

# MATH 300: Homework 5

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# 1 Question 1

Show  $\forall n \in \mathbb{N}, 1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$

- Let  $P(n) : 1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$
- Observe  $P(1) : 1^3 = \left[ \frac{(1)((1)+1)}{2} \right]^2 = 1$  therefore  $P(1)$  is true
- Suppose  $k \in \mathbb{N}, P(k) : 1^3 + 2^3 + \dots + k^3 = \left[ \frac{k(k+1)}{2} \right]^2$
- Observe  $\left[ \frac{k(k+1)}{2} \right]^2 = \left[ \frac{k^4 + 2k^3 + k^2}{4} \right]$
- Observe  $\left[ \frac{(k+1)^4 + 2(k+1)^3 + (k+1)^2}{2} \right] = \left[ \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \right]$
- Add  $(k+1)^3$  to both sides of  $1^3 + 2^3 + \dots + k^3 = \left[ \frac{k(k+1)}{2} \right]^2$
- This yields  $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[ \frac{k(k+1)}{2} \right]^2 + (k+1)^3$
- Observe  $(k+1)^3 = k^3 + 3k^2 + 3k + 1 = \frac{4k^3 + 12k^2 + 12k + 4}{4}$
- Substituting  $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[ \frac{k(k+1)}{2} \right]^2 + \frac{4k^3 + 12k^2 + 12k + 4}{4}$
- Substituting  $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[ \frac{k^4 + 2k^3 + k^2}{4} \right] + \frac{4k^3 + 12k^2 + 12k + 4}{4}$
- Simplify  $1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[ \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \right]$
- Therefore  $P(k+1)$  is true
- By PMI  $P(n)$  is true  $\forall n \in \mathbb{N}$

## 2 Question 2

Show  $\forall n \in \mathbb{N}, 1 + 4 + 7 + \dots + (3n - 2) = \frac{3n^2 - n}{2}$

- Let  $P(n) : 1 + 4 + 7 + \dots + (3n - 2) = \frac{3n^2 - n}{2}$
- Observe  $P(1) : 1 = \frac{3(1)^2 - (1)}{2} = 1$  therefore  $P(1)$  is true
- Suppose  $k \in \mathbb{N}, P(k) : 1 + 4 + 7 + \dots + (3k - 2) = \frac{3k^2 - k}{2}$
- Add  $(3k + 1)$  to both sides of  $1 + 4 + 7 + \dots + (3k - 2) = \frac{3k^2 - k}{2}$
- Thus  $1 + 4 + 7 + \dots + (3k - 2) + (3k + 1) = \frac{3k^2 - k}{2} + (3k + 1)$
- Observe  $\frac{3(k + 1)^2 - (k + 1)}{2} = \frac{3k^2 + 5k + 2}{2}$
- Simplify  $1 + 4 + 7 + \dots + (3k - 2) + (3k + 1) = \frac{3k^2 - k}{2} + \frac{(6k + 2)}{2}$
- Observe  $1 + 4 + 7 + \dots + (3k - 2) + (3k + 1) = \frac{3k^2 + 5k + 2}{2}$
- Therefore  $P(k + 1)$  is true
- By PMI  $P(n)$  is true  $\forall n \in \mathbb{N}$

## 3 Question 3

Show  $\forall n \in \mathbb{N}, n < 2^n$

- Let  $P(n) : n < 2^n$
- Observe  $P(1) : 1 < 2^1$  thus,  $P(1)$  is true
- Suppose  $k \in \mathbb{N}, P(k) : k < 2^k$
- Add 1 to both sides of  $k < 2^k$  to get  $k + 1 < 2^k + 1$
- Observe  $2^{k+1} = 2^k * 2$
- Clearly  $2^k + 1 < 2^{k+1}$
- Observe  $k + 1 < 2^k + 1 < 2^{k+1}$
- Therefore  $P(k + 1)$  is true
- By PMI  $P(n)$  is true  $\forall n \in \mathbb{N}$

## 4 Question 4

Show  $\forall n \in \mathbb{N}, 6|(n^3 + 5n)$

- Let  $P(n) : 6|(n^3 + 5n)$
- Let  $P(1) : 6|(1^3 + 5n) \equiv 6|(1 + 5) \equiv 6|6$  therefore  $P(1)$  is true
- Suppose  $k \in \mathbb{N}, P(k) : (k^3 + 5k) \implies \exists c \in \mathbb{N} \ni 6c = k^3 + 5k$
- Observe  $(k + 1)^3 + 5(k + 1) = k^3 + 3k^2 + 8k + 5$
- Add  $3k^2 + 3k + 6$  to both sides of  $6c = k^3 + 5k$
- Thus  $6c + 3k^2 + 3k + 6 = k^3 + 3k^2 + 8k + 6$
- Group  $6c + (3k^2 + 3k) + 6 = k^3 + 3k^2 + 8k + 6$
- Simplify  $6c + 3k(k + 1) + 6 = (k + 1)^3 + 5(k + 1)$
- Let  $m \in \mathbb{N} \ni k = 2m \implies 6m(2m + 1) = 12m + 6$  therefore is divisible by two
- Let  $m \in \mathbb{N} \ni k = 2m + 1 \implies 6(2m + 1)(2m + 2) = (12m + 6)(2m + 2) = 24m^2 + 24m + 12m + 12$  therefore is divisible by two
- Let  $n \in \mathbb{N} \ni k = 2n$  Subsitute  $6c + 3k(2n) + 6 = (k + 1)^3 + 5(k + 1)$
- Simplify  $6c + 6kn + 6 = (k + 1)^3 + 5(k + 1)$
- Simplify  $6(c + kn + 1) = (k + 1)^3 + 5(k + 1)$
- Let  $(c + kn + 1) = z, z \in \mathbb{N}$
- Subsitute  $6z = (k + 1)^3 + 5(k + 1)$
- Therefore  $P(k + 1) : 6|(k + 1)^3 + 5(k + 1)$  is true.
- By PMI  $P(n)$  is true  $\forall n \in \mathbb{N}$