Lecture 19

Monday, November 13, 2023 10:16 AM

Section 4.1 Limits of Functions

Def Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if $\forall \delta > 0$, there exists at least one point $x \in A$, $x \neq c$, such that $|x - c| < \delta$.

Theorem A number $c \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (x_n) in A such that

$$\lim (x_n) = c \quad \text{and} \quad x_n \neq c \quad \forall n \in \mathbb{N}$$

Examples – find the cluster points

- $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

  0 is the only cluster point

- $A = (0, 1)$

  $[0, 1]$

- $A = \mathbb{Q}$

  $\mathbb{Q} \cup \mathbb{P}$ or $\mathbb{R}$

  (density property of rationals)

Def Let $A \subseteq \mathbb{R}$ and c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit** of f at c if:

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This is denoted as $\lim_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$.

If the limit of f at c does not exist, we say f **diverges** at c.

Neighborhood version of def:

Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.

Theorem If $f: A \to \mathbb{R}$ and c is a cluster point of A, then f can have at most one limit at c.

Ex $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$

Notes: $\left| \frac{1}{x} - \frac{1}{c} \right| < \varepsilon$

$\Rightarrow \frac{|x-c|}{|xc|} < \varepsilon$

$\frac{1}{|xc|} \cdot |x-c| < \varepsilon$ and we want $|x-c| < \delta$

We want $\frac{1}{|xc|}$ to be bounded by some constant (say D)

So, $\frac{1}{|xc|} < D$

So $D|x-c| < \varepsilon \Rightarrow |x-c| < \frac{\varepsilon}{D}$

So, we have $\delta = \min\{\_\_, \frac{\varepsilon}{D}\}$

[redacted] I might be a missing a step

If $|x-c| < \frac{c}{2}$, then $\frac{c}{2} < x < \frac{3c}{2}$, then $\frac{1}{|xc|} = \frac{1}{xc} < \frac{1}{\frac{c}{2} \cdot c} = \frac{2}{c^2}$.

So, choose $\delta = \min\left\{ \frac{c}{2}, \frac{\varepsilon}{\frac{2}{c^2}} \right\}$

Proof Let $\varepsilon > 0$ be given. Let $\delta = \min\left\{ \frac{c}{2}, \frac{\varepsilon}{\frac{2}{c^2}} \right\}$. If $x \in A$ and $0 < |x-c| < \delta$, then

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x-c|}{|xc|} < \frac{2}{c^2}|x-c| < \frac{\cancel{2}}{\cancel{c^2}} \cdot \frac{\varepsilon}{\frac{2}{c^2}} = \varepsilon. \quad \blacksquare$$

Ex $\lim_{x \to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}$

Notes: $\left| \frac{x^2-x+1}{x+1} - \frac{1}{2} \right| < \varepsilon$

$\Rightarrow \left| \frac{2x^2 - 3x + 1}{2(x+1)} \right| < \varepsilon$

This is 9d in 4.1

good ol' factoring: $\frac{|2x-1|}{2|x+1|} \cdot |x-1| < \varepsilon$

* missed a little here

$< \frac{2}{3}|x-1|$ if $|x-1| < \frac{1}{2}$

So choose $\delta = \min\left\{ \frac{1}{2}, \frac{\varepsilon}{2/3} \right\}$

Proof Let $\varepsilon > 0$ be given and let $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2/3}\right\}$. If $x \in A$

$0 < |x-c| < \delta$, then

$$\left| \frac{x^2 - x + 1}{x+1} - \frac{1}{2} \right| = \frac{|2x^2 - 3x + 1|}{2|x-1|} = \frac{|2x-1|}{2|x+1|} \cdot |x-1|$$

* could also use sequential criterion to prove this

Sequential Criterion

1) $\lim_{x \to c} f(x) = L$

2) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ $\forall n \in \mathbb{N}$, then $(f(x_n))$ converges to L.

Divergence Criterion (corollary of sequential criterion)

$f(x)$ does not converge to L if and only if there exists a sequence (x_n) in A that converges to c, and $x_n \neq c$ for all $n \in \mathbb{N}$, but $(f(x_n))$ does not converge to L.

Examples

- $\lim_{x \to 0} \frac{1}{x}$ does not exist.

  Choose $x_n = \frac{1}{n} \to 0$ and $x_n \neq 0$

  $f(x_n) = \frac{1}{\frac{1}{n}} = n \to \infty$ as $n \to \infty$

  So $f(x_n)$ does not converge to any L.

- $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Create two sequences

$x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ $\quad \sin\left(\frac{1}{x_n}\right) = 1$

$y_n = \frac{1}{2n\pi}$ $\quad \sin\left(\frac{1}{y_n}\right) = 0$

If $L \neq 1$, choose (x_n), then $f(x_n)$ does not converge to L.

If $L = 1$, choose (y_n), then $f(y_n)$ does not converge to L.

Exam on Monday, covers only Ch3

Wed will be a review

There will be class on the Wed before Thanksgiving

* he ended up cancelling