

MVE161
Ordinary differential equations
and mathematical modelling

Home assignment 1

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1 Question 1

To make an analysis of the system mass-spring (1.1) and mass-spring-damper (1.2) easier the equations get non-dimensionalized.

$$m\ddot{x} = -kx \quad (1.1)$$

$$m\ddot{x} = -c\dot{x} - kx \quad (1.2)$$

After the non-dimensionalization the constants has no units and the time is scaled so the eigenfrequency is equal to one.

1.1 Mass-Spring

For the case with the mass-spring when there is no friction, equation (1.1). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -x^* \quad (1.3)$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{aligned} \dot{x} &= A(t)x \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (1.4)$$

The only stationary point for the system is $x_1 = x_2 = 0$ and it is stable for $t \rightarrow +\infty$. To see if the system is bounded or unbounded the eigenvalues are calculated according to:

$$\begin{aligned} \lambda^2 - \text{Tr}(A(t))\lambda + \det(A(t)) &= 0 \\ \Rightarrow \lambda_{1,2} &= \pm i \end{aligned} \quad (1.5)$$

According to the eigenvalues calculated in equation (1.5) the system is bounded because the real part of the eigenvalues is zero, marginally stable.

To find the periodic solutions the monodromy matrix is needed and it is calculated with the fundamental matrix. To get the fundamental matrix assume that $x_1 = a \cos(t) +$

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$b \sin(t)$. The first and second column of the fundamental matrix solution (1.6) is given by the initial condition $x_1(0) = 1$ and $x_2(0) = 0 \rightarrow (a = 1, b = 0)$ respectively $x_1(0) = 0$ and $x_2(0) = 1 \rightarrow (a = 0, b = 1)$.

$$\Phi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \quad (1.6)$$

Because of that the fundamental matrix solution only contains sinus and cosinus terms the periodicity is tested for $T = 2\pi$:

$$\begin{aligned} \Phi(t+T) &= \Phi(t)e^{TR} \\ (t=0 \text{ and } T=2\pi) \\ \Phi(2\pi) &= \Phi(0)e^{2\pi R} \end{aligned} \quad (1.7)$$

Equation (1.7) shows that the fundamental matrix solution is the monodromy matrix and that the system is periodic in $T = 2\pi$.

1.2 Mass-Spring-Damper

For the case with the mass-spring-damper when there is friction, equation (1.2). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -\gamma \dot{x}^* - x^* \quad (1.8)$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{aligned} \dot{x} &= A(t)x \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (1.9)$$

The only stationary point for the system is $x_1 = x_2 = 0$ and it is stable for $t \rightarrow +\infty$. To see if the system is bounded or unbounded the eigenvalues are calculated according to:

$$\begin{aligned} \lambda^2 - \text{Tr}(A(t))\lambda + \det(A(t)) &= 0 \\ \Rightarrow \lambda_{1,2} &= -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - 1} \end{aligned} \quad (1.10)$$

1.2 Mass-Spring-Damper

The eigenvalues from the equation (1.10) is investigated for seven different cases:

1. $\gamma > 2$: λ is real and negative \Rightarrow stable
2. $\gamma = 2$: λ is real and negative \Rightarrow stable
3. $-2 < \gamma < 0$: λ is real and imaginary but the real part is positive \Rightarrow unstable
4. $\gamma = 0$: λ is imaginary \Rightarrow bounded
5. $0 < \gamma < 2$: λ is real and imaginary but the real part is negative \Rightarrow stable
6. $\gamma = -2$: λ is real and positive \Rightarrow unstable
7. $\gamma < -2$: λ is real and positive \Rightarrow unstable

2 Question 2

To make an analysis of the system mass-spring (2.1) with external force and mass-spring-damper (2.2) with external force easier the equations get non-dimensionalized.

$$m\ddot{x} = -kx + F \cos(\omega t) \quad (2.1)$$

$$m\ddot{x} = -c\dot{x} - kx + F \cos(\omega t) \quad (2.2)$$

After the non-dimensionalization the constants has no units and the time is scaled so the eigenfrequency is equal to one.

2.1 Mass-Spring with external force

For the case with the mass-spring and external force when there is no friction, equation (2.1). After the non-dimensionalization the system can be described with:

$$\ddot{x}^* = -x^* + \beta \cos(\omega t) \quad (2.3)$$

To get the equation on a state-space-form variables is assigned as $x_1 = x^*$ and $x_2 = \dot{x}^*$ which gives the system:

$$\begin{aligned} \dot{x} &= A(t)x \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \quad (2.4)$$

2.2 Mass-Spring-Damper with external force

3 Question 3

4 Question 4

5 Question 5