Exam 1 Review

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Topics

- finite/infinite sets
- countable/uncountable sets
  - surjection/bijection proof
- Algebraic Properties of $\mathbb{R}$
- Irrational Number proof ($\sqrt{2}$, $\sqrt{3}$, etc.)
- Ordering Properties of $\mathbb{R}$
  - $\mathbb{R}^+$
- Absolute Values
- Triangle Inequality $|a+b| \leq |a| + |b|$
- ε neighborhoods
- Supremum/infimum
  - Completeness Property
  - Archimedean Property
- Density Theorem
- Intervals

Section 1.3

A set S is finite if it is empty or has n elements
- There exists a bijection from $\mathbb{N}_n \to S$

Otherwise, S is infinite.

Suppose S, T are sets and $T \subseteq S$. If S is finite, T is finite

Proof If $T = \varnothing$, then by def T is finite.

Suppose $T \neq \emptyset$. We use proof by induction on the number of elements in S.

Base Case: Suppose S has 1 element. Then the only nonempty subset of S is itself, so T is finite.

I.H.: Assume that every nonempty subset T of S with k elements is finite.

I.S.: We want to show that every T with $k+1$ elements is finite.

Let S have $k+1$ elements. Then there exists a bijection f of $\mathbb{N}_{k+1}$ onto S. $(T \subseteq S)$

Case 1: $f(k+1) \notin T$ — If $f(k+1) \notin T$, then $T \subseteq S_1 = S \setminus \{f(k+1)\}$, which has k elements. Thus, T is finite.

Case 2: $f(k+1) \in T$ — If $f(k+1) \in T$, then $T_1 = T \setminus \{f(k+1)\} \subseteq S_1$, which has k elements so T_1 is finite. Thus $T = T_1 \cup \{f(k+1)\}$ is also finite. ■

A set is denumerable (countably infinite) if there exists a bijection of $\mathbb{N}$ onto the set.

A set is countable if it is either finite or denumerable; otherwise, it is uncountable.

$\Rightarrow$ A set is denumerable iff. $\exists$ a bijection of $\mathbb{N}$ onto S.

Theorem 1.3.9 Suppose that S and T are sets and $T \subseteq S$. If S is countable, then T is countable.

Theorem 1.3.10 S is countable iff. there exists a surjection of $\mathbb{N}$ onto S.

Theorem (again) If A_m is a countable set $\forall m \in \mathbb{N}$, then $A := \bigcup_{m=1}^{\infty} A_m$ is countable.

Proof: Easiest way is to draw out a surjection of $\mathbb{N}$ onto A_n $\longrightarrow$

$$\begin{array}{llll} A_1 = & a_{11} & a_{12} & a_{13} \cdots \\ A_2 = & a_{21} & a_{22} & a_{23} \cdots \\ \vdots & & & \\ A_n = & a_{n1} & a_{n2} & a_{n3} \cdots \end{array}$$

Since there is a surjection, it is countable. ■

Section 2.1

Algebraic Properties of $\mathbb{R}$: $(\forall a, b, c \in \mathbb{R})$

(A1) $a+b=b+a$ (commutative property of addition)

(A2) $(a+b)+c = a+(b+c)$ (associative property of addition)

(A3) There exists a 0 element $\in \mathbb{R}$ such that $0+a=a$ and $a+0=a$ (existence of a 0 element)

(A4) $\forall a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a+(-a)=0$ and $(-a)+a=0$ (existence of negative elements)

(existence of negative elements)

(M1) $a \cdot b = b \cdot a$ (commutative property of multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative property of multiplication)

(M3) $\exists$ an element $1 \in \mathbb{R}$ distinct from 0 such that $1 \cdot a = a \cdot 1 = a$

(existence of a unit element)

(M4) $\forall a \neq 0$, there exists $\frac{1}{a} \in \mathbb{R}$ such that $a \cdot (\frac{1}{a}) = 1$ (existence of reciprocals)

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

(distributive property of multiplication over addition)

From these, it follows that:

- If $z + a = a$, then $z = 0$
- $a \cdot 0 = 0$
- If $a \cdot b = 0$, then either $a = 0$ or $b = 0$

$\mathbb{Q}$ $\mathbb{P}$

Rational and irrational numbers!

TL; DR: use contradiction to show that for $r = \frac{p}{q}$, p and q are both even.

Theorem There does not exist a rational number r such that $r^2 = 2$.

Proof Suppose, by way of contradiction, that r is a rational number and $r^2 = 2$. Let $r = \frac{p}{q}$ where p and q are integers with no common factors. Then $\left(\frac{p}{q}\right)^2 = 2$, or $p^2 = 2q^2$. So, p is even, which implies there exists $n \in \mathbb{Z}$ such that $p = 2n$, so we have $(2n)^2 = 2q^2 \Rightarrow 4n^2 = 2q^2 \Rightarrow 2n^2 = q^2$. So, q^2 is even, which implies that q is even, which means that q is divisible by 2. Since p is even, 2 is also a factor of p. However, it was stated that p and q have no common factors. Thus, we have reached a contradiction, and there does not exist a rational number r such that $r^2 = 2$. ■

The Order Properties of $\mathbb{R}$

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$ called the positive real numbers that satisfies the following for $a, b \in \mathbb{P}$:

- $a + b \in \mathbb{P}$
- $a \cdot b \in \mathbb{P}$
- Exactly one of the following holds:

$a \in \mathbb{P} \quad a=0 \quad -a \in \mathbb{P}$

From these, it follows that $\forall a,b,c \in \mathbb{R}$

- If $a>b$ and $b>c$, then $a>c$
- If $a>b$, then $a+c>b+c$
- If $a>b$ and $c>0$, then $ac>bc$
  - ↳ $c<0$, then $ac<bc$
- If $a \neq 0$, $a^2>0$
- $1>0$

Not sure how important this is, but here's the Arithmetic - Geometric Mean Inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b)$$

And Bernoulli's Inequality: $(1+x)^n \geq 1+nx$

Section 2.2

Absolute Value

$$|a| = \begin{cases} a & a>0 \\ 0 & a=0 \\ -a & a<0 \end{cases}$$

Properties that follow:

$$|a \cdot b| = |a| \cdot |b|$$

$$|a|^2 = a^2$$

Triangle Inequality: $|a+b| \leq |a|+|b|$

↳ If $ab>0$, then $|a+b| = |a|+|b|$

Corollary: i) $\big||a|-|b|\big| \leq |a-b|$

ii) $|a-b| \leq |a|+|b|$

Proof (i): We rewrite a as $a+b-b$ and apply the Triangle Inequality:
$|a| = |(a-b)+b| \leq |a-b|+|b|$. Subtracting $|b|$ on both sides yields
$|a|-|b| \leq |a-b|+|b|-|b| = |a-b|$. Similarly, $|b| = |b-a+a| \leq |b-a|+|a|$
$\implies -|a-b| = -|b-a| \leq |a|-|b|$.

There's more to that specific proof but I just wanted to practice applying the triangle inequality

Section 2.3

A set is bounded above if $\exists\, u \in \mathbb{R}$ such that $s \le u \;\; \forall s \in S$
upper bound

A set is bounded below if $\exists\, w \in \mathbb{R}$ such that $w \le s \;\; \forall s \in S$
lower bound

If a set is bounded above and below, then it is bounded.

Supremum - least upper bound
(sup S)

so u is the supremum of S if it is smaller than every other upper bound ($u \le v \;\; \forall$ upper bounds v)

Infimum - greatest lower bound
(inf S)

w is the infimum of S if it is greater than every other lower bound ($w \ge t \;\; \forall$ lower bounds t)

The supremum and infimum of a set is unique

Worth noting that sup S and inf S is just a number.

Section 2.4

Some examples:

- $\sup(a+S) = a + \sup(S)$

  Proof Shown in class, but:
  Let $u = \sup S$. Then $s \le u \;\; \forall s \in S$, so $a+s \le u+s$.
  This implies that $a+u$ is an upper bound for $a+S$.
  To show that it is the least upper bound, let v be any upper bound of $a+S$. Then $a+s \le v \Rightarrow s \le v-a$, so $v-a$ is an upper bound of S.

- $\sup(aS) = a \inf(S)$

The Archimedean Property

If $x \in \mathbb{R}$, $\exists\ n_x \in \mathbb{N}$ such that $x \leq n_x$.

Application $\Rightarrow$ Prove that if $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\inf S = 0$.

Proof: Since S is bounded below by 0, the infimum exists. Let $w = \inf S$. Since S is bounded below by 0, $w \geq 0$. $\forall \varepsilon > 0$, the Archimedean Property implies $\exists\ n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} > n$, or $\frac{1}{n} > \varepsilon$. Thus,

$$0 \leq w \leq \frac{1}{n} < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $w = 0$. ∎

Existence of $\sqrt{2}$

There exists a positive real number x such that $x^2 = 2$.

TL;DR:

use the fact that $S = \{s \in \mathbb{R},\ s \geq 0,\ s^2 < 2\}$ is bounded above by 2, so it has a supremum.

Show that $x^2 \nless 2$ and $x^2 \ngtr 2$, so $x^2 = 2$.

Density Theorem

If $x, y \in \mathbb{R}$ with $x < y$, then $\exists\ r \in \mathbb{Q}$ s.t. $x < r < y$.

A corollary of this is that there is also an irrational number in between any two real numbers.

When doing proofs with irrationals, use known irrational numbers (such as $\sqrt{2}$).

2.4 Examples

9. Let $X = Y = \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x,y) = 2x + y$.

a) Find $f(x) = \sup\{h(x,y) : y \in Y\}$, then find $\inf\{f(x) : x \in X\}$

Solution: $f(x) = \sup\{2x + y : y \in Y\} = 2x + 1$

$\inf(2x+1) = 2(0) + 1 = \boxed{1}$

b) $\forall y \in Y$, find $g(y) = \inf\{h(x,y) : x \in X\}$, then find $\sup\{g(y) : y \in Y\}$.

Solution $g(y) = \inf\{2x + y : x \in X\} = y$

$\sup\{y\} = \boxed{1}$