

Lecture 6

Wednesday, September 20, 2023 10:50 AM

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Corollary 1 If $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\inf S = 0$.

Notes: $\frac{1}{n} > 0$

Suppose $\frac{1}{n} \leq 0$

$$n \cdot \frac{1}{n} \leq 0 \Rightarrow 1 \leq 0 \quad \text{contradiction!}$$

$$\frac{1}{\inf S} < n$$

Contradiction strikes again

Proof Note that 0 is a lower bound of S , so $\inf S \geq 0$.

Suppose $\inf S > 0$

By the Archimedean Property, for $\frac{1}{\inf S} \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{\inf S} < n$. Since $\inf S > 0$ and $n > 0$, by the order properties, $\frac{1}{n} < \inf S$. Since $\frac{1}{n} \in S$, this is a contradiction! \square

Corollary 2 If $t > 0$, there exists an $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$
(t is not a lower bound)

Two ways to prove it:

- Archimedean Property: $\frac{1}{n_t} < t \Rightarrow \frac{1}{t} < n_t$
- Since $\inf S = 0$ by the previous corollary and $t > 0$, then t is not a lower bound of S . Then there exists some element $\frac{1}{n_t}$ in S that is less than t .

Corollary 3 If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_{y-1} \leq y \leq n_y$.

Proof Consider $E = \{m \in \mathbb{N}, y < m\}$ and let n_y be the smallest element in E . Since $n_y \in E$, then $y < n_y$. Since n_y is the smallest element in E , then $n_{y-1} \notin E$. So, $y < n_{y-1}$ is not true; that is, $n_{y-1} \leq y$ \square

PIGEON

Showing the existence of an irrational number $\sqrt{2}$

We want to find a positive solution for $x^2 = 2$.

We use $\sup S$ where $x^2 < 2$

Proof Let $S = \{s \in \mathbb{R} : s > 0 \text{ and } s^2 < 2\}$

Since $1 \in S$, then $S \neq \emptyset$

Also, S is bounded above by 2.

By the Completeness Property, $x = \sup S$ exists in \mathbb{R} .

"you should think a little bit!" - Kang

Rule out $x^2 < 2$ and $x^2 > 2$ to get $x^2 = 2$

To do this, we use Kang's favorite proof technique, contradiction

$x^2 < 2$ Suppose $x^2 < 2$. Then there exists an $n > 0$ such that

$x + \frac{1}{n} \in S$. Then x is not an upper bound, hence

$x \neq \sup S$. Contradiction!