Lecture 16

Wednesday, November 1, 2023 11:12 AM

Cauchy Convergence Criterion

A sequence is convergent if and only if it is Cauchy.

Proof ($\rightarrow$) $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that $n \geq N(\varepsilon)$ implies $|x_n - x_m| < \frac{\varepsilon}{2}$.

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(provided $m, n \geq N(\varepsilon)$).

($\leftarrow$) Need to show that Cauchy sequences are bounded.

Since (x_n) is Cauchy, let $\varepsilon = 1$. $\exists N(1)$ such that $|x_n - x_{N(1)}| < 1$. So,

$$|x_n| \leq |x_n - x_{N(1)}| + |x_{N(1)}| \leq 1 + |x_{N(1)}|.$$

Let $M = \max\{|x_1|, |x_2|, \dots |x_{N(1)-1}|, 1 + |x_{N(1)}|\}$. Then $|x_n| \leq M$.

By the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence, say (x_{n_k}). Let $x^* = \lim(x_{n_k})$. We want to show that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Since (x_n) is Cauchy, $\forall \varepsilon > 0$, $\exists H(\frac{\varepsilon}{2})$ such that $|x_n - x_m| < \frac{\varepsilon}{2}$ $\forall m, n \geq H(\frac{\varepsilon}{2})$.

Since $x_{n_k} \rightarrow x^*$, $\exists K(\varepsilon)$ such that $|x_{n_k} - x^*| < \frac{\varepsilon}{2}$ $\forall k \geq K(\varepsilon)$ and $n_k \geq H(\frac{\varepsilon}{2})$.

So, $|x_n - x^*| \leq |x_n - x_{n_k}| + |x_{n_k} - x^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ■

Def Let (x_n) be a sequence of real numbers. (x_n) is contractive if there exists a constant c $(0 < c < 1)$ such that

$$|x_{n+2} - x_{n+1}| \leq c\,|x_{n+1} - x_n| \qquad \forall n \in \mathbb{N}$$

Theorem Every contractive sequence is Cauchy, and thus convergent.

Ex I could not care less rn.