Quiz 1 Practice:

1.3

Countable:
- $\emptyset$
- Finite

Denumerable:
- Countably infinite
- $\exists$ bijection $\mathbb{N} \to \underbrace{S}_{\text{set } S}$

$4n$

$4, 8, 12, 16, \ldots - 4n$
$\uparrow \ \uparrow \ \uparrow \ \uparrow$
$n: 1, 2, 3, 4, \ldots - n$

Show that set of integer is denumerable $\mathbb{Z}$.

$1, -1, 2, -2, 3, -3$
$1, 2, 3, 4, 5, 6$ } Denumerable

$$h: \begin{cases} f(\frac{n}{2}) \ \forall \ 2n \\ g(\frac{n+1}{2}) \ \forall \ 2n+1 \end{cases}$$ Constructing $h(n)$ is surjective

$|\mathbb{N}| \leq |\mathbb{Z}|$ surjective
$|\mathbb{N}| \geq |\mathbb{Z}|$ injective
$|\mathbb{N}| = |\mathbb{Z}|$ countably infinite

S, T be denumerable, $S \cup T$ is denumerable

A) S has injection onto $\mathbb{N}$
B) S has surjective onto $\mathbb{N}$
C) S is countable

state S and T are finite
constructing $h(n)$ is all you mostly need

If $c>1$, $m,n\in\mathbb{N}$. Show $c^m>c^n \Leftrightarrow m>n$.

① Induction $\boxed{\Rightarrow}$ (hard)
② Contradiction $\boxed{\Leftarrow}$ (easy)

$\boxed{\Leftarrow}$ Suppose, for the sake of contradiction, $c^M \le c^N$.

Case 1: $c^M<c^N \rightarrow M<N \rightarrow\leftarrow \dots M>N$
Case 2: $c^M=c^N \rightarrow M=N \rightarrow\leftarrow \dots M>N$
$\therefore c^M>c^N$

$\exists x$ where $M=x+1, N=x$ ... $c^{x+1}<c^x \Rightarrow c\cdot c^x<c^x$ $(c>1)$
$\exists x$ where $M=x+1, N=x$ ... $c^{x+1}=c^x \Rightarrow c\cdot c^x=c^x \rightarrow\leftarrow$ $(c>1)$

$\boxed{\begin{array}{ll} M=x+1 & M>N \\ N=x & \end{array}}$ $\therefore c^M>c^N$

you could also say $M=x+k$ $(\forall k\in\mathbb{N})$
$N=x$

$\boxed{\Rightarrow}$ State that $c^N>1$ b/c $n=1$ $c^1>1$
I.H. $n=k$ $c^k>1$
I.S. $c^{k+1}>1 \rightarrow c\cdot c^k>1 \rightarrow c\cdot c^k>1 \rightarrow c\cdot 1>1$
$\rightarrow c>1$ $c^{k+1}>1$, So By induction.

$c^N>1>0 \rightarrow c^M-c^N>0$

$c^M-c^N>1>0$
$\downarrow$
$M-N>1$
$\therefore M>N$

If $c>1$, $m,n\in\mathbb{N}$. Show that $c^m>c^n$ if and only if $m>n$.

$\boxed{\Leftarrow}$ For the sake of contradiction, $m\le n$. Here we have two cases where $m<n$ and $m=n$.

$m<n$
For case 1: $c^M<c^N$, let $\exists x$, where $M=x+1, N=x$, this results to $c^{x+1}<c^x \Rightarrow c\cdot c^x \not< c$, $(c>1)$
contradiction

$m=n$
For case 2: $c^M=c^N$, let $\exists x$, where $M=x+1, N=x$, this means $c^{x+1}=c^x \Rightarrow c\cdot c^x \ne c^x$
contradiction

$\therefore c^M>c^N$ iff $m>n$

$\boxed{\Rightarrow}$ If $c^m > c^N$ the $m > N$.
We will introduce induction.

Let's introduce $\exists k$ where $M-N=k$, $c^{M-N}=c^k$

Base case where $c^1>1$
$k=a$ case where $c^a>1$
$k=a+1$ case where $c^{a+1}>1$

So for $c^{a+1}>1 \Rightarrow c^a \cdot c>1$, we know $c^a>1$.
So by induction $c^k>1$ meaning $c^{M-N}>1>0$.

We found out from above that if $m \neq n$ and if $m<n$ then the inequality doesn't hold. Therefore this implies that $M>N$.

$\therefore$ $c^M > c^N$ iff $M>N$

S, T be denumerable, $S \cup T$ is denumerable

For the sake of the proof we will consider $S \cap T = \phi$ because any two sets S and T, $S \cup T = (S \setminus T) \cup T$ which is made disjoint. Since S and T are denumerable, then there exist bijective functions $f: \mathbb{N} \to S$, $g: \mathbb{N} \to T$.

Now we want to show that $S \cup T$ is denumerable then we have to produce a bijective function $h: \mathbb{N} \to S \cup T$.
We define

$$h(n) = \begin{cases} f(\frac{n}{2}), & n \text{ is even} \\ g(\frac{n+1}{2}), & n \text{ is odd} \end{cases}$$

Here h is clearly onto because every element in $S \cup T$ has a preimage under h.
Now if $h(n_1)=h(n_2)$ where n_1, n_2 are either both even or both odd then by the bijective functions of f and g we can conclude that $n_1=n_2$ by injectivity.

If n_1 and n_2 are simultaneously odd or even then it is a trivial case in which it doesn't change much of the proof.

Since h is both surjective and injective.
$\therefore$ h is bijective, hence $S \cup T$ is denumerable