UNIVERSITÉ LIBRE DE BRUXELLES
Faculté des Sciences
Département d'Informatique

The Coverability problem for parametric Petri nets

Alexis Reynouard

Promoter : Prof. Gilles Geeraerts

Master Thesis in Computer Sciences

Academic year 2018 – 2019

Contents

Chapter 1

Introduction

Petri nets (PNs) are a mathematical and graphical model introduced by Carl Adam Petri in 1962 [18, 19]. It was successfully used to analyse systems in a wide range of domains, especially for the formal verification of asynchronous systems.

In their standard definition, PNs are instantiated through many natural numbers which may represent, for example, the amount of resource needed for a given action to be carried out.

The introduction of parameters into the model to avoid the need to state these values explicitly[1] may have several benefits: it may allow to perform analysis of a whole family of PNs in an efficient way, like in [2], or to model dynamic changes in the system, as introduced by [3] as a subclass of reconfigurable nets.

The use of parameters increases the modelling power of PNs but also make some basic coverability problems undecidable in the general case [6].

We adopt the parametric Petri net model introduced by [6], which seems the most general, and we study the existing results and algorithms for plain Petri nets to find out whether or not they still hold or how to adapt them to the parametric model.

The rest of the document is as follows: In this first part, we define the plain Petri net model (*i.e.* the classical one) and the parametric model. We then briefly motivate our study, [TODO: give concrete examples of applications,] and give an overview of the previous works on parametrisation of PNs. Finally we place the PN model in a broader model family: the well-structured transition systems (WSTSs).
In a second part, we recall, first, the classical results that we will study on this new model, second, the results already obtained for the parameterized Petri net model as we have defined them.
Then, we focus on the parametric Petri net model to establish whether the results related to the coverability problem in the plain Petri net model still hold or if the algorithms may be adapted to this new model.

[1] One can find in the literature many other way to use parameters in PNs. For example, place and / or transitions may also be parameters in order to dynamically change the network structure, like in [5].

1.1 Definitions

Definition 1 (Petri net)**.** A Petri net (PN) $\mathcal{N}$ is a weighted oriented bipartite graph, whose the two subsets of vertices define a tuple $\langle P, T \rangle$ where:

- P is a finite set of places,
- T is a finite set of transitions.

For each transition $t \in T$ are defined (exactly) these two functions:

- $I_t : P \mapsto \mathbb{N}$ associates to each place the weight of the edge to t *(input weight)*,
- $O_t : P \mapsto \mathbb{N}$ associates to each place the weight of the edge from t *(output weight)*.

It is denoted by $t = \langle I_t, O_t \rangle$. Because these functions define the edges of the graph, a PN is completely defined by the tuple $\langle P, T \rangle$ and so is denoted by $\mathcal{N} = \langle P, T \rangle$.

Definition 2 (marking)**.** Given a set of place P, a marking of P is a function $\mathbf{m} : P \mapsto \mathbb{N}$ that associates $\mathbf{m}(p)$ tokens to each place $p \in P$.

An order on the markings is essential for the analysis of PNs. The order we will define is a well quasi-order and a partial order.

Definition 3 (quasi-order)**.** A quasi-order on a set $\mathcal{E}$ is a binary relation R that is:

$$\begin{aligned} &\text{reflexive:} && \forall x \in \mathcal{E},\ x\ R\ x \\ &\text{transitive:} && \forall (x, y, z) \in \mathcal{E}^3,\ (x\ R\ y \wedge y\ R\ z) \Rightarrow x\ R\ z \end{aligned}$$

Definition 4 (well quasi-order)**.** A well quasi-order $\sqsubseteq$ on a set $\mathcal{E}$ is a quasi-order on $\mathcal{E}$ such that, for any infinite sequence $s = e_0, e_1, e_2, \ldots$ of elements from $\mathcal{E}$, there exist indices $i < j$ with $e_i \sqsubseteq e_j$. That is, there is no infinite antichain in $\mathcal{E}$ for this relation.

Definition 5 (partial order)**.** A partial order on a set $\mathcal{E}$ is a quasi-order R that is

$$\text{antisymmetric:} \qquad \forall (x, y) \in \mathcal{E}^2,\ (x\ R\ y \wedge y\ R\ x) \Rightarrow x = y$$

Definition 6 (partial order $\preceq$ on the markings)**.** Given a set of places P, the partial order $\preceq \subseteq \mathbb{N}^{|P|} \times \mathbb{N}^{|P|}$ is such that for all pair of markings $(\mathbf{m}_1, \mathbf{m}_2) \in \mathbb{N}^{|P|} \times \mathbb{N}^{|P|}$ we have that $\mathbf{m}_1 \preceq \mathbf{m}_2$ if and only if for all place $p \in P : \mathbf{m}_1(p) \leq \mathbf{m}_2(p)$.

$\mathbf{m} \prec \mathbf{m}'$ denotes that $\mathbf{m} \preceq \mathbf{m}'$ and $\mathbf{m}' \not\preceq \mathbf{m}$.

Lemma 1 ([7])**.** *$\preceq$ is a well quasi-order.*

The following result will be useful in the sequel.

Lemma 2 ([4])**.** *The PN model is* strongly monotonic *with regard to $\preceq$. That is, for all PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$, for all transition $t \in T$ and for all markings $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ of $\mathcal{N}$ such that $\mathbf{m}_1 \preceq \mathbf{m}_2$ and $\mathbf{m}_1 \xrightarrow{t} \mathbf{m}_3$, there exists a marking $\mathbf{m}_4$ of $\mathcal{N}$ such that $\mathbf{m}_2 \xrightarrow{t} \mathbf{m}_4$ and $\mathbf{m}_3 \preceq \mathbf{m}_4$.*

In this work we will focus on an extension of the PN model, the parametric Petri net (PPN) model, that is extended thanks to the use of parameters as input and output weights.

Definition 7 (parametric Petri net [6])**.** A parametric Petri net (PPN) $\mathcal{S} = \langle P, T, \mathbb{P} \rangle$ is a weighted oriented bipartite graph with a finite set $\mathbb{P}$ of parameters. The two subsets of vertices are:

- P: a finite set of places,
- T: a finite set of transitions,

For each transition $t \in T$ are defined the following functions:

- $I_t : P \mapsto \mathbb{N} \cup \mathbb{P}$ associates to each place the weight of the edge to t *(input weight)*,
- $O_t : P \mapsto \mathbb{N} \cup \mathbb{P}$ associates to each place the weight of the edge from t *(output weight)*.

As for plain PNs, this is denoted $t = \langle I_t, O_t \rangle$.

Definition 8 (parametric marking)**.** Given a set of place P, a parametric marking of P is a function $\mathbf{m} : P \mapsto \mathbb{N} \cup \mathbb{P}$ that associates $\mathbf{m}(p)$ tokens to each place $p \in P$.

A marking of a PN $\mathcal{N} = \langle P, T \rangle$ is a marking of P. A marking of a PPN $\mathcal{S} = \langle P, T, \mathbb{P} \rangle$ is a *parametric* marking of P. Note that a marking $\mathbf{m}$ is a parametric marking where $\mathbf{m}(p) \in \mathbb{N}$ for all $p \in P$.

More generally, given a set

Definition 9 (initialized (parametric) PN)**.** An initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ (resp. PPN $\mathcal{S} = \langle P, T, \mathbb{P}, \mathbf{m}_0 \rangle$) is a PN (resp. PPN) with an initial marking $\mathbf{m}_0$.

This is sometimes called a *marked (parametric) PN*. We will often refer to an initialized (parametric) PN loosely as a (parametric) PN.

The figure 1.1 shows an example of PPN whose $\mathbb{P} = \{a, b\}$ and with an initial marking $\mathbf{m}_0$ such that $\mathbf{m}_0(p_1) = 1$, $\mathbf{m}_0(p_2) = a$ and $\mathbf{m}_0(p_3) = 0$. The circles represent the places, the rectangles are the transitions, and the dots are the tokens. If the number of token at a given place is parametric (*i.e.* depends on a parameter of $\mathbb{P}$), it is written inside the circle. An arrow from a place p and to a transition t denotes that $I_t(p) = 1$. The absence of an arrow from p to t indicates that $I_t(p) = 0$. If $I_t(p) \notin \{0, 1\}$, a label with the value of $I_t(p)$ is added to the arrow. Symmetrically, the arrows from the transitions to the places indicate the output weights.

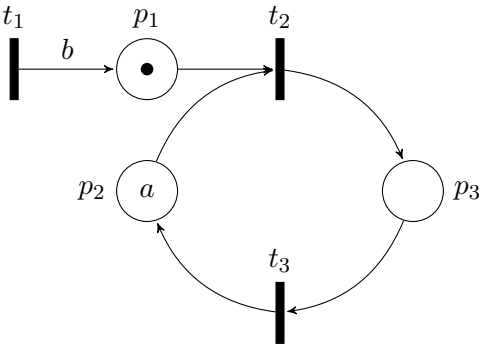


Figure 1.1: An initialized PPN

We usually set an order on the places. This allows to view the markings as vectors (here, $\mathbf{m}_0$ is the column vector $(1, a, 0)^T$, where $\cdot^T$ is the transpose operator) as well as the I and O functions. Likewise, we define an order on the transitions. Therefore, I_t and O_t denote respectively the I and O functions defined for the t^{th} transition (here, $I_1 = (0, 0, 0)^T$ and $O_1 = (b, 0, 0)^T$). Given a PPN $\mathcal{S} = \langle P, T, \mathbb{P} \rangle$, the backward and forward incidence matrices $\mathbf{I}_\mathcal{S} \in (\mathbb{N} \cup \mathbb{P})^{|P| \times |T|}$ and $\mathbf{O}_\mathcal{S} \in (\mathbb{N} \cup \mathbb{P})^{|P| \times |T|}$ are naturally defined by $\mathbf{I}_\mathcal{S}(p, t) = I_t(p)$ and $\mathbf{O}_\mathcal{S}(p, t) = O_t(p)$. ($\mathcal{S}$ is omitted when it is obvious from the context.) This allows to use linear algebra to analyse PNs.

$$\mathbf{I} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \mathbf{O} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 1.2: The incidence matrices of the PN from figure 1.1

1.1.1 Operational semantic of PNs

Given a PN $\mathcal{N} = \langle P, T \rangle$ and a marking $\mathbf{m}$ on $\mathcal{N}$, a transition $t \in T$ is said *enabled* by $\mathbf{m}$ if $\forall p \in P : \mathbf{m}(p) \geq I_t(p)$. An enabled transition can be *fired* to produce a new marking $\mathbf{m}'$ such that $\forall p \in P : \mathbf{m}'(p) = \mathbf{m}(p) - I_t(p) + O_t(p)$. This is denoted by $\mathbf{m} \xrightarrow{t} \mathbf{m}'$. It is important to note that the effect of a transition is to add or remove a constant number of tokens at each place and does not depend on the marking from which it is fired. A PN transition is said to have a *constant effect*.

Here are some additional notations:

- $\mathbf{m} \to \mathbf{m}'$ denotes that there exists $t \in T$ such that $\mathbf{m} \xrightarrow{t} \mathbf{m}'$.
- $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$ where σ is a sequence of transitions $\sigma = (t_1, \ldots, t_{n-1}), t_i \in T, i \in \{1, \ldots, n-1\}$ denotes that there exists a sequence of markings $\mathbf{m}_1, \ldots, \mathbf{m}_n$ such that : $\mathbf{m} = \mathbf{m}_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{n-1}} \mathbf{m}_n = \mathbf{m}'$.
- $\mathbf{m} \xrightarrow{*} \mathbf{m}'$ denotes that there exists a sequence of transition σ such that $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$. Note that the $\xrightarrow{*}$ relation is the reflexive and transitive closure of the relation $\to$.

Definition 10. Given an PN $\mathcal{N} = \langle P, T \rangle$ and a marking $\mathbf{m}$ of $\mathcal{N}$:

- $\mathsf{Post}(\mathbf{m}) = \{\mathbf{m}' \mid \mathbf{m} \to \mathbf{m}'\}$ is the set of one-step successors of $\mathbf{m}$,
- $\mathsf{Pre}(\mathbf{m}) = \{\mathbf{m}' \mid \mathbf{m}' \to \mathbf{m}\}$ is the set of one-step predecessors of $\mathbf{m}$,
- $\mathsf{Post}^*(\mathbf{m}) = \{\mathbf{m}' \mid \mathbf{m} \xrightarrow{*} \mathbf{m}'\}$ is the set of successors of $\mathbf{m}$, in any number of step. With $\mathbf{m}_0$ the initial marking of $\mathcal{N}$, $\mathsf{Post}^*(\mathbf{m}_0)$ is the *reachability set* of $\mathcal{N}$.
- $\mathsf{Pre}^*(\mathbf{m}) = \{\mathbf{m}' \mid \mathbf{m}' \xrightarrow{*} \mathbf{m}\}$ is the set of predecessors of $\mathbf{m}$, in any number of step.

These operators are naturally extended to sets of markings as the union of the sets obtained by applying the operator on each marking of the sets. That is, with M a set of markings of $\mathcal{N}$, $\mathsf{Post}(M) = \{\mathbf{m}' \mid \exists \mathbf{m} \in M : \mathbf{m} \to \mathbf{m}'\}$.

For example, regarding the PPN shown on figure 1.1, $\mathsf{Post}((0, 1, 0)) = \{(b, 1, 0)\}$ and $\mathsf{Post}^*((0, 1, 0)) = \{(i, 1, 0) \mid i \in \mathbb{N}\} \cup \{(i, 0, 1) \mid i \in \mathbb{N}\}$.

All of this applies to PPN through valuations of the parameters:

Definition 11 (Instantiation of PPNs)**.** Let $\mathcal{S} = \langle P, T, \mathbb{P}, \mathbf{m}_0 \rangle$ be a PPN and $v : \mathbb{P} \mapsto \mathbb{N}$ be a $\mathbb{N}$-*valuation*, or simply valuation, on $\mathbb{P}$. Then $v(\mathcal{S})$ is defined as the PN obtained by replacing each parameter $a \in \mathbb{P}$ by $v(a)$. Thus, we have $v(\mathcal{S}) = \langle P, T, \mathbf{m'}_0 \rangle$ such that :

- $\mathbf{I}_{v(\mathcal{S})}(p,t) = \begin{cases} \mathbf{I}_{\mathcal{S}}(p,t) & \text{if } \mathbf{I}_{\mathcal{S}}(p,t) \in \mathbb{N} \\ v(\mathbf{I}_{\mathcal{S}}(p,t)) & \text{if } \mathbf{I}_{\mathcal{S}}(p,t) \in \mathbb{P} \end{cases}$
- $\mathbf{O}_{v(\mathcal{S})}(p,t) = \begin{cases} \mathbf{O}_{\mathcal{S}}(p,t) & \text{if } \mathbf{O}_{\mathcal{S}}(p,t) \in \mathbb{N} \\ v(\mathbf{O}_{\mathcal{S}}(p,t)) & \text{if } \mathbf{O}_{\mathcal{S}}(p,t) \in \mathbb{P} \end{cases}$
- $\mathbf{m'}_0(p) = \begin{cases} \mathbf{m}_0(p) & \text{if } \mathbf{m}_0(p) \in \mathbb{N} \\ v(\mathbf{m}_0(p)) & \text{if } \mathbf{m}_0(p) \in \mathbb{P} \end{cases}$

Given $\mathcal{S}$ a PPN and a valuation v, one can thus instantiate a PN $v(\mathcal{S})$ from $\mathcal{S}$ and apply the semantic described above. When the PPN under consideration is clear from the context, $\mathbf{I}_v$ is used to denote $\mathbf{I}_{v(\mathcal{S})}$ and $\mathbf{O}_v$ to denote $\mathbf{O}_{v(\mathcal{S})}$. We write $\xrightarrow{t}_v$, $\rightarrow_v$, $\xrightarrow{\sigma}_v$, $\xrightarrow{*}_v$, Post_v, Pre_v, Post^*_v and Pre^*_v to denote $\xrightarrow{t}$, $\rightarrow$, $\xrightarrow{\sigma}$, $\xrightarrow{*}$, Post, Pre, Post^* and Pre^* on the plain PN $v(\mathcal{S})$.

This makes it possible to formally represent a system and interactions between its components. We will now define some properties that the model may have and that are usually of interest to show that the modelled system meets some requirements.

1.1.2 Behavioural properties of PNs

The markings basically indicate the state of the system. Thus, knowing if an initialized PN may reach a given marking, that represents for example a bad state, is essential to check properties of the modelled system. This is the *reachability problem*.

Definition 12 (Reachability)**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ and a marking $\mathbf{m}$ of $\mathcal{N}$, $\mathbf{m}$ is said reachable if $\mathbf{m}_0 \xrightarrow{*} \mathbf{m}$.

However, the verification of safety properties are more often reduced to a *coverability problem*, that is essentially asking if a PN can reach or exceed a given marking.

Definition 13 (Coverability)**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ and a marking $\mathbf{m}$ of $\mathcal{N}$, $\mathbf{m}$ is said coverable if there exists a marking $\mathbf{m'}$ such that $\mathbf{m} \preceq \mathbf{m'}$ and $\mathbf{m}_0 \xrightarrow{*} \mathbf{m'}$.

A set of markings is said coverable whenever one of them is coverable.

Definition 14 (Coverability problem)**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ and a set M of markings of $\mathcal{N}$, determine whether $\exists \mathbf{m} \in M$ and $\mathbf{m'} \in \mathsf{Post}^*(\mathbf{m}_0)$ such that $\mathbf{m} \preceq \mathbf{m'}$.

The coverability problem for a marking $\mathbf{m}$ is the coverability problem for the singleton $\{\mathbf{m}\}$.

The behaviour of a PPN is defined by the behaviours of all the PNs that can be obtained by a valuation of its parameters. So, for an initialized PPN $\mathcal{S}$, the coverability problem may be declined in an existential and an universal form. The existential coverability

problem ($\mathscr{E}$-cov) ask if there exists a valuation v such that $\mathbf{m}$ is coverable. The universal coverability problem ($\mathscr{U}$-cov) ask if $\mathbf{m}$ is coverable for all valuations v.

Definition 15 (Universal and existential coverability problems)**.** Given a PPN $\mathcal{S} = \langle P, T, \mathbb{P}, \mathbf{m}_0 \rangle$ and a set M of non-parametric markings of $\mathcal{S}$

- the *existential coverability problem* ask if there is a valuation v for $\mathbb{P}$ such that M is coverable,
- the *universal coverability problem* ask if M is coverable for all valuations of $\mathbb{P}$.

1.2 Motivations

1.2.1 Interests of PPNs

[TODO: sources and examples]

Today PNs are used in a wide range of areas. Most of the time they are used either to design a safe system, or to verify an existing one. These uses require that the system is complete. That is, for the design of a model, it must be entirely designed to be analyzable. On the other hand, when checking an existing system, if a desired property does not hold, the correction must be made "by hand".

With the introduction of parameters some variables unknown at the design stage can be integrated into the model without having to be set arbitrarily. Moreover, if during the verification a desired property turns out not to hold, it is possible to check if the change of parameters alone can solve the problem, or if the Petri net structure must be changed too. Going further, the use of parameters in the model can allow to determine the "safest values" for a system, or to synthesize the values that allow to respect a given strategy.

We can therefore say that parameters can simplify the *design* of a system. Indeed, since it is possible to keep unknown values, modelling can be done step by step, with the possibility to check the model at each step. In addition, the design can be partially automated by parameter synthesis. This approach gives a new interest for this model in fields as varied as chemistry, construction processes, financial loans...

There are also many advantages of using parameters when it comes to *verification*. For example, they allow to verify some properties simultaneously on many systems that differs only by parameters values.

1.2.2 Interest of the coverability problem in PPNs

[TODO: sources and examples]

As it provide evidence of safety properties on the studied systems, coverability problem is of primary interest for system design and verification. Therefore, for the reasons given in the previous section, it is worth being able to solve it efficiently on PPNs.

To give a more concrete intuition on the interest, consider a system that execute a *task* for others systems. At each instant (whatever an instant is), the system may receive

requests to perform the task from many other systems. We say that each request creates a *job*. We would like to have a system that is not too expensive to implement, but also capable of completing the tasks quickly enough. For this we make our system capable of performing a jobs at the same time, keeping a as low as possible to reduce costs. This system may be modeled as shown on the figure 1.1 with $\mathbf{m}_0 = (0, a, 0)$ as initial marking. p_1 represents the job queue and p_3 the execution unit.
We can now formally verify that, whatever the parameter values, the execution unit will not receive more than a jobs to perform at the same time, that is an instance of the $\mathscr{E}$-cov for the marking $(0, 0, a + 1)$. Indeed, it is easy to see that $\mathsf{Post}^*((0, a, 0)) = \{(i, j, k) \mid i, j \text{ and } k \in \mathbb{N}, j + k = a\}$. [TODO: maybe remove part on 'a as low as possible']

Before recalling the known result on PPN and plain PN that will be useful to our study, let us give a brief overview of some work already done on PPNs.

1.3 Previous works on parametrization of PNs

The use of parameters in formal verification systems is a well developed topic in the literature.

With regard to PN, parameters have been introduced with many different roles. Some works, like [5], use parameters as places or transitions, for example to make it possible to change a place into a more complex subnet and thus allow different levels of abstractions to be considered. In [16] parameters are used on the markings to obtain concise parametrised reachability trees, but not to realize formal verifications on these parametric systems.

[3] introduces parameters as the weight of arcs to model changes in a system. The parameters have a finite valuation domain and verifications are performed on these parametrized systems. Systems with quantitative parameters with infinite valuation domains are analysed in [2]. Similarly, [17] study PNs with parametric initial markings which represent sets of possible initial markings.

Close to the PPN model, ω-PNs ([13]) allow input and output weights to be ω. In this case, the transition consumes or produces a non-deterministic number of tokens. Note that in this model, the transitions does not have a constant effect anymore.

Our work is in the line with [6] which use discrete parameters as arc weights as well as in the markings. [6] provide also a proof for the non decidability of $\mathscr{U}$-cov and $\mathscr{E}$-cov, and define several subclasses of PPN for which these problems are decidable.

1.4 Overview of similar models

Many reactive systems are naturally modeled as infinite-state systems. Some infinite-states models are known to allow some automatic formal verification from model-checking techniques. Among them are PNs, but also Lossy FIFO systems, Broadcast protocols... [TODO: refs] All are well-structured transition systems (WSTSs) (but there exists other famous infinite-states systems, like timed-automata). [TODO: check whether timed-

automata are WSTSs.]

WSTSs are outside of the range of this work, but we will sometimes refer to them to distinguish between techniques specific to the PN model or usable in a wider range of contexts. [TODO: Check english writting.]

In a few words, WSTSs are transitions systems whose set of states are well-quasi-ordered and whose transition relations is monotonic with respect to the well quasi-ordering.

The monotonicity property differ from the strong monotonicity defined above by the fact that the second state may be found after many steps. Formally:

Definition 16 (Monotonicity)**.** A transition system is said *monotonic* whenever [TODO: or 'when'?] its transition relation is monotonic.

A transition relation $\rightarrow \subseteq (S \times S)$ over a $\leq$-well quasi-ordered set S is monotonic if, and only if, for all s_1, s_2, and s_3 from S such that $s_1 \leq s_2$ and $s_1 \rightarrow s_3$ there exists $s_4 \in S$ such that $s_2 \xrightarrow{*} s_4$.

Chapter 2

Preliminary results

2.1 Known results for coverability on plain PNs

We now present the results related to the coverability problem on the plain PNs which seem to us the most interesting.

To introduce these results, we need some additional definitions. They will be given for plain PNs, but most of them are naturally extended to PPNs.

A *coverability set* of an initialized PN is an over-approximation of the reachability set that is precise enough to solve the coverability problem, and is, therefore, interesting for our study. In order to define it formally, we need to know about the upward and downward closure of a (set of) marking(s).

Definition 17 (Upward- and downward-closure on markings)**.** Let $S \subseteq \mathbb{N}^{|P|}$ be a set of markings on the places P:

- The *upward-closure* of S, noted $\uparrow^{\preceq}(S)$, is the set $\{\mathbf{m} \in \mathbb{N}^{|P|} \mid \exists \mathbf{m'} \in S : \mathbf{m'} \preceq \mathbf{m}\}$,
- The *downward-closure* of S, noted $\downarrow^{\preceq}(S)$, is the set $\{\mathbf{m} \in \mathbb{N}^{|P|} \mid \exists \mathbf{m'} \in S : \mathbf{m} \preceq \mathbf{m'}\}$.

The closure of a marking $\mathbf{m}$ is the closure of the singleton $\{\mathbf{m}\}$.

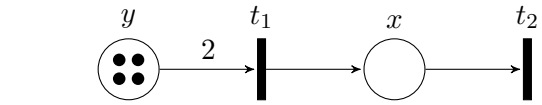
For instance, with $\mathbf{m} = (1, 2, 3)$, we have that its upward-closure is $\uparrow^{\preceq}(\mathbf{m}) = \{(i, j, k) \mid i \geq 1, j \geq 2, k \geq 3\}$.

Definition 18 (Upward- and downward-closed set of markings)**.** A set S of markings is said *upward-closed* if $S = \uparrow^{\preceq}(S)$. It is said *downward-closed* if $S = \downarrow^{\preceq}(S)$.

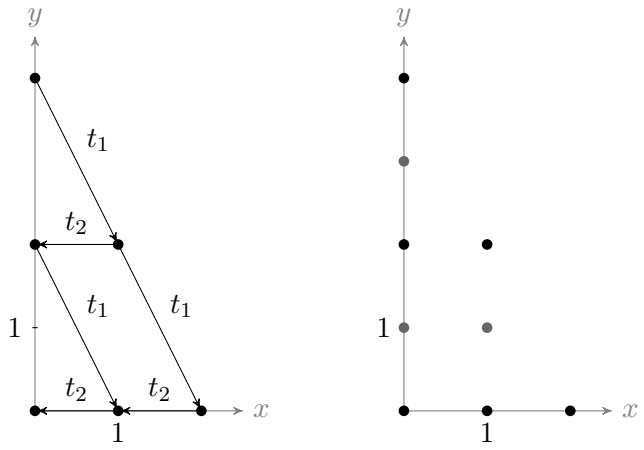
Definition 19 (Coverability set [8, 9])**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$, a *coverability set* S of $\mathcal{N}$ is a set of markings such that $\downarrow^{\preceq}(S) = \downarrow^{\preceq}(\mathsf{Post}^*(\mathbf{m}_0))$.

Obviously, the minimal coverability set is $\downarrow^{\preceq}(\mathsf{Post}^*(\mathbf{m}_0))$. It is called the *covering set* of $\mathcal{N}$ and is noted $\mathsf{Cover}(\mathcal{N})$.

Definition 20 (Covering set)**.** Let $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ be an initialized PN. The *covering set* S of $\mathcal{N}$, noted $\mathsf{Cover}(\mathcal{N})$, is the set $\{c \mid \exists c' \in \mathsf{Post}^*(\mathcal{N}) : c \leq c'\}$.



(a) A PN ($|P| = 2$)



(b) The reachable markings (c) The cover set

Figure 2.1: Reachability and minimal coverability sets

The figure 2.1a shows a marked PN with two places. One can therefore represents the markings as points on a plane. The figure 2.1b shows the reachable markings in the form of an accessibility graph. In 2.1c we see the minimal coverability set.

It worth notice that, in the context of PNs, the coverability problem for a set of markings S may be defined as follows:

Definition 21 (Coverability problem)**.** Given a PN $\mathcal{N}$ and an upward-closed set $U = \uparrow^{\preceq}(S)$ of markings of $\mathcal{N}$, detemine whether $\mathsf{Post}^*(\mathbf{m}_0) \cap U = \emptyset$.

Sometimes the number of token in a place is unbounded (*c.f.* the place boundedness problem). Therefore, the reachability and coverability sets are infinite. In a plain PN, this is due to the existence of an increasing self-covering sequence.

Definition 22 (Self-covering sequence)**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$, a self-covering sequence is a sequence of the form:

$$\mathbf{m}_0 \xrightarrow{\rho} \mathbf{m}_i \xrightarrow{\sigma} \mathbf{m}_j$$

with ρ and σ two sequences of transitions of T and with $\mathbf{m}_i \preceq \mathbf{m}_j$.

Note that, since $\mathbf{m}_i \preceq \mathbf{m}_j$, σ is firable from $\mathbf{m}_j$. In addition, the monotonicity of PNs ensures that, with $\mathbf{m'}_j$ given by $\mathbf{m}_j \xrightarrow{\sigma} \mathbf{m'}_j$, we have $\mathbf{m}_j \preceq \mathbf{m'}_j$. Thus, we see that it is a sufficient condition for the *non-termination* of the system (the system may be able to fire transitions infinitely often). In fact, because $\preceq$ is a well quasi-order (Lemma 1), one can find in any infinite sequence $\mathbf{m}_0 \to \mathbf{m}_1 \to \dots$ two markings $\mathbf{m}_i$ and $\mathbf{m}_j$ such that $\mathbf{m}_i \preceq \mathbf{m}_j$. Therefore, any infinite sequence is self-covering, and the existence of such a sequence is also necessary for the non-termination of the system.

Definition 23 (Increasing self-covering sequence)**.** Given an initialized PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$, an increasing self-covering sequence is a sequence of the form:

$$\mathbf{m}_0 \xrightarrow{\rho} \mathbf{m}_i \xrightarrow{\sigma} \mathbf{m}_j$$

with ρ and σ two sequences of transitions of T and with $\mathbf{m}_i \prec \mathbf{m}_j$.

Let $Q \subseteq P$ be the set of places $Q = \{q \in P \mid \mathbf{m}_i(q) < \mathbf{m}_j(q)\}$. $Q \neq \emptyset$ since $\mathbf{m}_i \prec \mathbf{m}_j$.

With a reasoning similar to the one above, we see that having such a sequence ensures that one can reach a marking $\mathbf{m'}_j$ given by $\mathbf{m}_j \xrightarrow{\sigma} \mathbf{m'}_j$ such that $\mathbf{m}_j \prec \mathbf{m'}_j$. Because of the constant effect of transitions, we know that $\forall q \in Q : \mathbf{m}_j(q) < \mathbf{m'}_j(q)$. The unboundedness of the places in Q follows. [TODO: A more formal proof that the existence of an increasing self-covering sequence is a necessary and sufficient condition for unboundedness on places is either to be done here or to be referenced.]

An ω-marking is a way to represent a set of markings which have the same number of tokens in some places, and may have any number of tokens, potentially an infinity, in the other places. They are useful to effectively represent potentially infinite downward-closed set, like a coverability set or the sets of markings of an increasing self-covering sequence.

Definition 24 (ω-marking)**.** We define ω to be such that: $\omega \notin \mathbb{N}$ and for all constant $c \in \mathbb{N}$:

- $c \leq \omega$
- $\omega + c = \omega$
- $\omega - c = \omega$

An ω-marking $\mathbf{m}$ over a set of places P is a function $\mathbf{m} : P \mapsto \mathbb{N} \cup \{\omega\}$ that associates $\mathbf{m}(p)$ tokens to each place $p \in P$.

With $\mathbb{P}$ a set of parameters, $\omega \notin \mathbb{P}$, *a parametric ω-marking* $\mathbf{m}$ over a set of places P is a function $\mathbf{m} : P \mapsto \mathbb{N} \cup \mathbb{P} \cup \{\omega\}$ that associates $\mathbf{m}(p)$ tokens to each place $p \in P$.

Note that an ω-marking $\mathbf{m}$ is a parametric ω-marking where $\mathbf{m}(p) \in \mathbb{N} \cup \{\omega\}$ for all places $p \in P$. Similarly, a parametric marking $\mathbf{m}$ is a parametric ω-marking where $\mathbf{m}(p) \neq \omega$ for all places $p \in P$. As for parametric markings, we often refer to a parametric ω-marking simply as ω-marking.

Having an ω-marking $\mathbf{m} \in \mathsf{Cover}(\mathcal{N})$ denotes that for all marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p) = \mathbf{m}(p) \forall p \in \{p \mid \mathbf{m}(p) \neq \omega\}$, there exists a marking $\mathbf{m}_2$ in the reachability set of $\mathcal{N}$ such that $\mathbf{m}_1 \prec \mathbf{m}_2$. Notice also that an ω-marking always stands for one and only one downward closed set. Symmetrically, any downward closed set may be represented by a finite set of ω-markings. [TODO: Indeed...] [TODO: See ... for a complete proof]

This allow to represent any infinite downward closed set of markings by a unique finite set of ω-markings.

An upward closed set is always infinite. It can be effectively represented through its unique set of minimal elements whose it is the upward closure [TODO: citep{someone}]. This is a direct consequence of the [TODO: someone] lemma: [TODO: Someone's lemma: every set of tuple has finitely many minimal elements.] Moreover this reprenstation is effective in the sense that the set may be manipulated through it. [TODO: cite{someone}]

give the way to perform the operations on upward closed sets through their minimal elements.

2.1.1 A general backward algorithm

There exists a simple way to solve the coverability problem for all the examples of WSTSs above mentioned. This algorithm was introduced by Abdulla *et al.* [1]. It is close of the one introduced earlier in [9]. Even if we do not see how to use it in the context of PPN, we mention it here because it helps to grasp, by comparison with the Karp and Miller algorithm presented in the following section, where does lie the difficulties of the coverability problem.

Relying on the definition 21 of the coverability problem, given an upward-closed set of markings U, the algorithm computes $\mathsf{Pre}^*(U)$ by iterating the Pre operator. The termination and correction of the algorithm were proven in [TODO: cite{someone}]. The termination is ensured by the existence of a fixed point in the sequence of upward closed set of markings $(R_i)_{i\geq 0}$:

$$R_0 = U$$
$$\forall i > 0 : R_i = R_{i-1} \cup \mathsf{Pre}(R_{i-1})$$

When this fixed point is reached, we have $\mathsf{Pre}^*(U)$. If $\mathbf{m}_0 \in \mathsf{Pre}^*(U)$, one can conclude positively. The result is negative otherwise.

This approach is elegant and general, but often inefficient in practice. It is well known that a forward exploration of the state of space (*i.e.*, in this context, using Post instead of Pre) is usually more efficient [TODO: citep{someone}]. We now present a forward algorithm, but whose the application is restricted to PN.

2.1.2 The Karp and Miller algorithm

Although it was not originally designed for this purpose, the Karp and Miller algorithm [15] is a classical algorithm to compute a coverability set of an initialized PN. More precisely, it constructs a coverability tree and uses an acceleration function to systematically detect self-covering sequences, and thus ensures the termination.

Definition 25 (Coverability tree)**.** Given a PN $\mathcal{N} = \langle P, T, \mathbf{m}_0\rangle$, a coverability tree $\mathcal{T}$ of $\mathcal{N}$ is a labelled tree $\mathcal{T} = \langle N, B, n_0, \Lambda\rangle$ where:

- N is the set of nodes of the tree.
- $n_0 \in N$ is the root of the tree, *i.e.* $\nexists n \in N$ such that $(n, n_0) \in B$.
- $\Lambda : N \mapsto (\mathbb{N} \cup \{\omega\})^{|P|}$ is a labelling function that associate to each node a ω-marking of $\mathcal{N}$.
- $B \subseteq N \times N$, the set of edges, is such that:
  - with $(n_1, n_2) \in N^2$, if there exists an edge $(n_1, n_2) \in B$ then there exists a sequence σ of transitions of T such that $\Lambda(n_1) \xrightarrow{\sigma} \Lambda(n_2)$,
  - for all node $n \in N \setminus \{n_0\}$, there exists a path from the root to n, that is, there exists a sequence of edges of B of the form $((n_0, n_1), (n_1, n_2), \ldots, (n_i, n))$, $i \geq 1$, and

– there is no cycles, that is, there is no sequences of edges of B of the form $((n_1, n_2), (n_2, n_3), \ldots, (n_i, n_1))$.

and such that $\downarrow^{\preceq}(\{\Lambda(n) \mid n \in N\}) = \mathsf{Cover}(\mathcal{N})$.

To keep N finite, the Karp and Miller algorithm exploits the strong monotonicity of PNs to introduce ω-markings through an *acceleration function* $\mathsf{Acceleration}$. This function takes a marking $\mathbf{m}$ to accelerate and a set of markings S as a base [TODO: ?] for the acceleration and returns a marking $\mathbf{m}_\omega$ such that:

$$\mathbf{m}_\omega(p) = \begin{cases} \omega & \text{if } \exists \mathbf{m}' \in S : \mathbf{m}' \prec \mathbf{m} \text{ and } \mathbf{m}'(p) < \mathbf{m}(p) \\ \mathbf{m}(p) & \text{otherwise} \end{cases}$$

We denote by $\mathscr{T}_n$ the path in the tree from the root to n.

The algorithm constructs the tree $\mathcal{T}$ as follows: The root n_0 of the tree is labelled with $\mathbf{m}_0$. A frontier F is defined to be the set of unprocessed nodes of the tree and is initialised to $\{n_0\}$. Then, while F is non empty, a node n is chosen from F to be processed: it is removed from F and, if there is no node n' in $\mathscr{T}_n$ such that $\Lambda(n) = \Lambda(n')$, for all ω-marking in $\{\mathsf{Acceleration}(\mathbf{m}, \mathscr{T}_n) \mid \mathbf{m} \in \mathsf{Post}(\Lambda(n))\}$, a node labelled with $\mathsf{Acceleration}(\mathbf{m}, \mathscr{T}_n)$ is added to the frontier and to the tree as a child of n.

The correctness and the termination of the algorithm lies on the strong monotonicity of PNs, and was proved by Karp and Miller in their work [15].

The Karp and Miller tree has a lot of convenient properties, and allows to answer coverability problem. [TODO: as well as ...] Furthermore, the Karp and Miller algorithm can easily be adapted to some parametric problems [6], as we will show in [TODO: section ?].

However, this tree, although finite, is often much larger than the minimal coverability set, and cannot be constructed in reasonable time. As a consequence, many improvement were proposed, as well as other algorithms with different approaches.

2.1.3 An efficient computation method of the coverability set of Petri nets

In [12, 14], Geeraerts proposes another approach to the computation of the coverability set. It is not based on the Karp and Miller algorithm and is not an alternative to it in the sense that it does not allow to answer the same set of questions than the Karp and Miller tree answers. However, this technique solve the coverability problem more efficiently in practice.

As in the Karp and Miller algorithm, an acceleration function exploits the strong monotonicity of PNs to allow termination. But here, the acceleration of a marking is performed with only one marking as the base (instead of a set of marking).

To choose the base to use, the algorithm works on pair of ω-markings. These pairs allow to record a relationship between the markings. More precisely, the algorithm constructs a pair of ω-markings $(\mathbf{m}_1, \mathbf{m}_2)$ only if $\downarrow^{\preceq}(\mathbf{m}_2) \subseteq \downarrow^{\preceq}(\mathsf{Post}^*(\mathbf{m}_1))$. This relationship is of interest because it ensures that there exists a sequence σ of transitions such that $\mathbf{m}_1 \xrightarrow{\sigma} \mathbf{m}_2$. If, in addition, $\mathbf{m}_1 \prec \mathbf{m}_2$, we know than an acceleration may occur.

To reduce the size of the set of pairs of ω-markings, only the pairs where the difference (as defined below) between $\mathbf{m}_1$ and $\mathbf{m}_2$ is maximal are kept. This will be the purpose of the order $\sqsubseteq$ we will define and it is justified by the intuitive idea that two more distant markings produce larger accelerations. Therefore, if the algorithm builds a pair $(\mathbf{m}_1, \mathbf{m}_2)$, it can forget about any other ($\sqsubseteq$-comparable) pair whose the elements are closer because

- by monotonicity, all potential successor of the elements of this pair will be covered by successor of $\mathbf{m}_1$ or $\mathbf{m}_2$, and
- any acceleration that can be created from this pair is covered by an acceleration one can build from $(\mathbf{m}_1, \mathbf{m}_2)$.

To describe the algorithm more formally, we will need the following definitions:

Given a pair of ω-markings $(\mathbf{m}_1, \mathbf{m}_2)$, we define:

- $\overline{\mathsf{Post}}((\mathbf{m}_1, \mathbf{m}_2)) = \{(\mathbf{m}_1, \mathbf{m}'), (\mathbf{m}_2, \mathbf{m}') \mid \mathbf{m}' \in \mathsf{Post}(\mathbf{m}_2)\}$,
- with $\mathbf{m}_1 \prec \mathbf{m}_2$, $\overline{\mathsf{Accel}}(\mathbf{m}_1, \mathbf{m}_2) = \{(\mathbf{m}_2, \mathsf{Acceleration}(\{\mathbf{m}_1\}, \mathbf{m}_2))\}$. $\overline{\mathsf{Accel}}(\mathbf{m}_1, \mathbf{m}_2)$ is not defined whenever $\mathbf{m}_1 \not\prec \mathbf{m}_2$,

With R a set of pair of markings, we define:

- $\overline{\mathsf{Post}}(R) = \bigcup_{(\mathbf{m}_1, \mathbf{m}_2) \in R} \overline{\mathsf{Post}}((\mathbf{m}_1, \mathbf{m}_2))$
- $\overline{\mathsf{Accel}}(R) = \bigcup^{\mathbf{m}_1 \prec \mathbf{m}_2}_{(\mathbf{m}_1, \mathbf{m}_2) \in R} \overline{\mathsf{Accel}}((\mathbf{m}_1, \mathbf{m}_2))$
- $\mathsf{Flatten}(R) = \{\mathbf{m} \mid \exists \mathbf{m}' : (\mathbf{m}', \mathbf{m}) \in R\}$

The efficient computation of the coverability set of the marked PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ lies on the sequence $\mathsf{CovSeq}(\mathcal{N}) = (V_i)_{i \geq 0}$ of pair of ω-markings, where, for all marked PN $\mathcal{N}$ we have:

$$V_0 = \{(\mathbf{m}_0, \mathbf{m}_0)\} \text{ and}$$
$$\forall i \geq 1 : V_i = V_{i-1} \cup \overline{\mathsf{Post}}(V_{i-1}) \cup \overline{\mathsf{Accel}}(V_{i-1})$$

One can shows that, first, for all node n of the Karp and Miller tree, there exists a value $k \geq 0$ of i such that $\Lambda(n) \in \mathsf{Flatten}(V_k)$, second, all the markings produced by $\overline{\mathsf{Post}}$ and $\overline{\mathsf{Accel}}$ are in the coverability set of $\mathcal{N}$. [TODO: Indeed...]

These two results lead us to the following lemma:

Lemma 3 ([14]). *Given a marked PN $\mathcal{N}$ such that $\mathsf{CovSeq}(\mathcal{N}) = (V_i)_{i \geq 0}$, there exists $k \geq 0$ such that for all $l \in \{0, ..., k-1\}$ we have that $\downarrow^{\preceq}(\mathsf{Flatten}(V_l)) \subset \downarrow^{\preceq}(\mathsf{Flatten}(V_{l+1}))$ and for all $l \geq k : \downarrow^{\preceq}(\mathsf{Flatten}(V_l)) = \mathsf{Cover}(\mathcal{N})$.*

Thus, the algorithm idea is to compute CovSeq until it stabilizes, *i.e.* to the lowest l such that $\downarrow^{\preceq}(\mathsf{Flatten}(V_l)) = \downarrow^{\preceq}(\mathsf{Flatten}(V_{l-1}))$ and to return $\downarrow^{\preceq}(\mathsf{Flatten}(V_l))$.

To perform it efficiently, one can use a well-chosen order $\sqsubseteq$ on the pair of markings to keep only the highest pairs with respect to this order. Intuitively, they are the pair with the more distant elements. Let us denote by $\ominus$ the componentwise difference between two markings and to extend it to ω-markings. Formally, given two ω-markings $\mathbf{m}_1$ and

$\mathbf{m}_2$ on a set of places P, $(\mathbf{m}_1 \ominus \mathbf{m}_2)(p)$ is defined for all $p \in P$ as:

$$\begin{cases} \omega & \text{whenever } \mathbf{m}_1(p) = \omega \\ -\omega & \text{whenever } \mathbf{m}_2(p) = \omega \text{ and } \mathbf{m}_1(p) \neq \omega \\ \mathbf{m}_1(p) - \mathbf{m}_2(p) & \text{otherwise} \end{cases}$$

Now we can define $\sqsubseteq$. Given two pairs $(\mathbf{m}_1, \mathbf{m}_2)$ and $(\mathbf{m'}_1, \mathbf{m'}_2)$ of ω-markings over a set of places P:

$$(\mathbf{m}_1, \mathbf{m}_2) \sqsubseteq (\mathbf{m'}_1, \mathbf{m'}_2) \Leftrightarrow \begin{cases} & \mathbf{m}_1 \preceq \mathbf{m'}_1 \\ \wedge & \mathbf{m}_2 \preceq \mathbf{m'}_2 \\ \wedge & \forall p \in P : (\mathbf{m}_2 \ominus \mathbf{m}_1)(p) \leq (\mathbf{m'}_2 \ominus \mathbf{m'}_1)(p) \end{cases}$$

For a set of pair of ω-markings R, $\mathsf{Max}^{\sqsubseteq}(R) = \{r \in R \mid \nexists r' \in R, r \sqsubseteq r'\}$ is the set of highest ω-marking of R with respect to $\sqsubseteq$.

This order has properties [14] that allows to keep the sets of markings of CovSeq small. Thus, one can compute $\mathsf{Cover}(\mathcal{N})$ of a PN $\mathcal{N} = \langle P, T, \mathbf{m}_0 \rangle$ by computing the sequence $(V_i)_{i \geq 0}$ defined below until $\downarrow^{\preceq}(\mathsf{Flatten}(V_i)) = \downarrow^{\preceq}(\mathsf{Flatten}(V_{i-1}))$.

$$V_0 = \{(\mathbf{m}_0, \mathbf{m}_0)\} \text{ and}$$
$$\forall i \geq 1 : V_i = \mathsf{Max}^{\sqsubseteq}(V_{i-1} \cup \mathsf{P\overline{ost}}(V_{i-1}) \cup \mathsf{A\overline{ccel}}(V_{i-1}))$$

At the end, we have that $\downarrow^{\preceq}(\mathsf{Flatten}(V_i)) = \mathsf{Cover}(\mathcal{N})$.

The correction and termination of the algorithm as well as useful properties of $\sqsubseteq$ are presented in [14, 10].

2.1.4 The Expand, Enlarge and Check (EEC) algorithm

EEC, introduced in [12, 11], is an iterative algorithm that allows, among other, to solve the coverability problem for PN. We present it restricted to this context, but it may be used for a wide range of well structured transitions systems, which PNs is part of, because it relies only on the monotonicity, and not on the strong monotonicity, of these models.

The idea is to compute and refine simultaneously an over- and an under-approximation of the covering set of the PN until one or the other allows to conclude.

The under-approximation is computed as follows: We define $(C_i)_{i \geq 0}$ to be the sequence of finite set of markings holding no more than i tokens in each places (plus $\mathbf{m}_0$):

$$\forall i \in \mathbb{N} : C_i = \{0, ..., i\}^{|P|} \cup \{\mathbf{m}_0\}$$

At step i, the algorithm computes $\mathsf{Sous}(\mathcal{N}, C_i)$ defined as the graph $\langle C_i, \mathbf{m}_0, \overset{\mathsf{Sous}}{\Longrightarrow} \rangle$ which is the transition system induced by the PN $\mathcal{N}$ restricted to the markings of C_i, *i.e.* $(\mathbf{m}_1, \mathbf{m}_2) \in \overset{\mathsf{Sous}}{\Longrightarrow}$ if, and only if, $\mathbf{m}_1 \to \mathbf{m}_2$. The under-approximation sought is the set of markings reachable through $\overset{\mathsf{Sous}}{\Longrightarrow}$ from $\mathbf{m}_0$ and is denoted $\mathcal{R}(\mathsf{Sous}(\mathcal{N}, C_i))$.

At step i, the algorithm also uses L_i from the sequence $(L_i)_{i\geq 0}$ of finite set of ω-markings such that $L_i = \{0, ..., i, \omega\}^{|P|} \cup \{\mathbf{m}_0\}$. That is, L_i contains all the markings with at most i tokens in any place, or ω (plus $\mathbf{m}_0$). This set is used to constructs the graph $\mathsf{Sur}(\mathcal{N}, C_i)$ defined as the graph $\langle L_i, \mathbf{m}_0, \overset{\mathsf{Sur}}{\Longrightarrow}\rangle$ where $(\mathbf{m}_1, \mathbf{m}_2) \in \overset{\mathsf{Sur}}{\Longrightarrow}$ if, and only if:

- either $\mathbf{m}_1 \to \mathbf{m}_2$,
- either $\mathbf{m}_1 \to \mathbf{m'}_2, \mathbf{m'}_2 \notin L_i, \mathbf{m'}_2 \preceq \mathbf{m}_2$, and $\nexists \mathbf{m''}_2 \in L_i$ such that $\mathbf{m'}_2 \prec \mathbf{m''}_2 \prec \mathbf{m}_2$.

In other words: if $\mathbf{m}_2 \notin L_i$, it is replaced by the lowest marking of L_i which over-approximate it. Note that this is an ω-marking which exists and is unique. [TODO: Indeed...] The over-approximation, then, is the set of markings of L_i reachable through $\overset{\mathsf{Sur}}{\Longrightarrow}$ from $\mathbf{m}_0$. It is denoted $\mathcal{R}(\mathsf{Sur}(\mathcal{N}, L_i))$.

We can say that they are under- and over-approximations thanks to the following lemmata:

Lemma 4 (Under-approximation [10])**.** *For all PN $\mathcal{N} = \langle P, T, \mathbf{m}_0\rangle$, for all upward-closed set $U \subseteq \mathbb{N}^{|P|}$, and for all $i \geq 0 : \mathcal{R}(\mathsf{Sous}(\mathcal{N}, C_i)) \cap U \neq \emptyset \Rightarrow \mathsf{Post}^*(\mathbf{m}_0) \cap U \neq \emptyset$.*

Lemma 5 (Over-approximation [10])**.** *For all PN $\mathcal{N} = \langle P, T, \mathbf{m}_0\rangle$, for all upward-closed set $U \subseteq \mathbb{N}^{|P|}$, and for all $i \geq 0 : \downarrow^{\preceq}(\mathcal{R}(\mathsf{Sur}(\mathcal{N}, L_i))) \cap U = \emptyset \Rightarrow \mathsf{Post}^*(\mathbf{m}_0) \cap U = \emptyset$.*

One can prove that one of the conditions mentioned in the lemmata will eventually happen. This ensures the termination of the algorithm.

Indeed, let S be the set of markings we wants to cover and let U be $\uparrow^{\preceq}(S)$. If U is reachable, we will eventually get a C_i that contains all the markings of a path from $\mathbf{m}_0$ to U. As this path will be present in $\mathsf{Sous}(\mathcal{N}, C_i)$, we will have that $\mathcal{R}(\mathsf{Sous}(\mathcal{N}, C_i)) \cap U \neq \emptyset$. Symmetrically, let j be such that L_j contains the maximal elements of the cover set of $\mathcal{N}$. Such a j exists and we have that $\downarrow^{\preceq}(\mathcal{R}(\mathsf{Sur}(\mathcal{N}, L_j))) = \mathsf{Cover}(\mathcal{N})$. Thus we know that $\downarrow^{\preceq}(\mathcal{R}(\mathsf{Sur}(\mathcal{N}, L_i))) \cap U = \emptyset$ will eventually happen for a $i \leq j$.

[REMOVED: A backward algorithm [9, 1]]

2.2 Known results on PPNs

By reduction from the halting problem as well as the counter boundedness problem, [6] has shown that $\mathscr{U}$-cov and $\mathscr{E}$-cov are undecidable on PPN. This motivates the introduction of two natural subclasses of PPNs where parametric coverability problems are decidable.

Namely, PreT-PPNs are PPNs where parameters are used only in **I**; PostT-PPNs are PPNs where parameters are used only in **O**. [6] provides an adaptation of the Karp and Miller Algorithm to solve the $\mathscr{U}$-cov problem on PreT-PPNs, and another to solve the $\mathscr{E}$-cov problem on PostT-PPNs. We will now present this two algorithms has they contain idea used in the sequel.

2.2.1 Karp and Miller algorithm for PostT-PPN

The adaptation of the Karp and Miller algorithm to solve the $\mathscr{E}$-cov problem on PreT-PPNs is the result of one key observation. In a PostT-PPN, a place may contains an arbitrary large number of tokens either because of the presence of a self-covering increasing sequence, as in a plain PN, or because of an arbitrary large valuation. In the latter case, the place is not necessary *unbounded*, that is to say, once a valuation is given, the number of tokens the place may contains may be bounded, even if the bound is arbitrary large due to the arbitrary large values of the valuation.

The adaptation of the Karp and Miller algorithm is therefore mainly the add of a way to distinguish between these two cases. It is done by introducing a new value allowed in the markings, noted $*$ and such that $* \notin \mathbb{N}$, $* \neq \omega$, and $\forall c \in \mathbb{N}$, we have:

- $c < * < \omega$,
- $* - n = *$,
- $* + n = *$,
- $\omega - * = \omega$, and
- $\omega + * = \omega$.

$*$ being defined, the propagation of the $*$ still have to be ensured in the cases where all the input places of a transition are marked by a $*$, even if the output of the transition is not a parameter. This is done by adapting the Acceleration function. This new Acceleration function, Acc, distinguishes three cases. Given a marking to accelerate $\mathbf{m}$ with the set S of markings as the base of the acceleration:

- Either there exists $\mathbf{m}' \in S$ such that $\mathbf{m}' \xrightarrow{\sigma} \mathbf{m}$, $\mathbf{m}' < \mathbf{m}$, and for all place p such that $\mathbf{m}'(p) \neq \omega$ we have $\mathsf{Effect}(\sigma)(p) \geq 0$.
  This case corresponds to the classic acceleration and the accelerated marking is given by:
  $$\mathsf{Acc}(\mathbf{m})(p) = \begin{cases} \omega & \text{if } \mathbf{m}'(p) < \mathbf{m}(p) \\ \mathbf{m}(p) & \text{otherwise} \end{cases}$$
- Or these first conditions does not hold but there exists a marking $\mathbf{m}' \in S$ such that $\mathbf{m}' \xrightarrow{\sigma} \mathbf{m}$, $\mathbf{m}' < \mathbf{m}$, and for all place p such that $\mathbf{m}'(p) \notin \{\omega, *\}$ we have $\mathsf{Effect}(\sigma)(p) \geq 0$.
  In this case the accelerated marking is given by:
  $$\mathsf{Acc}(\mathbf{m})(p) = \begin{cases} * & \text{if } \mathbf{m}'(p) < \mathbf{m}(p) \\ \mathbf{m}(p) & \text{otherwise} \end{cases}$$
  This case handle the propagation of the $*$ mentioned above.
- Or no one of the previous cases old. In this case the marking is not accelerated: $\mathsf{Acc}(\mathbf{m}) = \mathbf{m}$.

Intuitively, the second case formalise the idea that one can remove some tokens from a place with $*$ tokens without being an issue for the existence of a valuation that allows to cover a given marking.

Keeping in mind that we are looking for the existence of a valuation, as large as it may be, that allows to cover a (set of) marking(s) of the PPN $\mathcal{S}$, one can now perform the Karp and Miller algorithm, with the adapted acceleration function, on the plain PN $v(\mathcal{S})$ where v is the $*$-valuation that maps every parameter to $*$.

2.2.2 Karp and Miller algorithm for PreT-PPN

Here is a symmetrical situation. The parameters are used to indicate the number of tokens required for transitions to fire and we want to determine if a (set of) marking(s) is coverable for all the possible valuations of the parameters, as large as they may be. Therefore the adaptation of the Karp and Miller algorithm lies on considering that a transition with parametric input arcs is enabled if the corresponding places are marked by ω. This ensure that the transition is regarded as enabled only if it may actually fire whatever the valuation. The other direction of the implication holds too, that is to say that, if a transition has a parametric input arc whose the corresponding place is bounded, there exists a valuation that does not enable the transition. Indeed, recall that all simultaneous unbounded places in a PN appear marked by ω in at least one label of the Karp and Miller tree for this PN.

Chapter 3

Contributions

3.1 Adapting Geeraerts efficient computation method of the coverability set of Petri nets for PostT-PPNs

3.2 Adapting EEC for PreT-PPNs

In this section we provide an adaption of EEC to work with PPNs. It is inspired by the [TODO: KM^+] algorithm from [6].

The algorithm is the same as the one presented in the section 2.1.4, with the difference that the condition to add a marking to the over-approximation is more restrictive.

Conclusions

The conclusions are to be written with care, because it will be sometimes the part that could convince a potential reader to read the whole document.

Bibliography

[1] P. A. Abdulla, K. Cerans, B. Jonsson, and Yih-Kuen Tsay. General decidability theorems for infinite-state systems. In *Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science*, LICS '96, pages 313 – 321, Washington, DC, USA, 1996. IEEE Computer Society.

[2] Parosh Abdulla, Frédéric Haziza, and Lukáš Holík. All for the price of few: (parameterized verification through view abstraction). volume 7737, pages 476–495, 2013.

[3] Eric Badouel and Javier Oliver. Dynamic changes in concurrent systems: Modelling and verification. 1999.

[4] G. W. Brams. *Réseaux de Petri: théorie et pratique*. Number 1. 1983.

[5] Søren Christensen and Kjeld Høyer Mortensen. Parametrisation of coloured petri nets. *DAIMI Report Series*, 26(521), March 1997.

[6] Nicolas David. *Discrete Parameters in Petri Nets*. PhD thesis, Université de Nantes, Augustus 2017.

[7] Leonard Eugene Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413–422, 1913.

[8] Alain Finkel. A generalization of the procedure of karp and miller to well structured transition systems. In T. Ottmann, editor, *ICALP '87 Proceedings of the 14th International Colloquium, on Automata, Languages and Programming*, volume 267, pages 499 – 508. Springer-Verlag, 1987.

[9] Alain Finkel. Reduction and covering of infinite reachability trees. *Information and Computation*, 89(2):144 – 179, 1990.

[10] P. Ganty, G. Geeraerts, J.-F. Raskin, and L. Van Begin. Le problème de couverture pour les réseaux de petri: résultats classiques et développements récents. *Technique et Science Informatiques*, 28(9):1107 – 1142, 2009.

[11] G. Geeraerts, J.-F. Raskin, and L. Van Begin. Expand, enlarge and check: New algorithms for the coverability problem of wsts. *Journal of Computer and System Sciences*, 72(1):180 – 203, 2006.

[12] Gilles Geeraerts. *Coverability and Expressiveness Properties of Well-structured Transition Systems*. PhD thesis, Université Libre de Bruxelles, 2007.

[13] Gilles Geeraerts, Alexander Heussner, M Praveen, and Jean-François Raskin. ω-petri nets: Algorithms and complexity. 137:29–60, 01 2015.

[14] Gilles Geeraerts, Jean-François Raskin, and Laurent Van Begin. On the efficient computation of the minimal coverability set for petri nets. In *International Symposium on Automated Technology for Verification and Analysis*, pages 98–113. Springer Berlin Heidelberg, 2007.

[15] R. M. Karp and R. E. Miller. Parallel program schemata. *Journal of Computer and System Sciences*, 3(2):147 – 195, 1969.

[16] M. Lindqvist. *Parametrized Reachability Trees for Predicate / Transition Nets*, pages 351–372. Springer Berlin Heidelberg, Berlin, Heidelberg, 1991.

[17] Marco Ajmone Marsan, G. Balbo, Gianni Conte, S. Donatelli, and G. Franceschinis. *Modelling with Generalized Stochastic Petri Nets*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994.

[18] Carl Adam Petri. *Kommunikation mit Automaten*. PhD thesis, Universität Hamburg, 1962.

[19] Carl Adam Petri. *Communication with automata*. PhD thesis, Universität Hamburg, 1966.