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**PHYSICS OF ELEMENTARY PARTICLES  
AND ATOMIC NUCLEI. THEORY**

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# Square-Well Nuclear-Potential Depths and Bound Energy States from Neutron Scattering Data<sup>1</sup>

A. Czachor<sup>a</sup> and P. Pęczkowski<sup>a, b</sup>

<sup>a</sup>*Institute of Atomic Energy, Otwock-Świerk, Poland*

<sup>b</sup>*Institute of Experimental Physics, Department of Physics, UW, Warsaw*

**Abstract**—Using the neutron scattering lengths  $b$  determined experimentally for a majority of isotopes in last decades, one can in principle extract systematic information on some nuclear properties of elements. A significant “scatter” of experimental values of the (related to  $b$ ) nuclear radius  $R$  around the “classical” dependence  $R = r_0 A^{1/3}$ , where  $A$  is the mass number, is intriguing and requires a special attention. In this work, on extending the use of known formulas of the theory of neutron scattering on nucleus represented by a rectangular radial symmetry potential well (or barrier), we have determined the depths  $V_0$  of the potential well and for many isotopes the position of the bound-state energy level  $E_b$  in the well. The “scatter” mentioned above can be in part attributed to the four types of the s-type wave functions of slow neutron interacting with nucleus, which appear in this model. In several cases the bound-state energy level is close to the Fermi  $E_F$  level of the free-nucleon model of nuclear matter of the constant density, independent of  $A$ .

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## INTRODUCTION

Five decades of neutron experiments have resulted in determining the neutron scattering lengths  $b$  for about 250 isotopes. They have been collected in the Dianoux and Lander “Neutron Data Booklet” [1]. Let us remind essential notions.

The scattering length depends in general on the orientation of the neutron spin 1/2 with respect to the nucleus spin  $S$ , so we have two scattering lengths:  $b_{\uparrow\uparrow} \equiv b_+$ ,  $b_{\uparrow\downarrow} \equiv b_-$ . In practice one often deals with the coherent scattering length  $b_C$ :

$$b_C = \frac{S+1}{2S+1} b_+ + \frac{S}{2S+1} b_- \quad (1)$$

In case of a spinless nucleus  $S=0$  one returns to single scattering length  $b$  and then one can write  $b = b_+ = b_C$ . In this study we shall be mainly discussing the spinless nuclei, but for completeness we shall include the remaining ones, covering them by their coherent representation  $b_C$ . Generally, it is understood that  $b = b_{Z,A}$ .

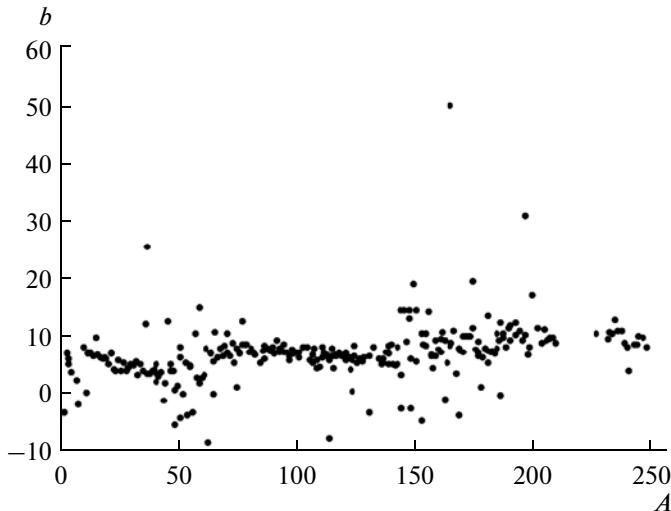
The present-day theory of nucleus is still somewhat heuristic and uncoherent. The way of thinking about nucleus is—which model is better for a given purpose? Criterions of acceptance of physical ideas in this field cannot be therefore considered firm or final. There is still a space for experimenting with models, when trying to cover the wealth of new experimental data. Even if one has at disposal the state-of-art description in the terms of the optical potential model, such as neutron–nucleus strong scattering radius  $R'$ , the bound level resonance energy  $E_b$ , gamma and neutron widths  $\Gamma_{\gamma b}$

and  $\Gamma_{nb}$  [2], neutron separation energies  $S_n$  [7], one may still be interested in the parameters of the simple-minded radial-symmetry rectangular potential well for nuclei, for the notion of the depth of such a potential well (in some cases the height of a potential barrier) is intuitively clear and acceptable. This is why we have attempted in this paper to try to fully exploit the idea of s-scattering and old radial-rectangular-potential model of nucleus, in order to systematize and interpret the experimentally determined collection of neutron scattering lengths  $b$  [1].

Very roughly, the nuclear radius parameters  $R$ , determined by the high-energy electron diffraction, grow with the mass number as  $A^{1/3}$ , and in this sense they may be treated as the radii of nuclei of fixed nucleon density. A better representation follows from the optical models of the nuclear theory, where certain oscillations in the radius vs.  $A$  plot reflect the nuclear orbital wave functions. It is displayed in the well-known plot of Mughabghab et al. [3], see also [1], showing the “spin-independent potential scattering” radius  $R'$  vs.  $A$ . There is a lot of bizarre scatter in these experimental data, which do not fit the theoretical curve.

Neutron scattering lengths  $b$ , shown in Fig. 1, depend on the mass number  $A$  and the charge number  $Z$  of nuclei even more erratically than the radius parameters. The parameters  $b$  and  $R$  are of different nature, although they are intimately related to each other by the matching conditions for the neutron wave function at the potential boundary. In this work we shall examine this relation in order to discover a possible regularity in the  $b$  vs.  $(Z, A)$  dependence.

<sup>1</sup> The article is published in the original.

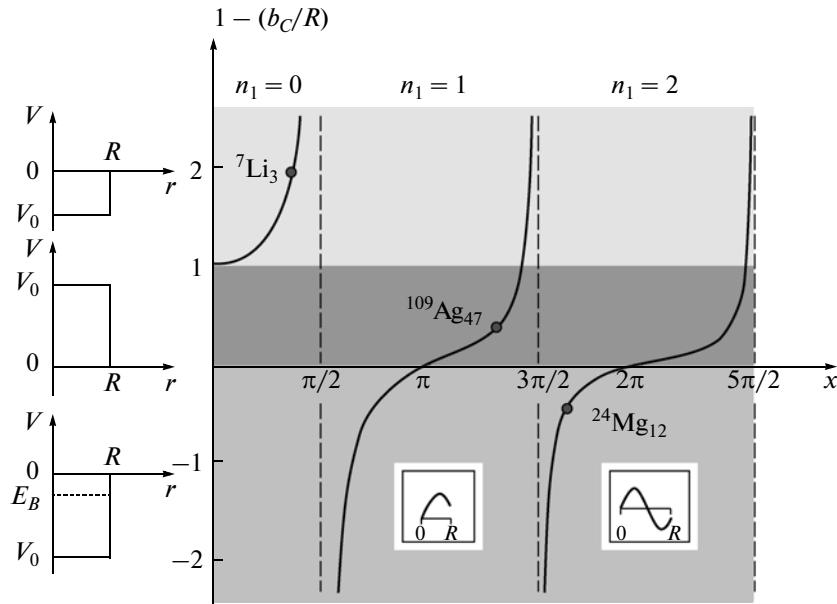


**Fig. 1.** Scattering lengths  $b$  ( $b_C$  in case  $S \neq 0$ ) of isotopes as a function of the mass number  $A$  according to [1], see also the table.

We believe that it is worthwhile to analyze the problem of this scatter within a reasonable and transparent model, with a special emphasis on the matter of depth of nuclear potential wells. As the density of the nuclear matter determined from diffraction data is known to be almost constant in all nuclei, one is often inclined to think that on adding nucleon to set another isotope one is increasing mainly the radius of potential well,

while its depth remains constant. The collection of the scattering length data offers an opportunity to verify it. Our main purpose was to determine, within the rectangular radial-symmetry potential-well model of nucleus, the depths  $V_0$  of such a well (or the height of potential barrier) for relevant isotopes. On analyzing the scattering lengths of about two hundred nuclei, we have found that the depth  $V_0$  does change vs.  $A$  significantly and there is certain regularity in its behavior. Besides, in cases  $b_{Z,A} > R_A$  (where  $R_A$ —radius of nucleus ( $Z, A$ )) we have arrived at an estimation of the bound-state energy-level positions  $E_b$  in deep potential wells and in many cases found its interesting coincidence with the Fermi level of the free nucleon model of nuclear matter.

A summary of the potential depth (barrier) data  $V_0 = V_{Z,A}$  and bound-state energy levels  $E_b$  (in MeV), evaluated within the rectangular radial-symmetry model of nucleus, using the experimental slow-neutron s-scattering lengths [1]. Colors—as in Fig. 2. Notation:  $Z, A$ —charge and mass numbers of nucleus,  $S$ —its spin,  $b_C$ —(coherent) scattering length  $b_{Z,A}$ ,  $n_1$  and  $n_2$ —domain indices of the solutions of Eqs. (9) and (17), respectively,  $E_B$ —bound-state energy level  $E_b$ ,  $E_F$ —Fermi level of the nuclear matter, Eq. (14). Last column—the  $(E_F, E_b)$  coincidence index, see Eq. (18). The  $V_0$  and  $E_b$  figures are to be trusted to approximately three digits,  $n_1$  gives (he number of oscillations of the scattered neutron wave function within the potential well, see Fig. 2.



**Fig. 2.** Graphical representation of three possible types of rectangular radial-symmetry potential wells (barriers)—a guide to the table. Colors show three domains of solutions of Eq. (9), depending on the ratio of neutron scattering length  $b$  (see the table) to the “classic” nucleon radius  $R$  given by Eq. (4). Yellow—shallow well,  $V_0 < 0$ ,  $b < 0$ ; violet—potential barrier,  $V_0 > 0$ ,  $0 < b < R$ ; green—deep well,  $V_0 < 0$ ,  $b > R$ , and there is a bound energy state in the well,  $E_b < 0$ . Inserts show qualitatively for the deep well the form of the inner function  $u(r)$  for  $n_1 = 1$  (single oscillation) and  $n_1 = 2$  (two oscillations); for  $n_1 = 3$  and 4 there are 3 and 4 oscillations, subsequently.

A ground for this study comes from the averaging properly of slow neutrons. They do not see the rich structure of the optical (or true) potential well of nucleus, but only its “average” depth, because the neutron wavelength is larger than the nucleus radius  $R$  by orders of magnitude. It is the depth of the true potential of nucleus seen by slow neutrons, for it is determined from the experimental scattering length  $b$ . Such a purpose justifies using the simple model of rectangular radial-symmetry nuclear potential. It needs just two nuclear parameters  $R$  and  $V_0$  and offers the necessary simplicity and transparency.

We believe that such a derivation of the potential depth (barrier height)  $V_0$  directly from the neutron scattering data is enlightening. It provides information which is both interesting and convincing. The derivation of the bound-state energy levels  $E_B$  for elements exhibiting deep potential well, given here, is perhaps more intuitive, but interesting too. The main purpose of the paper is to compare and systematize the  $V_0 = V_{Z,A}$  data for practically all isotopes, basing on the unifying ground of this model. It is old enough, but so far it has not been fully exploited for the purpose of such a global comparison.

## 1. OUTLINE OF THE THEORY OF ELASTIC S-SCATTERING

In this study we try to exploit in full the old idea of the s-scattering of neutron on the nucleus represented by the radial-symmetry rectangular potential well (or in some cases a rectangular radial-symmetry barrier) to systematize the data on neutron scattering lengths.

Quantum theory of the elastic slow-neutron scattering has been introduced long ago and it has been presented in several textbooks on nuclear physics. In the present contribution we shall shortly recall its results according to T. Mayer–Kuckuk [4], with occasional intercepts from P.J. Siemens and A.S. Jensen [5].

Basic notions of the theory are phase shifts  $\delta_l$  and scattering amplitudes  $f_l$  for the orbital momentum quantum number  $l$ , interrelated as follows:

$$f_l = \frac{1}{k} \exp(i\delta_l) \sin \delta_l, \quad (2)$$

where  $k = 2\pi/\lambda$  is the wavevector of the incoming neutron plane wave of the kinetic energy  $E = \hbar^2 k^2 / 2m$ . In case of the s-scattering ( $l=0$ ) and neutron long wavelength,  $k \rightarrow 0$ , on neglecting the complex factor (as related rather to absorption), the neutron scattering length is

$$b = -f_0. \quad (3)$$

For the study of slow-neutron scattering a good model of nucleus is the *radial rectangular potential well* of the depth  $V_0$ , and of the radius

$$R_A = r_0 A^{1/3}, \quad (4)$$

where we take as the single-nucleon radius  $r_0 = 1.3$  fm [5].  $A$  is the mass number of a nucleus,  $A = n + p$ , while  $Z = p$  is its charge number. On considering the equation of motion for the neutron of mass  $m_n$  ( $=939.57$  MeV/c $^2$ ) in the field of nuclear forces of nucleus of the mass  $m_A$ , one arrives at the *reduced mass*  $m$ :

$$\frac{1}{m} = \frac{1}{m_n} + \frac{1}{m_A} = \frac{A}{A+1}, \quad (5)$$

in the units  $m_n$ .

The Schrödinger equation for the s wave function ( $l=0$ )  $u(r)/r$  of neutron in the radial symmetry potential field  $V(r)$  of the nucleus is

$$\frac{d^2 u}{dr^2} + \frac{2m}{\hbar^2} (E - V(r)) u = 0. \quad (6)$$

On assuming the “attractive” rectangular potential well of the radius  $R = R_A$  and the depth  $V(0) = V_0 \equiv V_{Z,A} < 0$  (here we differ from the sign convention of the Mayer–Kuckuk book), one arrives under the conditions boundary  $u(0) = u(\infty) = 0$  at the following equation relating the potential depth  $V_0$ , the wavevector  $k \rightarrow 0$  (thus  $E \rightarrow 0$ ), and the phase shift  $\delta_0$  [4]:

$$\delta_0 = -kR_A + \arctan \left[ \frac{k}{k_i} \tan(k_i R_A) \right], \quad (7)$$

where

$$k_i = \sqrt{\frac{2m}{\hbar^2} (-V_{Z,A} + E)} \xrightarrow{k \rightarrow 0} \sqrt{-\frac{2m}{\hbar^2} V_{Z,A}}, \quad (8)$$

is the wavevector of the neutron wave function  $u(r)$  within the well (inner wave function), being proportional to  $\sin(kr)$ . On requesting again  $k \rightarrow 0$  in Eq. (7) we obtain that  $\delta_0$  is a small quantity (it is of the order  $10^{-4}$  for  $k = 10^{-10}$  m), so  $\sin(\delta_0) = \delta_0$  and we have for the scattering length

$$b = R_A - \frac{1}{k_i} \tan(k_i R_A). \quad (9)$$

This equation can be rewritten in the form

$$1 - \frac{b}{R_A} = \frac{\tan x}{x}, \quad (10)$$

where

$$x = \sqrt{-\frac{2m}{\hbar^2} V_{Z,A}} R_A = k_i R_A. \quad (11)$$

In case of repulsive potential barrier of the height  $V(0) > 0$  one formally puts in the formula  $V_{Z,Z} \rightarrow -V_{Z,A}$ . This results in the appearance of function  $\tanh(ix) = i \tan(x)$  and the final result is again Eq. (10), but now  $b < R_A$ .

Let us add by passing that, only assuming  $x$  small and  $b < 0$  (shallow potential well, see below), one

obtains the formula such as in the Born approximation for the rectangular radial potential well [4]:

$$b = b_{Z,A} = \frac{R_A x^2}{3} = \frac{2m}{3\hbar^2} V_{Z,A} R_A^3. \quad (12)$$

## 2. PROBLEMS WITH DETERMINATION OF THE POTENTIAL DEPTH $V_{Z,A}$

Knowing from experiment the value of scattering length  $b$ , one can immediately calculate from Eq. (10) one of its roots  $x$  (see Fig. 2), and then the depth of radial potential well  $V_0 \equiv V_{Z,A}$ :

$$|V_0| = \frac{\hbar^2 x^2}{2m R_A^2} = 12.26 \frac{x^2}{A^{2/3} 1 + A} \text{ MeV}, \quad (13)$$

where the sign is to be assigned to  $V_0$  depending on the value of  $b$ . In principle the value and sign of the scattering length  $b$  provides via this model not only the depth (or height in case of barrier) of the potential, but also basic qualitative physical features of the well. Namely, the construction of Eq. (10) allows three possibilities for the LHS, corresponding to three essentially different potential wells [4], as shown in Fig. 2:

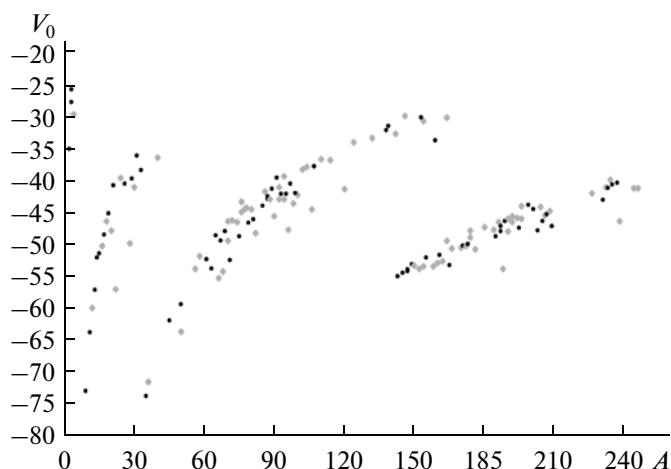
(a) LHS  $> 1$  shallow potential well, no neutron bound states,  $b_{Z,A} < 0$ ,  $V_0 < 0$ , yellow stripe.

(b) 1  $>$  LHS  $> 0$ , purely repulsive potential barrier,  $R_A > b_{Z,A} > 0$ ,  $V_0 > 0$ , violet stripe.

(c) LHS  $< 0$ , deep potential well with  $V_0 \ll 0$ , allowing the existence of neutron bound state  $E_b$ , green stripe. The inequality  $b_{Z,A} > R_A$  means the existence of the bound state.

Plots in the RHS of Fig. 2<sup>2</sup> show the function  $\tan(x)/x$  in the subsequent  $n_1$  domains,  $[(n_1 - 1/2)\pi < x < (n_1 + 1/2)\pi]$ ,  $n_1 = 0, 1, 2, 3, 4$ . Inserts show qualitatively the form of the function  $u(r)$  within the well for  $n_1 = 1$  and 2. As an example, the scattering lengths of three nuclei are represented in this figure by points. The so calculated  $x$  roots of Eq. (10) give via Eq. (13) the potential well depths  $V_0$  (heights of the barrier in case of violet strip data). One should mention here that for several isotopes this assignment depends critically on the chosen value of the “nucleon radius” parameter  $r_0$  in Eq. (4), while there is some arbitrariness here, compare the  $r_0$  values in [4] and [5].

This process of determining the  $V_0$  has its shortcomings. Due to the periodic nature of the function  $\tan x$ , there is an arbitrariness in choosing such a root of Eq. (10), which is acceptable from the physical point of view. Taking it formally, we have different  $x$  roots in each  $n_1$  domain,  $n_1 = 1, 2, 3, 4$  and each one of



**Fig. 3.** Potential well depths  $V_0 = V_{Z,A}$  for deep wells, green data. They could have been assigned to 4 domains of possible solutions of Eq. (10): from left to right there are branches  $n_1 = 1, 2, 3, 4$ . Formally, the depth range for all is  $[-73(\text{Be}), -26(\text{He})]$  MeV, but for majority of the nuclei there is  $-65 < V_{Z,A} < -35$  MeV. The  $S \neq 0$  isotopes are included as darker green points.

them results via Eq. (10) in different  $V_0$ . One needs an independent criterion to choose the one.

Following the nucleon Fermi gas model, one can generally argue that *the depth of the potential well  $V_0$  for heavier nuclei is no less than about 50 MeV* [5]. This is our main criterion for a reasonable choice of the  $n_1$  domain.

It says by the way that the  $x \rightarrow 0$  limit of Eq. (10) (i.e.,  $b_C \rightarrow R_A$ ), leading to the Born approximation Eq. (12), is not quite realistic—it can occur only in the domain  $n_1 = 0$  and usually provides the depth  $V_0$  of the order of a few MeV only. It follows that the case (a)—the “yellow” shallow-well case—is difficult to discuss in terms of the Fermi energy  $E_F$ . Here one should consider the value  $V_0$  evaluated assuming  $n_1 = 0$  merely as a quantitative indication, which is related to the scattering phenomenon, but provides no direct information on the true potential well of the nucleus in question and is not directly related to the nucleon density.

We have assumed that a reasonable choice for each nucleus with bound states (green field data) should be the lowest  $x$  root leading to the potential depth— $V_{Z,A} > E_F \approx 45$  MeV. The potential depth data  $V_0 = V_{Z,A}$  have been collected in the table. As one can see, the majority of cases are in the green field: the scattering length  $b > 0$  exceeds the nucleon radius  $R_A$ , i.e., the potential well is attractive and displays the bound-state level. The domain assignment of these deep-well (green color) nuclei,  $b > R_A$ , is shown in Fig. 3.

One can see that the nuclei which by their  $b$  values belong to the deep-well class (green) should be within this model further attributed to *four different  $n_1$  branches*, shown in Fig. 3, which is an interesting element of systematization. On comparing the defini-

<sup>2</sup> The colored version of the table and figures is only available at <http://wwwjinr.ru/publsh/>. In the present printed version the yellow color appears as white, green as light grey, and violet as dark grey.—Note of editor.

Table

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
1	1	H	0.5	-3.742312	0	-47.173			
1	2		1	6.66746	1	-35.656	1	-0.915	-0.231
1	3		0.5	4.79227	1	-28.293	1	-1.897	-0.415
2	3	He	0.5	5.747	1	-26.305	1	-1.210	-0.444
2	4		0	3.263	1	-30.199	1	-5.542	-0.454
3	6	Li	1	2.01	1	57.357			
3	7		1.5	-2.222	0	-4.937			
4	9	Be	1.5	7.791	2	-73.213	2	-0.554	0.609
5	10	B	3	-0.24	0	-0.677			
5	11		1.5	6.654	2	-64.105	2	-0.933	0.399
6	12	C	0	6.653514	2	-60.268	2	-0.950	0.314
6	13		0.5	6.199	2	-57.528	2	-1.193	0.247
7	14	N	1	9.372	2	-52.457	2	-0.388	0.153
7	15		0.5	6.443	2	-51.866	2	-1.104	0.124
8	16	O	0	5.8055	2	-50.623	2	-1.573	0.086
8	17		2.5	5.65	2	-48.983	2	-1.770	0.045
8	18		0	5.847	2	-46.832	2	-1.611	0.001
9	19	F	0.5	5.65412	2	-45.653	2	-1.858	-0.030
10	20	Ne	0	4.6316	2	-48.322	2	-4.345	-0.026
10	21		1.5	6.6619	2	-41.247	2	-1.109	-0.111
10	22		0	3.871	2	-57.358	2	-13.051	-0.019
11	23	Na	1.5	3.632	1	16.176			
12	24	Mg	0	5.4918	2	-40.097	2	-2.367	-0.165
12	25		2.5	3.6214	1	16.206			
12	26		0	4.8915	2	-41.063	2	-4.228	-0.184
13	27	Al.	2.5	3.4495	1	17.590			
14	28	Si	0	4.1066	2	-50.312	2	-12.562	-0.164
14	29		0.5	4.71	2	-40.172	2	-5.780	-0.238
14	30		0	4.588	2	-41.598	2	-6.998	-0.234
15	31	P	0.5	5.131	2	-36.610	2	-3.945	-0.277
16	32	S	0	2.8042	1	20.645			
16	33		1.5	4.7419	2	-38.860	2	-6.501	-0.283
16	34		0	3.483	1	16.574			
16	36		0	3.01	1	18.657			
17	35	Cl	1.5	11.709	3	-74.037	3	-0.250	0.634
17	37		1.5	3.086	1	18.111			
18	36	Ar	0	24.97	3	-71.772	2	-0.038	0.588
18	38		0	3.353	1	15.785			
18	40		0	1.7	1	20.393			
19	39	K	1.5	3.792	1	14.242			
19	40		4	3.11	1	17.372			
19	41		1.5	2.698	1	18.370			

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
20	40	Ca	0	4.785	2	-36.931			
20	42		0	3.361	1	16.049			
20	43		3.5	-1.569	0	-0.747			
20	44		0	1.426	1	19.420			
20	46		0	3.5521	1	14.741			
20	48		0	0.399	1	18.959			
21	45	Sc	3.5	12.11	3	-62.322			
22	46	Ti	0	4.725	1	9.359			
22	47		2.5	3.537	1	14.679			
22	48		0	-5.862	0	-1.411			
22	49		3.5	0.985	1	18.380			
22	50		0	5.881	3	-64.063			
23	50	V	6	7.66	3	-59.831			
23	51		3.5	-0.4022	0	-0.207			
24	50	Cr	0	-4.505	0	-1.217			
24	52		0	4.91415	1	8.631			
24	53		1.5	-4.203	0	-1.121			
24	54		0	4.551	1	10.024			
25	55	Mn	2.5	-3.75018	0	-1.025			
26	54	Fe	0	4.21	1	11.453			
26	56		0	10.12	3	-54.218			
26	57		0.5	2.31	1	15.611			
26	58		0	1.57	1	16.062			
27	59	Co	3.5	2.492	1	15.078			
28	58	Ni	0	14.41	3	-52.232			
28	60		0	2.81	1	14.527			
28	61		1.5	7.606	3	-52.794			
28	62		0	-8.72	0	-1.325			
28	64		0	-0.377	0	-0.156			
29	63	Cu	1.5	6.47713	3	-54.189			
29	65		1.5	10.2042	3	-49.096			
30	64	Zn	0	5.234	1	7.582			
30	66		0	5.985	3	-55.744			
30	67		2.5	7.588	3	-49.838			
30	68		0	6.043	3	-54.636			
30	70		0	6.91	3	-49.834			
31	69	Ga	1.5	8.04316	3	-48.422			
31	71		1.5	6.17011	3	-52.758			
32	70	Ge	0	10.01	3	-46.823			
32	72		0	8.511	3	-46.738			
32	73		4.5	5.024	1	8.185			
32	74		0	7.581	3	-46.932			
32	76		0	8.215	3	-45.422			
33	75	As	1.5	6.581	3	-49.180			
									0.018

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
34	74	Se	0	0.833	1	13.981			
34	76		0	12.21	3	-43.832	3	-0.269	-0.035
34	77			8.258	3	-45.013	3	-1.032	-0.026
34	78		0	8.249	3	-44.662	3	-1.046	-0.034
34	80		0	7.483	3	-44.981	3	-1.674	-0.041
34	82		0	6.348	3	-48.707	3	-4.789	-0.027
35	79	Br	1.5	6.797	3	-47.074	3	-2.818	-0.020
35	81		1.5	6.787	3	-46.534	3	-2.946	-0.035
36	78	Kr	0						
36	80		0						
36	82		0						
36	83		4.5						
36	84		0						
36	86		0	8.0726	3	-42.253	3	-1.243	-0.092
37	85	Rb	2.5	7.071	3	-44.450	3	-2.442	-0.070
37	87		2.5	7.2712	3	-43.358	3	-2.141	-0.087
38	84	Sr	0	5.02	1	8.096			
38	86		0	5.685	1	6.401			
38	87		4.5	7.417	3	-43.018	3	-1.918	-0.090
38	88		0	7.166	3	-43.379	3	-2.364	-0.092
39	89	Y	0.5	7.752	3	-41.846	3	-1.559	-0.108
40	90	Zr	0	6.51	3	-45.999	3	-4.702	-0.086
40	91		2.5	8.81	3	-40.129	3	-0.887	-0.131
40	92		0	7.52	3	-41.495	3	-1.896	-0.123
40	94		0	8.32	3	-39.776	3	-1.167	-0.145
40	96		0	5.51	1	6.784			
41	93	Nb	4.5	7.0543	3	-42.559	3	-2.813	-0.120
42	92	Mo	0	6.938	3	-43.335	3	-3.571	-0.120
42	94		0	6.827	3	-43.336	3	-3.496	-0.118
42	95		2.5	6.937	3	-42.619	3	-3.250	-0.128
42	96		0	6.226	3	-48.195	3	-7.803	-0.106
42	97		2.5	7.268	3	-40.990	3	-2.473	-0.147
42	98		0	6.607	3	-44.077	3	-4.941	-0.133
42	100		0	6.757	3	-42.738	3	-4.307	-0.149
43	99	Tc	4.5	6.83	3	-42.447	3	-3.897	-0.146
44	96	Ru	0						
44	98		0						
44	99		3						
44	100		0						
44	101		2.5						
44	102		0						
44	104		0						
45	100	Rh	0.5	5.904	1	5.925			
46	102	Pd	0	7.77	3	-38.721	3	-1.784	-0.182

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
46	104		0	7.77	3	-38.314			
46	105		2.5	5.53	1	6.698			
46	106		0	6.44	3	-45.012			
46	108		0	4.13	1	9.071			
46	110		0	7.77	3	-37.186			
47	107	Ag	0.5	7.55511	3	-38.235			
47	109		0.5	4.16511	1	8.987			
48	106	Cd	0	5.02	1	7.750			
48	108		0	5.3124	1	7.133			
48	110		0	5.788	1	6.143			
48	111		0.5	6.478	1	4.918			
48	112		0	6.346	1	5.123			
48	113		0.5	-8.01	0	-0.796			
48	114		0	7.485	3	-37.284			
48	116		0	6.269	1	5.248			
49	113	In	4.5	5.396	1	6.912			
49	115		4.5	4.003	1	8.915			
50	112	Sn	0	6.01	1	5.709			
50	114		0	6.03	1	5.670			
50	115		0.5	6.01	1	5.706			
50	116		0	6.101	1	5.539			
50	117		0.5	6.598	1	4.739			
50	118		0	6.234	1	5.306			
50	119		0.5	6.283	1	5.224			
50	120		0	6.674	3	-41.803			
50	122		0	5.933	1	5.833			
50	124		0	6.153	1	5.442			
51	121	Sb	2.5	5.716	1	6.238			
51	123		3.5	5.387	1	6.818			
52	120	Te	0	5.35	1	6.920			
52	122		0	3.82	1	8.787			
52	123		0.5	-0.0525	0	-0.012			
52	124		0	7.951	3	-34.577			
52	125		0.5	5.018	1	7.380			
52	126		0	5.557	1	6.490			
52	128		0	5.888	1	5.901			
52	130		0	6.017	1	5.662			
53	127	I	2.5	5.282	1	6.940			
54	124	Xe	0						
54	126		0						
54	128		0						
54	129		0.5						
54	130		0						
54	131		1.5						

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
54	132		0						
54	134		0						
54	136		0						
55	133	Cs	3.5						
56	130	Ba	0	-3.66	0	-0.481			
56	132		0	7.83	3	-33.857	3	-2.555	-0.307
56	134		0	5.71	1	3.435			
56	135		1.5	4.661	1	7.593			
56	136		0	4.908	1	7.293			
56	137		1.5	6.821	1	4.426			
56	138		0	4.838	1	7.328			
57	138	La	5	8.02	3	-32.665	3	-2.323	-0.328
57	139		3.5	8.244	3	-32.046	3	-1.949	-0.334
58	136	Ce	0	5.769	1	6.052			
58	138		0	6.659	1	4.645			
58	140		0	4.819	1	7.303			
58	142		0	4.729	1	7.350			
59	141	Pr	2.5	4.585	1	7.513			
60	142	Nd	0	7.73	3	-33.167	3	-3.251	-0.338
60	143		3.5	14.02	4	-55.430	4	-0.216	0.223
60	144		0	2.83	1	8.451			
60	145		3.5	14.02	4	-54.921	4	-0.217	0.211
60	146		0	8.72	3	-30.448	3	-1.476	-0.358
60	148		0	5.73	1	6.018			
60	150		0	5.282	1	6.587			
61	147	Pm	3.5	12.64	4	-54.624	4	-0.304	0.203
62	144	Sm	0	-3.04	0	-0.391			
62	147		3.5	14.03	4	-54.423	4	-0.217	0.200
62	148		0	-3.04	0	-0.382			
62	149		3.5	18.728	4	-53.605	4	-0.096	0.185
62	150		0	14.03	4	-53.699	4	-0.219	0.184
62	152		0	-5.06	0	-0.504			
62	154		0	8.01	3	-31.174	3	-2.874	-0.373
63	151	Eu	2.5						
63	153		2.5	8.2212	3	-30.661	3	-2.325	-0.373
64	152	Gd	0	10.03	4	-54.289	4	-0.753	0.185
64	154		0	10.03	4	-53.840	4	-0.762	0.175
64	155		1.5	13.83	4	-52.569	4	-0.232	0.159
64	156		0	6.34	1	5.085			
64	157		1.5	4.02	1	7.493			
64	158		0	9.02	4	-53.888	4	-1.341	0.164
64	160		0	9.155	4	-53.312	4	-1.252	0.153
65	159	Tb	1.5	7.342	4	-34.292	4	-7.240	-0.401
66	156	Dy	0	6.15	1	5.359			

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
66	158		0	6.04	1	5.506			
66	160		0	6.74	1	4.527			
66	161		2.5	10.34	4	-52.165	4	-0.690	0.140
66	162		0	-1.45	0	-0.204			
66	163		2.5	5.04	1	6.620			
66	164		0	49.45	4	-49.914	3	-19.666	-0.330
67	165	Ho	3.5	8.443	4	-53.650	4	-2.189	0.140
68	162	Er	0	9.0111	4	-53.103	4	-1.392	0.145
68	164		0	7.9514	3	-30.680	3	-3.507	-0.398
68	166		0	10.5119	4	-51.070	4	-0.658	0.116
68	167		3.5	3.065	1	7.620			
68	168		0	7.438	1	3.719			
68	170		0	9.616	4	-50.904	4	-1.028	0.104
69	169	Tm	0.5	7.073	1	4.112			
70	168	Yb	0	-4.072	0	-0.409			
70	170		0	6.81	1	4.430			
70	171		0.5	9.71	4	-50.637	4	-0.984	0.099
70	172		0	9.51	4	-50.641	4	-1.105	0.097
70	173		2.5	9.561	4	-50.412	4	-1.081	0.092
70	174		0	19.21	4	-48.312	4	-0.093	0.068
70	176		0	8.71	4	-51.172	4	-1.958	0.090
71	175	Lu	3.5	7.289	1	3.871			
71	176		7	6.12	1	5.296			
72	174	Hf	0	10.911	4	-49.376	4	-0.581	0.081
72	176		0	6.6118	1	4.674			
72	177		3.5	0.81	1	7.800			
72	178		0	5.92	1	5.519			
72	179		4.5	7.4616	1	3.694			
72	180		0	13.23	4	-47.710	4	-0.286	0.050
73	180	Ta	9	7.02	1	4.170			
73	181		3.5	6.917	1	4.291			
74	180	W	0	5.03	1	6.333			
74	182		0	7.044	1	4.142			
74	183		0.5	6.594	1	4.681			
74	184		0	7.556	1	3.602			
74	186		0	-0.734	0	-0.100			
75	185	Re	2.5	9.03	4	-49.143	4	-1.659	0.051
75	187		2.5	9.33	4	-48.375	4	-1.369	0.041
76	184	Os	0	10.02	4	-48.149	4	-0.904	0.046
76	186		0	12.017	4	-46.941	4	-0.412	0.030
76	187		0.5	10.02	4	-47.670	4	-0.920	0.035
76	188		0	7.83	4	-54.203	4	-5.897	0.070
76	189		1.5	11.03	4	-46.781	4	-0.591	0.023
76	190		0	11.43	4	-46.472	4	-0.510	0.018

Table. (Contd.)

Z	A	Symbol	S	$b_C$	$n_1$	$V_0$	$n_2$	$E_B$	$\frac{-V_0 + E_B - E_F}{E_F}$
76	192		0	11.94	4	-46.004	4	-0.431	0.009
77	191	Ir	1.5						
77	193		1.5						
78	190	Pt	0	9.01	4	-48.455	4	-1.757	0.034
78	192		0	9.95	4	-46.958	4	-0.982	0.018
78	194		0	10.558	4	-46.239	4	-0.737	0.008
78	195		0.5	8.919	4	-47.839	4	-1.972	0.016
78	196		0	9.898	4	-46.416	4	-1.034	0.005
78	198		0	7.81	1	3.370			
79	197	Au	1.5	6.261	1	5.011			
80	196	Hg	0	30.31	4	-44.397	3	-17.589	-0.406
80	198		0						
80	199		0.5	16.94	4	-44.291	4	-0.136	-0.022
80	200		0						
80	201		1.5	11.00243	4	-44.993	4	-0.628	-0.018
80	202		0						
80	204		0						
81	203	Tl	0.5	8.518	4	-48.311	4	-3.109	0.001
81	205		0.5	8.877	4	-46.852	4	-2.242	-0.012
82	204	Pb	0	10.89378	4	-44.624	4	-0.666	-0.027
82	206		0	9.22178	4	-45.956	4	-1.708	-0.020
82	207		0.5	9.28616	4	-45.718	4	-1.641	-0.024
82	208		0	9.4943	4	-45.266	4	-1.428	-0.029
83	209	Bi	4.5	8.5322	4	-47.682	4	-3.283	-0.017
84		Po							
85		At							
86		Rn							
87		Fr							
88	226	Ra	0	10.01	4	-42.519	4	-1.157	-0.084
89		Ac							
89		Ac							
90	232	Th	0	10.313	4	-41.576	4	-1.006	-0.102
91	231	Pa	1.5	9.13	4	-43.523	4	-2.276	-0.087
92	233	U	2.5	10.12	4	-41.650	4	-1.128	-0.103
92	234		0	12.43	4	-40.360	4	-0.419	-0.116
92	235		3.5	10.503	4	-41.102	4	-0.922	-0.110
92	238		0	8.4077	4	-46.792	4	-5.474	-0.085
93	237	Np.	2.5	10.551	4	-40.855	4	-0.909	-0.115
94	239	Pu	0.5	7.71	1	3.452			
94	240		0	3.51	1	5.961			
94	242		0	8.11	1	3.122			
95	243	Am	2.5	8.32	1	2.967			
96	244	Cm	0	9.53	4	-41.733	4	-1.810	-0.116
96	246		0	9.32	4	-41.737	4	-2.190	-0.124
96	248		0	7.72	1	3.441			

tions of  $x$  and  $k_i$  we can see that  $k_i = x/R_A$ . One finds that in all cases there occurs  $(n_1 - 1/2)\pi/R_A < k_i < (n_1 + 1/2)\pi/R_A$ , where the values of  $n_1 = 1, 2, 3, 4$  give the number of oscillations of the s-type wave function  $\approx \sin(k_i r)$  within the rectangular potential well of the radius  $R_A$ . It follows each isotope belongs within this model to one of the  $n_1$  classes. Let us remind that by construction there is a matching of the inner and outer parts of wave function at the point  $R_A$  [4]. The outer part declines exponentially with growing  $r$ .

However, in several cases we had to accept as the best choice the potential depths  $V_0$  different significantly from the criterion value of about  $-50$  MeV. Besides, there are many nuclei of the barrier type,  $0 < b < R_A$  (violet) and about 10 shallow-well nuclei displayed in the table in yellow field,  $b < 0$ .

### 3. DETERMINATION OF THE BOUND-STATE ENERGY $E_b$ — THE DATA OF THE GREEN STRIPE

As mentioned above, to fix the representative depths  $V_0$  for nuclei, one has to use some information following from other sources. The Fermi energy  $E_F$  for the nuclear matter in the free nucleon Fermi gas model is one of them [4]:

$$E_F = \frac{1}{2m_n} (3\pi^2)^{1/3} \hbar^2 d_n^{2/3}, \quad (14)$$

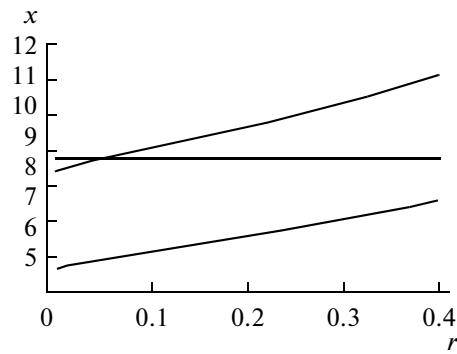
where the nucleon density in the nucleus is

$$d_n = \frac{A}{4/3\pi R_A^3} = \frac{1}{4/3\pi r_0^3}, \quad (15)$$

where the single-nucleon radius  $r_0 = 1.3$  fm [5]. Following it, one now generally accepts the Fermi energy for nuclear matter to be about 45 MeV. In most deep-well cases the calculated depth of the well exceeds (on absolute value) the value of  $V_{Z,A}$ , as expected.

Quantum mechanics says that three-dimensional potential well has the bound-state energy levels, if it is deep enough. In case of isotropic neutron scattering, the levels of interest belong to the s-type states. Using the green data, one obtains the equation for the bound-state energy level  $E_b$  in the potential well. The evaluation can be found in textbooks, e.g., in the “Quantum Mechanics” of Landau and Lifshitz [6], or as a slightly extended (to arbitrary  $A$ , via the mass parameter  $m$ , Eq. (5)) calculation for the radial-rectangular-well model of deuteron, in the Kuckuk book [4]:

$$\cot \left[ \left( \frac{m}{\hbar^2} (|V_{Z,A} + E_b|) R_A^2 \right)^{1/2} \right] = - \left( \frac{E_b}{|V_{Z,A} + E_b|} \right)^{1/2}. \quad (16)$$



**Fig. 4.** Example of graphical determination of the parameter  $r = E_b/V_0$ . Horizontal line corresponds to the value of  $x = 8.27$  for the isotope  $^{104}\text{Pd}$  see the table. There are two lines representing the LHS of Eq. (17) for  $n_2 = 2$  and  $n_2 = 3$ . The crossing of the  $n_2 = 3$  line with the  $x$  line gives the solution:  $r = 0.0476$ , i.e.,  $E_b = 1.826$  MeV.

With the notation  $r = E_b/V_{Z,A}$  and  $x$  given by Eq. (12), we have the following equation for  $r$ :

$$x = \frac{1}{\sqrt{1-r}} \left\{ -\arctan \left[ \sqrt{\frac{r}{1-r}} \right] + n_2 \pi \right\}, \quad (17)$$

$$n_2 = 1, 2, \dots,$$

where (he labels  $n_2$  need not be such as the labels  $n_1$  introduced above).

One can look for solutions  $r$  graphically, Fig. 4. To avoid unphysical negative  $x$ , we have to take  $n_2$  equal to 1 or more.

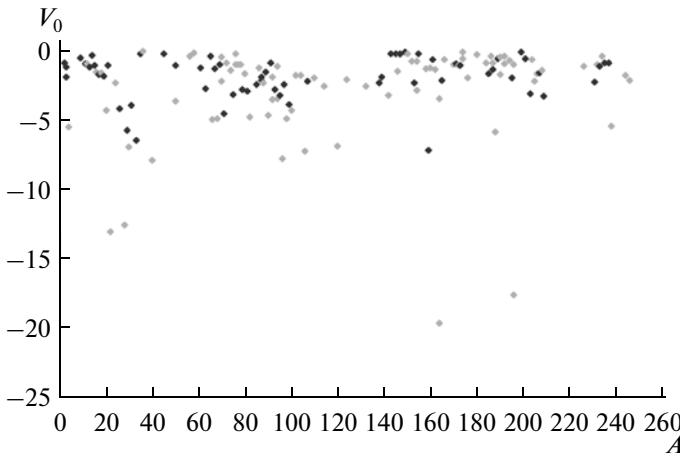
Several values of  $n_2$  had to be tried to obtain for the  $E_B$  the value being a real number, representing the energy level in the well,  $V_{Z,A} < E_B < 0$ . In most cases there is only one real and reasonable solution. It has been found that usually there is  $n_1 = n_2$ . It is not the case when three elements are characterized by exceptionally high values of the scattering length:  $^{36}\text{Ar}$ ,  $^{164}\text{Dy}$ ,  $^{196}\text{Hg}$ .

In principle one cannot identify the  $E_B$  with the neutron separation energy, which in the Fermi-gas model of nucleus is known to be close to the Fermi level [5]. Still we have found that the following approximate equality often holds:

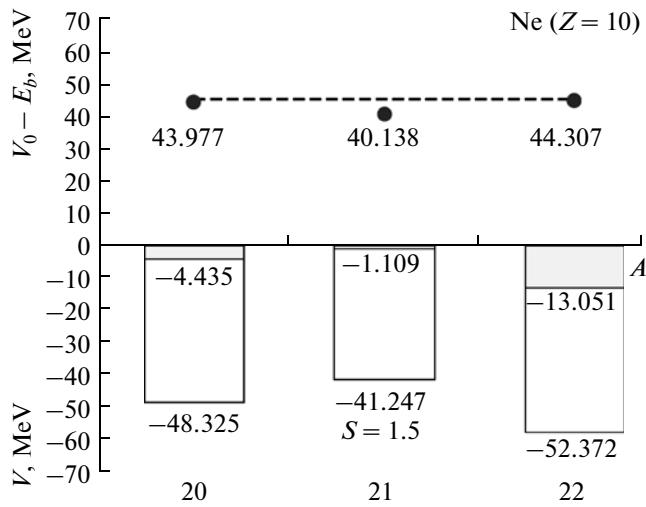
$$-V_{Z,A} + E_{b_{Z,A}} \approx E_F. \quad (18)$$

Having no better clues, we have sometimes treated it as a qualitative criterion for the green data on how to assign a given ( $A, Z$ ) nucleus to right ( $n_1, n_2$ ) class—choose the smallest numbers ( $n_1, n_2$ ) allowing one to approximately fulfill this equation.

The nucleus parameters  $V_{Z,A}$  derived here and shown in the table are essential characteristics of nuclei. We present also the numbers  $E_b \equiv E_{b_{Z,A}}$  for all nuclei exhibiting the deep potential well, see green



**Fig. 5.** The bound-state energy-level data  $E_b$  vs.  $A$  for the deep-well isotopes. The  $S \neq 0$  isotopes are included as darker green points. For the majority there is  $E_b > -5$  MeV, but no clear trends are visible in this plot.



**Fig. 6.** Graphical representation of the potential well depths in MeV vs.  $A$ , for the isotopes of Ne, as evaluated from the experimental neutron scattering lengths. The position and numerical value of the bound-state energy level in the well are marked with the blue color. The “classic” Fermi level  $E_F = 45$  MeV is shown with a dashed line, and the difference  $E_b - V_0$  as blue points. The  $S \neq 0$  isotope is indicated,  $n_1 = n_2 = 2$  for all isotopes of Ne.

field data. These numbers show no systematic features and fluctuate considerably vs.  $A$ , as shown in Fig. 5.

In the table the last column gives an index of coincidence between the  $E_b$  and the  $E_F$ ,  $(-V_{Z,A} + E_{b_{Z,A}} - E_F)/E_F$ , following from Eq. (18). One can see that for about 30 isotopes the coincidence is no worse than 2%, while for most of them it is about 10%. However, for some isotopes of He, Be, B, Cl, Ar, Ca, Sc, Ti, V, Ba, La, Nd, Sm, Eu, Gd, Tb, Dy, Hg there is no such a coincidence at all.

The above considerations should be treated with some restraint, though, for there is some uncertainty in attributing the so derived value  $E_b$  to a single bound-state s-type energy level. Rather, one should sometimes think of an *effective* bound-state level related to a few s-type states in the nucleus.

One should remember that at the root of the above estimation there lies the assumption of a constant density of nuclear matter, resulting in the monotonia dependence of the well radius on the mass number  $A$ , see Eq. (4). As far as one can trust the experimental data on scattering lengths, we can say that these fluctuations in the values of  $V_{Z,A}$  would be smaller if the nuclear densities were not quite fixed for all nuclei—if they were allowed to take on a part of variation in the scattering lengths vs.  $A$ .

In this context one should recall that the nucleon radius  $r_0$  in Eq. (4) is not a physical constant. It comes as a summary of many high-energy diffraction experiments and its value changed considerably over the last decades [4, 5, 7]. In principle one could try to determine it for each isotope separately from the experimental scattering-length data via the radial rectangular potential model. On relying on the model of radial rectangular well, one could gain thus from the scattering data a new information on the density of nuclear matter. This work will possibly be continued.

In Fig. 6 we show as an example the typical histogram of the potential well depths for the isotopes of Ne. As in this case  $n_1 = 2$ , there are just two oscillations in the wave function in the well region  $(0, R_A)$ , for all isotopes of Ne. Such histograms for all isotopes reported in the Dianoux–Lander booklet [1] are displayed in the IAE Report [8].

#### 4. SHALLOW POTENTIAL WELL— NO BOUND STATES—STRIPE DATA

The case of negative scattering length,  $b_C < 0$ , leads us immediately to the yellow stripe in Fig. 2. As, in order to set up a tractable form of the model, the energy of neutrons had been assumed very small, it means that when using Eq. (10) we have to limit ourselves to the  $n_1 = 0$  domain, and the evaluated depths of the well are to be small. Indeed, yellow  $V_0$ 's are most often (on the absolute value) of the order of 1 MeV or less.

#### 5. THE BARRIER ISOTOPES— VIOLET STRIPE DATA

Let us refer to the case of rectangular radial-symmetry *barrier* (rather than well), violet data. Almost a half of the nuclei belong to the barrier class,  $0 < b_{Z,A} < R_A$ . Having in mind that the nucleon-nucleon forces are mostly attractive, the very fact of existence of the repulsive-barrier nuclei deserves attention—at first glance they should not exist.

We cannot see here any link between the ideas of the Fermi free-micleon gas and the barrier height value. Let us only mention some evident features shown in the table and in Fig. 7:

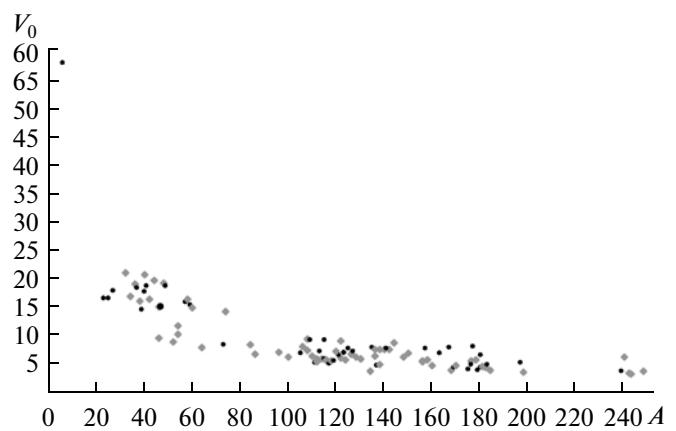
- (1) On excluding the  ${}^6\text{Li}$  nucleus, which escapes a systematic treatment, one observes that the height of the barrier, somewhat erratically but systematically, declines vs.  $A$  about 5 times: from about 20 MeV for  $A = 30$  to about 5 MeV at  $A = 240$ .
- (2) There are at least two broad gaps in the spectrum of the barrier displaying nuclei:  $7 < A < 22$  and  $196 < A < 237$ .
- (3) All isotopes of the elements Na, Al, K, Rh, In, Sn, Ce, Pr, Lu, Pu belong to the barrier class:  $0 < b_{Z,A} < R_A$ .
- (4) Statistically the  $S = 0$  and  $S > 0$  nuclei show a similar behavior vs.  $A$ .

## SUMMARY

The main reason for undertaking this effort of determining the depths  $V_0$  of potential wells for nuclei was our hope that within the radial rectangular potential model one can discover a source of erraticity of the observed neutron scattering lengths  $b$  vs.  $A$ . Partially, it comes true—one arrives here at the task of determining the roots of expressions involving trigonometric functions, and in principle there is a variety of discrete roots showing there.

For most of the isotopes there is  $b_C = b_{Z,A} > R_A$  (green field data) and in such cases the depth parameters  $V_0 \equiv V_{Z,A}$  can be attributed to 4 branches of the  $V_0$  vs.  $A$  plots in Fig. 3, corresponding to the roots of Eq. (10) in the domains:  $n_1 = 1, 2, 3, 4, \dots$ , see Fig. 2. We have therefore found that *the erraticity in scattering lengths  $b$  can be in part related to the boundary conditions of wave functions in the rectangular potential well, resulting in the  $n_1$  oscillations of the s-type wave function of scattered neutron in the range  $(0, R_A)$* . It should be emphasized that the scale of this erraticity could within this model diminish, if one decided to allow for a variation of the nucleon density in nuclei.

Once the depth  $V_0$  has been established, in case of the deep potential well one can look for the position of the bound-state energy level  $E_b$ . Equation (16) is similar to Eq. (10), so again one has to choose a domain of solution—the domain. One arrives at the  $E_b$  values of a few MeV. Basically, they show no systematic trends, see Fig. 5, but seem often be related to the  $E_F$  value of the Fermi-gas model of nucleus, see Eq. (18) and discussion therein. We have to note that we can see no clear correlation between the  $E_b$  values evaluated here and those evaluated by Aleksejev et al. within the optical potential model [2], or with the Yatnada and Matumoto results [7]. Our potential depths  $V_0 = V_{Z,A}$  are



**Fig. 7.** The barrier heights  $V_0$  vs.  $A$ , violet field data. The  $S \neq 0$  elements are shown as darker points. One observes here a systematic although erratic decline for growing  $A$  and  $A$ -gaps, see text.

representative solely of the slow-neutron-scattering capabilities of the nuclei.

The depth values  $V_0$  of the yellow strip are indeed small, so these shallow-well nuclei can often be treated in terms of the Born approximation for the scattering length. We have to remark that the Fermi free-nucleon gas model with  $E_F = 45$  MeV is not compatible with the idea of shallow potential.

In case of small positive  $b < R_A$ , violet field, one has to think of the repulsive potential barrier, not attractive potential. The barrier height parameter  $V_0$  declines vs.  $A$ , but no hint explaining the repulsive character of the nucleus follows from this behavior, as yet.

## CONCLUSIONS

(1) Using the experimental neutron scattering lengths collected in [1], the depths of potential wells within the rectangular radial potential well model (heights of potential barriers) have been estimated for the majority of isotopes. In a few cases the so estimated potential depth exceeds significantly the commonly accepted estimate about 50 MeV.

(2) We have found that the isotopes can be attributed to different branches  $n_1 = 1, 2, 3, 4$  of the s-wave-function-type solutions of the wave equation for the neutron-nucleus scattering problem. This partly explains the erraticity of the experimental scattering length values. Each  $n_1$  number gives the number of oscillations of the neutron wave function in the range  $(0, R_A)$ .

(3) For isotopes having deep potential well, one often finds a slight coincidence between the evaluated bound-state energy level and the Fermi level of the free-nucleon model of nuclear matter.

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## REFERENCES

1. *Neutron Data Booklet*, Ed. by A. J. Dianoux and G. Lander (ILL – Neutrons for Science, 2001).
2. A. Aleksejev et al., “Theoretical Evaluation of Neutron-Nucleus Scattering Parameters from Experimental Data in the  $6 < A < 60$  Mass Region,” *Z. Naturforsch. A: Phys. Sci.* **53**, 855 (1998).
3. S. F. Mughabghab, *Neutron Cross Sections* (Academic, New York, 1984).
4. T. Mayer-Kuckuk, *Kernphysik* (Teubner, Stuttgart, 1979).
5. P. J. Siemens and A. S. Jensen, *Elements of Nuclei. Many Body Physics with the Strong Interactions* (Addison-Wesley, Redwood City, CA, 1987).
6. L. Landau and E. Lifshitz, *Quantum Mechanics* (Nauka, Moscow, 1948; Pergamon, New York, 1977).
7. Yamada Masami and Matumoto Zyun-Itiro, “Nuclear Ground-State Energies,” *J. Phys Soc. Jap.* **16**, 1497 (1961).
8. A. Czachor and P. Pęczkowski, “Nuclear Rectangular Radial Potential Wells and Bound-State Energy Levels as Evaluated from Experimental Data on Neutron Scattering Lengths,” Report IEA B/59/2007.