CSC 226

Algorithms and Data Structures: II

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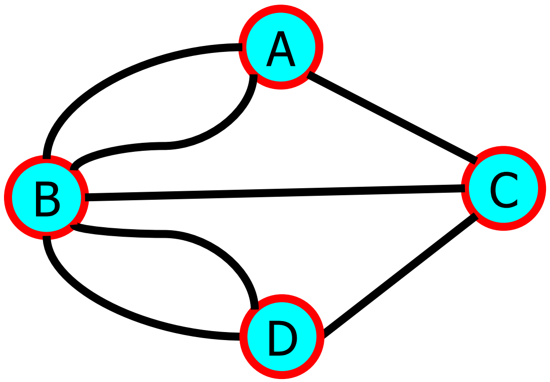
Abstract Meaning of the Term Graph

- A ***graph*** $G = (V, E)$ is a set V of ***vertices*** (*nodes*) and a collection E of pairs from V, called ***edges*** (*arcs*).

- **Graph Example:**

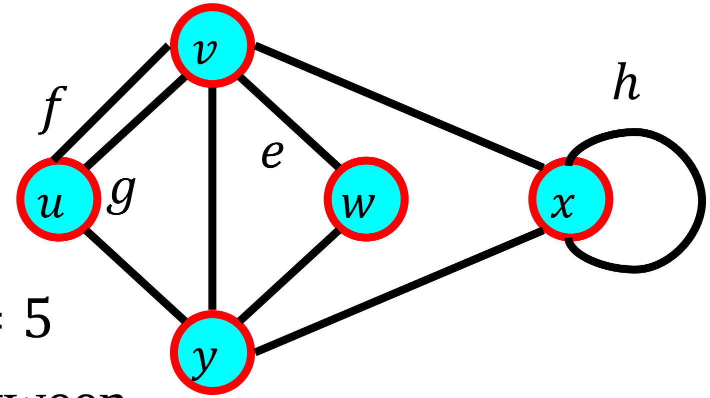


$$V = \{A, B, C, D\}$$

$$E = \left\{ \begin{array}{l} \{A, B\}, \{A, B\}, \{A, C\}, \\ \{B, C\}, \{B, D\}, \{B, D\}, \{C, D\} \end{array} \right\}$$

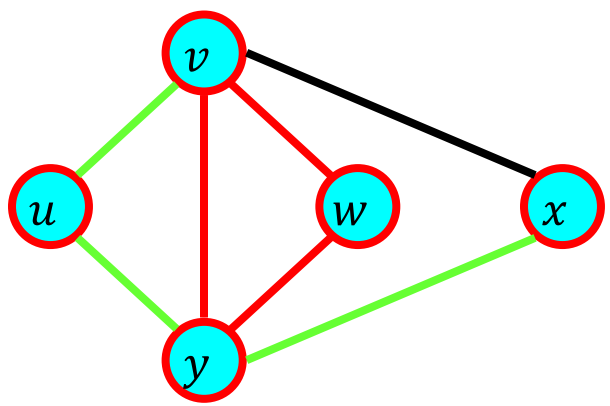
Undirected Edges

- An *undirected edge e* represents a *symmetric* relation between two vertices *v* and *w* represented by the vertices.
  - We usually write $e = \{v, w\}$, where $\{v, w\}$ is an unordered pair.
  - *v*, *w* are the *endpoints* of the edge
  - *v* is *adjacent* to *w*
  - *e* is *incident* upon *v* and *w*
  - The *degree* of a vertex is the number of incident edges, eg. $\deg(v) = 5$
  - *parallel* edges – more than one edge between a pair of vertices, eg. f and g
  - *self-loop* – edge that connects a vertex to itself, eg. *h*
  - Typically, the number of vertices is denoted by *n* and the number of edges by *m*.



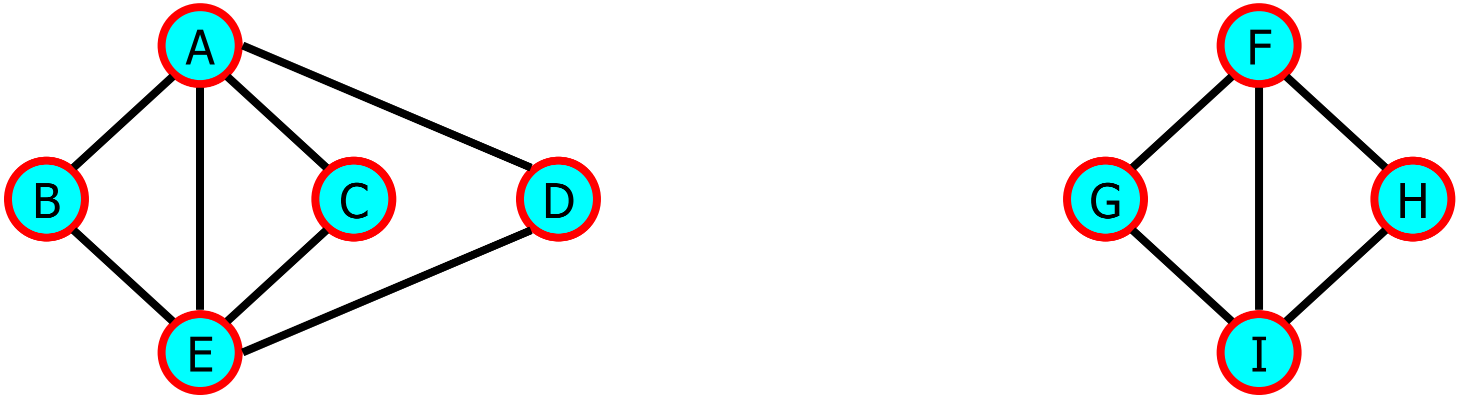
Undirected Paths

- A ***walk*** in a graph is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that there exist edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$
- The ***length*** of a walk is the number of edges
  - if $v_1 = v_n$, ***closed***, otherwise ***open***
- If no edge is repeated, it's a ***trail***
- A closed trail is a ***circuit***
- If no vertex is repeated, it's a ***path***
- A ***cycle*** is a path with the same start and end vertices



Connected Graphs

- A graph is ***connected*** if every pair of vertices is connected by a path.
- Example
  - ➢Two *connected components* of a graph
  - ➢*Unconnected* graph



Simple Graphs

- A ***simple graph*** is a graph with no self-loops and no parallel or multi-edges

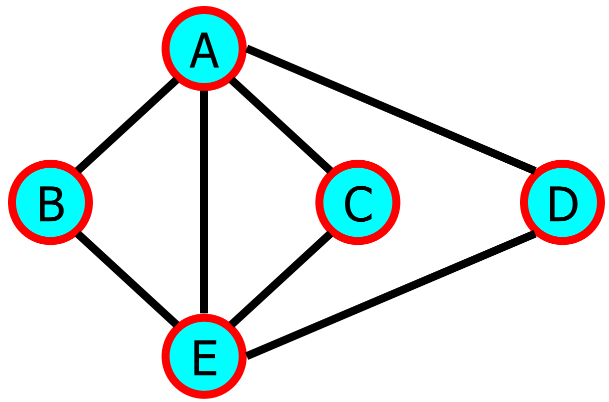
- **Theorem:** If $G = (V, E)$ is a graph with m edges, then

$$\sum_{v \in V} \deg(v) = 2m$$

- **Theorem:** Let G be a simple graph with n vertices and m edges. Then,

$$m \leq \frac{n(n-1)}{2}$$

- **Corollary:** A simple graph with n vertices has $O(n^2)$ edges.

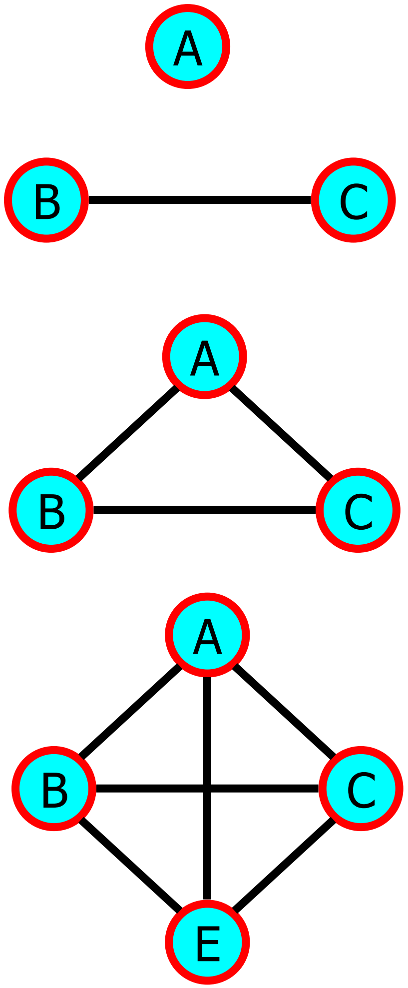


Complete Graphs

- A ***complete graph*** is a simple graph where an edge connects every pair of vertices

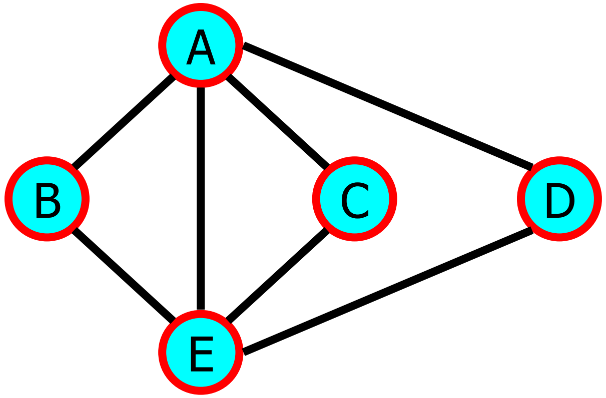
- The complete graph on n vertices has exactly $n(n - 1) / 2$ edges

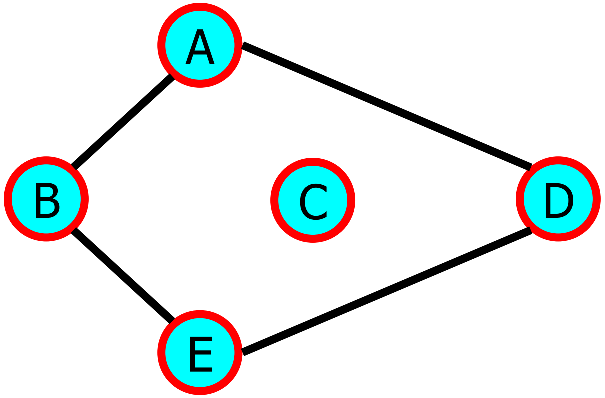
- A complete graph with at most one self loop per vertex on n vertices has exactly $n(n + 1) / 2$ edges



Subgraphs

- A ***subgraph*** of $G = (V, E)$ *is a graph* $G' = (V', E')$ where
  - V' is a subset of V
  - E' consists of edges $\{v, w\}$ in E such that both v and w are in V'
- A ***spanning subgraph*** of G contains all the vertices of G

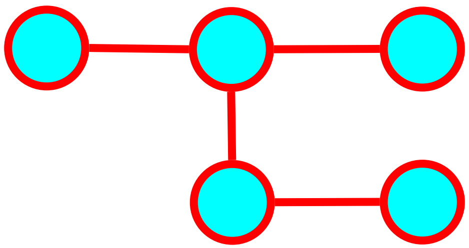




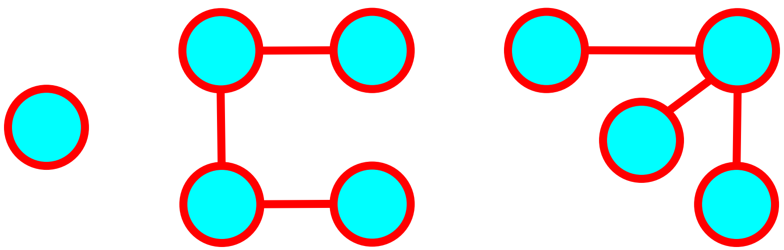
Trees and Forests

- A (*free*) *tree* is an undirected graph T such that
  - ➢ T is connected
  - ➢ T has no cycles

  This definition of tree is different from the one of a rooted tree
- A *forest* is an undirected graph without cycles
- The connected components of a forest are trees



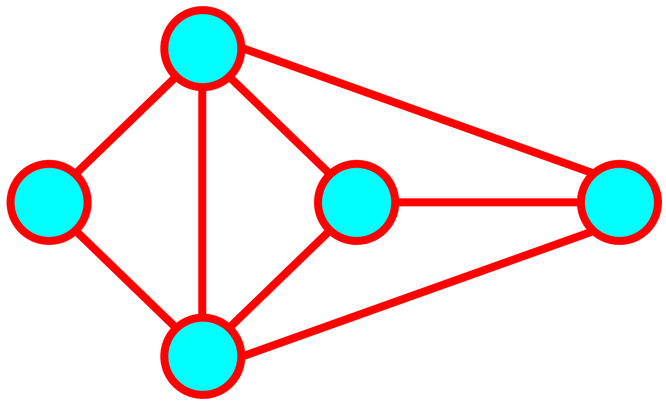
Tree



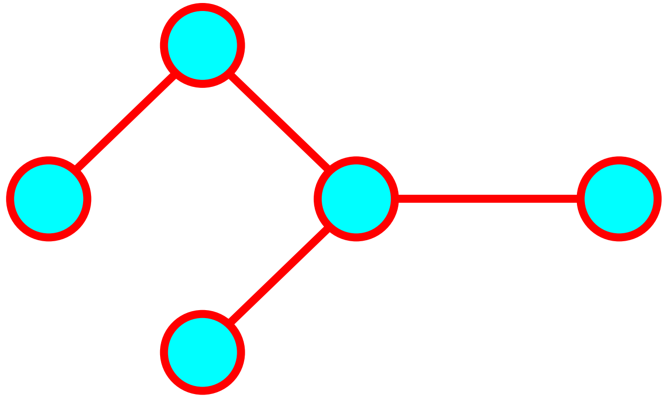
Forest

Spanning Trees and Forests

- A *spanning tree* of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A *spanning forest* of a graph is a spanning subgraph that is a forest



Graph



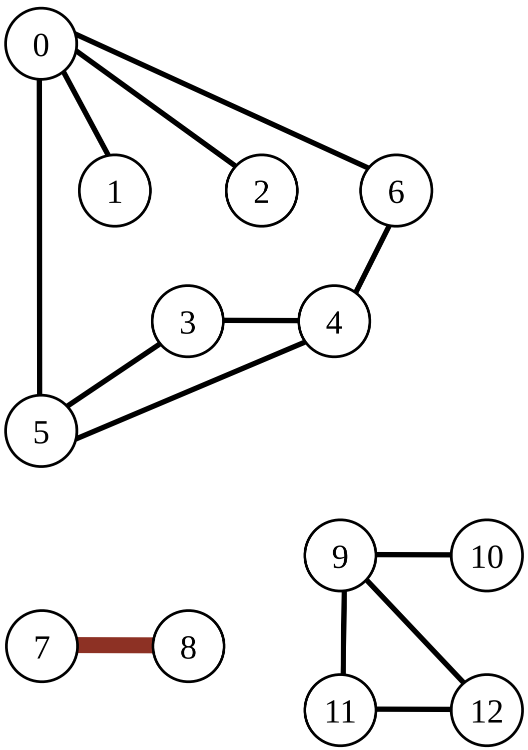
Spanning tree

Properties of Trees, Forests and Graphs

- Theorem: Let G be an undirected simple graph with n vertices and m edges. Then we have the following:
  - If G is connected, then $m \geq n - 1$.
  - If G is a tree, the $m = n - 1$.
  - If G is a forest, then $m \leq n - 1$.

Graph representation: set of edges

- Maintain a list of the edges (linked list or array).

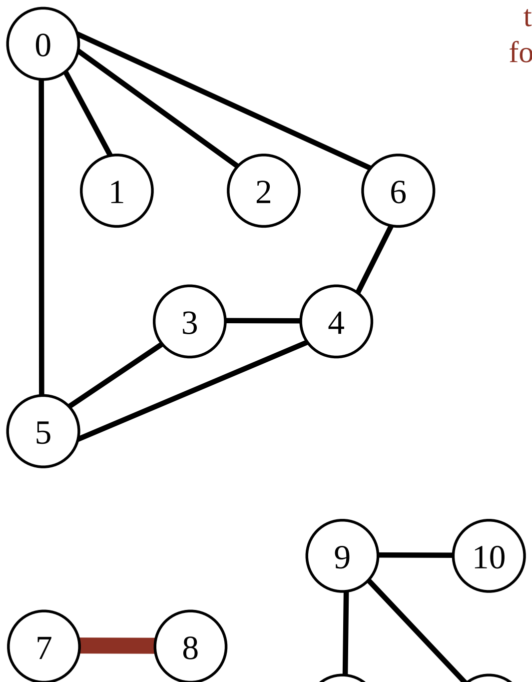


0	1
0	2
0	5
0	6
3	4
3	5
4	5
4	6
7	8
9	10
9	11
9	12
11	12

Q. How long to iterate over vertices adjacent to *v* ?

Graph representation: adjacency matrix

- Maintain a two-dimensional *n*-by-*n* boolean array;
  for each edge *v*–*w* in graph: adj[v][w] = adj[w][v] = true.



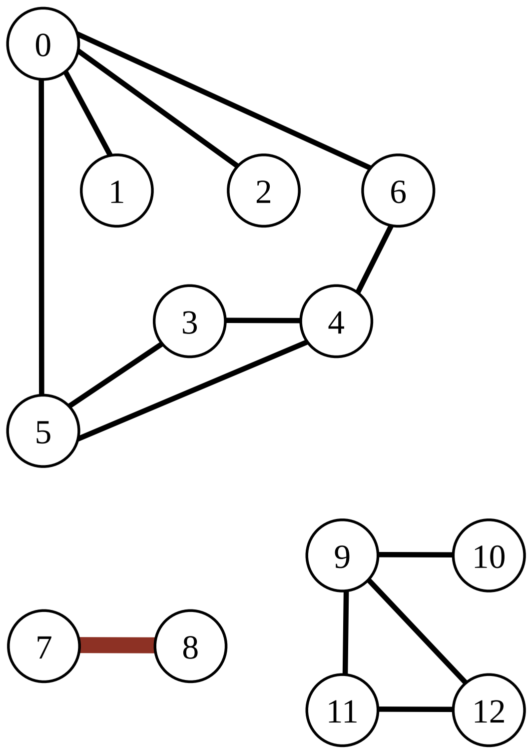
two entries for each edge

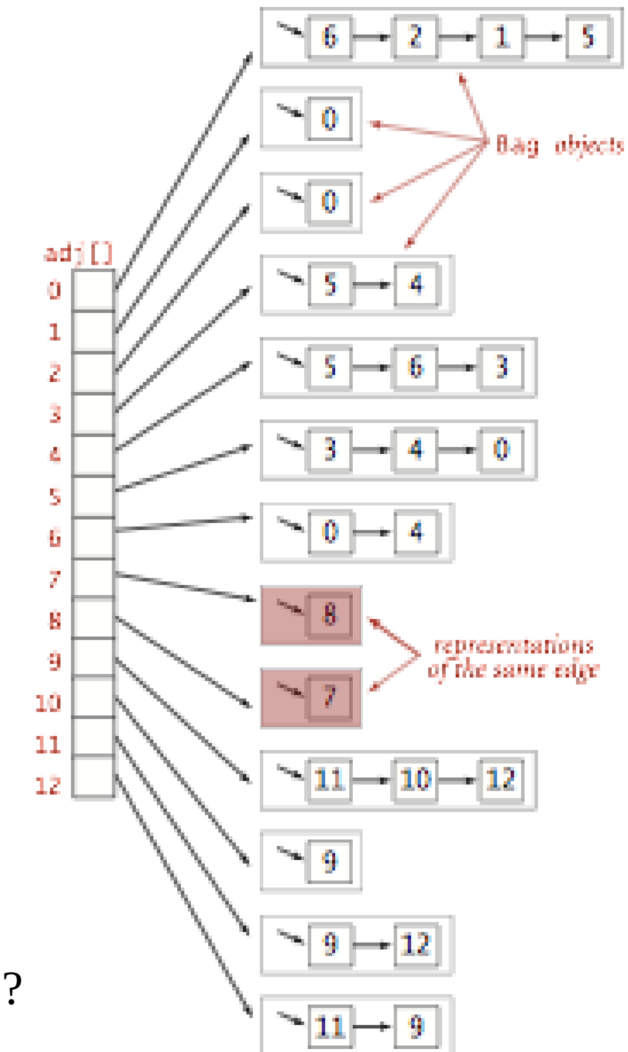
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0

Q. How long to iterate over vertices adjacent to *v* ?

Graph representation: adjacency lists

- Maintain vertex-indexed array of lists.





Q. How long to iterate over vertices adjacent to v ?

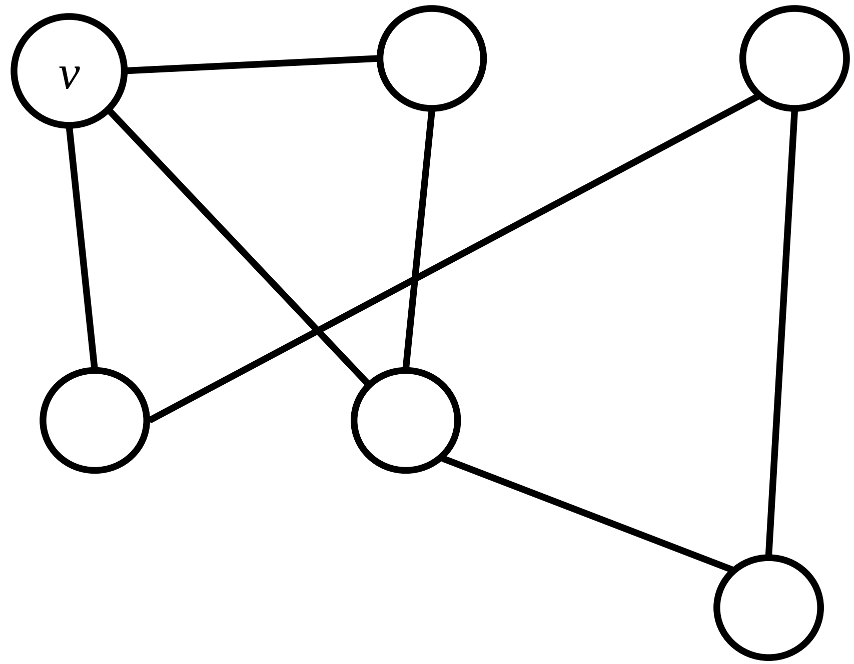
Algorithm DFS

Algorithm DFS(*G*, *v*):

Input: A graph *G* and a vertex *v* of *G*

*Output***:** A labeling of the edges in the connected component as discovery edges and back edges

Label *v* as explored
for each edge, *e*, incident to *v* **do**
  if *e* is unexplored **then**
    Let *w* be opposite node of *e*
    if *w* is unexplored **then**
      Label *e* as *discovery* edge
      DFS(G,*w*)
    else
      Label *e* as *back* edge



Algorithm BFS

Algorithm *BFS*(*G*, *s*):

Input: A graph *G* and a vertex *s* of *G*

***Output*:** A labeling of the edges as *discovery* edges and *cross* edges

Q ← new empty queue

Label *s* as explored

Q.enqueuer(*s*)

while *Q* is not empty **do**

v ← *Q.dequeue*()

for each edge, *e* = {*v*,*w*}, incident on *v* **do**

if *e* is unexplored **then**

if *w* is unexplored **then**

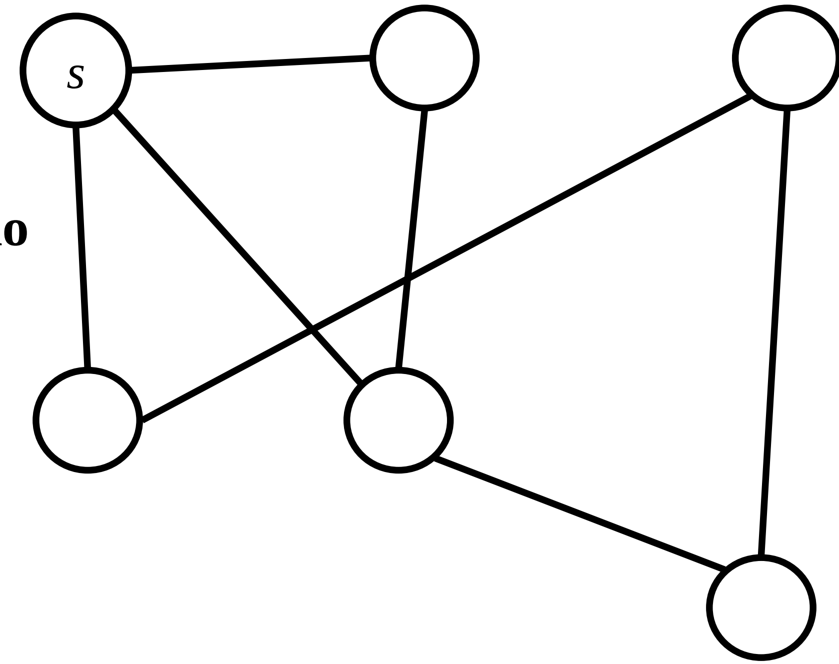
Label *e* as a *discovery* edge

Mark *w* as explored

Q.enqueue(*w*)

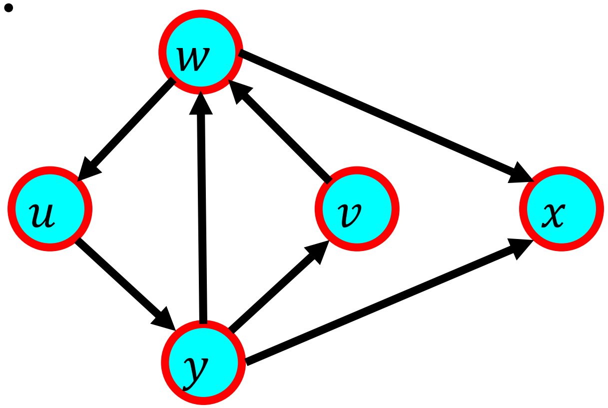
else

Label *e* as a *cross* edge



Directed Edges or Arcs

- A *directed edge* (*or arc*) *e* represents an *asymmetric* relation between two vertices *v* and *w*.
  e = (*v*, w) denotes an ordered pair.
  - *v*, *w* are the endpoints of the edge
  - *v* is *adjacent* to *w*
  - *e* is *incident* upon *v* and *w*
  - The arc goes from the *source* vertex *v* to the *destination* vertex *w*



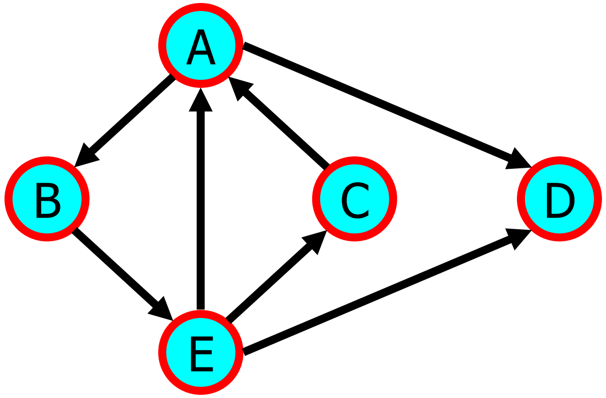
- The *indegree* of a vertex is the number of incoming arcs
- The *outdegree* of a vertex is the number of outgoing arcs

Simple Graphs

- A ***simple digraph*** is a graph with no self-loops and no parallel or multi-edges

- **Theorem:** If $G = (V, E)$ is a digraph with m edges, then

$$\sum_{v \in V} \operatorname{indeg}(v) = \sum_{v \in V} \operatorname{outdeg}(v) = m$$



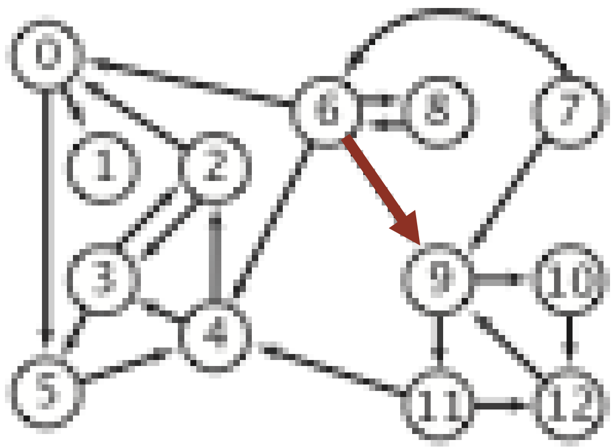
- **Theorem:** Let G be a simple digraph with n vertices and m edges. Then,

$$m \leq n(n-1)$$

- **Corollary:** A simple digraph with n vertices has $O\left(n^2\right)$ edges.

Digraph representation: set of edges

- Store a list of the edges (linked list or array).

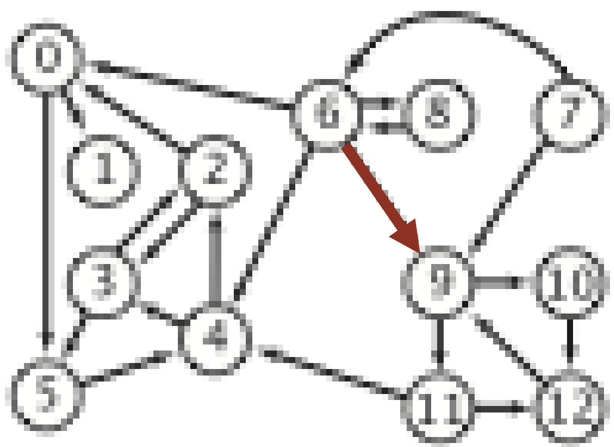


0	1
0	5
2	0
2	3
3	2
3	5
4	2
4	3
5	4
6	0
6	4
6	8
6	9
7	6
7	9
8	6
9	10
9	11
10	12
11	4
11	12
12	9

Digraph representation: adjacency matrix

- Maintain a two-dimensional V-by-V boolean array;
  for each edge $v \to w$ in the digraph: adj[v][w] = true.

Note: parallel edges disallowed



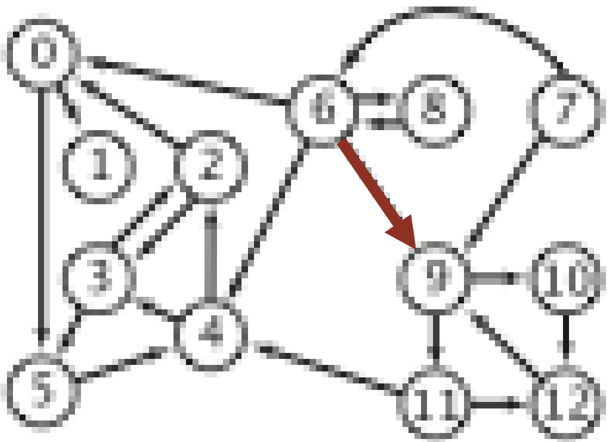
to

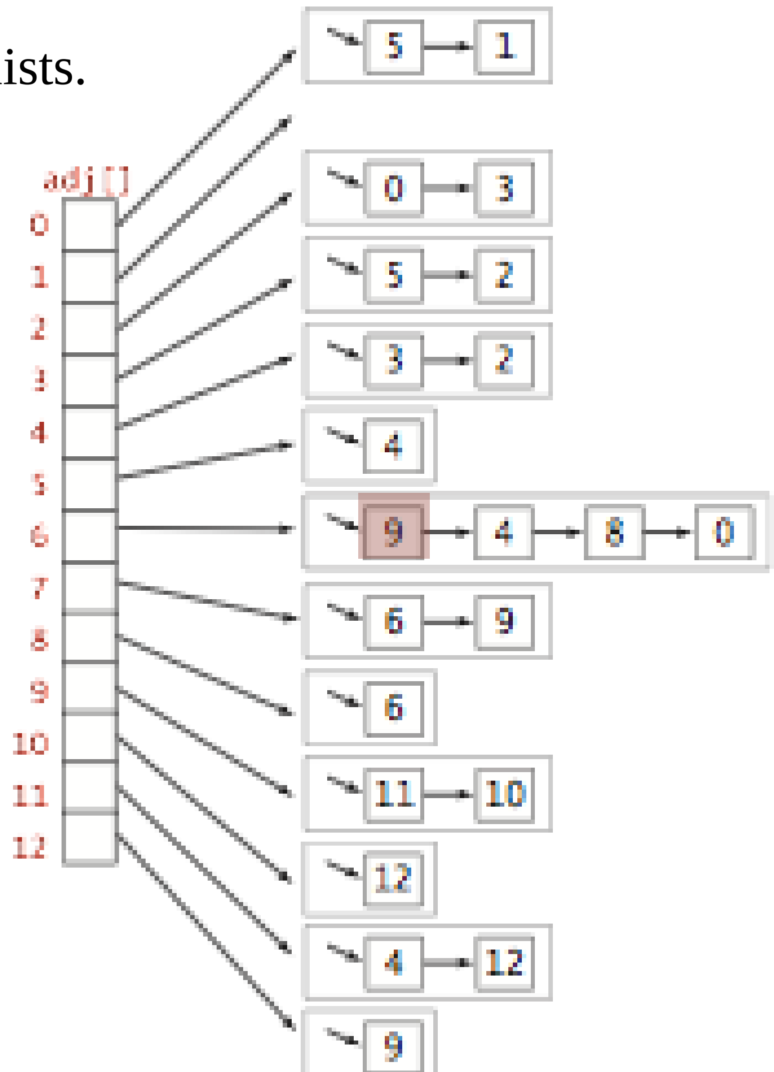
from

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	1	0	0	0	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	0	0	1	1	0	0	0
7	0	0	0	0	0	0	1	0	0	1	0	0	0

Digraph representation: adjacency lists

- Maintain vertex-indexed array of lists.





Algorithm Directed DFS

Algorithm DirectedDFS(G, v):

Input: A digraph G and a vertex v of G

Output: A label of the edges as *discovery*, *back*, *forward* or *cross* edges

Label v as active

for each outgoing edge e **do**

    if e is unexplored **then**

        Let w be the destination of e

        if w is unexplored and not active **then**

            Label e as a *discovery* edge

            DirectedDFS(G,w)

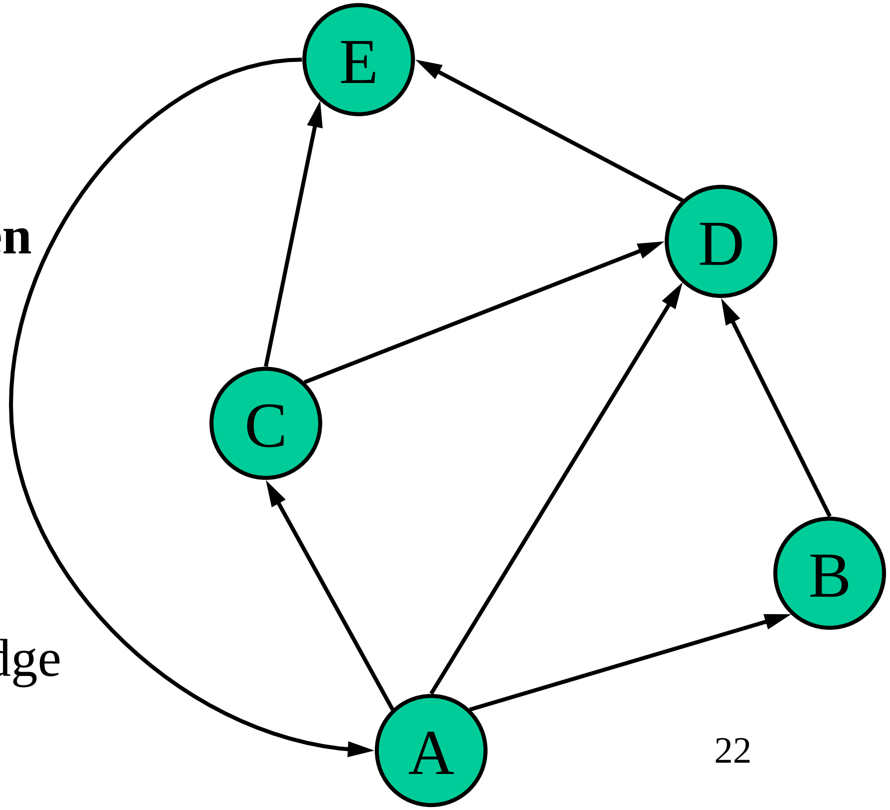
        else if w is active **then**

            mark edge e as a *back* edge

        else

            mark edge e as a *forward*/*cross* edge

Label v as explored



Algorithm Directed BFS

Algorithm *BFS*(*G*, *s*):

Input: A graph *G* and a vertex *s* of *G*

Output: A labeling of the edges as *discovery*, *back* or *cross* edges

$Q \leftarrow$ new empty queue

Label *s* as explored

Q.enqueuer(*s*)

while *Q* is not empty **do**

  $v \leftarrow$ *Q.dequeue*()

  for each outgoing edge, $e = (v,w)$, incident on *v* **do**

    if *e* is unexplored **then**

      if *w* is unexplored **then**

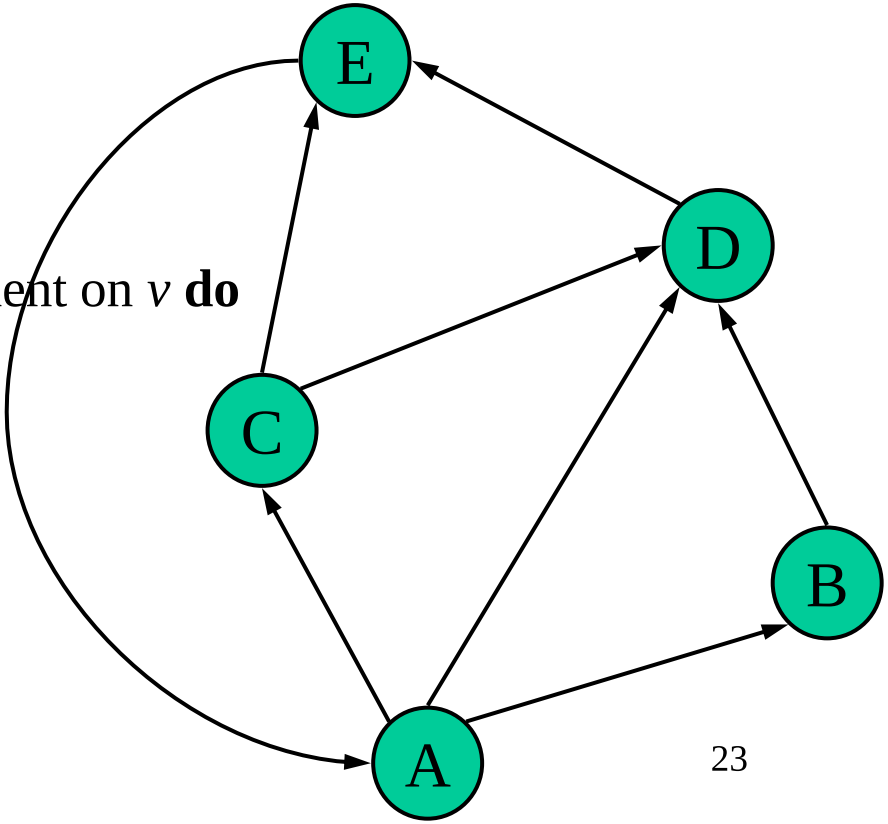
        Label *e* as a *discovery* edge

        Mark *w* as explored

        Q.enqueue(*w*)

    else

        Label *e* as a *back*/*cross* edge



Connected Digraphs

- Given vertices u and v of a digraph G, we say v is ***reachable*** from u if G has a directed path from u to v.

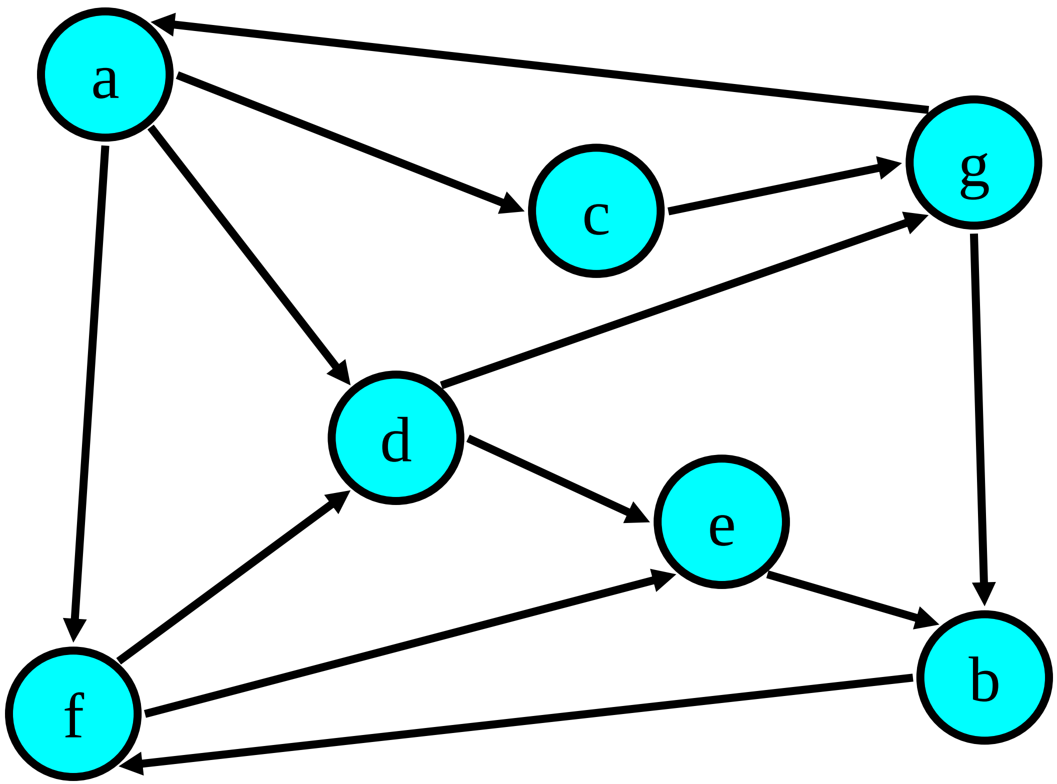
- A digraph G is ***connected*** if every pair of vertices is connected by an undirected path.

- A digraph G is ***strongly connected*** if for every pair of vertices u and v of G, u is reachable from v and v is reachable from u.

Strong Connectivity

- Each vertex can reach all other vertices

G:

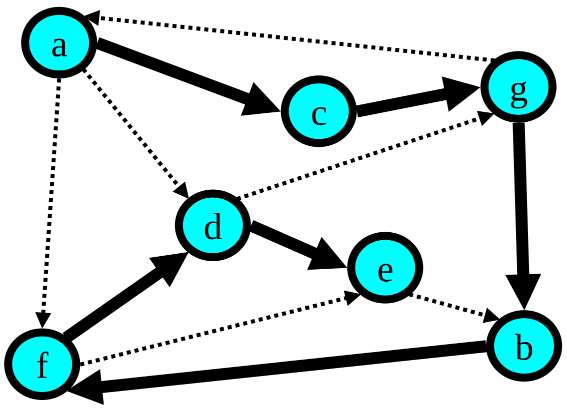


Strong Connectivity Algorithm

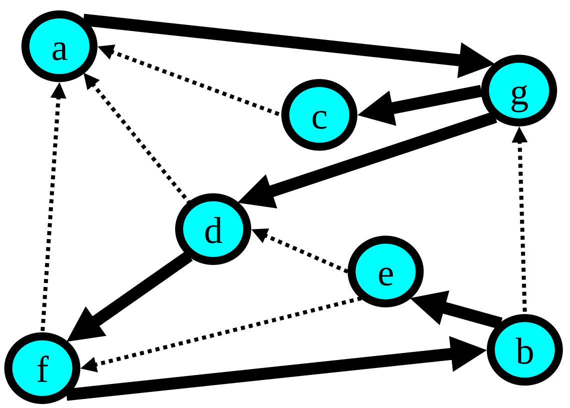
- Pick a vertex v in G.
- Perform a DFS from v in G.
  - ➢ If there's a w not visited, print "no".
- Let G' be G with edges reversed.
- Perform a DFS from v in G'.
  - ➢ If there's a w not visited, print "no".
  - ➢ Else, print "yes".

- Running time: $O(n + m)$.

G:

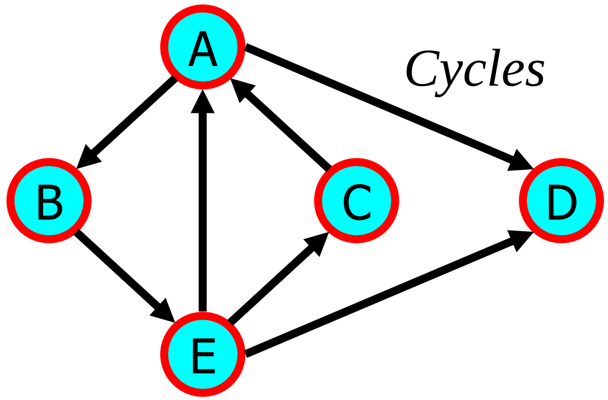


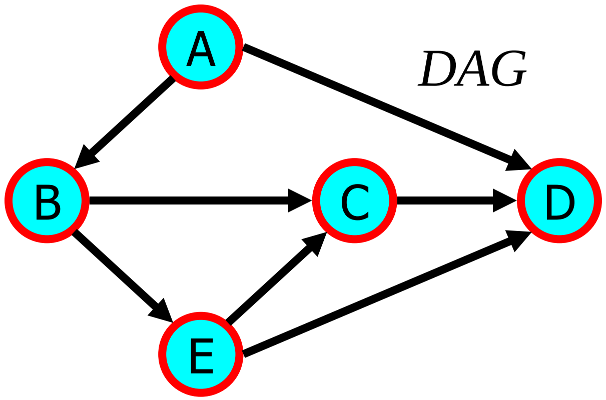
G′:



Directed Acyclic Graphs (DAGs)

- A ***directed acyclic graph*** (DAG) is a directed graph with no cycles.
- DAGs are more general than trees, but less general than arbitrary directed graphs.





DAGs and Topological Ordering

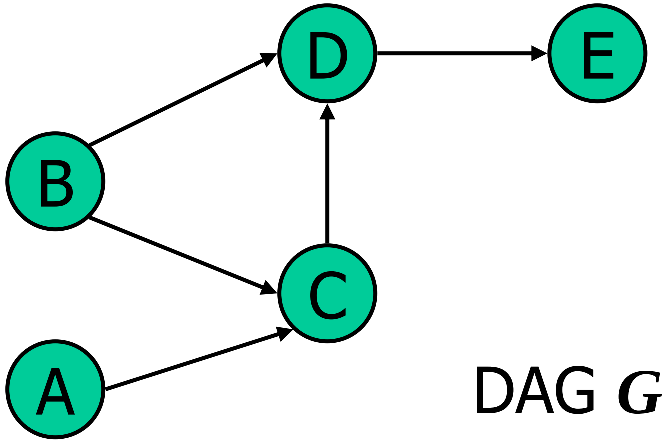
- A **topological ordering** of a digraph is a numbering

$$v_1, \ldots, v_n$$

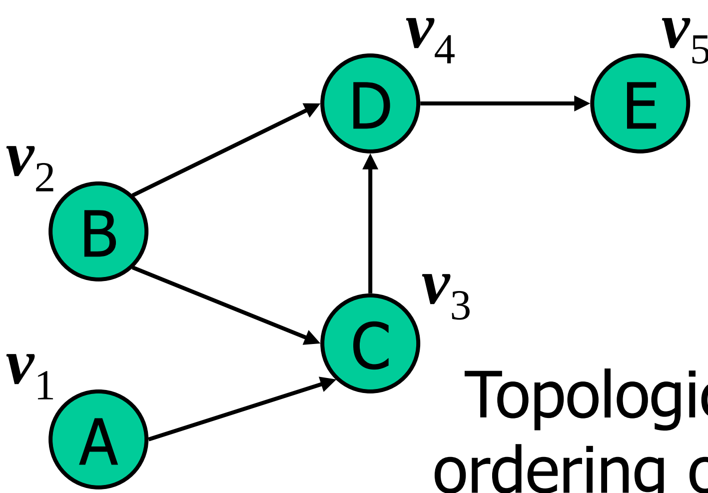
of the vertices such that for every edge (v_i, v_j), we have $i < j$

- **Theorem:**

A digraph admits a topological ordering if and only if it is a DAG



DAG G

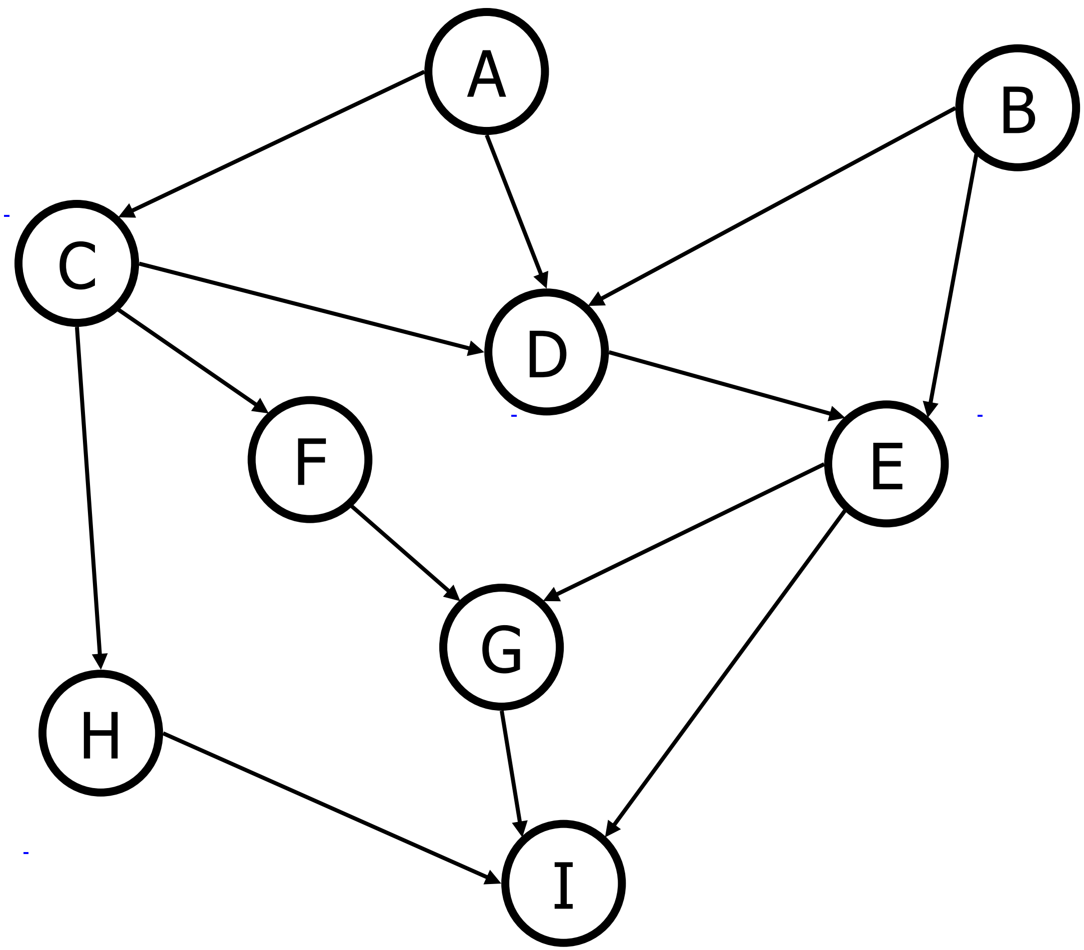


Topological ordering of G

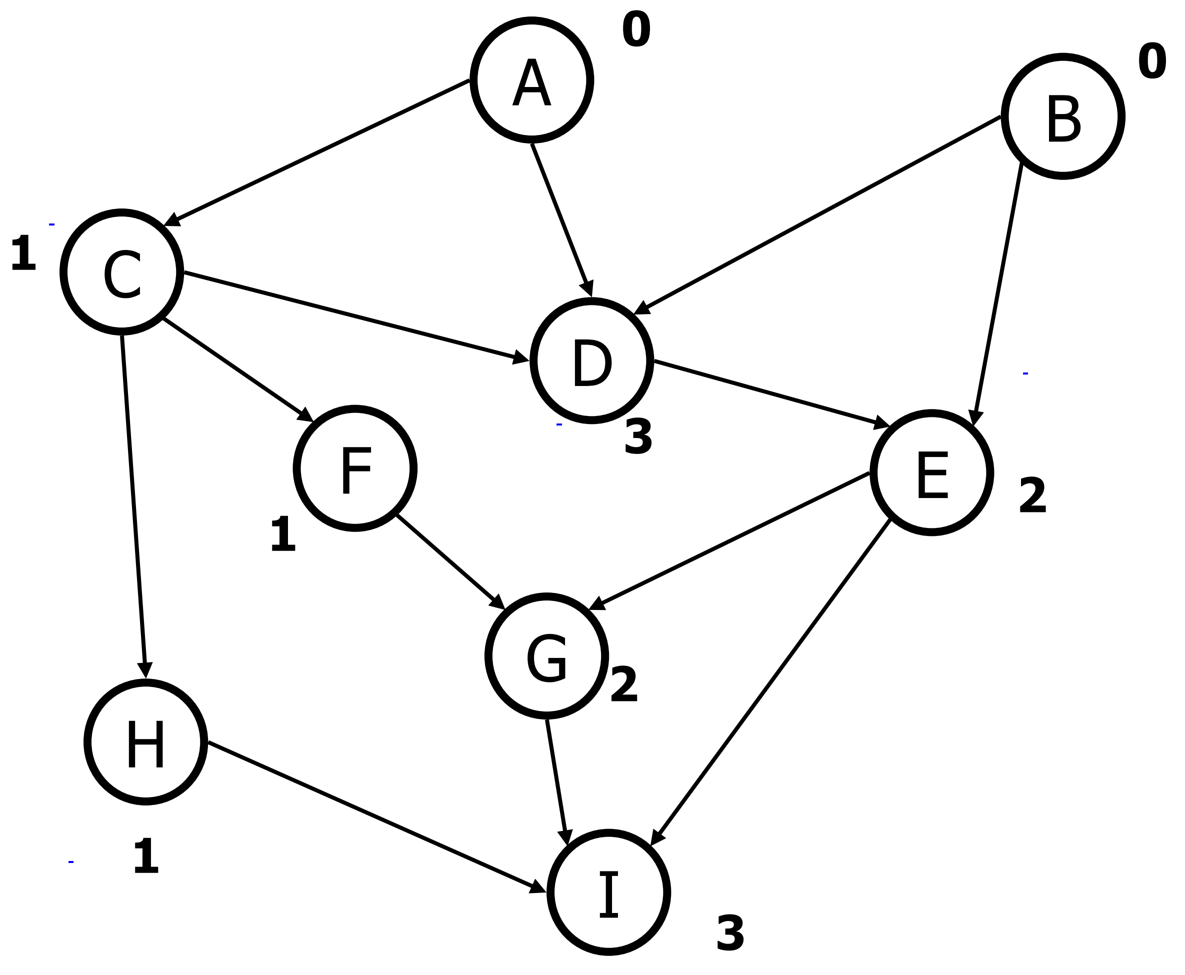
Algorithm Directed BFS

```
Algorithm topologicalSort(G)
    Input: digraph G with n vertices
    Output: topological ordering of G or an indication of a directed cycle
    S ← Empty stack
    for each vertex u ∈ G do
        incounter(u) ← indeg(u)
        if incounter(u) = 0 then
            S.push(u)
    i ← 1
    while S is not empty do
        u ← S.pop()
        number u as vertex v_i
        i ← i + 1
        for each outgoing edge  e ∈ G do
            w ← G.opposite(u,e)
            incounter(w) ← incounter(w) - 1
            if incounter(w) = 0 then
                S.push(w)
    if i > n then
        return v_1, v_2, ..., v_n
    return "G has a directed cycle"
```

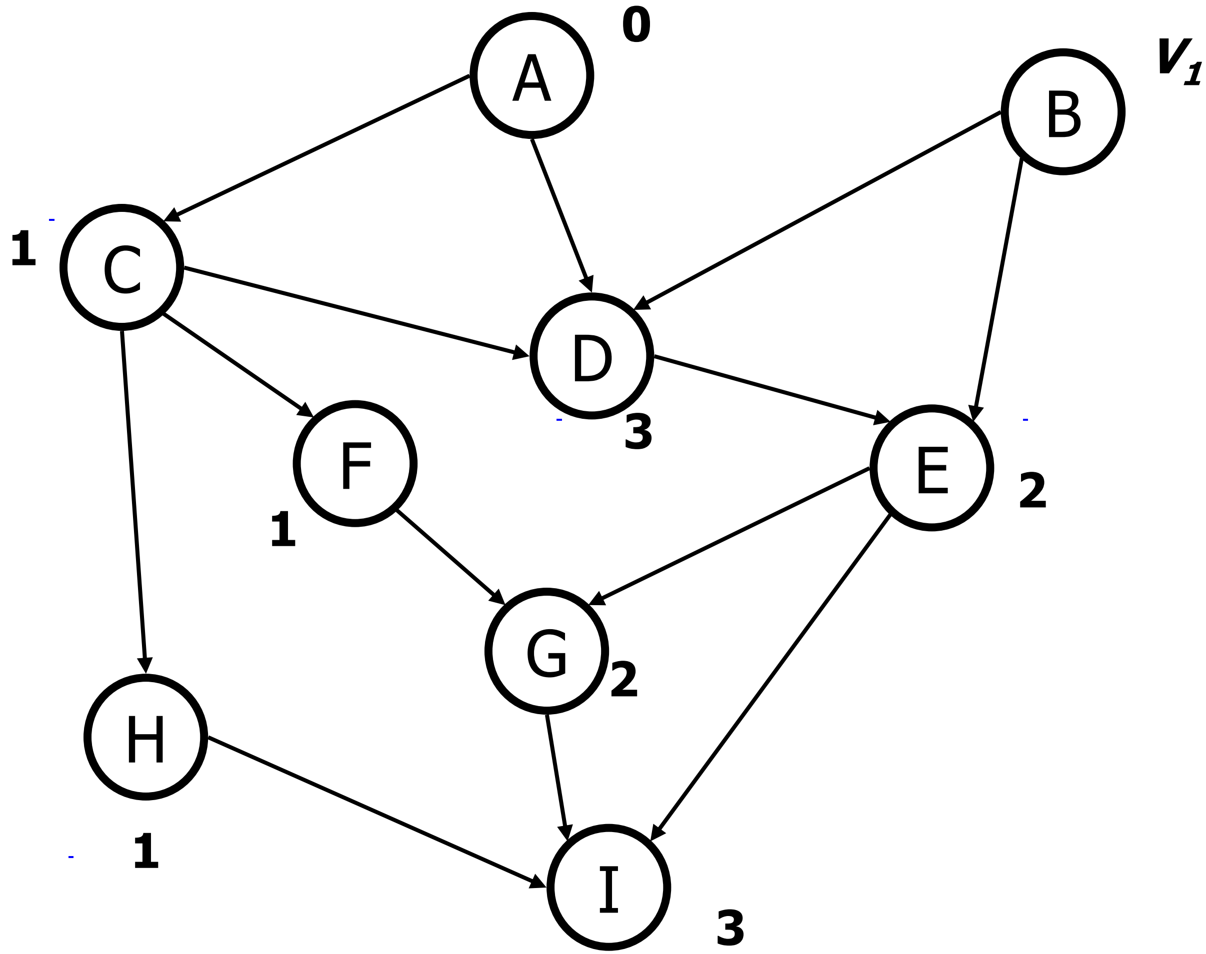
Topological Sorting Example



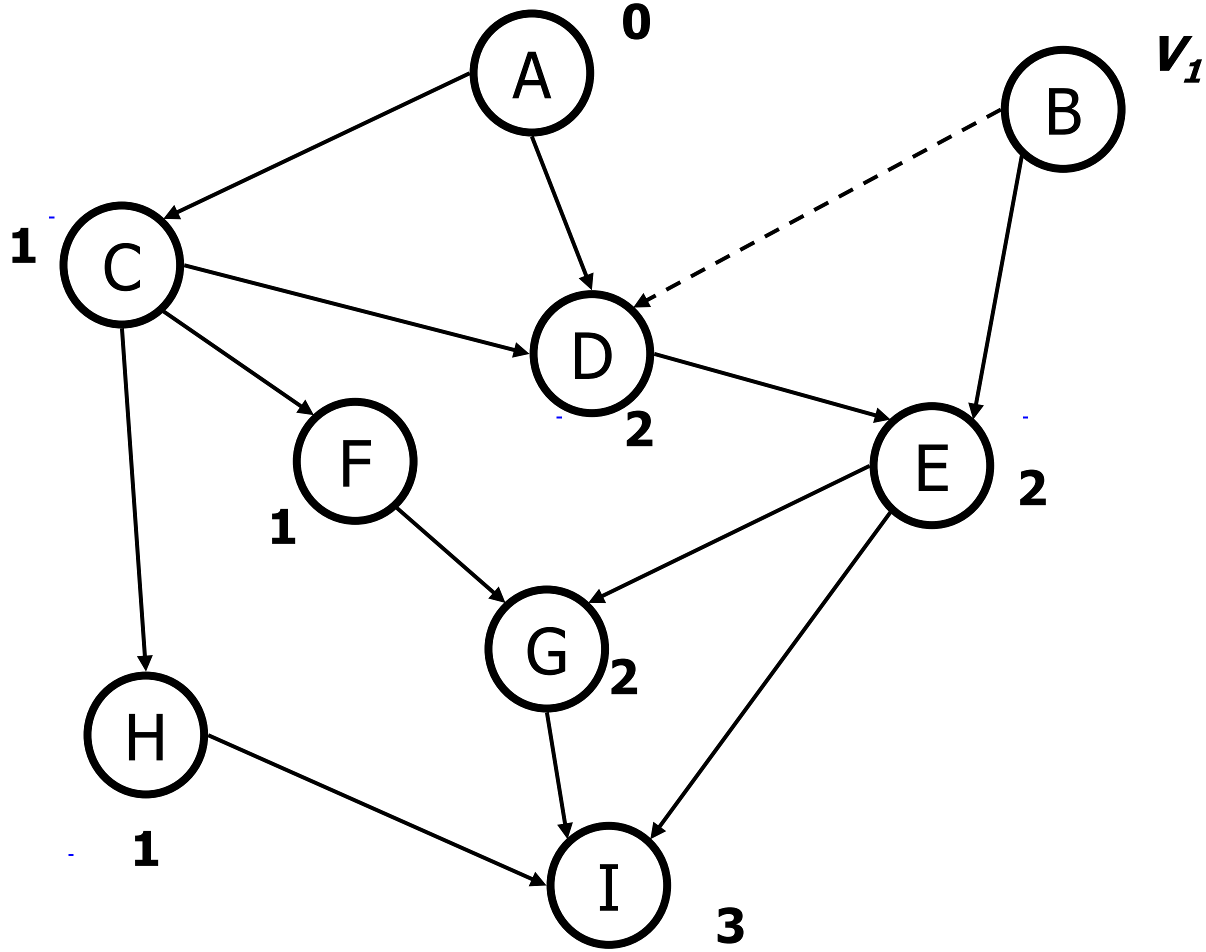
Topological Sorting Example



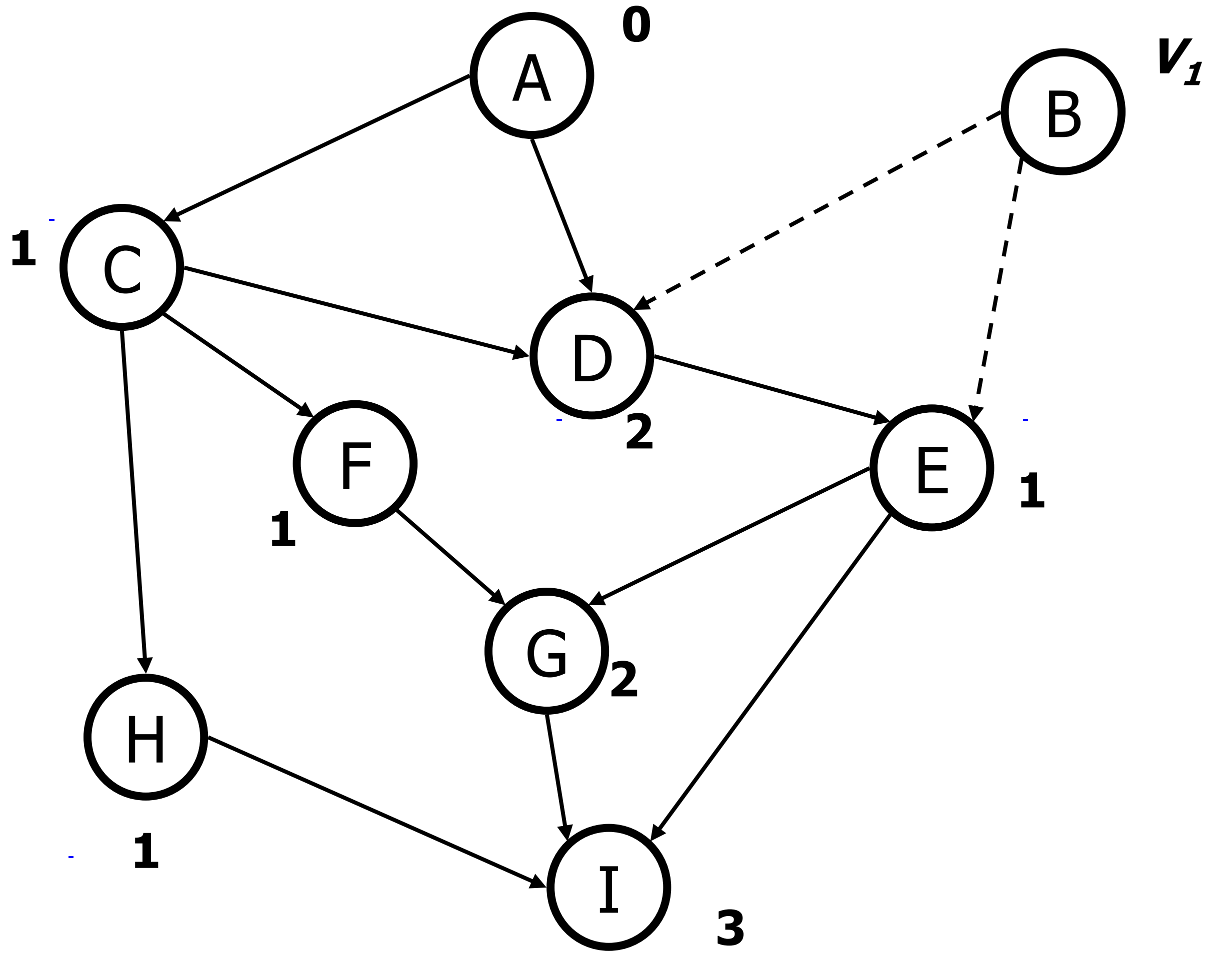
Topological Sorting Example



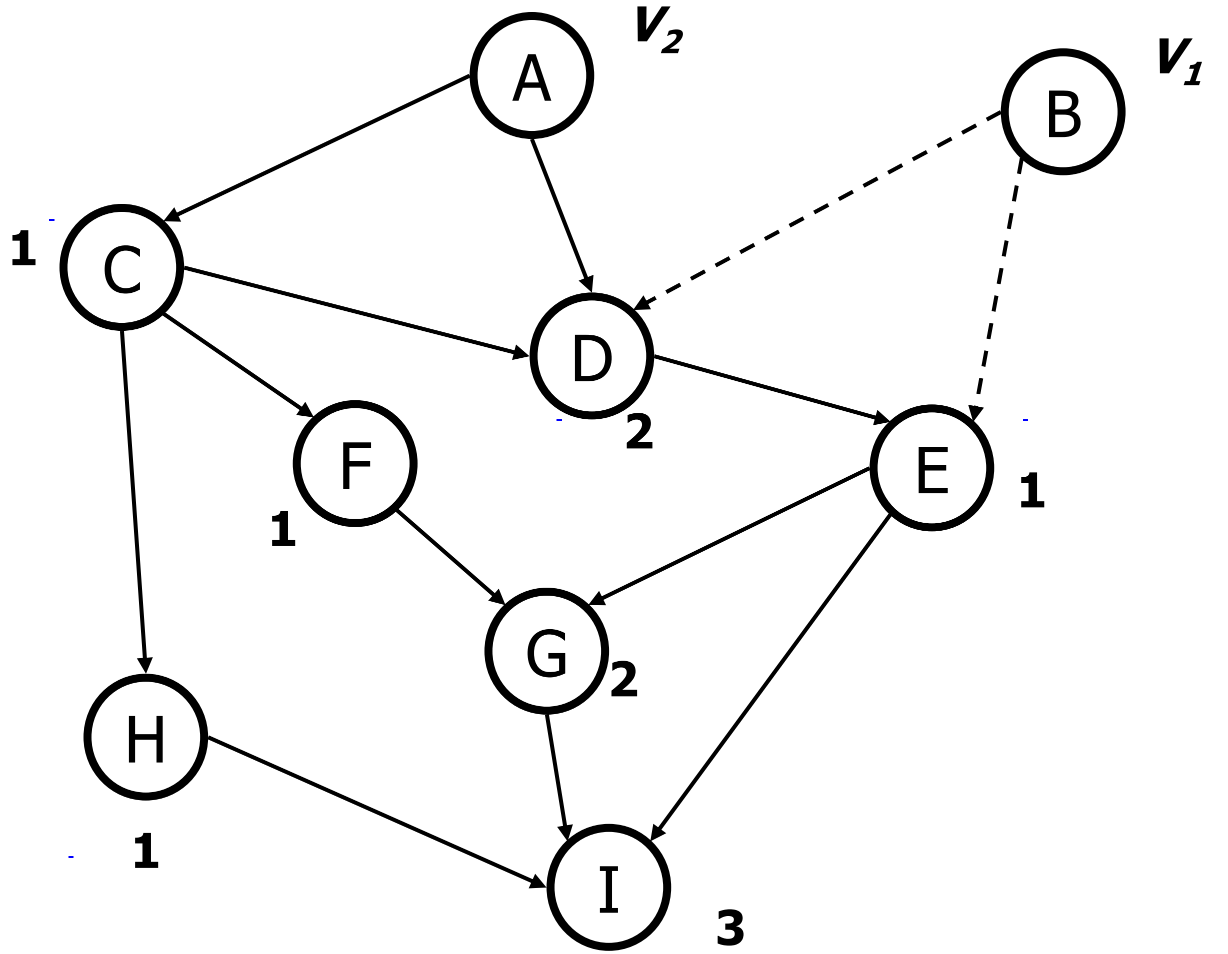
Topological Sorting Example



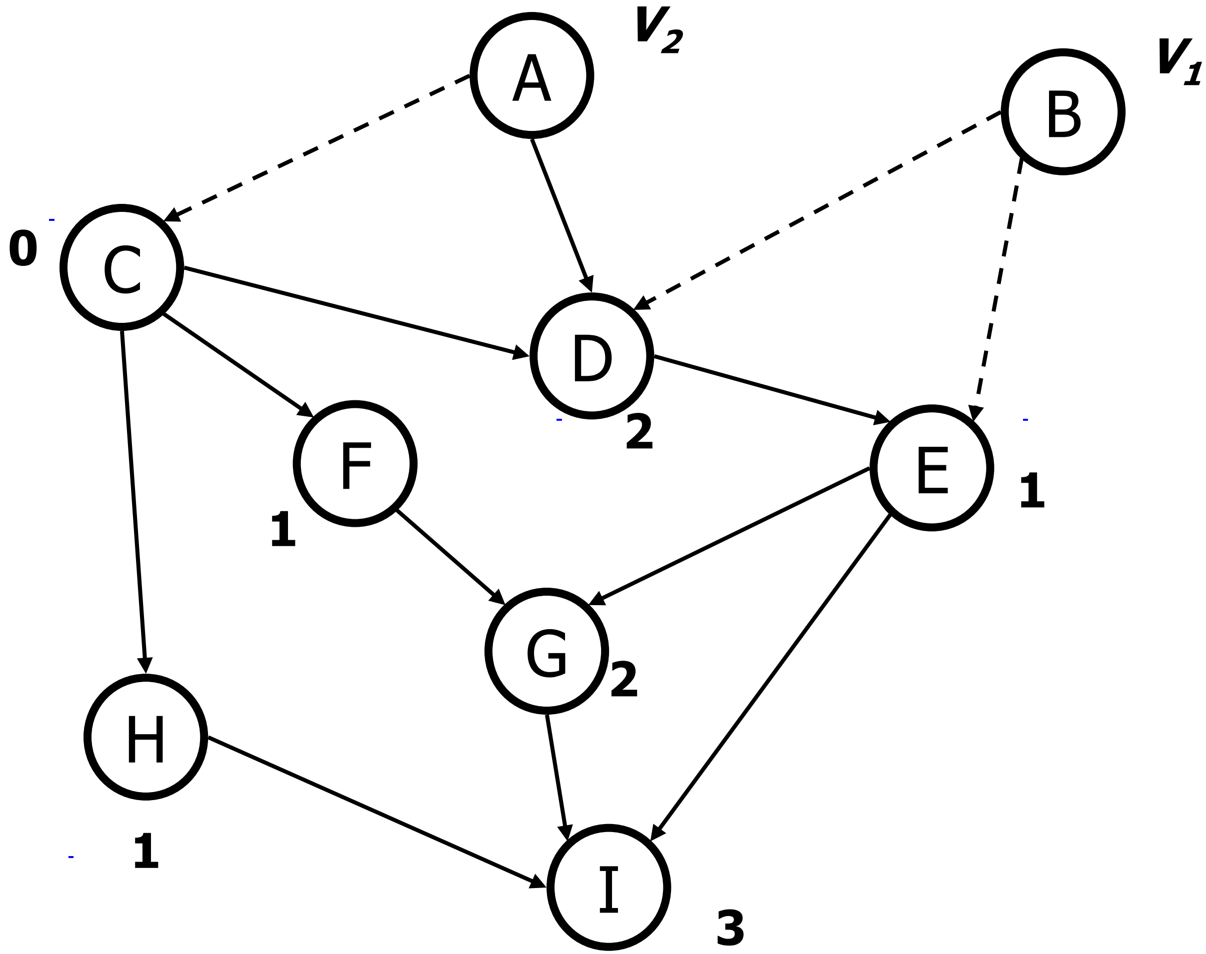
Topological Sorting Example



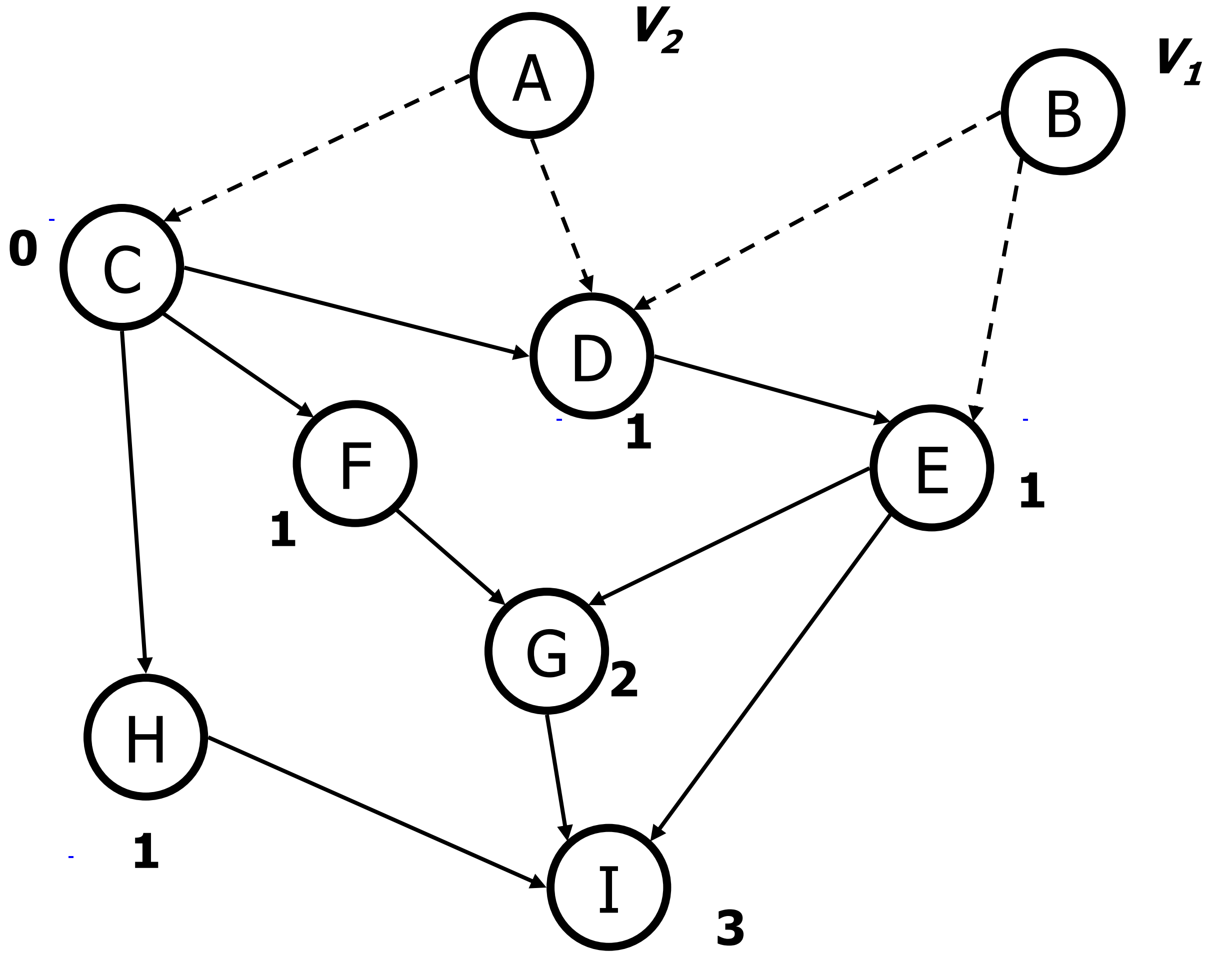
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



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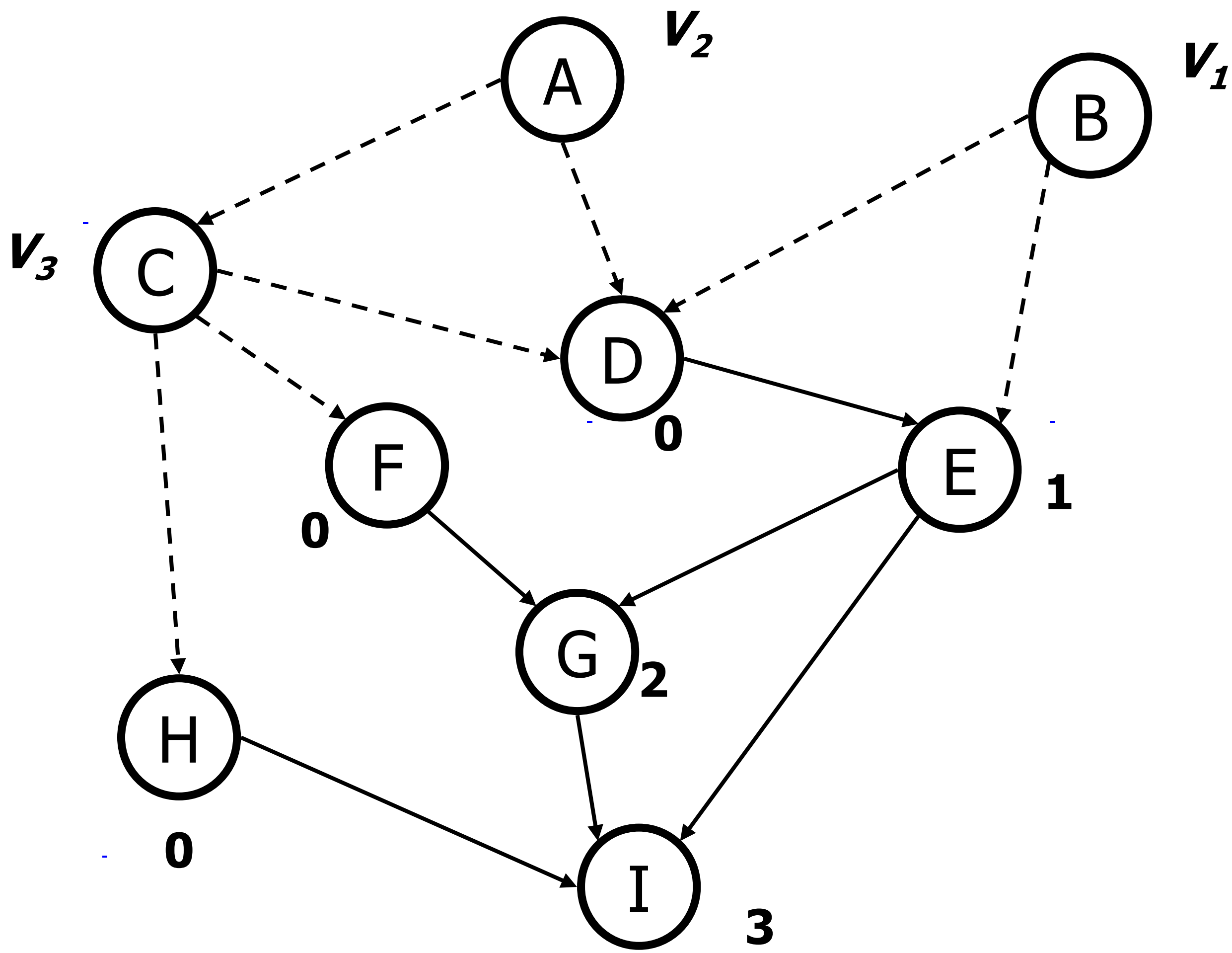

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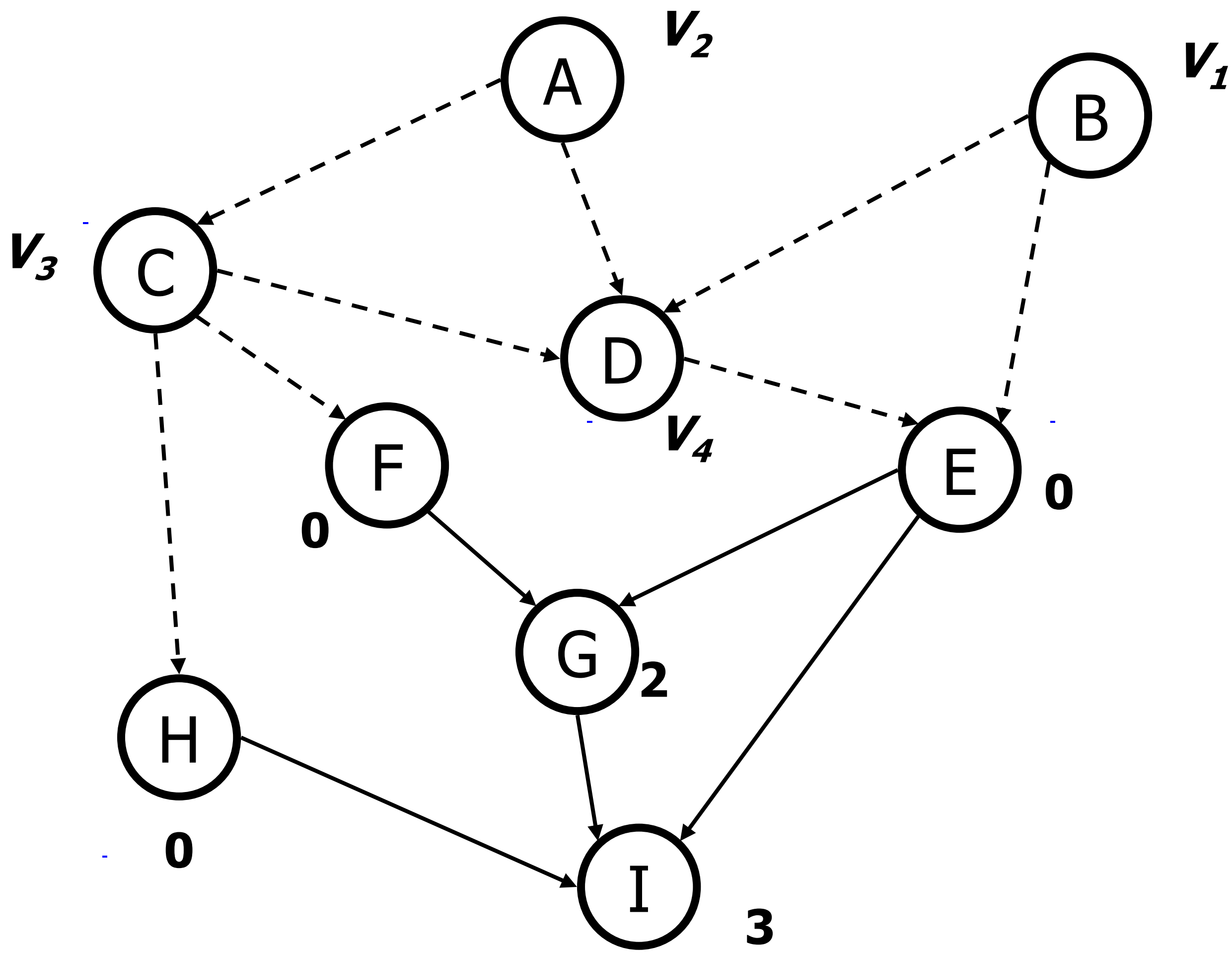

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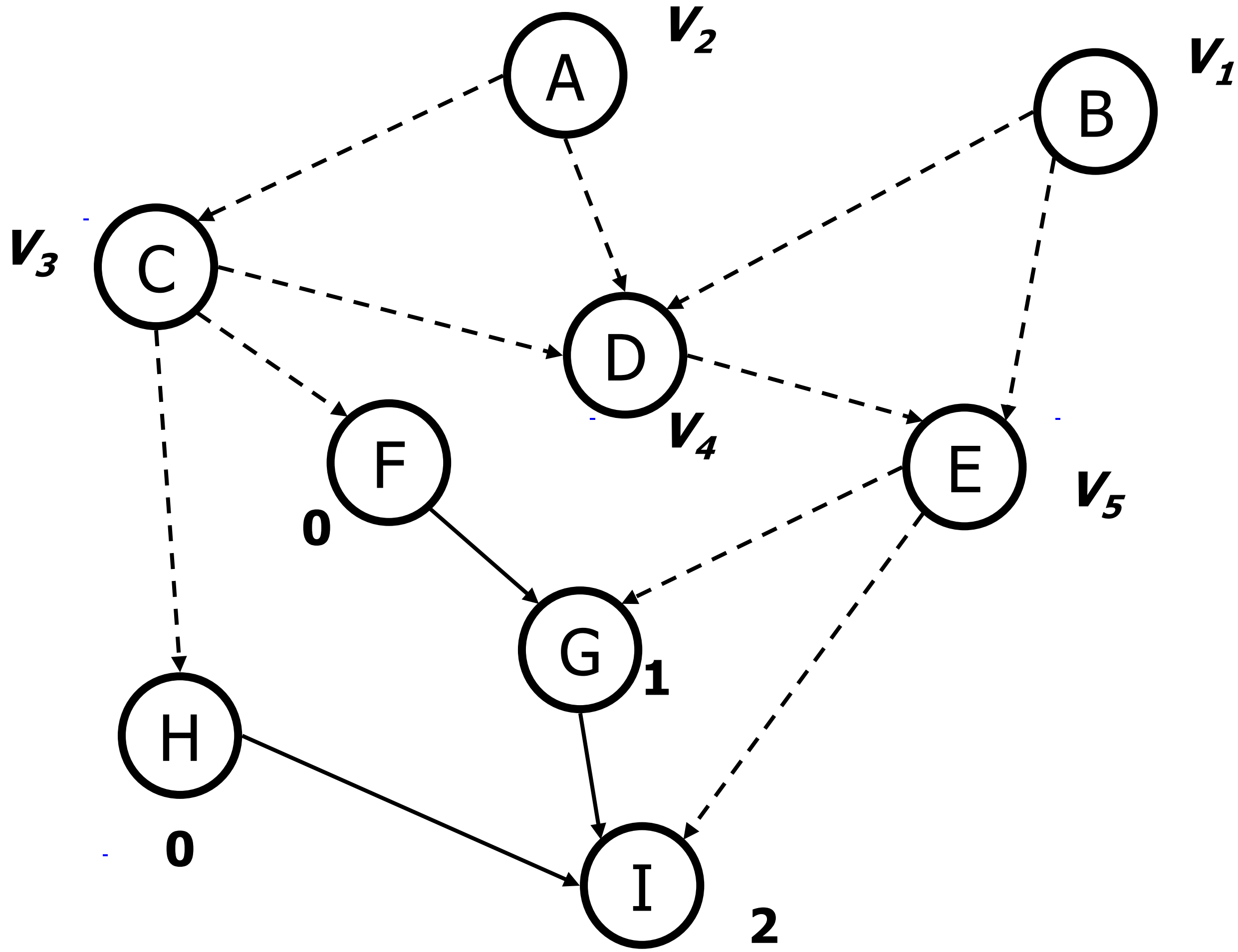

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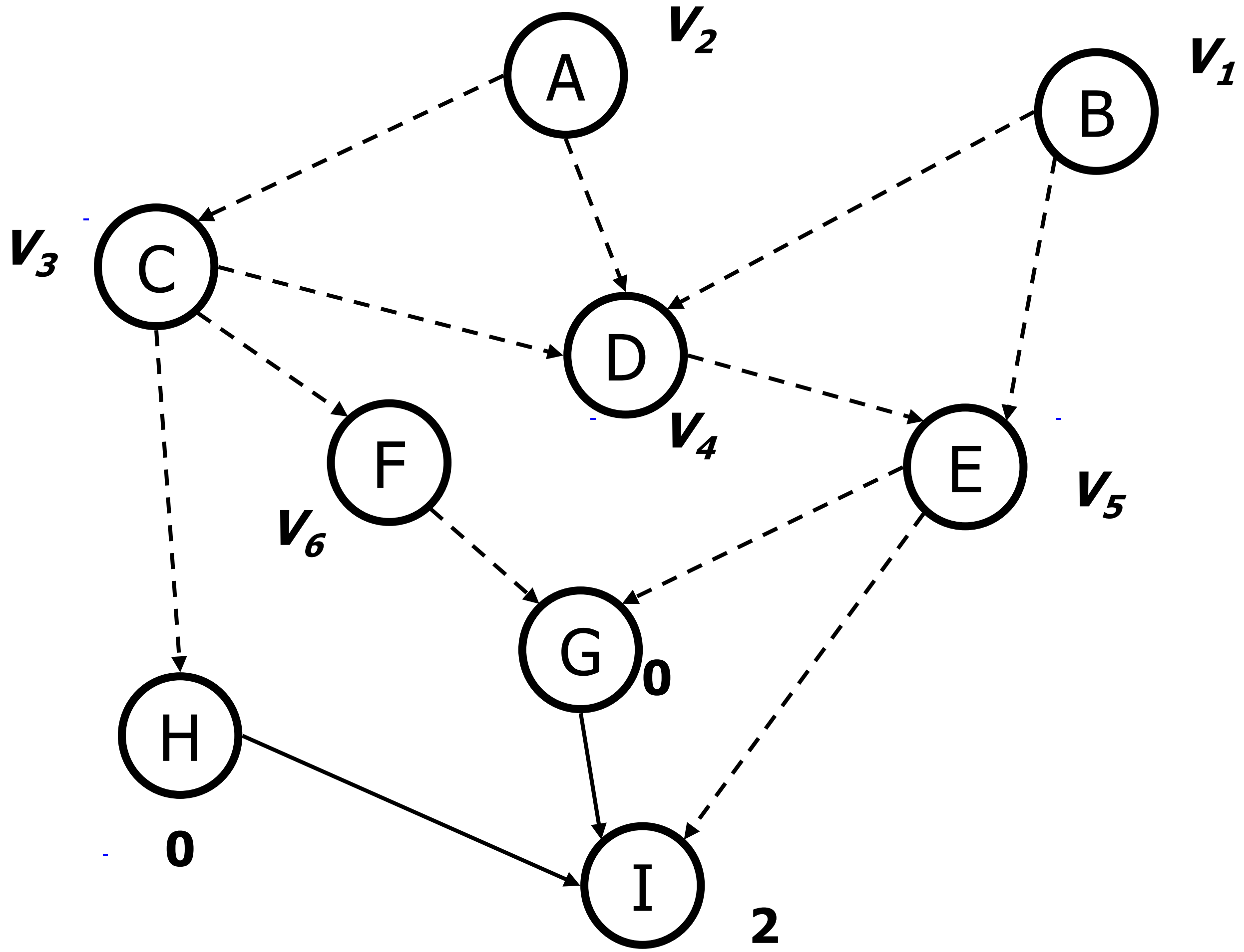

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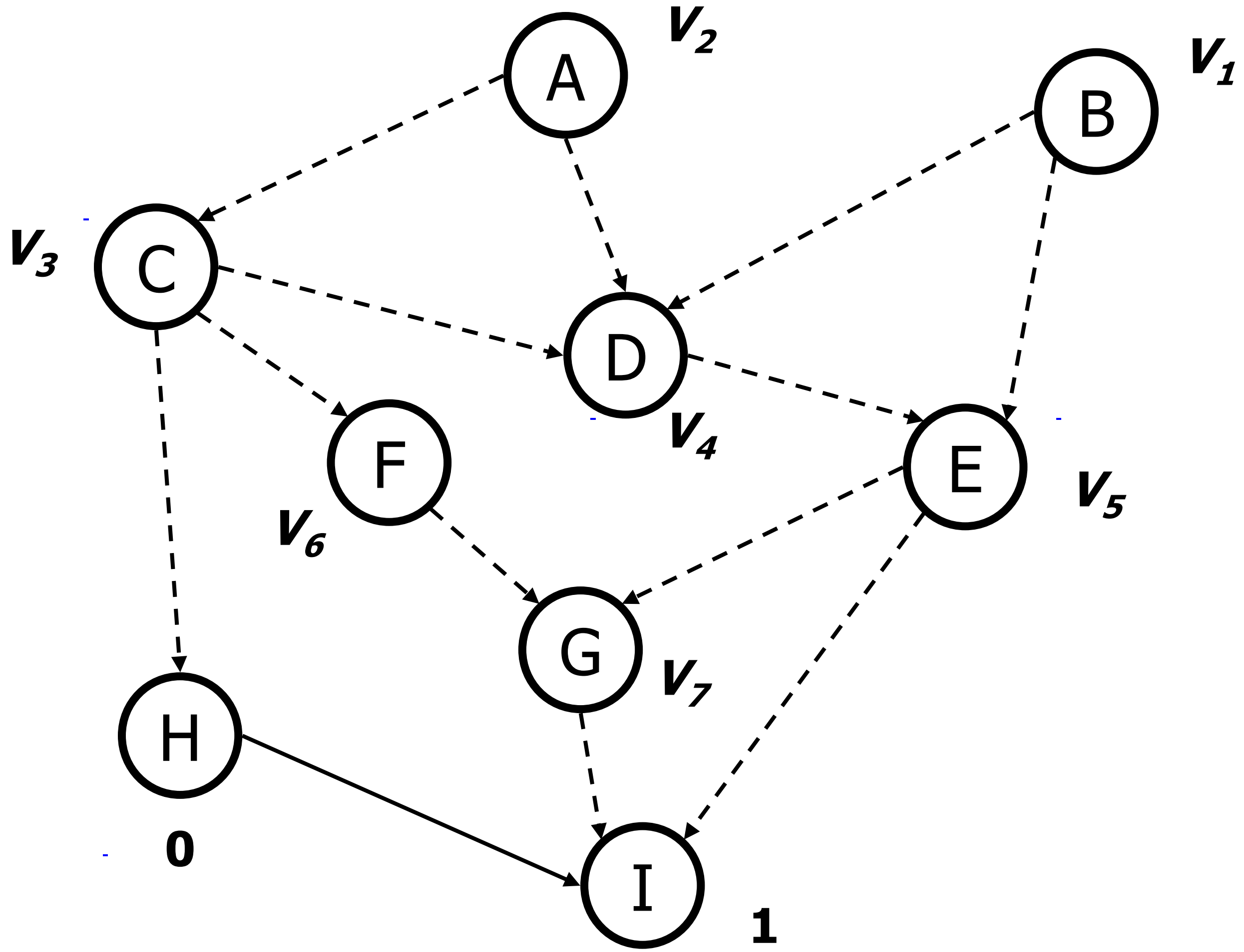

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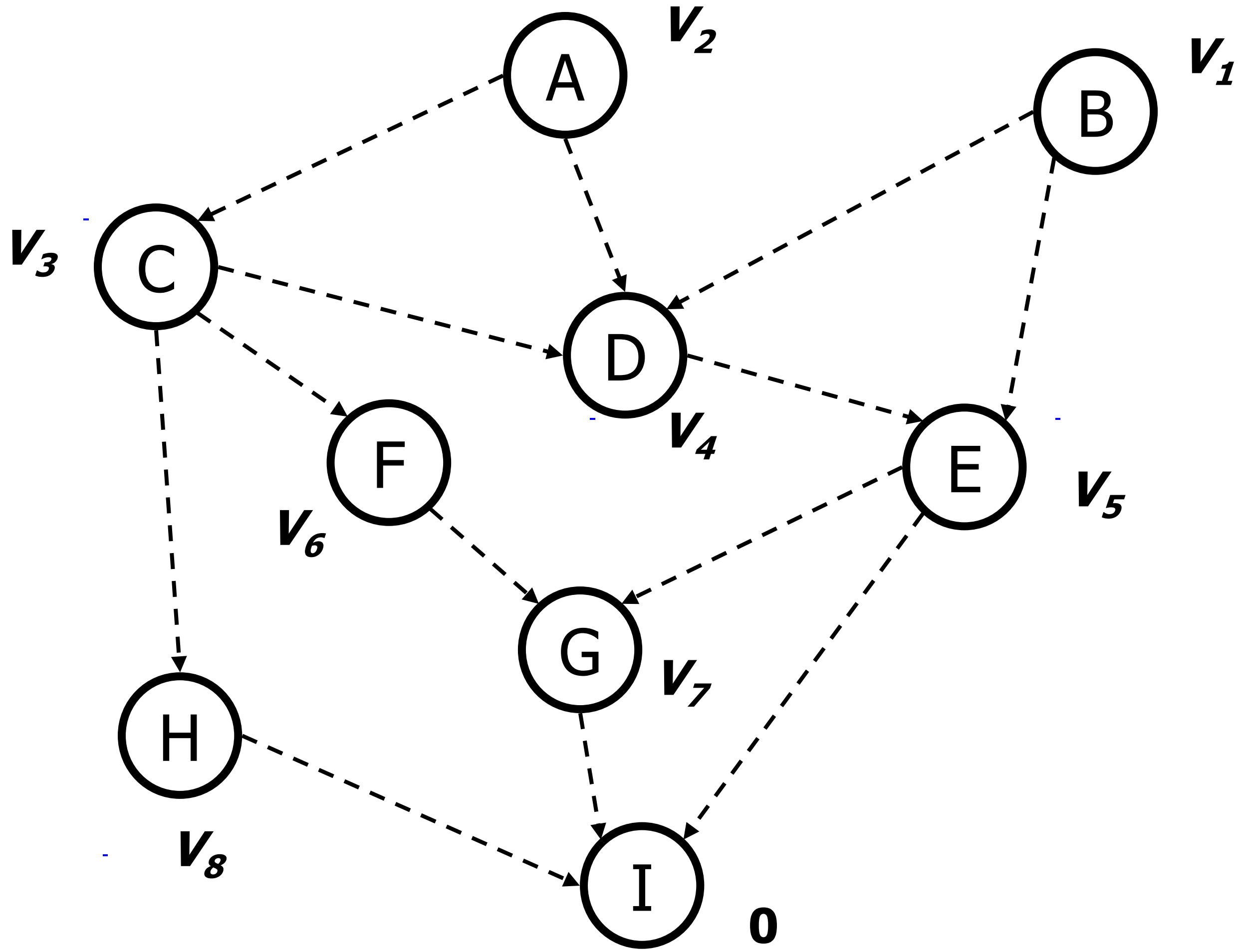

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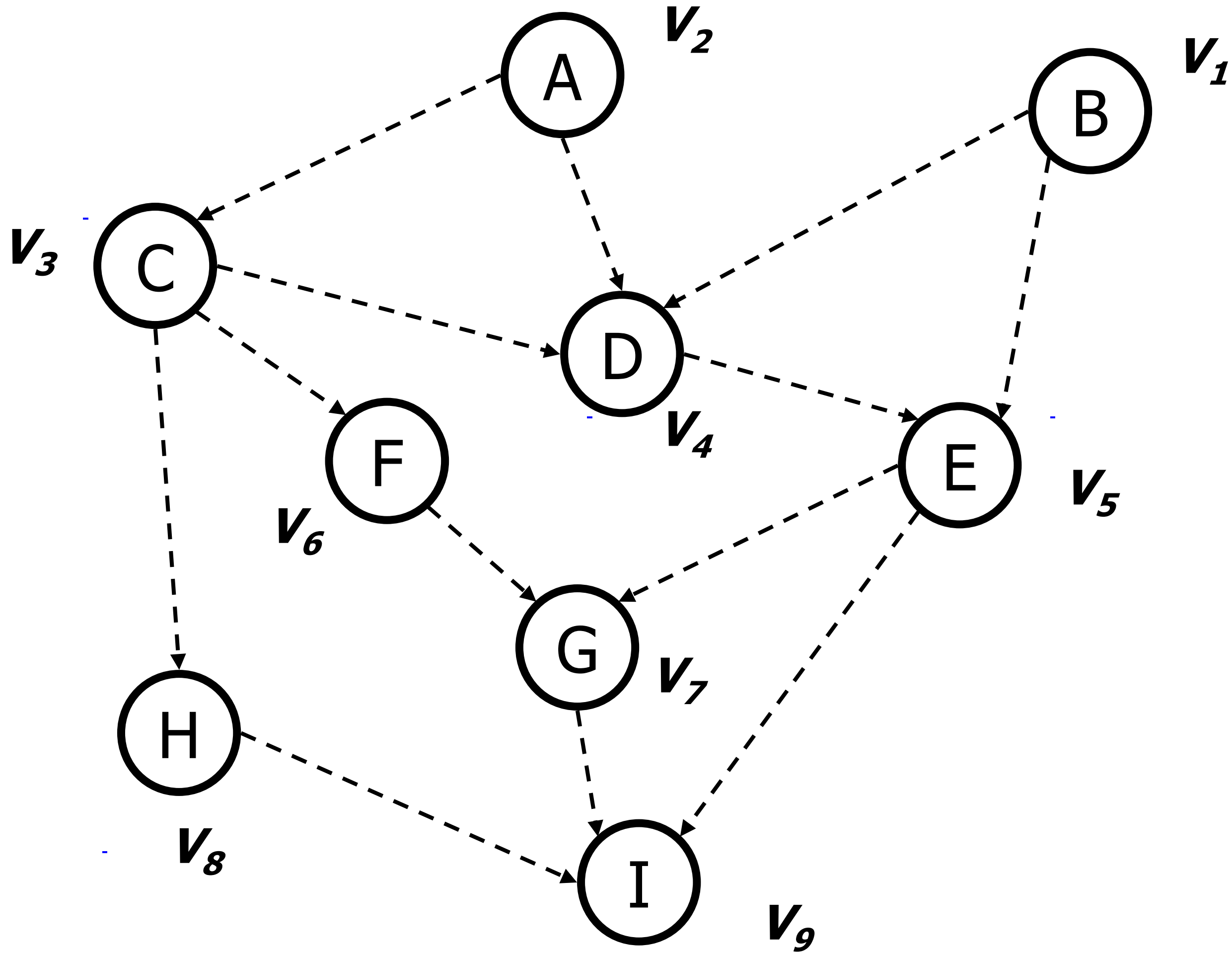

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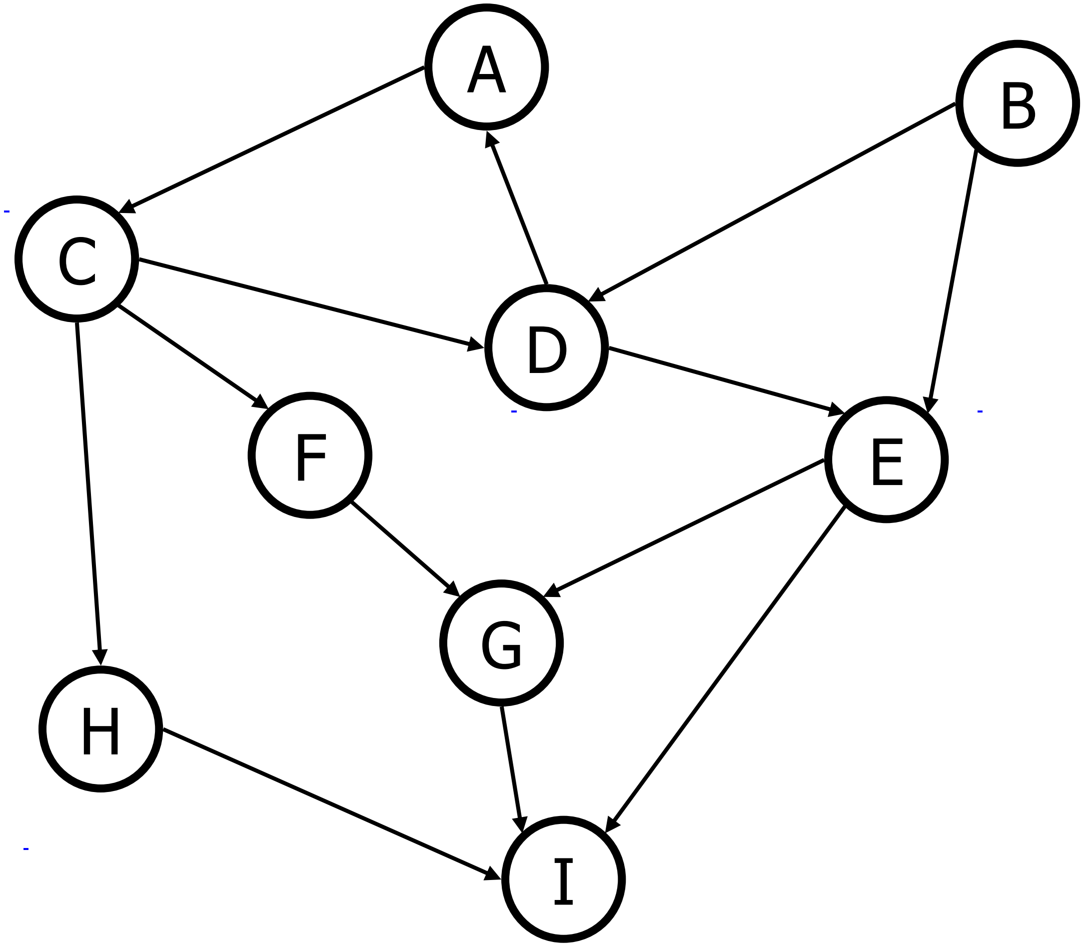

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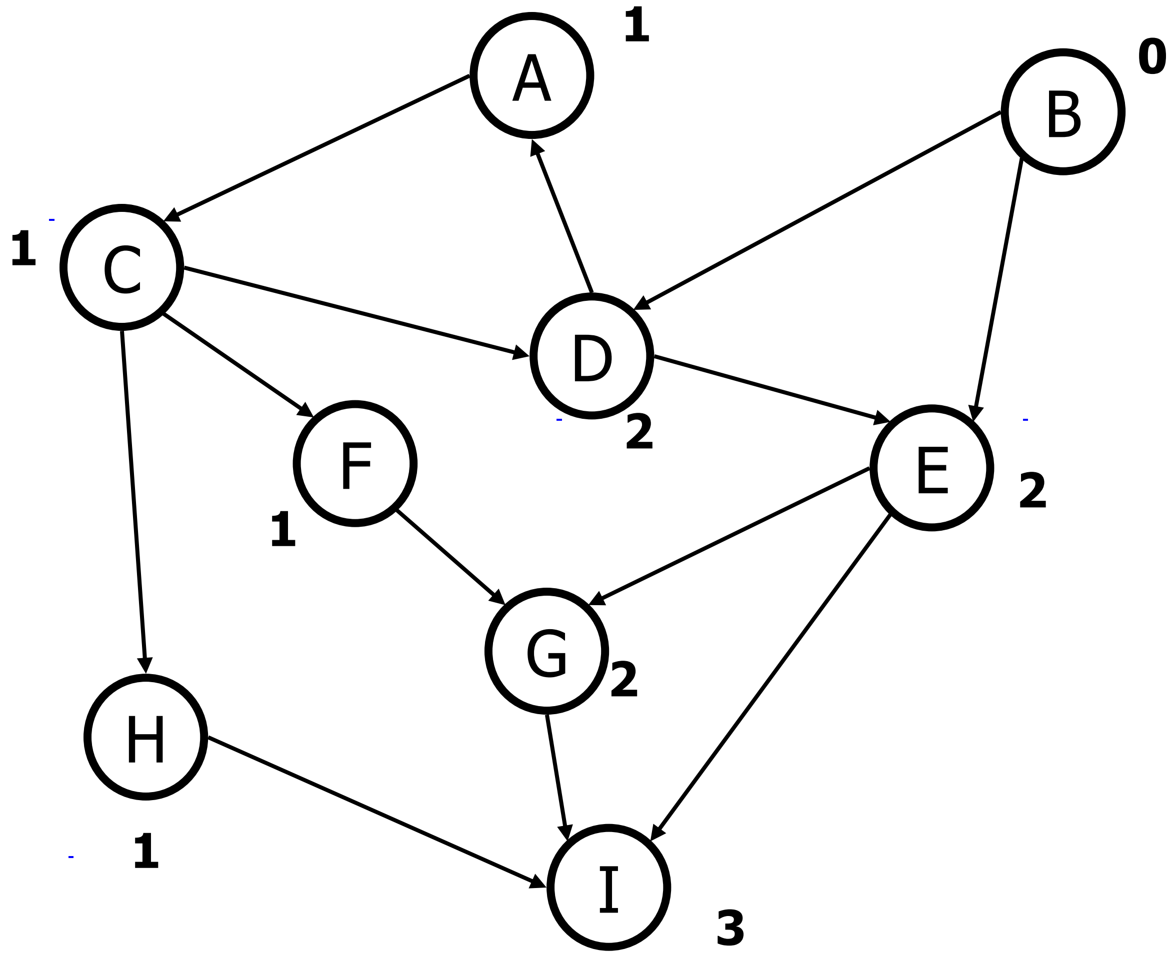

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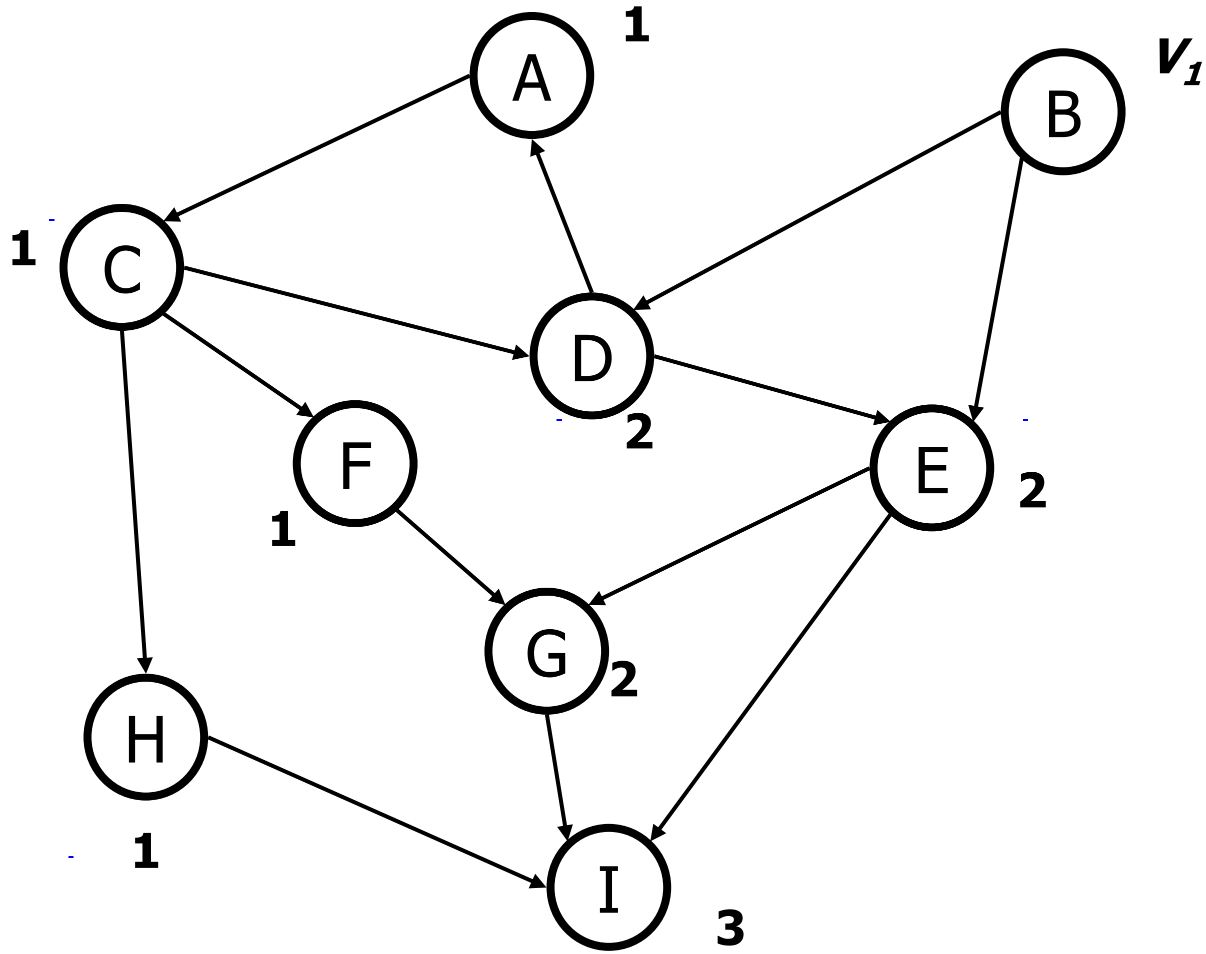

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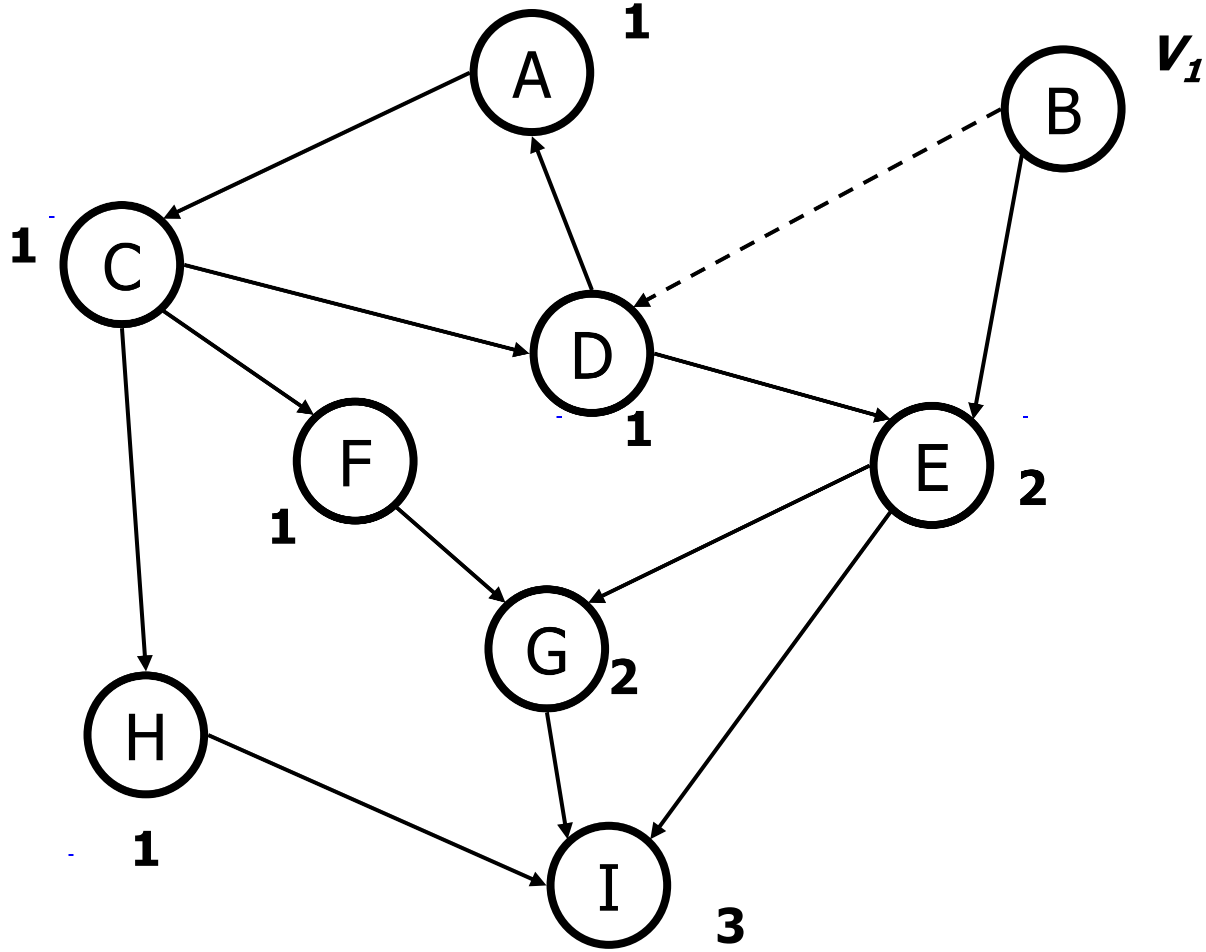

What about?