MATH 100, Fall, 2021
Tutorial #7
Derivative Tests and L'Hospital's Rule

Q1 Let $f(x) = (x+1)^2(x-1)(x+2)$. Note that f is defined on the domain $(-\infty, \infty)$, but we can also consider f defined on any subdomain of $(-\infty, \infty)$. Discuss, without making calculations, what the graph of f should look like.

1. Let $D = [-3, 1]$. Find (giving **exact** answers) all critical points.

Q2 Consider the same function as in Q1.

1. Find (with **exact** answers) inflection points. Sketch a graph and label the critical points from Q1 and the inflection points.

Discuss that $D = (-\infty, \infty)$, what are your global maxima and minima, if they exist? Explain in one sentence why they are the same or different from the global maxima and minima found in 1.

Q3 Let $D = (-\infty, \infty)$. Suppose a function f has the following properties: $f'(-1) = f'(0) = f'(1) = 0$, $f''(0) > 0$, $f''(-1) < 0$, and $f''(1) < 0$.

1. Sketch three different possible graphs for f. Be sure to label the points $x = -1, 0, 1$ on your x-axis. (Try to do something interesting!)

Discuss with your group: If in addition $f(\pm 1) = 0$ and $f(0) = -2$, how many different f's satisfy these requirements?

Q4 Let $k \in \mathbb{R}^+$.

1. Use L'Hospital's rule to show that $\lim\limits_{\eta \to \infty} \left(1 + \frac{k}{\eta}\right)^\eta = e^k$.

Discuss with your group how $\lim\limits_{\eta \to \infty} \left(1 + \frac{k}{\eta}\right)^\eta = e^k$ can be computed using only the fact that $\lim\limits_{\eta \to \infty} \left(1 + \frac{1}{\eta}\right)^\eta = e$.

Q5 Suppose $f(x) \neq 0$ for all $x \neq a$, and $\lim\limits_{x \to a} f(x) = 0$.

1. Evaluate $\lim\limits_{x \to a} \frac{f(x)}{f(x)}$.
2. Let $f(x) = e^{-1/x^2}$. What happens when we apply L'Hospital's rule to $\lim\limits_{x \to 0} \frac{f(x)}{f(x)}$? Show your work and explain your answer in a sentence.

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$$f(x) = (x+1)^2(x-1)(x+2)$$

$$f'(x) = 2(x+1)\cdot 1\cdot(x-1)(x+2) + (x+1)^2\cdot 1\cdot(x+2) + (x+1)^2(x-1)\cdot 1$$

$$= 2(x+1)(x-1)(x+2) + (x+1)^2(x+2+x-1)$$

$$= 2(x+1)(x-1)(x+2) + (x+1)^2(2x+1)$$

$$= (x+1)\left(2(x-1)(x+2) + (x+1)(2x+1)\right)$$

$$= (x+1)(2x^2+2x-4+2x^2+3x+1)$$

$$= (x+1)(4x^2+5x-3)$$

$f'(x)=0$ for $x=-1$ and the roots of $4x^2+5x-3$

$$-\frac{5 \pm \sqrt{25-4(4)(-3)}}{2\cdot 4}$$

$$= -\frac{5}{8} \pm \frac{\sqrt{25+48}}{8}$$

$$= -\frac{5}{8} \pm \frac{\sqrt{73}}{8}$$

$$\approx 0.443, -1.693$$

Each of these points lie in the interior of $D=[-3,1]$, hence all three points, $x=-1, -\frac{5}{8}\pm\frac{\sqrt{73}}{8}$, are critical points

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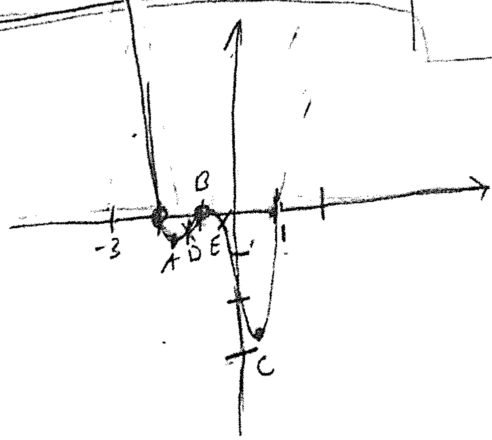
$$f''(x) = \frac{d}{dx}\left((x+1)(4x^2+5x-3)\right)$$
$$= 1\cdot(4x^2+5x-3) + (x+1)(8x+5)$$
$$= 4x^2+5x-3 + 8x^2+13x+5$$
$$= 12x^2+18x+2$$
$$= 2(6x^2+9x+1)$$

$$f''(x)=0, \text{ when } x = \frac{-9 \pm \sqrt{9^2-4(6)(1)}}{2\cdot 6}$$
$$= -\frac{9}{12} \pm \frac{\sqrt{81-24}}{12}$$
$$= -\frac{3}{4} \pm \frac{\sqrt{57}}{12}$$
$$\approx -0.12, -1.38$$

Inflection Points:
$$\left(-\frac{3}{4}-\frac{\sqrt{57}}{12}, \frac{27\sqrt{57}-265}{288}\right)$$
$$\left(-\frac{3}{4}+\frac{\sqrt{57}}{12}, -\frac{265+27\sqrt{57}}{288}\right)$$

Sketch:



Critical Values:
$A \approx (-1.693, -0.397)$
$B = (-1, 0)$
$C \approx (0.443, -2.833)$

Inflection points:
$D \approx (-0.12, -0.212)$
$E \approx (-1.38, -1.628)$

$$f''(-1) = 2(6\cdot(-1)^2+9(-1)+1)$$
$$= 2(6-9+1)$$
$$= 2(-2) < 0, \text{ concave down}$$

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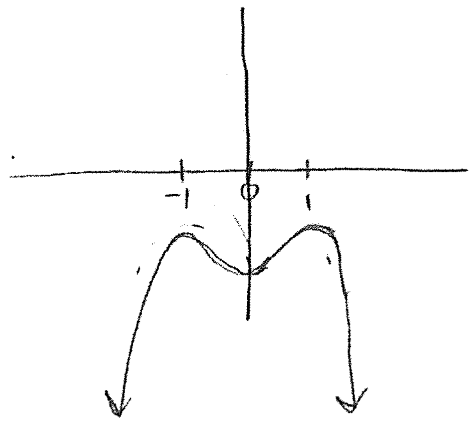
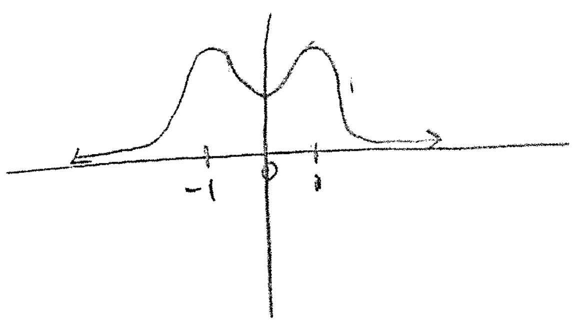
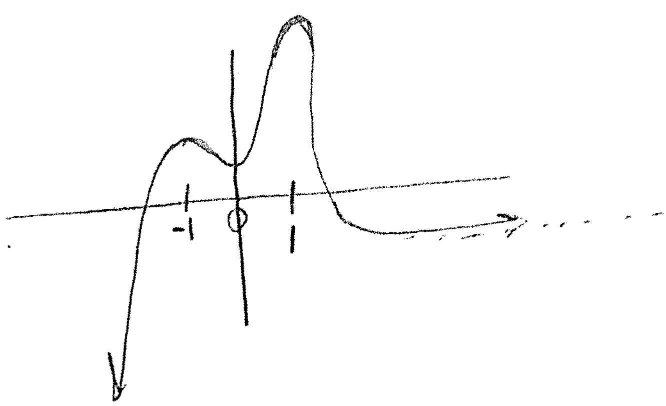
1)



2)



3)



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$$\lim_{\eta \to \infty} \left(1+\frac{k}{\eta}\right)^{\eta}$$

Consider $\displaystyle\lim_{\eta \to \infty} \eta \ln\left(1+\frac{k}{\eta}\right)$

$$= \lim_{\eta \to \infty} \frac{\ln(1+k/\eta)}{1/\eta},$$

$$\overset{\text{L'Hôpital's}}{=} \lim_{\eta \to \infty} \frac{\left(\frac{1}{1+k/\eta}\right)\cdot\left(-\frac{k}{\eta^2}\right)}{\left(-1/\eta^2\right)}$$

$$= \lim_{\eta \to \infty} \frac{k}{1+k/\eta}$$

$$= k$$

Thus, $\displaystyle\lim_{\eta \to \infty} \left(1+\frac{k}{\eta}\right)^{\eta} = e^{k}.$

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1) $$\lim_{x\to a} \frac{f(x)}{f(x)} = \lim_{x\to a} 1 = 1.$$

2) $$\lim_{x\to 0} \frac{e^{-1/x^2}}{e^{-1/x^2}} = ? \text{ via L'Hôpital ?}$$

$$\frac{f'(x)}{f'(x)} = \frac{e^{-1/x^2}\cdot\cancel{(-2x^{-3})}}{e^{-1/x^2}\cdot\cancel{(-2x^{-3})}}$$

$$= \frac{e^{-1/x^2}}{e^{-1/x^2}} = \frac{f(x)}{f(x)}$$

Method of L'Hôpital is circular!

Of course, the limit is = 1 as in part 1)