

$$x_n(y_n - y) \leq c |y_n - y| < \epsilon$$

$$|y_n - b| < \frac{\epsilon}{c}$$

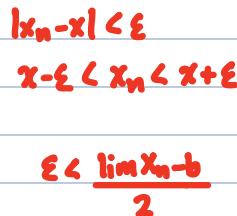
$$\frac{a_n}{b_n} < \frac{c}{d}$$

$$\frac{1}{y^n}$$

Thm: If (x_n) is a convergent sequence and

$$a \leq x_n \leq b \quad \text{for all } n \in \mathbb{N}; \text{ then } a \leq \lim x_n \leq b$$

(for all n large enough)



Thm: (squeeze theorem)

Suppose $x = (x_n)$, $y = (y_n)$, and $z = (z_n)$ are sequences such that

$$x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N} \quad \text{if } \lim x_n = x \text{ and } \lim z_n = z$$

(for all n large enough)

with $x = z$ then y converges with $\lim_{n \rightarrow \infty} y_n = z = x$

$$\frac{|y_n - z| < \epsilon}{\uparrow}$$

Proof: Since $x_n \rightarrow x = z$, but $z_n \rightarrow z$, for each $\epsilon > 0$, there exists $N_x(\epsilon)$ and $N_z(\epsilon)$, such that $|x_n - x| = |x_n - z| < \epsilon$ for all $n \geq N_x(\epsilon)$

$$|z_n - z| < \epsilon \quad \forall n \geq N_z(\epsilon)$$

$$\underline{x_n \leq y_n \leq z_n, x_n > z - \epsilon, z_n < z + \epsilon}$$

$$\uparrow \quad \uparrow$$

$$|\nu_n - x| < \epsilon \quad |z_n - z| < \epsilon$$

Let $N(\epsilon) = N_x(\epsilon) + N_z(\epsilon)$, for all $n \geq N(\epsilon)$, we have $x_n > z - \epsilon$, $z_n < z + \epsilon$.

Since $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$, then for all $n \geq N(\epsilon)$, $z - \epsilon < x_n \leq y_n \leq z_n < z + \epsilon$, So $|y_n - z| < \epsilon \quad \forall n \geq N(\epsilon)$ so $y_n \rightarrow z$, any $n \rightarrow \infty$.

$$\text{Ex: } \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$$

$$-1 \leq \sin n \leq 1$$

$$\underbrace{-\frac{1}{n}}_{0} \leq \frac{\sin n}{n} \leq \underbrace{\frac{1}{n}}_0$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

if a_n is bounded $b_n \rightarrow b$ as $n \rightarrow \infty$.

Thm: If $x_n \rightarrow x$, then $|x_n| \rightarrow |x|$

Proof: By the inequality $||x_n| - |x|| \leq |x_n - x|$

Thm: If $x_n \rightarrow x$ and $x_n \geq 0$ for all $n \in \mathbb{N}$
then $\lim \sqrt{x_n} = \sqrt{x}$

Case 1: $x=0$, Case 2: $x>0$

$$|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} < \varepsilon$$

\uparrow

$$\underline{x_n < \varepsilon^2}$$

$$|x_n - x| < \varepsilon$$

\uparrow

$$\underline{x_n < \varepsilon}$$

$$\frac{\sqrt{a} - \sqrt{b}}{1} = \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}}$$

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} = \frac{1}{\sqrt{x_n} + \sqrt{x}} |x_n - x| \leq \frac{1}{\sqrt{x}} |x_n - x| < \varepsilon$$

$$|x_n - x| < \sqrt{x} \varepsilon$$

$\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, such that $|x_n - x| < \varepsilon^2$, that is, $\underline{x_n < \varepsilon^2}$, $\forall n \geq N(\varepsilon)$ $\sqrt{x_n} + \sqrt{x} \leq \sqrt{x}$

$\forall \varepsilon > 0$, $\exists N(\varepsilon)$ s.t.

$|x_n - x| < \sqrt{x} \varepsilon \quad \forall n \geq N(\varepsilon)$

So this inequality implies

$$(x_n)^{\frac{1}{2}} < (\varepsilon^2)^{\frac{1}{2}} \text{ for all } n \geq N(\varepsilon)$$

$$\text{Then } |\sqrt{x_n} - \sqrt{x}| < \frac{1}{\sqrt{x}} |x_n - x| < \varepsilon$$

then $\sqrt{x_n} < \varepsilon$ for all $n \geq N(\varepsilon)$

for all $n \geq N(\varepsilon)$

$$|\sqrt{x_n} - \sqrt{x}| < \varepsilon$$

Thm: (Ratio Test)

Let (x_n) be a sequence of positive real numbers such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = L.$$

If $L > 1$, then (x_n) converges and $\lim x_n = 0$.

Ex: $x_n = b^n$, where $0 < b < 1$

$$b^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b < 1$$

Section 3.3 : Monotone Sequences

Let $X = (x_n)$ be a sequence

X is increasing if it satisfies

$$x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$$

(To verify X is increasing, we need to prove $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
(we use math induction)

X is decreasing if it satisfies

$$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq x_{n+1} \geq \dots$$

(To verify X is decreasing, we can prove $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ using induction)
we can also prove $-X = (-x_n)$ is increasing.

Thm: (Monotone Convergence Thm)

A monotone sequence of real numbers is convergent if and only if it is bounded and :

a) If $X = (x_n)$ is bounded increasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup \{x_n, n \in \mathbb{N}\}$$

b) If $X = (x_n)$ is bounded decreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf \{x_n, n \in \mathbb{N}\}$$

If a) is proved.

$$(x_n) \rightarrow (-x_n)$$

by $-(x_n)$ is convergent and

$$\lim_{n \rightarrow \infty} -x_n = \sup \{-x_n, n \in \mathbb{N}\}$$

$$x_n \quad -x_n$$

$$\uparrow$$

$$(-1)(-x_n)$$

$$\lim(x_n) = (-1) \lim(-x_n)$$

$$\inf \{x_n, n \in \mathbb{N}\}$$

$$\lim_{n \rightarrow \infty} x_n = (-1) \lim_{n \rightarrow \infty} (-x_n) = (-1) \sup \{-x_n, n \in \mathbb{N}\} \\ = (-1)(-\sup \{x_n, n \in \mathbb{N}\})$$

Proof of a) Since (x_n) is bounded increasing, then $\sup\{x_n, n \in \mathbb{N}\}$ exists in \mathbb{R} by the Completeness property
let $x^* = \sup\{x_n, n \in \mathbb{N}\}$

Note that $x_n \leq x^*$ for all $n \in \mathbb{N}$ since x^* is the supremum of $\{x_n, n \in \mathbb{N}\}$ then $x_n \leq x^* + \varepsilon$ for all $n \in \mathbb{N}$.
Since $x^* - \varepsilon$ is not an upper bound of $\{x_n, n \in \mathbb{N}\}$

$\exists N(\varepsilon)$ such that $x_{N(\varepsilon)} > x^* - \varepsilon$

For all $n \geq N(\varepsilon)$, $x_n \geq x_{N(\varepsilon)} > x^* - \varepsilon$ since x is increasing

So for all $n \geq N(\varepsilon)$, $x^* - \varepsilon < x_n < x^* + \varepsilon$ that is $|x_n - x^*| < \varepsilon$

So $x_n \rightarrow x^*$

Ex: Let $x = (x_n)$ be defined by $x_1 = 1$

$$x_{n+1} = \sqrt{2x_n} \text{ for all } n \in \mathbb{N}.$$

Show that $\lim_{n \rightarrow \infty} x_n = 2$.

Proof: We prove by induction that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

a) ($n=1$) $x_1 = 1 < 1.414 \approx \sqrt{2} = x_2$
 $(x_2 = \sqrt{2} > \sqrt{1} = 1 = x_1)$

$x_{n+1} \leq x_{n+2}$

b) Suppose $x_n \leq x_{n+1}$

c) $x_{n+1} = \sqrt{2x_n} \leq \sqrt{2 \cdot x_{n+1}}$

② We know (x_n) is bounded, $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$, by induction.

a) ($n=1$): $x_1 = 1$, $1 \leq x_n \leq 2$

b) suppose $1 \leq x_n \leq 2$

c) $2 \leq 2x_n \leq 4$, then $2^{\frac{1}{2}} \leq (\sqrt{2x_n})^{\frac{1}{2}} \leq 4^{\frac{1}{2}}$

So $\sqrt{2} \leq x_{n+1} \leq 2$. So $1 \leq x_{n+1} \leq 2$.

③ Let x^* be the limit.

$x^* = \sqrt{2x^*}$ then ~~$x^* \neq 0$~~ or $x^* = 2$.

x^* cannot be 0, because x_{n+1} is bounded between 1 and 2.

$\therefore x^* = 2$.