MATH 300: Homework 5

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1 Question 1

Show $\forall n \in \mathbb{N}, 1^3 + 2^3 + ... + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

- Let $P(n) : 1^3 + 2^3 + ... + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
- Observe $P(1) : 1^3 = \left[\dfrac{(1)((1)+1)}{2}\right]^2 = 1$ therefore $P(1)$ is true
- Suppose $P(k) : 1^3 + 2^3 + ... + k^3 = \left[\dfrac{k(k+1)}{2}\right]^2$
- Observe $\left[\dfrac{k(k+1)}{2}\right]^2 = \left[\dfrac{k^4 + 2k^3 + k^2}{2}\right]$
- Observe $\left[\dfrac{(k+1)^4 + 2(k+1)^3 + (k+1)^2}{2}\right] = \left[\dfrac{k^4 + 6k^3 + 13k^2 + 12k + 4}{2}\right]$
- Add $(k+1)^3$ to both sides of $1^3 + 2^3 + ... + k^3 = \left[\dfrac{k(k+1)}{2}\right]^2$
- This yields $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \left[\dfrac{k(k+1)}{2}\right]^2 + (k+1)^3$
- Observe $(k+1)^3 = k^3 + 3k^2 + 3k + 1 = \dfrac{4k^3 + 12k^2 + 12k + 4}{2}$
- Subsituting $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \left[\dfrac{k(k+1)}{2}\right]^2 + \dfrac{4k^3 + 12k^2 + 12k + 4}{2}$
- Subsituting $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \left[\dfrac{k^4 + 2k^3 + k^2}{2}\right] + \dfrac{4k^3 + 12k^2 + 12k + 4}{2}$
- Simplify $1^3 + 2^3 + ... + k^3 + (k+1)^3 = \left[\dfrac{k^4 + 6k^3 + 13k^2 + 4}{2}\right]$
- Therefore $P(k+1)$ is true
- By PMI $P(n)$ is true $\forall n \in \mathbb{N}$

2 Question 2

Show $\forall n \in \mathbb{N}, 1 + 4 + 7 + ... + (3n - 2) = \dfrac{3n^2 - n}{2}$

- Let $P(n) : 1 + 4 + 7 + ... + (3n - 2) = \dfrac{3n^2 - n}{2}$
- Observe $P(1) : 1 = \dfrac{3(1)^2 - (1)}{2} = 1$ therefore $P(1)$ is true
- Suppose $P(k) : 1 + 4 + 7 + ... + (3k - 2) = \dfrac{3k^2 - k}{2}$
- Add $(3k + 1)$ to both sides of $1 + 4 + 7 + ... + (3k - 2) = \dfrac{3k^2 - k}{2}$
- Thus $1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \dfrac{3k^2 - k}{2} + (3k + 1)$
- Observe $\dfrac{3(k + 1)^2 - (k + 1)}{2} = \dfrac{3k^2 + 5k + 2}{2}$
- Simplify $1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \dfrac{3k^2 - k}{2} + \dfrac{(6k + 2)}{2}$
- Observe $1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \dfrac{3k^2 + 5k + 2}{2}$
- Therefore $P(k + 1)$ is true
- By PMI $P(n)$ is true $\forall n \in \mathbb{N}$

3 Question 3

Show $\forall n \in \mathbb{N}, n < 2^n$

- Let $P(n) : n < 2^n$
- Observe $P(1) : 1 < 2^1$ thus, $P(1)$ is true
- Suppose $P(k) : k < 2^k$
- Add 1 to both sides of $k < 2^k$ to get $k + 1 < 2^k + 1$
- Observe $2^{k+1} = 2^k * 2$
- Clearly $2^k + 1 < 2^{k+1}$
- Observe $k + 1 < 2^k + 1 < 2^{k+1}$
- Therefore $P(k + 1)$ is true
- By PMI $P(n)$ is true $\forall n \in \mathbb{N}$

4 Question 4

Show $\forall n \in \mathbb{N}, 6|(n^3 + 5n)$

- Let $P(n) : 6|(n^3 + 5n)$
- Let $P(1) : 6|(1^3 + 5n) \equiv 6|(1 + 5) \equiv 6|6$ therefore $P(1)$ is true
- Suppose $P(k) : (k^3 + 5k) \implies \exists c \in \mathbb{N} \ni 6c = k^3 + 5k$
- Observe $(k + 1)^3 + 5(k + 1) = k^3 + 3k^2 + 8k + 5$
- Add $3k^2 + 3k + 6$ to both sides of $6c = k^3 + 5k$
- Thus $6c + 3k^2 + 3k + 6 = k^3 + 3k^2 + 8k + 6$
- Group $6c + (3k^2 + 3k) + 6 = k^3 + 3k^2 + 8k + 6$
- Simplify $6c + 3k(k + 1) + 6 = (k + 1)^3 + 5(k + 1)$
- Let $m \in \mathbb{N} \ni k = 2m \implies 6m(2m + 1) = 12m + 6$ therefore is divsible by two
- Let $m \in \mathbb{N} \ni k = 2m+1 \implies 6(2m+1)(2m+2) = (12m+6)(2m+2) = 24m^2+24m+12m+12$ therefore is divsible by two
- Let $n \in \mathbb{N} \ni k = 2n$ Subsitute $6c + 3k(2n) + 6 = (k + 1)^3 + 5(k + 1)$
- Simplify $6c + 6kn + 6 = (k + 1)^3 + 5(k + 1)$
- Simplify $6(c + kn + 1) = (k + 1)^3 + 5(k + 1)$
- Let $(c + kn + 1) = z, z \in \mathbb{N}$
- Subsitute $6z = (k + 1)^3 + 5(k + 1)$
- Therefore $P(k + 1) : 6|(k + 1)^3 + 5(k + 1)$ is true.
- By PMI $P(n)$ is true $\forall n \in \mathbb{N}$