Quiz Review (1.3-2.1)

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1.3

A set S is said to have n elements if there exists a bijection from $\mathbb{N}_n$ onto S

$\mathbb{N}_n = \{1, 2, \dots, n\}$

- S is finite if it has n elements (or is empty)
  Otherwise, it is infinite.

So, S has n elements iff. there is a bijection from S onto $\mathbb{N}_n$

Theorems

- If C is an infinite set and B is a finite set, then $C \setminus B$ is infinite (1.3.4(c))
- Suppose S and T are sets where $T \subseteq S$ (1.3.5)
  - If S is finite, then T is finite
  - If T is an infinite set, then S is infinite

  contrapositive of each other

Proof First, we rule out the case where $T = \emptyset$, since T would be finite by definition.
We use proof by induction on the number of elements n in S.
(Base case) Suppose S has 1 element. Then the only nonempty subset T of S would be S itself, so T is finite.

We assume the hypothesis holds for $n = k$. Let S be a set with $k+1$ elements. So, there exists a bijection f of $\mathbb{N}_{k+1}$ onto S.
If $f(k+1) \notin T$, then T is a subset of $S_1 = S \setminus \{f(k+1)\}$, which has k elements. So, by the inductive hypothesis, T is finite.
If $f(k+1) \in T$, then $T_1 = T \setminus \{f(k+1)\}$ is a subset of S_1. Since S_1 has k elements, T_1 is finite, which implies that $T = T_1 \cup \{f(k+1)\}$ is also finite.

Thus, by mathematical induction, if S is a finite set, then any any nonempty subset of S is also finite. $\square$

A set S is denumerable (countably infinite) if there exists a bijection of $\mathbb{N}$ onto S.

- S is countable if it is either finite or denumerable. Otherwise, it is uncountable

Notable examples

- $\mathbb{N}$ – denumerable
- $\mathbb{Q}$ – denumerable
- $\mathbb{R}$ - uncountable

More Theorems

- Suppose S and T are sets and $T \subseteq S$. (1.3.9)
  - If S is countable, then T is countable.
  - If T is uncountable, then S is uncountable
- If S is countable, then (1.3.10)
  - There exists a surjection of $\mathbb{N}$ onto S
  - There exists an injection of S into $\mathbb{N}$.

Proof If S is finite, there exists a bijection h of $\mathbb{N}_m$ onto S.

We define H on $\mathbb{N}$ by

$$H(k) := \begin{cases} h(k) & k = 1, 2, \ldots n \\ h(n) & k > n \end{cases}$$

Then H is a surjection of $\mathbb{N}$ onto S.

I don't feel like writing the whole proof. It's in the book

Another theorem - If A_m is a countable set for each $m \in \mathbb{N}$, then the union $A := \bigcup_{m=1}^{\infty} A_m$ is countable

The proof uses a surjection

Cantor's Theorem

If A is any set, then there is no surjection of A onto the power set $P(A)$ (set of all subsets of A)

This proof uses Kang's favorite technique, proof by contradiction.

2.1

The Algebraic Properties of $\mathbb{R}$ (also called field axioms)

(A1) $a+b=b+a$ (commutative property of addition)

(A2) $(a+b)+c = a+(b+c)$ (associative property of addition)

(A3) There exists an element 0 such that $0+a=a$ (existence of a zero element)

(A4) For each $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ such that $a+(-a)=0$ (existence of negative elements)

(M1) $a \cdot b = b \cdot a$ (commutative property of multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative property of multiplication)

(M3) There exists an element 1 distinct from 0 such that $1 \cdot a = a$ (existence of a unit element)

(M4) For each $a \neq 0$, $\exists \frac{1}{a} \in \mathbb{R}$ such that $a \cdot \left(\frac{1}{a}\right) = 1$ (existence of reciprocals)

(D) $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$ (distributive property)

Order Properties of $\mathbb{R}$

There is a subset P of $\mathbb{R}$ called the positive real numbers that satisfies:

i) If $a,b\in\mathbb{P}$, then $a+b\in\mathbb{P}$

ii) If $a,b\in\mathbb{P}$, then $ab\in\mathbb{P}$

iii) If $a\in\mathbb{R}$, exactly one of the following holds:

$a\in\mathbb{P}$ $\quad a=0 \quad -a\in\mathbb{P}$

① Trichotomy Property
since it divides $\mathbb{R}$ into 3 groups

Theorem 2.1.7

(a) If $a>b$ and $b>c$, then $a>c$.

(b) If $a>b$, then $a+c>b+c$

(c) If $a>b$ and $c>0$, then $ac>bc$

$c<0$, then $ac<bc$

Theorem 2.1.8

(a) If $a\neq 0$, then $a^2>0$

(b) $1>0$

Theorem 2.1.10

If $ab>0$, then either

- $a>0$ and $b>0$
- $a<0$ and $b<0$

Proof $ab>0 \Rightarrow a\neq 0$ and $b\neq 0$.

By the Trichotomy Property, either $a>0$ or $a<0$.

If $a>0$, then $\frac{1}{a}>0$, and therefore $b=\left(\frac{1}{a}\right)\cdot(ab)>0$. Similarly,

if $a<0$, then $\frac{1}{a}<0$, so that $b=\left(\frac{1}{a}\right)(ab)<0$. $\square$

Examples

- **If $a<b$, then $a<\frac{1}{2}(a+b)<b$**

$2a=a+a<b+a$

$a<b$ and $\uparrow \Rightarrow 2a<b+a$

$2b=b+b$ and we know $a+b<b+b$

so $a+b<2b$

$\Rightarrow 2a<a+b<2b$

Divide by $2 \Rightarrow a<\frac{1}{2}(a+b)<b$ $\square$.