

Lecture 3

Monday, September 11, 2023 9:59 AM

Theorem The following statements are logically equivalent:

- a) S is a countable set
- b) There exists a surjection of \mathbb{N} onto S
- c) There exists an injection of S into \mathbb{N}

* surjection is easiest
to construct

Proof a) \Rightarrow b)

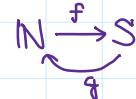
By def. of countable, there exists a bijection from $\mathbb{N}_n = \{1, 2, \dots, n\}$ onto S .

b) \Rightarrow c)

Let f be the surjection from \mathbb{N} onto S and let g be the mapping

from S into \mathbb{N} .

$g(s)$ is the least element in the set $f^{-1}(\{s\}) = \{n \in \mathbb{N} : f(n) = s, s \in S\}$



Claim: g is injective ($g(s_1) = g(s_2) \Rightarrow s_1 = s_2$)

Proof Suppose $g(s_1) = g(s_2)$

Then $s_1 = f(g(s_1)) = f(g(s_2)) = s_2$, so $s_1 = s_2$

So, g is injective from S into \mathbb{N} . ■

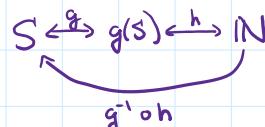
c) \Rightarrow a)

Let g be the injection of S into \mathbb{N} . Consider g as the mapping from S to $g(S) \subseteq \mathbb{N}$.

Since \mathbb{N} is countable and $g(S)$ is a subset of \mathbb{N} , it follows that $g(S)$ is countable.

Consider the case when $g(S)$ is denumerable.

Then there exists a bijection h from \mathbb{N} onto $g(S)$, then $g^{-1} \circ h$ is a bijection from \mathbb{N} onto S . So, S is denumerable.



Theorem 1.3.8 The set $\mathbb{N} \times \mathbb{N}$ is denumerable.

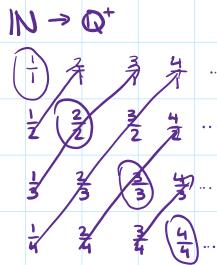
Theorem The set of rational numbers $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$ is denumerable.

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$

just focus on showing
that this is countable

$$\mathbb{Q}^+ = \left\{ \frac{m}{n} : m, n \in \mathbb{N} \right\}$$

Matrices:



"a picture is worth
more than 1000 words"

diagonal mapping

1 appears in every row
and column

This is a surjection of \mathbb{N} onto \mathbb{Q}^+

So \mathbb{Q}^+ is countable

It can also be shown in the same way that \mathbb{Q}^- is countable, and since $\{0\}$ is finite, $\mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ is countable.

Theorem If A_n is a countable set for each $n \in \mathbb{N}$, then $A = \bigcup_{n=1}^{\infty} A_n$
is countable.

$$A_1 = a_{11} \quad a_{12} \quad a_{13} \dots$$

$$A_2 = a_{21} \quad a_{22} \quad a_{23} \dots$$

:

$$A_n = a_{n1} \quad a_{n2} \quad a_{n3} \dots$$

another diagonal
mapping

Cantor's Theorem

Let A be any set. There is no surjection of A onto its power set $P(A)$.

* Kang did not mention this,
I just thought it was cool

Section 2.1 \mathbb{R}

The system of real numbers is a field with respect to addition & multiplication.

Algebraic Properties of \mathbb{R}

these are also called the field axioms of \mathbb{R}

Let $a, b, c \in \mathbb{R}$

(A1) $a + b = b + a$ (commutative property of addition)

(A2) $(a+b)+c = a+(b+c)$ (associative property of addition)

(A3) There exists an element $0 \in \mathbb{R}$ such that $0+a = a+0 = a$
(existence of a zero element)

(A4) $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$ such that $a+(-a) = (-a)+a = 0$

(existence of negative elements)

(M1) $a \cdot b = b \cdot a$ (commutative property of multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associative property of multiplication)

(M3) There exists an element $1 \in \mathbb{R}$ distinct from 0 such that $1 \cdot a = a \cdot 1 = a$
(existence of a unit element)

(M4) $\forall a \neq 0$, there exists an element $\frac{1}{a} \in \mathbb{R}$ such that $a \cdot (\frac{1}{a}) = (\frac{1}{a}) \cdot a = 1$
(existence of reciprocals)

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

(distributive property of multiplication over addition)

Examples (These are theorems in 2.1)

a) If $z, a \in \mathbb{R}$ with $z+a = a$, then $z = 0$ (Rule of cancellation).

(A4) $\exists -a \in \mathbb{R}$ such that $a+(-a) = 0$

So, $z+a = a$ implies $(z+a)+(-a) = a+(-a) = 0$

(A2) : $(z+a)+(-a) = z+(a+(-a)) = z+0 = 0$

(A3) : $z+0 = z = 0$

so $z = 0$ ✓

b) If $u, b \in \mathbb{R}$ and $b \neq 0$ with $u \cdot b = b$, then $u = 1$.

(M3) : $u = u \cdot 1$

(M4) : $b \cdot (\frac{1}{b}) = 1$, so $u = u \cdot (b \cdot \frac{1}{b})$

$\therefore u \cdot 1 = u \cdot (b \cdot \frac{1}{b})$

$$(M4): b \cdot \left(\frac{1}{b}\right) = 1, \text{ so } u = u \cdot \left(b \cdot \frac{1}{b}\right)$$

$$(M2): u = \underbrace{(u \cdot b)}_{=b} \cdot \left(\frac{1}{b}\right) = u = b \cdot \frac{1}{b}$$

By M3 again, $u = 1$

c) If $a \in \mathbb{R}$, $a \cdot 0 = 0$

$$(M3): a + a \cdot 0 = a \cdot 1 + a \cdot 0$$

$$(D): a \cdot (1+0)$$

$$(M3): a \cdot (1+0) = a \cdot 1 = a$$

$$\text{So, } a + (a \cdot 0) = a$$

By the rule of cancellation, $a \cdot 0$ must equal 0.

d) If $a \neq 0$ and $b \in \mathbb{R}$ with $a \cdot b = 1$, then $b = \frac{1}{a}$. (Uniqueness of Reciprocals)

Proof By (M4), $\exists \frac{1}{a} \in \mathbb{R}$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$. Since

$$a \cdot b = 1, \text{ then } \frac{1}{a} \cdot ab = \frac{1}{a} \cdot 1 = \frac{1}{a}.$$

By (M2), $\frac{1}{a} \cdot (a \cdot b) = (\frac{1}{a} \cdot a) \cdot b$ which by M4 again
is $1 \cdot b$, which by (M3) is equal to b .

$$\text{So, } b = \frac{1}{a}.$$

e) If $a \cdot b = 0$, then either $a=0$ or $b=0$.

Proof Case 1: $a=0$ done

Case 2: $a \neq 0$

By (M4), $\exists \frac{1}{a} \in \mathbb{R}$ such that $\frac{1}{a} \cdot a = 1$. Multiplying $ab=0$

$$\text{by } \frac{1}{a}, \text{ we get } \frac{1}{a} (a \cdot b) = 0 \cdot \frac{1}{a} = 0.$$

$$\text{By M2, } \frac{1}{a} \cdot (a \cdot b) = (\frac{1}{a} \cdot a) \cdot b, \text{ which by M4, is } 1 \cdot b.$$

$$\text{So, we have } b \cdot 1 = 0. \text{ By M3, } b \cdot 1 = b,$$

$$\text{so we have } b=0.$$



The operation of subtraction is defined by addition: $a - b = a + (-b)$

Similarly, division is defined in terms of multiplication: For $a, b \in \mathbb{R}$ with $b \neq 0$, $a/b = a \cdot (\frac{1}{b})$