Math 110 - Homework 9
Topic: Determinants; Row, Column, and Null spaces

Due at 6:00pm (Pacific) on Friday, November 19, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 4.5 and 4.6 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` is a square matrix then the determinant is calculated by `det(A)`.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $A = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}$.
   (a) Find a basis for $\text{row}(A)$.
   (b) Find a basis for $\text{col}(A)$.
   (c) Find a basis for $\text{null}(A)$.

Solution: For all three parts it is useful to first calculate $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 9/2 \end{bmatrix}$ (since this is a Part I questions students should do this by hand, showing steps).

For part (a), the non-zero rows of the RREF form a basis for $\text{row}(A)$. Thus a basis for $\text{row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 9/2 \end{bmatrix} \right\}$ (writing these as row vectors is also acceptable).

For part (b), the columns of A corresponding to pivot columns of $\text{RREF}(A)$ are a basis for $\text{col}(A)$. Thus a basis for $\text{col}(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Finally, for part (c), we view our matrix as the coefficient matrix of a homogeneous system. Then from the RREF we have $x_1 - 2x_4 = 0$, $x_2 - 6x_4 = 0$, and $x_3 + \frac{9}{2}x_4 = 0$. For vectors in the null space we therefore have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_4 \\ 6x_4 \\ -(9/2)x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ 6 \\ -9/2 \\ 1 \end{bmatrix}.$$

Therefore a basis for $\text{null}(A)$ is $\left\{ \begin{bmatrix} 2 \\ 6 \\ -9/2 \\ 1 \end{bmatrix} \right\}$.

2. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$.
   (a) Calculate $\det(A)$ without using any row operations.

Solution: We use cofactor expansion. Any row or column will work, so we pick column 1 to take advantage of the 0s there. We then use column 1 of each of the resulting matrices to continue.

$$\det\begin{bmatrix}1&2&0&1\\1&2&-1&1\\0&0&2&1\\0&0&2&-1\end{bmatrix} = \det\begin{bmatrix}2&-1&1\\0&2&1\\0&2&-1\end{bmatrix} - \det\begin{bmatrix}2&0&1\\0&2&1\\0&2&-1\end{bmatrix}$$
$$= 2\det\begin{bmatrix}2&1\\2&-1\end{bmatrix} - 2\det\begin{bmatrix}2&1\\2&-1\end{bmatrix}$$
$$= 0$$

(b) Calculate $\det(A)$ by row-reducing A and keeping track of how your steps modify the determinant.

Solution: We row-reduce, keeping track of the operations, until we reach a triangular matrix.

$$\begin{bmatrix}1&2&0&1\\1&2&-1&1\\0&0&2&1\\0&0&2&-1\end{bmatrix} \to_{R_2-R_1} \begin{bmatrix}1&2&0&1\\0&0&-1&0\\0&0&2&1\\0&0&2&-1\end{bmatrix}$$
$$\to_{R_4-R_3} \begin{bmatrix}1&2&0&1\\0&0&-1&0\\0&0&2&1\\0&0&0&-2\end{bmatrix}$$

We have only used row operations that do not change determinants, so the determinant of the triangular matrix we obtained is the same as the determinant of the original matrix. The determinant of a triangular matrix is the product of the diagonal entries, so the determinant of our final (and hence also original) matrix is $1(0)(2)(-2) = 0$.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Suppose that X is a 2×2 matrix such that for every 2×2 matrix A, $\det(X+A) = \det(X) + \det(A)$. Show that $X = 0_{2\times 2}$.
   Hint: Try picking some specific matrices for A and seeing what you can learn about the entries of X from the equation $\det(X+A) = \det(X) + \det(A)$.

Solution: Let $X = \begin{bmatrix}x&y\\z&w\end{bmatrix}$. Following the hint, let us use the equation $\det(X+A) = \det(X) + \det(A)$ for several specific choices of A.

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we have

$$\det(X + A) = \det \begin{bmatrix} x+1 & y \\ z & w \end{bmatrix} = (x+1)w - yz = xw - yz + w,$$

and

$$\det(X) + \det(A) = (xw - yz) + 0.$$

Thus from $\det(X+A) = \det(X) + \det(A)$ for this choice of A we conclude that $w = 0$. In a similar manner, using the matrices $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for the matrix A we learn (respectively) that $z = 0$, $y = 0$, and $x = 0$. Thus $X = 0_{2\times 2}$.

2. Let A be an $n \times n$ matrix where every entry of A is an integer.

   (a) Suppose that A is invertible and every entry of A^{-1} is also an integer. Explain why it must be the case that $\det(A) = \pm 1$.

   Solution: Observe that in the process of calculating a determinant using cofactor expansion the only things we do with the entries of A are add, subtract, and multiply. As a result, if every entry of A is an integer then so is $\det(A)$. Similarly, given that every entry of A^{-1} is an integer, $\det(A^{-1})$ must also be an integer. We know from class that $\det(A^{-1}) = 1/\det(A)$. Thus both $\det(A)$ and $1/\det(A)$ are integers. The only way this is possible is if $\det(A) = \pm 1$.

   (b) Suppose that A is an upper-triangular matrix, A is invertible, and A has a 2 somewhere on the diagonal. Explain why there will be at least one entry of A^{-1} that is not an integer.

   Solution: Since A is upper-triangular $\det(A)$ is the product of the diagonal entries of A. Since each entry of A is an integer and there is a 2 on the diagonal, this implies that $\det(A) \neq \pm 1$. Thus by part (a) it cannot be the case that every entry of A^{-1} is an integer.

3. Let $S = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \\ 1 \\ 3 \end{bmatrix} \right)$. Find $\dim(S)$.

Solution: Let $A = \begin{bmatrix} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 3 & 1 \\ 2 & 2 & 4 & 0 & -1 \\ 1 & 1 & 2 & 3 & -2 \\ 1 & 3 & 0 & 2 & 1 \\ -1 & 1 & -4 & 0 & 3 \end{bmatrix}$. Then $S = \operatorname{col}(A)$, so as we know from class, $\dim(S) = \dim(\operatorname{col}(A)) = \operatorname{rank}(A)$. We can ask MATLAB to find the rank of A either by using the command `rank` or by asking MATLAB for the RREF of A and then counting the non-zero rows by hand. Either way we found out that $\operatorname{rank}(A) = 4$, so $\dim(S) = 4$.