MATH 341: Homework 1

Aren Vista

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Question 1

Give an example where a combination of $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^4$, non-zero vectors, is a zero vector. Write in the form $Ax = 0$. What are the shapes of A and x and 0

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^4$ and be non-zero vectors. Let them be defined as follows:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Let A be defined as follows:

$$A = \left[\vec{a}, \vec{b}, \vec{c}\right]$$

Find values for elements of $\vec{a}, \vec{b}, \vec{c}, x$ so that the following relationship holds:

$$Ax = \vec{0}$$

Observe:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

Thus:

$$Ax = \vec{0} \implies A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Continuing:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (a_1x_1 + b_1x_2 + c_1x_3) \\ (a_2x_1 + b_2x_2 + c_2x_3) \\ (a_3x_1 + b_3x_2 + c_3x_3) \\ (a_4x_1 + b_4x_2 + c_4x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

An example of one such possible combination if the following:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Observe:

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} ((1*1) + (2*1)(-3*1)) \\ ((1*1) + (2*1)(-3*1)) \\ ((1*1) + (2*1)(-3*1)) \\ ((1*1) + (2*1)(-3*1)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the linear combination of non-zero vectors yields: $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Aside: The shapes of A is 4×3.
The vectors $[a - c], 0$ are 4×1.
The vector x is 3×1.
Recall $[4 \times 3][3 \times 1] = [4 \times 1]$

Question 2

Suppose A is a $[3 \times 3]$ matrix of all ones.
Find $\vec{x}$ and $\vec{y}$ that solve $Ax = 0$ and $Ay = 0$
Write $Ax = 0$ as a combination of the columns of A.
"Why don't I ask for a third independent vector with $Az = 0$?"

Let A be defined as:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \left[\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}\right] = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} & a_{(1,4)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} & a_{(2,4)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} & a_{(3,4)} \\ a_{(4,1)} & a_{(4,2)} & a_{(4,3)} & a_{(4,4)} \end{bmatrix}$$

Find $\vec{x}, \vec{y}$ s.t. $Ax = 0, Ay = 0$. Observe:

$$(1)\vec{a_1} + (1)\vec{a_2} + (-1)\vec{a_3} + (-1)\vec{a_4} = \vec{0}$$

This is the same as:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \vec{0}$$

Consider by Gausian Elim. the REF:

$$A_{ref} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In parametric vector form:

$$\begin{aligned} x_1 &= -x_2 - x_3 - x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Define in terms of free variables:

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the Null(A) is:

$$Nul(A) = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\forall\ \vec{x}, Nul(A)\vec{x} = \vec{b}$ we can guarentee $\vec{b}$ satisfies the equation $A\vec{b} = \vec{0}$. Notice the relationship

with the prior defined solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

$$\text{By definition of null space we would expect } \vec{x} \in Span(Nul(A)).$$
$$\exists\ \vec{x}_{nul} \ni Nul(A)\vec{x}_{nul} = \vec{x}$$

Observe

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

So that:

$$\begin{bmatrix} -1 & -1 & -1 & | & 1 \\ 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \implies \begin{bmatrix} -1 & -1 & -1 & | & 1 \\ 0 & -1 & -1 & | & 2 \\ 0 & 0 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \implies \vec{x}_{nul} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Checking

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1+1+1 & =1 \\ 1+0+0 & =1 \\ 0-1+0 & =-1 \\ 0+0-1 & =-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

We have shown that the vector $\vec{x}$ exists within the $Nul(A)$.

Question 3

Suppose $Col(A_{n\times m})$ is all of $\mathbb{R}^3$.
What can you say about n and the rank of the matrix?
(Comment: – I had flipped m,n)

Let A be defined as follows:

$$A = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & ... & a_{(1,m)} \\ a_{(2,1)} & a_{(2,2)} & & ... \\ ... & & ... & \\ a_{(n,1)} & & & a_{(n,m)} \end{bmatrix} = \left[\vec{a_1}, \vec{a_2}, ...\vec{a_m}\right]$$

Suppose: $Span(Col(A)) = \mathbb{R}^3$:

$$\implies \forall\ \vec{v} \in \mathbb{R}^3, \vec{v} \in Span(Col(A))$$
$$\implies \exists\ \{c_1, c_2, ..., c_m\} \subseteq \mathbb{R} \ni \vec{v} = c_1\vec{a_1} + c_2\vec{a_2} + ... + c_m\vec{a_m}$$

Observe: $Span(Col(A)) = \mathbb{R}^3 \implies \exists\ \vec{a_1}, \vec{a_2}, \vec{a_3} \in A$, where $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are distinct.

We say these vectors are linearly independent iff the following relationship holds:

Case 1: Proving independence of $\vec{a_1}, \vec{a_2}, \vec{a_3}$:
If $c_1, c_2, c_3 \in \mathbb{R}, \quad c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} + = \vec{0} \implies c_1 = c_2 = c_3 = 0$

Then we can say $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are independent towards each other.

Case 2: Proving dependence of $\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}$
Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$, consider $\vec{a_4} \in A$ where $\vec{a_4}$ represents an arbtrary distinct vector.
If $\exists\ c_1, c_2, c_3, c_4 \neq 0$ s.t. $c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} + c_4\vec{a_4} + = \vec{0}$

Then we can say $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are the only independent vectors in the set.

Let $\vec{a}$ represent a column vector in A, the $Span(Col(A))$ is defined to be as follows:

$$\forall\ \vec{a} \in A, \vec{a} \in Span(Col(A))$$

Thus by Rank-Nullity Theorem we say the following statments are equivalent:

1. $nullity(A) + rank(A) =$ (number of pivot columns) - (number of non pivot columns)
2. $rank(A) = dimCol(A) =$ (number of pivot columns)
3. $rank(A) = dimRow(A) =$ (number of pivot rows)

Therefore we can summarize the following staments as equivalent:

1. IFF $Span(Col(A)) = \mathbb{R}^3$
2. $\exists\ \vec{a_1}, \vec{a_2}, \vec{a_3} \in Col(A)$ which are distinct, linear indepedent vectors.
3. There are exactly three pivot columns in A.
4. $Rank(A) = 3 = n$

Question 4

If $A = CR$ then what are the CR factors of the matrix $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$

Observe:

$$\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & CR \\ 0 & CR \end{bmatrix}$$

Recall:

1. $\forall$ rank one matrix can be writter as an outter product of two, any only two, vectors.
2. $\forall$ rank r matricies can be written as the sum of r rank one matricies.

$$\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & A \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & CR \end{bmatrix}$$

Question 5

Suppose $\vec{a}, \vec{b}$ are column vectors with components $a_1, ..., a_m$ and $b_1, ..., b_p$
Is $a\vec{b}^T$ a valid operation?
What is the shape of $a\vec{b}^T$?
What is in the row i, column j of $a\vec{b}^T$?
What can you say about $a\vec{a}^T$?

Part 1: Is $a\vec{b}^T$ a valid operation? What is the shape of $a\vec{b}^T$?
Let $\vec{a}, \vec{b}$ be:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix} \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_p \end{bmatrix}$$

Consider $a\vec{b}^T$:

$$a\vec{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & ... & b_p & \end{bmatrix}$$

Note $\vec{a}$ has shape $[m \cdot 1]$ and $\vec{b}$ has shape $[1 \cdot p]$. Recall we can only multiply two matricies if their inner-products are the same. As both $\vec{a}, \vec{b}^T$ share the same inner product (1) this is a valid operation that yields a $[m \cdot p]$ matrix.
Part 2: What is in the row i, column j of $a\vec{b}^T$?
Consider $a\vec{b}^T$:

$$a\vec{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & ... & b_p & \end{bmatrix} = \begin{bmatrix} b_1a_1 + b_2a_1 + ... + b_pa_1 \\ b_1a_2 + b_2a_2 + ... + b_pa_2 \\ ... \\ b_1a_m + b_2a_m + ... + b_pa_m \end{bmatrix}$$

Part 2: What can you say about $a\vec{a}^T$
Consider $a\vec{a}^T$:

$$a\vec{a}^T = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix} \begin{bmatrix} a_1 & a_2 & ... & a_m & \end{bmatrix} = \begin{bmatrix} a_1a_1 + a_2a_1 + ... + a_ma_1 \\ a_1a_2 + a_2a_2 + ... + a_ma_2 \\ ... \\ a_1a_m + a_2a_m + ... + a_ma_m \end{bmatrix}$$

Observation: yields a square $[m \cdot m]$ matrix.

Question 6

If A has columns $\vec{a_1}, \vec{a_2}, \vec{a_3}$ and $B = I$ is the identity matrix.
what are the rank one matrices: $a_1\vec{b}_1^T, a_2\vec{b}_2^T, a_3\vec{b}_3^T$?
They should add to $AI = A$

Observe:

$$B = \begin{bmatrix} b_{(1,1)} & c_{(1,2)} & ... & c_{(1,n)} \\ c_{(2,1)} & b_{(2,2)} & & c_{(2,n)} \\ ... & & ... & ... \\ c_{(n,1)} & & & b_{(n,n)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & & ... \\ ... & & ... & \\ 0 & & & 1 \end{bmatrix}$$

Thus transposed:

$$B^T = \begin{bmatrix} b_{(n,n)} & c_{(n,n-1)} & ... & c_{(n,1)} \\ c_{(n-1,n)} & b_{(n-1,n-1)} & & c_{(n,2)} \\ ... & & ... & ... \\ c_{(1,n)} & & & b_{(1,1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & & ... \\ ... & & ... & \\ 0 & & & 1 \end{bmatrix}$$

Observe the transposition of $I^T = I = B = B^T$.
Thus we can see the following:

$$A = \begin{bmatrix} a_{(1,1)} & a_{(1,2)} & ... & a_{(1,m)} \\ a_{(2,1)} & a_{(2,2)} & & ... \\ ... & & ... & \\ a_{(n,1)} & & & a_{(n,m)} \end{bmatrix} = \left[\vec{a_1}, \vec{a_2}, ...\vec{a_m}\right]$$

Which