

Beyond Random Walks: Revealing the Fractal Memory of Financial Markets

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April 30, 2025

Efficient Market Hypothesis

- **Definition:** Prices fully reflect all available information at any time.
- **Three Forms:**
 - *Weak form:* past prices and volumes are already incorporated.
 - *Semi-strong form:* all publicly available information is embedded.
 - *Strong form:* even insider/private information is captured.
- Therefore, no trading strategy based on information can consistently achieve excess returns.
- **Challenges & Critiques:**
 - Empirical anomalies (momentum, mean-reversion, bubbles).
 - Long-range dependence and multifractality in returns.
 - Behavioral biases and heterogeneous agents.

Assumptions of Brownian Motion

1 Independent increments

For any $0 \leq t_0 < t_1 < \dots < t_n$, the increments

$$B(t_1) - B(t_0), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$$

are mutually independent.

2 Stationary, Gaussian increments

For all $s < t$,

$$B(t) - B(s) \sim \mathcal{N}(0, t - s),$$

meaning the distribution of the increment depends only on the time lag $t - s$.

3 Continuous paths

Almost surely, the function $t \mapsto B(t)$ is continuous on $[0, \infty)$, with no jumps.

Introduction: Fractional Brownian Motion (FBM)

- **Mean at scale s :**

$$\mathbb{E}[X_H(s)] = 0.$$

- **Variance at scale s :**

$$\text{Var}[X_H(s)] = \sigma^2 s^{2H}.$$

where $H \in (0, 1)$ is the Hurst exponent and σ^2 is a scale parameter.

- **Definition:** FBM $X_H(t)$ is a continuous-time Gaussian process with $X_H(0) = 0$ and covariance function

$$\text{Cov}(X_H(t), X_H(s)) = \frac{\sigma^2}{2} (t^{2H} + s^{2H} - |t - s|^{2H}),$$

- **Self-Similarity:** FBM is self-similar; that is, for any scaling factor $c > 0$,

$$X_H(ct) \stackrel{d}{=} c^H X_H(t),$$

which means the process scales with exponent H .

- **Increment Distribution:** The increments are normally distributed:

$$X_H(t) - X_H(s) \sim \mathcal{N}\left(0, \sigma^2(t - s)^{2H}\right) \quad \text{for } 0 \leq s < t.$$

Note that when $H = 0.5$, FBM reduces to standard Brownian motion.

Modeling Long Memory: Simulation of fBm

Fractional Brownian Motion & Autocorrelation

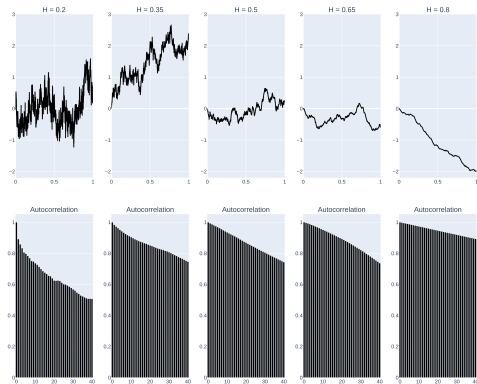


Figure: Simulated fractional Brownian motion for different H values, [0.2, 0.35, 0.5, 0.65, 0.8]

Hurst Exponent Estimation Methods

Overview

There are many ways to estimate the Hurst exponent from a financial time series. In our study, we focus on capturing long-term trends rather than high-frequency fluctuations, since long-term trends reveal persistent behavior and memory effects, while high-frequency data are often dominated by noise.

Methods and Their Authors:

- **Rescaled Ranged (R/S) Analysis:** Originally developed by Harold Edwin Hurst, and later refined by Mandelbrot & Wallis.
- **Modified R/S (M-R/S) Analysis:** Introduced by Lo (1991) and further discussed by Andrews (1991) to correct for short-term autocorrelation.

Hurst Exponent Estimation: R/S Analysis

The Hurst exponent H is estimated using the classical Rescaled Range (R/S) analysis. The procedure is as follows:

Then, construct the cumulative series (profile):

$$Y(t) = \sum_{i=1}^t (x_i - \bar{x}), \quad t = 1, 2, \dots, N.$$

For a segment of length T , define:

$$R(T) = \max_{1 \leq t \leq T} Y(t) - \min_{1 \leq t \leq T} Y(t)$$

and the standard deviation:

$$S(T) = \sqrt{\frac{1}{T} \sum_{i=1}^T (x_i - \bar{x})^2}.$$

The rescaled range is then:

$$Q_T = \frac{R(T)}{S(T)}$$

Scaling Law: Empirically, one finds (Mandelbrot and Wallis (1969b)):

$$\tilde{Q}_T \sim T^H$$

Hurst Exponent Estimation: Modified R/S Analysis

The Modified R/S Analysis, introduced by Lo (1991), refines the classical R/S method by adjusting for short-term autocorrelation, by adding the autocovariance terms.

The Modified R/S statistic is given by:

$$\tilde{Q}_T = \frac{R(T)}{\hat{\sigma}_T},$$

with the adjusted standard deviation:

$$\hat{\sigma}_T = \sqrt{\frac{1}{T} \sum_{j=1}^T (x_j - \bar{x})^2 + \frac{2}{T} \sum_{j=1}^T w_j(q) \sum_{i=j+1}^T (x_i - \bar{x})(x_{i-j} - \bar{x})},$$

where the weights, as introduced by Lo (1991), are defined as:

$$w_j(q) = 1 - \frac{j}{q+1}, \quad j = 1, 2, \dots, q.$$

$q = \lfloor k_T \rfloor$, $k_T = \left(\frac{3T}{2}\right)^{1/3} \left(\frac{2\rho}{1-\rho^2}\right)^{2/3}$, ρ is the first-order autocorrelation coefficient.

Significance Testing of the Modified R/S Statistic

Unlike the classical R/S analysis, the modified R/S statistic has a known limiting distribution. In particular, the statistic

$$V = \tilde{Q}_T / \sqrt{T}$$

converges to the range of a Brownian bridge on the unit interval. This property enables us to perform a formal statistical test of the null hypothesis of short memory versus the alternative hypothesis of long memory by comparing V with the critical values provided by Lo (1991).

Empirical Analysis: Data and Results

- Description of the data: indices (S&P 500, Russell 2000, FTSE 100, NIKKEI 225, DAX) and time period.
- Data processing: logarithmic returns and stationarity tests.

Estimation Results

Ticker	R/S	Hurst Exponent	Modified Hurst Exponent	Critical Value	Long Memory
S&P 500	30.166	0.558	0.501	1.007	False
Russell 2000	51.373	0.645	0.588	1.714	True
FTSE 100	40.236	0.605	0.548	1.341	False
Nikkei 225	22.234	0.508	0.508	1.048	False
DAX	26.985	0.540	0.540	1.278	False

Table: Results for R/S, Hurst exponent, modified Hurst exponent, critical value at 10% (1.620), and rejection of the null hypothesis of long memory. The Hurst exponent can be equal for the R/S and modified R/S methods in the case where the autocorrelation coefficients are less than zero. In this case, we set $q = 0$ and therefore the R/S and modified R/S share the same formula.

What Is Multifractality?

- **Beyond a single exponent:** Unlike a simple fractal (monofractal) characterized by one scaling exponent, a *multifractal* process exhibits a *spectrum* of local scaling behaviors.
- **Local regularity:** At each time t , the trajectory has its own Hölder exponent $\alpha(t)$, measuring how “smooth” or “rough” the series is around t .
- **Relevance in finance:**
 - Heavy-tailed returns and volatility clustering → multiple types of singularities.
 - Multifractality reveals heterogeneity across scales (short shocks vs. long trends).
- **From theory to practice:** MF-DFA provides a robust framework to estimate generalized exponents $h(q)$ and derive $D(\alpha)$ from real financial data.

Given a time series $\{x_k\}_{k=1}^N$, first compute the mean:

$$\bar{x} = \frac{1}{N} \sum_{k=1}^N x_k$$

Then, construct the cumulative profile:

$$Y(i) = \sum_{k=1}^i (x_k - \bar{x}), \quad i = 1, \dots, N.$$

- Divide the profile $Y(i)$ into $N_s = \lfloor N/s \rfloor$ non-overlapping segments of equal length s .
- To use all data, repeat the procedure starting from the opposite end.

For each segment ν , fit a polynomial $y_\nu(i)$ (usually of order m) to the data in the segment.

$$y_\nu(i) = \sum_{j=0}^m a_j^{(\nu)} i^j, \quad i = 1, \dots, s.$$

Step 3: Calculating the Fluctuation Function

For each segment ν , compute the variance:

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^s \{Y[(\nu - 1)s + i] - y_\nu(i)\}^2.$$

Then, average over all $2N_s$ segments for a given order q :

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} [F^2(\nu, s)]^{q/2} \right\}^{1/q},$$

with the special case for $q = 0$ defined by a logarithmic averaging:

$$F_0(s) = \exp \left\{ \frac{1}{4N_s} \sum_{\nu=1}^{2N_s} \ln [F^2(\nu, s)] \right\}.$$

Once the fluctuation function $F_q(s)$ is computed for different scales s , a scaling relation is established:

$$F_q(s) \sim s^{h(q)}.$$

Taking the logarithm of both sides yields:

$$\log F_q(s) \sim h(q) \log s + \text{constant}.$$

By plotting $\log F_q(s)$ against $\log s$, a linear relationship is observed, with the slope corresponding to the generalized Hurst exponent $h(q)$. This approach allows us to characterize the scaling behavior of the fluctuations for various orders q , providing insight into the multifractal structure of the signal.

Graphical view of the MF-DFA steps

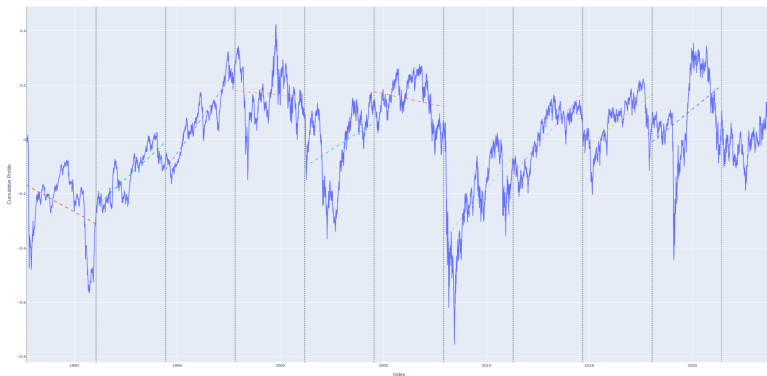


Figure: Simulation of the MF-DFA steps with the Russell 2000

Generalized Hurst Exponents: Russell 2000

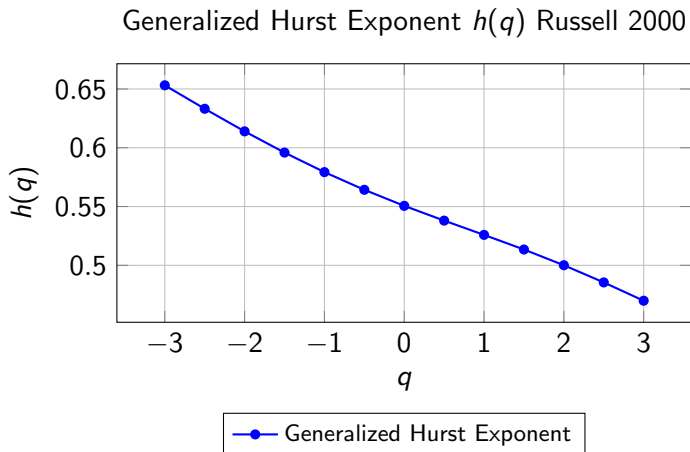


Figure 1: The generalized Hurst exponent $h(q)$ for Russell 2000 is plotted as a function of q . Values of q range from -3 to 3 , and scales used in the computation are logarithmically spaced between 10 and 500.

Generalized Hurst Exponents S&P 500

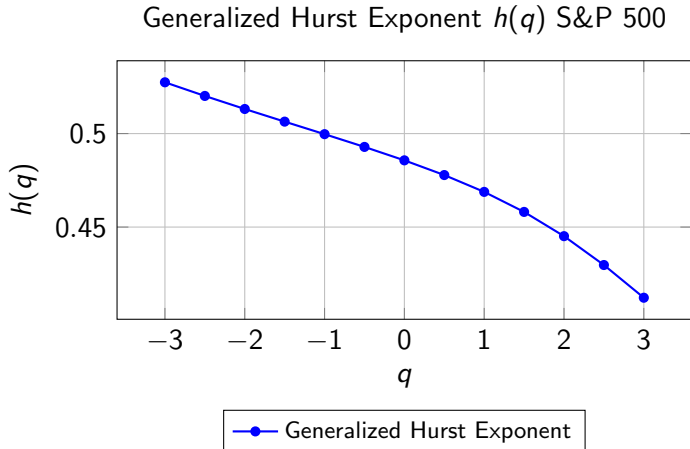


Figure 2: The generalized Hurst exponent $h(q)$ for S&P 500 is plotted as a function of q . Values of q range from -3 to 3 , and scales used in the computation are logarithmically spaced between 10 and 500.

Hölder Exponent $\alpha(q)$

Definition

The Hölder exponent $\alpha(q)$ quantifies the **local regularity** or **singularity** of a signal.

Computation

It is derived via the Legendre transform of the generalized Hurst exponent $h(q)$:

$$\alpha(q) = h(q) + q \cdot h'(q)$$

Interpretation

- **Low** α : sharp variations, spikes, or singular behavior (irregular regions).
- **High** α : smooth, regular zones with low local variability.

Insight

The function $\alpha(q)$ reveals the **heterogeneity** of the time series: different segments may exhibit drastically different local behavior.

Multifractal Spectrum $f(\alpha)$

Definition

Once $\alpha(q)$ is computed, the multifractal spectrum is given by:

$$f(\alpha) = q \cdot [\alpha(q) - h(q)] + 1$$

It describes the fractal dimension of all subsets with the same local regularity α .

Interpretation

- **Wider** spectrum \Rightarrow Stronger multifractality (more variation in scaling behavior).
- **Narrow** spectrum \Rightarrow Monofractal or homogeneous signal.

Sources of Multifractality (Kantelhardt 2002)

$$M(q) \propto \underbrace{f_{\text{tail}}(q)}_{\text{Type I: heavy-tailed value distribution}} + \underbrace{f_{\text{corr}}(q)}_{\text{Type II: long-range temporal correlations}}$$

- 1 **Type I (distribution-driven)** Arise when the return distribution has fat tails (extreme events).
- 2 **Type II (correlation-driven)** Stem from persistent or anti-persistent patterns across scales.

Decomposition via shuffling:

- Shuffle the series to destroy long-range correlations, leaving only the heavy-tail effect.
- Compare the multifractal spectra (or Hurst exponents) of original vs. shuffled data.
- The difference quantifies the Type I contribution to overall multifractality.

Multifractal Spectrum $f(\alpha)$ — Russell 2000

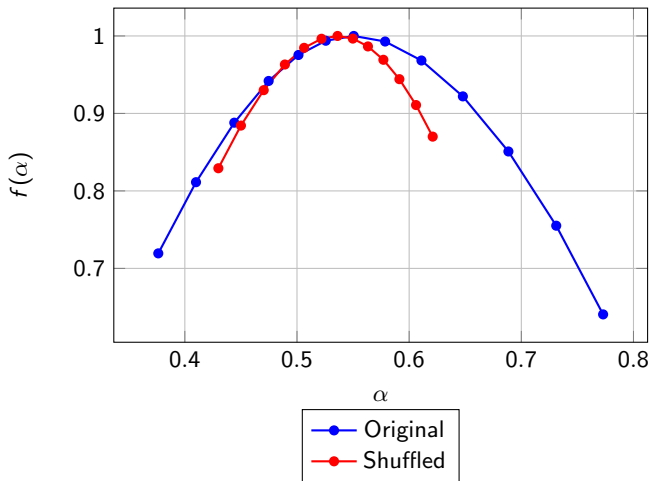


Figure: The multifractal spectrum $f(\alpha)$ characterizes the distribution of singularities in the Russell 2000 return series. A wider curve implies stronger multifractality (more heterogeneity in local scaling exponents).

Multifractal Spectrum $f(\alpha)$ — S&P 500

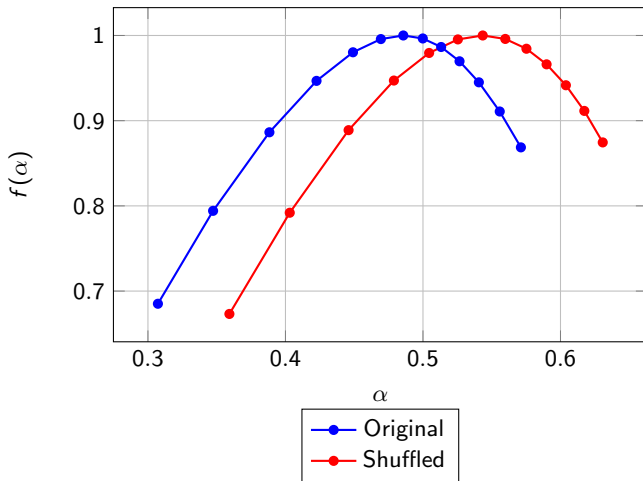


Figure: The multifractal spectrum $f(\alpha)$ for the S&P 500 return series. The width of the curve reflects the range of local singularities broader spectra indicate stronger multifractality.

Comparison of Multifractal Spectra — S&P 500 vs Russell 2000 vs Random Walk

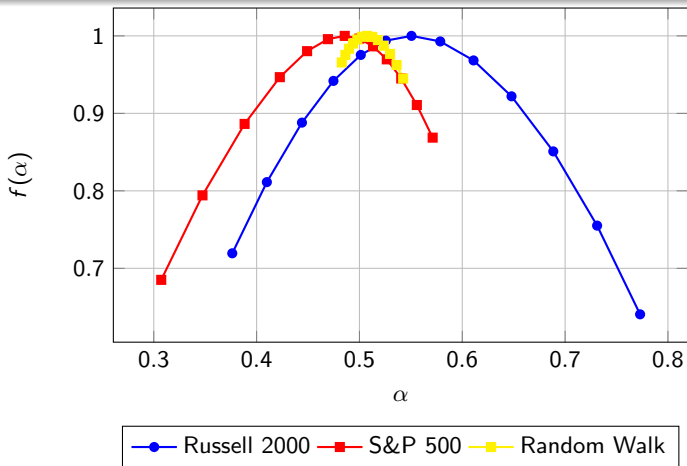


Figure: Comparison of multifractal spectra $f(\alpha)$ for the S&P 500 and Russell 2000. Wider curves indicate richer multifractal behavior. Differences in width and peak shape reflect distinct market dynamics.

Proposition of an Inefficiency Index

We propose combining two key structural components to capture market inefficiency:

- 1 **Fractal Difference** The width of the multifractal spectrum:

$$\Delta\alpha = \alpha_{\max} - \alpha_{\min}$$

where α is the singularity exponent from MF-DFA. Quantifies how complex the behavior of our series is.

- 2 **Deviation of the Rolling Hurst** In an efficient market:

$$H = 0.5$$

So the deviation is quantified as:

$$|H_{\text{rolling}} - 0.5|$$

Inefficiency Index

$$I = \Delta\alpha_{\text{diff}} \times |H_{\text{rolling}} - 0.5|$$

where $\Delta\alpha_{\text{diff}}$ is the absolute difference in spectrum widths (e.g., S&P 500 vs. Russell 2000).

Trading Strategy Overview

Exploit inefficiencies between the S&P 500 and Russell 2000 by combining:

- Rolling **modified Hurst exponent** H_{rolling}
- **Multifractal spectrum width difference** $\Delta\alpha_{\text{diff}}$, rolling 4 years
- A simple **momentum signal**

Approach :

- 1 Compute the **log-return difference** between Russell 2000 and S&P 500
- 2 Estimate a rolling Hurst exponent using modified R/S over a 6-month window
- 3 Calculate momentum: mean return over past 220 days, shifted by 20 days
- 4 Construct the Inefficiency Index:

$$I = \Delta\alpha_{\text{diff}} \times |H_{\text{rolling}} - 0.5|$$

Strategy Rules and Portfolio Allocation

Initial allocation: Start with 50% in S&P 500 and 50% in Russell 2000.

Adjust allocations as follows:

- **Overweight S&P 500 (80%/20%)** if:
 - $H_{\text{rolling}} > 0.5$
 - Momentum > 0 (S&P 500 outperforming)
 - I positive
 - \rightarrow Indicates persistent behavior + higher complexity in S&P 500
- **Overweight Russell 2000 (80%/20%)** if:
 - $H_{\text{rolling}} > 0.5$
 - Momentum < 0 (Russell 2000 outperforming)
 - I negative
 - \rightarrow Indicates persistent behavior + higher complexity in Russell 2000
- **Neutral (50%/50%)** if:
 - $H_{\text{rolling}} < 0.5$
 - \rightarrow Indicates no strong persistence; market closer to efficient

Cumulative log returns strategy



Figure: Comparison of the performance of the ModifOverlap120 strategy (blue) with the 50/50 portfolio (black). The backtest period is from 1988-10-18 to 2025-02-28.

Performance Metrics

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap120	9.39	19.34	0.49	−56.26
ModifOverlap120NoFilter	9.4	19.58	0.48	−57.1
Long Only SP500	8.2	17.88	0.46	−56.78
Long Only Russell	7.37	21.73	0.34	−59.89
50/50 Portfolio	7.94	19.13	0.42	−57.99

Table: Performance metrics of the ModifOverlap120 strategies (5 bps transaction fees) compared to the long-only S&P 500, long-only Russell 2000, and 50/50 Russell/S&P 500 portfolios.

Conclusions and Perspectives

- Explored several methods for estimating the Hurst exponent.
- Analyzed the series dynamics to illustrate how the Hurst exponent varies across different frequencies.
- Proposed an inefficiency index combining deviation from a random walk ($H = 0.5$) with multifractal spectrum width.
- Future work will extend this approach to portfolios of clustered multifractal pairs and refine the corresponding trading strategy.

Appendix

Cumulative log returns S&P 500 and Russell 2000



Figure: Comparison of the log returns between S&P 500 (red) and Russell 2000 (blue)

RMD 8192 points on returns

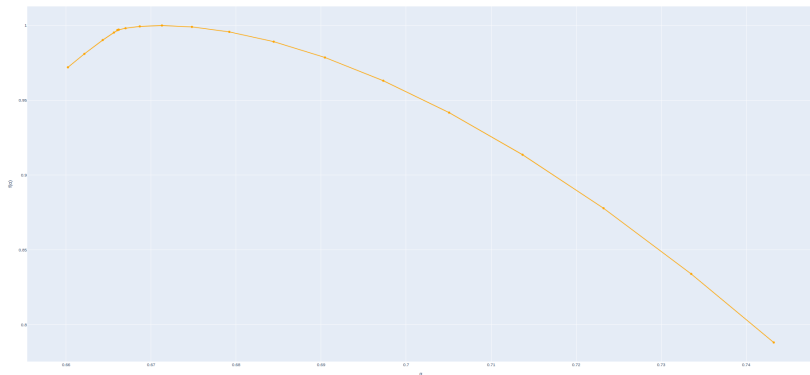


Figure: MF-DFA on simulated fractals, 8192 points on returns, peak is 0.673

RMD 65536 points on prices

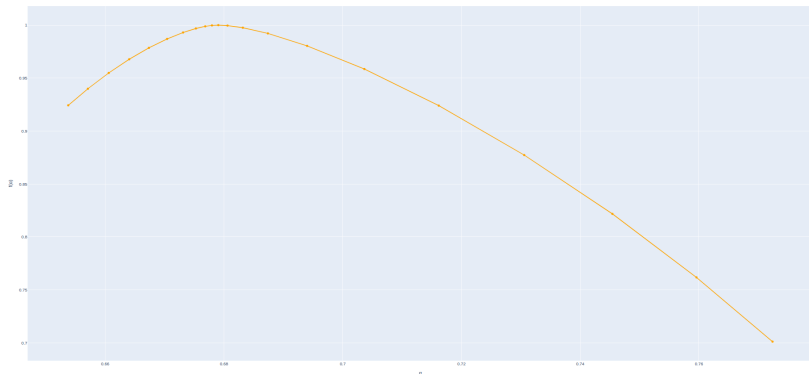


Figure: MF-DFA on simulated fractals, 65536 points on prices, peak is 0.678

RMD 65536 points on returns

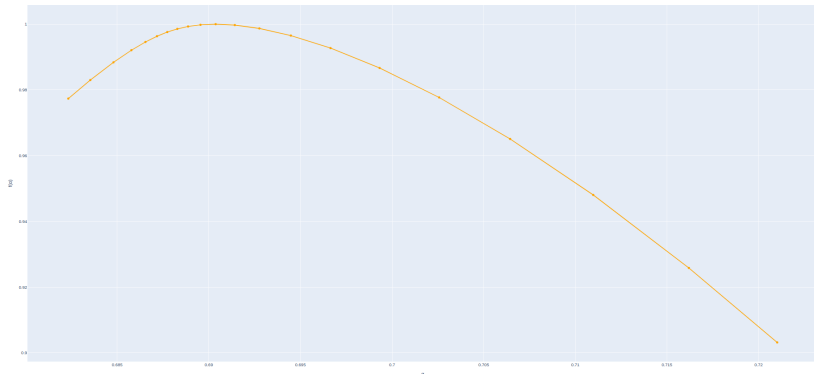


Figure: MF-DFA on simulated fractals, 65536 points on returns, peak is 0.691

Performance Metrics: Window size 120 days

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap120	9.39	19.34	0.49	−56.26
Long Only SP500	8.2	17.88	0.46	−56.78
Long Only Russell	7.37	21.73	0.34	−59.89
50/50 Portfolio	7.94	19.13	0.42	−57.99

Table: 1991-11-15 to 2025-02-28 — Window size 120 days

Performance Metrics: Window size 252 days

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap252	9.12	19.35	0.47	−57.12
Long Only SP500	8.2	17.88	0.46	−56.78
Long Only Russell	7.37	21.73	0.34	−59.89
50/50 Portfolio	7.94	19.13	0.42	−57.99

Table: 1991-11-15 to 2025-02-28 — Window size 252 days

Performance Metrics: Window size 504 days

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap504	8.79	19.39	0.45	−56.68
Long Only SP500	8.2	17.88	0.46	−56.78
Long Only Russell	7.37	21.73	0.34	−59.89
50/50 Portfolio	7.94	19.13	0.42	−57.99

Table: 1991-11-15 to 2025-02-28 — Window size 504 days

Performance Metrics: Window size 1260 days

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap1260	9.02	19.47	0.46	−56.27
Long Only SP500	8.23	18	0.46	−56.78
Long Only Russell	7.47	21.91	0.34	−59.89
50/50 Portfolio	8.01	19.28	0.42	−57.99

Table: 1992-08-22 to 2025-02-28 — Window size 1260 days

Performance Metrics: Window size 2520 days

Strategy	Annualized Return	Annualized Volatility	Sharpe	Max Drawdown
ModifOverlap2520	7.3	20.81	0.35	−55.98
Long Only SP500	6.85	19.04	0.36	−56.78
Long Only Russell	6.15	23.42	0.26	−59.89
50/50 Portfolio	6.67	20.56	0.33	−57.99

Table: 1997-06-18 to 2025-02-28 — Window size 2520 days

Problem

Show that the multifractal detrended fluctuation function

$$F_q(s) = \left[\frac{1}{2N_s} \sum_{v=1}^{2N_s} (F_v^2(s))^{q/2} \right]^{1/q}$$

has the finite limit

$$F_0(s) = \exp \left[\frac{1}{2N_s} \sum_{v=1}^{2N_s} \ln F_v(s) \right]$$

as $q \rightarrow 0$.

We will use a logarithm trick followed by L'Hospital's rule.

Step 1 — Take the natural logarithm

Define

$$S(q) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} \exp\left(\frac{q}{2} \ln F_v^2(s)\right), \quad \Rightarrow \quad \boxed{\ln F_q(s) = \frac{\ln S(q)}{q}}$$

- $S(0) = 1 \Rightarrow \ln S(0) = 0$.
- Both numerator and denominator $\rightarrow 0$ as $q \rightarrow 0$: an indeterminate form $\frac{0}{0}$.

Step 2 — Apply L'Hospital's rule

$$\lim_{q \rightarrow 0} \frac{\ln S(q)}{q} = \left. \frac{d}{dq} \ln S(q) \right|_{q=0} = \left. \frac{S'(q)}{S(q)} \right|_{q=0}.$$

Differentiate $S(q)$:

$$S'(q) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} \frac{\ln F_v^2(s)}{2} \exp\left(\frac{q}{2} \ln F_v^2(s)\right).$$

$$\Rightarrow S'(0) = \frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln F_v^2(s).$$

Hence

$$\lim_{q \rightarrow 0} \ln F_q(s) = \frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln F_v^2(s).$$

Step 3 — Exponentiate

$$F_0(s) = \exp\left[\frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln F_v^2(s)\right] = \exp\left[\frac{1}{2N_s} \sum_{v=1}^{2N_s} \ln F_v(s)\right].$$

Interpretation

$F_0(s)$ equals the *geometric mean* of all segment fluctuations $F_v(s)$.

Multifractal Setup and Partition Function

Step 1: Divide the Signal

We analyze a time series $X(t)$ by dividing it into non-overlapping segments (boxes) of size s . Let μ_i be the local fluctuation in the i -th box (e.g., absolute increment):

$$\mu_i = |X(t_{i+1}) - X(t_i)|$$

Step 2: Partition Function

The q -th order partition function is:

$$Z(q, s) = \sum_i \mu_i^q \sim s^{\tau(q)}$$

It measures how moments of the distribution scale with box size s .

Link to $\tau(q) = qh(q) - 1$

Step 3: Scaling of Local Fluctuations

Assume that the fluctuation $\mu_i \sim s^{h(q)}$, and the number of boxes is:

$$N(s) \sim \frac{1}{s}$$

Step 4: Combine the Scaling Laws

$$Z(q, s) \sim N(s) \cdot s^{qh(q)} = s^{-1} \cdot s^{qh(q)} = s^{qh(q)-1}$$

Conclusion

So the scaling exponent is:

$$\tau(q) = q \cdot h(q) - 1$$

Step 2 — Deriving the Multifractal Spectrum $f(\alpha)$

Idea: Local singularity scaling

Assume that in each box of size s , the measure scales as:

$$\mu_i \sim s^\alpha$$

The number of such boxes behaves like:

$$N_s(\alpha) \sim s^{-f(\alpha)}$$

Total contribution to the partition function

The partition function is approximated by:

$$Z(q, s) \sim \sum_{\alpha} N_s(\alpha) \cdot (s^\alpha)^q \sim \sum_{\alpha} s^{-f(\alpha)} \cdot s^{q\alpha} = \sum_{\alpha} s^{q\alpha - f(\alpha)}$$

Dominant contribution for small s

As $s \rightarrow 0$, the sum is dominated by the exponent that decays slowest:

$$\tau(q) = \inf_{\alpha} (q\alpha - f(\alpha)) \Rightarrow \boxed{f(\alpha) = q \cdot \alpha(q) - \tau(q)}$$

Why $N_s(\alpha) \sim s^{-f(\alpha)}$?

Fractal Dimension of a Singularity Set

Let α be a Hölder exponent. The set of points in the signal that share this singularity forms a fractal subset of dimension $f(\alpha)$.

Box-Counting Interpretation

To cover a fractal set of dimension $f(\alpha)$ using boxes of size s , we need:

$$N_s(\alpha) \sim s^{-f(\alpha)}$$

This is a standard result from fractal geometry (box-counting dimension).

Conclusion

$f(\alpha)$ gives the "size" of the set of points with singularity α . This assumption links the geometric structure of the signal to the scaling of the partition function.