Lecture 20 - Sample Test 2

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DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 301 Intro to Real Analysis
Fall 2023, KANG

Sample Test 2 NAME : ______________________

1. (a) Use the ϵ-δ definition of limit to establish the following limit.

$$\lim_{x\to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}.$$

(b) Show that the following limit do not exist

$$\lim_{x\to 0} \sin(4/x^2)$$

2. Find the following limit

$$\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x + 2x^2} \text{ where } x > 0.$$

3. Let (a_n) be an increasing sequence of real numbers.

   (a) If (a_n) has a bounded subsequence, show that (a_n) is itself bounded.

   (b) If (a_n) has a convergent subsequence, show that (a_n) is itself convergent.

4. Show that if $\lim(a_n/n) = L$, where $L > 0$, then $\lim(a_n) = +\infty$.

5. Let the sequence (x_n) be recursively defined by $x_1 > 0$ and

$$x_{n+1} = (2 + x_n)^{-1}, \; n \geq 1.$$

Show that (x_n) is a contractive sequence and evaluate its limit.

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3. (a_n) is increasing

a) If a_n has a bounded subsequence, then (a_n) is bounded

Proof Let (a_{n_k}) be a bounded subsequence of (a_n). Let M be an upper bound of (a_{n_k}). We want to show that M is an upper bound of (a_n); that is, $a_n \leq M \;\; \forall n \in \mathbb{N}$.

Let n be any natural number $(x_n \leq M)$. For this n, there exists $k \in \mathbb{N}$ such that $n < n_k$ since (n_k) is a strictly increasing sequence of natural numbers. Since (a_{n_k}) is increasing, $a_n \le a_{n_k}$ and $a_{n_k} \le M$, so $a_n \le M$. ■

b) If (a_n) has a convergent subsequence, show that (a_n) itself is convergent.

Proof

Use Monotone Convergence Theorem
+ part a)

4) $\lim\left(\frac{a_n}{n}\right) = L$, $L > 0$, then $\lim(a_n) = +\infty$

By the A.P., $(n) \to \infty$ as $n \to \infty$

Because $\forall \alpha > 0$, $\exists N(\alpha)$ such that $x_n > \alpha$ $\forall n \ge N(\alpha)$

So, $n > \alpha \Rightarrow N(\alpha) > \alpha$

Afterwards, show using ratio or comparison test

What if we have n^2?

$\forall \alpha > 0$, $\exists N(\alpha)$ such that $x_n > \alpha$ $\forall n \ge N(\alpha)$

So, $n^2 > \alpha \Rightarrow N(\alpha) > \sqrt{\alpha}$

5) Show (x_n) is contractive and find the limit.

For problems like this, we can also use the Monotone Convergence Theorem

$x_1 > 0$, $x_{n+1} = \frac{1}{2 + x_n}$

$|x_{n+1} - x_n| < c\,|x_n - x_{n-1}|$ where $0 < c < 1$

$$\left|\frac{1}{2+x_n} - \frac{1}{2+x_{n-1}}\right| = \left|\frac{x_{n-1} - x_n}{(2+x_n)(2+x_{n-1})}\right| = \left|\frac{1}{(2+x_n)(2+x_{n-1})}\right| \cdot |x_n - x_{n-1}|$$

So, $\{x_n\}$ is contractive, which implies Cauchy which implies convergent

Let x be its limit. $x = \frac{1}{2+x}$, then $x^2 + 2x - 1 = 0$

Quadratic Formula: $\frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

$x_1 = -1 + \sqrt{2}$, $x_2 = -1 - \sqrt{2}$

Not this because negative

So $\lim = x_1 = -1 + \sqrt{2}$