① 4.1.1

a) $|x^2-1| < \frac{1}{2}$

$|x^2-1| = |x+1||x-1| \leq (|x+1|)|x-1|$

$|x| = |x-1|+1$

Choose $\delta_1 = 1$

$|x-1| < \delta_1 = 1$

$|x| \leq |x-1|+1 < 1+1 = 2$

$|x^2-1| = |x+1||x-1| \leq (|x+1|)|x-1| \leq (2+1)|x-1|$

$\delta = \inf\left(1, \frac{1}{3\cdot 2}\right) = \frac{1}{6}$

$|x^2-1| \leq 3|x-1| < 3\cdot\left(\frac{1}{3\cdot 2}\right) = \frac{1}{2}$

$|x-1| < \frac{1}{6}$

$\boxed{|x^2-1| < \frac{1}{2}}$

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② 4.1.2

a) $|\sqrt{x}-2| < \frac{1}{2}$

$|\sqrt{x}-2| = \frac{|\sqrt{x}-2||\sqrt{x}+2|}{|\sqrt{x}+2|} = \frac{|x-4|}{|\sqrt{x}+2|} < \frac{1}{2}$

$|x-4| < \frac{|\sqrt{x}+2|}{2}$, Choose $|x-4| < \frac{1}{2}(4)$, so $\frac{1}{2}(4) < x < \frac{3}{2}(4)$.

Then $\frac{1}{\sqrt{x}+2} < \frac{1}{\sqrt{\frac{4}{2}}+2} = \frac{4}{5(2)} = \frac{4}{10} = \frac{2}{5} \Rightarrow \frac{5}{2}|\sqrt{x}-2| < |x-4|$.

$\delta(\varepsilon) = \inf\left\{2, \frac{5}{2}\varepsilon\right\}$, $\varepsilon = \frac{1}{2}$, $0 < |x-4| < \frac{10}{8}$ will be $|\sqrt{x}-2| < \frac{1}{2}$

③ 4.1.4

$\boxed{\Rightarrow}$ Assume that $\lim_{x\to c} f(x) = L$. This means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x in the domain of f, in $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$.

We want to show that $\lim_{x\to 0} f(x+c) = L$. Let $y = x+c$. Then as $x \to 0$, $y \to c$. So the condition $0 < |x-c| < \delta$ because $0 < |(y-c)-c| < \delta$ which simplifies to $0 < |y-2c| < \delta$, because we are looking at the limit as y approaches c, we can use the same δ from the original limit to satisfy the condition for the limit as $x \to 0$. Hence, $|f(y) - L| < \epsilon$, which shows that $\lim_{x\to 0} f(x+c) = L$.

$\boxed{\Leftarrow}$ Assume that $\lim_{x\to 0} f(x+c) = L$. This means that for every $\epsilon > 0$, there exists a $\delta > 0$ s.t. $\forall x \in f$, if $0 < |x| < \delta$, then $|f(x+c) - L| < \epsilon$. We want to show that $\lim_{x\to c} f(x) = L$. Let $y = x - c$. Then as $y \to 0$, $x \to c$. The condition $0 < |x| < \delta$ becomes $0 < |(y+c) - c| < \delta$, which simplifies to $0 < |y| < \delta$. Hence, for every y that satisfies $0 < |y| < \delta$, we have that $|f(y+c) - L| = |f(x) - L| < \epsilon$, which shows that $\lim_{x\to c} f(x) = L$.

$\boxed{\Rightarrow}\boxed{\Leftarrow}$ $\therefore$ the statement holds

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⑤ 4.1.6

Suppose $\exists$ constants $K, L \ni |f(x) - L| \le K|x-c|$
Let $\varepsilon > 0$, $\delta > 0$ s.t. $|x-c| < \delta$.

Choose $\delta = \frac{\varepsilon}{K} > 0$

Let $x \in I$. Consider $|f(x) - L| \le K|x-c| < K\frac{\varepsilon}{K} = \varepsilon \Rightarrow |f(x) - L| < \varepsilon$.
Hence for $\varepsilon > 0$, $\exists \delta > 0 \ni |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$, for all $x \in I$

$\therefore \lim_{x\to c} f(x) = L$

⑥ 4.1.9

a) $\lim_{x\to 2} \frac{1}{1-x} = -1$

Use seq crit. Suppose $x_n \to 2$. Then $\lim \frac{1}{1-x_n} = \frac{1}{1-\lim x_n} = -1$, by limit laws.

c) $\lim \frac{x^2}{|x|} = 0$

Suppose $\epsilon > 0$. Then $\left|\frac{x^2}{|x|}\right| = \frac{|x|^2}{|x|} = |x|$. So choose $|x| < \delta$ and the result follows.

$\therefore \lim_{x\to 0} \frac{x^2}{|x|} = 0$

⑦ 4.1.12

b) $\lim\limits_{x \to 0} \frac{1}{\sqrt{x}}$ $(x \geq 0)$

$\lim\limits_{x \to 0^-}$ = $\lim\limits_{h \to 0} f(0-h)$

$= \lim\limits_{h \to 0} \frac{1}{\sqrt{0-h}}$

$= \lim\limits_{h \to 0} \frac{1}{\sqrt{-h}}$

$= \infty \Rightarrow DNE$

$\lim\limits_{x \to 0^+} = \lim\limits_{h \to 0} f(0+h)$

$= \lim\limits_{h \to 0} \frac{1}{\sqrt{0+h}}$

$= \lim\limits_{h \to 0} \frac{1}{\sqrt{h}}$

$= \infty \Rightarrow DNE$

d) $\lim\limits_{x \to 0} \sin\left(\frac{1}{x^2}\right)$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0-h)$

$= \lim\limits_{h \to 0} \sin \frac{1}{(0-h)^2}$

$= \lim\limits_{h \to 0} \sin\left(\frac{1}{h^2}\right)$

$= \sin \infty$

$\lim\limits_{x \to 0^-} f(x) =$ undefined, $-1 \leq \sin \infty \leq 1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0+h)$

$= \lim\limits_{h \to 0} \sin\left(\frac{1}{(0+h)^2}\right)$

$= \lim\limits_{h \to 0} \sin\left(\frac{1}{h^2}\right)$

$= \sin \infty$

$\lim\limits_{x \to 0^+} f(x) =$ undefined

8 **4.1.14**

a) Given that $\lim_{x \to c} (f(x))^2 = L$ and $L=0$, we have $\lim_{x \to c} (f(x))^2 = 0$. This implies that $(f(x))^2 \to 0$ as $x \to c$.

Since $(f(x))^2 \geq 0$ $\forall x$, and its limit as $x \to c$ is 0, then $(f(x))^2$ would not approach 0.

$\therefore \lim_{x \to c} f(x) = 0$.

b) As example can be constructed using the function $f(x) = \sin\left(\frac{1}{x-c}\right)$ when $x \neq c$ and $f(c) = 0$ for some constant c. Let's analyze the limit:

The function $(f(x))^2 = \sin^2\left(\frac{1}{x-c}\right)$ will oscillate between 0 and 1 as $x \to c$, since $\sin^2$ function oscillate between 0 and 1. As $x \to c$ frequency increases. However $(f(x))^2$ averages out to $\frac{1}{2}$ over any interval around c, because $\sin^2$ is symmetric about $\frac{1}{2}$.

$\therefore$ L in $\lim_{x \to c} (f(x))^2 = L$ can be considered as $\frac{1}{2}$ if we are talking about the limit in the sense of average value of $(f(x))^2$ around c.

$\therefore$ $f(x)$ itself does not have a limit as x approaches c.

9 **4.2.1**

c) $\lim_{x \to 2} \left(\frac{1}{x+1} - \frac{1}{2x}\right)$

$= \lim_{x \to 2} \frac{1}{x+1} - \lim_{x \to 2} \frac{1}{2x}$

$= \frac{1}{2+1} - \frac{1}{2(2)} = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$ ✓

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(10) 4.2.2

b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim_{x\to 2} x+2$

$= \boxed{4}$

d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

$= \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}$

$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \boxed{\frac{1}{2}}$

(11) 4.2.4

By the sequential criterion, we need to find a sequence $x_n \to 0$ so that $\cos(1/x_n)$ has no limit. Define $x_n = \frac{1}{\pi n}$. Then $\cos(1/x_n) = \cos(n\pi) = (-1)^n$, which does not converge. On the other hand $-1 \le \cos(1/x) \le 1$, so $-x \le x\cos(1/x) \le x$, so by Squeeze Thm. and the sequential criterion $\lim_{x\to 0} x\cos(1/x) = 0$.

(12) 4.2.10

Consider $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin \mathbb{Q}$ and consider $g(x) = -f(x)$. Then it is clear that f and g do not have limits as $x \to c$, for any c. On the other hand $(f+g)(x) = 0$, and $fg(x) = -f^2(x) = -1$. So the sum and product functions have limits for every c.

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