hw 4

2.5.2

$\boxed{\Rightarrow}$ If S is bounded, then there exists $u = \sup(S)$ and $v = \inf(S)$.
We can define interval $I = [u,v]$ and S is contained in I.

$\boxed{\Leftarrow}$ If there exists an interval I such that $S \subseteq I = [a,b]$ then $a \leq s$ for all $s \in S$ and likewise, $b \geq s$ for all $s \in S$. Since S is bounded above and below, S, by definition, is bounded. ∎

2.5.9

Assume there is an element, x in the infinite intersection. Then by the Archimedean Property, there exists $k \in \mathbb{N}$ such that $x < k$. This means that $x \notin I_k$ for any $n \geq k$. Hence, x cannot be in the infinite intersection. ∎

3.1.5a

We are given $\frac{n}{n^2+1}$

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}$$

Let $\varepsilon > 0$ be any number and let us choose N so that

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \varepsilon, \text{ whenever } n > N$$

Thus, the inequality holds if we choose $N > 1/\varepsilon$

3.1.5d

We are given $\frac{n^2-1}{2n^2+3}$

Let $\varepsilon > 0$ be any number and let us choose N so that

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2-(2n^2+3)}{4n^2+6}\right| = \left|\frac{\cancel{2n^2}-2-\cancel{2n^2}-3}{4n^2+6}\right| = \left|\frac{-5}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2} \leq \frac{5}{4n}$$

Thus the inequality holds if we choose $N > 5/4\varepsilon$.

3.1.8

$\boxed{\Rightarrow}$

Assume $\lim(x_n) = 0$, which means for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$, $|x_n - 0| < \varepsilon$.
Since $|x_n| = |x_n - 0|$, it follows directly that $|x_n| < \varepsilon, \forall n \geq N$, which means $\lim(|x_n|) = 0$.

$\boxed{\Leftarrow}$

Assume $\lim(|x_n|) = 0$, which means for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$, $||x_n| - 0| < \varepsilon$.
Since $||x_n| - 0| = |x_n|$, it follows that for all $n \geq N$, $|x_n| < \varepsilon$, but $|x_n| < \varepsilon$ is the same as saying $|x_n - 0| < \varepsilon$, which means $\lim(x_n) = 0$.

$\therefore \lim(x_n) = 0 \Leftrightarrow \lim(|x_n|) = 0$

A counterexample would be to use $(-1)^n$ here the convergence of $|(-1)^n| = 1$ and $(-1)^n$ does not converge.

3.1.9

Given that $x_n \geq 0$ for all n and $\lim\limits_{n\to\infty} x_n = 0$, we want to show $\lim\limits_{n\to\infty} \sqrt{x_n} = 0$.
We know from the properties of limits and the fact that the square root function is continuous that if $\lim\limits_{n\to\infty} x_n = L$, then $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{L}$. Since it is given that $\lim\limits_{n\to\infty} x_n = 0$, by substituting $L = 0$, we get

$\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{0} = 0$. Hence, we have shown that if $x_n \to 0$ $\forall n \in \mathbb{N}$ that $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.