Lecture 2

Wednesday, September 6, 2023 9:54 AM

Consider a sequence

$a_1 = 1$ $\quad a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$a_2 = 2$ $\quad (n \geq 4)$

$a_3 = 3$

Show that $a_n < 2^n$ for all $n \in \mathbb{N}$.

Step 1: Prove base case

$a_1 = 1 < 2^1$ ✓

Step 2: Assume it holds for $n = k$

Suppose $a^k < 2^k$

Extension: Suppose $a^n < 2^n$ for all $n \geq k$

Step 3: Show $a_{k+1} < 2^{k+1}$

$$a_{k+1} = a_k + a_{k-1} + a_{k-2} < \underbrace{2^k + 2^{k-1} + 2^{k-2}}_{2^{k-2}(2^2 + 2^1 + 2^0) = 2^{k-2} \cdot 7}$$

we know that $2^{k+1} = 2^{k-2} \cdot 2^3 = 2^{k-2} \cdot 8$

and $2^{k-2} \cdot 7 < 2^{k-2} \cdot 8$, so $a_{k+1} < 2^{k+1}$

By mathematical induction, $a_n < 2^n$ ∎

Finite and Infinite Sets (1.3)

The empty set $\emptyset$ has no elements

A set is finite if it is either empty or has n elements for some $n \in \mathbb{N}$.

A set S is said to have n elements if there exists a bijection from $\mathbb{N}_n := \{1, 2, 3, \ldots, n\}$ onto S

A set is infinite if it is not finite.

Theorem Suppose S and T are sets and $T \subseteq S$.

- If S is a finite set, then T is a finite set
- If T is an infinite set, then S is an infinite set

Since point 2 is the contrapositive of the first point, we only need to prove one of them

We choose to prove the first point using induction

Base case: If $T = \emptyset$, then it is finite

Now suppose $T \neq \emptyset$

New statement to prove:

Let S be a finite set with n elements. Any nonempty subset of S is finite.

Proof We proceed by mathematical induction

Base Case: $n = 1$

The only nonempty subset T of S is S itself, So T is finite

Inductive Hypothesis: Suppose S has k elements and is finite.

Inductive Step: Suppose S has $k+1$ elements

Let f be the bijection from $\{1, 2, \dots, k, k+1\}$ onto S

we're going to kick out a term

Let T be a nonempty subset of S

- Case 1: $f(k+1) \in S$ but $\notin T$

  Then T is a nonempty subset of $S \setminus \{f(k+1)\}$.

  By the I.H., T is finite.

- Case 2: $f(k+1) \in T$

  Then $T \setminus \{f(k+1)\}$ is a subset of S which has k elements, so $T \setminus \{f(k+1)\}$ is finite.

  Then $T = T \setminus \{f(k+1)\} \cup \{f(k+1)\}$ is also finite.

If $T \setminus \{f(k+1)\} = \emptyset$, then $T = \{f(k+1)\}$, so there exists a bijection g from $\{1\}$ onto T. Since T has one element, it is finite.

If $T \setminus \{f(k+1)\} \neq \emptyset$, then $T \setminus \{f(k+1)\}$ has say m elements $(m \leq k)$.

Then there exists a bijection g from $\{1, 2, \dots, m\}$ onto $T \setminus \{f(k+1)\}$

Show T has $m+1$ elements, and show that it has a bijection from $\{1, 2, \dots, m, m+1\}$ onto T.

Define $\bar{g}$ from $\{1, 2, \dots, m, m+1\}$:

$$\bar{g}(l) = \begin{cases} g(l) & l \leq m \\ f(k+1) & l = m+1 \end{cases}$$

Then $\bar{g}$ is a bijection

So, T has $m+1$ elements and is finite. ∎

Then $\bar{g}$ is a bijection

So, T has $m+1$ elements and is finite. ■

Countable Sets

A set S is denumerable (countably infinite) if there exists a bijection of $\mathbb{N}$ onto S

A set is countable if it is either finite or denumerable

A set is uncountable if it is not countable

Examples:

$\mathbb{N}$ - countable, denumerable (maps itself)

$\mathbb{Q}$ - countable, denumerable

$\mathbb{R}$ - uncountable

$\mathbb{R}\setminus\mathbb{Q}$ (irrationals) - uncountable

Theorem 1 Suppose that S and T are sets and $T \subseteq S$

i) If S is countable, then T is countable

ii) If T is uncountable, then S is uncountable

Theorem 2 Suppose that S and T are sets.

i) If S or T (just one) is countable, then $S \cap T$ is countable.

ii) If S and T are countable, then $S \cup T$ is countable

Proof of i) follows from i) of Theorem 1

Proof of ii):

Consider the case when both S and T are denumerable

We want to show $S \cup T$ is denumerable.

$f: \mathbb{N} \to S$

$g: \mathbb{N} \to T$

$$h(n) = \begin{cases} f(k) & n = 2k \\ g(k) & n = 2k-1 \end{cases}$$

$\underbrace{S \setminus T}_{\text{disjoint}} \cup T$

h is a bijection from $\mathbb{N}$ onto $(S\setminus T) \cup T = S \cup T$

☆ HW Problem

Some Hints:

If one, say S, is empty, then $S \cup T = T$

If S is finite but T is denumerable, and $S \cap T \neq \emptyset$,

then $S \cup T$ is denumerable

$f: \{1, 2, \dots, n\}$ onto S

$g: \mathbb{N}_n$ onto T

Construct h from $\mathbb{N}$ onto $S \cup T$ (or from f onto g ?)

$$h(k) = \begin{cases} f(k) & k \leq n \\ g(k-n) & k > n \end{cases}$$

h is a bijection I think