Lecture 7

Monday, September 25, 2023 10:08 AM

he's adopting the policy that our final exam score can replace a midterm (if final exam > midterm)

Tally of "backwards search" (started halfway through)

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Theorem There exists a positive real number x such that $x^2 = 2$.

Let $S = \{ s \in \mathbb{R} : 0 \leq s,\ s^2 < 2 \}$

$x = \sup S$ exists in $\mathbb{R}$ by the Completeness Property

Show that $x^2 = 2$ by ruling out $x^2 < 2$ and $x^2 > 2$

$\underline{x^2 < 2}$ (Find something $\in S$ that is $> \infty$)

$$\left(x + \frac{1}{n}\right)^2 < 2$$

Factor out $(x+\frac{1}{n})^2$

$$x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$$

he keeps saying backwards search

Subtract x^2 and then pull out $\frac{1}{n}$

$$\frac{1}{n}\left(2x + \frac{1}{n}\right) < 2 - x^2$$

Solve for $\frac{1}{n}$

$$\frac{1}{n} < \frac{2-x^2}{2x+\frac{1}{n}} < \frac{2-x^2}{2x+1}$$

Solve for n

$$\Rightarrow n > \frac{2x+1}{2-x^2}$$

By the Archimedean Property, for $\frac{2x+1}{2-x^2} \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ such that $n > \frac{2x+1}{2-x^2}$. Rearranging this gives

$$\frac{2x+1}{n} < 2 - x^2$$

$$\frac{2x}{n} + \frac{1}{n} < 2 - x^2$$

Since $\frac{1}{n^2} < \frac{1}{n}$, we have that $\frac{2x}{n} + \frac{1}{n} < \frac{2x}{n} + \frac{1}{n^2}$.

So, $x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$, or $(x + \frac{1}{n})^2 < 2$.

So, $x + \frac{1}{n} \in S$, so x is not the supremum.

$\underline{x^2 > 2}$

Want to show $(x - \frac{1}{n^2}) > 2 \Rightarrow x - \frac{1}{n}$ is an upper bound

For $x^2 > 2$, by the Archimedean Property, for $\frac{2x}{x^2-2} \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ such that $n > \frac{2x}{x^2-2}$, then $x^2 - 2 > \frac{2x}{n}$, so $x^2 - 2 > \frac{2x}{n} - \frac{1}{n^2}$, or $x^2 - \frac{2x}{n} + \frac{1}{n^2} > 2$, then $(x - \frac{1}{n})^2 > 2$. So, $x - \frac{1}{n}$ is an upper bound of S. Contradiction!

So, $x^2 = 2$. Since $x \geq 1$ $(1 \in S)$, x is positive.

Let x_1, x_2 be two positive solutions to $x^2 = 2$.

$$x_1^2 = x_2^2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$

Since $x_1 + x_2 > 0$, $x_1 - x_2$ must be 0

So, $x_1 = x_2$. $\square$

The Density Theorem If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Notes

Let $r = \frac{m}{n}$. $x < \frac{m}{n} < y$

$nx < m < ny$

If $ny - nx > 1$, $\exists k \in \mathbb{Z}$ such that

$nx \leq k \leq ny$

pigeoncage???!?

Need to show $ny > k+1$ (use Kang's favorite technique)

Suppose $ny \leq k+1$. Then $k \leq nx < ny \leq k+1$, which implies that $ny - nx \leq 1$. Contradiction!

Proof Since $y - x > 0$, then by the Archimedean Property, there exists an $n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$, or $ny - nx > 1$.

Now we want to show that there exists an $m \in \mathbb{Z}$ such that $nx < m < ny$. By the last corollary of the Archimedean Property, for $nx \in \mathbb{R}$, $\exists\, k \in \mathbb{Z}$ such that $k \leq nx < k+1$.

Suppose on the contrary that $k+1 \geq ny$. Then $k < nx < ny \leq k+1$, then $ny - nx \leq 1$, which contradicts $ny - nx > 1$.

So, $nx < k+1 < ny$, then $m = k+1$ and $r = \frac{m}{n}$ exists. $\square$

☆ I lost track with his conclusion so this could be total BS

Corollary If $x, y \in \mathbb{R}$ with $x < y$, then there exists an irrational number number z such that $x < z < y$.

Proof we use a known irrational, $\sqrt{2}$:

$$\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}, \text{ then } \exists\, r \in \mathbb{Q} \text{ s.t.}$$

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \Rightarrow x < r\sqrt{2} < y$$

Then we prove that $r\sqrt{2}$ is irrational by contradiction:

Assume $r\sqrt{2}$ is rational. Then $r\sqrt{2} = \frac{m}{n}$, $r = \frac{p}{q}$ $(m, n, p, q \in \mathbb{Z})$,

so:
$$\frac{p}{q}\sqrt{2} = \frac{m}{n}$$

$$\Rightarrow \frac{mq}{np} = \sqrt{2}$$

So, $r\sqrt{2}$ is irrational. $\square$

Notes

- We don't know much about the irrationals, just that they are not rational

Section 2.5 Intervals

Types of intervals : (a, b) open, $[a,b]$ closed, $[a,b)$, $(a,b]$ ← fully bounded

$[a,\infty)$ (a,∞) $(-\infty,b]$ $(-\infty,b)$ ← only bounded above or below

$(-\infty,\infty)$

$$[a,b] = [a,\infty) \cap (-\infty,b]$$

The Characterization Theorem If $S \in \mathbb{R}$ contains at least two points and has the following property:

If $x, y \in S$ and $x < y$, then $[x,y] \in S$

then S is an interval.

Proof There are 4 cases

Case 1 - S is bounded

Let $a = \inf S$ and $b = \sup S$. We show that $(a,b) \in S$ (Note that $S \subseteq [a,b]$)

Let $z \in (a,b)$, then $z > a$ and $z < b$. Since $z < a$, then z is not a lower bound of S, so there exists some $s \in S$ such that $s < z$. Since $z < b$, then z is not an upper bound of S, so there exists some $c \in S$ such that $z < c$.

So we have $s < z < c$, where $s, c \in S$, so $[s,c] \subseteq S$.

So, S has to be an interval on one of the four cases.

* The other cases are :

ii) S is bounded above but not below

iii) S is bounded below but not above

iv) S is unbounded

Def We say a sequence of naturals $\{I_n : n \in \mathbb{N}\}$ is nested if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

What can we say about $\bigcap_{n=1}^{\infty} I_n$?

Ex $I_n = [0, \frac{1}{n}]$

$\bigcap_{n=1}^{\infty} I_n = \{0\}$

$x > 0 \in \bigcap_{n=1}^{\infty} I_n$

$x \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$

x is a lower bound of $\{\frac{1}{n} : n \in \mathbb{N}\}$

$x \leq \inf\{\frac{1}{n}\} = 0$

Sir can you move

$I_n = [0, \frac{1}{n})$

$\bigcap_{n=1}^{\infty} I_n = \varnothing$

$I_n = [n, \infty)$

$\bigcap_{n=1}^{\infty} I_n = \varnothing$