Section 3.1: Sequences and their limits

Definition: A sequence of real numbers is a mapping from $\mathbb{N} = \{1, 2, \dots\}$ onto a subset of $\mathbb{R}$, denoted by $X: \mathbb{N} \to \mathbb{R}$. The value of X of $n \in \mathbb{N}$ is denoted by $X(n)$ or x_n. The sequence is denoted by X, (x_n), $\{x_n, n \in \mathbb{N}\}$

Ex: $\{(-1)^n, n \in \mathbb{N}\}$, $\{\frac{1}{n}, n \in \mathbb{N}\}$

Other times, a sequence can be defined inductively

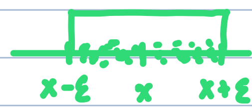
Ex: $x_1 = 1$, $x_2 = 2$, $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$, $n \geq 3$

Ex: $x_1 = 1$, $x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$, $n \geq 3$
(Fibonacci Sequence)

The limit of a sequence:

Definition: A sequence $\{x_n, n \in \mathbb{N}\}$ is said to converge to $x \in \mathbb{R}$ or x is a limit of $\{x_n, n \in \mathbb{N}\}$ if ($\varepsilon \to N$ language) for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that $\forall n \geq N(\varepsilon)$, $|x_n - x| < \varepsilon$.

(x_n is in the $\underbrace{\varepsilon\text{-neighborhood of } x}_{\{y : |y-x| < \varepsilon\}}$)



If a sequence has a limit, then the sequence is convergent.
Otherwise, the sequence is divergent.

$\lim_{n \to \infty} x_n = x$

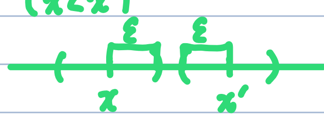
$x_n \to x$ as $n \to \infty$

Theorem: A sequence in $\mathbb{R}$ can have at most one limit.

Proof: Suppose $\{x_n, n \in \mathbb{N}\}$ has two limits, x, x'. ($x \neq x'$)
↑ For the sake of contradiction

Let ε be a positive real number less than $\frac{x' - x}{2}$. $(x < x')$

For $\varepsilon/2 > 0$, there exists $N(\varepsilon/2) > 0$ s.t. $|x_n - x| < \varepsilon/2$.



there exists $N'(\varepsilon/2) > 0$ such that $|x - x'| < \varepsilon/2$ $\forall n \geq N'(\varepsilon/2)$

$N(\varepsilon)$, $\varepsilon < \frac{x' - x}{2}$ $N'(\varepsilon)$

for all $n \geq N(\varepsilon/2) + N(\varepsilon/2)$, then $|x_n - x| < \varepsilon/2$ and $|x_n - x'| < \varepsilon/2$.

then $|x - x'| = |x - x_n + x_n - x'|$
$\leq |x - x_n| + |x_n - x'| < \varepsilon/2 + \varepsilon/2 = \varepsilon < x' - x$

Contradiction!

Ex: $\lim_{n\to\infty} \frac{1}{n} = 0$

For each $\varepsilon > 0$. We want to find $N(\varepsilon)$ s.t.
$|\frac{1}{n} - 0| < \varepsilon \quad \forall n \geq N(\varepsilon)$

$|\frac{1}{n} - 0| < \varepsilon$

$\Uparrow$

$\frac{1}{n} < \varepsilon$

$\Uparrow$

$n > \frac{1}{\varepsilon}$

$n \geq N(\varepsilon) > \frac{1}{\varepsilon}$

$N(\varepsilon) > \frac{1}{\varepsilon}$

Proof: For each $\varepsilon > 0$, by the Archimedean property, there exists a natural number $N(\varepsilon) > 0$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ for all $n \geq N(\varepsilon)$, $n > \frac{1}{\varepsilon}$, then $|\frac{1}{n} - 0| < \varepsilon$. So $\lim_{n\to\infty} \frac{1}{n} = 0$.

Ex: $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \infty$

For each $\varepsilon > 0$, we want to find $N(\varepsilon) > 0$, such that

$$|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon$$

$$\sqrt{n+1} - \sqrt{n} < \varepsilon$$

$$\frac{(\sqrt{n+1} - n)(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} < \varepsilon$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon^2 \Rightarrow N(\varepsilon) > \frac{1}{\varepsilon^2}$$

For each $\varepsilon > 0$, by the Archimedean property, there exists $N(\varepsilon) > \frac{1}{\varepsilon^2}$, then from $n \geq N(\varepsilon)$.

$n > \frac{1}{\varepsilon^2}$, then $\sqrt{n} > \frac{1}{\varepsilon}$, then $\frac{1}{\sqrt{n}} < \varepsilon$, then $\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$,

then $\sqrt{n+1} - \sqrt{n} < \varepsilon$, then $|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon$

Ex: $\{(-1)^n, n \in \mathbb{N}\}$

Tails of sequences

Definition: If $X = \{x_1, x_2, x_3, \ldots\}$ is a sequence and if m is a given natural number then the m-tail of X is the sequence $X_m = \{x_{m+1}, n \in \mathbb{N}\} = \{x_{m+1}, x_{m+2}, \ldots\}$

Theorem: X converges if and only if X_m converges and in this case, $\lim X_m = \lim X$

a) $c>1$, $m,n\in\mathbb{N}$

$c^m > c^n$ iff $m>n$

Proof "$\Leftarrow$"

$c^m > c^n$

$m = n+k$, $k\in\mathbb{N}$

$c^{n+k} > c^n$

$c^n \cdot c^k > c^n$

$\Uparrow$

$c^k > 1$

$\Uparrow$ induction

$c>1$

We prove the claim by induction

1 $n=1$, $c^n = c^1 = c > 1$ by assumption

2 $c^{k+1} = c\cdot c^k > c\cdot 1 = c > 1$
So the claim is proved.

Since $c^k > 1 > 0$, then $c^m = c^{n+k} = c^n \cdot c^k > c^k \cdot 1 = c^n$

$\Rightarrow$ $c^m > c^n$ implies $m>n$

Proof: Suppose $m \le n$

Case 1 ($m=n$) $c^m = c^n$ which contradicts the assumption by applying the first derivation for

Case 2 ($m<n$) ($m<n$)

then we have $c^n > c^m$ which contradicts the assumption again.

So $m>n$.