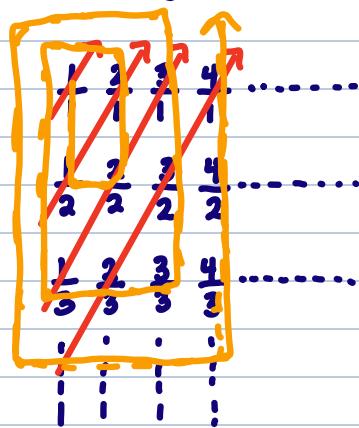


$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}$$

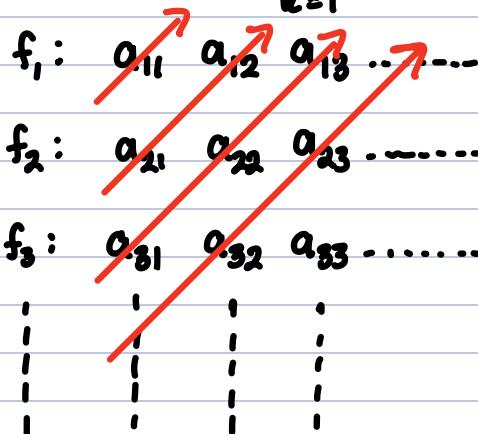
$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$



Use "arrows" or "spiral"

Theorem: If  $A_n$  is a countable set for each  $n \in \mathbb{N}$ ,

then  $A = \bigcup_{k=1}^{\infty} A_k$  is countable.



We have a surjective mapping for  $\mathbb{N}$  onto  $A = \bigcup_{n=1}^{\infty} A_n$

$$h(1) = a_{11}$$

$$h(2) = a_{21}$$

$$h(3) = a_{12}$$

$f_1, f_2, \dots$  are bijections from  $\mathbb{N}$  onto  $A$   
 $g_1, g_2, \dots$  are bijections from  $\mathbb{N}$  onto  $B$

$A \cup B$

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right), & n \text{ is even} \\ g\left(\frac{n+1}{2}\right), & n \text{ is odd} \end{cases}$$

$f: b_1, b_2, b_3, \dots$       45° counting  
 $g: a_1, a_2, a_3, \dots$

$$h(1) = b_1 = g(1)$$

$$h(2) = a_1 = f\left(\frac{2}{2}\right)$$

$$h(3) = b_2 = g\left(\frac{3+1}{2}\right)$$

## Algebraic Properties of $\mathbb{R}$ .

Two binary operations, denoted by "+" and " $\cdot$ ", and defined in  $\mathbb{R}$ , called "addition" and "multiplication", respectively.

A1)  $a+b=b+a$  (commutativity property of addition)

A2)  $(a+b)+c=a+(b+c)$  (associative property of addition)

A3) There exists an element  $0$  in  $\mathbb{R}$  s.t.  $a+0=a \forall a \in \mathbb{R}$ .

A4) For each  $a \in \mathbb{R}$ , there exists an element  $(-a)$  in  $\mathbb{R}$  such that  $a+(-a)=0$

M1)  $a \cdot b = b \cdot a$

M2)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M3) There exists an element  $1$  in  $\mathbb{R}$  such that  $a \cdot 1=a \forall a \in \mathbb{R}$

M4) For each  $a \neq 0$  in  $\mathbb{R}$ , there exists one element  $\frac{1}{a}$  in  $\mathbb{R}$  s.t.  $a \cdot \frac{1}{a}=1$

D)  $a \cdot (b+c) = a \cdot b + a \cdot c$  (Distributive Property)

Example: a) if  $z, a \in \mathbb{R}$  with  $z+a=a$ , then  $z=0$  (rule of cancellation)

Since  $z+a=a$  by A4) on  $a$ , there exists  $-a$  in  $\mathbb{R}$ , such that  $a+(-a)=0$ .

$$\frac{(z+a)+(-a)=a+(-a)=0}{\text{A2 II} \quad \text{A4} \quad \text{A4}}$$

$$z+(a+(-a))$$

II (A4)

$$z+0$$

II (A3)

$$\underline{\underline{z}} =$$

b) Similarly, if  $u$  is  $b \neq 0$  are in  $\mathbb{R}$  with  $u \cdot b=b$ , then  $u=1$ .

$$(u \cdot b) \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

II M4

$b \cdot \frac{1}{b} = 1$

M2

$$u \cdot (b \cdot \frac{1}{b})$$

II M4

$$u \cdot 1$$

II M3

$$u$$

Example c) If  $a \in \mathbb{R}$ , then  $a \cdot 0 = 0$

Proof:

$$\begin{aligned} a + a \cdot 0 &= a \cdot 1 + a \cdot 0 && \text{let's say } 0+0=0 \\ &\stackrel{M3}{=} a \cdot (1+0) && a(0+0)=a \cdot 0 \\ &\stackrel{D}{=} a \cdot 1 && a \cdot 0 + a \cdot 0 = a \cdot 0 \\ &\stackrel{A3}{=} a && \downarrow \\ &\stackrel{M3}{=} a \end{aligned}$$

Example a),  $a \cdot 0 = 0$

d) If  $a \neq 0$ , and  $b \in \mathbb{R}$  are such that  $a \cdot b = 1$ , then  $b = \frac{1}{a}$ .

Proof: Since  $a \neq 0$ , by M4),  $\exists \frac{1}{a} \in \mathbb{R}$  such that  $a \cdot \frac{1}{a} = 1$

$$\underline{\frac{1}{a} \cdot (a \cdot b)} = \underline{\frac{1}{a} \cdot 1} \stackrel{M3}{=} \underline{\frac{1}{a}}$$

$\parallel M2$

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

$\parallel M4$

$$\begin{aligned} &1 \cdot b \\ &\parallel M3 \\ &b \end{aligned}$$

e) If  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$

Suppose  $a \neq 0$  (Goal  $b = 0$ )

$$\frac{1}{a} (a \cdot b) = \frac{1}{a} \cdot 0 = 0$$

$\parallel M2 \quad \text{Ex c)}$

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

$$\begin{aligned} &\parallel M4 \\ &1 \cdot b \\ &\parallel M3 \\ &b \end{aligned}$$

$$\text{Ex: } -a = (-1) \cdot a$$

$$(-1) = (-1) + 0$$

$$(-1) = (-1) + a + (-a)$$

$$a + (-a) = 0 = a \cdot 0 \quad M3$$

$$A4 \quad = a \cdot (1 + (-1)) \quad A4$$

$$= a \cdot 1 + a \cdot (-1) \quad D$$

$$= a + a(-1) \quad M3$$

$$\underline{-a = a \cdot (-1)}$$

Theorem: There doesn't exist a rational number  $r$  such that  $r^2 = 2$ .