Math 122 Assignment 5 Solution Ideas

1. (a) $(1331)_b = b^3 + 3b^2 + 3b + 1 = (b+1)^3$, and since b is an integer, we have the cube of an integer. Now for a valid base we need $b > 1$, and also since we have a digit of 3 we need $b > 3$. So for $b \geq 4$ we have that $(1331)_b$ is the cube of an integer.

   (b)

$$\begin{aligned}
(31)_b \times (14)_b &= (464)_b \\
(3b+1)(b+4) &= 4b^2 + 6b + 4 \\
3b^2 + 13b + 4 &= 4b^2 + 6b + 4 \\
b^2 - 7b &= 0 \\
b(b-7) &= 0
\end{aligned}$$

   Therefore $b = 0$ or $b = 7$. Since we can't have a base of 0, we have that $b = 7$ (and note that this is OK since all our digits are less than 7).

   (c) Say $(56)_b = n^2$ and $(66)_b = (n+1)^2$. Then $(5b+6) = n^2$ and $(6b+6) = (n+1)^2 = n^2 + 2n + 1$. Subtract the first equation from the second to obtain $b = 2n+1$. Sub this expression for b into the equation $5b + 6 = n^2$ to obtain $n^2 = 5(2n+1) + 6 = 10n + 11$, which rearranges to $n^2 - 10n - 11 = 0$. Factoring, we get $(n-11)(n+1) = 0$, from which we have $n = 11$ or $n = -1$. Since we only want positive values for our base, this means that $n = 11$ and so $b = 23$ (and note that this is OK since all our digits are less than 23).

   (d) We set the base 5 and base 7 expansions equal to each other and solve for x:

$$\begin{aligned}
(24x3)_5 &= (x02x)_7 \\
2(5^3) + 4(5^2) + 5x + 3 &= 7^3x + 0 + 2(7) + x \\
250 + 100 + 5x + 3 &= 343x + 14 + x \\
339 &= 339x \\
1 &= x
\end{aligned}$$

2. (a) Note that there is more than one way to prove our statement. Here is one method: Here we proceed by a proof by induction. We wish to prove the statement $S(n) : 6|n^3+5n$ for all $n \geq 1$. Basis: When $n = 1$ we have $n^3 + 5n = 1 + 5(1) = 6$ and $6|6$.
   Induction Hypothesis: Suppose that $6|k^3 + 5k$ for some integer $k \geq 1$.
   Induction Step: We want to show that $6|(k+1)^3 + 5(k+1)$.

$$\begin{aligned}
(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\
&= k^3 + 3k^2 + 8k + 6 \\
&= k^3 + 5k + 3k^2 + 3k + 6
\end{aligned}$$

   Now $6|k^3 + 5k$ by the induction hypothesis, and $6|6$, so we now turn our attention to $3k^2 + 3k = 3(k^2 + k)$. If k is an even integer we have that $k^2 + k$ is also an even integer, and if k is an odd integer we have that $k^2 + k$ is an even integer. Therefore $k^2 + k$ is

even in all cases and so $2|(k^2+k)$. Thus there exists $l \in \mathbb{Z}$ such that $k^2+k=2l$, and so $3k^2+3k=3(k^2+k)=3(2l)=6l$. Since $l \in \mathbb{Z}$ we have that $6|3k^2+3k$.
Now with $6|k^3+5k$, $6|3k^2+3k$, and $6|6$ we have that $6|k^3+5k+3k^2+3k+6$ and so $6|(k+1)^3+5(k+1)$.
Conclusion: Therefore, by induction, we have that $6|n^3+5n$ for all $n \geq 1$.

Another way to prove our statement would be to use modular arithmetic and work modular 6. Here we proceed by cases:
Our value n can be congruent to any of 0, 1, 2, 3, 4, 5 (mod 6).

- If $n \equiv 0 \pmod 6$ we have $n^3+5n \equiv 0+5(0) \equiv 0 \pmod 6$.
- If $n \equiv 1 \pmod 6$ we have $n^3+5n \equiv 1+5(1) \equiv 6 \equiv 0 \pmod 6$.
- If $n \equiv 2 \pmod 6$ we have $n^3+5n \equiv 2^3+5(2) \equiv 8+10 \equiv 18 \equiv 0 \pmod 6$.
- If $n \equiv 3 \pmod 6$ we have $n^3+5n \equiv 3^3+5(3) \equiv 27+15 \equiv 42 \equiv 0 \pmod 6$.
- If $n \equiv 4 \pmod 6$ we have $n^3+5n \equiv 4^3+5(4) \equiv 64+20 \equiv 84 \equiv 0 \pmod 6$.
- If $n \equiv 5 \pmod 6$ we have $n^3+5n \equiv 5^3+5(5) \equiv 125+25 \equiv 150 \equiv 0 \pmod 6$.

In all cases we have that $n^3+5n \equiv 0 \pmod 6$, which means that $6|n^3+5n$. Therefore $6|n^3+5n$ for all $n \geq 1$. (In fact, this proves that $6|n^3+5n$ for all $n \in \mathbb{Z}$!)

(b) Suppose that for nonzero integers a and b we have that $\gcd(a,b)=1$ and $a \mid c$ and $b \mid c$. Since $a \mid c$ there exists $k \in \mathbb{Z}$ such that $c=ak$, and since $b \mid c$ there exists $l \in \mathbb{Z}$ such that $c=bl$. Since $\gcd(a,b)=1$ there exist integers x and y such that $ax+by=1$. Multiplying by c gives $acx+bcy=c$. From there we have that $a(bl)x+b(ak)y=c$ and so $ab(lx+ky)=c$. Since $l,x,k,y \in \mathbb{Z}$ we have that $lx+ky \in \mathbb{Z}$, and so $ab \mid c$.

3. Notice that $24=2^3 \cdot 3$ and $42=2 \cdot 3 \cdot 7$. Thus the prime power decomposition of our number will be comprised of the primes 2, 3, and 7. Now, values that are a fourth power will have exponents that are multiples of 4 and in the prime power decomposition, and values that are a sixth power will have exponents that are multiples of 6 and in the prime power decomposition. Thus our exponents in the prime power decomposition must be multiples of both 4 and 6. Since $\operatorname{lcm}(4,6)=12$ we can use exponents of 12 as our smallest possible value. Thus the smallest integer that is divisible by 24 and 42, and is simultaneously a fourth power and a sixth power is $2^{12}3^{12}7^{12}$.

4. (a) The prime power decomposition of n^3 will consist of the same primes in the prime power decomposition as n, but the primes in the decomposition of n^3 will have exponents that are 3 times that of the exponents in the decomposition of n. So if p is a prime number such that $p \mid n^3$, then p will be in the prime power decomposition of n^3 and p will be in the prime power decomposition of n. Therefore $p|n$. Since $p|n$ there exists $k \in \mathbb{Z}$ such that $n=pk$, and so we can write $n^3=p^3k^3$. Since $k \in \mathbb{Z}$, we have that $k^3 \in \mathbb{Z}$, and so $p^3|n^3$.

(b) Say $\gcd(a, 63) = d$. By definition, this means that $d|n$ and $d|63$. We know that $63 = 3^2 7$, which (by FTA) says that the possible divisors of 63 are:

$$\begin{aligned} 1 &= 3^0 7^0 \\ 3 &= 3^1 7^0 \\ 9 &= 3^2 7^0 \\ 7 &= 3^0 7^1 \\ 21 &= 3^1 7^1 \\ 63 &= 3^2 7^1 \end{aligned}$$

Thus the possibilities for $\gcd(a, 63)$ are only: 1, 3, 7, 9, 21, 63.

(c) Consider the positive integers n and $n+9$ and the positive integer d where $d|n$ and $d|n+9$. This means that there exists $k \in \mathbb{Z}$ such that $n = dk$ and there exists a $l \in \mathbb{Z}$ such that $n + 9 = dl$. Thus we have

$$\begin{aligned} n + 9 &= dl \\ dk + 9 &= dl \\ 9 &= dl - dk \\ 9 &= d(l - k) \end{aligned}$$

Since $k, l \in \mathbb{Z}$ we have $l - k \in \mathbb{Z}$ and so $d|9$. Since $9 = 3^2$, the FTA says that the only possible divisors are $1 = 3^0$, $3 = 3^1$ and $9 - 3^2$. Thus $d = 1$ or $d = 3$ or $d = 9$.

5. (a) Suppose $a \equiv b \pmod{12}$ and $b \equiv c \pmod{18}$. This means that $12|(a - b)$ and $18|(b - c)$. That is, there exists $k \in \mathbb{Z}$ such that $(a - b) = 12k$, and there exists $l \in \mathbb{Z}$ such that $(b - c) = 18l$.

Now $(a - c) = (a - b) + (b - c) = 12k + 18l = 3(4k + 6l)$. Since $k, l \in \mathbb{Z}$ we have that $4k, 6l \in \mathbb{Z}$ and so $4k + 6l \in \mathbb{Z}$. Therefore $3|(a - c)$. This in turn gives that $a \equiv c$ (mod 3).

(b) There are many combinations that are possible here for x and m. One such example is $m = 8$ and $x = 3$. Here we have $x^2 \equiv 3^2 \equiv 9 \equiv 1 \pmod 8$, but $3 \not\equiv 1 \pmod 8$ and $x \not\equiv -1 \equiv 7 \pmod 8$.

(c) To find the last digit of 37^{37} in base 10 we want to find the remainder of 37^{37} when dividing by 10, so we will work (mod 10).

$37^{37} \equiv 7^{37} \equiv 7 \cdot (7)^{36} \equiv 7 \cdot (7^2)^{18} \equiv 7 \cdot (49)^{18} \equiv 7 \cdot (9)^{18} \equiv 7 \cdot (-1)^{18} \equiv 7 \pmod{10}$.
Since $0 \leq 7 < 10$, we now have that 7 is the remainder of 37^{37} when divided by 10. Therefore the last digit of 37^{37} is 7.

To find the last digit of 37^{37} in base 7 we want to find the remainder of 37^{37} when dividing by 7, so we will work (mod 7).
$37^{37} \equiv 2^{37} \equiv 2 \cdot (2)^{36} \equiv 2 \cdot (2^3)^{12} \equiv 2 \cdot (8)^{12} \equiv 2 \cdot (1)^{12} \equiv 2 \pmod 7$.
Since $0 \leq 2 < 7$, we now have that 2 is the remainder of 37^{37} when divided by 7. Therefore the last digit of 37^{37} in base 7 is 2.