CSC 226

Algorithms and Data Structures: II

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Kruskal's Algorithm Correctness

- Initialize each vertex in it's own tree

Cut property

- Merge two trees by finding lightest edge not yet in a tree and merging two trees together; repeat until all vertices are in one tree

Cycle property

- If the lightest edge connects two vertices in one tree already; discard that edge

- *Example of greedy algorithm*

Pseudocode: Kruskal's Algorithm

Algorithm KruskalMST(G):
Input: A weighted connected graph G with n vertices and m edges
Output: an MST T for G
Data structures: Disjoint set C; Priority Queue Q; and tree T

```
for each vertex v in G do
    C(v) ← {v}
Let Q be a min priority queue storing all edges in G by weight
T ← ∅          // initialize  tree T
while T has less than n − 1 edges do
    (u, v) ← Q.removeMin()
    Let v ∈ C(v)
    Let u ∈ C(u)
    if C(v) ≠ C(u) then
        Add edge (u, v) to T
        Union C(v) and C(u)
return T
```

C :disjoint set of vertices called clusters

Q :a priority queue for the edges according to edge weights

Implementing Kruskal's Algorithm

- Two new ideas arise out of Kruskal's algorithm here
  - Efficient heap construction – bottom-up heap
  - Efficient cycle detection – union-find data structure

Idea 1: Bottom-up Heap

- Avoid sorting the edge weights by storing the edges in a heap …..

Building up a heap

- m standard insert-operations for a heap result in $\mathrm{O}(m \log(m))$ time.

- Can we build up a heap for m given elements faster? Is $\mathrm{O}(m)$ possible?

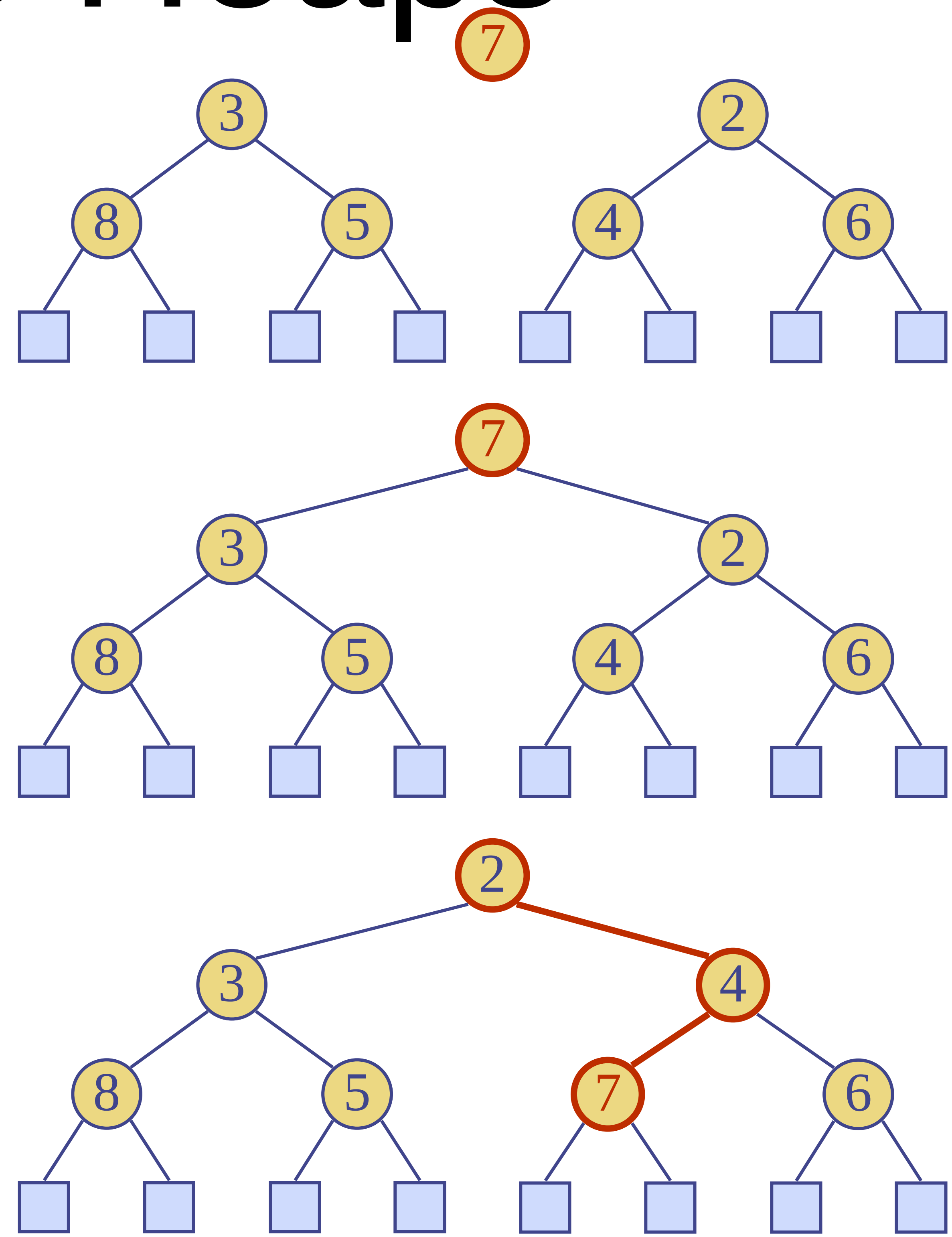
Bottom-Up Heap (pgs. 176-178)

```
Algorithm BottomUpHeap(S):
   Input: A list S storing m keys
   Output: A heap T storing the m keys

   if S is empty then
      return external node
   remove the first key, k, from S
   split S in half, lists S1 and S2
   T1 ← BottomUpHeap(S1)
   T2 ← BottomUpHeap(S2)
   T ← merge(k,T1,T2)
   DownHeap(T,root)
   return T
```

Merging Two Heaps

- We are given two heaps and a key *k*
- We create a new heap with the root node storing *k* and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



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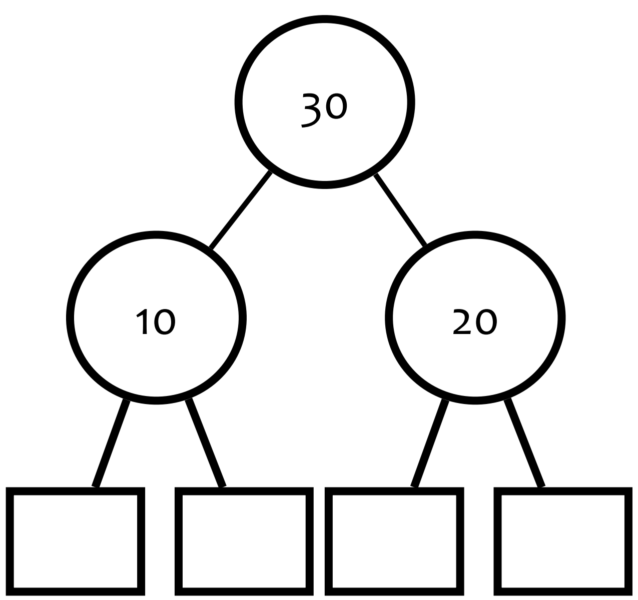

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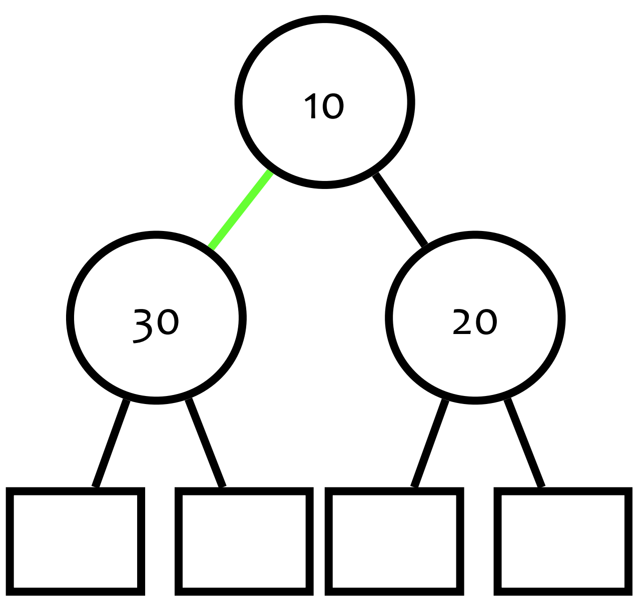

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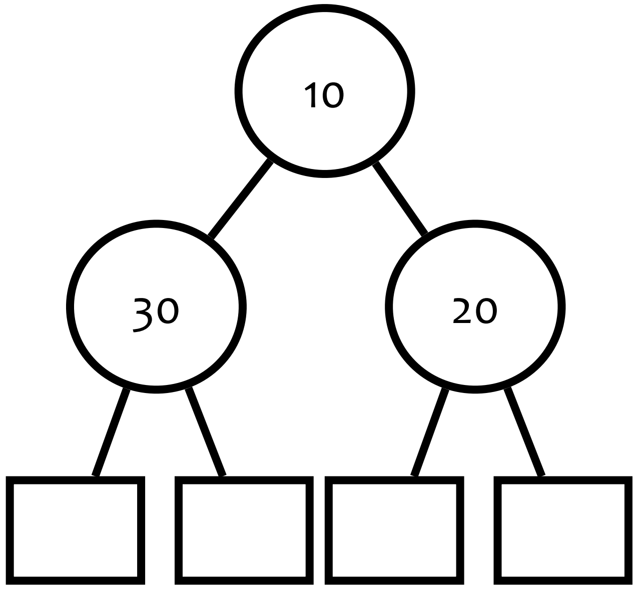

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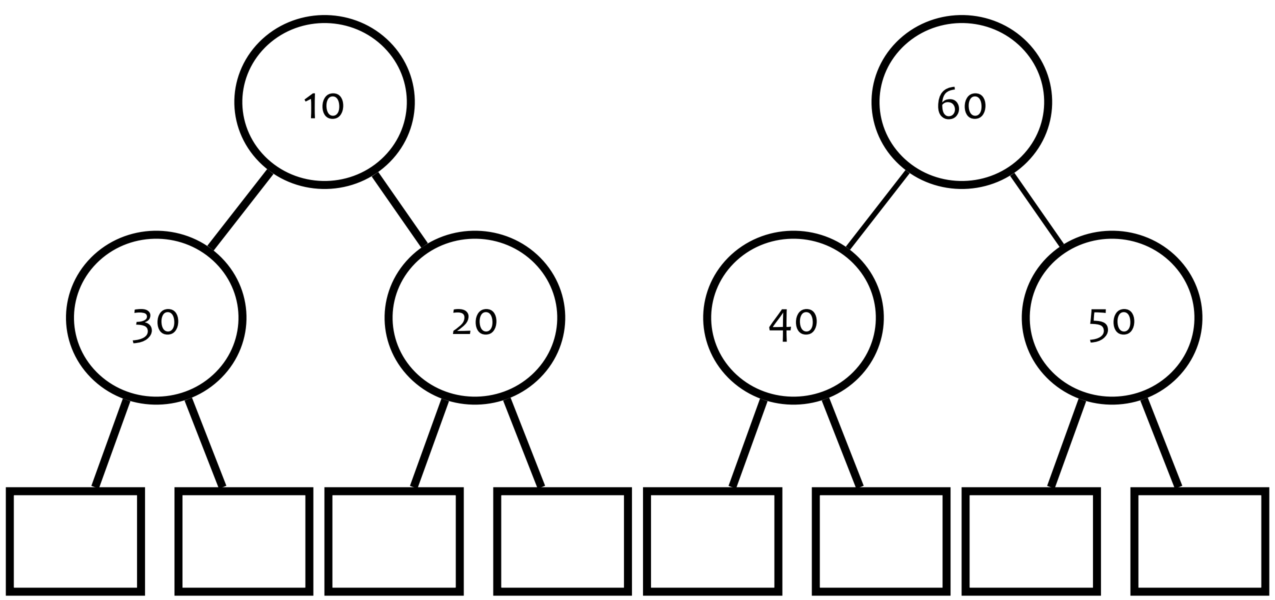

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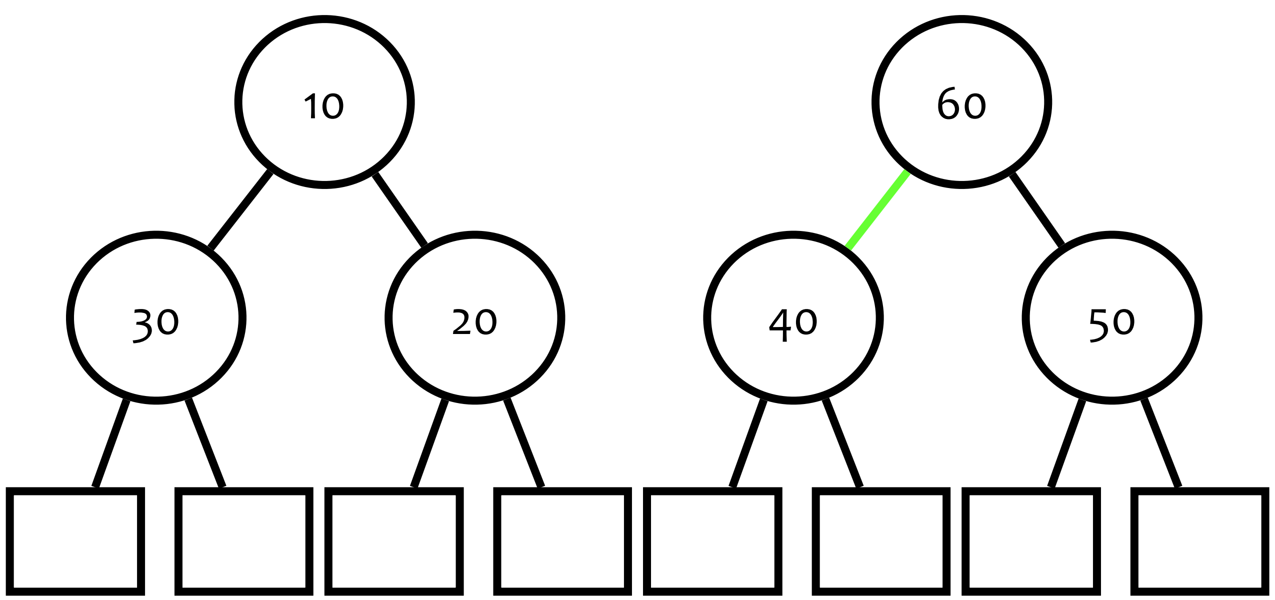

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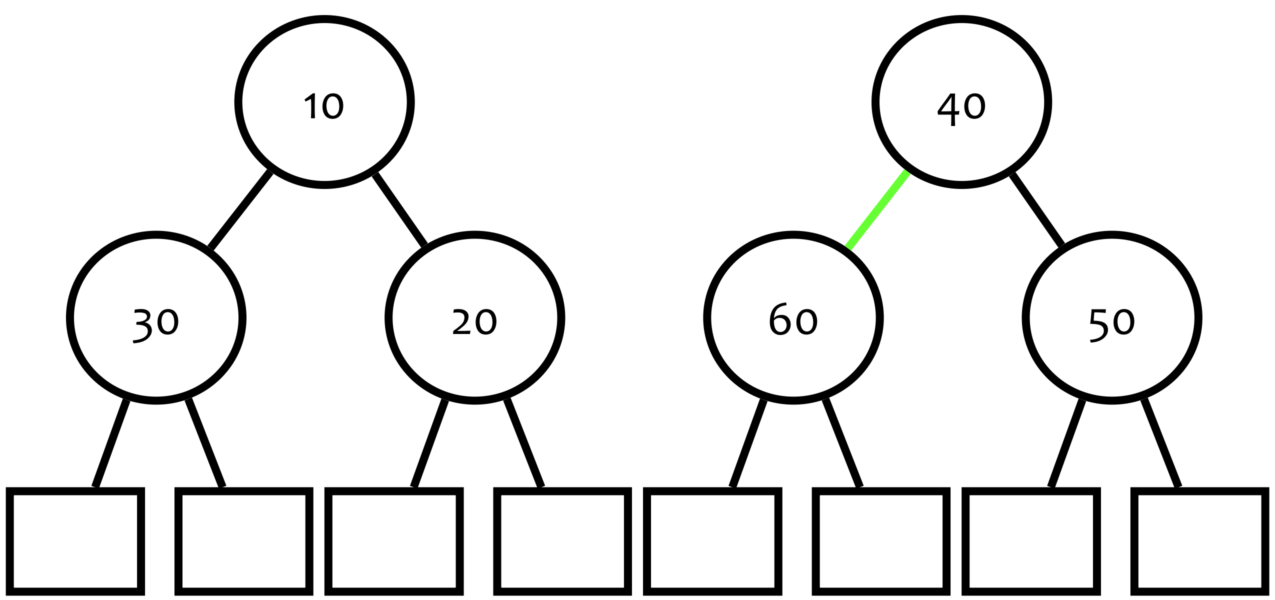

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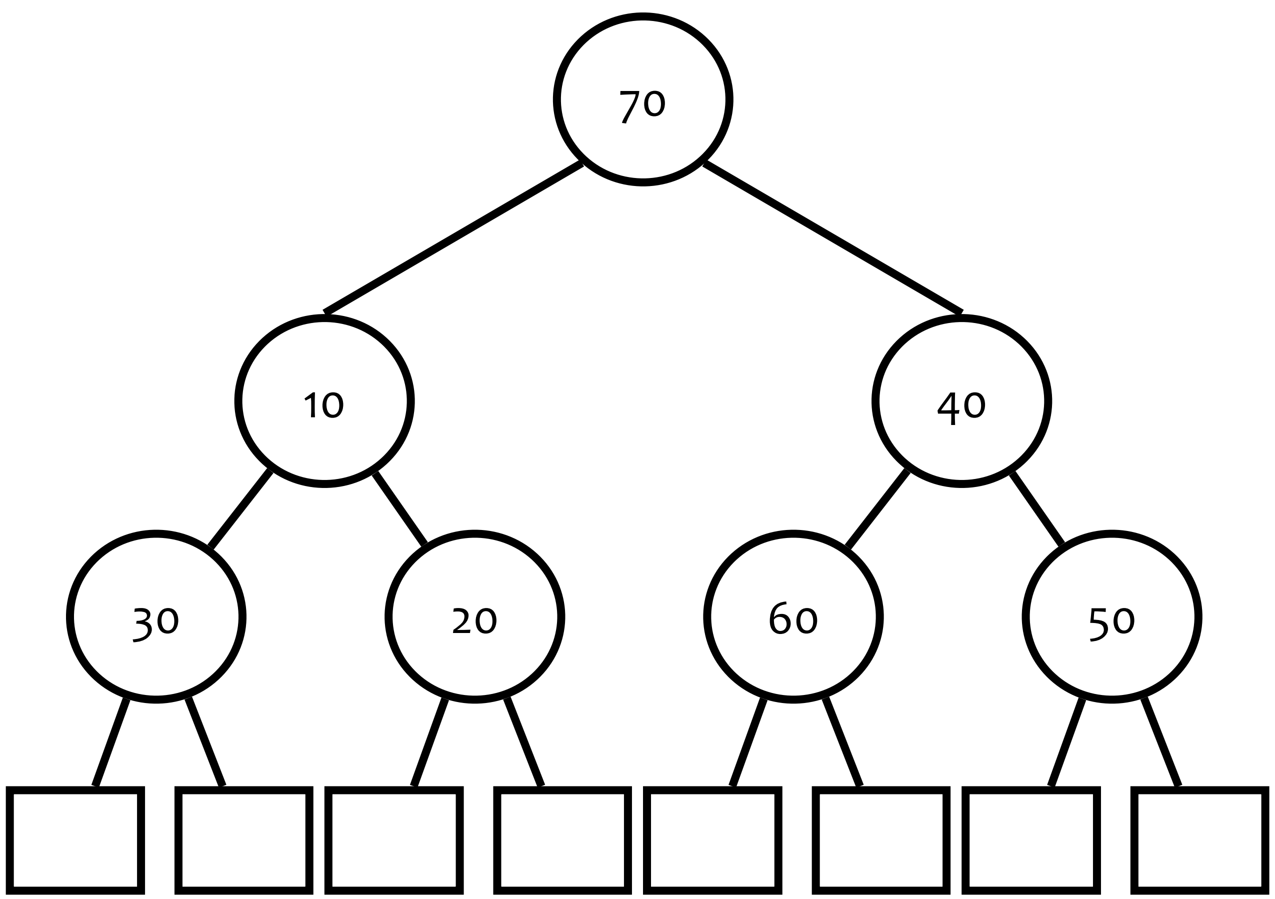

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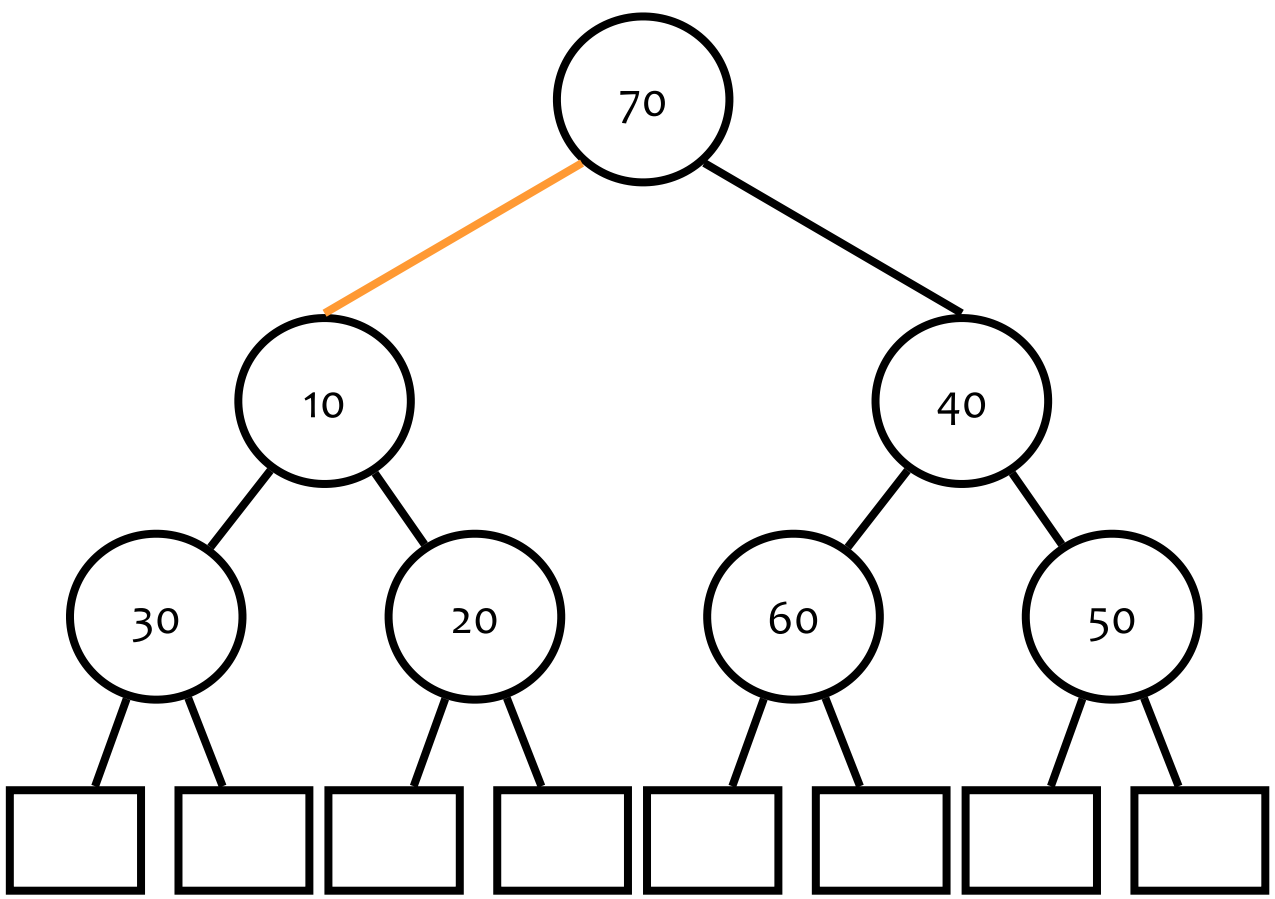

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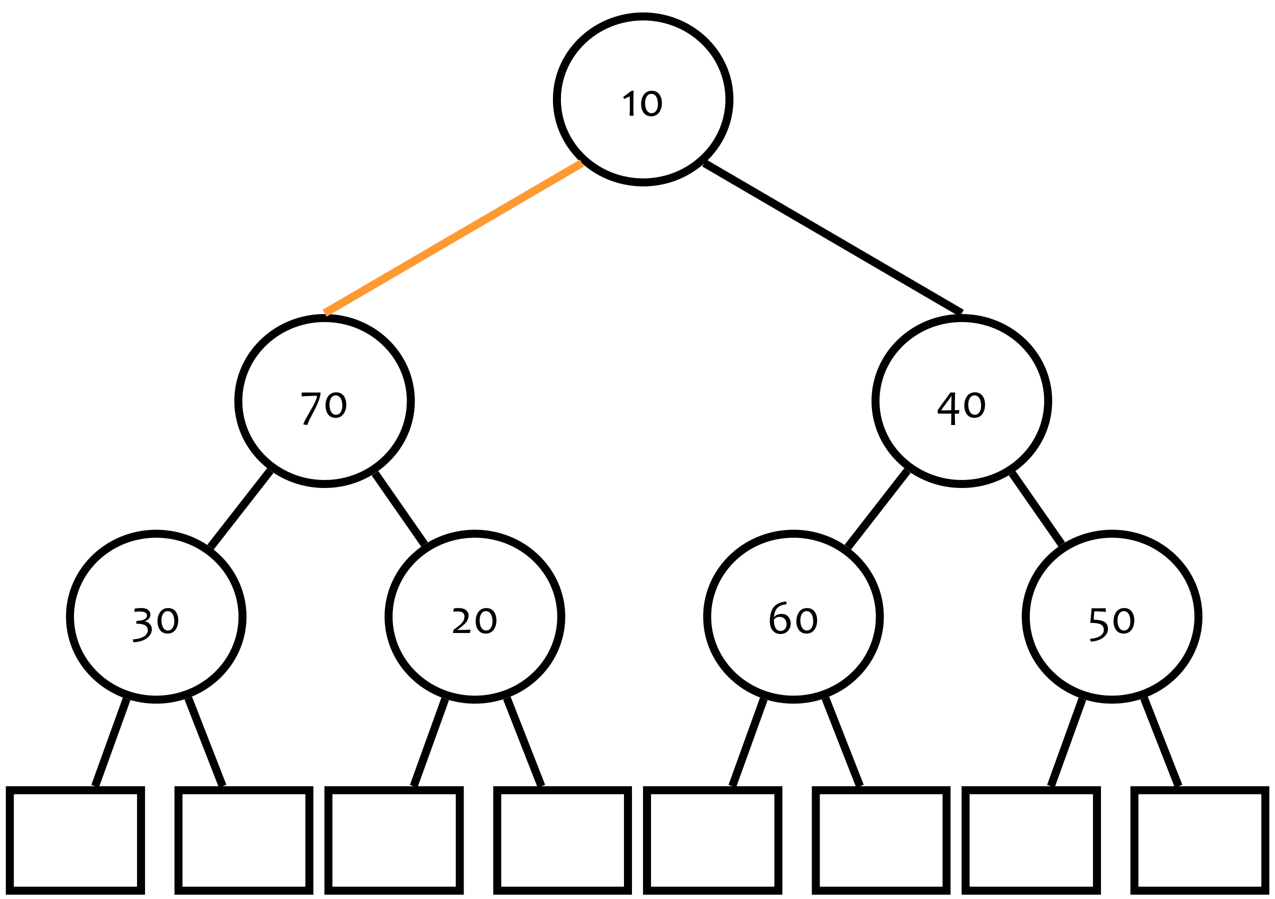

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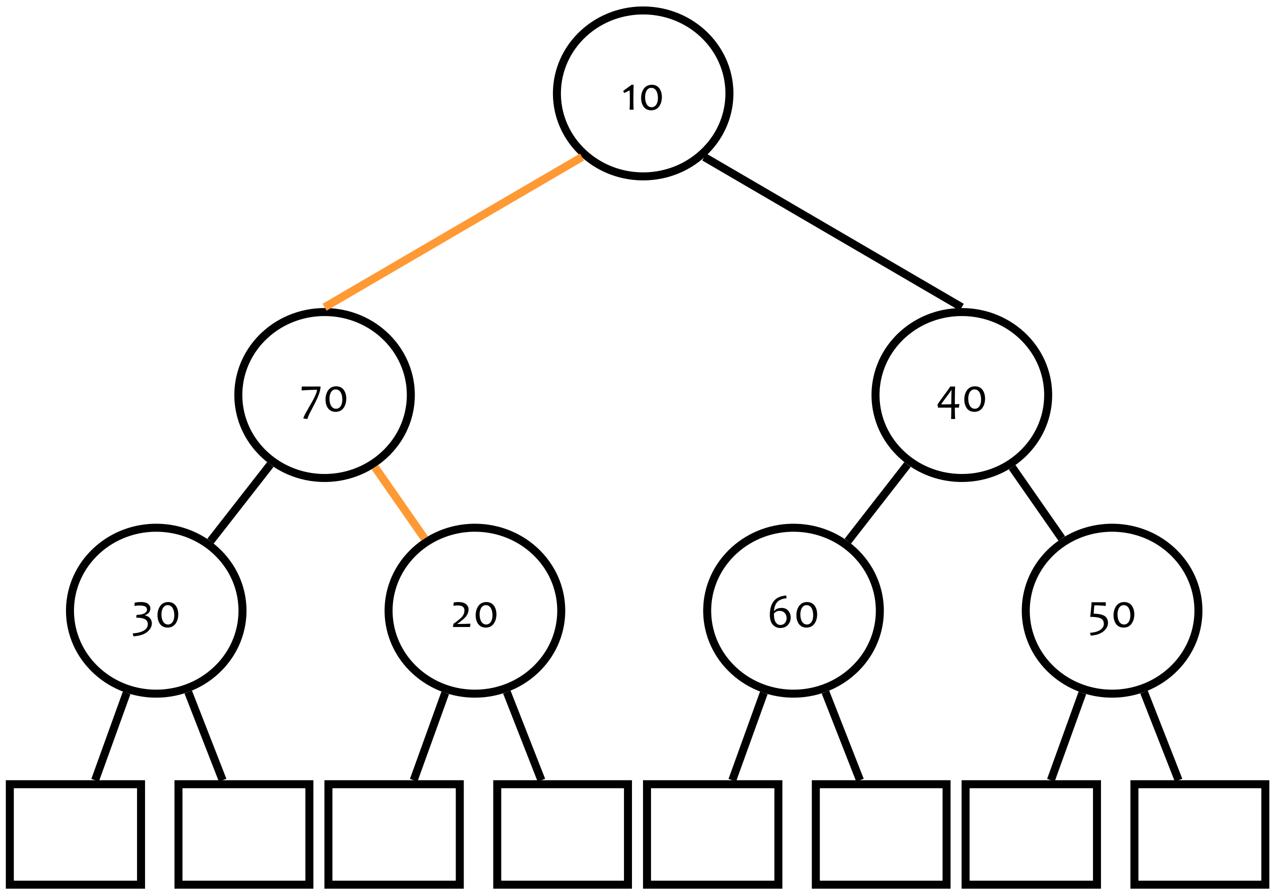

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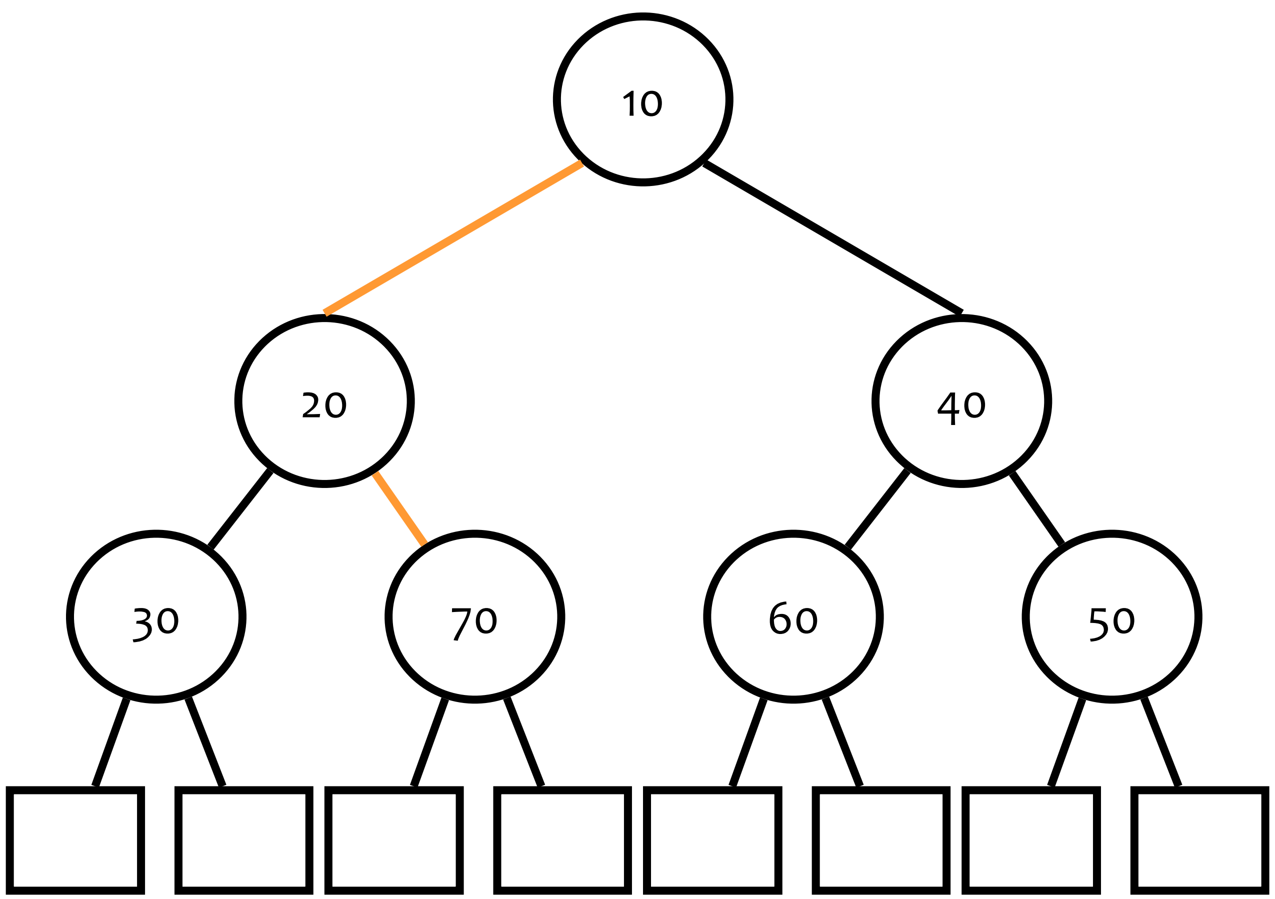

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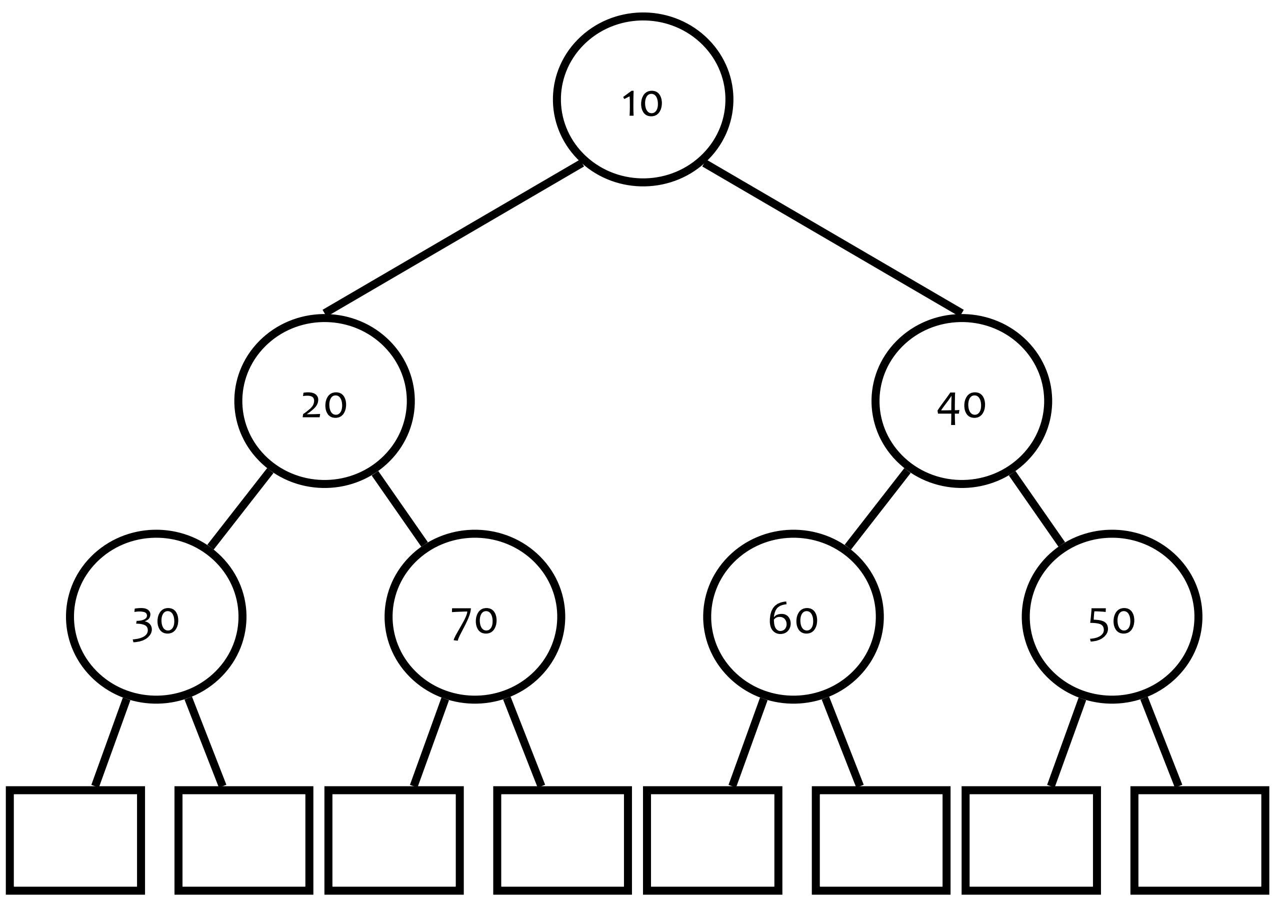

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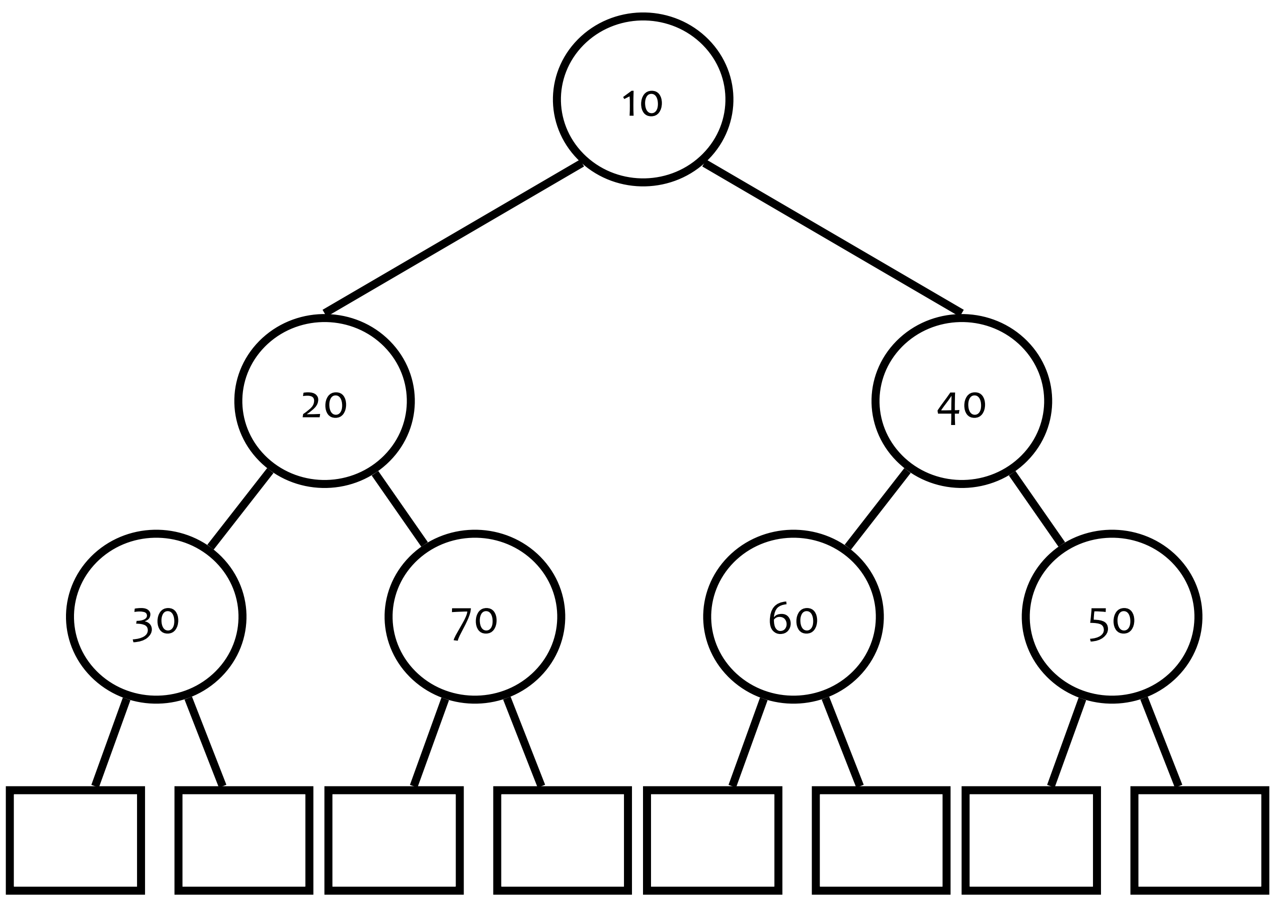

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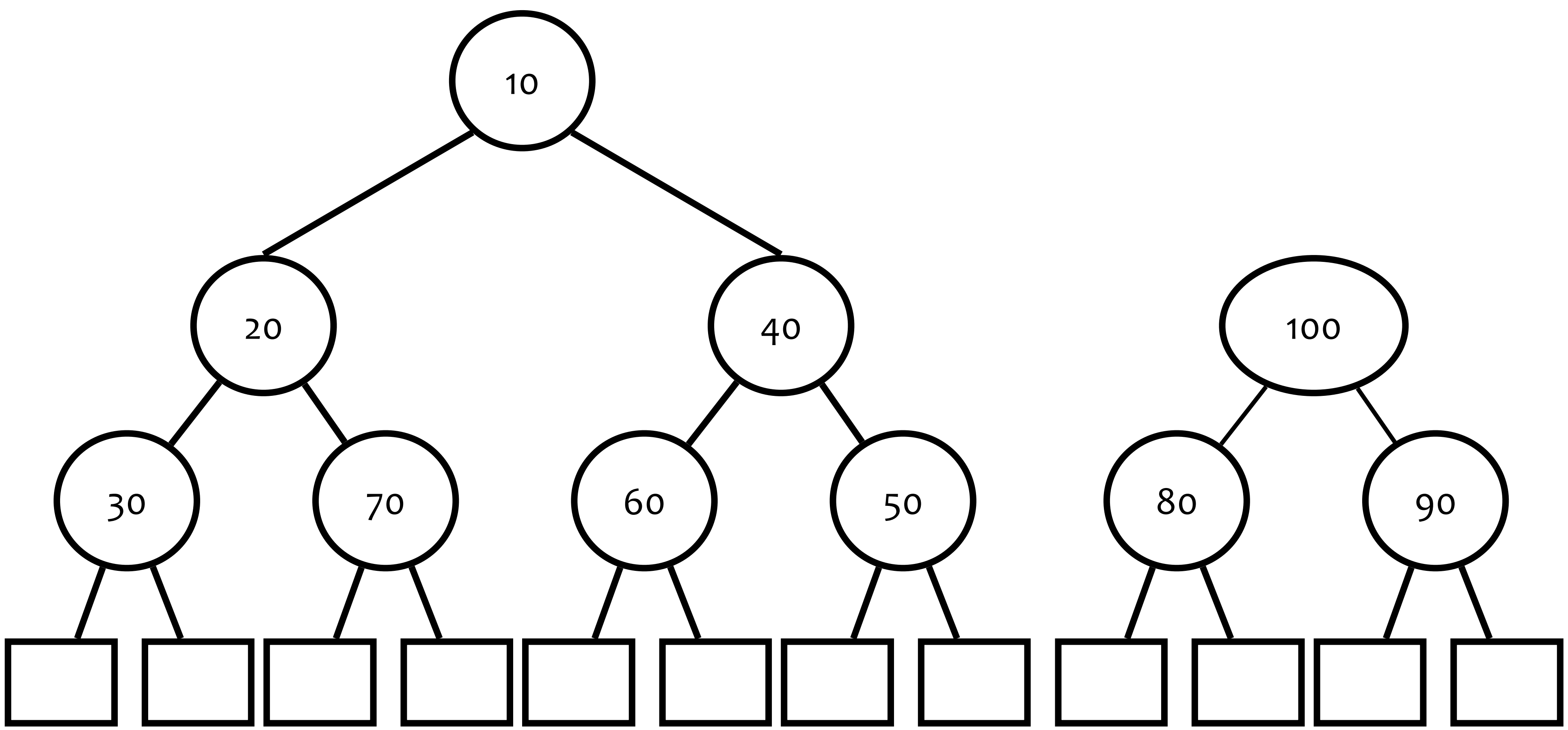

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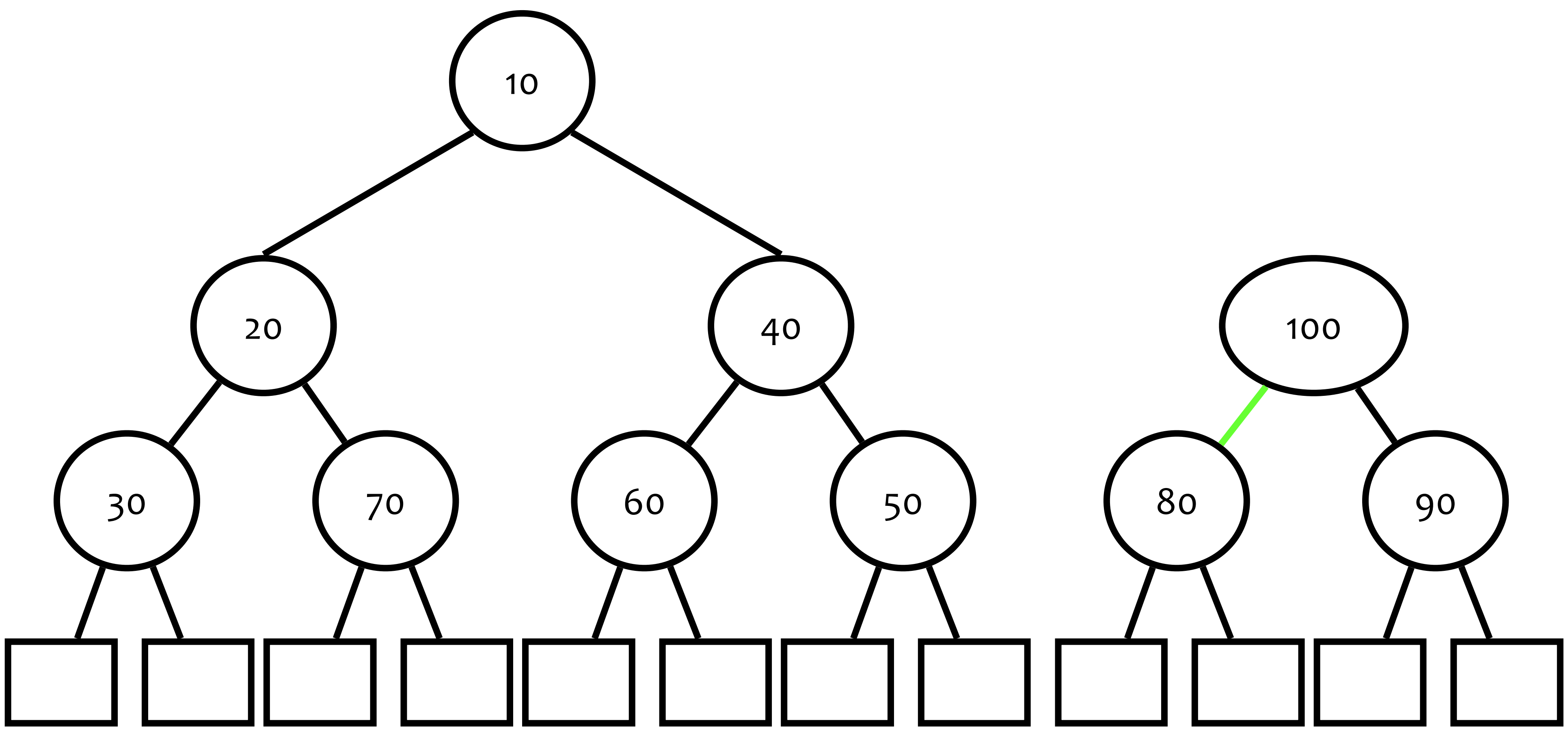

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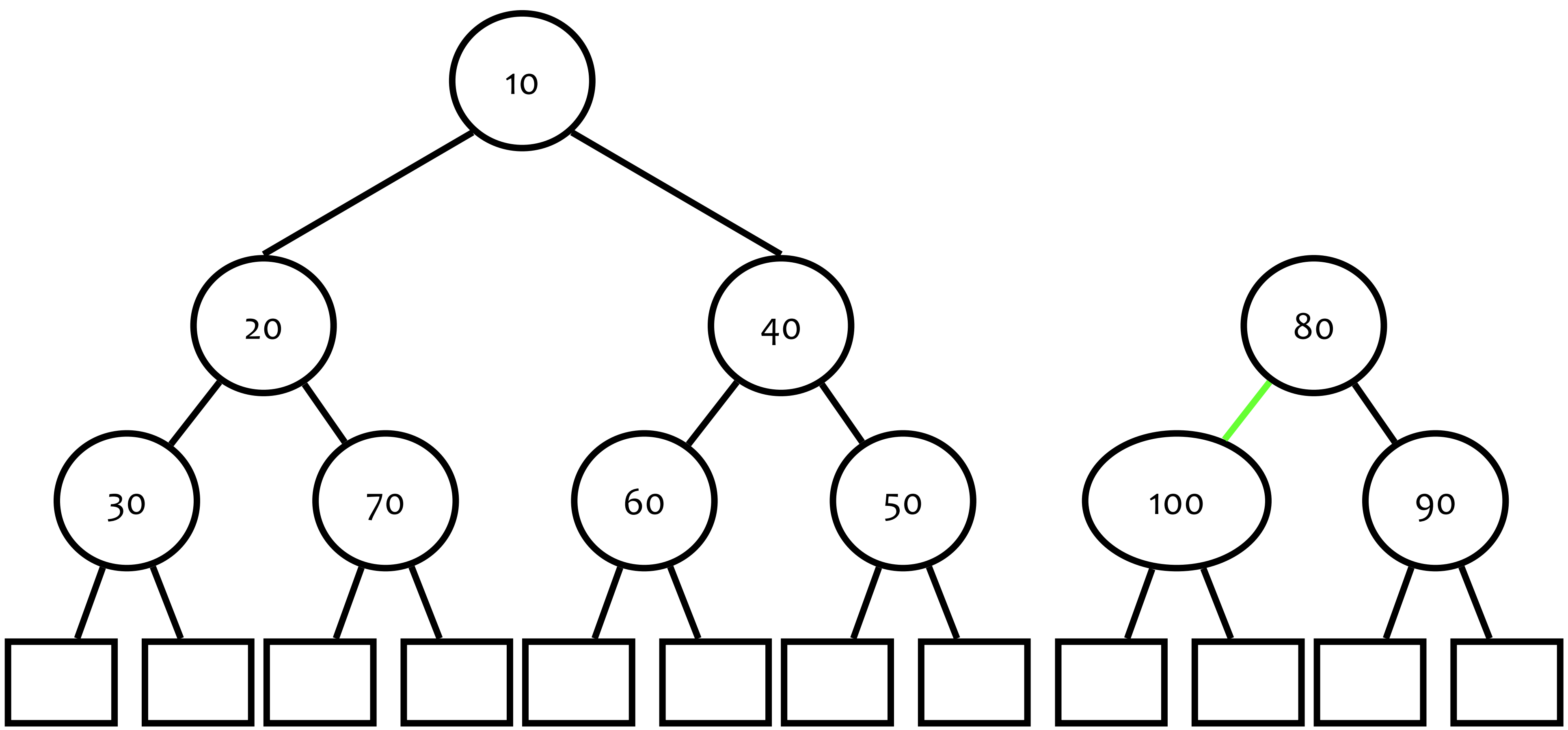

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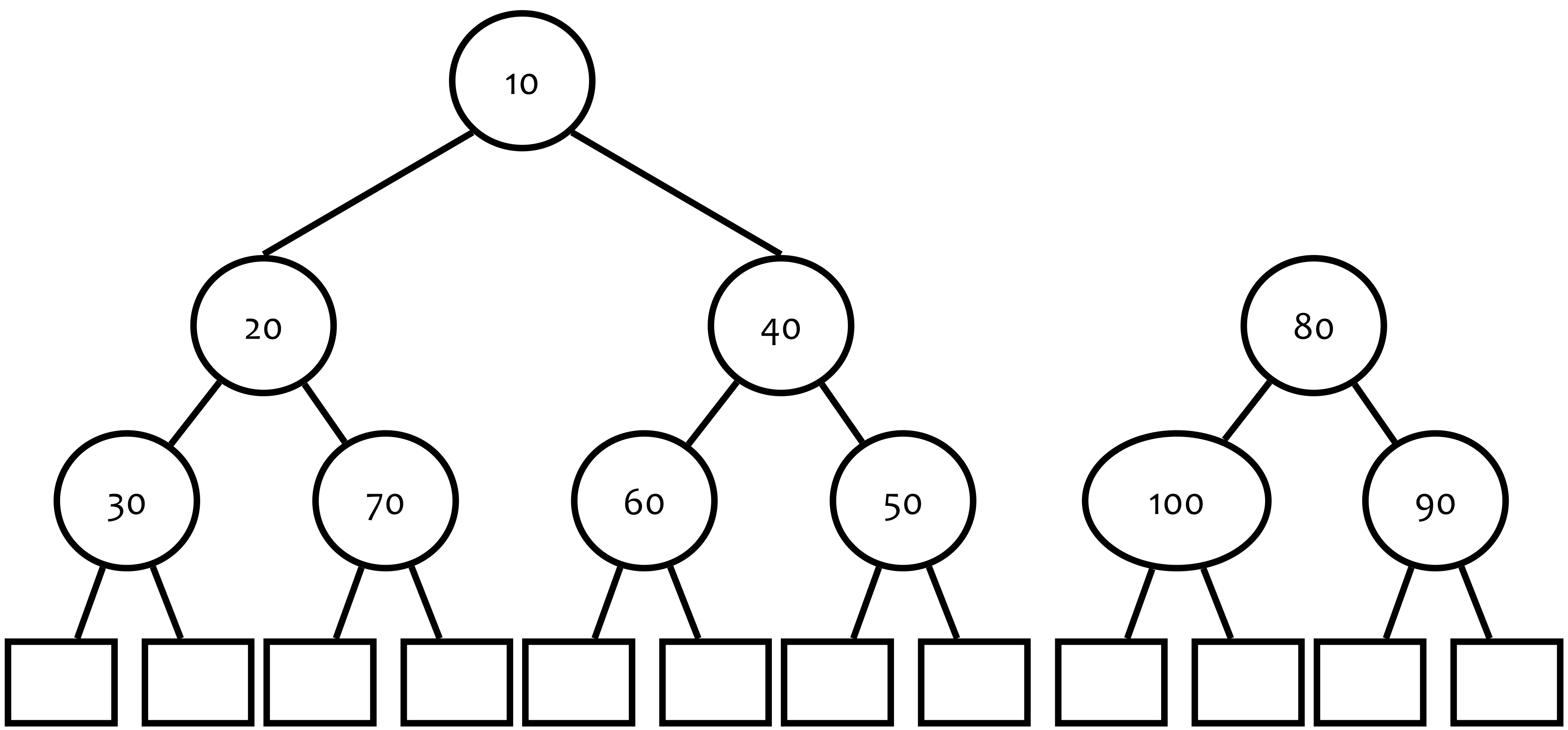

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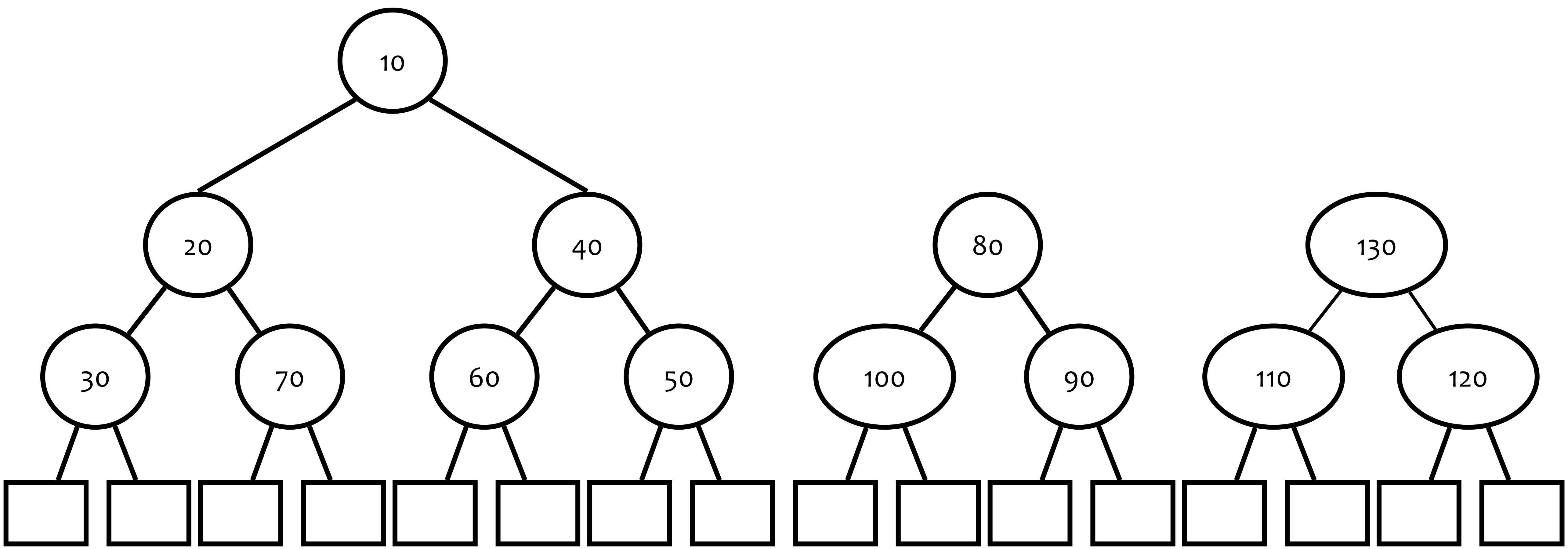

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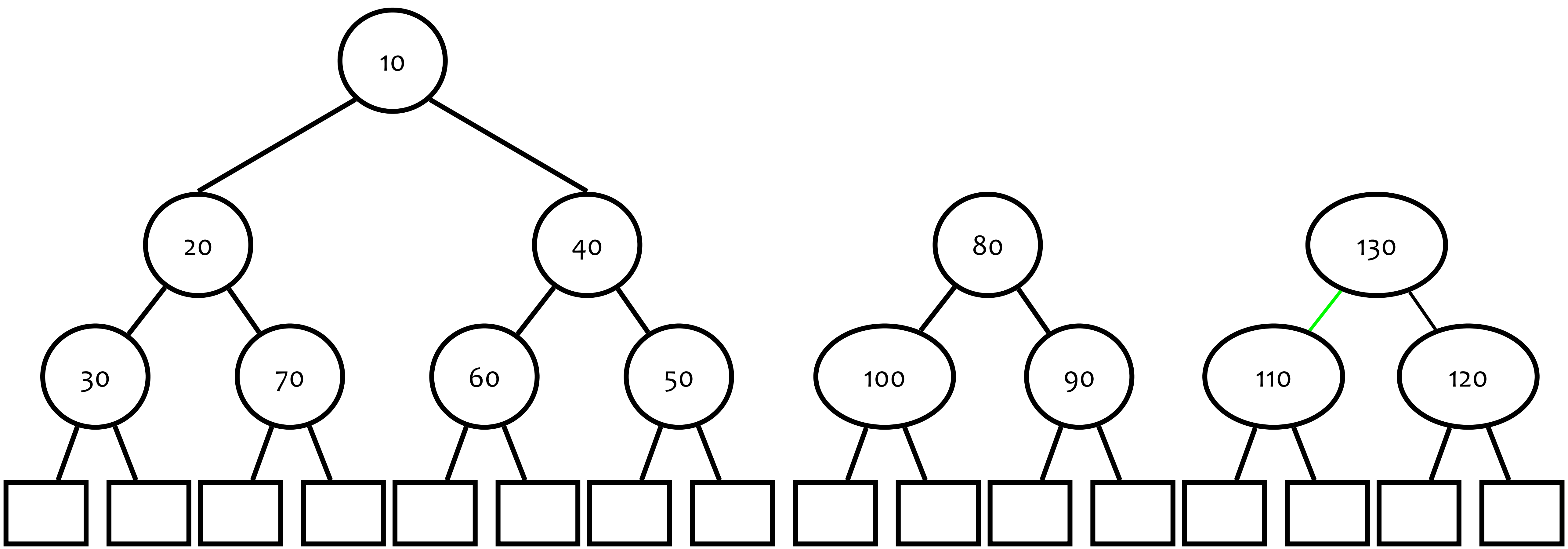

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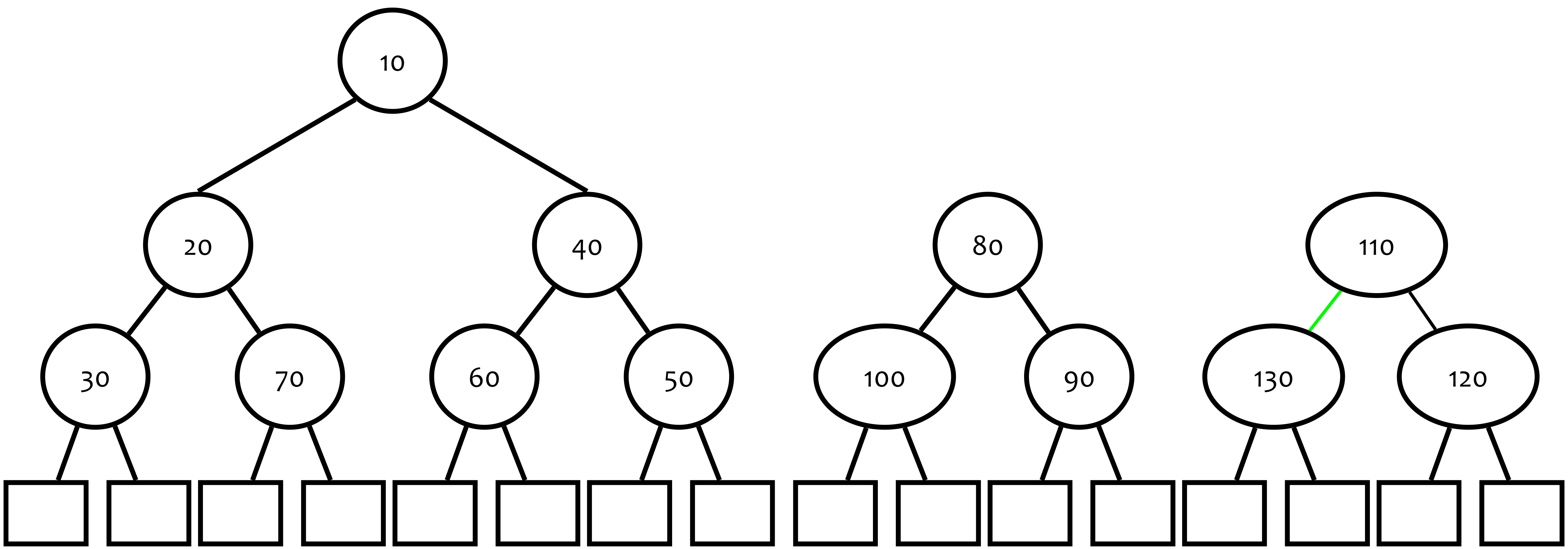

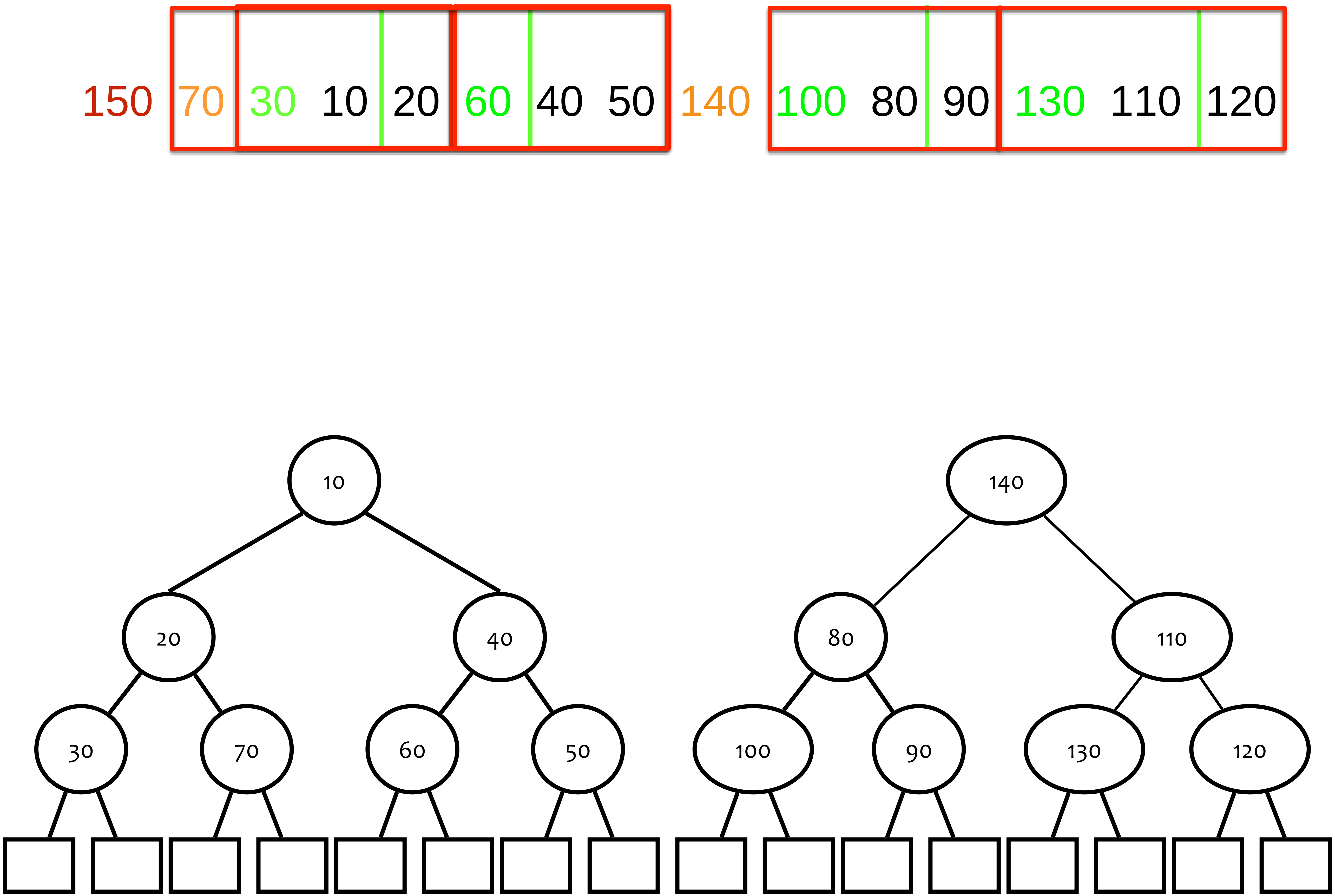

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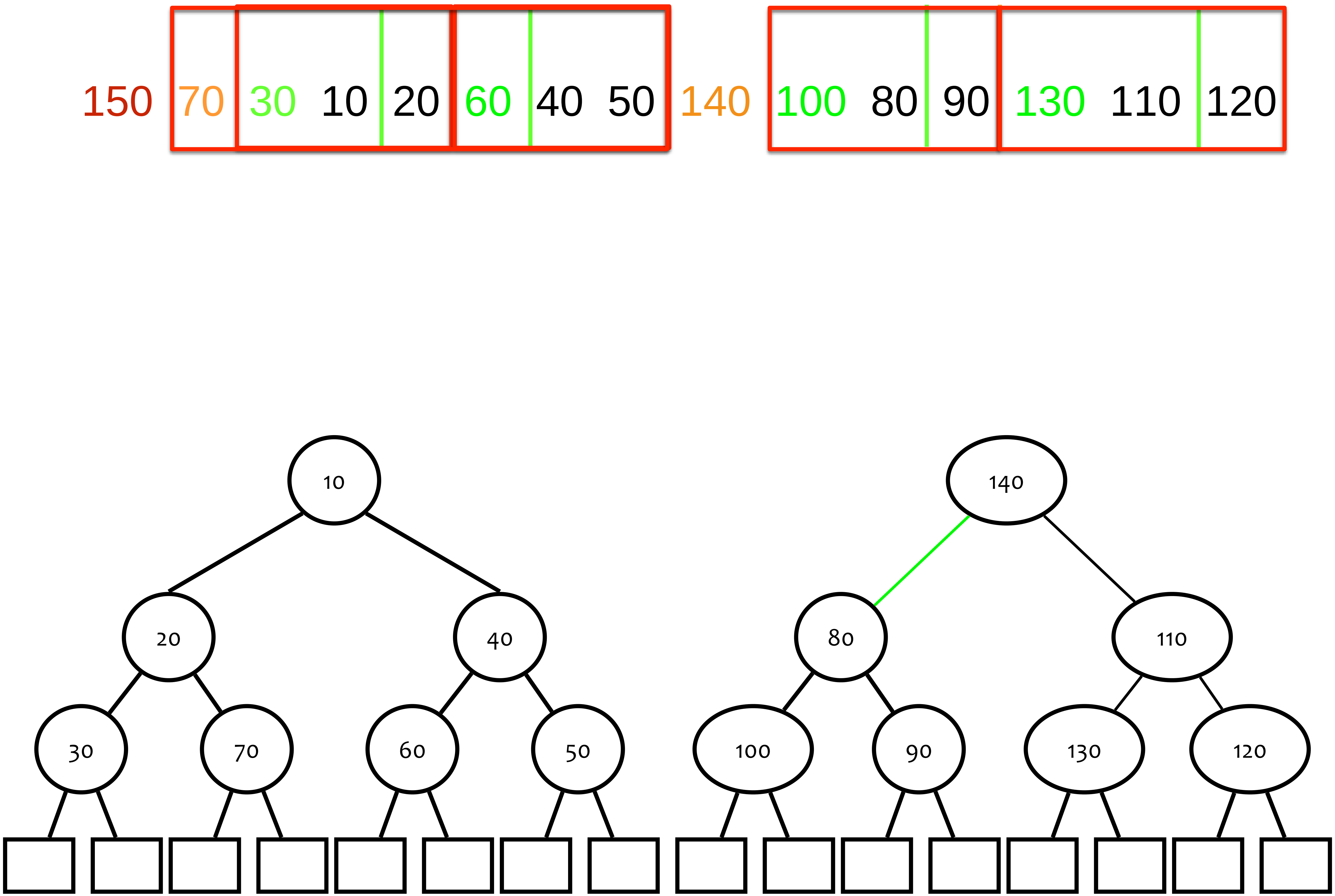
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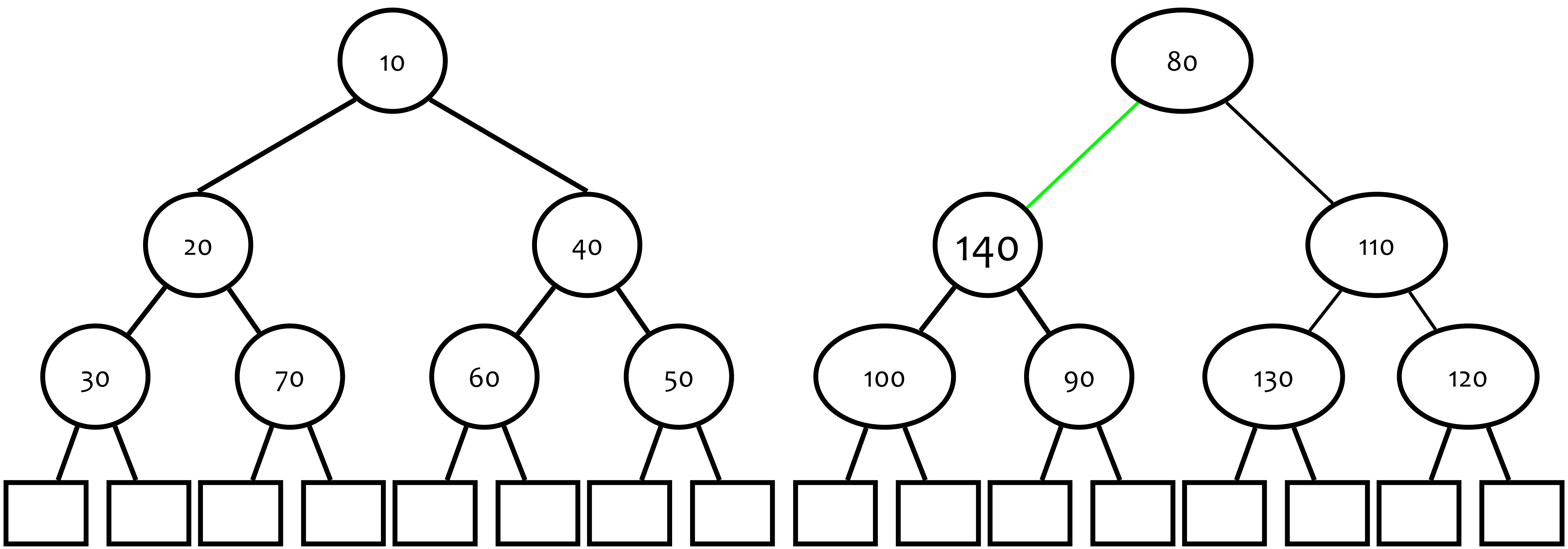

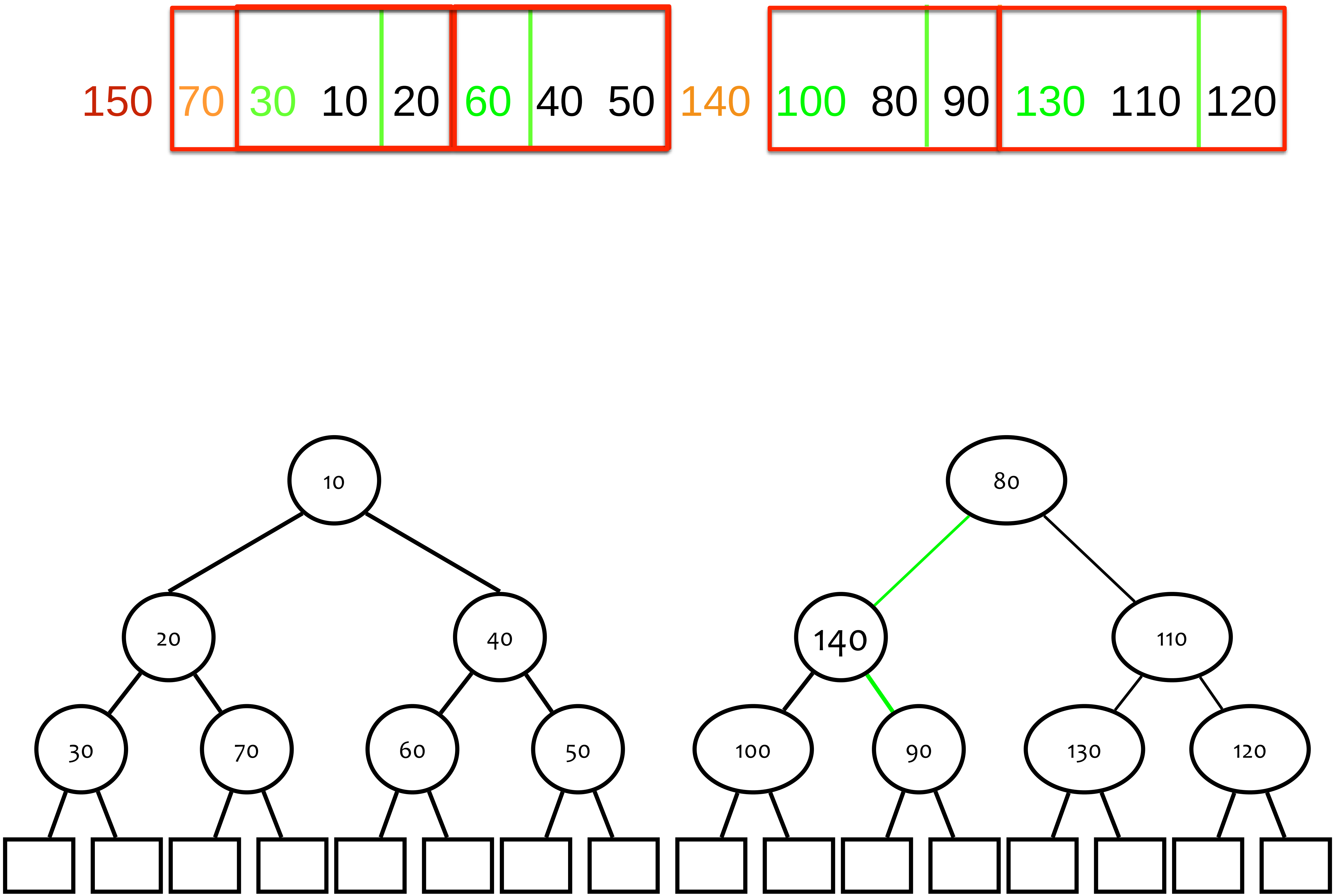
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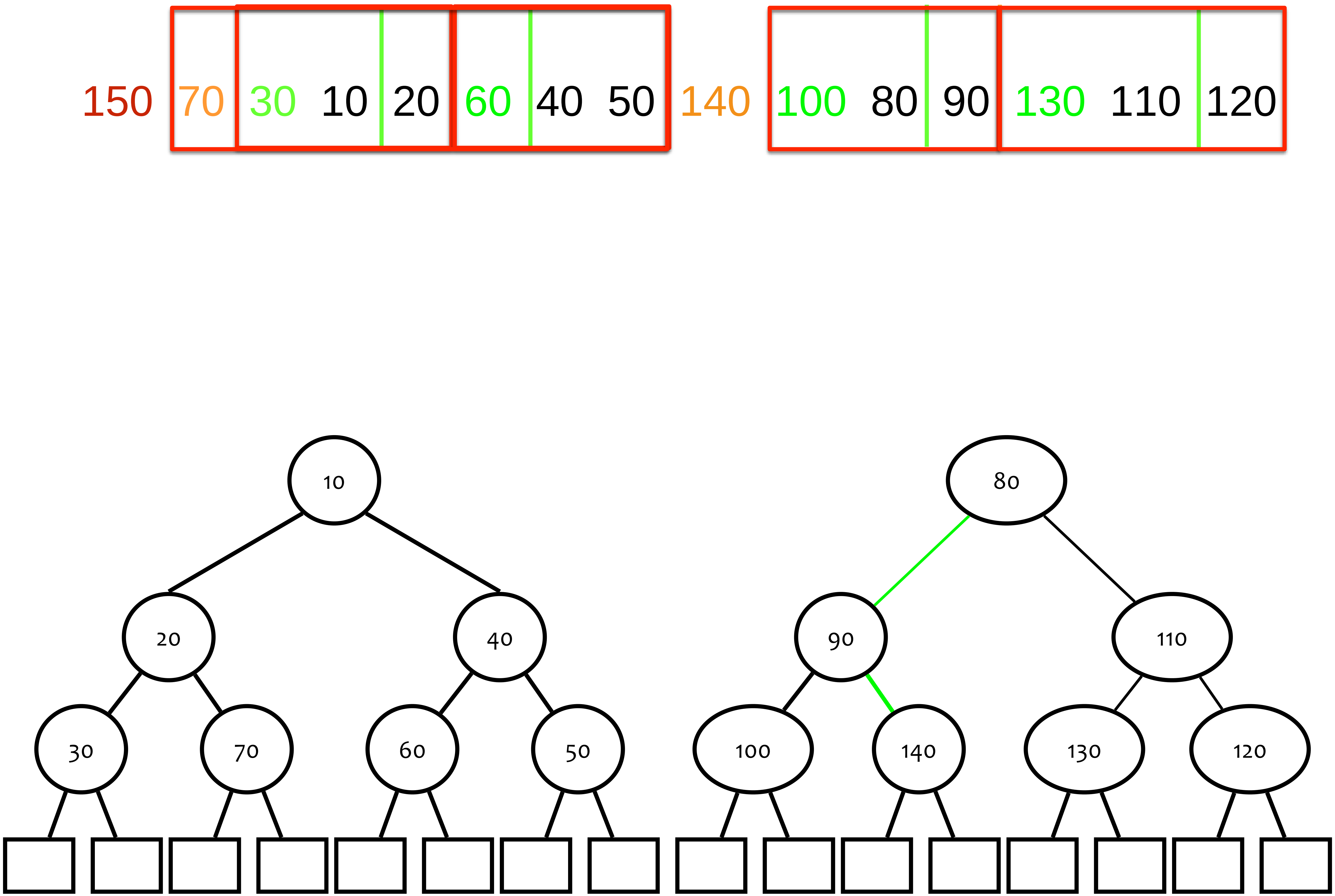
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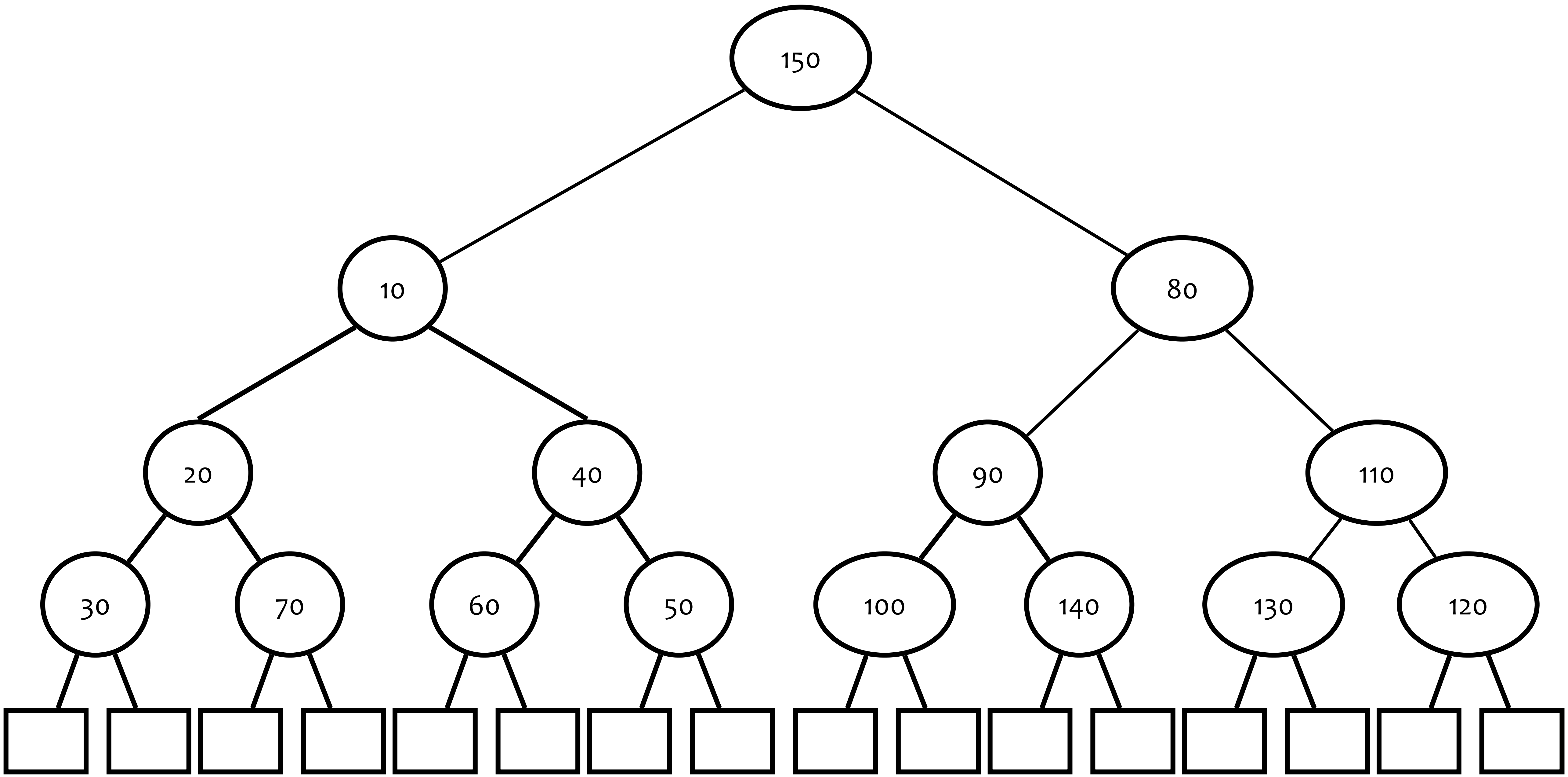

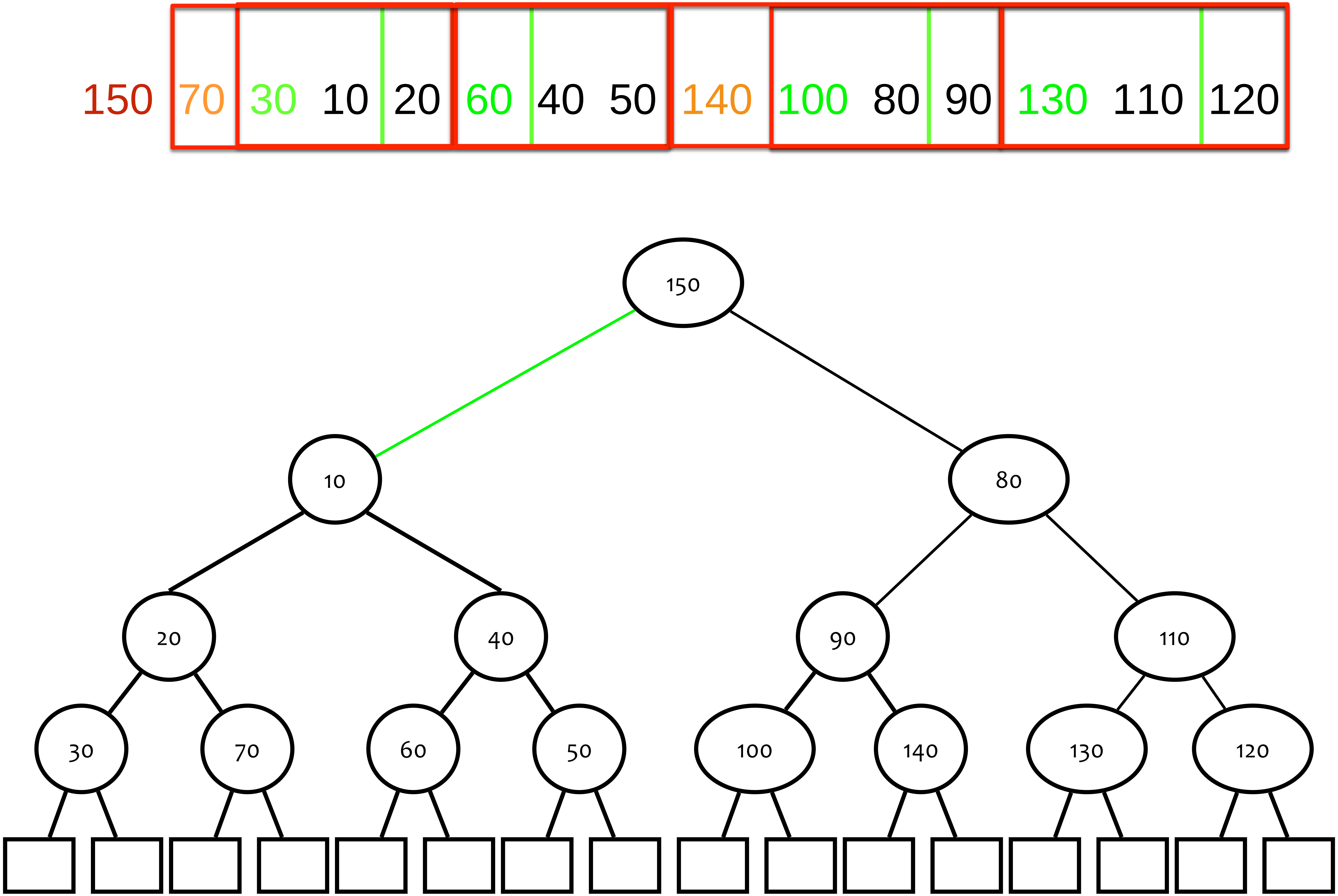
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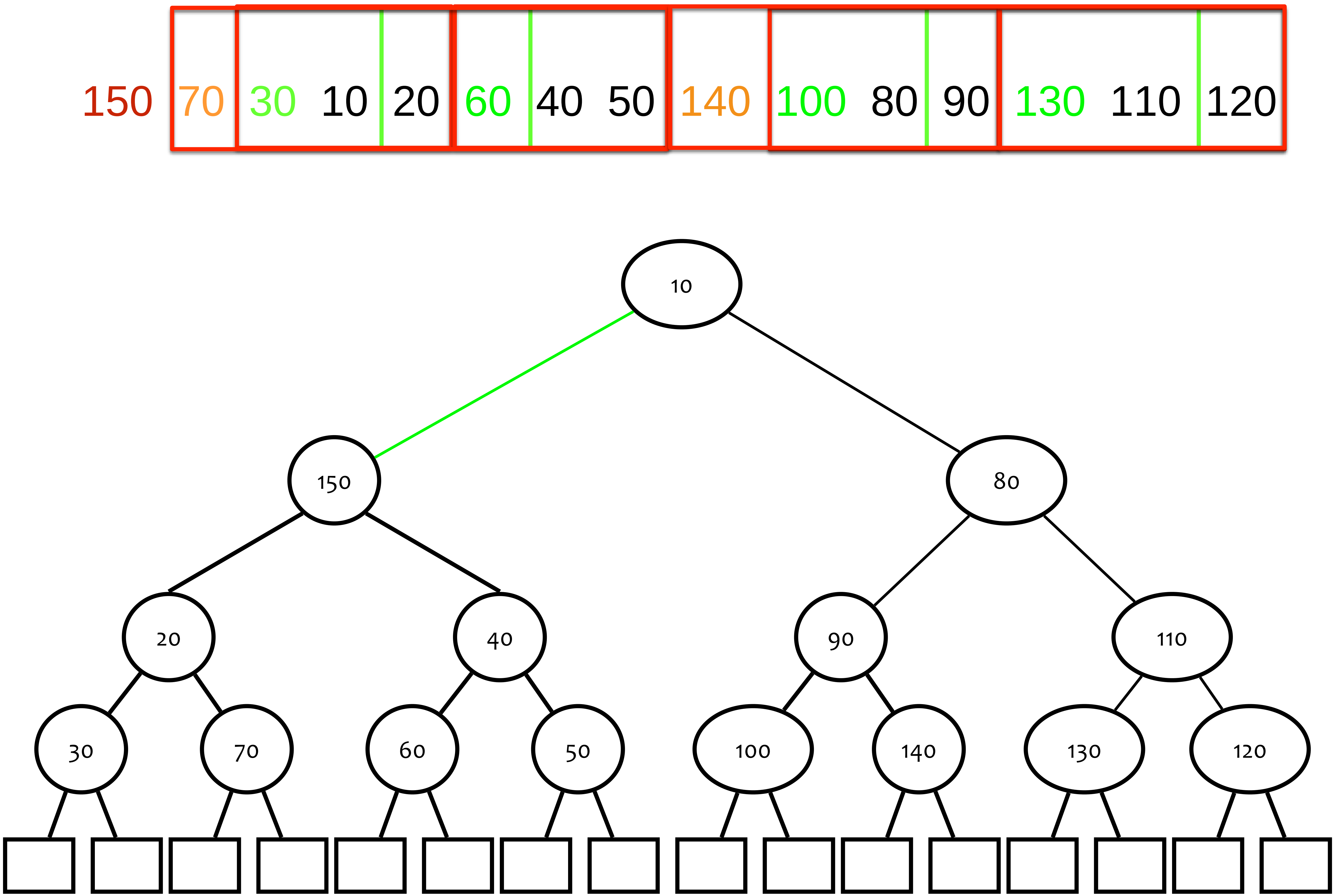
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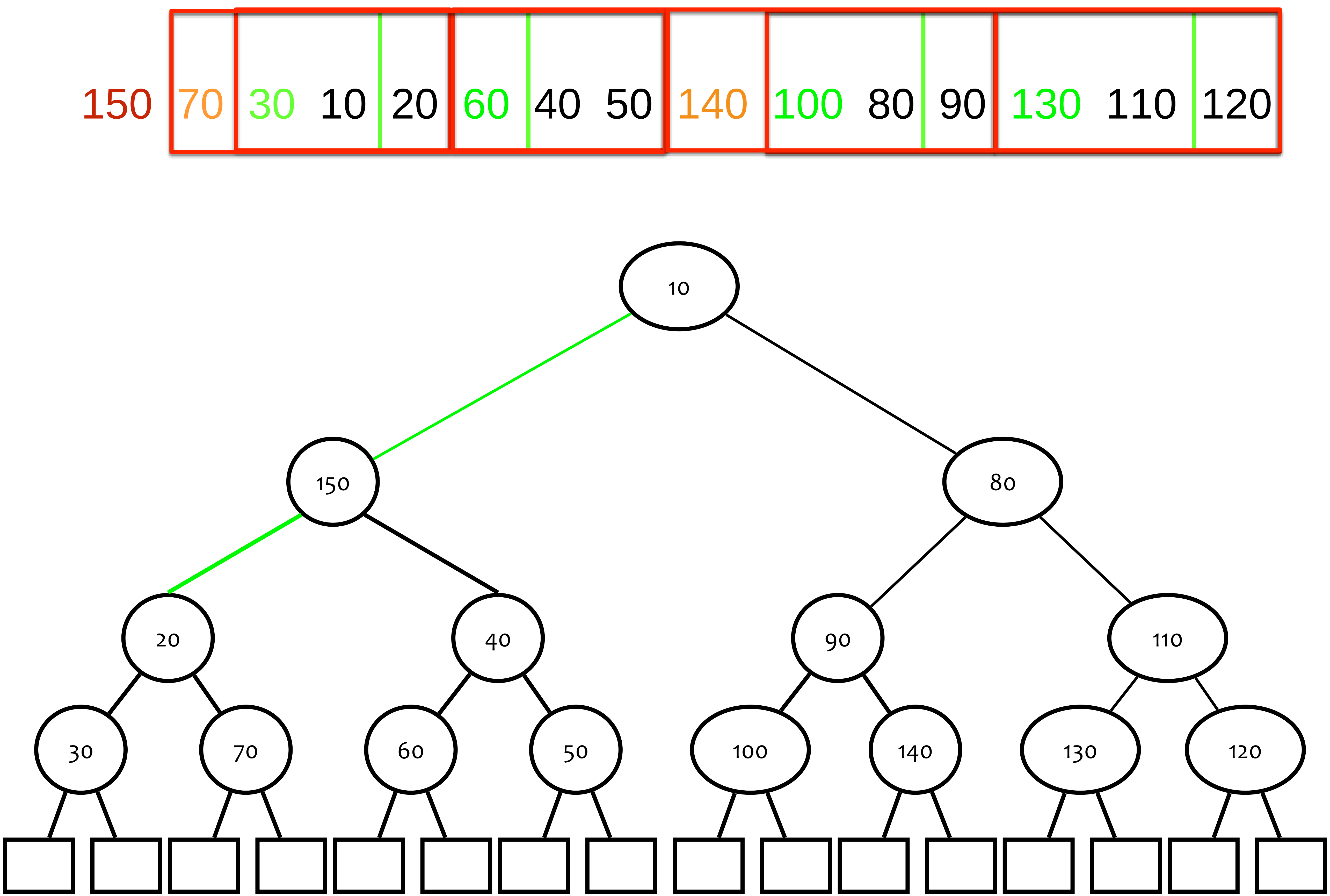
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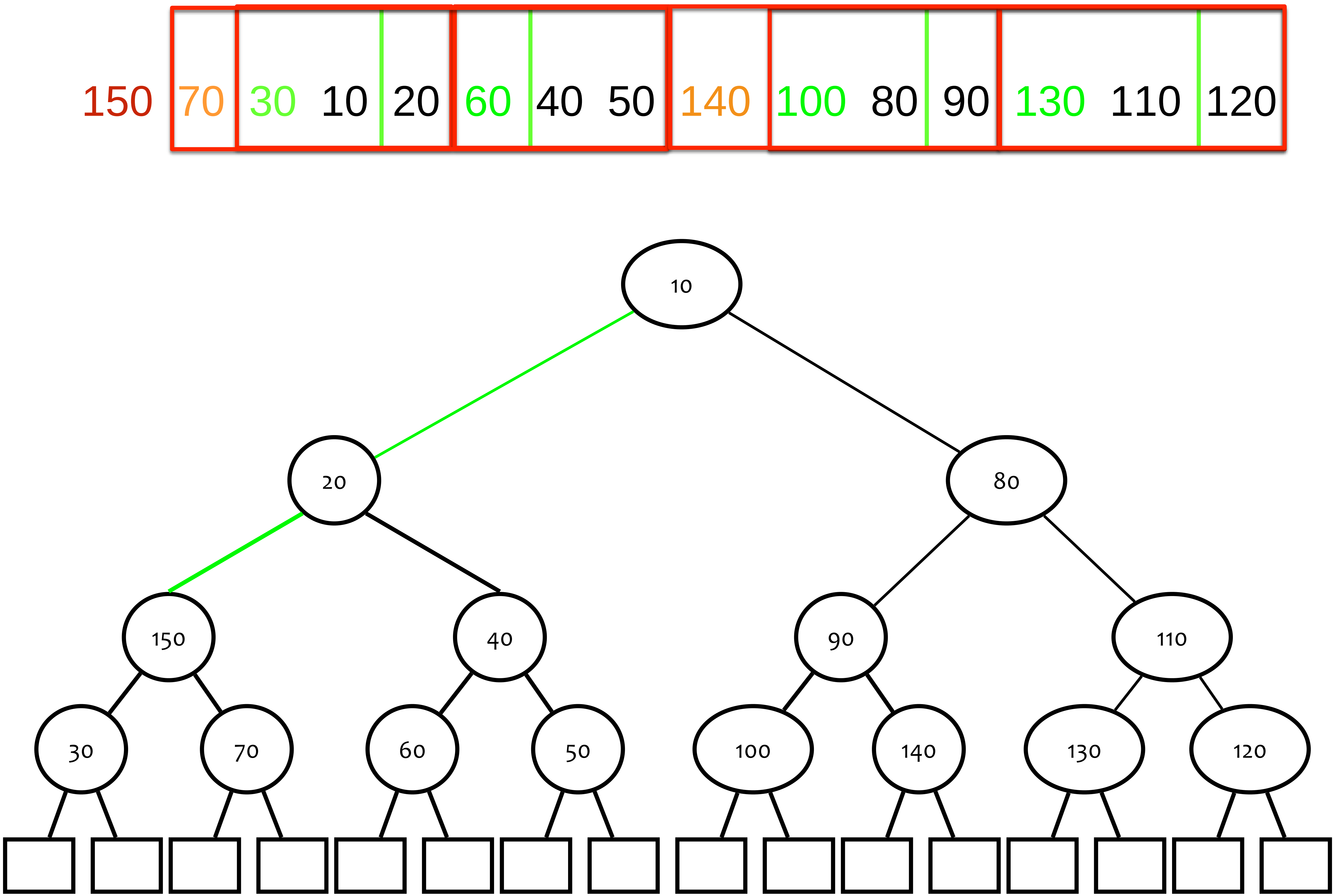

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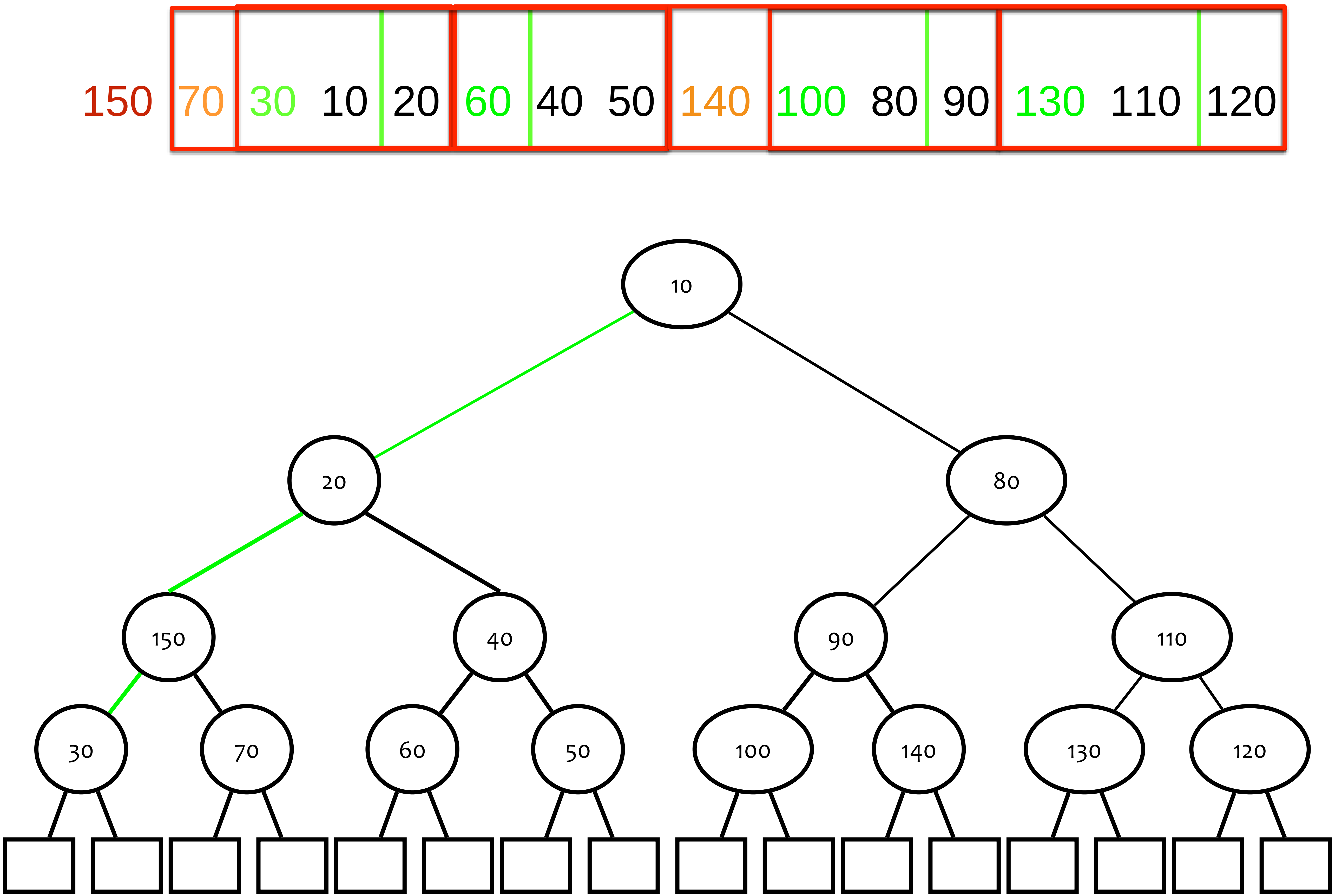
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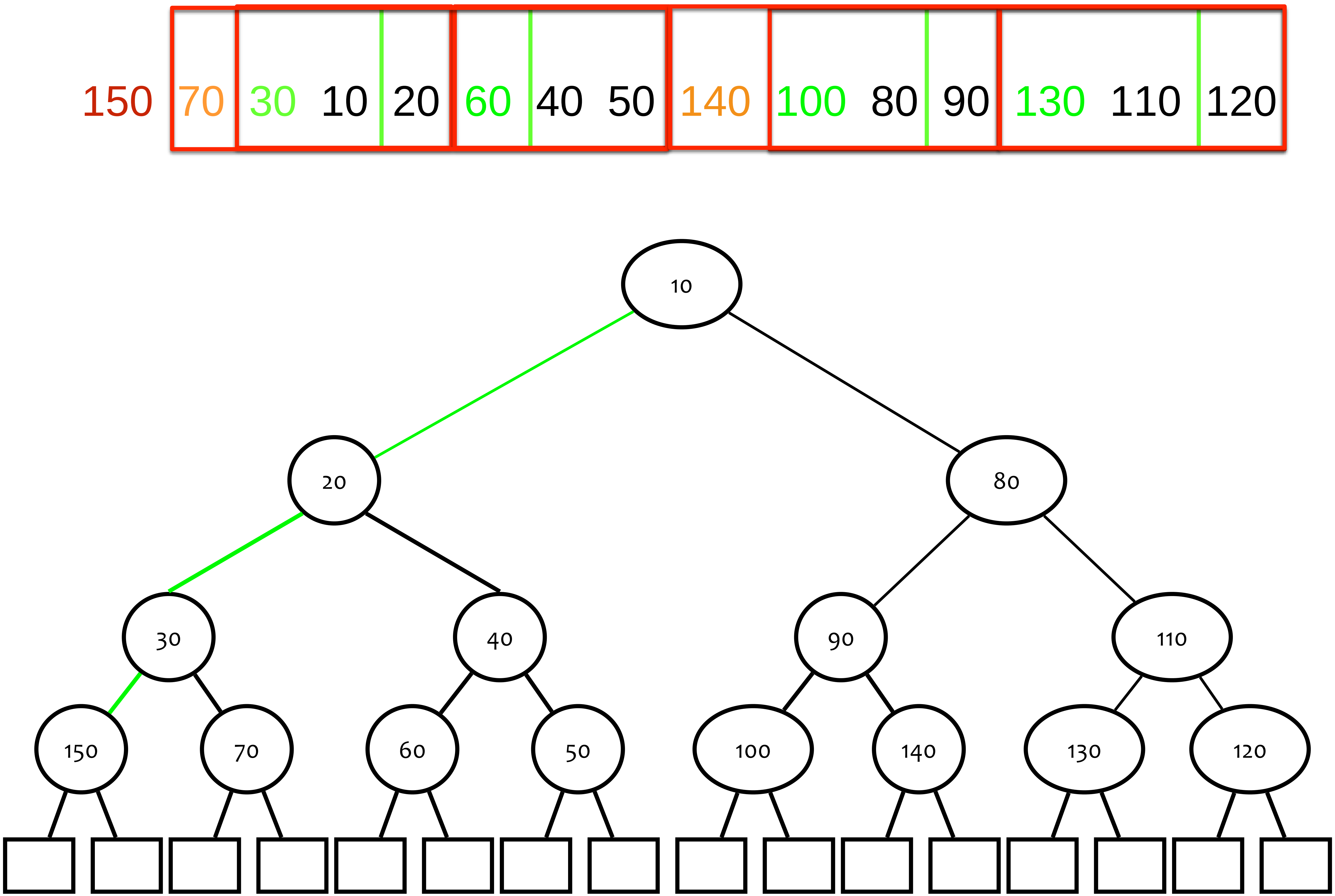

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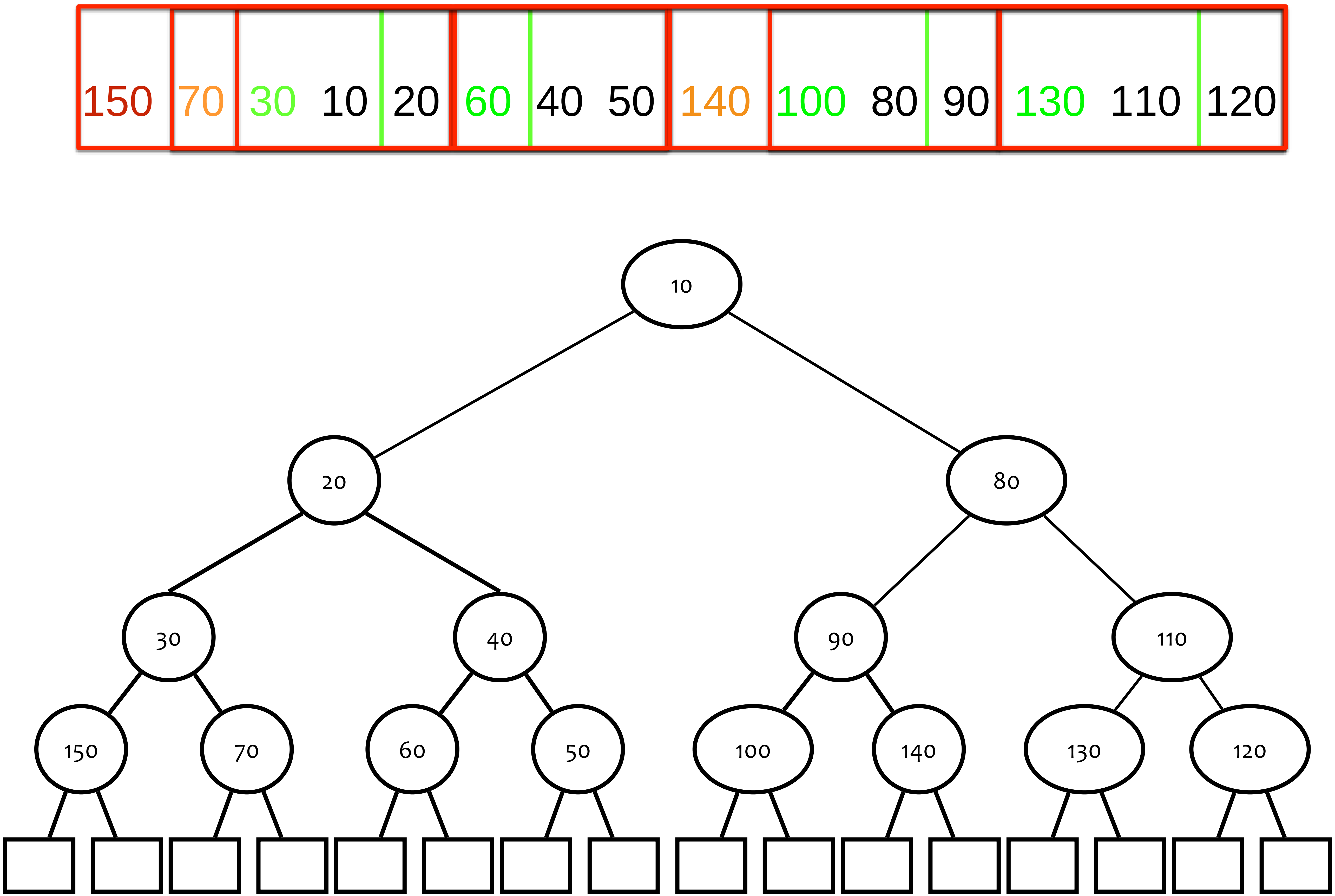
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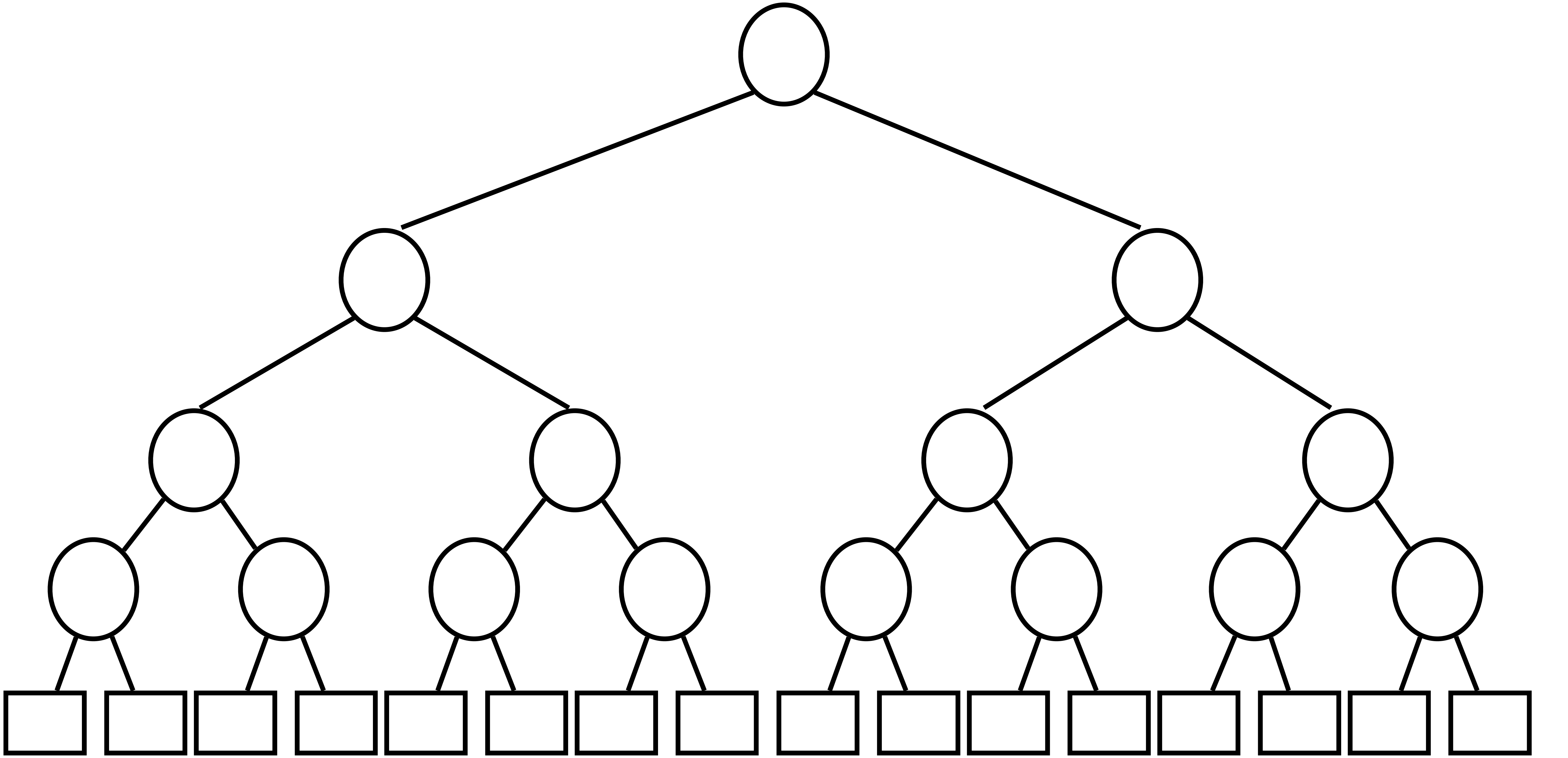
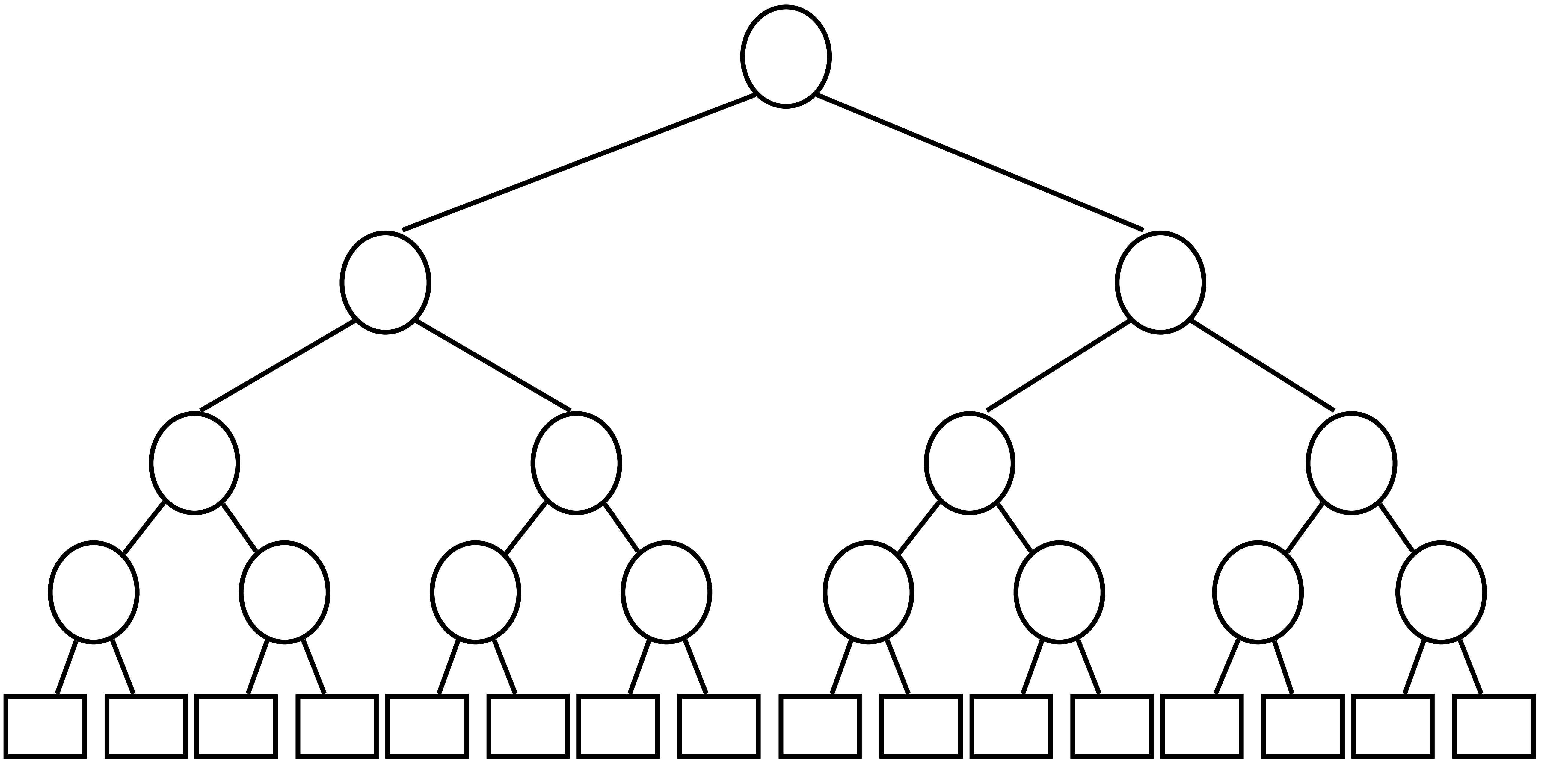
Did we really insert all m elements in $O(m)$ time??



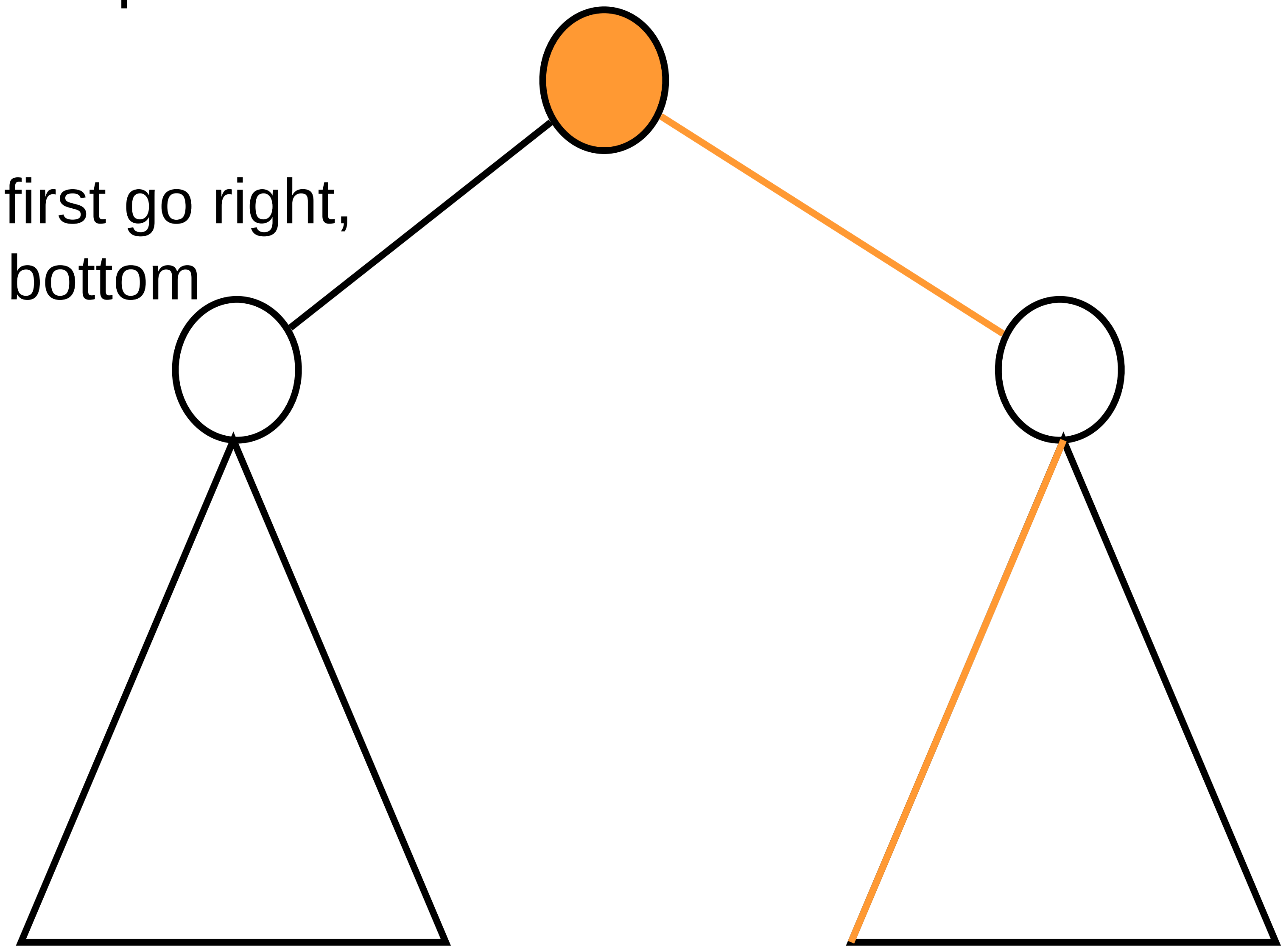
We show: max# bubble down ops < # heap edges



Proof idea

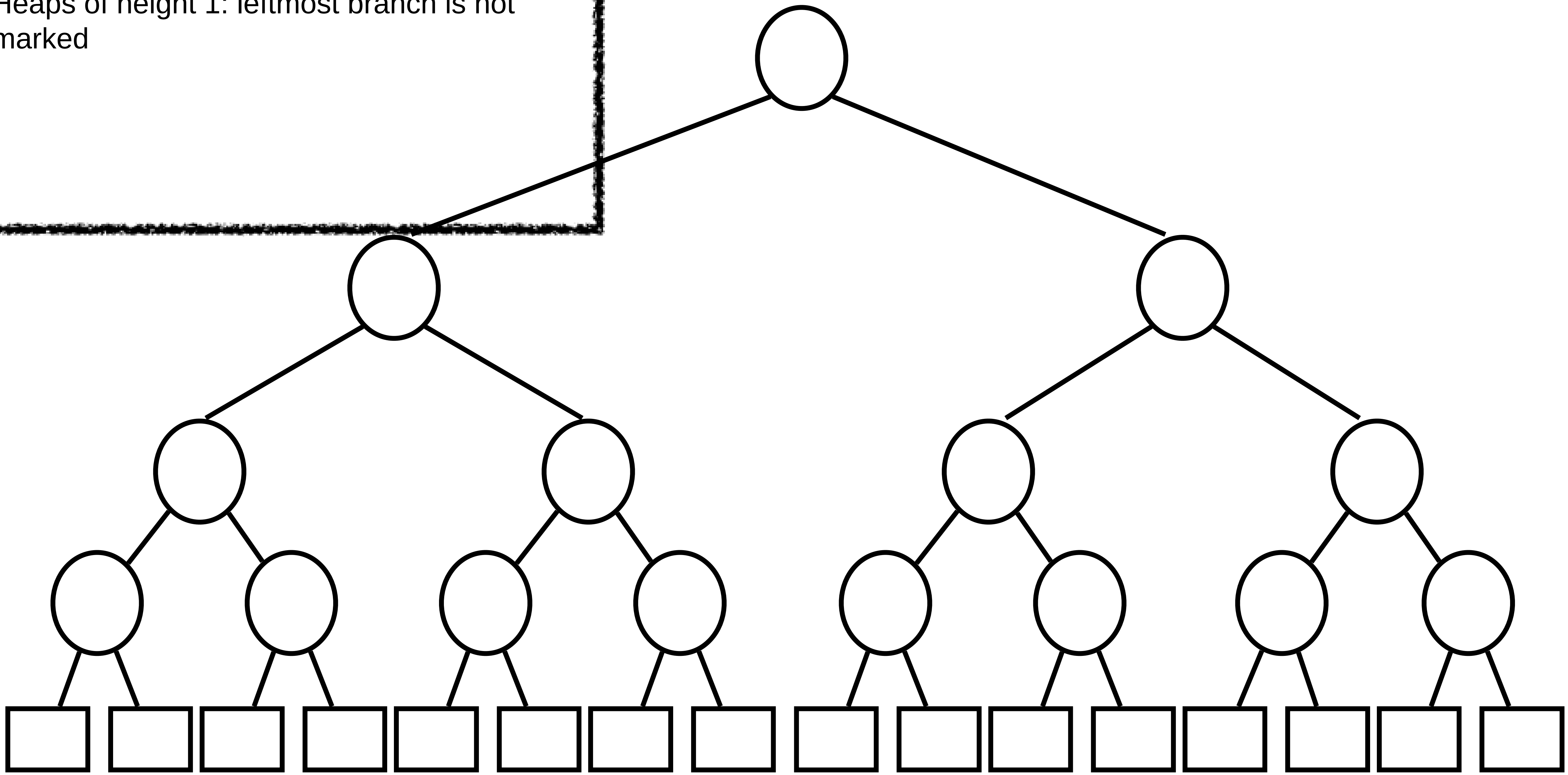
For each new node joining two heaps: mark path of maximum number of bubble-down operations

For marking: first go right, then left until bottom



For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 1: leftmost branch is not marked



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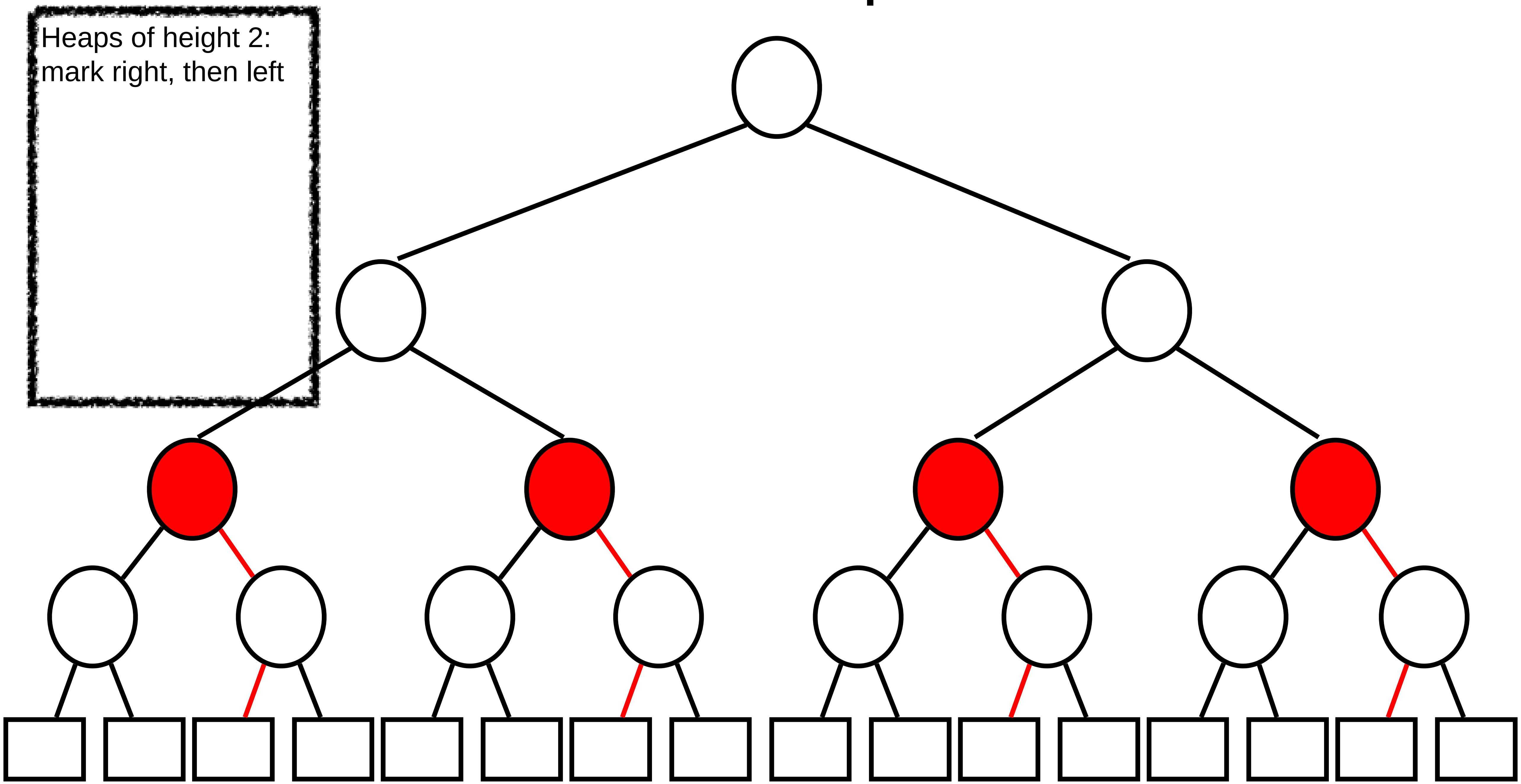
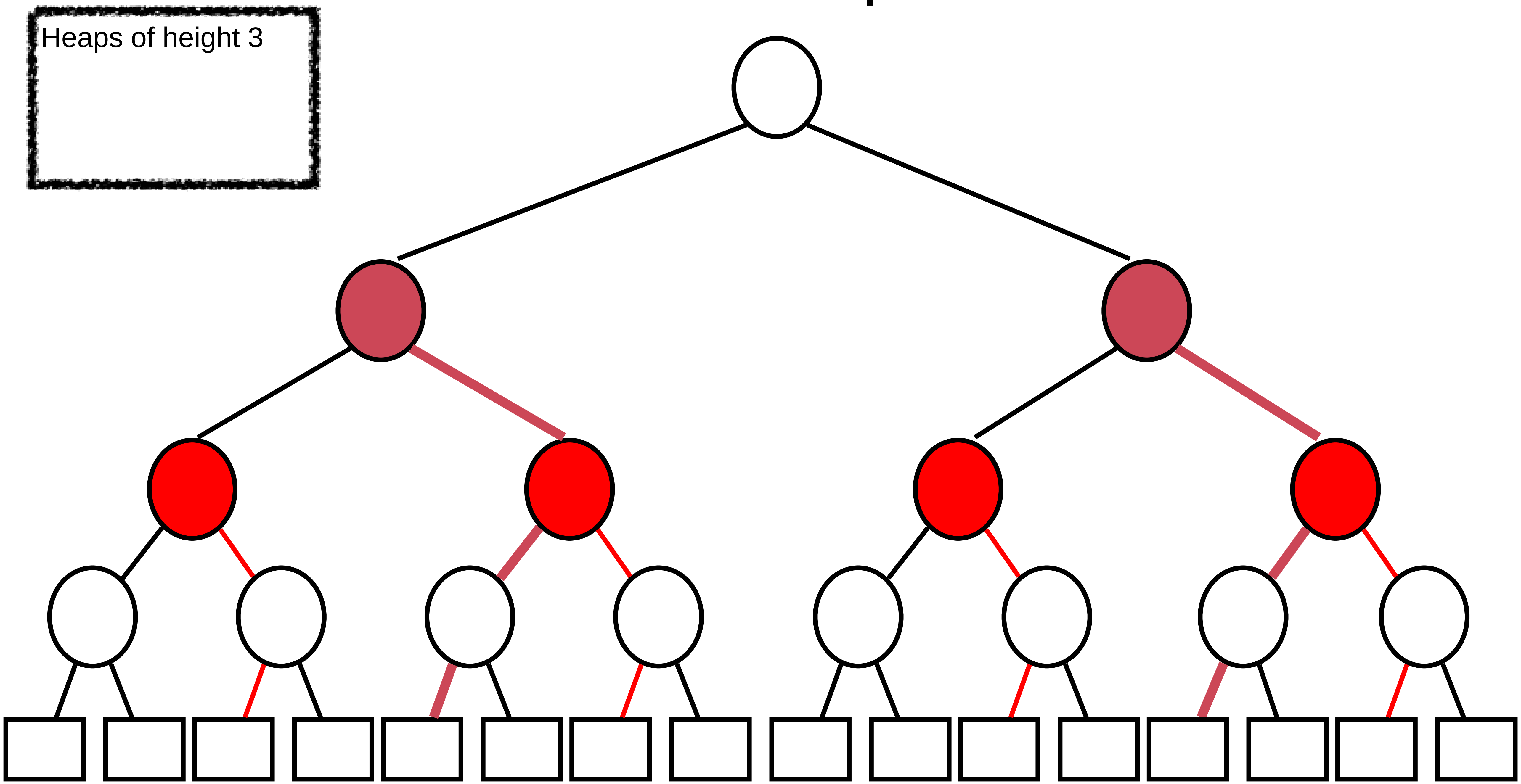


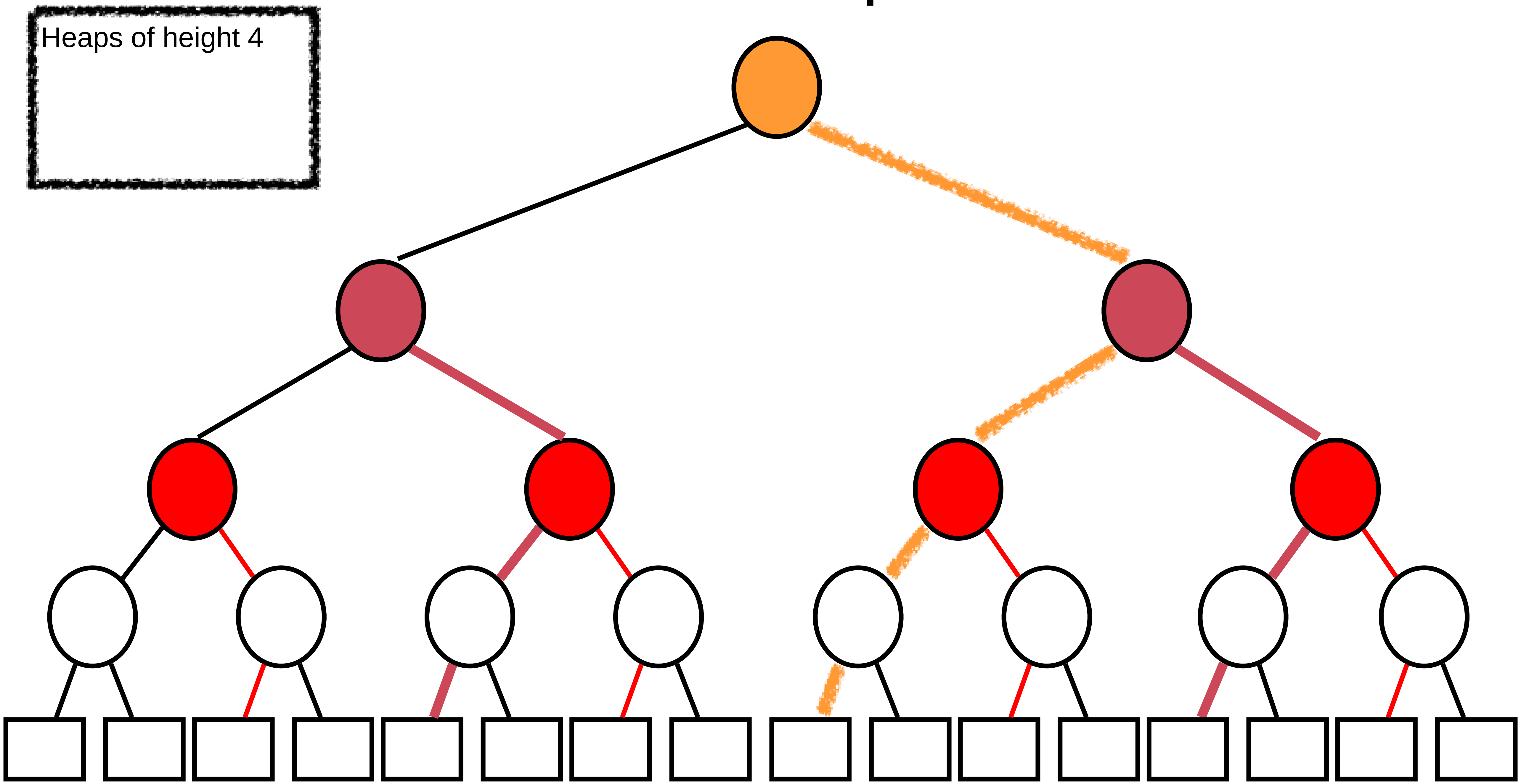
For each height-2 heap, leftmost branch not marked

For each new node joining two heaps: mark path with maximum number of bubble-down operations



For each height-3 heap, leftmost branch not marked

For each new node joining two heaps: mark path with maximum number of bubble-down operations



For height-4 heap, leftmost branch not marked

Inductive argument: marking procedure will never mark all edges in heap, since the leftmost branch is never marked

- Note: leftmost branch in height-h heap: not marked
- When joining 2 heaps of height h to heap of height $h + 1$: new edges to be marked are
  - edge joining new node and right heap of height h, and
  - edges on left path in the right heap of height h
- We conclude: leftmost branch in height ($h+1$) heap is not marked

Build Heap In-place

Algorithm buildHeap(A,n):
  for $i \leftarrow \lfloor n/2 \rfloor$ **to** 1 **do**
    downHeap(A,i)

Algorithm downHeap(A,i):
  $l \leftarrow 2i$
  $r \leftarrow 2i + 1$
  if $l \leq n \wedge A[l] < A[i]$ **then**
    $min \leftarrow l$
  else
    $min \leftarrow i$
  if $r \leq n \wedge A[r] < A[min]$ **then**
    $min \leftarrow r$
  if $i \neq min$ **then**
    swap(i,min)
    downHeap(A,min)