MATH 300: Homework 5

Aren Vista

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1 Question 1

Suppose A, B, C are pairwise dijoint sets. Prove or disprove $|A \cup B \cup C| = |A| + |B| + |C|$

Proof. Suppose A, B, C are pairwise disjoint sets. Then:

$$\begin{aligned} |A \cap B| &= 0, \\ |A \cap C| &= 0, \\ |B \cap C| &= 0, \\ |A \cap B \cap C| &= 0. \end{aligned}$$

By the principle of inclusion-exclusion, we have:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Substituting the values from above, we get:

$$|A \cup B \cup C| = |A| + |B| + |C| - 0 - 0 - 0 + 0.$$

Therefore, $|A \cup B \cup C| = |A| + |B| + |C|$. $\square$

2 Question 2

Let A, B be nonempty sets. Prove $A \times B = B \times A$ iff $A = B$

Proof. Suppose $A = B$.
Observe $A \times B = A \times A = B \times B = B \times A$ by definition of $A = B$.

Suppose $A \times B = B \times A$.
Observe $A \times B = \{(a, b) | a \in A, b \in B\} = C$
Observe $B \times A = \{(b, a) | a \in A, b \in B\} = D$
Then $A \times B = B \times A \equiv \{(a, b) | a \in A, b \in B\} = \{(b, a) | a \in A, b \in B\}$

Observe $\exists\ b_0 \in B \ni \forall a \in A, (a, b_0) \in A \times B$
By supposition $(a, b_0) \in B \times A$
By defn. of $B \times A \implies a \in B \wedge b_0 \in A$
$\therefore A \subset B$

Observe $\exists\ a_0 \in A \ni \forall b \in B, (a_0, b) \in A \times B$
By supposition $(a_0, b) \in B \times A$
By defn. of $B \times A \implies a_0 \in B \wedge b \in A$
$\therefore B \subset A$

$A \subset B \wedge B \subset A \implies A = B$
Thus both directions of the biconditional have been proven and we can conclude. $A \times B = B \times A$ iff $A = B$ $\square$

3 Question 3

For each of the following statements, determine whether it is true or false. If the statement is true, provide a proof; if it is false, present a counterexample.

$$|A - B| = |A| - |B|$$

Proof.

Suppose $B = \{b_1, b_2\}$ and $A = \{a_1, b_1\}$

Observe $A - B = \{a_1\}$

Observe $|A - B| = 1$ and $|A| - |B| = 2 - 2 = 0$

$1 \neq 0 \therefore |A - B| \neq |A| - |B|$

$|A - B| = |A| - |B|$ is false by counterexample $\square$

4 Question 4

Let $X = \{1, 2, 3\}$. For two subsets $A, B \in X$, define a relation ARB if $|A| = |B|$. Show this is an equivalence relation. What is the equivalence class contianing $\{1\}$
Suppose $A, B \in X \implies a \in A \wedge a \in X, b \in B \wedge b \in X$
$ARB = \{(A, B) : |A| = |B|\}$
$|A| = |B| \iff |B| = |A| \implies ARB \wedge BRA$ therefore symmetric
$ARB \wedge BRC \implies |A| = |B| \wedge |B| = |C| \implies |A| = |C| \therefore ARC \in R$ therefore is transitive
$\forall A, |A| = |A| \implies ARA \in R$ therefore is reflexive
Since is symmetric, transitive, and reflexive, R is an equivalence relation.

$$[\{1\}] = \{\{1\}, \{2\}, \{3\}\}$$

5 Question 5

Given two sets A and B with $|A| = 3$ and $|B| = 4$ what is the cardinality of the power set of $A \times B$
$|A \times B| = |A| \times |B| =$ 3*4=12
$P|A \times B| = 2^{|A \times B|} = 2^{12}$