CSC 226

Algorithms and Data Structures: II

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All-Pairs Shortest Paths

A dynamic programming approach

All-Pairs shortest paths in weighted digraphs without negative-weight cycles

- Variation of Floyd-Warshall algorithm
- Algorithm-design technique: dynamic programming

Algorithmic Paradigms

- **Greedy** – Build up a solution incrementally, myopically optimizing some local criterion.

- **Divide-and-Conquer** – Break up a problem into independent subproblems, solve each subproblem, and combine solutions to subproblems to form solution to original problem.

- **Dynamic Programming** – Break up a problem into a series of overlapping subproblems, and build up solutions to larger and larger subproblems.

fancy name for caching intermediate results in a table

Dynamic Programming History

- **Bellman** – Pioneered the systematic study in dynamic programming in the 1950s.
- **Etymology**
  - Dynamic programming = planning over time.
  - Secretary of Defense was hostile to mathematical research.
  - Bellman sought an impressive name to avoid confrontation.

THE THEORY OF DYNAMIC PROGRAMMING

RICHARD BELLMAN

1. Introduction. Before turning to a discussion of some representative problems which will permit us to exhibit various mathematical features of the theory, let us present a brief survey of the fundamental concepts, hopes, and aspirations of dynamic programming.

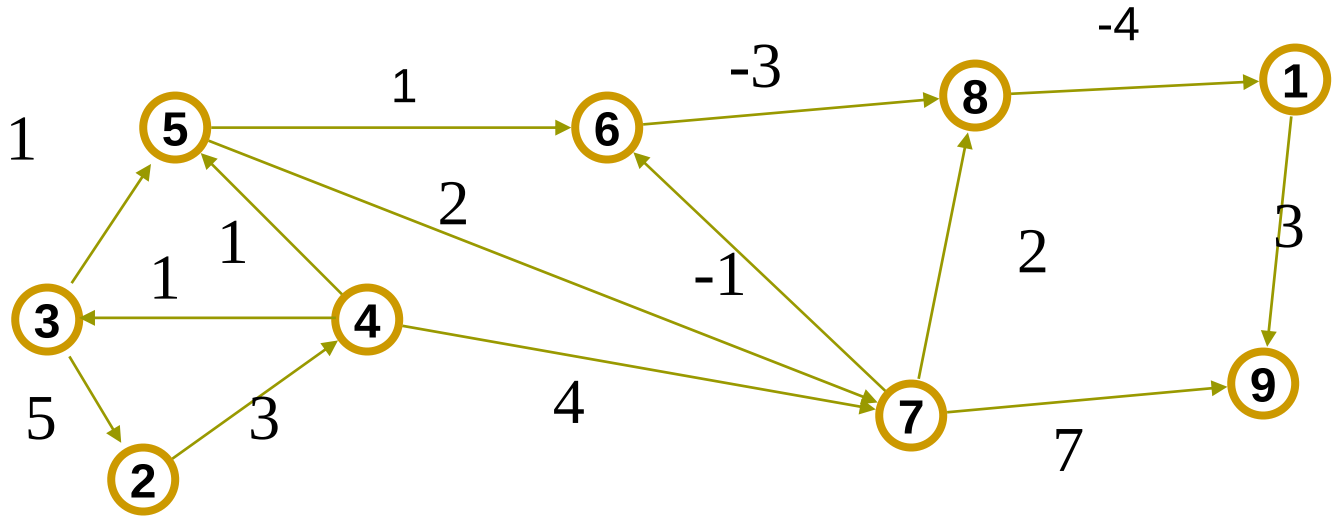
To begin with, the theory was created to treat the mathematical problems arising from the study of various multi-stage decision processes, which may roughly be described in the following way: We have a physical system whose state at any time t is determined by a set of quantities which we call state parameters, or state variables. At certain times, which may be prescribed in advance, or which may be determined by the process itself, we are called upon to make decisions which will affect the state of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of maximizing some function of the parameters describing the final state.

Examples of processes fitting this loose description are furnished by virtually every phase of modern life, from the planning of industrial production lines to the scheduling of patients at a medical clinic; from the determination of long-term investment programs for universities to the determination of a replacement policy for machinery in factories; from the programming of training policies for skilled and unskilled labor to the choice of optimal purchasing and inventory policies for department stores and military establishments.

Algorithm Design Technique
Dynamic Programming

- **Simple subproblems**
  - There is a way to break down the problem into simple subproblems of similar structure
  - The subproblems can easily be defined with a few indices (i.e., i, j, k)
- **Subproblem optimality**
  - An optimal solution to the global problem is a composition of optimal subproblem solutions using a simple combining operation
- **Subproblem overlap**
  - Optimal solutions to unrelated subproblems can have subproblems in common. Such overlap is recognized and improves the efficiency significantly

Algorithm AllPairsShortestPaths(G)



Algorithm AllPairsShortestPaths(G)

Input: An edge-weighted digraph $G = (V, E)$ without negative-weight cycles. Further $V = \{1,2, \ldots, n\}$

Output: Matrix D s.t. for all $i,j \in V$, $D[i,j]$ denotes the length of a shortest path from i to j

Terminology and Notation

- Let $v_1, v_2, \ldots, v_l$ be the vertices of a path p in directed graph G.
- Then the vertices $v_2, v_3, \ldots, v_{l-1}$ are called *intermediate vertices* of p.
- Let $d_{ij}^{(k)}$ be the length of a shortest path from i to j such that all intermediate vertices that are on the path are members of the set $\{1, 2, \ldots, k\}$.
- Then $d_{ij}^{(0)}$ is the weight of the edge from i to j if such an edge exists, $+\infty$ otherwise
- $d_{ij}^{(n)}$is the shortest path from i to j

Observations

1. A shortest path does not contain the same vertex twice.

Proof. Otherwise the path would contain a cycle, and removing the cycle would shorten the path.

Observations

2. If vertex k *is not* on a shortest path from i to j with intermediate vertices from $\{1, 2, \ldots, k\}$ then $d_{ij}^{(k)} = d_{ij}^{(k-1)}$.

Proof. Since k is *not* on a shortest path from i to j with intermediate vertices from $\{1, 2, \ldots, k\}$, adding k to the set $\{1, 2, \ldots, k\text{-}1\}$ of intermediate vertices will not improve the length of the shortest path.

Observations

3. If vertex k *is* on a shortest path from i to j with intermediate vertices from $\{1, 2, \ldots, k\}$ then

$$d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$$

Proof. k is somewhere on the shortest path from i to j. Let us consider the shortest paths from (1) i to k and (2) k to j. The internal vertex set for these is $\{1, 2, \ldots, k-1\}$ since k is an external vertex for both paths. Thus,

$$d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$$

We conclude for a shortest path from *i* to *j* with intermediate vertices from {1, 2, …, *k*}:

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

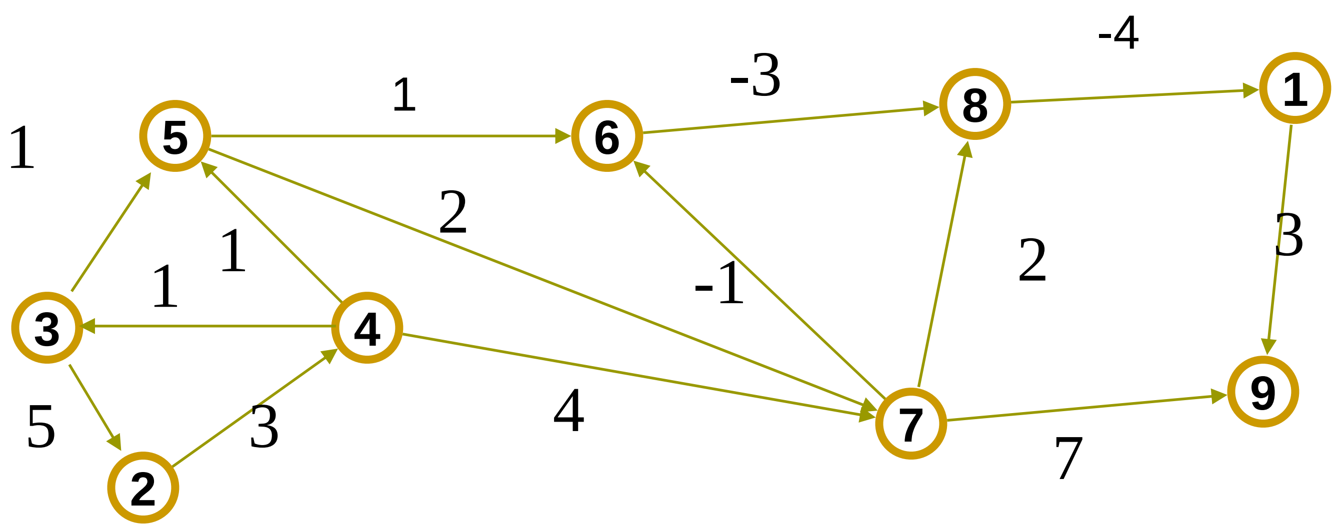
Developing the Dynamic Programming Algorithm

- Let D be an $n \times n$ matrix for each pair of vertices in the graph
- Initially, each cell $D[i, j]$ contains the weight of edge (i, j), if existent, 0 if $i = j$, and $+\infty$ otherwise.
- We then iterate over the included intermediate vertices and update each cell for the (possibly) improved paths.
- Once all vertices are included in the set of intermediate vertices, $D[i, j]$ contains the weight of a shortest path from i to j if existent, 0 if $i = j$, and $+\infty$ otherwise.

Algorithm AllPairsShortestPaths(G)

```
// Initializing
for i ← 1 to n do
   for j ← 1 to n do
       if i=j then
             D[i,j] ← 0
       if (i,j)∈G then
             D[i,j] ← w((i,j))
       else
             D[i,j] ← +∞
```

Algorithm AllPairsShortestPaths(G)



Algorithm AllPairsShortestPaths(G)

	1	2	3	4	5	6	7	8	9
1	0								
2		0							
3			0						
4				0					
5					0				
6						0			
7							0		
8								0	
9									0

Algorithm AllPairsShortestPaths(G)

	1	2	3	4	5	6	7	8	9
1	0								3
2		0		3					
3		5	0		1				
4			1	0	1		4		
5					0	1	2		
6						0		-3	
7						-1	0	2	7
8	-4							0	
9									0

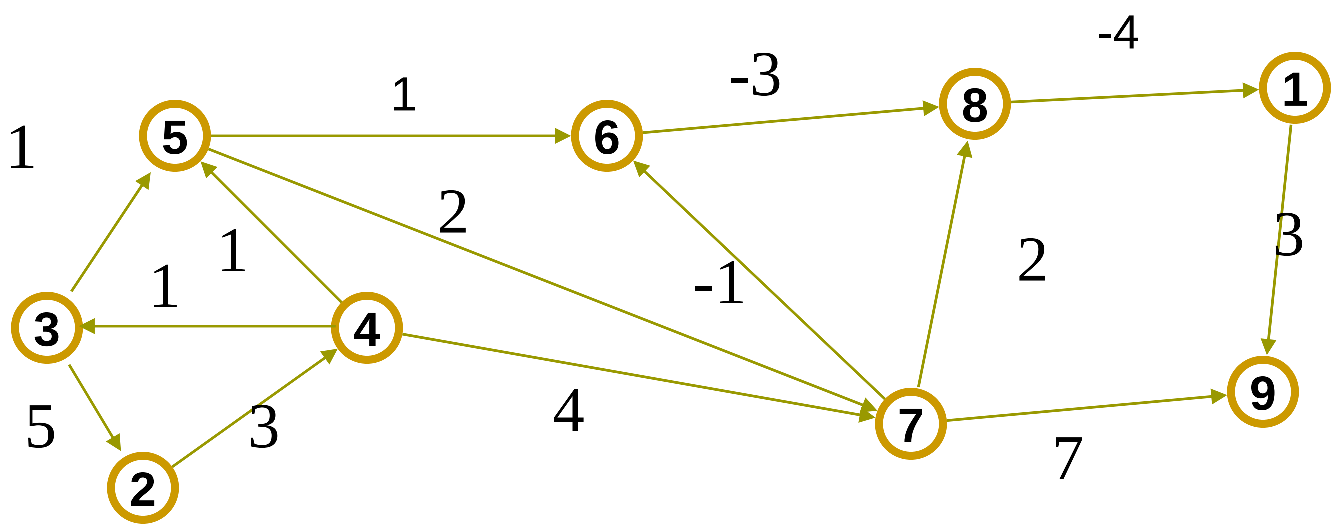
Algorithm AllPairsShortestPaths(G)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
3	$+\infty$	5	0	$+\infty$	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
4	$+\infty$	$+\infty$	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

Algorithm AllPairsShortestPaths(G)

```
//Compute shortest paths
for k ← 1 to n do //k is the latest included vertex
  for i ← 1 to n do //path starts at i
    for j ← 1 to n do //path ends at j
      if i≠k and j≠k then
        D[i,j] ← min{D[i,j],D[i,k]+D[k,j]}
return D
```

Algorithm AllPairsShortestPaths(G)



Algorithm AllPairsShortestPaths(G)

	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
3	$+\infty$	5	0	$+\infty$	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
4	$+\infty$	$+\infty$	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

Algorithm AllPairsShortestPaths(G)

j

$K = 1$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
3	$+\infty$	5	0	$+\infty$	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
4	$+\infty$	$+\infty$	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty/$ -1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

i

Algorithm AllPairsShortestPaths(G)

j

$K = 2$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
3	$+\infty$	5	0	$+\infty$ /8	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
4	$+\infty$	$+\infty$	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

i

Algorithm AllPairsShortestPaths(G)

j

$K=3$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty$	3	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
3	$+\infty$	5	0	8	1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
4	$+\infty$	$+\infty/6$	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
i 9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

Algorithm AllPairsShortestPaths(G)

j

i

$K=4$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	$+\infty/4$	3	$+\infty/4$	$+\infty$	$+\infty/7$	$+\infty$	$+\infty$
3	$+\infty$	5	0	8	1	$+\infty$	$+\infty/12$	$+\infty$	$+\infty$
4	$+\infty$	6	1	0	1	$+\infty$	4	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

Algorithm AllPairsShortestPaths(G)

j

i

$K = 5$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	4	3	4	$+\infty/5$	$7/6$	$+\infty$	$+\infty$
3	$+\infty$	5	0	8	1	$+\infty/2$	$12/3$	$+\infty$	$+\infty$
4	$+\infty$	6	1	0	1	$+\infty/2$	$4/3$	$+\infty$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	2	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

Algorithm AllPairsShortestPaths(G)

j

$K = 6$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	4	3	4	5	6	$+\infty/2$	$+\infty$
3	$+\infty$	5	0	8	1	2	3	$+\infty/-1$	$+\infty$
4	$+\infty$	6	1	0	1	2	3	$+\infty/-1$	$+\infty$
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	$+\infty/-2$	$+\infty$
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	$2/-4$	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

i

Algorithm AllPairsShortestPaths(G)

j

$K = 7$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty$	0	4	3	4	5	6	2	$+\infty$/ 13
3	$+\infty$	5	0	8	1	2	3	-1	$+\infty$/ 10
4	$+\infty$	6	1	0	1	2	3	-1	$+\infty$/ 10
5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	1	2	-2	$+\infty$/9
6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty$
7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	-4	7
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

i

Algorithm AllPairsShortestPaths(G)

j

$K=8$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	$+\infty/$ -2	0	4	3	4	5	6	2	$13/1$
3	$+\infty/$ -5	5	0	8	1	2	3	-1	$10/-2$
4	$+\infty/$ -5	6	1	0	1	2	3	-1	$10/-2$
5	$+\infty/$ -6	$+\infty$	$+\infty$	$+\infty$	0	1	2	-2	$9/-3$
6	$+\infty/$ -7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	$+\infty/$ -4
7	$+\infty/$ -8	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	-4	$7/-5$
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0

i

Algorithm AllPairsShortestPaths(G)

j

i

$K=9$	1	2	3	4	5	6	7	8	9
1	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	3
2	-2	0	4	3	4	5	6	2	1
3	-5	5	0	8	1	2	3	-1	-2
4	-5	6	1	0	1	2	3	-1	-2
5	-6	$+\infty$	$+\infty$	$+\infty$	0	1	2	-2	-3
6	-7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	$+\infty$	-3	-4
7	-8	$+\infty$	$+\infty$	$+\infty$	$+\infty$	-1	0	-4	-5
8	-4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	-1
9	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0