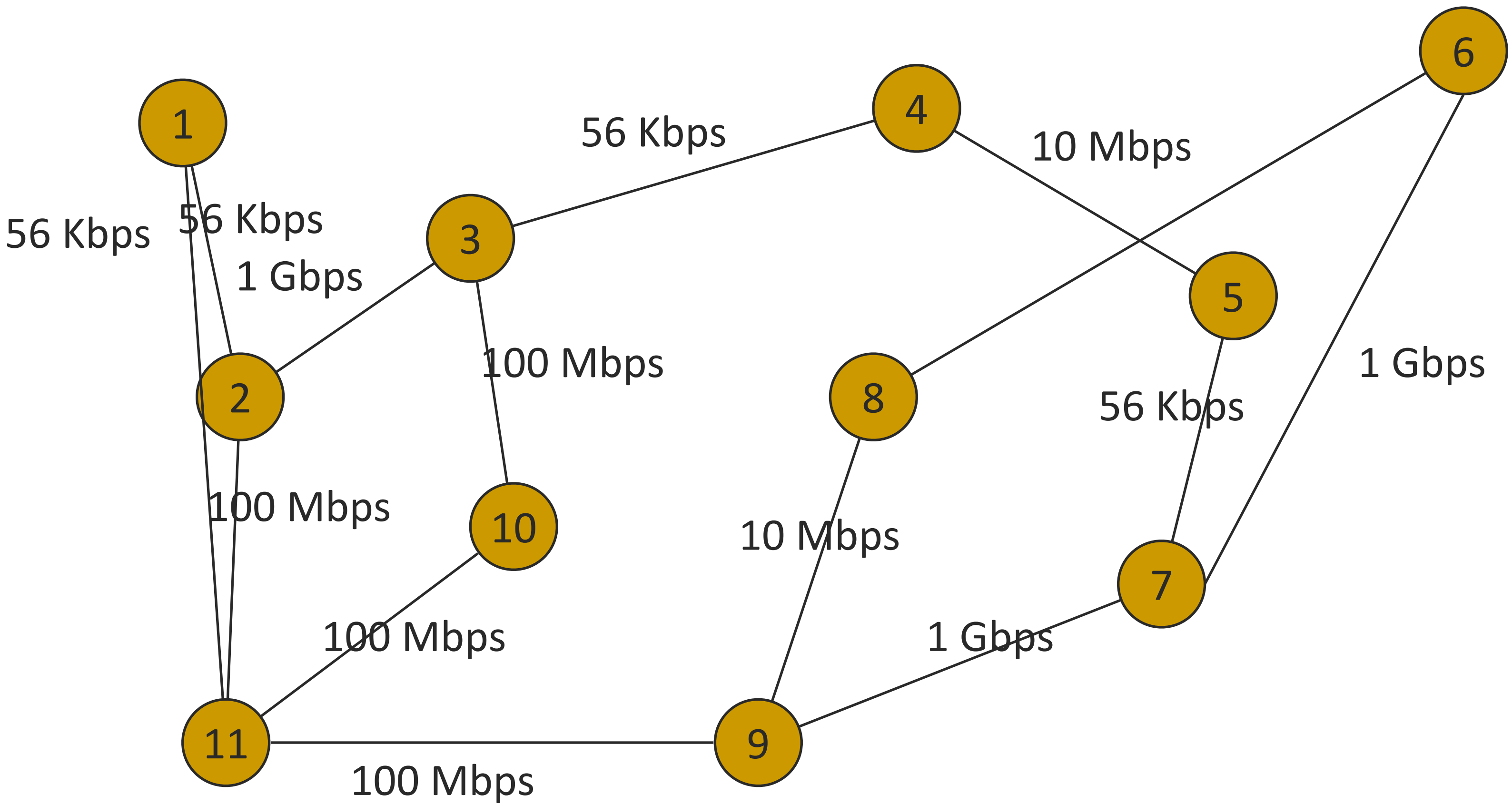
Shortest Paths

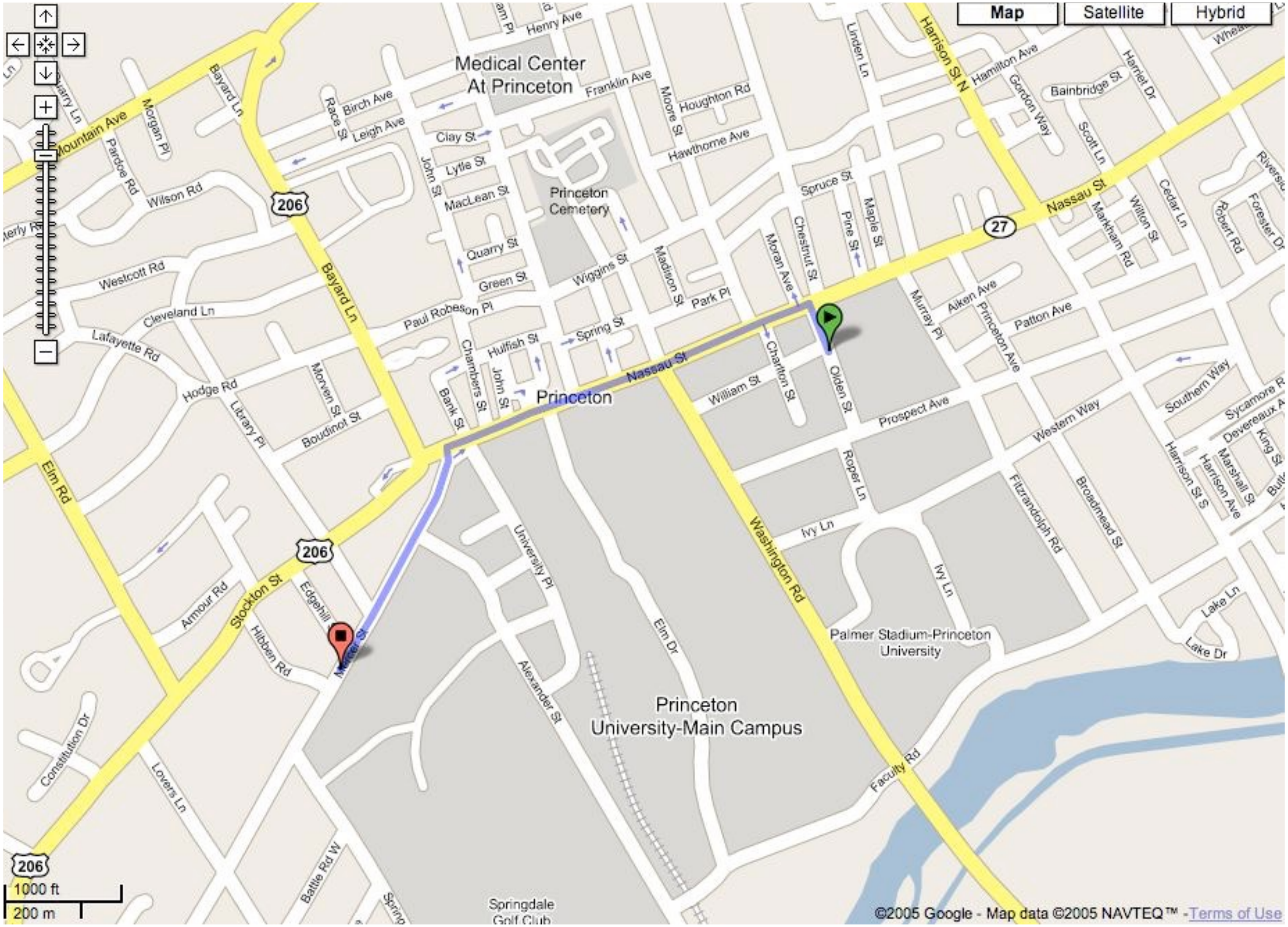
in edge-weighted graphs

Communication Speeds in a Computer Network

Find fastest way to route a data packet between two computers



Google maps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Satellite
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Palmer Stadium-Princeton
University
Princeton
University-Main Campus
Springdale
Golf Club
206
27
1000 ft
200 m
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Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest Path problems

- Find a shortest path between two given vertices
- Single source shortest paths
- Single sink shortest paths
- All pair shortest paths

Single Source Shortest Path problems

- Undirected graphs with non-negative edge weights
- Directed graphs with non-negative edge weights
- Directed graphs with arbitrary weights

Single Source Shortest Paths

- If graph is not weighted (all edge-weights are unit-weight): BFS works
- Now assume: graph is edge-weighted
  - Every edge is associated with a positive number
    - Possible weights: integers, real numbers, rational numbers
    - Edge-weights can represent: distance, connection cost, affinity

Single Source Shortest Paths and shortest distances

- Input: An edge-weighted undirected graph and a source node v with: for every edge e edge-weight w(e) > 0
- Output: All single-source shortest paths (and their weight) for v in G: for every node $w \neq v$ in G a shortest path from v to w.
  - Here, a *path* p from v to w consisting of edges $e_0, e_1, \dots, e_{k-1}$ is shortest in G, if its length

$$w(p) = \sum_{i=0}^{k-1} w(e_i)$$

is minimum (i.e., there is no path from v to w in G that is shorter).

Let d(u) = length of shortest path from v to u

Algorithm
`DijkstraShortestPaths(G,v)`

Input: A simple undirected graph G with nonnegative edge-weights, a distinguished vertex v in G

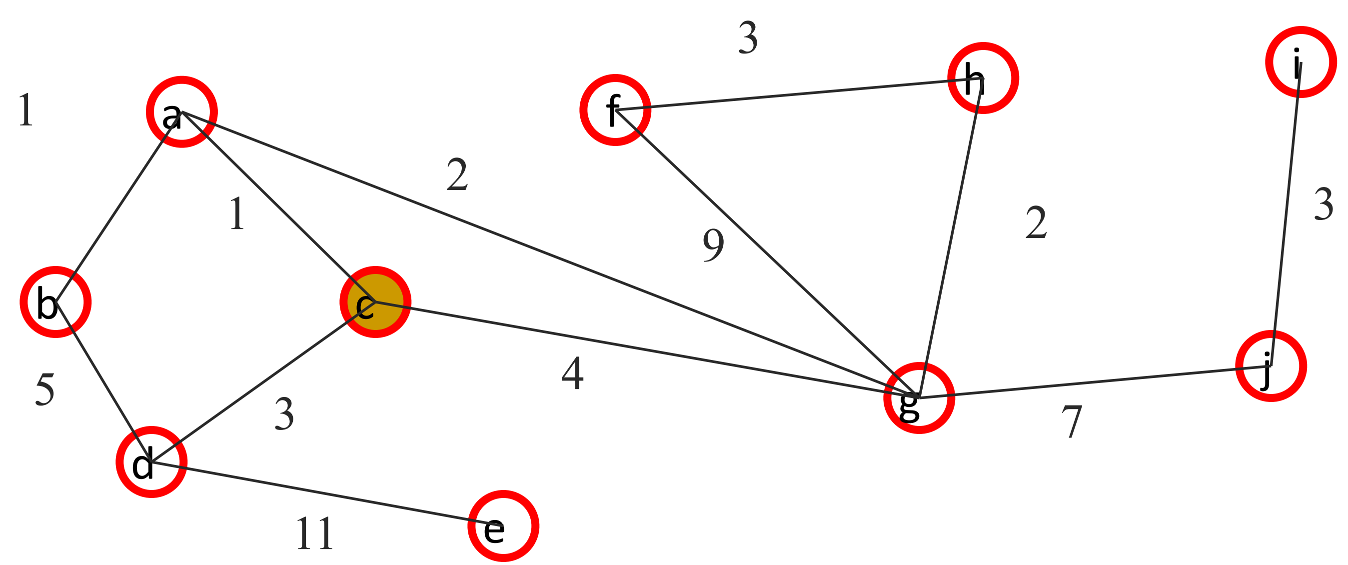
Output: A label D[u] for each vertex u in G such that D[u]=d(u)

Algorithm DijkstraShortestPaths(G,v)

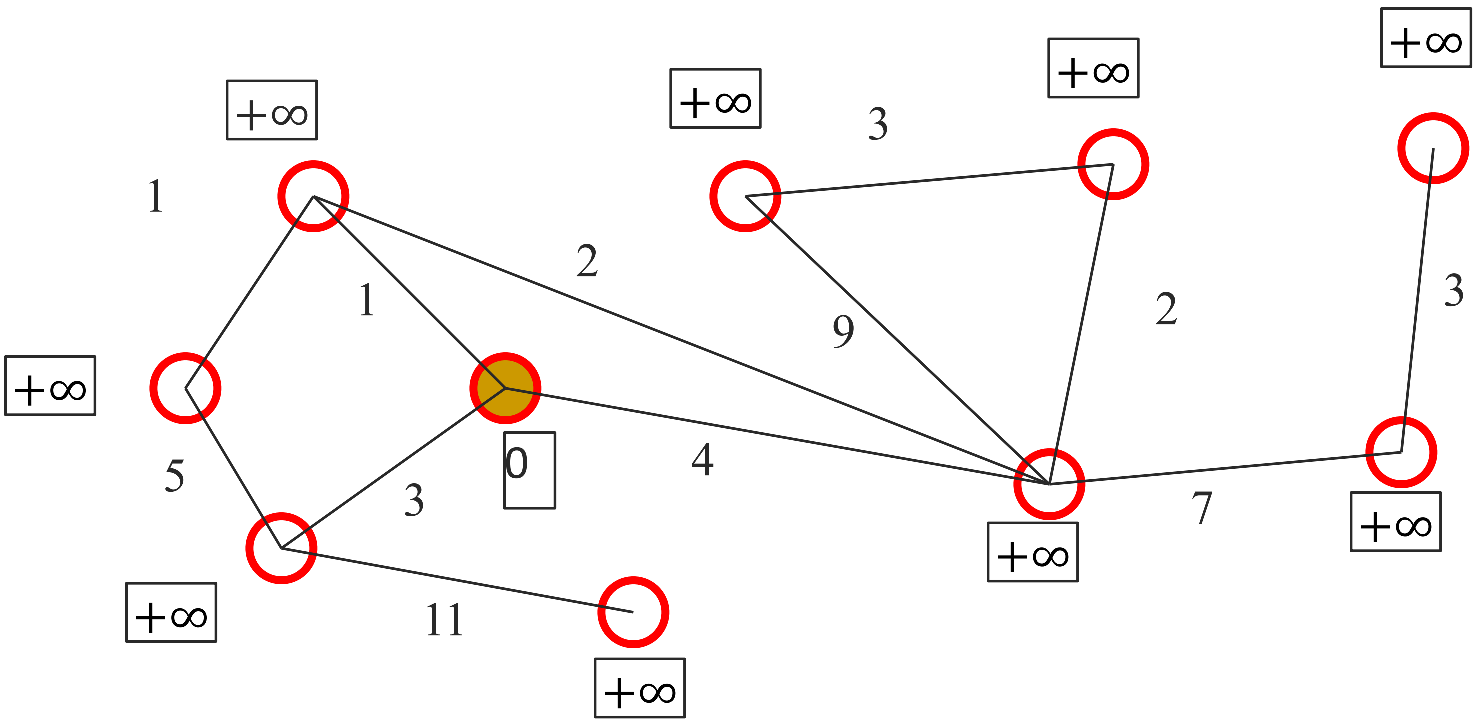
```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to T
   for each vertex z∈N(u) with z∈Q do
      if D[u]+w((u,z)) < D[z] then
         D[z]←D[u]+w((u,z))
         update z's key in Q to D[z]
return D
```

Relaxation

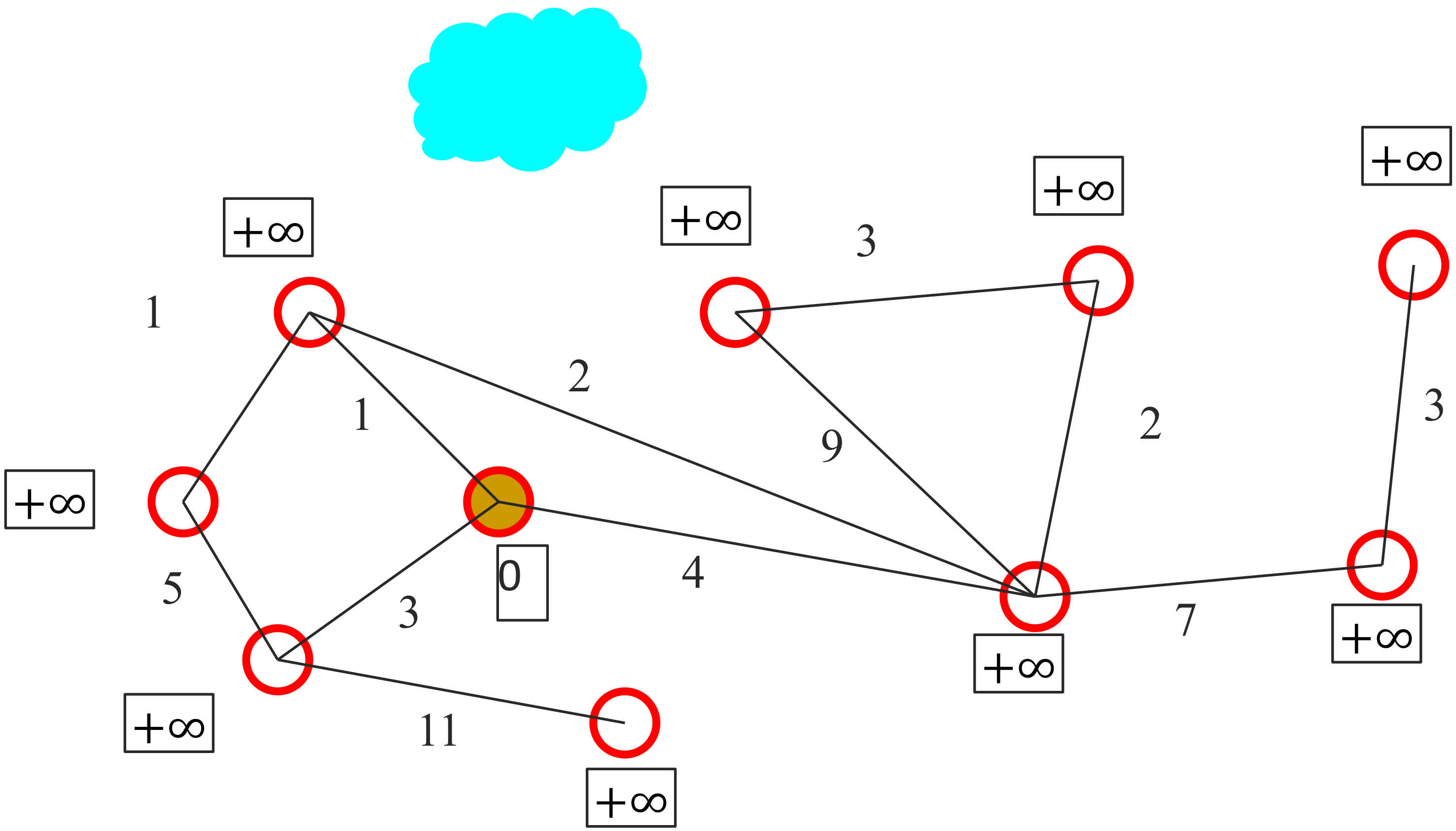
Dijkstra’s algorithm: a greedy algorithm



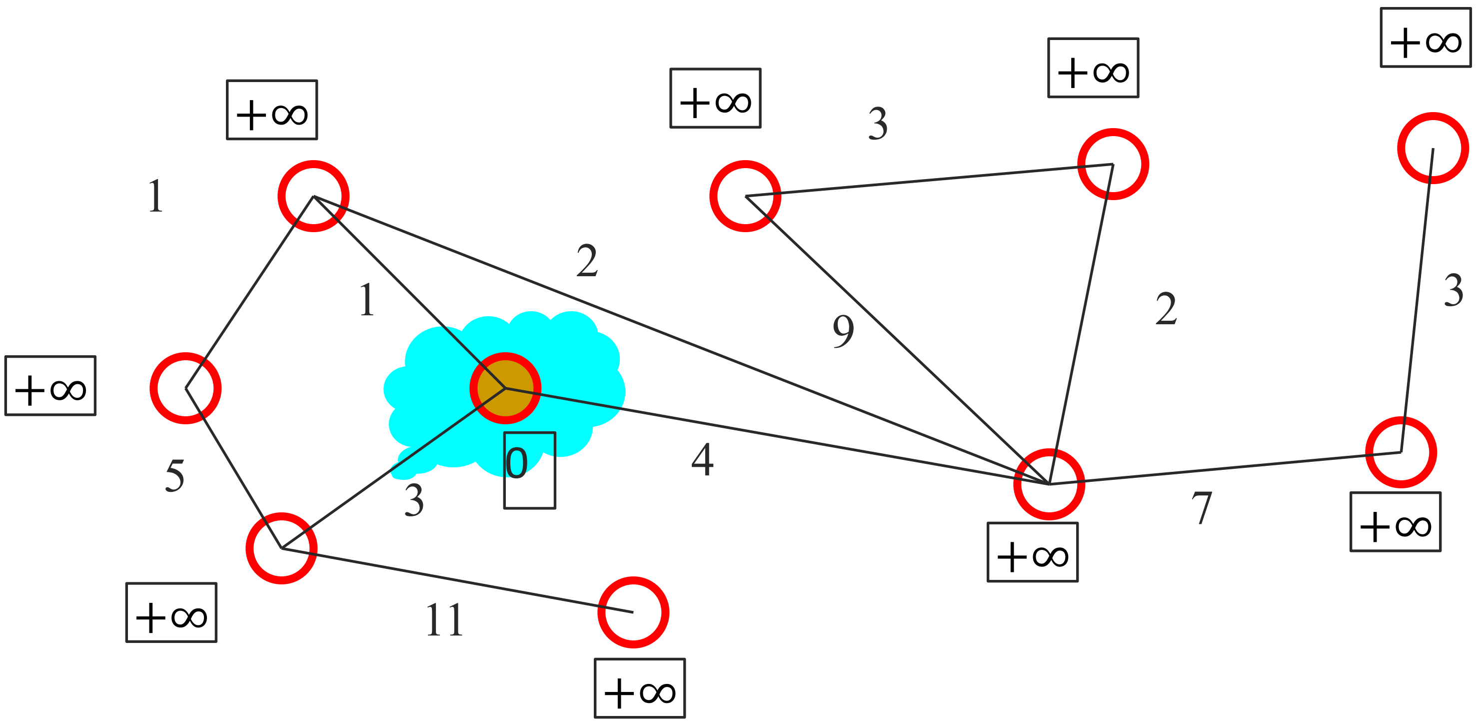
Dijkstra’s algorithm: Initializing



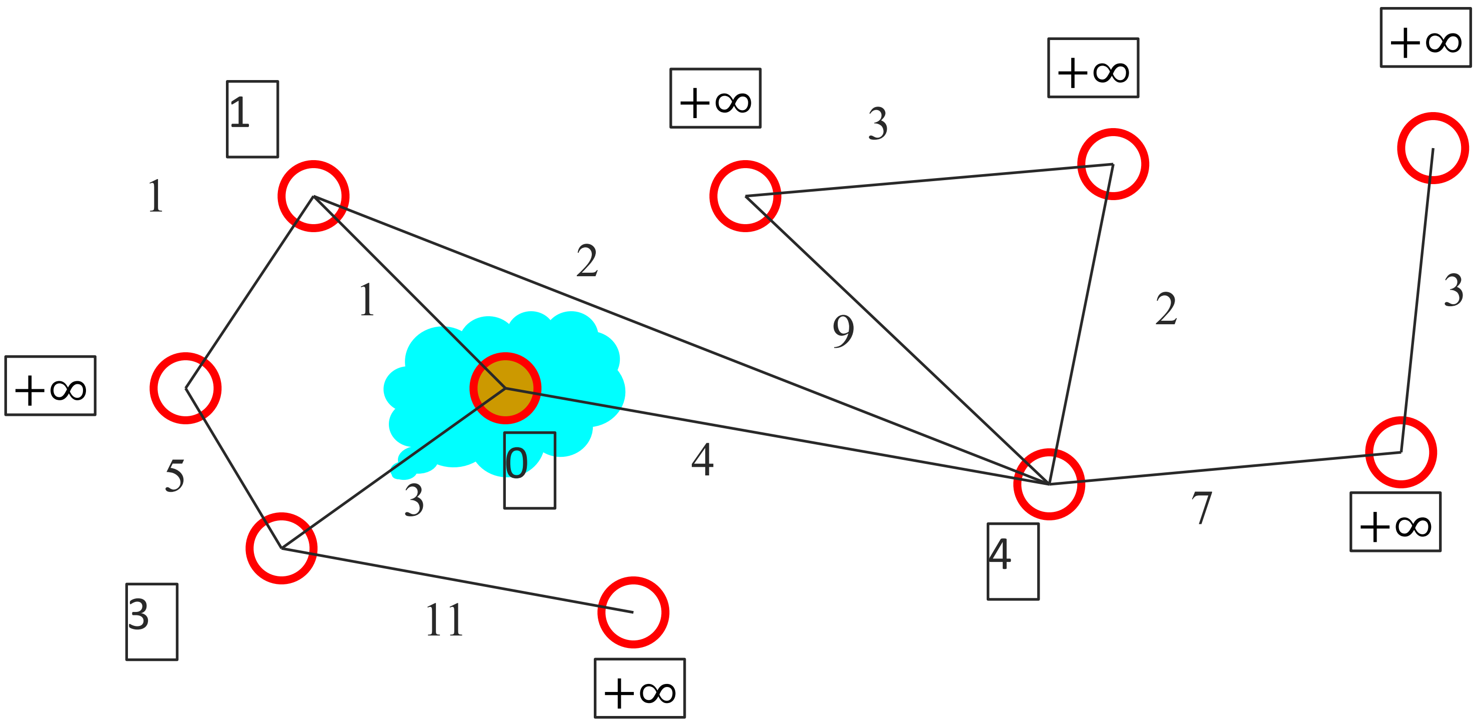
Dijkstra's algorithm: Initializing Cloud C (consisting of "solved" subgraph)



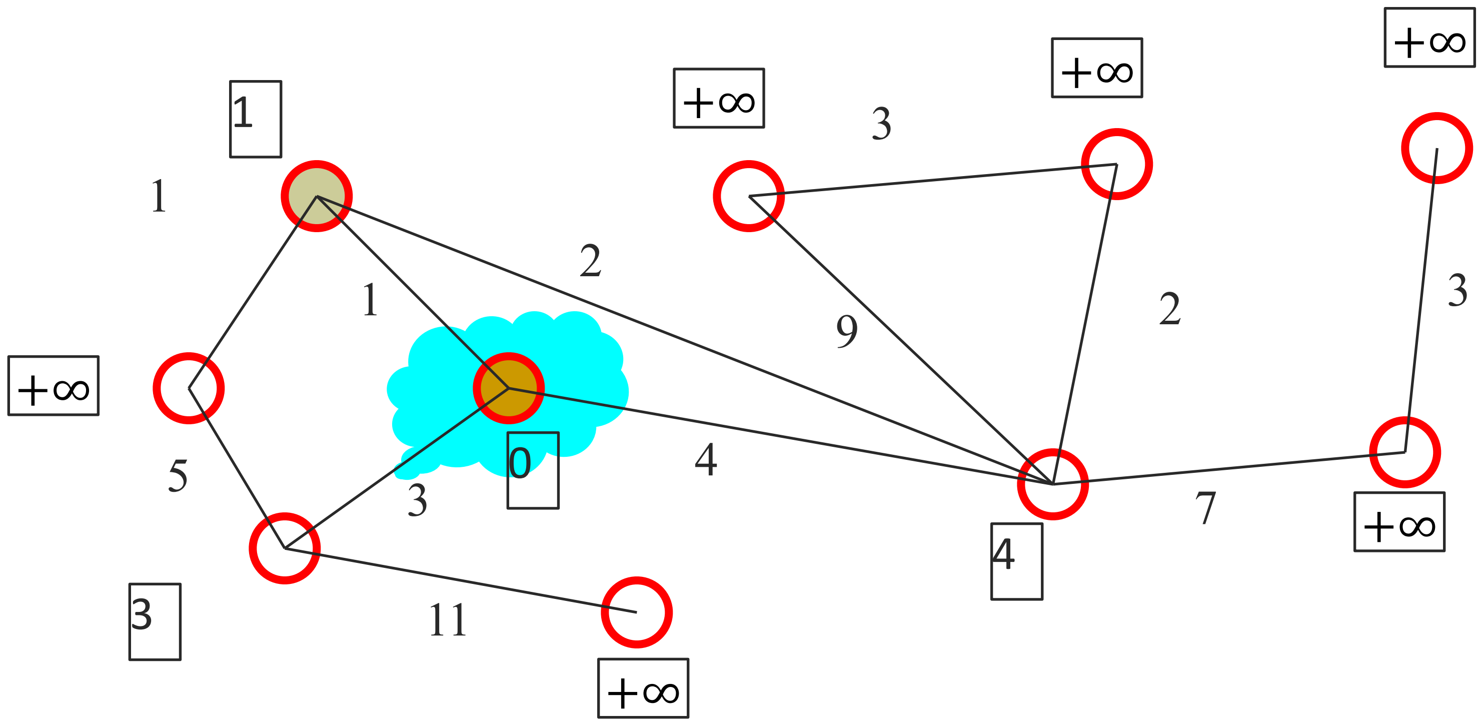
Dijkstra's algorithm: pull v into C



Dijkstra's algorithm: update *C's* neighborhood



Dijkstra’s algorithm: pick closest vertex u outside C



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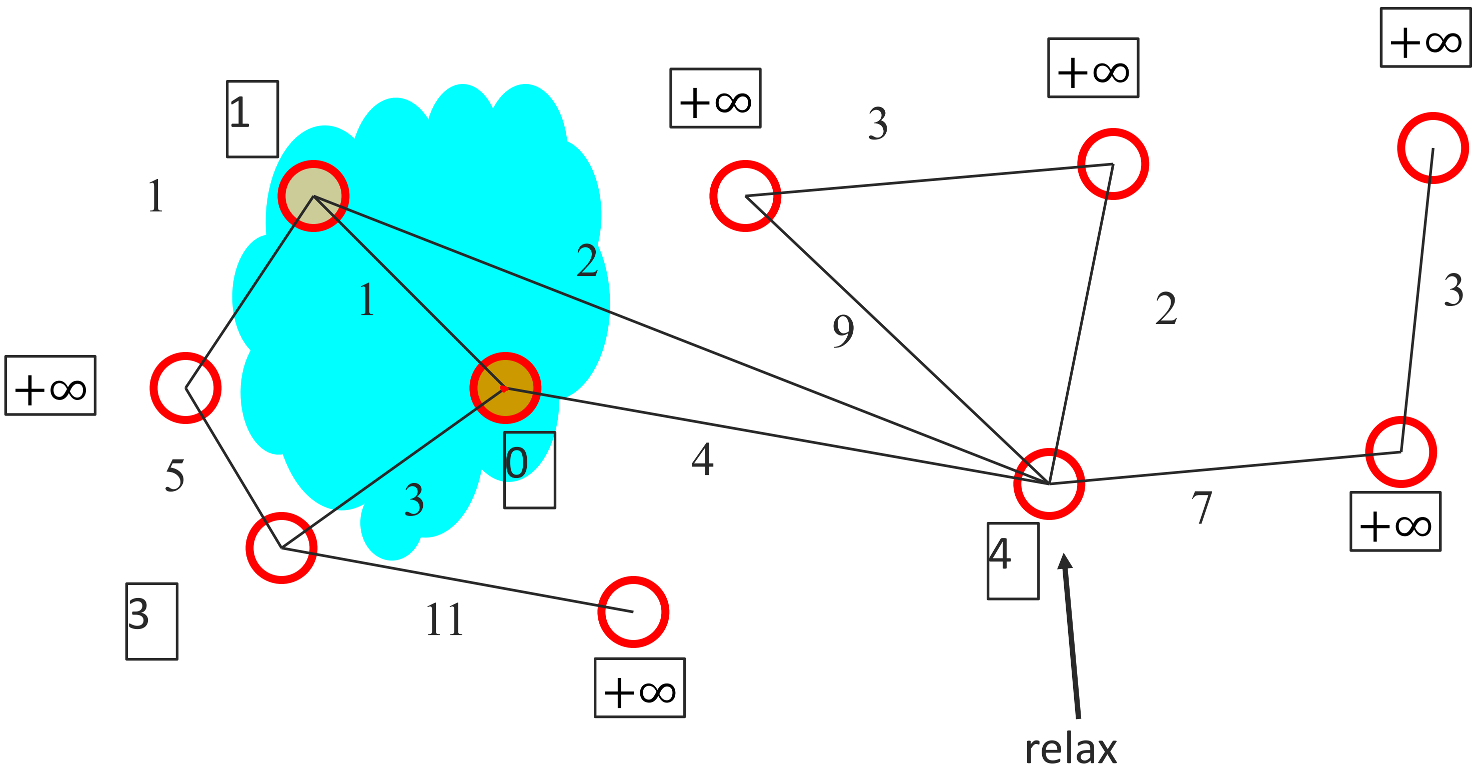

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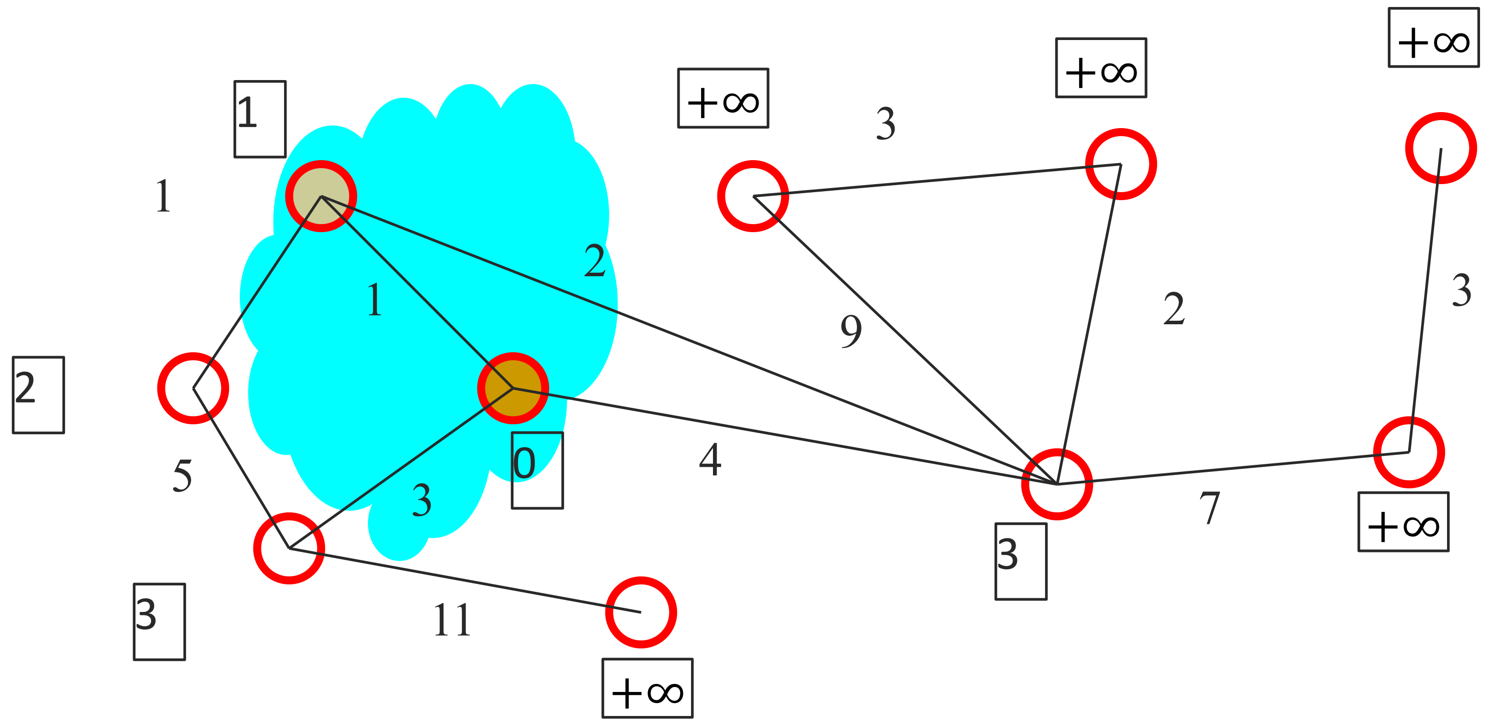

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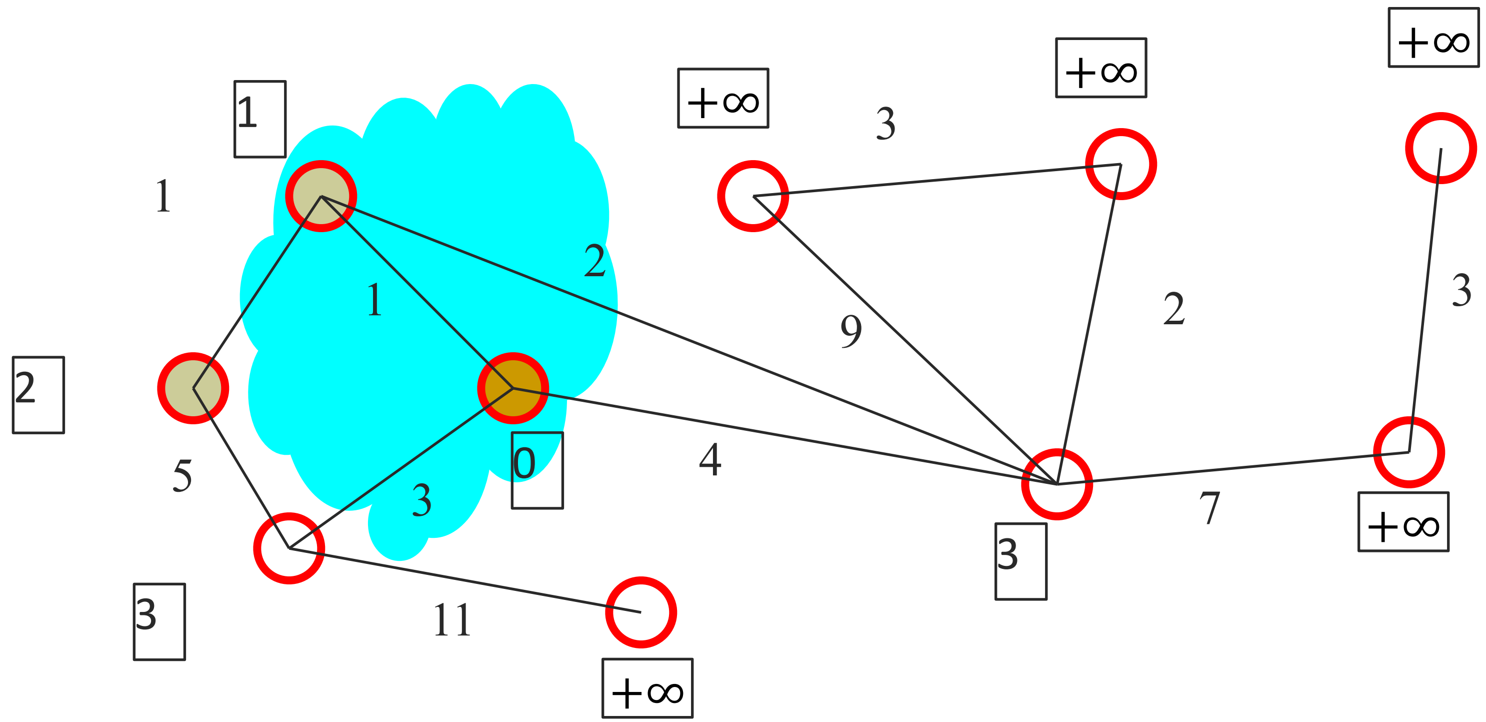

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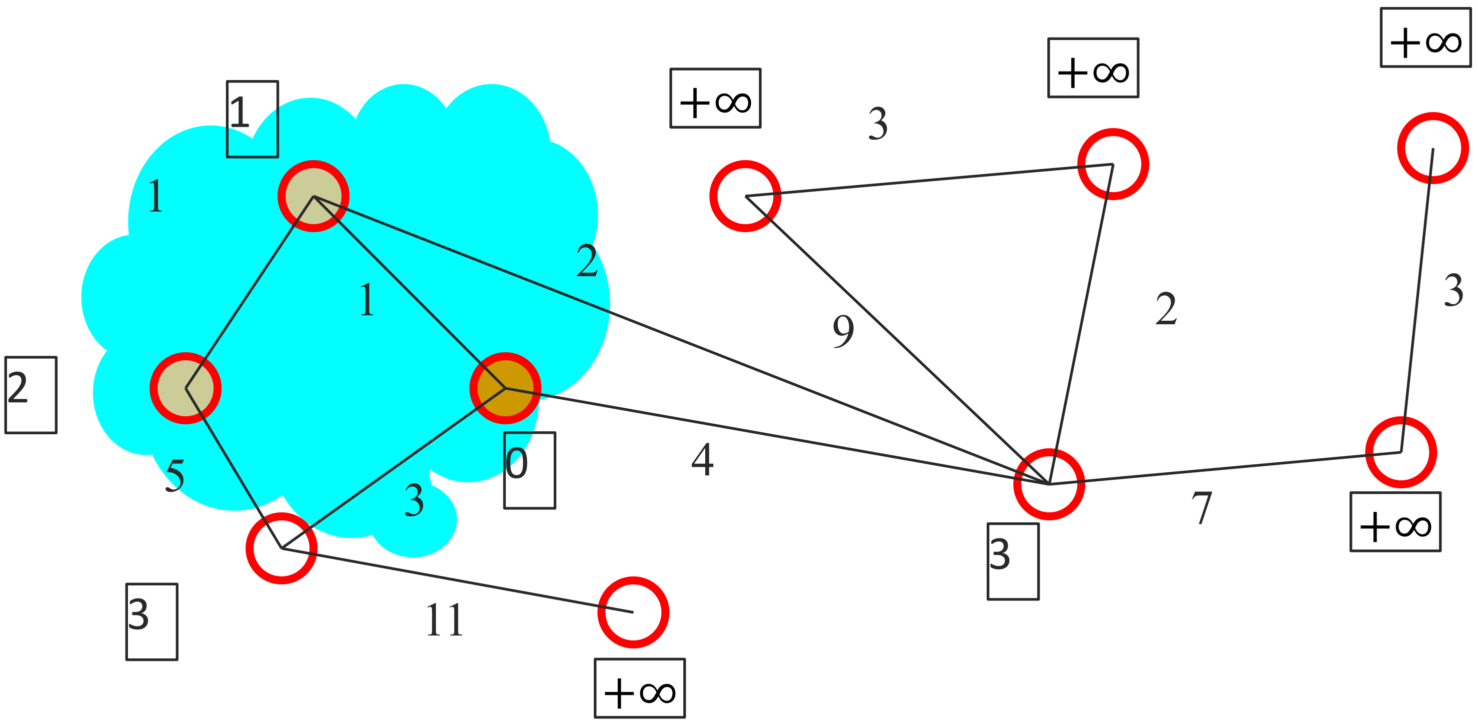

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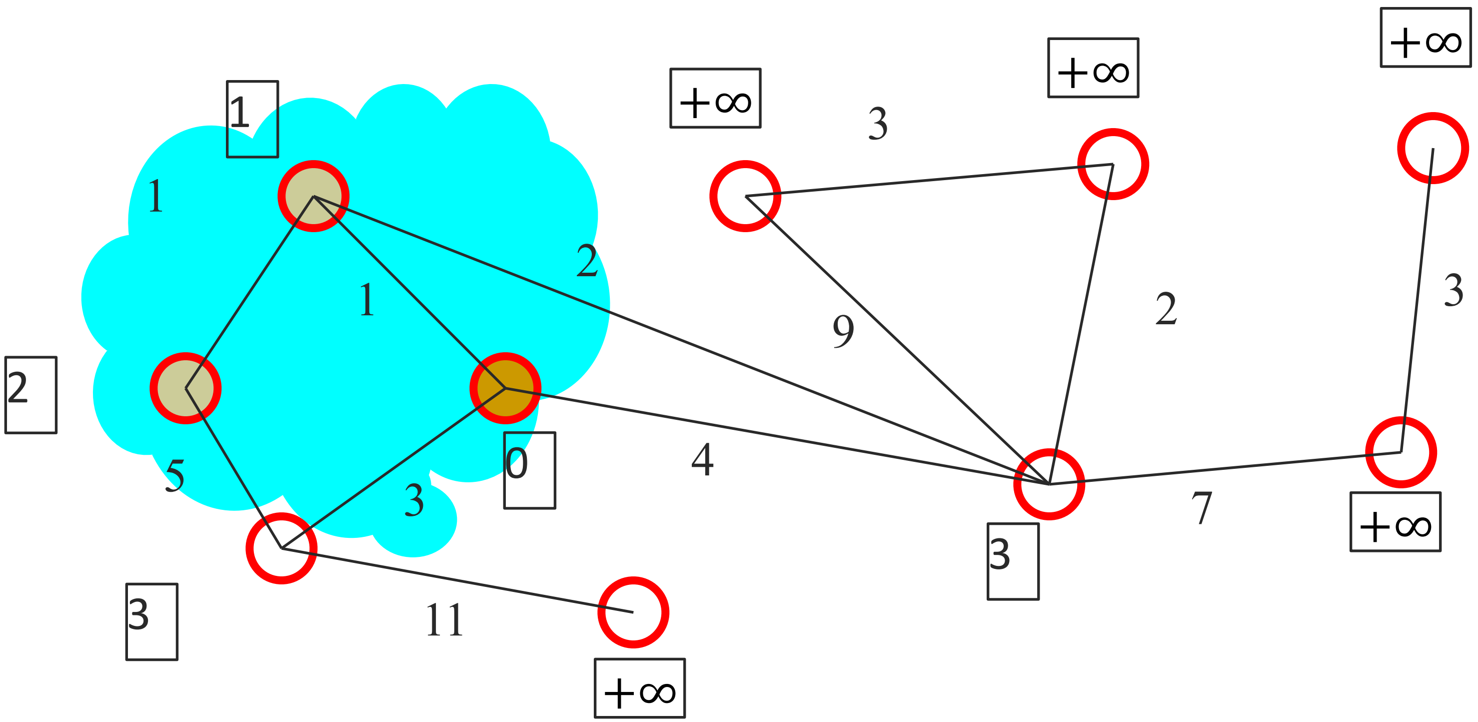

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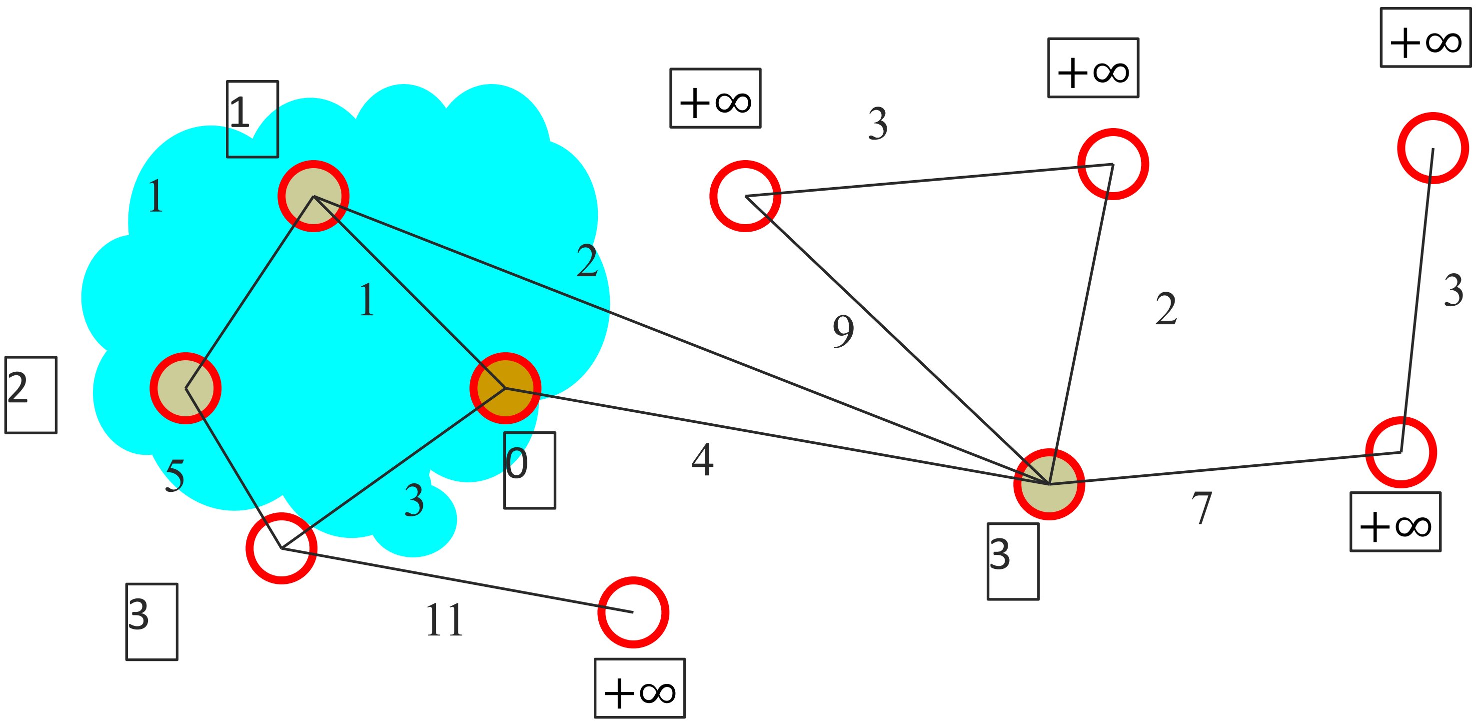

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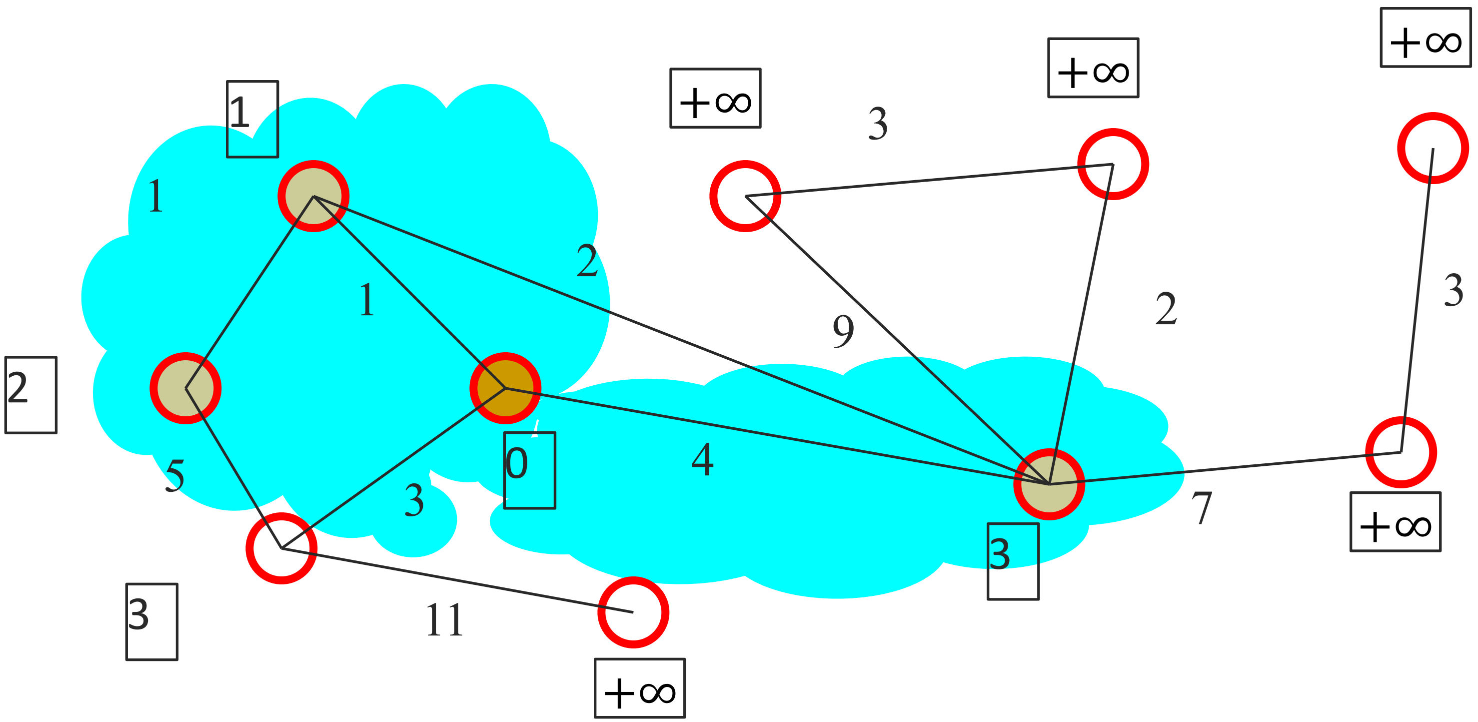

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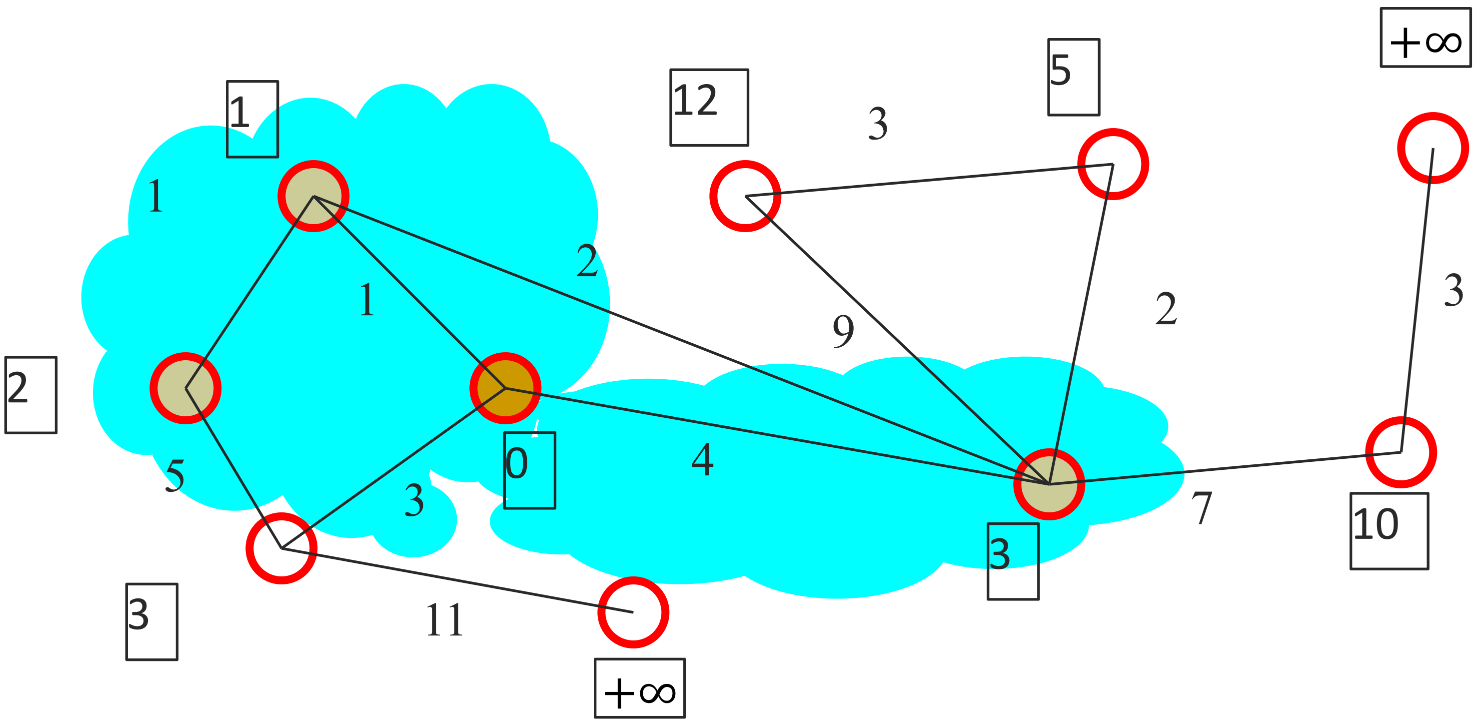

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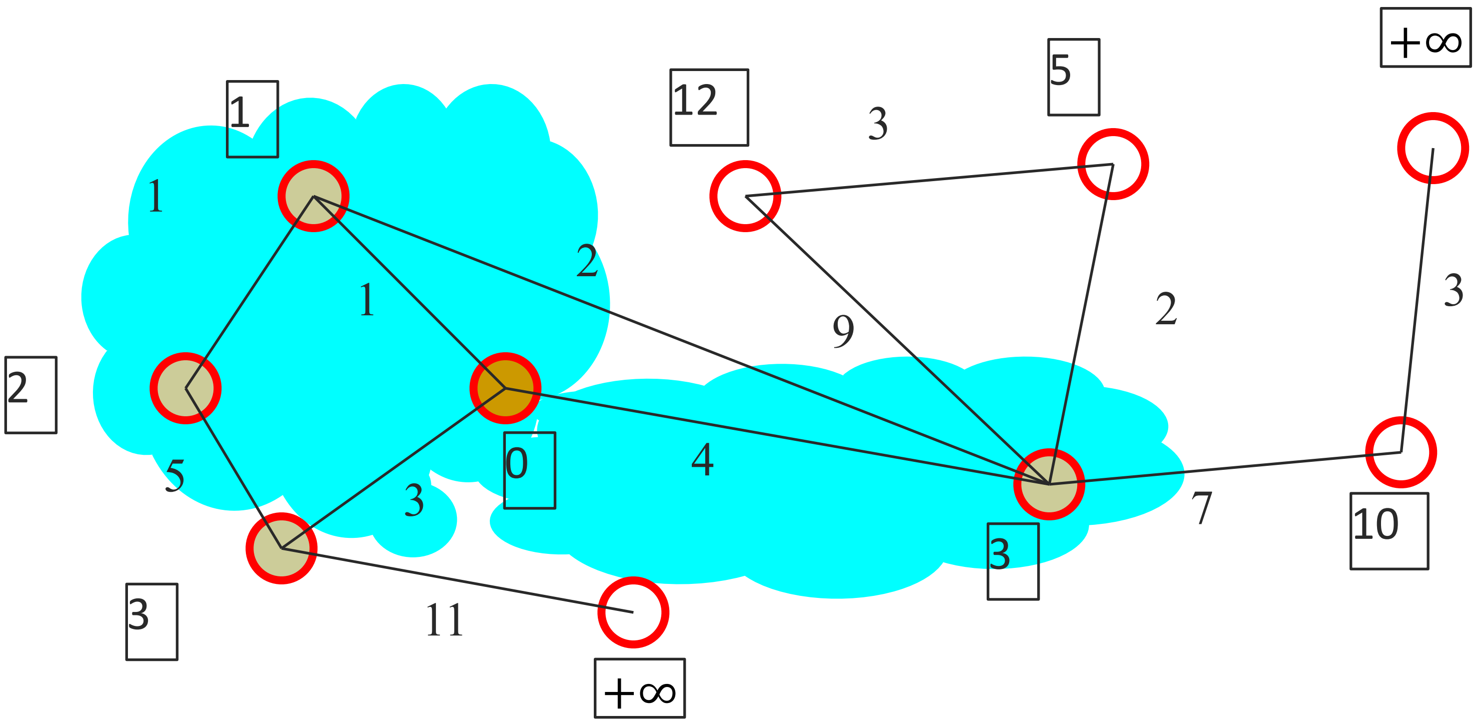

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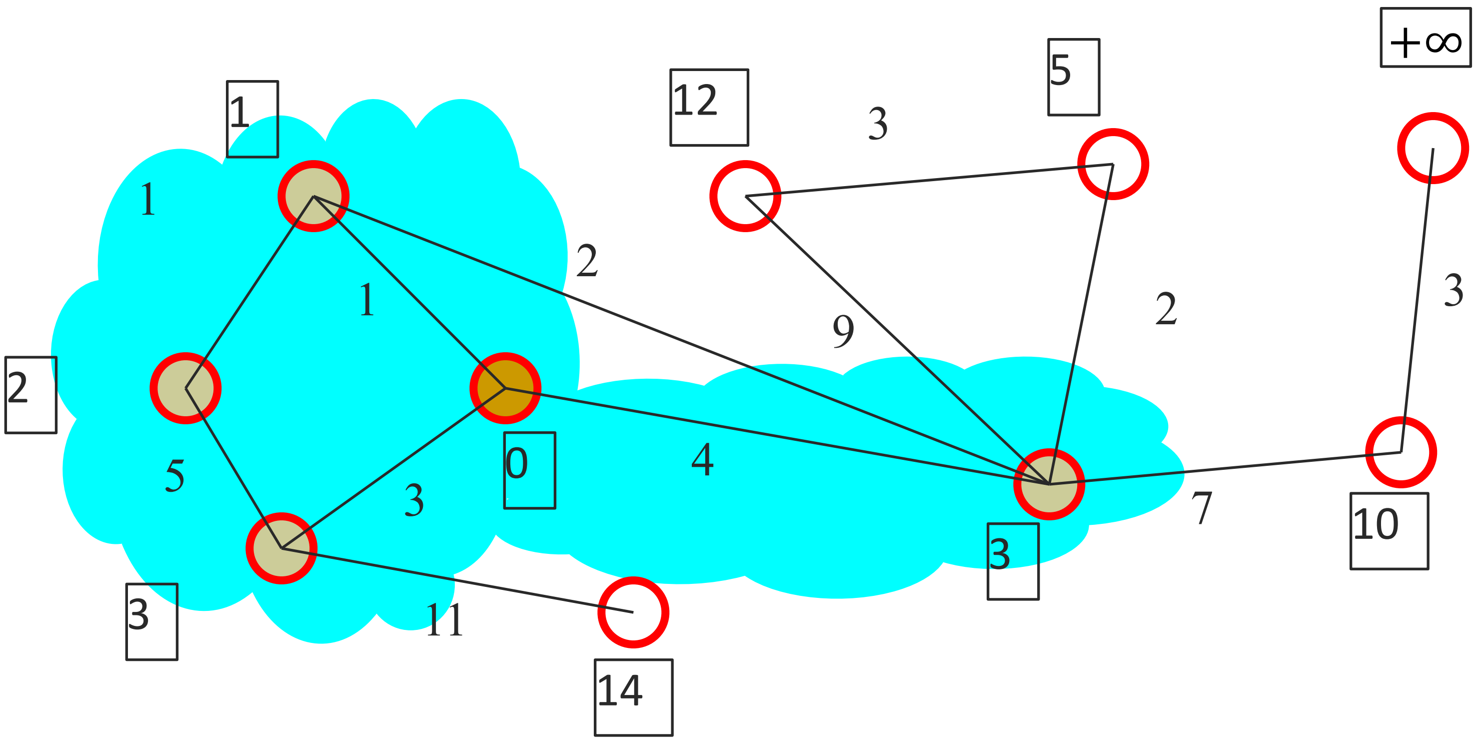

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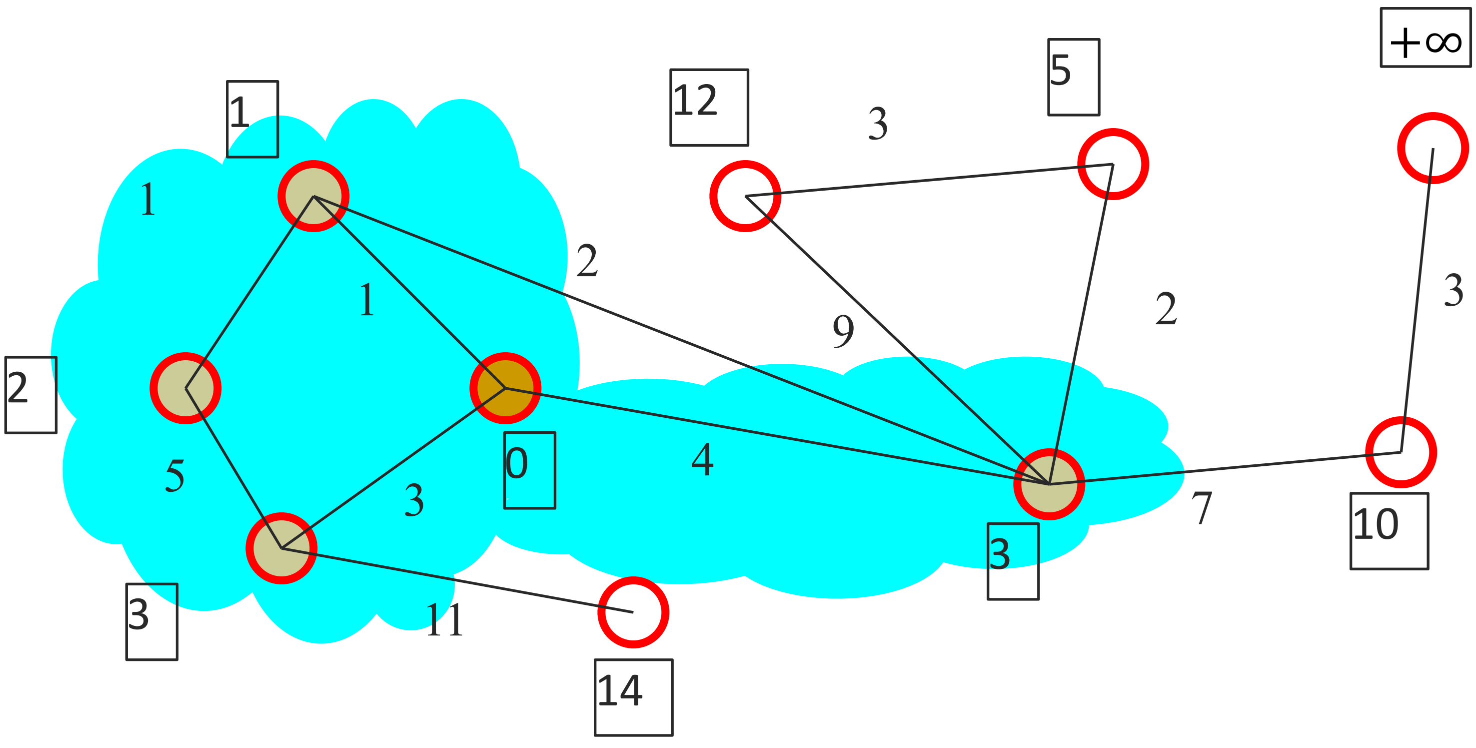

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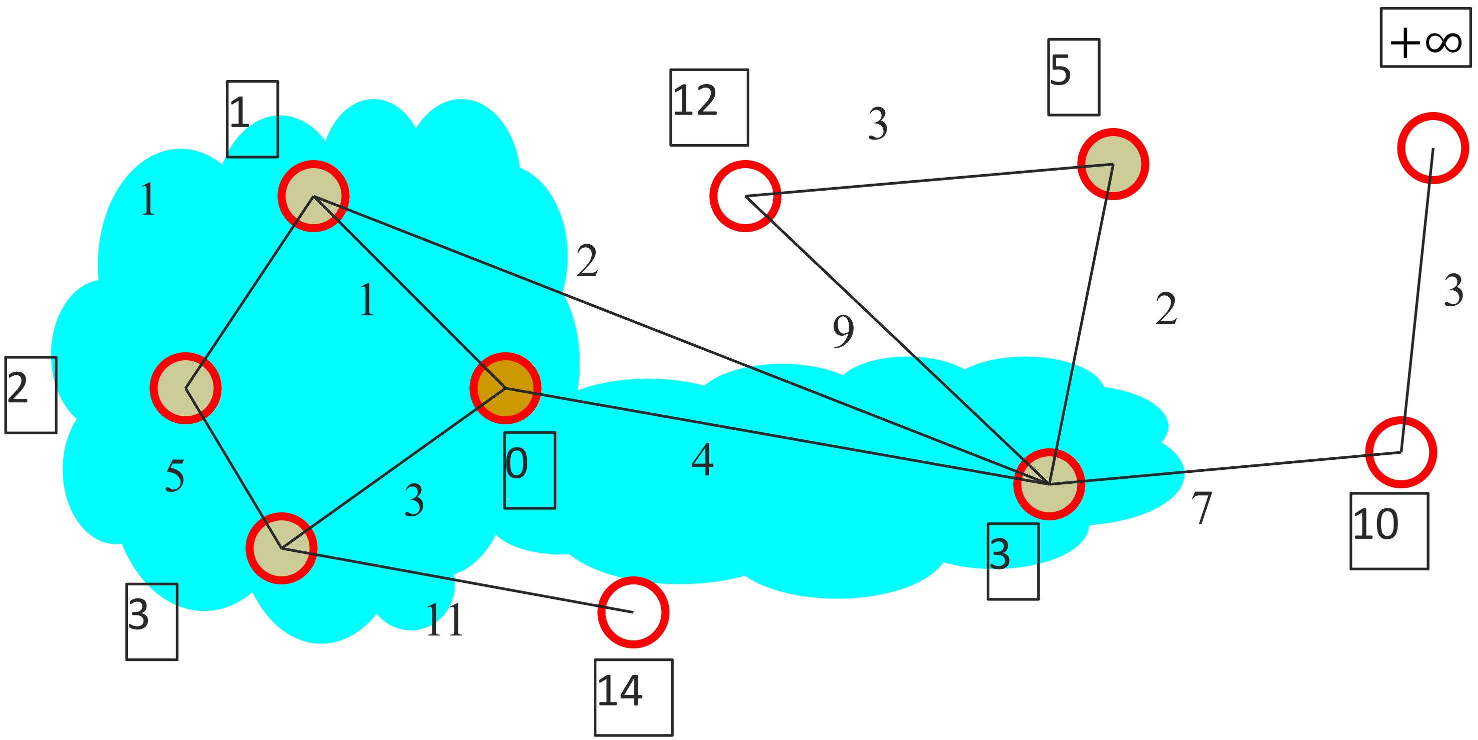

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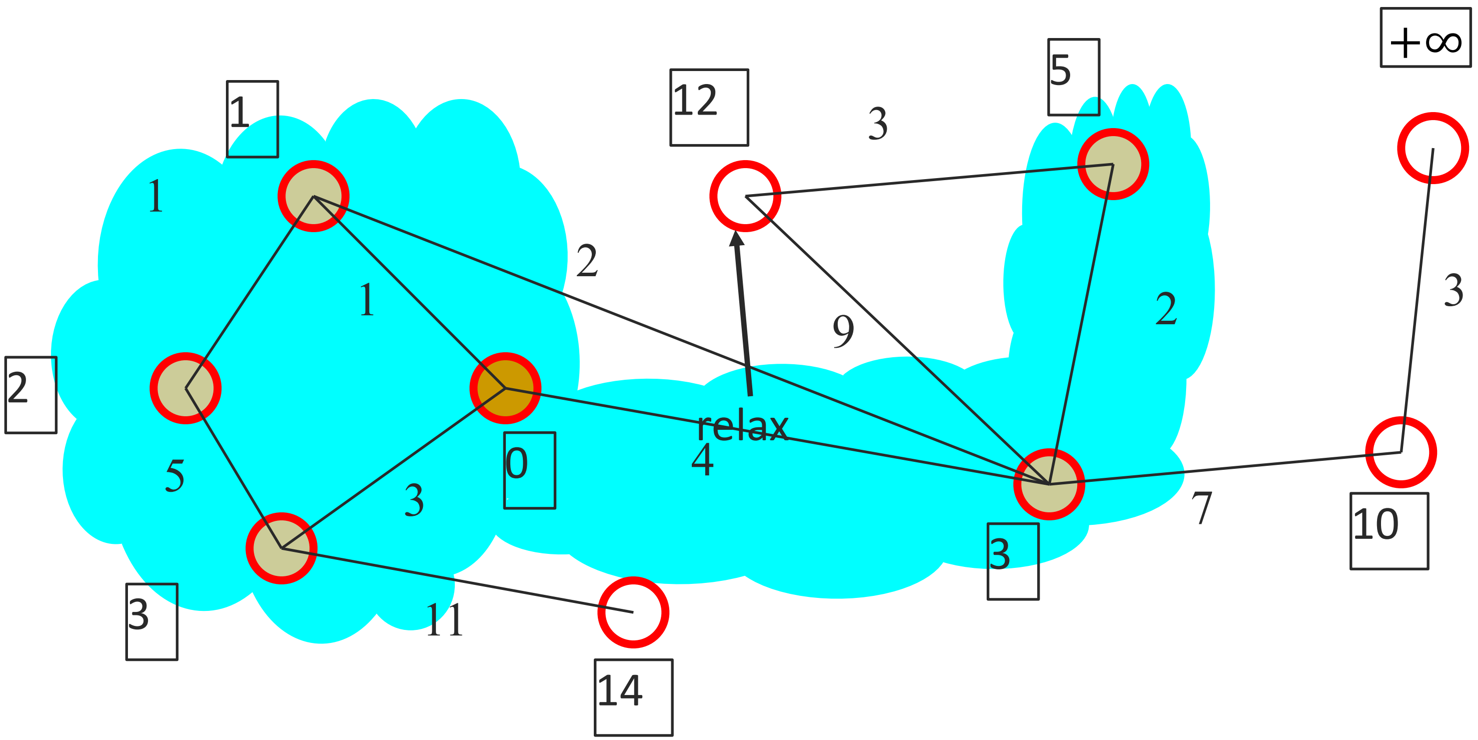

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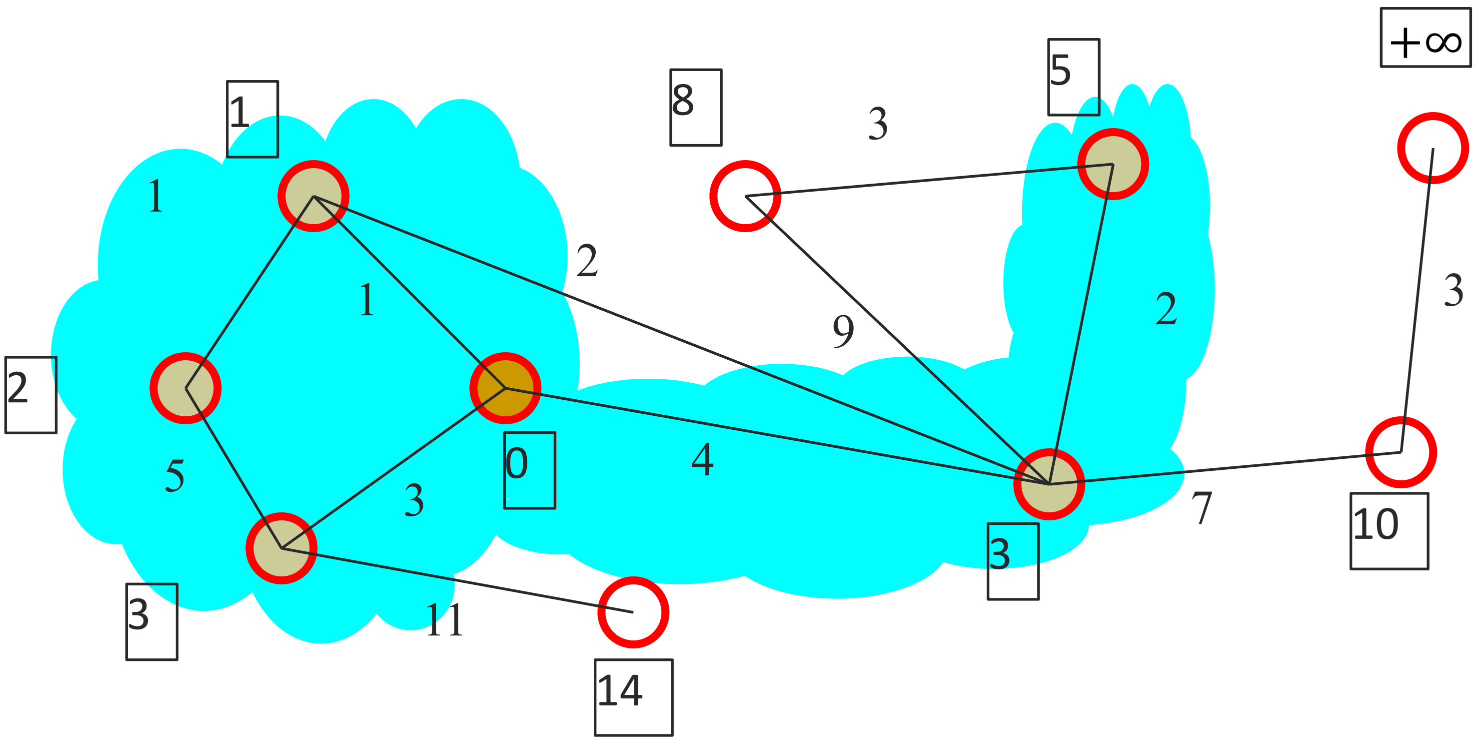

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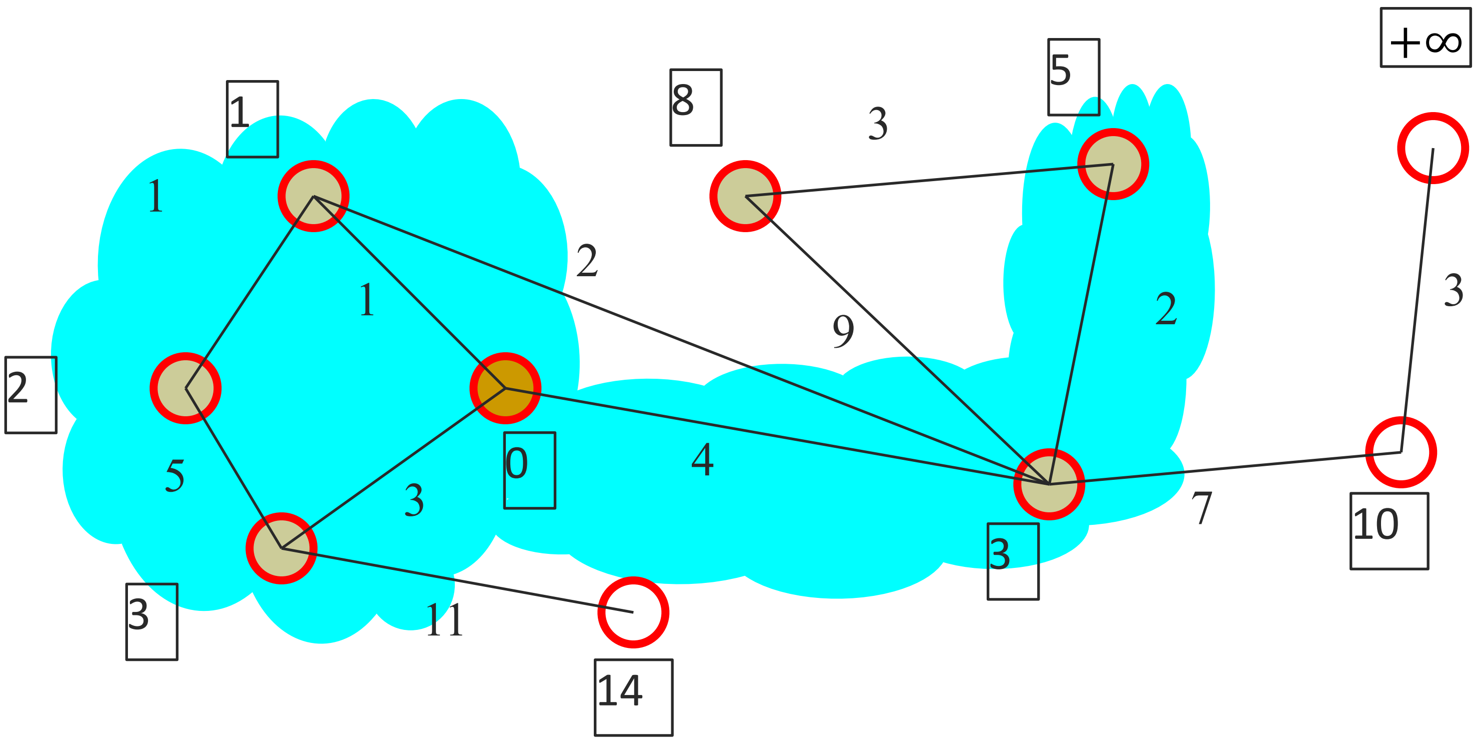

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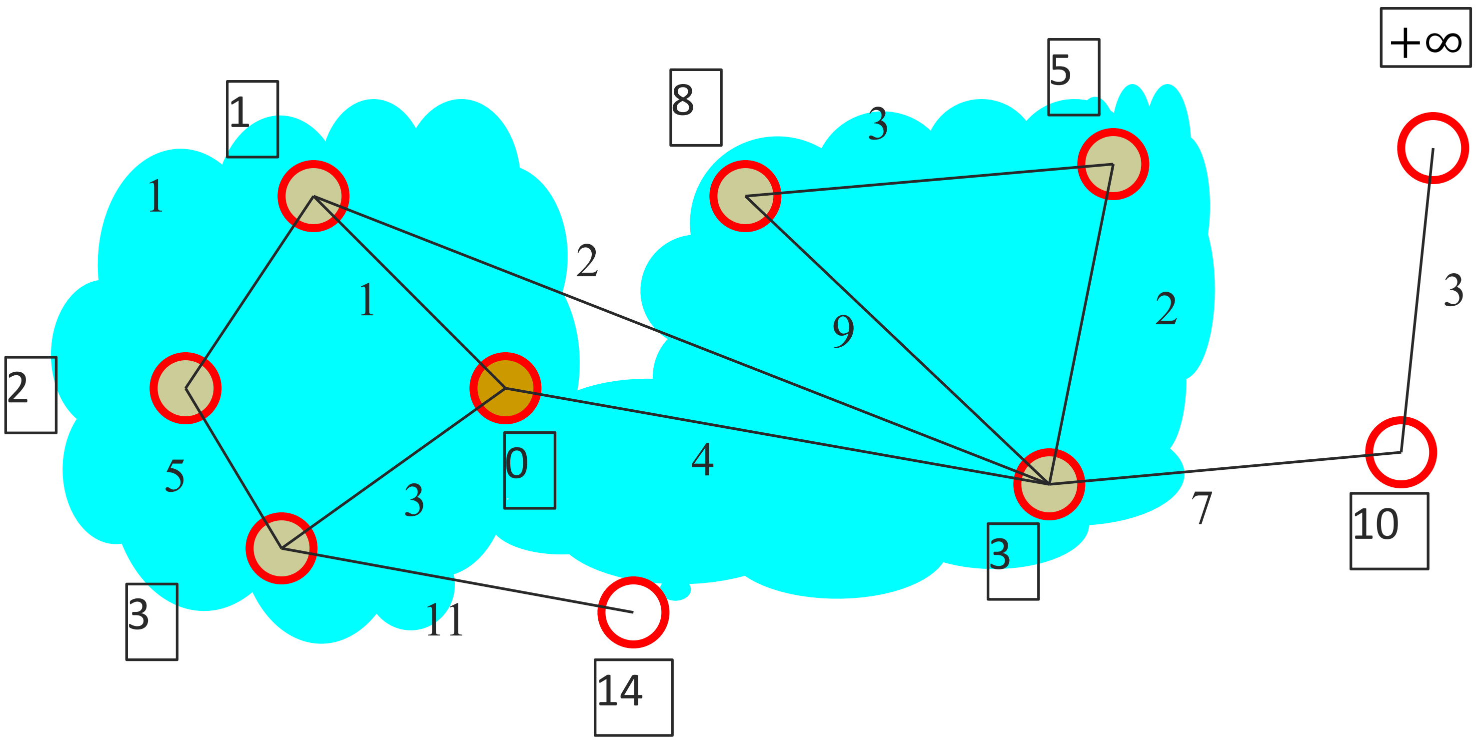
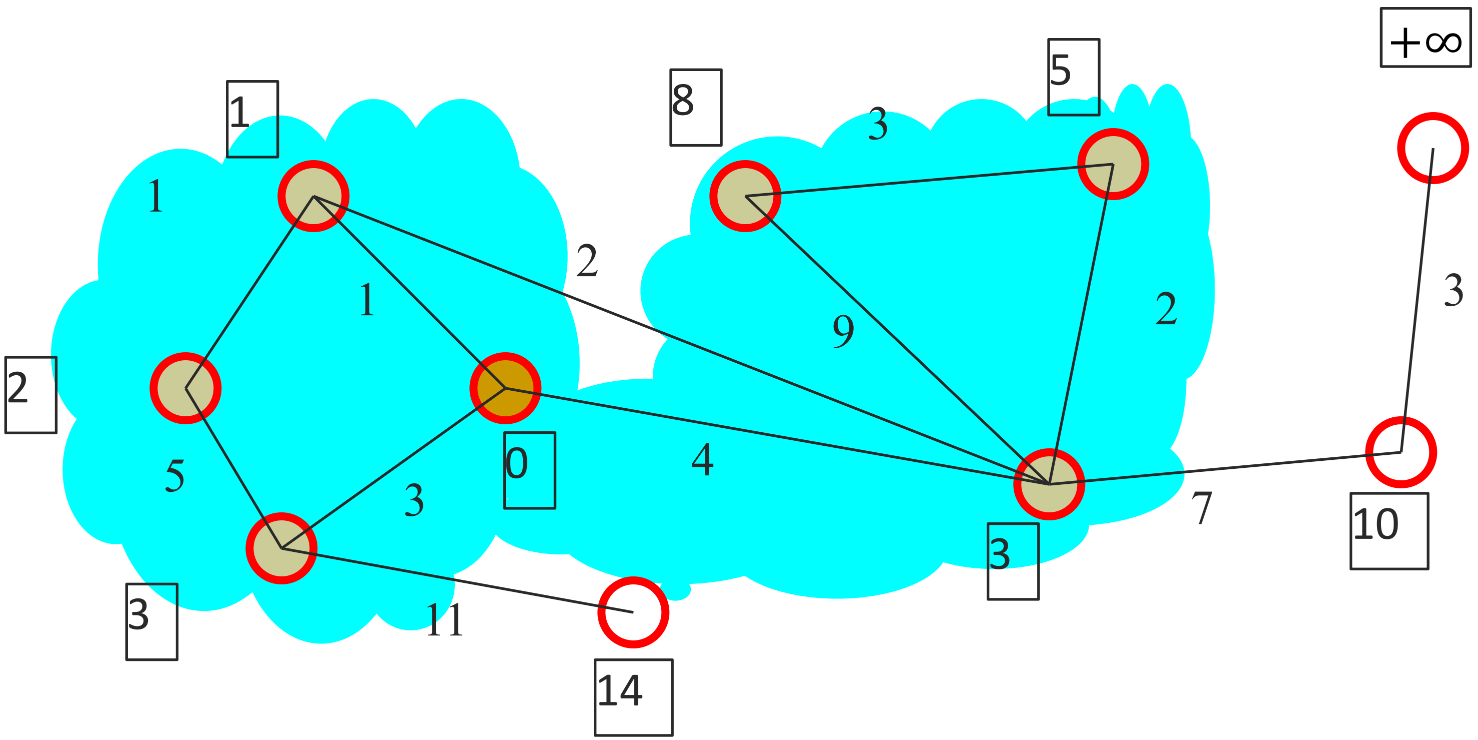


Dijkstra's algorithm: update *C's* neighborhood



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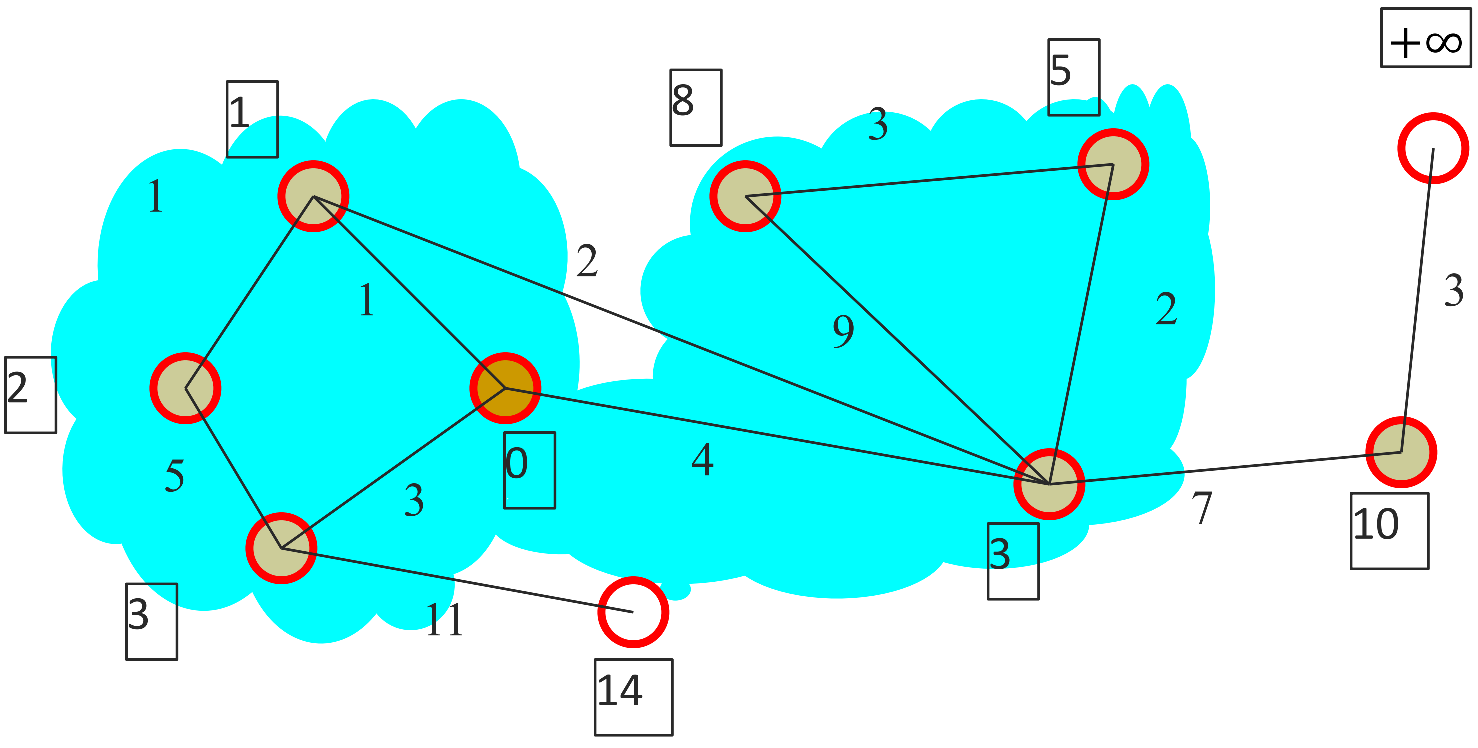

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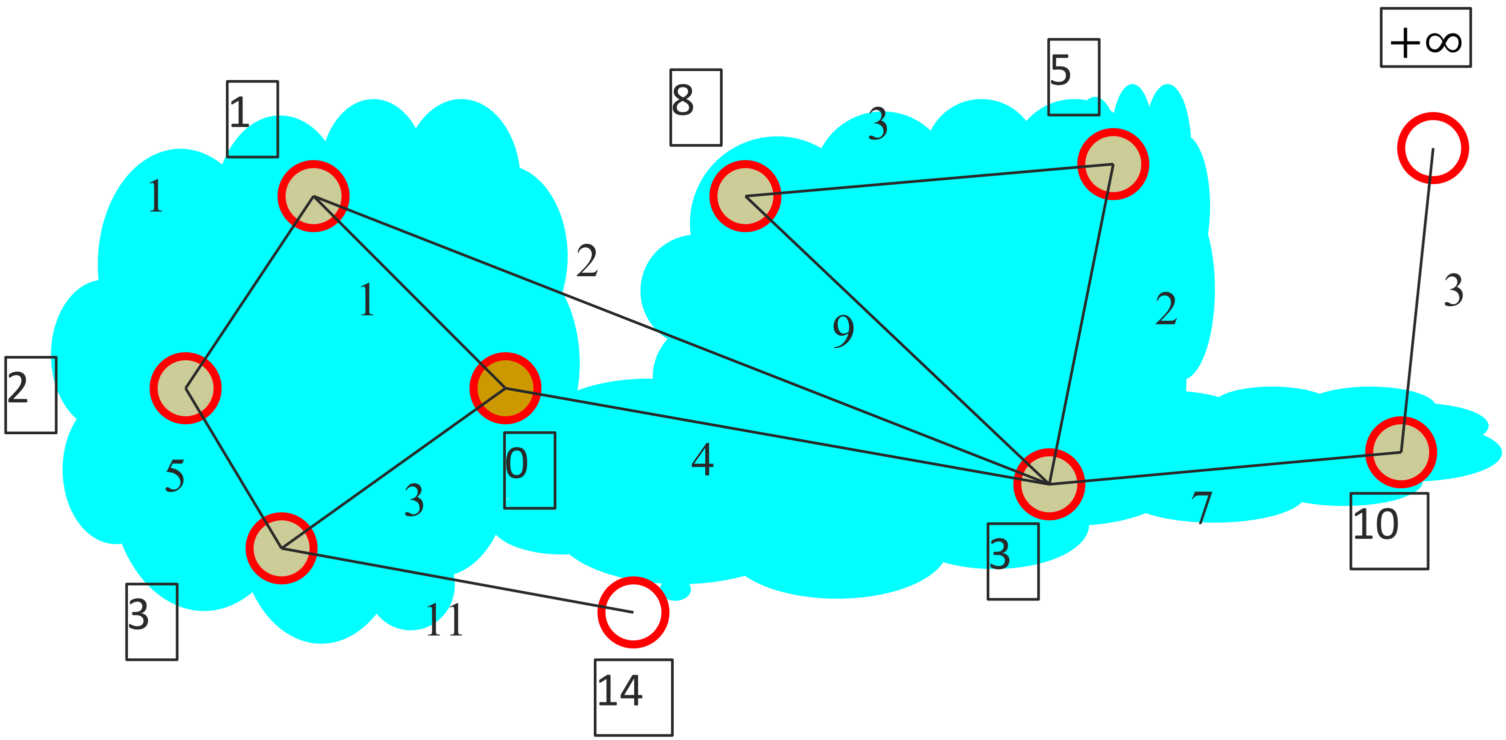

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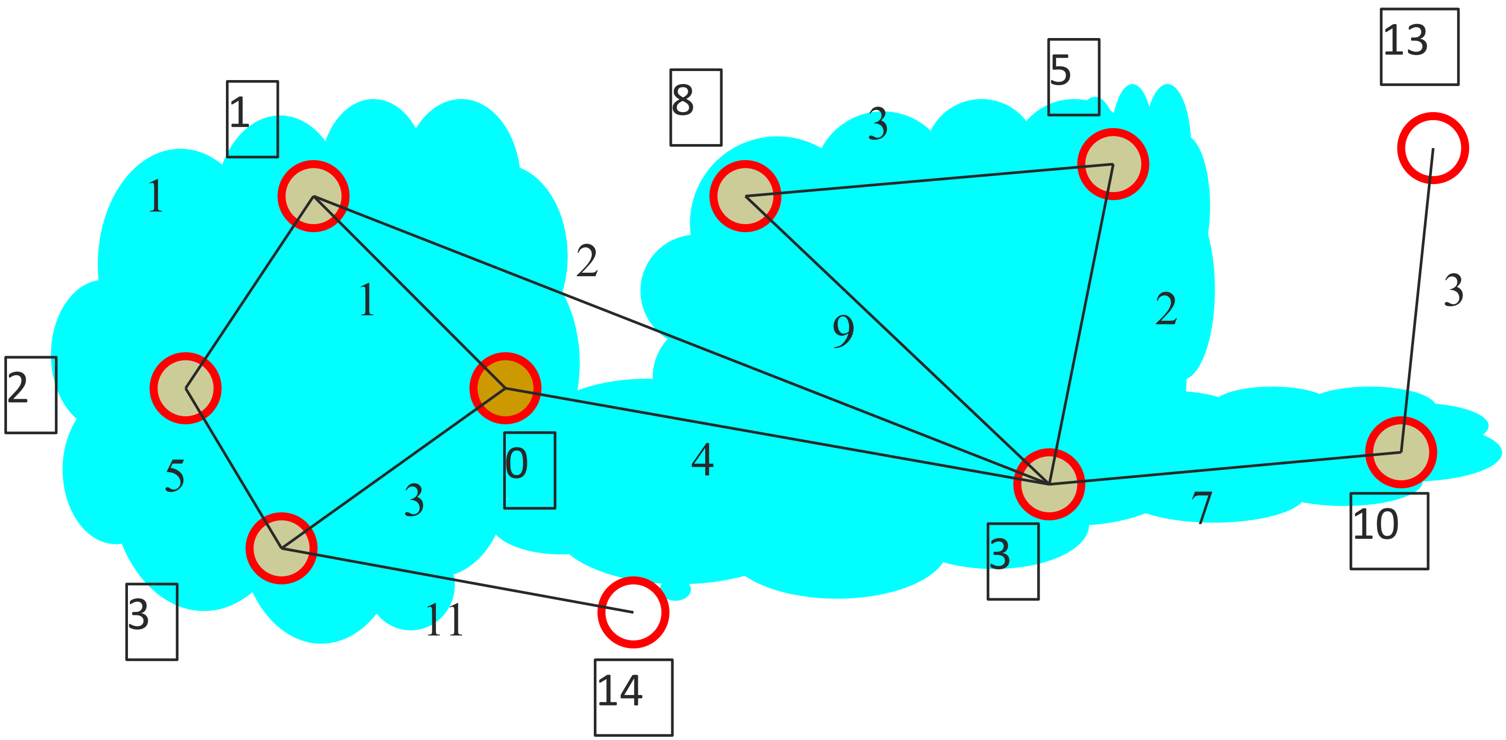

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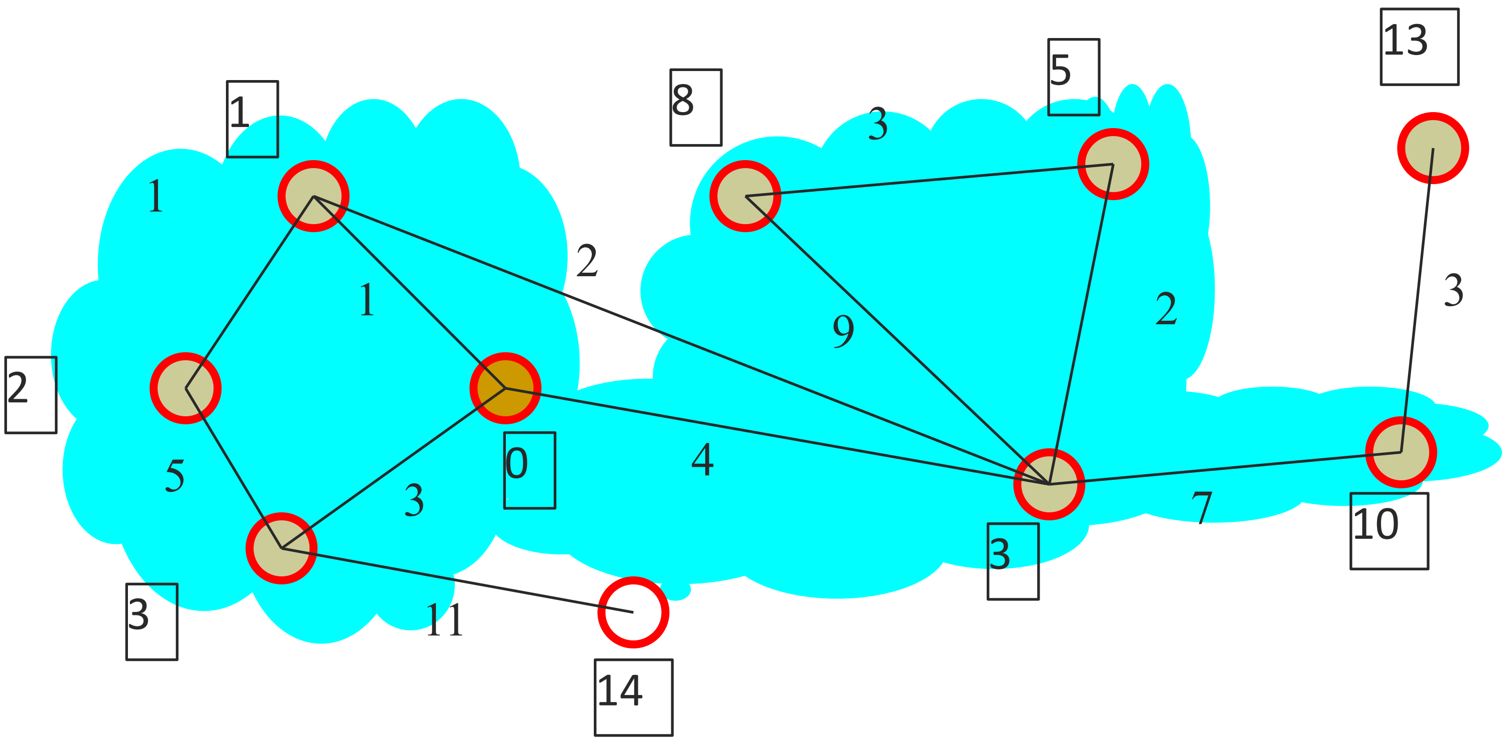

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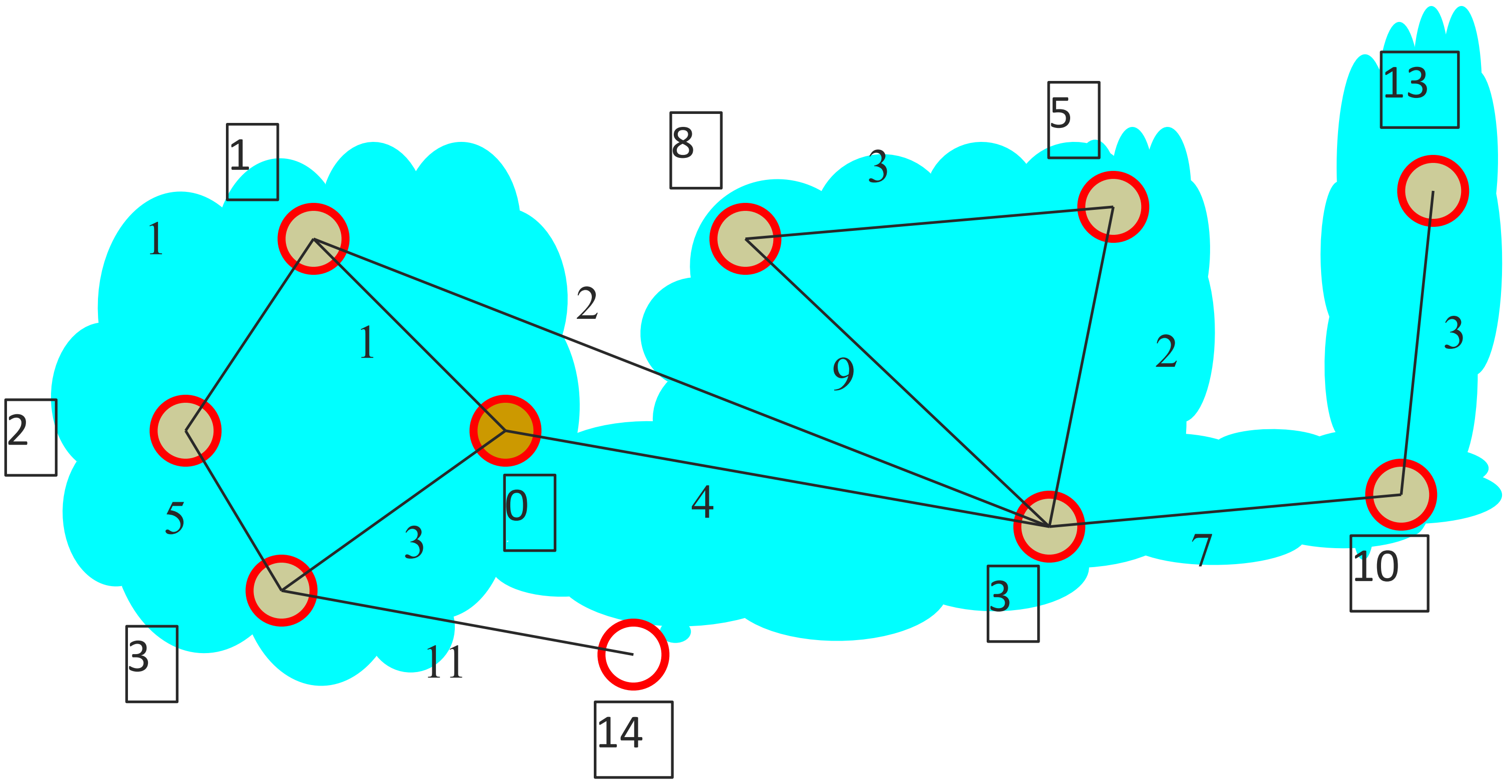

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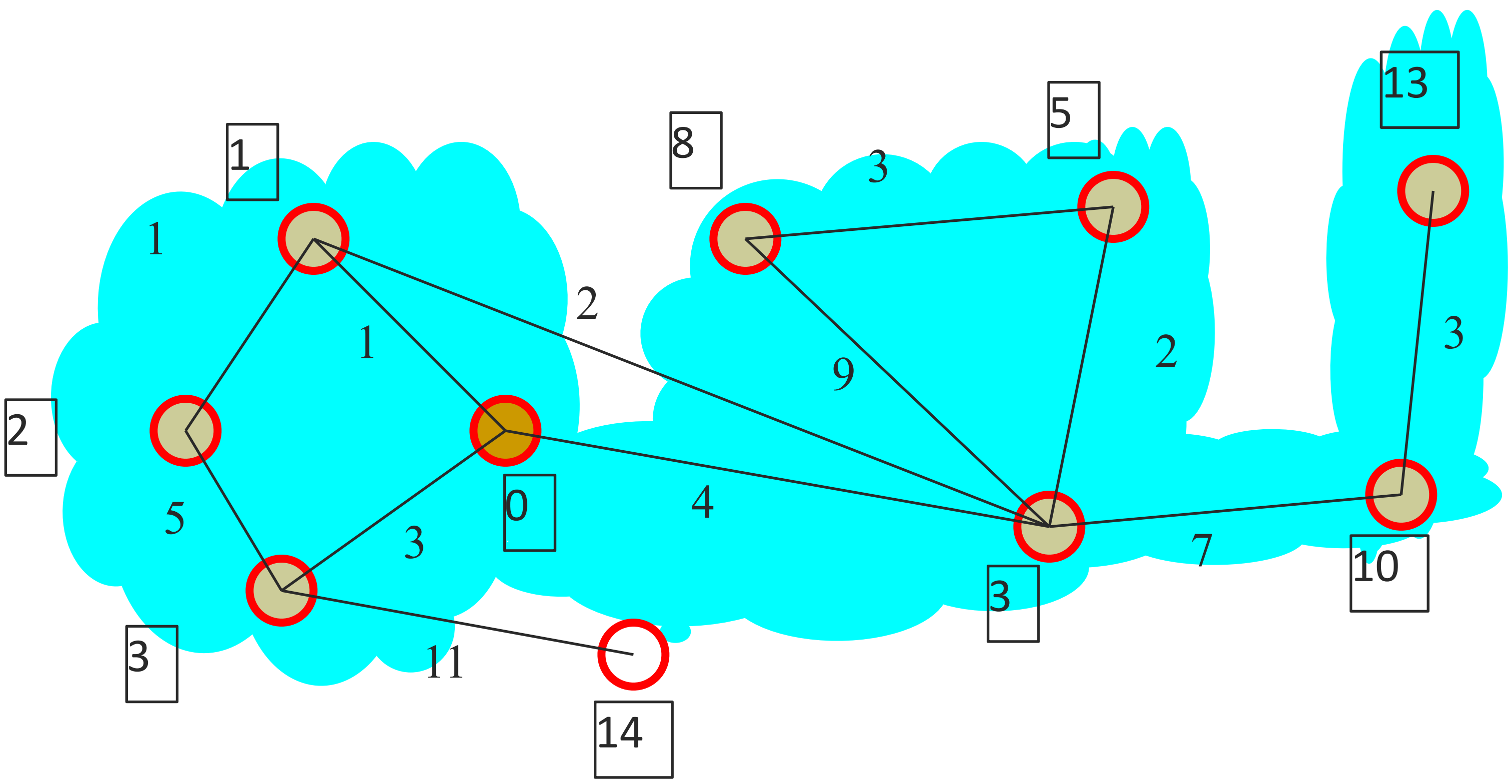

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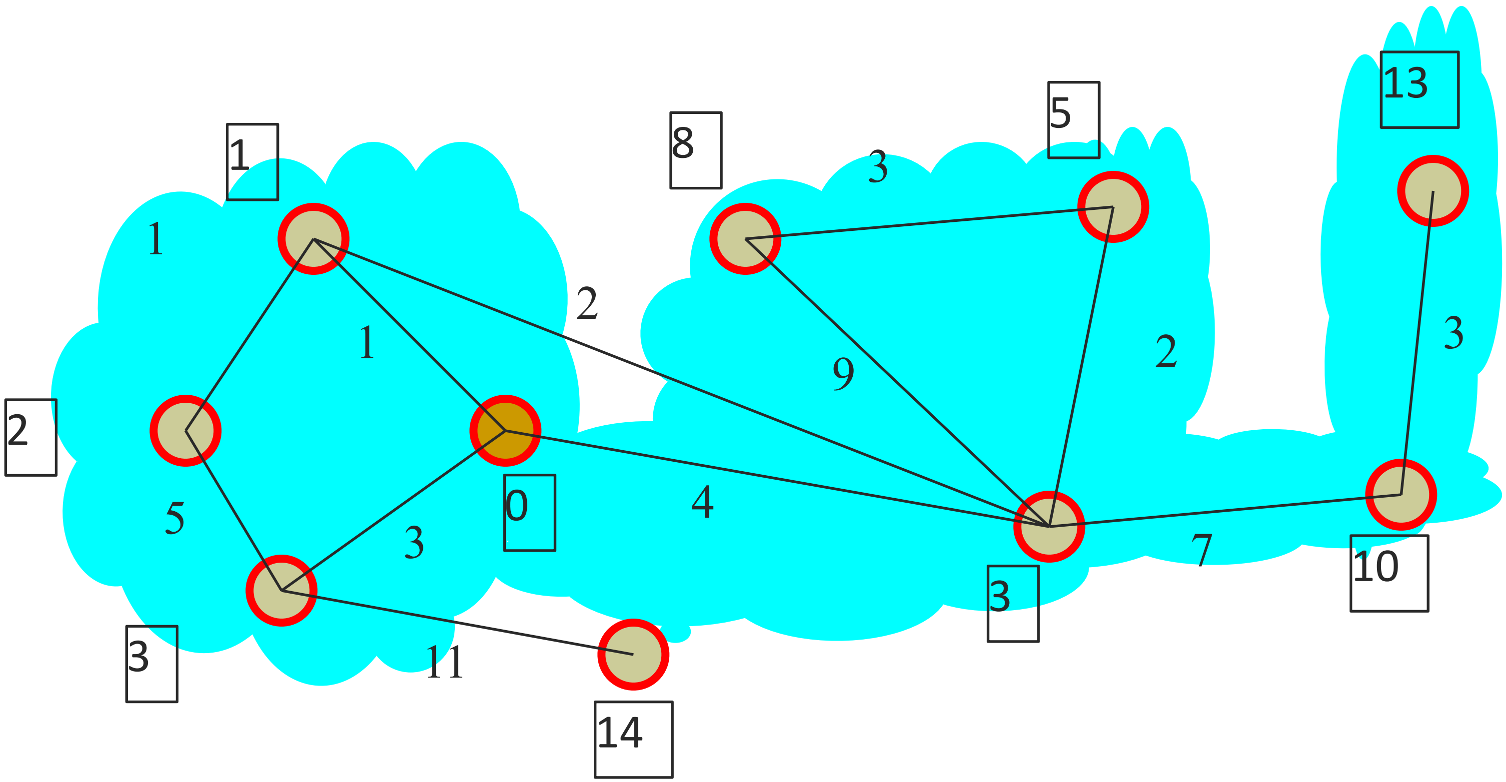

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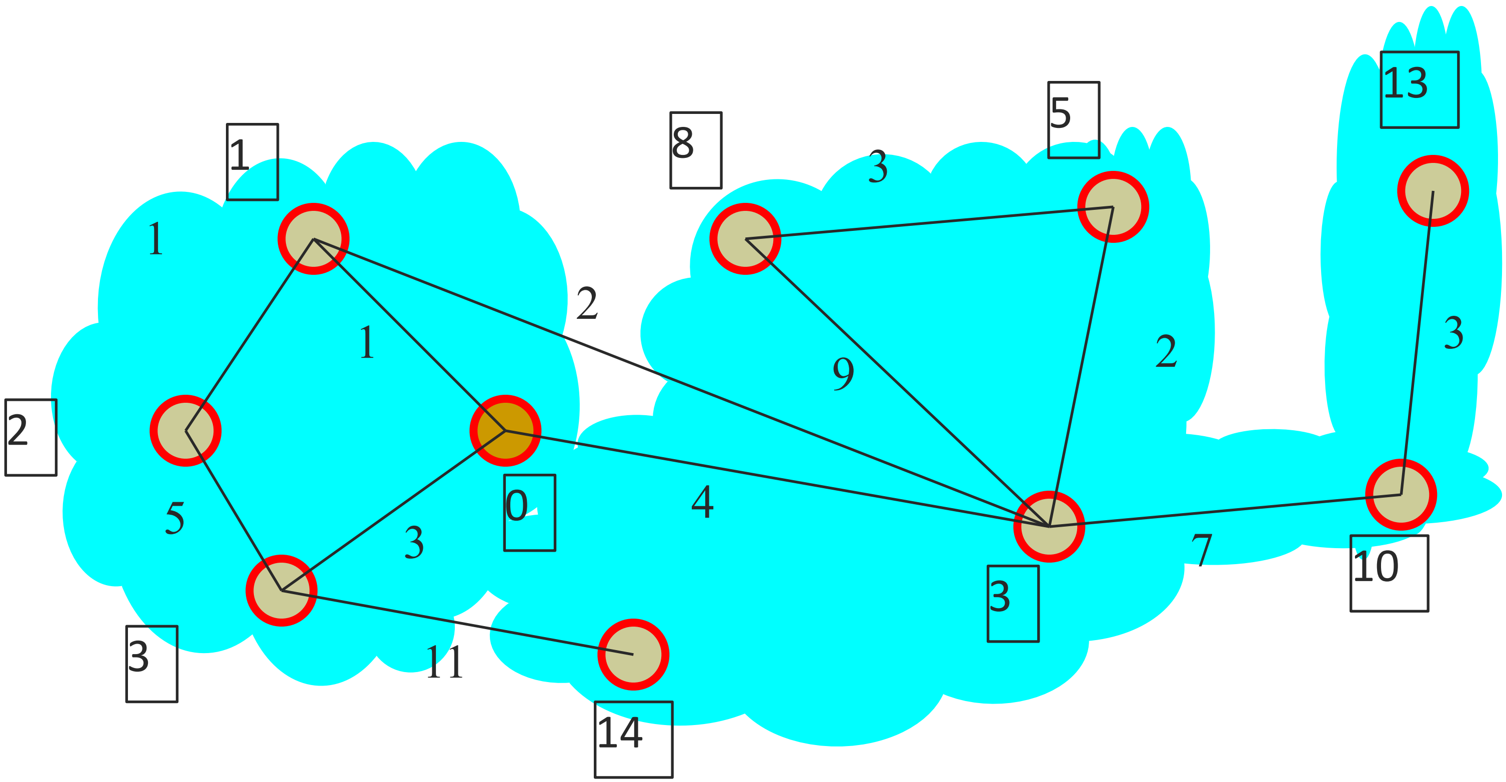
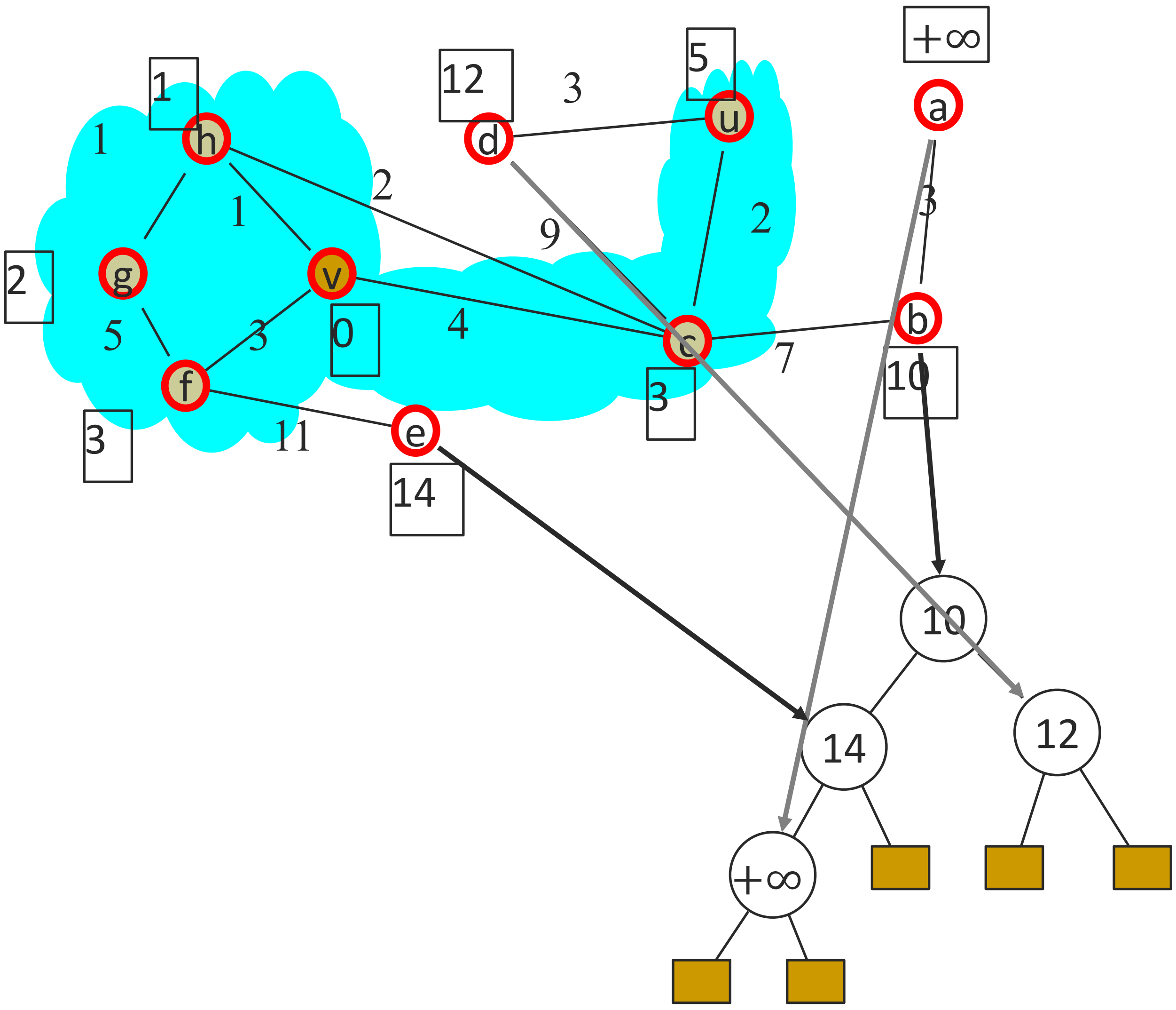


Algorithm
DijkstraShortestPaths(G,v)

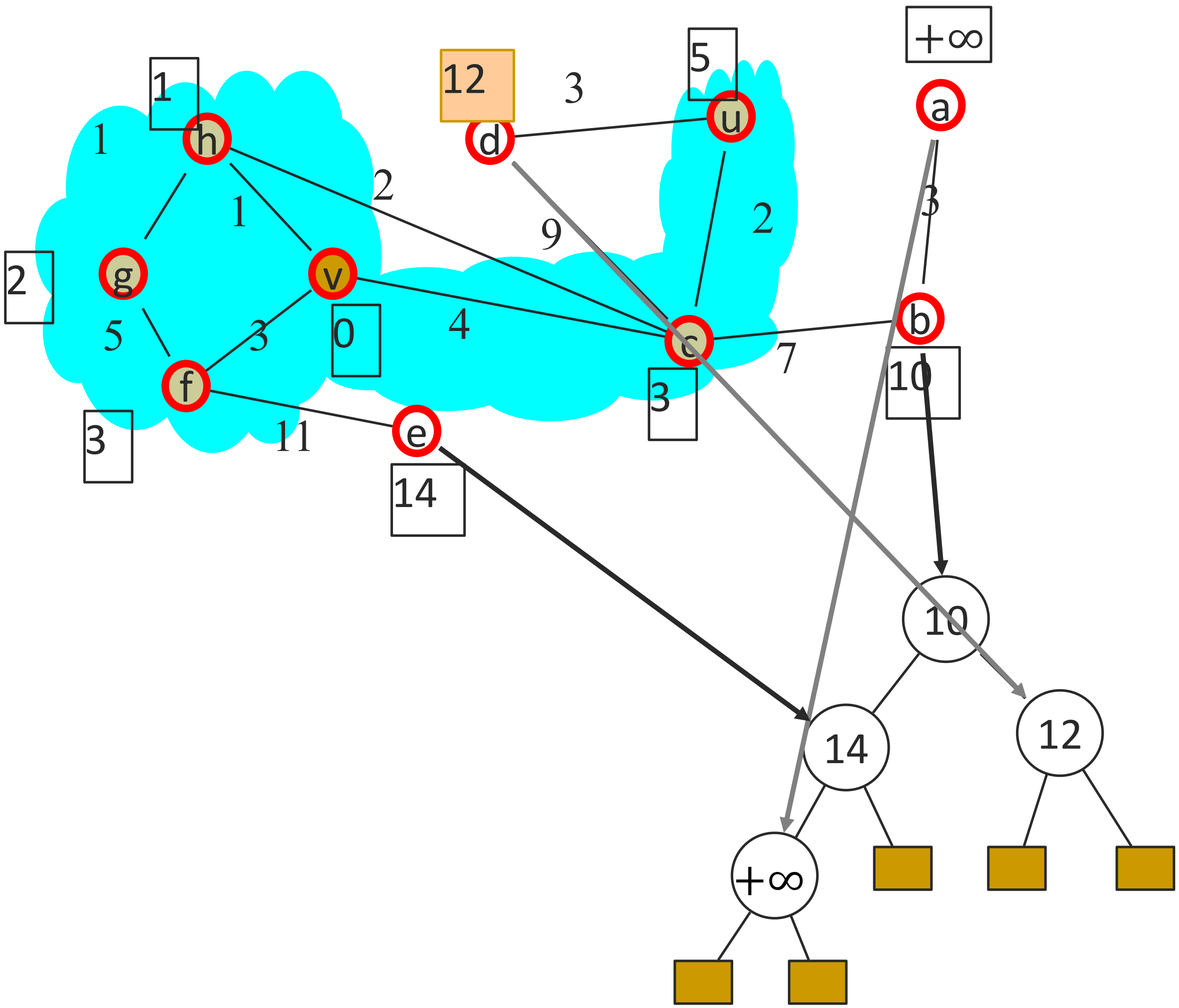
```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to T
   for each vertex z∈N(u) with z∈Q do
      if D[u]+w((u,z)) < D[z] then
         D[z]←D[u]+w((u,z))
         update z's key in Q to D[z]
return D
```

Relaxation

Dijkstra’s algorithm using heaps



Dijkstra's algorithm using heaps



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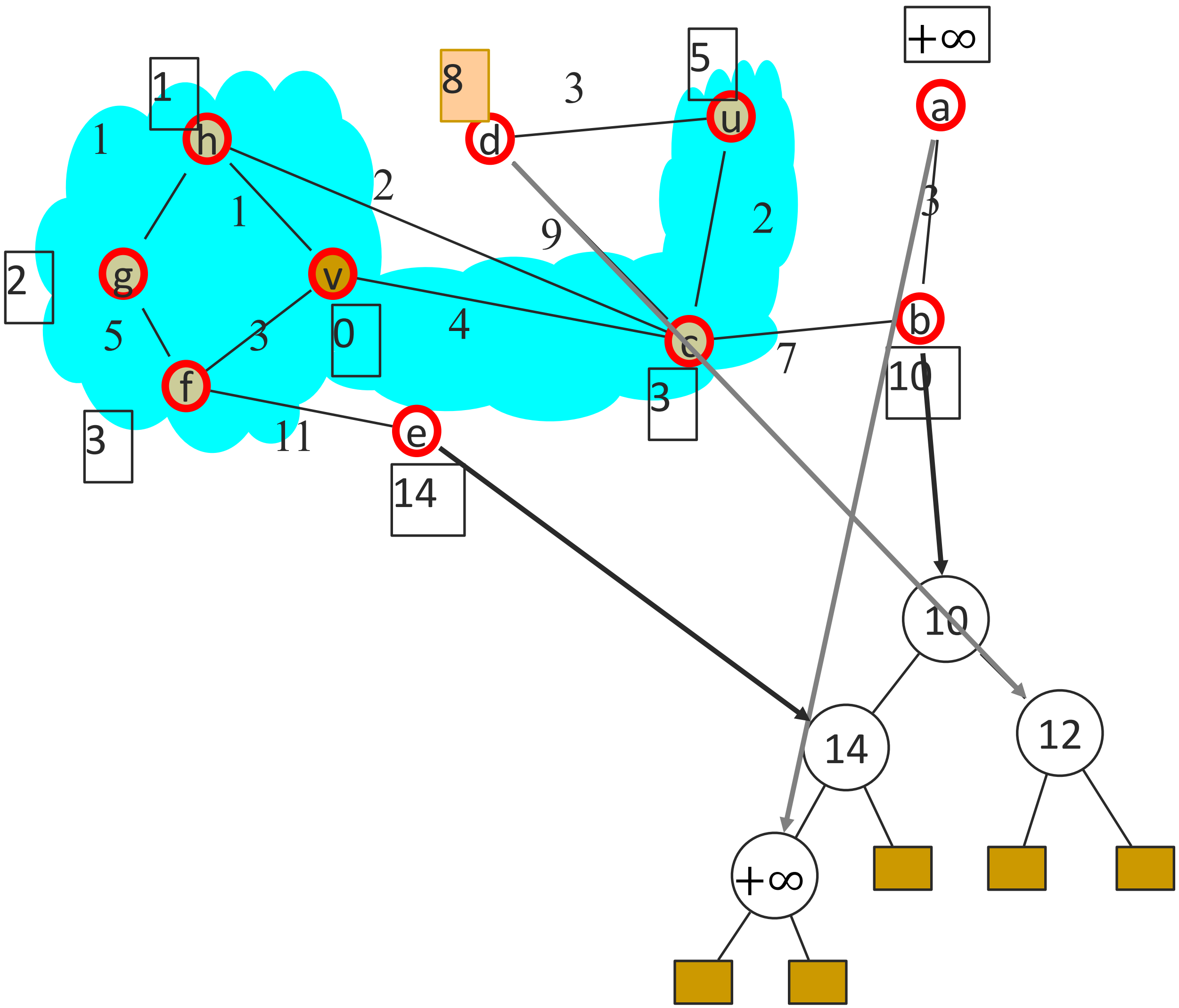

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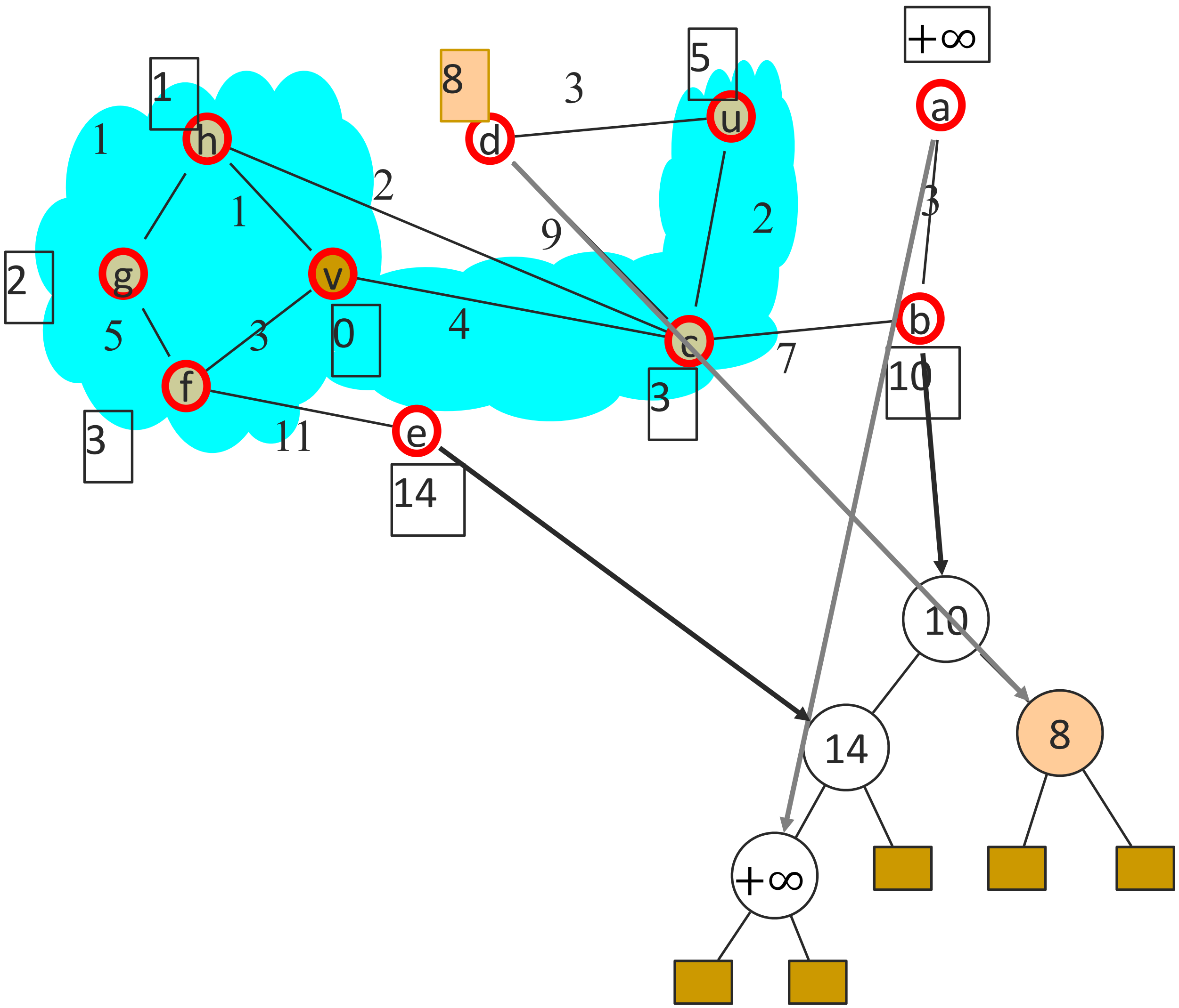
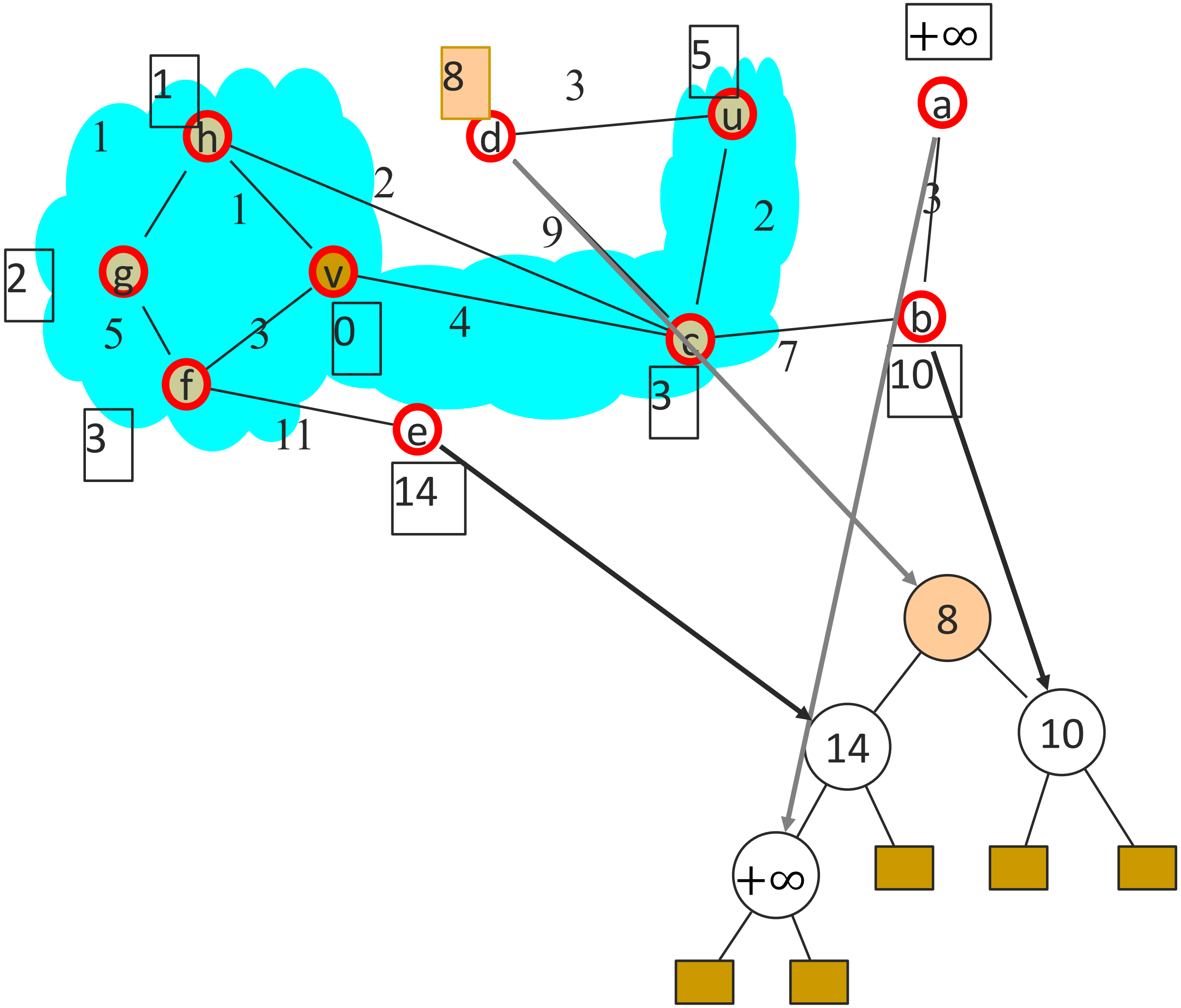


Dijkstra’s algorithm using heaps



Running time

```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to cloud
   for each vertex z∈N(u) with z∈Q do
      if D[u]+w((u,z)) < D[z] then
         D[z]←D[u]+w((u,z))
         update z's key in Q to D[z]
return D
```

Relaxation

Running time for G=(V,E) with $|V|$=n and $|E|$=m

- Insertion of vertices in priority queue Q
  - $O(n)$ when using bottom-up heap construction
- While loop:
  - Per iteration:
    - Remove vertex from Q $O(\log n)$
    - Relaxation $O(\deg(u) \log(n))$
  - $\sum_{u \in G}(1 + \deg(u)) \log n$ is $O((n + m)\log n)$
- Overall running time: $O(m \log n)$

In real life applications

- Often the graphs are <u>sparse</u>
- Then $O(m \log n)$ may be $O(n \log n)$

Algorithm DijkstraShortestPaths(G,v)

Correctness

```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to T
   for each vertex z∈adj(u) with z∈Q do
      if D[u]+w((u,z)) < D[z] then
         D[z]←D[u]+w((u,z))
         update z's key in Q to D[z]
return D
```

Relaxation

Invariants

Let d(u)=distance of *u* from *v*

1. For each node u *in* T, $D[u] =$ d(u)
2. For each node *u* not in T, *D[u]*= length of shortest path from v to u without the use of other nodes outside of T

   $+\infty$ denotes that the node cannot be reached yet from v via T nodes only

→ For all nodes u *in* T, $D[u] \geq$ d(u)

Proof that invariants hold:

Initially they hold:
$D(v)=0$, $D(u)=\infty$

Consider the first time they fail:
CASE A: If Invariant 1 holds, i.e, $D(u)=d(u)$ for all u in T , but Invariant 2 fails for some z outside T when we add u to T, i.e, $D(z)$ fails to be the length of the shortest path from v to z passing only through nodes in T.

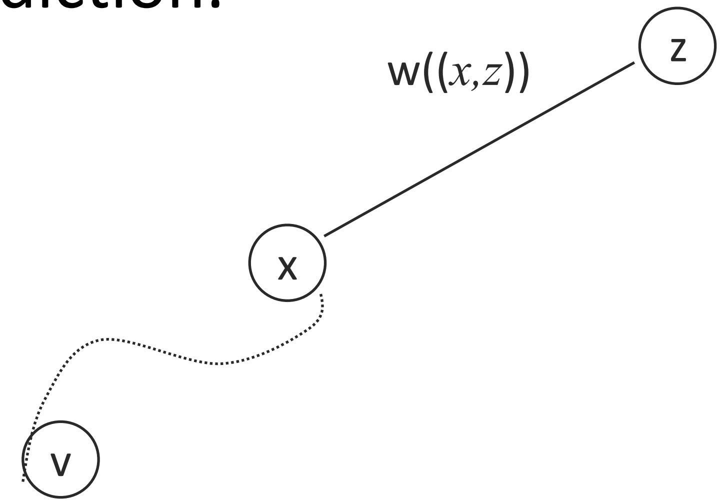
<u>Claim:</u> This can only happen if there is a shorter path of length $d'<D(z)$ from v to z whose last edge is $\{u,z\}$:
Why?

Claim: Assuming Invariant I holds, Invariant 2 is violated only if there is a shorter path from v to z whose last edge is {u,z}: Why?

Proof of Claim: Suppose to the contrary, the last edge of the shorter path is {x,z}, x≠u, x in T.

We would have previously set D(z) to D(x)+w(x,z) when we relaxed x.

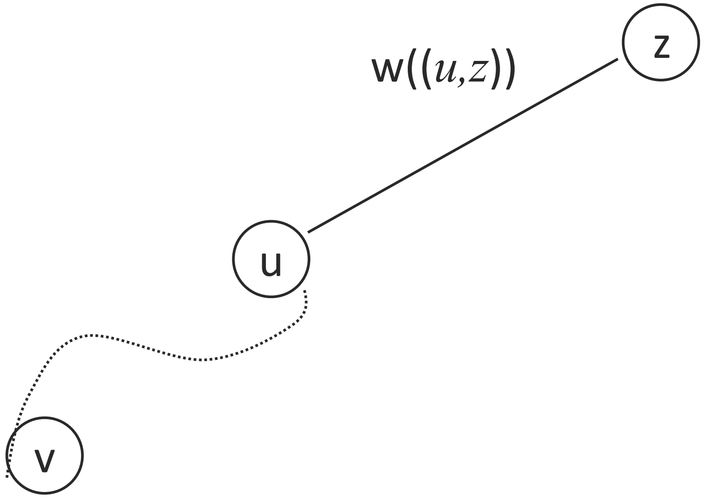
By Invariant 1, D(x)=d(x) since x is in T, so D(z) would have been set to d(x)+w(x,z)=d' < D(z) giving a contradiction.



Since u is the last node in the shortest path to z passing through only T, then
the relaxation step correctly sets

- $D(z) = d(v,u) + w((u,z))$ so Invariant 2 is NOT violated

after the relaxation with respect to u is completed.



Proof that invariants hold (cont'd):

<u>CASE B: </u>(Invariant 1 fails first)

Consider the first time Invariant 1 fails:

u is added to T, D(u) is the length of the shortest path from u to v through T but D(u) ≠ d(u)

Let P =shortest path from v to u

- Note: For all u, $D(u) \geq d(u)$
- Let y =last node in T on path P
- By Invariant 1, $D[y] = d(y)$

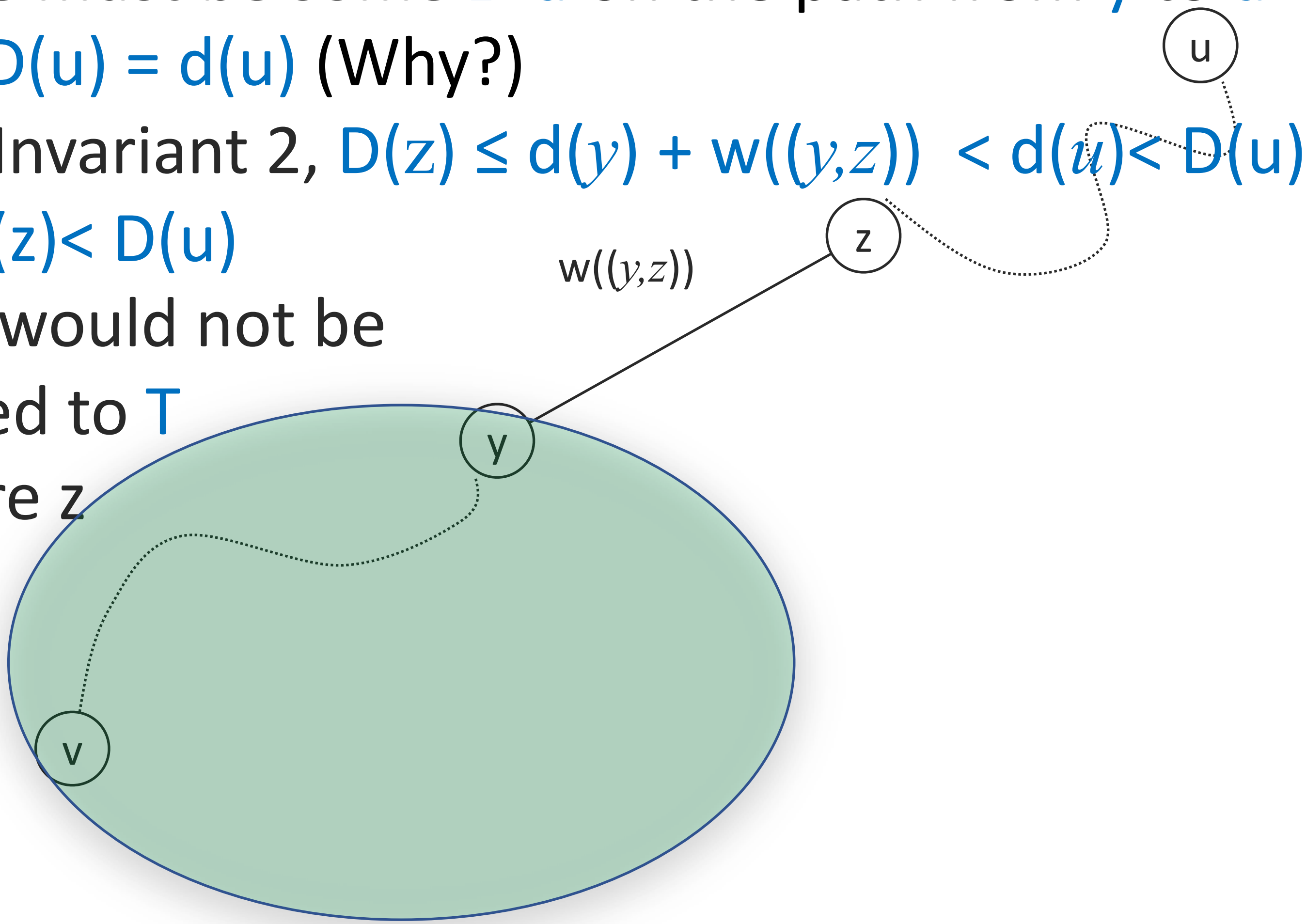
There must be some $z \neq u$ on the path from y to u

Else $D(u) = d(u)$ (Why?)

- By Invariant 2, $D(z) \leq d(y) + w((y,z)) < d(u) < D(u)$

→ $D(z) < D(u)$

→ u would not be added to T before z



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We've just show both invariants always hold

Let $d(u)$=distance of u from v

1. For each node u *in* T, $D[u] = d(u)$

2. For each node u not in T, D[u]= length of shortest path from v to u without the use of other nodes outside of T

$+\infty$ denotes that the node cannot be reached yet from v via T nodes only

→ When the algorithm is finished, all nodes are in T and all D[u]=d(u)