Math 110 - Homework 3
Topic: Dot products, lines in $\mathbb{R}^2$

Due at 6:00pm (Pacific) on Friday, October 1, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 2.2 and 2.3 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `v` and `w` are vectors, their dot product is calculated by `dot(v, w)`.
- To calculate the length of the vector `v`, use either $\sqrt{\texttt{dot(v,v)}}$ or `norm(v)`.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\overrightarrow{v_2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\overrightarrow{v_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\overrightarrow{v_4} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For each of the following parts, determine whether or not the given expression makes sense. If it does, calculate it. If it does not, briefly explain why not.

   (a) $\overrightarrow{v_1} \cdot \overrightarrow{v_3}$

   Solution: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1(0) + 2(0) = 0.$

   (b) $\overrightarrow{v_3} \cdot \overrightarrow{v_4}$

   Solution: Does not make sense, because $\overrightarrow{v_3}$ and $\overrightarrow{v_4}$ do not have the same number of entries.

   (c) $\overrightarrow{v_2} \cdot (2\overrightarrow{v_1} - 3\overrightarrow{v_2})$

   Solution: $\begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \left(2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -5 \end{bmatrix} = (-2)(8) + (3)(-5) = -31.$

   (d) $\overrightarrow{v_2}(2\overrightarrow{v_1} - 3\overrightarrow{v_2})$

   Solution: Does not make sense, because it asks us to multiply two vectors.

   (e) $\text{proj}_{\overrightarrow{v_1}}(\overrightarrow{v_2})$

   Solution: $\text{proj}_{\overrightarrow{v_1}}(\overrightarrow{v_2}) = \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} = \frac{1(-2) + 2(3)}{1(1) + 2(2)} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 8/5 \end{bmatrix}.$

2. There is exactly one line in $\mathbb{R}^2$, call it L, that passes through the point $(3, -5)$ and is parallel to the line $x - 3y = 5$. Describe L by giving:

   (a) A general equation.

   Solution: The fact that L is parallel to $x - 3y = 5$ tells us that L has the same direction vector as $x - 3y = 5$, and therefore also has the same normal vector as $x - 3y = 5$. Therefore L has general equation of the form $x - 3y = C$ for some constant C. To find C, we plug in the point $(3, -5)$, which we know is on L, and thus find $C = 3 - 3(-5) = 18$. Therefore the general equation for L is

   $$x - 3y = 18.$$

   Any other general equation for this line is obtained from this one by multiplying the whole equation by a non-zero constant.

(b) A vector equation.

Solution: Starting from the general equation $x - 3y = 18$ that we found in part (a), we have $x = 18 + 3y$. Thus we have:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 + 3y \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The above equation is the desired vector equation. Note that there are many other possible vector equations. They all have direction vector that is a non-zero scalar multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, but the point $\begin{bmatrix} 18 \\ 0 \end{bmatrix}$ can be replaced by any point on L.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. Note that there may be questions that can be solved without doing any calculations, or where MATLAB is not helpful for the calculations you need to do; in such cases, do the calculations by hand. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Is there a vector in $\mathbb{R}^6$, other than $\vec{0}$, that is orthogonal to all three of

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}?$$

If so, find one, and explain how you found it (guessing and checking is not a sufficient explanation). If not, explain why not.

Solution: We would like to find a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$ that satisfies:

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = x_1 + 3x_2 + 2x_3 + x_4 + x_5,$$

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix} = -x_1 + 2x_2 + x_3 - 4x_5 + 5x_6,$$

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = x_1 + x_2 + x_3 + 2x_4 + 2x_5 - x_6.$$

This is a system of linear equations to which we want to find a solution. We enter the corresponding augmented matrix into MATLAB and ask for the reduced row echelon form:

$$\texttt{A} = [1\ 3\ 2\ 1\ 1\ 0\ 0; -1\ 2\ 1\ 0\ \ -4\ 5\ 0; 1\ 1\ 1\ 2\ 2\ \ -1\ 0]$$

$$\texttt{rref(A)}$$

The response from MATLAB is:

$$\texttt{ans} = \begin{bmatrix} 1 & 0 & 0 & -1 & 3 & -4 & 0 \\ 0 & 1 & 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 1 & 7 & -1 & 5 & 0 \end{bmatrix}$$

The corresponding system of equations, after rearranging, is:

$$x_1 = x_4 - 3x_5 + 4x_6$$

$$x_2 = 4x_4 + 2x_6$$

$$x_3 = -7x_4 + x_5 - 5x_6$$

We can now choose any values of x_4, x_5, x_6 and find a corresponding solution. Any choice except for all 0 is fine, and the question only asks for one solution, so we'll just pick $x_4 = x_5 = x_6 = 1$, and obtain the vector $\begin{bmatrix} 2 \\ 6 \\ -11 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

2. Let $\overrightarrow{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find all vectors $\overrightarrow{w}$ in $\mathbb{R}^2$ such that both $\|\overrightarrow{w}\| = 1$ and $\overrightarrow{v} \cdot \overrightarrow{w} = \frac{1}{2}$.
Be sure to explain how you know that you found *all* possible solutions!

Solution: Let $\overrightarrow{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. The assumption that $\overrightarrow{v} \cdot \overrightarrow{w} = \frac{1}{2}$ means that

$$-2x + y = \frac{1}{2},$$

or equivalently,

$$y = 2x + \frac{1}{2}.$$

Thus we have

$$\vec{w} = \begin{bmatrix} x \\ 2x + 1/2 \end{bmatrix}.$$

Now the requirement $\|\vec{w}\| = 1$ tells us:

$$1 = x^2 + y^2 = x^2 + (2x + 1/2)^2 = 5x^2 + 2x + \frac{1}{4}.$$

We therefore need to solve

$$5x^2 + 2x - \frac{3}{4} = 0.$$

Using the quadratic formula (or any other method you like), the solutions are

$$x = -\frac{1}{5} \pm \frac{\sqrt{19}}{10}.$$

We therefore find that there are two solutions to the problem:

$$\begin{bmatrix} -\frac{1}{5} + \frac{\sqrt{19}}{10} \\ \frac{1}{10} + \frac{\sqrt{19}}{5} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -\frac{1}{5} - \frac{\sqrt{19}}{10} \\ \frac{1}{10} - \frac{\sqrt{19}}{5} \end{bmatrix}.$$

3. Suppose that $\vec{v}$ and $\vec{w}$ are non-zero vectors in $\mathbb{R}^n$, and $\vec{v} \perp \vec{w}$. Show that, for every vector $\vec{x}$ in $\mathbb{R}^n$,

$$\mathrm{proj}_{\vec{w}}(\mathrm{proj}_{\vec{v}}(\vec{x})) = \vec{0}.$$

Solution: We calculate from the left side, using the formulas we know for orthogonal projection. Along the way, we will use the hypothesis that $\vec{v} \cdot \vec{w} = 0$.

$$\begin{aligned}
\mathrm{proj}_{\vec{w}}(\mathrm{proj}_{\vec{v}}(\vec{x})) &= \mathrm{proj}_{\vec{w}}\left(\frac{\vec{v} \cdot \vec{x}}{\vec{v} \cdot \vec{v}}\vec{v}\right) \\
&= \frac{\left(\frac{\vec{v} \cdot \vec{x}}{\vec{v} \cdot \vec{v}}\vec{v}\right) \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} \\
&= \frac{(\vec{v} \cdot \vec{x})(\vec{v} \cdot \vec{w})}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}\vec{w} \\
&= \frac{(\vec{v} \cdot \vec{x})(0)}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}\vec{w} \\
&= 0\vec{w} \\
&= \vec{0}
\end{aligned}$$