① **3.3.2**

Here we want to show induction.
Since $x_1>1$, we have that $\frac{1}{x_1}<1$. This means $x_2=2-\frac{1}{x_1}>1$.

We want to show $x_n>1$ for all n. By induction we see that if $x_k>1$, then $x_{k+1}=2-\frac{1}{x_k}>1$.
So here we see that we are bounded above by 2 and below by 1.
Now we want show that the monotone is decreasing. We still use induction.
$0<(x_1-1)^2 \Rightarrow 0<x_1^2-2x_1+1 \Rightarrow 2x_1<x_1^2+1 \Rightarrow 2<x_1+\frac{1}{x_1} \Rightarrow 2-\frac{1}{x_1}<x_1 \Rightarrow x_2<x_1$. This is for the first base case. We use induction to finish the monotone.
$x_{k+1}<x_k \Rightarrow \frac{1}{x_{k+1}}>\frac{1}{x_k} \Rightarrow \frac{-1}{x_{k+1}}<\frac{-1}{x_k} \Rightarrow 2-\frac{1}{x_{k+1}}<2-\frac{1}{x_k} \Rightarrow x_{k+2}<x_{k+1}$. Now with the limits as used in the book $x_{n+1}=2-\frac{1}{x_n} \Rightarrow x=2-\frac{1}{x} \Rightarrow x=1$. ■

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② **3.3.2**

Since we know that $x_1\geq 2$, we have that $x_1-1\geq 1$. We want to show x_n is bounded below 2.
So we can write $x_1-1\geq 1$ and relate it to $x_{n+1}=1+\sqrt{x_n-1}$.
$x_1-1\geq 1$ this means $\sqrt{x_1-1}\geq 1 \Rightarrow 1+\sqrt{x_1-1}\geq 1+1=2$. We can recognize that x_2 would be $1+\sqrt{x_1-1}$. Hence, $x_2\geq 2$, but that was our base case. We need to use induction. So we can use the same method but just for k. If $x_{k+1}\geq 2$ then $x_{k+1}-1\geq 1$. Let's relate $x_{k+2}=1+\sqrt{x_{k+1}-1}$.
Now we have $\sqrt{x_k-1}\geq 1 \Rightarrow \sqrt{x_k-1}+1\geq 1+1=2$. We recognize that $x_{k+2}=1+\sqrt{x_{k+1}-1}$.
Hence, $x_{k+2}\geq 2$. We need to do monotone. We do the same method of induction. We start out with a base case. We notice $x_1-1>1$, then $x_1-1>\sqrt{x_1-1} \rightarrow x_1>1+\sqrt{x_1-1}=x_2$. That was our base case.
Now onto our inductive case for k+1. We know that $x_k>x_{k+1} \Rightarrow x_k-1>x_{k+1}-1 \Rightarrow \sqrt{x_k-1}>\sqrt{x_{k+1}-1}$
$1+\sqrt{x_k-1}>1+\sqrt{x_{k+1}-1} \Rightarrow x_{k+1}>x_{k+2}$. Now we use limits $x_{n+1}=2-\frac{1}{x_n} \Rightarrow x=2-\frac{1}{x} \Rightarrow x=1$, but this is impossible $\therefore x=2$. ■

③ **3.3.9**

We let $u=\sup A$, here for every $\varepsilon>0$ there exists an element a in A such that $u-\varepsilon<a<u$, that is $u-a<\varepsilon$. Now take $\varepsilon=1$ then we can find a_1 in A s.t. $u-a_1<1$. Now take $\varepsilon=\frac{1}{2}$. We can find a b in A such that $u-b<\frac{1}{2}$. Now if $a_1>b$ then $u-a_1<u-b<\frac{1}{2}$. In this case let $a_2=a_1$. Otherwise let $a_2=b$. In eithercase we have $a_1\leq a_2$ and $u-a_2<\frac{1}{2}$. Proceeding like this for each n, we will get a sequence of elements $a_1,a_2,a_3,\ldots$ of A satisfying $u-a_n<\frac{1}{n}$ and $a_1\leq a_2\leq a_3\leq\ldots$ Since for every n, $u-a_n<\frac{1}{n}$ we have that $\lim a_n=u$. So we find an increasing sequence which converges to u. ■

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④ 3.3.11

Let $x_n = 1 + \frac{1}{2^n} + \dots + \frac{1}{n^2}$

$x_{n+1} = 1 + \frac{1}{2^n} + \dots + \frac{1}{n^2} + \frac{1}{(n+1)^2}$

$x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$

$x_{n+1} > x_n$, $\therefore \{x_n\}$ is increasing

$\{x_n\} = \{1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}\}$ is bounded

$n > n-1$, $n \geq 2$

$n^2 > n(n-1)$

$\therefore \frac{1}{n^2} < \frac{1}{n(n-1)}$

$\therefore \{x_n\}$ is bounded above

Since $\{x_n\}$ is monotone increasing and bounded above then $\{x_n\}$ convergent.

⑤ 3.4.2

If $x_n = c^{1/n}$, where $0 < c < 1$, then x_n is increasing and bounded, so it has a limit x. Since $x_{2n} = \sqrt{x_n}$, the limit satisfies $x = \sqrt{x}$, so $x = 0$ or $x = 1$. Since $x = 0$ is impossible, we have $x = 1$.

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⑥ 3.4.4

a) $(1 - (-1)^n + 1/n)$

Let $x_n = (1 - (-1)^n + 1/n)$. Let $n_k = 2k$ and $m_k = 2k+1$ for $k \in \mathbb{N}$.
So $x_{n_k} = (1/2k)$ converges to 0, while the subsequence $x_{m_k} = 2 + \frac{1}{(2k+1)} \to 2$

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$\therefore$ The subsequence is divergent.

b) $x_n = \sin(n\pi/4)$. Let $n_k = 2 + 8k$ and $m_k = 8k$ $\forall k \in \mathbb{N}$
The limit of the subsequence $(x_{n_k} = \sin(\pi/2 + 2k\pi) = 1)$ is 1, while the other subsequence $(x_{m_k} = \sin(2k\pi) = 0)$ has limit equal to 0. $\therefore x_n$ is divergent.

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⑦ 3.4.6

a) $X_n = n^{\frac{1}{n}} \Rightarrow X_{n+1} = (n+1)^{\frac{1}{n+1}}$

$$\frac{X_{n+1}}{X_n} = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = \frac{n^{\frac{1}{n+1}}\left(1+\frac{1}{n}\right)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = \frac{\left(1+\frac{1}{n}\right)^{\frac{1}{n+1}}}{n^{\frac{1}{n(n+1)}}}$$

$$\because \left(1+\frac{1}{n}\right)^n < n \Leftrightarrow \left(1+\frac{1}{n}\right) < n^{\frac{1}{n}} \Leftrightarrow \left(1+\frac{1}{n}\right)^{\frac{1}{n+1}} < n^{\frac{1}{n(n+1)}} \Leftrightarrow \frac{\left(1+\frac{1}{n}\right)^{\frac{1}{n+1}}}{n^{\frac{1}{n(n+1)}}} < 1 \Leftrightarrow \frac{X_{n+1}}{X_n} < 1$$

$X_{n+1} < X_n \quad \therefore X_{n+1} < X_n$ iff $\left(1+\frac{1}{n}\right)^n < n$

Now $\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{1}{n^n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots = e < 3$$

$\therefore X_{n+1} < X_n$ iff $\left(1+\frac{1}{n}\right)^n < n, \forall n \geq 3$

$\{X_n\}$ is monotonically decreasing and bounded above by 2.

$\therefore$ Sequence $\{X_n\}$ is convergent

b) Every subsequence of convergent and converges to the same limit

$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = x$; $\lim_{n\to\infty} (2n)^{\frac{1}{2n}} = x$, but $\lim_{n\to\infty} (2n)^{\frac{1}{2n}} = x \Rightarrow \lim_{n\to\infty} (2n)^{\frac{1}{n}} = x^2 \Rightarrow \lim_{n\to\infty} (2)^{\frac{1}{n}} \lim_{n\to\infty} (n)^{\frac{1}{n}} = x^2$

$\Rightarrow \lim_{n\to\infty} (n)^{\frac{1}{n}} = x^2 \quad \therefore x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$

Since $x_n \geq 1$, $x \neq 0$, must be $x = 1$.