Lecture 17

Monday, November 6, 2023 9:42 AM

* was 45 min late due to an event but it seems like he didn't do much

Review :

Theorem Every contractive sequence is Cauchy, and thus convergent.

Cool thing!

The sequence of Fibonacci fractions $x_n := \frac{f_n}{f_{n+1}}$ is contractive $\left(c = \frac{4}{9}\right)$
and it converges to $x = \frac{-1+\sqrt{5}}{2}$ ★

The Golden Ratio: $\frac{1+\sqrt{5}}{2} \approx 1.618\ldots$

Section 3.6 Properly Divergent Sequences

What does it mean for something to "tend to $\pm\infty$"?

Def Let (x_n) be a sequence.

- (x_n) tends to $+\infty$, written as $\lim(x_n) = +\infty$, if for every $\alpha > 0$, there exists a natural number $K(\alpha)$ such that $x_n > \alpha$ if $n \geq K(\alpha)$.

- (x_n) tends to $-\infty$, written as $\lim(x_n) = -\infty$, if for every $\beta > 0$, there exists a natural number $K(\beta)$ such that $x_n < \beta$ if $n \geq K(\beta)$.

If (x_n) satisfies either of these, then it is properly divergent.

the sequence is "going wild"

you want the stock market to tend towards ∞

Properties

① If (x_n) tends to ∞, then $\frac{1}{x_n} \to 0$ as $n \to \infty$

$\left| \frac{1}{x_n} - 0 \right| < \varepsilon \Rightarrow x_n > \frac{1}{\varepsilon}$

The converse is not necessarily true

Ex. $x_n = (-1)^n n$

Follow up from the Monotone Convergence Theorem

② A monotone sequence is divergent if and only if it is unbounded.

- If (x_n) is an unbounded increasing sequence, then (x_n) tends to $+\infty$
- If (x_n) is an unbounded decreasing sequence, then (x_n) tends to $-\infty$

③ Let $(x_n),(y_n)$ be two sequences such that $x_n \le y_n$ $\forall n \in \mathbb{N}$. *

* he said "for all n sufficiently large enough" but the book says this holds for all $n \in \mathbb{N}$

- If $\lim x_n = \infty$, then $\lim y_n = \infty$
- If $\lim y_n = -\infty$, then $\lim x_n = -\infty$

Limit Comparison Theorem!

④ If $\lim\left(\frac{x_n}{y_n}\right) = L > 0$, then $\lim(x_n) = \pm\infty$ if and only if $\lim(y_n) = \pm\infty$

Choose $\varepsilon = \frac{L}{2}$

$$\left| \frac{x_n}{y_n} - L \right| < \frac{L}{2}$$

$$-\frac{L}{2} < \frac{x_n}{y_n} - L < \frac{L}{2}$$

$$\frac{L}{2} < \frac{x_n}{y_n} < \frac{3L}{2}$$

We are not going to cover Infinite Series (3.7)

Kang says there's nothing useful

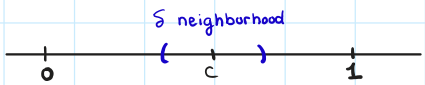
Section 4.1 Limits of Functions

$\lim_{x \to c} f(x)$

Def Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for any $\delta > 0$, there exists at least one point $x \in A$, $x \neq c$, such that $|x - c| < \delta$.

- If $c \notin A$, it must be the supremum or infimum of A

Ex If $A = (0, 1)$, find the collection of cluster points.



We can choose a small enough δ so that the δ-neighborhood is also in $(0, 1)$.

$\Longrightarrow \quad c + \delta < 1 \quad$ and $\quad c - \delta < 1$

$\Longrightarrow \quad \delta > c$