

# MATH 301: Homework 2

Aren Vista

December 27, 2025

## Question 1

Prove Theorem 1.3.4 (b)

If a set  $A$  with  $m \in \mathbb{N}$  elements and  $C \subseteq A$  is a set with one element then  $A - C$  is a set with  $m - 1$  elements

*Proof.* Suppose a set  $A$  with  $m \in \mathbb{N}$  elements

$$A = \{a_1, a_2, \dots, a_{m-1}, a_m\} \implies |A| = m$$

Suppose  $C \subseteq A$  is a set with one element

$$C = \{a\}, a \in A$$

Observe  $a$  is an arbitrary element of  $A$

WLOG let  $a = a_m$  s.t.

$$C = \{a_m\}$$

Thus

$$A - C = \{a_1, a_2, \dots, a_{m-1}\} \implies |A - C| = m - 1$$

□

## Question 2

Prove Theorem 1.3.4 (c)

If  $C$  is an infinite set and  $B$  is a finite set

Then  $C - B$  is an infinite set

Recall an infinite set is a set which is not finite (defn.)

*Proof.* Suppose  $C$  is an infinite set

$$C = \{c_1, c_2, \dots\}$$

Suppose  $B$  is a finite set

$$B = \{b_1, b_2, \dots, b_n\}, n \in \mathbb{N}$$

Suppose for the sake of contradiction  $C - B$  is finite s.t

$$C - B = \{x_1, x_2, \dots, x_m\} m \in \mathbb{N}$$

Observe at most  $C \cap B$  must be finite ( $n$  number of elements) by definition of set intersection

$$C \cap B = \{b_1, b_2, \dots, b_n\}$$

By rearrangement of the principle of exclusion

$$C = C - B \cup (C \cap B)$$

If  $C - B$  is finite and  $C \cap B$  is finite then  $C$  must be finite. This is a logical inconsistency.

Thus,  $C - B$  must be infinite.

□

### Question 3

Prove that if  $x$  is a rational number and  $y$  is an irrational number, then  $x + y$  is an irrational number.

If in addition  $x \neq 0$  then show that  $xy$  is an irrational number.

*Proof.* Suppose  $x$  is a rational number and  $y$  is an irrational number

Suppose for the sake of contradiction  $x + y$  is a rational number.

Then given  $m, n \in \mathbb{Z}$

$$x + y = \frac{m}{n}$$

By defn.  $x, k, p \in \mathbb{Q} \implies x = \frac{k}{p}$

Thus

$$\begin{aligned} \frac{k}{p} + y &= \frac{m}{n} \\ y &= \frac{m}{n} - \frac{k}{p} \\ y &= \frac{mp}{np} - \frac{kn}{pn} \\ y &= \frac{mp - kn}{np} \end{aligned}$$

By set inclusion of integers,  $mp - kn, np \in \mathbb{Z}$ . This would imply  $y$  is rational.

This is a logical inconsistency, thus  $x + y$  is irrational.  $\square$

### Question 4

Modify the proof of Theorem 2.1.4

Show that there does not exist a rational number  $t$  so that  $t^2 = 3$

*Proof.* Let  $t \in \mathbb{Q}$

Suppose for the sake of contradiction  $t^2 = 3$

Let  $x, y \in \mathbb{Z}$ ,  $y$  defn of  $t = \frac{x}{y}$ , where  $x, y$  are the lowest possible factors of  $t$

Thus

$$\begin{aligned} \left(\frac{x}{y}\right)^2 &= 3 \\ \frac{x^2}{y^2} &= 3 \\ x^2 &= 3y^2 \end{aligned}$$

Notice  $x^2$  is a multiple of 3

By properties of prime

$$3|x^2 \implies 3|x$$

Thus for some  $k \in \mathbb{Z}$

$$x^2 = (3k)^2 = 3y^2$$

Thus  $3k = y^2$

This concludes  $y|3$  and  $x|3$ .

Note, we stated  $x, y$  are the lowest possible factors of  $t$  implying there does not exist any more common factors which could reduce  $x, y$

This is a logical inconsistency, thus there can not exist a  $t \in \mathbb{Q}$  s.t.  $t^2 = 3$

□

## Question 5

Recall the **Triangle Inequality**

If  $a, b \in \mathbb{R}$

Then  $|a + b| \leq |a| + |b|$

Prove this inequality holds iff  $ab \geq 0$

Statement P:  $|a + b| \leq |a| + |b| \iff ab \geq 0$

**Case 1:**  $a$  and  $b$  have the same sign

$a$  and  $b$  have the same sign  $\implies ab \geq 0$ : Observe

$a$  and  $b$  are positive

$$\implies |a + b| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$$

$a$  and  $b$  are negative

$$\implies |(-a) + (-b)| = |-(a + b)| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$$

**Case 2:**  $a = 0 \vee b = 0$

$a = 0 \vee b = 0 \implies ab \geq 0$  Observe

If  $a = 0$

$$\implies |0 + b| = |b| \equiv b \iff b \leq 0 + |b| \iff |a + b| \leq |a| + |b|$$

If  $b = 0$

$$\implies |0 + a| = |a| \equiv a \iff a \leq 0 + |a| \iff |a + b| \leq |a| + |b|$$

## Conclusion

Thus By Case 1,2  $|a + b| \leq |a| + |b| \iff ab \geq 0$  holds

## Question 6

Show that if  $a, b \in \mathbb{R}$  and  $a \neq b$

Then there exists  $\epsilon$ -neighborhoods  $U$  of  $a$  and  $V$  of  $b$  such that  $U \cap V = \emptyset$