MVE161
Ordinary differential equations and mathematical modelling

Home assigment 1

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1 Question 1

To make an analysis of the system mass-spring (1.1) and mass-spring-damper (1.2) easier the equations get non-dimensionalized.

$$m\ddot{x} = -kx \tag{1.1}$$

$$m\ddot{x} = -c\dot{x} - kx \tag{1.2}$$

After the non-dimensionalization the constants has no units and the time is scaled so the eigenfrequency is equal to one.

1.1 Mass-Spring

For the case with the mass-spring when there is no friction, equation (1.1). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -x^* \tag{1.3}$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{gathered} \dot{x} = A(t)x \\ \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{gathered} \tag{1.4}$$

The only stationary point for the system is $x_1 = x_2 = 0$ and it is stable for $t \to +\infty$. To see if the system is bounded or unbounded the eigenvalues are calculated according to:

$$\begin{gathered} \lambda^2 - Tr(A(t))\lambda + det(A(t)) = 0 \\ \Rightarrow \lambda_{1,2} = \pm i \end{gathered} \tag{1.5}$$

According to the eigenvalues calculated in equation (1.5) the system is bounded because the real part of the eigenvalues is zero, marginaly stable.

To find the periodic solutions the monodromy matrix is needed and it is calculated with the fundamental matrix. To get the fundamental matrix assume that $x_1 = a\cos(t)+$

$b\sin(t)$. The first and second column of the fundamental matrix solution (1.6) is given by the initial condition $x_1(0) = 1$ and $x_2(0) = 0 \rightarrow (a = 1, b = 0)$ respectively $x_1(0) = 0$ and $x_2(0) = 1 \rightarrow (a = 0, b = 1)$.

$$\Phi(t) = \begin{bmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{bmatrix} \tag{1.6}$$

Because of that the fundamental matrix solution only contains sinus and cosinus terms the periodicy is testet for $T = 2\pi$:

$$\begin{gathered} \Phi(t+T) = \Phi(t)e^{TR} \\ (t = 0 \text{ and } T = 2\pi) \\ \Phi(2\pi) = \Phi(0)e^{2\pi R} \end{gathered} \tag{1.7}$$

Equation (1.7) shows that the fundamental matrix solution is the monodromy matrix and that the system is periodic in $T = 2\pi$.

1.2 Mass-Spring-Damper

For the case with the mass-spring-damper when there is friction, equation (1.2). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -\gamma\dot{x}^* - x^* \tag{1.8}$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{gathered} \dot{x} = A(t)x \\ \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{gathered} \tag{1.9}$$

The only stationary point for the system is $x_1 = x_2 = 0$ and it is stable for $t \rightarrow +\infty$. To see if the system is bounded or unbounded the eigenvalues are calculated according to:

$$\begin{gathered} \lambda^2 - Tr(A(t))\lambda + det(A(t)) = 0 \\ \Rightarrow \lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - 1} \end{gathered} \tag{1.10}$$

The eigenvalues from the equation (1.10) is investigated for seven different cases:

1. $\gamma > 2$: λ is real and negative $\Rightarrow$ stable
2. $\gamma = 2$: λ is real and negative $\Rightarrow$ stable
3. $-2 < \gamma < 0$: λ is real and imaginary but the real part is positive $\Rightarrow$ unstable
4. $\gamma = 0$: λ is imaginary $\Rightarrow$ bounded
5. $0 < \gamma < 2$: λ is real and imaginary but the real part is negative $\Rightarrow$ stable
6. $\gamma = -2$: λ is real and positive $\Rightarrow$ unstable
7. $\gamma < -2$: λ is real and positive $\Rightarrow$ unstable

2 Question 2

To make an analysis of the system mass-spring (2.1) with external force and mass-spring-damper (2.2) with external force easier the equations get non-dimensionalized.

$$m\ddot{x} = -kx + F\cos(\omega t) \tag{2.1}$$

$$m\ddot{x} = -c\dot{x} - kx + F\cos(\omega t) \tag{2.2}$$

After the non-dimensionalization the constants has no units and the time is scaled so the eigenfrequency is equal to one.

2.1 Mass-Spring with external force

For the case with the mass-spring and external force when there is no friction, equation (2.1). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -x^* + \beta\cos(\omega t) \tag{2.3}$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{gathered}\dot{x} = A(t)x \\ \begin{bmatrix}\dot{x_1} \\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \beta\cos(\omega t)\end{bmatrix}\end{gathered} \tag{2.4}$$

The A-matrix is identical to the A-matrix in equation (1.1) so the same conclutions as in section **??** is true for the system. Due to the external force the system can become instable even though the A-matrix is stable.

2.2 Mass-Spring-Damper with external force

For the case with the mass-spring-damper and external force when there is friction, equation (2.2). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -\gamma\dot{x}^* - x^* + \beta\cos(\omega t) \tag{2.5}$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{aligned} \dot{x} &= A(t)x \\ \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta\cos(\omega t) \end{bmatrix} \end{aligned} \tag{2.6}$$

The A-matrix is identical to the A-matrix in equation (1.1) so the same conclutions as in section **??** is true for the system. Due to the external force the system can become instable even though the A-matrix is stable.

3 Question 3

4 Question 4

5 Question 5