ECON 180

Introduction to Principles of Microeconomics and Financial Project Evaluation

Lecture 8: Discounted Cash Flow Analysis II
Gradients

September 24, 2021

Version 1

Required Reading and Viewing

- Learning OnDemand. (2018, March 31). Engineering Economic Analysis - Gradient Series [Video File]. https://youtu.be/3YeDeCawZog
- Section 17.5 Gradients on pages 373-377 of
- Higbee, C. (1995). *Engineering Cost Analysis*. Oregon: Geo-Heat Center. http://digitallib.oit.edu/digital/collection/geoheat/id/10700/rec/2
  - **This 48-page PDF makes an excellent free textbook on the basics of Engineering Economics, and includes solved examples.**

Recommended Reading

- *Engineering Economics*, Chapter 3, Sections 3.6 – 3.8
- Shaw, M. & Snyder, D. E. (2001). Selection of wood pole alternatives by means of present-worth analysis. *Rural Electric Power Conference, 29 Apr – 01 May 2001*, pp. C5/1 – C5/9. Retrieved from https://ieeexplore-ieee-org.ezproxy.library.uvic.ca/document/949522
  - **The source for the case study. Note that in the version presented in class, I've made a few changes to make it slightly more suitable for teaching.**

Learning Objectives

- Continue becoming familiar with the notation for conversion factors.
- Know how to deal with regular payments that are spaced apart.
- Become familiar with the basic cash flow series elements (Geometric and Arithmetic Gradients) and be able to convert between them and other cash flows at will using appropriate conversion factors.
- To become more familiar with breaking down realistic cash flows into appropriately timed cash flow elements.

Relevant Solved Problems I

- From *Engineering Economics*, 6th edition:
- Arithmetic Gradient: Example 3.6, 3.9.b., 3.10, 3.30, 3.34, 3.53
- Geometric Gradient: Example 3.7, Example 3.8, 3.11, 3.12, 3.35, 3.37, 3.40, 3.46
- Repeated Payments with Gaps (e.g. $100 every 5 years, or $100 every working day): Example 3.9, Example 3.10, Review Problem 3.2, 3.13, 3.15, 3.31
- Challenging 'everything together' practice problems: 3.14, 3.32, 3.36, 3.37, 3.38, 3.49, 3.50, 3.51

Relevant Solved Problems II

- From Stuart Nielsen’s *Engineering Economics: The Basics* (2nd edition):
- Chapter 6 (all)

Notation Dictionary

(Not provided on quiz/final formula sheet)

- A = Annuity
- F = Future Value
- g = growth rate
- G = Gradient Element
- GGS = Geometric Gradient Series
- i = interest rate
- i^o = growth-adjusted interest rate
- N = the N'th time period
- P = Present Value
- S = Salvage
- Green Text = Excel Formula

- Conversion factors are of the form (X/Y,z)
- Read as: X, given Y and z.
- X is the element we want.
- Y is the element we have.
- z represents additional parameters.
- e.g. (P/F,i,N)
- Present Value, given a Future Value at time N and interest rate i.

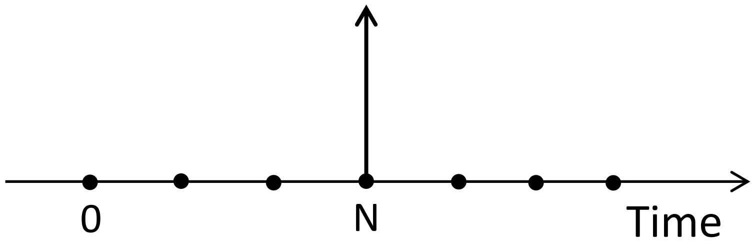
$f(x)$

Equations

- Notation: The orange symbol on a slide indicates a formula is introduced there.
- (A/G,i,N) = $\frac{1}{i} - \frac{N}{(1+i)^N - 1}$
- (P/A,g,i,N) = $\frac{(P/A, i^o, N)}{1+g}$
- $i^o = \frac{1+i}{1+g} - 1$

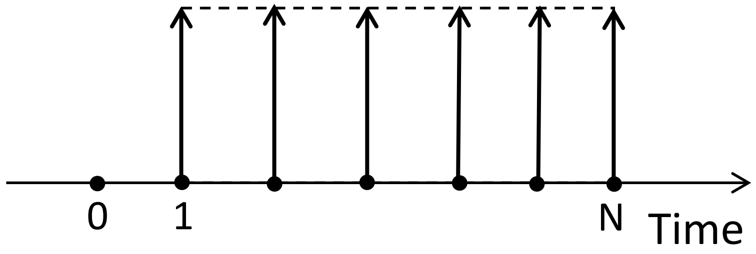
ESSENTIALS (11 slides)

Reminder: Four basic cash flow elements



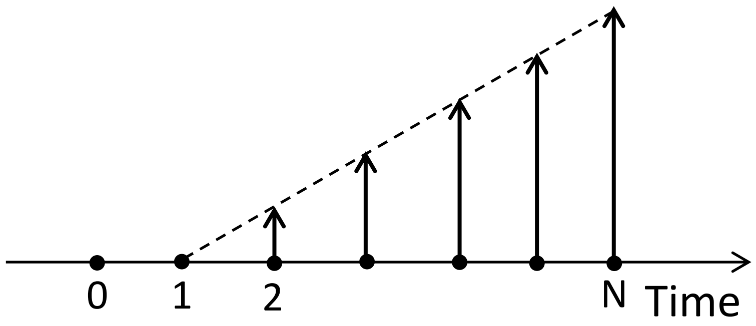
Impulse (Future Value, F)

Positive in Year N only



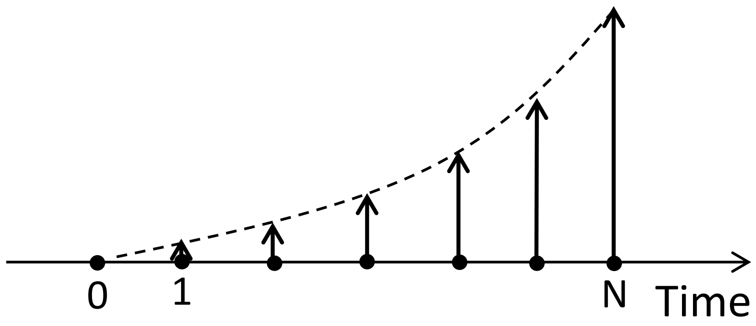
Step (Annuity, A)

Positive from Years 1 to N (<u>not</u> 0)



Ramp (Arithmetic Gradient, G)

Positive from Years 2 to N

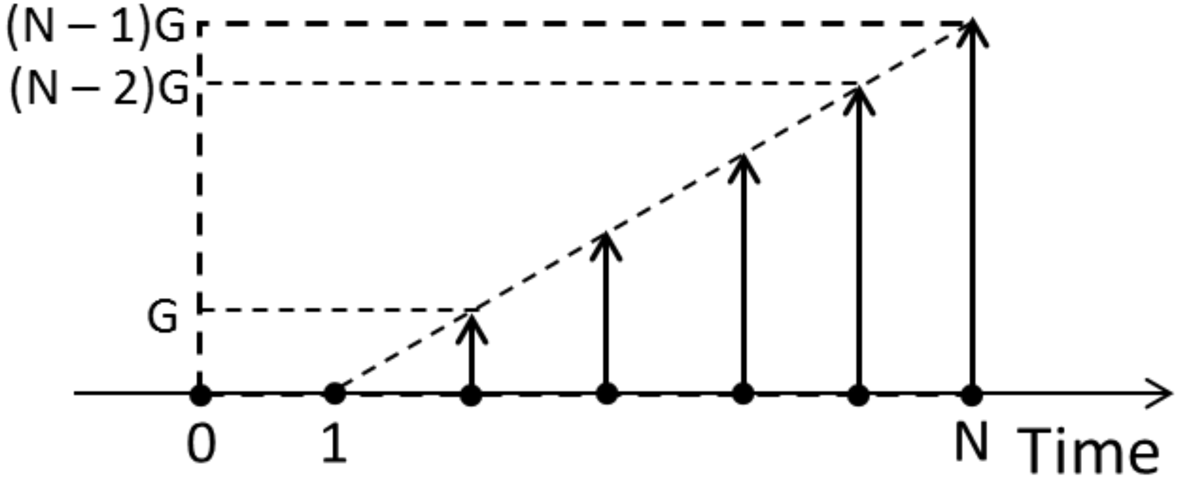


Geometric Gradient Series (Growth/Decay)

Positive from Years 1 to N

Introducing the arithmetic gradient

- A function that starts at 0 in period 1 rises by G each period until period N. G may be positive or negative.
- Series: 0, 0, G, 2G, 3G, 4G, … (N-1)G = (t – 1)G for t = 1 .. N

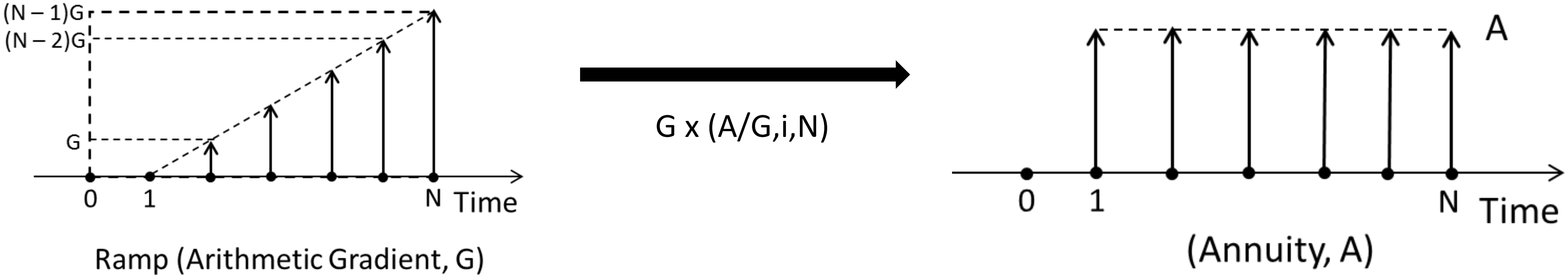


Ramp (Arithmetic Gradient, G)

Arithmetic gradient to annuity conversion factor (A/G,i,N) $f(x)$

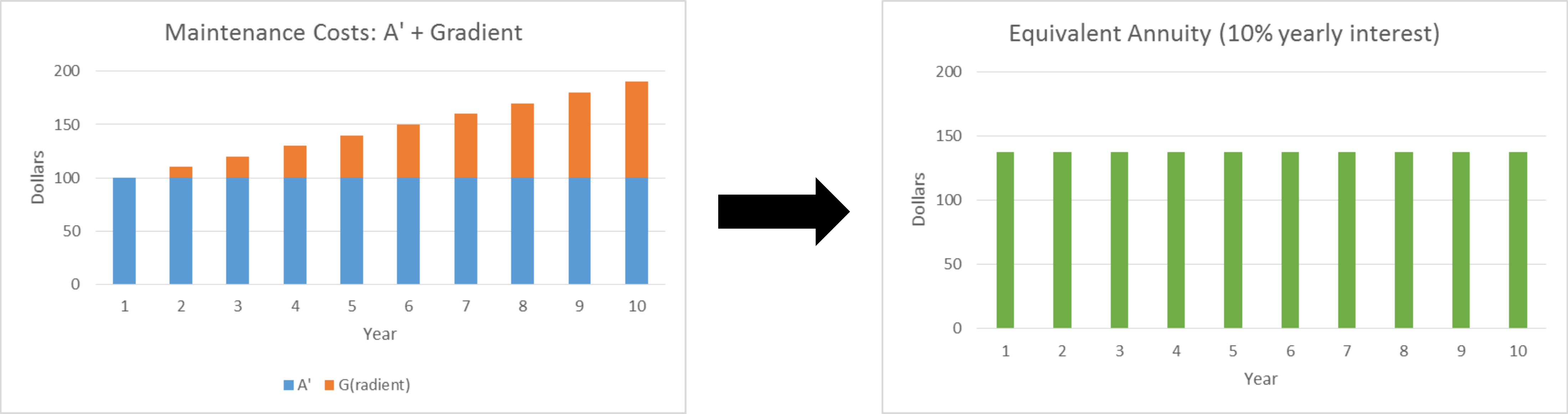
- Let's try this A/GiN.
- Converts G to A: A = G x (A/G,i,N) No easy Excel shortcut, sorry!
- That's right: to get an annuity, you need to be (A/G,i,N)G.

$$(A/G,i,N) = \frac{1}{i} - \frac{N}{(1+i)^N - 1}$$

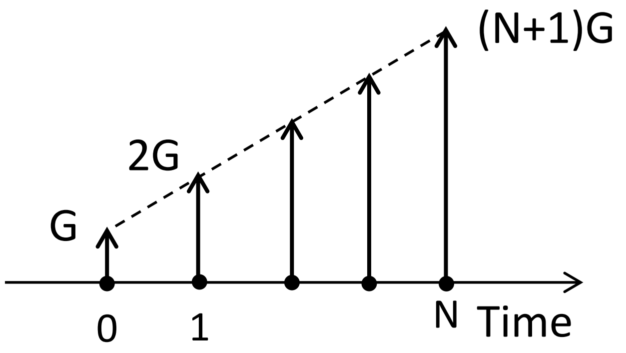


What if our gradient is positive at t=1?

- Often, a gradient will be sitting on top of an annuity A’, like a hat.
- A gradient that is positive at t=1 is equivalent to such a series.
- e.g. Maintenance that starts at $100 /yr and increases by $10 /yr.
- To get the total equivalent annuity, just add them up: A_{tot}= A’ + G(A/G,i,N)



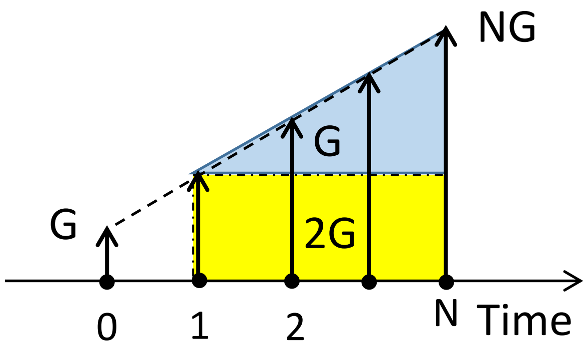
What if the 'gradient' starts at time 0?



This looks very much like an arithmetic gradient G…
BUT gradients aren't positive until Year 2.

Solution: Divide the cash flow into a 2G annuity, a payment of G today and a gradient G.

P = G + 2G(P/A,i,N) + G(P/G,i,N)



If we don't have P/G, we can build it:
(P/G) = (P/A) x (A/G)
→ G(P/G,i,N) = G(P/A,i,N)(A/G,i,N)

→P = G + 2G(P/A,i,N) + G(P/A,i,N)(A/G,i,N)

Brute Force Testing: G=5, i=0.1, N=5

Year	Flow	PV
0	10	$10.00
1	20	$18.18
2	30	$24.79
3	40	$30.05
4	50	$34.15
5	60	$37.26
Total		**$154.43**
	P	$10.00
	A	$75.82
	A/G	18.10126
	P/A	$68.62
Total		**$154.43**

Excel Formulas Used

PV = PV(i,Year,,-Flow)

P = G

A = PV(i,5,-2G) [Present value of the annuity]

A/G = i*(1/I - N/((1+i)^5 - 1))

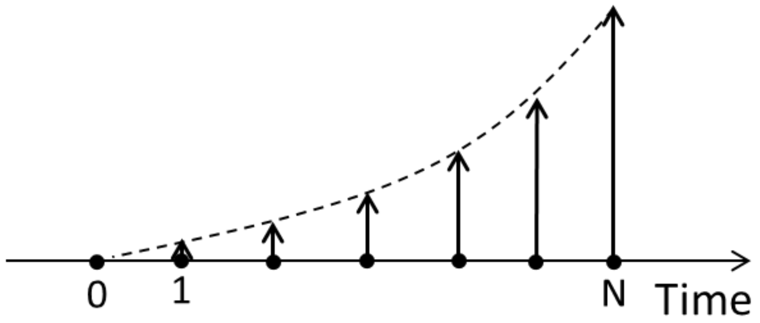
P/A = PV(i,N,-A/G)

Total = P + A + P/A

By '**Brute Force**' I mean, 'Forget all these other fancy discount factors! Let's just find the present values one by one and add them up!'

Introducing geometric gradient series (GGS)

- A GGS is an annuity that grows at a rate g. The rate of 'growth' may be positive or negative.
- Like an annuity (which is a special case with g=0), a GGS has a value of 0 in period 0, and A in period 1.
- The value of a GGS in period t is $A(1+g)^{t-1}$
- This kind of series can help you when you need to adjust cash flows for inflation.



Geometric Gradient Series (Growth/Decay)

Growth-adjusted interest rate, i^O

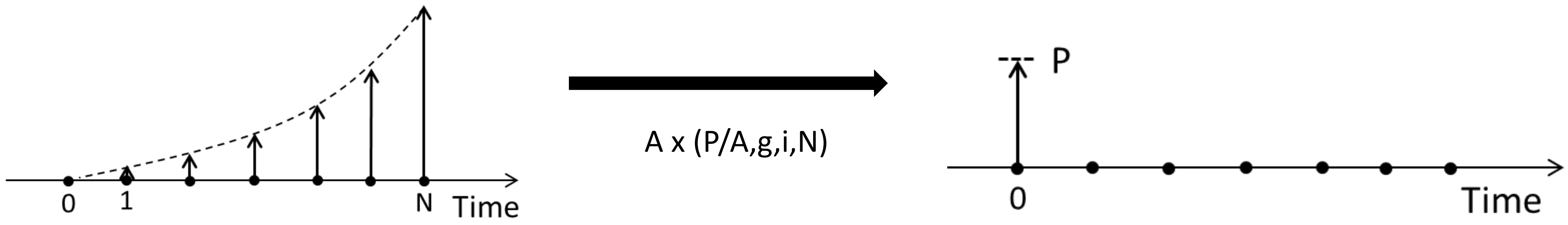
- Consider the standard case, where you can either keep \$P in a box for a year, after which you have \$P, or put it in the bank for a year at interest i, after which you have \$P x (1 + i).
- The \$P x (1 + i) is greater than the \$P you would have had by keeping the money in a box by a factor (1 + i), and this corresponds to an interest rate of i (= (1 + i) – 1). Now let's spice things up a bit…
- Suppose we observe that P becomes P(1 + i) at the end of one period, in a setting where there is growth g. What's the return JUST due to interest?
- In the absence of interest, in 1 period, P would have become P x (1 + g) on its own.
- Instead, it became P x (1 + i). This is greater than P x (1 + g) by a factor of (1 + i)/(1 + g).
- This corresponds to a growth-adjusted interest rate of i^O, where $i^O = \frac{1+i}{1+g} - 1$
- P x (1 + g) x (1 + i^O) = P x (1 + i)
- We can use this to correct interest rates for inflation, which we'll look at later.
- The real interest rate is a growth-adjusted interest rate were g = inflation
- (See section 9.4.1 on p. 309 of the text for the inflation version.)

Wait, WHAT? Suppose prices are rising…

- You only care about apples. Apples today cost \$1 each, and the price of apples doubles in 1 year, so apples will cost \$2 one year from now.
- The price of apples *grows* at a rate g = 100%/year.
- Your bank offers i = 20% interest/year. You put \$100 in the bank today, and take out \$120 one year from now.
- What you put in: \$100/\$1 = 100 apples.
- What you took out: \$120/\$2 = 60 apples.
- Due to the growth in the price of apples, the bank's interest rate is not an accurate description of what happens to your purchasing power.
- You need the **growth adjusted interest rate**: (1 + i)/(1 + g) – 1
- Here, (1+20%)/(1+100%) – 1 = -40%.
- Check: 100 x (1 – 40%) = 100 x 60% = 60

Geometric Gradient to Present Worth Conversion Factor (P/A,g,i,N)

- If this looks familiar, it's because (P/A,i,N) = (P/A,0,i,N)
- Annuities are just really simple geometric gradient series with g = 0.
- Converts a GGS to P: P = A x (P/A,g,i,N) = PV(i^O,N,-A)/(1+g)
- (P/A,g,i,N) = (P/A, i^O,N)/(1 + g)



Geometric Gradient Series (Growth/Decay)

Present Value, P

What if i = g?

- This is a question that often trips students up.
- What if I'm looking at a geometric gradient, and the growth-adjusted interest rate is *zero*?
- For that matter, what if I'm looking at ANY conversion factor, and i=0?
- Then you're lucky – your calculations just got a LOT easier!
- If interest = 0, that means a dollar today is a dollar tomorrow is a dollar a million years from now, or 4 billion years ago.
- A x (P/A,0,N) = N x A
- i = g → A x (P/A,g,i,N) = A x (P/A,0,N)/(1+g) = NA/(1 + g)
- …and so on.
- Intuition: Before starting this course, you probably thought about most income and costs in terms of interest = 0. That shouldn't make the math blow up – quite the opposite.

AFTER HOURS

- Real-world case study (power lines) (18 marks)
- Demonstrates uses of present value, future value, annuities & arithmetic gradients

Breaking news…

https://www.tcec.coop/content/tcec-makes-major-progress-restoring-power

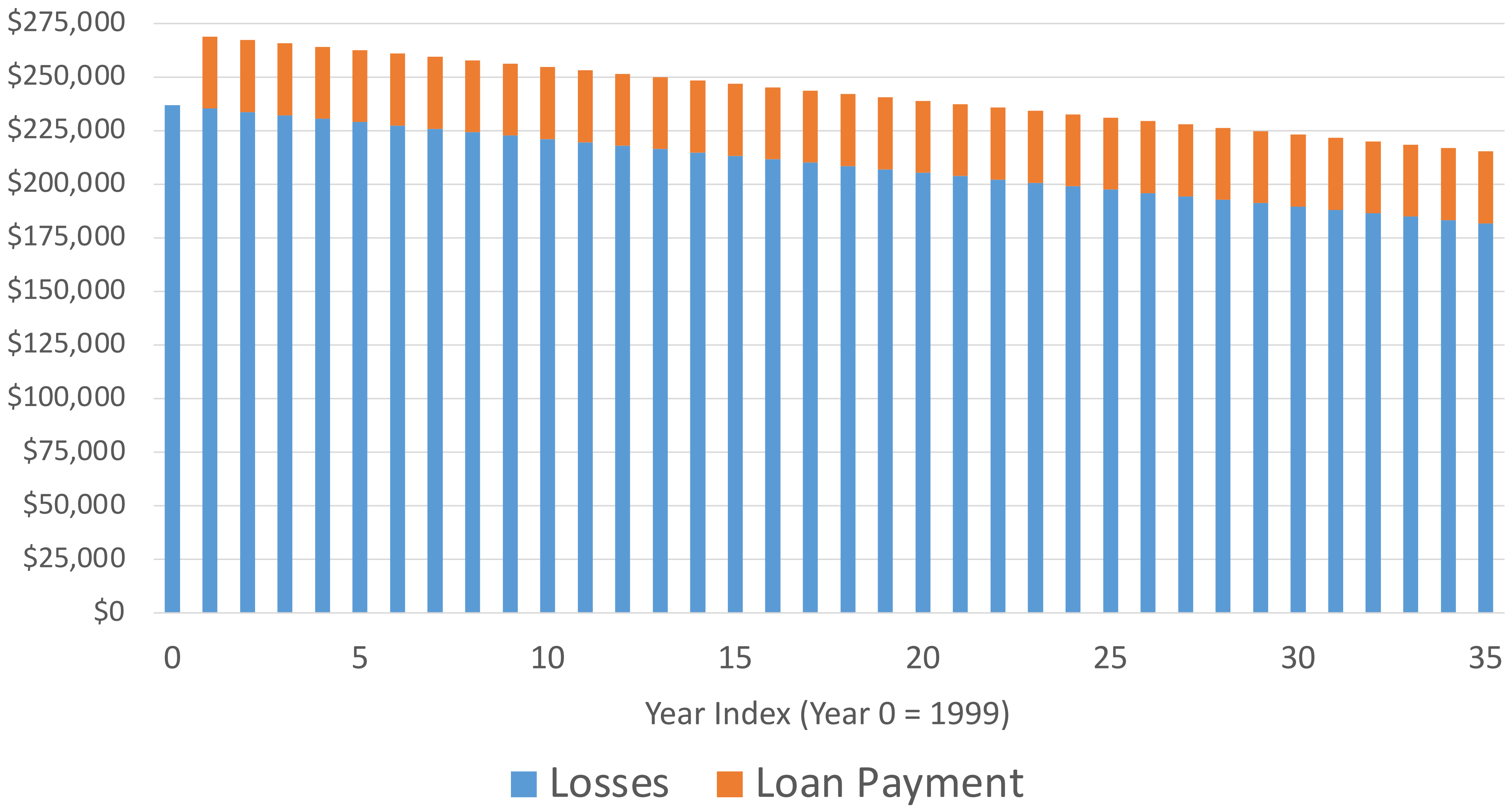
Trouble with Power Lines

- In 1999, the Tri-County Electric Company of Oklahoma ran into trouble with a 9-mile segment of a transmission link between Boise City and a substation.
- Limited power flow and service loss were costing Tri-County $236,884 annually in system losses, and between $4,000 and $10,000 in repairs to old wooden poles along the line.
- Tri-County considered several options – today, we'll look at two of them:
- A. Repair the old line, and add a voltage regulator.
- B. Build a new, more reliable and durable line.
- We'll compare the *annual worth* of costs for each of the two alternatives.
- Tri-County Electric uses an interest rate of 6% and a planning period of 35 years for its calculations.
- (Note if you're reading the original article: we're ignoring carrying charges.)

Plan B: Build a new line

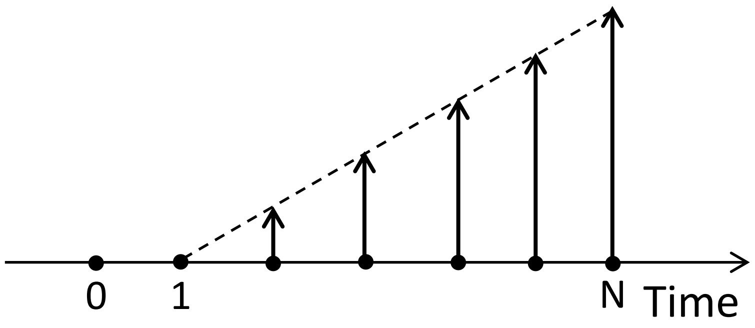
- Tri-County would take out a 35-year loan to pay for the new line.
- The loan would be paid in 35 yearly payments of $33,571, starting in year 1 (2000, with 1999 as the present).
- The new line would not have any significant repair costs during the study period.
- Currently (Year 0), the cost of system losses is $236,884/year.
- If the new line were built, these losses would *fall* by $1,575 each year for the next 35 years, starting in Year 1.

Total Costs by Year, New Line

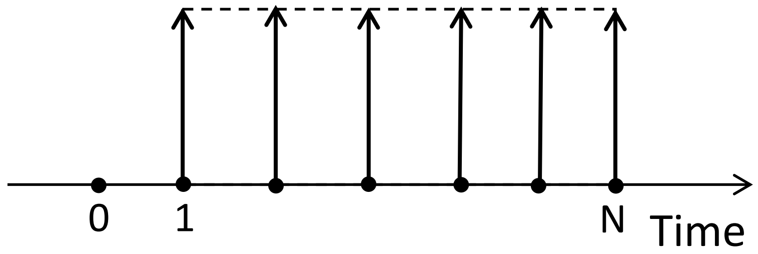


Dividing the cash flow into elements

- Looking at the cash flow and information, we have:
- A present-value cost, P = 236,884
- An arithmetic gradient with (negative) step size G = -1,575
- An annuity A from Year 0 to Year 35 (annuities are first positive in year 1)



Ramp (Arithmetic Gradient, G)

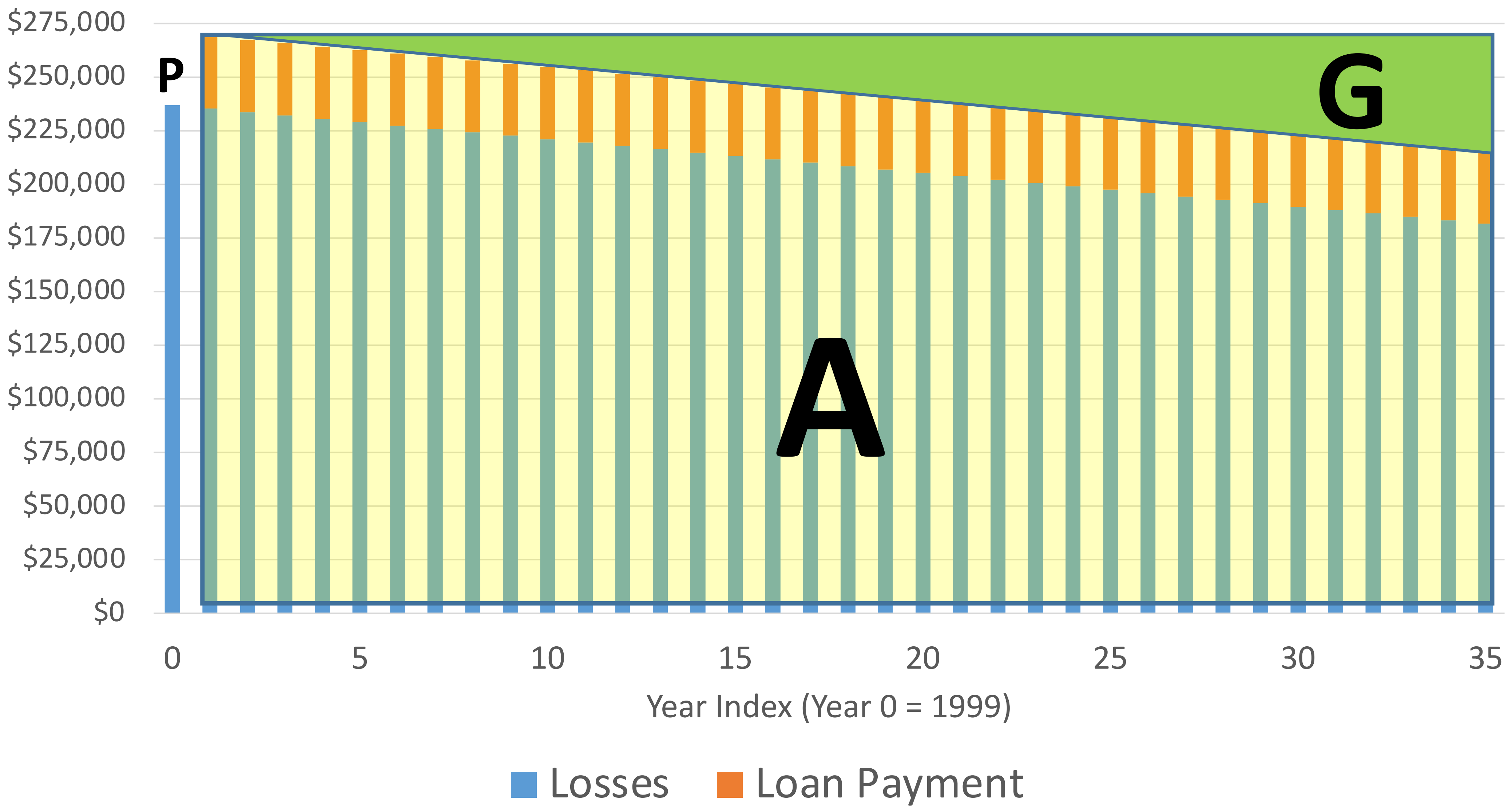


Step (Annuity, A)

Working out the timing

- Arithmetic gradients are first positive (or in this case, negative) in time period 2. That works well with our diagram, and we don’t even have to play around with timing.
- The total costs at time 1 are the ‘floor’ of the gradient.
- The total costs at time 1 are also the value of our annuity, A.
- Since annuities are first positive in Year 1, that leaves the total costs at time 0 as a one-off cost, which is in present value terms.
- Recall that we want to find the *annual* value of all this. Let’s put it together….
- …after taking a quick look at what it means, on the diagram.

Total Costs by Year, New Line

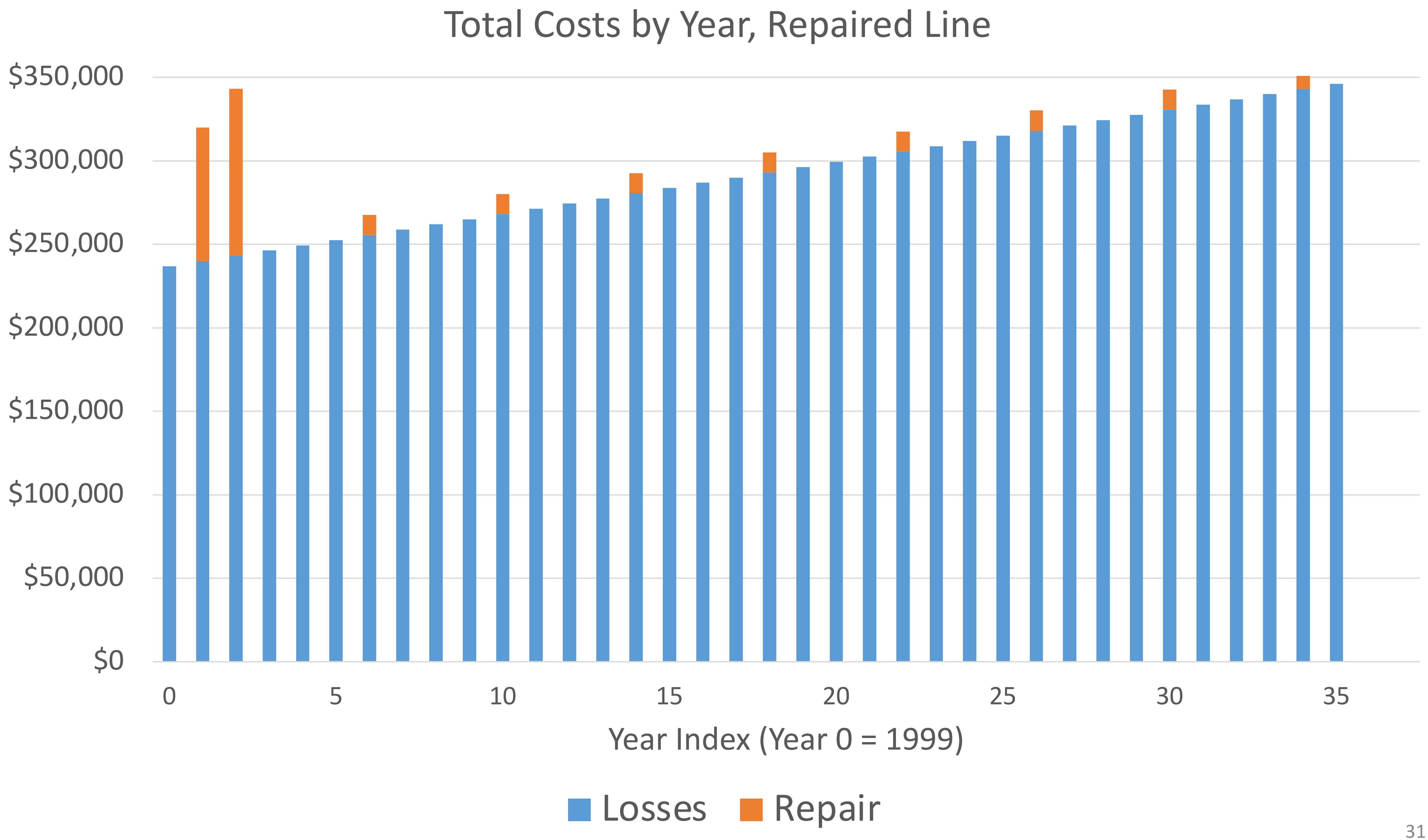


Annual worth calculations

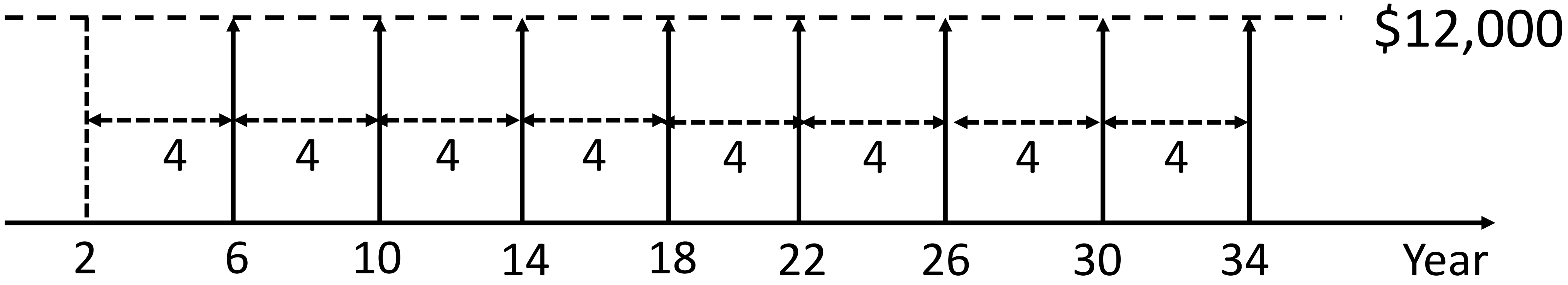
- What we know: N = 35, i = 6%, G = -1,575, Initial Cost P = 236,884
- A = total costs in Year 1 = 236,884 – 1,575 + 33,571 = 268,880
- Annual worth of…
- Initial Cost: \$236,884 x (A/P,6%,35) = \$16,338.80
- Gradient: -\$1,575 x (A/G,6%,35) = -\$18,005.27
- Annuity: A = \$268,880
- → Total annual costs = \$267,213.54 (sum of the above)

Plan A: Repair the Line

- Tri-County would not have to borrow money to repair the line, but there are other costs to keep track of.
- Under a repaired line, system losses would increase by $3,123 a year starting in Year 1.
- Initial ‘catch-up’ repairs would cost $80,000 in Year 1 and $100,000 in Year 2.
- Regular repairs would cost $12,000 every fourth year, starting in Year 6 and with the last repair session in the study period being in Year 34.
- The first two items are simple enough: we have one-off payments in Years 0, 1 and 2, plus an annuity and gradient, just as in Plan B.
- The third item is the weird one. We COULD treat it as a series of future payments, but is there any way to make use of its regularity?


Total Costs by Year, Repaired Line
$350,000
$300,000
$250,000
$200,000
$150,000
$100,000
$50,000
$0
0
5
10
15
20
25
30
35
Year Index (Year 0 = 1999)
Losses
Repair

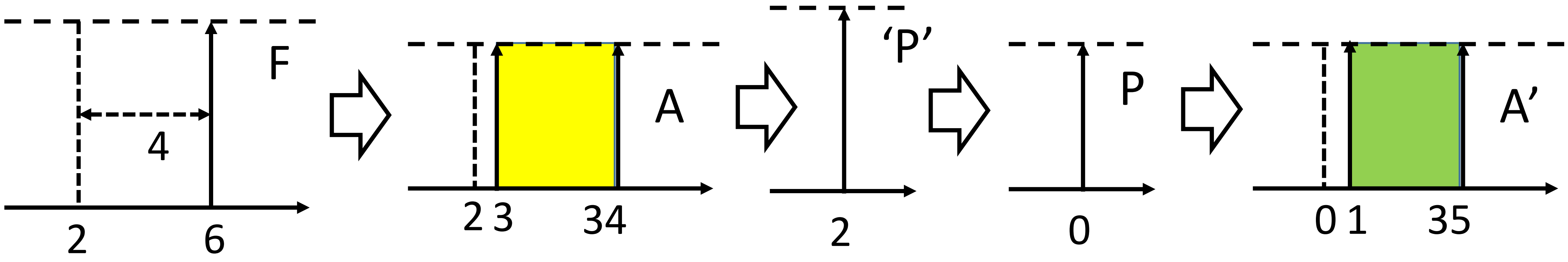
Let’s look at that repeating $12,000



- We have a conversion factor, (A/F,i,N), that will turn a payment in Year N into an annuity that is positive from years 1 to N.
- We only need to do it for ONE of the four-year periods, and (by symmetry) we’ll obtain an annualized value, A, that we can use for the entire time period.
- That gives us an annuity with Year ‘0’ at Year 2, and N = 34 – 2 = 32.
- To find the annualized value with the correct ‘present time’ and N = 35, we’ll have to first use (P/A,6%,32) to find a Year 2 value, and then (P/F,6%,2) to find a Year 0 present value.
- Finally, we’ll use (A/P,6%,35) to find the correct annualized value.

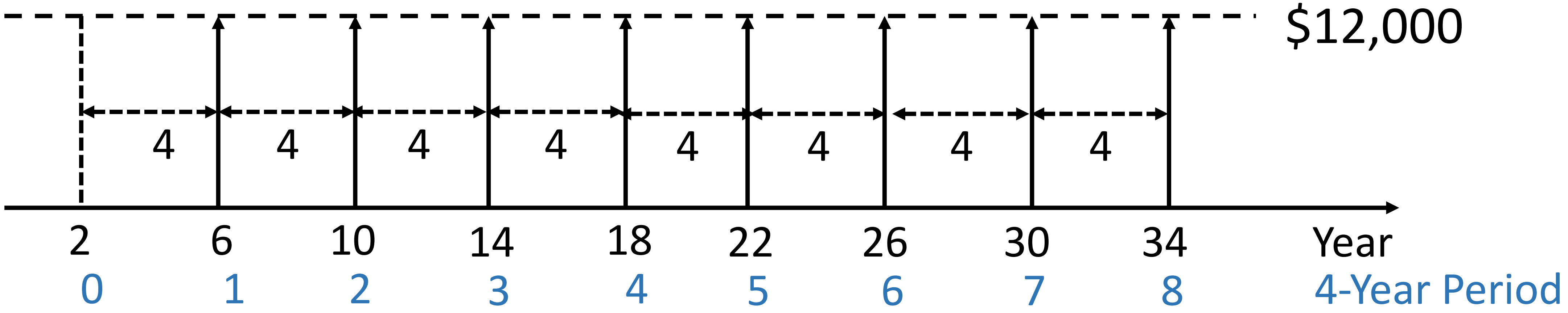
Let's get to it.

- A = \$12,000 x (A/F,6%,4) = \$2,743.10
- Year 2 P = \$2,743.10 x (P/A,6%,32) = \$38,633.91
- Year 0 P = \$38,633.91 x (P/F,6%,2) = \$34,384.04
- → Annual Worth is \$34,384.04 x (A/P,6%,35) = \$2,371.60



Still confused? This is extremely similar to the main example in section 3.8 of the textbook.

An easier way (and possible ‘d’oh!’ moment)



- Everything is in terms of **4 years**… why are we using a one-year interest rate?
- If our time period is 4 years, we have a standard, 8-period annuity with Year 2 as its ‘0’. We should just...
- Find the 4-year interest rate.
- Use it with (A/P,i,N) to find the discounted worth in Year 2.
- Use (P/F,i,2) with the *annual* interest rate to find the present worth (Year 0).
- Use (A/P,i,N) with the *annual* interest rate to find the annual worth.

Going through it…

- Interest = 6%/year, so after 4 years, \$1 becomes \$1 x $(1 + 6\%)^4$.
- → After 4 years, \$1 becomes \$1.262, so 4-year interest is 26.2% (rounded).
- A = \$12,000, N = 8 (4-year-periods), with Year 2 as the element's 'Year 0'.
- →Year 2 worth = \$12,000 x (P/A,26.2%,8) = \$12,000 x 3.22 = \$38,633.91
- Any difference from the above when you check the math is a rounding error.
- Small differences in interest significant figures can have big impacts over time!
- Present Worth = \$38,633.91 x (P/F,6%,2) = \$34,384.04
- Annual Worth (over 35 years) = \$34,384.04 x (A/P,6%,35) = \$2,371.60

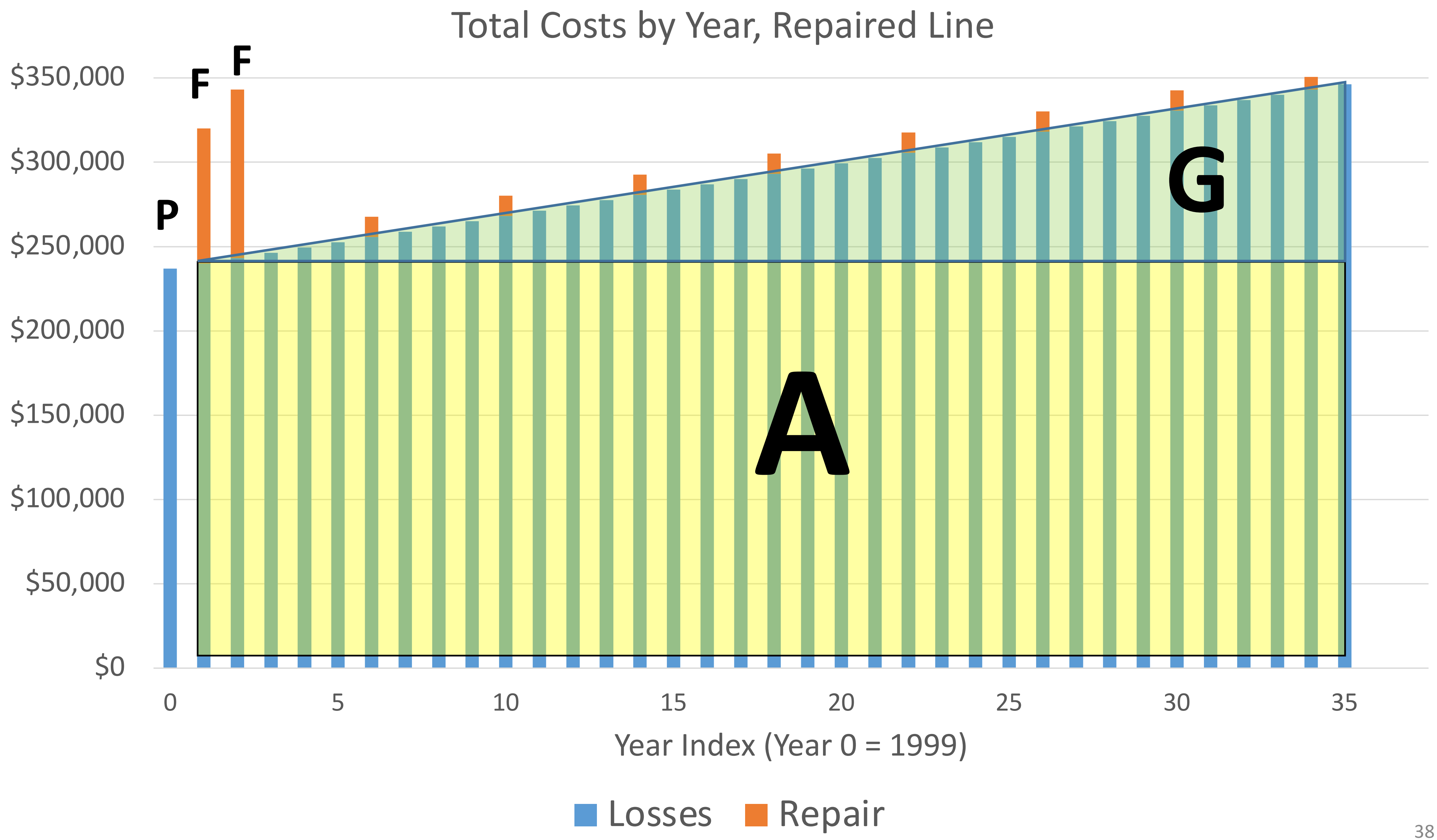
In a nutshell…

- When faced with a regular payment every X years, you can…
- A: Find the equivalent annuity for each X years, and work from there.
- B: Change the interest rate to be an every X-period interest rate, and switch back to an every-period rate when convenient.

- When dealing with *geometric gradients* with gaps (say, a payment every 5 years that is 10% higher each time), option B is often much faster.

Now for the rest

- Compared to that, the rest is simple.
- Gradients are first positive in Year 2, and annuities are first positive in Year 1. We have an arithmetic gradient with G = 3,123 that is first positive at year 2, sitting on top of an annuity that is first positive in Year 1, and has a value of (236,884 + 3,123) = $240,007.
- Apart from that, we have three one-shot payments: 236,884 in Year 0, 80,000 in Year 1 and 100,000 in Year 2.
- First, we'll revisit our diagram, then we'll perform the breakdown and annual worth calculations.



Total Costs by Year, Repaired Line
$350,000
$300,000
$250,000
$200,000
$150,000
$100,000
$50,000
$0
P
F
F
G
A
0
5
10
15
20
25
30
35
Year Index (Year 0 = 1999)
Losses
Repair

Annual equivalent worth of costs

- What we know: N = 35, i = 6%, G = 3,123, Initial Cost P = 236,884, Year 1 Payment = 80,000, Year 2 Payment = 100,000, Annual Worth of regular repairs = $2,371.60
- A = total costs in Year 1 = 236,884 + 3,123 = 240,007
- Annual worth of…
- Initial Cost: $236,884 x (A/P,6%,35) = $16,338.80
- Gradient: $3,123 x (A/G,6%,35) = 35,701.87
- Annuity: A = $240,007
- Year 1 Costs: $80,000 x (P/F,6%,1) x (A/P,6%,35) = $6,138.65
- Year 2 Costs: $100,000 x (P/F,6%,2) x (A/P,6%,35) = $2,371.60
- Regular Repairs: $2,371.60
- Total Annual Costs = $305,763.50 (Sum of the above)

In this case, replacement is cheaper than repairs over a 35-year period.