CSC 226

Algorithms and Data Structures: II

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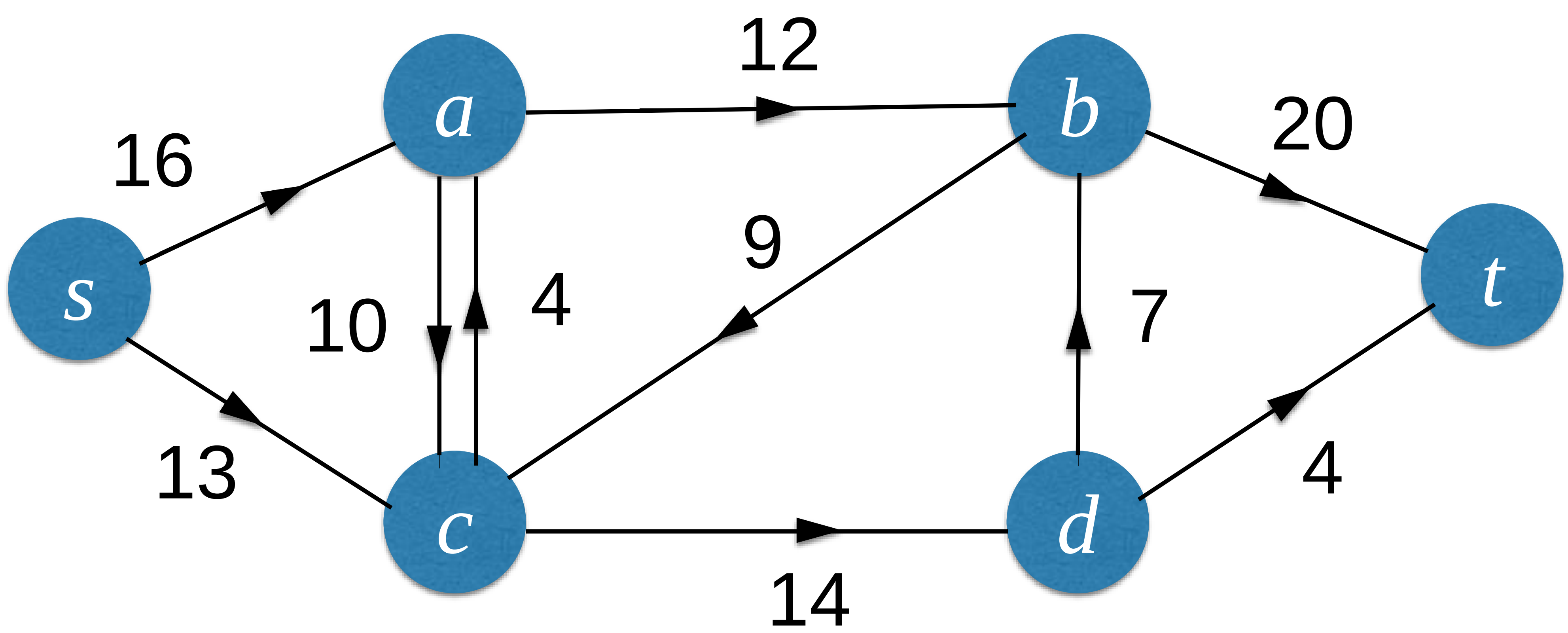
Network Flow

Example of an *st*-flow network



Network Flow (Definitions)

- A ***flow network*** is an edge-weighted, directed graph with positive edge weights, called **capacities**, denoted $c(e)$ for each edge e (capacities of non-existing edges are zero).

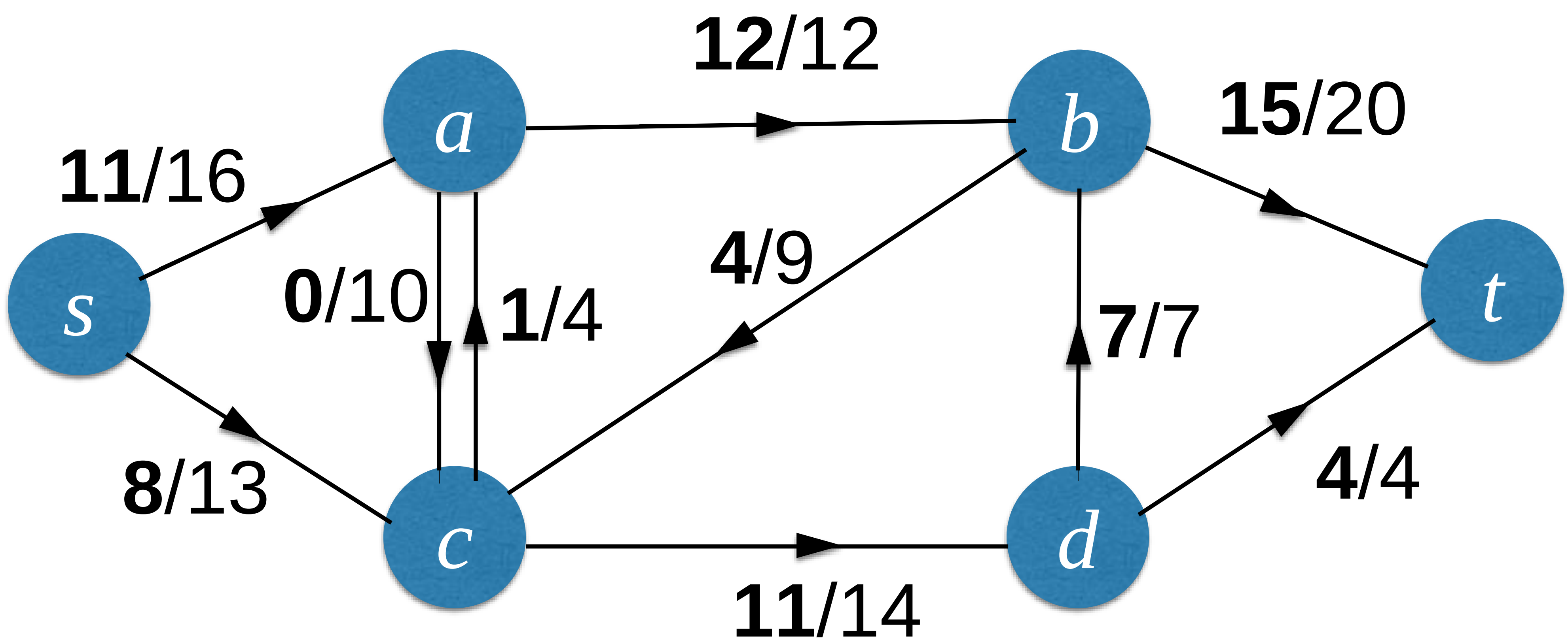
- An ***st-flow network***, N, is a flow network that has two identified vertices, namely the ***source***, s, and the ***sink***, t.

- An ***st-flow***, f, in an st-flow network, N, is a set of nonnegative values (*edge flows*, denoted $f(e)$) associated with each edge. Furthermore, we define
  - ***inflow***: total flow of edges into a specific vertex
  - ***outflow***: total flow of edges from a specific vertex
  - ***netflow***: inflow minus outflow of a specific vertex

Flow Network (Definitions)

- An st-flow, f, is *feasible* if it satisfies the conditions that
  - no edge's flow is greater than that edge's capacity
    - i.e. $0 \leq f(e) \leq c(e)$
  - the *netflow* of every vertex v (except s and t) in the st-flow network is zero:
    - i.e $netflow(v) = 0$ or $inflow(v) = outflow(v)$
- st-flow ***value***, $|f|$, for st-flow network, N, with st-flow, f, is the sink's inflow, (or the source's outflow.)
  - i.e. $|f| = inflow(t) = outflow(s)$
- ***Maximum*** *st-flow* (or ***maxflow***): a feasible st-flow with maximum st-flow value over all feasible flows

Example of a feasible *st*-flow in an *st*-flow network



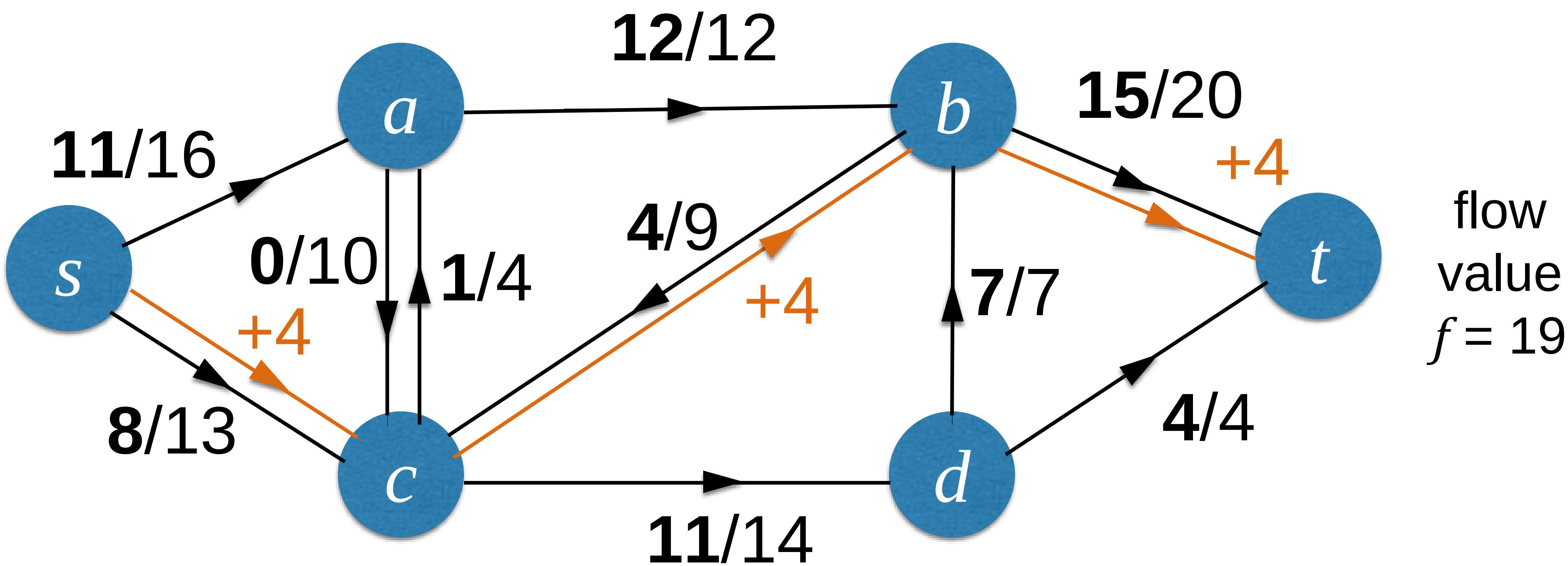
Maximum Flow Problem: *maxflow*

- Input: An *st*-flow network
- Output: A maximum *st*-flow

Key idea: Augmenting paths in *st*-flow networks

- An ***augmenting path*** in an *st*-flow network, with feasible *st*-flow, is an undirected path from source *s* to sink *t* along which we can push more flow, obtaining an *st*-flow with higher *st*-flow value.

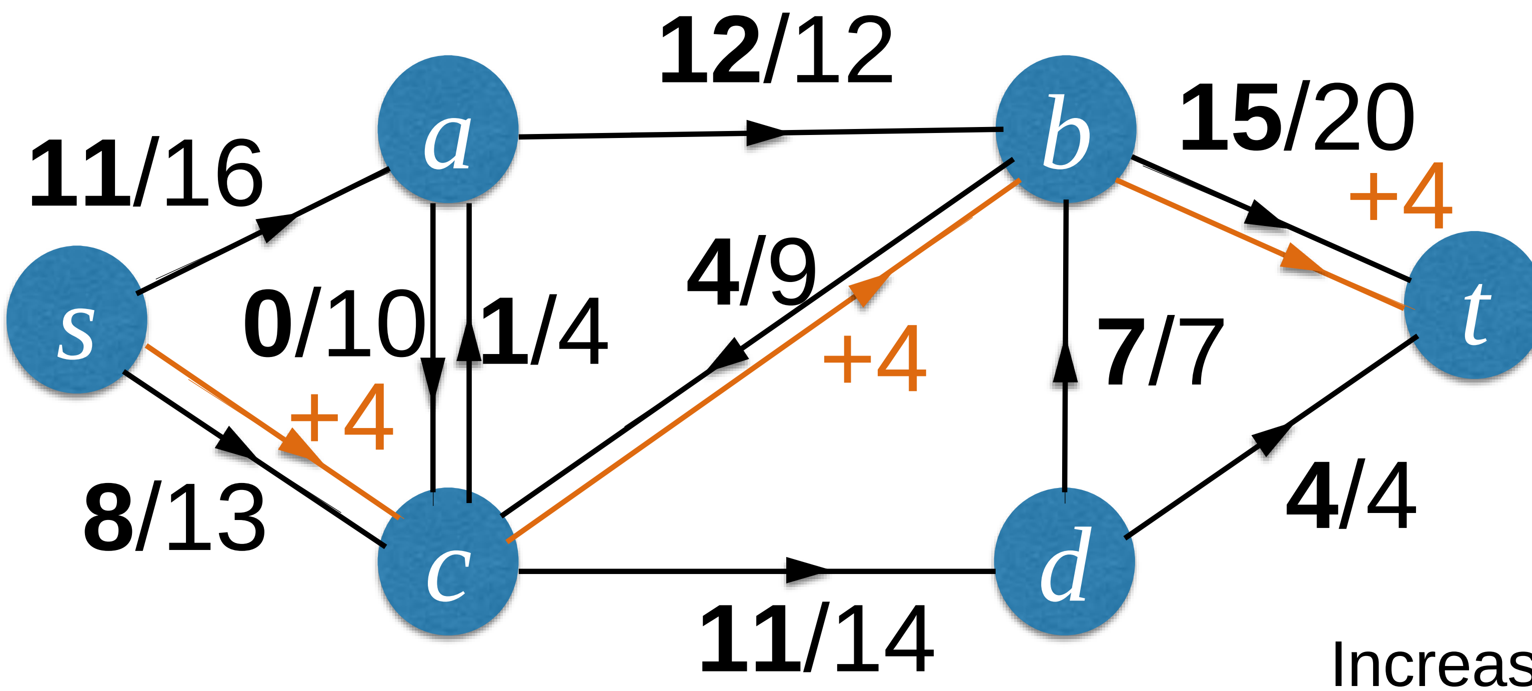
Example of an augmenting path that improves the flow: $scbt$



Arc bc is a *backward* arc on the path $scbt$.

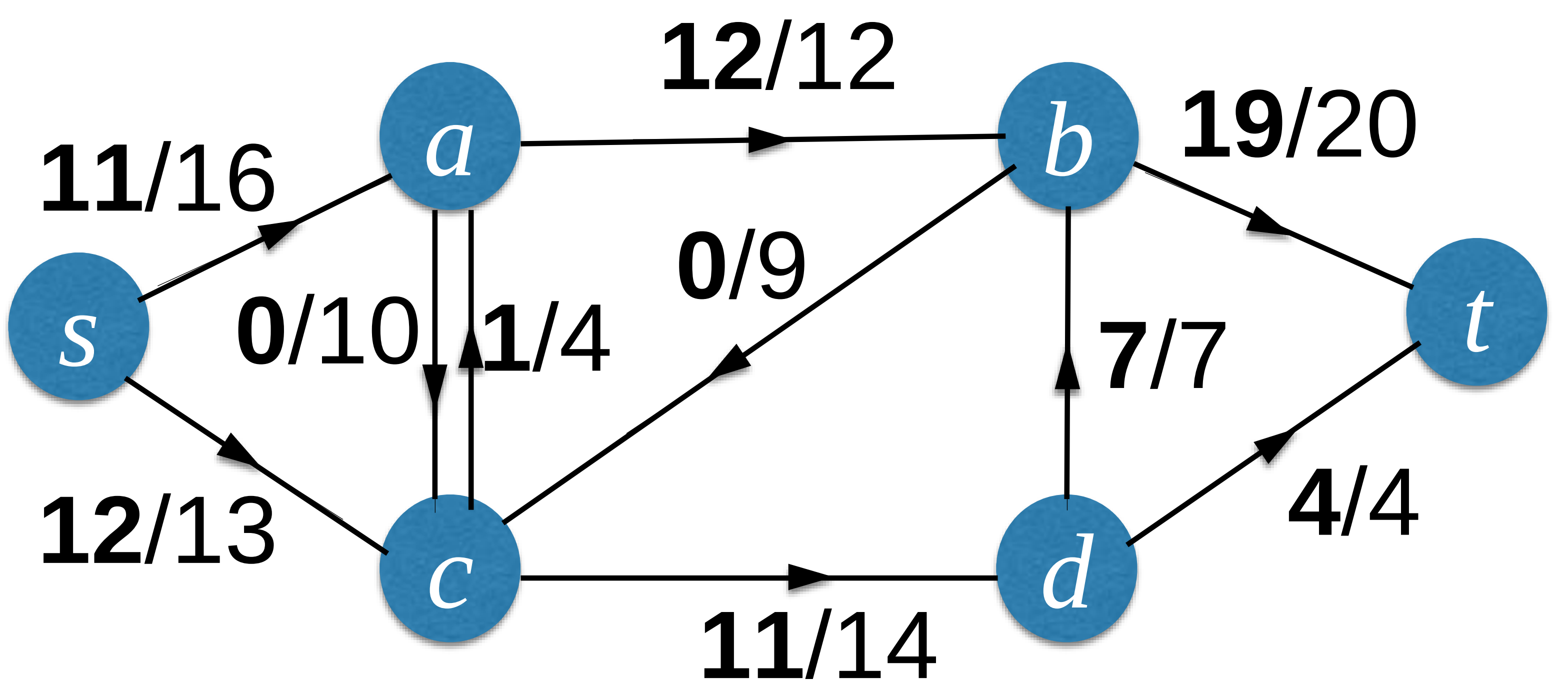
bottleneck capacity = min{(13-8), 4, (20-15)} = +4

Improved flow: can increase by +4



old flow
value
$|f| = 19$

Increase the flow by 4 in each forward arc and decrease the flow by 4 in each backward arc.



new flow
value
$|f| = 23$

Ford-Fulkerson’s maxflow method

1. Initialize with a 0 flow: st-flow value $|f\,| = 0$
2. Increase the flow along any augmenting path from s to t
3. Repeat step 2 as long as an augmenting path exists

Finding Augmenting Paths: the residual network G_f of a flow f

- Consider an st-flow f in st-flow network G and a directed edge (u,v) in G
- The amount of additional flow we can push from u to v along (u,v) in G is called the *residual capacity* $c_f(u,v)$ of edge (u,v) -- it depends on f.
- That is: for edge (u,v) with capacity $c(u,v)$ and flow value $f(u,v)$ from u to v we have the residual capacity $c_f(u,v) = c(u,v) - f(u,v)$; this creates a directed edge (u,v) in the residual network G_f with capacity $c_f(u,v)$.
- Of course in G we could instead *reduce* the flow in (u,v) by $f(u,v)$; this creates a directed edge (v,u) in the residual network G_f with capacity

  $c_f(v,u) = f(u,v)$. (Note the order of the vertices.)

Residual Network

- Given an st-flow network $G = (V,E)$ and a flow f, the *residual network* of G *induced by* f is $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$

Question:

Consider an edge (u,v) in G. How many edges does (u,v) create in the residual network G_f?

A. 1 edge: (u,v)
B. 2 edges: (u,v) and (v,u)
C. Can’t tell – it depends on the flow f, which can change

Question:

Consider an edge (u,v) in G , where

capacity $c(u,v) = 5$

and

flow $f(u,v) = 5$.

How many edges does (u,v) create in the residual network G_f?

A. 0
B. 1
C. 2

Question:

Consider an edge (u,v) in G , where

capacity $c(u,v) = 5$

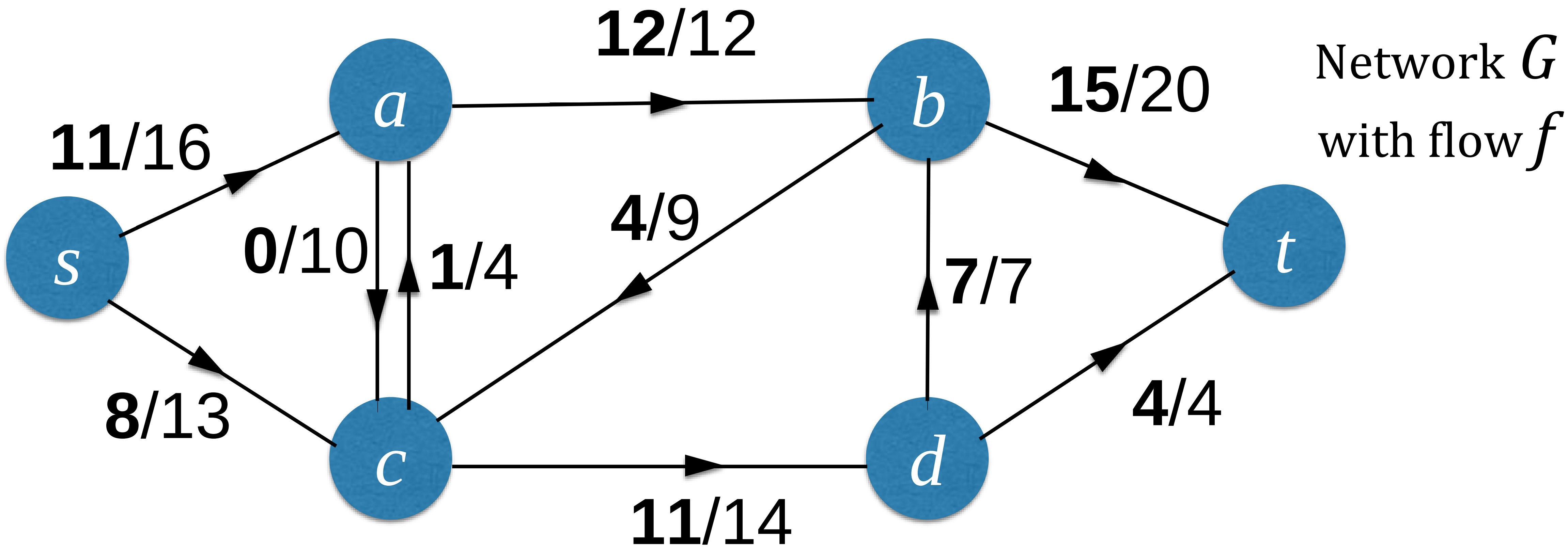
and

flow $f(u,v) = 3$.

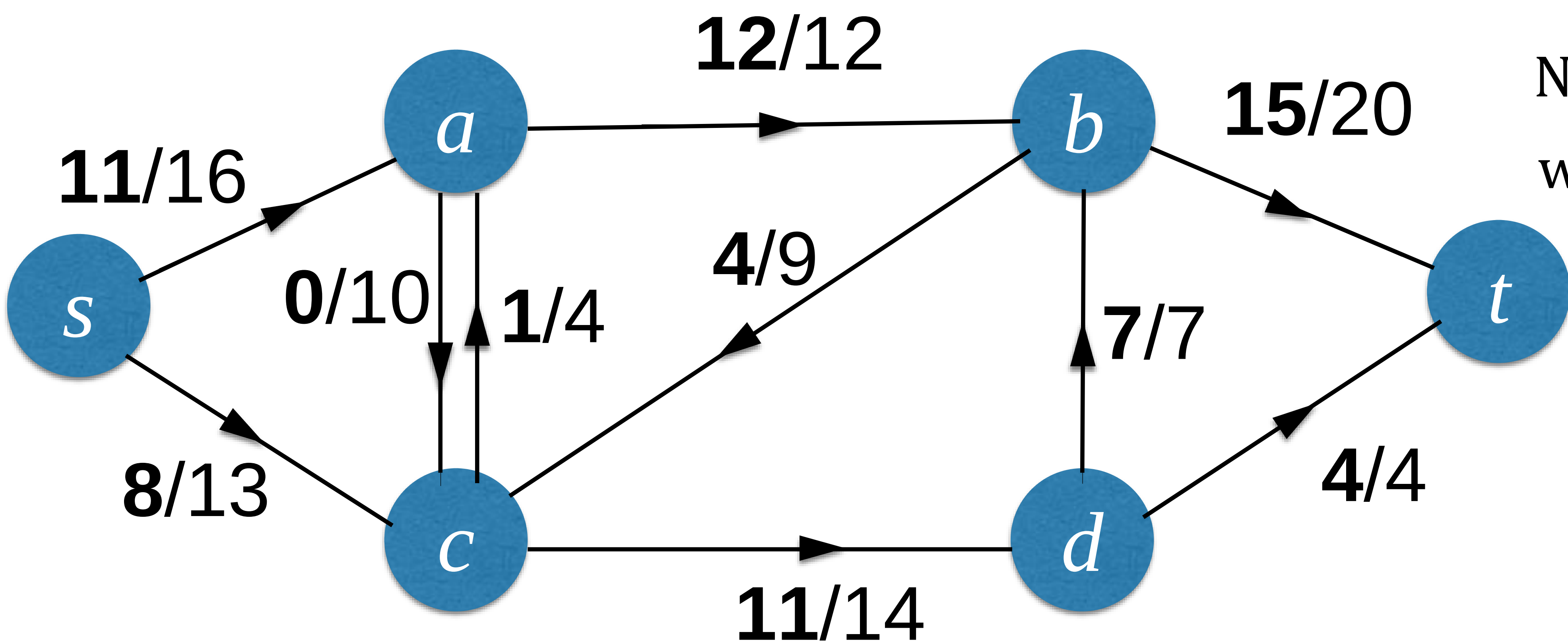
How many edges does (u,v) create in the residual network G_f?

A. 0
B. 1
C. 2

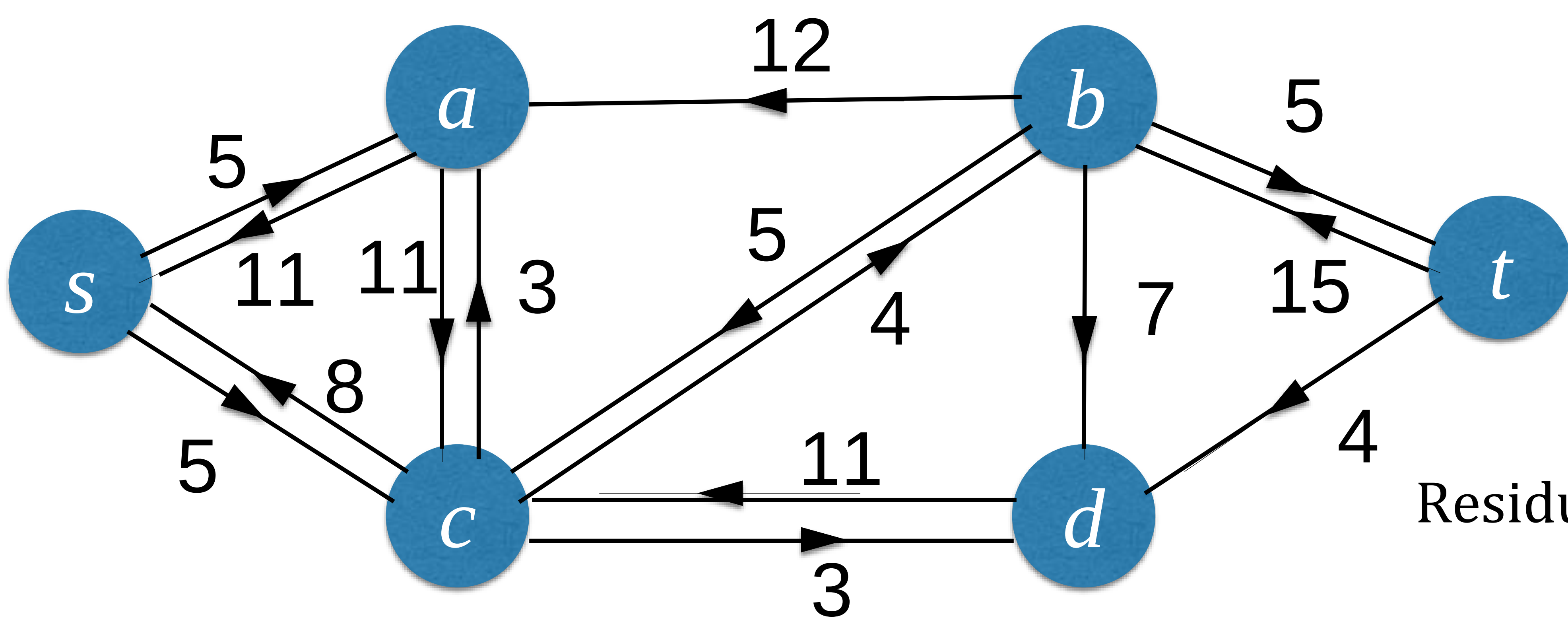
Example of a residual network



Example of a residual network



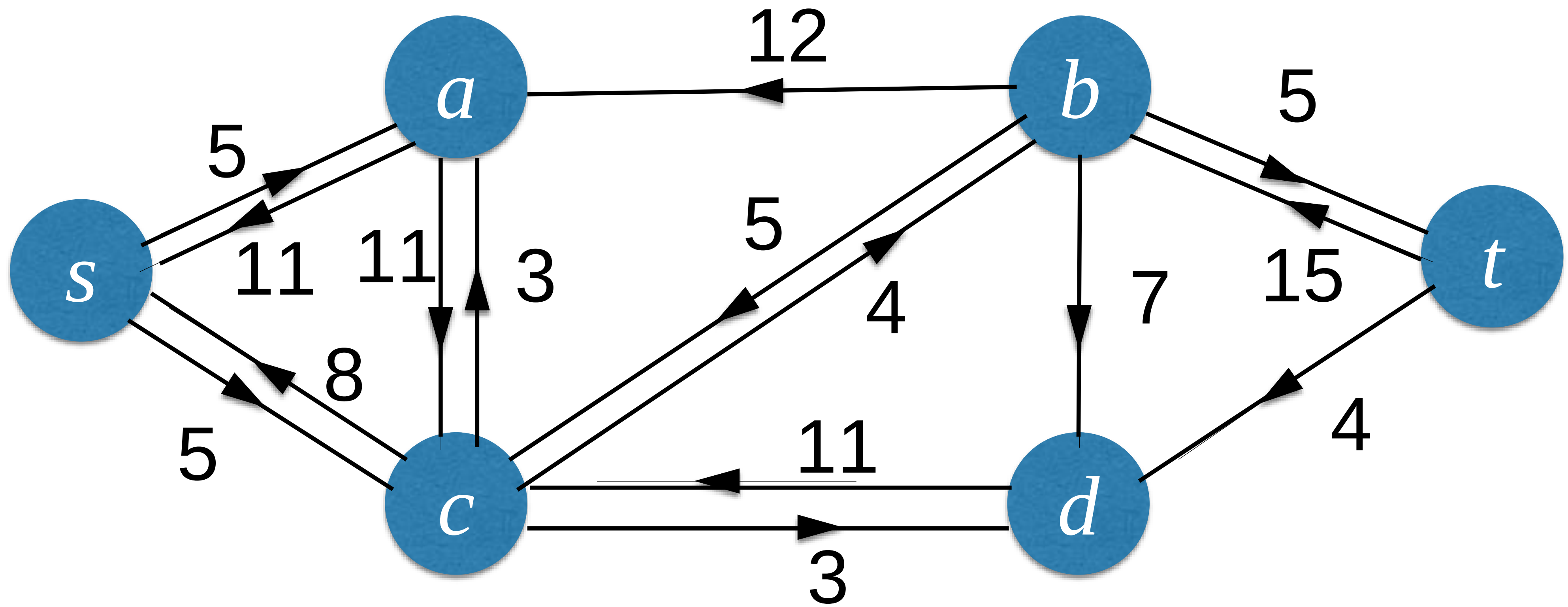
Network G with flow f



Residual network G_f

Augmenting path from s to t in residual network G_f : $scbt$

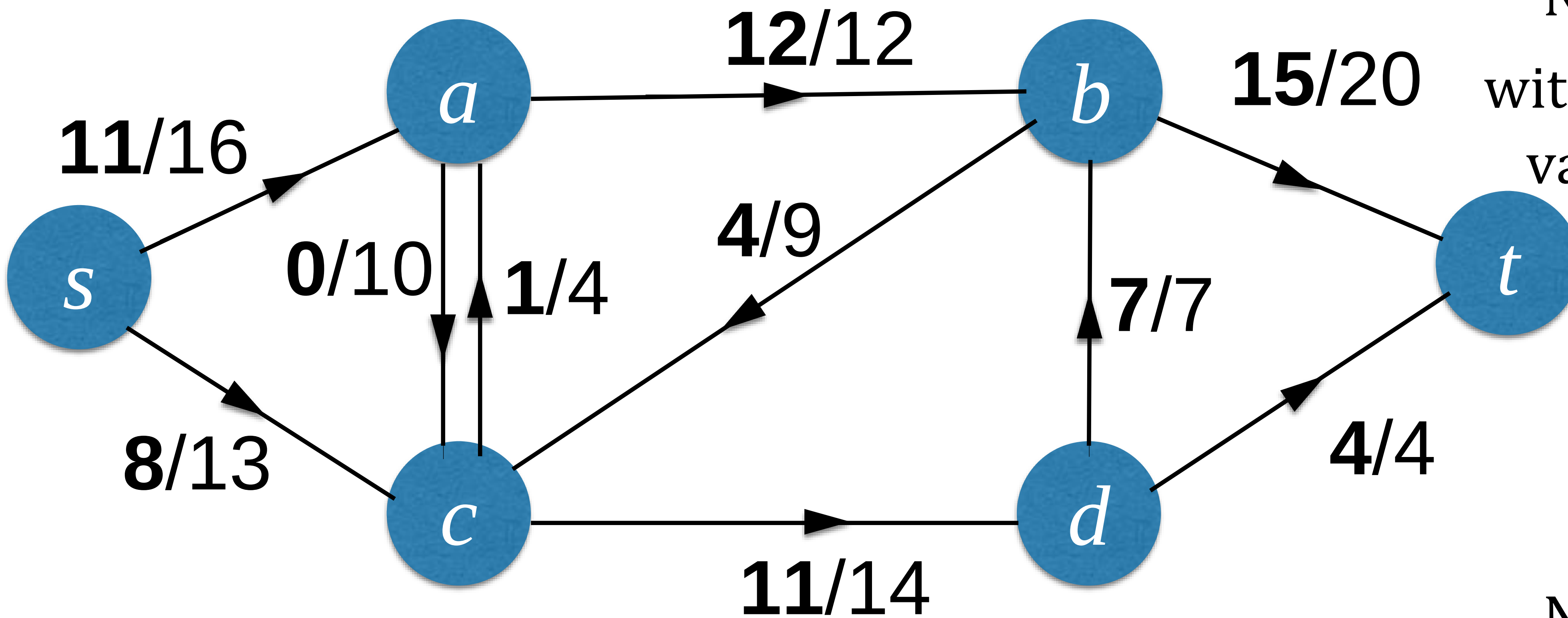
Residual network G_f



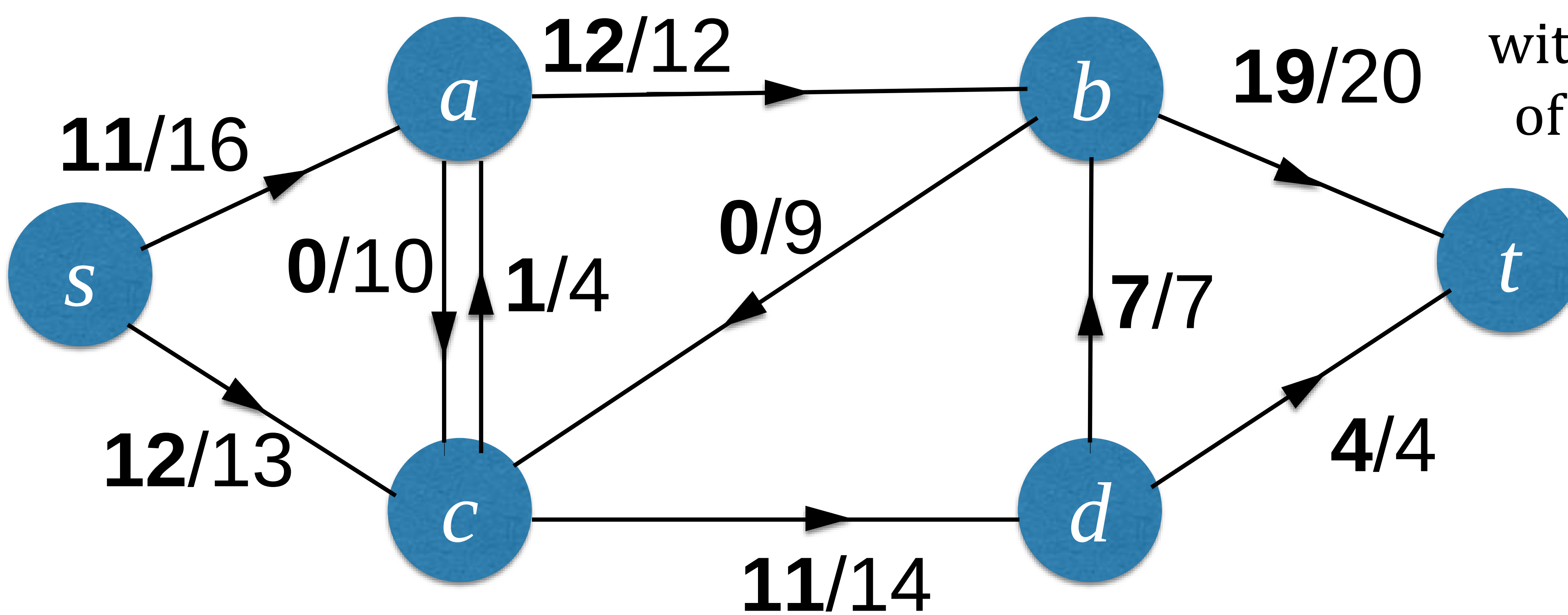
Residual capacity of path $scbt$ is *min*{5,4,5} = 4

In G , increase flow in each forward arc by 4, decrease flow in each backward arc by 4 to get new flow

Augment the flow along path *scbt* by 4

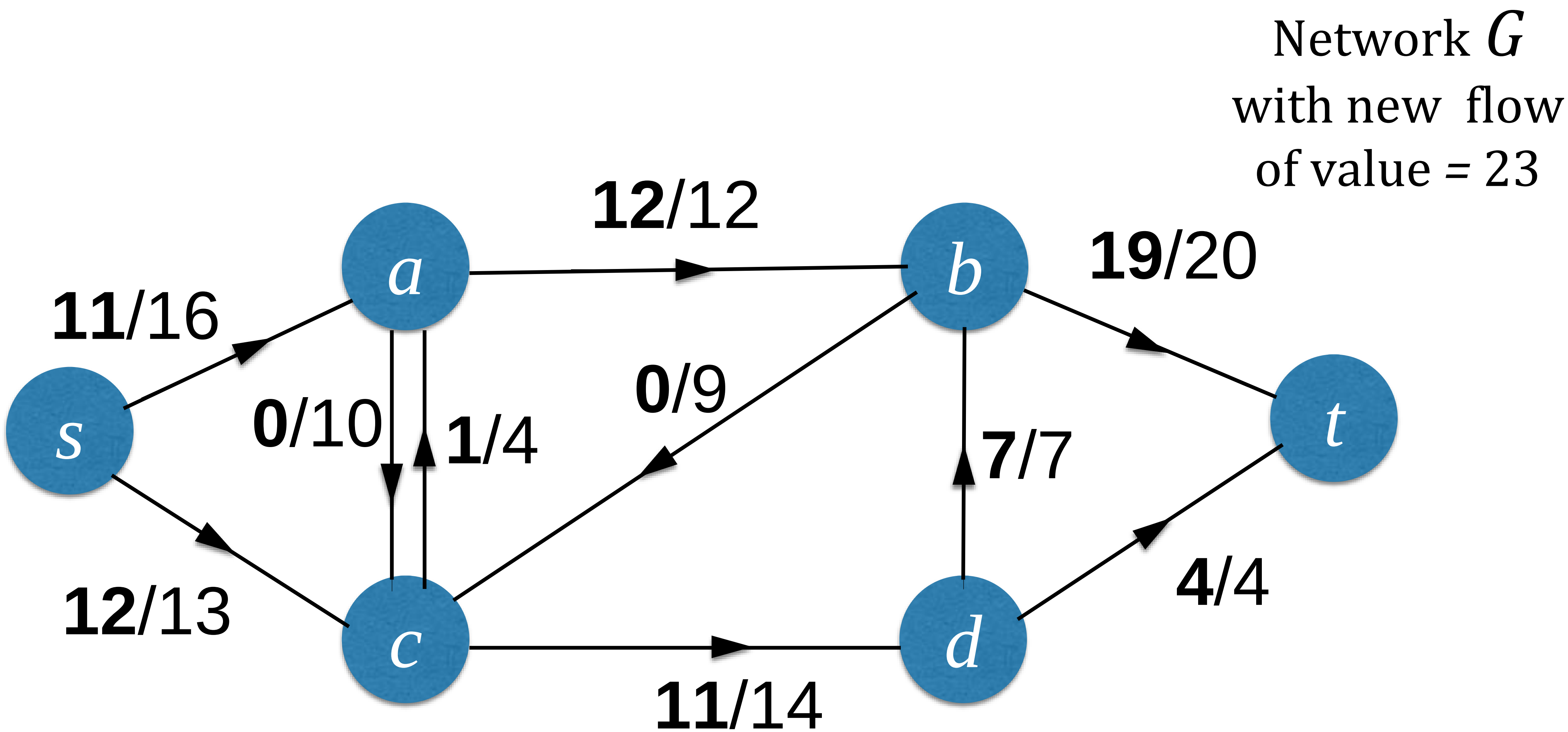


Network G
with old flow f;
value(f) = 19



Network G
with new flow
of value = 23

Claim: the new flow is a maxflow.



How would you prove this claim?
(two ways) Maxflow-mincut.