Lecture 8

Wednesday, September 27, 2023 1:37 AM

Kang is a whole minute late smh

We have a grader now!

Tally of "backwards search"

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"backtracking"

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"think a little bit"

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"use definitions, not your words"

Quiz

1) Two ways of doing it:

Surjection

$f: \mathbb{N} \to S$

$g: \mathbb{N} \to T$

$$h(n) = \begin{cases} f(k) & n = 2k \\ g(k) & n = 2k+1 \end{cases}$$

h is a surjection from $\mathbb{N}$ onto $S \cup T$

So, $S \cup T$ is denumerable

Bijection

Case 1: $S \setminus T = \emptyset$

Case 2: $S \setminus T$ has n elements ($n \in \mathbb{N}$)

Case 3: $S \setminus T$ is denumerable

2) $c^m < c^n$ iff. $m > n$

Proof ($\to$) Suppose $c^m < c^n$ and $m \le n$

Case 1: $m = n$. Then $c^m = c^n$. Contradiction!

Case 2: $m < n$. Then $n - m > 0$

Show that $c > 1$ implies $c^k > 1$ using induction

Base case: $k = 1$

$c^k = c > 1$ ✓

Suppose $c^k > 1$

Multiply by c: $c^k \cdot c > 1 \cdot c$

$c^{k+1} > c > 1$, so $c^{k+1} > 1$

Therefore, $c^{n-m} > 1$

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$c^m > c^n$ implies $c^m - c^n > 0$

$c^n(c^{m-n} - 1) > 0$

$c^n > 0$ and $(c^{m-n} - 1) > 0$

So, $c^{m-n} > 1$

So $c^n = c^{n-m} \cdot c^m > 1 \cdot c^m$

Contradiction!

($\Leftarrow$) Suppose $m > n$. $m - n > 0$ and $\in \mathbb{N}$

$c^{m-n} > 1$ by the claim

$c^m = c^{m-n} \cdot c^n > 1 \cdot c^n = c^n$

$\Rightarrow c^m > c^n$ $\square$

he also went over the hw problem with $c^{\frac{1}{m}} < c^{\frac{1}{n}}$

2.5

Review: Def A set is an interval if:

1. It has at least 2 points
2. For some $x, y \in S$ and $x < y$, $[x, y] \in S$

"contradiction argument is my favorite"

"get coffee, review lecture notes"

Theorem (Nested Interval Property)

If $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that

$$\xi \in \bigcap_{n=1}^{\infty} I_n \quad (\xi \in I_n \ \forall n \in \mathbb{N})$$

\xi in LATEX

Proof Consider $S = \{a_1, a_2, \dots\}$. Then S is nonempty.

want to show $a_n \le \xi \le b_n$

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Proof Consider $S = \{a_1, a_2, \dots\}$. Then S is nonempty.

Note that for each $n \in \mathbb{N}$, $a_n \le b_n \le b_1$, so b_1 is an upper bound of S. Since S is bounded above, by the Completeness Property, $\sup S$ exists $\in \mathbb{R}$.

Let $\xi := \sup S$. Then $a_n \le \xi$.

(want to show that $\xi \le b_n$)

Suppose, by way of contradiction, that $\xi > b_n$. For each $m \in \mathbb{N}$, if $m \ge n$, $a_m \le b_m \le b_n$.

If $m < n$, $a_m \le a_n \Rightarrow I_n \subseteq I_m \Rightarrow a_n \in I_m$.

So, b_n is an upper bound of S, which implies $\sup S \le b_n$.

Contradiction! Thus, $\xi \in I_n$.

$$\sup\{a_n\} = \inf\{b_n\} = \bigcap_{n=1}^{\infty} I_n$$

Theorem If $I_n = [a_n, b_n]$ $\forall n \in \mathbb{N}$ is a nested sequence of closed and bounded intervals such that $b_n - a_n$ of I_n satisfy

$$\inf\{b_n - a_n, n \in \mathbb{N}\} = 0$$

Then $\xi \in \bigcap_{n=1}^{\infty} I_n$ is unique

Proof Suppose, by way of contradiction, that $\xi \in \bigcap_{n=1}^{\infty} I_n$ is not unique.

Let $\xi_1, \xi_2 \in \bigcap_{n=1}^{\infty} I_n$, then $\xi_1, \xi_2 \in I_n$ $\forall n \in \mathbb{N}$. Let $\xi_1 < \xi_2$.

Then $|\xi_2 - \xi_1| \le b_n - a_n$. So, $|\xi_2 - \xi_1|$ is a lower bound of $\{b_n - a_n\}$. Contradiction!

Cool thing! The fact that $\mathbb{R}$ is uncountable can be proven using the Nested Intervals Property.

Also, $\mathbb{R}$ can be divided into two types of numbers: algebraic and transcendental.

A real number is an algebraic number if it is a solution to a polynomial equation $P(x)=0$, where all the coefficients of $P(x)$ are integers. Transcendental real numbers are not algebraic.

π and e are transcendental numbers.

The set of algebraic numbers is countably infinite, and the set of transcendental numbers is uncountable.

3.1 Sequences

Def A **sequence of real numbers** is a mapping from $\mathbb{N}$ onto a subset of $\mathbb{R}$, denoted by $X: \mathbb{N} \to \mathbb{R}$.

The value of X at $n \in \mathbb{N}$ is denoted by $x(n)$ or x_n.

The sequence is denoted by

$$X \qquad (x_n) \qquad \{x_n : n \in \mathbb{N}\}$$

A sequence can be defined inductively

Ex $x_{n+2} = x_{n+1} + x_n$