# Multifractal Analysis on the Return Series of Stock Markets Using MF-DFA Method

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# FIG.Multifractal Analysis on the Return Series of Stock Market Using MF-DFA Method

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**Abstract.** Analyzing the daily returns of NASDAQ Composite Index by using MF-DFA method has led to findings that the return series does not fit the normal distribution and its leptokurtic indicates that a single-scale index is insufficient to describe the stock price fluctuation. Furthermore, it is found that the long-term memory characteristics are a main source of multifractality in time series. Based on the main reason causing multifractality, a contrast of the original return series and the reordered return series is made to demonstrate the stock price index fluctuation, suggesting that the both return series has multifractality. In addition, the empirical results verify the validity of the measures which illustrates that the stock market fails to reach the weak form efficiency.

**Keywords:** Stock market fluctuations, Generalized Hurst, Exponent, multifractal, MF-DFA, Time series

## 1 Introduction

Since the stock market was established, there are numerous bull and bear markets. It is hard to find a proper way to understand the unusual patterns, but the efficient market theory provides a widely applicable opinion. The efficient market theory is the cornerstone of modern finance, which is first proposed by Bachelier [1]. Fama establishes EMH (Efficient Market Hypothesis) theory [2]. However, the efficient theory cannot explain many actual phenomena that appear in stock markets, such as Black Monday or the breakdown of U.S. stock market in Oct. 1987. Also, the statistical characteristics of financial data appear the fat tail, long-term memory characteristics, volatility clustering, self-similarity, and so on. For these visions, it is necessary to develop a new method to capture the characteristics of stock price fluctuations in order to perform better risk estimation, prevention and control.

## 2 Literature Review

Mandelbrot first proposes fractal theory in the 1970s [3]. In 1997, Mandelbrot proposes a multifractal model of asset returns to describe the variation of financial asset prices and he points out that the multifractal analysis can be reproduced volatile financial transactions and provide information on the predicted value of market trends, thus showing some regularity of various financial markets [4]. It has been verified that multifractality widely exists in financial markets such as stock markets, future markets, spot markets, foreign exchange markets, derivative markets, interest rate markets and so on [5-9].

MF-DFA, short for multifractal detrended fluctuation analysis, is first proposed by Kantelhardt [10], which can describe different statistical characteristics of time series on different time scales, also is an efficient way to test whether non-stationary time series is multifractal. It considers the average volatility of time series of each interval as a statistical point to calculate volatility functions, and then determines the generalized Hurst exponents based on the power law of volatility functions. Its advantages are the ability to discover the long-term correlation in non-stationary time series and to avoid the misjudgment of correlation. Norouzzadeh et al. studies the Iranian Rial-US dollar exchange rate logarithmic variations through MF-DFA [11]. They find that the time series exhibits multifractality, which is mostly due to different long-range correlations for small and large fluctuations. Ying et al. measures multifractality in Shanghai stock market using MF-DFA [12] and finds that the generalized Hurst exponent can capture multifractality better. Panigrahi et al. characterizes price index behavior through fluctuation dynamics, involving companies listed on New York Stock Exchange [13]. They use wavelet based multifractal detrended fluctuation analysis to analyze companies' self-similar and non-statistical properties.

However, most recent studies are focused on the original price series or its deformation; thus they have not detected the contribution of long-term memory characteristics on multifractality. This paper uses both the original return series and the reordered return series in order to study stock price fluctuations and discuss their connection, therefore, providing a better way to understand the stock price fluctuations.

## 3 Methodology

For series $\{x(i)\}$, where $i = (1, 2, \cdots, N)$ and $N$ is the length of $\{x(i)\}$, the MF-DFA method is as following.

Through the sum process, the original series $\{x(i)\}$ merges into a new series $\{y(j)\}$, with $\bar{x}$ indicating the mean value of series $\{x(i)\}$.

$$y(j) = \sum_{i=1}^{j}[x(i) - \bar{x}],\ i = (1, 2, \cdots, N) \tag{1}$$

Next, divide the new series $\{y(j)\}$ into $N_s = int\left(\frac{N}{s}\right)$ non-overlapping segments of equal length *s*. Usually time scale *s* is not an integer multiple of length *N*, so repeating the same procedure from the opposite end to get the whole part of series $\{y(j)\}$ other than disregard extra parts. Therefore, total $2N_s$ segments are obtained.

$$y_v(j) = y(l + j), j = (1, 2, \cdots, s),\ v = (1, 2, \cdots, 2N_s),\ l = (v - 1)s \tag{2}$$

Fit local trend function $\tilde{y_v}(j)$ on $2N_s$ sub segments *v* by the least squares method in order to eliminate the local trends in each sub segments *v* and get the residuals series $\varepsilon_v(j)$.

$$\varepsilon_v(j) = y_v(j) - \tilde{y_v}(j),\ j = (1, 2, \cdots, s) \tag{3}$$

Then calculate the mean squared value of $2N_s$ sub segments without local trends.

$$F^2(s, v) = \frac{1}{s}\sum_{j=1}^{s} \varepsilon_v^2(j) = \frac{1}{s}\sum_{j=1}^{s}\{y[(v - 1)s + j] - \tilde{y_v}(j)\}^2,\quad v = (1, 2, \cdots, N_s) \tag{4}$$

$$F^2(s, v) = \frac{1}{s}\sum_{j=1}^{s}\{y[N - (v - N_s)s + j] - \tilde{y_v}(j)\}^2,\ v = (N_s + 1, N_s + 2, \cdots, 2N_s) \tag{5}$$

Also average all segments to get the *q-th* order fluctuation function.

$$\begin{cases} F_q(s) = \left\{\dfrac{1}{2N_s}\displaystyle\sum_{v=1}^{2N_s}[F^2(s, v)]^{\frac{q}{2}}\right\}^{\frac{1}{q}},\ q \neq 0 \\ F_0(s) = \exp\left\{\dfrac{1}{4N_s}\displaystyle\sum_{v=1}^{2N_s}\ln[F^2(v, s)]\right\},\ q = 0 \end{cases} \tag{6}$$

To any fixed *q*, determine the scaling exponent of fluctuation function, and the relationship between $F_q(s)$ and *s* is obtained.

$$F_q(s) \propto S^{h(q)} \tag{7}$$

For every time scale *s*, we can get a correspondent value $F_q(s)$. The *q-th* order generalized Hurst exponent is the slope of $Ln(F_q(s))\sim Ln(s)$. Here, if $h(q)$ is a constant and independent from *q*, the series $\{x(i)\}$ is monofractal; and if $h(q)$ is a function of *q*, the series $\{x(i)\}$ is multifractal.

The multifractal spectrum $f(\alpha)$ is another efficient way to describe the multifractal time series. The $h(q)$ generated by MF-DFA is related to Renyi exponent $\tau(q)$.

$$\tau(q) = qh(q) - 1 \quad (8)$$

Thus the multifractal spectrum $f(\alpha)$ can be generated by formula (9) and (10).

$$\alpha = h(q) + qh'(q) \quad (9)$$

$$f(\alpha) = q[\alpha - h(q)] + 1 \quad (10)$$

Furthermore, we define the range $\Delta h$ of generalized Hurst exponents $h(q)$ to measure the degree of multifractality. The greater the $\Delta h$ is, the stronger the degree of multifractality is and the more severe the fluctuation is.

$$\Delta h = h_{\max}(q) - h_{min}(q) \quad (11)$$

# 4 Empirical Research

## 4.1 Data Description

The daily closing prices of NASDAQ Composite Index (IXIC) from 31 December 2008 to 31 December 2013 are selected as the sample data. Total 1259 data are derived from Yahoo! Finance and calculated in MATLAB R2012b. Based on the original prices, we assume $I_t$ represents the closing price in time *t*; thus the logarithmic rate of return $R_t$ in time *t* is as formula (12).

$$R_t = \ln(I_t) - \ln(I_{t-1}) \quad (12)$$

Therefore, total 1258 daily logarithmic rates of return $R_t$ $(t = 1, 2, \cdots, 1258)$ are obtained, shown as Fig. 1.

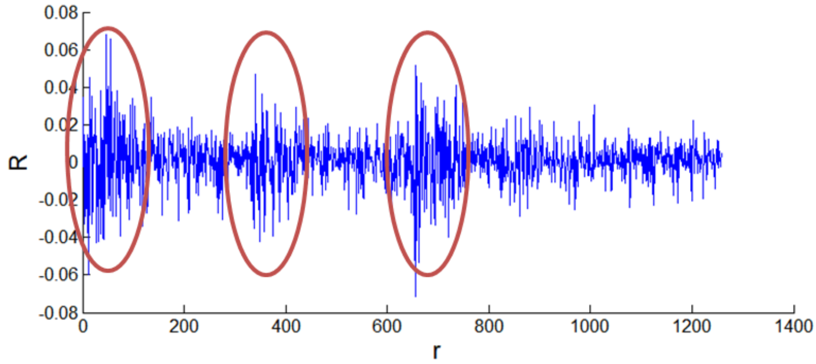



**Fig. 1.** Daily logarithmic rates of return R of NASDAQ Composite Index (IXIC) from 01 January 2009 to 31 December 2013 have three obvious aggregated fluctuations, illustrating the long-term memory characteristics of NASDAQ.

**Table 1.** The fundamental statistics of *R*

| Series | Mean | Median | Variance | Kurtosis | Skewness |
|---|---|---|---|---|---|
| R | 0.000744 | 0.0011 | 0.0132 | 6.3116 | -0.2028 |

The table 1 illustrates the skewness of return series is not equal to 0 and the kurtosis is much larger than it of normal distribution, which is approximately equal to 3. The fundamental statistics show that the return series are not normally distributed and have leptokartic characteristics, indicating that traditional EMH is not a proper way to describe the return series.

However, the long-term memory characteristics of low or high volatility in time series also cause multifractal behaviors. In this paper, random reordering process is used to eliminate the data correlation and keep the volatility, demonstrating the volatility of reordered series is the same as the original one, without long-term memory characteristics. Figure 2 is generated by the following random reordering procedures.

First, to generate a random pair of natural numbers *(a, b)*, in which *a* and *b* are less than or equal to the length *N* of time series.

Second, to change the *a-th* and the *b-th* number in time series.

Third, to repeat the above two procedures *20*N* times, in order to make sure the order fully disrupted.

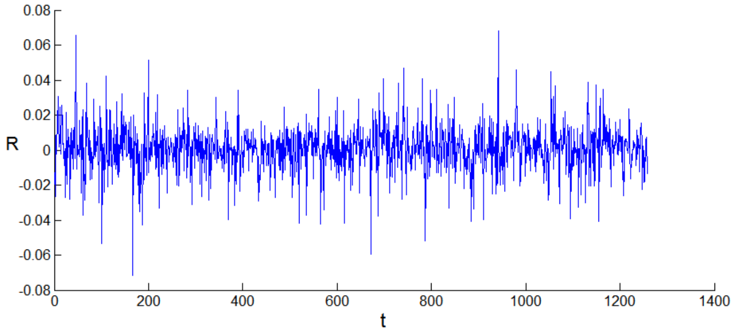



**Fig. 2.** Reordered return series $R_{reordered}$ of NASDAQ Composite Index (IXIC) from 01 January 2009 to 31 December 2013 has the same fluctuation distribution as the original one. Also, there are no obvious aggregated fluctuations, indicating the reordering process is effective.

## 4.2 Empirical Research Results

The MF-DFA method is applied to the daily logarithmic rates of return $R$ and its reordered series $R_{reordered}$. Here we define parameter $q$ is [-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10] and $s$ is an integer array, ranging from 3 to $\frac{N}{5}$, where $N$ is the length of $R$ and $R_{reordered}$. Fig. 3 shows the fluctuation function $F_q(s)$ of both $R$ and $R_{reordered}$ series in NASDAQ. Fig. 4 shows the generalized Hurst exponents $H_q$ of two series, which depends on $q$, by $3 \leq s \leq \frac{N}{5}$. The different values of $h(q)$, the results of MF-DFA, are illustrated in table 2, when $q$ changes from -10 to 10.

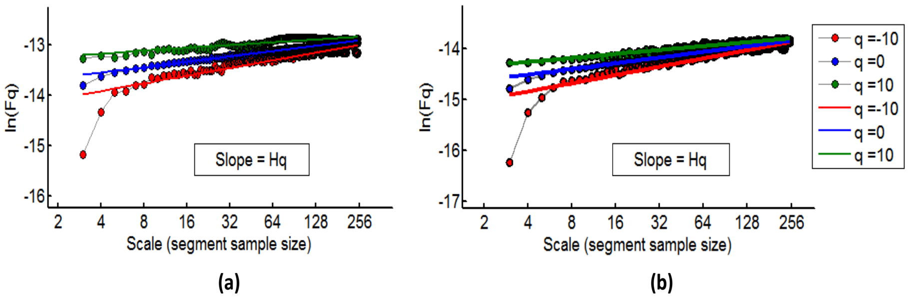
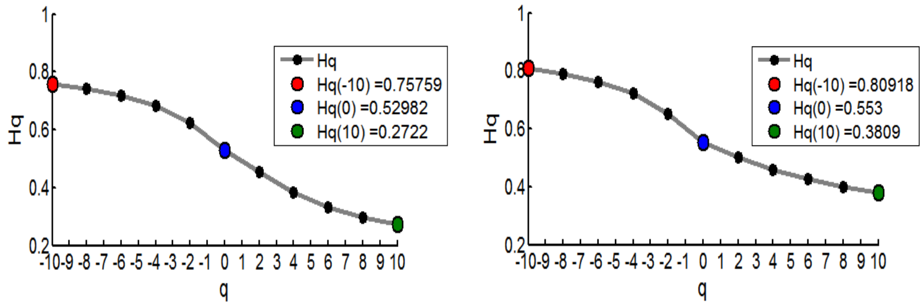



**Fig. 3.** The multifractal fluctuation function $F_q(s)$ of the return series $R$ for NASDAQ is figure (a); and $F_q(s)$ of $R_{reordered}$ is figure (b). The upper, the middle and the lower curves are the curves of *q=10*, *q=0*, and *q=-10*. This figure also shows the generalized Hurst exponents $H_q$ of $R_{reordered}$ is slightly greater than those of $R$; and the goodness of fit is better. E.g. When *q=-10*, $H_q(R) = 0.757591165$ and $H_q(R_{reordered}) = 0.809184708$.




(a) (b)

**Fig. 4.** The generalized Hurst exponents $H_q$ of $R$ and $R_{reordered}$ are not constants but the function with respect to $q$, indicating multifractality in $R$ and $R_{reordered}$. Figure (a) is for $R$ and figure (b) is for $R_{reordered}$.

**Table 2.** The generalized Hurst exponents $h(q)$ of $R$ and $R_{reordered}$, when $q$ is from -10 to 10.

| Order $q$ | $h(q)$ of $R$ | $h(q)$ of $R_{reordered}$ |
|---|---|---|
| -10 | 0.757591 | 0.809185 |
| -8 | 0.741504 | 0.789698 |
| -6 | 0.718603 | 0.761953 |
| -4 | 0.684111 | 0.720634 |
| -2 | 0.623898 | 0.651603 |
| 0 | 0.529818 | 0.552996 |
| 2 | 0.452512 | 0.500296 |
| 4 | 0.382411 | 0.458481 |
| 6 | 0.330822 | 0.424952 |
| 8 | 0.295986 | 0.399629 |
| 10 | 0.272200 | 0.380904 |
| $\Delta h$ | 0.485391 | 0.428281 |

As can be seen in table 2, when $q$ changes from 10 to10, the $h(q)$ of original return series descends from 0.757591 to 0.272200 and the $h(q)$ of reordered return series descends from 0.809185 to 0.380904. Both $h(q)$ are not constant, indicating that there is obvious multfractality. So using monofractal model to describe is not appropriate.

Comparing both return series, the $h(q)$ of reordered one is closer to 0.5 than it of original one, because $Var(h(R)) = 0.035$ and $Var(h(R_{reordered})) = 0.027$. Therefore, the relevance of $R_{reordered}$ is higher than it of $R$.

When $q$ is a negative or relatively small positive number, $h(q) > 0.5$. The small fluctuations of the rates of return are amplified, expressing the persistent feature. Correspondingly, when $q$ is a relatively large positive number, $h(q) < 0.5$, indicating that the large fluctuations are dominant; therefore, anti-persistent feature is clear.

For a given $q$, each $h(q)$ of original return series is less than it of reordered one, indicating that the persistent characteristic of $R_{reordered}$ is stronger and the anti-persistent characteristic is more weaken. Meanwhile, after reordering, $\Delta h$ reduces and the multifractality weakens. Because $h(q)$ changes slightly with $q$, which is not significant, it is a proof that the long-term memory characteristics of returns can lead to multifractality. $\Delta h(R) > \Delta h(R_{reordered})$, so the multifractality of $R$ is more obvious.

Furthermore, the characteristics of multifractality are analyzed by combining MF-DFA and multifractal spectrum. According to formula (8), the relationship between $\tau(q)$ and $q$ can be obtained, shown as Fig. 5. For monofractal, $\tau(q)$ is linear; for multifractal, $\tau(q)$ is nonlinear. And the stronger the nonlinearity is, the stronger the multifractality is. As can be seen from Fig. 5, the nonlinearity in $\tau(q)$ of reordered return series is obviously more weaken than it of original one. It also illustrates our interpretation of Fig. 4 and table 2.

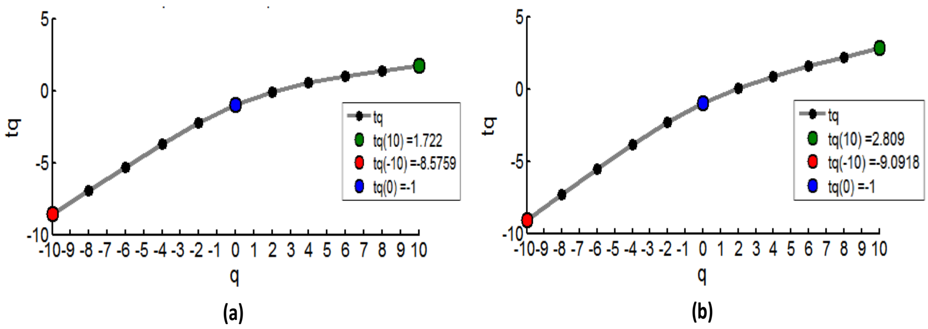



**Fig. 5.** Multifractal $\tau(q) \sim q$: Figure (a) is for $R$ and figure (b) is for $R_{reordered}$.

Finally, according to formula (9) and (10), the multifractal spectrums of both series can be captured, shown as Fig. 6. The multifractal spectrum width of reordered return series is less than it of original one, illustrating the interpretation of Fig. 5. The variety of reordered return series confirms that persistent relevance is an important factor to the multi-scaling changes in price volatility.

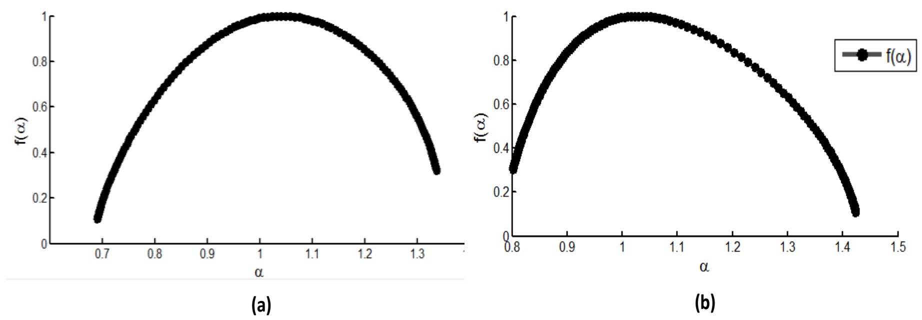



**Fig. 6.** Multifractal Spectrum $f(\alpha) \sim \alpha$: Figure (a) is for $R$ and figure (b) is for $R_{reordered}$.

## 5 Conclusions

The multifractal model is more appropriate to describe the price variance. MF-DFA, which is executed in this paper, captures the multifractality in both the original return series and the reordered one.

Through the statistical research on NASDAQ Composite Index using MF-DFA, it is discovered that the long-term memory characteristics of the return volatility are a main reason of multifractality in the stock market. The empirical results also suggest that the entire stock market is not a random process, but a process affected by large and small fluctuations in some periods. The correlation of return volatility, especially the persistent relevance, contributes to the multi-scaling changes in price volatility.

In fact, the long-term memory enables us to recall noise, opaque prices in stock market and other complex information. In addition, investors also respond to the market in a non-linear way. They begin to act only when the information accumulated to a certain extent and to trade at an accepted price other than a fair price. Their behaviours lead to the stock price walking in a biased random way; therefore, the stock market is hard to reach the weak form efficiency. How to understand the multifratcality in financial markets and how to characterize the multifractal indicators of financial risks are the two key issues in the future. A fresh look at financial markets will help make more accurate risk estimation and control.

## References

1. Barchelier, L.: The theory of speculation, Vol. 3-17. Annales Scientifiques de l'École Normale Supérieure. (1990) 21–86
2. Fama, F.E.: Efficient Capital Markets: A Review of Theory and Empirical Work, Vol. 25-2. The Journal of Finance. (1970) 385-417
3. Mandelbrot, B.B.: Fractals: Form, Chance and Dimension. Wheeler Freeman, San Francisco, USA. (1977)
4. Mandelbrot, B.B., Fisher, J.A., Calvet, E.L.: A Multifractal Model of Asset Return. Working paper in Yale University. (1997)
5. Karuppiah, J., Los, C.A.: Wavelet Multiresolution Analysis of High-frequency Asian FX Rates, Summer 1997, Vol 14. International Review of Financial Analysis. (2005) 211–246
6. Norouzzadeh, P., Jafari, G.R.: Application of Multifractal Measures to Tehran Price Index, Vol. 356. Physica A. (2005) 609–627
7. Cajueiro, D.O., Tabak, B.M.: Long-range Dependence and Multifractality in the Term Structure of LIBOR Interest Rates, Vol. 373. Physica A. (2007) 603–614
8. Lee, K.E., Lee, L.W.: Probability Distribution Function and Multiscaling Properties in the Korean Stock Market, Vol. 383-1. Physica A. (2007) 65–70
9. Lim, G., Kim, S.Y., Lee, H., Kim, K., Lee, D.I.: Multifractal Detrended Fluctuation Analysis of Derivative and Spot Markets, Vol. 386. Physica A. (2007) 259–266
10. Kantelhardt, J.W., Zschiegner, S.A., Koscielny-Bunde, E., et al.: Multifractal Detrended Fluctuation Analysis of Non-stationary Time Series, Vol. 361. Physica A. (2002) 87-114
11. Norouzzadeh, P., Rahmani, B.: A Multifractal Detrended Fluctuation Description of Iranian Rial-US Dollar Exchange Rate, Vol. 367. Physica A. (2006) 328-336

12. Ying, Y., Zhuang, X.T., Jin, X.: Measuring Multifractality of Stock Price Fluctuation Using multifractal Detrended Fluctuation Analysis, Vol. 388. Physica A. (2009) 2189-2197
13. Panigrahi, K.P., Ghosh, S., Banerjee, A., Bahadur, J., Manimaran, P.: Econophysics of Systemic Risk and Network Dynamics. Springer-Verlag Italia, Milan (2013) 287-295