Lecture 16:

Reading: Bendat and Piersol, Ch. 9.1-9.2

Recap

We've looked at a couple examples of coherence calculations along with some (incomprehensible) figures from published cases. The key feature of coherence is that it allows you to decide if two records vary in a consistent way at any given frequency. This is a subtle point: if you view the world from the perspective of a Fourier transform, everything is sinusoidal, and naturally all data records vary sinusoidally, although the phasing of record x could differ from the phasing of record y. When we compute coherence, we ask whether the phasing (at frequency σ or wavenumber k) between x and y is consistent between different chunks of the data records. To compute coherence we need to segment our data; for the same reasons that we detrend and window when we compute spectra, we should detrend and window when we compute coherence.

Coherence calculations produce two results: a coherence that varies between 0 and 1, and a phase that varies between $-\pi$ and π (or equivalently from 0 to 360 degrees or -180 to 180 degrees.)

Frequency-wavenumber details First a digression to think about how to compute frequency-wavenumber spectra. On the surface this seems easy:

1. Break data into segments in time or space, probably time.
2. Demean, detrend, window as appropriate.
3. Fourier transform in time.
4. Fourier transform in space.
5. Average amplitudes for all realizations.
6. Use fftshift to put frequency and wavenumber zero at the center of your data matrix.
7. Plot with log of values. Show half the data plane to cover positive and negative propagation possibilities.

The devil is in the details. Do you detrend in time only, or in time and space? Do you window at all, in time, in space, both at once?

Uncertainties of coherence and phase: What do we believe?

Bendat and Piersol provide a good discussion of bias and uncertaintites in spectral estimators. As a starting point, the variance of the quantity that we want to estimate is

$$\text{var}[\tilde{A}] = E[\tilde{A}^2] - A^2, \tag{1}$$

where A is the true value, and $\tilde{A}$ is the unbiased estimate (so $E[\tilde{A}] = A$. For spectral estimators we tend to talk about the normalized error:

$$\epsilon^2 = \frac{\text{var}\tilde{A}}{A^2}. \tag{2}$$

Bendat and Piersol first derive relationships for the variance of the spectrum and cross-spectrum in the case of one segment and two degrees of freedom (see appendix). They then note that variance scales with $1/n$, where n is the number of degrees of freedom, so that variance can

be inferred for spectra and cross-spectra with any number of degrees of freedom (by dividing by n_d the number of segments).

We compute a significance level for coherence several ways. The standard approach that we discussed previously is to set a threshold for evaluating whether a calculated coherence exceeds what we might expect from random white noise. We started with the uncertainty for the squared coherence:

$$\beta = 1 - \alpha^{1/(n_d-1)}, \tag{3}$$

where n_d is the number of degrees of freedom and C is the coherence (see Thomson and Emery). An alternate formulation is presented by Bendat and Piersol, who report the standard deviation of the coherence to be:

$$\delta_{\gamma^2_{xy}} = \frac{\sqrt{2}(1-\gamma^2_{xy})\gamma_{xy}}{\sqrt{n_d}}. \tag{4}$$

These are different metrics. One tells us whether the derived coherence is statistically different from zero; the second evaluates the range of values that would be consistent with an observed coherence.

Uncertainty for phase is often reported with the formula I provided in class

$$\delta_\phi = \sin^{-1}\left[t_{\alpha,2n_d}\sqrt{\frac{1-C^2_{xy}}{2n_dC^2_{xy}}}\right] \tag{5}$$

where $t_{\alpha,2n_d}$ is the "Student t distribution". But when we plot this up, it doesn't look entirely plausible. Bendat and Piersol provide a different formulation:

$$\text{std}\left[\phi_{xy}(f)\right] \approx \sqrt{\left[\frac{1-\gamma^2_{xy}(f)}{\gamma^2_{xy}(f)2n_d}\right]}. \tag{6}$$

and Zwiers and Von Storch quote Hannan and provide:

$$\delta_\phi = \sin^{-1}\left[t_{(1+\alpha)/2,2n_d-2}\frac{C^{-2}_{xy}-1}{2n_d-2}\right] \tag{7}$$

In Matlab, these become:

```
% cab is covariance between a and b
alpha = .05;
delta_phase = asin((1-betainc(2*nd/(2*nd+alpha^2),nd,.5))*...
  sqrt((1-abs(cab).^2)./(abs(cab).^2*sqrt(2*nd))));
delta_phase2 = sqrt((1-cab.^2)./(abs(cab)^2*2*nd));
delta_phase3 = asin((1-betainc(2*nd/(2*nd+alpha^2),nd,.5))...
    *(1 ./cab.^2-1)/(2*nd-2));
```

The expressions are similar, though not identical. Which is most appropriate? We can test this out by creating a fake data set with a known phase relationship:

```
a=randn(100,1000)+ cos(2*pi/10*(1:100)')*ones(1,1000);
b=randn(100,1000) + sin(2*pi/10*(1:100)')*ones(1,1000);
fa=fft(a);
fb=fft(b);
```

```
fab=conj(fa).*fb;
faa=conj(fa).*fa;
fbb=conj(fb).*fb;

cab=abs(mean(fab,2)) ./sqrt(abs(mean(faa,2)) .* abs(mean(fbb,2)));

m=10;
clear phase_c
for i=1:1000/m
 phase_c(:,i)=atan2(-imag(mean(fab(:,(i-1)*m+1:i*m),2)),...
      real(mean(fab(:,(i-1)*m+1:i*m),2)));
end

nd=m;
delta_phase = asin(tinv(.95,2*nd)*...
  sqrt((1-abs(cab).^2)./(abs(cab).^2*sqrt(2*nd))));
delta_phase2 = sqrt((1-cab.^2)./(abs(cab).^2*2*nd));
delta_phase3 = asin(tinv(.975,2*nd-2)*(1 ./cab.^2-1)/(2*nd-2));
% compare results
[delta_phase(11) delta_phase2(11) delta_phase3(11)]
std(phase_c(11,:))
```

It's clear from these tests that (a) the distribution of the phases should be roughly Gaussian, (b) Bendat and Piersol's representation for the standard deviation of the phase (delta_phase2) is relatively reliable, (c) the inverse sine formulations should produce phase errors representing the 95th percentile, and they are not consistent with that, suggesting that some detail has been lost in translation.

Appendix: More detail on variance of cross-spectra

If we have two degrees of freedom, the cross spectrum is

$$\begin{aligned} |\hat{C}_{XY}(\sigma)|^2 &= \hat{C}^*_{XY}\hat{C}_{XY} && (8)\\ &= |X^*(\sigma)Y(\sigma)|^2 && (9)\\ &= X(\sigma)Y^*(\sigma)X^*(\sigma)Y(\sigma) && (10)\end{aligned}$$

So trying all combinations to get the sum of the 4-term product:

$$\begin{aligned} \langle|\hat{C}_{XY}(\sigma)|^2\rangle &= \langle<XX^*\rangle\langle<YY^*\rangle + \langle<XY^*\rangle\langle<X^*Y\rangle + \langle<XY\rangle\langle<X^*Y^*\rangle && (11)\\ &= C_{XX}C_{YY} + |C_{XY}|^2 && (12)\end{aligned}$$

where C here is the total cross-spectrum. The variance is then

$$\begin{aligned} \text{var}[\hat{C}_{XY}] &= \langle|\hat{C}_{XY}(\sigma)|^2\rangle - |\hat{C}_{XY}(\sigma)|^2 && (13)\\ &= C_{XX}C_{YY} && (14)\\ &= \frac{|C_{XY}|^2}{\gamma^2_{xy}} && (15)\end{aligned}$$

where γ_{xy} is the coherence.

With more degrees of freedom, error scales with the square root of the number of samples, just like the standard error:

$$\text{var}C_{XX} = \frac{C_{xx}^2}{n_d} \tag{16}$$

$$\text{var}C_{YY} = \frac{C_{YY}^2}{n_d} \tag{17}$$

$$\text{var}C_{XY} = \frac{|C_{XY}|^2}{\gamma_{xy}^2 n_d} \tag{18}$$

This scaling gives us the uncertainty for the coherence and phase after some manipulation. (See Bendat and Piersol, Ch. 9 for details.)

This means that the normalized uncertainty of C_{XY} is

$$\epsilon[|\hat{C}_{XY}|] = \frac{\text{std}[C_{XY}]}{C_{XY}} = \frac{1}{|\gamma_{xy}|\sqrt{n_d}} \tag{19}$$