Review

Archimedean Property: $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x \leq n$

Triangle Inequality: $|a+b| \leq |a| + |b|$

supremum: the least upper bound of a set

infimum: the greatest lower bound of a set

* In general, for definition questions, make sure to say "a sequence of real numbers"

Section 3.1 – Sequences

Def | A sequence of real numbers is a function defined on $\mathbb{N}$ whose range is contained in $\mathbb{R}$

It is a mapping from $\mathbb{N}$ onto a subset of $\mathbb{R}$

A sequence (x_n) is said to converge to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $|x_n - x| < \varepsilon$ (also x is said to be a limit of (x_n)).

If a sequence has a limit, it is convergent; otherwise, it is divergent.

Limits are unique (Uniqueness Theorem)

Negation of limit def:

x is not the limit of (x_n) if there exists $\varepsilon_0 \geq 0$ such that for every $N(\varepsilon) \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $n \geq N(\varepsilon)$ implies $|x_n - x| \geq \varepsilon_0$.

Ex Show that $\lim_{n\to\infty} \frac{1}{n} = 0$

We need to show that $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that

$\forall n \geq N(\varepsilon)$, $\left|\frac{1}{n} - 0\right| < \varepsilon$

$$\Longrightarrow \quad n > \frac{1}{\varepsilon}$$

By the Archimedean Property, $\exists N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ (so choose this)

Proof Let $\varepsilon > 0$ be given. Since $\varepsilon > 0$, by the Archimedean Property there exists $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$. For all $n \geq N(\varepsilon)$, we have $n > \frac{1}{\varepsilon}$, or $\frac{1}{n} < \varepsilon$. Then $\left|\frac{1}{n} - 0\right| < \varepsilon$. Thus, $\left(\frac{1}{n}\right)$ converges to 0. ■

Ex $\lim\left(\sqrt{n+1} - \sqrt{n}\right) = 0$

$$\left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} - \sqrt{n}} \leq \frac{1}{\sqrt{n}} < \varepsilon$$

$$\frac{1}{\varepsilon} < \sqrt{n} \Longrightarrow \frac{1}{\varepsilon^2} < n$$

So choose $N(\varepsilon) > \frac{1}{\varepsilon^2}$

Proof Let $\varepsilon > 0$ be given. Since $\varepsilon \in \mathbb{R}$, $\frac{1}{\varepsilon^2} \in \mathbb{R}$, and by the Archimedean Property, there exists an $N(\varepsilon) \in \mathbb{N}$ such that $N(\varepsilon) > \frac{1}{\varepsilon^2}$. Then for all $n \geq N(\varepsilon)$, $n > \frac{1}{\varepsilon^2}$, then $\frac{1}{\sqrt{n}} < \varepsilon$. This implies that $\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$, so $\sqrt{n+1} - \sqrt{n} < \varepsilon$. Thus, $\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon$ ■

Section 3.2 – Limit Theorems

Def A sequence (x_n) is said to be bounded if there exists a real number $M > 0$ such that $|x_n| \leq M \ \forall n \in \mathbb{N}$

A convergent sequence is bounded

Algebraic Properties of Sequences

Assuming (x_n) converges to x and (y_n) converges to y:

- $(x_n) + (y_n) \rightarrow x + y$
- $(x_n) - (y_n) \rightarrow x - y$
- $(x_n) \cdot (y_n) \rightarrow x \cdot y$
- $\frac{(x_n)}{(y_n)} \longrightarrow \frac{x}{y}$
- $c(x_n) \rightarrow cx$

Squeeze Theorem Let $(x_n), (y_n), (z_n)$ be sequences such that

$x_n \leq y_n \leq z_n$ and $\lim(x_n) = \lim(z_n)$.

Then (y_n) converges and $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

More fun theorems:

- If $(x_n) \rightarrow x$, then $(|x_n|) \rightarrow |x|$

  So if $\lim(x_n) = x$, then $\lim(|x_n|) = |x|$
- If $(x_n) \rightarrow x$, then $(\sqrt{x_n}) \rightarrow \sqrt{x}$

  So if $\lim(x_n) = x$, then $\lim(\sqrt{x_n}) = \sqrt{x}$
- **Ratio Test:** Let (x_n) be a sequence such that $\lim\left(\frac{x_{n+1}}{x_n}\right) = L$.

  If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$

Section 3.3 – Monotone Sequences

Def | A sequence is monotone if it is either increasing or decreasing

Monotone Convergence Theorem

A monotone sequence is convergent if and only if it is bounded.

Also:

- If (x_n) is a bounded increasing sequence, $\lim(x_n) = \sup\{x_n\}$
- If (x_n) is a bounded decreasing sequence, $\lim(x_n) = \inf\{x_n\}$

Section 3.4 – Subsequences

Def | Let (x_n) be a sequence and let $n_1 < n_2 < \dots < n_k$ be a strictly increasing sequence of natural numbers. Then (x_{n_k}) is called a subsequence of (x_n).

Theorem Any subsequence of a convergent sequence converges to the same value.

A sequence is divergent if one of the following is true:

- (x_n) has two subsequences that converge to different values

  Ex $(-1)^n$

  $(-1)^{2n} \to 1$, $(-1)^{2n+1} \to -1$

- (x_n) is unbounded

Monotone Subsequence Theorem

Every sequence of real numbers has a monotone convergent subsequence

Bolzano - Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence

If every subsequence of a sequence converges to the same value, then the sequence also converges to that same value.

Def Let (x_n) be a bounded sequence of real numbers.

- $\limsup(x_n)$ or $\overline{\lim}(x_n)$ is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$.
- $\liminf(x_n)$ or $\underline{\lim}(x_n)$ is the supremum of the set W of $w \in \mathbb{R}$ such that $w > x_n$ for at most a finite number of $n \in \mathbb{N}$.

A bounded sequence (x_n) is convergent if and only if $\limsup(x_n) = \liminf(x_n)$.

Examples to do

- $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$
- $\limsup\left(\frac{1}{x_n}\right) = \frac{1}{\liminf(x_n)}$

Section 3.5 - The Cauchy Criterion

Def A sequence (x_n) of real numbers is said to be Cauchy if for every $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq N(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

Useful since the limit is not needed to show convergence

Cauchy implies convergent!

It also implies that the sequence is bounded.

Cauchy Convergence Criterion

it's the CCC

A sequence is convergent if and only if it is Cauchy

Def A sequence is contractive if there exists a constant C $(0 < C < 1)$ such that $|x_{n+2} - x_{n+1}| \leq C |x_{n+1} - x_n| \quad \forall n \in \mathbb{N}$.

Every contractive sequence is Cauchy, and therefore convergent

For recursively defined sequences, there are two ways to handle them:

- Monotone Convergence Theorem
- Contractive Sequences

Section 3.6 - Properly Divergent Sequences

Def: A sequence is properly divergent if its limit tends to $\pm\infty$

A monotone sequence is properly divergent if and only if it is unbounded.

If a sequence is not bounded from above, it tends to ∞.

If it is not bounded from below, it tends to $-\infty$.

Comparison Theorems

- Let (x_n) and (y_n) be sequences such that $x_n \leq y_n \quad \forall n \in \mathbb{N}$
    - If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$
    - If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$

- Let (x_n) and (y_n) be sequences and suppose for some $L \in \mathbb{R}$, $L > 0$,
  we have $\lim\left(\frac{x_n}{y_n}\right) = L$.
  We have $\lim(x_n) = +\infty$ if and only if $\lim(y_n) = +\infty$.

* He said that we will not have to prove the comparison tests
but we should know how to apply them.