$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\}$

$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$

$$\begin{array}{ccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Use $\nearrow$ "arrows" or $\boxed{\lrcorner}$ "spiral

Theorem: If A_n is a countable set for each $n \in \mathbb{N}$,

then $A = \bigcup_{k=1}^{\infty} A_n$ is countable.

$$\begin{array}{lcccc} f_1: & a_{11} & a_{12} & a_{13} & \cdots \\ f_2: & a_{21} & a_{22} & a_{23} & \cdots \\ f_3: & a_{31} & a_{32} & a_{33} & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

We have a surjective mapping for $\mathbb{N}$ onto $A = \bigcup_{n=1}^{\infty} A_n$

$h(1) = a_{11}$
$h(2) = a_{21}$
$h(3) = a_{12}$

f ↘ are bijections from $\mathbb{N}$ onto ↗ A
g ↗ ↘ B

$A \cup B$

$$\boxed{h(n) = \begin{cases} f(\frac{n}{2}), & n \text{ is even} \\ g(\frac{n+1}{2}), & n \text{ is odd} \end{cases}}$$

$$\begin{array}{lcccc} f: & b_1 & b_2 & b_3 & \cdots \\ g: & a_1 & a_2 & a_3 & \cdots \end{array}$$

45° counting

$h(1) = b_1 = g(1)$
$h(2) = a_1 = f(\frac{2}{2})$
$h(3) = b_2 = g(\frac{3+1}{2})$

Algebraic Properties of $\mathbb{R}$.

Two binary operations, denoted by "$+$" and "$\cdot$", and defined in $\mathbb{R}$, called "addition" and "multiplication", repeatively.

A1) $a+b=b+a$ (commutativity property of addition)

A2) $(a+b)+c=a+(b+c)$ (associative property of addition)

A3) There exists an element 0 in $\mathbb{R}$ s.t. $a+0=a$ $\forall a\in\mathbb{R}$.

A4) For each $a\in\mathbb{R}$, there exists an element $(-a)$ in $\mathbb{R}$ such that $a+(-a)=0$

M1) $a\cdot b=b\cdot a$

M2) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$

M3) There exists an element 1 in $\mathbb{R}$ such that $a\cdot 1=a$ $\forall a\in\mathbb{R}$

M4) For each $a\neq 0$ in $\mathbb{R}$, there exists one element $\frac{1}{a}$ in $\mathbb{R}$ s.t. $a\cdot\frac{1}{a}=1$

D) $a\cdot(b+c)=a\cdot b+a\cdot c$ (Distributive Property)

Example: a) if $z,a\in\mathbb{R}$ with $z+a=a$, the $z=0$ (rule of cancellation)

Since $z+a=a$ by A4) on a, there exists $-a$ in $\mathbb{R}$, such that $a+(-a)=0$.

$$\underline{(z+a)+(-a)} \overset{A4}{=} a+(-a) \overset{A4}{=} 0$$

$$\overset{A_2}{=} z+(a+(-a)) \overset{A4}{=} z+0 \overset{A3}{=} z$$

b) Similarly, if u is $b\neq 0$ are in $\mathbb{R}$ with $u\cdot b=b$, then $u=1$.

$$(u\cdot b)\cdot\frac{1}{b} \overset{M4}{=} b\cdot\frac{1}{b} \overset{M4}{=} 1$$

$$\overset{M2}{=} u\cdot\left(b\cdot\frac{1}{b}\right) \overset{M4}{=} u\cdot 1 \overset{M3}{=} u$$

Example c) If $a \in \mathbb{R}$, then $a \cdot 0 = 0$

Proof:

$$a + a\cdot 0 = a \cdot 1 + a \cdot 0$$
M3
$$= a \cdot (1+0)$$
D
$$= a \cdot 1$$
A3
$$= a$$
M3

Example a), $a \cdot 0 = 0$

let's say $0+0=0$

$$a(0+0) = a \cdot 0$$
$$a \cdot 0 + a \cdot 0 = a \cdot 0$$
$$\Downarrow$$
$$a \cdot 0 = 0$$

d) If $a \neq 0$, and $b \in \mathbb{R}$ are such that $a \cdot b = 1$, then $b = \frac{1}{a}$.

Proof: Since $a \neq 0$, by M4), $\exists \frac{1}{a} \in \mathbb{R}$ such that $a \cdot \frac{1}{a} = 1$

$$\underline{\frac{1}{a} \cdot (a \cdot b)} = \underline{\frac{1}{a} \cdot 1} \overset{M3}{=} \underline{\frac{1}{a}}$$

|| M2

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

|| M4

$$1 \cdot b$$

|| M3

$$b$$

e) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$

Suppose $a \neq 0$ (Goal $b = 0$)

$$\frac{1}{a}(a \cdot b) = \frac{1}{a} \cdot 0 = 0$$
Ex c)

|| M2

$$\left(\frac{1}{a} \cdot a\right) \cdot b$$

|| M4

$$1 \cdot b$$

|| M3

$$b$$

Ex: $-a = (-1)\cdot a$

$(-1) = (-1) + 0$

$(-1) = (-1) + a + (-a)$

$a + (-a) = 0 = a \cdot 0$ M3

A4 $= a\cdot(1+(-1))$ A4

$= a\cdot 1 + a(-1)$ D

$= a + a(-1)$ M3

$\underline{-a = a\cdot(-1)}$

Theorem: There doesn't exist a rational number r such that $r^2 = 2$.