# Math 122 Lecture Notes Section 2.1 - Open Statements

When a statement includes a variable, we need to know something about the variable before we can tell the truth value of the statement.

### Example 1:

- $x^2 + 3x + 2 = 0$
- for all x in the real numbers,  $x^2 + 3x + 2 = 0$
- for some x in the real numbers,  $x^2 + 3x + 2 = 0$
- for x = -1,  $x^2 + 3x + 2 = 0$

An **open statement** is an assertion containing one or more variables.

#### **Notation:**

p(x) is a statement that contains the variable x p(x,y) is a statement that contains the variables x and y

The Laws of Logic apply to open statements (because they will apply once the variables are assigned values).

For example, we can say that the contrapositive of  $p(x) \to q(x)$  is:

Also,  $\neg(p(x) \lor q(x))$  is equivalent to:

### Math 122 Lecture Notes Section 2.2 - Quantifiers

We need to give values to variables in statements in order to decide the truth value. The options are:

- Give a specific value. (e.g. x = 3)
- Specify the quantity of allowed replacements to make the statment true.

That second option leads us to the use of quantifiers.

The **universe** of a variable is the collection of values it is allowed to take. (e.g. the real numbers, the integers, positive rational numbers, etc.)

The universal quantifier  $\forall$  says that the statement is true for all allowed replacements of the variable.

Watch for: "for all", "all", "every", "for each".

The existential quantifier  $\exists$  says that there exists at least one allowed replacement to make the statement true.

Watch for: "there exists", "there is", "there are", "some", "at least one".

**Example 1:** For each statement, write in symbolic form using quantifiers and decide on the truth value of the statement.

(a) statement: "for all integers  $n, n^2$  is not negative".

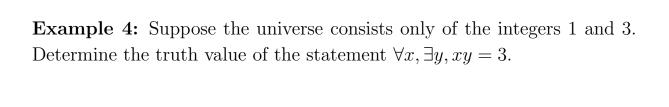
(b) statement: "for the universe of integers 1, 2, and 3, for all x in this universe, the value  $x^2$  is odd".

(c) statement: "there is an integer n such that  $n^2 + 3n + 2 = 0$ ".

We can nest quantifiers. Here the ordering is important, and we read the statement from left to right.

**Example 2:** Determine the truth value of the statement  $\forall x, \exists y, x+y=0$ .

**Example 3:** Determine the truth value of the statement  $\exists y, \forall x, x+y=0$ .



**Example 5:** Suppose the universe consists only of the integers 1 and 3. Determine the truth value of the statement  $\exists y, \forall x, xy = 3$ .

Be careful! Sometimes quantifiers can be hidden!

Look at the quadratic formula:

if 
$$a \neq 0$$
 and  $ax^2 + bx + c = 0$ , then  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 

This rule is mean to apply to **all** real numbers x, so the quantifier  $\forall$  is hidden.

Explicitly stated:

$$\forall x \in \mathbb{R}$$
, if  $a \neq 0$  and  $ax^2 + bx + c = 0$ , then  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 

Final note: if the universe contains at least one element and we know the statement " $\forall x, p(x)$ " is true, then we know that the statement " $\exists x, p(x)$ " is also true.

# Math 122 Lecture Notes Section 2.3 - Negating Statements Involving Quantifiers

The negation of "for all something, p" is: "there exists something such that  $\neg p$ ".

#### For all - Quantifier

The negation of "there exists something such that p" is: "for all something,  $\neg p$ ".

### Example 1: Write the negation of each statement

(a) statement: " $\forall$  integers  $x \geq 2$ , x is divisible by a prime".

(b) statement: "∃ a dog that is purple".

(c) statement: " $\forall x, \exists y, ((x^2 > y) \land (x < y))$ "

# Math 122 Lecture Notes Section 2.4 - Some Examples of Written Proofs

In this section we'll talk about some common methods of demonstrating a mathematical proof:

- direct proof
- contrapositive
- proof by contradiction
- proof by cases

The Direct Proof Method: Here we build our conclusion directly from the premises.

**Proposition 2.4.1:** If the integer n is even, then  $n^2$  is even.

**The Contrapositive Method:** If our statement was  $p \Rightarrow q$ , then instead we try to show that  $\neg q \Rightarrow \neg p$ . Our premise then is  $\neg q$  (the negation of the conclusion in our original statement) and we try build  $\neg p$  (the negation of the original premise).

**Proposition 2.4.2:** If the integer  $n^2$  is even, then n is even.

The Proof by Contradiction Method: Here we assume the negation of the conclusion along with the original premises. We want to show that we have a contradiction. Therefore, our extra assumption (the negation of the conclusion) was wrong, so we actually have the intended conclusion.

**Proposition 2.4.3:**  $\sqrt{2}$  is not rational.

The Proof by Cases Method: Here we do a proof for each case to demonstrate that the statement hold in all cases.

**Proposition 2.4.4:** If the integer  $n^2$  is a multiple of 3, then n is a multiple of 3.