Quiz 2 Review (3.1–3.4)

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Review of stuff from before:

- Triangle Inequality: $|a+b| \leq |a|+|b|$
- Archimedean Property: For all $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $x \leq n$

* It is worth noting that I am very lazy. So, please assume that all sequences are sequences of real numbers.

Section 3.1

Def A sequence of real numbers is a function on $\mathbb{N}$ whose range is in $\mathbb{R}$

Limit Def: $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $|x_n - x| < \varepsilon$.

- If x exists, x_n converges to x; otherwise, (x_n) diverges.
- This also implies $x - \varepsilon < x_n < x + \varepsilon$.
- Limits are unique

Ex $\lim\left(\frac{1}{n}\right) = 0$

Proof: Let $\varepsilon > 0$ be given. Then $\frac{1}{\varepsilon} > 0$, and by the Archimedean Property, there exists a natural number $N(\varepsilon)$ such that $\frac{1}{N(\varepsilon)} < \varepsilon$. Then, if $n \geq N(\varepsilon)$, we have $\frac{1}{n} \leq \frac{1}{N(\varepsilon)} < \varepsilon$, so $\frac{1}{n} < \varepsilon$. Thus, we have

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

So, the sequence converges to 0. ∎

Def Let X be a sequence of real numbers and $m \in \mathbb{N}$.
The <u>m-tail</u> of X is $X_m := (x_{m+n} : n \in \mathbb{N})$

Basically the m-tail of a sequence is the sequence itself but starting somewhere else

The m-tail of a sequence converges if and only if the sequence also converges, and $\lim(X_m) = \lim(X)$ by some fancy theorem

Section 3.2

Def A sequence is said to be <u>bounded</u> if there exists a real number $M > 0$ such that $|x_n| \leq M \quad \forall n \in \mathbb{N}$

A convergent sequence is bounded

Some other properties
(x_n converges to x and y_n converges to y)

Sequence	Converges to
$x_n + y_n$	$x + y$
$x_n - y_n$	$x - y$
$x_n \cdot y_n$	$x \cdot y$
$\frac{x_n}{y_n}$	$\frac{x}{y}$

More Limit Theorems

- If x_n and y_n are convergent sequences and if $x_n \leq y_n \ \forall n \in \mathbb{N}$, then

$$\lim(x_n) \leq \lim(y_n)$$

- Squeeze Theorem: Suppose that x_n, y_n, and z_n are sequences such that

$$X_n \leq Y_n \leq Z_n$$

If $\lim(x_n) = \lim(z_n)$, then y_n is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

- If x_n converges to x, then $\sqrt{x_n}$ converges to $\sqrt{x}$

$$\lim(x_n) = x \implies \lim(\sqrt{x_n}) = \sqrt{x}$$

- If x_n converges to x, then $|x_n|$ converges to $|x|$

$$\lim(x_n) = x \implies \lim(|x_n|) = |x|$$

Ex) Prove that (n) is divergent.

Proof: Suppose, by way of contradiction, that (n) is convergent. Then by Theorem 3.2.2, it must also be bounded, so there exists an $M \in \mathbb{R}$ such that $|n| < M \ \forall n \in \mathbb{N}$. However, this violates the Archimedean Property. Thus we have reached a contradiction, and (n) is divergent. ■

★ will do another example with Squeeze Theorem

Section 3.3

A sequence is monotone if it is either increasing or decreasing.

Monotone Convergence Theorem – A monotone sequence is convergent if and only if it is bounded

If x_n is bounded increasing: $\lim x_n = \sup(x_n)$

If x_n is bounded decreasing: $\lim x_n = \inf(x_n)$

Section 3.4

Subsequences:

Subsequences of convergent sequences converge to the same value as the sequence

Negation of limit def: $\exists\ \varepsilon > 0$ such that $\forall\ N(\varepsilon) \in \mathbb{N}$, $\exists\ n \in \mathbb{N}$ such that $n \geq N(\varepsilon)$ implies $|x_n - x| \geq \varepsilon$.

A sequence is divergent if one of the following is true

- x_n has two subsequences that converge to different values

  Ex: $(-1)^n$

  $(-1)^{2n} \to 1,\ (-1)^{2n-1} \to -1$

- x_n is unbounded

Monotone Subsequence Theorem

Every sequence of real numbers has a monotone convergent subsequence

Bolzano - Weierstrass Theorem

Every bounded sequence has a convergent subsequence

If every subsequence of a sequence converges to a certain value (say x), then the sequence also converges to x.

Defs: $\lim\sup(x_n)$ or $\overline{\lim}(x_n)$

The infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$

$\lim\inf(x_n)$ or $\underline{\lim}(x_n)$

The supremum of the set W of $w \in \mathbb{R}$ such that $w > x_n$ for at most a finite number of $n \in \mathbb{N}$

A sequence x_n is bounded if and only if $\lim\sup(x_n) = \lim\inf(x_n)$