)/(c_k N_{1,f}^{(1+\gamma_1)\frac{2+\delta}{2}})}{\left(1 + O(1)(c_N - c_k)/(c_k N_{1,f}^{\gamma_1})\right)^{\frac{2+\delta}{2}}} = \frac{c_N^{\frac{2+\delta}{2}}}{c_k^{1+\delta}} o(1) = o(1)$$

where the first equality follows from $\frac{c_N - c_k}{N_f} \frac{N_f}{c_k N_{1,f}^{1+\gamma_1}} = o(1)$ in the previous paragraph and $N_f \geq c_k N_{1,f}$ (which implies $N_{1,f}/N_f \leq 1/c_k$). The final equality follows from condition (e) of the theorem.

2) Now consider the case $c_k/c_N \to 1$. First part of the proof follows by the same reasoning as in Theorem 1 using assumptions (e)(ii)-(iii). Also, by similar reasoning as the one there:

$$\left(\frac{c_N}{N_f}\right)^{\frac{2+\delta}{2}} \frac{\sum_{c=1}^{c_N} N_{c,f}^{(1+\gamma_c)\frac{2+\delta}{2}}}{\left(\sum_{c=1}^{c_N} N_{c,f}^{\gamma_c}\right)^{\frac{2+\delta}{2}}} \leq \left(\frac{c_N}{N_f}\right)^{\frac{2+\delta}{2}} N_{1,f}^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \frac{\left(\frac{N_{1,f}^{\gamma_1}}{N^\epsilon}\right)^{\frac{2+\delta}{2}} + o(1)}{(1 + o(1))^{\frac{2+\delta}{2}}}$$

Since:

$$\left(\frac{c_N}{N_f}\right)^{\frac{2+\delta}{2}} N_{1,f}^{\frac{2+\delta}{2}} c_k^{-\frac{\delta}{2}} \left(\frac{N_{1,f}^{\gamma_1}}{N^\epsilon}\right)^{\frac{2+\delta}{2}} \leq \left(\frac{c_N}{c_k}\right)^{\frac{\delta}{2}} \frac{c_N}{N} \left(\frac{N_{1,f}^{1+\gamma_1} N^{\frac{2}{2+\delta}}}{N_f}\right)^{\frac{2+\delta}{2}}$$

A sufficient condition for the last expression to converge to zero is $N_{1,f}^{1+\gamma_1} N^{\frac{2}{2+\delta}}/N_f \to 0$. As in the proof of part (b), we have $N_f \geq N(N-1)/c_N$, which gives:

$$\frac{N_{1,f}^{1+\gamma_1} N^{\frac{2}{2+\delta}}}{N_f} \leq \frac{c_N}{N} \left(\frac{N_1^{2(1+\gamma_1)\frac{2+\delta}{\delta}}}{N}\right)^{\frac{\delta}{2+\delta}}$$

and the latter converges to zero by condition $\log_{d_{max}-1} a > 2(1+\gamma_c)\frac{2+\delta}{\delta}$.

3) Here $c_k/c_N \to 0$. Condition (a) of Theorem 5 follows by an argument mirroring the one in

the proof of Theorem 1. To verify condition (c) note that:

$$\begin{aligned}
\frac{c_N}{N_f}\frac{\sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_c}}{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_k}} &\leq \left(\frac{c_N N_{1,f}}{N_f}\right)^{\frac{2+\delta}{2}} c_N^{-\frac{\delta}{2}} \frac{\left(\frac{\sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_c}}{N_{1,f}\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}}\right)^{\frac{2+\delta}{2}} + \frac{c_N - c_k}{N_{1,f}\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}} O(1)}{\left(1 + \frac{c_N - c_k}{\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}} O(1)\right)^{\frac{2+\delta}{2}}} \\
&= O(1)\left(\frac{c_N N_{1,f}}{N_f}\right)^{\frac{2+\delta}{2}} c_N^{-\frac{\delta}{2}} \leq O(1)\frac{c_N}{N}^{2+\frac{\delta}{2}} \left(\frac{N_1^{\frac{2(2+\delta)}{\delta}}}{N}\right)^{\frac{\delta}{2}} = o(1)
\end{aligned}$$

where we have used:

$$\frac{\sum_{c=1}^{c_k} N_{c,f}^{1+\gamma_c}}{N_{1,f}\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c}} \leq 1, \quad \frac{1}{c_N} N_{1,f}\sum_{c=1}^{c_k} N_{c,f}^{\gamma_c} \to \infty, \quad N_f \geq N(N-1)/c_N$$

and the last equality follows from $\log_{d_{max}-1} a > \frac{2(2+\delta)}{\delta}$. The result now follows by Theorem 5.

G Proof of Theorem 4

For a sequence of random variables $\{W_1, \ldots, W_N\}$ define the U_N operator as:

$$U_N h = \frac{1}{N(N-1)}\sum_{i \neq j} h(W_i, W_j)\mathbb{1}\{l(i,j) < \infty\}.$$

where h is a symmetric kernel. By Hoeffding decomposition:

$$\frac{\sqrt{N}\overline{Y}_c}{B_{N,c}} = B_{N,c}^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} h_1(Y_i)\frac{N_c(i)-1}{N-1} + \frac{1}{2}B_{N,c}^{-1}\sqrt{N}U_N h_2 \tag{6}$$

where $h_2(y_1, y_2) = h(y_1, y_2) - h_1(y_1) - h_1(y_2)$. Let us first show that $B_{N,c}^{-1}\sqrt{N}U_N h_2 = o_p(1)$.

We have:

$$\begin{aligned}
Var(\sqrt{N}U_N h_2) &= \frac{1}{N(N-1)^2}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{k=1}^{N}\sum_{l\neq k} E[h_2(Y_i, Y_j)h_2(Y_k, Y_l)]\mathbb{1}\{l(i,j) < \infty\}\mathbb{1}\{l(k,l) < \infty\} \\
&= \frac{1}{N(N-1)^2}\sum_{c=1}^{c_N}\sum_{i\in C_c}\sum_{j\in C_c, j\neq i}\sum_{k\in C_c}\sum_{l\in C_c, l\neq k} E[h_2(Y_i, Y_j)h_2(Y_k, Y_l)]
\end{aligned}$$

since the term under the sum is only nonzero if (i,j,k,l) belong to the same component (note that

$Eh_2(Y_i, Y_j) = 0)$.

As in the proof of Theorem 3.1 in Kojevnikov et al. (2021), let $H_{N_c}(s, m)$ be defined as the sets of nodes $\{i, j, k, l\}$ where $\{i, j\}$ and $\{k, l\}$ are both in the m-neighbourhood from each other and the network distance between $\{i, j\}$ and $\{k, l\}$ is at least s, formally: $H_{N_c}(s, m) = \{(i, j, k, l) : l(i, j) \leq m, l(k, l) \leq m, l(\{i, j\}, \{k, l\}) \geq s\}$. We have $H_{N_c}(s, m) \leq 4N_c c_{N_c}(s, m; 2)$ (ibid.). Now by Lemma 2 we can bound:

$$N^3 Var(\sqrt{N} U_N h_2) = \sum_{c=1}^{c_N} \sum_{s \geq 0} \sum_{\substack{\{i,j,k,l\} \in H_{N_c}(s, N_c) \\ j \neq i, l \neq k}} E[h_2(Y_i, Y_j) h_2(Y_k, Y_l)] \leq \sum_{c=1}^{c_N} \sum_{s \geq 0} |H_{N_c}(s, N_c)| \alpha_c(s)$$
$$\leq 4 \sum_{c=1}^{c_N} N_c \sum_{s \geq 0} c_{N_c}(s, N_c; 2) \alpha_c(s).$$

Now Assumption WDEP(b) implies that $Var(B_{N,c}^{-1} \sqrt{N} U_N h_2) = o(1)$.

Finally, the asymptotic normality of the first element in (6) follows from Theorem 3.2 in Kojevnikov et al. (2021). To see that define $X_{i,N} = h_1(Y_i) \frac{N_c(i)-1}{N-1}$ note that the triangular array $\{X_{i,N}\}_{i=1}^{\infty}$ is strong mixing with coefficients $\alpha_N(\cdot)$ and the conditions of their theorem are implied by Assumptions WDEP(a), (c) (note that $N_c/N \leq 1$).

H Additional MC simulations

Table 3: Simulated coverage, edge-specific means (contrasts), normal errors, 95% level

	BA model					SW model				
N	m	z_a	d_{max}	Δ_N	Coverage oracle	p	k	d_{max}	Δ_N	Coverage oracle
	Equal Components									
500	1	0	10	10	0.944	0.05	2	6	12	0.957
1000	1	0	10	11	0.953	0.05	2	6	15	0.951
5000	1	0	10	13	0.953	0.05	2	7	20	0.946
10000	1	0	10	14	0.942	0.05	2	7	20	0.948
500	1	1	10	16	0.946	0.05	5	13	5	0.931
1000	1	1	10	16	0.941	0.05	5	13	6	0.940
5000	1	1	10	23	0.950	0.05	5	14	7	0.945
10000	1	1	10	19	0.963	0.05	5	15	8	0.939
500	1	2	10	14	0.953	0.05	10	24	3	0.946
1000	1	2	10	15	0.941	0.05	10	24	4	0.935
5000	1	2	10	18	0.947	0.05	10	26	5	0.948
10000	1	2	10	21	0.955	0.05	10	26	5	0.949
500	2	0	20	7	0.959	0.10	2	8	9	0.956
1000	2	0	20	8	0.958	0.10	2	7	10	0.946
5000	2	0	20	11	0.943	0.10	2	8	14	0.942
10000	2	0	20	11	0.951	0.10	2	9	15	0.956
500	2	1	20	7	0.953	0.10	5	14	4	0.953
1000	2	1	20	7	0.933	0.10	5	14	5	0.952
5000	2	1	20	9	0.941	0.10	5	16	6	0.955
10000	2	1	20	11	0.945	0.10	5	16	7	0.935
500	2	2	20	6	0.948	0.10	10	25	3	0.932
1000	2	2	20	7	0.955	0.10	10	25	4	0.949
5000	2	2	20	9	0.940	0.10	10	28	4	0.948
10000	2	2	20	9	0.936	0.10	10	30	5	0.933
	Growing + Fixed									
500	1	0	10	10	0.956	0.05	2	6	12	0.949
1000	1	0	10	13	0.952	0.05	2	6	15	0.952
5000	1	0	10	12	0.953	0.05	2	7	20	0.947
10000	1	0	10	13	0.941	0.05	2	7	20	0.962
500	1	1	10	11	0.958	0.05	5	13	5	0.944
1000	1	1	10	14	0.960	0.05	5	13	6	0.941
5000	1	1	10	16	0.961	0.05	5	14	7	0.939
10000	1	1	10	17	0.955	0.05	5	15	8	0.946
500	1	2	10	16	0.962	0.05	10	24	3	0.948
1000	1	2	10	17	0.961	0.05	10	24	4	0.959
5000	1	2	10	23	0.954	0.05	10	26	5	0.959
10000	1	2	10	22	0.950	0.05	10	26	5	0.947
500	2	0	20	7	0.928	0.10	2	8	9	0.943
1000	2	0	20	8	0.949	0.10	2	7	10	0.940
5000	2	0	20	9	0.948	0.10	2	8	14	0.947
10000	2	0	20	10	0.933	0.10	2	9	15	0.944
500	2	1	20	7	0.952	0.10	5	14	4	0.958
1000	2	1	20	7	0.953	0.10	5	14	5	0.953
5000	2	1	20	10	0.955	0.10	5	16	6	0.951
10000	2	1	20	10	0.948	0.10	5	16	7	0.959
500	2	2	20	6	0.943	0.10	10	25	3	0.952
1000	2	2	20	7	0.945	0.10	10	25	4	0.944
5000	2	2	20	9	0.949	0.10	10	28	4	0.944
10000	2	2	20	9	0.953	0.10	10	30	5	0.947

Note: 1000 Monte Carlo simulations, 1000 bootstrap replications. "Oracle" – known variance.

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