Lecture 13: Frequency/Wavenumber Spectra

Recap

We've looked at multiple strategies for computing spectra in the frequency domain or by extension in the wavenumber domain. We've considered uncertainties, resolution and multiple methods. Last time we went through the details of aliasing. Now, we'll consider what happens when we want to consider sinusoidal patterns of variability in both time and space.

Examples.

First consider some example frequency-wavenumber spectra. What frequencies and wavenumbers are resolved? What is plotted? What is the Nyquist frequency and Nyquist wavenumber? Is the full frequency-wavenumber space represented?

As a reminder, frequency represents cycles per unit time, and wavenumber represents cycles per unit distance.

We use frequency-wavenumber spectra as a means to track propagating sinusoidal patterns. If it has a characteristic wavelength and frequency, we might suppress that variability if we didn't think about the full structure of the propagating wave. Westward propagating Rossby waves and eastward propagating Kelvin waves have characteristic frequencies and characteristic wavenumbers.

Basics.

Consider a data set $y(x,t)$ where $-x_f < x < x_f$ and $-T < t < T$ (where here we're using $\pm x_f$ for the end points in space, and following our previous examples $\pm T$ for the end points in time.) We know from our definition of the Fourier transform that we can represent y as

$$y(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi f_n t}, \tag{1}$$

or

$$y(x,t) = \sum_{m=-\infty}^{\infty} a_m e^{i2\pi k_m t}, \tag{2}$$

and by extension we can do this process in two dimensions:

$$y(x,t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm} e^{i2\pi(f_n t + k_m x)}, \tag{3}$$

where

$$2\pi k_m = \frac{2\pi m}{2x_f} \tag{4}$$

$$2\pi f_n = \frac{2\pi n}{2T} \tag{5}$$

The corresponding spectral density estimate can be calculated from the squared coefficients:

$$\hat{E}(k_m, f_n) = \frac{|a_{nm}|^2}{\Delta k \Delta f} \tag{6}$$

Practicalities.

Suppose we take a time-space data set and Fourier transform it in both directions. Here are some issues that might concern us:

1. *Does it matter whether we Fourier transform time or distance first?* All other things being equal, no. Time and space are orthogonal, and one will not influence the other. To see this consider the following:

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y(x,t)e^{i2\pi kx}\,dx\right]e^{i2\pi ft}\,dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y(x,t)e^{i2\pi ft}\,dt\right]e^{i2\pi kx}\,dx = \tag{7}$$

   (How you demean or detrend might matter, and you can ponder these issues.)

2. *Can we compute a frequency-wavenumber spectrum by Fourier tranforming a frequency spectrum in space?* No. If you look at the above equation, you'll see that you need the full complex structure of the Fourier transformed variables in order to determine the spectrum. You have to retain the phase information. If you computed spectra first, you would suppress some of the signal that you wanted.

3. *When we compute frequency spectra, we usually only plot positive frequencies? Does the same strategy apply for frequency-wavenumber spectra?* No, at a given frequency we can have forward and backward propagating signals, so we'll often want to plot a half plane for frequency with positive and negative wavenumbers, or vice versa.

4. *In practical terms, how do we implement this?* If you Fourier transform in time and space, you'll end up with a domain that should be structured as in Figure **??**. But Matlab will of course line up the frequencies and wavenumbers starting with positive and then negative. You can use the Matlab command "fftshift" to swap the presentation around.

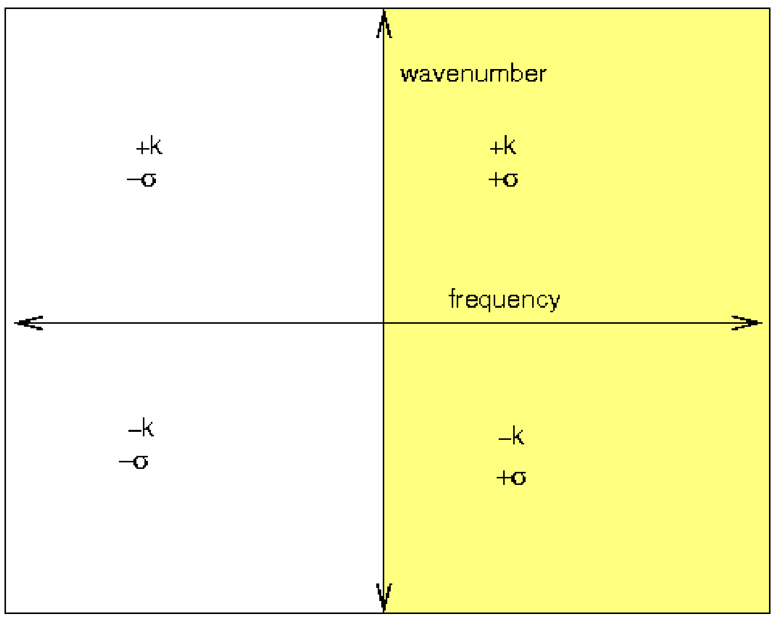

Figure 1: Fourier transform of time/space domain to form frequency/wavenumber domain. Note that $+k, +f$ is the complex conjugate of $-k, -f$ and similarly $+k, -f$ is the complex conjugate of $-k, +f$ so we normally plot only half the domain.