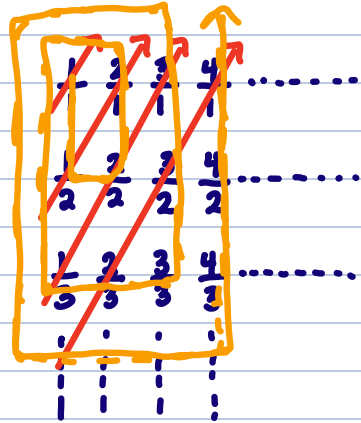




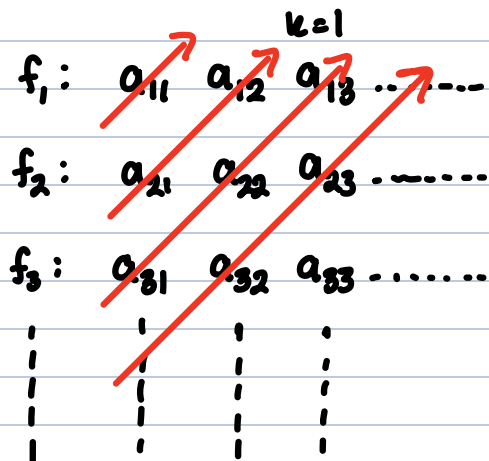
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \right\}$$

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$



Use  or  "arrows" "spiral"

Theorem: If A_n is a countable set for each $n \in \mathbb{N}$,
then $A = \bigcup_{k=1}^{\infty} A_k$ is countable.



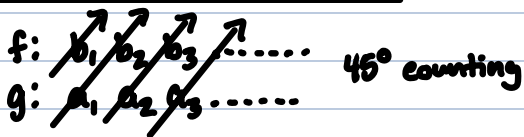
We have a surjective mapping for \mathbb{N} onto $A = \bigcup_{n=1}^{\infty} A_n$

$$\begin{aligned} h(1) &= a_{11} \\ h(2) &= a_{21} \\ h(3) &= a_{12} \end{aligned}$$

f, g are bijections from \mathbb{N} onto $\begin{matrix} \rightarrow A \\ \rightarrow B \end{matrix}$

$A \cup B$

$$h(n) = \begin{cases} f(\frac{n}{2}), & n \text{ is even} \\ g(\frac{n+1}{2}), & n \text{ is odd} \end{cases}$$



$$\begin{aligned} h(1) &= b_1 = g(1) \\ h(2) &= a_1 = f(\frac{2}{2}) \\ h(3) &= b_2 = g(\frac{3+1}{2}) \end{aligned}$$

Algebraic Properties of \mathbb{R} .

Two binary operations, denoted by "+" and "·", and defined in \mathbb{R} , called "addition" and "multiplication", respectively.

A1) $a+b=b+a$ (commutativity property of addition)

A2) $(a+b)+c=a+(b+c)$ (associative property of addition)

A3) There exists an element 0 in \mathbb{R} s.t. $a+0=a \quad \forall a \in \mathbb{R}$.

A4) For each $a \in \mathbb{R}$, there exists an element $(-a)$ in \mathbb{R} such that $a+(-a)=0$

M1) $a \cdot b = b \cdot a$

M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M3) There exists an element 1 in \mathbb{R} such that $a \cdot 1 = a \quad \forall a \in \mathbb{R}$

M4) For each $a \neq 0$ in \mathbb{R} , there exists one element $\frac{1}{a}$ in \mathbb{R} s.t. $a \cdot \frac{1}{a} = 1$

D) $a \cdot (b+c) = a \cdot b + a \cdot c$ (Distributive Property)

Example: a) if $z, a \in \mathbb{R}$ with $z+a=a$, then $z=0$ (rule of cancellation)

Since $z+a=a$ by A4) on a , there exists $-a$ in \mathbb{R} , such that $a+(-a)=0$.

$$\begin{array}{l} (z+a)+(-a) = a+(-a) = 0 \\ \hline \text{A2} \quad \text{A4} \quad \text{A4} \end{array}$$

$$z+(a+(-a))$$

$$\parallel \text{A4}$$

$$z+0$$

$$\parallel \text{A3}$$

$$z =$$

b) Similarly, if u is $b \neq 0$ are in \mathbb{R} with $u \cdot b = b$, then $u=1$.

$$(u \cdot b) \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

$$\parallel \text{M2}$$

$$u \cdot (b \cdot \frac{1}{b})$$

$$\parallel \text{M4}$$

$$u \cdot 1$$

$$\parallel \text{M3}$$

$$u$$

Example c) if $a \in \mathbb{R}$, then $a \cdot 0 = 0$

Proof:

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0$$

M3

$$= a \cdot (1 + 0)$$

0

$$= a \cdot 1$$

A3

$$= a$$

M3

Example a), $a \cdot 0 = 0$

let's say $0 + 0 = 0$

$$a(0 + 0) = a \cdot 0$$

$$a \cdot 0 + a \cdot 0 = a \cdot 0$$

\Downarrow

$$a \cdot 0 = 0$$

d) If $a \neq 0$, and $b \in \mathbb{R}$ are such that $a \cdot b = 1$, then $b = \frac{1}{a}$.

Proof: Since $a \neq 0$, by M4), $\exists \frac{1}{a} \in \mathbb{R}$ such that $a \cdot \frac{1}{a} = 1$

$$\frac{\frac{1}{a} \cdot (a \cdot b)}{\frac{1}{a} \cdot 1} = \frac{\frac{1}{a} \cdot 1}{\frac{1}{a}}$$

\parallel M2

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

\parallel M4

$$1 \cdot b$$

\parallel M3

$$b$$

e) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$

Suppose $a \neq 0$ (Goal $b = 0$)

$$\frac{1}{a} (a \cdot b) = \frac{1}{a} \cdot 0 = 0$$

\parallel M2

Exc)

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

\parallel M4

$$1 \cdot b$$

\parallel M3

$$b$$

$$\text{Ex: } -a = (-1) \cdot a$$

$$(-1) = (-1) + 0$$

$$(-1) = (-1) + a + (-a)$$

$$a + (-a) = 0 = a \cdot 0 \quad M3$$

$$A4 \quad = a \cdot (1 + (-1)) \quad A4$$

$$= a \cdot 1 + a(-1) \quad D$$

$$= a + a(-1) \quad M3$$

$$\underline{-a = a \cdot (-1)}$$

Theorem: There doesn't exist a rational number r such that $r^2 = 2$.