Aircraft Structures

Ing. Simon Bergé

Maître assistant, ISIB

Introduction to plates

Plates and shells

- Plate
  - Flat 2D surface
  - Small thickness (constant or slowly-varying)
- Shell
  - Curved 2D surface

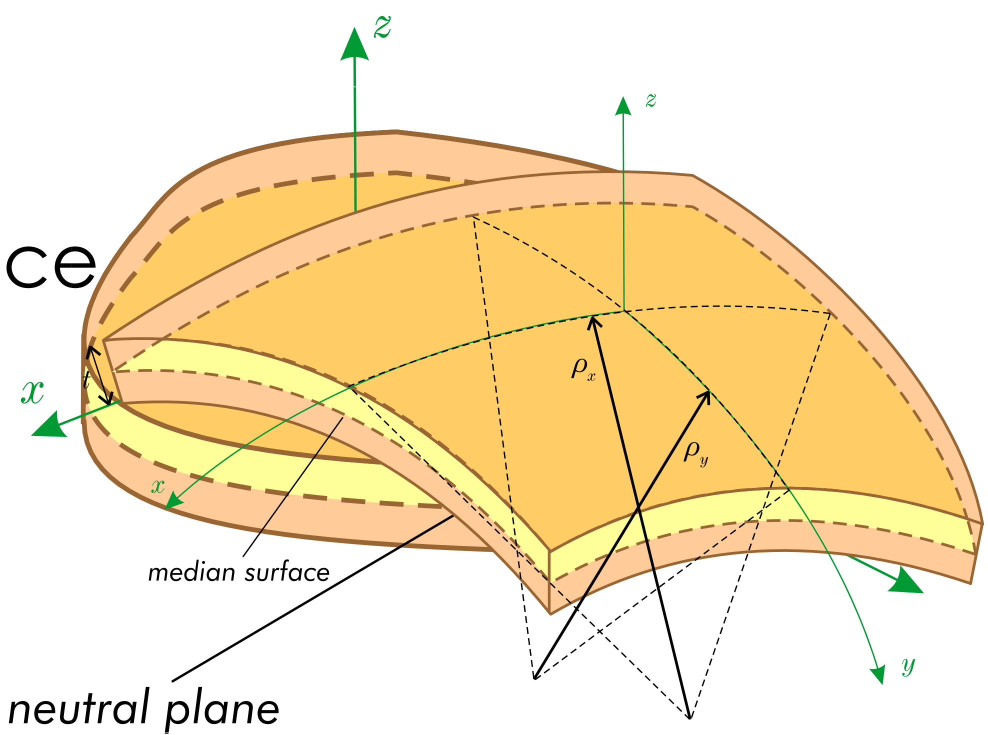


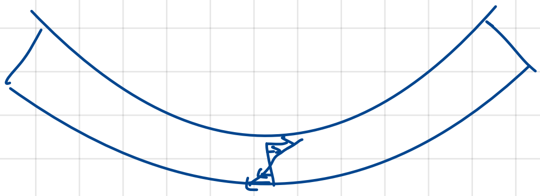
Plate theories

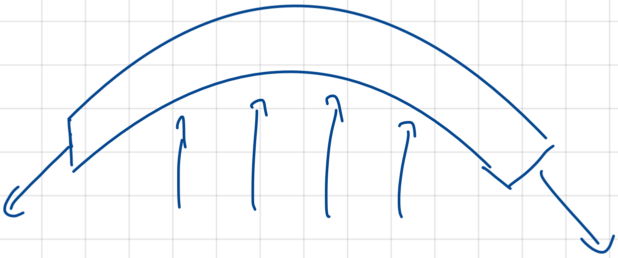
- Kirchhoff
  - No (bending) stress in the neutral plane
  - Small deflection of neutral plane
    (uncoupled bending/extension, no extension)
  - No normal stresses due to transverse loads
  - No strain due to shear stresses
    (transverse loads = shear by force equilibrium)

  - Displacement or equilibrium formulations
  - Fundamental plate theory

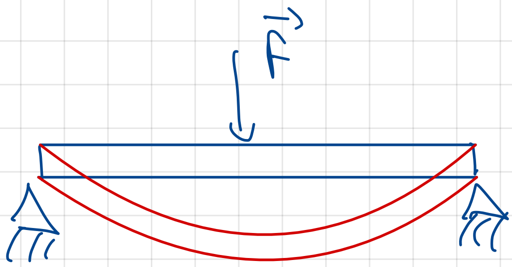
Plate

thin plate 2D structure





Shell



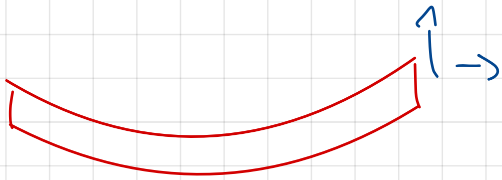


Plate theories

- Thick plates
  - No (bending) stress in the neutral plane
  - Small deflection of neutral plane (uncoupled bending/extension, no extension)
  - No normal stresses due to transverse loads
  - ~~No strain due to shear stresses~~
  - Displacement formulation (Hencky, cst shear)
  - Equilibrium formulation (Reissner, parab. shear)

Plate theories

- Thicker plates
  - No (bending) stress in the neutral plane
  - Small deflection of neutral plane (uncoupled bending/extension, no extension)
  - ~~No normal stresses due to transverse loads~~
  - ~~No strain due to shear stresses~~
  - 3D elasticity problem
  - Kromm, Woinosky-Krieger

Plate-membrane theory

- Von Kármán
  - No bending stress in the neutral plane
  - ~~Small deflection of neutral plane~~
    (bending/extension coupled, membrane stresses)
  - No normal stresses due to transverse loads
  - No strain due to shear stresses
    (transverse loads = shear by force equilibrium)

  - Equilibrium formulation
  - System of nonlinear PDE

Membrane theory

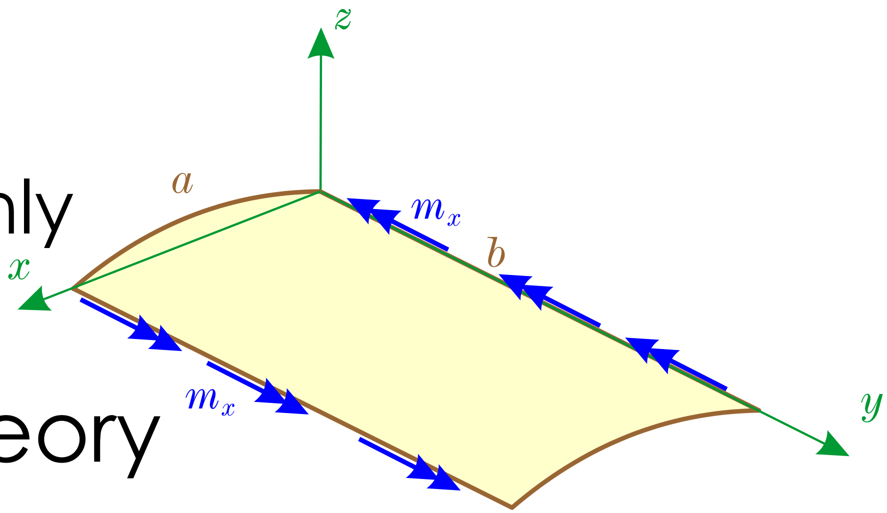
- Föppl
  - ~~No bending stiffness~~
  - ~~Small deflection of neutral plane~~
    (bending/extension coupled, membrane stresses)
  - No normal stresses due to transverse loads
  - No strain due to shear stresses
    (transverse loads = shear by force equilibrium)

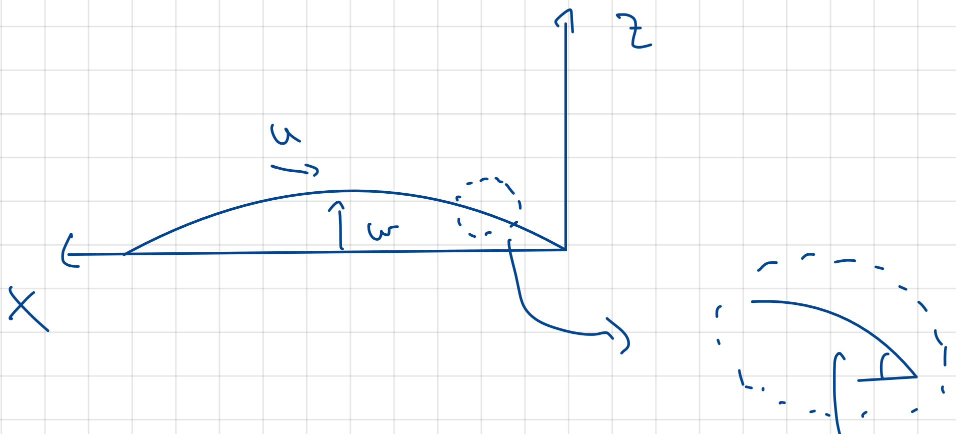
  - Equilibrium formulation
  - Nonlinear PDE

Long plates

Problem statement

- Long plate
  - $b >>> a$
  - Two loaded sides only
  - Cylindrical shape
- We use Kirchhoff theory
  - Plate deflection w
  - Cross-section remains plane
  - (u,v) only due to rotation of cross-section
  - No v displacement (cylindrical shape)





1st derivative of the deflection w

$$u = -z\frac{\partial w}{\partial x} \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

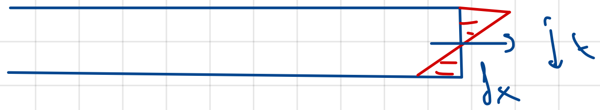
$$v = -z\frac{\partial w}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = 0$$

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu\\ \nu & 1\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \underbrace{\varepsilon_{yy}}_{=0}\end{pmatrix}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\,\varepsilon_{xx}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\,\varepsilon_{xx}\cdot\nu = \sigma_{xx}\cdot\nu$$

f_x [N/m]



$$f_x = \int_{-t/2}^{t/2} \sigma_{xx} \, dz$$

$$= \int_{-t/2}^{t/2} -\frac{E}{1-\nu^2} \cdot z \frac{\partial^2 w}{\partial x^2} \cdot dz = -\frac{E}{1-\nu^2} \frac{\partial^2 w}{\partial x^2} \underbrace{\int_{-t/2}^{t/2} z \, dz}_{=0} = 0$$

0 because no normal stress in Kirchhoff theory, we assume small displacements

moment resultant

$$m_x = \int_{-t/2}^{t/2} \sigma_{xx} \, z \, dz$$

$$= -\frac{E}{1-\nu^2} \frac{\partial^2 w}{\partial x^2} \int_{-t/2}^{t/2} z^2 \, dz$$

$$\left[\frac{z^3}{3}\right]_{-t/2}^{t/2} = \frac{t^3}{24} - \left(-\frac{t^3}{24}\right) = \frac{t^3}{12}$$

$$m_x = -\frac{E}{1-\nu^2} \frac{t^3}{12} \frac{\partial^2 w}{\partial x^2} \Rightarrow m_x = -D w''$$

↳ D similar "EI" for a beam

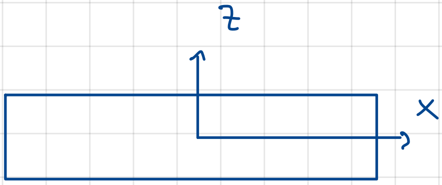
↳ flexural rigidity

Von Karmann

$$u = -z\frac{\partial w}{\partial x} + u_0$$

small

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\cancel{\left(\frac{\partial u}{\partial x}\right)^2} + \cancel{\left(\frac{\partial v}{\partial x}\right)^2} + \left(\frac{\partial w}{\partial x}\right)^2\right]$$



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) = \frac{E}{1-\nu}\left[-z\cdot\frac{\partial^2 w}{\partial x^2} + \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$f_x = \frac{E}{1-\nu^2}\int_{-t/2}^{t/2}\left[-z\frac{\partial^2 w}{\partial x^2} + \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dy$$

$$f_x = \frac{E}{1-\nu^2}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\cdot t \Rightarrow f_x = \underbrace{\frac{Et}{1-\nu^2}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)}_{cte} \neq 0$$

$$m_x = -\frac{E}{1-\nu^2}\frac{t^3}{12}\frac{\partial^2 w}{\partial x^2} \Rightarrow m_x = -D\,w''$$

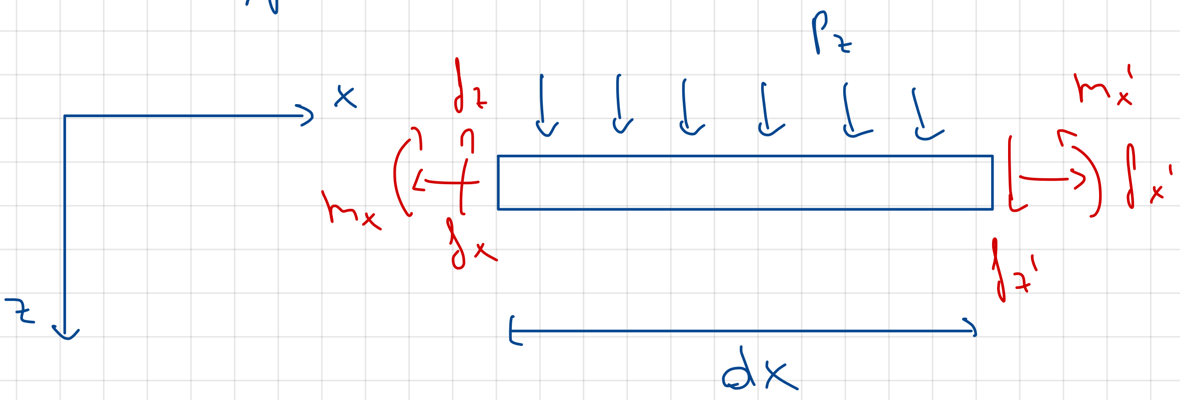
Kirchhoff $f_x = 0$

$$m_x = \frac{-E}{1-\nu^2}\cdot\frac{t^3}{12}\frac{\partial^2 w}{\partial x^2}$$

Von Karmann $f_x = \frac{Et}{1-\nu^2}\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)$

$$m_x = -\frac{E}{1-\nu^2}\frac{t^3}{12}\frac{\partial^2 w}{\partial x^2}$$

Kirchhoff



$/\vec{e}_x : f_x' - f_x = 0 \Rightarrow \dfrac{\partial f_x}{\partial x} = 0 \Rightarrow f_x = cte = 0$

$f_x' = f_x + \dfrac{\partial f_x}{\partial x} dx$

$/\vec{e}_y : \underbrace{f_z' - f_z}_{} + p_z dz = 0$

$\dfrac{\partial f_z}{\partial x} dx + p_z dx = 0 \Rightarrow \dfrac{\partial f_z}{\partial x} = -p_z$

$\circlearrowright^{+} A \quad m_x' - m_x - f_z dx + p_z \cdot dx \cdot \dfrac{dx}{2} = 0$

$\dfrac{\partial m_x}{\partial x} dx - f_z dx + p_z \cdot \dfrac{dx^2}{2} = 0$

$\dfrac{\partial m_x}{\partial x} = f_z$ and $\dfrac{\partial f_z}{\partial x} = -p_z$

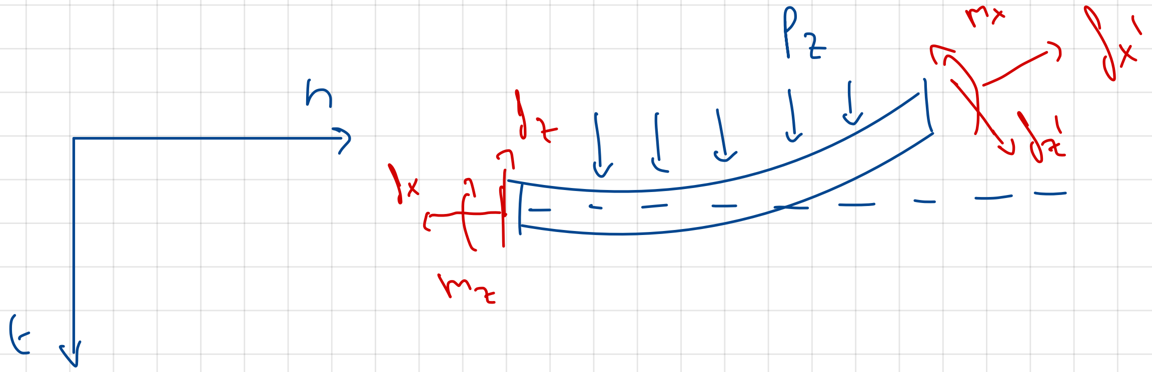
$\dfrac{\partial^2 m_x}{\partial x^2} = -p_z \qquad m_x = -D \dfrac{\partial^2 w}{\partial x^2}$

$-D \dfrac{\partial^4 w}{\partial x^4} = -p_z$

$$\frac{\partial^4 w}{\partial x^4} = \frac{p_z}{D}$$

↳ Lagrange equation

Kirchhoff $\boxed{\Delta^2 w = \frac{p_z}{D}}$



$$\Delta\theta = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx$$

$$\Delta\theta \ll 1 \begin{cases} \sin\Delta\theta = \Delta\theta \\ \cos\Delta\theta = 1 \end{cases}$$

$$/\vec{e}_n : -f_x + f_x' \underbrace{\cos\Delta\theta}_{1} + f_z' \underbrace{\sin\Delta\theta}_{\Delta\theta} = 0$$

$$\frac{\partial f_x}{\partial x}dx + \left(f_z + \cancel{\frac{\partial f_z}{\partial x}dx}\right)\left(-\frac{\partial^2 w}{\partial x^2}dx\right) = 0$$

= 0, hypothesis von KARMANN

$$\frac{\partial f_x}{\partial x} = f_z \frac{\partial^2 w}{\partial x^2} = 0 \Rightarrow f_x = cte$$

$$/\vec{e}_z : -f_x' \sin\Delta\theta + f_z' \cos\Delta\theta - f_z + p_z\, dx = 0$$

$$\frac{\partial f_z}{\partial x}dx + p_z\, dx - \left(f_x + \frac{\partial f_x}{\partial x}dx\right)\underbrace{\Delta\theta}_{-\frac{\partial^2 w}{\partial x^2}dx} = 0$$

$$\frac{\partial f_z}{\partial x} + p_z + f_x \frac{\partial^2 w}{\partial x^2} = 0 \qquad\qquad -D\frac{\partial^4 w}{\partial x^4} = -p_z - f_x \frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial f_z}{\partial x} = -p_z - f_x \frac{\partial^2 w}{\partial x^2}$$

(A^+) $$\frac{\partial m_x}{\partial x} = f_z = 0 \Rightarrow \frac{\partial^2 m_x}{\partial x} = -p_z - f_x \frac{\partial^2 w}{\partial x^2}$$

Kirchhoff

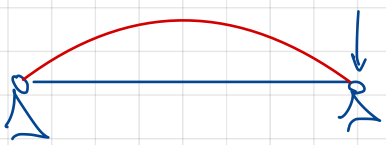
$$w'''' = \frac{p_z}{D}$$

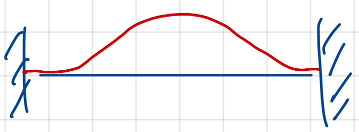
Von Karmann

$$\boxed{w'''' = \frac{p_z}{D} + \frac{f_x}{D} w''}$$

to solve these you need 4 B.C.

no deformation



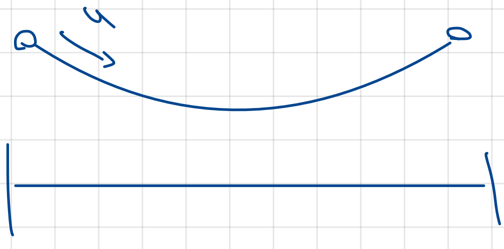


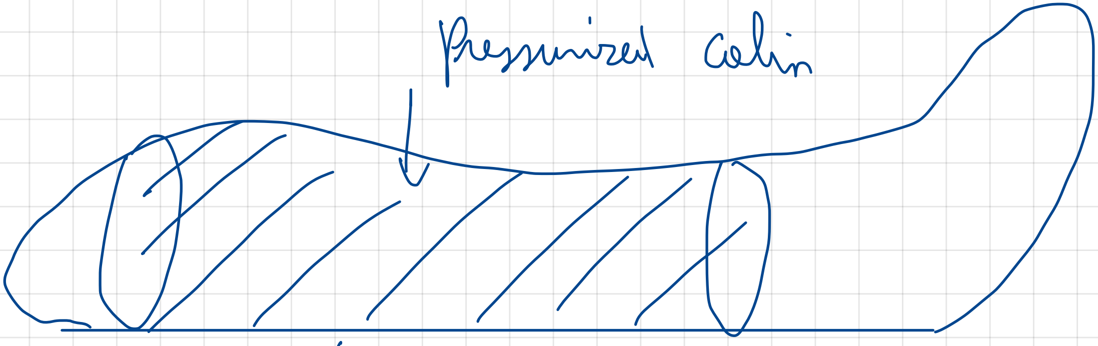
0	L	0	L
$w=0$	$w=0$	$w=0$	$w=0$
$w''=0$	$w''=0$	$w'=0$	$w'=0$

$$w'''' - \frac{f_x}{D} w'' = \frac{p_z}{D}$$

$$\eta'' - \frac{f_x}{D}\eta = \frac{p_z}{D}$$

let $w'' = \eta$





6000 ft

FL 370

$$\delta = \frac{p}{p_0} = 0{,}8014$$

$p_0 = 101\,325\ \text{Pa}$

$$\hookrightarrow \delta = \frac{p}{p_0} = 0{,}2138$$

$$p = p_0\,(0{,}8014 - 0{,}2138)$$
$$= 59\,539\ \text{Pa}$$

Solution procedure (1)

- Assumed displacement field
- Strain-displacement relations
  - In this case, only ε_{xx} exists
- Stress-displacement relations
  - Use Hooke's law for stress-strain
  - Substitute strains for stress-displacement
  - In this case, $\sigma_{yy} = \nu\, \sigma_{xx}$

Solution procedure (2)

- Global equil. (stress ⇒ internals)
  - Distributed resulting force f_x is zero
    - Fixes location of neutral plane for heterogeneous beams
  - Distributed resulting moment m_x is unknown
    - Provides a moment-displacement relation
    - Defines plate *flexural rigidity* D
- Global equil. (internals ⇒ externals)
  - Force equilibrium under pressure load
  - Moment equilibrium under pressure load

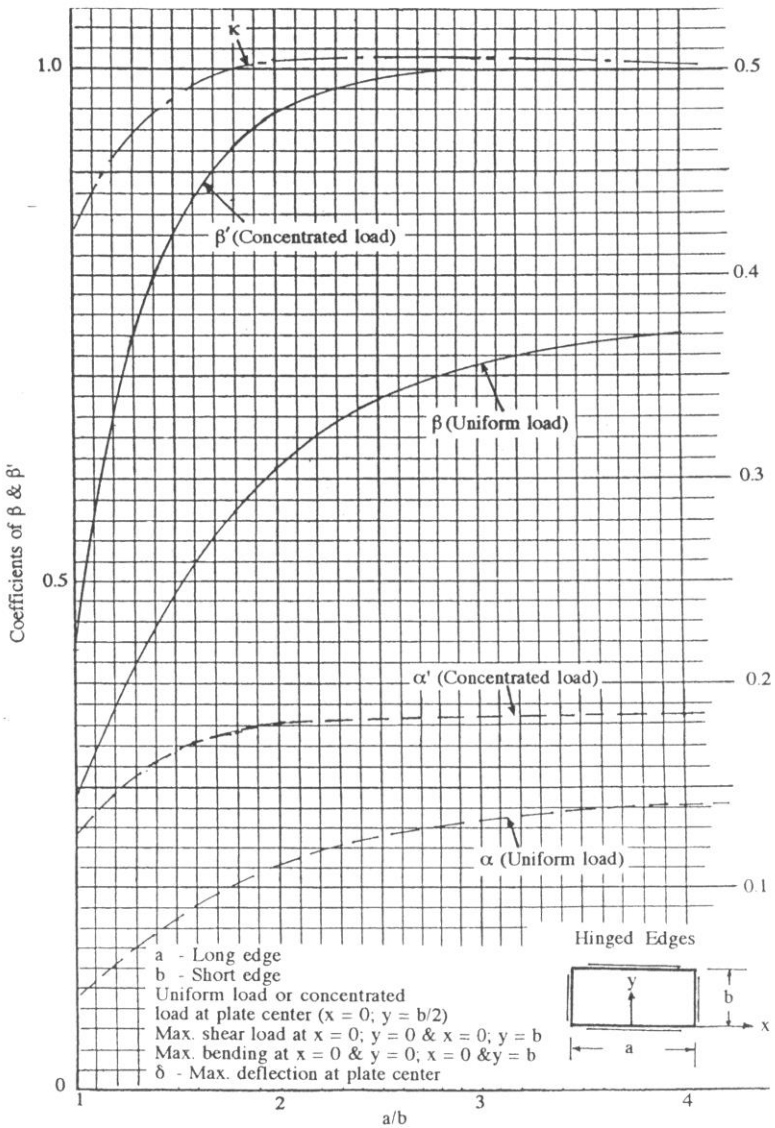
Solution procedure (3)

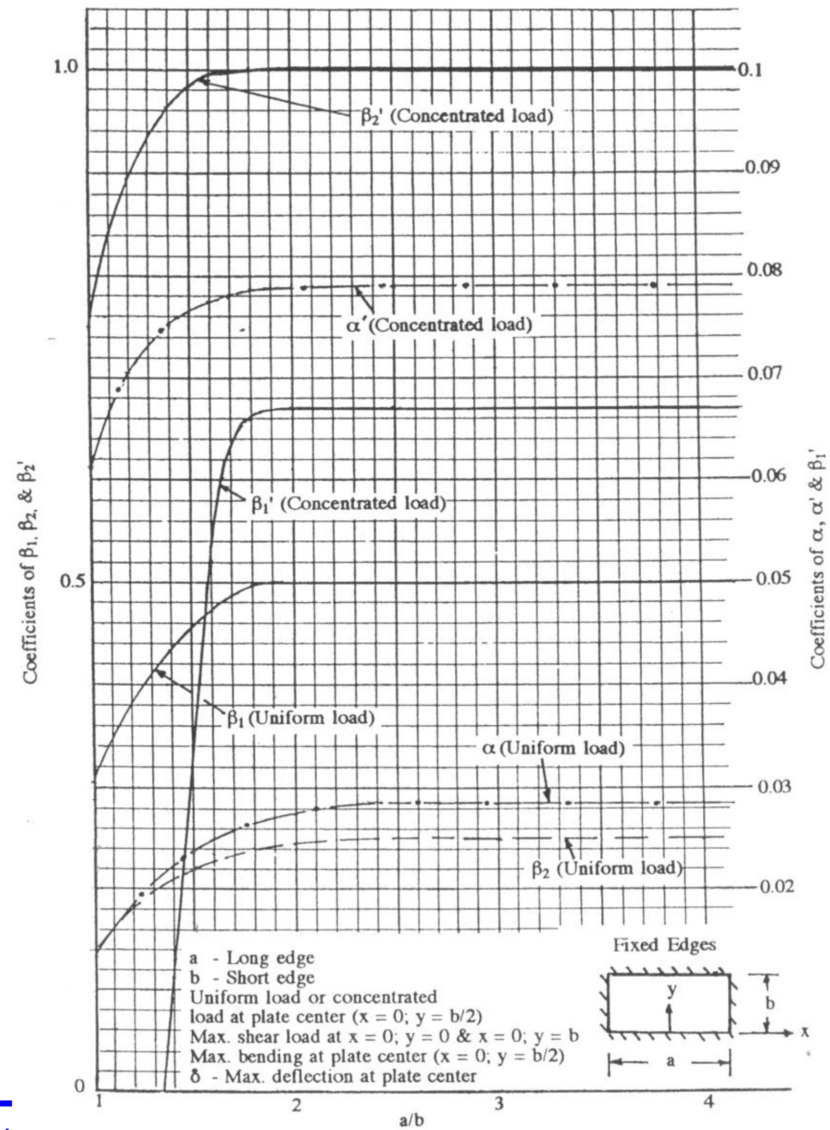
- Obtain plate differential equation
  - Fourth-order equation in w(x)
  - Simplified version of Lagrange equation
- Solve to obtain deflection
  - Fixed edge: no deflection or slope
  - Hinged edge: no deflection or moment
- Work backwards
  - Internals from internals – displacement laws
  - Stresses from stress – displacement laws

Example

- Compute the solution of the long plate problem in the hinged-hinged (and fixed-fixed) cases and compare with the numerical coefficients provided in Niu

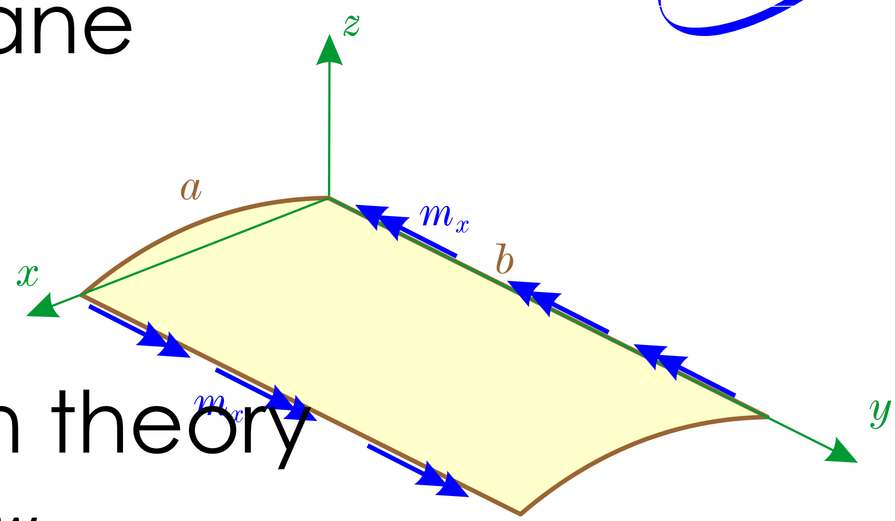
Plate design charts





Problem statement

- Long plate-membrane
  - $b >>> a$
  - Two loaded sides only
  - Cylindrical shape



- We use von Kármán theory
  - Large plate deflection *w*
    - Nonlinear strain-displacement relations
  - Cross-section remains plane
  - Include *u* field due to membrane extension
  - No *v* displacement (cylindrical shape)

Solution procedure (1)

- Assumed displacement field
- Strain-displacement relations
  - In this case, only ε_{xx} exists
  - Simplified nonlinear strain-displacement laws
- Stress-displacement relations
  - Use Hooke's law for stress-strain
  - Substitute strains for stress-displacement
  - In this case, $\sigma_{yy} = \nu\ \sigma_{xx}$

Solution procedure (2)

- Global equil. (stress ⍰ internals)
  - Distributed resulting force f_x is unknown
    - Provides a force-displacements relation
    - Defines plate *extensional stiffness* K
  - Distributed resulting moment m_x is unknown
    - Provides a moment-displacement relation
    - Defines plate *flexural rigidity* D
- Global equil. (internals ⍰ externals)
  - Include in-plane force equilibrium
  - Include effect of plate deflection

Solution procedure (3)

- Obtain plate differential equation
  - System of coupled PDEs
    - Fourth-order equation in w(x)
    - Second-order equation in u(x)
    - Simplified version of von Kármán’s equation
- Solve to obtain deflections
  - Very difficult solution procedures required
  - Very few analytical solutions exist
    - Approximate solution methods (energy, assumed deflections)
    - Numerical solutions

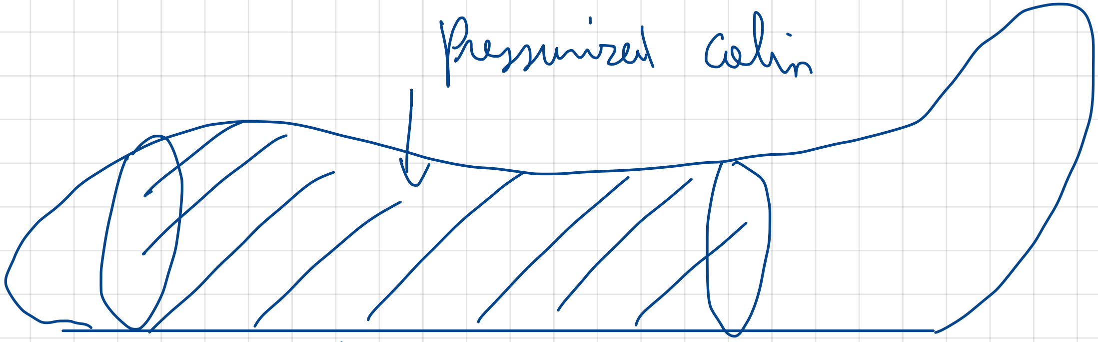
Example

- Compute the solution of the long plate-membrane problem in the hinged-hinged (and fixed-fixed) cases and compare with the numerical coefficients provided in Niu

Example

➢ The fuselage of an ERJ-type aircraft (238 cm diameter) is sealed off at the rear by a flat pressure bulkhead. Size the bulkhead based on the three following materials, including weight.

	E [GPa]	ν	ρ [kg/m^3]	σ [MPa]
Aluminium	72.4	0.32	2768	400
Steel	200	0.32	7833	689
Titanium	110	0.31	4429	827



6000 ft

$\delta = \frac{p}{p_0} = 0{,}8074$

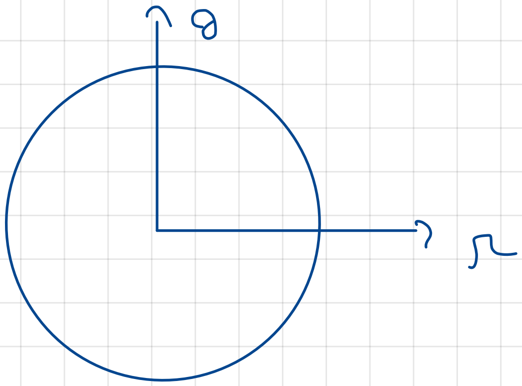
101 325 Pa

FL 370

$\delta = \frac{p}{p_0} = 0{,}2738$

$p = p_0 (0{,}8074 - 0{,}2738)$
$= 59\ 539 \text{ Pa}$

Kirchoff



$$\Delta^2 w = \frac{p}{D}$$

$$\Delta = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\partial}{\partial r}\right)\right)\right)$$

$$\cancel{\frac{1}{r}} \cdot \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right)\right)\right] = \frac{p}{D}\, r$$

Aluminium:

$E = 72{,}4$ GPa

$\nu = 0{,}32$

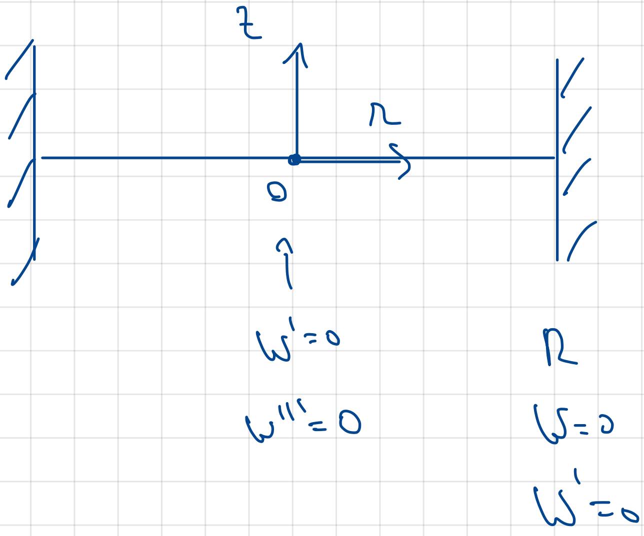
$\sigma_y = 400$ MPa

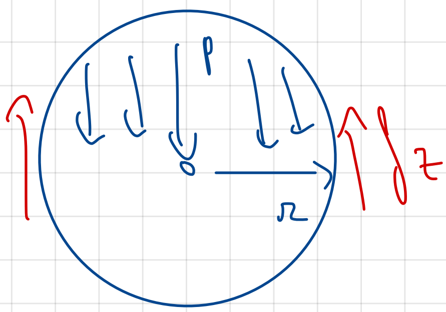
$$\cancel{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right) = \frac{p}{D}\frac{r^{\cancel{2}}}{2} + \frac{C_1}{r}$$

$$\cancel{\frac{1}{r}}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{p}{D}\frac{r^{3}}{4} + r\,\cancel{C_1 \ln r} + C_2 r$$

$$\cancel{r}\frac{\partial w}{\partial r} = \frac{p}{D}\frac{r^{3}}{16} + \cancel{\frac{\int \left(r C_1 \ln(r)\right) dr}{r}} + C_2\frac{r^{\cancel{2}}}{2} + \frac{\cancel{C_3}}{r}$$

$$w = \frac{p}{D}\frac{r^4}{64} + \int \cancel{\frac{\int r C_1 \ln r \, dr}{r}}\, dr + C_2\frac{r^2}{4} + \cancel{C_3}\ln r + C_4$$





$$\pi r^2 p = Q_z 2\pi r \Rightarrow Q_z = p\frac{r}{2}$$

at $r = 0$ $Q_z = 0$ $\quad Q_z = \frac{\partial M_x}{\partial x} = -Dw'''$

$$0 = \frac{p}{D} \cdot \frac{0}{2} + \frac{C_1}{D} \Rightarrow \boxed{C_1 = 0}$$

$$\Rightarrow 0 = \frac{p}{D} \cdot \frac{0}{16} + C_2 \frac{0}{2} + \frac{C_3}{D} \rightarrow \boxed{C_3 = 0}$$

at $r = R$, $w' = 0$

$$0 = \frac{p}{D} \frac{R^3}{16} + C_2 \frac{R}{2}$$

$$C_2 = -\frac{p}{D} \frac{R^2}{8}$$

at $r = R$, $w = 0$

$$0 = \frac{p}{D} \frac{R^4}{64} - \frac{p}{D} \frac{R^2}{8} \frac{R^2}{4} + C_4$$

$$C_4 = \frac{p}{D} \frac{R^4}{32} - \frac{p}{D} \frac{R^4}{64}$$

$$C_4 = \frac{p}{D} \frac{R^4}{64}$$

$$w = \frac{p}{D} \frac{r^4}{64} - \frac{p}{D} \frac{R^2}{8} \frac{r^2}{4} + \frac{p}{D} \frac{R^4}{64}$$

$$= \frac{p}{64D} \left(r^4 - 2 \cdot R^2 \cdot r^2 + R^4 \right)$$

$$= \frac{p}{64D} \left(r^2 - R^2 \right) \longrightarrow \delta : \frac{p}{64D} R^4$$

$$\sigma_{rr}(0) = \sigma_{\theta\theta}(0) = \frac{3}{8} p \frac{R^2}{t^2}(1+\nu)$$

$$\boxed{\sigma_{rr}(R) = -\frac{3p}{4} \frac{R^2}{t^2} \qquad \sigma_{\theta\theta}(R) = \nu\,\sigma_{rr}(R)}$$

Bigger stress at the edge, not at centre of plate!

$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

Von Mises

$\sigma_1 = \sigma_{rr}$

$\sigma_2 = \sigma_{\theta\theta}$

$\sigma_3 = 0$

$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_{rr}-\nu\,\sigma_{rr})^2+\nu^2\sigma_{rr}^2+\sigma_{rr}^2}$$

$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{\sigma_{rr}^2\left[(1-\nu)^2+1+\nu^2\right]}$$

$$\sigma_{rr} = \frac{\sigma_{VM}\cdot\sqrt{2}}{\sqrt{1-\nu^2+1+\nu^2}}$$

$\sigma_{VM} \rightarrow$ 400 MPa (yield)

$\sigma_{rr} = 452$ MPa

$$t^2 = \frac{3pR^2}{4\sigma_m} \Rightarrow t = \frac{R}{2}\sqrt{\frac{3p}{\sigma_m}} = 1,2 \text{ cm (thickness)}$$

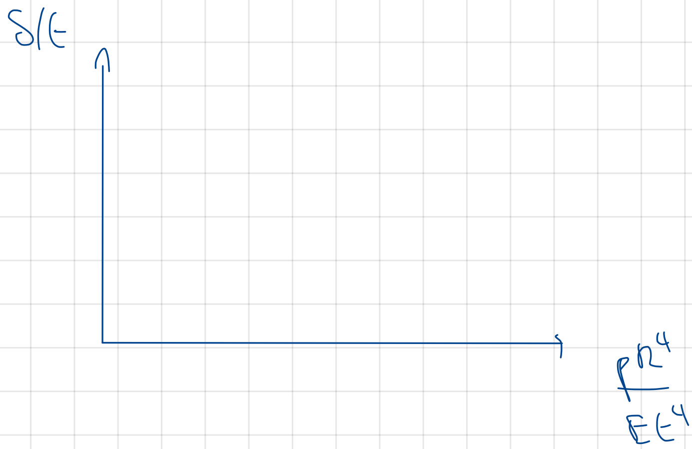
↳ min thickness
You need to stay under elastic strain

$$\delta = \frac{p}{64D} R^4 = 16,8 \text{ cm}$$

↳ $D = \frac{E}{1-\nu^2}\frac{t^3}{12}$

$\frac{\delta}{t} = 14$

$$\delta = \frac{p}{64Et^3}(1-\nu^2)\,12 \cdot R^4$$



$$\overset{\nearrow\ 3,4}{\frac{\delta}{t}} = \underbrace{\left(\frac{1-\nu^2 \cdot 12}{64}\right)}_{0,17} \overset{\nearrow\ 20}{\frac{pR^4}{Et^4}}$$

y   0,17   x

↳ linear function

$$\frac{pR^4}{Et^4} = \frac{59359 \cdot 1,19^4}{72,4 \cdot 10^9 \cdot 0,012^4}$$

$$= 19,3$$

$$\frac{\delta}{t} = 2,9$$

Kirchoff valid if deflection $\frac{\delta}{t} < 1$!

⇒ Von Karman better!

$\sigma_b^* = 30 \rightarrow \sigma_{bending}$ (plate)

$\frac{\sigma^*}{\sigma} = \frac{R^2}{Et^2}$

$\sigma_m^* = 5 \rightarrow \sigma_{membrane}$ (shell)

$$\sigma_{rb} = \sigma^* \frac{Et^2}{R^2} = \frac{30 \cdot 72{,}4 \cdot 10^9 \cdot (1{,}2 \cdot 10^{-2})^2}{1{,}19^2}$$

$$= 221 \text{ MPa}$$

$$\sigma_{rm} = \sigma_m^* \cdot \frac{Et^2}{R^2} = 37 \text{ MPa}$$

$$\sigma_r = 258 \text{ MPa} \rightarrow 452 \text{ MPa}$$

<u>iteration</u> :

Kirchoff : $\sigma_r = 458$ MPa (limit elastic range)

Von K : $\sigma_{rr} = 258$ MPa $\quad t = 1{,}2$ cm

$t = ?$

$$t = \sqrt[4]{\frac{\sigma_{r\,VK}}{\sigma_{r\,K}}} \quad t_{old} = 1 \text{ cm}$$

$\hookrightarrow$ based on Von Mises

$$\frac{pR^4}{Et^4} = 140 \xrightarrow{\text{chart}} \frac{\delta}{t} = 3{,}6$$

$\sigma^*_{\theta} = 50 \Rightarrow \sigma_{rr_{\theta}} = \dfrac{50 \cdot 72{,}4 \cdot 10^9 (0{,}01)^2}{1{,}79^2}$

$\sigma^*_m = 8$

$= 255\ MPa$

$\sigma_{rr,m} = 40\ MPa$

$\hookrightarrow$ 295 MPa (Sum because it's the same direction r)

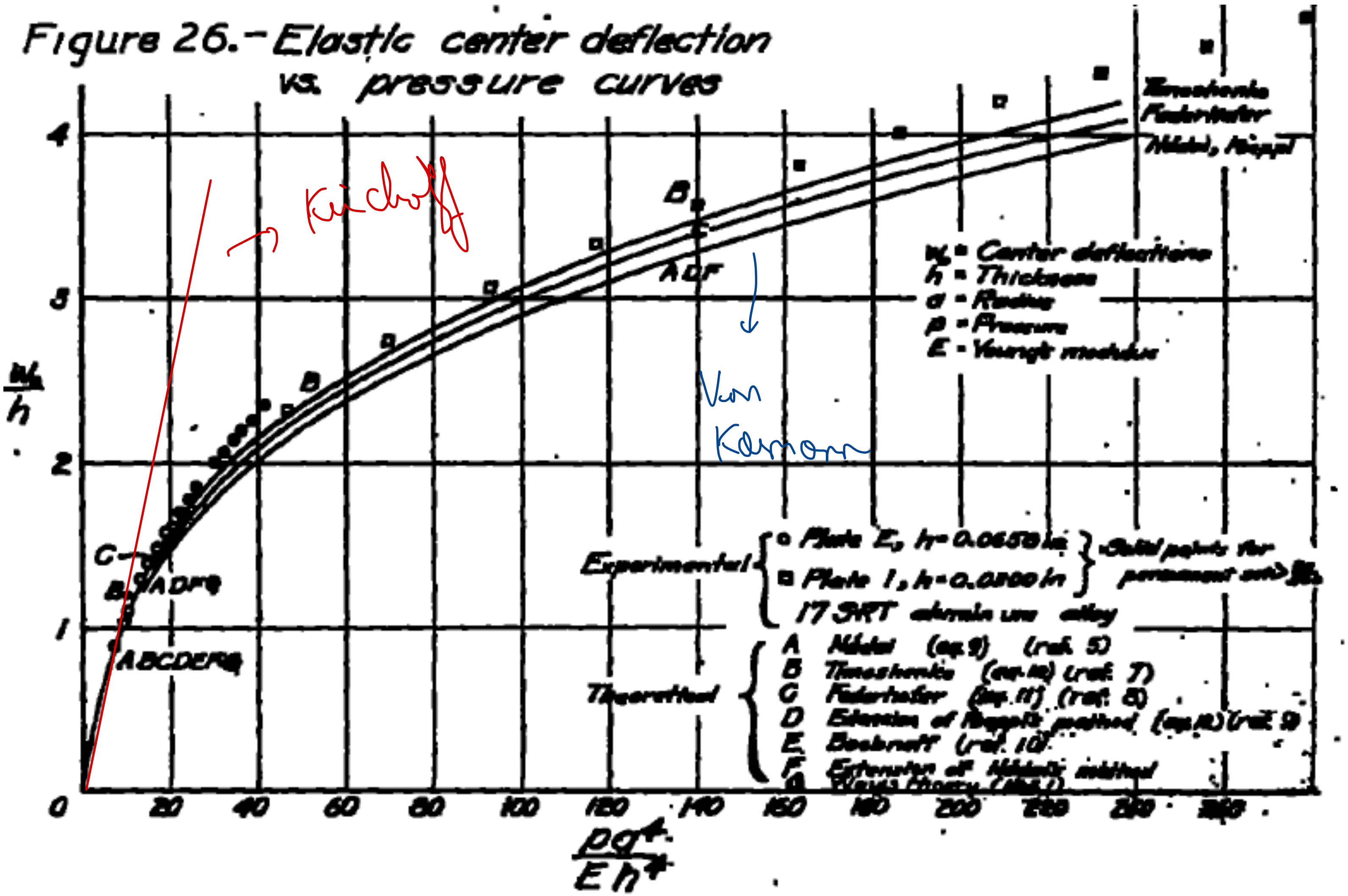
$+255$

$\pi R^2 \cdot t = V$

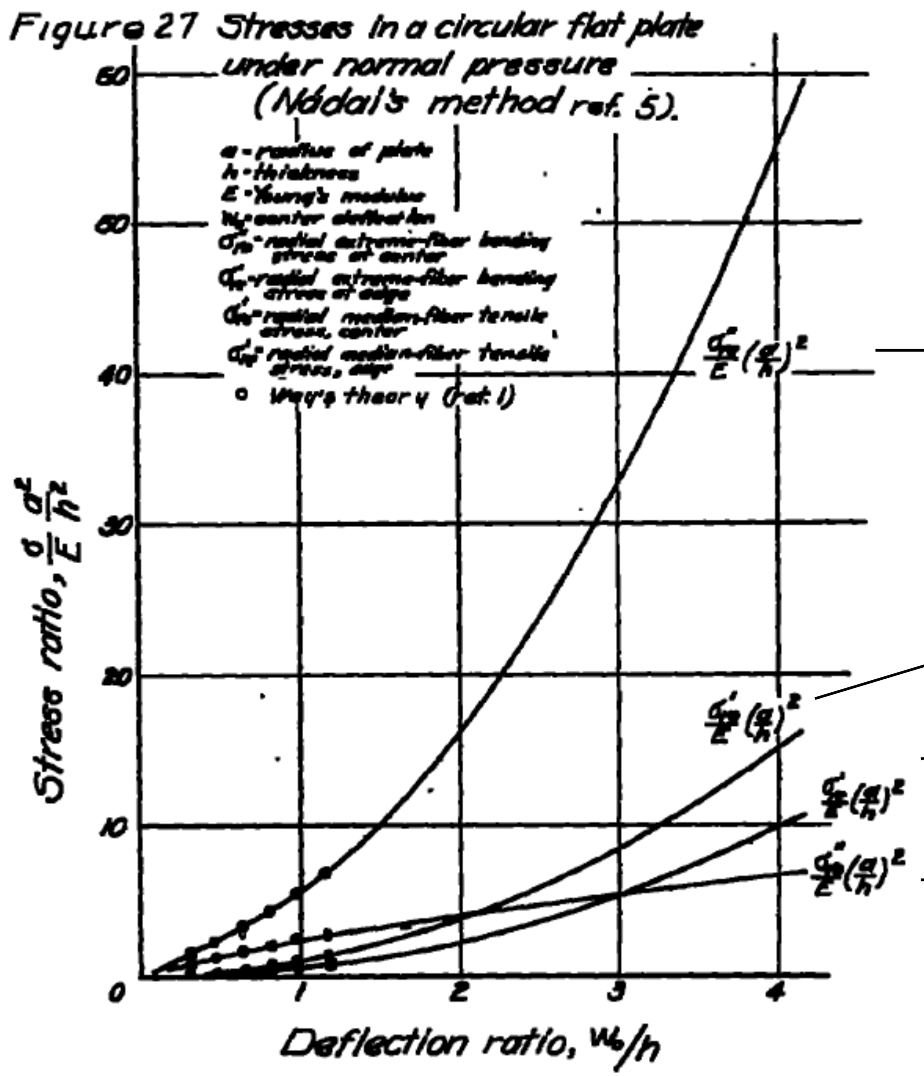
$V = \pi \cdot 1{,}79^2 \cdot 0{,}01 = 0{,}044\ m^2$

$m = \rho V = 123\ kg$

Circular plate-membrane



Circular plate-membrane



Bending stress - edge

Membrane stress - centre

Membrane stress - edge

Bending stress - centre