Hello, my name is Alexandre Remiat. I am a second-year Master’s student and currently on a work-study contract in quantitative research at the asset management firm Kanopy-AM.

Thank you for being here. Today I’ll present research I had the pleasure of conducting on long-memory analysis using the Hurst exponent. Before diving into the work, I’d like to make three preliminary remarks:

1. I don’t claim that this presentation offers an exhaustive overview, but it does lay the foundations of the topic.
2. This is not a final product; I still consider it a working document that can evolve based on your feedback, comments, and suggestions.
3. I would like to acknowledge and thank Jean Jardel and Mathieu Garcin, who guided me throughout this research.

I’ll keep this brief, as I plan to let you go in about ten hours—for the most resilient among you!

We will dive into the analysis of the long term memory processes estimation with the Hurst exponents and multifractal analysis of time series.

First of all, in this context what we are looking for is trying to see if the past returns have an impact on current ones. It is a big question since it put in cause the Efficient market Hypothesis.  
A quick reminder on the EMH, it was originally described by ?? and stated that the price of a secrutiy at any given time is described by all the public information available at time t. Therefore no arbitrage seems possible since market are “efficient”. Years later, Lo revoked this hypothesis and describe rather the AMH, Adapatvie market hypothesis that stated that market are efficient in the long run but not in the short term.

Because markets can be efficient over long periods yet still exhibit predictable bursts or trends in the short term, we need a tool capable of capturing those irregular, scale-invariant patterns in price data. Fractal analysis offers exactly that: by treating price paths as self-similar, “rough” shapes, it lets us quantify deviations from pure randomness and thus measure market inefficiency.

A **fractal** is a shape or pattern that looks “rough” or “broken up” no matter how closely you zoom in it repeats similar details at every scale. Consider the Mandelbrot set as a guiding example. You generate it by repeatedly applying a simple rule to every point in the complex plane—yet its boundary is infinitely intricate. Zoom in on any “bulb” or filament and you’ll find miniature copies of the whole set, swirling seahorse tails and spirals that echo the main cardioid. No matter how far you magnify it, new detail keeps appearing, and small patches resemble the global shape.

In the same way, financial markets can hide persistent structure at many scales: quiet periods give way to sudden bursts of volatility, trends emerge and reverse, and clusters of returns look surprisingly similar whether you’re looking at minutes, days, or months. By treating price paths like the edge of the Mandelbrot set self-similar, richly detailed, and far from pure randomness we can use fractal analysis to uncover and quantify those hidden patterns, measuring just how “inefficient” a market really is.

Starting from this Mandelbrot, revoked one of the main assumption of the Brownian motion.  
They are 3 assumptions of the Brownian motion, increments are iid, gaussian, and assum a continuous path.  
Therefore Mandelbrot revoked the first assumptions that increments are not iid, introducing dependcy in the increments, leading to a new Brownian motion call the fractional Brownian motion.

Introduction : Fractional Brownian motion

Mean = 0

Var = sigma squared, \* s^2H, with H between 0 and , it acutally comes from the Hydrologist harold Hurst in the 1950 to quantifies long term memory in river flows,

The covariance of the fbm is given by this expression where it is time dependent, which is the main revolution compared to a standard Brownian motion, the process also exhibits self-similairy as we have seen with fractals, which means that in the context of financial time series, if you look at daily, weekly, monthly data, the patterns must be the same.

Finaly, increments are normally distributed with mean 0 and variaces sigma squared, s^2H.  
When, you have an Hurst exponent equals to 0.5, your process behaves like a standard Brownian motion with var s, it behaves like a random walks.  
  
Therefore, to give a better idea of the processus we can simulated it’s paths with several values of H. In this simulation we use a chloesky decomposition based on the convariance function of the fBm.  
Starting from the left is a lower Hurst exponents, 0.2 and right is 0.8.  
You can see that the graph with the lowest hurst exponents is more erratic and it’s autocorrelation function decreases rapidly, from a higher hurst exponents, the trends is smooth and the autocorrelation decay less rapidly. The intuition behind using this process is that past returns will still have an impact and may be used to predict future returns.  
  
Therefore, many ways were used to estimate the Hurst expoenents on a given time series, we will look into two of them first of all the most common methods is the Rescaled Range anaylysis developed by Hurst and refined by Mandelbrot and walis and the Modified Rescaled Range analysis introduced by Lo 1991 to correct for biais in the R/S methods.  
  
Just a quick summary of the methods used, here, we first compute the profile which is the cumulative sum of the series, we define the range as the difference between the max of our profile and the min. We normalize by the volatility. The rescaled range is given by the division between the two. Finaly we use the scaling law of our process and Mandelbrot showed that the Rescaled Range grows at the rate T^H, If we take the logarithms we end up the Hurst exponent which can be define as the slope of the line when you plot log (R/S) versus log T, again if the H is 0.5, is scaled as the square root of T, with is the standard deviation of a Standard Brownian motion. This methods as several bias, first of all it does not account for short term autocorrelation noise, for example When returns are positively autocorrelated, each move tends to follow the previous one, so their running totals drift more smoothly and produce a larger range R than truly random data. Because R/S measures that oversized range against the standard deviation S, it interprets the extra “wandering” as long-term memory, falsely inflating the Hurst estimate.

And secondly it does not make a statistical test to compute for long term memory.  
Therefore, another method was developed by Lo (1991), to account for short term autocorrelation.  
If you, increase the denominator the Q variable will be lower, hence the autoccorelation biais reduced, moreover we can compute a statistical test, the t-stat will be given by this expression and we can compare it with critival values of 1.620 at 10% and 1.74 at 5%.  
  
Therefore what will do now is compute the Hurst expoenents over several indices S&P 500 (US), Russell 2000 (US), FTSE 100 (UK), Nikkei 225 (Japan), DAX (Germany).

The data is monthly, the returns also exhibits stationary after differenting them once.

Here is our first results, first of all, all processes seems to exhibits long term memory caracteristices snce the Hurst expoenent is greater than 0.5, moreover we see that the Hurst Exponent tends to be overestimated with the R/S analysis since all Hurst exponents are greater than or equal than the M-R/S. Finally if we test the hypothesis of Long term memory only one indices exhibits a statistical, and it Is the Russell small and mid caps 2000. This analysis gives us insight on the behavior of the series, but it is subjects to periods and frequency estimations. Moreover we can not do anything with this information of long term memory solely based on one Hurst exponents. Nevertheless, we can obsver that the Sp500 is the series that is the closest to a random walk with a M-R/S close to 0.5 and the Russell being the less efficient market with a M-R/S hurst at 0.588.  
  
Since we cannot do something with those results we dive into what we call a multifractal analysis where we are able to compute an hurst exponent for a specific market periods.  
A methods disgned for the multifractal analysis of time series is the Miltifractal Detrended Fluctations analysys. The goal of this methods is to estimate several Hurst Exponents given different market periods such as high volitlity, low volitlity. It gives us insights on how the Hurst exponents behaves.

The MF-DFA originally developed by Jan kandelhardt consist of five steps.  
First compute the mean of the series, and the profile like in the R/S.  
Then you split your data into N non overlapping segment of length s, s is the scale you can see it as a frequency variables, bigger scales gives you lower frequencies lower scales gives you higher frequencies. Since N\_s is an integer we must repear the process starting from the end of the seires, that way we make sure that every parts of the series is taken into account. That is why we divide by 2Ns.

For each segment we fit a polynomial trend, not needed for stationary time series. Then we compute the local variances of each non overlapping segment v inside the scale s.  
Finally we sum all of those variances and introduce a variable q, that represents the intensity of the fluctuations. For small q, small fluctuations are more representative of the series, since you weighted them more than higher fluctuations, and for higher q, higher fluctuations are more representative since those variances will dominate the lower ones.

One we summed all the fluctuations, we use the scaling rule to get the hurst exponents given an order fluctuationq, and we call it the generalized Hurst exponents.

A quick representation of the steps to better understand it, for different scaling s, we are able to compute the variances of our series and quantify it’s Hurst exponents.  
  
For this analysis e will use the Russell 2000 and Sp500 since they seems from the R/S analysis to differ in terms of local behaviors. We use the daily differentiated log returns of those series which corresponds to 10000 data points.

Let’s have a look at the multfracctal spectrum of the Russell 2000, you have the Hurst exponents of the y axis and the local fluctuations q on the x axis,  
While looking at this spectrum we are interested in several details, first of all the slope of the line, it is donward sloping with q. As we said lower values of q represents calm parts of the series, lo volatility and higher q, represents high volatility of regimes switches. Therefore the Hurst exponents is higher given small fluctuations, it seems logic with our first representation of the fBm, where the trends for a higher Hurst exponents seemed smoother than for lower exponents. At q =-3, so for small fluctuations the Hurst exponents is the higher at 0.65, and for high fluctuations the Hurst exponents is the lower at 0.47. What we are interested in also is the width of the generalized Hurst exponents values, if they are well separated the series likely exhibits multifractality here the width is close the 0.18, for a claissical random walk estimated it would be close to 0.05. Finalyy at q = 0, it represents the most local hurst expoenents of the series here it is situated at 0.55, which is a bit higher than a random walk at 0.5  
  
If we compare it with the SP500 we can have several results, donward sloping too meaning that the series is likely to exhibits long term memory in low volatility periods. The width of the GHE is lower around 0.11, and q = 0.485.   
When we compared the two results, the SP500 is likely to be less multifractal and more efficient than the Russell 2000, it seems logical in several ways, the liquidity of the Sp500 is way higher than the Russell 2000, the companies are well known with higher market cap, the Sp500 is the most traded index. The multifractal analysis corroborates our results from the R/S analysis Sp500 efficient market, Russell 2000 non efficient with long term memeory behavior.

Given the GHE we can derive more results from the MF-DFA methods, first of all we can compute the Holder exponent which quantifies the local regularity of a signal. In multifractal analysis of financial data, **local regularity** (or **pointwise Hölder exponent**) and **singularity** refer to how “smooth” or “rough” the price path is at each instant:

It is derived from the legendre transformations and depend on the generalized hurst exponent and it’s derivation. That extra q h′(q) term is exactly what encodes multifractality it measures how the scaling exponent itself changes with the moment order q. In regions where h(q) is steep, q h′(q) blows up α, signaling patches of unusually rough or smooth behavior.

If h(q) were flat (so h′(q)=0) then small and large moves would all follow exactly the same scaling law—just like in a pure random walk, where there’s no difference in the clustering of small versus large price changes.

In financial terms, observing h′(q)≠0 is a clear sign of multifractality—markets aren’t “scale‐invariant” in the simple way a random walk would be. Instead, the presence of heavy tails, volatility clustering, or bursts of activity shows up as a changing h(q) and, equivalently, a nonzero derivative h′(q)This is exactly why we interpret it as evidence that large price moves and small price moves follow different statistical laws.

Moreover, from the Hölder exponent we can compute the multifractal spectrum f(α) which represents the fractal dimension (or “size”) of the set of time‐points where the local regularity equals α. In other words, for each level of roughness α, f(α) tells you how prevalent that roughness is in the series – a wide spectrum means the price path alternates frequently between very smooth and very jagged behavior, whereas a narrow spectrum (collapsing to a point) would indicate a uniform, monofractal process.

Kantelhardt definied the multifractality as a combinaition of several things the first one being Type 1 which comes from long term correlation of the series, and the second one come from type II, Heavy tails value distribution. Therefore if we shuffle our return we lose the Type 1. We all know that financial time series are non gaussian with higher kurtosis than normal. Therefore when we compute the multifractal spectrum, we should observe a shrinking in the width of the spectrum signalling that the multifractalité is less than before.

Multifractal Spectrum f (α) — Russell 2000

In bleu you have the original series without shuffling and in red the shuffled serie,

First of all, the Width is larger, when I’m talking about the width I mean the difference between the alpha max and the alpha min that you have. It is a measure of inefficiency since the wider the spectrum the less the series behaves as a random walk it was shown that a simulated random walk exhibits a spectrum width of 0.10 to 0.15. Whereas for the Russell 2000 the width is close to 0.4, the alpha pic repsents the most local holder exponents which is in this case 0.5506 so greater than 0.5 meaning that the series exhibits long memory. The Shuffled series is as it should is narrower close to 0.2, with a alpha peak also lower 0.53. Therefore, we can assume that the Russell 2000 exhbitis multifractality and that it comes from the long term memory inside the series.

For the Sp500, the sectrum is different, first of all the width is lower on the original series it is close to 0.2 and when you shuffle the width doesn’t shrink, the multifractality doesn’t come from the long term memeory but rather from the strongly non gaussian, when estimating the Kurtosis on Sp500 it was estimated at 28.41 and on the Russell it was 12.71.

From this analysis we can see that a clear measure of inefficiency would be the width of the spectrum, a wider width would mean more heterogenous and complex behavior, a narrower spectrum would mean that price would follow a random walk, the price is rather define as a combinaition of 1 behavior over time rather than a complex sum of irregular behavior.

Therefore we propose a way of defining this inefficiency by combining two things.  
First we use thespectrum width the difference between alpha max and min. That way we quantifie for market inefficiency moreover we use the deviation from the Hurst exponent, a Hurst close to 0.5 would mean effiency. The classical Ineffiency Index as you would find it in the literature would only take into account the deviation from Hurst, but we width of the spectrum to also quantifie for it. In the way we compute it we use the difference of the width between two stocks or indices that way we know wich market is less efficient than the other.

Therefore, based on our analysis we have everything to make a trading strategy.  
The trading strategy is as follow, we first computed a rolling modified R/S of 6 months, we use the M-R/S to account for short term autocorrelation biais, We calculate the width of the spectrum for each index and we compute the difference between them to see which one is more inefficienct. Finaly we compute a classical momentum over the last twelwe months without taking into the account the last month, based on the difference between the return o the two indices a positive value mean that the SP500 is outperforming the Russell and a negative value means that the Russell is outperforming the Sp500. In this approach we use the inefficiency index as a filter for our trading signal.

Strategy Rules and Portfolio Allocation

We start with a 50/50 porotfolio in each index.

If the Hurst exponent is greater than 0.5 signalling long memory, that we also have a positive momentum meaning that over the last period the SP500 is outperforming the Russell, and also that the Inefficiency index is positive meaning that the width of the SP500 is larger than the one of the Russell meaning more complex behavior and greater market ineffiency we overweight ourselves on the SP500.  
  
Otherwise is H > 0.5, momentum negative, and I negative we overweight on the russel,

If we don’t have a long memory signal we stay at 50/50 in each index.

First of all, our strategy out perfoms the benchmark in the long run, most of the performance is driven by the last 5 years but still. Moreover, we never underperform the benchmark for long period of time, we have a strategy that can outperforms over long periods of time and in bad time stick to the index. And what we wanted to show here is the fact that we are capable of capturing the overperfoming index. If you look at the cumulative returns of both indices, you observed period where the SP500 outperforms and Russell 2000 underperfoms, in here, here and here and those are periods were we outperforms our benchmark.

So this is the most relevant metrics of al the strategies, our strategy achieved better statistics than all the strategies expect at a higher volatility. The ModifOverlap120NoFilter corresponds to the strategy without the proposed inefficnecy index. Therefore we see that the ineffiency is used as a risk management tool to achieve same performance but lower vol and max drawdown.

Overall we took a look at long memory estimation methods namely the R/S ad M-R/S, we dived into the multifractality of time series and explained how the Hurst exponents behaved given market fluctuations. We proposed a way to quantify market inefficiency by combineing deviation from the Hurst exponent and multifractal spectrum width. And we proposed a trading strategy that combined our whole analysis.  
Future work will be based on working with clustered multifractal pairs and trying to refine the trading strategy.  
I thank you all for your attention, hope you found this presentation interesting and I am now eager to take your comments, suggestions and improvements based on my work. Thanks