Spring 2024 Real Analysis Final Exam Questions/Topics:

9. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

14. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x+y) = g(x)\,g(y)$ for all x, y in $\mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

3. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)| \le \frac{1}{2}|f(x)|$. Prove there exists a point c in I such that $f(c) = 0$.

4. Show that the function $f(x) := 1/(1+x^2)$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

13. Given any $x \in \mathbb{R}$, show that there exists a *unique* $n \in \mathbb{Z}$ such that $n - 1 \le x < n$.

11. Use the definition of limit to prove the following.

(a) $\displaystyle\lim_{x \to 3} \frac{2x+3}{4x-9} = 3,$ (b) $\displaystyle\lim_{x \to 6} \frac{x^2 - 3x}{x+3} = 2.$

1. Show that if (x_n) is an unbounded sequence, then there exists a properly divergent subsequence.

Given the sequence defined by the recurrence relation $v_{n+2} = \frac{1}{2}(v_n + v_{n+1})$ with initial conditions $v_1 = 1$ and $v_2 = 2$, we aim to show that the sequence is bounded by $1 \le v_n \le 2$ for all n.