Hurst exponents and delampertized fractional Brownian motions

Matthieu Garcin

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Abstract

The inverse Lamperti transform of a fractional Brownian motion is a stationary process. We determine the empirical Hurst exponent of such a composite process with the help of a regression of the log absolute moments of its increments, at various scales, on the corresponding log scales. This perceived Hurst exponent underestimates the Hurst exponent of the underlying fractional Brownian motion. We thus encounter some time series having a perceived Hurst exponent lower than 1/2, but an underlying Hurst exponent higher than 1/2. This paves the way for short- and medium-term forecasting. Indeed, in such series, mean reversion predominates at high scales, whereas persistence is overriding at lower scales. We propose a way to characterize the Hurst horizon, namely a limit scale between these opposite behaviours. We show that the delampertized fractional Brownian motion, which mixes persistence and mean reversion, is relevant for financial time series, in particular for high-frequency foreign exchange rates. In our sample, the empirical Hurst horizon is always above 1 hour and 23 minutes.

Keywords – fractional Brownian motion, Hurst exponent, Lamperti transform, Ornstein-Uhlenbeck process, foreign exchange rates

1 Introduction

The fractional Brownian motion (fBm) was introduced by Mandelbrot and van Ness [25]. With a Hurst exponent $H \in (0,1)$ and a volatility parameter σ, the fBm X is defined as the only zero-mean Gaussian process with zero at the origin and with the following covariance function, for all $(s,t) \in \mathbb{R}^2$:

$$\mathbb{E}\{X_t X_s\} = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{1}$$

By definition, X is H-self-similar: $\forall t, \lambda > 0$, X_t and $\lambda^{-H} X_{\lambda t}$ have the same probability distribution. An important fact is that any H-self-similar process is derived from a stationary process S, thanks to the Lamperti transform of S, noted $\mathcal{L}_H S$. This transformation is invertible. Therefore, the inverse Lamperti transform of the fBm X or of any H-self-similar process, noted $\mathcal{L}_H^{-1} X$, is a stationary process [21, 14]. For $t \in \mathbb{R}$, the Lamperti transform of S is defined by $(\mathcal{L}_H S)_t = t^H S_{\ln(t)}$ and the inverse Lamperti transform of X by $(\mathcal{L}_H^{-1} X)_t = e^{-Ht} X_{exp(t)}$. The Lamperti transform is convenient, as it allows using well-established tools from the framework of stationary processes for

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problems related to self-similar processes. For instance, tests of stationarity can be used to define self-similarity tests [22].

The delampertized fBm, $\mathcal{L}_H^{-1}X$, is thus a stationary process built from a fBm. However, delampertizing is not the only way to stationarize a fBm. Indeed, the Ornstein-Uhlenbeck process, which is a standard example of stationary process, can be generalized to the fractional Ornstein-Uhlenbeck (fOU) process [19]. The fOU process is stationary, thanks to a mean reversion effect that appears explicitly in the equation defining this process.

It is worth noting that the delampertized fBm does not share with the fBm exactly the same properties. Neither does it with the fOU process, even though both are stationary. Among all these properties, two seem particularly relevant to understand and interpret the parameters of the delampertized fBm:

- ▷ How is statistically driven the stationarity of the delampertized fBm?
- ▷ What does the estimated Hurst exponent of a delampertized fBm tell about the properties of this process?

These questions are pivotal when facing the statistical study of fractional dynamics. Indeed, in the empirical literature, the analysis of fractional dynamics is mostly restricted to the framework of the fBm. The guideline is usually the following: assuming that the fBm is well specified for the observed series, the Hurst exponent is estimated and decisions are taken accordingly. Indeed, for a fBm, a Hurst exponent above (respectively below) 1/2 means persistence (resp. anti-persistence) of the increments. Therefore, in fields as diverse as finance [11, 27, 15, 20] or medicine [33, 34], researchers hunt for values of Hurst exponents above 1/2, in order to leverage the persistence of the series and make predictions. Or, on the contrary, values of Hurst exponents below 1/2 are expected to make us benefit from the mean reversion of the series, as in pair trading [32]. However, the good specification of the fBm model in these practical examples is rarely challenged. We believe that, for time series of interest rates, volatilities, or foreign exchange rates, a stationary delampertized fBm would be more appropriate than a simple geometric fBm, which is very unlikely to remain between bounds. Then, the estimator for the Hurst exponent of the underlying fBm should be different in these two models and, for a same estimated value, different conclusions regarding the persistence of the series could arise between the fBm case and the delampertized fBm case.

The relevance of these questions is confirmed by the findings presented in this paper. In particular, we show that estimating the Hurst exponent of a delampertized fBm with the standard absolute-moment method designed for the fBm leads to a bias. A fBm-based method would indeed underestimate the Hurst exponent of a delampertized fBm.[1] The interpretation of the estimated Hurst exponent is tricky as well, since a value above 1/2 does not necessarily mean persistence.

We define several notions of Hurst exponents. First, the *perceived Hurst exponent* is simply the Hurst exponent estimated as if the series was a fBm, using a standard regression across scales. The *underlying Hurst exponent* is the Hurst exponent of the fBm before its delampertization. For a dynamics with an underlying Hurst exponent above 1/2, the persistent aspect predominates at small scales, whereas the mean reversion is overriding at higher scales. The *Hurst horizon* is an approximation of the scale separating these two opposite sides.

We apply the delampertized fBm to financial data. We indeed found one other paper proving empirically, with the Hurst approach, that some financial time series are persistent at low scales and anti-persistent at higher scales [23]. We confirm here this assertion for high-frequency foreign exchange rates and we provide a mathematical framework for this stylized fact. We explain why

[1] That is to say, it will underestimate the Hurst exponent of the underlying fBm.

the delampertized fBm is relevant for our sample, in particular more than the fOU process. We also show that the perceived Hurst exponent is almost always below 1/2, whereas the underlying Hurst exponent is above this threshold. This makes predictions feasible up to the Hurst horizon, which has a magnitude of several hours, whereas such series were reputed to be unpredictable due to their perceived Hurst exponent. We also study in more detail the case of GBP/EUR, which has a very long Hurst horizon of several days. We analyse as well the impact of the Brexit referendum on our estimations.

The rest of the paper is organized as follows. Section 2 addresses the question of the mean reversion mechanism in the delampertized fBm. Section 3 is about the interpretation of the Hurst exponent of the underlying fBm in its delampertized version. Section 4 applies this mathematical framework to financial time series. Section 5 concludes.

2 Mean reversion

The fBm is a stochastic process with stationary increments, but the process itself is not stationary. The standard Brownian motion, as a particular case of the fBm, has the same property. The Ornstein-Uhlenbeck process is a convenient way to define a stationary process using the non-stationary standard Brownian motion. It is based on an explicit mean reversion, characterized by a parameter depicting how intense the mechanism is. Substituting the standard Brownian motion by the fBm transforms the Ornstein-Uhlenbeck process in a fOU process, which is stationary as well. A fOU process Z is defined by the following equation:

$$dZ_t = -\alpha\left(Z_t - \mu\right)dt + \sigma dX_t,$$

where X is a fBm, $\sigma > 0$ is the volatility parameter, $\mu \in \mathbb{R}$ is the long-term average parameter, and $\alpha > 0$ is the speed of reversion. The parameter α and the corresponding mean reversion effect explain how a non-stationary process can be transformed in a stationary one. Indeed, when Z_t is above (resp. below) the long-term average μ, a negative (resp. positive) drift $-\alpha\left(Z_t - \mu\right)$ tends to attract Z towards μ. Such effect is balanced by σdX_t, so that the intensity of the mean reversion is well illustrated by α/σ. Of course, the farther Z_t is from μ, the more intense is the mean reversion. The fOU has been used for instance for modelling temperature [4], fluid turbulence [8] or the volatility of financial asset prices [9].

Nevertheless, the fOU process is not the only way to model stationarity with the help of the fBm. Another solution is based on the inverse Lamperti transform. It is obtained by an exponential contraction of the time and an appropriate scaling in the state space. The inverse Lamperti transform of the fBm X, of Hurst exponent H, is indeed defined by $(\mathcal{L}_H^{-1}X)_t = e^{-Ht}X_{exp(t)}$. Moreover, for $\theta > 0$, the process Y, defined by

$$Y_t = (\mathcal{L}_H^{-1}X)_{\theta t}, \tag{2}$$

is stationary as well.

A natural question arises: is the delampertized fBm the same as a fOU process? Both are indeed stationary and based on a fBm. In addition, if the Hurst exponent of the fBm X is $H = 1/2$, which corresponds to the case of a standard Brownian motion, $\mathcal{L}_H^{-1}X$ and the standard Ornstein-Uhlenbeck process based on X are the same [6]. Nevertheless, the likeness stops here. Indeed, if $H \neq 1/2$, the autocovariance function of $\mathcal{L}_H^{-1}X$ decays exponentially, whereas the autocovariance function of the corresponding fOU process is a power function [6]. As a result, when $H \neq 1/2$, the delampertized fBm is not a fOU process.

The equation defining the delampertized fBm is not based on an explicit mean reversion, contrary to the fOU process. Nevertheless, as a stationary process, a mean reversion exists in some way, even if it is only implicitly stated. We now want to depict more explicitly this mechanism. In the following theorem, we show that the delampertized fBm is indeed based on a mean reversion like the fOU process, despite the differences between the two processes. If Y is the delampertized fBm, we show that any increment of Y is negatively correlated to Y_s itself, for s before the observed increment. This assertion is true whatever H and depicts the reversion effect toward 0 implied by this process. On the contrary, the sign of this covariance depends on H in the case of a fBm.

Theorem 1. *Let $H \in (0,1)$, $0 < s \leq t < u$ and X be a fBm of Hurst exponent H.*

1. *For the fBm X, we have:*

$$Cov(X_s, X_u - X_t) = \frac{\sigma^2}{2}\left(u^{2H} - t^{2H} - (u-s)^{2H} + (t-s)^{2H}\right).$$

   This covariance is strictly positive for $H > 1/2$, strictly negative for $H < 1/2$, and equal to zero for $H = 1/2$.

2. *Let $\theta > 0$ and $Y_t = (\mathcal{L}_H^{-1}X)_{\theta t}$. Then:*

$$Cov(Y_s, Y_u - Y_t) = \frac{\sigma^2}{2}\left[h(\theta(u-s)) - h(\theta(t-s))\right],$$

   where $h : x \geq 0 \mapsto 2\cosh(Hx) - \left(2\sinh\left(\frac{x}{2}\right)\right)^{2H}$. This covariance is strictly negative, whatever $H \in (0,1)$.

The proof of this theorem, as well as of other results, is reported in appendix.

Mean reversion is verified statistically, for a process of mean 0, when an increment is negatively correlated to a previous state of the process. A mean-reverting process is not necessarily stationary but, by essence, a non-constant stationary process should be mean-reverting. For instance, in Theorem 1, we see that a fBm of Hurst exponent strictly lower than 1/2 is statistically mean-reverting, whereas it is not stationary. Also, as we can expect from a stationary process, the delampertized fBm Y is statistically mean-reverting, whatever the value of H.

In addition, Theorem 1 provides us with more details about how this mean reversion works. We see, indeed, that the covariance of an increment with a previous state of Y, distant of a time horizon τ, decreases from 0 to $-\sigma^2 h(\theta\tau)/2$ for a duration of the increment varying from 0 to infinity. For an increment of a given duration δ, this covariance is asymptotically zero when the reference time s is far in the past, but it decreases until $-\sigma^2(2 - h(\theta\delta))/2$ when the reference time becomes closer to the beginning of the increment.

Until now, there is no surprise: the stationary process obtained by delampertizing a fBm is statistically mean-reverting and we gave some details about this effect. This result holds whatever the value of H. We can mention that the slope of the function h, in Theorem 1, is steeper when H is lower, what means that the covariance $\text{Cov}(Y_s, Y_u - Y_t)$ is greater, in absolute value, for a lower H. In other words, the mean reversion is stronger for a lower H. Contrary to the fBm, two distinct values of H do not represent distinct features of the process, but they only depict a varying intensity in the same feature, at least regarding Theorem 1. Nevertheless, as we will see in the next section, when considering the correlation between increments, instead of between an increment and a reference state of the process, opposite interpretations about the persistence of the process can appear depending on the value of H.

3 About Hurst exponents

The main purpose of this paper is to stress the fact that the Hurst exponent estimated directly on a time series Y does not necessarily reflect the Hurst exponent of the underlying fBm X, if the time series is obtained by a transformation of X, in particular by an inverse Lamperti transform of X. In order to avoid any confusion, we define more precisely the two kinds of Hurst exponents we will work with:

- H is the **underlying Hurst exponent** of Y: it is the Hurst exponent of the fBm X from which Y is the inverse Lamperti transform.
- If $\hat{H}(X)$ is an estimator of the Hurst exponent of the fBm X, the same estimator applied to Y instead of X, that is to say $\hat{H}(Y)$, provides us with an estimate of the **perceived Hurst exponent** of Y. The perception depends on the scale, or on the range of scales, used in the estimation.

The meaning of the Hurst exponent also varies between X and Y. For example, a fBm with a Hurst exponent above 1/2 is persistent, that is to say increments are positively autocorrelated. But the inverse Lamperti transform Y of the same fBm does not have systematically positively autocorrelated increments. This doesn't mean that the perceived Hurst exponent, estimated directly on the series Y, is more appropriate to interpret the persistence of the series. In fact, estimating the Hurst exponent directly on Y is meaningless, since the fBm model is not well specified for Y.

In this section, we will first discuss challenges linked to the estimation of Hurst exponents, then we will see how the underlying Hurst exponent can be interpreted.

3.1 Estimation of underlying and perceived Hurst exponents

The process X is still a fBm, of Hurst exponent H, and Y, defined by equation (2), is a delampertization of X.

In this subsection, we adopt a statistical point of view. Therefore, the process Y is now assumed to be observed in discrete time in the interval $[0, T/N]$, with a time step $1/N$, where $N \in \mathbb{N}$. For $k \in \mathbb{N}$, let $M_{k,N,T}(X)$ and $M_{k,N,T}(Y)$ be the empirical absolute moment of order k of the increments of X:

$$M_{k,N,T}(X) = \frac{1}{T}\sum_{i=1}^{T} |X_{i/N} - X_{(i-1)/N}|^k$$

and of the increments of Y:

$$\begin{array}{rcl} M_{k,N,T}(Y) & = & \frac{1}{T}\sum_{i=1}^{T} |Y_{i/N} - Y_{(i-1)/N}|^k \\ & = & \frac{1}{T}\sum_{i=1}^{T} |e^{-\theta H i/N} X_{exp(\theta i/N)} - e^{-\theta H (i-1)/N} X_{exp(\theta (i-1)/N)}|^k. \end{array}$$

For the fBm, among a rich litterature about estimators of Hurst exponents [26, 1, 24, 31], we know a simple estimator of H based on absolute moments [29, 10, 15]:

$$\check{H}_{k,N,T}(X) = -\frac{\log(\sqrt{\pi} M_{k,N,T}(X)/[2^{k/2}\Gamma(\frac{k+1}{2})\sigma^k])}{k\log(N)},$$

which converges almost surely towards H, when $T \to +\infty$. However, this estimator depends on the explicit knowledge of σ. If the volatility parameter σ is unknown, one can define another estimator that is consistent with the moment-based estimators at two different scales, say for time steps $1/N$ and $2/N$:

$$\hat{H}_{k,N,T}(X) = \frac{1}{k}\log_2\left(\frac{M_{k,\frac{N}{2},\frac{T}{2}}(X)}{M_{k,N,T}(X)}\right).$$

Like $\check{H}_{k,N,T}(X)$, the estimator $\hat{H}_{k,N,T}(X)$ converges almost surely towards H, when $T \to +\infty$.

The estimator $\hat{H}_{k,N,T}(X)$ is defined in a way that ensures consistency with observations of the process X at two different scales. More generally, this approach can be extended to a greater number of scales [10]. The estimator of the Hurst exponent is then defined with the help of a logarithmic regression. Indeed, $-kH$ is the slope of the regression of $\log(M_{k,N,T}(X))$ on $\log(N)$.

The property of convergence of $\check{H}_{k,N,T}(.)$ and $\hat{H}_{k,N,T}(.)$ toward H holds if applied to a fBm X of Hurst exponent H, but not if applied to Y, a delampertized fBm of underlying Hurst exponent H. In this latter case, such estimators are biased. The theoretical formula of the expected value of the empirical absolute moment gives a clear support to the existence of a bias. Indeed, we show, in the following theorem, that $\mathbb{E}[M_{k,N,T}(Y)]$ writes in such a way that neither $\check{H}_{k,N,T}(Y)$, nor $\hat{H}_{k,N,T}(Y)$, nor an estimator based on a regression on several scales, can be a good estimator of H, in the case of the delampertized fBm Y.

Theorem 2. *Let $k, N, T \in \mathbb{N}$, $H \in (0,1)$, $\theta > 0$, X be a fBm of Hurst exponent H and volatility σ, and $Y_t = (\mathcal{L}_H^{-1}X)_{\theta t}$. We have*

$$\mathbb{E}\left[M_{k,N,T}(X)\right] = A(\sigma,k)N^{-kH}$$

and

$$\mathbb{E}\left[M_{k,N,T}(Y)\right] = A(\sigma,k)\left[2 - h\left(\frac{\theta}{N}\right)\right]^{k/2},$$

where $A(\sigma,k) = \frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sigma^k$ and $h : x \geq 0 \mapsto 2\cosh(Hx) - \left(2\sinh\left(\frac{x}{2}\right)\right)^{2H}$. Moreover, we have asymptotically, when $\frac{\theta}{N} \to 0$:

$$\mathbb{E}\left[M_{k,N,T}(Y)\right] = A(\sigma,k)\theta^{kH}N^{-kH} + \mathcal{O}\left(\frac{\theta}{N}\right)^{kH+\min(1,2-2H)}.$$

The consequences of Theorem 2 are the following, also summarized graphically in Figure 1:

- ▷ Low scales, that is to say higher values of θ/N: The locally perceived Hurst exponent is very low and tends toward zero, since $\mathbb{E}\left[M_{k,N,T}(Y)\right]$ decays exponentially towards $2^{k/2}A(\sigma,k)$ when N tends toward 0.
- ▷ High scales, that is to say lower values of θ/N: The locally perceived Hurst exponent becomes close to the underlying Hurst exponent as $\theta/N \to 0$.
- ▷ Role of θ: This parameter changes the scale of the process and thus shifts the frontier between flattened and accurate perception of the Hurst exponent.

In other words, if one was tempted to estimate directly a Hurst exponent on the series Y, as if it was a fBm, the perceived Hurst exponent would depend on the scale considered and it would in general underestimate the underlying Hurst exponent. The bias is even stronger that the resolution is low (or the scale high), which implies that Y is perceived as a process with very low Hurst exponent: the mean reversion effect, the anti-persistence, or the stationarity are predominating. On the

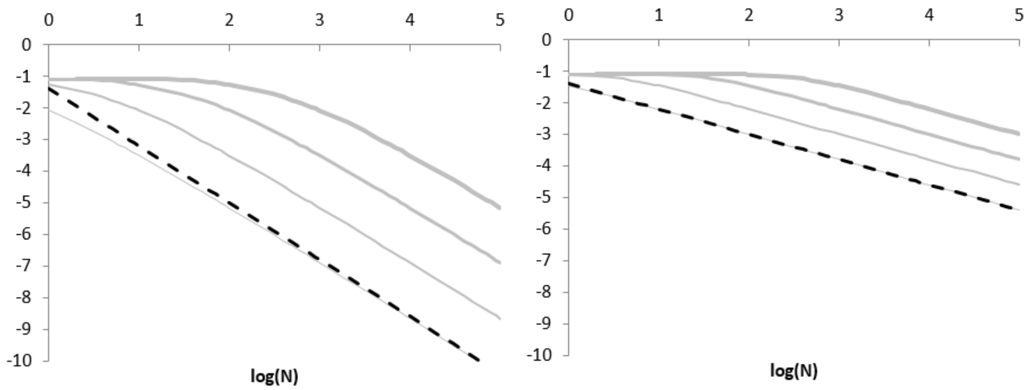

Figure 1: Value of $\log\left(\mathbb{E}\left[M_{k,N,T}(X)\right]\right)$ (black) and $\log\left(\mathbb{E}\left[M_{k,N,T}(Y)\right]\right)$ (grey, for several values of θ, from the thinnest to the thickest: 1, 10, 100, 1000) as a function of $\log(N)$. The underlying Hurst exponent H is equal to 0.9 (left) and 0.4 (right), whereas $k = 2$ and $\sigma = 0.2$. For a given resolution, the perceived Hurst exponent should be $-1/k$ times the slope of the curve at this resolution.

opposite, for high resolutions (or low scales), the perception of Y is consistent with X, the bias in the perceived Hurst exponent disappears, and the fBm feature plays a major role.

We are thus facing a stochastic process whose aspect, that is to say the Hurst statistic, depends on the resolution at which it is observed. We may perceive a Hurst exponent either close to the underlying one, or close to zero. Between these extremes, a short range of resolutions provides the statistician with a mitigated bias. The location of this transition zone depends monotonically on θ. This kind of model is not disconnected from reality. For example, the series of the durations of cardiac interbeat intervals, which is expected to be stationary when the patient remains at rest, shows a crossover phenomenon: two Hurst exponents are perceived, one for small scales and the other for larger scales [30]. This empirical observation could be astonishing without the theoretical justification by a delampertized fBm.

We put forward the possibility to estimate the Hurst exponent of a fBm by taking into account a wide range of scales instead of only two. The estimated Hurst exponent is then obtained by a regression: it is $-1/k$ times the average slope of the curve $\log(N) \mapsto \log\left(\mathbb{E}\left[M_{k,N,T}(Y)\right]\right)$, for the range of considered resolutions. Such an estimator, in the case of a delampertized fBm, thus does not help avoiding the bias. This bias will be even stronger that θ is high. In fact, fitting a linear regression, whereas the theoretical shape of $\log(N) \mapsto \log\left(\mathbb{E}\left[M_{k,N,T}(Y)\right]\right)$ is far from a line, is unreasonable.

The perceived Hurst exponent and the underlying one may differ, not only because of the inaccuracy intrinsic to statistical estimators, but mostly because of the bad specification of the model. In the literature, most estimators used are related to a perceived Hurst exponent, based on the fBm. But for other models, in particular for a delampertized fBm, if one follows the custom and estimates undeservedly the Hurst exponent with the standard fBm-based method, the result will be meaningless, as it will depend on the scale, and the perceived Hurst exponent may strongly underestimate the underlying one. Nevertheless, a properly estimated underlying Hurst exponent says more about the persistence of the underlying fBm than about the behaviour of its delampertized version. The interpretation of this statistic is thus challenging.

3.2 Interpreting the underlying Hurst exponent

For a fBm, the Hurst exponent depicts the persistence of the process:

- ▷ if $H > 1/2$, the process is persistent: increments are positively autocorrelated,
- ▷ if $H < 1/2$, the process is anti-persistent: increments are negatively autocorrelated,
- ▷ if $H = 1/2$, the process is a martingale: increments are not autocorrelated.

For a delampertized fBm of underlying Hurst exponent H, the interpretation of H is different from the fBm case. Indeed, a positive autocorrelation is inconsistent with the stationary property of the delampertized fBm. Nevertheless, from the previous subsection, we know that this process behaves differently at low and high resolutions. When focusing on small scales, it appears indeed as a fBm, whereas the mean reversion leads the dynamic at higher scales. Therefore, for $H > 1/2$, we expect the autocorrelation to be positive between close increments of short duration and negative for non-overlapping increments of long duration. The following findings will confirm this intuition.

In Theorem 1, since $X_0 = 0$, $\mathrm{Cov}(X_s, X_u - X_t)$ is the covariance of non-overlapping increments of duration s and $u - t$. It is therefore not surprising to find a positive covariance for $H > 1/2$, a negative one for $H < 1/2$, and a zero one for $H = 1/2$. On the contrary, $Y_0 = X_1$, so that $\mathrm{Cov}(Y_s, Y_u - Y_t)$ does not match with the covariance of increments of duration s and $u - t$. This covariance of increments may in fact be positive, depending on s, t, u, θ and H. We show this property of Y in the following proposition, restricted to the case of adjacent increments of same duration.

Proposition 1. *Let $H \in (0, 1)$, $s, \theta > 0$, X be a fBm of Hurst exponent H and volatility σ, and $Y_t = (\mathcal{L}_H^{-1} X)_{\theta t}$. We then have the following properties:*

1. *The covariance of adjacent increments of Y, of duration s, is:*

$$C(s) = Cov(Y_s - Y_0, Y_{2s} - Y_s) = \frac{\sigma^2}{2}\left[2h(\theta s) - 2 - h(2\theta s)\right],$$

where $h : x \geq 0 \mapsto 2\cosh(Hx) - \left(2\sinh\left(\frac{x}{2}\right)\right)^{2H}$, with the asymptotic property:

$$\lim_{s \to +\infty} C(s) = -\sigma^2.$$

2. *The correlation of adjacent increments of Y, of duration s, is:*

$$\rho(s) = corr(Y_s - Y_0, Y_{2s} - Y_s) = -1 + \frac{2 - h(2\theta s)}{4 - 2h(\theta s)},$$

with the asymptotic property:

$$\lim_{s \to +\infty} \rho(s) = -\frac{1}{2}.$$

The correlation of increments of same duration is displayed in Figure 2, for various Hurst exponents. The usual interpretation of the Hurst exponent does not hold in the case of a delampertized fBm: autocorrelation is of course negative for $H < 1/2$ but, for $H > 1/2$, it is positive for increments of short duration and negative else.

For a delampertized fBm of parameters H and θ, the Hurst exponent theoretically perceived ranges between 0 and H, depending on the scale at which the process is observed. If $H > 1/2$, there is a

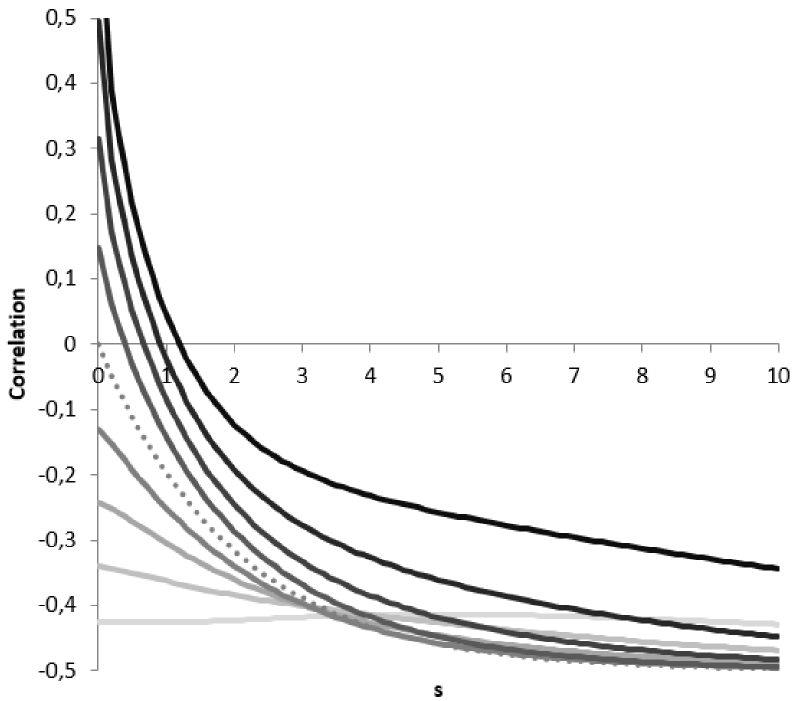

Figure 2: Correlation of adjacent increments of a delampertized fBm of underlying Hurst exponent H equal to 0.1 (the lightest), 0.2, ..., 0.9 (the darkest). The dotted curve corresponds to $H = 0.5$. The parameter θ is here equal to 1.

scale $\eta(H,\theta)$ under which the theoretically perceived Hurst exponent is greater than 1/2. We call this threshold scale the Hurst horizon. We define it as the scale $1/N$ such that the slope of our regression corresponds to a perceived Hurst exponent of 1/2:

$$\frac{1}{2} = -\frac{1}{k}\frac{\partial \log\left(\mathbb{E}\left[M_{k,N,T}(Y)\right]\right)}{\partial \log(N)}.$$

Therefore, $\eta(H,\theta)$ is the solution of the following equation in x: $h(\theta x) - 2 = h'(\theta x)\theta x$. Moreover, it is straightforward that $\eta(H,\theta) = \eta(H,1)/\theta$: when the time scale of the series is contracted by a factor θ, the Hurst horizon is contracted as well.

Intuitively, the Hurst horizon provides us with a rough idea of the scale at which increments are positively correlated. It is only an approximation. For instance, if we consider two adjacent increments of same duration, the maximum duration s for which the pair of increments is positively correlated is below $\eta(H,\theta)$, whatever H and θ, whereas the size of the interval containing both increments, $2s$, is above $\eta(H,\theta)$, as exposed in Figure 3. Nevertheless, s and $\eta(H,\theta)$ evolve concomitantly, regarding H, what confirms the intuition behind the Hurst horizon.

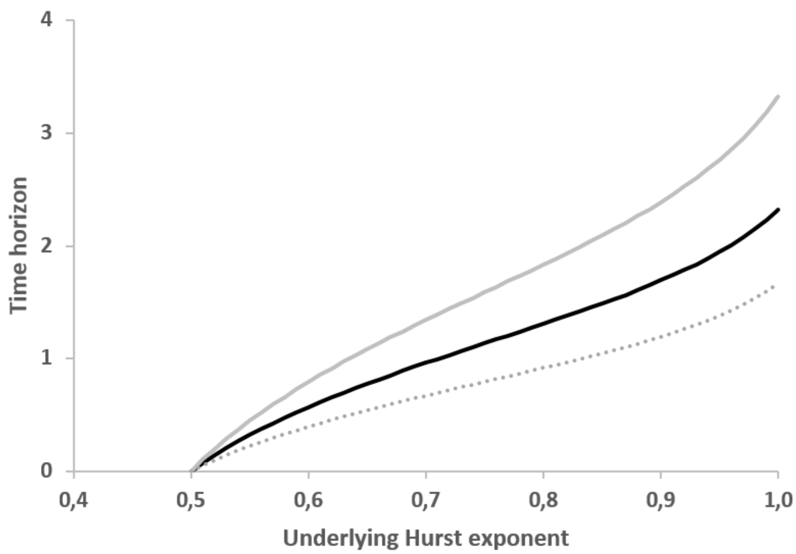

Figure 3: Hurst horizon (black), maximal duration s such that an increment of duration s of a delampertized fBm is positively correlated to the adjacent increment of same duration (grey dots), duration of the interval containing these two increments (grey line), as a function of the underlying Hurst exponent. The grey dotted line corresponds in fact to the intercept of the various curves represented in Figure 2. A linear regression with a R^2 equal to 0.9997 provides us with the relation $s = 0.7052 \times \eta(H,\theta)$. The parameter θ is here equal to 1.

Even though a delampertized fBm may have positively correlated increments, it does not hold long-range dependence, due to the exponential decay of the autocorrelation, as stated in the following proposition.

Proposition 2. *Let $H \in (0,1)$, $s,\theta > 0$, $d \geq 0$, X be a fBm of Hurst exponent H, and $Y_t = (\mathcal{L}_H^{-1}X)_{\theta t}$. The correlation of adjacent increments of Y, of duration s, separated by an interval of* *duration* $(d-1)s$ *is then:*

$$\rho(s,d) = corr(Y_s - Y_0, Y_{(d+1)s} - Y_{ds}) = \frac{2h(d\theta s) - h((d-1)\theta s) - h((d+1)\theta s)}{4 - 2h(\theta s)},$$

where $h : x \geq 0 \mapsto 2\cosh(Hx) - \left(2\sinh\left(\frac{x}{2}\right)\right)^{2H}$, *with the asymptotic property:*

$$\rho(s,d) \overset{d\to+\infty}{\sim} \begin{cases} \left(\frac{1-\cosh(H\theta s)}{2-h(\theta s)}\right) e^{-dH\theta s} & \text{if } H \leq \frac{1}{2} \\ 2H\left(\frac{1-\cosh((1-H)\theta s)}{2-h(\theta s)}\right) e^{-d(1-H)\theta s} & \text{else.} \end{cases}$$

This is consistent with the fact that this kind of process looks like a fBm for small scales and like a mean-reverting process at higher scales. On the opposite, a fBm has either only negatively correlated increments and no long-range dependence, or only positively correlated increments and long-range dependence, depending on the value of its Hurst exponent. Indeed the correlation between two increments of a fBm, of duration s, and separated by an interval of duration $(d-1)s$, is $((d+1)^{2H} - 2d^{2H} - (d-1)^{2H})/2$.

Provided that θs is small enough, we also observe positive correlations for $H > 1/2$, at least for increments close in time. However, for larger values of θs, the mean-reverting aspect is predominant and autocorrelations are always negative, whatever the value of H and of d. This is illustrated by Figure 4.

It is worth noting that the fOU process, despite its stationarity, holds a long memory if its underlying Hurst exponent is above 1/2 [6]. It is one of the main differences between the fOU approach and the delampertized fBm. This difference can lead to the preference of one of the two models, depending on the existence or not of a long-memory feature in the data. In particular, we show in the next section that the FX rates series we are working with do not hold long memory, whereas their underlying Hurst exponent is above 1/2. This makes the use of the delampertized fBm more relevant than the fOU process.

4 Application to high-frequency FX rates

We propose an application to financial time series of the delampertized fBm model, in which the perceived Hurst exponent may depend monotonically on the scale. In particular, we are interested in foreign exchange rates. Many studies of their predictability have already been published. Among them, we can cite techniques based on fractional dynamics and Hurst exponents [15, 13], on multiscale decomposition [2, 18], or on chaos theory [12, 17, 13]. In the latter case, one of the challenges consists in determining the Lyapunov horizon, under which forecasts are supposed reliable. In the case of fractional dynamics, the reliability of the forecasts usually only relies on the value of the estimated Hurst exponent: if it is below 0.5, it is better to avoid any forecast, whereas greater values of Hurst exponents should make the forecasts reliable for a quite long horizon, because of a long-memory feature. The present paper brings two remarks concerning this belief:

- ▷ The perceived Hurst exponent may strongly underestimate the underlying one, so that time series reputed to be unpredictable, because of a low Hurst exponent, may in fact be a transform of a fBm with a Hurst exponent above 0.5. In other words, asserting that these series are unpredictable may be wrong.
- ▷ The inverse Lamperti transform of a fBm is not a long-memory process. We introduced the Hurst horizon, beyond which forecasts should be unreliable because increments become

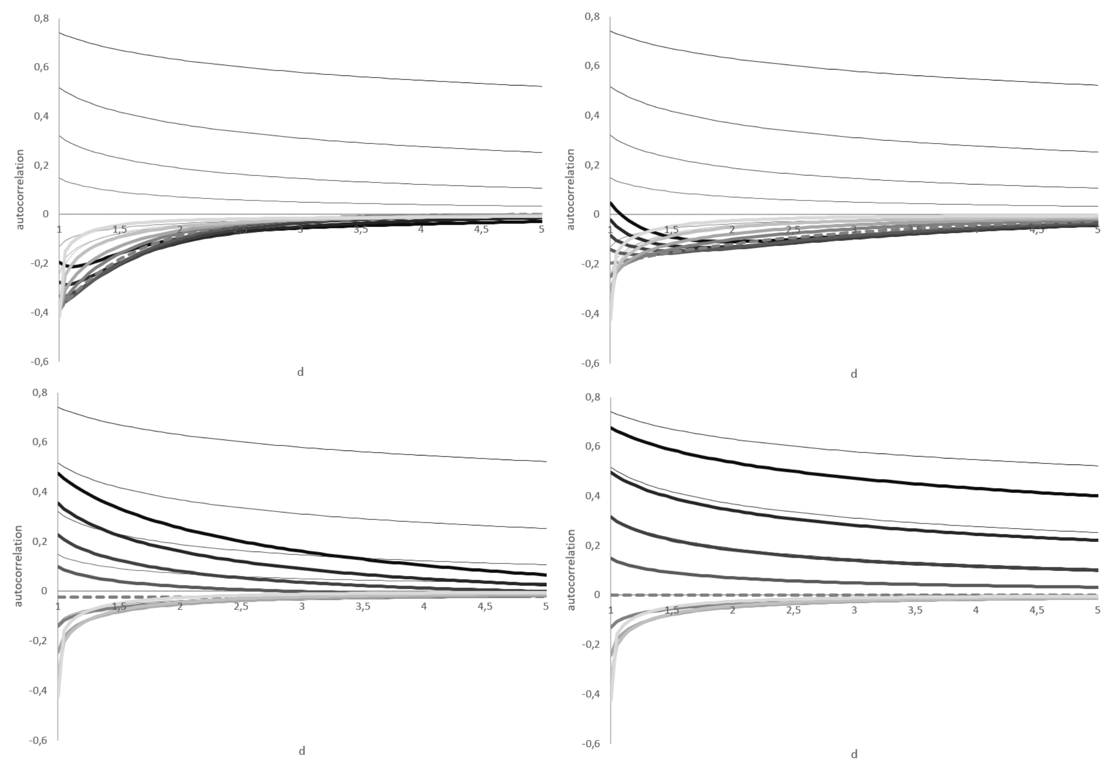

Figure 4: Correlation of increments of duration 1 separated by an interval of duration $(d-1)$ for a fBm (thin lines) and a delampertized fBm (thick lines) of underlying Hurst exponent H equal to 0.1 (the lightest), 0.2, ..., 0.9 (the darkest). The dotted curves correspond to $H = 0.5$. The parameter θ is here equal to 3 (top left), 1 (top right), 0.1 (bottom left) or 0.001 (bottom right).

negatively correlated. Therefore, whatever the estimated Hurst exponent, predictions might be feasible for small scales up to the Hurst horizon.

Our application is based on nine FX rates *versus* Euro: the British pound (GBP), the Swiss franc (CHF), the Swedish krona (SEK), the United States dollar (USD), the Canadian dollar (CAD), the Australian dollar (AUD), the Japanese yen (JPY), the Chinese yuan (CNY), and the Singapore dollar (SGD). The series begin the 7th March 2016 and finish the 7th September 2016, with a time step of 15 minutes while the market is open. The exchange rate considered is the average between the highest and the lowest rate in the 15-minute time interval. For each exchange rate, we thus have roughly 12,930 observations. The CNY/EUR rate is an exception, with only 7,422 observations, because of discontinuous trading.

4.1 Relevance of the delampertized fBm

In order to apply the delampertized fBm to FX rates, we have to justify that this model is relevant. In particular, two features are of importance: the stationarity of the process and the presence or not of long memory. For its part, the fractional nature of FX rates series is now well established [35, 5, 15, 36] and makes it possible to generalize dynamics based on the standard Brownian motion.

For the stationarity, we performed an augmented Dickey-Fuller (ADF) test, with trend and intercept. The null hypothesis of this test is the presence of a unit root and therefore of non-stationarity. We see in Table 1, that this hypothesis is rejected for two pairs: SEK/EUR and CNY/EUR. This proves the significance of the stationarity of these two series, even though, for the seven other series, the stationarity is not proved statistically. Using a stationary process, like the delampertized fBm, to model FX rates is thus relevant, at least for some pairs of currencies.

Among fractional stationary processes, we can cite the delampertized fBm and the fOU process. The preference to one or the other process should be lead by the presence or absence or long memory. The delampertized fBm is not a long-memory process, since its autocorrelation function is an exponential function [6]. On the contrary, the fOU process with a Hurst exponent higher than 0.5 has a long memory, with a power function for its autocorrelation [6]. As a consequence, we computed the empirical autocorrelation $R(l)$ for each FX rate series,[2] with a lag l ranging from 1 to 1,000 time steps.[3] We want to determine which of the two following models best fits the empirical autocorrelation:

- ▷ $\ln(R(l)) = -\beta l + \varepsilon_l$, which corresponds to an exponential decay of the autocorrelation,
- ▷ $\ln(R(l)) = -\beta \ln(l) + \varepsilon_l$, which corresponds to a power decay of the autocorrelation.

In both models, $\beta > 0$ and ε_l are Gaussian i.i.d. variables. We make two linear regressions of $\ln(R(l))$ on either l or $\ln(l)$, for $l = 1, 2, ..., 1,000$, with a maximum likelihood estimation. We determine the information loss induced by each model, namely the Akaike information criterion (AIC), which, in our framework of Gaussian residuals, is equal to 2 minus 2 times the sum of squared errors of the regression.[4] The lower is the AIC, the more suitable is the model. As we can see in Table 1, all the pairs have a lower AIC for the exponential model of autocorrelation decay than for the power model. This means that the exponential decay is more likely than the power one. The extent to which the exponential decay is more likely is quantified by the relative likelihood,

[2] We worked directly with the rate series, not with the series of the variations of the rate.

[3] In other words, the lag is between 15 minutes and more than 10 days.

[4] As the two models have the same number of parameters, the BIC would lead to the same conclusions as the AIC.

$\exp(\frac{1}{2}[AIC_{exp} - AIC_{pow}])$. For example, for SGD/EUR, the relative likelihood is 2.95×10^{-4}, what means that the power decay of the autocorrelation function is 2.95×10^{-4} as likely as the exponential decay.

	ADF tau-stat.	**AIC exp. decay**	**AIC power decay**	**Relative likelihood**
GBP/EUR	-1.42	2.03	10.48	1.46×10^{-2}
CHF/EUR	-2.57	2.37	32.77	2.51×10^{-7}
SEK/EUR	$-3.20^{\star}$	2.26	5.50	1.99×10^{-1}
USD/EUR	-3.12	8.86	99.38	2.20×10^{-20}
CAD/EUR	-2.91	9.12	300.32	5.84×10^{-64}
AUD/EUR	-1.68	5.22	52.03	6.83×10^{-11}
JPY/EUR	-2.36	2.08	4.39	3.15×10^{-1}
CNY/EUR	$-3.99^{\star\star\star}$	12.63	251.27	1.52×10^{-52}
SGD/EUR	-2.37	2.55	18.81	2.95×10^{-4}

Table 1: Statistic of the stationarity test with p-value ≤ 0.1 (*) and ≤ 0.01 (***). AIC for an exponential and power decay, and the corresponding relative likelihood of the power decay with regard to the exponential decay.

The long-memory feature is thus clearly refuted in our dataset, what is not always the case in the literature for other samples of FX rates [7, 28]. However, other studies confirm our findings with a twofold behaviour of the series, and without long memory at low frequency [5]. Thus, we are inclined to work with the delampertized fBm rather than with the fOU process.

4.2 Results

All our estimations of Hurst exponents are based on the absolute-moment estimator of order 2.[5] For each FX rate, we estimate:

- ▷ the perceived Hurst exponent, calculated as half the slope of $\log\left(M_{2,2^{-i}/\tau,T}(X)\right)$ on $\log\left(\tau 2^i\right)$, for i ranging from 0 to 11 and τ equal to the minimum time step, namely 15 minutes (therefore $\tau 2^i$ is between 15 minutes and 21 days),
- ▷ the underlying Hurst exponent, estimated as the Hurst exponent at the finest scale (the limit of half the slope mentioned above when $i \to 0$),
- ▷ the empirical Hurst horizon, estimated as the smallest scale for which the above-mentioned half slope reaches 0.5 (we also call thereafter *local Hurst exponent* the perceived Hurst exponent for a given isolated scale). It is linearly interpolated between $\tau 2^{i-1}$ and $\tau 2^i$, where $\tau 2^i$ is the first scale with a Hurst exponent below 0.5.

Results are gathered in Table 2. The standard regression-based estimation technique leads to a Hurst exponent lower than 0.5, except for GBP/EUR. This means that the perceived Hurst exponent almost always encourages not to make any forecast. Nevertheless, we see in two examples, Figures 5 and 6, that the scaling rule varies with the scale and therefore that a global Hurst exponent is not enough to depict how the series behaves across scales. Indeed, for small scales, we found

[5] In Figures 5, 6, and 7, we also display estimations for orders 1, 3, and 4, which confirm the findings related to order 2.

systematically a local Hurst exponent above 0.5, whereas lowering the resolution leads to lower Hurst exponents.

	Perceived Hurst exponent	**Underlying Hurst exponent**	**Empirical Hurst horizon**
GBP/EUR	0.542	0.680	7d 15h 24m
CHF/EUR	0.448	0.652	2h 20m
SEK/EUR	0.449	0.668	1h 33m
USD/EUR	0.466	0.694	4h 53m
CAD/EUR	0.457	0.668	4h 19m
AUD/EUR	0.498	0.669	2h 16m
JPY/EUR	0.493	0.703	5h 28m
CNY/EUR	0.418	0.604	3h 1m
SGD/EUR	0.456	0.645	1h 49m

Table 2: Estimations of Hurst exponents and of Hurst horizons (in days, hours, and minutes). The perceived Hurst exponent is based on a regression for scales $2^i \times 15$ minutes, for i ranging from 0 to 11.

This finding is also confirmed graphically in Figures 5 and 6 for other orders of absolute moments. For $k = 2$, the absence of linearity between the log absolute moment and the log resolution is not striking at first glance, though it is real as does certify the slight concavity of the curve compared to the linear regression, or the computation of the local Hurst exponent. But for higher orders, the variation of the slope with the scale is obvious. The time series of foreign exchange rates thus cannot be a simple fBm.

This scaling law is in fact typical of a process acting as a fBm of high Hurst exponent at low scales, and as a mean-reverting process at higher scales, exactly like the delampertized fBm. Such a twofold behaviour also appears graphically in Figures 5 and 6, with the historical FX rate at two scales. At a low resolution, the series seems unpredictable, made of peaks and mean reversions, whereas, at the scale of a day, clear patterns, based on a positive autocorrelation, make predictions feasible.

In this framework, we also display in Table 2 the underlying Hurst exponent and the empirical Hurst horizon. The underlying Hurst exponent is much higher than the perceived one, always above 0.6. This leads to the counter-intuitive fact that reliable predictions could be made for all these FX rates, even though the perceived Hurst exponent is lower than 0.5. Nevertheless, the estimated horizon of reliability of these predictions is quite short, between 1 hour 33 minutes (SEK/EUR) and 5 hours 28 minutes (JPY/EUR).

The case of GBP/EUR is different, with a long Hurst horizon of more than a week. We explain this by the fact that it is the only pair studied with a perceived Hurst exponent above 0.5. This high value of the perceived Hurst exponent may suggest that there is no mean reversion in this series and, thus, that the delampertized fBm is not relevant for GBP/EUR. In fact, this perceived Hurst exponent is an average of local Hurst exponents on several scales. Contrary to many FX rates, the scale above which the local Hurst exponent is below 0.5 is very high, as we can see in Table 2 and as the local Hurst exponents displayed in Figure 7 confirm. Therefore, with our sample limited to 6 months, our perceived Hurst exponent of GBP/EUR is an average of many Hurst exponents above 0.5 and of only some below. This means that, if we increased the set of scales considered or if we overweighted high scales, we would get a perceived Hurst exponent lower than 0.5. Mean reversion thus exists for GBP/EUR too, but at a much higher scale than for other series.

In Figures 5, 6, and 7, we observe a peak in each series of rates in June 2016. These peaks

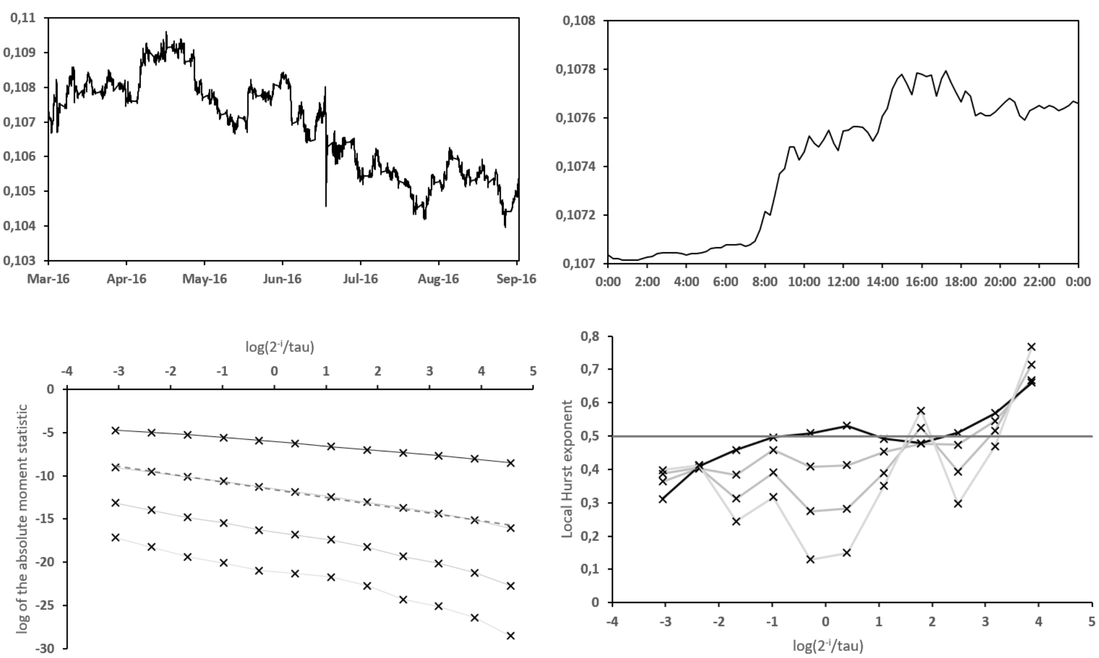

Figure 5: SEK/EUR rate of all our sample (top left) and with a focus on the 9th March 2016 (top right). Bottom left: log empirical absolute moment on the log of the inverse time step, for moments of order $k = 1$ (black) to $k = 4$ (lightest grey); a linear regression is added as a dotted line for $k = 2$. Bottom right: corresponding local Hurst exponents, defined, for the scale $\log(2^{-i}/\tau)$, as $-1/k$ times the slope of the bottom-left curve between scales $\log(2^{-i}/\tau)$ and $\log(2^{-(i-1)}/\tau)$.

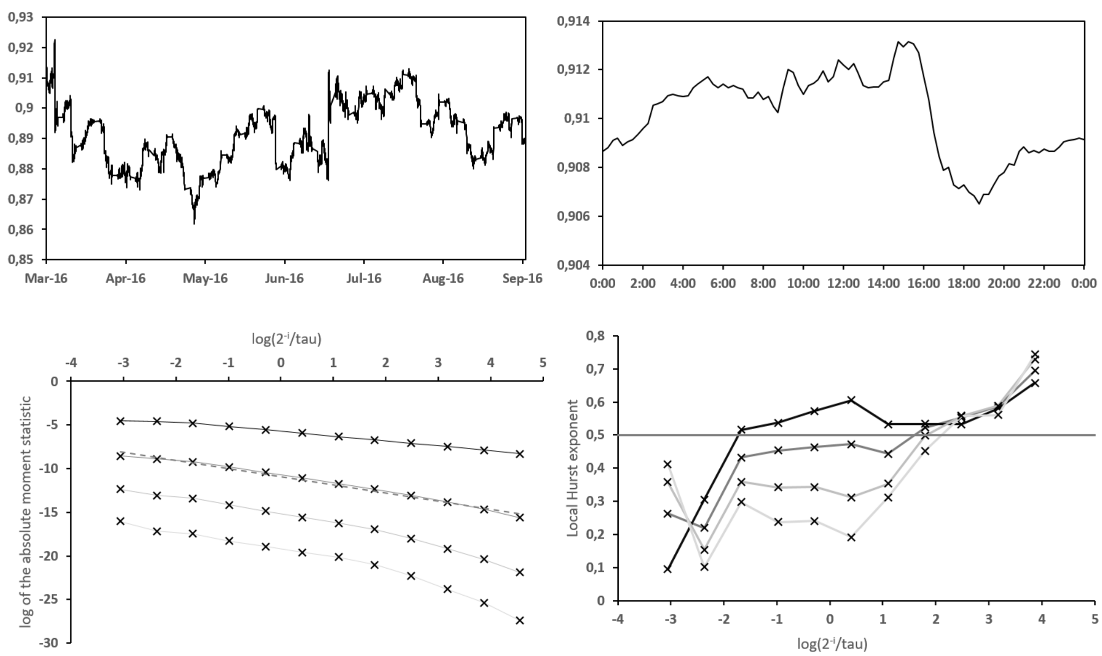

Figure 6: USD/EUR rate of all our sample (top left) and with a focus on the 9th March 2016 (top right). Bottom left: log empirical absolute moment on the log of the inverse time step, for moments of order $k = 1$ (black) to $k = 4$ (lightest grey); a linear regression is added as a dotted line for $k = 2$. Bottom right: corresponding local Hurst exponents, defined, for the scale $\log(2^{-i}/\tau)$, as $-1/k$ times the slope of the bottom-left curve between scales $\log(2^{-i}/\tau)$ and $\log(2^{-(i-1)}/\tau)$.

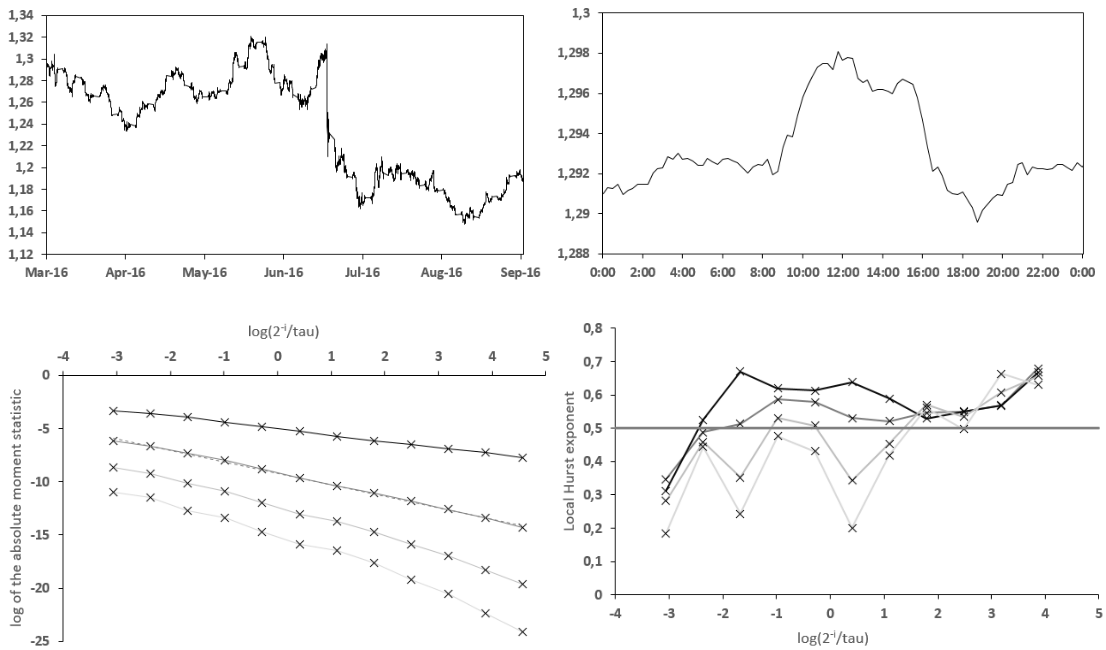

Figure 7: GBP/EUR rate of all our sample (top left) and with a focus on the 9th March 2016 (top right). Bottom left: log empirical absolute moment on the log of the inverse time step, for moments of order $k = 1$ (black) to $k = 4$ (lightest grey); a linear regression is added as a dotted line for $k = 2$. Bottom right: corresponding local Hurst exponents, defined, for the scale $\log(2^{-i}/\tau)$, as $-1/k$ times the slope of the bottom-left curve between scales $\log(2^{-i}/\tau)$ and $\log(2^{-(i-1)}/\tau)$.

correspond to a market reaction after the Brexit referendum of the 23rd June, 2016. If we stop the sample on the 23rd June, just before the publication of the outcome of the referendum, we get perceived and underlying Hurst exponents very close to those estimated on the full sample, as one can see in Table 3. In particular, underlying Hurst exponents are still all above 0.5. Perceived ones are still below 0.5, except for GBP/EUR and for AUD/EUR. However, for this latter pair, the perceived Hurst exponent had not changed much (0.503 for the partial sample *versus* 0.498 for the full sample). The magnitude of the empirical Hurst horizon remains the same for all the pairs, except for CAD/EUR and JPY/EUR. Indeed, in these cases, the horizon is increased by a few hours. However, the global picture remains the same for all the series, with mean reversion at high scales, and persistence at lower scales.

	Perceived Hurst exponent	**Underlying Hurst exponent**	**Empirical Hurst horizon**
GBP/EUR	0.516	0.668	7d 14h 15m
CHF/EUR	0.482	0.653	1h 23m
SEK/EUR	0.454	0.647	1h 28m
USD/EUR	0.471	0.704	3h 57m
CAD/EUR	0.478	0.677	12h 48m
AUD/EUR	0.503	0.682	2h 18m
JPY/EUR	0.486	0.706	23h 44m
CNY/EUR	0.442	0.588	1h 42m
SGD/EUR	0.439	0.662	1h 43m

Table 3: Estimations, on a partial sample, of Hurst exponents and of Hurst horizon (in days, hours, and minutes). The perceived Hurst exponent is based on a regression for scales $2^i \times 15$ minutes, for i ranging from 0 to 11. The sample stops the 23rd June, 2016.

Our study thus confirms that the delampertized fBm is relevant for modelling FX rates, with or without Brexit referendum. This result is complementary to a recent analysis which showed that the Hurst exponent of GBP/USD was higher by the Brexit referendum [36]. Indeed, a peak spread on several hours is characterized by a high autocorrelation of high-frequency increments. In our work, we show that, at a higher scale, the effect of the Brexit referendum on the persistence of FX rates is negligible.

5 Conclusion

In this paper, we studied the inverse Lamperti transform of a fBm. This process looks like a fBm at small scales and like a mean-reverting process at higher scales. Thanks to this transform, the fBm gains stationarity but loses its long memory in the same time.

Therefore, the usual analysis of the time series with a global Hurst exponent is flawed. In this framework, we have to distinguish the underlying Hurst exponent, which is the Hurst exponent of the fBm before it is transformed, and the perceived Hurst exponent, directly estimated on the delampertized fBm. The latter Hurst exponent is scale-dependent. The perceived Hurst exponent, estimated by the usual linear regression of the log absolute moments on the log scales, provides us with an underestimation of the underlying Hurst exponent.

We have thus several notions of Hurst exponent. A natural question is: on which of these Hurst exponents should we base the usual analysis of persistence of the series? Unfortunately, we can interpret neither the underlying Hurst exponent, nor the perceived one, as the usual persistence indicator. We simply state that a delampertized fBm of underlying Hurst exponent above 1/2 is persistent up to a given scale, called the Hurst horizon. This incites us to believe that, in this framework, we could build reliable predictions in the limit of this Hurst horizon. The way we may make such predictions is the purpose of a future paper [16].

This model seems well suited to financial data. We found underlying Hurst exponents higher than 1/2 on series of FX rates, whereas the standard analysis provided a perceived Hurst exponent lower than 1/2. In other words, without this framework, we would have limited our analysis to the perceived Hurst exponent. We would thus have concluded that predictions are unreliable, because of a lack of persistence in the time series. On the contrary, with our new contribution, we can assert that predictions at small scales are relevant.

This is consistent with the findings of Bouchaud *et alii* in a recent paper [3], in which they show empirically that returns are predictable at small scales and not at higher scales, at least on their dataset, which consists in daily prices of commodities, stocks, and currencies. In addition, they define a model to describe this effect: an Ornstein-Uhlenbeck process with a positively autocorrelated noise. The rationale of this model is close to ours, but an important strength of our approach is that it directly relates to the Hurst literature, which is massive in econophysics.

A Proof of Theorem 1

Proof. 1. Thanks to equation (1), we can write:

$$\begin{aligned}\mathrm{Cov}(X_s, X_u - X_t) &= \mathbb{E}[X_s X_u] - \mathbb{E}[X_s X_t] \\ &= \frac{\sigma^2}{2}\left(u^{2H} - t^{2H} - (u-s)^{2H} + (t-s)^{2H}\right).\end{aligned} \tag{3}$$

Then, three cases arise:

- ▷ If $H = 1/2$, then equation (3) immediately leads to $\mathrm{Cov}(X_s, X_u - X_t) = 0$.
- ▷ If $H > 1/2$, then the function $\kappa : t \in [s,u) \mapsto 2\mathrm{Cov}(X_s, X_u - X_t)/\sigma^2$ is differentiable with a derivative equal to $\kappa'(t) = 2H\left((t-s)^{2H-1} - t^{2H-1}\right)$, which is equal to zero if and only if $s = 0$, what is not allowed by the assumptions of the theorem. The function κ is thus strictly decreasing. Moreover, $\lim_{t \to u} \kappa(t) = 0$. Therefore, whatever $t \in [s,u)$, $\kappa(t)$ is strictly positive. Finally, $\mathrm{Cov}(X_s, X_u - X_t) > 0$.
- ▷ If $H < 1/2$, symmetrically to the case $H > 1/2$, the same function κ is strictly increasing with the same limit when t tends toward u. Therefore, κ is strictly negative because $2H - 1 < 0$ and $\mathrm{Cov}(X_s, X_u - X_t) < 0$.

2. Thanks to equation (1), we can write:

$$\begin{aligned} & \mathrm{Cov}(Y_s, Y_u - Y_t) \\ = \; & e^{-\theta H(s+u)}\mathbb{E}[X_{\exp(\theta s)} X_{\exp(\theta u)}] - e^{-\theta H(s+t)}\mathbb{E}[X_{\exp(\theta s)} X_{\exp(\theta t)}] \\ = \; & \frac{\sigma^2}{2}\left[e^{-\theta H(s+u)}\left(e^{2\theta H s} + e^{2\theta H u} - (e^{\theta u} - e^{\theta s})^{2H}\right)\right. \\ & \left. - e^{-\theta H(s+t)}\left(e^{2\theta H s} + e^{2\theta H t} - (e^{\theta t} - e^{\theta s})^{2H}\right)\right] \\ = \; & \sigma^2\left[\cosh(\theta H(u-s)) - \cosh(\theta H(t-s)) - 2^{2H-1}\left(\sinh\left(\frac{\theta(u-s)}{2}\right)\right)^{2H} + 2^{2H-1}\left(\sinh\left(\frac{\theta(t-s)}{2}\right)\right)^{2H}\right] \\ = \; & \frac{\sigma^2}{2}\left[h(\theta(u-s)) - h(\theta(t-s))\right].\end{aligned}$$

It is strictly negative since both $\theta(u-s) > \theta(t-s)$ and h is a strictly decreasing function.

Indeed, $h(x)$ also writes $e^{Hx}\left[1+e^{-2Hx}-(1-e^{-x})^{2H}\right]$ and, for $x>0$ and $H\in(0,1)$:

$$\begin{array}{rcl} h'(x) & = & He^{Hx}\left\{\left[1+e^{-2Hx}-(1-e^{-x})^{2H}\right]+\left[-2e^{-2Hx}-2e^{-x}(1-e^{-x})^{2H-1}\right]\right\} \\ & = & He^{Hx}\left[1-e^{-2Hx}-(1+e^{-x})(1-e^{-x})^{2H-1}\right] \\ & \leq & He^{Hx}\left[1-e^{-2Hx}-(1+e^{-x})(1-e^{-x})\right] \\ & \leq & He^{Hx}\left[e^{-2x}-e^{-2Hx}\right] \\ & < & 0, \end{array}$$

in which the first inequality is quite straightforward: let $y=1-e^{-x}\in(0,1)$; if $H\leq 1/2$ then $2H-1\leq 0$ and $y^{2H-1}\geq 1\geq y$; if $H>1/2$ then $2H-1\in(0,1)$ and $y^{2H-1}\geq y$.

□

B Proof of Theorem 2

Proof. First, for a Gaussian random variable Z of mean 0 and variance 1, we have:

$$\mathbb{E}\left[|Z|^k\right]=\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}. \tag{4}$$

- ▷ Therefore, since, for all $t\in\mathbb{R}$, $(X_t-X_{t-1/N})/(\sigma N^{-H})$ is a Gaussian random variable of mean 0 and variance 1, we get:

  $$\mathbb{E}\left[|X_t-X_{t-1/N}|^k\right]=\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sigma^k N^{-kH}.$$

  And, by linearity, the same result holds for $\mathbb{E}\left[M_{k,N,T}(X)\right]$.

- ▷ We note that any increment Y_t-Y_s of the delampertized fBm, for $(s,t)\in\mathbb{R}^2$, is a Gaussian variable of mean 0 and of variance:

  $$\begin{array}{rcl} \mathbb{E}\left[(Y_t-Y_s)^2\right] & = & e^{-2\theta Ht}\mathbb{E}[X^2_{exp(\theta t)}]+e^{-2\theta Hs}\mathbb{E}[X^2_{exp(\theta s)}]-2e^{-\theta H(t+s)}\mathbb{E}[X_{exp(\theta t)}X_{exp(\theta s)}] \\ & = & \sigma^2+\sigma^2-\sigma^2e^{-\theta H(t+s)}\left(e^{2\theta Ht}+e^{2\theta Hs}-|e^{\theta t}-e^{\theta s}|^{2H}\right) \\ & = & \sigma^2\left[2-e^{\theta H(t-s)}\left(1+e^{-2\theta H(t-s)}-|1-e^{-\theta(t-s)}|^{2H}\right)\right], \end{array}$$

  thanks to equation (1). Increments are therefore stationary Gaussian variables and their variance, for the time step $1/N$, is, for $H\in(0,1)$:

  $$\mathbb{E}\left[(Y_t-Y_{t-1/N})^2\right]=\sigma^2\left[2-e^{\theta H/N}\left(1+e^{-2\theta H/N}-|1-e^{-\theta/N}|^{2H}\right)\right].$$

  Then, using equation (4), we get:

  $$\mathbb{E}\left[(Y_t-Y_{t-1/N})^k\right]=\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sigma^k\left[2-e^{\theta H/N}\left(1+e^{-2\theta H/N}-|1-e^{-\theta/N}|^{2H}\right)\right]^{k/2}.$$

  Again, by linearity, the same result holds for $\mathbb{E}\left[M_{k,N,T}(Y)\right]$.

- ▷ When $\frac{\theta}{N}\to 0$, we have, as $H\in(0,1)$:

  $$\begin{array}{rcl} \mathbb{E}\left[M_{k,N,T}(Y)\right] & = & A(\sigma,k)\left[2-\left(1+\frac{\theta H}{N}+\mathcal{O}\left(\frac{\theta}{N}\right)^2\right)\left(2-\frac{2\theta H}{N}+\mathcal{O}\left(\frac{\theta}{N}\right)^2-\left(\frac{\theta}{N}\right)^{2H}\left(1+\mathcal{O}\left(\frac{\theta}{N}\right)\right)\right)\right]^{k/2} \\ & = & A(\sigma,k)\left[\left(\frac{\theta}{N}\right)^{2H}+\mathcal{O}\left(\frac{\theta}{N}\right)^{\min(2H+1,2)}\right]^{k/2} \\ & = & A(\sigma,k)\left(\frac{\theta}{N}\right)^{kH}\left[1+\mathcal{O}\left(\frac{\theta}{N}\right)^{\min(1,2-2H)}\right]^{k/2} \\ & = & A(\sigma,k)\left(\frac{\theta}{N}\right)^{kH}+\mathcal{O}\left(\frac{\theta}{N}\right)^{kH+\min(1,2-2H)}. \end{array}$$

□

C Proof of the propositions of Section 3.2

Proof. The proof of the formulae concerning covariance and correlation in Proposition 1 and 2 is straightforward using Theorem 1 and the bilinearity of the covariance. The asymptotic results are a consequence of the asymptotic behaviour of the function h. For this purpose, we show that, for $x \to +\infty$:

$$\left(e^x - e^{-x}\right)^{2H} = e^{2Hx}\left(1 - 2He^{-2x} + o\left(e^{-2x}\right)\right)$$

Therefore, when $x \to +\infty$,

$$\begin{array}{rcl} h(x) & = & e^{Hx} + e^{-Hx} - \left(e^{x/2} - e^{-x/2}\right)^{2H} \\ & = & e^{-Hx} + 2He^{(H-1)x} + o\left(e^{(H-1)x}\right), \end{array} \tag{5}$$

or also $h(x) = \mathcal{O}\left(e^{\xi x}\right)$, where $\xi = \max(-H, H-1)$. As $\xi \in (-1, 0)$, $h(x)$ tends toward zero when x tends toward infinity. This proves the asymptotic results in Proposition 1.

Concerning Proposition 2, we distinguish two cases:

- ▷ If $H \leq 1/2$, then, for $x \to +\infty$, we have, from equation (5), $h(x) = e^{-Hx} + \mathcal{O}\left(e^{(H-1)x}\right)$, so that

  $$\rho(s,d) = \frac{e^{-dH\theta s}\left(2 - e^{H\theta s} - e^{-H\theta s}\right) + \mathcal{O}\left(e^{d(H-1)\theta s}\right)}{4 - 2h(\theta s)}$$

  when $d \to +\infty$.

- ▷ If $H > 1/2$, then, for $x \to +\infty$, we have, from equation (5), $h(x) = 2He^{(H-1)x} + o\left(e^{(H-1)x}\right)$, so that

  $$\rho(s,d) = \frac{2He^{d(H-1)\theta s}\left(2 - e^{-(H-1)\theta s} - e^{(H-1)\theta s}\right) + o\left(e^{d(H-1)\theta s}\right)}{4 - 2h(\theta s)}$$

  when $d \to +\infty$.

These two cases directly lead to the equivalence formulae provided in Proposition 2. □

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