U is a supremum of S

① $s \le u \ \forall s \in S$

② If v is an upper bound of S then $u \le v$.

②' for each $v < u$, there exists $s_v \in S$, such that $v < s_u$

②'' for each $\varepsilon > 0$, there exists $s_\varepsilon \in S$, s.t. $u - \varepsilon < s$
$v = u - \varepsilon$

$\boxed{\cancel{\sup S \in S}}$ not always in the set

Ex: $S = (0,1)$
$\sup S = 1$
$\inf S = 0$
$S = [0,1]$

The Completeness Property of $\mathbb{R}$:

Every nonempty subset of $\mathbb{R}$ that has an upper and lower bound also has a supremum and infinum in $\mathbb{R}$.

Ex: Let S be a nonempty subset of $\mathbb{R}$ and define the shifted S by a as

$$a + S = \{a + s : s \in S\}$$

$(1,2) = 1 + (0,1)$ Then $\sup(a+S) = a + \sup S$

$(-1,0) = (-1) + (0,1)$ Proof: By the completeness property, $\sup S \in \mathbb{R}$.
We need to show $a + \sup S$ is the supremum of $a+S$.

① Show $a + \sup S$ is an upperbound of $a+S$
$\Uparrow$
$(a + s \le a + \sup S \ \forall s \in S)$
$\Uparrow$
$s \le \sup S \ \forall s \in S$
$\Uparrow$
$\sup S$ is the supremum of S,
hence an upper bound of S

② Show that for each upperbound v of $a+S$, $a + \sup S \le v$
Fix an upperbound v of $a+S$

Goal: $a + \sup S \le v$

$\Uparrow$

$\sup S \le v - a$

$\Uparrow$

show that $v-a$ is an upper bound

Since v is an upper bound of $a+S$.
$a+s \le v \ \forall s \in S$, then $s \le v-a \ \forall s \in S$.

$v-a$ is an upper bound of S.
then $\sup S \le v-a$
then $a + \sup S \le v$

So $a + \sup S$ ① the least upperbound of $a+S$
Then $\sup(a+S) = a + \sup S$.

Define the scaled S by $a (a \neq 0)$ as

$$aS = \{as : s \in S\}$$

$$\sup(aS) = \begin{cases} a \sup S & \text{if } a > 0 \\ a \inf S & \text{if } a < 0 \end{cases}$$

Proof "$a<0$" $a \inf S$ is the supremum of aS ($\sup(aS) = a \inf S$)

① $a \inf S$ is an upperbound of aS

$\Downarrow$

$as \le a \inf S \ \forall s \in S$

$\Uparrow$

$s \ge \inf S \ \forall s \in S$

$\Uparrow$

$\inf S$ is a lower bound of S

② $a \inf S$ is the least upper bound of aS

$\Uparrow$

$a \inf S \le v \ \forall$ upper bounds v of S.

$\Uparrow$

$\inf S \ge v/a$ for all upperbounds v of aS.

$\Uparrow$

for each upperbound v of aS, $\frac{v}{a}$ is a lower bound of S.

Fix an upperbound v of aS, then $as \le v \ \forall s \in S$
then $1/a \cdot as \ge \frac{1}{a} \cdot v$, $\forall a \in S$ then $s \ge \frac{v}{a} \ \forall s \in S$, then
v/a is a lower bound of S.

$\boxed{* \text{ if } a<0, \text{ then } \frac{1}{a} < 0}$

Suppose $1/a \ge 0$
then $a \cdot \frac{1}{a} \le a \cdot 0 = 0$

then $1 \le 0$ $\therefore$ contradiction

Ex: Given two nonempty sets A and B of $\mathbb{R}$ if $a \leq b$ $\forall a \in A$ and $b \in B$,

then $\sup A \leq \inf B$

By the Completeness Property $\sup A$ and $\sup B$ are in $\mathbb{R}$.

Every element b in $\mathbb{R}$ is an upper bound of A.

<u>$\sup A \leq b$ for each $b \in B$</u>

By definition $\sup A$ is a lower bound of B.

$\therefore \sup A \leq \inf B$

<u>The Archimedean Property</u>

Theorem: If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that
$$x \leq n_x$$

Proof:

Suppose there is no $n_x \in \mathbb{N}$ such that $\mathbb{N} \leq n_x$.
Then $n < x$ for all $n \in \mathbb{N}$.
By the completeness property $\mathbb{N}$ has the supremum $\sup \mathbb{N} \in \mathbb{R}$.

$\sup \mathbb{N} - 1$ is not an upper bound of $\mathbb{N}$.
Then there exists some $m \in \mathbb{N}$ such that
$$\sup \mathbb{N} - 1 < m.$$
$$\sup \mathbb{N} < m+1$$

Note that $m+1 \in \mathbb{N}$. So $\sup S$ is not an upperbound of $\mathbb{N}$.
This is a contradiction to the definition of supremum.

Corollary: If $S = \{\frac{1}{n}, n \in \mathbb{N}\}$ then $\inf S = 0$

① show that 0 is a lower bound of S

$$0 \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}$$

Suppose $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < 0$.

Since $m > 0$, then $m \cdot \frac{1}{m} < m \cdot 0 = 0$, then $1 < 0$. (Contradiction)

② Let $\varepsilon > 0$, show that ε is not a lower bound of S.

$\Uparrow$

there exists $\frac{1}{n} \in S$ such that $\frac{1}{n} < \varepsilon$

$\Updownarrow$

there exists $n \in \mathbb{N}$ such that $n > 1/\varepsilon$

$\Uparrow$

has to be true by the Archimedean Property

For each $\varepsilon > 0$, $\frac{1}{\varepsilon} \in \mathbb{R}$, by the Archimedean property, $\exists n \in \mathbb{N}$, s.t. $n > \frac{1}{\varepsilon}$, that is $\frac{1}{n} < \varepsilon$

Then ε is not a lower bound of S since $1/n \in S$.

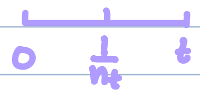
Then 0 is the greatest lower bound of S.

Corollary: If $t > 0$, $\exists n_x \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$

Corollary: If $y > 0$, $\exists n_y \in \mathbb{N}$ s.t.

$n_y - 1 \le y < n_y$

Consider $E = \{m \in \mathbb{N}, y < m\}$



By the Archimedean property, E is non-empty

y is a lower bound of E, then $\inf E$ exists in $\mathbb{R}$

then $\inf E \in E$. Since E is the collection of integers

Let n_a = not E

$n_y - 1 \le y < n_y$