

MATH 301: Homework 2

Aren Vista

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Question 1

Prove Theorem 1.3.4 (b)

If a set A with $m \in \mathbb{N}$ elements and $C \subseteq A$ is a set with one element then $A - C$ is a set with $m - 1$ elements

Proof. Suppose a set A with $m \in \mathbb{N}$ elements

$$A = \{a_1, a_2, \dots, a_{m-1}, a_m\} \implies |A| = m$$

Suppose $C \subseteq A$ is a set with one element

$$C = \{a\}, a \in A$$

Observe a is an arbitrary element of A

WLOG let $a = a_m$ s.t.

$$C = \{a_m\}$$

Thus

$$A - C = \{a_1, a_2, \dots, a_{m-1}\} \implies |A - C| = m - 1$$

□

Question 2

Prove Theorem 1.3.4 (c)

If C is an infinite set and B is a finite set

Then $C - B$ is an infinite set

Recall an infinite set is a set which is not finite (defn.)

Proof. Suppose C is an infinite set

$$C = \{c_1, c_2, \dots\}$$

Suppose B is a finite set

$$B = \{b_1, b_2, \dots, b_n\}, n \in \mathbb{N}$$

Suppose for the sake of contradiction $C - B$ is finite s.t

$$C - B = \{x_1, x_2, \dots, x_m\} m \in \mathbb{N}$$

Observe at most $C \cap B$ must be finite (n number of elements) by definition of set intersection

$$C \cap B = \{b_1, b_2, \dots, b_n\}$$

By rearrangement of the principle of exclusion

$$C = C - B \cup (C \cap B)$$

If $C - B$ is finite and $C \cap B$ is finite then C must be finite. This is a logical inconsistency.

Thus, $C - B$ must be infinite.

□

Question 3

Prove that if x is a rational number and y is an irrational number, then $x + y$ is an irrational number.

If in addition $x \neq 0$ then show that xy is an irrational number.

Proof. Suppose x is a rational number and y is an irrational number

Suppose for the sake of contradiction $x + y$ is a rational number.

Then given $m, n \in \mathbb{Z}$

$$x + y = \frac{m}{n}$$

By defn. $x, k, p \in \mathbb{Q} \implies x = \frac{k}{p}$

Thus

$$\begin{aligned}\frac{k}{p} + y &= \frac{m}{n} \\ y &= \frac{m}{n} - \frac{k}{p} \\ y &= \frac{mp}{np} - \frac{kn}{pn} \\ y &= \frac{mp - kn}{np}\end{aligned}$$

By set inclusion of integers, $mp - kn, np \in \mathbb{Z}$. This would imply y is rational.

This is a logical inconsistency, thus $x + y$ is irrational. □

Question 4

Modify the proof of Theorem 2.1.4

Show that there does not exist a rational number t so that $t^2 = 3$

Proof. Let $t \in \mathbb{Q}$

Suppose for the sake of contradiction $t^2 = 3$

Let $x, y \in \mathbb{Z}$, y defn of $t = \frac{x}{y}$, where x, y are the lowest possible factors of t

Thus

$$\begin{aligned}\left(\frac{x}{y}\right)^2 &= 3 \\ \frac{x^2}{y^2} &= 3 \\ x^2 &= 3y^2\end{aligned}$$

Notice x^2 is a multiple of 3

By properties of prime

$$3|x^2 \implies 3|x$$

Thus for some $k \in \mathbb{Z}$

$$x^2 = (3k)^2 = 3y^2$$

Thus $3k = y^2$

This concludes $y|3$ and $x|3$.

Note, we stated x, y are the lowest possible factors of t implying there does not exist any more common factors which could reduced x, y

This is a logical inconsistency, this there can not exist a $t \in \mathbb{Q}$ s.t. $t^2 = 3$ □

Question 5

Recall the **Triangle Inequality**

If $a, b \in \mathbb{R}$

Then $|a + b| \leq |a| + |b|$

Prove this inequality holds iff $ab \geq 0$

Statement P: $|a + b| \leq |a| + |b| \iff ab \geq 0$

Case 1: a and b have the same sign

a and b have the same sign $\implies ab \geq 0$: Observe

a and b are positive

$$\implies |a + b| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$$

a and b are negative

$$\implies |(-a) + (-b)| = |-(a + b)| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$$

Case 2: $a = 0 \vee b = 0$

$a = 0 \vee b = 0 \implies ab \geq 0$ Observe

If $a = 0$

$$\implies |0 + b| = |b| \equiv b \iff b \leq 0 + |b| \iff |a + b| \leq |a| + |b|$$

If $b = 0$

$$\implies |0 + a| = |a| \equiv a \iff a \leq 0 + |a| \iff |a + b| \leq |a| + |b|$$

Conclusion

Thus By Case 1,2 $|a + b| \leq |a| + |b| \iff ab \geq 0$ holds

Question 6

Show that if $a, b \in \mathbb{R}$ and $a \neq b$

Then there exists ϵ -neighborhoods U of a and V of b such that $U \cap V = \emptyset$