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Modelling stock price movements: multifractality or multifractionality?

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The scaling properties of two alternative fractal models recently proposed to characterize the dynamics of stock market prices are compared. The former is the Multifractal Model of Asset Return (MMAR) introduced in 1997 by Mandelbrot, Calvet and Fisher in three companion papers. The latter is the multifractional Brownian motion (mBm), defined in 1995 by Péltier and Lévy Véhel as an extension of the very well-known fractional Brownian motion (fBm).

We argue that, when fitted on financial time series, the partition function as well as the scaling function of the mBm, i.e. of a generally non-multifractal process, behave as those of a genuine multifractal process. The analysis, which concerns the daily closing prices of eight major stock indexes, suggests to evaluate prudently the recent findings about the multifractal behaviour in finance and economics.

Keywords: Multifractals; MMAR; Multifractionality; Stock prices

1. Introduction

In the last few years the analysis of the scaling properties of financial time series has become the subject of an increasing number of works. Since the pioneering papers due to Mandelbrot in the 1960s, both empirical and theoretical issues have been addressed in the field. Concerning the former, Müller *et al.* (1990) find evidence of a scaling law behaviour in four FX spot rates against the US Dollar and observe that the distributions of the price changes strongly differ for different interval sizes. More recently, Schmitt *et al.* (2000) find nonlinearity in the log variations of the daily US Dollar/French Franc exchange rate and argue the multifractal nature of the FX data. Bonanno *et al.* (2000) report $1/f$ behaviour† of the spectral density of the logarithm of stock price and $1/f$-like behaviour of the spectral density of the daily number of trades for several stocks traded in the New York Stock Exchange. Potgieter (2001) refers to significant deviations from the unifractal scaling for three series of exchange rates of the South African rand; from the behaviour of the partition and scaling functions the author concludes in favour of the multifractal model. Turiel and Pérez-Vicente (2002) show that the fluctuations of returns in stock market time series exhibit multifractal properties and exploit the geometry of the series. Analysing the scaling properties of the DAX, Górski *et al.* (2002) conclude that even more complicated dynamics than the multifractal one should be used to model price fluctuations. Analysing the qth order moments of the German index DAX, Ausloos and Ivanova (2002) find a hierarchy of power law exponents. Plotting the averaged absolute returns as a function of time intervals for different powers, Gençay and Xu (2003) find that the 5 min returns of the USD-DEM series show different slopes for these powers and argue that the nonlinearity of the scaling exponent indicates multifractality of returns. Similar results are found by Matia *et al.* (2003), who analyse the daily prices of 29 commodities and 2449 stocks and find that the former have a significantly broader multifractal spectrum than the latter. Fillol (2003) investigates the multifractal properties of the French Stock Market and provides Monte Carlo simulations showing that the Multifractal Model of Asset Returns (MMAR) is a better model to replicate the scaling properties observed in the CAC40 series than alternative specifications like GARCH or FIGARCH.

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†Shortly, it is called '$1/f$ noise', the type of noise whose power spectra $P(f)$ as a function of the frequency f behaves like $P(f)=f^{-\alpha}$, for some positive real α.

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At the same pace of the empirical analyses, advances have been reached within the theoretical framework. Muzy *et al*. (2000) provide an interpretation of multifractal scaling laws in terms of volatility correlations and show that in this context $1/f$ power spectra naturally emerge. Brachet *et al*. (2000) address the question of scaling transformation of the price process by establishing a new connection between nonlinear group theoretical methods and multifractal methods developed in mathematical physics. Using two sets of financial time series, they show that the scaling transformation is a nonlinear group action on the moments of the price increments, whose linear part has a spectral decomposition that puts in evidence a multifractal behaviour in the price increments. Calvet and Fisher (2001) develop analytical methods to forecast the distribution of future returns for the Poisson multifractal and show that the process captures the thick tails, the volatility persistence and the moment scaling exhibited by many financial time series. Pochart and Bouchaud (2002) generalize the construction of the multifractal random walk due to Bacry *et al*. (2001) to take into account the asymmetric character of the financial returns. They show how one can include in this class of models the observed correlation between past returns and future volatilities, in such a way that the scale invariance properties of the motion are preserved. Through a market simulation model which displays multifractality, Yamasaki and Machin (2003) reproduce many stylized facts of speculative markets. From this model they analytically derive the MMAR for the macroscopic limit. More recently, Calvet and Fisher (2004) have introduced a discrete-time stochastic volatility model in which regime switching captures features such as the low-frequency variations, the smooth autoregressive transitions and the outliers that are generally considered as distinct in the literature.

In addition to those above recalled, many other works seem to agree about the multifractal nature of financial time series. Anyway, as usual, things are more complicated, at least because the continuum of local scales characterizing multiscaling processes has been defined in two different, non-equivalent ways. On the one hand, the *Multifractal Model of Asset Returns* (MMAR), introduced by Mandelbrot *et al*. (1997), is based on the concept of global scaling and qualifies as *multifractal* those stationary increment processes whose absolute moments satisfy a proper scaling relation. Differently, Péltier and Lévy Véhel (1995) and independently Benassi *et al*. (1997) define the *multifractional Brownian motion* (mBm) as a generalization of the very well-known *fractional Brownian motion* (fBm), obtained by replacing the parameter H with the functional parameter $H(t)$. The basic idea founding this extension is that in many fields data cannot be modelled by standard stationary increment processes simply because the pointwise regularity (or smoothness) of the time series changes widely—and often abruptly—from point to point.

This paper deals with these two different stock price models: we show that the mBm, which generally is not multifractal, can display the same features (same partition and scaling functions and same spectrum) of genuine multifractal processes. It follows that the partition function-based techniques, commonly used to argue multifractality, can be deceived by a non-multifractal process such as the mBm. Really, the problem of misidentification of these techniques is not totally new, both from a theoretical and an empirical point of view: Bouchaud *et al*. (2000) have built an asymptotically monofractal model displaying apparent multiscaling behaviour as a consequence of a slow crossover phenomenon on finite time scales; by simulation, Lux (2004) concludes that the scaling estimators used to deduce multifractality are unable to discriminate between actual and spurious multifractal processes obtained by fat-tailed distributed innovations.

The analysis we perform in this paper has been carried out through a dedicated software written by the authors in MatLab 6.5 and freely downloadable from the site: http://multifractal.unicas.it.

The paper is organized as follows: in section 2 we shortly recall the definition of multifractality and the technique used to test for it; section 3 concerns a brief description of the mBm and of an estimator of its functional parameter. The knowledge of $H(t)$ is indeed necessary to fit the model on the analysed financial time series. Section 4 deals with the empirical analysis, which has taken into consideration eight world stock indexes. Section 5 concludes.

2. Multifractal processes and the MMAR

Mandelbrot *et al*. (1997) define the stochastic process $\{X(t)\}$ multifractal if it has stationary increments and satisfies the scaling relation

$$\mathbb{E}(|X(t)|^q) = c(q)t^{\tau(q)+1} \text{ for } t \in \mathcal{T},\ q \in \mathcal{Q} \tag{1}$$

with $\mathcal{T}$ and $\mathcal{Q}$ intervals on the real line such that $0 \in \mathcal{T}$, $[0, 1] \in \mathcal{Q}$ and q such that $\mathbb{E}(|X(t)|^q) < \infty$.

The deterministic functions $c(q)$ and $\tau(q)$ are respectively called *prefactor* and *scaling function*; the key quantity is $\tau(q)$, which takes into account the influence of time on the absolute moment of order q and synthesizes all the information about the rate of growth of the moments of $X(t)$ as t varies.

In the particular case of an uniscaling (or monofractal) process the function $\tau(q)$ is linear; an example is given by the fBm of exponent H, for which—as a consequence of its self-similarity—relation (1) holds with $\tau(q) = Hq - 1$ and prefactor $c(q) = \mathbb{E}(|X(1)|^q)$.

Differently, as the genuine multifractal processes show scaling properties implying a concave $\tau(q)$, multifractality is generally tested through relation (1) as follows.

Given $X(t)$ defined on $[0, T]$, divide the interval into N subintervals of length Δt and define the partition function as

$$S_q(T, \Delta t) = \sum_{i=0}^{N-1} |X((i+1)\Delta t) - X(i\Delta t)|^q. \tag{2}$$

Because of the assumed stationarity of $X(t)$, the addends are identically distributed and—whenever the qth moment exists—the scaling law (1) yields $\mathbb{E}[S_q(T,\Delta t)] = Nc(q)(\Delta t)^{\tau(q)+1}$, from which it trivially follows

$$\log(\mathbb{E}[S_q(T,\Delta t)]) = \tau(q)\log \Delta t + \log Tc(q). \quad (3)$$

For each fixed admissible q, equation (3) provides testable moment conditions requiring the partition function to vary linearly with the increment size Δt.

Within this context, the Multifractal Model of Asset Return (MMAR) assumes the log-prices $\{X(t) = \ln P(t) - \ln P(0);\ 0 \le t \le T\}$ to be a compound process satisfying the following assumptions (Mandelbrot *et al.* 1997, Calvet and Fisher 2002):

1. $X(t) = B_H[\vartheta(t)]$, where B_H is an fBm of exponent H and $\vartheta(t)$ is a stochastic multifractal trading time, that is a multifractal transformation of chronological time into what can be thought as a 'trading time' (see Assumption 2). As both the fBm and the multifractal trading time are self-similar, self-similarity is preserved when the two are compounded.
2. The multifractal trading time $\vartheta(t)$, which governs the istantaneous volatility of the log-price process, is the cumulative distribution function of a multifractal measure† μ defined on $[0, T]$ (see below).
3. B_H and $\vartheta(t)$ are independent.

As known, multifractal measures can be obtained by iterating a procedure called a *multiplicative cascade*, a detailed description of which can be found for example in Calvet and Fisher (2002). Roughly speaking, the basic procedure consists in spreading a mass on the subintervals generated by dividing in successive steps the unit interval. With regard to the specification of the multifractal measure μ, Calvet *et al.* (1997) show that a quadratic spectrum implies a log normal distribution of the masses involved in the multifractal-generating mechanism.

MMAR has several attractive features. It includes as special cases three models (see, e.g. Mandelbrot (2001)): the standard Brownian motion when $H = 1/2$ and $\vartheta(t) = t$; the fractional Brownian motion when $H \neq 1/2$ and $\vartheta(t) = t$; and the L-stable model, when $H = 1/2$ and $\vartheta(t)$ is a stable subordinator.‡ Moreover, the intensity of the long-term memory displayed by $B_H[\vartheta(t)]$ depends on the powers of returns, as for actual financial time series (see, e.g. Ding *et al.* (1993) or Lobato and Savin (1998)).

Under Assumptions 1–3, the scaling function $\tau(q)$ is such that

$$\tau\left(\frac{1}{H}\right) = 0 \quad (4)$$

and

$$\tau_X(q) = \tau_\vartheta(Hq). \quad (5)$$

Thus, relation (4) provides the estimate H of the parameter of the fBm in Assumption 1. Linking the scaling function τ_X of the price series to the scaling function τ_ϑ of the trading time, relation (5) gives information about the properties of the trading time $\vartheta(t)$ (if τ_X is quadratic the stochastic multifractal measure μ must be built using lognormal masses).

It is also possible to detect a multifractal process by studying the multifractal spectrum $f(\alpha)$, defined as the Legendre transform of the scaling function $\tau(q)$ (Calvet *et al.* 1997)

$$f(\alpha) = \inf_q [q\alpha - \tau(q)]. \quad (6)$$

In fact, a truly multifractal process is expected to present a concave spectrum, such that the particular value α_0 satisfying $f(\alpha_0) = 1$ is the point of maximum of the multifractal spectrum, i.e. the most probable Hölder exponent of the time series (see next section).

Remark 1: As stated above, multifractality implies the linearity of the partition function when drawn in log–log scale and the concavity of $\tau(q)$. Notice that the reverse is not necessarily true and this argument will be used in section 4 to show that financial data do display the same behaviour (linearity and concavity) even when they are modelled by a large class of non-multifractal Gaussian processes.

3. Multifractionality and the mBm

It is well known that the celebrated fractional Brownian motion, $B_H(t)$—introduced by Mandelbrot and Van Ness (1968)—is characterized by a slowly decaying autocorrelation function depending on the parameter $H \in (0, 1]$, named *Hurst* (or *Hölder*) exponent. When $H \in (1/2, 1]$ the motion displays paths more and more persistent as H increases to 1; for $H = 1/2$ it reduces to the ordinary Brownian motion and, finally, when $H \in (0, 1/2)$ the sequences show an antipersistent (or mean-reverting) behaviour, the more evident the more H decreases to 0. From a geometrical point of view, H determines the (constant) roughness of the sample paths of the fBm and is linked to the fractal dimension D of the graph by the relation $D = 2 - H$.

The intensity of the long-range dependence of its increments depends on both H and the lag; so, with the latter fixed, the autocorrelation only depends on the Hölder exponent. The most intuitive generalization of the fBm

†The notion of multifractal measure substantially founds the whole literature related to multifractal analysis. For an introduction and motivation see, e.g. Mandelbrot (1999) or Harte (2001).

‡Expressed briefly, Mandelbrot and Taylor (1967) proved that a sequence with stable increments having a characteristic exponent α less than 2 can be written as a subordinated process with normal increments, the variance process of which follows a stable distribution of its own with characteristic exponent $\alpha - 1$ (meaning that the expected value of the variance process is infinite, or undefined).

therefore can be obtained on replacing H by the (Hölderian) function $H: t \in [0, \infty) \to (0, 1]$.† This extension, known as *multifractional Brownian motion* (mBm) (see Péltier and Lévy Véhel (1995), Benassi *et al.* (1997), and Ayache and Lévy Véhel (2000)), is no longer stationary nor self-similar but, compared to the fBm, has the advantage of being very flexible since the function $H(t)$ can model phenomena whose sample paths display a time changing regularity. As for the MMAR, in many situations $H(t)$ can indeed be thought itself as a random process (Ayache and Taqqu 2003); in the financial context it can be interpreted as the market's pointwise memory which obviously depends on the new information, i.e. on a process that is ultimately random. The standard mBm admits the following moving average representation‡

$$M(t) = V_{H(t)}^{1/2} \int_{\mathbb{R}} f_t(s) \mathrm{d}B(s) \tag{8}$$

with

$$f_t(s) = \frac{1}{\Gamma\left(H(t) + \frac{1}{2}\right)} \{ |t - s|^{H(t) - \frac{1}{2}} \mathbb{I}_{]-\infty, t]}(s) - -s^{H(t) - \frac{1}{2}} \mathbb{I}_{]-\infty, 0]}(s) \}.$$

In (8) $V_{H(t)} = \Gamma(2H(t) + 1) \sin(\pi H(t))$ is a normalizing factor such that the process variance at unit time is 1, $\mathbb{I}$ is the indicator function and B denotes the Brownian motion.

From (8) it can be seen that the mBm is Gaussian with mean 0 and variance $\mathbb{E}[M^2(t)] = t^{2H(t)}$ and that there exists a positive continuous function σ_t, defined for $t \geq 0$, such that the increments $M(t+h) - M(t)$ are normally distributed with variance equal to $h^{2H(t)} \sigma_t^2$ as h tends to zero. This property will be exploited to estimate the function $H(t)$.

What is important here is to stress that generally, unless $H(t)$ is chosen both multifractal and everywhere smaller than the process' Hölder regularity, the mBm *is not* multifractal.§ This can be easily seen using two very simple arguments:

(a) the increments of $M(t)$ do not generally form a stationary sequence;
(b) it is well known that if Y normally distributes with mean 0 and variance σ^2 then

$$\mathbb{E}(|Y|^q) = \frac{2^{q/2} \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \sigma^q.$$

Therefore, setting

$$c(q) = \frac{2^{q/2} \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

for the absolute moments of the mBm one immediately gets $\mathbb{E}(|M(t)|^q) = c(q) t^{qH(t)}$. Hence, combining this with (1), it would follow $\tau(q) = qH(t) - 1$, that is the scaling function would depend also on t, which contradicts relation (1) stating that $\tau(q)$ is a deterministic function of the sole q.

Although the mBm is not generally multifractal, we will show in the next section that—once fitted on financial data—this process can mimic the partition function, the scaling function and the spectrum of a genuine multifractal process, making questionable the interpretation of the results provided by the main tools used in multifractal analysis.

3.1. Estimating the function *H(t)* of the mBm

To fit the mBm on actual financial data the functional parameter is needed: given $H(t)$ one can in fact simulate the paths of the mBm having the dependence structure assigned by $H(t)$ itself. The problem of estimating $H(t)$ is addressed by Bianchi (2005) who, exploiting the Gaussianity of $M(t)$, estimates the pointwise Hölder exponent by using the following sliding window-like estimator

$$H_{\delta,N}^k(i) = -\frac{\log\left(\frac{\sqrt{\pi}}{2^{k/2} \delta \Gamma\left(\frac{k+1}{2}\right)} \sum_{j=i-\delta}^{i-1} \left| X_{j+1,N} - X_{j,N} \right|^k \right)}{k \log(N-1)},$$
$$(i = \delta + 1, \ldots, N). \tag{9}$$

†Given the metric spaces (X, d_X) and (Y, d_Y), the function $f: X \to Y$ is called a Hölder function with exponent $\beta > 0$ if, for each $x, y \in X$ such that $d_X(x, y) < 1$ there exists a constant k satisfying the condition

$$d_Y(f(x), f(y)) \leq k d_X(x, y)^\beta. \tag{7}$$

The hölderianity is sufficient (but not necessary) for the continuity for the process.

‡The representation of the mBm given by Benassi *et al.* (1997) is of 'harmonizable' type and differs from the one considered here, which is of 'moving-average' type. The authors in fact assume the following:

Definition. Let $\alpha: x \in R^d \to (0, 1)$ be a function in $C^A(R^d)$ where $A = \sup \alpha(x)$. The multifractional Brownian motion of order $\alpha(x)$ is defined by

$$B_\alpha(x) = \int \frac{\mathrm{e}^{\mathrm{i}x\xi} - 1}{|\xi|^{\alpha(x) + d/2}} \mathrm{d}W(\xi),$$

where $\mathrm{d}W(\xi)$ is the white noise.

§See Ayache (2000) for a discussion on how to generalize the mBm in order to obtain possibly multifractal scaling.

Table 1. Actual frequencies of rejection (at $\alpha = 10\%$, 5% and 1%) of the Shapiro–Wilk test for different values of the lag δ.

$\delta =$		200	150	100	50	40	30	20	10
Americas									
Dow Jones Ind.	10%	0.7344	0.6245	0.4976	0.3099	0.2618	0.2120	0.1651	0.1224
	5%	0.6836	0.5733	0.4290	0.2345	0.1955	0.1476	0.1036	0.0673
	1%	0.5832	0.4646	0.3179	0.1430	0.1079	0.0708	0.0415	0.0194
Bovespa	10%	0.6018	0.5305	0.4393	0.2683	0.2200	0.1629	0.1355	0.1011
	5%	0.5394	0.4454	0.3680	0.2115	0.1643	0.1150	0.0852	0.0625
	1%	0.4503	0.3577	0.2740	0.1256	0.0930	0.0610	0.0351	0.0202
Asia/Pacific									
All Ordinaries	10%	0.6275	0.5327	0.4445	0.2782	0.2355	0.2005	0.1551	0.1170
	5%	0.6047	0.5021	0.3902	0.2119	0.1714	0.1362	0.1014	0.0668
	1%	0.5424	0.4356	0.3086	0.1343	0.1032	0.0776	0.0450	0.0218
Hang Seng	10%	0.8487	0.7294	0.5535	0.3210	0.2681	0.2222	0.1684	0.1143
	5%	0.8118	0.6571	0.4835	0.2580	0.2172	0.1668	0.1173	0.0723
	1%	0.6861	0.5350	0.3897	0.1891	0.1383	0.0990	0.0630	0.0269
Nikkei 225	10%	0.8668	0.7611	0.5953	0.3427	0.2840	0.2220	0.1599	0.1167
	5%	0.8274	0.6940	0.5033	0.2750	0.2213	0.1599	0.1087	0.0677
	1%	0.6967	0.5457	0.3705	0.1604	0.1202	0.0900	0.0490	0.0214
Europe									
CAC 40	10%	0.6046	0.4851	0.3095	0.1824	0.1723	0.1545	0.1389	0.1023
	5%	0.5215	0.3831	0.2223	0.1302	0.1105	0.0968	0.0804	0.0541
	1%	0.3672	0.2239	0.1400	0.0629	0.0517	0.0446	0.0241	0.0167
Footsie 100	10%	0.5038	0.4068	0.3093	0.2051	0.1803	0.1459	0.1360	0.1099
	5%	0.3902	0.3000	0.2420	0.1393	0.1110	0.0872	0.0793	0.0559
	1%	0.2604	0.1933	0.1348	0.0660	0.0482	0.0350	0.0205	0.0145
MibTel	10%	0.5964	0.4901	0.3184	0.1730	0.1454	0.1362	0.1170	0.0957
	5%	0.5234	0.4202	0.2688	0.1362	0.1067	0.0865	0.0677	0.0571
	1%	0.4170	0.3365	0.2082	0.0957	0.0762	0.0532	0.0305	0.0188

In (9), $k \geq 1$ is an integer, N denotes the length of the sample and δ the length of the window within which the Hölder function is assumed constant (so is locally the variance of the mBm). As usual, X indicates the log price process.

When $H(t) < 3/4$, which is indeed the case of interest for financial time series, $H^k_{\delta,N}(t)$ is an unbiased estimator of $H(t)$, normally distributed with variance proportional to δ^{-1}. So, if X is an mBm, one can use the sequence $H^k_{\delta,N}$ to simulate new traces of mBm displaying almost the same statistical properties of the original series X. The adverb *almost* is basically due to two opposite but non-necessarily compensating effects. For small values, because of the estimator's variance, the sequence $H^k_{\delta,N}$ tends to be more jagged than $H(t)$ itself and, on the other hand, having $H^k_{\delta,N}$ with the form of a δ-moving average, as the lag increases the sequence $H^k_{\delta,N}$ is likely to be smoother than $H(t)$. When the estimator is applied to financial data, this trade-off problem has a natural upward limit in the requirement of normality of the increments $X_{j+1,N} - X_{j,N}$ (see Bianchi (2005) for a discussion of these and other technicalities of the estimator (9)). Table 1 clearly shows how for the stock indexes considered in our analysis Gaussianity arises as δ decreases. The table has been obtained as follows: given the sample T of the log variations of a stock index, for each fixed lag δ the Shapiro–Wilk normality test has been sequentially run over all the $T - \delta + 1$ overlapping subsequences $[1, \delta], [2, \delta + 1], \ldots, [T - \delta + 1, T]$. The actual frequencies of rejection converge to the nominal ones (for $\alpha = 10\%$, 5% and 1%) as δ decreases (see in particular column $\delta = 10$), indicating that indeed the log variations of the sequences are locally Gaussian. Besides justifying the application of estimator (9), this suggests that the mBm could really be a good model for financial dynamics.

An example of the filtering obtained by using (9) is provided in figure 1, which concerns the index Footsie 100 with $\delta = 30$ and $k = 2$.

4. The empirical application

In this section the partition-based methodology used to detect multifractal scaling in finance will be employed to show how non-univocal its outputs are. As recalled above, multiscaling is deduced to hold when, for any q, the estimated partition function is found to be linear with Δt and the scaling function is found concave with q.

The analysis is concerned with the closing prices of the eight major stock indexes listed in table 2.

Following Calvet and Fisher (2002), we have set $\Delta t = 1, \ldots, 125$ and $q = 0.5, \ldots, 5$ (with step 0.01).

Figures 2–9 summarize (*a*) the partition functions, (*b*) the scaling functions and (*c*) the spectra of the original time series. Even with different degrees, in all cases the partition functions look linear and the scaling functions concave. For all the indexes but Bovespa, H is very close to 1/2; this would mean that the compound model $X(t) = B_H[\vartheta(t)] = B_{1/2}[\vartheta(t)]$, namely that the stochastic trading time is compounded with an ordinary Brownian motion. In this case, theorem 2 in Calvet and

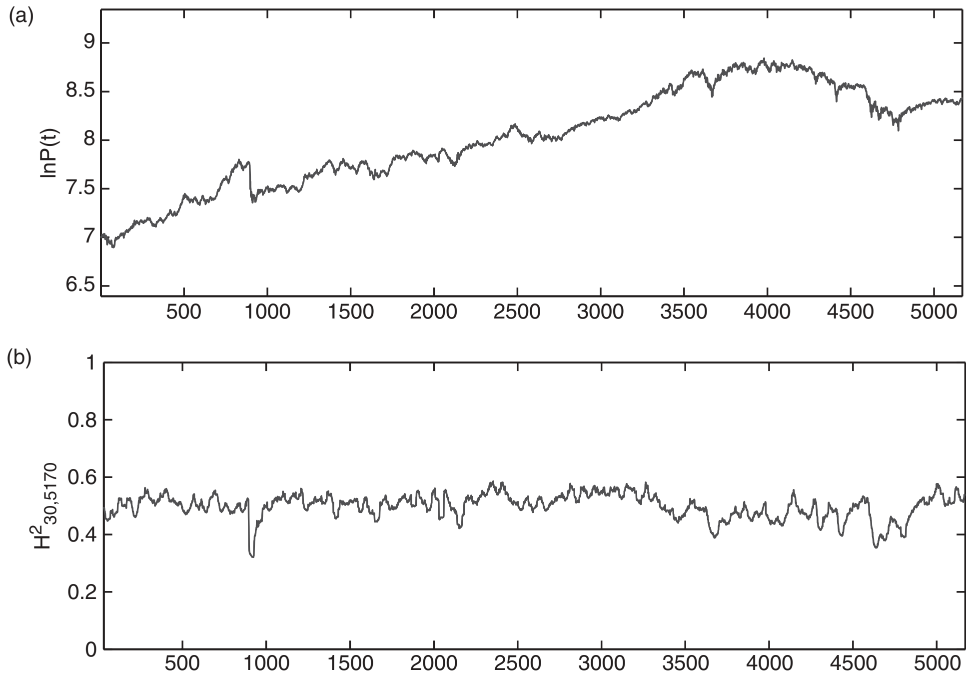

Figure 1. (a) Footsie 100 Index (log scale) and (b) estimated functional parameter $H^2_{30,\,5170}(t)$.

Table 2. Analysed stock indexes.

Americas			
Dow Jones Ind.	19281001	20040923	19076
Bovespa	19930427	20040920	2819
Asia/Pacific			
All Ordinaries	19840803	20040920	5088
Hang Seng	19861231	20040920	4384
Nikkei 225	19840104	20040917	5099
Europe			
CAC40	19900301	20040920	3658
Footsie 100	19840402	20040920	5170
MibTel	19930719	20041001	2821

Fisher (2002) holds and the price process $P(t)$ is a semi-martingale with respect to its natural filtration and $X(t)$ is a martingale with finite variance and uncorrelated increments.

At this point each sequence has been filtered as follows.

1. The functional parameter $H(t)$ has been estimated by $H^2_{30,\,N}(t)$. Although not optimal according to table 1, the lag $\delta = 30$ has been set in order to maintain the estimator's variance at an acceptably low level. In addition, visual inspection of the considered time series suggests that the actual frequencies of rejection differ from the nominal ones when $\delta = 30$ mainly because of the contribution of specific clusters along the sample paths. In other words, the segments of data for which normality does not hold are very concentrated in time and generally correspond to well-known financial crises.
2. The sequence $H^2_{30,\,N}(t)$ has been used to simulate, for each index, an mBm of length equal to the number of data in the original time series.†

Once the surrogate mBms have been fitted on the corresponding actual series, the simulated samples have been analysed according to the partition-based technique. The results are graphically displayed in figures 10–17. Table 3 shows how close are the values of H and α_0 estimated by actual and surrogate data. On average, the difference between the two sets is 0.04 (0.01 excluding the Bovespa) for H and 0.05 (0.02 excluding the Bovespa) for α_0. So the surrogate series look virtually indistinguishable from the original ones, both in terms of estimated H and α_0 and graphical outputs.

Also notice that the linearity of the partition function, measured by Pearson's coefficient of determination in table 4, is for the surrogate time series as good as that of the actual ones and in all cases it is very close to 1.

One might object that the results here obtained could be originated by the multifractality of the surrogate mBms. In fact, as recalled in section 3, if the function $H(t)$ were multifractal and everywhere smaller than the process' Hölder regularity, the mBm

†The simulation has been carried out by using the circulant embedding method (Wood and Chan 1994).

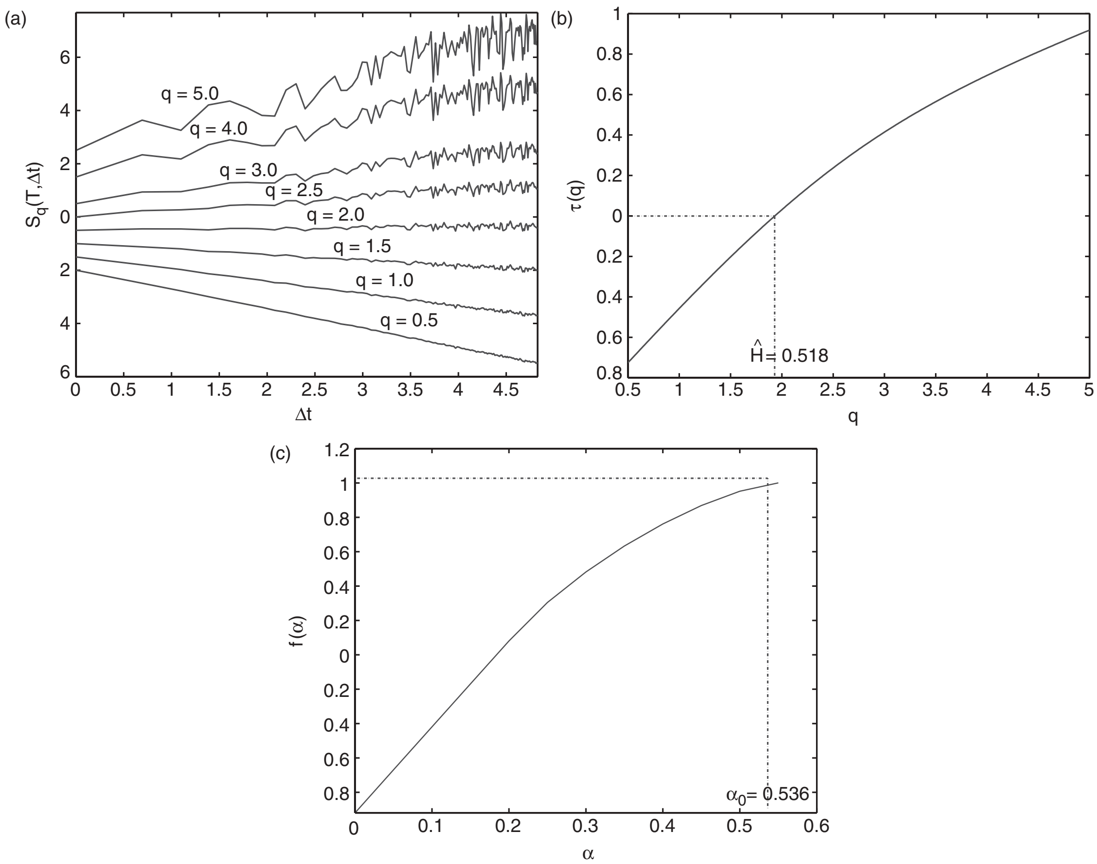

Figure 2. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, Dow Jones Index.

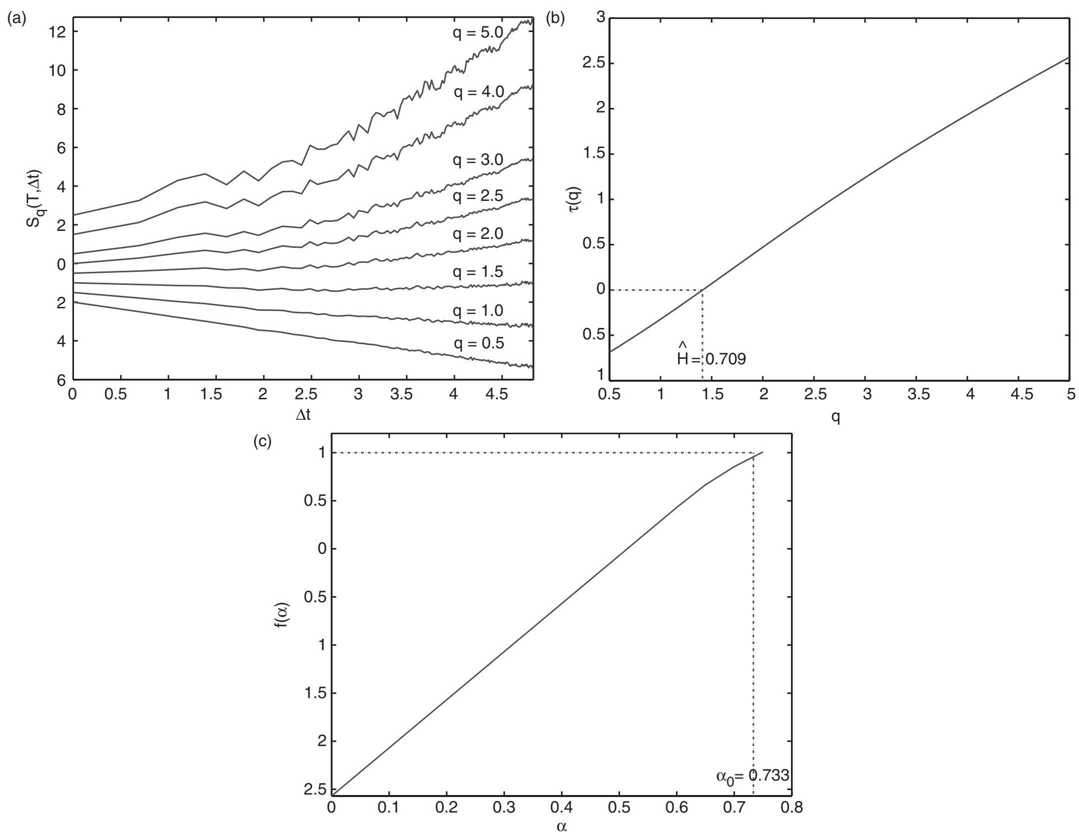

Figure 3. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, Bovespa.

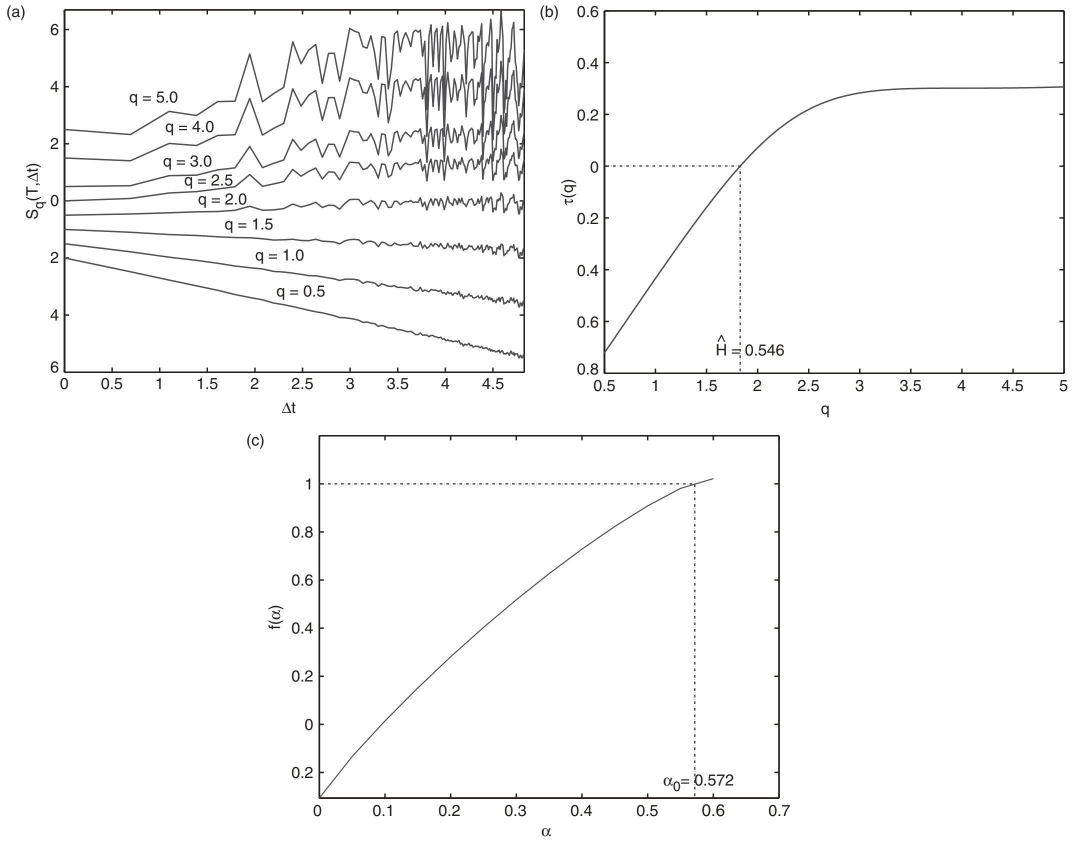

Figure 4. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, All Ordinaries.

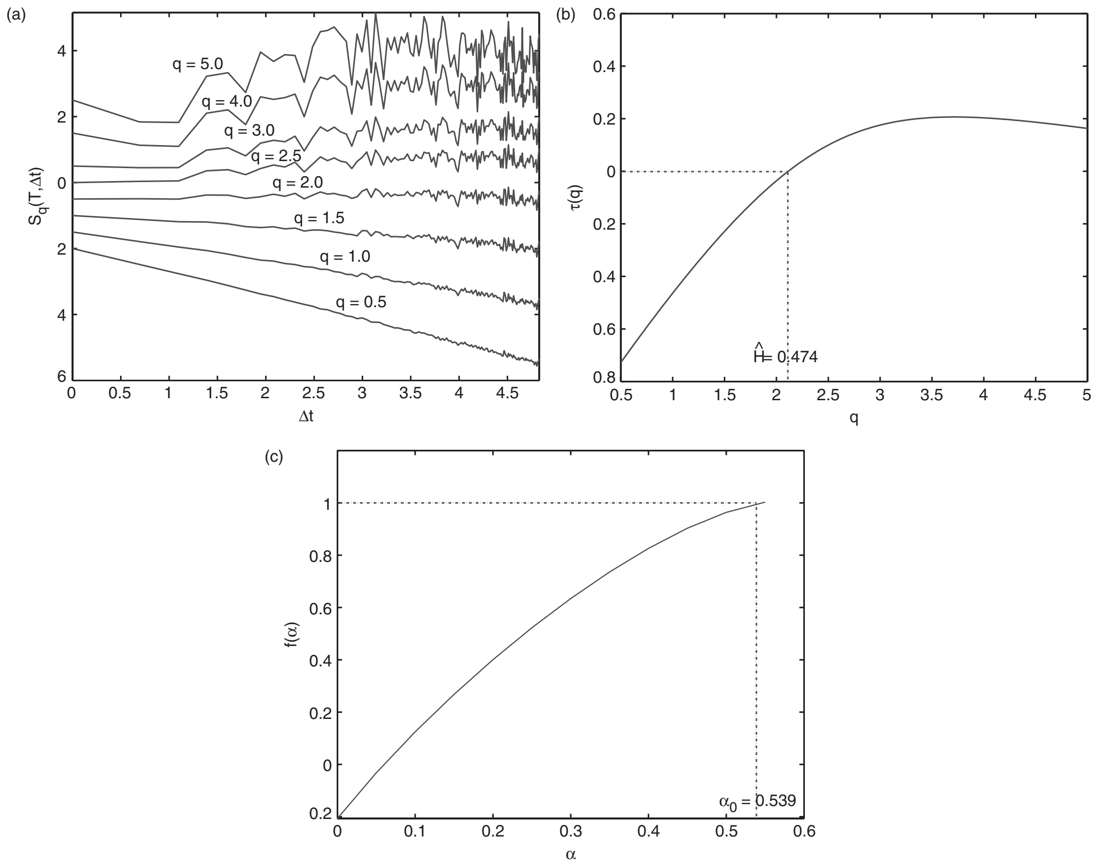

Figure 5. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, Hang Seng.

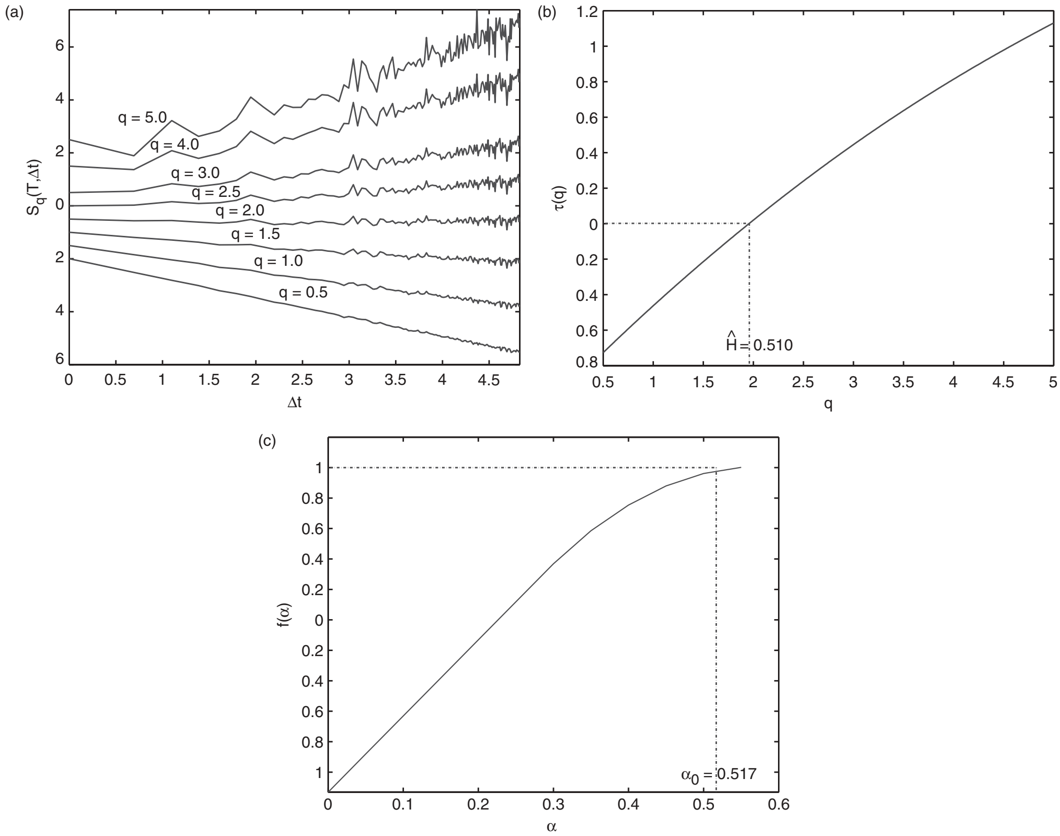

Figure 6. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, Nikkei 225.

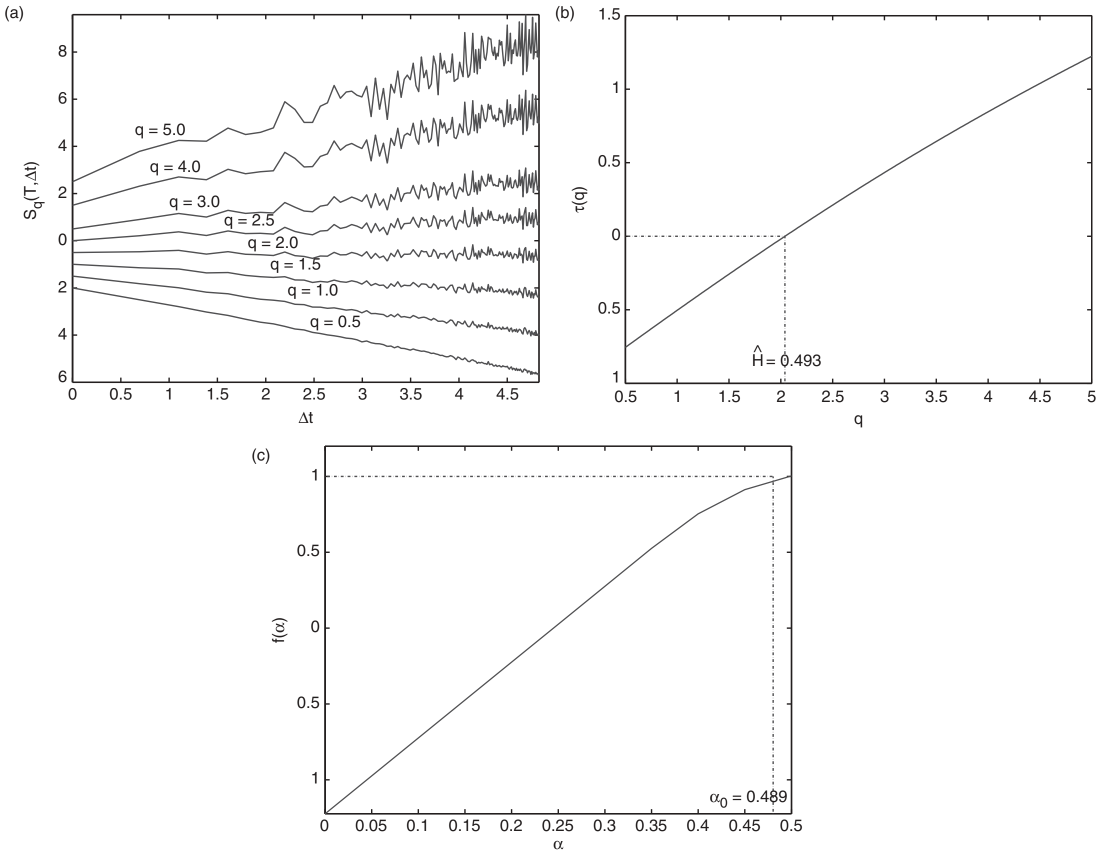

Figure 7. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, CAC 40.

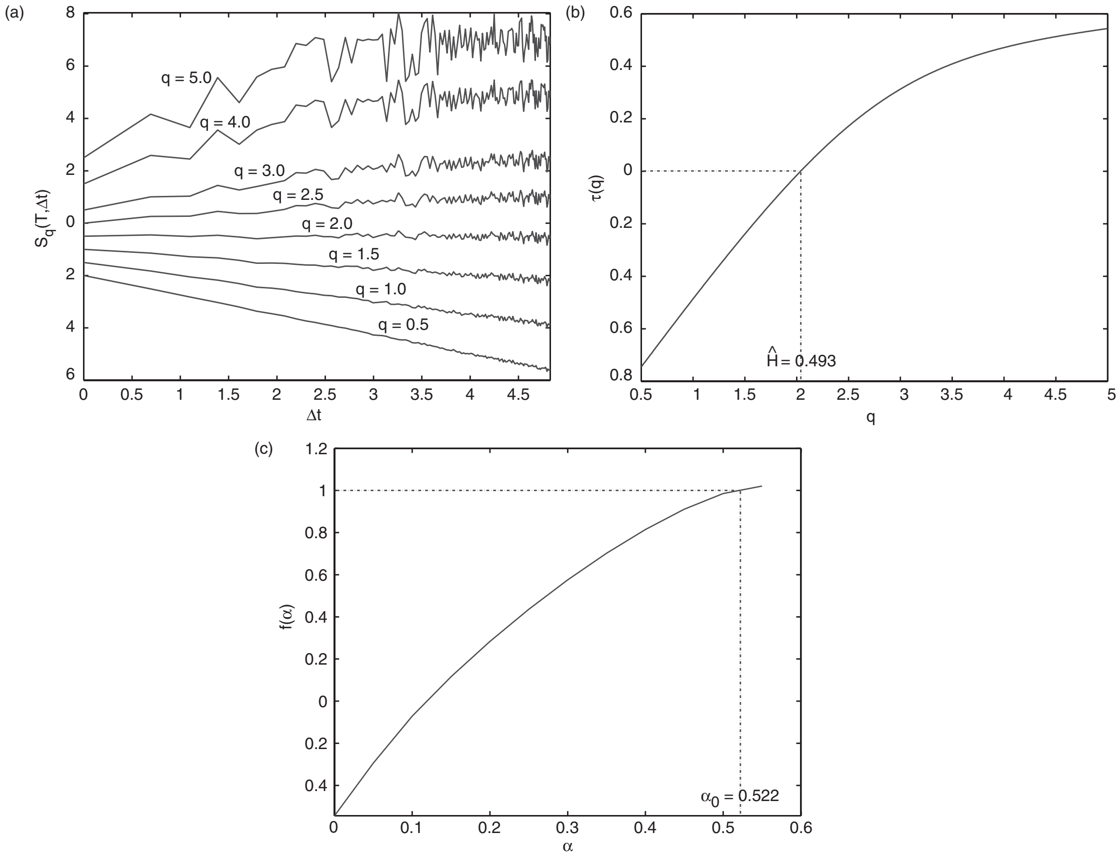

Figure 8. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, Footsie 100.

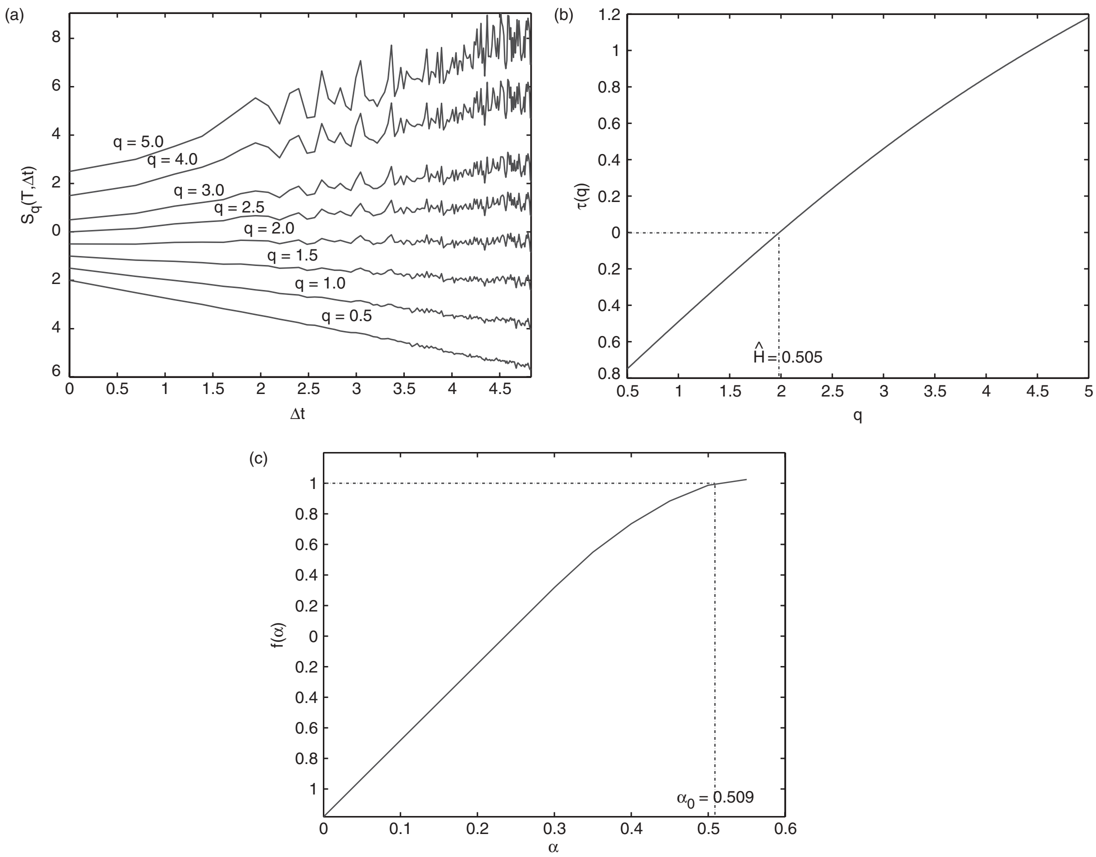

Figure 9. (a) Partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the considered stock index, MibTel.

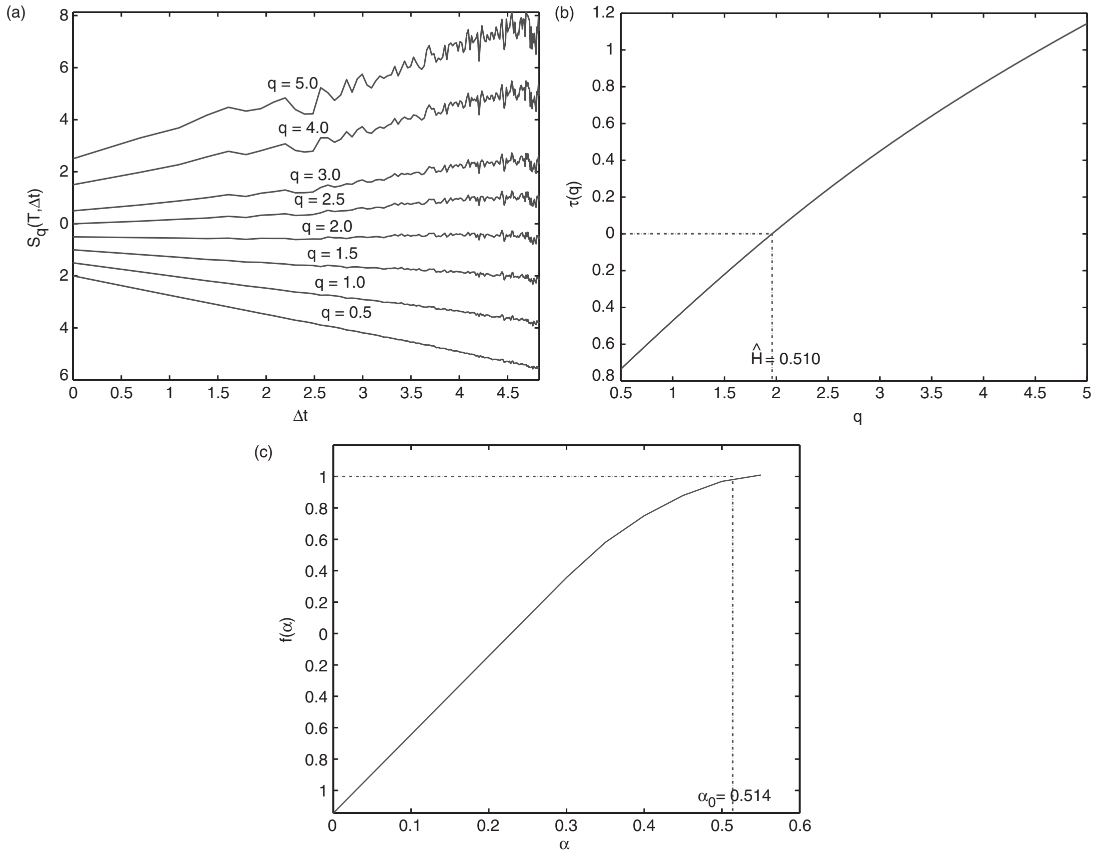

Figure 10. Dow Jones Index: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

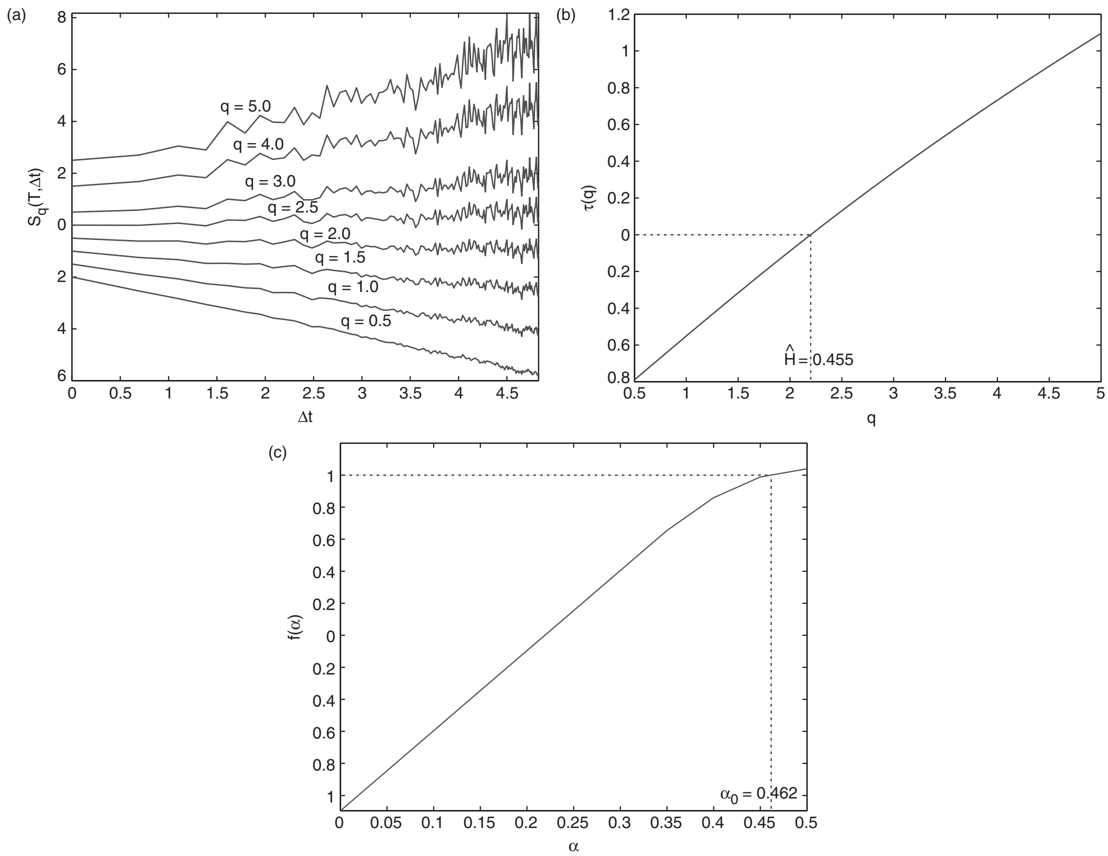

Figure 11. Bovespa: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

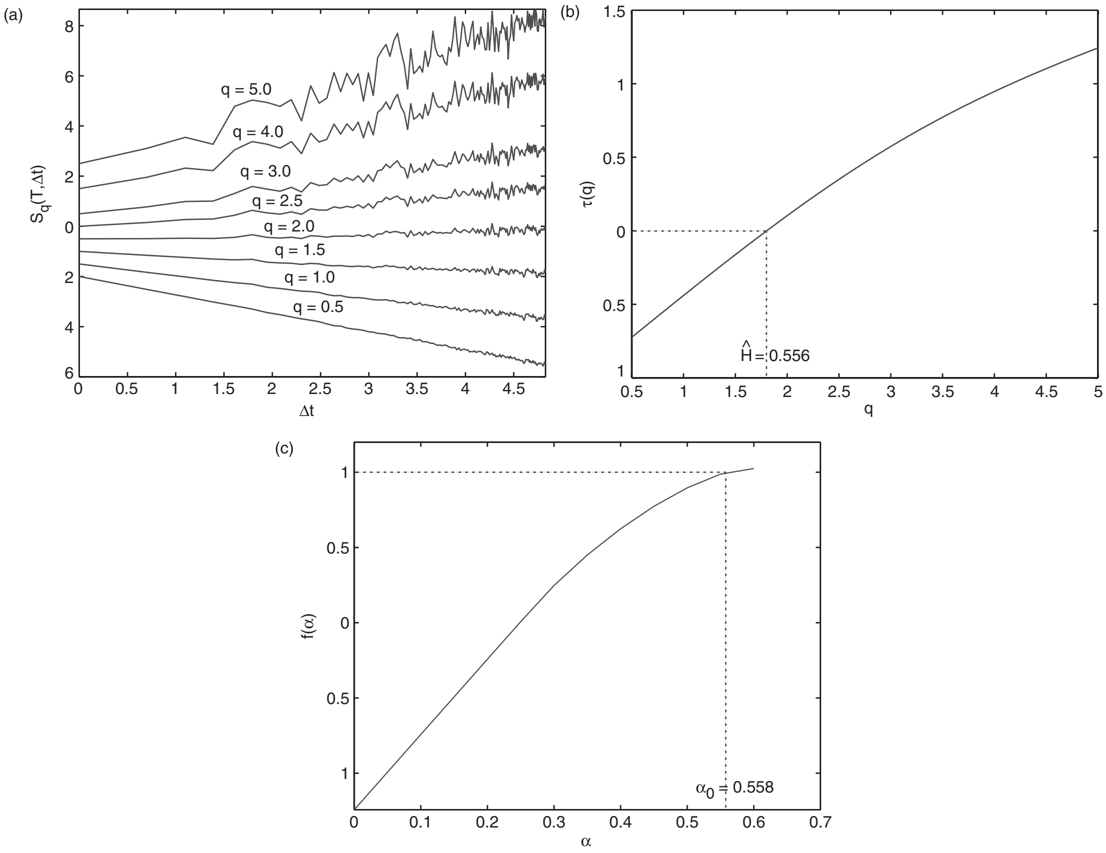

Figure 12. All Ordinaries: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

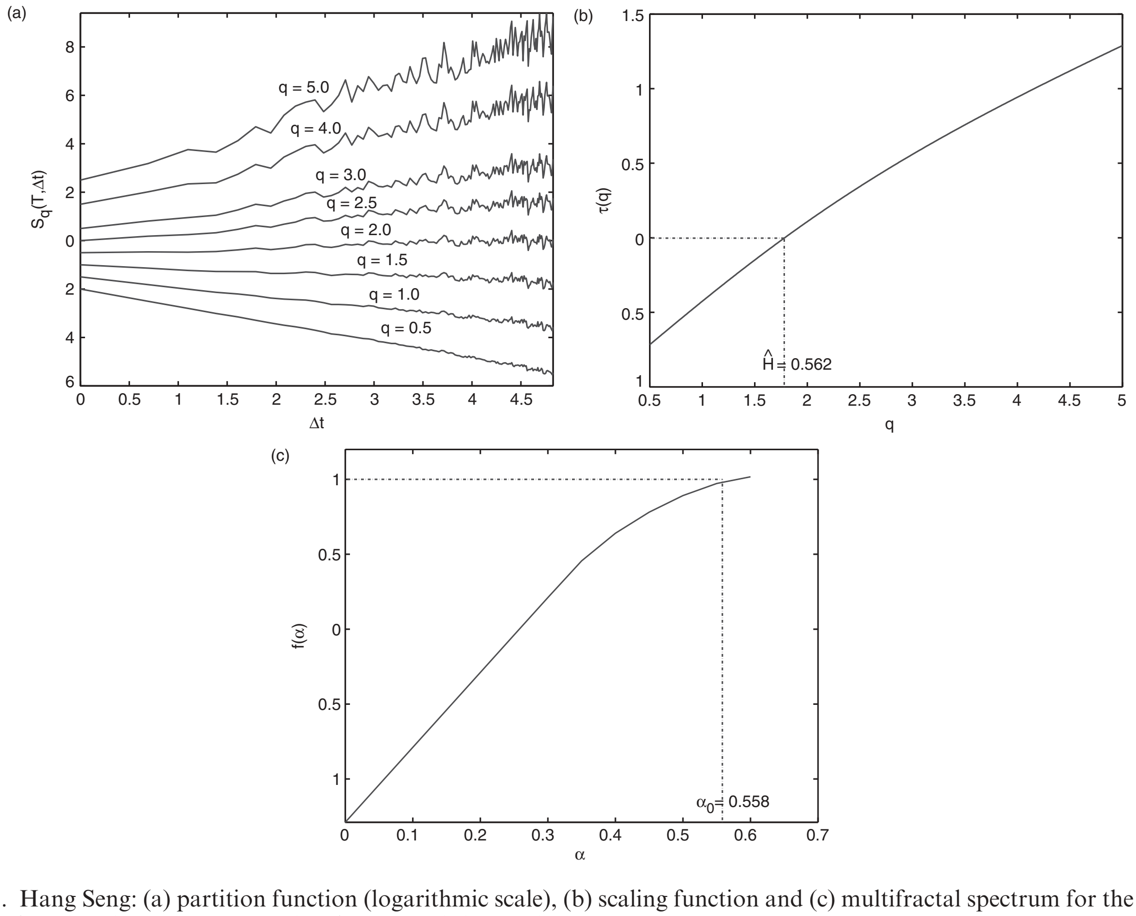

Figure 13. Hang Seng: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

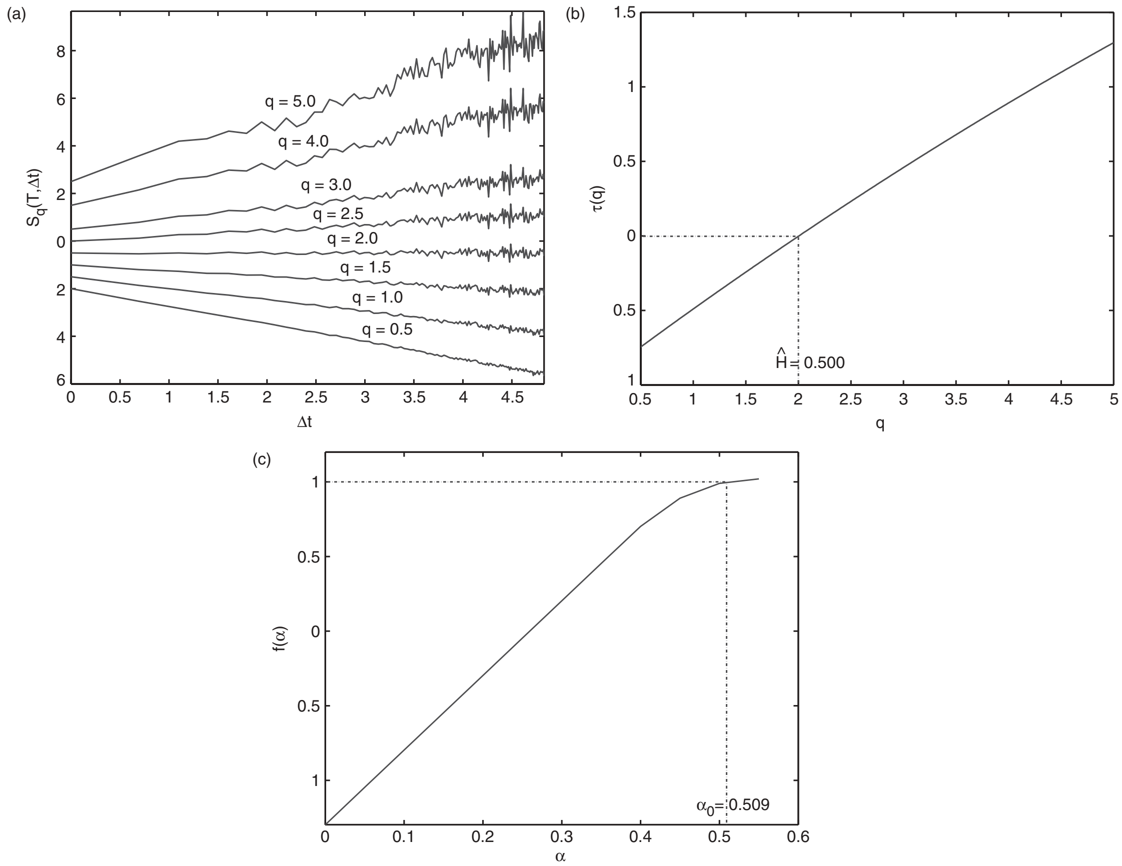

Figure 14. Nikkei 225: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

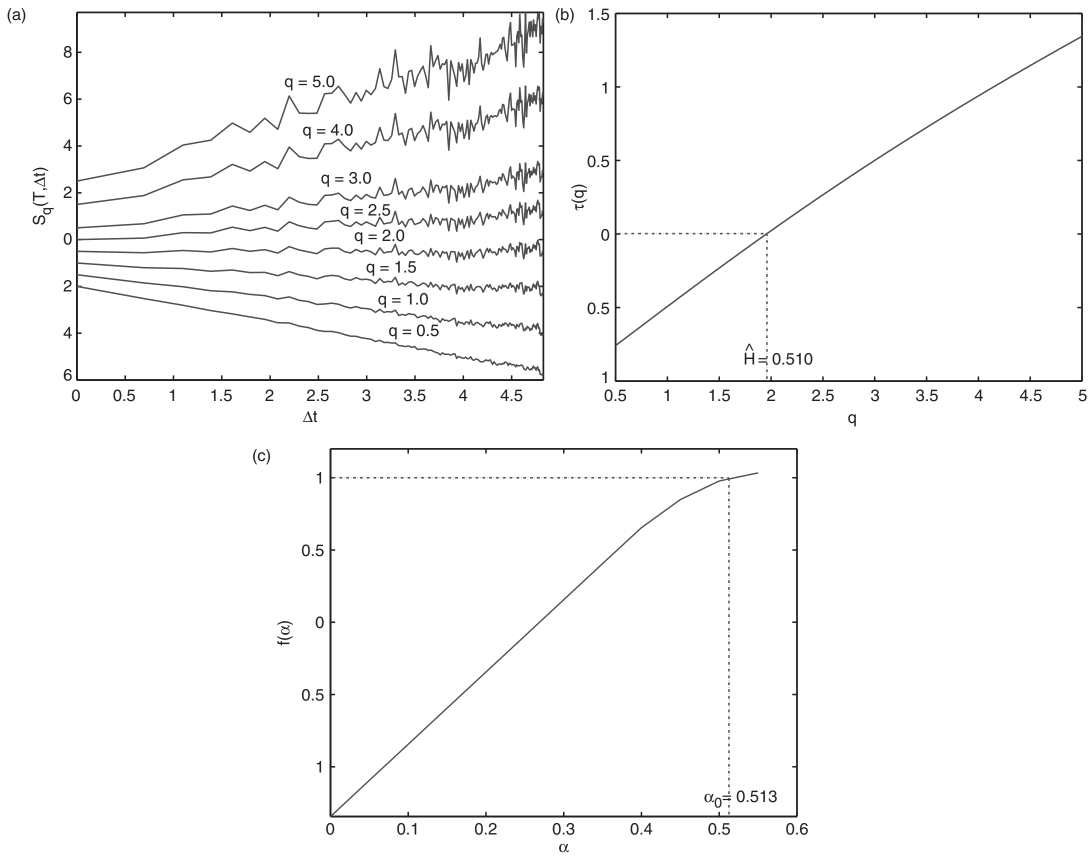

Figure 15. CAC 40: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

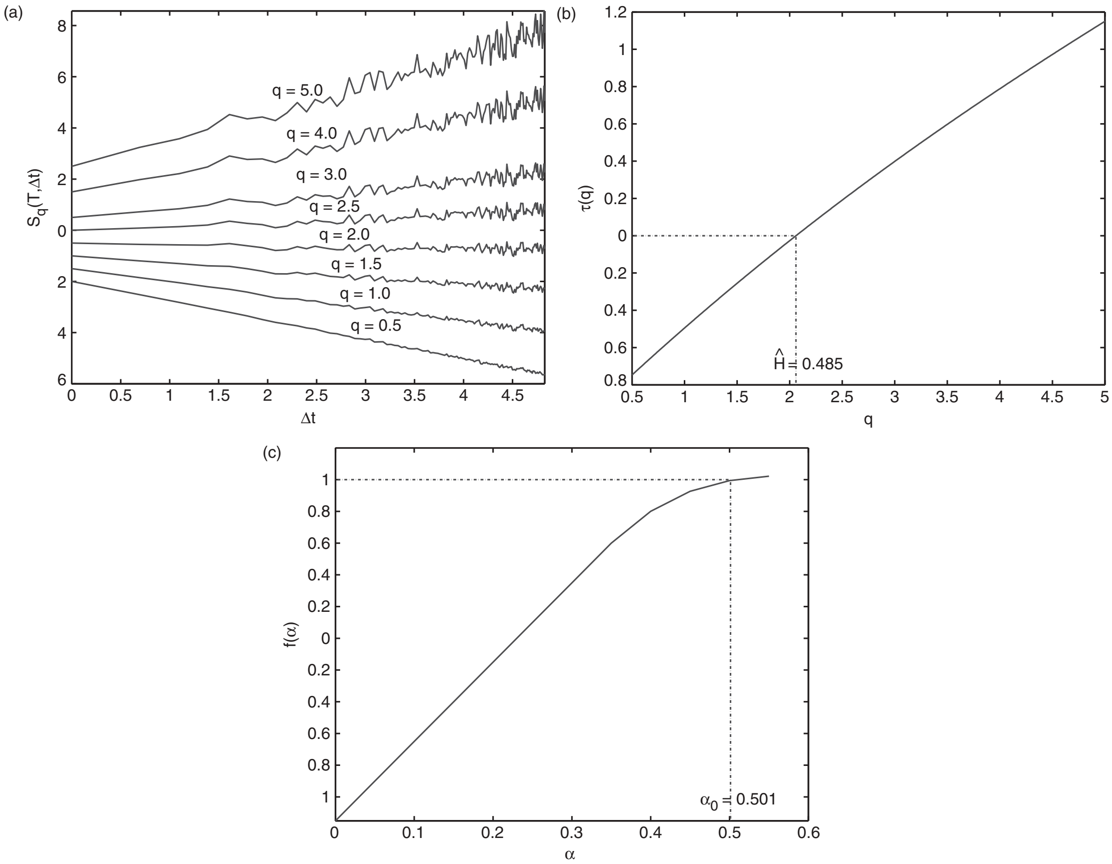

Figure 16. Footsie 100: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

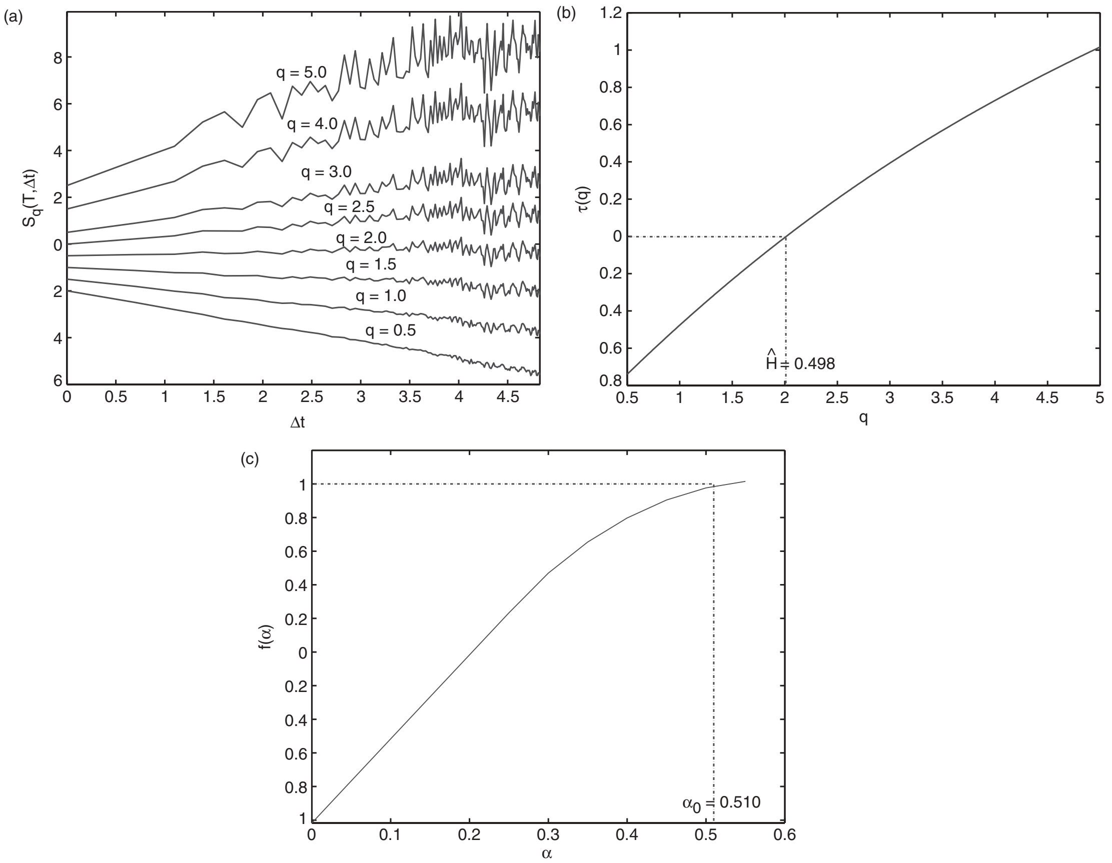

Figure 17. MibTel: (a) partition function (logarithmic scale), (b) scaling function and (c) multifractal spectrum for the surrogate mBm with functional parameter $H(t)$ fitted on actual time series.

Table 3. Actual and simulated $\widehat{H}$'s and $\widehat{\alpha}_0$'s.

Index	Actual		Surrogate	
	$\widehat{H}$	$\widehat{\alpha}_0$	$\widehat{H}$	$\widehat{\alpha}_0$
Dow Jones	0.518	0.536	0.510	0.514
Bovespa	0.709	0.733	0.455	0.462
All Ordinaries	0.546	0.572	0.556	0.558
Hang Seng	0.474	0.539	0.457	0.467
Nikkei 225	0.510	0.517	0.500	0.509
CAC40	0.493	0.489	0.510	0.513
Footsie 100	0.493	0.522	0.485	0.501
Mibtel	0.505	0.509	0.498	0.510

itself would be multifractal. Figure 18 shows that this is not the case: it displays the coefficients of determination of the partition functions calculated with respect to the sequence $H^2_{30,N}$ of each considered index. If $H(t)$ were multifractal, then the corresponding partition function would be linear for any q, whereas it is apparent that for all the indexes the linearity heavily gets worse as the value of q increases.

On the other hand, figure 18 alone is not conclusive because it is not based on the true pointwise Hölder exponents but only on the estimated ones; the length δ of the estimation window could affect (e.g. deteriorate) the linearity of the partition function even of a true multifractal process. To show that this is not the case, a test has been drawn to evaluate the effects of estimating $H(t)$ for a genuine multifractal process.

The test has been carried out for each considered stock index but here it is reported for the sake of shortness just for the Dow Jones, which is the longest analysed time series. The procedure can be described through the following steps:

1. $H(t)$ has been chosen to be a true multifractal function.
   As recalled above, a quadratic spectrum implies the multifractal-generating mechanism to use masses with lognormal distribution, whose mean (λ) and standard deviation (σ) can be estimated through H and $\widehat{\alpha}_0$. For the DJIA, these, respectively, equal 0.518 and 0.536 (see table 3). In this way, as proved in Calvet *et al.* (1997), we get
   $$\widehat{\lambda} = \frac{\widehat{\alpha}_0}{\widehat{H}} \quad \text{and} \quad \widehat{\sigma}^2 = \frac{2(\widehat{\lambda}-1)}{\log b}, \tag{12}$$
   where b is the number of equal sized intervals subdividing each interval of the previous step in the generation of the multiplicative cascade (in this application we have set $b=2$ and $T=16,384$). So, with the obtained $\widehat{\lambda} \cong 1.035$ and $\widehat{\sigma} \cong 0.3167$, the multifractal measure $\mu(t)$ has been generated. The multifractal process driven by $\mu(t)$ has been obtained as
   $$H_1(t) = \sum_{s=1}^{t} \mu(s)^{\widehat{H_0}} \zeta,$$
   where $t = 1, \ldots, T$; $\zeta \sim \mathcal{N}(0,1)$.
   Since for all the indexes, $H^2_{30,N}(t)$ ranges from 0.35 to 0.65 the sequence $H_1(t)$ has been normalized† and detrended by removing the best linear fit. In this way we have obtained the sequence $H(t)$ having a mean nearly equal to the mean of $H^2_{30,N}(t)$. Obviously, rescaling and detrending do not affect the multifractality of the process.
2. An mBm has been simulated with functional parameter $H(t)$. The partition function, the scaling function and the spectrum of the simulated series are reproduced in figures 19(a)–(c). As expected, the plots look approximately the same as the ones originated by the corresponding actual data (figures 2(a)–(c)), with $\widehat{H} = 0.503$ and $\widehat{\alpha}_0 = 0.513$. Being the surrogate mBm multifractal (because such is its functional parameter), its partition function is characterized by very high R^2's, in the range $[0.9876, 0.9960]$ for q ranging from 0.5 to 5. Again, similar results have been obtained for each of the considered indexes.
3. $H^2_{30,N}(t)$ has been estimated from the path of the mBm surrogated at point 2. Figure 20 reproduces the sequences $H(t)$ and $H^2_{30,N}(t)$; this has been shifted downwards of the quantity 0.1 to facilitate the reading of the graph. As testified by the mean square error, which equals 0.0111, even using a very small window (δ is just 30 trading days), the estimator works well in shadowing the true $H(t)$.
4. The partition function of both $H(t)$ and $H^2_{30,N}(t)$ have been calculated and their linearity has been tested as above through the behaviour of the coefficients determined for increasing values of q.

Figure 21 replicates figure 18 with the addition of the values of Pearson's coefficients calculated at point 4. Despite what happens for the actual time series, the R^2's of the partition functions of both $H(t)$ and $H^2_{30,N}(t)$ remain close to one even when q increases. More precisely, although the R^2's estimated for the series $H^2_{30,N}(t)$ are systematically below the corresponding values calculated for the truly multifractal sequence $H(t)$, the difference decreases for increasing values of q, just the opposite of what happens for the actual financial series, for which the R^2's decline very quickly. In addition,

† The following normalization has been considered:

$$H_2(t) = \frac{H_1(t) - \min(H_1(t))}{\max(H_1(t)) - \min(H_1(t))} \times 0.3 + 0.35.$$

Table 4. R^2's of the partition functions (in parentheses the r^2's values of the corresponding fitted mBms).

$q=$	0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
Americas								
Dow Jones Ind.								
	0.9953	0.9964	0.9961	0.9938	0.9888	0.9809	0.9578	0.9298
	(0.9910)	(0.9926)	(0.9924)	(0.9913)	(0.9896)	(0.9872)	(0.9801)	(0.9695)
Bovespa	0.9850	0.9909	0.9906	0.9887	0.9868	0.9849	0.9809	0.9764
	(0.9395)	(0.9602)	(0.9644)	(0.9639)	(0.9612)	(0.9574)	(0.9479)	(0.9374)
Asia/Pacific								
All Ordinaries	0.9956	0.9900	0.9880	0.9780	0.9525	0.9066	0.7722	0.6352
	(0.9830)	(0.9907)	(0.9926)	(0.9921)	(0.9898)	(0.9858)	(0.9723)	(0.95322)
Hang Seng	0.9841	0.9868	0.9849	0.9794	0.9691	0.9519	0.8894	0.7905
	(0.9748)	(0.9818)	(0.9836)	(0.9835)	(0.9821)	(0.9795)	(0.9715)	(0.9610)
Nikkei 225	0.9881	0.9910	0.9913	0.9905	0.9888	0.9864	0.9783	0.9656
	(0.9768)	(0.9841)	(0.9861)	(0.9857)	(0.9840)	(0.9812)	(0.9736)	(0.9640)
Europe								
CAC40	0.9791	0.9835	0.9818	0.9777	0.9723	0.9658	0.9510	0.9352
	(0.9407)	(0.9636)	(0.9704)	(0.9716)	(0.9701)	(0.9672)	(0.9593)	(0.9503)
Footsie 100	0.9868	0.9902	0.9905	0.9889	0.9835	0.9712	0.9173	0.8353
	(0.9812)	(0.9866)	(0.9868)	(0.9854)	(0.9835)	(0.9812)	(0.9754)	(0.9697)
MibTel	0.9621	0.9750	0.9781	0.9768	0.9725	0.9655	0.9447	0.9173
	(0.9442)	(0.9540)	(0.9528)	(0.9470)	(0.9383)	(0.9274)	(0.9007)	(0.8705)

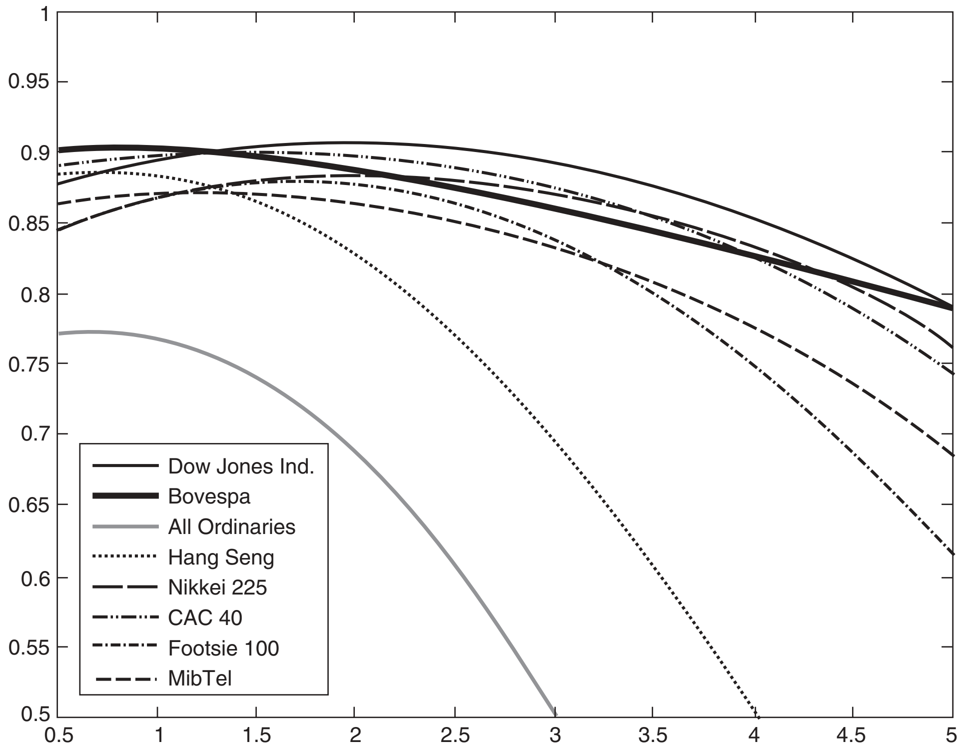

Figure 18. R^2's of the partition functions of $H^2_{30,N}(t)$ for the considered indexes.

even if lower, the R^2's of $H^2_{30,N}(t)$ remain decidedly significant and never drop under 0.91 in the considered interval.

The situation does not modify if a different distribution of the masses is considered: figure 22 reproduces the analysis described above for a multifractal process generated using, in place of lognormal masses, the binomial measure on $[0,1]$ with $m_0 = 0.4$ and $m_1 = 0.6$. It is clear that changing the measure does not affect the values of Pearson's coefficients, which also in this case are always close to 1 (as above, to facilitate the comparison, the behaviour of the R^2's of the analysed financial indexes is again reproduced in figure 22).

So, when a genuine multifractal process is considered, the results summarized above suffice to exclude the estimator (9) being responsible for the loss of significance of the linearity shown in figure 18.

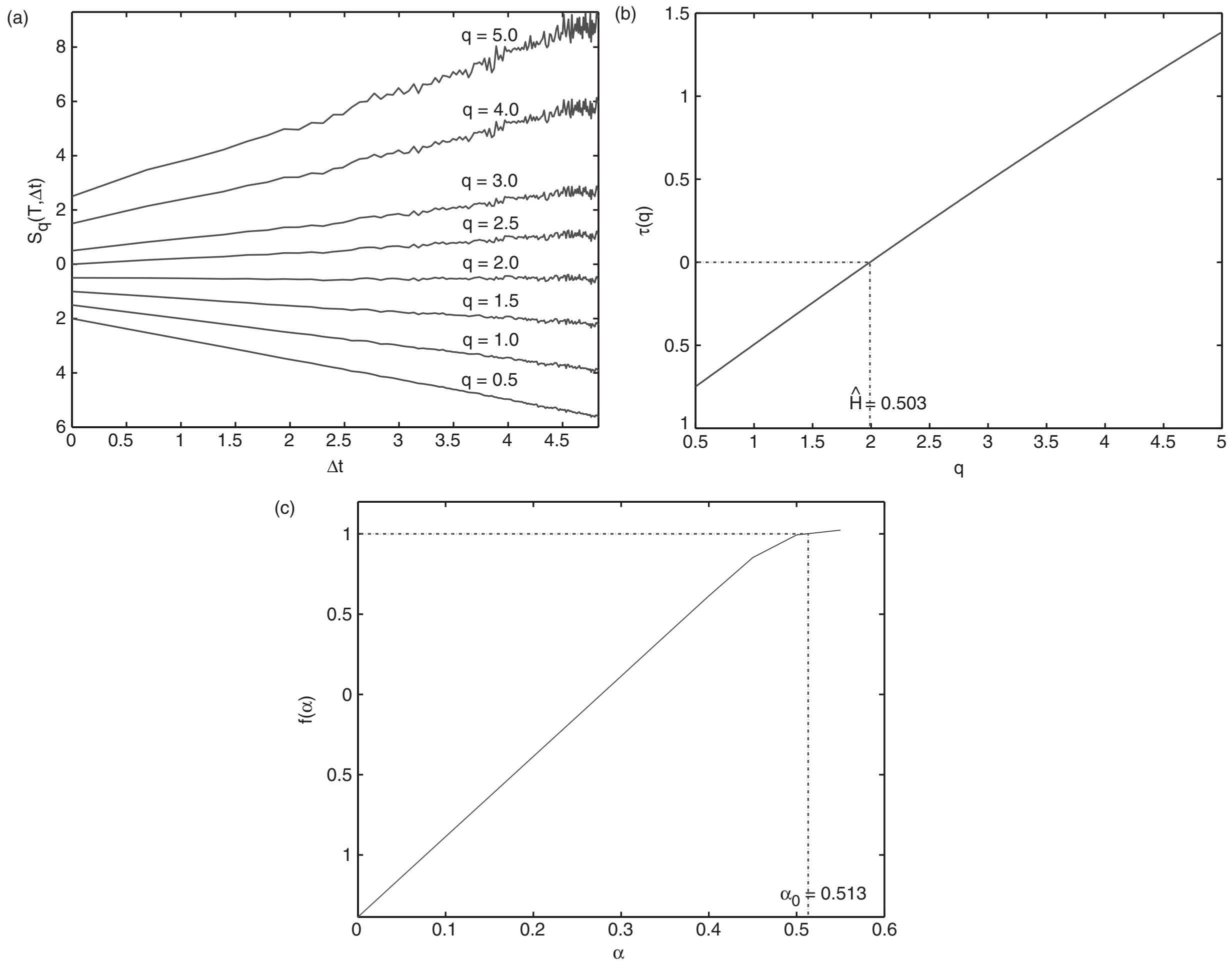

Figure 19. (a) Partition function (logarithmic scale), (b) scaling function and (c) spectrum of an mBm with multifractal functional parameter.

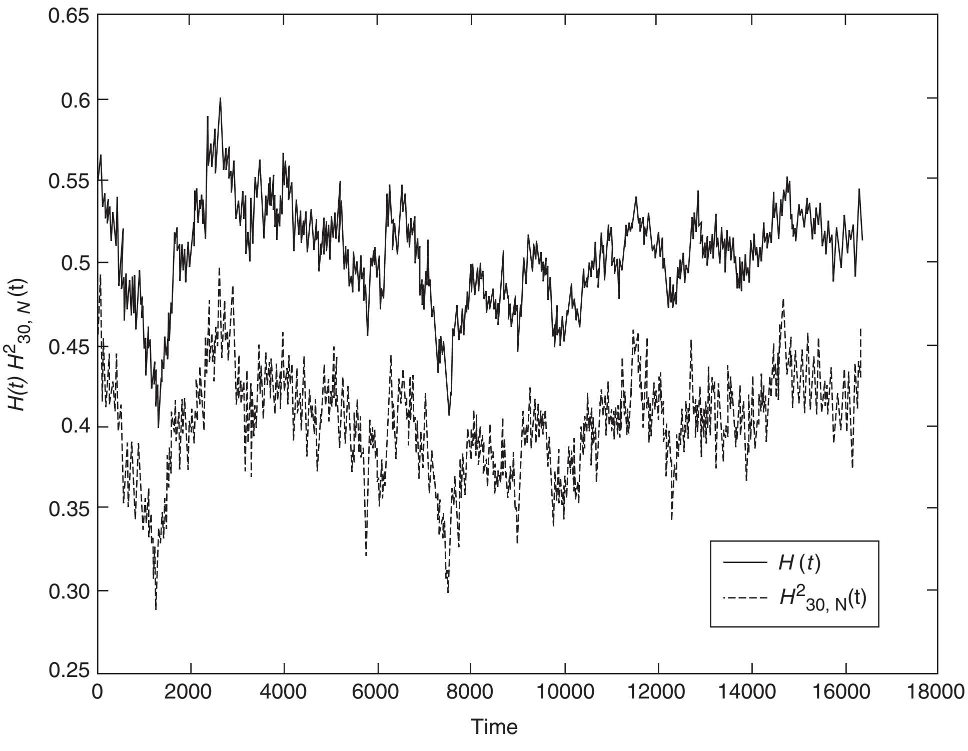

Figure 20. True and estimated $H(t)$. The estimated sequence $H^2_{30,N}(t)$ has been shifted downwards of the quantity 0.1.

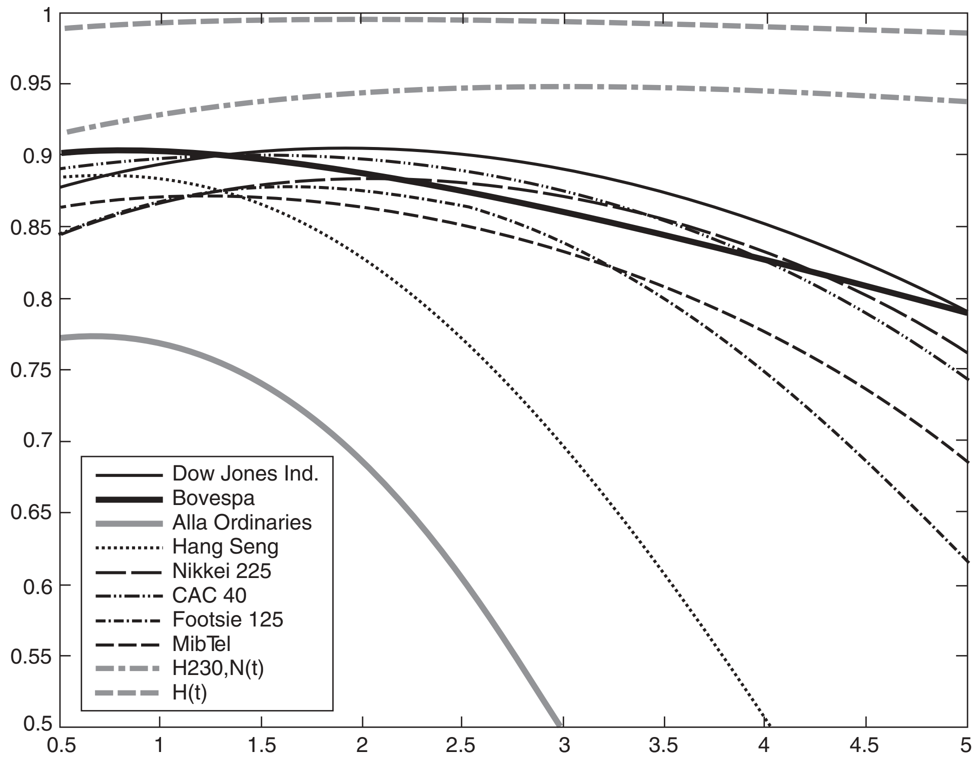

Figure 21. R^2's of the partition functions for actual and simulated (multifractal) series with lognormal measure.

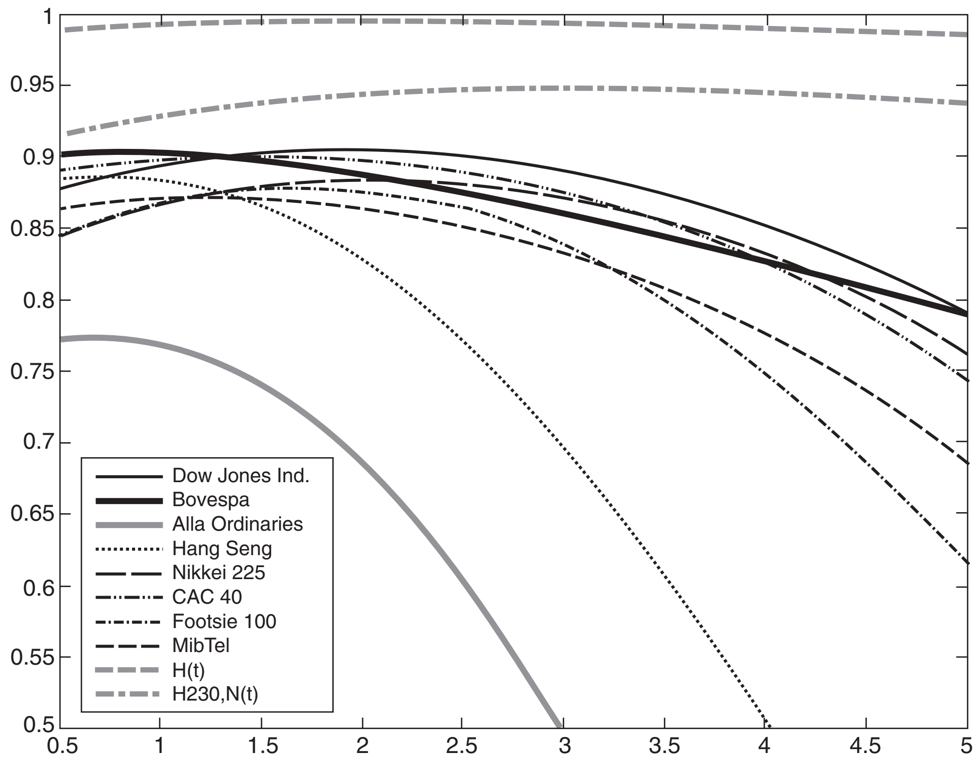

Figure 22. R^2's of the partition functions for actual and simulated (multifractal) series with binomial measure.

5. Conclusions

Evidence has been offered here that the tools generally used to detect multifractal scaling in finance are unable to discriminate a non-multifractal process such as the mBm, once this has been fitted on actual financial data. As a consequence, the conclusion achieved in several empirical works in favour of multifractality looks questionable and therefore more refined analyses than the partition function-based technique would need to assess the multifractality of financial time series (see, e.g. Calvet and Fisher 2003).

On the other hand, our analysis suggests that, because of its capability to reproduce the scaling features observed in many empirical works, the multifractional Brownian motion deserves more attention as a model of financial dynamics.

Acknowledgments

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