

## Lecture 6

Wednesday, September 20, 2023 10:50 AM

Fuck fuck fuck fuck fuck

Corollary 1 If  $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ , then  $\inf S = 0$ .

Notes:  $\frac{1}{n} > 0$

Suppose  $\frac{1}{n} \leq 0$

$$n \cdot \frac{1}{n} \leq 0 \Rightarrow 1 < 0 \quad \text{contradiction!}$$

$$\frac{1}{\inf S} < n$$

Proof Note that 0 is a lower bound of  $S$ , so  $\inf S \geq 0$ .

Contradiction strikes again

Suppose  $\inf S > 0$

By the Archimedean Property, for  $\frac{1}{\inf S} \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  such that  $\frac{1}{\inf S} < n$ . Since  $\inf S > 0$  and  $n > 0$ , by the order properties,  $\frac{1}{n} < \inf S$ . Since  $\frac{1}{n} \in S$ , this is a contradiction!  $\square$

Corollary 2 If  $t > 0$ , there exists an  $n_t \in \mathbb{N}$  such that  $0 < \frac{1}{n_t} < t$   
( $t$  is not a lower bound)

Two ways to prove it:

- Archimedean Property:  $\frac{1}{n_t} < t \Rightarrow \frac{1}{t} < n_t$
- Since  $\inf S = 0$  by the previous corollary and  $t > 0$ , then  $t$  is not a lower bound of  $S$ . Then there exists some element  $\frac{1}{n_t}$  in  $S$  that is less than  $t$ .

Corollary 3 If  $y > 0$ , there exists  $n_y \in \mathbb{N}$  such that  $n_{y-1} \leq y \leq n_y$ .

Proof Consider  $E = \{m \in \mathbb{N}, y < m\}$  and let  $n_y$  be the smallest element in  $E$ . Since  $n_y \in E$ , then  $y < n_y$ . Since  $n_y$  is the smallest element in  $E$ , then  $n_{y-1} \notin E$ . So,  $y < n_{y-1}$  is not true; that is,  $n_{y-1} \leq y$   $\square$

PIGEON

## Showing the existence of an irrational number $\sqrt{2}$

We want to find a positive solution for  $x^2 = 2$ .

We use  $\sup S$  where  $x^2 < 2$

Proof Let  $S = \{s \in \mathbb{R} : s > 0 \text{ and } s^2 < 2\}$

Since  $1 \in S$ , then  $S \neq \emptyset$

Also,  $S$  is bounded above by 2.

By the Completeness Property,  $x = \sup S$  exists in  $\mathbb{R}$ .

Rule out  $x^2 < 2$  and  $x^2 > 2$  to get  $x^2 = 2$

"you should think  
a little bit!" - Kang

To do this, we use Kang's favorite proof technique, contradiction

$x^2 < 2$  Suppose  $x^2 < 2$ . Then there exists an  $n > 0$  such that

$x + \frac{1}{n} \in S$ . Then  $x$  is not an upper bound, hence

$x \neq \sup S$ . Contradiction!