MATH 301: Quiz l

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Question 1

Theorem: If S and T are denumerable, disjoint sets, then $S \cup T$ is denumerable.

Proof. Suppose S and T are denumerable. By definition, there exist bijective functions:

$$f : \mathbb{N} \to S \quad \text{and} \quad g : \mathbb{N} \to T$$

To show $S \cup T$ is denumerable, we must construct a bijection $h : \mathbb{N} \to S \cup T$.

Define $h : \mathbb{N} \to S \cup T$ as:

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ g\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

Clearly S, T must be disjoint; otherwise $\exists\ n_1, n_2 \in \mathbb{N}, n_1 \neq n_2 \ni h(n_1) = f(n_1) = h(n_2) = g(n_2)$
For the sake of the proof we must presume S, T to be disjoint s.t $S \cap T = \emptyset$

We must show that h is both surjective and injective.

1. Surjectivity (onto):
We want to show $\forall y \in S \cup T, \exists n \in \mathbb{N}$ such that $h(n) = y$.

- **Case 1:** Let $y \in S$. Since f is surjective, there exists $k \in \mathbb{N}$ such that $f(k) = y$. Let $n = 2k$. Then n is even, and:

$$h(n) = f\left(\frac{2k}{2}\right) = f(k) = y$$

- **Case 2:** Let $y \in T$. Since g is surjective, there exists $k \in \mathbb{N}$ such that $g(k) = y$. Let $n = 2k - 1$. Then n is odd, and:

$$h(n) = g\left(\frac{(2k-1)+1}{2}\right) = g\left(\frac{2k}{2}\right) = g(k) = y$$

Thus, h is surjective.

2. Injectivity (one-to-one):
We want to show that $h(n_1) = h(n_2) \implies n_1 = n_2$. Suppose $h(n_1) = h(n_2) = y$.

- **Case 1: Same Parity.**
  - If n_1, n_2 are both even, then $f(\frac{n_1}{2}) = f(\frac{n_2}{2})$. Since f is injective, $\frac{n_1}{2} = \frac{n_2}{2} \implies n_1 = n_2$.
  - If n_1, n_2 are both odd, then $g(\frac{n_1+1}{2}) = g(\frac{n_2+1}{2})$. Since g is injective, $\frac{n_1+1}{2} = \frac{n_2+1}{2} \implies n_1 = n_2$.
- **Case 2: Mixed Parity.**
  Suppose one is even and one is odd (e.g., n_1 even, n_2 odd). Then $h(n_1) \in \text{Im}(f) = S$ and $h(n_2) \in \text{Im}(g) = T$. Since S and T are disjoint ($S \cap T = \emptyset$), $h(n_1) \neq h(n_2)$. Therefore, $h(n_1) = h(n_2)$ is impossible if parities differ.

Thus, h is injective.

Since h is both surjective and injective, it is a bijection. Therefore, $S \cup T$ is denumerable. $\square$

Question 2

Proposition: Let $c > 1$ and $m, n \in \mathbb{N}$. Then $c^m > c^n \iff m > n$.

Proof. Let $c > 1$ and $m, n \in \mathbb{N}$. We prove the biconditional by proving both directions separately.

Direction 1: ($\Longleftarrow$)
Prove that if $m > n$, then $c^m > c^n$.
Suppose $m > n$. Then we can write $m = n + k$ for some integer $k \geq 1$.
Consider the expression for c^m:

$$c^m = c^{n+k} = c^n \cdot c^k$$

Since $c > 1$ and $k \geq 1$, it follows that $c^k > 1$.
Because $c > 1$, we know c^n is positive ($c^n > 0$). Multiplying the inequality $c^k > 1$ by c^n gives:

$$\begin{aligned} c^k &> 1 \\ c^n \cdot c^k &> c^n \cdot 1 \\ c^{n+k} &> c^n \\ c^m &> c^n \end{aligned}$$

Thus, $m > n \implies c^m > c^n$.

Direction 2: ($\Longrightarrow$)
Prove that if $c^m > c^n$, then $m > n$.
We proceed by contradiction (using the Trichotomy Law). Assume $c^m > c^n$, but suppose for the sake of contradiction that $m \not> n$. This leaves two cases: $m = n$ or $m < n$.

- **Case 1: $m = n$.**
  If $m = n$, then obviously $c^m = c^n$. This contradicts the hypothesis that $c^m > c^n$.
- **Case 2: $m < n$.**
  If $m < n$, then by the result we proved in Direction 1 (swapping m and n), we must have $c^m < c^n$. This explicitly contradicts the hypothesis that $c^m > c^n$.

Since both $m = n$ and $m < n$ lead to contradictions, it must be true that $m > n$.

Conclusion:
We have shown both directions:

1. $m > n \implies c^m > c^n$
2. $c^m > c^n \implies m > n$

Therefore, $c^m > c^n \iff m > n$. $\square$