

$$x_n(y_n - y) \leq C |y_n - y| < \varepsilon$$

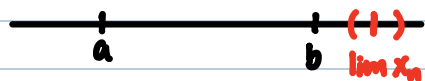
$$|y_n - b| < \frac{\varepsilon}{C}$$

$$\frac{a_n < C}{b_n > d}$$

$$\frac{1}{y^n}$$

Thm: If (x_n) is a convergent sequence and

$$a \leq x_n \leq b \quad \underbrace{\text{for all } n \in \mathbb{N}}_{\substack{\text{(for all } n \text{ large)} \\ \text{enough}}}; \text{ then } a \leq \lim x_n \leq b$$



$$|x_n - x| < \varepsilon$$

$$x - \varepsilon < x_n < x + \varepsilon$$

$$\varepsilon < \frac{\lim x_n - b}{2}$$

Thm: (squeeze theorem)

Suppose $x = (x_n)$, $y = (y_n)$, and $z = (z_n)$ are sequences such that

$$x_n \leq y_n \leq z_n \quad \underbrace{\forall n \in \mathbb{N}}_{\substack{\text{(for all } n \text{ large)} \\ \text{enough}}} \quad \text{if } \lim x_n = x \text{ and } \lim z_n = z$$

with $x = z$ then y converges with $\lim_{n \rightarrow \infty} y_n = z = x$

Proof: Since $x_n \rightarrow x = z$, but $z_n \rightarrow z$, for each $\varepsilon > 0$, there exists $N_x(\varepsilon)$ and $N_z(\varepsilon)$, such that $|x_n - x| = |x_n - z| < \varepsilon$ for all $n \geq N_x(\varepsilon)$

$$|z_n - z| < \varepsilon \quad \forall n \geq N_z(\varepsilon)$$

Let $N(\varepsilon) = N_x(\varepsilon) + N_z(\varepsilon)$, for all $n \geq N(\varepsilon)$, we have $x_n > z - \varepsilon$, $z_n < z + \varepsilon$.

Since $x_n \leq y_n \leq z_n \quad \forall n \in \mathbb{N}$, then for all $n \geq N(\varepsilon)$, $z - \varepsilon < x_n \leq y_n \leq z_n < z + \varepsilon$, So $|y_n - z| < \varepsilon \quad \forall n \geq N(\varepsilon)$ so $y_n \rightarrow z$, any $n \rightarrow \infty$.

$$\frac{|y_n - z| < \varepsilon}{\updownarrow}$$

$$z - \varepsilon < y_n < z + \varepsilon$$

$$\uparrow$$

$$\underline{x_n \leq y_n \leq z_n, \quad x_n > z - \varepsilon, \quad z_n < z + \varepsilon}$$

$$\uparrow$$

$$\uparrow$$

$$|y_n - x| < \varepsilon$$

$$|z_n - z| < \varepsilon$$

Ex: $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$

$$-1 \leq \sin n \leq 1$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

$$\underbrace{-\frac{1}{n}}_0 \leq \underbrace{\frac{\sin n}{n}}_0 \leq \underbrace{\frac{1}{n}}_0$$

if a_n is bounded $b_n \rightarrow b$ as $n \rightarrow \infty$.

Thm: If $x_n \rightarrow x$, then $|x_n| \rightarrow |x|$

Proof: By the inequality $||x_n| - |x|| \leq |x_n - x|$

Thm: If $x_n \rightarrow x$ and $x_n \geq 0$ for all $n \in \mathbb{N}$
then $\lim \sqrt{x_n} = \sqrt{x}$

$$|\sqrt{x_n} - \sqrt{x}| < \varepsilon, \quad |x_n - x| < \varepsilon$$

Case 1: $x = 0$, Case 2: $x > 0$

$$\frac{\sqrt{a} - \sqrt{b}}{1} = \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}}$$

$$|\sqrt{x_n} - \sqrt{x}| = \sqrt{x_n} < \varepsilon$$

$$\uparrow$$

$$\underline{x_n < \varepsilon^2}$$

$$|x_n - x| < \varepsilon$$

$$\uparrow$$

$$\underline{x_n < \varepsilon}$$

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} = \frac{1}{\sqrt{x_n} + \sqrt{x}} |x_n - x| \leq \frac{1}{\sqrt{x}} |x_n - x| < \varepsilon$$

$\forall \varepsilon > 0, \exists N(\varepsilon) > 0$, such that $|x_n - x| < \varepsilon^2$, that is, $x_n < \varepsilon^2$, $\forall n \geq N(\varepsilon)$

$|x_n - x| < \sqrt{x} \varepsilon$
 $\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t.
 $|x_n - x| < \sqrt{x} \varepsilon \quad \forall n \geq N(\varepsilon)$

So this inequality inequality implies

$$(x_n)^{\frac{1}{2}} < (\varepsilon^2)^{\frac{1}{2}} \text{ for all } n \geq N(\varepsilon)$$

$$\text{Then } |\sqrt{x_n} - \sqrt{x}| < \frac{1}{\sqrt{x}} |x_n - x| < \varepsilon$$

then $\sqrt{x_n} < \varepsilon$ for all $n \geq N(\varepsilon)$

for all $n \geq N(\varepsilon)$

$$|\sqrt{x_n} - \sqrt{x}| < \varepsilon$$

Thm: (Ratio Test)

Let (x_n) be a sequence of positive real numbers such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = L.$$

If $L > 1$, then (x_n) converges and $\lim x_n = 0$.

Ex: $x_n = b^n$, where $0 < b < 1$

$$b^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{b^n} = b < 1$$

Section 3.3: Monotone Sequences

Let $X = (x_n)$ be a sequence

X is increasing if it satisfies

$$x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq x_{n+1} \leq \dots$$

(To verify X is increasing, we need to prove $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
(we use math induction)

X is decreasing if it satisfies

$$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq x_{n+1} \geq \dots$$

(To verify X is decreasing, we can prove $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ using induction)
we can also prove $-x = (-x_n)$ is increasing.

Thm: (Monotone Convergence Thm)

A monotone sequence of real numbers is convergent if and only if it is bounded and:

a) If $X = (x_n)$ is bounded increasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup \{x_n, n \in \mathbb{N}\}$$

b) If $X = (x_n)$ is bounded decreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf \{x_n, n \in \mathbb{N}\}$$

If a) is proved.

$$(x_n) \rightarrow (-x_n)$$

by $-(x_n)$ is convergent and

$$\lim_{n \rightarrow \infty} -x_n = \sup \{-x_n, n \in \mathbb{N}\}$$

$$\lim_{n \rightarrow \infty} x_n = (-1) \lim_{n \rightarrow \infty} (-x_n) = (-1) \sup \{-x_n, n \in \mathbb{N}\} \\ = (-1)(-1) \sup \{x_n, n \in \mathbb{N}\}$$

$$\inf \{x_n, n \in \mathbb{N}\}$$

$$x_n \quad \text{(-}x_n\text{)}$$

$$\uparrow \quad (-1)(-x_n)$$

$$\lim(x_n) = (-1) \lim(-x_n)$$

Proof of a) Since (x_n) is bounded increasing, then $\sup\{x_n, n \in \mathbb{N}\}$ exists in \mathbb{R} by the Completeness property
Let $x^* = \sup\{x_n, n \in \mathbb{N}\}$

Note that $x_n \leq x^*$ for all $n \in \mathbb{N}$ since x^* is the supremum of $\{x_n, n \in \mathbb{N}\}$ then $x_n \leq x^* + \varepsilon$ for all $n \in \mathbb{N}$.
Since $x^* - \varepsilon$ is not an upper bound of $\{x_n, n \in \mathbb{N}\}$

$\exists N(\varepsilon)$ such that $x_{N(\varepsilon)} > x^* - \varepsilon$

For all $n \geq N(\varepsilon)$, $x_n \geq x_{N(\varepsilon)} > x^* - \varepsilon$ since x is increasing

So for all $n \geq N(\varepsilon)$, $x^* - \varepsilon < x_n < x^* + \varepsilon$ that in $|x_n - x^*| < \varepsilon$

So $x_n \rightarrow x^*$

Ex: Let $x = (x_n)$ be defined by $x_1 = 1$
 $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$.

Show that $\lim_{n \rightarrow \infty} x_n = 2$.

Proof: We prove by induction that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

$$\begin{aligned} \text{a) } (n=1) \quad x_1 &= 1 < 1.414 \approx \sqrt{2} = x_2 \\ (x_2 &= \sqrt{2} > \sqrt{1} = 1 = x_1) \end{aligned}$$

$$x_{k+1} \leq x_{k+2}$$

b) Suppose $x_k \leq x_{k+1}$

$$\text{c) } x_{k+1} = \sqrt{2x_k} \leq \sqrt{2 \cdot x_{k+1}}$$

② We know (x_n) is bounded, $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$, by induction.

$$\text{a) } (n=1): x_1 = 1, 1 \leq x_1 \leq 2$$

$$\text{b) suppose } 1 \leq x_n \leq 2$$

$$\text{c) } 2 \leq 2x_k \leq 4, \text{ then } 2^{\frac{1}{2}} \leq (2x_k)^{\frac{1}{2}} \leq 4^{\frac{1}{2}}$$

$$\text{So } \sqrt{2} \leq x_{k+1} \leq 2. \text{ So } 1 \leq x_{k+1} \leq 2.$$

③ Let x^* be the limit.

$$x^* = \sqrt{2x^*} \text{ then } x^* = 0 \text{ or } x^* = 2.$$

x^* cannot be 0, because x_{k+1} is bounded between 1 and 2.

$$\therefore x^* = 2.$$