Aircraft Structures

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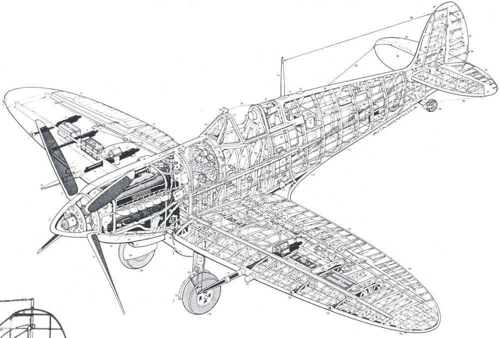
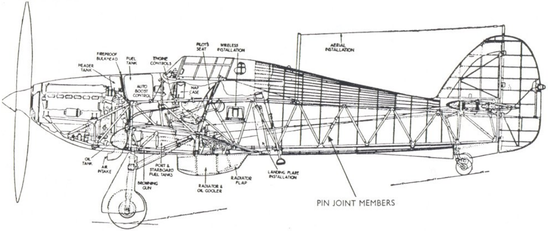
The monocoque aeroplane

Dawn of the aeroplane

- ➢WWII: both conceptions exist
  - ➢ Robust trussed aeroplanes
  - ➢ More capable monocoques

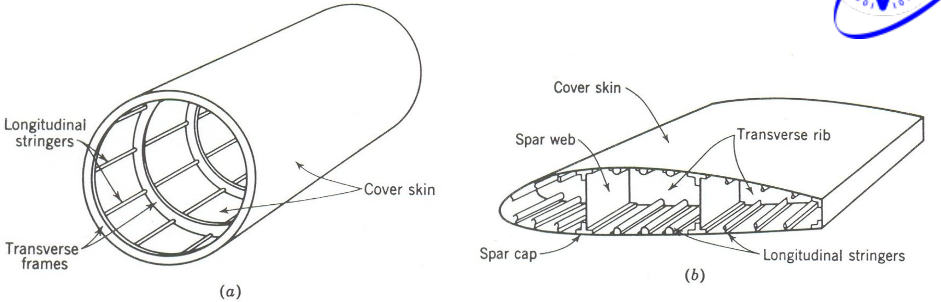


Vickers-Supermarine Spitfire



Hawker Hurricane

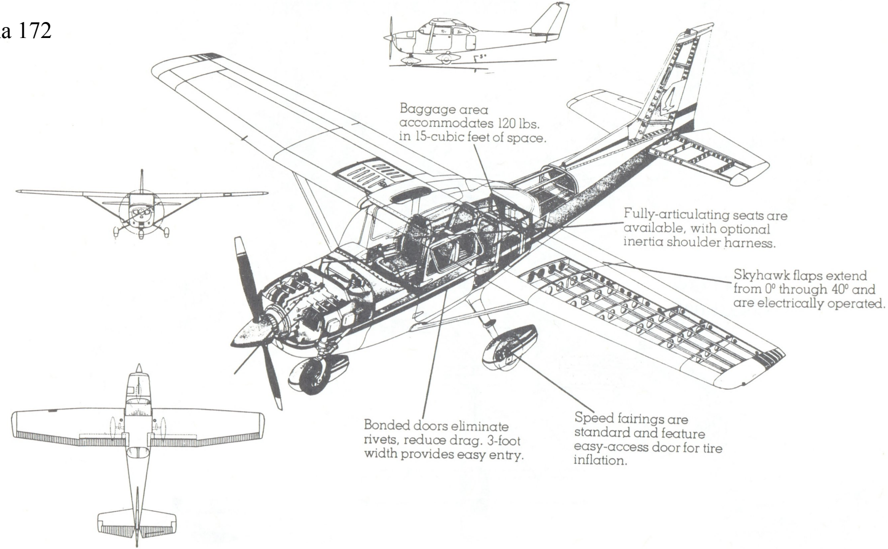
Semi-monocoque design



- Semi-monocoque structure:
  - Shear-resistant skin/spar webs
  - Bending-resistant stringer/spars
  - Torsion-resistant “torque box”
  - Pressure-resistant frames/bulkheads
  - Load transfer by ribs/frames

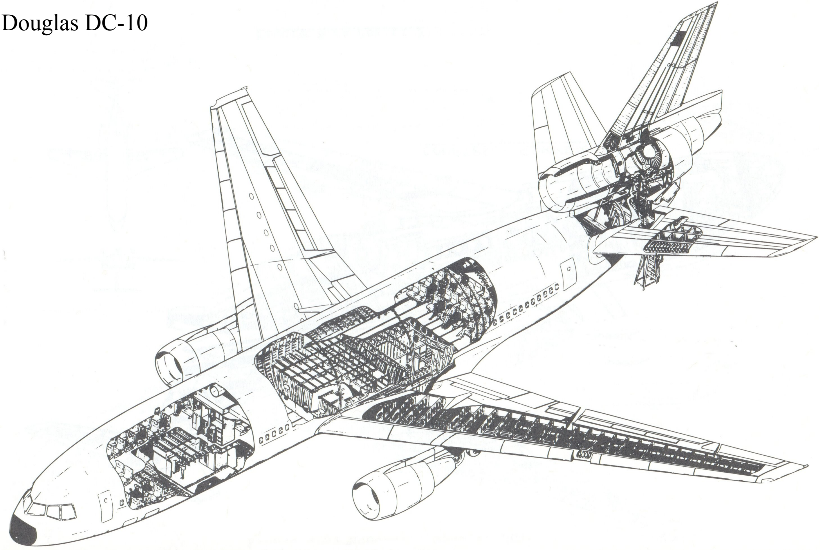
Semi-monocoques (1)

Cessna 172



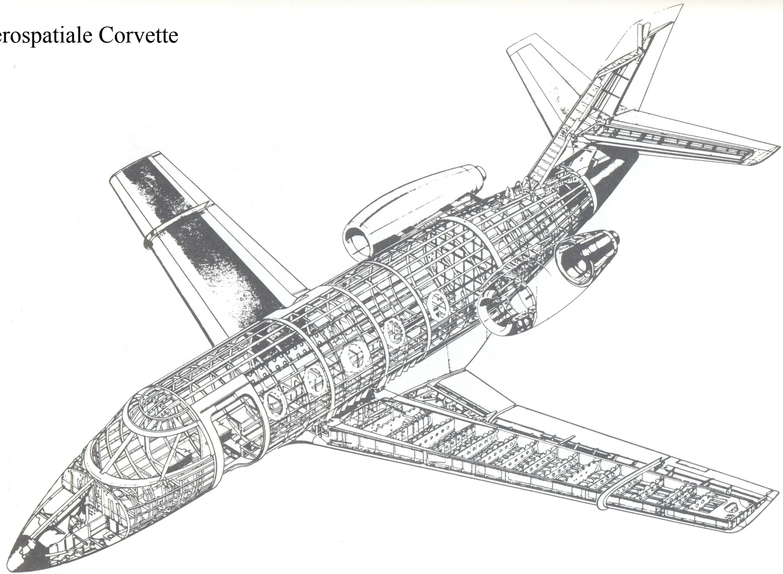
Semi-monocoques (2)

Douglas DC-10



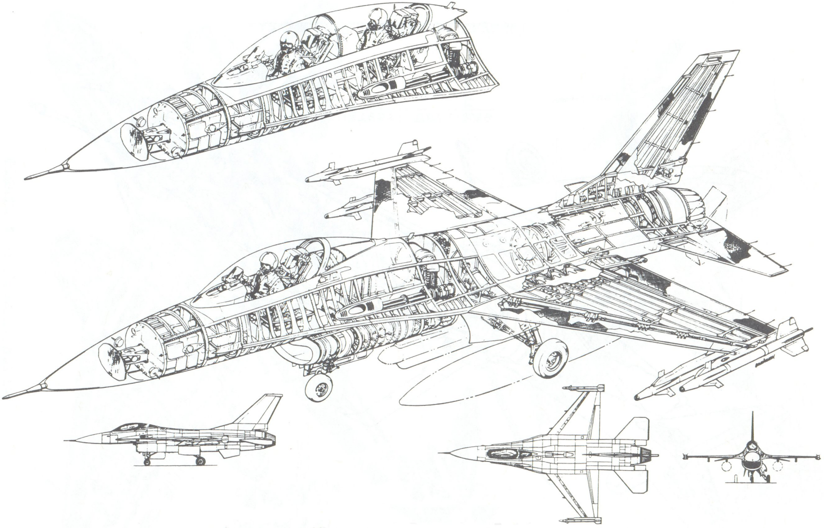
Semi-monocoques (3)

Aerospatiale Corvette

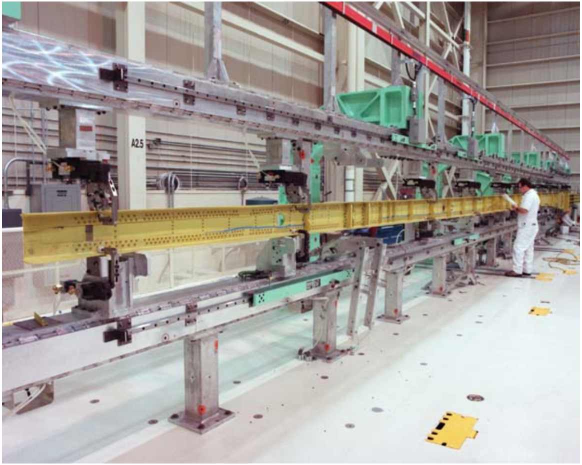


Semi-monocoques (4)

General Dynamics F-16

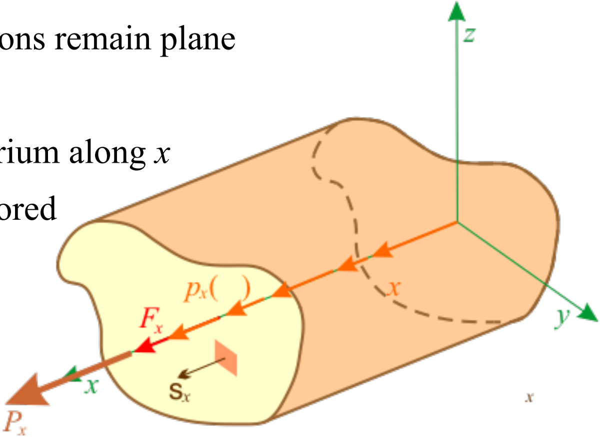


Extension of beams



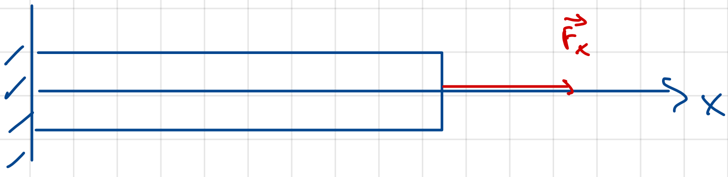
Rod or bar hypothesis

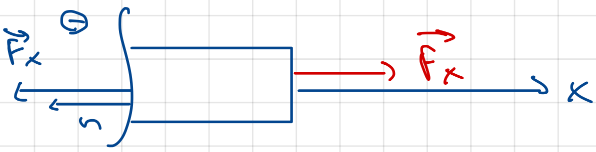
- Pure extension
  - Transverse stresses negligible (axial stress field)
  - Transverse sections remain plane
  - No body forces
  - Average equilibrium along *x*
  - Equilibrium ignored along *y* and *z*



Pure extension

- Well-known simple problem
  - Useful to introduce new concepts
- Heterogeneous structures
  - Several materials with different properties
  - Concept of modulus-weighted properties
- Inclusion of thermal effects
  - Concept of thermal stresses and strains
  - Duhamel-Neumann stress-strain relations
  - Thermal loads (forces, and later moments)





$$F_x = P_x \qquad F_x = \int \sigma_{xx}\, dA$$

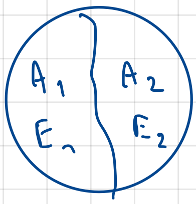
Hooke : $\sigma = \varepsilon E$

$$\Rightarrow P_x = \int_A \varepsilon E\, dA$$

$$= \varepsilon E A \Rightarrow \varepsilon = \frac{P_x}{EA}$$

$$\frac{du}{dx} = \varepsilon = \frac{P_x}{EA} \Rightarrow u = \int \frac{P_x}{EA}\, dx = \frac{P_x x}{EA} \qquad \Delta L = \frac{P_x L}{EA}$$

Heterogeneous :



$$\varepsilon_1 = \varepsilon_2$$

$$\varepsilon_1 = \frac{\sigma_1}{E_1} \qquad \varepsilon_2 = \frac{\sigma_2}{E_2}$$

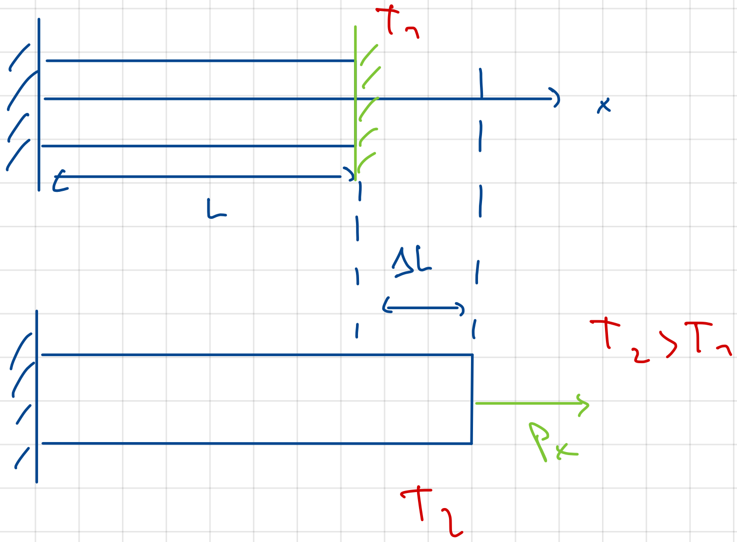
You can choose whatever you want for E_0

$$F_x = \int_A \sigma_{xx}\, dA = \int_A \varepsilon E\, dA = \varepsilon \int_A E\, dA = \varepsilon \underbrace{\left(\int_A \frac{E}{E_0}\, dA \right)}_{A^*} E_0$$

$$A^* = \frac{1}{E_0} \int_A E\, dA = \int_A \frac{E}{E_0}\, dA$$

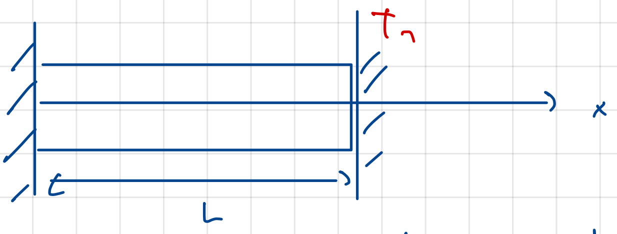
$$A^* = \int_{A_1} \frac{E_1}{E_0}\, dA + \int_{A_2} \frac{E_2}{E_0}\, dA$$

$$= \frac{E_1}{E_0} A_1 + \frac{E_2}{E_0} A_2$$



$$\varepsilon^T = \alpha \Delta T$$

$\sigma = 0 \rightarrow$ no stress due to thermal load because one end is free



$$\varepsilon = \underbrace{\frac{\sigma}{E}}_{\to \text{ mechanical action}} + \alpha \Delta T$$

$$\sigma = \varepsilon E - \alpha E \Delta T$$ Duhamel - Neumann relation

you can use this even when you don't have mechanical load (only thermal)

$$F_x = \int_A \sigma_{xx} \, dA = \int_A (\varepsilon E - \alpha E \Delta T) \, dA$$

$$= \int_A \varepsilon_{xx} E \, dA - \underbrace{\int_A \alpha E \Delta T \, dA}_{F_x^T = \alpha E \Delta T A}$$

$$F_x = 0 = \int_A \varepsilon_{xx} E \, dA - \int_A \alpha E \Delta T A$$

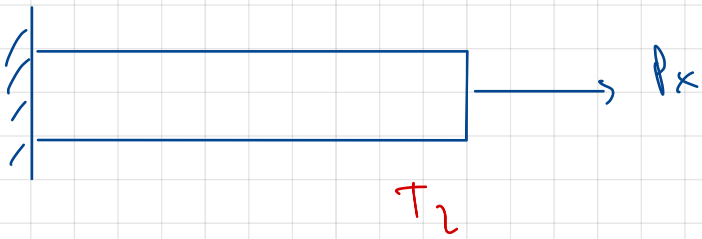
$$= \varepsilon_{xx} E A - F_x^T$$

$$F_x^T = \varepsilon_{xx} E A \Rightarrow \varepsilon_{xx} = \frac{F_x^T}{EA}$$

$$\sigma = \frac{F_x^T}{\cancel{E}A}\cancel{E} - \alpha E \Delta T \quad \text{(Duhamel - Neumann)}$$

$$= \frac{\alpha E \Delta T \cancel{A}}{\cancel{A}} - \alpha E \Delta T = 0$$

With mechanical load:



$$F_x = P_x$$

$$F_x^T = \alpha E \Delta T A$$

$$F_x^x = P_x + \alpha E \Delta T A$$

$$\varepsilon_{xx} = \frac{F_x^x}{EA}$$

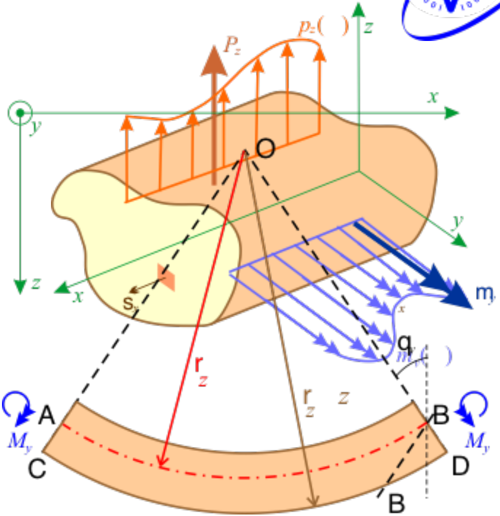
$$= \frac{P_x}{EA} + \alpha \Delta T$$

$$\sigma_{xx} = \frac{P_x}{A} + \cancel{\alpha E \Delta T} - \cancel{\alpha E \Delta T}$$

Bending of advanced beams

Bernoulli-Euler hypothesis

- Simple bending
  - Cross-sections remain planar
  - Constant bending moment
  - Axial stress field
  - No body forces
  - Average equilibrium along x and z



Simple bending solution

➢

$$\varepsilon_{xx} = \frac{z}{\rho}$$

$$\sigma_{xx} = E\,\varepsilon$$

$$\sigma_{xx} = \frac{M_y}{I_y}$$

$$\frac{\partial u}{\partial x} = \varepsilon_{xx}$$

Combined bending

➢

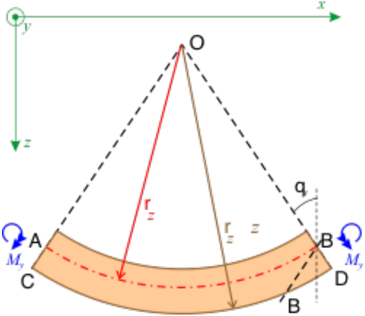
$$\varepsilon_{xx} = \varepsilon_0 + \frac{z}{\rho_z} -$$

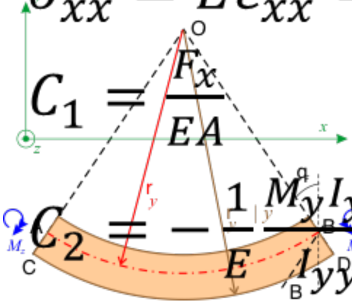
$$\varepsilon_{xx} = C_1 + C_2 y$$

$$\sigma_{xx} = E\varepsilon_{xx} = E$$

$$C_1 = \frac{F_x}{EA}$$

$$C_2 = -\frac{1}{E}\frac{M_y I_{yz} + M}{I_{yy} I_{zz} -}$$





Solution

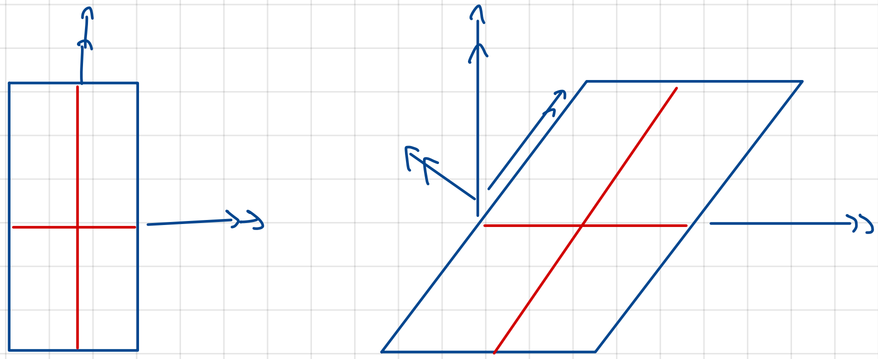
➢

$$\sigma_{xx} = \frac{F_x}{A} - \frac{M_y I_{yz} + M \ldots}{I_{yy} I_{zz} - \ldots}$$

$$\varepsilon_{xx} = \frac{F_x}{EA} - \frac{1}{E}\,\frac{M_y I_{yz} + \ldots}{I_{yy} I_{zz} \ldots}$$

$$\frac{z}{y} = \tan\lambda = \frac{M_y I_{yz} + \ldots}{M_z I_{yz} + \ldots}$$

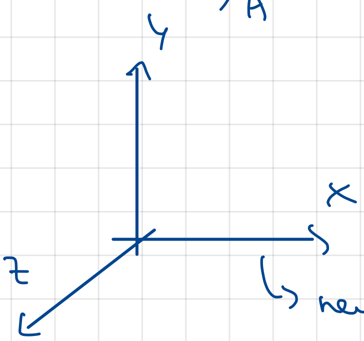
$$\frac{\partial u}{\partial x} = \varepsilon_{xx}$$



$$\sigma_{xx} = E C_1 + E C_2 y + E C_3 z$$

$$F_x = \int_A \sigma_{xx} \, dA$$

$$= \int_A (E C_1 + E C_2 y + E C_3 z) \, dA$$



↳ neutral fiber (neutrale as)

$$z_F = \frac{1}{A} \int_A z \, dA = 0$$

$$y_F = \frac{1}{A} \int_A y \, dA = 0$$

$$\Rightarrow F_x = \int_A E C_1 \, dA + \int_A E C_2 y \, dA + \int_A E C_3 z \, dA$$

$$= E C_1 A \Rightarrow C_1 = \frac{F_x}{EA}$$

$$E C_2 \overbrace{\int_A yz \, dA}^{\text{Product of inertia } I_{yz}}$$

$$M_y = \int \sigma_{xx} \, dA$$

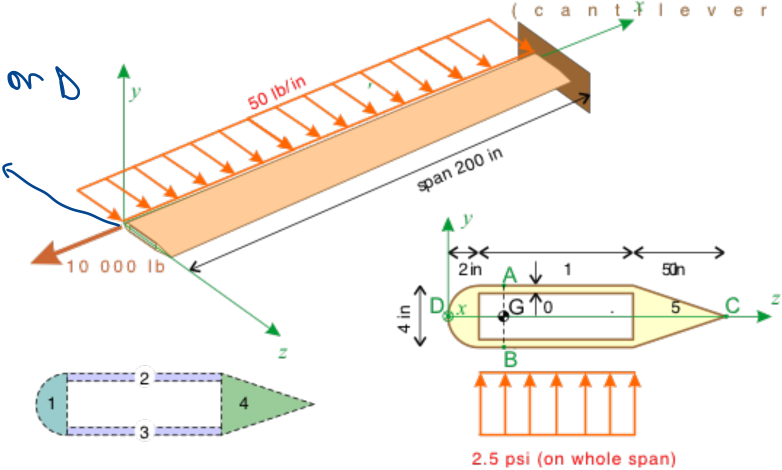
$$= \int_A E C_1 z \, dA + \int_A E C_2 y z \, dA + \int_A E C_3 z^2 \, dA$$

$$M_y = E C_2 I_{yz} + E C_3 I_{zz}$$

Application

➢ the blade is homogeneous with E = 107 psi

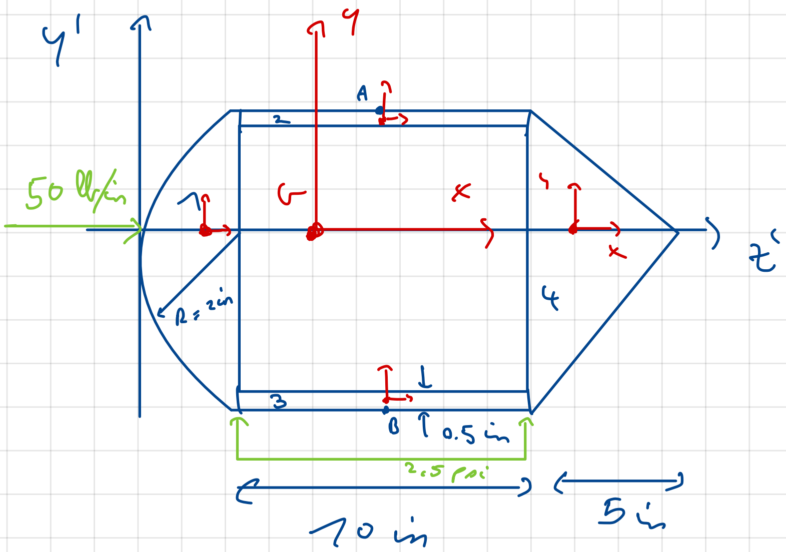
force applied on D



1) Stress resultant "MNT" → F_x, M_y, M_z

2) Center of gravity of the section
product and moment of inertia } geometry

3) Bending stress



use x', y', z' to find center of gravity (CG)

$$y_G = \frac{1}{A}\int_A y\, dA = \frac{\sum_i y_i A_i}{\sum_i A_i}$$

z_G z z_i

$z' = \frac{4R}{3\pi}$ (0 starting from the base)

$z' = \frac{b}{3}$ (1 starting from base)

	A_i	y_i'	z_i'	$A_i y_i'$	$A_i z_i'$
1	6,28	0	1,15	0	7,23
2	5	1,75	7	8,75	35
3	5	-1,75	7	-8,75	35
4	10	0	13,67	0	136,7
Σ	26,28	/	/	0	213,93

$$y_G = \frac{0}{26,28} = 0 \text{ in} \qquad z_G = \frac{273,9}{26,28} = 8,74 \text{ in}$$

$$I_{y'y'} = I_{yy} + z'^2_G A$$
$$I_{z'z'} = I_{zz} + y'^2_G A$$

$$I_{y'z'} = I_{yz} + y'_G z'_G A$$

local frame of reference

	I_{yy}	I_{zz}	I_{yz}	$z'^2_G A_i$	$y'^2_G A_i$	$z'_G y'_G A_i$
1	1,756 *	6,28 △	0	8,33	0	0
2	47,67	0,704	0	245	15,31	67,25
3	47,67	0,704	0	245	15,31	-67,25
4	13,89	6,67	0	1867	0	0
Σ	98,98	13,16	0	2366	30,62	0

$$\triangle = \frac{\pi R^4}{8} \qquad I_{yy\,\square} = \frac{t b^3}{12} \qquad I_{zz\,\square} = \frac{t^3 b}{12}$$

$$* = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) R^4$$

$$I_{y'y'} = I_{yy} + z'^2_G A = 98,98 + 2366,1 = 2465,08 \text{ in}^4$$

$$I_{yy} = I_{y'y'} - z'^2_G A = 2465,08 - 8,74^2 \cdot 26,28 = 724,3 \text{ in}^4$$

$$I_{zz} = 43,78 \text{ in}^4$$

$$I_{yy} = 0 \text{ in}^4$$

$$y'_G = 0 \text{ in}$$

$$z'_G = 8,74 \text{ in}$$

distributed force

$p_x = 0$

$p_y = 25$ lb/in

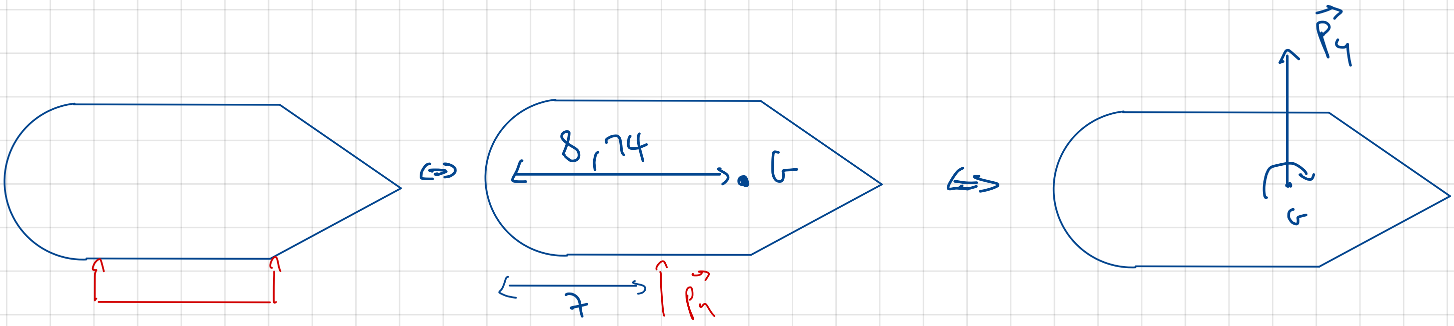
$p_z = 50$ lb/in

distributed moment

$m_x = 28,45$ lb in/in

$m_y = 0$ lb in/in

$m_z = 0$ lb in/in



$p_y(8,74 - 7)$

Concentrated load

$P_x = -10\,000$ lb at $x = 0$

$M_y = 8,74 \cdot 10\,000 \cdot$ lb in

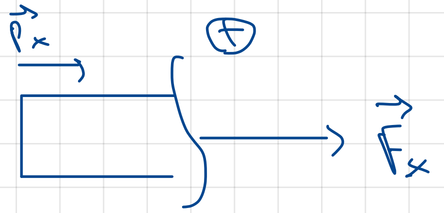
$= 87400$ lb in

$$\frac{dF_x}{dx} = -p_x \qquad \frac{dM_x}{dx} = -m_x$$

$$\frac{dF_y}{dx} = -p_y \qquad \frac{dM_y}{dx} = -m_y + F_z$$

$$\frac{dF_y}{dx} = -p_z \qquad \frac{dM_z}{dz} = -m_z - F_y$$

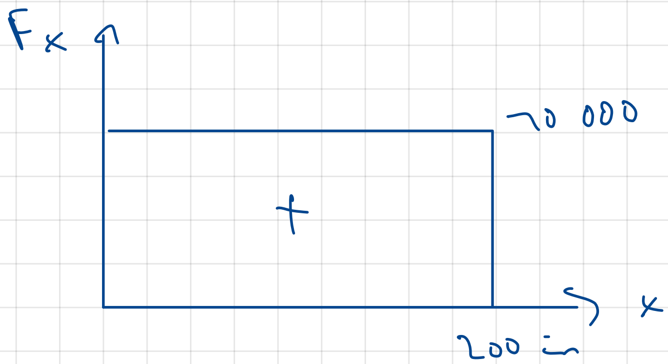
- $\frac{dF_x}{dx} = -p_x = 0 \Rightarrow F_x = A \Rightarrow A = 70\,000$ lb



$\Rightarrow p_x + F_x = 0 \Rightarrow F_x = -p_x$

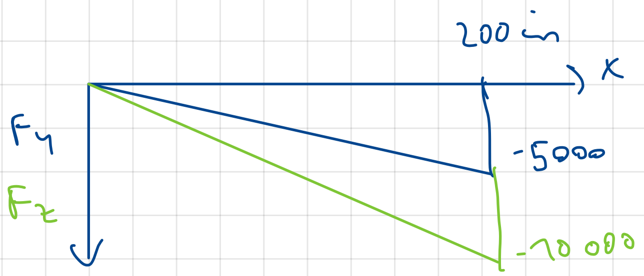
$F_i = -p_i$

$M_i = -M_i$



$$\frac{dF_y}{dx} = -25 \Leftrightarrow F_y = -25x + \underbrace{B}_{=0} \qquad x = 0 \Rightarrow F_y = 0$$

$$\frac{dF_z}{dz} = -50 \rightarrow F_z = -50x + \underbrace{C}_{=0}$$



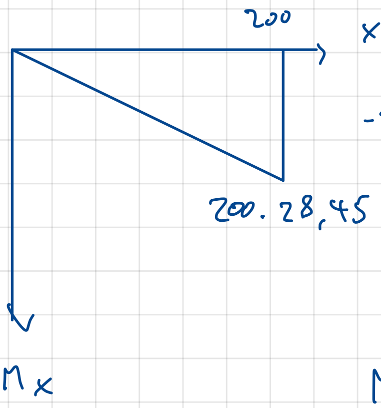
$$\frac{dM_x}{dx} = -28,45 \Rightarrow M_x = -28,45\,x + \underbrace{D}_{=0}$$

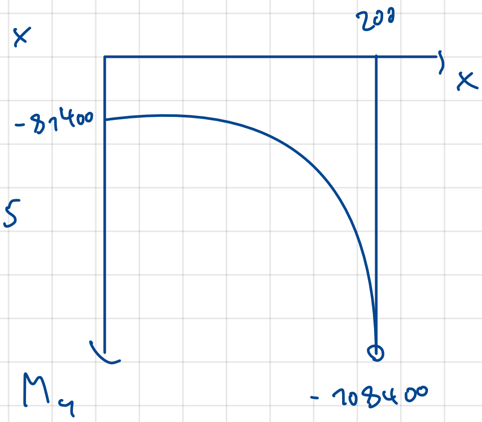
$$\frac{dM_y}{dx} = 0 - 50\,x \Rightarrow M_y = -25\,x^2 + E$$

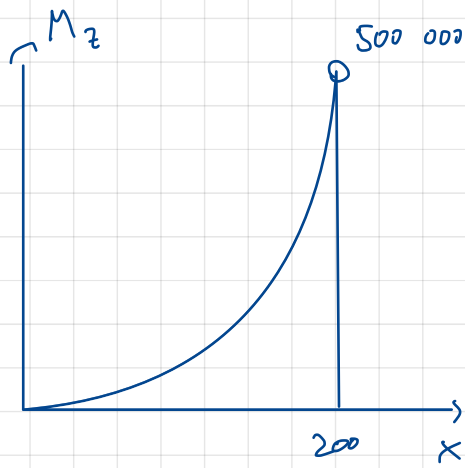
↳ at $x = 0 \rightarrow M_y = -81400 = E$

$$M_y = -25\,x^2 - 81400$$

$$\frac{dM_z}{dx} = 0 + 25x \Rightarrow M_z = 12,5\,x^2 + \underbrace{\cancel{F}}_{=0}$$







$F_x = 10\,000$ lb ; $M_y = -10.81400$ lb in ; $M_z = -500\,000$ lb in

	y	z	σ [psi]
A	2	0	− 22 459
B	−2	0	23 220
C	0	8,86	−12 850
D	0	−8,14	12532

$$\sigma_{xx} = \frac{F_x}{A} - \frac{M_z}{I_{zz}}\,y + \frac{M_y}{I_{yy}}\cdot z$$

Modulus-weighed properties

➢

$$\sigma_{xx} = \varepsilon_{xx} E$$

$$\varepsilon_{xx} = \varepsilon_0 +$$

$$\varepsilon_{xx} = C_1 +$$

$$F_x = \int_A \sigma_{xx}\, dA = \int_A [E C_1 -$$

$$A^* = \int_A \frac{E}{E_0}$$

Modulus-weighed properties

➢

$$M_y = \int_A z\sigma_{xx}\, dA = E_0 \int_A \left[\frac{E}{E_0} C_1 z + \frac{E}{E_0} C \right.$$

$$M_z = \int_A y\sigma_{xx}\, dA = E_0 \int_A \left[\frac{E}{E_0} C_1 y + \frac{E}{E_0} C \right.$$

$$I^*_{yy} = \int_A z$$

$$I^*_{zz} = \int_A y$$

Thermal loads

- $$F_x^T = \int_A E$$

  $$M_y^T = \int_A z$$

  $$M_z^T = -\int_A$$

  $$F_x + F_x^T =$$

  $$M_y + M_y^T =$$

$$M_z + M_z^T =$$

Final relations

$$\varepsilon_{xx} = \frac{F_x^*}{E_0 A^*} - \frac{1}{E_0} \frac{M_y^* I_{yz}^* + M_z^* I_{yy}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}$$

$$\sigma_{xx} = \frac{E F_x^*}{E_0 A^*} - \frac{E}{E_0} \frac{M_y^* I_{yz}^* + M_z^* I_{yy}^*}{I_{yy}^* I_{zz}^* - I_{yz}^{*2}}$$

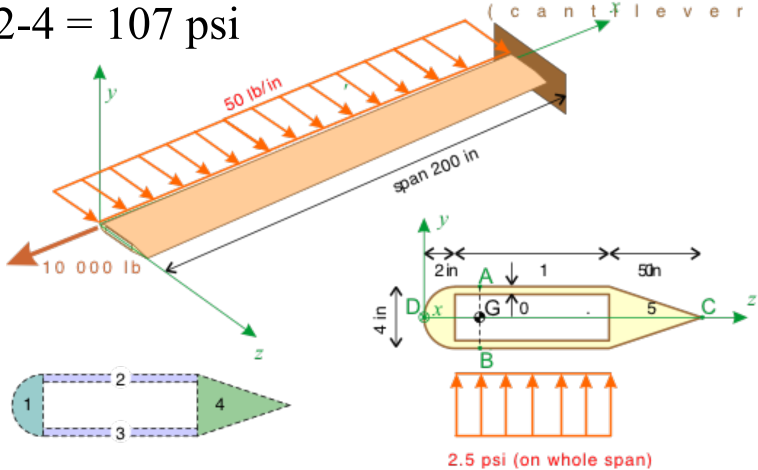
$$\sigma_{xx} = E(\varepsilon_{xx} - \alpha \Delta T$$

$$\frac{\partial u}{\partial x} = \varepsilon_{xx}$$

$$\frac{\partial^2 v}{\quad} = \frac{1}{\quad} \frac{M_y^* I_{yz}^* + M_z^* I_{y}^*}{\quad}$$

Application

➢ the blade is heterogeneous
with $E1$ = 3 107 psi
and $E2$-4 = 107 psi



$M_y = 58800$ lb . in

$M_y = -10.58800$

$m_x = -(7 - 5{,}88) . 25 = -28{,}05$

$I^*_{yy} = 1142{,}8$

$I^*_{zz} = 56{,}85 \text{ in}^4$

$I^*_{yz} = 0 \text{ in}^4$

$y'_\sigma = 0$ in

$z'_\sigma = 5{,}88$ in

$A^* = 50{,}85 \text{ in}^2$

	y^*	z^*	σ [psi]
A	2	0	-17 366
B	-2	0	18 127
C	0	17 - 5,88 = 11,12	- 9921
D	0	-5,88	17484

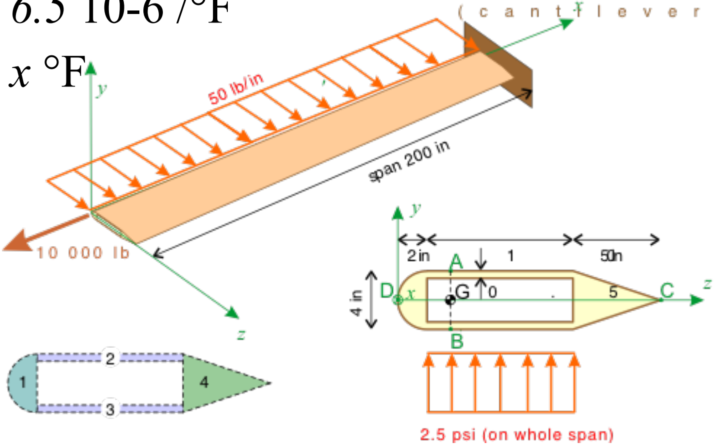
difference due to the term $\frac{F_x}{A}$

$F^*_x = F_x + \cancel{F^T_x}$ ↪ no thermal load

$$\sigma_{xx} = \frac{E}{E_0}\frac{F_x}{A} - \frac{E}{E_0}\frac{M_z}{I_{zz}} \cdot y^* + \frac{E}{E_0}\frac{M_y}{I_{yy}} \cdot z^*$$

Application

- the blade is heterogeneous
  with a1 = 5 10-6 /°F
  and a2-4 = *6.5* 10-6 /°F
- D$T = 0.10\ x$ °F



$\alpha_1 = 5 \cdot 10^{-6}\ {}^\circ F$
$\alpha_{2-4} = 6{,}5 \cdot 10^{-6}\ {}^\circ F$
$\Delta T = 0{,}1x\ {}^\circ F$

$$F_x^T = \int_A E\alpha\Delta T\, dA = \Delta T \int_A E\alpha\, dA$$

$$= \Delta T \int_A \frac{E}{E_0}\alpha\, dA \cdot E_0$$

$$= E_0 \Delta T \left(\underbrace{\int_{A_1} \frac{E_1}{E_0}\alpha_1\, dA}_{\alpha_1 A_1^*} + \underbrace{\int_{A_2} \frac{E_2}{E_0}\alpha_2\, dA}_{\alpha_2 A_2^*} + \int \dots \right)$$

	α_i	A_i^*	$\alpha_i A_i^*$
1	$5 \cdot 10^{-6}$	18,85	$9{,}4 \cdot 10^{-5}$
2	$6{,}5 \cdot 10^{-6}$	5	$3{,}25 \cdot 10^{-5}$
3	$6{,}5 \cdot 10^{-6}$	5	$3{,}25 \cdot 10^{-5}$
4	$6{,}5 \cdot 10^{-6}$	10	$6{,}5 \cdot 10^{-5}$
Σ			$2{,}24 \cdot 10^{-4}$

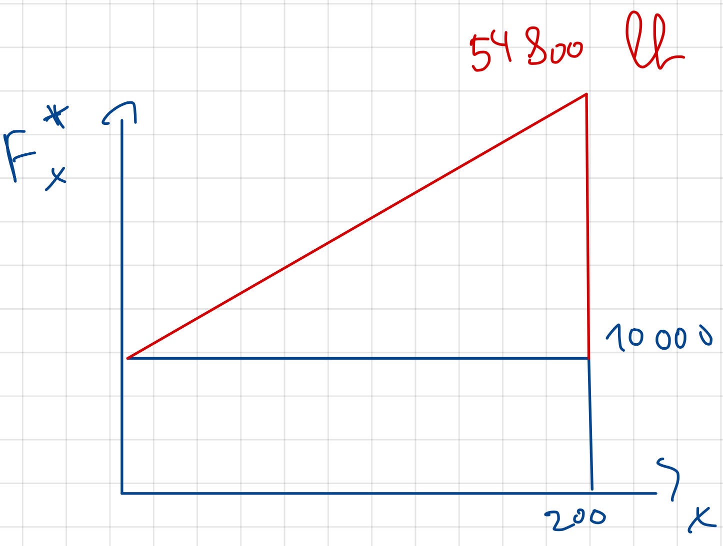
$$= E_0 \Delta T \cdot \sum_i A_i^* \alpha_i = E_0 \Delta T \cdot 2{,}24 \cdot 10^{-4}$$

$$= 10^7 \cdot 10^{-1} \cdot x \cdot 2{,}24 \cdot 10^{-4}$$

$$= 224x$$

$$F_x^* = F_x + F_x^T$$

$$= 10\,000 + 224x$$



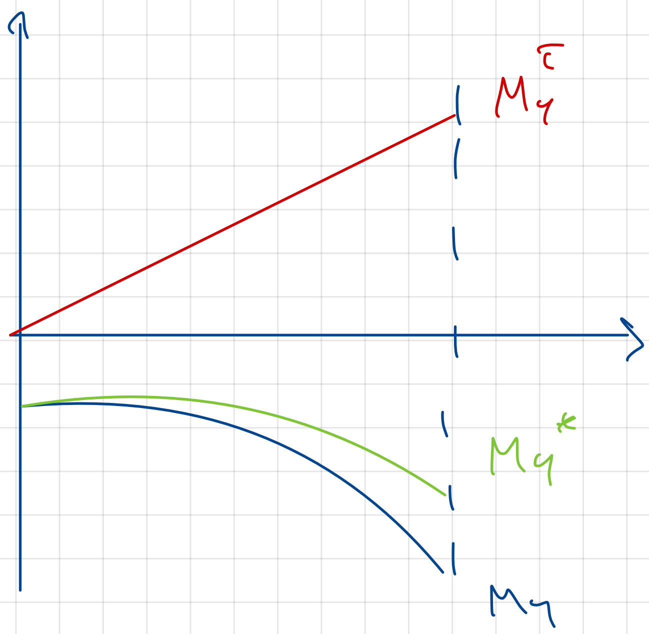
	$\alpha_i A_i^* y_i^*$	$\alpha_i A_i^* z_i^*$
1	0	$1{,}08 \cdot 10^{-4}$
2	$5{,}69 \cdot 10^{-5}$	$2{,}27 \cdot 10^{-4}$
3	$-5{,}69 \cdot 10^{-5}$	$2{,}27 \cdot 10^{-4}$
4	0	$8{,}88 \cdot 10^{-4}$
Σ	0	$1{,}45 \cdot 10^{-3}$

$$M_y^T = \int_A z E \alpha \Delta T \, dA$$

$$= 1{,}45 \cdot 10^{-3} \cdot \Delta T E_0 = 1{,}45 \cdot 10^{-3} \cdot 10^{-1} x \cdot 10^{7}$$

$$= 1450\, x = 290000$$

$$M_z^T = 0 \cdot \Delta T \cdot E_0 = 0$$



$$F_x^* = 10\,000 + 224 \cdot 200 = 54\,800 \text{ lb}$$

$$M_y^* = -1058800 + 1450 \cdot 200 = -721\,360 \text{ lb in}$$

$$M_z^* = 500\,000 \text{ lb in}$$