

Long Term Memory Processes Using Hurst Estimation

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Abstract

This paper presents a study on long-term memory processes in financial time series using Hurst estimation methods, specifically the traditional R/S statistic (Rescaled Range analysis) and the Modified R/S statistic. We analyze the long-term memory properties of five major stock market indices: S&P 500 (GSPC), FTSE 100 (FTSE), SBF 250 (SBF250), TOPIX (TOPX), and Toronto Stock Exchange 300 (GSPTSE) using daily returns data spanning from January 2, 1995, to December 31, 2024.

The R/S statistic and the Modified R/S statistic are computed to determine the presence of long-range dependence in the time series data. The Hurst exponent, derived from both statistics, is used to characterize the memory behavior of the series.

The results indicate that, except for GSPC, all indices exhibit short-term memory or randomness with Hurst exponents below the critical threshold of 1.620. Specifically, the Hurst exponent for the GSPC index exceeds 1.620, indicating the presence of long-term memory (persistent behavior). These findings suggest that certain stock markets, like the US stock market represented by GSPC, may exhibit more pronounced long-term memory effects compared to others.

1 Introduction

The Hurst exponent is a key measure in the study of time series data, used primarily to analyze the long-term memory and self-similarity of stochastic processes. First introduced by the British hydrologist Harold Hurst in the 1950s to study river flow data, the Hurst exponent has since found widespread application in various fields, including finance, physics, and environmental science. The exponent provides insights into the persistence or mean-reversion behavior of a time series: a value greater than 0.5 indicates long-range dependence and persistence, while a value less than 0.5 suggests mean-reverting behavior.

The most commonly used method to estimate the Hurst exponent is the Rescaled Range (R/S) analysis, introduced by Hurst and later refined by Mandelbrot. However, the traditional R/S statistic has limitations, particularly in its sensitivity to short-term memory effects, which can obscure long-range dependence in the data. To address these issues, Lo (1991) proposed the Modified R/S statistic, which improves the estimation of the Hurst exponent by accounting for short-term autocorrelation.

This paper presents an analysis of time series data using both the traditional R/S statistic and the Modified R/S statistic, with a focus on their ability to detect long memory in financial data. We apply these methods to a range of stock market indices and examine the significance of the Hurst exponent in characterizing market dynamics. Additionally, we discuss the critical values for the Modified R/S test, based on Lo's (1991) table, to help identify whether a series exhibits long memory behavior.

Through this analysis, we aim to highlight the advantages of the Modified R/S statistic in overcoming the limitations of the traditional R/S method, providing a more robust framework for studying time series with long memory. This is particularly valuable in the context of financial markets, where long memory and fractal-like behavior are often observed.

2 Methods

2.1 Fractional Brownian Motion

Fractional Brownian motion (fBm) is a generalization of standard Brownian motion that introduces dependence in increments, making it suitable for modeling processes with memory effects. It is a continuous-time Gaussian process $X_H(t)$ where H corresponds to the Hurst exponent and $H \in [0, 1]$ with the following properties:

- $X_H(0) = 0$.
- The increments $X_H(t) - X_H(s)$ follow a normal distribution with mean zero and variance:

$$\mathbb{E} [(X_H(t) - X_H(s))^2] = \sigma^2 |t - s|^{2H}, \quad (1)$$

where H is the Hurst exponent.

- The process exhibits self-similarity, meaning that for any scaling factor c , $c \in \mathbb{R}^+$, the rescaled process satisfies:

$$X_H(ct) \stackrel{d}{=} c^H X_H(t). \quad (2)$$

- When $H = 0.5$, fBm reduces to classical Brownian motion.
- For $H > 0.5$, the process exhibits long-term positive autocorrelation, meaning that an increase in the past tends to be followed by further increases.

- For $H < 0.5$, the process has anti-persistent behavior, where an increase in the past is more likely to be followed by a decrease.

The covariance function of fBm is given by:

$$C_H(t_1, t_2) = \frac{\sigma^2}{2} (t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}), \quad (3)$$

which accounts for the dependence structure of the process. The Hurst exponent H plays a critical role in determining the smoothness and correlation properties of fBm:

- **For small H values** ($H < 0.5$), the process is highly erratic, with rapid changes and weak memory effects.
- **For large H values** ($H > 0.5$), the trajectory becomes smoother, and the process exhibits long-range dependence.

Fractional Brownian motion is widely used in finance, telecommunications, and physics to model phenomena exhibiting memory effects and self-similarity, such as stock market fluctuations, internet traffic, and geological formations.

2.2 Simulation of Fractional Brownian Motion

To generate the fractional Brownian motion (fBm), we use a Cholesky decomposition-based approach. The covariance matrix of fBm is given by (??):

where H is the Hurst exponent, which determines the degree of long-term dependence in the process.

The steps of the simulation are as follows:

1. Define a time grid of N points between 0 and T .
2. Compute the covariance matrix using the formula above.
3. Apply Cholesky decomposition to obtain a lower triangular matrix L .
4. Generate a vector W of standard normal random variables.
5. Obtain the fBm path by computing $X = LW$.

The params used for this simulation are $N = 1000$, $T = 1$, and Hurst exponents $H = 0.2, 0.3, 0.5, 0.6, 0.8$, the number of lag for the autocorrelation is 40.

Fractional Brownian Motion & Autocorrelation

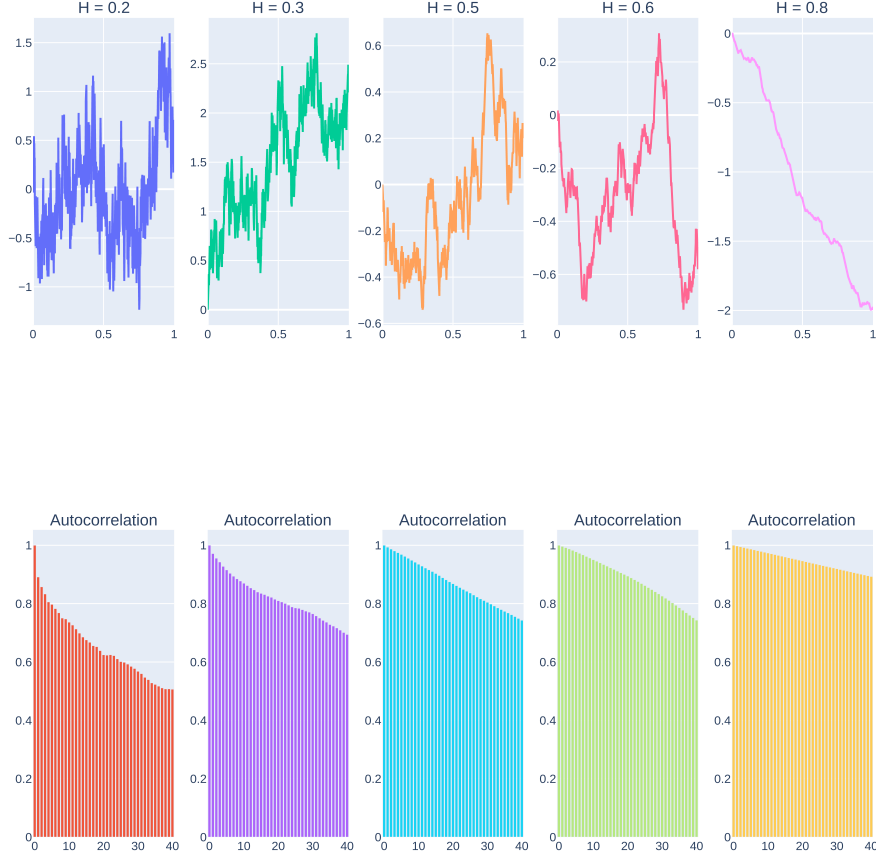


Figure 1 – Simulation of Fractional Brownian Motion with different Hurst exponent and its autocorrelation function.

The behavior of the fractional Brownian motion (fBm) varies significantly with the Hurst exponent H .

When H is small (close to 0), the fBm exhibits high local variability, resulting in a highly granular trajectory with frequent fluctuations. The autocorrelation of increments decays rapidly, indicating that future values are weakly influenced by past values. This suggests a short-memory process, similar to standard Brownian motion.

As H increases, the autocorrelation decays more slowly, meaning that past values have a more significant impact on future values. This introduces a form of long-term dependence, where the process exhibits persistent trends. Consequently, the fBm trajectory appears smoother, with larger coherent movements and fewer abrupt changes.

In summary, a lower H leads to a more irregular and noisy path, characteristic of short-memory processes, while a higher H results in a smoother trajectory with stronger persistence.

2.3 Hurst Exponent Calculation

2.4 R/S and Modified R/S Analysis

The R/S (Rescaled Range) analysis, introduced by Hurst and developed in various works by Mandelbrot, is certainly the most well-known method for estimating the Hurst exponent H . This statistic is defined as the range of the partial sums of deviations from the mean of a time series divided by its standard deviation. Consider a time series Y_t , $t = 1, \dots, T$, with mean \bar{Y} . The range R is defined as:

$$R = \max_{1 \leq j \leq T} (Y_j - \bar{Y}) - \min_{1 \leq j \leq T} (Y_j - \bar{Y})$$

The R/S statistic is then computed by dividing the range by the standard deviation s_T of the series:

$$Q_T = \frac{R}{s_T} = \frac{\max_{1 \leq j \leq T} (Y_j - \bar{Y}) - \min_{1 \leq j \leq T} (Y_j - \bar{Y})}{s_T}$$

The R/S statistic, Q_T , is always positive. In various papers, Mandelbrot and Wallis (1969e), Mandelbrot (1973), and Mandelbrot and Taqqu (1979) have emphasized the superiority of R/S analysis over more traditional methods of detecting long memory, such as autocorrelation studies, variance ratio tests, and spectral analysis. Mandelbrot and Wallis (1968) show that R/S analysis can detect long memory even in highly non-Gaussian time series. Mandelbrot and Wallis (1969d) further note that, unlike spectral analysis which only detects periodic cycles, the R/S statistic can detect non-periodic cycles. Finally, Mandelbrot and Wallis (1969e) demonstrate that the R/S statistic is independent of the marginal distribution.

The R/S analysis leads to the Hurst exponent, where T is the number of observations in the series.

2.5 Modified R/S Analysis

The Modified R/S statistic, denoted by \tilde{Q}_T , is defined as:

$$\tilde{Q}_T = \frac{R}{\hat{\sigma}_T(q)}$$

where

$$\hat{\sigma}_T(q) = \sqrt{\frac{1}{T} \sum_{j=1}^T (Y_j - \bar{Y})^2 + \frac{2}{T} \sum_{j=1}^T w_j(q) \left[\sum_{i=j+1}^T (Y_i - \bar{Y})(Y_{i-j} - \bar{Y}) \right]}$$

and

$$w_j(q) = 1 - \frac{j}{q+1}$$

This statistic differs from the traditional R/S statistic only by its denominator. In the presence of autocorrelation, the denominator does not only represent the sum of the variances of the individual terms, but also includes autocovariances. These are weighted according to lags q , with the weights $w(q)$ suggested by Newey and West (1987). Moreover, Andrews (1991) proposed a rule for choosing q :

$$q = [k_T] \quad \text{where} \quad k_T = \left(\frac{3T}{2}\right)^{1/3} \left(\frac{2\rho_1}{1-\rho_1^2}\right)^{2/3}$$

where $[k_T]$ is the integer part of k_T , and ρ_1 is the first-order autocorrelation coefficient.

Unlike the classical R/S analysis, the limiting distribution of the Modified R/S statistic is known, and the statistic V , defined by

$$V = \frac{\tilde{Q}_T}{\sqrt{T}},$$

converges to the range of a Brownian bridge over the unit interval. It is therefore possible to perform a statistical test for the null hypothesis of short memory against the alternative hypothesis of long memory by referring to the critical value table provided by Lo (1991).

2.6 Critical Value Table from Lo (1991)

The critical values for the Modified R/S test are provided in the table below. These values are used to assess whether the series exhibits long memory behavior based on the Modified R/S statistic.

Significance Level	Critical Value (Modified R/S Statistic)
0.005	2.098
0.05	1.747
0.10	1.620

Table 1 – Critical values for the Modified R/S Statistic (Lo, 1991)

2.7 Multifractal Detrended Fluctuation Analysis

The Multifractal Detrended Fluctuation Analysis (MF-DFA) is a generalization of the standard DFA approach designed to detect multifractality in time series. The procedure can be summarized in five steps, as described below:

1. **Profile construction.** Given a series $\{x_k\}_{k=1}^N$, we define its mean $\langle x \rangle$. We build the profile

$$Y(i) = \sum_{k=1}^i (x_k - \langle x \rangle), \quad i = 1, 2, \dots, N.$$

This cumulative sum helps to capture the local fluctuations in the data.

2. **Division into segments.** We split the profile $Y(i)$ into $N_s \equiv \lfloor N/s \rfloor$ non-overlapping segments, each of length s . Since N may not be a multiple of s , we repeat this procedure also starting from the opposite end, yielding $2N_s$ segments in total.
3. **Detrending.** For each of the $2N_s$ segments, we fit a polynomial trend (often linear or quadratic) and subtract it from $Y(i)$ in that segment. Let $y_\nu(i)$ be the fitting polynomial in segment ν . We then define the local variance

$$F^2(s, \nu) = \frac{1}{s} \sum_{i=1}^s \left[Y((\nu-1)s + i) - y_\nu(i) \right]^2.$$

This detrending removes possible polynomial trends in the data.

4. **Generalized fluctuation function.** We compute, for each scale s , the q th-order fluctuation function,

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} [F^2(s, \nu)]^{q/2} \right\}^{1/q}.$$

Varying q allows us to emphasize large ($q > 0$) or small ($q < 0$) fluctuations.

5. **Scaling behavior.** Finally, on double-logarithmic axes, we examine the dependence of $F_q(s)$ sur s . If

$$F_q(s) \sim s^{h(q)},$$

we say that $h(q)$ is the generalized Hurst exponent. In a *multifractal* system, $h(q)$ varies with q , indicating different scaling behaviors for large versus small fluctuations.

For *monofractal* series, $h(q)$ is approximately constant for tous q . For *multifractals* series, $h(q)$ strongly depends on q , revealing heterogeneity in the scaling of fluctuations.

Thus, the two methods are complementary:

- *Modified R/S*: Explicitly tests the hypothesis of *short memory* vs. *long memory* based on a critical statistic.
- *MF-DFA*: Detects and quantifies *multifractality*, revealing how fluctuations scale differently depending on the order q .

In practice, an analyst can first employ the *Modified R/S* statistic to validate or reject the presence of long memory and then refine the analysis by computing the function $h(q)$ via MF-DFA, highlighting the complexity (multifractal or not) of the examined time series.

2.8 Holder Exponent and Singularity Spectrum

2.9 Holder Exponent

The Holder exponent $\alpha(q)$ characterizes the local singularity strength of a signal and is obtained using the Legendre transform of $h(q)$:

$$\alpha(q) = h(q) + qh'(q). \quad (4)$$

where $h'(q)$ is the derivative of $h(q)$ with respect to q . This exponent quantifies the intensity of local singularities: lower values of α indicate highly irregular (or sharply singular) behavior, while higher values correspond to smoother regions of the signal. Thus, the Hölder exponent reveals the heterogeneity of fluctuations within the signal. This exponent describes the degree of singularity in different parts of the series, revealing the heterogeneity of fluctuations.

2.10 Singularity Spectrum

The multifractal spectrum $f(\alpha)$ provides a measure of the fractal dimension of subsets characterized by a given α :

$$f(\alpha) = q[\alpha(q) - h(q)] + 1. \quad (5)$$

This spectrum describes the distribution of singularities in the time series. A wider spectrum indicates stronger multifractality.

The analysis using the Hölder exponent and singularity spectrum is a powerful tool for studying complex systems. In particular, it enables one to:

- Identify and quantify regions of strong singularity, which may correspond to extreme events or sudden changes in dynamics.
- Describe the distribution and frequency of irregular behaviors in time series or spatial data.
- Enhance the modeling of signals exhibiting multifractal structures, such as financial time series, physiological signals, or natural phenomena.

Thus, the multifractal approach offers a detailed and nuanced description of a signal’s local variability, providing essential insights for understanding and predicting its underlying dynamics.

For each index, the closing price time series was transformed using the natural logarithm to obtain a series of log returns. These log returns were then used to calculate the R/S and Modified R/S statistics and estimate the Hurst exponent. The purpose of using this data is to evaluate the long-term memory properties of financial markets, which can indicate persistence or mean-reversion in market behavior.

Additionally, a stationarity test was conducted on the log return series using the Augmented Dickey-Fuller (ADF) test. The results indicated that all series were non-stationary, suggesting the presence of unit roots. To address this, the series were differenced once, after which they exhibited stationarity. This step ensures that the analysis of long-term memory is conducted on properly transformed data, free from non-stationary distortions.

3 Data

The data used in this analysis consists of the historical closing prices of five major stock market indices: the S&P 500 (GSPC), FTSE 100 (FTSE), SBF 250 (SBF250), TOPIX (TOPX), and the Toronto Stock Exchange 300 (GSPTSE). The data spans the period from September 10th, 1987, to February 28, 2025, and was downloaded from Yahoo Finance.

4 Results

The following table summarizes the results of the R/S statistic, Modified R/S statistic, and the estimated Hurst exponents for each of the five indices analyzed:

Ticker	R/S	Hurst Exponent	Modified Hurst Exponent	Critical Value	Long Memory
GSPC	30.166	0.558	0.501	1.007	False
RUT	51.373	0.645	0.588	1.714	True
FTSE	40.236	0.605	0.548	1.341	False
N225	22.234	0.508	0.508	1.048	False
GSPTSE	35.912	0.586	0.586	1.693	True

Table 2 – Results for R/S, Hurst Exponent, Modified Hurst Exponent, Critical Value at 10%, and rejection of the null hypothesis of Long Memory.

Based on the results obtained from applying the traditional R/S method, all the series appear to exhibit long-term memory, as the Hurst exponents are consistently greater than 0.5. However, the asymptotic distribution of this statistic is unknown, which prevents us from determining whether the estimated Hurst exponent is significantly greater than 0.5. This issue can be addressed by using the modified R/S method. In this case, we simply compare the estimated values of the statistic V to the critical values provided by Lo (1991), which are 1.620 and 1.747 for the 10% and 5%

significance levels, respectively, in the case of a one-tailed test. It is observed that only one serie actually exhibit persistence: the returns of the Russell 2000 small and mid cap (US). For all other series, despite the Hurst exponent being greater than 0.5, the null hypothesis of short memory cannot be rejected. From now on, we will focus on the Russell 2000 index to further investigate the long memory properties of the series.

4.1 Pitfalls and more granular analysis

Unfortunately those results are not very conclusive, the Hurst exponent is not a very robust statistic and can be influenced by the length of the series, the presence of trends, or the presence of structural breaks. To highlight these shortcomings, we adjust both the analysis frequency and the study period. Specifically, we perform a detailed investigation across different frequencies (daily, weekly, monthly) and various study periods, comparing the outcomes. This combined approach enables us to detect long-memory behavior within particular timeframes and ensures more robust and reliable conclusions.

Frequency	Hurst Exponent	Modified Hurst Exponent	Critical Value	Long Memory
D	0.488	0.488	0.893	False
W	0.504	0.504	1.028	False
M	0.645	0.588	1.714	True

Table 3 – Results for Hurst Exponent, Modified Hurst Exponent, Critical Value at 10% and rejection of the null hypothesis of Long Memory, different frequencies daily, weekly and monthly.

Period	Hurst Exponent	Modified Hurst Exponent	Critical Value	Long Memory
1987-09-10 - 2025-02-28	0.645	0.588	1.714	True
1990-01-02 - 2025-02-28	0.637	0.546	1.319	False
1995-01-02 - 2025-02-28	0.606	0.548	1.324	False
2005-01-02 - 2025-02-28	0.545	0.545	1.281	False
2010-01-02 - 2025-02-28	0.509	0.509	1.047	False
2015-01-02 - 2025-02-28	0.562	0.562	1.346	False

Table 4 – Results for Hurst Exponent, Modified Hurst Exponent, Critical Value at 10% and rejection of the null hypothesis of Long Memory, different estimation periods.

4.2 Rolling Critical value

To further investigate the long memory properties of the Russell 2000 index, we perform a ten year rolling analysis of the Modified R/S statistic and compare it to the critical value at the 10% significance level (1.620).

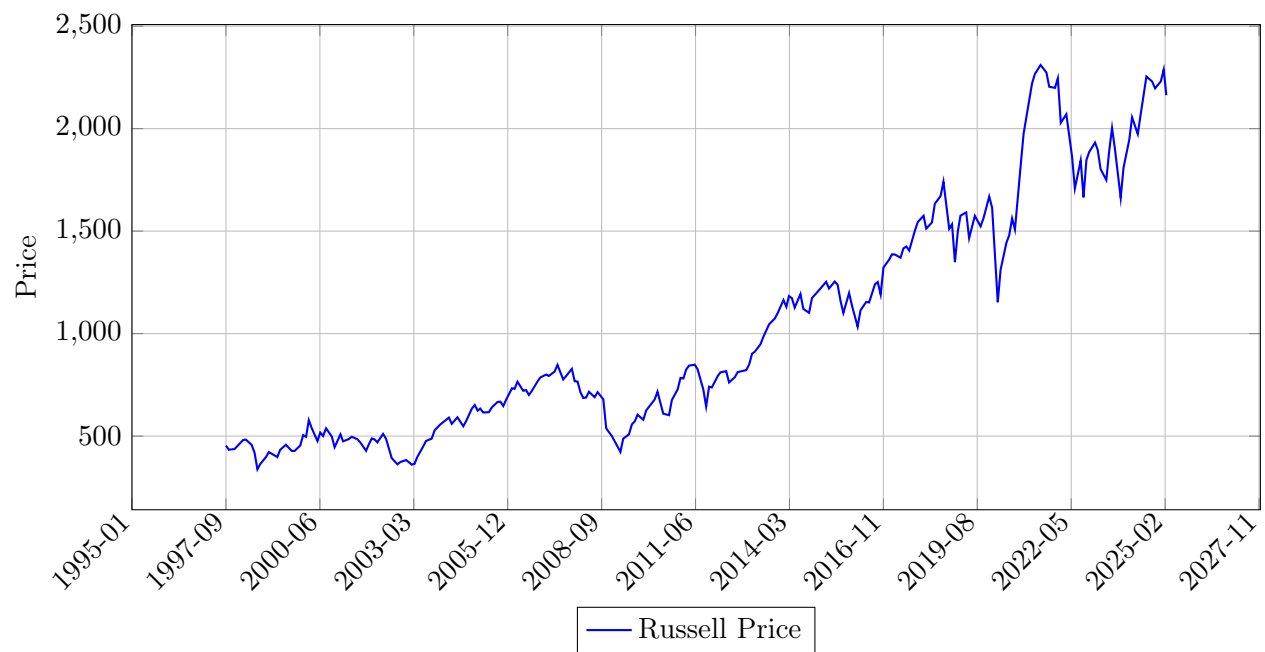


Figure 2 – Russell Price Evolution

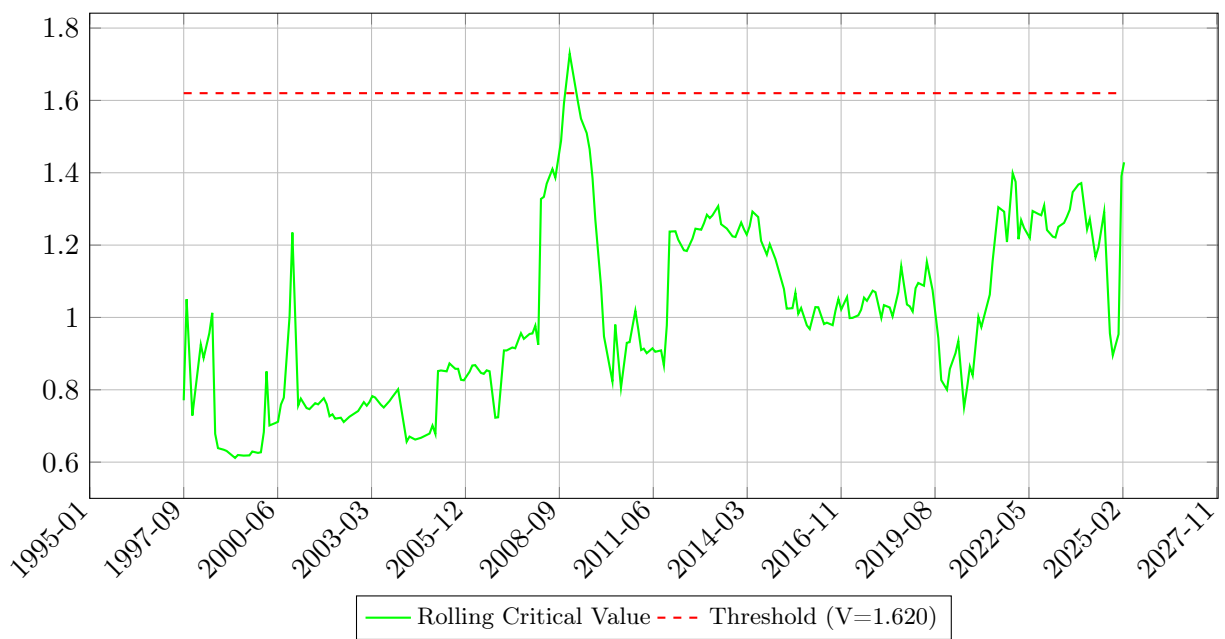


Figure 3 – Rolling Critical Value

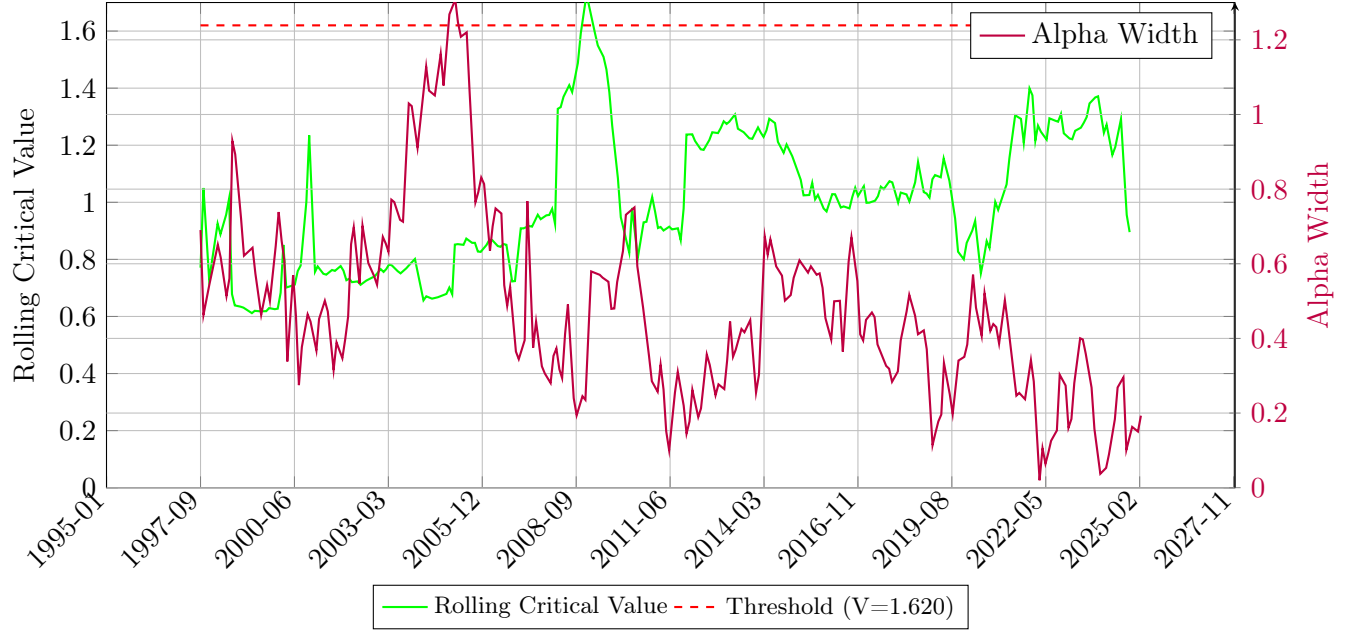


Figure 4 – Rolling critical value (green) and alpha width (purple). Both metrics are computed using a ten-year rolling window on monthly data. The alpha width represents the difference between the maximum and minimum alpha over the estimation window.

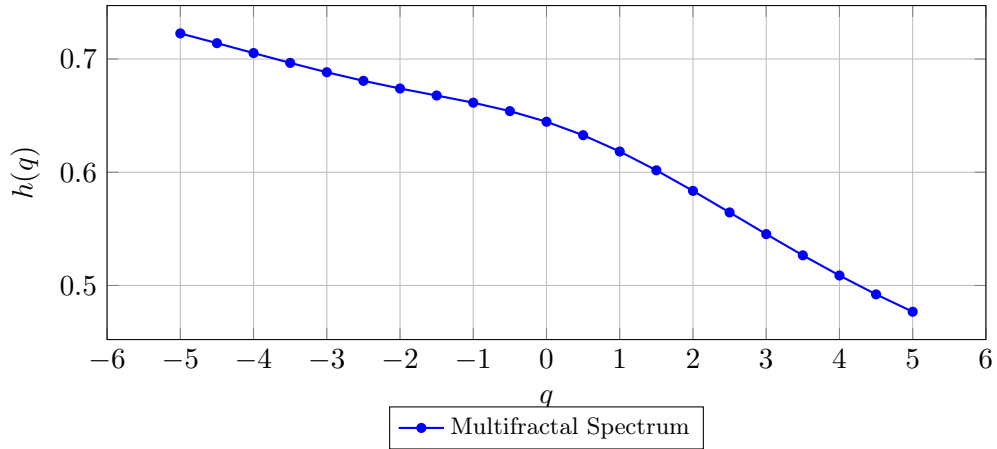


Figure 5 – Multifractal Spectrum $h(q)$ for the Russell 2000 returns.

5 Appendix

Ticker	P-Value on prices	P-Value on log differentiated return
GSPC	0.863	0.0
RUT	0.695	0.0
FTSE	0.226	0.0
N225	0.66	0.0
GSPTSE	0.755	0.0

Table 5 – P-values from the Augmented Dickey-Fuller (ADF) test for stationarity. The P-value of prices refers to the Augmented Dickey Fuller test (ADF) on the original series, while the P-value of log-differentiated returns indicates the ADF test on log-differentiated returns. The null hypothesis is non-stationarity.

Table 6 – Hurst Statistics for ticker

Ticker	Count	Mean	Std	Min	25	50	75	Max
AAPL	7,299	1.19	0.23	0.65	1.02	1.18	1.34	1.88
JPM	7,299	1.13	0.21	0.66	0.99	1.12	1.26	1.84
JNJ	7,299	1.09	0.2	0.64	0.94	1.08	1.22	1.85
XOM	7,299	1.08	0.22	0.59	0.94	1.06	1.17	1.82
WMT	7,299	1.11	0.17	0.61	1	1.11	1.22	1.75
BA	7,299	1.13	0.18	0.59	0.99	1.15	1.26	2.08
DIS	7,299	1.18	0.24	0.63	1.01	1.16	1.36	1.98
LIN	7,299	1.03	0.2	0.53	0.88	1	1.17	1.74
NEE	7,299	1.1	0.19	0.6	0.97	1.11	1.24	1.76
SPG	7,299	1.11	0.2	0.63	0.95	1.1	1.28	1.78

6 References

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