Absolute Values

Definition: The absolute value of $a \in \mathbb{R}$ is definitely

$$|a| \equiv \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

$|a| = 0$ if and only if $a = 0$.

$$0 \overset{A4}{=} \underbrace{a - a} = 0 + a \overset{A3}{=} a$$

Properties 1.1:

1. $|ab| = |a| \cdot |b|$

Case $a > 0, b > 0$, $|a \cdot b| = ab$

$|a| = a$, $|b| = 0$, then $|ab| = |a||b|$

Others can be prove similarly

2. $|a|^2 = a^2 \quad \forall a \in \mathbb{R}$

From (1) when $a = b$, $|a^2| = |a|^2$

When $a = 0$, $a^2 = 0$, then $|a^2| = 0$.

$|a| = 0$, so $|a|^2 = 0$

when $a \neq 0$, $a^2 > 0$, then $|a|^2 = |a^2| = a^2$

3. If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

Proof: "$\Rightarrow$" $|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases} \leq c$

Then $a \leq c$, and $\boxed{-a \leq c}$

$\Leftrightarrow -c \leq a$

Then $-c \leq a \leq c$

4. $-|a| \leq a \leq |a| \quad \forall a \in \mathbb{R}$

Let $c = |a| \geq 0$, $|a| \leq |a|$ if and only if $-|a| \leq a \leq |a|$

<u>by 3</u>

Since $|a| = |a|$, then $|a| \leq |a|$

⑤ Triangle Inequality:

$|a+b| \le |a| + |b|$

Let $c = |a| + |b| \ge 0$. So $|a+b| \le |a|+|b|$ is equivalent to $-(|a|+|b|) \le a+b \le |a|+|b|$

$-|a| \le a \le |a|$
$-|b| \le b \le |b|$

If $a \le b$ and $c \le d$, then $a+c \le b+d$

$$\left.\begin{array}{l} a \le b \Rightarrow a+c \le b+c \\ c \le d \Rightarrow b+c \le b+d \end{array}\right\} \Rightarrow a+c \le b+d$$

$-|a| \le a \le |a|$
$-|b| \le b \le |b|$
$-|a|-|b| \le a+b$
$-|a|-|b| \le a+b$
$-1(|a|+|b|)$
$\quad \shortparallel$
$-(|a|+|b|)$

⑥ If $a,b \in \mathbb{R}$, then $|a-b| \le |a|+|b|$ and $\big||a|-|b|\big| \le |a-b|$

$|a-b| = |a+(-b)| \le |a|+|-b|$

$$|-b| = \begin{cases} -b & \text{if } -b>0 \Leftrightarrow b<0 \\ 0 & \text{if } -b=0 \Leftrightarrow b=0 \\ \underset{-(-b)}{b} & \text{if } -b<0 \Leftrightarrow b>0 \end{cases} = |b|$$

$-(-b) = (-1)(-1)\cdot b = 1 \cdot b = b$

$\big||a|-|b|\big| \le |a-b|$

$\Leftrightarrow -|a-b| \le |a|-|b| \le |a-b|$ by ③ with $c = |a|-|b|$
$c = |a-b|$

for $|a|-|b| \le |a-b|$, note that it is the same as
$|a| \le |b| + |a-b|$

$|a| = |a-b+b| \le |b| + |a-b|$

for $-|a-b| \le |a|-|b| \Leftrightarrow |b| \le |a| + |a-b|$
$b = b-a+a$
$\boxed{|b| \le |b-a| + |a|}$

$|a-b| = |-(b-a)|$
$\quad = |b-a|$

7. $|a_1+a_2+\cdots+a_n| \le |a_1|+|a_2|+\cdots+|a_n|$

Section 2.3: The Completeness Property of $\mathbb{R}$.

a) The set S is said to be bounded above. If there exists a number $u \in \mathbb{R}$ such that $s \le u$ for all $s \in S$.

(1,2) Each such number u is called <u>an upper bound</u> that S.

b) The set S is said to be <u>bounded below</u>. If there exists a number $w \in \mathbb{R}$ s.t. $s \ge w$ $\forall s \in S$.
Each such number w is called a lower bound of S.

c) S is said to be <u>bounded</u> if it is bounded above and below. Otherwise, it is unbounded.

Definition Let S be a non-empty subset of $\mathbb{R}$.

a) If S is bounded above, then a number u is said to be a <u>supremum</u>. (or a least upper bound) of S

If it satisfies the two condition:

1. U is an upper bound of S. ($s \le u$ $\forall s \in S$)

2. If u is any upper bound of S, then $u \le v$.
(any number less than u is not an upperbound)

b) If S is bounded below, then a number w is said to be an infimum (or greatest lower bound) of S.
If it satisfies two conditions

1. w is c lower bounded of S
2. If t is a lower bounded of S, then $t \le w$.

These can be only one supremum (infimum) of a given subset S of $\mathbb{R}$.

Suppose there are two (smallest) lowest upper bound of S, called U_1, U_2.

$$\begin{matrix} U_1 \le U_2 \\ U_2 \le U_1 \end{matrix} \Rightarrow U_1 = U_2$$

If the supremum of S exists, it is denoted by $\sup S$.
If the infimum of S exists, it is denoted by $\inf S$.

Lemma: A number u is the supremum if a non-empty subset S of $\mathbb{R}$, if and only if

① $s \leq u \quad \forall s \in S$
(u is an upper bound of S)
② if $v < u$, there exists $s' \in S$
such that $v < s'$