Math 122 Assignment 4 Solution Ideas

1. (a) We are given that $a_3 = b_3 = 1$ and that $a_4 = 2$ and $b_4 = 1$.
Let $n \geq 5$. Then $f_n = a_n f_2 + b_n f_1$ by definition. Since $n \geq 5$ we can use the Fibonacci recurrence to write

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} \\ &= (a_{n-1}f_2 + b_{n-1}f_1) + (a_{n-2}f_2 + b_{n-2}f_1) \\ &= (a_{n-1} + a_{n-2})f_2 + (b_{n-1} + b_{n-2})f_1 \end{aligned}$$

Therefore, $a_n = a_{n-1} + a_{n-2}$ and $b_n = b_{n-1} + b_{n-2}$ for $n \geq 5$.
Our recursive definition then for a_n is: $a_3 = 1$, $a_4 = 2$, and $a_n = a_{n-1} + a_{n-2}$, $n \geq 5$, and our recursive definiton for b_n is: $b_3 = 1$, $b_4 = 1$, and $b_n = b_{n-1} + b_{n-2}$, $n \geq 5$.

We claim that $a_n = f_{n-1}$. We can see this since $a_3 = f_2 = 1$ and $a_4 = f_3 = 2$ and a_n follows the same recurrence definition as the Fibonacci numbers.
By the same reasoning $b_n = f_{n-2}$ since $b_3 = f_1 = 1$, $b_4 = f_2 = 1$, and b_n follows the same recurrence definition as the Fibonacci numbers.

(b) We are given that $c_1 = c_2 = 0$. Now to calculate the value of f_n we can apply the recurrence definition once to arrive at $f_n = f_{n-1} + f_{n-2}$. Now to determine f_{n-1} we would need to apply the recurrence definition c_{n-1} times, and to determine f_{n-2} we would need to apply the recurrence defintion c_{n-2} times. In total this gives that $c_n = c_{n-1} + c_{n-2} + 1$.
Thus, our recursive defintion is $c_1 = 0$, $c_2 = 0$, and $c_n = c_{n-1} + c_{n-2} + 1$, $n \geq 3$.

(c)

$$\begin{aligned} c_7 &= c_6 + c_5 + 1 \\ &= (c_5 + c_4 + 1) + c_5 + 1 \\ &= 4 + 2 + 1 + 4 + 1 \\ &= 12 \end{aligned}$$

2. Basis: When $n = 6$ we have $LHS = 5^6 = 15625$ and $RHS = 6^5 = 7776$ so indeed $LHS > RHS$.

Induction Hypothesis: Suppose that $5^k > k^5$ for some integer $k \geq 6$.

Induction Step: We want to show that $5^{k+1} > (k+1)^5$.

$$
\begin{array}{rcll}
LHS & = & 5^{k+1} & \\
& = & 5 \cdot 5^k & \\
& > & 5 \cdot k^5 & \text{(by the IH)} \\
& = & k^5 + k^5 + k^5 + k^5 + k^5 & \\
& = & k^5 + k \cdot k^4 + k^2 \cdot k^3 + k^3 \cdot k^2 + k^4 \cdot k & \\
& > & k^5 + 5k^4 + 5^2 \cdot k^3 + 5^3 \cdot k^2 + 5^4 \cdot k & \text{(since } k > 5) \\
& = & k^5 + 5k^4 + 25k^3 + 125k^2 + 625k & \\
& = & k^5 + 5k^4 + 25k^3 + 125k^2 + 624k + k & \\
& > & k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 & \text{(since } k > 1) \\
& = & (k+1)^5 & \\
& = & RHS &
\end{array}
$$

Conclusion: Therefore, by induction, we have that $5^n > n^5$ for all $n \geq 6$.

Now, we can say that $n = 6$ is the smallest possible value n_0 for which $5^n > n^5$ for all $n \geq n_0$ since for $n = 5$ we have that $LHS = 5^5$ and $RHS = 5^5$ so we have that $LHS \not> RHS$.

3. Basis: When $n = 0$ we have $a_0 = 5(-2)^0 + (-3)^0 = 6$. When $n = 1$ we have $a_1 = 5(-2)^1 + (-3)^1 = -13$. Therefore the statement is true when $n = 0$ and $n = 1$.

   Induction Hypothesis: Suppose that there exists an integer $k \geq 1$ such that $a_n = 5(-2)^n + (-3)^n$ for all $n = 0, 1, \ldots, k$.

   Induction Step: We want to show that $a_{k+1} = 5(-2)^{k+1} + (-3)^{k+1}$. Since $k \geq 1$ we know that $k + 1 \geq 2$, and we can therefore use the recursion to write

$$
\begin{array}{rcll}
a_{k+1} & = & (-5)a_k - 6a_{k-1} & \\
& = & (-5)[5(-2)^k + (-3)^k] - 6[5(-2)^{k-1} + (-3)^{k-1}] & \text{(by the IH)} \\
& = & -25(-2)^k - 5(-3)^k - 30(-2)^{k-1} - 6(-3)^{k-1} & \\
& = & 50(-2)^{k-1} + 15(-3)^{k-1} - 30(-2)^{k-1} - 6(-3)^{k-1} & \\
& = & 20(-2)^{k-1} + 9(-3)^{k-1} & \\
& = & 5(-2)^2(-2)^{k-1} + (-3)^2(-3)^{k-1} & \\
& = & 5(-2)^{k+1} + (-3)^{k+1} & \text{as wanted.}
\end{array}
$$

   Conclusion: Therefore, by induction, $a_n = 5(-2)^n + (-3)^n$ for all integers $n \geq 0$.

4. Basis: When $n = 1$ we have $f_1^2 = (1)^2 = (1)(1) = f_1 f_2$. Therefore the statement is true when $n = 1$.

   Induction Hypothesis: Suppose that there exists an integer $k \geq 1$ such that $f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$.

   Induction Step: We want to show that $f_1^2 + f_2^2 + \cdots + f_{k+1}^2 = f_{k+1} f_{k+2}$.

   Now

$$
\begin{array}{rcll}
LHS & = & f_1^2 + f_2^2 + \cdots + f_{k+1}^2 & \\
& = & f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 & \\
& = & f_k f_{k+1} + f_{k+1}^2 & \text{(by the IH)} \\
& = & f_k f_{k+1} + f_{k+1} f_{k+1} & \\
& = & f_{k+1}(f_k + f_{k+1}) & \\
& = & f_{k+1} f_{k+2} & \text{(since } k \geq 1\text{, and so } k + 2 \geq 3)
\end{array}
$$

Conclusion: Therefore, by induction, $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all integers $n \geq 1$.

5. $t_0 = b$
$t_1 = at_0 + b = a \cdot b + b$
$t_2 = at_1 + b = a(a \cdot b + b) + b = a^2 \cdot b + a \cdot b + b$
$t_3 = at_2 + b = a(a^2 \cdot b + a \cdot b + b) + b = a^3 \cdot b + a^2 \cdot b + a \cdot b + b$
$t_4 = at_3 + b = a(a^3 \cdot b + a^2 \cdot b + a \cdot b + b) + b = a^4 \cdot b + a^3 \cdot b + a^2 \cdot b + a \cdot b + b$

From this we can guess that when $a \neq 1$

$$\begin{aligned} t_n &= a^n \cdot b + a^{n-1} \cdot b + \cdots + a^2 \cdot b + a \cdot b + b \\ &= b(a^n + a^{n-1} + \cdots + a^2 + a + 1) \\ &= b\left(\frac{a^{n+1} - 1}{a - 1}\right) \end{aligned}$$

When $a = 1$ we would have

$$\begin{aligned} t_n &= a^n \cdot b + a^{n-1} \cdot b + \cdots + a^2 \cdot b + a \cdot b + b \\ &= b + b + \cdots + b + b + b \\ &= (n+1)b \end{aligned}$$

Now we prove that our guess of t_n is correct. First consider the case when $a \neq 1$, so we have a guess of $t_n = b\left(\frac{a^{n+1} - 1}{a - 1}\right)$.

Basis: When $n = 0$ we have $t_0 = b\left(\frac{a^1 - 1}{a - 1}\right) = b$. Therefore the statement is true when $n = 0$.

Induction Hypothesis: Suppose that there exists an integer $k \geq 0$ such that $t_k = b\left(\frac{a^{k+1} - 1}{a - 1}\right)$.

Induction Step: We want to show that $t_{k+1} = b\left(\frac{a^{k+2} - 1}{a - 1}\right)$. Since $k \geq 0$ we know that $k + 1 \geq 1$, and we can therefore use the recursion to write

$$\begin{aligned} t_{k+1} &= a \cdot t_k + b \\ &= a \cdot b\left(\frac{a^{k+1} - 1}{a - 1}\right) + b && \text{(by the IH)} \\ &= a \cdot b\left(\frac{a^{k+1} - 1}{a - 1}\right) + b\left(\frac{a - 1}{a - 1}\right) \\ &= b\left(\frac{a^{k+2} - a}{a - 1}\right) + b\left(\frac{a - 1}{a - 1}\right) \\ &= b\left(\frac{a^{k+2} - a + a - 1}{a - 1}\right) \\ &= b\left(\frac{a^{k+2} - 1}{a - 1}\right) && \text{as wanted.} \end{aligned}$$

Conclusion: Therefore, by induction, $t_n = b\left(\dfrac{a^{n+1}-1}{a-1}\right)$ for all integers $n \geq 0$ when $a \neq 1$.

Now we consider the case when $a = 1$ so our guess is $t_n = (n+1)b$. In this case, note that the recursive definition becomes $t_n = at_{n-1} + b = t_{n-1} + b$.

Basis: When $n = 0$ we have $t_0 = (0+1)b = b$. Therefore the statement is true when $n = 0$.

Induction Hypothesis: Suppose that there exists an integer $k \geq 0$ such that $t_k = (k+1)b$.

Induction Step: We want to show that $t_{k+1} = (k+2)b$. Since $k \geq 0$ we know that $k+1 \geq 1$, and we can therefore use the recursion to write

$$\begin{aligned} t_{k+1} &= t_k + b \\ &= (k+1)b + b \quad \text{(by the IH)} \\ &= (k+2)b \quad \text{as wanted.} \end{aligned}$$

Conclusion: Therefore, by induction, $t_n = (n+1)b$ for all integers $n \geq 0$ when $a = 1$.