Hand-in Homework 1 **Alice Ren**
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$$h_t + h_a = -\frac{\mu h}{1+\alpha t}$$

(i)

$$\int_0^\infty h(a,t)da = N$$

$$\frac{dN}{dt} = b(t) - \int_0^\infty \frac{\mu}{1+\alpha t} h(a,t)da = 0$$

$$b(t) = \frac{\mu}{1+\alpha t}\int_0^\infty h(a,t)da$$

Since μ is not a function of a.

$$b(t) = \frac{\mu}{1+\alpha t}N$$

(ii) The average age should equal the average lifespan.

$$N = b * (avg \quad life \quad span)$$

$$\frac{N}{b} = \frac{1+\alpha t}{\mu}$$

$$\bar{a} = \frac{1+\alpha t}{\mu}$$

(iii)

$$\frac{dh}{dt} = \frac{-\mu h}{1+\alpha t}$$

$$\frac{da}{dt} = 1$$

$$a = t - c$$

$$c = a - t = characteristic$$

$$\frac{dh}{h} = \frac{-\mu}{1+\alpha t}dt$$

$$ln(h) = \frac{\mu}{\alpha}ln(1+\alpha t) + f(\xi)$$

$$h = f(\xi)(1+\alpha t)^{-\frac{\mu}{\alpha}}$$

$$h(a,0) = b_0 e^{-\mu a}$$

$$f(\xi) = b_0 e^{-\mu(a-t)}$$

$$h(a,t) = (1+\alpha t)^{-\nu} b_0 e^{-\mu(a-t)}$$

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(i)

$$\rho_t + (c(x)\rho)_x = 0$$

(ii) The general solution

$$\frac{dx}{dt} = c \quad with \quad x(0) = \xi$$

$$\frac{dx}{c} = dt$$

$$\int_0^x \frac{dx'}{c(x')} = t + \int_0^{\xi(x,t)} \frac{dx'}{c(x')}$$

$$\frac{d\rho}{dt} = -c(x(\xi,t))_{(\xi\frac{d\xi}{dx})}\rho$$

However, the problem can be solved if we take the total integral with respect to x rather than t. Transforming the derivative:

$$\frac{dt}{dx}\frac{d\rho}{dt} = \frac{d\rho}{dx}$$

$$\frac{dt}{dx} = \frac{1}{c(x)}$$

$$s(x) = \int_0^x \frac{dx'}{c(x')} \qquad s_x(x) = \frac{1}{c(x)} = \frac{dt}{dx}$$

$$\frac{d\rho}{dx} = -\rho\frac{c_x(x)}{c(x)}$$

$$\frac{d\rho}{\rho} = -\frac{c_x(x)}{c(x)}dx$$

$$\rho = A(\xi)\frac{1}{c(x)}$$

where A denotes an arbitrary function. To invert x and ξ:

$$s(x) = t + s(\xi) \qquad s^{-1}(s(x) - t) = \xi$$

General solution:

$$\rho = A(s(x) - t)\frac{1}{c(x)}$$

(iii) With the initial condition $\rho(x, 0) = \rho_0(\xi)$

$$\rho_0(\xi) = A(s(x))\frac{1}{c(x)}$$

At $t = 0$, $s(x) = s(\xi)$ and thus $x = \xi$:

$$A(s(\xi))\frac{1}{c(\xi)} = \rho_0(\xi)$$

$$A(s(\xi)) = c(\xi)\rho_0(\xi)$$

General solution:

$$\rho = \frac{c(\xi)\rho_0(\xi)}{c(x)} = \frac{c(s^{-1}(s(x)-t))\rho_0(s^{-1}(s(x)-t))}{c(x)}$$

where s^{-1} denotes the inverse of s.

(iv) To confirm the solution we plug in for $c(\xi)$, $\rho_0(\xi)$, and $c(x)$ at x = 0.

First, find ξ:

$$s(x) - t = ln(e^{2x} + 2e^{x}) - t = s(\xi) = ln(e^{2\xi} + 2e^{\xi})$$

At x = 0:

$$ln(3) - t = ln(e^{2\xi} + 2e^{\xi})$$
$$3e^{-t} = e^{2\xi} + 2e^{\xi}$$

Complete the square:

$$3e^{-t} + 1 = e^{2\xi} + 2e^{\xi} + 1$$
$$\sqrt{3e^{-t}+1} = e^{\xi} + 1$$
$$\xi = ln(\sqrt{3e^{-t}+1} - 1)$$

Finding $c(\xi)$ also takes a few lines of algebra:

$$c(\xi) = \frac{e^{2(ln(\sqrt{3e^{-t}+1}-1))} + 2e^{ln(\sqrt{3e^{-t}+1}-1)}}{2e^{2(ln(\sqrt{3e^{-t}+1}-1))} + 2e^{ln(\sqrt{3e^{-t}+1}-1)}}$$

$$= \frac{(\sqrt{3e^{-t}+1}-1)^2 + 2(\sqrt{3e^{-t}+1}-1)}{2(\sqrt{3e^{-t}+1}-1)^2 + 2(\sqrt{3e^{-t}+1}-1)}$$

$$= \frac{(3e^{-t}+1) - 2(\sqrt{3e^{-t}+1}-1) + 1 + 2\sqrt{3e^{-t}+1} - 2}{2(3e^{-t}+1) - 4(\sqrt{3e^{-t}+1}-1) + 2 + 2\sqrt{3e^{-t}+1} - 2}$$

$$c(\xi) = \frac{3e^{-t}}{2(3e^{-t}+1) - 2\sqrt{3e^{-t}+1}}$$

The others are easy:

$$\rho_0(\xi) = e^{-\xi^2}$$
$$\rho_0(\xi) = e^{-ln^2(\sqrt{3e^{-t}+1}-1)}$$
$$c(0) = \frac{1+2}{2+2} = \frac{3}{4}$$
$$\frac{1}{c(0)} = \frac{4}{3}$$

The solution, which matches the prompt, is:

$$\rho(0,t) = \frac{2e^{-t}e^{-ln^2(\sqrt{3e^{-t}+1}-1)}}{3e^{-t}+1-\sqrt{3e^{-t}+1}}$$