Lecture 5

Monday, September 18, 2023 9:59 AM

Quiz on Wed

- 30 min, 2-3 problems
- 1.3 - 2.1
- Similar to hw
  - one problem on countable/denumerable sets, another on algebraic/order properties of $\mathbb{R}$

Review of Last Time

supremum - smallest upper bound

infimum - largest lower bound

supremum of S - $\sup S$

infimum of S - $\inf S$

There can only be one supremum or infimum of a given subset S of $\mathbb{R}$

(supremum and infimum of a set is unique)

Can prove this either directly or by contradiction

Proof Suppose, by way of contradiction, that there are at least two different suprema of S, called u_1 and u_2 ($u_1 \neq u_2$). Since u_2 is a supremum of S, then it must be an upper bound of S (part 1 of def). Since u_1 is a supremum of S and u_2 is an upper bound of S, then by definition $u_1 \leq u_2$ (part 2 of def). Similarly, we have that $u_2 \leq u_1$. Thus, $u_2 = u_1$, which is a contradiction. So, the supremum of a set must be unique. ■

A Lemma Dilemma

Lemma A number u is the supremum of a nonempty subset S of $\mathbb{R}$ iff.

① $s \leq u \quad \forall s \in S$

② If $v < u$, then there exists some $s_v \in S$ such that $s_v > v$. $\quad v = u - \varepsilon$

- $\forall \varepsilon > 0$, $\exists s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$

Examples (from text)

- $S := \{x : 0 \le x \le 1\}$

  $\sup S = 1, \ \inf S = 0$

- $T := \{x : 0 < x < 1\}$

  $\sup T = 1, \ \inf T = 0$

sup and inf do not have to be in the set

The Completeness Property of $\mathbb{R}$

Every <u>nonempty</u> subset of $\mathbb{R}$ that has an upper bound also has a supremum in $\mathbb{R}$

Similarly, every nonempty subset of $\mathbb{R}$ that has a lower bound also has an infimum in $\mathbb{R}$

basically this guarantees the existence of a sup and inf

<u>Ex</u> Let S be a nonempty subset of $\mathbb{R}$ and define the shifted S by a as $a + S = \{a + s : s \in S\}$

Prove that $\sup(a + S) = a + \sup(S)$

* This applies to a scale $(aS = \{as : s \in S\})$ only if $a > 0$

This might be HW

<u>Proof</u> If S is bounded above, then $\sup S \in \mathbb{R}$ by the Completeness property of $\mathbb{R}$.

1) Show that $a + \sup S$ is an upper bound of $a + S$

2) Show that $a + \sup S$ is the least upper bound of $a + S$

<u>1</u>: Since $\sup S$ is an upper bound of S, then $s \le \sup S \ \forall s \in S$.
Then $a + s \le a + \sup S$, so $a + \sup S$ is an upper bound of $a + S$.

<u>2</u>: Let v be any upper bound of $a + S$. Then $a + s \le v$ for all $s \in S$.
Then $s \le v - a$, so $v - a$ is an upper bound of S. So, $\sup S \le v - a$
so $a + \sup S \le v$. Thus, $a + \sup S$ is the least upper bound of $a + S$. $\square$

Thought process for 2:

$\boxed{a + \sup S \text{ is the least upper bound of } S}$

$\Uparrow$

$\boxed{\text{if } v \text{ is an upper bound of } a + S, \text{ then } a + \sup S \le v}$

$\Uparrow$

$\boxed{\sup S \le v - a}$

$\Uparrow$

$\boxed{v - a \text{ is an upper bound of } S}$

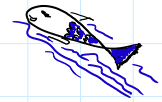
$\Uparrow$

$\boxed{s \le v - a \ \ \forall s \in S}$

$\Uparrow$

$\boxed{a + s \le v \ \ \forall s \in S}$

Ex Given two sets A and B of $\mathbb{R}$, if $a \le b$ for all $a \in A$ and $b \in B$, then $\sup A \le \inf B$.



- Dolphin, my mom

$\boxed{\sup A \le \inf B}$

$\boxed{\text{inf B is an upper bound of A}} \Rightarrow \boxed{\sup A \le \inf B}$

$\boxed{\text{sup A is a lower bound of B}} \Rightarrow \boxed{\sup A \le \inf B}$

$\boxed{a \le b} \Rightarrow \boxed{\text{a is a lower bound of B}} \Rightarrow \boxed{a \le \inf B} \Rightarrow \boxed{\text{inf B is an upper bound of A}}$

Proof Let a be any element in A and b be any element in B. Since $a \le b$, then a is a lower bound of B, so $a \le \inf B$.

Since $a \le \inf B$, then $\inf B$ is an upper bound of A, so $\sup A \le \inf B$. $\square$

Theorem | The Archimedean Property

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$.

Kang loves proof by contradiction

Proof Suppose, by way of contradiction, that there is no $n_x \in \mathbb{N}$ such that $x \le n_x$; that is, $n < x$ for all $n \in \mathbb{N}$. Then x is an upper bound of $\mathbb{N}$, so $\sup \mathbb{N}$ exists in $\mathbb{R}$ by the completeness property of $\mathbb{R}$. So, $\sup \mathbb{N} - 1$ is not an upper bound. Then there exists $u \in \mathbb{N}$ such that $\sup \mathbb{N} - 1 < u$. So, $\sup \mathbb{N} < u + 1 \in \mathbb{N}$, so $\sup \mathbb{N}$ is not an upper bound of $\mathbb{N}$, which is a contradiction. Thus, $x \le n_x$. $\square$