hw 7

① **3.4.19** $\times$ +0

If $v > \limsup(x_n)$ and $u > \limsup(y_n)$, then there are at most finitely many n such that $x_n > v$ and at most finitely many n such that $y_n > v$. Therefore, there are at most finitely many n such that $x_n + y_n > v + u$, which implies $\limsup(x_n + y_n) \le v + u$, which implies $\limsup(x_n + y_n) \le v + u$. This proves that stated inequality. So, here is a counterexample, one can take $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

② **3.5.2b**

$$\left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$$

Prove that $\frac{1}{n!} < \frac{1}{2^n}$ as long as $n \ge 4$. Then by using induction on m, prove that

$$\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m}} \le \frac{1}{2^{n-1}}$$

Now if $\epsilon > 0$ is given, choose N so that if $n > N$ then $\frac{1}{2^n} < \epsilon$. Then if $n > m > N$, we have

$$\left|1 + \frac{1}{2!} + \cdots \frac{1}{n!} - \left(1 + \frac{1}{2!} + \cdots + \frac{1}{m!}\right)\right| = \left|\frac{1}{(m+1)!} + \cdots + \frac{1}{(m+n)!}\right| \le \left|\frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{m+n}}\right|$$

$\therefore \frac{1}{2^m} \le \epsilon$, so the sequence is Cauchy +10

③ **3.5.3c**

Take $m = 2n$, so $x_m - x_n = x_{2n} - x_n = \ln(2n) - \ln(n) = \ln 2$, for all n

④ **3.5.4**

Let $x = \lim x_n$ and $y = \lim y_n$. We need to show that $e > 0$ there is natural N so that $n = N$, then $|(x_n + y_n) - (x + y)| = e$. Given any $e > 0$ we have $\frac{e}{3} > 0$ so from the definition of convergence there is an natural number N_x so that $|x_n - x| = \frac{e}{3}$ for all $n > N_x$; similarly we can choose N_y, $|y_n - y| = \frac{e}{3}$ for all $n > N_y$.
Let $N = \max(N_x, N_y)$. If $n > N$, then by triangle inequality we have $|x_n + y_n - (x + y)| = |(x_n - x) + (y_n - y)| < \frac{e}{3} + \frac{e}{3} < \frac{2}{3}e < e$.

Let $x = \lim x_n$ and $y = \lim y_n$. Since these sequences are convergent they are bounded. Let M_x be a bound for x_n and let M_y be a bound for y_n, by increasing these quantities of necessary we may also assume $M_x > x$ and $M_y > y$. Given $e > 0$ there exists some N_x and N_y such that $|x_n - x| < e/2M_y$ for $n > N_x$ and $|y_n - y| < e/2M_x$ for $n > N_y$ then for every $n > \max(N_x, N_y)$, $|x_n y_n - xy| = |(x_n - x) y_n + x(y_n - y)| \to = |x_n - x| M_y + |y_n - y| M_x = e/2 + e/2 = e$ +10

⑤ 3.5.7

Suppose that $\{x_n\}$ is a Cauchy sequence and that $x_n \in \mathbb{Z}, \forall n \in \mathbb{N}$. Let $\epsilon = \frac{1}{2}$. Then there exists $K \in \mathbb{N}$ such that $|x_m - x_n| < \frac{1}{2} = \epsilon$, for all $m, n \geq K$, but $|x_m - x_n|$ is not negative integer, so $|x_m - x_n| < \frac{1}{2}$ is possible only if $|x_m - x_n| = 0$, if $x_m = x_n$. $\therefore x_n = x_K, \forall n \in K$, which means that the sequence is ultimately constant sequence.

⑥ 3.6.1

Assume that x_n is a monotone and unbounded sequence. So, the sequence x_n is unbounded above. That means, x_n has no upper bound. Then for each integer k, there is some integer n_k s.t. $x_{n_k} > k$. That is, $x_1 < x_2 < \ldots\ldots$, and $|x_{n_k}| > k, \forall k = 1, 2, 3, \ldots\ldots$

Let y_k be an element in x_n. Then the sequence y_k is a subsequence of x_n which is divergent. Hence, an unbounded monotone sequence must be a monotone sequence which is properly divergent.

+5

⑦ 3.6.3

Given that $x_n > 0$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} x_n = 0$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $|x_n| < \varepsilon, \forall n \geq N$. Then $\left|\frac{1}{x_n}\right| > \frac{1}{\varepsilon}$ for all $n \geq N$. Then $\left\{\frac{1}{x_n}\right\}$ diverges and $\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = +\infty$.

Conversely suppose that $\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = +\infty$, i.e. $\left\{\frac{1}{x_n}\right\} \to \infty$ as $n \to \infty$, for all $\varepsilon > 0$, there exists $m \in \mathbb{N}$ s.t. $\left|\frac{1}{x_n}\right| > \frac{1}{\varepsilon}$ for all $n \geq m$. Then $\frac{1}{\left(\left|\frac{1}{x_n}\right|\right)} < \varepsilon$ for all $n \geq m$. Then $|x_n| < \varepsilon$ for all $n \geq m$.

+10

$\therefore x_n$ converges to 0 as $n \to \infty$. $\therefore \lim_{n\to\infty} x_n = 0$.

⑧ 3.6.7

a) Let $\varepsilon = 1$ then there exists $N \in \mathbb{N}$ s.t. $\left|\frac{x_n}{y_n} - 0\right| < \varepsilon, \forall n > N$. $\frac{x_n}{y_n} < 1$, Since $\varepsilon = 1$ and x_n, y_n positive. $x_n < y_n, \forall n > N$. $\lim(x_n) < \lim(y_n) \Rightarrow +\infty < \lim(y_n) \Rightarrow \lim(y_n) = +\infty$.

b) Suppose y_n is bounded that there exist, $M > 0$ such that $y_n < M, \forall n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. $\frac{\varepsilon}{M} > 0$. There exist $N \in \mathbb{N}$ such $\frac{x_n}{y_n} < \frac{\varepsilon}{M}$. Since $\lim \frac{x_n}{y_n} = 0$. Now, $x_n = \frac{x_n}{y_n} y_n \leq \frac{\varepsilon}{M} \cdot M = \varepsilon, \forall n > N$.

So for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $x_n < \varepsilon \; \forall n > N$. Hence $\lim(x_n) = 0$.