Functions

Codomain + Range

Ex: $f(x) = x^2$

$D(f) : \mathbb{R} \to \underbrace{\mathbb{R}}_{\text{codomain}}$

$R(f) : [0, \infty)$

Direct image of f

$= \{f(a) : x \in D(f)\} = R(f)$

$$\boxed{\begin{array}{l}\text{Direct image A under f} \\ \quad = \{f(x) : x \in A\}\end{array}}$$

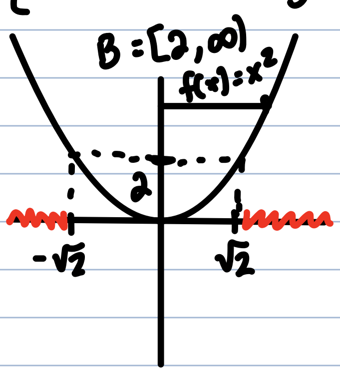
$f(x) = x^2, \; A = [1, \infty)$

Ex: Direct image of A under f

$= [1, \infty)$

inverse image of B under f, where B is a subset of the co-domain.

$= \{x \in D(f) : f(x) \in B\}$



inverse image in $(\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

$f(f^{-1}(B)) = B, \; f^{-1}(f(A)) = A$?

Injective (one-to-one): if $f(x_1)=f(x_2)$
Surjective (onto) : if the direct image of the domain is the codomain
Bijective : function is both injective and surjective

Types of Proofs :
Direct : if A then B
Contrapositive ; if not B then not A
Contradiction : assume A not B $\Rightarrow$ C, where C is false
Induction: start case of n the generalize with k

Section 1.3 : Finite and Infinite Sets

The empty set "$\emptyset$" has 0 elements.
A set S is said to have n elements if $\underbrace{\text{there exists}}_{\text{we need to construct}}$ a bijection from $\{1,2,\dots,n\}$ onto S.

A set S is finite if it is either empty or it has n elements for some $n \in \mathbb{N}$.
A set S is infinite if it is not finite.
Theorem: Suppose that S and T are sets and $T \subseteq S$.
a) if S is a finite set, then T is a finite set.
b) if T is infinite, then S is infinite.

Proof: If $T=\emptyset$, then T is a finite set.

If $T \neq \emptyset$, then $S \neq \emptyset$ since $T \subseteq S$.
We prove by induction on the number n of elements in S.
① $(n=1)$: If S has one element. Then the only non-empty subset of S is $T=S$.
Since S is finite, then T is finite as well.

② Suppose S has k elements and every non-empty subset of S is finite.

③ Now let S be a set having k+1 elements, then there exists a bijection f from $\{1,2,\dots,k+1\}$ onto S
of $f(k+1) \notin T$, then T is a subset of $S \setminus \{f(k+1)\}$.

Note that : $S \setminus \{f(k+1)\}$ has k elements since f is still a bijection from $\{1,2,\dots,k\}$ onto $S\setminus\{f(k+1)\}$
By the induction hypothesis, T is finite

If $f(k+1) \in T$, then $T \setminus \{f(k+1)\}$ is a subset of $S \setminus \{f(k+1)\}$.
So $T \setminus \{f(k+1)\}$ is finite by the induction hypothesis.

Then $T = (T \setminus \{f(k+1)\}) \cup \{f(k+1)\}$ is also finite.

if $T \setminus \{f(k+1)\} = \emptyset$, then $T = \{f(k+1)\}$ (Thus one element), then T is finite
if $T \setminus \{f(k+1)\}$ has m elements $(m \le k)$, let g be the bijection from $\{1, 2, \ldots, m\}$ onto
Extend g from $\{1, 2, \ldots m\}$ to a bijection $\bar{g}$ from $\{1, 2, \ldots, m+1\}$ by

$$\begin{cases} \bar{g}(i) = g(i) & i \le m \\ \bar{g}(m+1) = f(k+1) & i = m+1 \end{cases}$$

Countable Sets

Definition: A set S is denumerable (or countably infinite)

A set is countable if it is either finite or denumerable.
A set is uncountable if it is not countable.

Ex: $\mathbb{N} = \{1, 2, \ldots\}$, the set of natural numbers is countable.
(denumerable)

$\mathbb{R}$ is uncountable