MATH 300: Homework 1

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Supposition

Let $f : X \to Y$ be a function, $A \subset X$ and $C \subset Y$ Then:

$$f(A) := \{f(a)|a \in A\}$$

$$f^{-1}(C) = \{x \in X|f(x) \in C\}$$

P1

Show: $f : X \to Y$ is a function $\implies$ $\forall$ sets $A \subset X, A \subset f^{-1}(f(A))$

Observe $f : X \to Y$ is a function $\implies \forall x \in X, f(x) \in Y$

From supposition

$$f^{-1}(C) = \{x \in X|f(x) \in C\}$$

Recall $A \subset X \implies \forall\ a \in A, a \in X$

Thus, $f : A \to Y$ is a function $\implies \forall a \in X, f(a) \in Y$ must hold

Therefore

$$B = \{a \in A|f(a) \in C\}, B \subset f^{-1}(C)$$

Thus

$$\forall a \in A, a \in B$$

P1*

Let a be an arbitrary element in A

By the defn of the image set

$$f(a) \in f(A)$$

As $A \subset X, f(X) \in C$

$$f(A) \subset C$$

By the defn of the preimage

$$f^{-1}(C) = \{x \in X|f(x) \in C\}$$

By subsititon the preimage of must also hold for

$$f^{-1}(f(A)) = \{x \in X|f(x) \in f(A)\}$$

Since $f(a) \in f(A)$:

$$a \in f^{-1}f(A)$$

P2

Show: If $f : X \to Y$ is injective, then for any two sets A and B in X

$$f(A \cap B) = f(A) \cap f(B)$$

Case 1: Prove $f(A \cap B) \subset f(A) \cap f(B)$

Let

$$y \in f(A \cap B)$$

Thus

$$\exists\ x \in A \cap B \ni f(x) = y$$

By defn

$$x \in A \cap B \implies x \in A \wedge x \in B$$

Since $x \in A \wedge f(x) = y$

$$y \in f(A)$$

Since $x \in B \wedge f(x) = y$

$$y \in f(B)$$

Since $y \in f(B) \wedge y \in f(A)$

$$y \in f(A) \wedge f(B)$$

Thus we can conclude:

$$(A \cap B) \subset f(A) \cap f(B)$$

Case 2: Prove $f(A) \cap f(B) \subset (A \cap B)$
Let

$$y \in f(A) \cap f(B)$$

Then

$$y \in f(A) \wedge y \in f(B)$$

Since

$$y \in f(A) \implies \exists\ a \in A \ni f(a) = y$$

Since

$$y \in f(B) \implies \exists\ b \in B \ni f(b) = y$$

Naturally

$$f(a) = f(b) = y$$

Since f is injective

$$f(a) = f(b) \implies a = b$$

Let

$$x = a = b$$

Since

$$x \in A \wedge x \in B, x \in A \cap B$$

Thus we can conclude:

$$f(A) \cap f(B) \subset (A \cap B)$$

By Case 1 and Case 2 we conclude:

$$f(A) \cap f(B) = (A \cap B)$$

P3

A is a set with 5 elements. If $f : A \to A$ is a function, what are the minimum and maximum values of $|f(A)|$. When is the maximum achieved?

Let

$$A = \{a_1, a_2, a_3, a_4, a_5\}$$

If $f : A \to A$ is a function

$$f(a) = a_x \wedge f(a) = a_y \implies a_x = a_y$$

Thus at most:

$$f(A) = \{a_1, a_2, a_3, a_4, a_5\}$$

Thus at least

$$f(A) = \{a_1\}$$

Meaning the min cardinality is 1, and the max is 5.

P4

Show the function f is bijective:

$$f : (0, \inf) \to (0, 1),$$

$$f(x) = \frac{x^2}{1 + x^2}$$

f is bijective iff f is injective and f is surjective
Prove injectivity:
Let $a, b \in (0, \inf)$

$$\frac{a^2}{1 + a^2} = \frac{b^2}{1 + b^2}$$

$$a^2(1 + b^2) = b^2(1 + a^2)$$

$$a^2 + a^2b^2 = b^2 + a^2b^2$$

$$a^2 = b^2$$

$$a = b$$

Thus f is injective
Prove surjectivity:
Let $a \in (0, \inf), b \in (0, 1)$

$$\frac{a^2}{1 + a^2} = b$$

$$a^2 = b(1 + a^2)$$

$$a^2 = b + ba^2$$

$$a^2 - ba^2 = b$$

$$a^2(1 - b) = b$$

$$a^2 = \frac{b}{(1 - b)}$$

$$a = \sqrt{\frac{b}{(1 - b)}}$$

Thus f is surjective.
As f is both injective and surjective. It must be bijective as defn.