Lecture 18

Wednesday, November 8, 2023 10:05 AM

Reviewing the Quiz

1\. a) $\lim \frac{\sqrt{n}}{n+1}$

Kang is really disappointed

Proof Given any $\varepsilon > 0$, let $N(\varepsilon) > \frac{1}{\varepsilon^2}$. If $n \geq N(\varepsilon)$, then $n > \frac{1}{\varepsilon^2}$, then $\sqrt{n} < \frac{1}{\varepsilon}$, then $\frac{\sqrt{n}}{n+1} < \varepsilon$, then $\left|\frac{\sqrt{n}}{n+1} - 0\right| < \varepsilon$.

b) $\lim(\sqrt{n^2+1} - n) = 0$

Proof. $\forall \varepsilon > 0$, Let $N(\varepsilon) > \frac{1}{\varepsilon}$, if $n \geq N(\varepsilon)$ then $n > \frac{1}{\varepsilon}$, then $\frac{1}{n} < \varepsilon$, then $\frac{1}{\sqrt{n^2+1} - \sqrt{n}} < \varepsilon$, then $\sqrt{n^2+1} - \sqrt{n} < \varepsilon$. Then $|\sqrt{n^2+1} - n - 0| < \varepsilon$.

2\. (x_n) : $x_1 = \sqrt{2}$

$x_{n+1} = \sqrt{2 + x_n}$ $n \geq 1$

Proof I don't feel like writing it
but need to prove that it is bounded above by 2.

3\. $(a_n), (b_n)$ bounded, positive

$\overline{\lim}(a_n b_n) \leq \overline{\lim}(a_n)\, \overline{\lim}(b_n)$

Proof For (a_n), let $V_a = \{v \in \mathbb{R} : v < a_n$ for only a finite many n's$\}$.

For (b_n), let $V_b = \{w \in \mathbb{R} : w < b_n$ for only a finite many n's$\}$

For $(a_n b_n)$, let $V_{ab} = \{z \in \mathbb{R} : z < a_n b_n$ for only a finite many n's$\}$.

Let $v \in V_a$, $w \in V_b$. Then $a_n \leq v$ for all n after some integer N
$b_n \leq v$ for all n after some integer M

Then $a_n b_n \leq vw$ for all n after the integer $\max\{N, M\}$

Then $a_n b_n \le vw$ for all n after the integer $\max\{N, M\}$

So $vw \in V_{ab}$.

So $\inf V_{ab} \le vw$

So $\frac{\inf V_{ab}}{v} \le w$, so $\frac{\inf V_{ab}}{v}$ is a lower bound of V_b

So $\frac{\inf V_{ab}}{v} \le \inf V_b$

So $\inf V_{ab} \le v \cdot \inf V_b$

Case 1: $\inf V_b = 0$

So $\inf V_{ab} \le 0$. Note that V_{ab} is the set of positive numbers

then $\inf V_{ab} \ge 0$

So if $V_{ab} = 0 = (\inf V_b)(\inf V_a)$

Case 2: $\inf V_b > 0$

$$\frac{\inf V_{ab}}{\inf V_b} \le v$$

So $\frac{\inf V_{ab}}{\inf V_b}$ is a lower bound of V_a

Then $\frac{\inf V_{ab}}{\inf V_b} \le \inf V_a$

Then $\inf V_{ab} \le (\inf V_a)(\inf V_b)$

Exam coming up (Nov 20th)

Probably going to be just Ch3

Homework

3.b) Show that $\ln(n)$ is not Cauchy

$|\ln(n) - \ln(m)| < \varepsilon$

Choose $n = m^2$ and $\varepsilon = 1$

$|2\ln(m) - \ln(m)| < 1$

$\ln(m) \to \infty$ $\quad m > e$

4. Show that if (x_n), (y_n) are Cauchy, then $(x_n \cdot y_n)$ is Cauchy.

$|x_n y_n - x_m y_m| < \varepsilon$

By triangle inequality

$= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \leq |x_n y_n - x_m y_n| + |x_m y_n - x_m y_m|$

$= y_n |x_n - x_m| + x_m |y_n - y_m|$

Since y_n is Cauchy, it is convergent and bounded.

Let M be a bound, so $|y_n| < M \quad \forall n \in \mathbb{N}$

Since (x_n) is Cauchy, $|x_n| < N$

bound

7. (x_n) is Cauchy, (x_n) is an integer. Show that (x_n) is ultimately constant.

$\exists N$ such that $x_n = a$ for some constant a

Use proof by contradiction

Proof Suppose BWOC the sequence is not ultimately constant. Then there is a subsequence (x_{n_k}) such that $(x_n) \neq (x_{n_k}) \quad \forall k \in \mathbb{N}$.

$\Rightarrow 1 \leq |x_{n_k} - x_{n_{k-1}}|$. Since (x_n) is Cauchy, for $\varepsilon = \frac{1}{2}$, there exists $N(\frac{1}{2})$ such that $|x_n - x_m| < \frac{1}{2}$ for all $n \geq N(\frac{1}{2})$ and

$1 \leq |x_{n_k} - x_{n_{k-1}}| < \frac{1}{2}$ which is a contradiction ■

he's going over 3.3 #2 rn