Assume the existence of roots, show that if $c>1$ and $m,n \in \mathbb{N}$ such that $m>n$, $c^{\frac{m}{n}-1}>1$.

Suppose $c^{\frac{m}{n}-1} \leq 1$, then by the order property

$$\left(c^{\frac{m}{n}-1}\right)^n \leq 1^n$$

then $c^{m-n}<1$

$$c^n \cdot c^{m-n} < 1 \cdot c^n$$

$$c^m < c^n$$

Note that from quiz 1 $c^m > c^n$

$\therefore$ This is a contradiction

Let $S \subseteq \mathbb{R}$ be nonempty. Prove that if a number u in $\mathbb{R}$ has the properties:

a) for every $n \in \mathbb{N}$ $u-\frac{1}{n}$ is not an upper bound of S,
b) for every number $n \in \mathbb{N}$ the number $u+\frac{1}{n}$ is an upper bound of S, then $\sup S = u$

Proof: 1 Show that u is an upperbound of S.
$s \leq u$ for all $s \in S$. Let s be any element in S.
by b) $s \leq u + \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then $s-u \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. So $s-u$ is a lower bound of the set $\{\frac{1}{n}, n \in \mathbb{N}\}$
Note that if $\{\frac{1}{n}, n \in \mathbb{N}\} = 0$

So $s-u \leq 0$, that is, $s \leq u$. So u is an upperbound in S.

2 Show that u is the smallest upperbound of S.
Let v be any number such that $v<u$.

v is not an upper bound of S

$\Updownarrow$

$S_n \in S$ s.t. $S_n > v$

$v < u - \frac{1}{n} < u$, $\quad \inf\{\frac{1}{n}, n \in \mathbb{N}\} = 0$

Backwards Search

If we can find $v < u - \frac{1}{n}$, this means $\frac{1}{n} < u-v \Rightarrow n > \frac{1}{u-v}$

By the Archimedean Prop

So by the Archimedean Property for $\frac{1}{u-v} > 0$, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{u-v}$ than $\frac{1}{n} < u-v$
then $v < u - \frac{1}{n}$. Since $u-\frac{1}{n}$ is not an upperbound of S,
There exists $s \in S$ such that $s' > u - \frac{1}{n}$ then $s' > v$. Then v is not an upper bound of S. So $u = \sup S$.

Alternate Solution :

Note that $S \neq \emptyset$ and S is bounded above by $u+\frac{1}{n}$ for any $n\in\mathbb{N}$. So by the completeness property, $\sup S$ exists in $\mathbb{R}$. Note that $\sup S \leq u+\frac{1}{n}$ for all $n\in\mathbb{N}$.

For each $n\in\mathbb{N}$, Since $u-\frac{1}{n}$ is not an upper bound of S there exists $s_n\in S$ such that $u-\frac{1}{n}<s_n$
Note that $s_n \leq \sup S$, then $u-\frac{1}{n}\leq \sup S$ for all $n\in\mathbb{N}$.

$$u-\frac{1}{n}\leq \sup S \leq u+\frac{1}{n} \text{ for all } n\in\mathbb{N}$$

Note that $\{[u-\frac{1}{n}, u+\frac{1}{n}], n\in\mathbb{N}\}$ is a nested sequence of closed and bounded intervals such that the length $u-\frac{1}{n}-(u+\frac{1}{n})=\frac{2}{n}$

$\inf\{u+\frac{1}{n}-(u-\frac{1}{n}), n\in\mathbb{N}\} = \inf\{\frac{2}{n}, n\in\mathbb{N}\}=0$

So by the nested interval property there exists the unique ξ in each interval. Note that $\sup S$ and u are in each interval, then $\sup S \leq u$.

3.2.3

X, Y
X is convergent
Y is convergent
Show that Y is convergent

$\underline{\underline{Y}}=(X+Y)-X$

3.1.14

$0<b<1$
$\lim(nb^n)=0$

Ratio Test : $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L<1$

$\lim\limits_{n\to\infty} a_n = 0$

$a_n = nb^n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)b^{n+1}}{nb^n} = \frac{(n+1)}{n}b = \frac{1+\frac{1}{n}}{1}b = 1\cdot b = b<1$$

Binomial Thm:

$(1+x)^n = 1+nx+\frac{1}{2}n(n-1)x^2+\cdots \geq 1 \geq 1+nx \geq 1+nx+\frac{1}{2}n(n+1)x^2+\cdots$

$nb^n = \frac{n}{(\frac{1}{b})^n} = \frac{n}{(1+x)^n}$, where $x=\frac{1}{b}-1>0$

$|nb^n| < \varepsilon$

$nb^n < \varepsilon$

$\frac{n}{(1+x)^n} < \varepsilon$

$\frac{n}{(1+x)^n} \leq \frac{n}{\boxed{}} < \frac{2}{(n+1)x^2} < \varepsilon$

$\frac{2}{(n+1)x^2} < \varepsilon$

$(n+1)x^2 > \frac{2}{\varepsilon}$

$n+1 > \frac{2}{x^2\varepsilon}$

$nb^n = \frac{n}{(\frac{1}{b})^n} = \frac{n}{(1+x)^n}$, where $x=\frac{1}{b}-1>0$

3.2.5

$2^n < \varepsilon$

$2^n = (1+1)^n \geq n$

$2^n > n$

$n < \varepsilon$

$a_n = 2^n$, lets say if $\lim(a_n) = a$

$a_{n+1} = 2a_n$

$a = 2a$

$a = 0$

but the $\inf\{a_n\} = 2 \neq 0$

$\therefore$ Contradiction!

3.2.15

$z_n = (a^n + b^n)^{\frac{1}{n}}$, $0 < a < b$

then $\lim z_n = b$

$z_n = b^n\left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}}$

$= b^n\left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}}$

$\lim\limits_{n\to\infty}\left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}} = 1$

$1 \leq \left(\frac{a}{b}\right)^n + 1 \leq 2$

$1^{\frac{1}{n}} \leq \left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} \to 1$

$\parallel$

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