Theorem: There does not exist a rational number r such that $r^2 = 2$.

Proof: Suppose there exists a rational $r = \frac{m}{n}$ such that $r^2 = 2$. (no common integer factors other than 1)
So $\frac{m^2}{n^2} = 2$, then $m^2 = 2n^2$, then m^2 is even m is even that $m = 2p$

So $(2p)^2 = 2n^2$, then $4p^2 = 2n^2$ then $n^2 = 2p^2$ then n^2 is even, then n is even, that is $n = 2q$. So m, n have a common integer factor 2, which is different from 1)

Contradiction because we assumed there are no common factors.

The Ordering properties of $\mathbb{R}$
Let P be a nonempty subset of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties.

① If $a, b \in P$, then $a + b \in P$

② If $a, b \in P$, then $ab \in P$

③ If $a \in \mathbb{R}$, then exactly one of the following holds
$a \in P$, $a = 0$, $-a \in P$

If $a \in P$, we say a is positive, written as $a > 0$.

If $-a \in P$, we say a is negative, written as $a < 0$.

If $a \in P \cup \{0\}$, we say a is non-negative, written as $a \geq 0$

If $-a \in P \cup \{0\}$, we say a is non-positive, written as $a \leq 0$

Definition: Let $a, b \in \mathbb{R}$

a) If $a - b \in P$, then we write $a > b$ and $b < a$

b) If $a - b \in P \cup \{0\}$, we write $a \geq b$ or $b \leq a$.

So for any two $a, b \in \mathbb{R}$, exactly one of the following will hold:

$a > b$, $a = b$, $a < b$

*

$\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$

$\frac{m}{n} \cdot \frac{p}{q} = m \cdot \frac{1}{n} \cdot p \cdot \frac{1}{q} = (m \cdot p) \cdot \left(\frac{1}{n} \cdot \frac{1}{q}\right)$

Claim $\frac{1}{n} \cdot \frac{1}{q} = \frac{1}{nq}$

$a \neq 0$ $\exists \frac{1}{a}$ such that

$a \cdot \frac{1}{a} = 1$

$\frac{1}{a}$ and b

$a \cdot \frac{1}{a} = 1$

$a \cdot b = 1$

$a \cdot \frac{1}{a} = a \cdot b$

$\frac{1}{a} \cdot a \cdot \frac{1}{a} = \frac{1}{a} \cdot (a \cdot b)$

$\left(\frac{1}{a} \cdot a\right) \cdot \frac{1}{a} = \left(\frac{1}{a} \cdot a\right) \cdot b$

$1 \cdot \frac{1}{a} = 1 \cdot b$

$\frac{1}{a} = b$

$\frac{1}{nq} = \frac{1}{n} \cdot \frac{1}{q}$ such that two numbers

$nq \cdot \frac{1}{nq} = 1$

$nq \cdot \left(\frac{1}{n} \cdot \frac{1}{q}\right) = 1$

$* \ a - b \in \mathbb{R}$

Order Properties:

a) If $a>b$, $b>c$, then $a>c$.

To show $a>c$, we need $a-c \in P$
Since $a>b$, $b>c$, then $a-b \in P$, $b-c \in P$.
then $a-c=(a-b)+(b-c) \in P$ by the axioms
So $a>c$.

b) If $a>b$, then $a+c>b+c$

$$\underbrace{a+c-(b+c)=(a-b)}_{\in P}+\underbrace{(c-c)}_{0}=a-b$$

$a>b$

$a+c-(b+c) \in P \Rightarrow a+c>b+c$

c) If $a>b$ and $c>0$, then $ca>cb$

$ca-cb \in P$

$c(a-b) \in P$

Since $a>b$, $c>0$, then $a-b \in P$, $c \in P$, then by ② of set P
$c(a-b) \in P$, that is $ca-cb \in P$ so $ca>cb$.

d) If $a \in \mathbb{R}$ and $a \neq 0$ then $a^2>0$. Since $a \in \mathbb{R}$, and $a \neq 0$, then by ③ of set P, either $a \in P$ or $-a \in P$. If $a \in P$, then $a^2 \in P$ by ② of set P.
If $a \in P$, then $a^2=(-a)(-a)=a^2 \in P$.

e) $1>0$

Proof $1 \in \mathbb{R}$ and $1 \neq 0$.
by d) $1^2>0$

Note that $1 \cdot 1=1$ by M3 than $1>0$.

f) If $n \in \mathbb{N}$, then $n>0$.

Proof by Induction:
① $n=1$, then $1>0$ by e)
② Suppose $k>0$
③ $k \in P$ and $1 \in P$, then $k+1 \in P$ by ① and set P then $k+1>0$.

g) If $a \cdot b>0$, then either $a>0$, $b>0$ or $a<0$, $b<0$.
If $a \cdot b<0$, then either $a>0$, $b<0$ or $a<0$, $b<0$.

1) $a \in P, b \in P$

~~2) $a = 0, b \in P$~~

3) $-a \in P, b \in P$ $\rightarrow$ $-a \cdot b \in P$

$a \cdot b < 0$

~~4) $a \in P, b = 0$~~

~~5) $a = 0, b = 0$~~

~~6) $-a \in P, b = 0$~~

7) $a \in P, -b \in P$

~~8) $a = 0, -b \in P$~~

9) $-a \in P, -b \in P$

1) and 9) when $a \cdot b > 0$

3) and 7) when $a \cdot b < 0$