ComS 331 Spring 2024 Name: Aren Ashlock

1. Reproduce the text and figure...

This is an inline equation: $x + y = 3$.
This is a displayed equation:

$$x + \frac{y}{z - \sqrt{3}} = 2$$

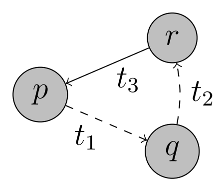
This is how you can define a piece-wise linear function:

$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ 7x + 2 & \text{if } x \geq 0 \text{ and } x < 10 \\ 5x + 22 & \text{otherwise} \end{cases}$$

This is a matrix:

9	9	9	9
6	6	6	
3		3	3

This is a graph with two types (solid and dashed) of labeled edges:



2. Prove that $\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integer numbers) are equinumerous.

We can prove that two sets are equinumerous through the existence of a bijection.
First, we must define a function $f(n)$ that maps the values of one set to another.
$f(n) = \begin{cases} -\frac{n}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd} \end{cases}$
One-to-one: $\forall a, b \in \mathbb{N}, f(a) = f(b) \implies a = b$
N is even: $-\frac{a}{2} = -\frac{b}{2} \to a = b$
N is odd: $\frac{a+1}{2} = \frac{b+1}{2} \to a = b$
Onto: $\forall a, b \in \mathbb{N}, f(a) = f(b) \implies a = b$
N is even: $f(a) = b \to -\frac{a}{2} = b \to a = -2b$
N is odd: $f(a) = b \to \frac{a+1}{2} = b \to a = 2b - 1$
Since, the function is both one-to-one and onto, this proves that there is a bijection between $\mathbb{N}$ and $\mathbb{Z}$. This means that they are equinumerous.

3. Prove that the relation R defined by $\forall m, n \in \mathbb{N}, (m, n) \in R \Leftrightarrow (m - n) \bmod 3 = 0$ is an equivalence relation, and describe its equivalence classes.

We prove this by showing that the relation is reflexive, symmetric, and transitive.
Reflexive: $\forall x \in \mathbb{N}, (x, x) \in R$:
$(x - x) \bmod 3 = 0 \bmod 3 = 0$
Symmetric: $\forall x, y \in \mathbb{N}, (x, y) \in R \implies (y, x) \in R$:
$(x - y) \bmod 3 = 0 \to x - y = 3k$
$(y - x) = 3(-k)$: $3(-k) \bmod 3 = 0$
$\therefore (y, x) \in R$
Transitive: $\forall x, y, z \in \mathbb{N}, (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$:

$(x-y) \bmod 3 = 0 \wedge (y-z) \bmod 3 = 0$
$(x-y+y-z) \bmod 3 = 0$
$(x-z) \bmod 3 = 0$
Since, the relation R is reflexive, symmetric, and transitive, that means it IS an equivalence relation.
Equivalence Classes:
$[0]_R = \{3k | k \in \mathbb{N}\}$
$[1]_R = \{1+3k | k \in \mathbb{N}\}$
$[2]_R = \{2+3k | k \in \mathbb{N}\}$

4. Show that $\sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$

   We prove this by using induction on n.
   Basis: This holds for $n=0$, since $\sum_{i=1}^{0} i^2 = 0$ and $(2(0)+1)(0+1)0/6 = 0$.
   Induction Hypothesis: Assume that $\forall m \leq n$, $\sum_{i=1}^{m} i^2 = (2m+1)(m+1)m/6$.
   Inductive Step: Then, the property holds for $n+1$ as well, since...
   $\sum_{i=1}^{n+1} i^2 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   $\sum_{i=1}^{n} i^2 + (n+1)^2 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   $(2n+1)(n+1)n/6 + n^2 + 2n + 1 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   $(2n^3+3n^2+n)/6 + (6n^2+12n+6)/6 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   $(2n^3+9n^2+13n+6)/6 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   $(2(n+1)+1)((n+1)+1)(n+1)/6 = (2(n+1)+1)((n+1)+1)(n+1)/6$
   The equality above is **TRUE**. Therefore, $\sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$.

5. Show that, for $n \geq 1$, $\sum\limits_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.

   We prove this by using induction on n.
   Basis: This holds for $n=1$, since $\sum\limits_{i=1}^{1} \frac{1}{i^2} = 1$ and $2 - \frac{1}{1} = 1$ and $1 \leq 1$.
   Induction Hypothesis: Assume that $\forall m \leq n$, $\sum\limits_{i=1}^{m} \frac{1}{i^2} \leq 2 - \frac{1}{m}$ when $m \geq 1$.
   Inductive Step: Then, the property holds for $n+1$ as well, since...
   $\sum\limits_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}$
   $\sum\limits_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$
   $2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$
   $-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1}$
   $-\frac{(n+1)^2}{n(n+1)^2} + \frac{n}{n(n+1)^2} \leq -\frac{n(n+1)}{n(n+1)^2}$
   $\frac{n-n^2-2n-1}{n(n+1)^2} \leq \frac{-n^2-n}{n(n+1)^2}$
   $\frac{-n^2-n-1}{n(n+1)^2} \leq \frac{-n^2-n}{n(n+1)^2}$
   The equality above is **TRUE**. Therefore, for $n \geq 1$, $\sum\limits_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.