Lecture 21

Monday, January 8, 2024 9:21 PM (11/27)

Review

cluster point - c is a cluster point of A if for each $\delta > 0$, there exists an $x_0 \in A$ with $x_0 \neq c$ such that $x_0 \in (\varepsilon - \delta, \varepsilon + \delta)$

4.2 Limit Theorems

Def Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded** on a neighborhood of c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that $|f(x)| < M$ for all $x \in A \cap V_\delta(c)$.

Theorem If $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.

Algebraic Properties for Limits of Functions

Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let c be a cluster point of A.

If $\lim\limits_{x\to c} f = L$ and $\lim\limits_{x\to c} g = M$, then:

Sum	$\lim\limits_{x\to c}(f+g) = L+M$
Difference	$\lim\limits_{x\to c}(f-g) = L-M$
Product	$\lim\limits_{x\to c}(f\cdot g) = L\cdot M$
Multiple	$\lim\limits_{x\to c}(bf) = b\cdot L \quad (b\in\mathbb{R})$
Quotient	$\lim\limits_{x\to c}\left(\frac{f}{g}\right) = \frac{L}{M} \quad$ (provided $M\neq 0$)

Squeeze Theorem

Let $A\subseteq\mathbb{R}$, let $f, g, h: A\to\mathbb{R}$, and let $c\in\mathbb{R}$ be a cluster point of A.
If $f(x)\leq g(x)\leq h(x)$ $\forall$ $x\in A$, $x\neq c$, and if $\lim\limits_{x\to c} f(x) = L = \lim\limits_{x\to c} h(x)$,
then $\lim\limits_{x\to c} g(x) = L$.

Ex $\quad \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Since $-1\leq \sin(x)\leq 1$, we have

$$-|x| \leq x\cdot\sin\left(\tfrac{1}{x}\right)\leq |x| \qquad \forall x\in\mathbb{R}$$

$\lim\limits_{x\to 0}|x| = \lim\limits_{x\to 0} -|x| = 0$, so by the Squeeze Theorem $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$