MATH 301: Homework 1

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Question 1

Proof. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Case 1: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

Consider an arbtrary element x:

$$x \in A \cap (B \cup C) \implies x \in A \wedge x \in (B \cup C)$$

Logically:

$$x \in A \wedge x \in (B \cap C) \implies x \in A \wedge (x \in B \vee x \in C)$$

By distributive property

$$x \in A \wedge (x \in B \vee x \in C) \equiv (x \in A \wedge x \in C) \vee (x \in A \wedge x \in B)$$

Redefinig as logical statment to set inclusion yields:

$$(x \in A \wedge x \in C) \vee (x \in A \wedge x \in B) \implies x \in (A \cap C) \cup (A \cap B)$$

As x is an arbtrary element Case 1: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ holds

Case 2: $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$

Notice all implications can be done in reverse as:
Consider an arbtrary element x:

$$x \in A \cap (B \cup C) \leftrightarrow x \in A \wedge x \in (B \cup C)$$

Logically:

$$x \in A \wedge x \in (B \cap C) \leftrightarrow x \in A \wedge (x \in B \vee x \in C)$$

By distributive property

$$x \in A \wedge (x \in B \vee x \in C) \equiv (x \in A \wedge x \in C) \vee (x \in A \wedge x \in B)$$

Redefinig as logical statment to set inclusion yields:

$$(x \in A \wedge x \in C) \vee (x \in A \wedge x \in B) \leftrightarrow x \in (A \cap C) \cup (A \cap B)$$

Thus as x is an arbtrary element Case 2: $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ holds

Conclusion

As Case 1,2 holds the two sets must be equal □

Question 2

Proof. Show that is $f : A \to B$ is surjective and $H \subseteq B$ then $f(f^{-1}(H)) = H$.
Suppose $f : A \to B$ is surjective and $H \subseteq B$
Observe $f : A \to B$

$$f = \{f(a) : a \in A\} \implies f(a) = b, b \in B$$

As f is surjective

$$\forall\ b \in B, \exists\ a \in A \ni f(a) = b$$

Observe

$$H \subseteq B \implies \forall\ h \in H, h \in B$$

Thus for some $X \subset A$

$$f(X) = f(x)|x \in X = H$$

By definition of pre-image

$$f^{-1}(H) = x|f(x) \in H = X$$

Thus

$$f(f^{-1}(H)) = f(X) = H$$

□

Proof. Show that is $f : A \to B$ then $f(f^{-1}(H)) \subseteq H$
Suppose $f : A \to B$ Observe $f : A \to B$

$$f = \{f(a) : a \in A\} \implies f(a) = b, b \in B$$

As f is not surjective

$$\exists\ b \in B \ni \forall\ a \in A, f(a) \neq b$$

Consider some $X \subset A$

$$f(X) = f(x) \in H|x \in X$$

As f is not surjective $f(X)$ may not equal H
Thus there may exists an $h \in H$ s.t. $f^{-1}(h)$ is not defined.
Therefore, $f(f^{-1}(H)) \subseteq H$ □

Question 3

Theorem 1.1.14:

Let $f : A \to B$ be a function

Let $g : B \to C$ be function

Let $H \subset C$

Prove $(gof)^{-1}(H) = f^{-1}(g^{-1}(H))$

Proof. $= f^{-1}(g^{-1}(H))$ Let $f : A \to B$ be a function

$$f = \{f(a) \in B : a \in A\}$$

Let $g : B \to C$ be functions

$$g = \{g(b) \in C : b \in B\}$$

Let $H \subset C$

$$\forall\ h \in H, h \in C$$

Case 1: $(gof)^{-1}(H) \subset f^{-1}(g^{-1}(H))$

Let x be an arbitrary element $x \in (gof)^{-1}(H)$

$$\begin{aligned} x \in (g \circ f)^{-1}(H) &\iff (g \circ f)(x) \in H && \text{(By definition of inverse image)} \\ &\iff g(f(x)) \in H && \text{(By definition of composition)} \\ &\iff f(x) \in g^{-1}(H) && \text{(By definition of inverse image for } g\text{)} \\ &\iff x \in f^{-1}(g^{-1}(H)) && \text{(By definition of inverse image for } f\text{)} \end{aligned}$$

Case 2: $f^{-1}(g^{-1}(H)) \subset (gof)^{-1}(H)$

Let x be an arbitrary element $(gof)^{-1}(H)$ and (traverse statements in reverse order)

$$\begin{aligned} x \in (g \circ f)^{-1}(H) &\iff (g \circ f)(x) \in H && \text{(By definition of inverse image)} \\ &\iff g(f(x)) \in H && \text{(By definition of composition)} \\ &\iff f(x) \in g^{-1}(H) && \text{(By definition of inverse image for } g\text{)} \\ &\iff x \in f^{-1}(g^{-1}(H)) && \text{(By definition of inverse image for } f\text{)} \end{aligned}$$

Conclusion

Thus, as Case 1,2 holds $(gof)^{-1}(H) = f^{-1}(g^{-1}(H))$ holds. □

Question 4

Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$; skipping too easy

Question 5

Conjecture a formula for the sum of the first n odd natural numbers $1 + 3 + ... + (2n - 1)$ Prove via induction.

$$P(n) = \sum_{i=1}^{n} (2i - 1) = n^2$$

Proof. Let $P(n) = \sum_{i=1}^{n} (2i - 1) = n^2$
Observe

$$P(1) = 1 = 1^2$$

Thus $P(1)$ holds
Suppose $P(k), k \in \mathbb{N}$ holds. Thus,

$$P(k) = \sum_{i=1}^{k} (2i - 1) = k^2$$

Add $2(k + 1) - 1 = 2k + 1$ to both sides of the eqution

$$\sum_{i=1}^{k} (2i - 1) + (2k + 1) = k^2 + (2k + 1)$$

$$\sum_{i=1}^{k+1} (2i - 1) = k^2 + 2k + 1$$

Observe

$$P(k+1) : \sum_{i=1}^{k+1} (2i-1) = (k+1)^2 = k^2 + 2k + 1$$

Thus $P(k+1)$ holds.
By PMI $P(n)$ holds $\forall\ n \in \mathbb{N}$ $\square$