ComS 331 Spring 2024 Name: Aren Ashlock

1. Consider the language $L = \{a^r b^s c^t : t \le r \cdot s\}$. Use the pumping lemma to prove that L is not a context-free language.

   If L is a context-free language, then $\exists m, \forall |w| \ge m, \exists uvxyz = w, |vy| > 0, |vxy| \le m, \exists k, uv^kxy^kz \in L$. I'll choose $w = a^m b^m c^{m^2}$. Since $|vxy| \le m$, then w can span at most two of the runs. Here are a few cases this string could be broken down into...

   Case 1: $v = a^i, y = a^j$ with $k = 0$, we get $a^{m-i-j}b^m c^{m^2}$. The lemma definition makes it so that at least either i or j is ≥ 1, meaning we have at least $a^{m-1}b^m c^{m^2}$. Since $m^2 \le m^2 - m$ is not true, $w \notin L$. A simlar situation occurs when $v = b^i$, $y = b^j$ with $k = 0$, we get $a^m b^{m-i-j} c^{m^2}$. Since either i or j is ≥ 1, we have at least $a^m b^{m-1} c^{m^2}$. Since $m^2 \le m^2 - m$ is not true, $w \notin L$.

   Case 2: $v = a^i, y = b^j$ with $k = 0$, we get $a^{m-i}b^{m-j}c^{m^2}$. The lemma definition makes it so that at least either i or j is ≥ 1 and both are $< m$, meaning $m^2 \le m^2 - mi - mj + ij$ is not true. Thus, $w \notin L$.

   Case 3: v or $y = a^i b^j$ or $b^i a^j$. With $k = 2$, pumping gets us either a string containing $a^i b^j a^i b^j$ or $b^i a^j b^i a^j$, which is not accepted based on the language definition. Therefore, $w \notin L$.

   Case 4: $v = c^i, y = c^j$ with $k = 2$, we get $a^m b^m c^{m^2+i+j}$. The lemma definition makes it so that at least either i or j is ≥ 1, meaning we have at least $a^m b^m c^{m^2+1}$. Since $m^2 + 1 \le m^2$ is not true, $w \notin L$.

   Case 5: $v = b^i, y = c^j$ with $k = 0$, we get $a^m b^{m-i} c^{m^2-j}$. Looking at the language definition, we must satisfy $m^2 - j \le m^2 - mi$. Reworking it, we get $j \ge mi$. Since we know that $j < m$, this is only satisfied when $i = j = 0$. This is not true because the lemma states that at least either i or j is ≥ 1. As a result, $w \notin L$.

   Since none of these cases work, you cannot pump w meaning L is not a context-free language.

2. Consider the language

   $$L = \{w \in \{a, b\}^* : \text{the longest run of } a\text{'s in } w \text{ is longer than any run of } b\text{'s in } w\}.$$

   For example, $abbbbaabbbaaaaaa \in L$ because the longest run of b's in it has length four, while the longest run of a's has length six. Prove that L is not context-free.

   If L is a context-free language, then $\exists m, \forall |w| \ge m, \exists uvxyz = w, |vy| > 0, |vxy| \le m, \exists k, uv^kxy^kz \in L$. I'll choose $w = b^m a^{m+1} b^m$. Since $|vxy| \le m$, then w can span at most two of the runs. Here are a few cases this string could be broken down into...

   Case 1: $v = b^i, y = b^j$ with $k = 2$, we get $b^{m+i+j}a^{m+1}b^m$ OR $b^m a^{m+1} b^{m+i+j}$. The lemma definition makes it so that at least either i or j is ≥ 1, meaning we have at least $b^{m+1}a^{m+1}b^m$ OR $b^m a^{m+1} b^{m+1}$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

   Case 2: $v = a^i, y = a^j$ with $k = 0$, we get $b^m a^{m+1-i-j} b^m$. The lemma definition makes it so that at least either i or j is ≥ 1, meaning we have at least $b^m a^m b^m$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

   Case 3: $v = b^i a^j, y = a^k$ with $k = 0$, we get $b^{m-i} a^{m+1-j-k} b^m$. Since vxy is spanning across two runs, we cannot have ϵ for either v or y (meaning they both have a length of at least 1). This means we have at least $b^{m-1}a^{m-1}b^m$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

   Case 4: $v = a^i, y = a^j b^k$ with $k = 0$, we get $b^m a^{m+1-i-j} b^{m-k}$. Since vxy is spanning across two runs, we cannot have ϵ for either v or y (meaning they both have a length of at least 1).

This means we have at least $b^m a^{m-1} b^{m-1}$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

Case 5: $v = b^i, y = a^j$ with $k = 0$, we get $b^{m-i} a^{m+1-j} b^m$. Since vxy is spanning across two runs, we cannot have ϵ for either v or y (meaning they both have a length of at least 1). This means we have at least $b^{m-1} a^m b^m$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

Case 6: $v = a^i, y = b^j$ with $k = 0$, we get $b^m a^{m+1-i} b^{m-j}$. Since vxy is spanning across two runs, we cannot have ϵ for either v or y (meaning they both have a length of at least 1). This means we have at least $b^m a^m b^{m-1}$. Since the run of a's is not longer than the longest run of b's, $w \notin L$.

Since none of these cases work, you cannot pump w meaning L is not a context-free language.

3. Define a NPDA for the language $L = \{a^{2n} b^n : n \in \mathbb{N}\}$.

   NPDA $M = (\{q_0, q_1, q_2, q_f\}, \{a, b\}, \{A, z\}, \delta, q_0, z, \{q_f\})$
   $\delta(q_0, \epsilon, z) = \{(q_f, \epsilon)\}$
   $\delta(q_0, a, z) = \{(q_1, Az)\}$
   $\delta(q_1, a, A) = \{(q_0, A)\}$
   $\delta(q_0, a, A) = \{(q_1, AA)\}$
   $\delta(q_0, b, A) = \{(q_2, \epsilon)\}$
   $\delta(q_2, b, A) = \{(q_2, \epsilon)\}$
   $\delta(q_2, \epsilon, z) = \{(q_f, \epsilon)\}$

4. Define a NPDA for the language $L = \{uv \in \{0, 1\}^* : |u| = |v| \wedge u \neq v^R\}$.

   NPDA $M = (\{q_0, q_1, q_2, q_f\}, \{0, 1\}, \{A, B, z\}, \delta, q_0, z, \{q_f\})$
   $\delta(q_0, 0, z) = \{(q_0, Az)\}$
   $\delta(q_0, 0, A) = \{(q_0, AA), (q_1, \epsilon)\}$
   $\delta(q_0, 0, B) = \{(q_0, AB), (q_2, \epsilon)\}$
   $\delta(q_0, 1, z) = \{(q_0, Bz)\}$
   $\delta(q_0, 1, A) = \{(q_0, BA), (q_2, \epsilon)\}$
   $\delta(q_0, 1, B) = \{(q_0, BB), (q_1, \epsilon)\}$
   $\delta(q_1, 0, A) = \{(q_1, \epsilon)\}$
   $\delta(q_1, 1, B) = \{(q_1, \epsilon)\}$
   $\delta(q_1, 0, B) = \{(q_2, \epsilon)\}$
   $\delta(q_1, 1, A) = \{(q_2, \epsilon)\}$
   $\delta(q_2, 0, A) = \{(q_2, \epsilon)\}$
   $\delta(q_2, 1, B) = \{(q_2, \epsilon)\}$
   $\delta(q_2, 0, B) = \{(q_2, \epsilon)\}$
   $\delta(q_2, 1, A) = \{(q_2, \epsilon)\}$
   $\delta(q_2, \epsilon, z) = \{(q_f, \epsilon)\}$