Lecture 9

Monday, October 2, 2023 9:56 AM

Notes

- Midterm next week (10/11)
  - covers Ch 1-2 (3.1 will <u>not</u> be on it)
- He will upload a practice exam today or tomorrow
- Monday is a review day

"Backwards Search" tally

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Def A sequence is said to **converge** to $x \in \mathbb{R}$, or x is a limit of the sequence, if for each $\varepsilon > 0$, $\exists$ a natural number $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, $|x_n - x| < \varepsilon$.

* This def is used to verify a limit, but cannot be used to find one

$V_\varepsilon(x) := \{ y : |y - x| < \varepsilon \}$ is the ε-neighborhood of x

All of the elements of the sequence after $x_{N(\varepsilon)}$ are in $V_\varepsilon(x)$

If a sequence has a limit, it is **convergent**; otherwise, it is **divergent**.

Uniqueness Theorem A sequence in $\mathbb{R}$ can have at most one limit.

Proof "leave as practice"

use contradiction

$n > N(\varepsilon)$ and $n > N^*(\varepsilon)$

Ex $\lim_{n \to \infty} \frac{1}{n} = 0$

Notes We need to show for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $|\frac{1}{n} - 0| < \varepsilon$.

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$N(\varepsilon) \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $|\frac{1}{n} - 0| < \varepsilon$.

$\frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

if $n \geq N(\varepsilon)$, then $n > \frac{1}{\varepsilon}$

so choose $N(\varepsilon) > \frac{1}{\varepsilon}$

this is given by the Archimedean Property

he quite literally wrote "Backwards search" on the board

Proof Let $\varepsilon > 0$ be given.

Since $\frac{1}{\varepsilon} \in \mathbb{R}$, by the Archimedean Property, there exists $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$. For all $n \geq N(\varepsilon)$, $n \geq N(\varepsilon) > \frac{1}{\varepsilon}$, so $n > \frac{1}{\varepsilon}$ then $\frac{1}{n} < \varepsilon$, so $|\frac{1}{n} - 0| < \varepsilon$.

he did not write "Let $\varepsilon > 0$ be given" but the book does and Hoffman always did in Math300 so I added it.

Ex $\lim_{n \to \infty} \sqrt{n+1} - \sqrt{n} = 0$

Notes $n+1 > n$

$\Rightarrow (n+1)^{\frac{1}{2}} > n^{\frac{1}{2}}$

Suppose $(n+1)^{1/2} \leq n^{1/2}$. Then $((n+1)^{1/2})^2 \leq (n^{\frac{1}{2}})^2$

$\Rightarrow n+1 \leq n$

$\Rightarrow 1 \leq 0$ Contradiction!

$|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon$

$\sqrt{n+1} - \sqrt{n} < \varepsilon$

$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$ Multiply by conjugate

$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$

$\frac{1}{n} < \varepsilon^2 \Rightarrow N(\varepsilon) > \frac{1}{\varepsilon^2}$

Proof Let $\varepsilon > 0$ be given. Since $\frac{1}{\varepsilon^2} \in \mathbb{R}$, by the Archimedean Property, there exists $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\varepsilon^2}$. For all $n \geq N(\varepsilon)$, $n > \frac{1}{\varepsilon^2}$. Then $\frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$, so $\sqrt{n+1} - \sqrt{n} < \varepsilon$.

Thus, $|\sqrt{n+1} - \sqrt{n} - 0| < \varepsilon$. $\square$

Thus, $|\sqrt{n+1}-\sqrt{n}-0| < \varepsilon$. $\square$

sir why are you being so aggressive towards Kang

Ex $\lim_{n\to\infty} (-1)^n$ does not exist.

Negation of limit def: x is not the limit of x if there exists $\varepsilon_0 \geq 0$ such that for every $N \in \mathbb{N}$, there exists $n(N) \in \mathbb{N}$ such that $n(N) \geq N$ implies $|x_{n(N)} - x| \geq \varepsilon_0$

We have infinitely many elements in x_n that are outside of $V_{\varepsilon_0}(x)$.

Tails of Sequences

Def| If $X = (x_1, x_2, \dots)$ is a sequence of real numbers and if m is a given natural number, then the m-tail of X is

$$X_m = (x_{m+1}, x_{m+2}, \dots)$$

If $m=0$, the m-tail is the original sequence

literally just shifting the starting point of the sequence

Theorem Let $X = (x_1, x_2, \dots)$ be a sequence of real numbers and X_m be the m-tail of X. X converges if and only if X_m converges.

$$\lim_{n\to\infty} X = \lim_{n\to\infty} X_m$$

Section 3.2 Limit Theorems

Def| A sequence is bounded if there exists an $M \in \mathbb{R}, M > 0$ such that $|x_n| \leq M$ (for all $n \in \mathbb{N}$)

$\sup |x_n| < M$

Theorem A convergent sequence is bounded.

Notes Using the limit def, $|x_n - x| < \varepsilon$

$$|x_n| = |x_n + x - x|$$
$$\leq |x_n - x| + |x| \quad \leftarrow \text{triangle inequality}$$

create something from nothing

$$|x_n - x| + |x| < \varepsilon + |x|$$

Proof Since x_n is convergent, let x be its limit.

For $\varepsilon = 1$, there exists $N(1) > 0$ such that for all $n \geq N(1)$,

$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| \leq 1 + |x|$. Let

$M = \max\{1 + |x|, |x_1|, |x_2|, \dots, |x_{N(1)+1}|\} \in \mathbb{R}$.

For all $n \in \mathbb{N}$, $|x_n| \leq M$. $\square$

The converse is not true.