# Aircraft Structures


**Ing Simon Bergé**

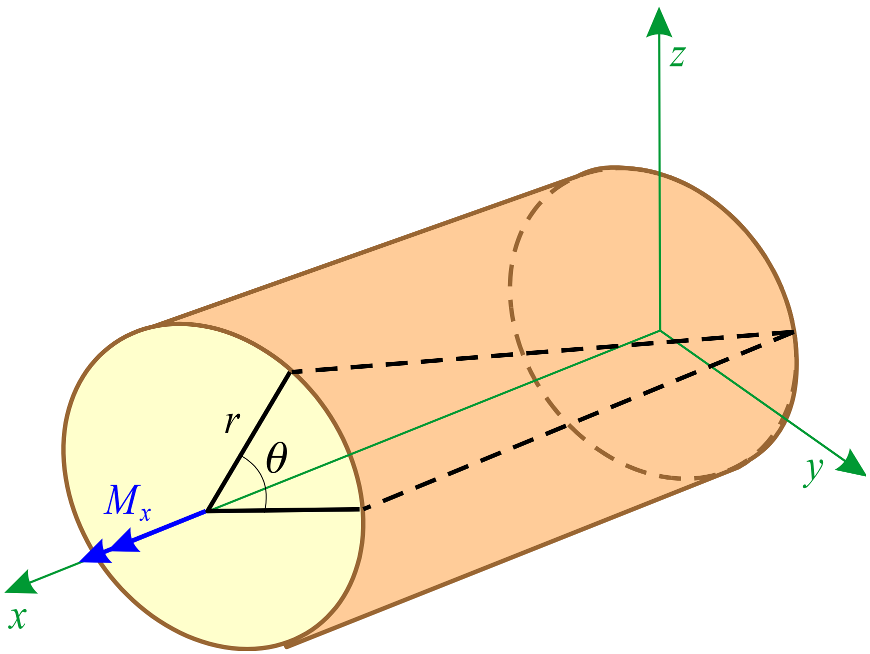
Maître assistant, ISIB


*Masters en Sciences de l'Ingénieur Industriel - finalité Mécanique (Génies Mécanique et Aéronautique)*

# Torsion of thin-walled beams

*Aft Section 46 of the 767-400ER, built upside-down for ergonomics, is placed into a turn fixture to rotate right-side-up.*

# Circular beam




$$\begin{cases} u_r = 0 \\ u_\theta = Crx \\ u_x = 0 \end{cases}$$

$$\varepsilon_{\theta x} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{r}\frac{\partial u_x}{\partial \theta}\right) = \frac{Cr}{2}$$

$$\sigma_{\theta x} = \frac{E}{(1+\nu)}\frac{Cr}{2} = GCr$$

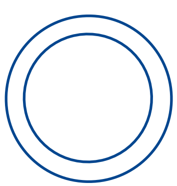
$$M_x = \int_0^{2\pi}\int_0^{R} r\sigma_{\theta x}(r\,dr\,d\theta) = \frac{GC\pi R^4}{2}$$

$$\tau = \sigma_{\theta x} = \frac{M_x r}{I_p} = \frac{M_x r}{J}$$

$$J = \frac{\pi R^4}{2}$$

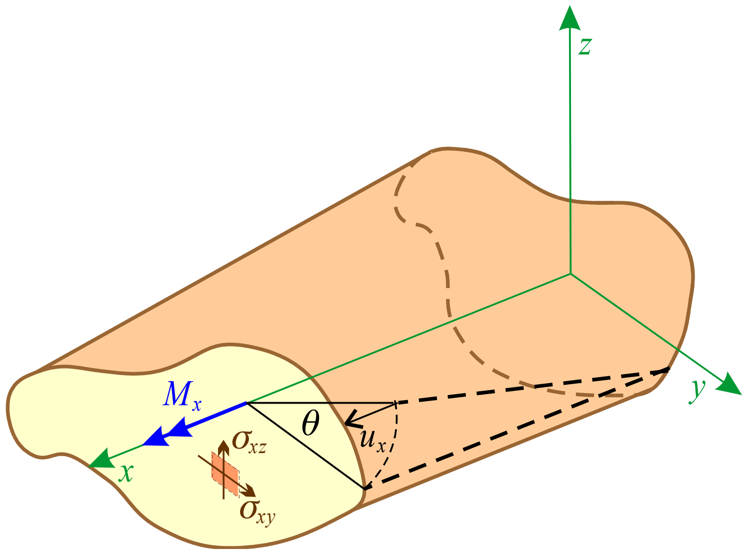
$$\gamma = \frac{M_x}{GJ}r,\ \ u_\theta = \frac{M_x}{GJ}rx$$

$$\theta = \frac{u_\theta}{r} = \frac{M_x}{GJ}x$$



$$M_x = GC\pi\frac{R_o^4 - R_i^4}{2}$$

# Non-circular beam




- Warping of cross-sections
- Stress formulation
- Hydrodynamic analogy (Prandtl)

Prandtl's stress function definition

$$\begin{cases} \sigma_{xy} = \dfrac{\partial\psi}{\partial z} \\ \sigma_{xz} = -\dfrac{\partial\psi}{\partial y} \end{cases}$$

Compatibility

$$\Delta\sigma_{xy} = 0 \Rightarrow \frac{\partial}{\partial z}\Delta\psi = 0$$

$$\Delta\sigma_{xz} = 0 \Rightarrow \frac{\partial}{\partial y}\Delta\psi = 0$$

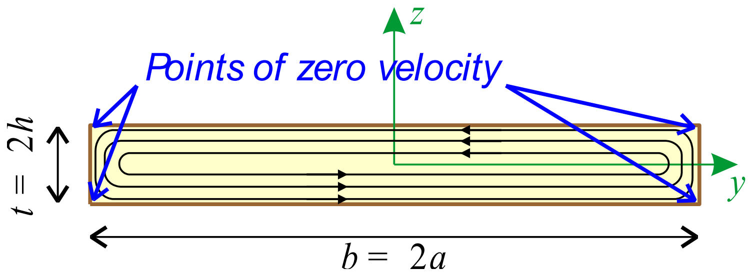
$$\Delta\psi = F$$

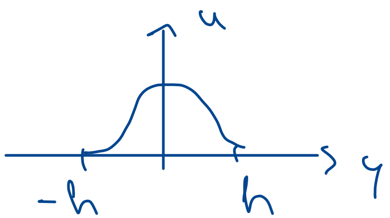
Boundary conditions

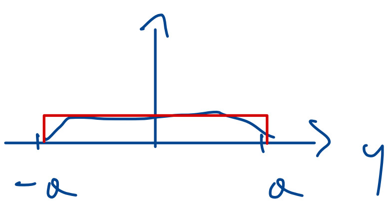
$$\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial s} - \frac{\partial\psi}{\partial y}\left(-\frac{\partial y}{\partial s}\right) = \frac{\partial\psi}{\partial s} = 0$$

$$\psi_{contour} = 0$$

# Open rectangular beam




$$\Delta\psi = \frac{\partial^2\psi}{\partial z^2} = F$$

$$\psi = \frac{F}{2}z^2 + Az + B$$

$$\psi = \frac{F}{2}(z^2 - h^2)$$

$$M_x = 2\int_A \psi\, dA = \int_{-h}^{h}\int_{-a}^{a} F(z^2 - h^2)\, dy\, dz$$

$$M_x = -\frac{8}{3}Fh^3a = -\frac{1}{2}\frac{bt^3}{3}F$$

$$J = -\frac{2M_x}{F} = \frac{bt^3}{3}$$

$$\sigma_{xy} = -\frac{2M_x z}{J}$$

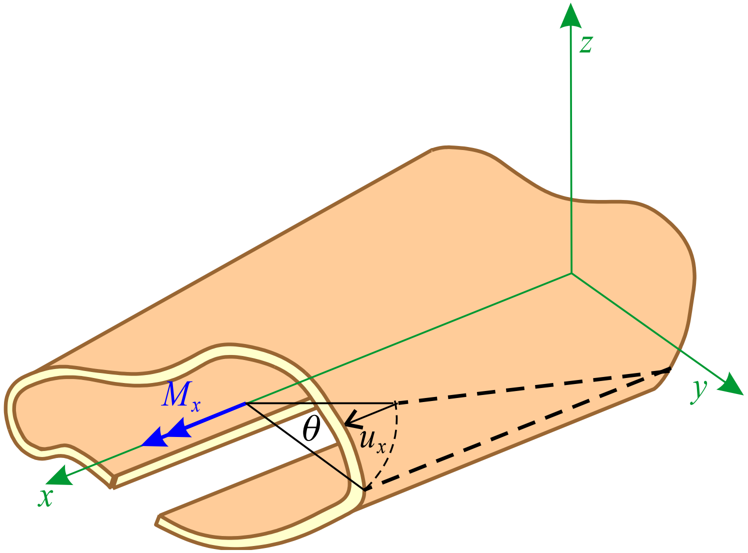
$$\theta' = \frac{M_x}{GJ}$$

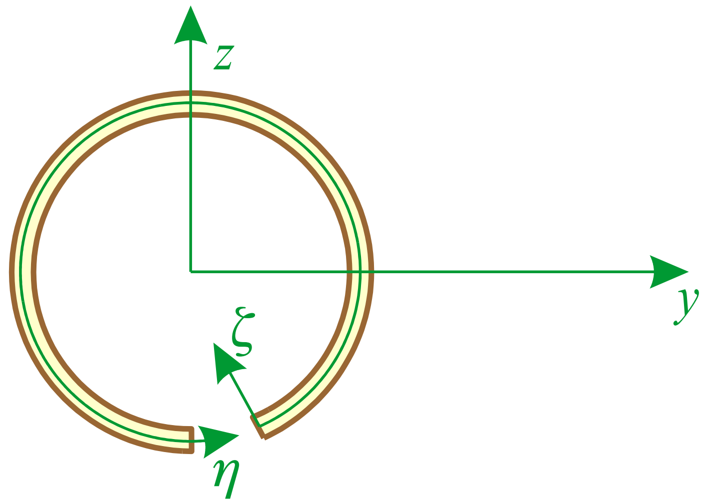
# Open non-rectangular beam

$$M_x = -\frac{1}{2} F \int_0^b \frac{t^3(\eta)}{3} d\eta$$

$$J = \int_0^b \frac{t^3(\eta)}{3} d\eta$$

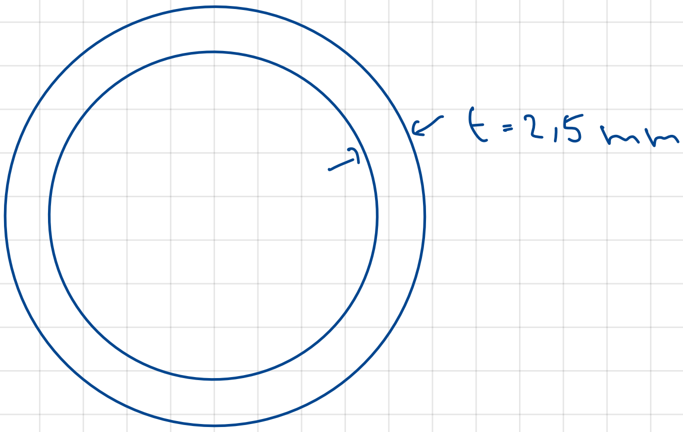
Thickness over the contour



# Exercise 1 (a)

- Compute the torsional rigidity, unit twist and maximum shear stress of a beam of hollow circular section of thickness 2.5 mm and mean radius 50 mm in the open case (slotted beam). The beam is 30 cm long and the torque is 912 Nm. Consider $G$ = 27.6 GPa.




$L = 300\,mm$ $\quad \langle GJ \rangle = ?\ ,\ \theta' = \frac{\theta}{x} = ?\ ,\ \tau_{max} = ?$

$M_x = 912\,Nm$

$G = 27,6\,GPa$

$R_i = \bar{R} - \frac{t}{2} = 48,75\,mm$

$R_o = \bar{R} + \frac{t}{2} = 51,25\,mm$

$J = \frac{\pi}{2}(51,25^4 - 48,75^4) = 1,96 \cdot 10^6\,mm^4$

$= 1,96 \cdot 10^{-6}\,m^4$

$\langle GJ \rangle = 27,6 \cdot 10^9 \cdot 1,96\,m^{-6} = 54096\,Nm^2$

$\tau_{max} = \frac{M_x \cdot R_o}{J} = \frac{912 \cdot 0,05125}{1,96 \cdot 10^{-6}} = 23,8\,MPa$

$\theta' = \frac{M_x}{GJ} = \frac{912}{54096} = 1,68 \cdot 10^{-2}\ rad/m$

## Open case:

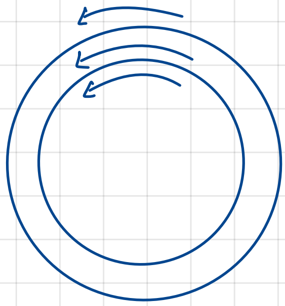
$$J = \int_0^b \frac{t^3(\eta)}{3}\, d\eta = \frac{t^3}{3}\int_{\eta_0}^{\eta_f} d\eta = \frac{t^3}{3}(\underbrace{\eta_f}_{2\pi R} - \underbrace{\eta_0}_{=0})$$

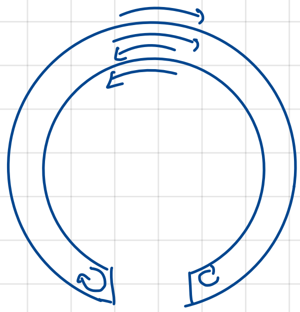
$$= \frac{2{,}5^3}{3} \cdot 2\pi \cdot 50 = 1{,}64 \cdot 10^3\ mm^4$$
$$= 1{,}64 \cdot 10^{-9}\ m^4$$

$$\tau = \frac{-2M_x \cdot \zeta}{J} = \frac{-972 \cdot 0{,}0025}{1{,}64 \cdot 10^{-9}} = -1{,}39\ GPa$$

$$\theta' = \frac{M_x}{GJ} = \frac{972}{27{,}6 \cdot 10^9 \cdot 1{,}64 \cdot 10^{-9}} = 20{,}2\ rad/m$$

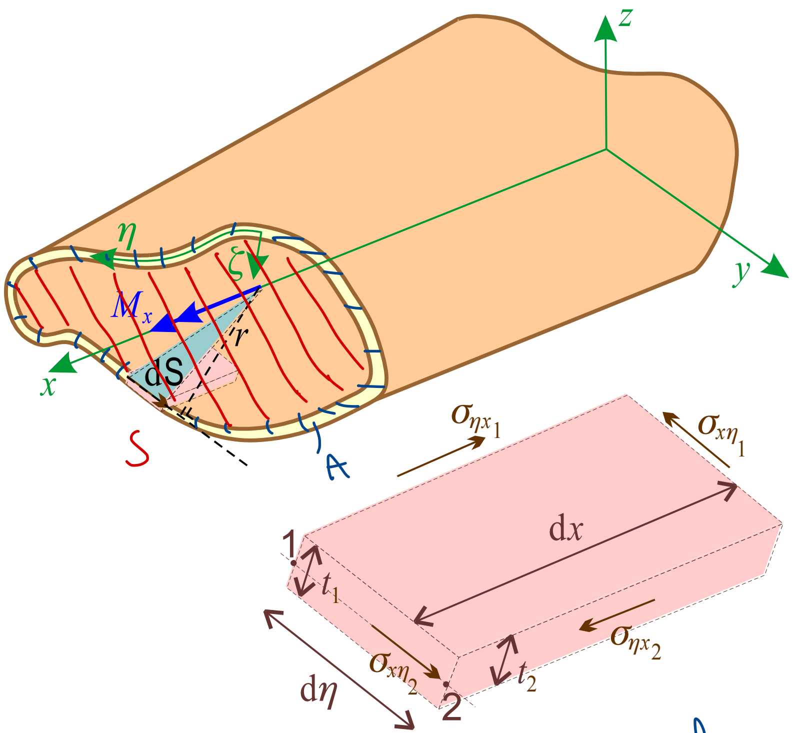
$\theta = \theta' \cdot L = 347°$ → it's not a small displacement so the theory is not valid





↳ open beam has less resistance to torsion

# Closed single-cell beams




Equilibrium of skin element

$$\sigma_{\eta x_1} t_1 dx = \sigma_{\eta x_2} t_2 dx$$

$$q = \sigma_{x\eta}(\eta) t(\eta) = C^t$$

Torsion moment

$$M_x = \int_A r\sigma_{x\eta}\, dA$$

$$M_x = \oint_C r\sigma_{x\eta}\, t\, d\eta = 2q \oint_C dS = 2qS$$

First Bredt formula

$$q = \frac{M_x}{2S}$$

shear flow

Second Bredt formula

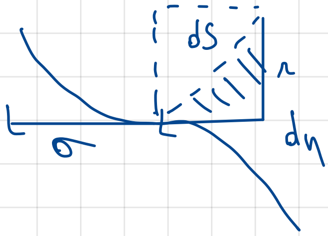
$$\theta' = \frac{1}{2S} \oint_C \frac{q}{Gt}\, d\eta$$

$$J = \frac{M_x}{G\theta'}$$

$V = U \cdot dx \cdot t$

$A = ty$

$dA = t\,dy$




$\frac{r\,dy}{2} = dS$

$M_x = q \oint_c r\,dy = q \oint_c 2\,dS \Rightarrow M_x = 2qS$ $\quad \left( q = \sigma t \Rightarrow \sigma = \frac{q}{t} \right)$

$q = \frac{M_x}{2S}$

# Exercise 1 (b)

- Compute the torsional rigidity, unit twist and maximum shear stress of a beam of hollow circular section of thickness 2.5 mm and mean radius 50 mm in the closed case (tubular beam). The beam is 30 cm long and the torque is 912 Nm. $G$ = 27.6 GPa.

$$q = \frac{M_x}{2S}, \quad \sigma = \frac{q}{t}, \quad \theta' = \frac{1}{2S}\oint_c \frac{q}{Gt}\,d\eta, \quad J = \frac{M_x}{G\theta}$$

$$q = \frac{M_x}{2\pi R^2} = \frac{912}{2\pi(0{,}05)^2} = 58\,000 \ \frac{N}{m}$$

$$\sigma = \frac{q}{t} = \frac{58060}{2{,}5\cdot 10^{-3}} = 23{,}2 \ MPa$$

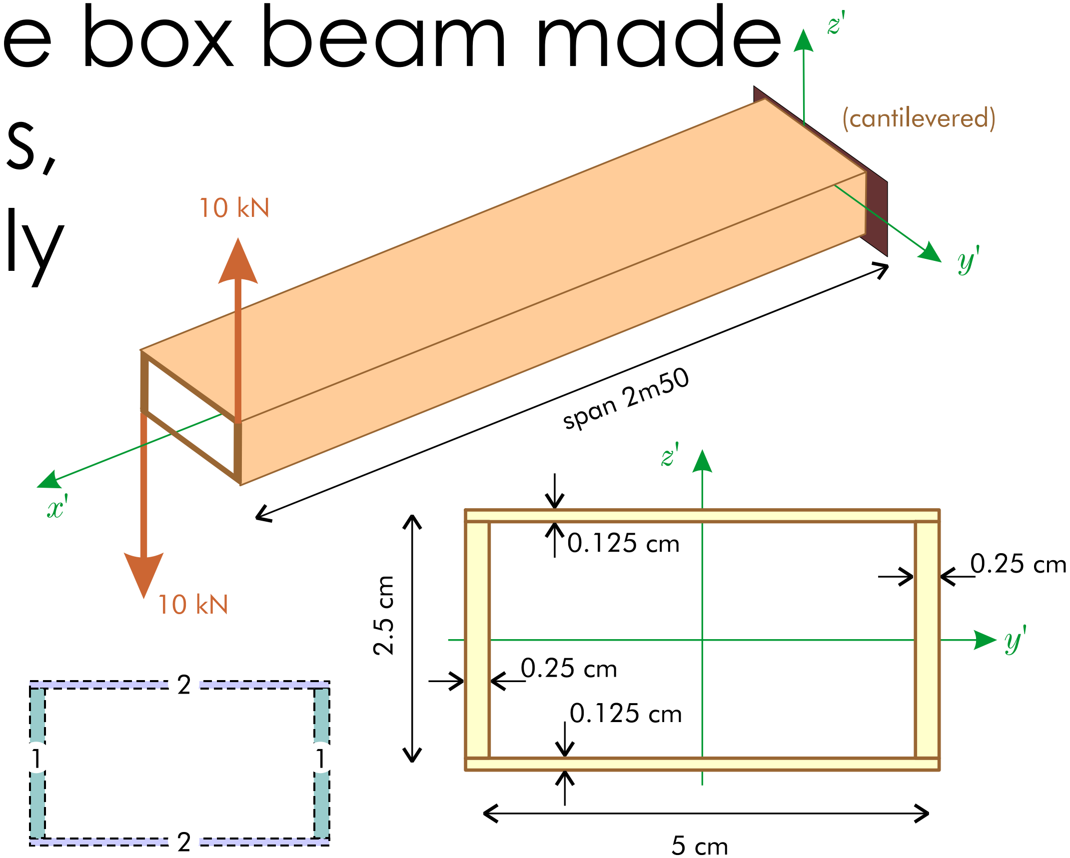
$$\theta' = \frac{1}{2\pi R^2}\frac{q}{Gt}\underbrace{\oint_c d\eta}_{2\pi R} = \frac{1}{R}\frac{q}{Gt} = \frac{58060}{50\cdot 10^{-3}\cdot 27{,}6\cdot 10^{9}\cdot 2{,}5\cdot 10^{-3}}$$

$$= 1{,}68\cdot 10^{-2} \ rad/m$$

↳ same result

# Exercise 2

- Determine the stress distribution, the twist angle and the torsional constant of this composite box beam made of two materials, with respectively
  $G_1 = 26$ GPa
  $G_2 = 83$ GPa.



$q = \frac{M_x}{2S}$, $\tau = \frac{q}{t}$, $\theta' = \frac{1}{2S}\oint_c \frac{q}{Gt}\,dy$, $J = \frac{M_x}{G\theta}$

1) $M_x$?

2) $q$?

3) $\tau_1, \tau_2, \theta'$

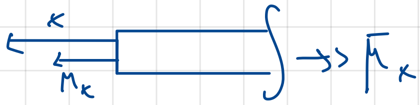
1) $M_x = 10\,kN \cdot 5\,cm = 10 \cdot 10^3\,N \cdot 5 \cdot 10^{-2}\,m = 500\,Nm$

2) $q = \frac{M_x}{2S} = \frac{\quad}{2 \cdot 5} =$

Distributed load /

Constant load: @ $x = 2{,}5\,m$ $\quad M_x = 10\,000 \cdot 0{,}05 = 500\,Nm$

$\frac{dM_x}{dx} = -m_x = 0 \Rightarrow M_x = A$




$M_x = \bar{M}_x \Rightarrow M_x = 500\,Nm$

$$q = \frac{M_x}{2 \cdot S} = \frac{500}{2 \cdot 0{,}05 \cdot 0{,}025} = 200\,000\ N/m$$

$$\tau_1 = \frac{q}{t_1} = \frac{200\,000}{0{,}25 \cdot 10^{-2}} = 80\,MPa$$

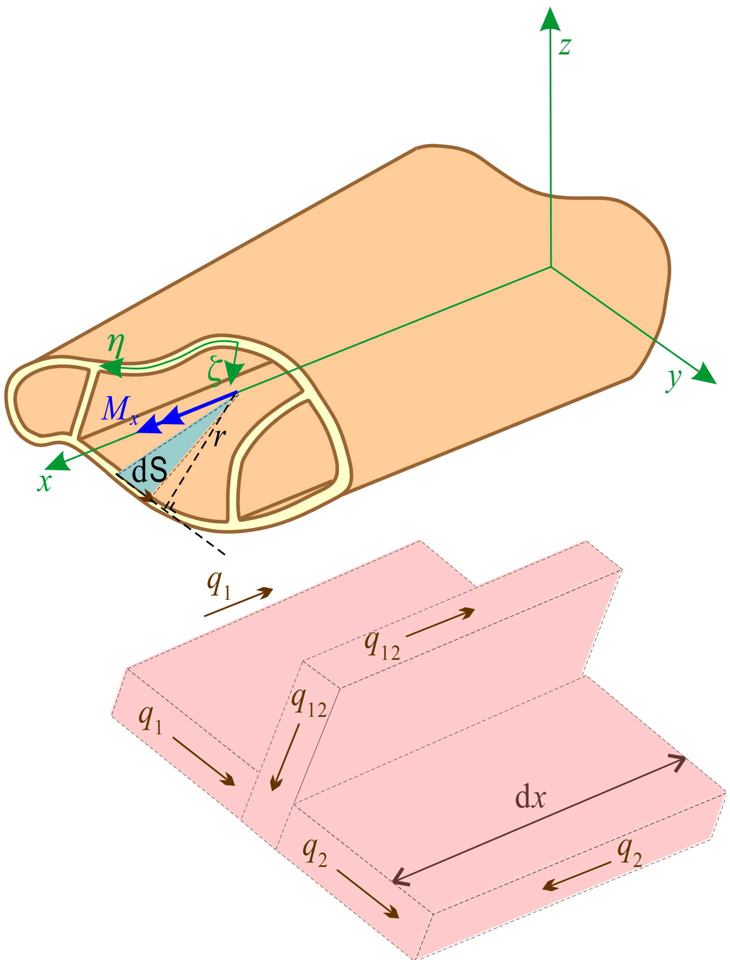
$$\tau_2 = \frac{q}{t_2} = \frac{200\,000}{0{,}125 \cdot 10^{-2}} = 160\,MPa$$

$$\theta' = \frac{1}{2S}\oint \frac{q}{Gt}\,dy = \frac{1}{2S} \cdot \sum_i \frac{q \cdot L_i}{G_i t_i} = \frac{1}{2S}\left(\frac{2qL_1}{G_1 t_1} + \frac{2qL_2}{G_2 t_2}\right)$$

$$= \frac{2 \cdot 200\,000}{2 \cdot 5 \cdot 10^{-2} \cdot 2{,}5 \cdot 10^{-2}}\left(\frac{2{,}5 \cdot 10^{-2}}{26 \cdot 10^9 \cdot 0{,}25 \cdot 10^{-2}} + \frac{5 \cdot 10^{-2}}{83 \cdot 10^9 \cdot 0{,}125 \cdot 10^{-2}}\right) = 0{,}139\ rad/m$$

$$\langle GJ \rangle = \frac{M_x}{\theta'} = \frac{500}{0{,}139} = 3600\ Nm^2 = 3600\ Nm/(rad/m)$$

# Closed multiple-cell beams




Equilibrium of skin element

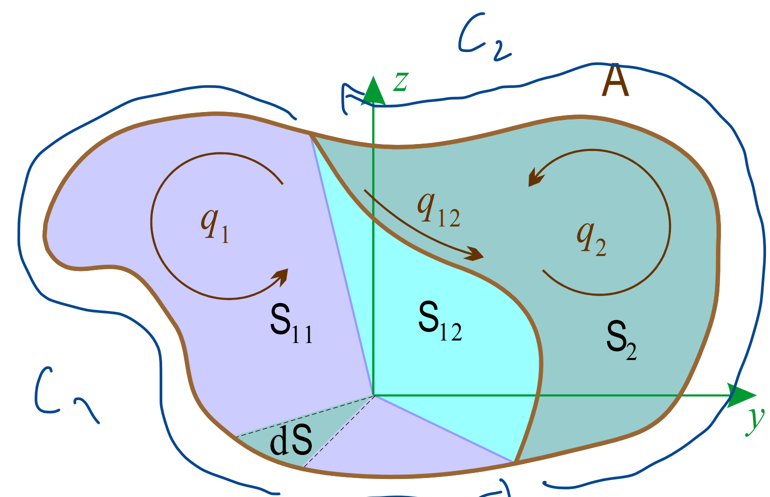
$$q_2 dx = q_1 dx + q_{12} dx$$

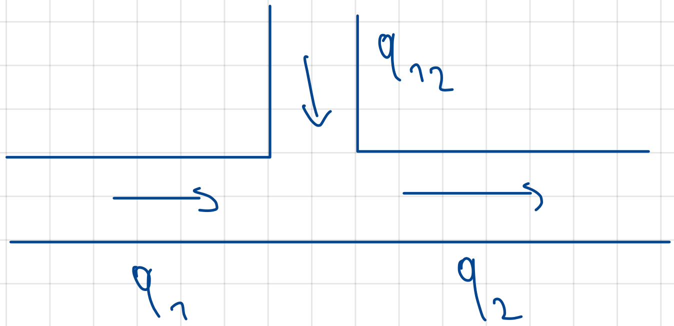
Torsion moment

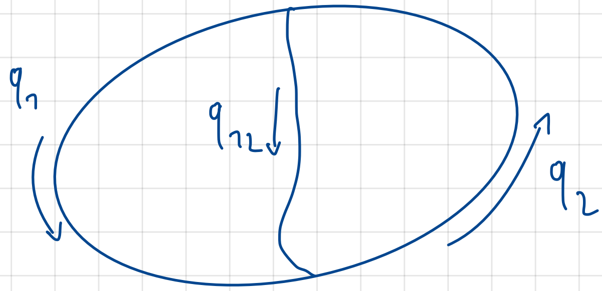
$$M_x = 2\, q_1 \oint_{C_1} dS - 2 q_{12} \oint_{C_{12}} dS + 2\, q_2 \oint_{C_2} dS$$

First Bredt formula (multiple-cell)

$$M_x = 2 \sum_{i=1}^{n} q_i S_i$$






$$q_1 + q_{12} = q_2$$

$$\boxed{q_{12} = q_2 - q_1} \qquad M_x = 2 \sum_{i=1}^{n} q_i S_i$$

$$M_x = 2q_1 \underbrace{\oint_{c_1} dS}_{S_{11}} - 2q_{12} \underbrace{\oint_{c_{12}} dS}_{S_{12}} + 2q_2 \underbrace{\oint_{c_2} dS}_{S_2 + S_{12}}$$

$$= 2q_1 S_{11} + 2q_1 S_{12} - 2q_2 S_{12} + 2q_2 S_2 + 2q_2 S_{12}$$

$$M_x = 2q_1 (\underbrace{S_{11} + S_{12}}_{S_1}) + 2q_2 S_2$$

$$\theta'_{I} = \theta'_{II} \; (= \theta'_{III})$$

$$\theta'_i = \frac{1}{2S_i} \oint_{c_i} \frac{q}{Gt}\, ds$$

# Closed multiple-cell beams

Second Bredt formula (multiple-cell)

$$\theta_i' = \frac{1}{2\,S_i}\oint_{C_i}\frac{q}{Gt}\,d\eta$$

Heterogeneous beam formulas

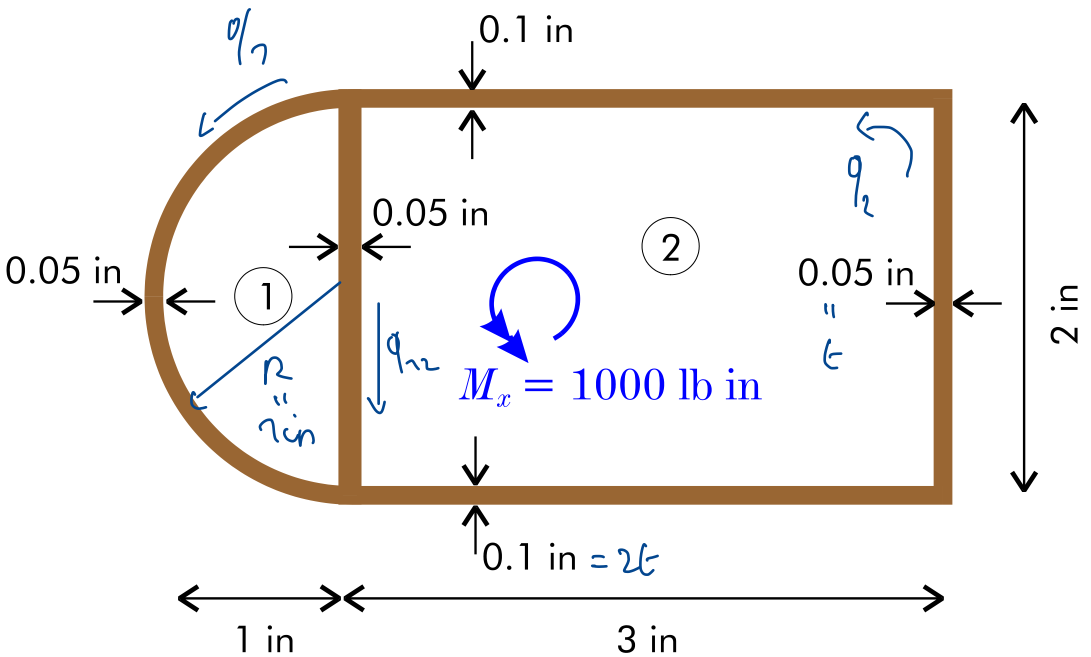
$$\theta_i' = \frac{1}{2\,S_i G_{ref}}\oint_{C_i}\frac{q}{t^*}\,d\eta$$

$$t^* = \frac{G}{G_{ref}}\,t$$

$$J = \frac{M_x}{G_{ref}\theta\prime}$$

# Exercise 3

- Consider this two-cell box beam which has $G = 4\ 10^6$ psi. Determine the stress distribution in all panels, the twist angle and the torsional constant.



$$q_2 = q_{12} + q_1 \Rightarrow q_{12} = q_2 - q_1$$

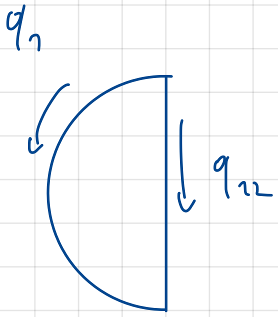
$$S_I = \frac{\pi R^2}{2}$$

$$S_{II} = 6R^2$$

$$M_x = 2 \cdot q_1 \frac{\pi R^2}{2} + 2 \cdot q_2 \cdot 6R^2$$

$$M_x = q_1 \pi R^2 + 12 R^2 q_2 \Rightarrow \boxed{\pi q_1 + 12 q_2 = 1000}$$

$$\begin{pmatrix} \pi & 12 \\ \frac{7}{6} + \frac{2}{\pi} & -\left(\frac{7}{12} + \frac{2}{\pi}\right) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}$$



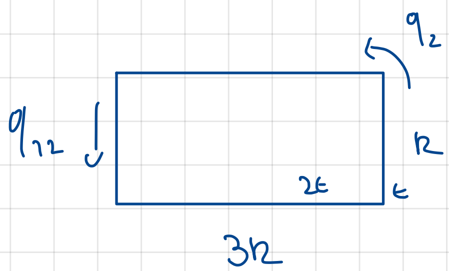

$$\theta_I' = \frac{1}{2S_I} \oint_{C_I} \frac{q}{Gt} \, dy$$

$$= \frac{1}{2S_I} \left( \frac{q_1}{Gt} \cdot \pi R - \left( \frac{q_{12}}{Gt} \cdot 2R \right) \right)$$

$$S_I = \frac{\pi R^2}{2}$$

$$RGt\theta_I' = \frac{1}{\pi} \left( q_1 \pi - (q_2 - q_1) \cdot 2 \right)$$

$$RGt\theta_I' = q_1 \left( 1 + \frac{2}{\pi} \right) - \frac{2}{\pi} q_2$$




$$\theta_{II}' = \frac{1}{2S_{II}} \oint_{C_2} \frac{q}{Gt(y)} \, dy$$

$$\theta_{II}' = \frac{1}{12R^{\cancel{2}}} \left( \frac{q_{12}}{Gt} \cdot 2\cancel{R} + \frac{q_2}{G \cdot 2t} \cdot 3\cancel{R} \cdot 2 + \frac{q_2}{Gt} \cdot 2\cancel{R} \right)$$

$$GtR\theta_{II}' = \frac{1}{12} \left( (q_2 - q_1) \cdot 2 + 3q_2 + 2 \cdot q_2 \right)$$

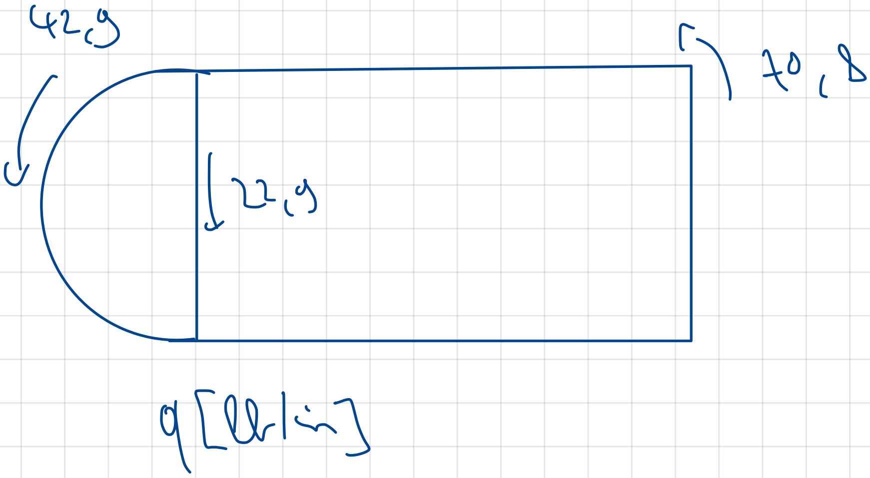
$$GtR\theta_{II}' = \frac{1}{12} \left( 7q_2 - 2q_1 \right)$$

$$q_1 \left( 1 + \frac{2}{\pi} \right) - \frac{2}{\pi} q_2 = \frac{7}{12} q_2 - \frac{q_1}{6} \Rightarrow \underline{\left( \frac{7}{6} + \frac{2}{\pi} \right) q_1 - \left( \frac{2}{\pi} + \frac{7}{12} \right) q_2 = 0}$$

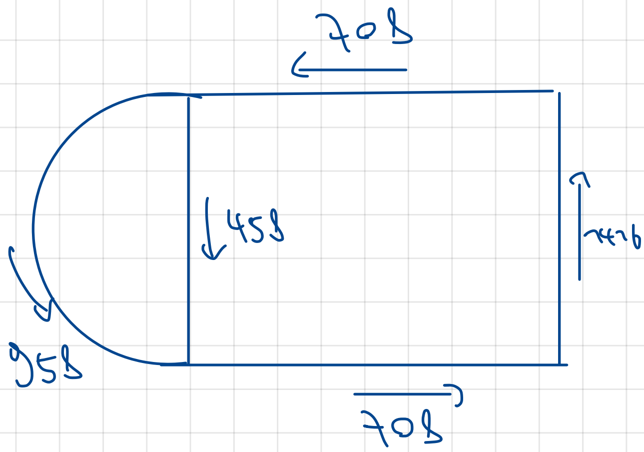
$$\begin{pmatrix} \pi & 12 \\ \frac{7}{6}+\frac{2}{\pi} & -\left(\frac{7}{12}+\frac{2}{\pi}\right) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}$$

$q_1 = 47{,}9$ lb/in

$q_2 = 70{,}8$ lb/in




q [lb/in]



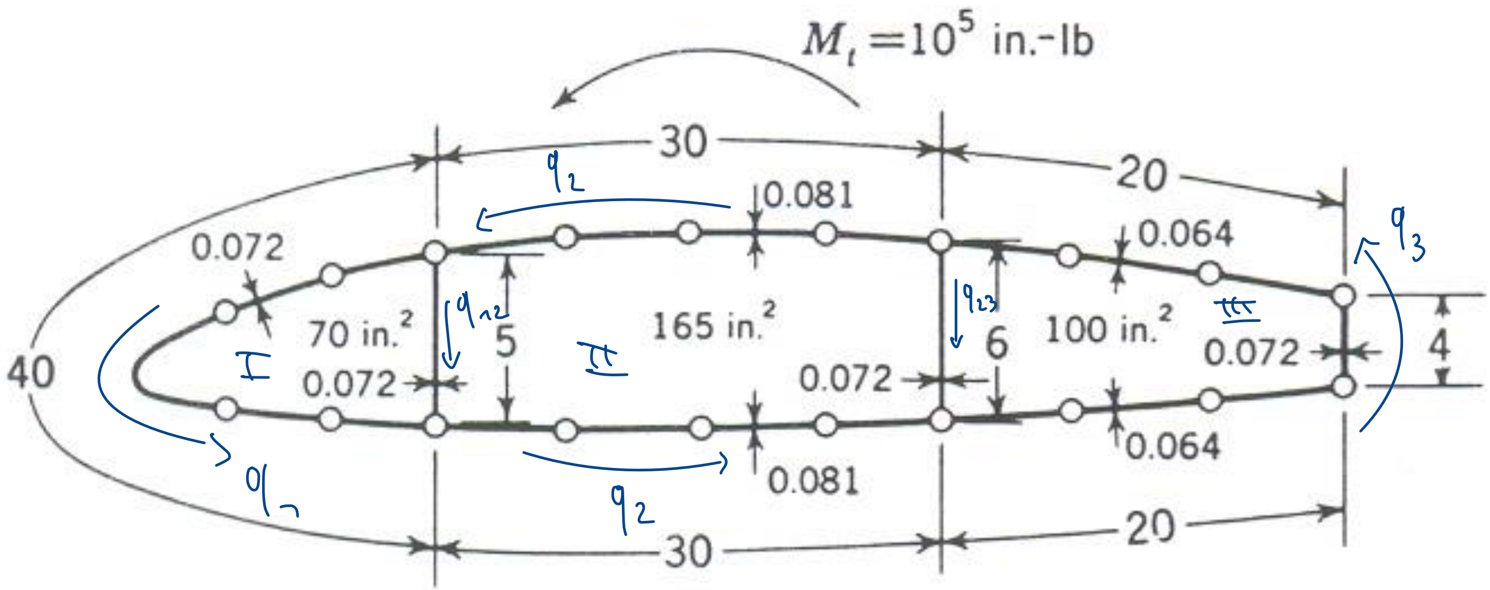

τ (psi)

$$G t R \theta'_{II} = \frac{1}{R}\left(7q_2 - 2q_1\right) = 1{,}70 \cdot 10^{-4} \text{ rad/m}$$

$$\langle GJ \rangle = \frac{M_x}{\theta'} = \frac{1000}{1{,}7 \cdot 10^{-4}} = 5{,}9 \cdot 10^{6} \text{ lb}\cdot\text{in}/(\text{rad/m})$$

# Exercise 4 (home)

- Consider the idealised wing section shown below. Determine the shear flow distribution in all panels and the torsional constant



$$M_x = 2q_1S_1 + 2q_2S_2 + 2q_3S_3$$

$$M_x = 2\cdot q_1\cdot 70 + 2q_2 165 + 2q_3 100$$

$$10^5 = 140q_1 + 330q_2 + 200q_3$$

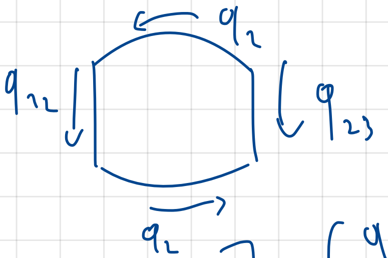
Left side:

$$\theta_I' = \frac{1}{2S_I}\oint_{C_I}\frac{q}{Gt}\,dy$$

$$= \frac{1}{140}\left(\frac{q_1\cdot 40}{G\cdot 0,072} - \frac{q_{12}\cdot 5}{G\cdot 0,072}\right) \Rightarrow G\theta_I' = \frac{1}{140}\left(\frac{q_1\cdot 40}{0,072} + \frac{5\cdot q_{12}}{0,072} - q_2\frac{5}{0,072}\right)$$

$$G\theta_I' = 4,464\,q_1 - 0,496\,q_2$$

middle

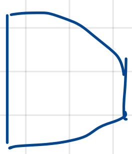



$$\theta_{II}' = \frac{1}{330}\left(\frac{q_{12}\cdot 5}{G\cdot 0,072} + \left(\frac{q_2\cdot 30}{G\cdot 0,087}\right)\cdot 2 - \frac{q_{23}\cdot 6}{G\cdot 0,072}\right)$$

$$G\theta_{II}' = \frac{1}{330}\left(\frac{q_2\cdot 5}{0,072} - \frac{q_1\cdot 5}{0,072} + \frac{60\cdot}{0,087}q_2 - q_3\frac{6}{0,072} + q_2\frac{6}{0,072}\right)$$

$$G\theta_{II}' = -0,217\cdot q_1 + 2,707\,q_2 - 0,252\,q_3$$

right



$$\theta_{III}' = \frac{1}{200}\left(\frac{q_{23}\cdot 6}{G\cdot 0,072} + \left(\frac{q_3\cdot 20}{G\cdot 0,064}\right)\cdot 2 + \frac{q_3 4}{G\cdot 0,072}\right)$$

$$G\theta_{III}' = 3,873\cdot q_3 - 0,417\,q_2$$

$$G\theta_I' = G\theta_{II}' \qquad G\theta_I' = 4,464q_1 - 0,496q_2$$

$$4,674q_1 - 3,203\,q_2 + 0,252\,q_3 = 0$$

$$G\theta_I' = G\theta_{III}'$$

$$4,464q_1 - 0,079q_2 - 3,873\cdot q_3 = 0$$

# For your background…

- « Building the dream »
  - A movie showing the assembly lines of the Boeing Everett plant (Washington, USA)