MATH 301: Homework 2

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Question 1

Prove Theorem 1.3.4 (b)
If a set A with $m \in \mathbb{N}$ elements and $C \subseteq A$ is a set with one element then $A - C$ is a set with $m - 1$ elements

Proof. Suppose a set A with $m \in \mathbb{N}$ elements

$$A = \{a_1, a_2, ..., a_{m-1}, a_m\} \implies |A| = m$$

Suppose $C \subseteq A$ is a set with one element

$$C = \{a\}, a \in A$$

Observe a is an arbtrary element of A
WLOG let $a = a_m$ s.t.

$$C = \{a_m\}$$

Thus

$$A - C = \{a_1, a_2, ..., a_{m-1}\} \implies |A - C| = m - 1$$

□

Question 2

Prove Theorem 1.3.4 (c)
If C is an infinite set and B is a finite set
Then $C - B$ is an infinite set
Recall an infinite set is a set which is not finite (defn.)

Proof. Suppose C is an infinite set

$$C = \{c_1, c_2, ...\}$$

Suppose B is a finite set

$$B = \{b_1, b_2, ..., b_n\}, n \in \mathbb{N}$$

Suppose for the sake of contradiction $C - B$ is finite s.t

$$C - B = \{x_1, x_2, ..., x_m\} m \in \mathbb{N}$$

Observe at most $C \cap B$ must be finite (n number of elements) by definition of set intersection

$$C \cap B = \{b_1, b_2, ..., b_n\}$$

By rearangement of the principle of exclusion

$$C = C - B \cup (C \cap B)$$

If $C - B$ is finite and $C \cap B$ is finite then C must be finite. This is a logical inconsistency.
Thus, C-B must be infinite. □

Question 3

Prove that if x is a rational number and y is an irrational number, then $x + y$ is an irrational number.
If in addition $x \neq 0$ then show that xy is an irrational number.

Proof. Suppose x is a rational number and y is an irrational number
Suppose for the sake of contradiction $x + y$ is a rational number.
Then given $m, n \in Z$

$$x + y = \frac{m}{n}$$

By defn. $x, k, p \in \mathbb{Q} \implies x = \frac{k}{p}$
Thus

$$\frac{k}{p} + y = \frac{m}{n}$$
$$y = \frac{m}{n} - \frac{k}{p}$$
$$y = \frac{mp}{np} - \frac{kn}{pn}$$
$$y = \frac{mp - kn}{np}$$

By set inclusion of integers, $mp - kn, np \in \mathbb{Z}$. This would imply y is rational.
This is a logical inconsistency, thus $x + y$ is irrational. □

Question 4

Modify the proof of Theorem 2.1.4
Show that there does not exist a rational number t so that $t^2 = 3$

Proof. Let $t \in \mathbb{Q}$
Suppose for the sake of contracdiction $t^2 = 3$
Let $x, y \in \mathbb{Z}$, y defn of $t = \frac{x}{y}$, where x, y are the lowest possible factors of t
Thus

$$(\frac{x}{y})^2 = 3$$
$$\frac{x^2}{y^2} = 3$$
$$x^2 = 3y^2$$

Notice x^2 is a multiple of 3
By properties of prime

$$3|x^2 \implies 3|x$$

Thus for some $k \in \mathbb{Z}$

$$x^2 = (3k)^2 = 3y^2$$

Thus $3k = y^2$
This concludes $y|3$ and $x|3$.
Note, we stated x, y are the lowest possible factors of t implying there does not exist any more common factors which could reduced x, y
This is a logical inconsistency, this there can not exist a $t \in \mathbb{Q}$ s.t. $t^2 = 3$ $\square$

Question 5

Recall the **Triangle Inequality**
If $a, b \in \mathbb{R}$
Then $|a + b| \leq |a| + |b|$
Prove this inequality holds iff $ab \geq 0$
Statement P:$|a + b| \leq |a| + |b| \iff ab \geq 0$

Case 1: a and b have the same sign

a and b have the same sign $\implies ab \geq 0$: Observe

a and b are positive
$\implies |a + b| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$

a and b are negative
$\implies |(-a) + (-b)| = | - (a + b)| \equiv a + b \iff a + b \leq |a| + |b| \iff |a + b| \leq |a| + |b|$

Case 2: $a = 0 \vee b = 0$

$a = 0 \vee b = 0 \implies ab \geq 0$ Observe

If $a = 0$
$\implies |0 + b| = |b| \equiv b \iff b \leq 0 + |b| \iff |a + b| \leq |a| + |b|$

If $b = 0$
$\implies |0 + a| = |a| \equiv a \iff a \leq 0 + |a| \iff |a + b| \leq |a| + |b|$

Conclusion

Thus By Case 1,2 $|a + b| \leq |a| + |b| \iff ab \geq 0$ holds

Question 6

Show that if $a, b \in \mathbb{R}$ and $a \neq b$
Then there exists ϵ-neighborhoods U of a and V of b such that $U \cap V = \emptyset$

Proof. Suppose $a, b \in \mathbb{R}$ with $a \neq b$.
This implies the distance between them is strictly positive, i.e., $|a - b| > 0$.
We choose a radius ϵ defined as half the distance between the points:

$$\epsilon = \frac{|a - b|}{2}$$

Since $|a-b| > 0$, it follows that $\epsilon > 0$.
Let U and V be the ϵ-neighborhoods of a and b respectively:

$$U = \{x \in \mathbb{R} : |x-a| < \epsilon\}$$
$$V = \{y \in \mathbb{R} : |y-b| < \epsilon\}$$

We claim that $U \cap V = \emptyset$.
To prove this by contradiction, assume that $U \cap V \neq \emptyset$.
Then there exists an element $z \in \mathbb{R}$ such that $z \in U$ and $z \in V$.
By the definition of the neighborhoods, this implies:

$$|z-a| < \epsilon \quad \text{and} \quad |z-b| < \epsilon$$

Now, consider the distance between a and b. By the Triangle Inequality:

$$|a-b| = |(a-z)+(z-b)| \leq |a-z| + |z-b|$$

Using the property $|a-z| = |z-a|$ and substituting our inequalities for z:

$$|a-b| < \epsilon + \epsilon = 2\epsilon$$

However, recall our choice of $\epsilon = \frac{|a-b|}{2}$
This implies $2\epsilon = |a-b|$. Substituting this back into the inequality yields:

$$|a-b| < |a-b|$$

This is a contradiction (a number cannot be strictly less than itself).
Therefore, our assumption that such a z exists must be false. Thus, $U \cap V = \emptyset$. $\square$