Lecture 14

Monday, October 23, 2023 9:55 AM

Plan is to finish subsequences today

$\lim_{n\to\infty} b_n = 0$ if $0 < b < 1$

Proof For each $n \in \mathbb{N}$, $b^{n+1} = b^n \cdot b < b^n$ (can show using induction)

So, (b^n) is decreasing. Note that $b^n > 0$ (also induction)

So, (b^n) is bounded from below. By the Monotone Convergence Theorem,

b^n converges to some number x ($x = \inf\{b^n\}$).

Take $\{b^{2n}\}$. This is a subsequence of (b^n). Then $b^{2n} \to x$ as $n \to \infty$.

$b^{2n} = (b^n)^2$, so this converges to x. So,

$$x = \lim(b^{2n}) = (\lim(b^n))^2 = x^2$$

So, $x^2 = x$, so $x = 0$ or $x = 1$. $x \neq 1$,* so $x = 0$. ■

Slay

why does he keep bringing up ln

Kang really likes natural log lmao

☆ Still a little unclear to me but here is Kang's explanation

$b^n \leq b$

$n=1 \Rightarrow b \leq b$ ✓

By the Order Properties...

$\lim b^n \leq b < 1$

so $x \neq 1$

Divergence of Sequences

Theorem Let $X = (x_n)$ be a sequence. Then the following are logically equivalent:

① (x_n) does not converge to $x \in \mathbb{R}$ (x is not a limit)

② Negation of limit def:

There exists $\varepsilon_0 > 0$ for all $n \in \mathbb{N}$ such that there exists $k_n \geq n$ such that $|x_{k_n} - x| \geq \varepsilon_0$.

③ There exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \varepsilon_0$ ($\forall k \in \mathbb{N}$)

this man has never been told to shut up and it shows

lowkey might be me soon

We know that ① implies ②.

Showing that $2 \Rightarrow 3$ (Select a $\{k_n\}$ such that it is strictly increasing)

Proof By 2, when $n = 1$, $\exists\, k_1 \geq 1$ such that $|x_{k_1} - x| \geq \varepsilon_0$.

When $n = k_1 + 1$, $\exists\, k_2 \geq k_1 + 1$ such that $|x_{k_2} - x| \geq \varepsilon_0$.

So we have a sequence $\{k_n\}$ such that $k_{n+1} \geq k_n + 1$ and

When $n = k_1 + 1$, $\exists\, k_2 \geq k_1 + 1$ such that $|x_{k_2} - x| \geq \varepsilon_0$.

So, we have a sequence $\{k_n\}$ such that $k_{n+1} \geq k_n + 1$ and

and $|x_{k_n} - x| \geq \varepsilon_0 \;\forall n \in \mathbb{N}$.

So, (x_{k_n}) is a subsequence of (x_n). ■

Divergence Criteria

X is divergent if:

① X has two convergent subsequences $x' = (x_{n_k})$ and $x'' = (x_{m_k})$ whose limits are not equal.

LMAO he won't call on [redacted] anymore

② X is unbounded

- this comes from the contrapositive of X is convergent $\Rightarrow$ X is bounded

mans cannot handle not speaking

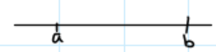
Bolzano Weierstrass Theorem!

"Let's take a break, we haven't done that in a while"

A bounded sequence of real numbers has a convergent subsequence.

Proof: Since (x_n) is bounded, there exists an interval $I_1 = [a, b]$ such that $x_n \in I_1$ for all $n \in \mathbb{N}$. We now bisect I_1 into two equal subintervals, I_1' and I_1''.

So, one of I_1' and I_1'' contains infinitely many elements of (x_n), denoted by I_2, and let n_2 be the smallest index of x_n that is bigger than n_1 $(n_2 > n_1)$.



We now bisect I_2 into two equal subintervals I_2' and I_2''. One of these contains infinitely many elements of (x_n), denoted by I_3, and let n_3 be the smallest index of x_n in I_3 $(n_3 > n_2)$.

was n_3 really necessary

Continuing like this, we have a nested sequence of closed and bounded intervals with $\inf\{I_n\} = 0$ and a subsequence (x_{n_k}) such that $x_{n_k} \in I_k$ $\forall k \in \mathbb{N}$.*

Since the length of I_k is equal to $\frac{(b-a)}{2^{k-1}}$, it follows that there is a unique common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Since x_{n_k} and ξ both are in I_k, we have

$$|x_{n_k} - \xi| \leq \frac{(b-a)}{2^{k-1}}$$

from which it follows that the subsequence (x_{n_k}) of (x_n) converges to ξ. ■

* everything after this is directly from the book since he erased it immediately after writing -_-

Def Let $X = (x_n)$

a) The **limit superior** of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of elements of $n \in \mathbb{N}$

Why did he sound French when he said lim sup ???

(at most a finite number of elements of (x_n) can be $> v$)

$\limsup(x_n) = \inf\{V\}$ where $V = \{v \in \mathbb{R} : v < x_n$ for at most a finite number of $n \in \mathbb{N}\}$

Ex $\{1 + \frac{1}{n} : n \in \mathbb{N}\}$

Collection of upper bounds $= [2, \infty)$