Beyond Random Walks: Revealing the Fractal Memory of Financial Markets

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Efficient Market Hypothesis

- **Definition:** Prices fully reflect all available information at any time.
- Therefore, no trading strategy based on information can consistently achieve excess returns.
- **Challenges & Critiques:**
  - Empirical anomalies (momentum, mean-reversion, bubbles).
  - Long-range dependence and multifractality in returns.

Benoît Mandelbrot and the Geometry of Nature

- **Benoît Mandelbrot** (1924–2010) was a Franco-American mathematician known for developing the concept of **fractals**.
- He coined the term **"fractal"** in 1975 to describe shapes that exhibit self-similarity at different scales.
- Mandelbrot challenged traditional Euclidean geometry by modeling complex, irregular forms found in nature (e.g., coastlines, clouds, mountains, market fluctuations).
- His most famous contribution is the **Mandelbrot Set**, a mathematical object that reveals infinite complexity and beauty under zoom.

Zoom in the Mandelbrot Set

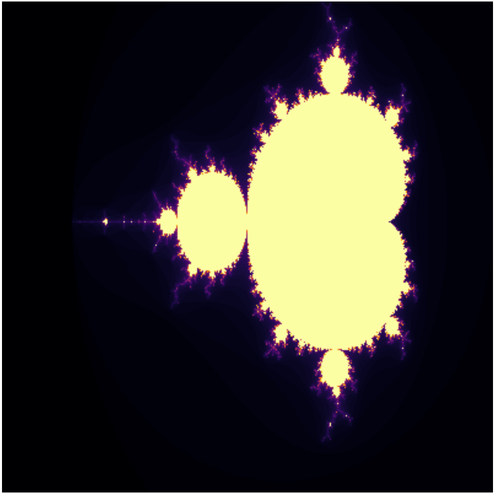


Figure: Zoom animation of the Mandelbrot set (30 frames at 10 fps).

Zoom in the Mandelbrot Set

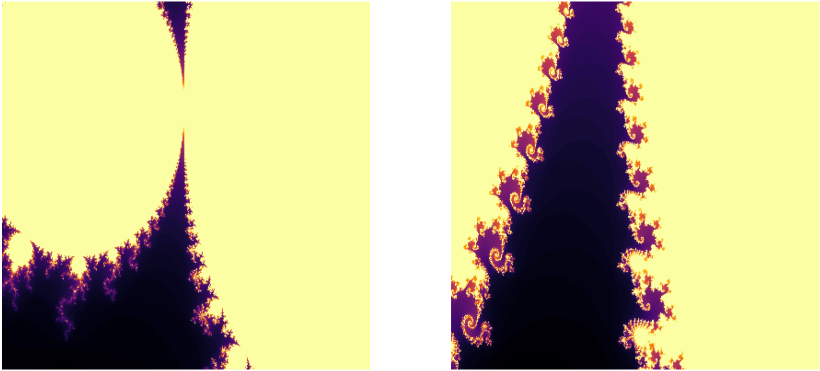


Figure: Zoom in the Mandelbrot Set

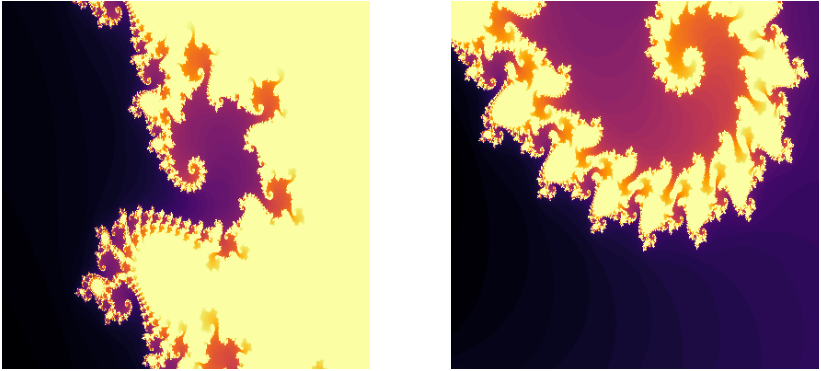


Figure: Zoom in the Mandelbrot Set

Introduction: Fractional Brownian Motion (FBM)

- **Mean at scale s:**
$$\mathbb{E}[X_H(s)] = 0.$$
- **Variance at scale s:**
$$\mathrm{Var}[X_H(s)] = \sigma^2 s^{2H}.$$
where $H \in (0, 1)$ is the Hurst exponent and σ^2 is a scale parameter.
- **Self-Similarity:** FBM is self-similar; that is, for any scaling factor $c > 0$,
$$X_H(ct) \stackrel{d}{=} c^H X_H(t),$$
which means the process scales with exponent H.
- Note that when $H = 0.5$, FBM reduces to standard Brownian motion.

Modeling Long Memory: Simulation of fBm

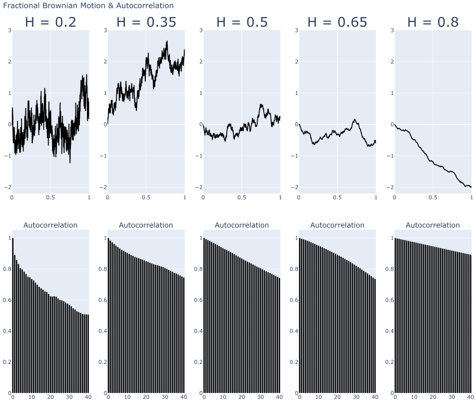


Figure: Simulated fractional Brownian motion for different H values, [0.2, 0.35, 0.5, 0.65, 0.8]

Hurst Exponent Estimation Methods

Overview

There are many ways to estimate the Hurst exponent from a financial time series. In our study, we focus on capturing long-term trends rather than high-frequency fluctuations, since long-term trends reveal persistent behavior and memory effects, while high-frequency data are often dominated by noise.

Methods and Their Authors:

- **Rescaled Ranged (R/S) Analysis:** Originally developed by Harold Edwin Hurst, and later refined by Mandelbrot & Wallis.
- **Modified R/S (M-R/S) Analysis:** Introduced by Lo (1991) and further discussed by Andrews (1991) to correct for short-term autocorrelation.

Empirical Analysis: Data and Results

- Description of the data: indices (S&P 500, Russell 2000, FTSE 100, NIKKEI 225, DAX), the data is monthly and from september 10th, 1987 to february 28th, 2025.
- Data processing: logarithmic returns and stationarity tests.

Estimation Results

Ticker	R/S	Hurst Exponent	Modified Hurst Exponent	Critical Value	Long Memory
S&P 500	30.166	0.558	0.501	1.007	False
Russell 2000	51.373	0.645	0.588	1.714	True
FTSE 100	40.236	0.605	0.548	1.341	False
Nikkei 225	22.234	0.508	0.508	1.048	False
DAX	26.985	0.540	0.540	1.278	False

Table: Results for R/S, Hurst exponent, modified Hurst exponent, critical value at 10% (1.620), and rejection of the null hypothesis of short memory. The Hurst exponent can be equal for the R/S and modified R/S methods in the case where the autocorrelation coefficients are less than zero. In this case, we set $q = 0$ and therefore the R/S and modified R/S share the same formula.

What Is Multifractality?

- **Beyond a single exponent:** Unlike a simple fractal (monofractal) characterized by one scaling exponent, a *multifractal* process exhibits a *spectrum* of local scaling behaviors.
- **Local regularity:** At each time t, the trajectory has its own Hölder exponent $\alpha(t)$, measuring how "smooth" or "rough" the series is around t.
- **From theory to practice:** MF-DFA (Kantelhardt 2002) provides a robust framework to estimate generalized exponents $h(q)$ and derive $f(\alpha)$ from real financial data.

Graphical view of the MF-DFA steps

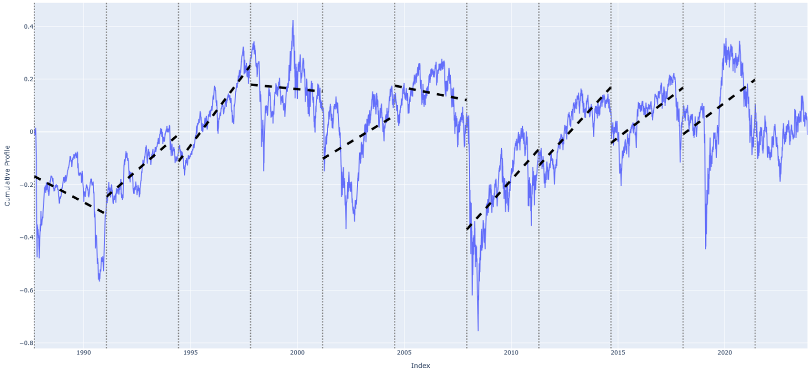


Figure: Simulation of the MF-DFA steps with the Russell 2000. Cumulative profile on the y-axis, Date from 1987 to 2025 on the x-axis

Step 3: Calculating the Fluctuation Function

For each segment ν, compute the variance:

$$F^2(\nu, s) = \frac{1}{s}\sum_{i=1}^{s}\Big\{Y\big[(\nu-1)s+i\big] - y_\nu(i)\Big\}^2.$$

Then, average over all $2N_s$ segments for a given order q:

$$F_q(s) = \left\{\frac{1}{2N_s}\sum_{\nu=1}^{2N_s}\big[F^2(\nu, s)\big]^{q/2}\right\}^{1/q}.$$

$$\text{Scaling:} \quad F_q(s) \propto s^{h(q)},$$

where $h(q)$ is the generalized Hurst exponent of order q.

Generalized Hurst Exponents: Russell 2000

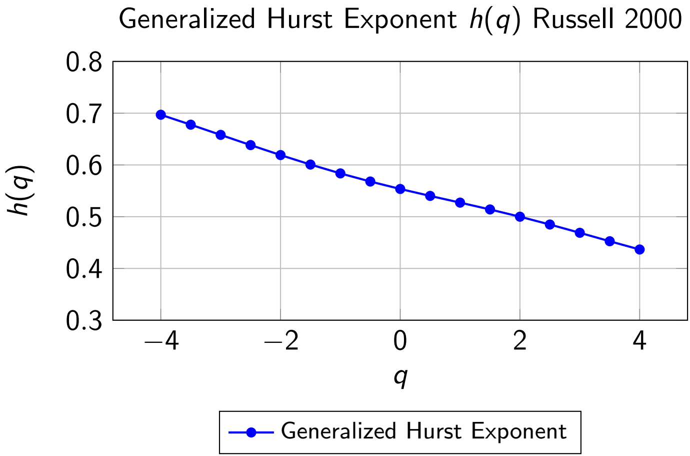


Figure 1: *The generalized Hurst exponent $h(q)$ for Russell 2000 is plotted as a function of q. Values of q range from -4 to 4 representing markets flucutations, and scales used in the computation are logarithmically spaced between 10 and 500.*

Generalized Hurst Exponents comparison

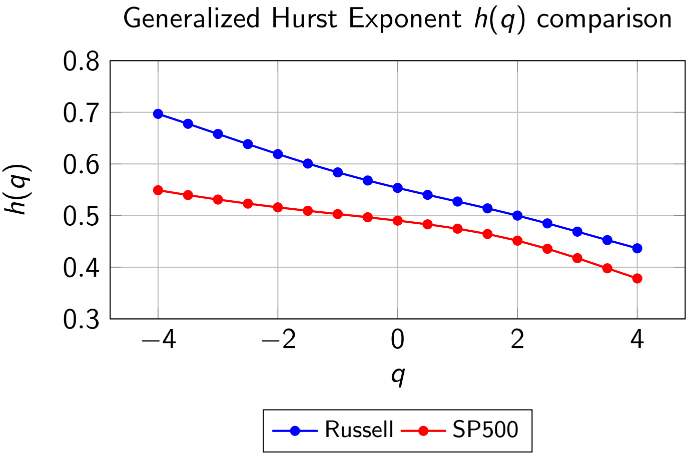


Figure 1: *The generalized Hurst exponent $h(q)$ is plotted as a function of q. Values of q range from -4 to 4 representing markets flucutations, and scales used in the computation are logarithmically spaced between 10 and 500.*

Hölder Exponent $\alpha(q)$

Definition

The Hölder exponent $\alpha(q)$ quantifies the **local regularity** or **singularity** of a signal.

Quick definition

$$\alpha(q) = \frac{d\,\tau(q)}{dq}, \qquad \tau(q) = q\,h(q) - 1$$

It is the slope of the scaling function $\tau(q)$; larger slopes mean smoother behaviour, smaller slopes mean sharper changes. Hence,

$$\alpha(q) \;=\; h(q) \;+\; q\,h'(q).$$

Interpretation

- **Low** α: sharp variations, spikes, or singular behavior (irregular regions).
- **High** α: smooth, regular zones with low local variability.

Multifractal Spectrum $f(\alpha)$

Definition

Once $\alpha(q)$ is computed, the multifractal spectrum reads:

$$f(\alpha) = q\left[\alpha(q) - h(q)\right] + 1.$$

It gives the fractal dimension of the set of points sharing the same local regularity α.

$f(\alpha)$ captures multifractality

- For a **monofractal** signal, only one Hölder exponent exists; the spectrum collapses to a single point (a narrow peak).
- A **broad, concave spectrum** evidences a whole family of exponents: smooth regions (α large) coexist with highly singular ones (α small).

Sources of Multifractality (Kantelhardt 2002)

$$M(q) \;\propto\; \underbrace{f_{\mathrm{tail}}(q)}_{\text{Type I: heavy-tailed value distribution}} \quad \textit{and} \quad \underbrace{f_{\mathrm{corr}}(q)}_{\text{Type II: long-range temporal correlations}}$$

1. **Type I (distribution-driven)** Arise when the return distribution has fat tails.
2. **Type II (correlation-driven)** Stem from persistent or anti-persistent patterns across scales.

Decomposition via shuffling and surrogate:

- Shuffle the series to destroy long-range correlations, leaving only the heavy-tail effect.
- Generate surrogate series (e.g. AAFT) to preserve the original autocorrelation structure while Gaussianizing the marginals, isolating genuine long-range dependence.

Multifractal Spectrum $f(\alpha)$ — Russell 2000

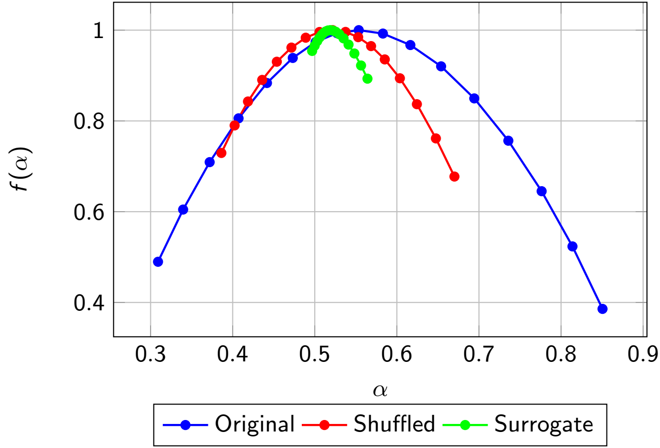


Figure: The multifractal spectrum $f(\alpha)$ characterizes the distribution of singularities in the Russell 2000 return series. A wider curve implies stronger multifractality (more heterogeneity in local scaling exponents).

Multifractal Spectrum $f(\alpha)$ — S&P 500

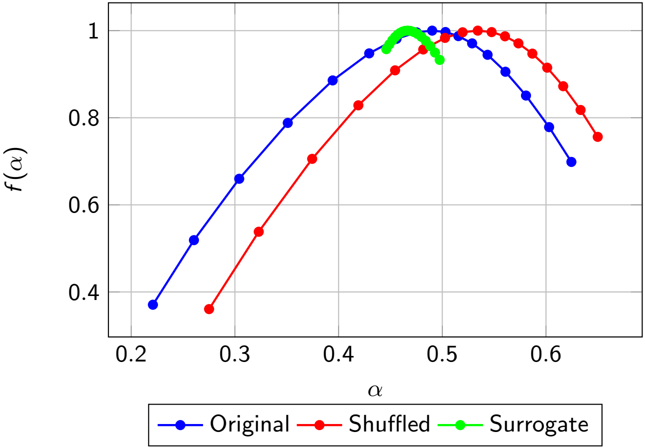


Figure: The multifractal spectrum $f(\alpha)$ for the S&P 500 return series. The width of the curve reflects the range of local singularities broader spectra indicate stronger multifractality.

Proposition of an Inefficiency Index

We propose combining two key structural components to capture market inefficiency:

1. **Fractal Difference** The width of the multifractal spectrum that carries long term correlation:

$$\Delta\alpha_{\text{surrogate}} = \alpha_{\text{max surrogate}} - \alpha_{\text{min surrogate}}$$

where α is the singularity exponent from MF-DFA. Quantifies the multifractality linked to the long term correlation.

2. **Deviation of the Rolling Hurst** In an efficient market:

$$H = 0.5$$

So the deviation is quantified as:

$$|H_{\text{rolling}} - 0.5|$$

Inefficiency Index

$$I = \Delta\alpha_{\text{surrogate}} \times |H_{\text{rolling}} - 0.5|$$

where $\Delta\alpha_{\text{surrogate}}$ is the multifractality due to long term correlation.

Inefficiency Index S&P500

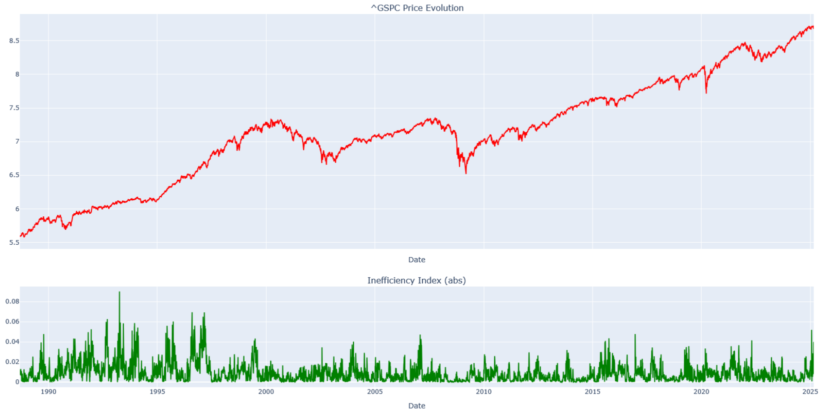


Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the S&P 500

Inefficiency Index FTSE

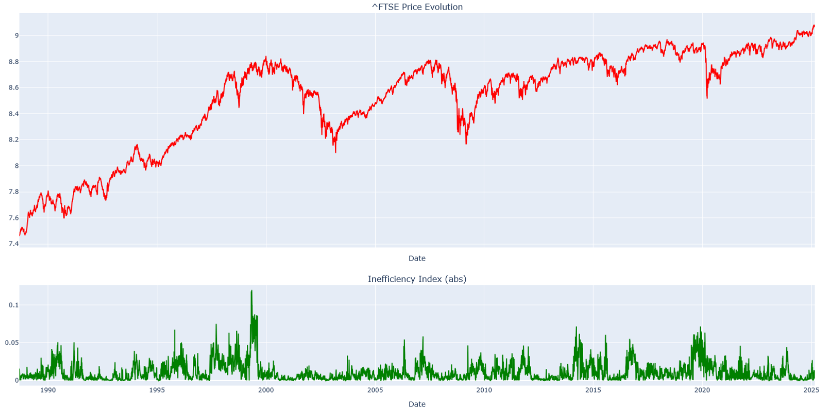


Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the FTSE

Inefficiency Index Nikkei

Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the Nikkei

Inefficiency Index SSEC

Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the SSEC

Inefficiency Index BTC

Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the BTC

Conclusions and Perspectives

- Explored several methods for estimating the Hurst exponent.
- Analyzed the series dynamics to illustrate how the Hurst exponent varies across different frequencies.
- Proposed an inefficiency index combining deviation from a random walk ($H = 0.5$) and the multifractality coming from long term correlations.
- Future work will focus on different Hurst estimation methods, frequencies and rolling length.

Appendix

Inefficiency Index rolling 4 years S&P 500

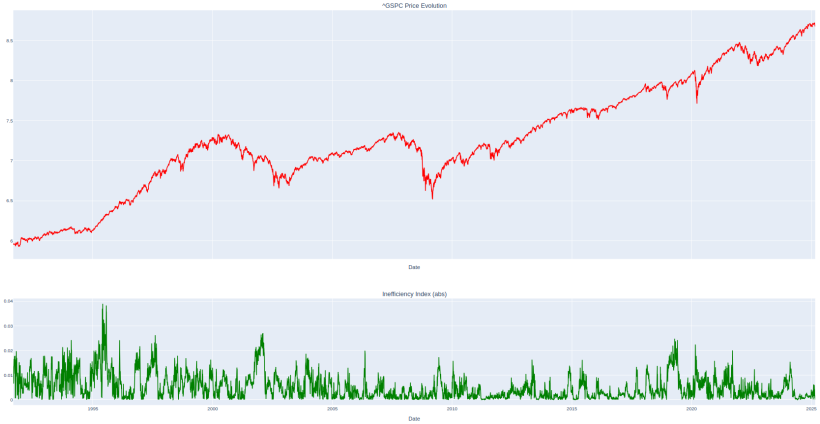


Figure: Log Prices (upper graph) and Inefficiency Index rolling 4 years (lower graph) of the S&P 500

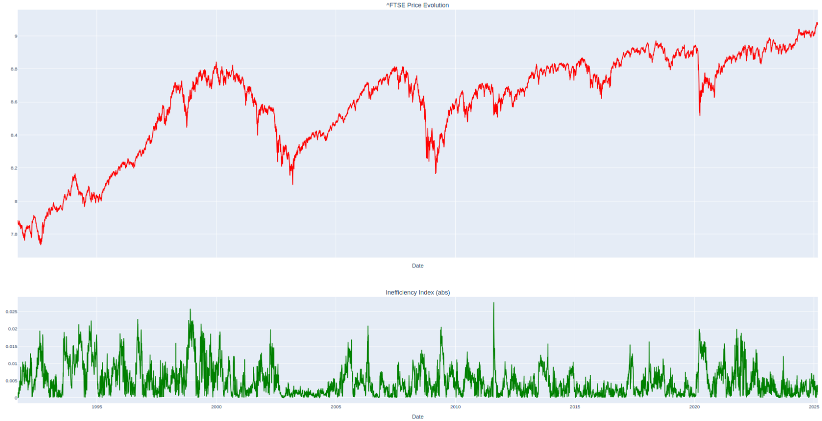


Figure: Log Prices (upper graph) and Inefficiency Index rolling 4 years (lower graph) of the FTSE

Inefficiency Index rolling 4 years CAC

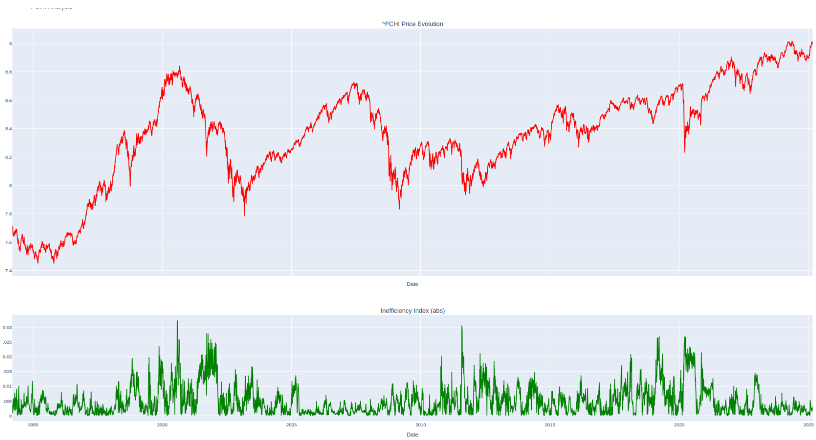


Figure: Log Prices (upper graph) and Inefficiency Index rolling 4 years (lower graph) of the CAC

Inefficiency Index rolling 4 years SSEC

Figure: Log Prices (upper graph) and Inefficiency Index (lower graph) of the SSEC

Figure: Log Prices (upper graph) and Inefficiency Index rolling 4 years (lower graph) of the BTC

Multifractal Spectrum Width ($\Delta\alpha$)

Index	$\Delta\alpha_{\text{orig}}$	$\Delta\alpha_{\text{shuffle}}$	$\Delta\alpha_{\text{surrogate}}$
Russell 2000	0.397	0.210	0.044
S&P 500	0.264	0.232	0.035
DAX	0.289	0.198	0.013
Nikkei 225	0.195	0.121	0.052
FTSE 100	0.204	0.140	0.049

Pointwise Hölder Exponent Condition

Condition for α to be a local Hölder exponent

There exist constants $C > 0$ and $\delta > 0$ such that, for all t with

$$|t - t_0| < \delta, \qquad |f(t) - f(t_0)| \leq C\,|t - t_0|^{\alpha}.$$

- If this inequality holds for a given α, then f is at least C^{α} at t_0.
- The pointwise Hölder exponent $h_f(t_0)$ is the *supremum* of all such α.

Illustration Holder Exponent

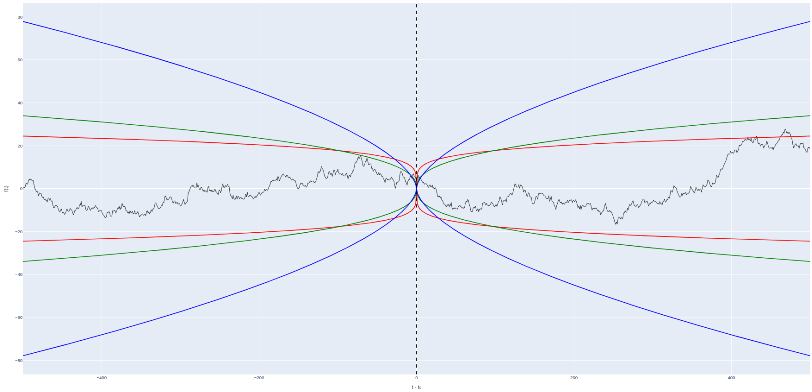


Figure: Local regularity for different Holder exponent, 0.6 (blue), 0.4 (green) and 0.2 (red)

Comparison: FBM vs MBM

FBM (Fractional Brownian Motion)

- Fixed Hurst exponent $H \in (0, 1)$
- Variance: $\mathrm{Var}[X_H(s)] = \sigma^2 s^{2H}$
- Covariance: $\frac{\sigma^2}{2}[t^{2H} + s^{2H} - |t - s|^{2H}]$
- Gaussian, stationary increments; correlated if $H \neq 0.5$
- Monofractal spectrum

MBM (Multifractal Brownian Motion)

- Time-varying Hurst exponent $H(t)$ or scale-dependent
- Local variance: $\approx \sigma^2 s^{2H(t)}$
- Rich multiscale covariance structure
- Non-Gaussian increments; heavy-tailed distributions
- Wide multifractal spectrum $f(\alpha)$

Hurst Exponent Estimation: R/S Analysis

The Hurst exponent H is estimated using the classical Rescaled Range (R/S) analysis. The procedure is as follows:
Then, construct the cumulative series (profile):

$$Y(t) = \sum_{i=1}^{t} (x_i - \bar{x}), \quad t = 1, 2, \ldots, N.$$

For a segment of length T, define:

$$R(T) = \max_{1 \le t \le T} Y(t) - \min_{1 \le t \le T} Y(t)$$

and the standard deviation:

$$S(T) = \sqrt{\frac{1}{T} \sum_{i=1}^{T} (x_i - \bar{x})^2}.$$

The rescaled range is then:

$$Q_T = \frac{R(T)}{S(T)}$$

Scaling Law: Empirically, one finds (Mandelbrot and Wallis (1969b)):

$$\tilde{Q}_T \sim T^H$$

Hurst Exponent Estimation: Modified R/S Analysis

The Modified R/S Analysis, introduced by Lo (1991), refines the classical R/S method by adjusting for short-term autocorrelation, by adding the autocovariance terms.

The Modified R/S statistic is given by:

$$\tilde{Q}_T = \frac{R(T)}{\hat{\sigma}_T},$$

with the adjusted standard deviation:

$$\hat{\sigma}_T = \sqrt{\frac{1}{T}\sum_{j=1}^{T}(x_j - \bar{x})^2 + \frac{2}{T}\sum_{j=1}^{T} w_j(q) \sum_{i=j+1}^{T}(x_i - \bar{x})(x_{i-j} - \bar{x})},$$

where the weights, as introduced by Lo (1991), are defined as:

$$w_j(q) = 1 - \frac{j}{q+1}, \quad j = 1, 2, \ldots, q.$$

$$q = \lfloor k_T \rfloor, \quad k_T = \left(\frac{3T}{2}\right)^{1/3} \left(\frac{2\rho}{1-\rho^2}\right)^{2/3},$$ ρ is the first-order autocorrelation coefficient.

Significance Testing of the Modified R/S Statistic

Unlike the classical R/S analysis, the modified R/S statistic has a known limiting distribution. In particular, the statistic

$$V = \tilde{Q}_T/\sqrt{T}$$

converges to the range of a Brownian bridge on the unit interval. This property enables us to perform a formal statistical test of the null hypothesis of short memory versus the alternative hypothesis of long memory by comparing V with the critical values provided by Lo (1991).

MF-DFA steps

Given a time series $\{x_k\}_{k=1}^{N}$, first compute the mean:

$$\bar{x} = \frac{1}{N} \sum_{k=1}^{N} x_k$$

Then, construct the cumulative profile:

$$Y(i) = \sum_{k=1}^{i} (x_k - \bar{x}) \, , \quad i = 1, \ldots, N.$$

MF-DFA steps

- Divide the profile $Y(i)$ into $N_s = \lfloor N/s \rfloor$ non-overlapping segments of equal length s.
- To use all data, repeat the procedure starting from the opposite end.

For each segment ν, fit a polynomial $y_\nu(i)$ (usually of order m) to the data in the segment.

$$y_\nu(i) = \sum_{j=0}^{m} a_j^{(\nu)}\, i^j, \quad i = 1, \dots, s.$$

Step 3: Calculating the Fluctuation Function

For each segment ν, compute the variance:

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} \{ Y[(\nu - 1)s + i] - y_\nu(i) \}^2 .$$

Then, average over all $2N_s$ segments for a given order q:

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} \left[F^2(\nu, s) \right]^{q/2} \right\}^{1/q} ,$$

with the special case for $q = 0$ defined by a logarithmic averaging:

$$F_0(s) = \exp \left\{ \frac{1}{4N_s} \sum_{\nu=1}^{2N_s} \ln \left[F^2(\nu, s) \right] \right\} .$$

MF-DFA steps

Once the fluctuation function $F_q(s)$ is computed for different scales s, a scaling relation is established:

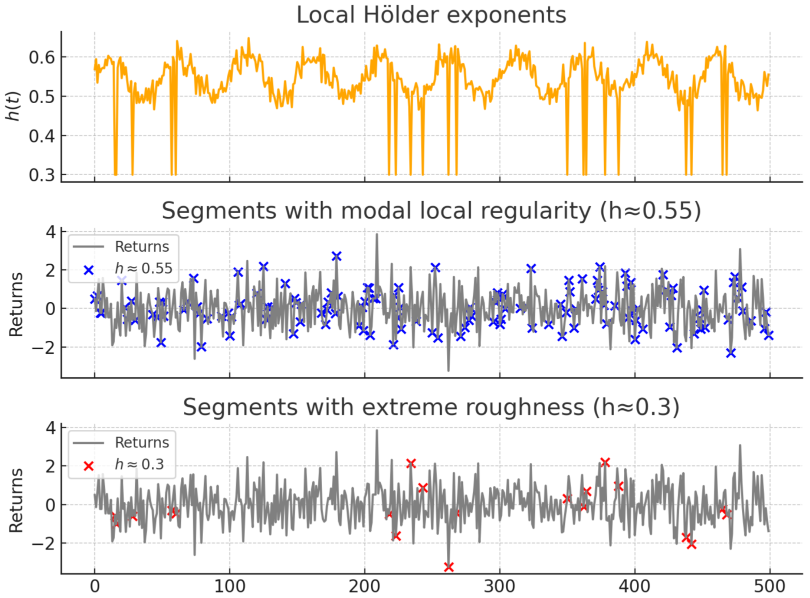
$$F_q(s) \sim s^{h(q)}.$$

Taking the logarithm of both sides yields:

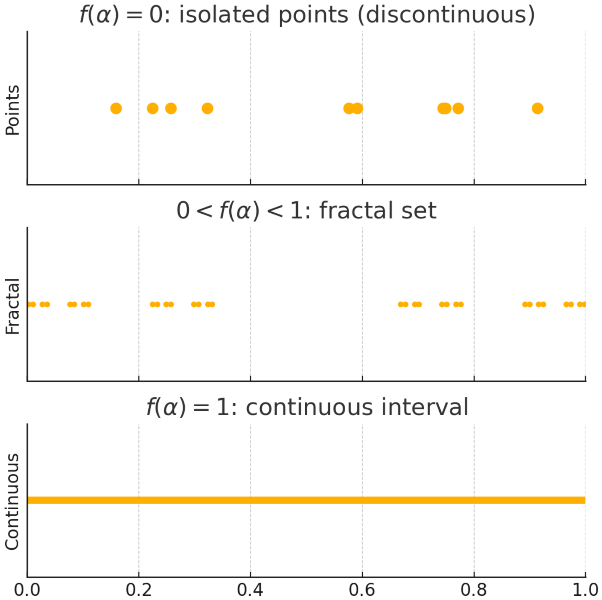
$$\log F_q(s) \sim h(q) \log s + \text{constant}.$$

By plotting $\log F_q(s)$ against $\log s$, a linear relationship is observed, with the slope corresponding to the generalized Hurst exponent $h(q)$. This approach allows us to characterize the scaling behavior of the fluctuations for various orders q, providing insight into the multifractal structure of the signal.

Holder Exponent



Fractal Dimension



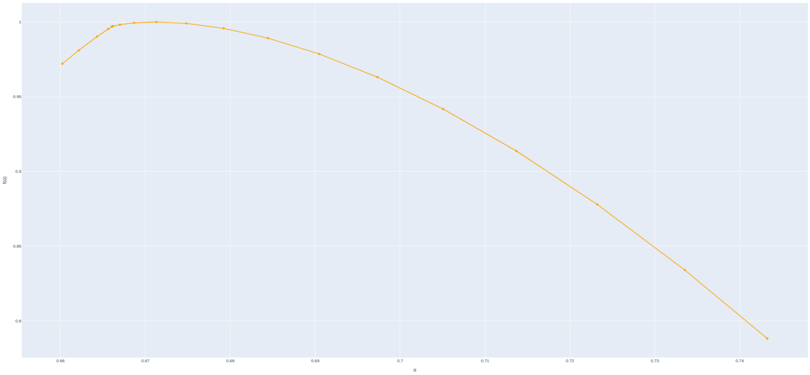


Figure: MF-DFA on simulated fractals, 8192 points on returns, peak is 0.673

RMD 65536 points on prices

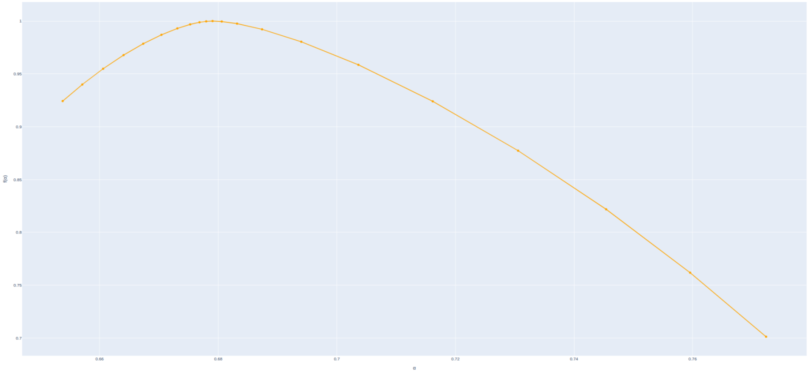


Figure: MF-DFA on simulated fractals, 65536 points on prices, peak is 0.678

RMD 65536 points on returns

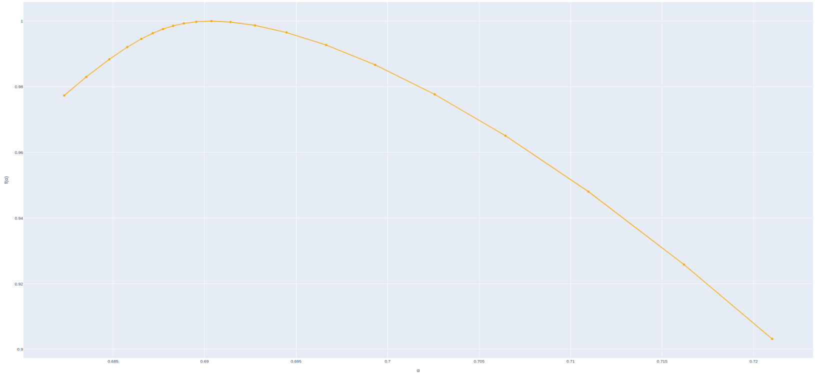


Figure: MF-DFA on simulated fractals, 65536 points on returns, peak is 0.691

Goal

Problem

Show that the multifractal detrended fluctuation function

$$F_q(s) = \left[\frac{1}{2N_s} \sum_{v=1}^{2N_s} \left(F_v^2(s) \right)^{q/2} \right]^{1/q}$$

has the finite limit

$$\boxed{F_0(s) = \exp\left[\frac{1}{2N_s} \sum_{v=1}^{2N_s} \ln F_v(s) \right]}$$

as $q \to 0$.

We will use a logarithm trick followed by L'Hospital's rule.

Step 1 — Take the natural logarithm

Define

$$S(q) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} \exp\!\Big(\tfrac{q}{2} \, \ln F_v^2(s) \Big), \qquad \Longrightarrow \qquad \boxed{\ln F_q(s) = \frac{\ln S(q)}{q}}$$

- $S(0) = 1 \;\Rightarrow\; \ln S(0) = 0.$
- Both numerator and denominator $\to 0$ as $q \to 0$: an indeterminate form $\frac{0}{0}$.

Step 2 — Apply L'Hospital's rule

$$\lim_{q\to 0} \frac{\ln S(q)}{q} = \frac{\mathrm{d}}{\mathrm{d}q} \ln S(q)\bigg|_{q=0} = \frac{S'(q)}{S(q)}\bigg|_{q=0}.$$

Differentiate $S(q)$:

$$S'(q) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} \frac{\ln F_v^2(s)}{2} \exp\Big(\tfrac{q}{2} \ln F_v^2(s)\Big).$$

$$\Longrightarrow \quad S'(0) = \frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln F_v^2(s).$$

Hence

$$\boxed{\lim_{q\to 0} \ln F_q(s) = \frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln F_v^2(s)}.$$

Step 3 — Exponentiate

$$F_0(s) = \exp\left[\frac{1}{4N_s}\sum_{v=1}^{2N_s} \ln F_v^2(s)\right] = \exp\left[\frac{1}{2N_s}\sum_{v=1}^{2N_s} \ln F_v(s)\right].$$

Interpretation

$F_0(s)$ equals the *geometric mean* of all segment fluctuations $F_v(s)$.

Multifractal Setup and Partition Function

Step 1: Divide the Signal

We analyze a time series $X(t)$ by dividing it into non-overlapping segments (boxes) of size s. Let μ_i be the local fluctuation in the i-th box (e.g., absolute increment):

$$\mu_i = |X(t_{i+1}) - X(t_i)|$$

Step 2: Partition Function

The q-th order partition function is:

$$Z(q, s) = \sum_i \mu_i^q \sim s^{\tau(q)}$$

It measures how moments of the distribution scale with box size s.

Link to $\tau(q) = qh(q) - 1$

Step 3: Scaling of Local Fluctuations

Assume that the fluctuation $\mu_i \sim s^{h(q)}$, and the number of boxes is:

$$N(s) \sim \frac{1}{s}$$

Step 4: Combine the Scaling Laws

$$Z(q,s) \sim N(s) \cdot s^{qh(q)} = s^{-1} \cdot s^{qh(q)} = s^{qh(q)-1}$$

Conclusion

So the scaling exponent is:

$$\boxed{\tau(q) = q \cdot h(q) - 1}$$

Step 2 — Deriving the Multifractal Spectrum $f(\alpha)$

Idea: Local singularity scaling

Assume that in each box of size s, the measure scales as:

$$\mu_i \sim s^{\alpha}$$

The number of such boxes behaves like:

$$N_s(\alpha) \sim s^{-f(\alpha)}$$

Total contribution to the partition function

The partition function is approximated by:

$$Z(q,s) \sim \sum_{\alpha} N_s(\alpha) \cdot (s^{\alpha})^q \sim \sum_{\alpha} s^{-f(\alpha)} \cdot s^{q\alpha} = \sum_{\alpha} s^{q\alpha - f(\alpha)}$$

Dominant contribution for small s

As $s \to 0$, the sum is dominated by the exponent that decays slowest:

$$\tau(q) = \inf_{\alpha} \left(q\alpha - f(\alpha) \right) \Rightarrow \boxed{f(\alpha) = q \cdot \alpha(q) - \tau(q)}$$

Why $N_s(\alpha) \sim s^{-f(\alpha)}$?

Fractal Dimension of a Singularity Set

Let α be a Hölder exponent. The set of points in the signal that share this singularity forms a fractal subset of dimension $f(\alpha)$.

Box-Counting Interpretation

To cover a fractal set of dimension $f(\alpha)$ using boxes of size s, we need:

$$N_s(\alpha) \sim s^{-f(\alpha)}$$

This is a standard result from fractal geometry (box-counting dimension).

Conclusion

$f(\alpha)$ gives the "size" of the set of points with singularity α. This assumption links the geometric structure of the signal to the scaling of the partition function.