$x^2=2$

Theorem: There exists a unique positive real number x such that $x^2=2$.

$x^2<2$
$s^2<2$

Proof: Let $S=\{s\in\mathbb{R} : 0\le s,\ s^2<2\}$

We want to show that S is nonempty

$s=1$, $0<1$

Note that $1\in S$, then S is non-empty.

Claim: 2 is an upperbound of S.

For this case, suppose 2 is not an upper bound of S, then there exists some $s'\in S'$ such that $2<s'$, then $(s')^2 \ge 2\cdot s' > 2\cdot 2 = 4$

Note that $s'\in S$, then $(s')^2<2$, then, $4<2$, then $2<0$. Contradiction!

So by the completeness property, $x=\sup S$ exists in $\mathbb{R}$. We need to prove that $x^2=2$ by ruling out the other 2 cases $x^2<2$ and $x^2>2$.

For $x^2<2$, suppose it is true. We show that there exists $n\in\mathbb{N}$ such that $x+\frac{1}{n}\in S$.
For $\frac{2x+1}{2-x^2}\in\mathbb{R}$, by the Archimedean Property, there exists $n\in\mathbb{N}$ such that $\frac{2x+1}{2-x^2}<n$
then $\frac{2x+1}{n}<2-x^2$, then $x^2+\frac{2}{n}x+\frac{1}{n}<2$, then $x^2+\frac{2}{n}x+\frac{1}{n^2}<2$,
then $(x+\frac{1}{n})^2<2$, then $x+\frac{1}{n}\in S$. So x can not be the supremum of S, Contradiction!

$x+\frac{1}{n}\in S$

$\Uparrow$

We know $x^2<2$ $\quad (x+\frac{1}{n})^2<2$

$x^2+\frac{2}{n}x+\frac{1}{n^2}<2$

$\frac{1}{n^2}\le\frac{1}{n}$

$\Updownarrow$

$n\le n^2$

$\Uparrow$

$x^2+\frac{2}{n}x+\frac{1}{n}<2$

$x^2+\frac{2x+1}{n}<2$

$\frac{2x+1}{n}<2$

For $x^2>2$, suppose it is true, we show that there exists $n\in\mathbb{N}$ such that $x-\frac{1}{n}$ is an upperbound of S, then x can not be the supremum of S.

$(x-\frac{1}{n})^2>2>s^2$

$x-\frac{1}{n}>s$

$x-\frac{1}{n}\le s$

$(x-\frac{1}{n})^2\le(x-\frac{1}{n})\cdot s\le s\cdot s$

We need to show this is true:

$(x-\frac{1}{n})^2>2$

$\Uparrow$

$x^2-\frac{2}{n}x+\frac{1}{n^2}>2$

$\Uparrow$

$x^2-\frac{2}{n}x>2$

$\Uparrow$

$\frac{2x}{n}<x^2-2$

$\Uparrow$

$n>\frac{2x}{x^2-2}$

For $\frac{2x}{x^2-2} \in \mathbb{R}$, by the Archimedean Property, $\exists n \in \mathbb{N}$ such that $n > \frac{2x}{x^2-2}$ then $\frac{2x}{n} < x^2-2$, then $x^2 - \frac{2x}{n} > 2$, then

$x^2 - \frac{2}{n}x + \frac{1}{n^2} > 2$ then $(x-\frac{1}{n})^2 > 2$ then $(x-\frac{1}{n}) > s$ from all $s \in S$ then $x - \frac{1}{n}$ in $\mathbb{R}$

Uniqueness: Let x_1, x_2 be two solutions to $x^2 = 2$, then $x_1^2 = x_2^2$.

$(x_1+x_2)(x_1-x_2) = 0$

Since $x_1 > 0$, $x_2 > 0$, then $x_1 + x_2 > 0$ then $x_1 - x_2 = 0$, then $x_1 = x_2$.

Hence $x = \sqrt{2}$ is not a rational number hence it is irrational.

Theorem: If x and y are two rational numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

$* \; x < \frac{m}{n} < y$

$nx < m < ny$

$ny - nx > 1$

$n(y-x) > 1$

$n > \frac{1}{y-x}$

Archimedean Property