Section 2.3: 5, 6, 8, 10

Section 2.4: 2, 7, 9, 14, 15

① 2.3.5

a) $A := \{x \in \mathbb{R} : 2x+5 > 0\}$

$A = (-5/2, \infty) = \{x \in \mathbb{R} : -5/2 < x < \infty\}$

inf $A = -5/2$

sup A = does not exist

b)

$B := \{x \in \mathbb{R} : x+2 \geq x^2\}$

$x+2 \geq x^2$

$-x^2+x+2 \geq 0$

$x^2-x-2 \leq 0$

$(x+1)(x-2) \leq 0$

$-1 \leq x \leq 2$

$B = [-1, 2] = \{x \in \mathbb{R} : -1 \leq x \leq 2\}$

inf $B = -1$

sup $B = 2$

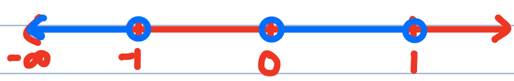
c)

$C := \{x \in \mathbb{R} : x < 1/x\}$

$x \neq 0$, $x^2-1 < 0$

$(x+1)(x-1) < 0$

$0 < x < 1$, $x < -1$



$C = (-\infty, -1) \cup (0, 1)$

inf C = does not exist

sup $C = 1$

d)

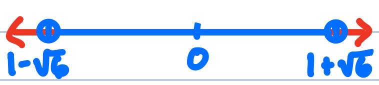
$D := \{x \in \mathbb{R} : x^2-2x-5 < 0\}$

$x^2-2x-5 < 0$

$x^2-2x-5 = 0$

$$x = \frac{+2 \pm \sqrt{(-2)^2-4(1)(-5)}}{2} = \frac{2 \pm \sqrt{4+20}}{2} = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2} = 1 \pm \sqrt{6}$$

$1-\sqrt{6} < x < 1+\sqrt{6}$



inf $D = 1-\sqrt{6}$

sup $D = 1+\sqrt{6}$

② 2.3.6

If S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$

Proof:
Let $S' = \{-s : s \in S\}$. Let $w = \inf S$. We need to show that $-w = \sup S'$. First we will show that $-w$ is an upperbound of S'. By definition of infinum, $w = \inf S$ in which implies that for every $s \in S$, $w \leq s$. Thus, for every $s \in S$, $-w \geq -s$. Thus for every $-s \in S'$, $-w \geq -s$. This shows that $-w$ is an upperbound for S'.

Secondly, we will show that for every $r \in \mathbb{R}$ such that $r < -w$ there exists $-s \in S'$ such that $r < -s$. Let r be an arbitrary real number such that $r < -w$. Then, $w < -r$. Since $w = \inf S$, $\exists s \in S$ s.t. $s < -r$. Thus, $\exists s \in S$ such that $r < -s$, but this also means $\exists -s \in S'$ such that $r < -s$. This shows $-w$ is the least upper bound for S'

$\therefore \inf S = -\sup\{-s : s \in S\}$

+10

③ 2.3.8

Proof:
$\boxed{\Rightarrow}$

Assume u is an upper bound of S. Let $t \in \mathbb{R}$ such that $u < t$. Then, for every $s \in S$, $s \leq u < t$. Thus, $t \notin S$.

$\boxed{\Leftarrow}$

Let $u \in \mathbb{R}$ such that for every $t \in \mathbb{R}$, $u < t$ implies $t \notin S$. If there exists an $s \in S$ where $s > u$, then $s \notin S$. Thus, $s \leq u$

$\therefore \Rightarrow \Leftarrow$ the statement holds

④ 2.3.10

Let $u = \sup A$, $v = \sup B$, and $w = \sup\{u, v\}$. If $x \in A$, then $x \leq u \leq w$, and if $x \in B$, then $x \leq v \leq w$. Hence $A \cup B$ is a bounded set.

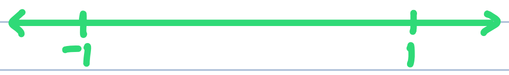
If z is any upperbound of $A \cup B$, then z is an upperbound of A and of B, so that $u \leq z$ and $v \leq z$. Hence $w \leq z$.
$\therefore w = \sup(A \cup B)$

+7

⑤ 2.4.2

$S = \{1/n - 1/m : n, m \in \mathbb{N}\}$

$\frac{1}{1000000...} - \frac{1}{1} \approx -1$



$\inf S = -1$
$\sup S = 1$

For $\inf S = -1$. Let $1/n - 1/m$ be an arbitrary element in S. Then $1/n - 1/m \geq 1/n - 1 > -1$. So -1 is a lower bound of S. Let $\epsilon > 0$. By the Corollary of 2.4.5, there exists $n_0 \in \mathbb{N}$ s.t. $1/n_0 < \epsilon$. Now, $1/n_0 - 1 < \epsilon - 1 = -1 + \epsilon$ and $1/n_0 - 1 \in S$. Thus, $-1 = \inf S$. ✓

We claim $\sup S = 1$. From the proof of 2.3.6 we proved that $\inf S = -\sup\{-s : s \in S\}$. So we get $-1 = \inf S = -\sup\{-s : s \in S\} = -\sup S$ which implies $\sup S = 1$. ✓

+10

⑥ 2.4.7

Proof:

Let $u = \sup A$, $v = \sup B$, $w = \sup(A+B)$. If $x \in A$ and $y \in B$, then $x + y \leq u + v$, so that $w \leq u + v$. Now, fix $y \in B$ and note that $x \leq w - y$ $\forall x \in A$; thus $w - y$ is an upperbound for A so that $u \leq w - y$. Then $y \leq w - u$ $\forall y \in B$, so $v \leq w - u$ and hence $u + v \leq w$. By combining the inequalities we get the following $u + v \leq w \leq u + v$, meaning $w = u + v$. $\therefore \sup(A+B) = \sup(A) + \sup(B)$.

Let $u = \inf S$, $v = \inf B$, $w = \inf(A+B)$. If $x \in A$ and $y \in B$, then $x + y \geq u + v$, so that $w \geq u + v$. Now, fix $y \in B$ and note that $x \geq w - y$ $\forall x \in A$; thus $w - y$ is a lower bound for A so that $u \geq w - y$. Then $y \geq w - u$ $\forall y \in B$, so $v \geq w - u$ and hence $u + v \geq w$. By combining the inequalities we get the following $u + v \geq w \geq u + w$, meaning $w = u + v$. $\therefore \inf(A+B) = \inf(A) + \inf(B)$.

⑦ 2.4.9

a) $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$, $h : X \times Y \to \mathbb{R}$ by $h(x,y) := 2x + y$
$x \in X$, $f(x) := \sup\{h(x,y) : y \in Y\}$, $\inf\{f(x) : x \in X\}$

let $y = 1$, where 1 is the least upperbound of set Y, $\sup\{h(x,1)\} = f(x) = 2x + 1$
let $x = 0$ which is the greatest upperbound of set X, $\inf\{f(0)\} = 1$

b) $y \in Y$, $g(y) := \inf\{h(x,y) : x \in X\}$, $\sup\{g(y) : y \in Y\}$

let $x = 0$, where 0 is the greatest lowerbound of set X, $\inf\{h(0,y)\} = g(y) = y$
let $y = 1$, where 1 is the least upperbound of set Y, $\sup\{g(1)\} = 1$

It seems that the $\inf\{f(x)\} = \sup\{g(y)\}$ in some sense where the upper and lower bounds are constructed in a way of individulaizing the function by swapping inf and sup.

⑧ 2.4.14

By the Archimedean Property of $\mathbb{R}$

If y be any positive real number, then there exists a positive integer n such that $n>y$

$\frac{1}{n}<y \;\; \forall n \in \mathbb{N}$

Now show: $n<2^n \;\forall n \in \mathbb{N}$

Prove by induction

Base Case: $n=1$, $1<2$

Try for $n=k$, $k<2^k$

now apply $n=k+1$, $k+1<2^{k+1}$

We get $k<2^k$

$k+1<2^k+1<2^k \cdot 2$

$<2^{k+1}$

Hence $n<2^n \;\forall n \in \mathbb{N}$

$\frac{1}{n}>\frac{1}{2^n}$

Now we combine inequalities: $\frac{1}{2^n}<\frac{1}{n}<y \;\; \forall n \in \mathbb{N}$

$\therefore \frac{1}{2^n}<y$

+10

⑨ 2.4.15

+2

Consider $S:=\{y \mid y \in \mathbb{R}, y^3<2\}$

1 is in S so S is nonempty. The set S also has an upper bound, 2 is an upperbound for S, for if y is in S and $y>2$, then $y^3>2^3=8$, a contradiction. Therefore the supremum property implies S has at least upper bound; say, $c=\sup(S)$. Clearly $c\geq 1$. We claim $c^3=2$ by contradiction.

Assume that $c^3>2$. I will find an $\alpha>0$ so that $(c+\alpha)^3<2$ implying that c is an upperbound. To find α, consider

$(c+\alpha)^3<2 \Leftrightarrow c^3+3\alpha c^2+3\alpha^2 c+\alpha^3<2$

$\Leftrightarrow 3c^2\alpha+3c\alpha^2+\alpha^3<2-c^3$

$\Leftrightarrow 0<\alpha<1 \Rightarrow 3c^2\alpha+3c\alpha+\alpha<2-c^3$ as $\alpha^2<\alpha$ and $\alpha^3<\alpha$

$\Leftrightarrow \alpha(3c^2+3c+1)<2-c^3$

Since $2-c^3>0$, we can choose α so that $0<\alpha<1$ and $\alpha<\frac{(2-c^3)}{(3c^2+3c+1)}$

With this α, we have $(c+\alpha)^3<2$ a contradiction.

Now suppose $c^3>2$. I will find a $b>0$ so that $(c-b)^3>2$. If so then $y^3<2<(c-b)^3\ \forall y$. As a result, $y<c-b\ \forall y$. This means that $c-b$ is an upper bound violating the minimality of the supremum.
To find a b, consider: $(c-b)^3>2 \Leftrightarrow c^3-3bc^2+3cb^2-b^3>2$

$$\Leftrightarrow 3bc^2-3cb^2+b^3<c^3-2$$
$$\Leftrightarrow 0<b<1 \Rightarrow 3bc^2-3bc+b<c^3-2$$
$$\Leftrightarrow b(3c^2-3c+1)<c^3-2$$

Then, since $c^3-2<0$ we may choose b such that $0<b<1$ and $b<\frac{(c^3-2)}{(3c^2+3c+1)}$. With this b, we have $(c-b)^3<2$, a contradiction.

Hence $c^3=2$.
$\therefore\ y^3=2$