Ex: S = set of all natural numbers less than 20.

$=\{1,3,5,7,9,11,13,15,17,19\}$

$n=10$

Find $\underline{\underline{f}}=\{1,2,\ldots,10\}$ onto S.

$f(n)=2n-1$ a good choice of a bijection

$g(n)=20-(2n-1)=21-2n$ this is another choice

set S is <u>denumerable</u>

Thm: Suppose that S and T are sets and $T\subseteq S$

a) If S is countable, then T is countable
b) If T is uncountable, then S is uncountable.

Thm: Suppose that S and T are sets
a) If one of S and T is countable, then $S\cap T$ is countable. ($S\cap T$ is a subset of that countable set of S and T)
b) If both S and T are countable, then $S\cup T$ is countable.

Proof of b) Case 1 (S and T are denumerable)

$\exists f:\mathbb{N}\mapsto S,\ g:\mathbb{N}\mapsto T$
<u>Goal: Find a bijection $h:\mathbb{N}\mapsto S\cup T$</u>

$$h(n)=\begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd } \frac{n+1}{2} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even } \frac{n}{2}\end{cases}$$

$$\begin{array}{cccc} h(1) & h(2) & h(3) & h(4) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ f(1), & g(1), & f(2), & g(2), \ldots \end{array}$$

This mapping h is onto, but may not be one-to-one

$S\cup T=\underline{(S\setminus T)\cup T}$ is a disjoint union

h is onto (surjection)

Assume $S\setminus T$ is denumerable. $\exists$ a bijection $\tilde{f}: \mathbb{N} \to S\setminus T$. Since T is denumerable $\exists$ a bijection $g: \mathbb{N} \to T$.
Define a mapping $h: \mathbb{N} \to (S\setminus T) \cup T$ as

$$h(n) = \begin{cases} \tilde{f}\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

Then h is onto and one-to-one ($(S\setminus T)\cup T$ is disjoint union).
$(S\cup T)\cup T$ is denumerable that is $S \cup T$ is denumerable

Case 2: (S is finite and T is denumerable).

S has n elements $\exists \underline{\underline{k}} = \{1, 2, \dots, n\}$ onto S
$S \cup T = (S\setminus T) \cup T$ $\qquad \exists \underline{\underline{g}} : \mathbb{N} \mapsto T$

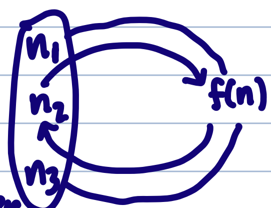
$$h(m) = \begin{cases} k(m) & \text{if } m \le n \\ g(m-n) & \text{if } m > n \end{cases}$$

Theorem: The following statements are equivalent.

a) S is countable (S is denumerable)
b) $\exists$ a surjection of $\mathbb{N}$ onto S
c) $\exists$ an injection of S into $\mathbb{N}$.

a) $\Rightarrow$ b) <u>Simple definition</u> $\qquad \mathbb{N} \underset{g?}{\overset{f}{\rightleftarrows}} S$
b) $\Rightarrow$ c)

If f is a surjection of $\mathbb{N}$ onto S, the let g be a mapping from S into $\mathbb{N}$ by.



$g(s)$ = the least natural number in the set $\underbrace{f^{-1}(\{s\})}_{\text{inverse image}} = \{n \in \mathbb{N} : f(n) = s\}$, for each $s \in S$.

Claim: g is injective
(if $g(s_1) = g(s_2)$, $f(\underbrace{g(s_1)}_{s_1}) = f(\underbrace{g(s_2)}_{s_2}) \Rightarrow s_1 = s_2$ $\therefore$ injection)

c) $\Rightarrow$ a) if f is an injection of S into $\mathbb{N}$, then it is a bijection of S onto $\underbrace{f(S)}_{}$ $\subseteq \mathbb{N}$.
(range or direct image)

Since $\mathbb{N}$ is denumerable, then $f(S)$ as a subset of $\mathbb{N}$, is denumerable
(since we assumed S is infinite)

So f is a bijection from S onto a denumerable set $f(S)$ claim: S is denumerable

$$\begin{tikzcd} \mathbb{N} \arrow[rr, "f^{-1}\circ g"] \arrow[dr, "g"'] & & S \\ & f(S) \arrow[ur, "f^{-1}"'] & \end{tikzcd}$$

Since $f(S)$ is denumerable, then $\exists$ a bijection g from $\mathbb{N}$ onto $f(S)$,
Then $f^{-1}\circ g$ is a bijection from $\mathbb{N}$ onto S.

So S is denumerable.

Thm: $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\right\}$ is denumerable

$$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{R}^+$$

$$\mathbb{Q}^+ = \left\{\frac{m}{n} : m, n \in \mathbb{N}\right\}$$
positive rationals

$\mathbb{Q}^-$ is the set of all negative rationals

It suffices to show $\mathbb{Q}^+$ is denumerable.

$$\begin{bmatrix} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1}, & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2}, & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3}, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$\mathbb{N} \to \mathbb{Q}^+$