Lecture 24

Monday, January 8, 2024 9:21 PM

(12/06)

Location of Roots Theorem

Let $I = [a,b]$ and $f: I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$.

Bolzano's Intermediate Value Theorem

Let I be an interval and $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ such that $f(a) < k < f(b)$, then there exists a number c such that $f(c) = k$.

Proof Suppose that $a < b$ and let $g(x) := f(x) - k$. Then $g(a) < 0 < g(b)$.
By the previous theorem, $\exists\, c$ with $a < c < b$ such that $g(c) = f(c) - k$.
Therefore, $f(c) = k$.
If $b < a$, let $h(x) := k - f(x)$. ■

Corollary If $k \in \mathbb{R}$ satisfies

$$\inf\{f(I)\} \leq k \leq \leq \sup\{f(I)\}$$

then there exists a number c such that $f(c) = k$.

Section 5.4 Uniform Continuity

Def Let $A \subseteq \mathbb{R}$ and $f: I \to \mathbb{R}$.

f is uniformly continuous on A if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ satisfy $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

independent of u

analog to Cauchy for sequences

Theorem Let I be a closed bounded interval and $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Negation of def: $\exists \varepsilon_0 > 0$, $\forall \delta > 0$, if $\exists x_\delta, u_\delta \in A$ such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

Sequential Criterion for Non-Uniform Continuity

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. f is not uniformly continuous on A if there exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ $\forall n \in \mathbb{N}$.

Uniform Continuity Theorem

Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

pop
six
squish
uh-uh
cicero
Lipschitz

Lipschitz!

Def Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$|f(x) - f(u)| \leq K|x - u|$$

$\forall x, u \in A$, then f is a Lipschitz function.

Fun aside: the mathematician's name was Rudolf Lipschitz.
He was Dirichlet's student

Lipschitz is like the analog of contractive sequences, just less strict – the constant for contractive must be between 0 and 1, but for Lipschitz it just has to be greater than 0.

Theorem If $f: A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

This is where Kang stopped in the textbook, he did not cover Theorem 5.4.7