Angular Momentum

Key observation:
- Where a force is exerted on an extended object &
- What force was

Both impact how object spins.

Consider a point mass, subject to $\vec{F}_{net}$, mass is at $\vec{r}$

2nd law $\quad \vec{F}_{net} = \frac{d}{dt}\vec{p}$

$$\vec{r} \times \vec{F}_{net} = \vec{r} \times \left(\frac{d}{dt}\vec{p}\right)$$

$$\vec{\tau} = \vec{r} \times \left(\frac{d}{dt}\vec{p}\right)$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

$$f(x)\left(\frac{d}{dx}g(x)\right) = \frac{d}{dx}\left(f(x)g(x)\right) - \left(\frac{d}{dx}f(x)\right)g(x)$$

(see integration by parts)

$$f(t)\left(\frac{d}{dt}g(t)\right) = \frac{d}{dt}\left(f(t)g(t)\right) - \left(\frac{d}{dt}f(t)\right)g(t)$$

$$\vec{f}(t)\times\left(\frac{d}{dt}\vec{g}(t)\right) = \frac{d}{dt}\left(\vec{f}(t)\times\vec{g}(t)\right) - \left(\frac{d}{dt}\vec{f}(t)\right)\times\vec{g}(t)$$

$$\vec{\tau} = \vec{r}\times\left(\frac{d}{dt}\vec{p}\right)$$

$$= \frac{d}{dt}\left(\vec{r}\times\vec{p}\right) - \cancel{\underbrace{\left(\frac{d}{dt}\vec{r}\right)\times\vec{p}}_{\substack{0! \\ \vec{v}\times(m\vec{v})}}}$$

$$\vec{\tau} = \frac{d}{dt}\left(\vec{r}\times\vec{p}\right)$$

For a point particle

$$\vec{L} = \vec{r}\times\vec{p}$$

$\vec{L}$ = "angular momentum"

units: $kg\,m^2/s$

measured around origin

Will turn out to express "rotation"

8-3-Example-AM1

Angular Momentum - I

A ball of mass $3kg$ travels with constant velocity $3\frac{m}{s}\hat{\imath} + 2\frac{m}{s}\hat{k}$. At time $t = 0s$ it is at $2m\hat{\imath}$.

- What is the angular momentum measured around the origin? *
- What is the angular momentum measured around $-1m\hat{\imath} - 2m\hat{k}$?
- What is the angular momentum measured around $1m\hat{\imath} + 2m\hat{\jmath}$?

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{p} = m\vec{v}$

from location we measure $\vec{L}$ around to object

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0[t - 0s]$$
$$= 2m\hat{\imath} + (3\,m/s\,\hat{\imath} + 2\,m/s\,\hat{k})t$$

$$\vec{p}(t) = m\vec{v}(t)$$
$$= (3kg)(3\,m/s\,\hat{\imath} + 2\,m/s\,\hat{k})$$
$$= 9\,kg\,m/s\,\hat{\imath} + 6\,kg\,m/s\,\hat{k}$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= \left[2\,\text{m}\,\hat{i} + \left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)t\right] \times \left[(3\,\text{kg})\left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)\right]$$

$$= 2\,\text{m}\,\hat{i} \times \left[(3\,\text{kg})\left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)\right]$$
$$+ \left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)t \times \left[(3\,\text{kg})\left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)\right]$$

$$= 0 + \left(-12\,\text{kg}\,\tfrac{\text{m}^2}{\text{s}}\,\hat{j}\right)$$
$$+ (3\,\text{kg})t\left[0 + \cancel{6\,\tfrac{\text{m}^2}{\text{s}^2}\,\hat{j} + 6\,\tfrac{\text{m}^2}{\text{s}^2}(-\hat{j})} + 0\right]$$

$$= (\vec{r}_0 + \vec{v}_0 t) \times (m\vec{v}_0)$$

$$= -12\,\text{kg}\,\tfrac{\text{m}^2}{\text{s}}\,\hat{j}$$

$\vec{L}$

around $-1\text{m}\,\hat{i} - 2\text{m}\,\hat{k}$

$$= \left[\underbrace{2\,\text{m}\,\hat{i} + \left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)t}_{\vec{r}(t)} - \left(-1\,\text{m}\,\hat{i} - 2\,\text{m}\,\hat{k}\right)\right] \times$$
$$\times \left[(3\,\text{kg})\left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)\right]$$

$$= \left(3\,\text{m}\,\hat{i} + 2\,\text{m}\,\hat{k}\right) \times \left[(3\,\text{kg})\left(3\,\tfrac{\text{m}}{\text{s}}\,\hat{i} + 2\,\tfrac{\text{m}}{\text{s}}\,\hat{k}\right)\right]$$

$$= 0$$

$$\vec{L}_{\text{around } 1m\hat{i} + 2m\hat{j}} = \left[2m\hat{i} + \left(3\tfrac{m}{s}\hat{i} + 2\tfrac{m}{s}\hat{k}\right)t - \left(1m\hat{i} + 2m\hat{j}\right)\right] \times \left[\left(9\,kg\tfrac{m}{s}\hat{i} + 6\,kg\tfrac{m}{s}\hat{k}\right)\right]$$

$$= \left[1m\hat{i} - 2m\hat{j}\right] \times \left[9\,kg\tfrac{m}{s}\hat{i} + 6\,kg\tfrac{m}{s}\hat{k}\right]$$

$$= 0 + \left(-6\,kg\tfrac{m^2}{s}\hat{j}\right) + \left(18\,kg\tfrac{m^2}{s}\hat{k}\right) + \left(-12\,kg\tfrac{m^2}{s}\hat{i}\right)$$

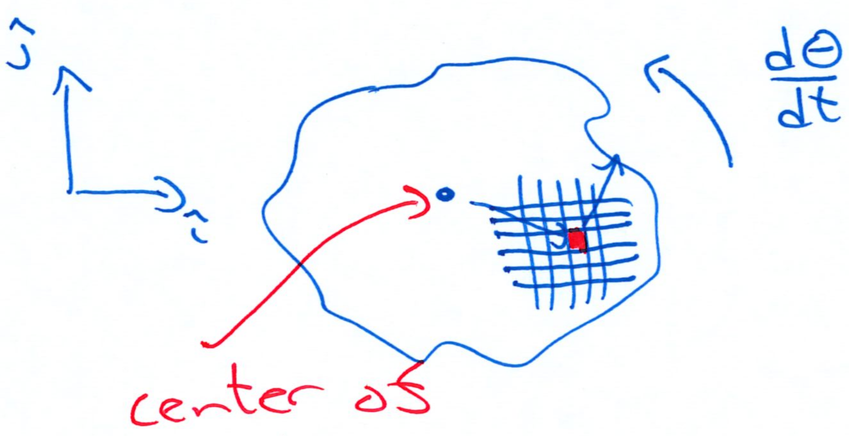
Angular momentum

For point particle is $\vec{r} \times \vec{p}$

What is $\vec{L}$ for extended rotating object?

[Only considering objects going around their axis of symmetry]

(In this case $\vec{L}$ lines up with rotation axis)



What is $\vec{L}$ measured around here?

Imaging extended object broken into lots of pieces (small enough to pretend they are point objects)

- For each piece, calculate $\vec{r}$

  For each piece, calculate $\vec{v}$
- For each piece, find mass (dm)

  this gives $\vec{p}$
- Calulate $\vec{L}$ for each piece
- Add it up.

$$\vec{L}_{total} = \sum_{\substack{all\\pieces}} \vec{r}_{\substack{for\\piece}} \times (dm\, \vec{v})$$

$\vec{r}$ at 90° to $\vec{v}$

$$|\vec{r} \times \vec{v}| = |\vec{r}||\vec{v}|$$

$$= \sum_{\substack{all\\pieces}} dm\, |\vec{r}||\vec{r}\frac{d\theta}{dt}| \text{ (along axis of rot}^n\text{)}$$

$$= \sum_{\substack{all\\pieces}} (dm)|\vec{r}|^2 \left|\frac{d\theta}{dt}\right| \text{ (along axis of rot}^n\text{)}$$

$$= \left|\frac{d\theta}{dt}\right| \left\{ \sum_{\substack{all\\pieces}} dm\, |\vec{r}|^2 \right\} \text{ (along axis of rot}^n\text{)}$$

$$I = \sum_{\substack{\text{all}\\\text{bits}}} dm\,|\vec{r}|^2$$

$$= \int dm\,|\vec{r}|^2$$

"Moment of inertia"
Combo of mass &
how spread out it is.