Lecture 12

Monday, October 16, 2023 9:59 AM

Kang got a haircut!

Also our favorite guy shaved 0.0

Backwards search

(Ratio Test)

Theorem Let (x_n) be a sequence of positive real numbers such that $\lim \frac{x_{n+1}}{x_n} = L$. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.

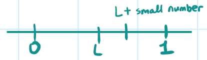
Notes $|x_n - 0| < \varepsilon$

$\Rightarrow x_n < \varepsilon$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$

$\left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1-L}{2}$

L+ small number $< 1 \Rightarrow \frac{1-L}{2}$



$\Rightarrow \exists M\left(\frac{1-L}{2}\right) \quad \forall n \geq M\left(\frac{1-L}{2}\right)$

$\frac{x_{n+1}}{x_n} < \frac{1+L}{2} < 1$ (adding L)

$x_{n+1} < \left(\frac{1+L}{2}\right) x_n$

Use m-tail: $x_{M_L+k} < \left(\frac{1+L}{2}\right)^k x_{M_L}$

And squeeze theorem

Proof We first show that $\left(\frac{1+L}{2}\right)^k \to 0$ as $k \to \infty$.

Since $\lim \frac{x_{n+1}}{x_n} = L < 1$, $\exists M_L = M\left(\frac{1-L}{2}\right) > 0$ such that $\left|\frac{x_{n+1}}{x_n} - L\right| < \frac{1-L}{2}$ for all $n \geq M_L$. Note that

$$\frac{x_{n+1}}{x_n} = \left|\frac{x_{n+1}}{x_n} - L + L\right| \leq \left|\frac{x_{n+1}}{x_n} - L\right| + L \leq \frac{1-L}{2} + L = \frac{1+L}{2}$$

by the triangle inequality. Then by mathematical induction, $x_{M_L+k} < \left(\frac{1+L}{2}\right)^k < x_{M_L}$ $\forall k \in \mathbb{N}$. So, $0 < x_{M_L+k} < \left(\frac{1+L}{2}\right)^k x_{M_L}$

By the Squeeze Theorem, $\lim x_{M_L+k} = 0$. Since x_{M_L+k} is the m-tail of x_n, $x_n \to 0$.

More Notes (once we learn log rules):

$\left(\frac{1+L}{2}\right)^k < \varepsilon$

$k \log\left(\frac{1+L}{2}\right) < \log(\varepsilon)$

$k < \frac{\log(\varepsilon)}{\log\left(\frac{1+L}{2}\right)}$ so by the Archimedean Property, $\exists N > \frac{\log(\varepsilon)}{\log\left(\frac{1+L}{2}\right)}$

Section 3.3 Monotonic Sequences

Def Let (x_n) be a sequence.

- (x_n) is **increasing** if it satisfies

  $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots$

  (To show a sequence is increasing, use induction to show $x_n \leq x_{n+1}$)

- (x_n) is **decreasing** if it satisfies

  $x_1 \geq x_2 \geq x_3 \geq \ldots \geq x_n \geq x_{n+1} \geq \ldots$

- (x_n) is **monotonic** if it is either increasing or decreasing

$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq x_{n+1} \geq \dots$

- (x_n) is **monotonic** if it is either increasing or decreasing

(x_n) is decreasing if $(-x_n)$ is increasing

Monotone Convergence Theorem

A monotone sequence of real numbers is convergent if and only if it is bounded. In addition:

i) If (x_n) is bounded increasing, $\lim(x_n) = \sup\{x_n\}$

ii) If (x_n) is bounded decreasing, $\lim(x_n) = \inf\{x_n\}$

Proof i) We need to show $x^* = \sup\{x_n\}$ is the limit of (x_n).

By the Completeness Property, $x^* \in \mathbb{R}$.

$\forall \varepsilon > 0$, since $x^* + \varepsilon$ is an upper bound of (x_n), $\exists N(\varepsilon) \in \mathbb{N}$ such that $x_{N(\varepsilon)} > x^* - \varepsilon$. Then $\forall n \geq N(\varepsilon)$, since (x_n) is increasing, $x_n > x^* - \varepsilon \Rightarrow x^* - \varepsilon < x_n < x^* + \varepsilon$. This implies that $|x_n - x^*| < \varepsilon$. So, $\lim(x_n) \to x^*$.

Ex Let (x_n) be defined as $x_1 = 1$, $x_{n+1} = \sqrt{2x_n}$. Show that $\lim(x_n) = 2$.

Proof We first show that (x_n) is increasing (use induction)

Then show that $x_n \leq 2$ $\forall n \in \mathbb{N}$ using the Monotone Convergence Theorem

Section 3.4

Def| Let (x_n) be a sequence of real numbers and let $n_1 < n_2 < \dots < n_k < \dots$ be a strictly increasing sequence of natural numbers ($x_n : \mathbb{N} \to \mathbb{R}$).

Then the sequence (x_{n_k}) given by $x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots$ is called a **subsequence** of (x_n).

Ex $(x_n = \frac{1}{n})$

$\frac{1}{2n}$ can be viewed as a subsequence of $\frac{1}{n}$

Or do $\frac{1}{2}(\frac{1}{n})$.