2) • $Y_1 \sim \text{Poisson}(\alpha\beta)$, $Y_2 \sim \text{Poisson}((1-\alpha)\beta)$

• Y_1 e Y_2 são condicionalmente independentes.

• A priori: $\alpha \sim \text{Beta}(p,q)$, $\beta \sim \text{Gamma}(p+q,1)$

• α e β são independentes.

a)

PMF da Poisson: $P(Y=k,\lambda) = \dfrac{e^{-\lambda}\lambda^{k}}{k!}$

Como há independência a veross. conjunta é o produto dos PMF's:

$$L(y_1,y_2|\alpha,\beta) = \left(\frac{e^{-\alpha\beta}(\alpha\beta)^{y_1}}{y_1!}\right)\left(\frac{e^{-(1-\alpha)\beta}((1-\alpha)\beta)^{y_2}}{y_2!}\right)$$

$$= \frac{\alpha^{y_1}(1-\alpha)^{y_2}\beta^{y_1+y_2}e^{-\alpha\beta-(1-\alpha)\beta}}{y_1!\,y_2!} \propto \underline{\alpha^{y_1}(1-\alpha)^{y_2}\beta^{y_1+y_2}e^{-\beta}}$$

—11—

b) a priori $\pi(\alpha,\beta)$ → constante de normalização.

$$\pi(\alpha) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\cdot\alpha^{p-1}(1-\alpha)^{q-1} \propto \alpha^{p-1}(1-\alpha)^{q-1}$$

$$\pi(\beta) = \frac{1^{p+q}}{\Gamma(p+q)}\beta^{p+q-1}e^{-\beta} \propto \beta^{p+q-1}e^{-\beta}$$

Como temos independência → $\pi(\alpha,\beta) = \pi(\alpha)\pi(\beta)$

$$\propto \underline{\alpha^{p-1}(1-\alpha)^{q-1}\beta^{p+q-1}e^{-\beta}}$$

c) $\pi(\alpha,\beta|y_1,y_2) \propto L(y_1,y_2|\alpha,\beta)\cdot\pi(\alpha,\beta)$ Post. conjunta

$\therefore \pi(\alpha,\beta|y_1,y_2) \propto \alpha^{y_1}(1-\alpha)^{y_2}\beta^{y_1+y_2}e^{-\beta}\cdot(\alpha^{p-1}(1-\alpha)^{q-1}\beta^{p+q-1}e^{-\beta})$

agrupando os termos:

$$\pi(\alpha,\beta|y_1,y_2) \propto \alpha^{y_1+p-1}(1-\alpha)^{y_2+q-1}\beta^{(y_1+y_2)+(p+q-1)}e^{-2\beta}$$

—//—

d) Posterior Marginal α.

$$\pi(\alpha|y_1,y_2) = \int_{\mathbb{R}^+}\pi(\alpha,\beta|y_1,y_2)\,d\beta$$

$$\therefore \pi(\alpha|y_1,y_2) \propto \alpha^{y_1+p-1}(1-\alpha)^{y_2+q-1}\cdot\int_{\mathbb{R}}\beta^{(y_1+y_2)+(p+q-1)}e^{-2\beta}\,d\beta$$

A integral em questão se assemelha ao núcleo de uma distribuição Gamma: $x^{\alpha-1}e^{-\beta x}$

$\therefore$ A integral resulta em uma constante que não depende de α, portanto:

$$\therefore \pi(\alpha|y_1,y_2) \propto \alpha^{y_1+p-1}(1-\alpha)^{y_2+q-1} \sim \text{Beta}(y_1+p,\, y_2+q)$$

—//—

e) Posterior Marginal β (mesma ideia da anterior)

$$\pi(\beta|y_1,y_2) = \int_0^1\pi(\alpha,\beta|y_1,y_2)\,d\alpha$$

$$\therefore \pi(\beta|y_1,y_2) \propto \beta^{y_1+y_2+p+q-1}e^{-2\beta}\int_0^1\alpha^{y_1+p-1}(1-\alpha)^{y_2+q-1}\,d\alpha$$

Como já vimos essa integral é núcleo de uma Beta logo consiste numa constante em relação a β.

$$\therefore \pi(\beta|y_1,y_2) \propto \beta^{y_1+y_2+p+q-1}e^{-2\beta} \sim \text{Gamma}(y_1+y_2+p+q,\, 2)$$

③ X := Número de defeituosas / dia

$(X|Y,\theta) \sim Bin(Y,\theta)$, $Y \sim Poisson(\lambda)$ sendo λ conhecido

a prior: $(\theta|Y=y) \sim Beta(\alpha,\gamma)$. α e γ conhecidos e independentes de Y.

$X|\theta \sim Poisson(\lambda\theta)$. $\theta \sim Beta(\alpha,\gamma)$

Problema: Estimar θ

a) a Posterior: $\pi(\theta|X=x)$

$\pi(\theta|x) \propto \underbrace{p(x|\theta)}_{?} \cdot \underbrace{\pi(\theta)}_{B(\alpha,\gamma)}$. → Temos $p(x|y,\theta)$ então tiraremos y da equação.

$$p(x|\theta) = \sum p(x|y,\theta) \cdot p(y) = \sum \left(\binom{y}{x} \theta^{x} (1-\theta)^{y-x} \right) \left(\frac{e^{-\lambda} \lambda^{y}}{y!} \right)$$

∴ $p(x|\theta)$ é uma mistura Binomial (Y,θ) com Poisson (λ)

o que resulta em $\sim Poisson(\lambda\theta) \rightarrow p(x|\theta) = \dfrac{(\lambda\theta)^{x} e^{-\lambda\theta}}{x!}$

$$\therefore \pi(\theta|X=x) \propto \frac{(\lambda\theta)^{x} e^{-\lambda\theta}}{x!} \left(\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)} \theta^{\alpha-1} (1-\theta)^{\gamma-1} \right)$$

Removendo constantes:

$$\therefore \pi(\theta|X=x) \propto \left(\theta^{x+\alpha-1}\right)(1-\theta)^{\gamma-1} e^{-\lambda\theta}$$

b) Posterior Conjunta $\pi(Y, \theta \mid X = x)$

$$\pi(Y, \theta \mid X = x) \propto p(X = x \mid Y = y, \theta) \cdot p(Y = y) \cdot \pi(\theta)$$

veross. de X · dist Y · priori (θ)

$$\propto \binom{y}{x} \theta^{x} (1-\theta)^{y-x} \cdot \frac{e^{-\lambda} \lambda^{y}}{y!} \cdot \theta^{\alpha-1} (1-\theta)^{\gamma-1}$$

$$\propto \frac{1}{x!\,(y-x)!} \theta^{x} (1-\theta)^{y-x} \cdot e^{-\lambda} \lambda^{y} \cdot \theta^{\alpha-1} (1-\theta)^{\gamma-1}$$

$$\therefore \pi(Y, \theta \mid X = x) \propto \frac{\lambda^{y}}{(y-x)!} \theta^{x+\alpha-1} (1-\theta)^{y-x+\gamma-1}$$

portanto é válido p/ $y \geq x$ e $\theta \in [0,1]$