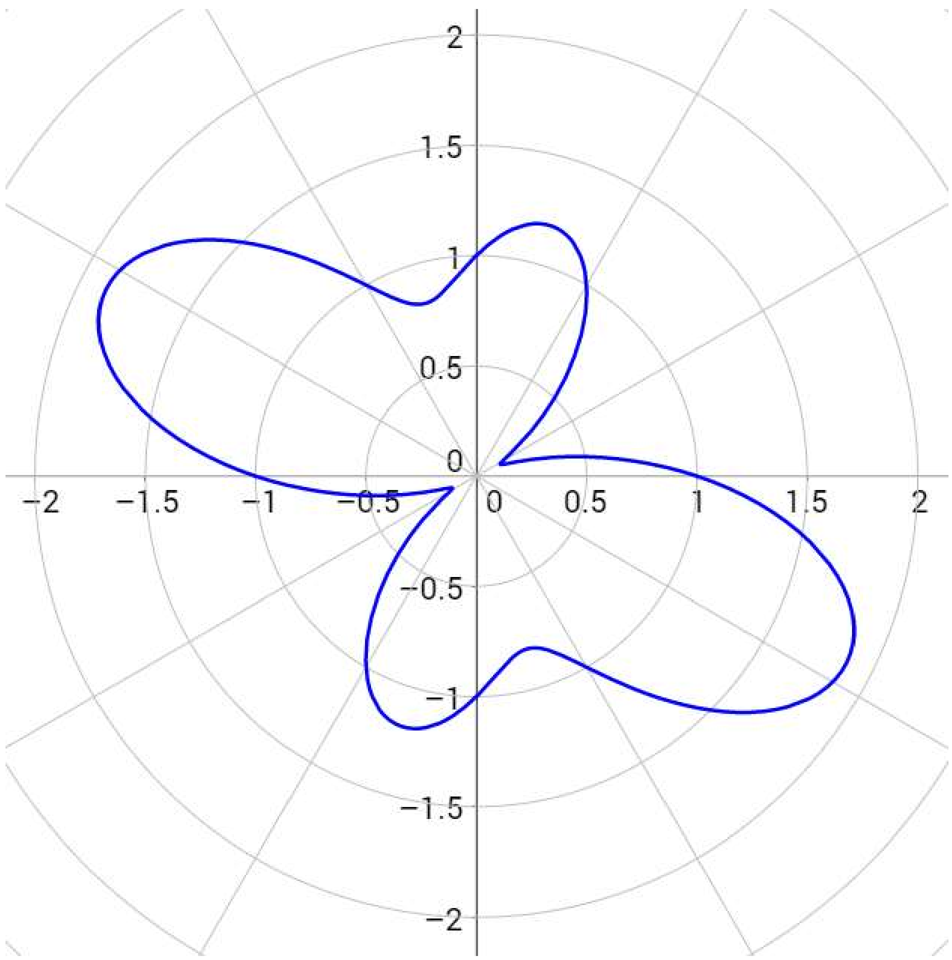
Chapter 3

Coordinate Systems



3.1 (Appendix A7) Complex Numbers

3.1.1 Introducing Complex Numbers and Algebraic Operations

$\mathbb{R}$

Everything we've dealt with is over the collection of measurable quantities, the real numbers $\mathbb{R}$. The complex numbers $\mathbb{C}$ are an extension of the reals $\mathbb{R}$ such that we allow solutions to the equation

$\mathbb{C}$

$$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$$

The "positive" solution $x = +\sqrt{-1} = i$ gives rise to "imaginary" parts of numbers. The space $\mathbb{C}$ consists of all numbers of the form

$$a + bi \qquad \mathbb{C}$$

where a and b are real.

Example: The equation $x^2 + 4x + 5 = 0$ has solutions

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4}\sqrt{-1}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

Addition and multiplication are defined as expected.

$$= -2 \pm i$$

$$\Rightarrow -2+i, -2-i$$ //

Example: Compute $(3 + 4i) - (-1 + 2i)$ and $(2 + i)(1 + 3i)$.

$$(3+4i)-(-1+2i) = 3+4i+1-2i = 4+2i$$

$$(2+i)(1+3i) = 2+6i+i+3i^2$$

$$= 2+7i+3(-1) = -1+7i$$

Definition

The **conjugate** of $z = a + bi$ is denoted and defined as $\bar{z} = a - bi$.

Definition

The **norm/modulus** of $z = a + bi$ is defined as

$$\sqrt{(a+bi)(a-bi)} = \sqrt{a^2 + \cancel{abi} - \cancel{abi} - b^2 i^2}$$

$$= \sqrt{a^2+b^2}$$

//

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$$

$$\frac{3+4i}{1-i} = a+bi \qquad \frac{3+4\sqrt{5}}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}}$$

Definition

We define **division** for $z = a + bi$, $w = c + di$ as

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Essentially you compute it by rationalizing the denominator.

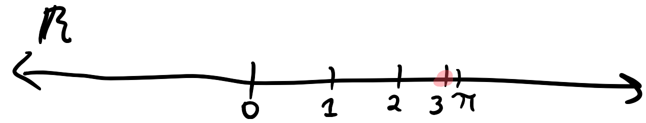
Example: Compute $\frac{2+3i}{1-7i}$. $= \frac{2+3i}{1-7i} \cdot \frac{1+7i}{1+7i}$

$$= \frac{(2+3i)(1+7i)}{(1-7i)(1+7i)} = \frac{2+14i+3i+21i^2}{1+\cancel{7i}-\cancel{7i}-49i^2}$$

$$= \frac{2+17i-21}{1-49(-1)}$$

$$= \frac{-19+17i}{50} = -\frac{19}{50} + \frac{17}{50}i$$

$\sim a+bi$



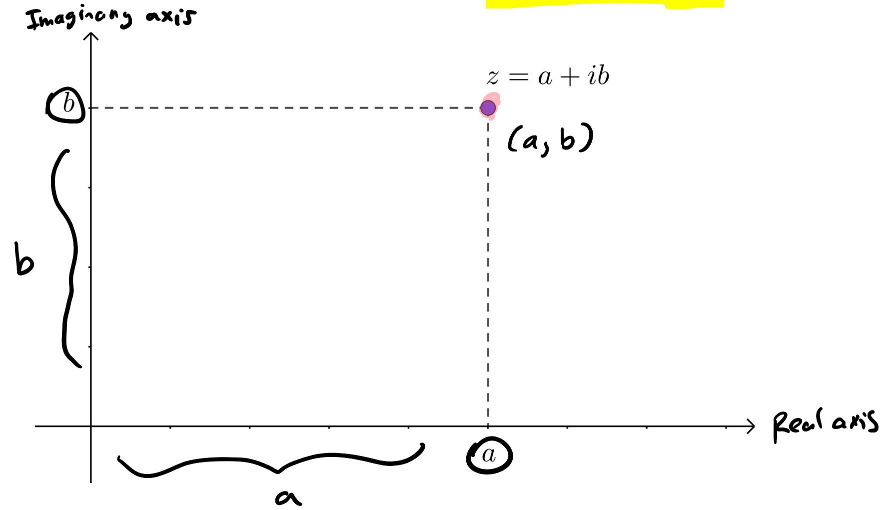
$\mathbb{C}$ $a + bi$

measure

measure

3.1.2 Argand Diagrams and Polar Coordinates

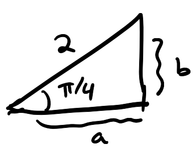
There are two components to a complex number we need to measure instead of just the one component that real numbers have. Thus real numbers are represented by a single quantity while complex numbers are represented as a pair of numbers (thus a point). We call this representation an Argand diagram.



We can also represent them by "polar coordinates". The point makes an angle with the x-axis and a radius out from the origin.

(a,b) instead (r,θ)

$(r,\theta) = (2, \frac{\pi}{4})$

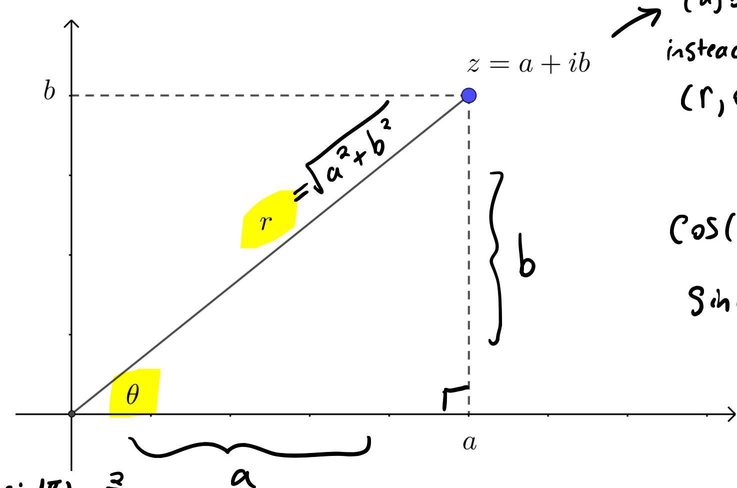


$\Rightarrow \cos(\frac{\pi}{4}) = \frac{a}{2}$

$\Rightarrow a = 2\cos(\frac{\pi}{4})$

$= \frac{2}{\sqrt{2}}$

$\sin(\frac{\pi}{4}) = \frac{b}{2} \Rightarrow b = 2\sin(\frac{\pi}{4}) = \frac{2}{\sqrt{2}}$



$\cos(\theta) = \frac{a}{r} \Rightarrow a = r\cos(\theta)$

$\sin(\theta) = \frac{b}{r} \Rightarrow b = r\sin(\theta)$

From the diagram $a = r\cos(\theta)$ and $b = r\sin(\theta)$ where $r = |z|$.

Definition

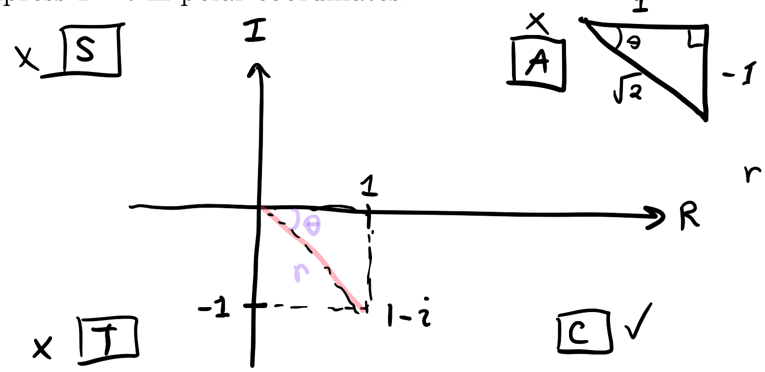
Consider the complex number $z = a + ib$ and let θ be the angle between the segment connecting point representation of z with the origin and the positive x-axis. Provided that r represents the length of this segment, the **Polar Representation** of the point is

$$z = r(\cos(\theta) + i\sin(\theta))$$

$z = a + ib = r\cos(\theta) + i r\sin(\theta)$

$\to a=1,\ b=-1$

Example: Express $1-i$ in polar coordinates.

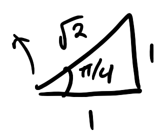


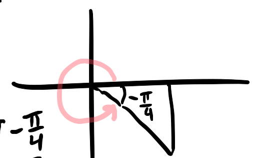
$$r = |1-i| = \sqrt{(1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Let's find the angle

$$\Rightarrow \begin{cases} \cos(\theta) = \frac{1}{\sqrt{2}} \\ \sin(\theta) = -\frac{1}{\sqrt{2}} \end{cases}$$

reference angle of $\phi = \frac{\pi}{4}$





$2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ OR

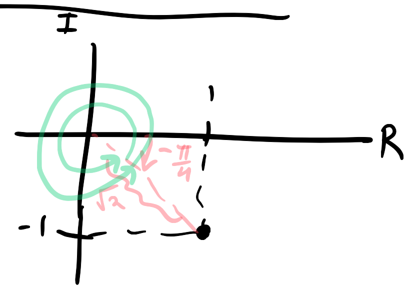
so $\theta = -\frac{\pi}{4}$

and thus $z = 1-i = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$

$z = 1-i \Rightarrow r = \sqrt{2}$, $\theta = -\frac{\pi}{4}$

$= r(\cos(\theta) + i\sin(\theta))$

$= \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$



$1-i \Rightarrow \theta = -\frac{\pi}{4}$

$\theta = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$\theta = -\frac{\pi}{4} + 2\pi + 2\pi$

$= -\frac{\pi}{4} + 4\pi$

$z = 1 - i = \sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$
$= \sqrt{2}\, e^{-i\pi/4}$

$\Rightarrow$ Page 158 $\Leftarrow$

3.1.3 Euler's Formula

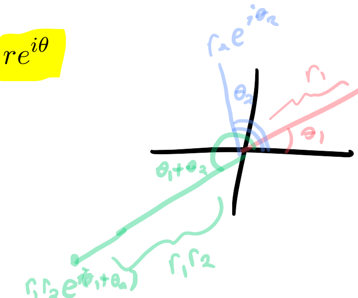
Euler's Formula

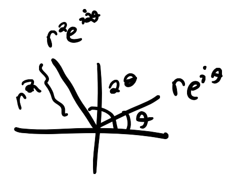
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This allows us to write the polar form as

$$a + bi = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

This form allows us to exploit exponent properties.





<u>Products</u>: Taking the product gives $(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$

<u>Quotients</u>: Taking the quotient gives $\dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

<u>Powers</u>: Taking the power gives $(re^{i\theta})^n = r^ne^{i\theta n}$.

$z^n = (a+ib)^n = (re^{i\theta})^n = r^ne^{i(n\theta)}$

That last point is the most interesting because it gives us the following theorem.

De Moivre's Theorem

$r=1 \Rightarrow (e^{i\theta})^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$
$= (\cos(\theta) + i\sin(\theta))^n$

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

$\Rightarrow (e^{i\theta})^n = e^{i(n\theta)}$

Proof: Combine $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ with $(e^{i\theta})^n = e^{in\theta}$. ■

\+ noting $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

You can use this to generate identities for trigonometric functions!

Example: Use De Moivre's Theorem to generate identities for $\cos(2\theta)$ and $\sin(2\theta)$.

what happens when $n=2$?

$(e^{i\theta})^2 = e^{i(2\theta)} \Rightarrow (\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta)$

expand

$\Rightarrow \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + i^2\sin^2(\theta) = \cos(2\theta) + i\sin(2\theta)$

$(a,b) = (c,d)$

$\Rightarrow (\cos^2(\theta) - \sin^2(\theta)) + i(2\sin(\theta)\cos(\theta)) = \cos(2\theta) + i\sin(2\theta)$

Real Part

$\Rightarrow \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$

Imaginary Part

$\Rightarrow 2\sin(\theta)\cos(\theta) = \sin(2\theta)$

3.1.4 Roots of Complex Numbers

$z^2 = 1 \Rightarrow z = \pm 1$

Every polynomial of degree n always has n-roots over $\mathbb{C}$ (called the Fundamental Theorem of Algebra). We can use the exponential form to 'easily' solve

$$z^n = C \qquad z^n = C$$

for z where C is a constant (possibly a complex number $C = a + bi$).

Procedure for Finding All Roots of a Complex Number

Start with an equation of the form $z^n = C$.

1. Express C in exponential form as $C = re^{i\theta}$. Then

$$z^n = re^{i\theta}$$

2. Use the periodicity of the complex exponential (sine and cosine). Since sine and cosine are periodic with period 2π then

$$z^n = r\exp(i(\theta + 2\pi k))$$

$$z^n = re^{i\theta} = re^{i(\theta+2\pi k)}$$

any integer k

for k any integer.

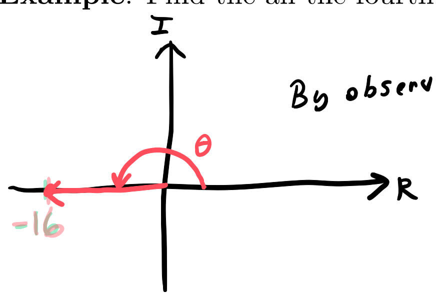
3. Take the n-th root of both sides,

$$z = r^{1/n}\exp\left(i\left(\frac{\theta + 2\pi k}{n}\right)\right) \qquad z = r^{1/n} e^{i\left(\frac{\theta+2\pi k}{n}\right)}$$

for $k = 0, 1, 2, 3, ..., n-1$.

The reason it goes from k being all integers to just the first n values is because afterwards it repeats. So we can obtain the n roots from the first n values of k starting from zero.

Example: Find the all the fourth roots of -16. $\Rightarrow z^4 = -16 = -16 + 0i$



By observation $\theta = \pi$ and $r = 16$ $\Rightarrow -16 = 16e^{i\pi}$

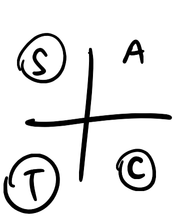
1. $z^4 = 16e^{i\pi}$

2. $z^4 = 16e^{i(\pi+2\pi k)}$

3. $z = 16^{1/4} e^{i\left(\frac{\pi+2\pi k}{4}\right)}$ ← $(re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n}$

$= 2e^{i\left(\frac{\pi+2\pi k}{4}\right)}$

we can expect 4 roots as $z^4 = -16$ is fourth degree

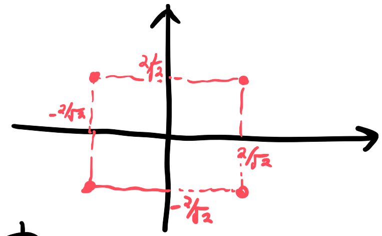


$k=0 \Rightarrow z_0 = 2e^{i\left(\frac{\pi+0}{4}\right)} = 2e^{i\left(\frac{\pi}{4}\right)} = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}}$

$k=1 \Rightarrow z_1 = 2e^{i\left(\frac{\pi+2\pi}{4}\right)} = 2e^{i\left(\frac{3\pi}{4}\right)} = -\frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}}$

$k=2 \Rightarrow z_2 = 2e^{i\left(\frac{\pi+4\pi}{4}\right)} = 2e^{i\left(\frac{5\pi}{4}\right)} = -\frac{2}{\sqrt{2}} - i\frac{2}{\sqrt{2}}$

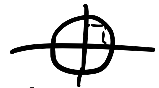
$k=3 \Rightarrow z_3 = 2e^{i\left(\frac{\pi+6\pi}{4}\right)} = 2e^{i\left(\frac{7\pi}{4}\right)} = \frac{2}{\sqrt{2}} - i\frac{2}{\sqrt{2}}$



3.2 (Section 11.1) Parametric Equations

3.2.1 Defining Parametric Curves

$x^2+y^2-1=0 \Rightarrow$

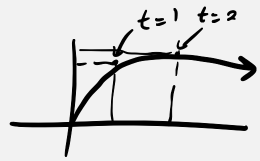


A curve C given by $f(x, y) = 0$ gives us a graph but it falls short on a few aspects.

Definition

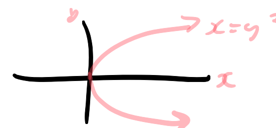
If C is a curve and (x, y) is any point on C then provided there exists functions such that

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$



we call it a **parametric representation** of C.

Example: Consider $y^2 = x$. Then $y = t$ and $x = t^2$ is a parametric representation of this curve. All points can be mapped out by choosing values of t.



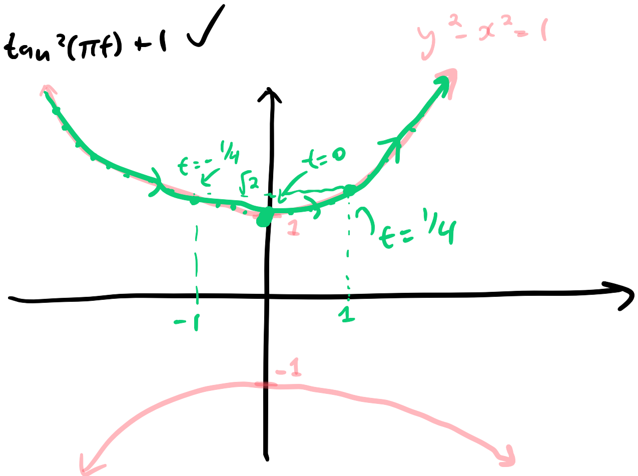
$y=t$
$\Rightarrow x=y^2=t^2$
$\Rightarrow \begin{cases} x=t^2 \\ y=t \end{cases}$

Why is this useful? For modeling the position of a particle by time.

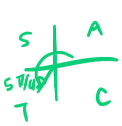
Example: Suppose that the position of a particle follows the trajectory of the curve $y^2 = x^2 + 1$ given by the equations $y = \sec(\pi t)$ and $x = \tan(\pi t)$ where $-1/2 < t < 1/2$. Determine the position of the particle after 0.25 seconds. Graph and Determine the orientation of the particle flow along the curve.

$y^2=x^2+1 \Rightarrow \sec^2(\pi t) = \tan^2(\pi t)+1$ ✓

$\boxed{\begin{matrix} y=\sec(\pi t) \\ x=\tan(\pi t) \end{matrix}}$



t	x	y
0	$\tan(0)=0$	$\sec(0)=1$
1/4	$\tan(\frac{\pi}{4})=1$	$\sec(\frac{\pi}{4})=\frac{\sqrt{2}}{1}$



3.2.2 Graphing a Parametric Curve

Technique #1: Plotting Points

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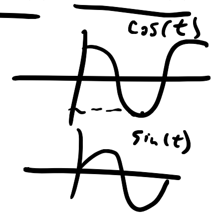
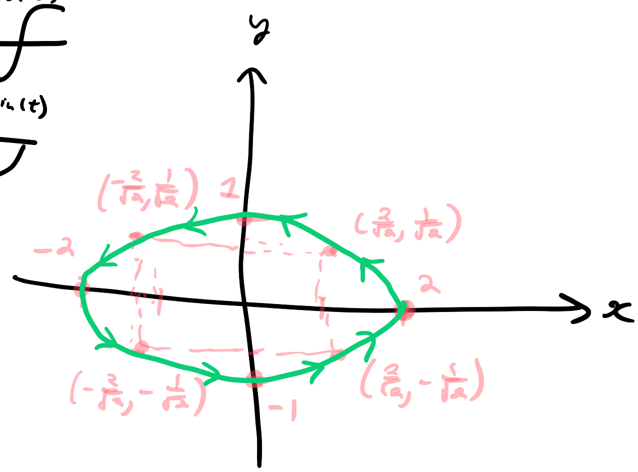
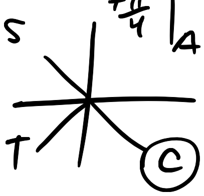
This is the worst way and I don't really condone it.

Example: Attempt to graph $x = 2\cos(t)$, $y = \sin(t)$ for $0 \le t \le 2\pi$ by plotting a few points.

t	x	y
0	$2\cos(0) = 2$	$\sin(0) = 0$
$\frac{\pi}{2}$	$2\cos(\frac{\pi}{2}) = 0$	$\sin(\frac{\pi}{2}) = 1$
π	-2	0
$\frac{3\pi}{2}$	0	-1
2π	2	0
$\frac{\pi}{4}$	$\frac{2}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{4}$	$-\frac{2}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5\pi}{4}$	$-\frac{2}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7\pi}{4}$	$\frac{2}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$







$$\begin{cases} x = a\cos(t) \\ y = b\sin(t) \end{cases}$$ ellipse

If you do it this way, you might as well use GeoGebra.

Graphing Using GeoGebra

Using GeoGebra Classic 6, you can graph parametric curves given by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$ using the command

```
Curve(f(t),g(t),t,a,b)
```

Technique #2: Finding the Curve C

This is usually done by solving one equation for t and plugging it in the other.

Example: Consider the curve given by $\underline{x = t^2 + t}$, $\underline{y = 2t - 1}$. $\Rightarrow$ $\frac{y+1}{2} = t$

t	x	y
0	$0^2+0=0$	$2(0)-1=-1$
1	$1^2+1=2$	$2(1)-1=1$

Sub into $x = t^2 + t$

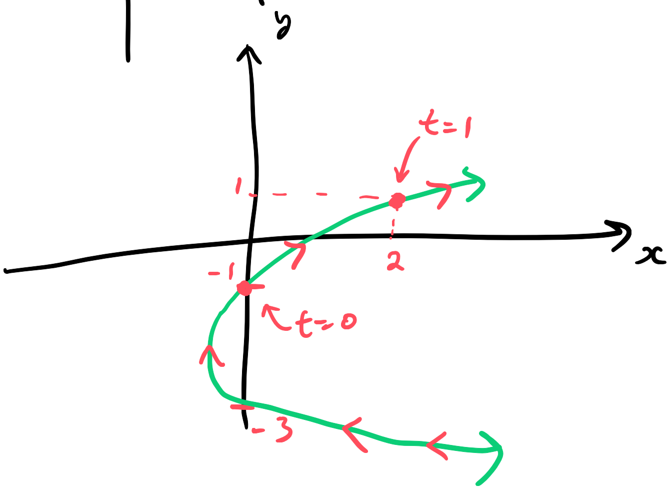
$$= \left(\frac{y+1}{2}\right)^2 + \left(\frac{y+1}{2}\right)$$

$$= \frac{1}{4}(y^2+2y+1) + \frac{y+1}{2}$$

$$= \frac{1}{4}(y^2+2y+1+2y+2)$$

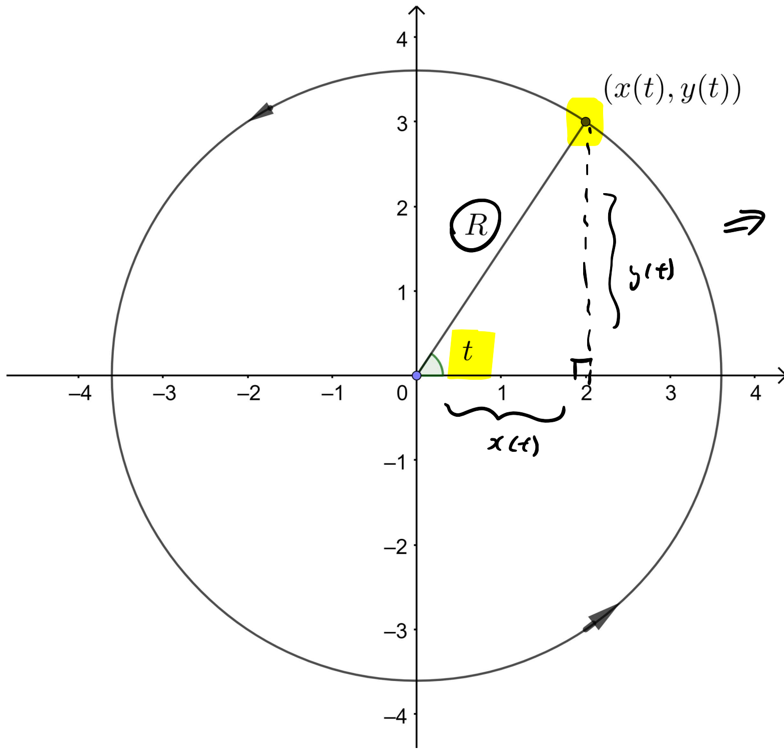
$$= \frac{1}{4}(y^2+4y+3)$$

$$= \frac{1}{4}(y+3)(y+1)$$



3.2.3 Standard Representation of a Circle

Consider the circle $x^2 + y^2 = R^2$.



$\Rightarrow \cos(t) = \dfrac{x(t)}{R}$

$\sin(t) = \dfrac{y(t)}{R}$

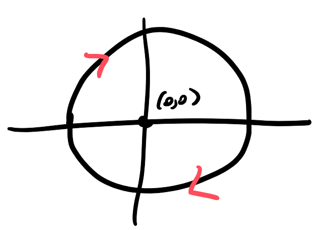
We note $\cos(t) = \dfrac{x}{R}$, $\sin(t) = \dfrac{y}{R}$. So a parametric representation is

$$\begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases}$$

for $0 \leq t < 2\pi$. The orientation is counter-clockwise as that represents the direction of the increasing angle (which is variable the circle is parametrized by).

3.2.4 Transformations

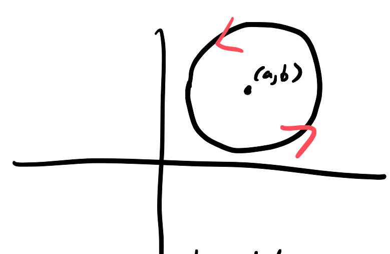
Reverse Orientation



$\begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases} \xRightarrow{\text{reverse}} \begin{cases} x = R\cos(-t) \\ y = R\sin(-t) \end{cases}$

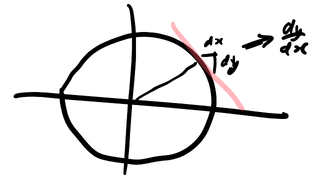
$t \mapsto -t$

Translation by a Point



$\begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases} \xRightarrow{\text{translate}} \begin{cases} x = a + R\cos(t) \\ y = b + R\sin(t) \end{cases}$

3.3 (Section 11.2) Calculus of a Parameterized Curve



3.3.1 The Slope of a Parametric Curve

Let C be a curve given parametrically by $x = f(t)$, $y = g(t)$. We still want to discuss the slope of the tangent line $\frac{dy}{dx}$.

$u = f(x)$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

By the chain rule $du = f'(x)dx$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{y'(t)\cancel{dt}}{x'(t)\cancel{dt}} = \frac{y'(t)}{x'(t)}$$

Notation

People commonly use the following,

$y'(x)$ $\qquad$ $\dot{y}(t)$

$$y' = \frac{dy}{dx} \qquad \dot{y} = \frac{dy}{dt}$$

to denote differentiation with respect to space and time respectively. Thus the above may be stated as $y' = \frac{\dot{y}}{\dot{x}}$ //

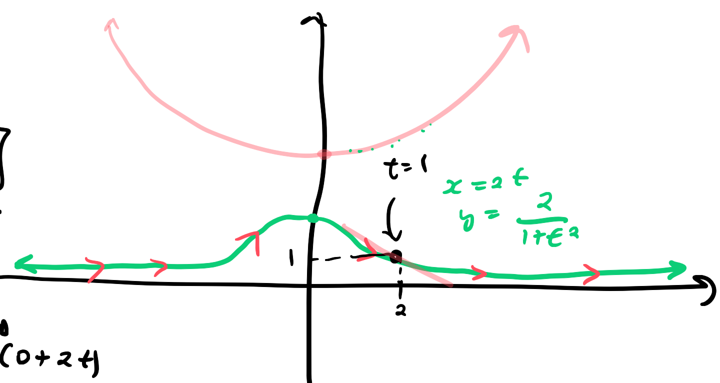
Theorem

Provided $\dot{x} \neq 0$ we have $y' = \frac{\dot{y}}{\dot{x}}$. Provided $\dot{y} \neq 0$ we have $x' = \frac{\dot{x}}{\dot{y}}$.

Example: (Witch of Agnesi) Consider the curve given by $x = 2t$, $y = \frac{2}{1+t^2}$.

$$t = \frac{x}{2} \Rightarrow y = \frac{2}{1+(\frac{x}{2})^2} = \frac{8}{4+x^2} \;//$$

$4+x^2$



$$y' = \frac{\dot{y}}{\dot{x}} = \frac{\frac{d}{dt}\left[\frac{2}{1+t^2}\right]}{\frac{d}{dt}[2t]}$$

$$= \frac{-\cancel{2}(1+t^2)^{-2}(0+2t)}{\cancel{2}}$$

$$= -\frac{2t}{(1+t^2)^2} \;///$$

at $t = 1 \Rightarrow x = 2(1) = 2$

$y = \frac{2}{1+1^2} = 1$

$$y'\Big|_{t=1} = \frac{-2(1)}{(1+1^2)^2} = -\frac{2}{4} = -\frac{1}{2}$$

⇒ Page 165 t

Example: Find the tangent line to $x = t^5 - 4t^3$, $y = t^2$ at $(0, 4)$.

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

Need t where $\begin{cases} x = 0 \\ y = 4 \end{cases}$

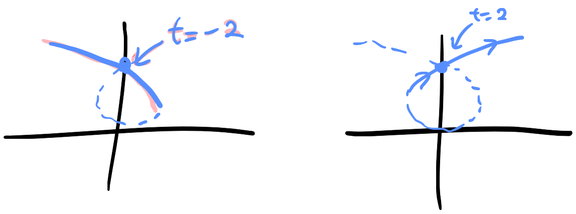
$$\Rightarrow \begin{cases} t^5 - 4t^3 = 0 \Rightarrow t^3(t^2 - 4) = 0 \\ t^2 = 4 \Rightarrow t^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} t^3(t-2)(t+2) = 0 \\ (t-2)(t+2) = 0 \end{cases}$$

$t = 0, \pm 2$ $\leftarrow$

$t = \pm 2$ $\leftarrow$

overlap is $t = \pm 2$

What???



at $t = -2, 2 \Rightarrow (x_0, y_0) = (0, 4)$ but the slopes are

$$y'\Big|_{t=-2} = \frac{2}{5(-2)^3 - 12(-2)} = \frac{2}{-40 + 24} = \frac{1}{-20 + 12} = -\frac{1}{8}$$

$$y'\Big|_{t=2} = \frac{2}{5(2)^3 - 12(2)} = \frac{2}{40 - 24} = \frac{1}{20 - 12} = \frac{1}{8}$$

$$y - y_0 = m(x - x_0)$$

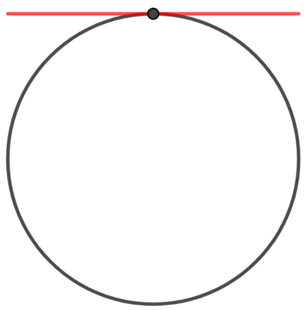
at $t = -2 \Rightarrow$ $y - 4 = -\frac{1}{8}(x - 0) \Rightarrow y = -\frac{1}{8}x + 4$

$t = 2 \Rightarrow$ $y - 4 = \frac{1}{8}(x - 0) \Rightarrow y = \frac{1}{8}x + 4$

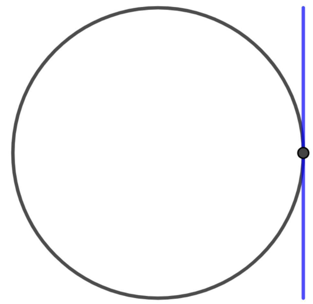
3.3.2 Higher Order Derivatives and Vertical/Horizontal Tangency

Horizontal: $\dot{y} = 0$, (provided $\dot{x} \neq 0$)

$$y' = 0 = \frac{\dot{y}}{\dot{x}}$$



Vertical: $\dot{x} = 0$, (provided $\dot{y} \neq 0$)



$$x' = \frac{\dot{x}}{\dot{y}} = 0$$

We can also talk about concavity. Since

$$\frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{\dot{x}} \quad \to \quad \frac{d}{dx}[\ddot\smile] = \frac{\frac{d}{dt}[\ddot\smile]}{\dot{x}}$$

then by the same procedure,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\dot{x}}$$

Theorem

Let $x(t)$ and $y(t)$ be parametric equations of a curve. Then

$$y'' = \frac{\frac{d}{dt}[y']}{\dot{x}} \qquad y''' = \frac{\frac{d}{dt}[y'']}{\dot{x}}$$

Example: Consider $x = 1 - t^2$, $y = t^7 + t^5$.

$\ddot\smile = y'$

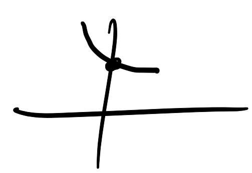
$$\frac{d}{dx}[\ddot\smile] = \frac{\frac{d}{dt}[\ddot\smile]}{\dot{x}}$$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{7t^6 + 5t^4}{0 - 2t} = -\frac{7}{2}t^5 - \frac{5}{2}t^3$$

$$y'' = \frac{\frac{d}{dt}[y']}{\dot{x}} = \frac{-\frac{35}{2}t^4 - \frac{15}{2}t^2}{0 - 2t} = \frac{35}{4}t^3 + \frac{15}{4}t \, //$$

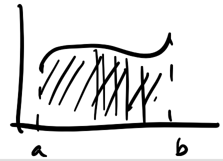
e.x. $t = 1 \Rightarrow x = 1 - 1^2 = 0$ $\quad (0, 2)$

$y = 1^7 + 1^5 = 2$



$$y'|_{t=1} = -\frac{7}{2}(1)^5 - \frac{5}{2}(1)^3 = -\frac{12}{2} = -6$$

$$y''|_{t=1} = \frac{35}{4}(1)^3 + \frac{15}{4}(1) = \frac{50}{4} = \frac{25}{2}$$



$A = \int_a^b y\,dx$

3.3.3 Integration

$a \le x \le b$

Definition

The signed area between $y(x)$ and the x-axis over $[a, b]$ is $\int_a^b y dx$. The signed area between $x(y)$ and the y-axis over $[c, d]$ is $\int_c^d x dy$.

Example: Compute the area trapped between the curve $x = \cos^3(t)$, $y = \sin^3(t)$ and the x-axis over $0 \le t \le 2\pi$.

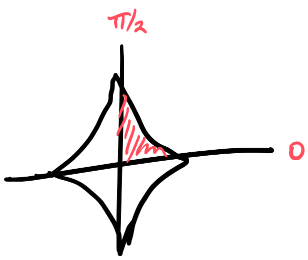
$x = \cos^3(t) \Rightarrow dx = 3\cos^2(t)(-\sin(t))\,dt$

$= -3\cos^2(t)\sin(t)\,dt$

$$A = \int_{x=0}^{x=1} y\,dx = 4\int_{\pi/2}^{0} \sin^3(t)(-3\cos^2(t)\sin(t))\,dt$$

$$= -3\int_{\pi/2}^{0} \sin^4(t)\cos^2(t)\,dt$$

Based on Geogebra



Can finish using section 8.3 techniques online

(Continued...)

$$\Rightarrow \int_{y_S}^{y_B} x\,dy$$

Example: Find the area enclosed by the y-axis and the curve $x = t - t^2$, $y = 1 + e^{-t}$.

Set $x = t - t^2 = 0$

$\Rightarrow t(1-t) = 0 \rightarrow t = 0, 1$

check to obtain

$y(0) = 1 + e^0 = 2$

$y(1) = 1 + e^{-1} < y(0)$

$$A = \int_{1+e^{-1}}^{2} x\,dy = \int_1^0 (t-t^2)\underbrace{(-e^{-t})\,dt}_{dy = (0-e^{-t})dt}$$

$$= -\int_1^0 (t-t^2)e^{-t}\,dt = \int_0^1 (t-t^2)e^{-t}\,dt$$

LIPET

+/-	u	dv
+	$t-t^2$	e^{-t}
-	$1-2t$	$-e^{-t}$
+	-2	e^{-t}
-	0	$-e^{-t}$

Tabular Method

Section 8.2

$$= (t-t^2)(-e^{-t})\Big|_0^1 - (1-2t)e^{-t}\Big|_0^1 + (-2)(-e^{-t})\Big|_0^1 - \int_0^1 0\,dt$$

$$= \left[(1-1^2)(-e^{-1}) - (0-0^2)(-e^0)\right] - \left[(1-2)e^{-1} - (1-0)e^0\right] + (+2)\left[e^{-1} - e^0\right] - 0$$

$$= 0 - \left[-e^{-1} - 1\right] + 2\left[e^{-1} - 1\right]$$

$$= e^{-1} + 1 + 2e^{-1} - 2$$

$$= \underline{\underline{3e^{-1} - 1}}$$

3.3.4 Arc-Length

We learned that the length of $y(x)$ over $[a,b]$ is

$$L = \int_a^b \sqrt{1+y'^2}dx$$

Let's examine the integrand. Since $y' = \dfrac{\dot{y}}{\dot{x}}$ then

$$\sqrt{1+y'^2}dx = \sqrt{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2}dx = \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}dx$$

However $\dfrac{1}{\dot{x}}dx = t'dx = dt$. So the length of $x(t)$, $y(t)$ over $t=c$ to $t=d$ is the following.

Theorem

The arc length of a parametrized curve $x(t)$, $y(t)$ over $c \le t \le d$ is

$$L = \int_c^d \sqrt{\dot{x}^2+\dot{y}^2}dt$$

Example: Compute the length of a circle of radius R using the standard parametrization.

$$\Rightarrow \begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases} \text{ for } 0 \le t \le 2\pi$$

$$L = \int_0^{2\pi} \sqrt{(-R\sin(t))^2 + (R\cos(t))^2}\, dt$$

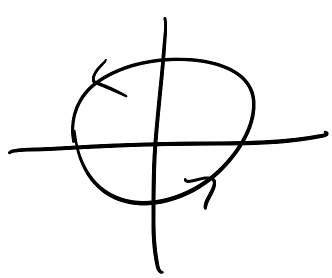
$$= \int_0^{2\pi} \sqrt{R^2\cos^2(t) + R^2\sin^2(t)}\, dt = \int_0^{2\pi} \sqrt{R^2}\sqrt{\cos^2(t)+\sin^2(t)}\, dt$$

$$= \int_0^{2\pi} R\cdot(1)\, dt$$

$$= Rt\Big|_0^{2\pi}$$

$$= 2\pi R - \cancel{0}R$$

$$= 2\pi R$$



$$y = \sqrt{R^2 - x^2}$$

$$L = 2\int_{-R}^{R} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y' = \frac{-x}{\sqrt{R^2-x^2}}$$

$$= 2\int_{-R}^{R} \sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$$

Example: Compute the length of the curve $x = \ln(\sec(t) + \tan(t)) - \sin(t)$, $y = \cos(t)$ over $0 \le t \le \pi/3$.

$$\dot{x} = \frac{1}{\sec(t)+\tan(t)}\left(\sec(t)\tan(t) + \sec^2(t)\right) - \cos(t)$$

$$= \frac{\sec(t)\left(\sec(t)+\tan(t)\right)}{\sec(t)+\tan(t)} - \cos(t) = \sec(t) - \cos(t)$$

$$\dot{y} = -\sin(t)$$

SO...

$$L = \int_0^{\pi/3} \sqrt{(\sec(t) - \cos(t))^2 + (-\sin(t))^2}\, dt$$

$$= \int_0^{\pi/3} \sqrt{\sec^2(t) - 2\sec(t)\cos(t) + \cos^2(t) + \sin^2(t)}\, dt$$

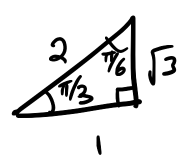
$$= \int_0^{\pi/3} \sqrt{\underbrace{\sec^2(t) - 2 + 1}_{=\sec^2(t)-1}}\, dt$$

$0 < t < \pi/2$

$$= \int_0^{\pi/3} \sqrt{\tan^2(t)}\, dt = \int_0^{\pi/3} |\tan(t)|\, dt$$

$$= \int_0^{\pi/3} \tan(t)\, dt$$

$$= \ln|\sec(t)| \Big|_0^{\pi/3}$$



$$= \ln|\sec(\pi/3)| - \ln|\sec(0)|$$

$$= \ln(2) - \ln(1) \quad (\ln(1) \to 0)$$

$$= \ln(2)$$

3.4 (Section 11.3) Polar Coordinates

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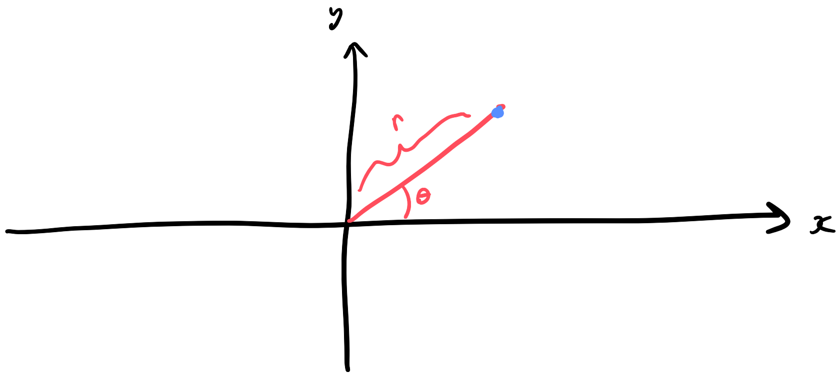
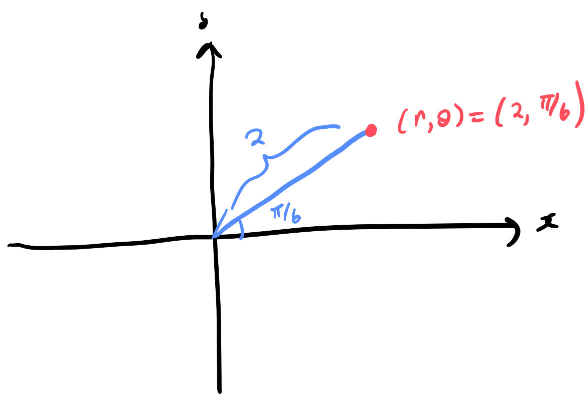
3.4.1 Revisiting Polar Coordinates and Non-Uniqueness

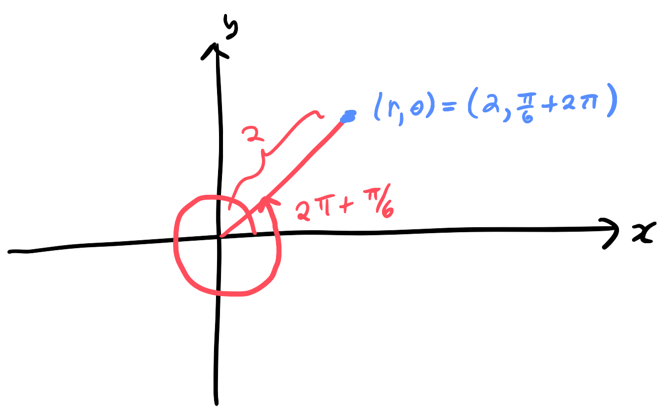
We already talked a bit about polar form with complex numbers. The idea is that we measure points in the xy-plane using an angle θ and radius r.



Consider the point $(r, \theta) = (2, \pi/6)$.



The expression for a point is not necessarily unique! For example, technically $(2, \pi/6) = (2, \pi/6 + 2\pi)$.

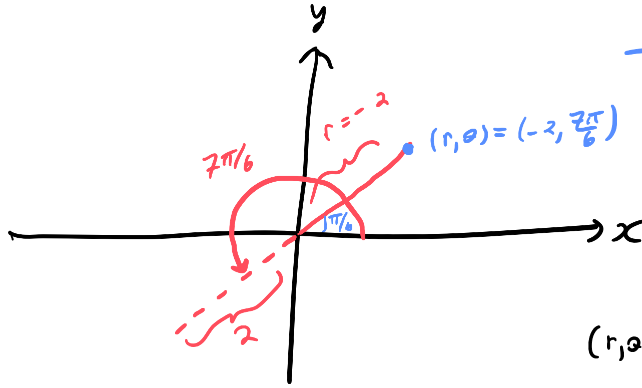


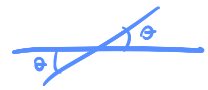
In fact, $(2, \pi/6) = (2, \pi/6 + 2\pi n)$ for any integer n. Polar coordinates also allows negative distance (unless specified otherwise)!

$(r,\theta) = (2, \frac{\pi}{6}) \cong (2, \frac{\pi}{6} + 2\pi n)$

for any integer n

For example consider the angle $\theta = \pi/6 + \pi = 7\pi/6$.





for any integers m and n.

$(r,\theta) = (2, \frac{\pi}{6}) \cong (2, \frac{\pi}{6} + 2\pi n)$

$\cong (-2, \frac{7\pi}{6} + 2\pi m)$

If we take the radius $r = -2$ then we measure the distance backwards, so $(2, \pi/6) = (-2, 7\pi/6) = (-2, 7\pi/6 + 2\pi n)$ for any integer n as well.

3.4.2 Polar to Cartesian (Points), Cartesian to Polar (Equations)

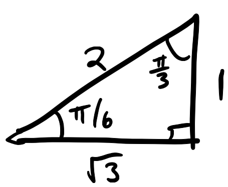
Expressing Cartesian Values in Terms of Polar Values

Consider a point (x, y) in Cartesian that makes an angle θ with the positive x axis and is a distance r from the origin. Then...

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

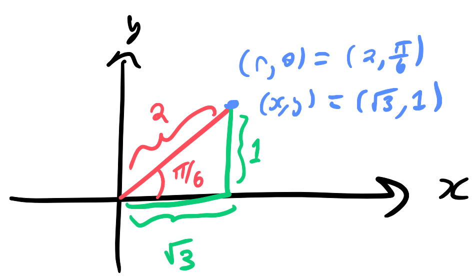
We can use this to convert equations to polar form. This is very useful for equation with the term $x^2 + y^2$. Given a specified r and θ this allows you to find the Cartesian equivalent of a point in polar coordinates.

Example: Find the Cartesian representation of the point $(r, \theta) = (2, \pi/6)$.



$x = r\cos(\theta) = 2\cos(\frac{\pi}{6}) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$y = r\sin(\theta) = 2\sin(\frac{\pi}{6}) = 2\left(\frac{1}{2}\right) = 1$



Example: Find the polar curve equation of $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$.

$x = r\cos(\theta)$
$y = r\sin(\theta)$ $\overset{\text{sub in}}{\Rightarrow}$ $(r\cos(\theta))^4 + (r\sin(\theta))^4 + 2(r\cos(\theta))^2(r\sin(\theta))^2$

$+ 2(r\cos(\theta))^3 + 2(r\cos(\theta))(r\sin(\theta))^2 - (r\sin(\theta))^2 = 0$

$\Rightarrow r^4(\cos^4(\theta) + \sin^4(\theta) + 2\cos^2(\theta)\sin^2(\theta))$

$+ r^3(2\cos^3(\theta) + 2\cos(\theta)\sin^2(\theta))$

$- r^2\sin^2(\theta) = 0$

$\cos^4(\theta) + 2\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$
$= (\cos^2(\theta) + \sin^2(\theta))^2$

$a^4 + 2a^2b^2 + b^4 = (a^2+b^2)^2$

$u^2 + 2uv + v^2 = (u+v)^2$

$\Rightarrow r^4(\cos^2(\theta) + \sin^2(\theta))^2$

$+ 2r^3\cos(\theta)(\cos^2(\theta) + \sin^2(\theta))$

$- r^2\sin^2(\theta) = 0$

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\Rightarrow r^4(1)^2 + 2r^3\cos(\theta)(1) - r^2\sin^2(\theta) = 0$

$\Rightarrow r^4 + 2r^3\cos(\theta) - r^2\sin^2(\theta) = 0$

$\Rightarrow r^2(r^2 + 2r\cos(\theta) - \sin^2(\theta)) = 0$

$\sin^2(\theta) = 1 - \cos^2(\theta)$

$\Rightarrow r^2(r^2 + 2r\cos(\theta) - 1 + \cos^2(\theta)) = 0$

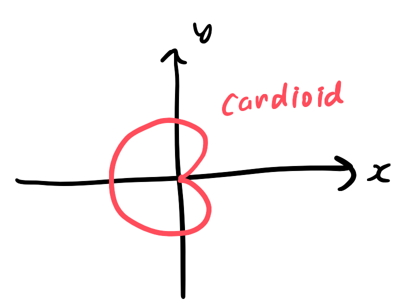
$\Rightarrow r^2[(r^2 + 2r\cos(\theta) + \cos^2(\theta)) - 1] = 0$

$\Rightarrow r^2[(r + \cos(\theta))^2 - 1^2] = 0$

$\Rightarrow r^2[(r + \cos(\theta) + 1)(r + \cos(\theta) - 1)] = 0$

$abc = 0$
$a = 0$ OR $b = 0$ OR $c = 0$

$r = 0$ OR $r = -1 - \cos(\theta)$ OR $r = 1 - \cos(\theta)$



3.4.3 Cartesian to Polar (Points), Polar to Cartesian (Equations)

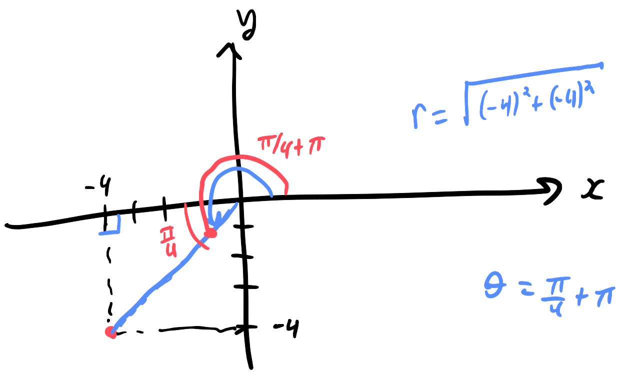
Expressing Polar Values in Terms of Cartesian Values

Solving the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ for r and θ gives

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan(\theta) = \dfrac{y}{x} \end{cases}$$

Where you solve for them explicitly depending on the "*Branch*" (this means what region the angle in the Cartesian plane: $0 \le \theta < 2\pi$, $2\pi \le \theta < 4\pi$, etc... and whether you use a positive or negative radius).

Example: Convert $(x, y) = (-4, -4)$ to polar coordinates.



$r = \sqrt{(-4)^2 + (-4)^2} = \sqrt{16+16}$

$= \sqrt{16}\sqrt{1+1}$

$= 4\sqrt{2}$

ref angle

$\tan(\theta) = \frac{y}{x} = \frac{-4}{-4}$

$= 1$ in QIII

$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

so $(r, \theta) = \left(4\sqrt{2}, \frac{5\pi}{4}\right)$ //

Example: Convert the equation $r = -8\cos(\theta)$ to Cartesian.

$\overset{\times r}{\Rightarrow}$ $r^2 = -8r\cos(\theta)$

$\Rightarrow x^2 + y^2 = -8r\cos(\theta) = -8x$

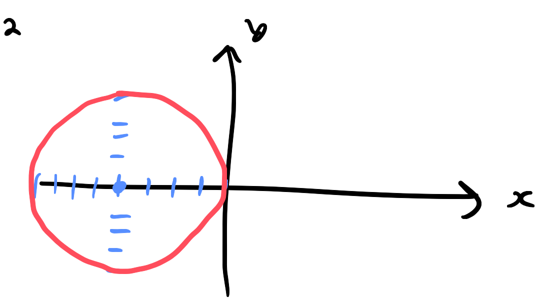
$\therefore x^2 + y^2 = -8x$

$\Rightarrow x^2 + 8x + y^2 = 0$

$\Rightarrow (x^2 + 8x + 16 - 16) + y^2 = 0$

$\Rightarrow (x+4)^2 - 16 + y^2 = 0$

$\Rightarrow (x+4)^2 + y^2 = 16 = 4^2$



3.4.4 Graphing Sinusoidal Curves

We will be talking about graphing polar curves from their equations in the next section. To do so, we will need to review how to graph sinusoidal curves.

<u>Review</u>:

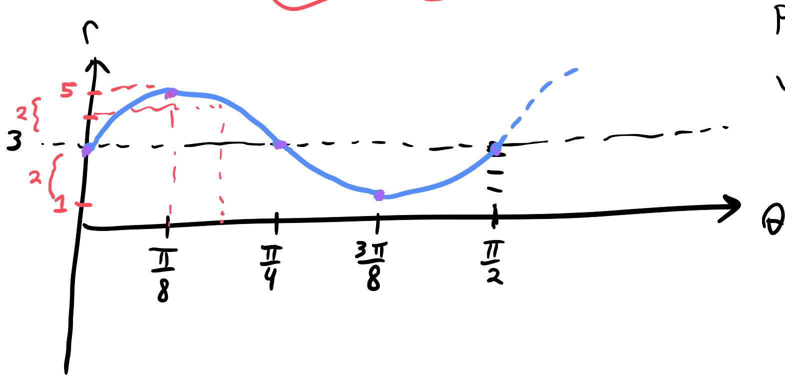
$$r = A\sin(B\theta + C) + D \qquad \text{or} \qquad r = A\cos(B\theta + C) + D$$

- The amplitude is $|A|$ (height of the curve is $2|A|$).
- The period is $\dfrac{2\pi}{|B|}$ (length of one wave).
- The phase shift is $-C/B$ (horizontal shift).
- The vertical shift is D (vertical translation).

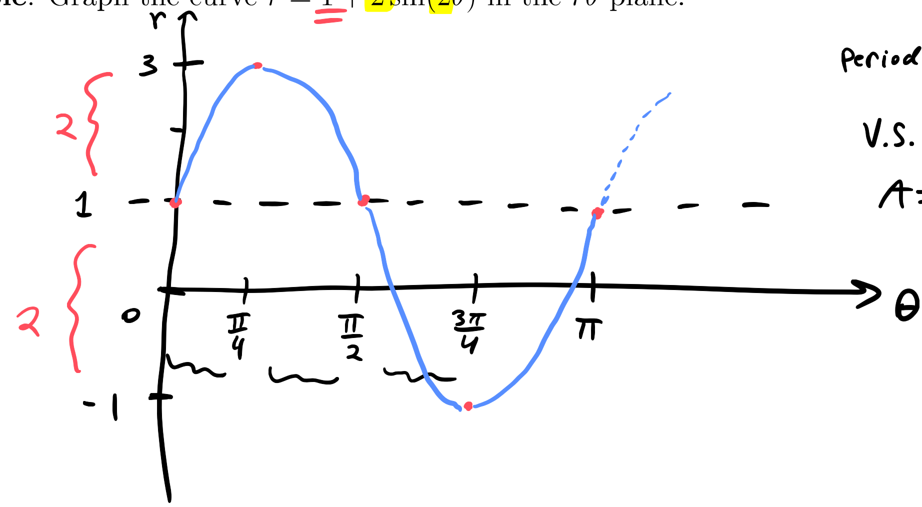
Example: Graph the curve $r = 2\sin(4\theta$ [redacted]$) + 3$ in the $r\theta$-plane.

Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

V.S. $= 3$



Example: Graph the curve $r = 1 + 2\sin(2\theta)$ in the $r\theta$-plane.



Period $= \frac{2\pi}{2} = \pi$

V.S. $= 1$

$A = 2$

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3.5 (Section 11.4) Graphing Polar Curves in the Cartesian Plane

3.5.1 Plotting Curves in the xy-Plane Based on Their $r\theta$-Graph

When graphing functions of the form $r = f(\theta)$ in the Cartesian plane, the best way is to follow along by graphing it initially in the $r\theta$-plane and then the xy-plane. This is best seen by example.

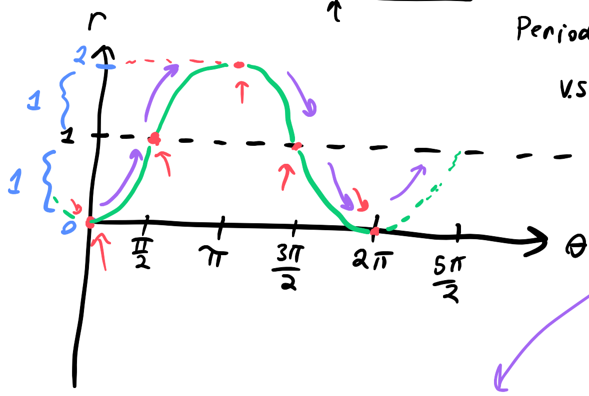
Example: Graph $r = 1 - \cos(\theta)$ in the Cartesian plane.

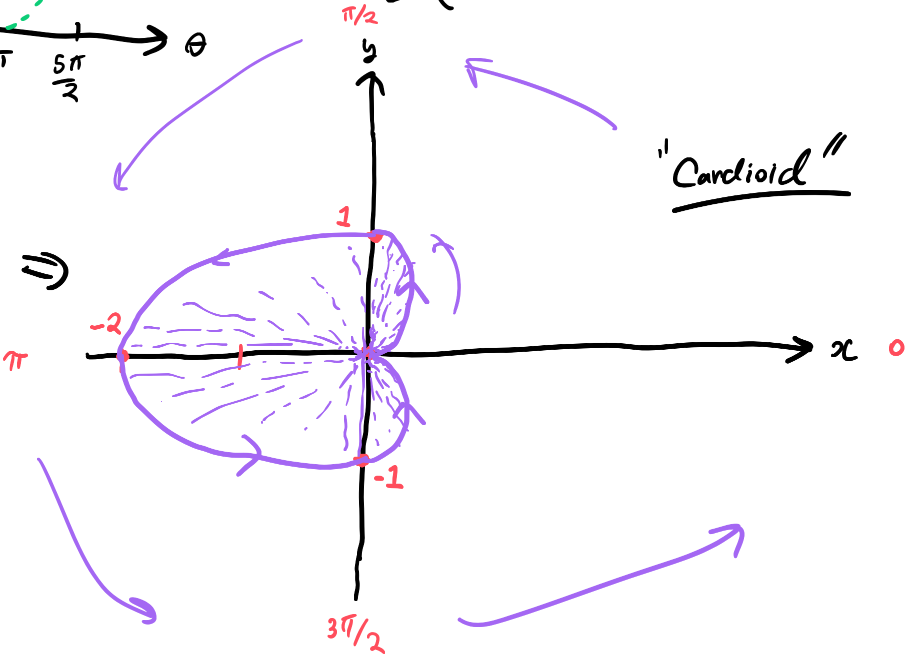
$\cos(\theta - \frac{\pi}{2}) = \sin(\theta)$

Period $= \frac{2\pi}{1} = 2\pi$

V.S. $= 1$

$A = -1 \rightarrow$ flip



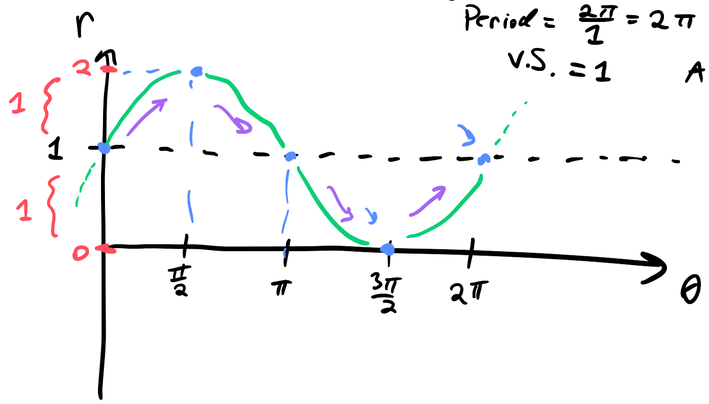


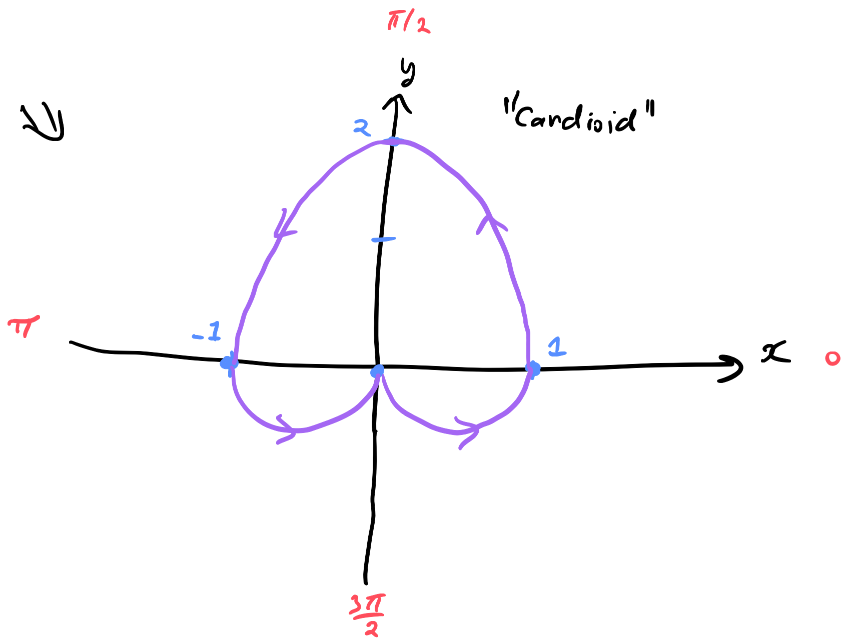
"Cardioid"

Example: Graph $r = 1 + \sin(\theta)$ in the Cartesian plane.

Period $= \frac{2\pi}{1} = 2\pi$

V.S. $= 1$ $\quad A = 1$



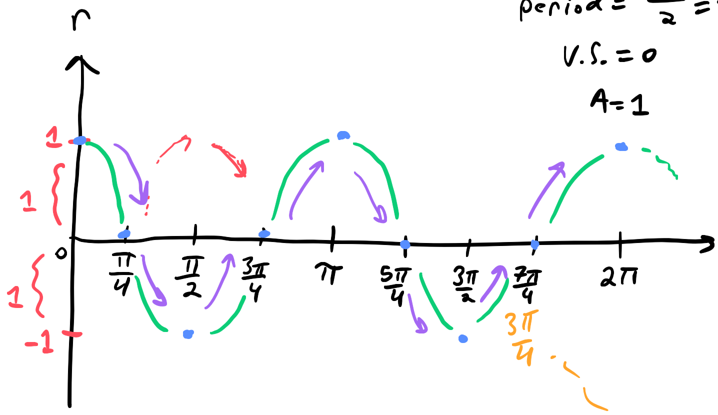


Example Graph $r = \cos(2\theta)$ in the Cartesian plane.

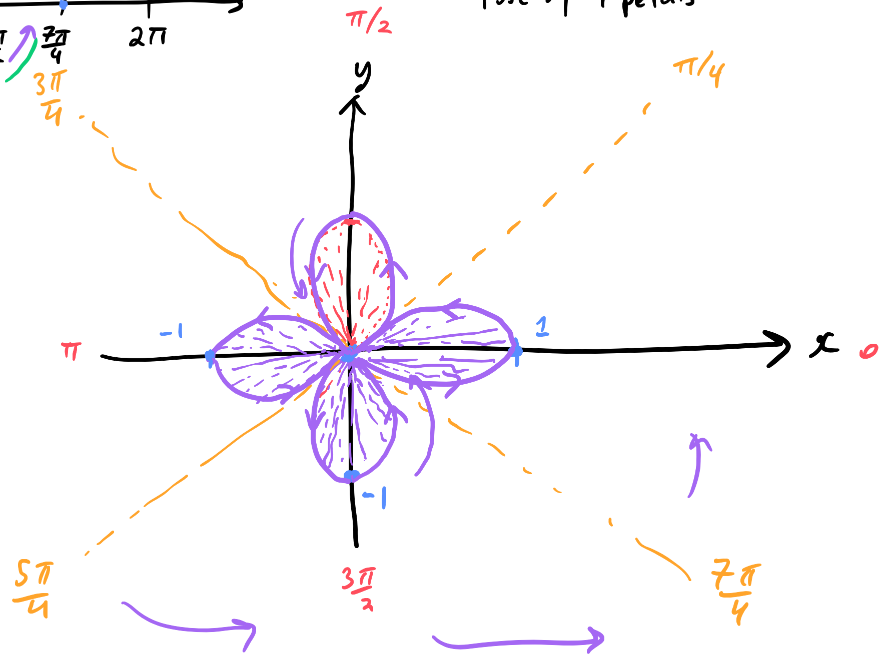
$\text{period} = \frac{2\pi}{2} = \pi$

V.S. $= 0$

$A = 1$



"rose of 4 petals"



Example: Graph $r = 1 + 2\sin(\theta)$ in the Cartesian plane.

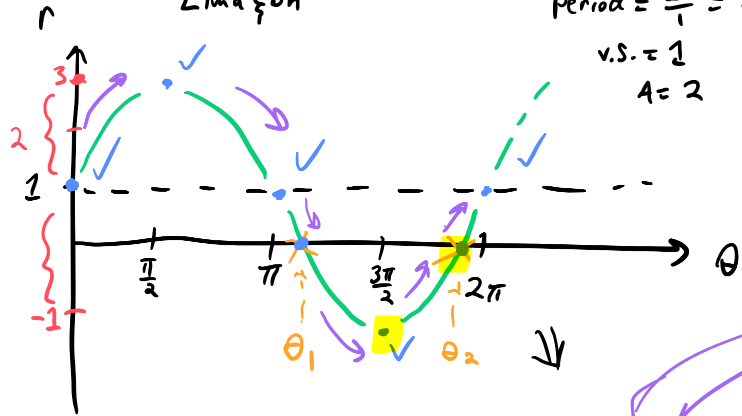
$\Rightarrow$ page 181 $\Leftarrow$

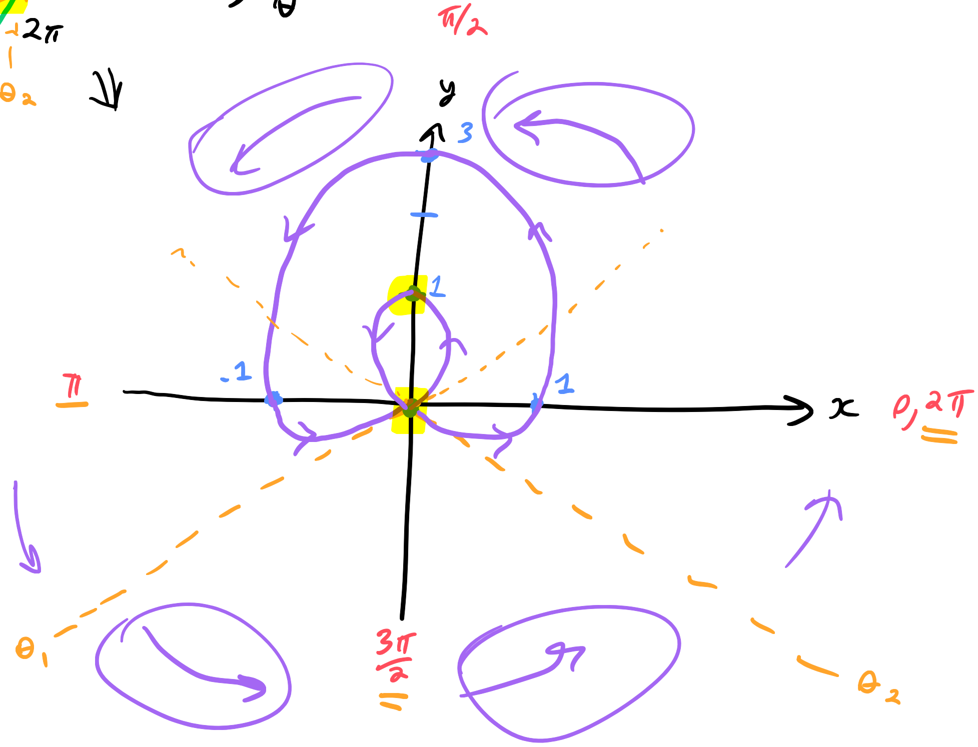
"Limaçon"

Period $= \frac{2\pi}{1} = 2\pi$

v.s. $= 1$

$A = 2$





Example: Graph $r^2 = 4\cos(2\theta)$ in the Cartesian plane.

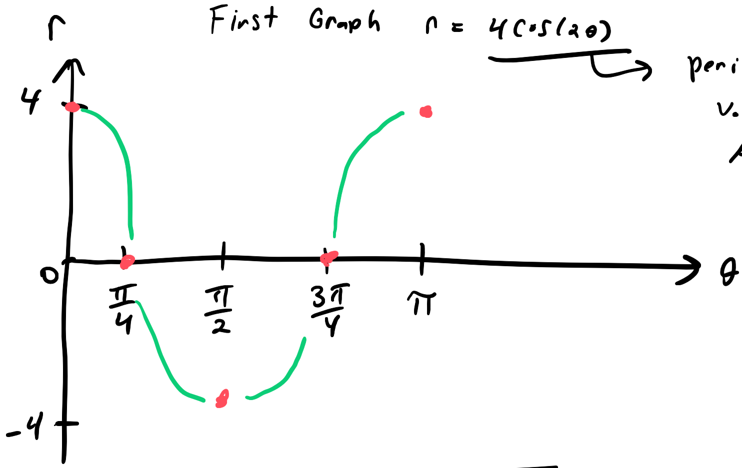
$$r = \pm 2\sqrt{\cos(2\theta)}$$

First Graph $r = 4\cos(2\theta)$

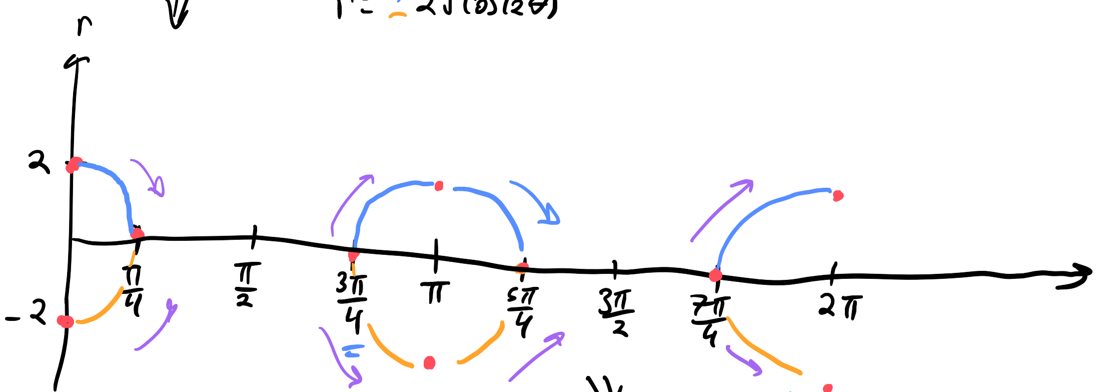
period $= \frac{2\pi}{2} = \pi$

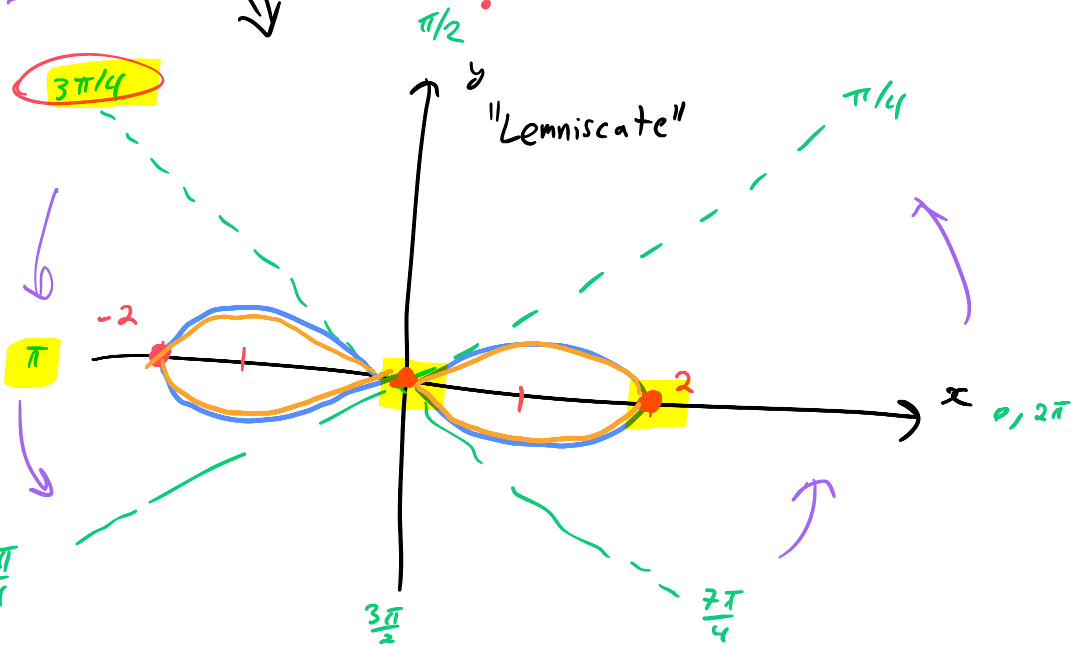
v.s. $= 0$

$A = 4$



$$r = \pm 2\sqrt{\cos(2\theta)}$$





3.5.2 Graphing Polar Curves Using GeoGebra

Using Geogebra Classic 6, there is a way to plot polar curves of the form $r = f(\theta)$ with ease. Below is the command that you use to obtain this.

GeoGebra Polar Graphing

To graph the curve $r = f(\theta)$ over the region $a \leq \theta \leq b$ use the command

```
Curve[(f(t);t),t,a,b]
```

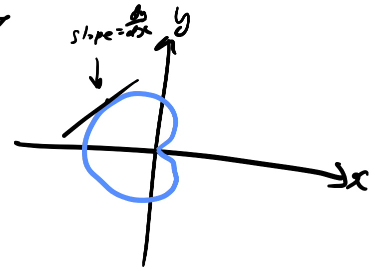
Note that the semicolon is important!

Example: Use GeoGebra Classic 6 to graph the polar curve $r(\theta) = \sin(\theta) + \sin^3(5\theta/2)$.

See video

$\Rightarrow$ page 184 $\Leftarrow$

3.5.3 Calculus in Polar Coordinates



We can use $r = f(\theta)$ to describe a curve parametrically (in terms of a parameter θ) as

$$\begin{cases} x = r\cos(\theta) = f(\theta)\cos(\theta) \\ y = r\sin(\theta) = f(\theta)\sin(\theta) \end{cases}$$

From this use $y' = \dfrac{\dot{y}}{\dot{x}} = \dfrac{dy/d\theta}{dx/d\theta}$ to obtain the following formula for the slope.

Theorem

Let $r = f(\theta)$ be a differentiable curve in polar coordinates. Then the Cartesian slope of the curve at (r, θ) is given by the following.

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

You can also apply the same logic to finding the arc-length by creating a parametrization but it always simplifies to the following formula for a curve $r = f(\theta)$ and is given by the following.

Theorem

Let $r = f(\theta)$ be a differentiable curve in polar coordinates. Then the Arc-Length of the curve over $\alpha \le \theta \le \beta$ is given by

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Cartesian

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Compute the slope of the tangent line to $r = 1 + 2\cos(\theta)$ at $\theta = \pi/4$.

"Limaçon"

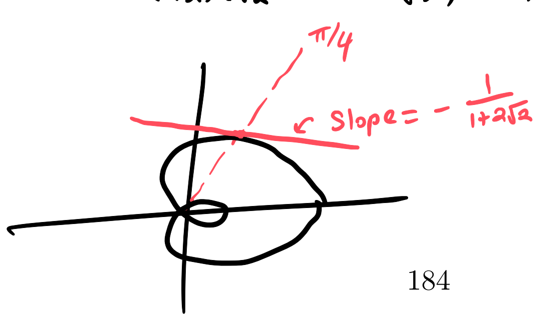
$$\boxed{L = \int_{\theta_1}^{\theta_2} \sqrt{1 + (\ddot{o})^2}}$$

$$\begin{matrix} x = r\cos(\theta) \\ y = r\sin(\theta) \end{matrix} \xrightarrow[r = 1+2\cos(\theta)]{\text{convert to parametric}} \begin{cases} x = (1+2\cos(\theta))\cos(\theta) \\ y = (1+2\cos(\theta))\sin(\theta) \end{cases}$$

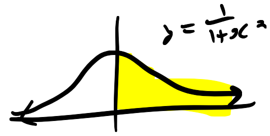
$$\frac{dx}{d\theta} = (-2\sin(\theta)\cos(\theta)) - (1+2\cos(\theta))\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-2\sin^2(\theta) + (1+2\cos(\theta))\cos(\theta)}{-2\sin(\theta)\cos(\theta) - (1+2\cos(\theta))\sin(\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{-2\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{-2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(1+\frac{2}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)} = \frac{-2\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}} + \cancel{\frac{2}{2}}}{-\frac{2}{2} - \frac{1}{\sqrt{2}} - \frac{2}{2}}$$

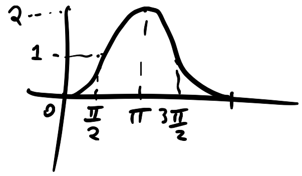


$$= \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} - 2} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{-1-2\sqrt{2}} = -\frac{1}{1+2\sqrt{2}}$$

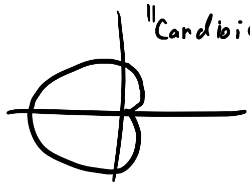


$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

Example: Find the length of the curve $r = 1 - \cos(\theta)$ for $0 \le \theta \le 2\pi$.



"Cardioid"



$$\frac{dr}{d\theta} = 0 - (-\sin(\theta))$$
$$= \sin(\theta)$$

So $L = \displaystyle\int_0^{2\pi} \underbrace{\sqrt{(1-\cos(\theta))^2 + (\sin(\theta))^2}}_{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}\,d\theta$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos(\theta) + \underbrace{\cos^2(\theta) + \sin^2(\theta)}_{=1}}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos(\theta)}\,d\theta$$

section 8.3

case of $\sqrt{1-\cos(n\theta)}$

$$= \int_0^{2\pi} \sqrt{2}\cdot\sqrt{1-\cos(\theta)}\,d\theta$$

use $\sin^2(A) = \frac{1-\cos(2A)}{2}$

set $A = \frac{\theta}{2}$

$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos(\theta)$

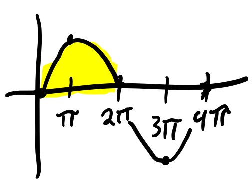
$$= \int_0^{2\pi} \sqrt{2}\,\sqrt{2\sin^2\left(\tfrac{\theta}{2}\right)}\,d\theta$$

$\sin\left(\frac{\theta}{2}\right) \ge 0$ in $[0, 2\pi]$

$$= 2\int_0^{2\pi} \left|\sin\left(\tfrac{\theta}{2}\right)\right|d\theta$$

$\sin\left(\frac{\theta}{2}\right) = r$

period $= \frac{2\pi}{1/2} = 4\pi$



$$= 2\int_0^{2\pi} \sin\left(\tfrac{\theta}{2}\right)d\theta$$

$$= -\frac{2}{1/2}\cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi}$$

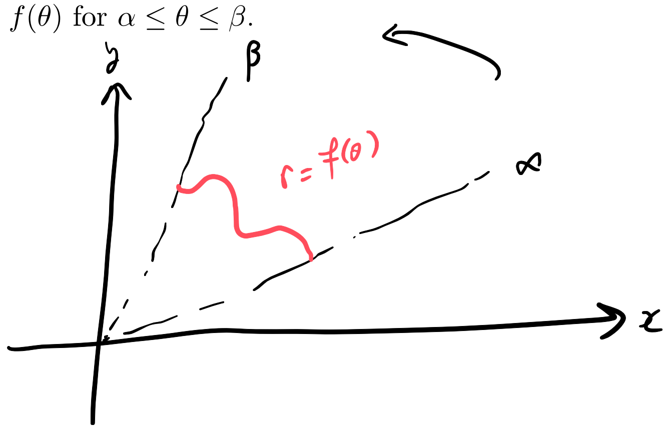
$$= -4\left(\cos(\pi) - \cos(0)\right)$$

$$= -4(-1-1) = 8$$

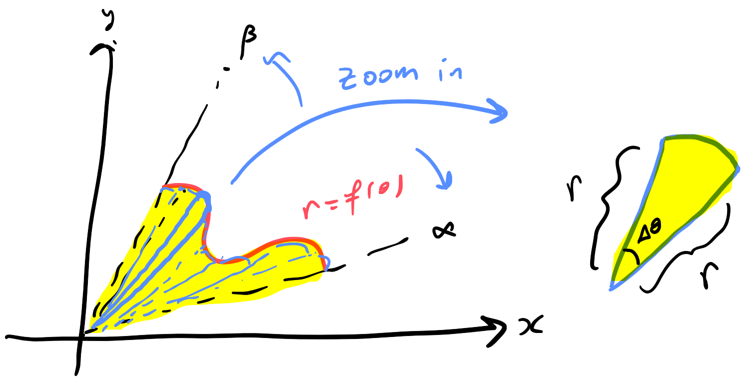
3.6 (Section 11.5) Area Trapped by Polar Curves

3.6.1 Formulating the Wedged Area Trapped by Polar Curves

Consider a curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$.

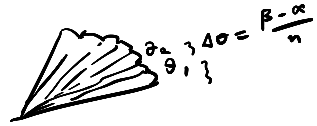


To find the area trapped above we take wedges instead of rectangles for simplicity.



sector

$$A = \frac{1}{2} r^2 \Delta\theta$$



We subdivide the angles into equal widths $\Delta\theta_n = \dfrac{\beta-\alpha}{n}$ each with a radius of $r_k = f(\theta_k)$ for some θ_k in each subregion. The formula for the area of each sector is

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_n = \frac{1}{2} f(\theta_k)^2 \Delta\theta_n$$

Then we take $n \to \infty$ and add them up to obtain

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta$$

Riemann Sum

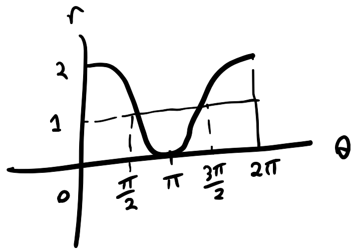
$$A \approx \sum_{k=1}^{n} \frac{1}{2} f(\theta_k)^2 \Delta\theta_n$$

$$= \sum_{k=1}^{n} \frac{1}{2} f\left(\alpha + k\frac{\beta-\alpha}{n}\right)^2 \frac{\beta-\alpha}{n}$$

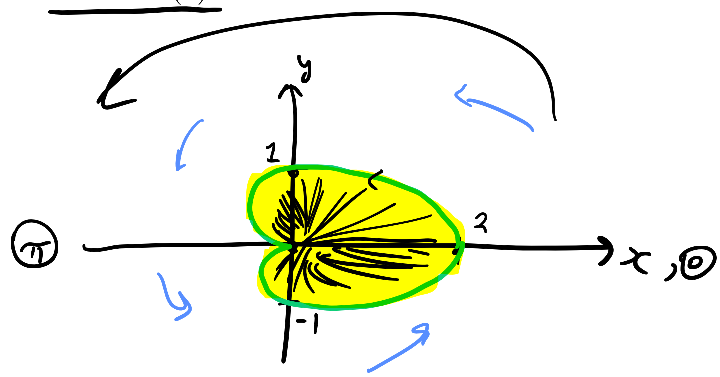
$$A = \lim_{n\to\infty} (\text{above})$$

3.6.2 Examples of Computing Area

Example: Compute the area trapped inside of $r = 1 + \cos(\theta)$.



$\Rightarrow$



$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos(\theta))^2 \, d\theta \overset{\text{By symmetry}}{=} 2 \times \left(\frac{1}{2} \int_0^{\pi} (1+\cos(\theta))^2 \, d\theta \right)$$

$$= \int_0^{\pi} (1 + 2\cos(\theta) + \cos^2(\theta))\, d\theta \quad \leftarrow \text{Section 8.3}$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$= \int_0^{\pi} \left(\frac{3}{2} + 2\cos(\theta) + \frac{1}{2}\cos(2\theta) \right) d\theta$$

$$= \frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{4}\sin(2\theta) \Big|_0^{\pi}$$

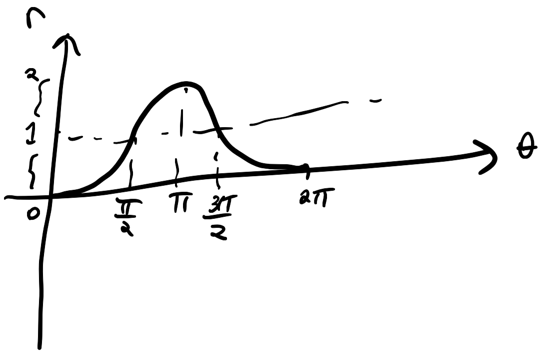
$$= \left[\frac{3}{2}\pi + 2\cancelto{0}{\sin(\pi)} + \frac{1}{4}\cancelto{0}{\sin(2\pi)} \right] - \left[\cancelto{0}{\frac{3}{2}(0)} + 2\cancelto{0}{\sin(0)} + \frac{1}{4}\cancelto{0}{\sin(0)} \right]$$

$$= \frac{3\pi}{2}$$

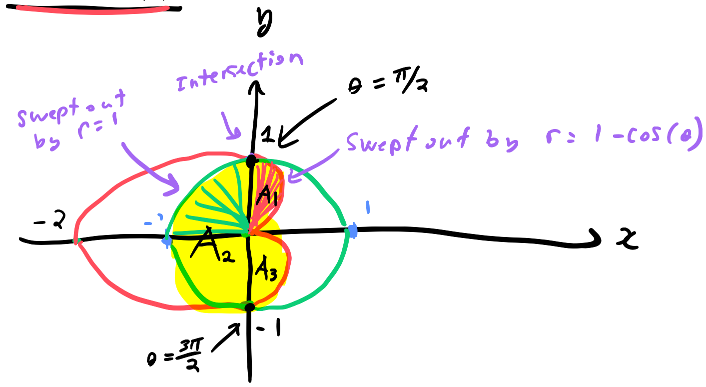
Cardioid $\begin{cases} a \pm a\cos(\theta) \\ \text{or } a \pm a\sin(\theta) \end{cases}$ $\Rightarrow$ Page 188 $\Leftarrow$

Example: Find the area trapped inside $r = 1$ and $r = 1 - \cos(\theta)$.

unit circle



$\Rightarrow$



when does the change occur? $\cancel{1} - \cos(\theta) = \cancel{1}$

$\Rightarrow \cos(\theta) = 0 \quad \Rightarrow \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

(Direct) Method 1:

$$A = \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(\theta))^2 d\theta}_{A_1} + \underbrace{\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1)^2 d\theta}_{A_2} + \underbrace{\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (1-\cos(\theta))^2 d\theta}_{A_3}$$

(Symmetry) Method 2:

$$2 \times \left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(\theta))^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1)^2 d\theta \right)$$

quarter of unit circle

Method 3: 

$$2 \times \left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(\theta))^2 d\theta + \frac{\pi}{4} \right)$$

$$= \int_0^{\frac{\pi}{2}} (1 - 2\cos(\theta) + \cos^2(\theta))\, d\theta + \frac{\pi}{2}$$

$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

$$= \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos(\theta) + \frac{1}{2}\cos(2\theta) \right) d\theta + \frac{\pi}{2}$$

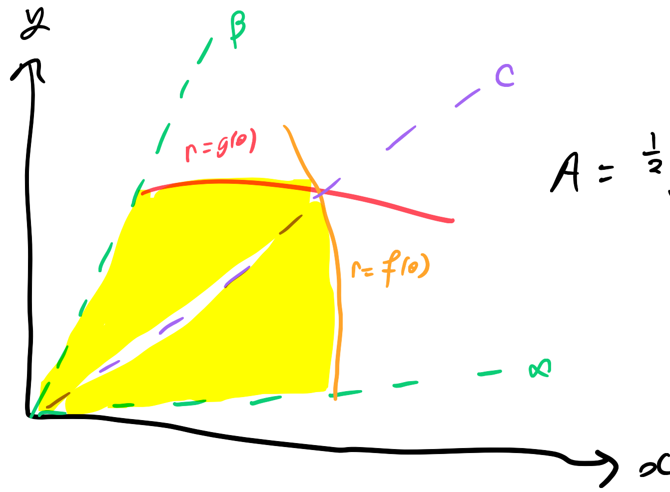
$$= \left(\frac{3}{2}\theta - 2\sin(\theta) + \frac{1}{4}\sin(2\theta) \right)\Big|_0^{\frac{\pi}{2}} + \frac{\pi}{2}$$

$$= \frac{3\pi}{4} - 2\underline{\sin\left(\tfrac{\pi}{2}\right)} + \frac{1}{4}\overset{0}{\cancel{\sin(\pi)}} - (0 - 0 + 0) + \frac{\pi}{2}$$

$$= \frac{3\pi}{4} - 2 + \frac{\pi}{2} = \frac{5\pi}{4} - 2$$

3.6.3 Splits in Integration

When integrating between two objects splits are a common occurence



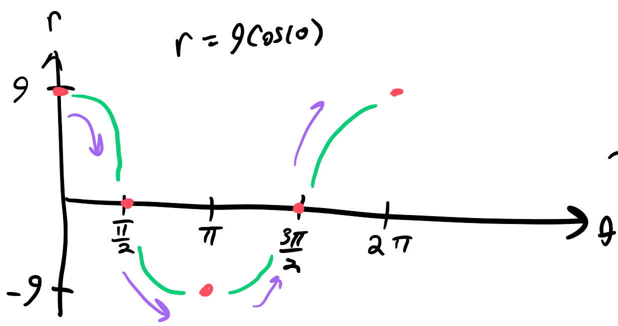
$$A = \frac{1}{2}\int_{\alpha}^{c} f(\theta)^2 d\theta + \frac{1}{2}\int_{c}^{\beta} g(\theta)^2 d\theta$$

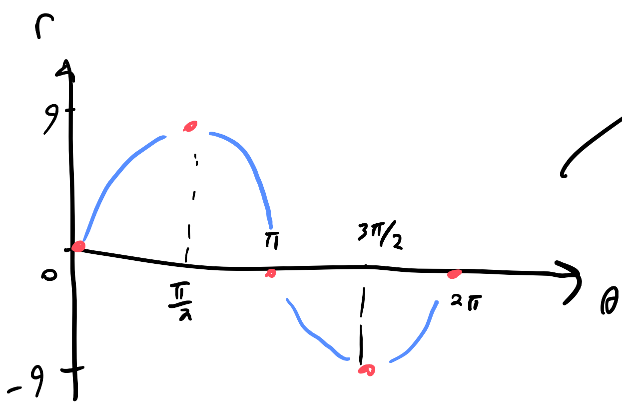
The behaviour changes at $\theta = c$.

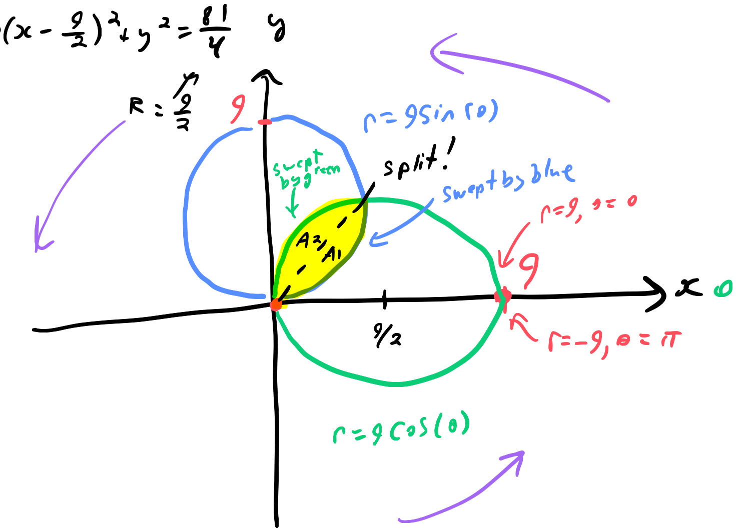
$$r^2 = 9r\cos(\theta)$$
$$x^2 + y^2 = 9x$$

Example: Find the area common to the curves $r = 9\cos(\theta)$ and $r = 9\sin(\theta)$.

$$x^2 - 9x + y^2 = 0$$
$$\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} + y^2 = 0$$
$$\Rightarrow \left(x - \frac{9}{2}\right)^2 + y^2 = \frac{81}{4}$$

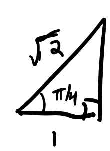






Intersection $\Rightarrow$ $9\sin(\theta) = 9\cos(\theta)$

$$\Rightarrow \sin(\theta) = \cos(\theta)$$
$$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = 1$$
$$\Rightarrow \tan(\theta) = 1$$
$$\Rightarrow \theta = \frac{\pi}{4}$$



Method 1: $A = A_1 + A_2 = \frac{1}{2}\int_0^{\frac{\pi}{4}} (9\sin(\theta))^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (9\cos(\theta))^2 d\theta$

 $=$  $\times 2$

Method 2:

$$A = 2\times A_1 \text{ or } 2\times A_2 = 2\times\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} 81\sin^2(\theta)\,d\theta\right)$$

$$= 81\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{81}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/4}$$

$$= \frac{81}{2}\left\{\frac{\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - 0 + 0\right\}$$

$$= \frac{81}{2}\left\{\frac{\pi}{4} - \frac{1}{2}\right\} = \frac{81}{2}\cdot\frac{\pi-2}{4}$$

$$= \frac{81(\pi-2)}{8} \,//$$

$\Rightarrow$ FIN $\Leftarrow$