Lecture 10

Wednesday, October 4, 2023 9:40 AM

Review: $M = \max\{1+|x|, |x_1|, |x_2|, \ldots\}$

"Backwards Search" Tally

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Algebraic Properties of Sequences

Sum	$X+Y = x_n + y_n$
Difference	$X-Y = x_n - y_n$
Product	$X \cdot Y = x_n \cdot y_n$
Constant	$cX = c(x_n)$
Quotient	$\frac{X}{Y} = \frac{x_n}{y_n}$

*if both X and Y are convergent, then $X+Y \to x+y$, etc.

$X+Y \to x+y$

$\forall \varepsilon > 0, \exists N(\varepsilon) > 0$ s.t. $\forall n \geq N(\varepsilon)$, $|(x_n+y_n)-(x+y)| < \varepsilon$

$|x_n - x| < \frac{\varepsilon}{2}$ $\quad |y_n - y| < \frac{\varepsilon}{2}$

Triangle Inequality: $|(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Proof For each $\varepsilon > 0$, since $X \to x$, $Y \to y$, there exists $N_x(\varepsilon) > 0$, $N_y(\varepsilon) > 0$ such that for all $n \geq N_x(\varepsilon)$, $|x_n - x| < \frac{\varepsilon}{2}$ and for all $n \geq N_y(\varepsilon)$, $|y_n - y| < \frac{\varepsilon}{2}$.

Let $N(\varepsilon) = N_x(\varepsilon) + N_y(\varepsilon)$. For all $n \geq N(\varepsilon)$, we have $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$. So, by the triangle inequality,

$|(x_n + y_n) - (x+y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

$XY \to x \cdot y$

Proof Since X is convergent, it is bounded. Let M_x be a bound such that $|x_n| < M_x$ for all $n \in \mathbb{N}$.

Case 1: $y \neq 0$ For each $\varepsilon > 0$, there exists $N_x(\varepsilon) > 0$ and $N_y(\varepsilon) > 0$ such that for all $n \geq N_x(\varepsilon)$, $|x_n - x| < \frac{\varepsilon}{2|y|}$, and for all $n \geq N_y(\varepsilon)$, $|y_n - y| < \frac{\varepsilon}{2M_x}$.

Let $N(\varepsilon) = N_x(\varepsilon) + N_y(\varepsilon)$. For all $n \geq N(\varepsilon)$, we have

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq M_x |y_n - y| + |y| \cdot |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Case 2: $y = 0$ For each $\varepsilon > 0$, there exists $N_T(\varepsilon) > 0$ such that

Case 2: $y=0$ For each $\varepsilon>0$, there exists $N_T(\varepsilon)>0$ such that for all $n \geq N_T(\varepsilon)$, $|y_n - y| < \frac{\varepsilon}{M_x}$. Let $N(\varepsilon) = N_T(\varepsilon)$. For all $n \geq N(\varepsilon)$:

$$|x_n y_n - xy| = |x_n y_n| = |x_n y_n - x_n y|$$
$$= x_n |y_n - y| < M_x |y_n - y| < \varepsilon. \quad \blacksquare$$

$\frac{1}{Y} \to \frac{1}{y}$

Proof For $\varepsilon = \frac{|y|}{2}$, there exists $N\left(\frac{|y|}{2}\right)$ such that for all $n \geq N\left(\frac{|y|}{2}\right)$, $|y_n - y| < \frac{|y|}{2}$. So, $|y_n| \geq |y| - \frac{|y|}{2} = \frac{|y|}{2} > 0$

this guarantees $|y_n|$ is bounded below by a positive number

For all $\varepsilon>0$, there exists an $N_y(\varepsilon)>0$ such that for all $n \geq N_y(\varepsilon)$, $|y_n - y| < \frac{\varepsilon}{\frac{2}{y^2}}$.

Let $N(\varepsilon) = N\left(\frac{|y|}{2}\right) + N_y(\varepsilon)$. For all $n \geq N(\varepsilon)$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{1}{|y_n|\cdot|y|} \cdot |y_n - y| \leq \frac{2}{y^2}|y_n - y| < \varepsilon. \quad \square$$

Theorem If $X=(x_n)$ and $Y=(y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n \; \forall n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$

Theorem If $X=(x_n)$ is a convergent sequence and if $a \leq x_n \leq b \;\; \forall n \in \mathbb{N}$, then $a \leq \lim(x_n) \leq b$.

Proof Let Y be the constant sequence $(b, b, b, \ldots)$. By the previous theorem, we know that $\lim(x_n) \leq \lim(y_n) = b$. Similarly, let Z be the constant sequence $Z = (a, a, a, \ldots)$. We then have $\lim(z_n) = a \leq \lim(x_n)$. Thus, $a \leq \lim(x_n) \leq b$.

I think Kang used contradiction for this proof but I kind of zoned out for a bit, so this is copied directly from the textbook

Squeeze Theorem

Suppose X, Y, Z are sequences such that $x_n \leq y_n \leq z_n$ ($\forall n \in \mathbb{N}$).

If $\lim x_n = x$ and $\lim z_n = x$, then $\lim y_n = x$.

To prove this, we need $|y_n - y| < \varepsilon$

$\Downarrow$

$x - \varepsilon < y_n < x + \varepsilon$

$\Downarrow$

$|x_n - x| < \varepsilon$

$|z_n - z| < \varepsilon$

$$x-\varepsilon < y_n < x+\varepsilon$$

$$\Downarrow$$

$$x-\varepsilon < x_n \le y_n \le z_n < x+\varepsilon$$

<u>Proof</u> For each $\varepsilon > 0$, since $x_n \to x$ and $z_n \to x$, there exists $N_x(\varepsilon), N_z(\varepsilon) > 0$ such that for all $n \ge N_x(\varepsilon)$, $|x_n - x| < \varepsilon$ and for all $n \ge N_z(\varepsilon)$, $|z_n - x| < \varepsilon$.

his shoes are so good

Then $x_n > x-\varepsilon$ and $z_n < x+\varepsilon$.

Let $N(\varepsilon) = N_x(\varepsilon) + N_z(\varepsilon)$. Then $x-\varepsilon < x_n \le y_n \le z_n < x+\varepsilon$.

So $x-\varepsilon < y_n < x+\varepsilon$, then $|y_n - x| < \varepsilon$.

<u>Theorem</u> If $x_n \to x$, then $|x_n| \to |x|$.

<u>Proof</u> Since $x_n \to x$, for each $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that for all $n \ge N(\varepsilon)$, $|x_n - x| < \varepsilon$. Then

$$|x_n| - |x| \le |x_n - x| \le \varepsilon$$

<u>Theorem</u> If $x_n \to x$ and $x_n \ge 0$, then $\sqrt{x_n} \to \sqrt{x}$.

Consider 2 cases: $x = 0$ and $x > 0$ (safe to assume that $x \not< 0$)

For $x > 0$, use conjugate of $\sqrt{x_n} - \sqrt{x}$ to prove

$$\sqrt{x_n} < \varepsilon$$

Ratio Test Next time