Student: Arfaz Hossain **Date:** 10/30/21	**Instructor:** UVIC Math **Course:** MATH 100 (A01, A02, A03) Fall 2021 **Assignment:** Assignment 6

For what values of a, m, and b does the function f(x) satisfy the hypotheses of the mean value theorem on the interval [0,7]?

$$f(x) = \begin{cases} -5 & x = 0 \\ -x^2 + 6x + a & 0 < x < 3 \\ mx + b & 3 \le x \le 7 \end{cases}$$

The hypotheses of the mean value theorem are that f is continuous on the closed interval [a,b] and that f is differentiable on the open interval (a,b).

Note that each of the three pieces of the function f(x) is a polynomial function. Recall that every polynomial function is continuous, since the limit as x approaches c of a polynomial function P(x) is equal to P(c). Therefore, f(x) is continuous and differentiable on the interval (0,3) and on the interval (3,7]. Determine the values of a, m, and b that make f(x) continuous and differentiable at the points $x = 0$ and $x = 3$.

A function $y = f(x)$ is continuous at a point c if f(c) and the limit of f(x) as x approaches c both exist and are equal.

Evaluate the given function at the point $x = 0$.

$$f(0) = -5$$

For f(x) to be continuous at $x = 0$, the limit of f(x) as x approaches 0 must exist. Find the limit as x approaches 0 from the right.

$$\begin{aligned} \lim_{x \to 0} f(x) &= \lim_{x \to 0} \left(-x^2 + 6x + a\right) \\ &= -0^2 + 6(0) + a \\ &= a \end{aligned}$$

For f(x) to be continuous at $x = 0$, the limit of f(x) as x approaches 0 must be equal to f(0).

$$\begin{aligned} \lim_{x \to 0} f(x) &= f(0) \\ a &= -5 \end{aligned}$$

Thus, for f(x) to be continuous at $x = 0$, a must be equal to -5.

$$f(x) = \begin{cases} -5 & x = 0 \\ -x^2 + 6x - 5 & 0 < x < 3 \\ mx + b & 3 \le x \le 7 \end{cases}$$

Now examine the function f(x) at the point $x = 3$. A function is differentiable on an open interval if it has a derivative at each point of the interval, and a function has a derivative at a point if and only if the one-sided derivatives at that point exist and are equal.

Use the power rule, constant multiple rule, and sum rule to find the derivative of $f(x) = -x^2 + 6x - 5$.

$$f'(x) = -2x + 6$$

Similarly, find the derivative of $f(x) = mx + b$.

$$f'(x) = m$$

To find the value of m that makes f(x) differentiable at $x = 3$, set the one-sided derivatives at this point equal to each other.

$$\begin{aligned} \text{left-hand derivative} &= \text{right-hand derivative} \\ -2x + 6 &= m \\ -2(3) + 6 &= m \\ 0 &= m \end{aligned}$$

For f(x) to be differentiable at $x = 3$, m must be equal to 0.

$$f(x) = \begin{cases} -5 & x = 0 \\ -x^2 + 6x - 5 & 0 < x < 3 \\ b & 3 \le x \le 7 \end{cases}$$

For f(x) to be continuous at x = 3, the limit as x approaches 3 from the left must be equal to the limit as x approaches 3 from the right, and this limit must be equal to f(3). Evaluate f(3).

$$f(3) = b$$

Note that f(3) = 0 + b is also the limit of f(x) as x approaches 3 from the right, since 0x + b is a polynomial function. Find the limit of f(x) as x approaches 3 from the left.

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} \left(-x^2 + 6x - 5\right) \\ &= -(3)^2 + 6(3) - 5 \\ &= 4 \end{aligned}$$

To find the value of b that makes f(x) continuous at x = 3, set the limit of f(x) as x approaches 3 from the right equal to the limit as x approaches 3 from the left.

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^-} f(x) \\ b &= 4 \end{aligned}$$

For f(x) to be continuous at x = 3, b must be equal to 4. The function f(x) that satisfies the hypotheses of the mean value theorem on the interval [0,7] is shown below, where a is -5, m is 0, and b is 4.

$$f(x) = \begin{cases} -5 & x = 0 \\ -x^2 + 6x - 5 & 0 < x < 3 \\ 4 & 3 \le x \le 7 \end{cases}$$