Lecture 15

Wednesday, October 25, 2023 9:59 AM

Def Let $X = (x_n)$ be a bounded sequence.

a) The **limit superior** of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$.

$$\overline{\lim_{n\to\infty}}(x_n) \quad \text{or} \quad \limsup_{n\to\infty}(x_n)$$

$\overline{\lim}(x_n) = \inf\{V\}$ where $V = \{v \in \mathbb{R} : v < x_n \text{ for at most a finite number of } n \in \mathbb{N}\}$

$v < x_n$ for a finite number of n but $v \geq x_n$ for an infinite number of n

- If x_n is decreasing and x is an upper bound of x_n, then $x_1 \in V$

b) The **limit inferior** of (x_n) is the supremum of the set W of $w \in \mathbb{R}$ such that $w > x_n$ for at most a finite number of $n \in \mathbb{N}$.

$$\underline{\lim_{n\to\infty}}(x_n) \quad \text{or} \quad \liminf_{n\to\infty}(x_n)$$

$\underline{\lim}(x_n) = \sup\{W\}$ where $W = \{w \in \mathbb{R} : w > x_n \text{ for at most a finite number of } n \in \mathbb{N}\}$

not bounded from above

V contains all upper bounds of the tails

If $v \in V$ and $w > v$, then $w \in V$

$\liminf(x_n) = -\limsup(-x_n)$

$\sup W \leq \inf V$

For a bounded sequence (x_n), $\overline{\lim}(x_n)$ and $\underline{\lim}(x_n)$ exist.

Furthermore: $\underline{\lim}(x_n) \leq \overline{\lim}(x_n)$

Proof Let w be any element in W and v be any element in V.

$\exists\, N_w \in \mathbb{N}$ such that $w \leq x_n$ for all $n \geq N_w$ and

$\exists\, N_v \in \mathbb{N}$ such that $x_n \leq v$ for all $n \geq N_v$.

Choose $n \geq N_w$ and $n \geq N_v$. Then $w \leq x_n \leq v$, so

$\sup W \leq \inf V$; that is,

$$\overline{\lim}\, x_n \leq \underline{\lim}\, x_n. \quad \blacksquare$$

limit exists when $\overline{\lim} = \underline{\lim}$

Theorem A bounded sequence is convergent if and only if
$\overline{\lim}\, x_n = \underline{\lim}\, x_n$.

Proof ($\rightarrow$) Let x be the limit of (x_n).

For any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$,
$|x_n - x| < \varepsilon$.

$$x - \varepsilon < x_n < x + \varepsilon$$

Since $x_n < x + \varepsilon$, $x + \varepsilon \in V$. So, $\inf V \leq x + \varepsilon$.

Since $x_n > x - \varepsilon$, $x - \varepsilon \in W$. So, $\sup W \geq x - \varepsilon$.

So, we have: $x - \varepsilon \leq \sup W \leq \inf V \leq x + \varepsilon$.

Let $\varepsilon \downarrow 0$ (via $\varepsilon = \frac{1}{n}$). Then

$\downarrow$ denotes decreasing

$$x \leq \sup W \leq \inf V \leq x$$

So, $x = \sup W = \inf V$.

($\leftarrow$) Let x be the common number of $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$,
and suppose BWOC (x_n) does not converge to x. Then $\exists\, \varepsilon_0 > 0$
and a subsequence (x_{n_k}) of (x_n) such that

$$|x_{n_k} - x| \geq \varepsilon_0 .$$

So, $x_{n_k} - x \geq \varepsilon_0$ or $x_{n_k} - x < -\varepsilon_0$ (for a finite number of $k \in \mathbb{N}$).

* x_n could converge to a different number or be divergent
It doesn't matter

If $x_{n_k} - x \geq \varepsilon_0$ is true for an infinite number of $k \in \mathbb{N}$, then
$x + \varepsilon_0 \notin V$, then $x + \varepsilon_0$ is a lower bound of V. Thus,
$x + \varepsilon_0 \leq \overline{\lim}\, x_n = x$, which is a contradiction.

If $x_{n_k} - x < -\varepsilon_0$ is true for an infinite number of $k \in \mathbb{N}$, then
$x - \varepsilon_0 \in W$, so $x - \varepsilon_0$ is an upper bound of W. So,
$x - \varepsilon_0 \geq \sup W = \underline{\lim}\, x_n = x$. Contradiction!

Weining "you should think a little bit" Kang

Hint for 19

19. a) $\overline{\lim}(x_n + y_n) \leq \overline{\lim}(x_n) + \overline{\lim}(y_n)$

b) $\underline{\lim}(x_n + y_n) \leq \underline{\lim}(x_n) + \underline{\lim}(y_n)$

$$\underline{\lim}(x_n + y_n) = -\overline{\lim}(-x_n + (-y_n))$$
$$= -\left(\overline{\lim}(-x_n) + \overline{\lim}(-y_n)\right)$$
$$= \underline{\lim}(x_n) + \underline{\lim}(y_n)$$

Section 3.5 Cauchy Criterion

Def) A sequence $X = (x_n)$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that $|x_n - x_m| < \varepsilon \ \forall m, n \geq N(\varepsilon)$.

One major difference is that the limit is not needed for this to show convergence

Ex $x_n = \frac{1}{n}$

To show this is Cauchy, we need to show $\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$.

Triangle Inequality! : $\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \varepsilon$.

$\Rightarrow \frac{1}{n} < \frac{\varepsilon}{2}$ and $\frac{1}{m} < \frac{\varepsilon}{2}$

For $\frac{2}{\varepsilon}$, $\exists \ N(\varepsilon)$ such that $N(\varepsilon) > \frac{2}{\varepsilon}$ (A.P.)

Then, if $m, n \geq N(\varepsilon)$, where $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\frac{1}{m} < \frac{\varepsilon}{2}$

Then $\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon$.

Ex $(-1)^n$

$x_n = (-1)^n$

$x_m = (-1)^{n+1}$ where $m = n+1$

$|x_n - x_{n+1}| = 2$ no matter what n is

So $(-1)^n$ diverges ✓

Cauchy implies convergent!

Ex $x_n = \sum \frac{1}{n}$

Let $m = 2n$.

$$|x_n - x_{2n}| = \frac{1}{n+1} + \dots + \frac{1}{2n} \geq \underbrace{\frac{1}{2n} + \dots + \frac{1}{2n}}_{n} = \frac{1}{2} \quad \forall n \in \mathbb{N}$$

He just told a fun story about a farmer collecting a single grain of rice and doubling it with every game he won or something

Anyways at the end, the farmer had a whole pound

It seems small, but when you accumulate it, it becomes a lot (like $\sum \frac{1}{2^n}$)

Ooo he also mentioned the quiz, will cover Ch3