Lecture 23

(12/04)

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Chapter 5 - Continuous Functions

Def Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in A$. We say f is **continuous** at c if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point in A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f is not continuous at c, then f is **discontinuous** at c.

Theorem

A function $f: A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$, there exists a δ-neighborhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$; that is,

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)).$$

f is continuous at $c \in A$ if and only if $\lim_{x \to c} f(x) = f(c)$.

So, if c is a cluster point of A, then the following conditions must hold for f to be continuous at c:

- f must be defined at c
- the limit of f at c must exist $\in \mathbb{R}$
- these two values must be equal.

Sequential Criterion for Continuity

A function $f: A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence $(x_n) \in A$ that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Discontinuity Criterion (Negation)

f is discontinuous at c iff. there exists a sequence (x_n) in A that converges to c but $(f(x_n))$ does not converge to $f(c)$.

Ex. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \quad \text{rational} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \quad \text{irrational} \end{cases}$ dense subsets of $\mathbb{R}$

Dirichlet's "discontinuous function"

f is not continuous at any point in $\mathbb{R}$

- c is rational: $f(c) = 1$, $\lim(f(x_n)) = 0$
- c is irrational: $f(c) = 0$, $\lim(f(y_n)) = 1$

Just use density properties

Ex. Thomae's Function

Let $A := \{x \in \mathbb{R} : x > 0\}$.

For $x \in \mathbb{R}\setminus\mathbb{Q}$, $h(x) := 0$.

For rational numbers $\frac{m}{n}$, $h\left(\frac{m}{n}\right) = \frac{1}{n}$ (also $h(0) = 1$).

h is continuous at every irrational number in A and discontinuous at every rational number in A.

5.2

The sum, difference, product, multiple, and quotient functions of continuous functions is continuous.

If f and g are continuous, then

- $f+g$
- $f\cdot g$
- $\frac{f}{g}$ $(g\neq 0)$
- $f-g$
- $b\cdot f$ $(b\in\mathbb{R})$

are all continuous.

Theorems Let $A\subseteq\mathbb{R}$ and $f:A\to\mathbb{R}$.

- If f is continuous, $|f|$ is continuous
- If f is continuous (provided $f(x)\geq 0$ $\forall x\in A$), then $\sqrt{f}$ is continuous.

The composition of continuous functions is also continuous

5.3 Continuous Functions on Intervals

Def| A function $f:A\to\mathbb{R}$ is said to be bounded on A if there exists a constant $M>0$ such that $|f(x)|\leq M$.

Boundedness Theorem

Let $I=[a,b]$ be a closed and bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof uses contradiction

Def Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. f has an **absolute maximum (minimum)** on A if there exists a point x^* (x_*) in A such that $f(x) \leq f(x^*)$ $\big(f(x) \geq f(x_*)\big)$ $\forall x \in A$.

Min/Max Theorem

Let $I := [a,b]$ be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

$\Rightarrow$ $f(A)$ is bounded