PHYS 110 Textbook
A Calculus Based Introduction to Mechanics

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Contents

Preface

A student entering the first year of a university Science or Engineering program is typically expected to undertake a multiple-term calculus-based survey of physics. This survey is usually accompanied by two terms worth of single-variable differential and integral calculus, and often either accompanied or followed by courses on multi-variable calculus, linear algebra, and differential equations. The topics covered in the survey of physics can vary from institution to institution, but they almost invariably start with a single-term introduction to mechanics. This is a text intended for such a course.

The course for which this text is designed has as its intended prerequisites a high school level algebra-based survey of physics, and a similar level pre-calculus course. It is intended to be taken concurrently with an introductory calculus course. As there are several mathematical topics that are essential to the understanding of mechanics, this text includes brief and self-contained introductions to vector algebra, differentiation, and integration.

The choices that we have made for topics and order of presentation deviate from many of the introductory texts that are available. In particular, we open with a discussion of vectors and then apply this concept directly to translational and rotational equilibrium. This is done in part because our experience is that manipulating vectors is novel for our students, and in need of reinforcement. It is also done in part because we expect the course to be taken concurrently with a calculus course and the typical introductory calculus course takes a little while to introduce derivatives. Our development of acceleration and velocity only touches briefly on one-dimensional motion, because of the time we spent developing vectors. We include discussions of Newtonian gravity, the Coulomb force, and the Lorentz force. Our treatment of work and energy is vector-calculus based.

We intend the text to be rigorous enough that a student who completes a course based on this will be well prepared for intermediate-level courses in physics, and for courses which build on the knowledge from first-year physics. A guiding principle in our development of the text is that we do not say anything which will need to be corrected in future courses.

This text evolved from lecture notes associated with introductory physics courses taught by the authors at the University of Victoria in the early 2010s. The authors would like to thank our dean of science and our department chair for their support and encouragement in this project. They would also like to express their appreciation for their students who asked the questions and made the comments that sparked them to write the text.

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Chapter 1

Vectors

1.1 Introduction

The central idea of physics is to provide a mathematical and predictive description of the natural world. Many properties such as *mass* and *temperature* are numerical and can be manipulated using familiar algebraic rules, whereas other physical quantities must be described making reference to both direction and amount. Numerical quantities are called *scalars*. Quantities which are characterized by a single direction and amount are known as *vectors* and have both a mathematical structure and interpretation that make them suitable for use describing phenomena such as *forces* and *accelerations*. A careful application of the rules associated with vectors allows physical laws to be more concisely stated.

1.2 Displacement - a motivating example

The rules describing the addition and scaling of vectors can be deduced from considering a specific type of vector with which most people are familiar: displacement. This quantity will serve as a prototype from which most of the important properties of vectors can be deduced.

An object which has moved from one location to another on a flat surface can be said to have undergone a displacement. This displacement can be characterized by two quantities: a distance and an angle. The distance is the *length* of the displacement, and the angle is expressed with respect to some reference direction. Two objects are said to have undergone the same displacement provided the distance and direction they moved was the same, even though they may have started and ended at different points, as

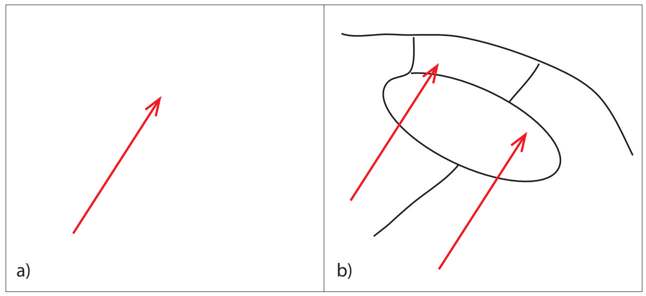

Figure 1.1: In part (a), a displacment. In part (b), the same displacement from different starting points results in a different final location.

illustrated in figure 1.1. Displacements can be added. The addition of two displacements can be understood as a concatenation of instructions: from a starting point, first move as described by the first displacement, and then, from that intermediate point, move as described by the second displacement. The length and direction of the resulting total displacement can be determined using trigonometry and considering the lengths and directions of the individual displacements. The operation of addition of displacements is not sensitive to order; the resulting displacement will be the same regardless of which happens first and which happens second. The addition of displacements can be expressed algebraically; denoting one displacement as $\Delta\vec{r}_1$ and the second displacement as $\Delta\vec{r}_2$ the total displacement ($\Delta\vec{r}_T$) is expressed as

$$\Delta\vec{r}_T \equiv \Delta\vec{r}_1 + \Delta\vec{r}_2 = \Delta\vec{r}_2 + \Delta\vec{r}_1, \tag{1.1}$$

where the equality reflects the fact that the sum does not depend on order. This is illustrated in figure 1.2 In a similar way it is possible to determine the difference of displacements. The difference between displacement $\Delta\vec{r}_1$ and $\Delta\vec{r}_2$ is defined as the displacement required, that when added to $\Delta\vec{r}_2$ will give $\Delta\vec{r}_1$. Denoting the difference as $\Delta\vec{r}_D$, this is written as

$$\Delta\vec{r}_1 = \Delta\vec{r}_2 + \Delta\vec{r}_D \rightarrow \Delta\vec{r}_D = \Delta\vec{r}_1 - \Delta\vec{r}_2. \tag{1.2}$$

This difference can be seen to be the equivalent to adding a displacement of the same length but the opposite direction as $\Delta\vec{r}_2$ to $\Delta\vec{r}_1$. This is illustrated in figure 1.3.

The addition rule for displacements implies a rule for multiplying displacements by numbers. Executing the same displacement twice gives a

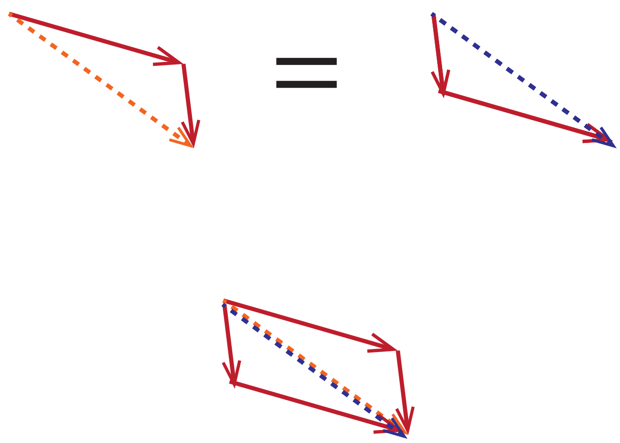

Figure 1.2: Two displacements, $\Delta\vec{r}_1$ and $\Delta\vec{r}_2$, added in the opposite order give the same result.

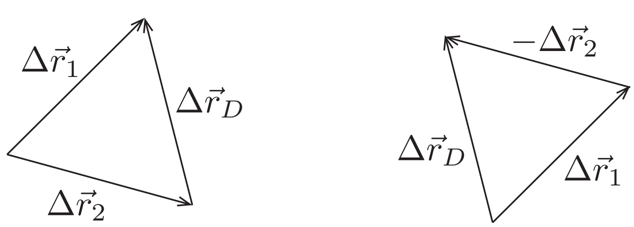

Figure 1.3: The difference between the displacements $\Delta\vec{r}_1$ and $\Delta\vec{r}_2$.

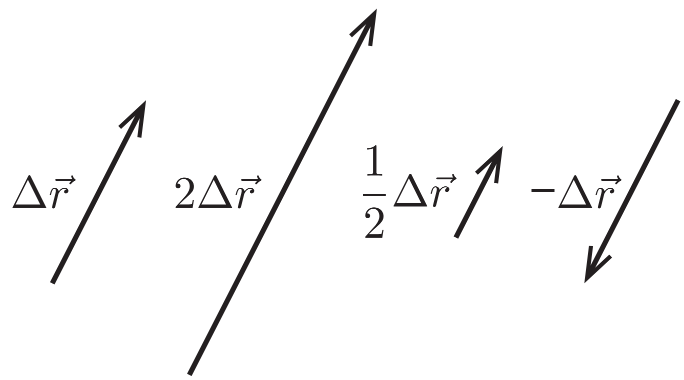

Figure 1.4: The vectors $\Delta\vec{r}_1$, $2\Delta\vec{r}_1$, $\frac{1}{2}\Delta\vec{r}_1$, and $-\Delta\vec{r}_1$.

displacement which is twice as long, but in the same direction, while algebraically

$$\Delta\vec{r}_1 + \Delta\vec{r}_1 = 2\Delta\vec{r}_1. \tag{1.3}$$

Similarly, it is possible to identify a displacement such that two copies of it add up to a known displacement:

$$2\Delta\vec{r}_1 = \Delta\vec{r}_2 \rightarrow \Delta\vec{r}_1 = \frac{1}{2}\Delta\vec{r}_2. \tag{1.4}$$

This establishes that the effect of multiplying a displacement by positive real number c is to create a displacement which is in the same direction as the original displacement while the length is c times the original length. Since subtracting a quantity is the same as adding -1 times that quantity, equation 1.2 establishes that multiplying a displacement by a negative number has both the effect of scaling the displacement and of making the direction opposite. These rules are illustrated in figure 1.4.

Displacements can, in addition to being expressed in distance and direction, be expressed as a combination of distances in certain cardinal directions. For example, a displacement which was to the East of North could be expressed as the sum of a displacement directly to the North and a displacement directly to the East. The length of the displacement in each of these cardinal directions, called *components*, can be determined from the total length of the displacement and the angle the displacement makes through trigonometry. When a displacement is multiplied by a real number the components are each individually multiplied by that number. When two

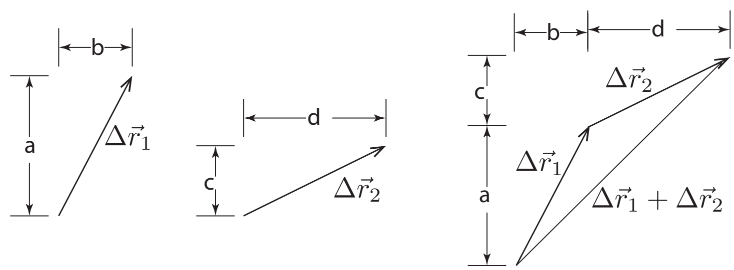

Figure 1.5: The vectors $\Delta\vec{r}_1$ and $\Delta\vec{r}_2$ with their components illustrated are added. The total components of the final displacement are the sum of the the components of the individual displacements.

displacements are added the resulting displacement has components which are the sums of the components of the individual displacements. Since multiplication of a displacement by a negative number reverses the direction a displacement to the West is the same as a negative displacement to the East. These displacements can be expressed as a combination a displacement in the North-South direction and a displacement in the East-West direction, as illustrated in figure 1.5.

1.3 Vectors and their properties

The example of displacement is instructive, and provides a geometric basis for the understanding of vectors in general. The techniques for adding displacements are geometric in origin and can be easily visualized and drawn. While diagrams can aid in the solution of many physical problems, it is convenient to have a mathematical way of manipulating quantities which does not rest on illustration.

Displacements are a prototype for a class of mathematical objects called *vectors*. The key properties are:

- Vectors are quantities which encode an 'amount' and a single direction. The name for the quantity, analogous to the length of a displacement, is the *magnitude*.

- The magnitude of a vector can have units. In the case of displacement, the units are meters ($[m]$).

- Vectors can be added, and this addition is commutative. Vectors can only be added to other vectors which have the same units.

- The effect of multiplying a vector by a positive real number is to scale the magnitude by that amount, while multiplying a vector by a negative real number will both reverse the direction and scale the magnitude.

- The effect of multiplying a vector by a dimensional number is to produce a vector with different units. Multiplying a velocity vector (units $\left[\frac{m}{s}\right]$) by a mass (units $[kg]$) will produce a vector with units $\left[kg\frac{m}{s}\right]$.

- Vectors can be uniquely expressed in terms of certain 'cardinal' directions. The *component* of any vector in one of those directions can be determined by geometry.

1.3.1 Coordinate systems

A coordinate system consists of a 'center', typically called the *origin*, and a defined set of cardinal directions. This defined set of directions is known as a set of *basis vectors*. All vectors expressed in the same coordinate system make use of the same set of basis vectors, regardless of their units. There are as many basis vectors as there are independent directions, and by convention the basis vectors are at 90 degrees to each other.

While there are many possible coordinate systems, this text will primarily use a *cartesian* coordinate system. An example is illustrated in figure 1.6. In this system positions are specified by their x, y, and z coordinates, and the basis vectors are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ which point along the positive x, y, and z axes respectively. In most cases in diagrams $\hat{\jmath}$ will point to the right, $\hat{k}$ will denote the vertical direction, and $\hat{\imath}$ is the direction perpendicular to the page. The basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are *unit vectors*. These vectors have *no* units – they are dimensionless – and they point in the directions indicated in the diagram. They are referred to as unit vectors because they have magnitude 1.

1.4 Manipulation of Vectors

In this coordinate system, a vector $\vec{A}$ can be uniquely expressed as

$$\vec{A} \equiv A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}. \tag{1.5}$$

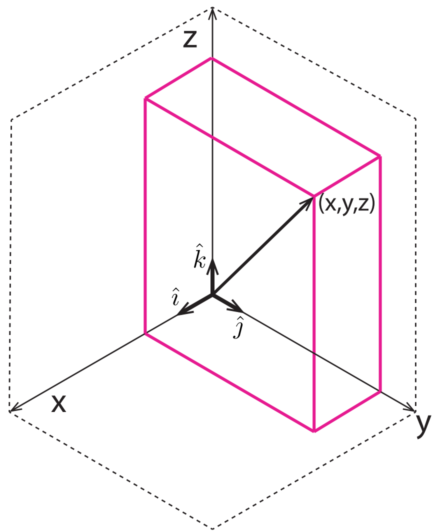

Figure 1.6: The 3-dimensional cartesian coordinate system.

All vectors are denoted with an 'arrow' above them, as in $\vec{A}$; all unit vectors have instead a 'circumflex', as in $\hat{\imath}$.

The rule for adding two vectors is, as it was for displacements, that the individual components add.

$$\begin{aligned} \vec{A}+\vec{B} &= \left(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}\right)+\left(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}\right) \\ &= \left(A_x+B_x\right)\hat{\imath}+\left(A_y+B_y\right)\hat{\jmath}+\left(A_z+B_z\right)\hat{k} \qquad (1.6) \end{aligned}$$

Any vectors with the same units can be added this way. Two vectors are equal if, and only if, all their components are the same.

The multiplication of a vector by a number can also be expressed in components.

$$\begin{aligned} c\vec{A} &= c\left(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}\right) \\ &= cA_x\hat{\imath}+cA_y\hat{\jmath}+cA_z\hat{k} \qquad (1.7) \end{aligned}$$

All components are scaled by the same factor; if the sign of one component is reversed the sign of all components is reversed.

The magnitude of a vector is related to the individual components of the vector via the Pythagorean theorem. The notation for the magnitude of vector $\vec{A}$ is $\left|\vec{A}\right|$. The magnitude is determined by the components as

$$\left|\vec{A}\right| \equiv \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.8}$$

The units of the magnitude $\left|\vec{A}\right|$ are the same as the units of any of the individual components.

The angle that a vector makes with a particular axis is determined by the component of the vector along that axis. Denoting θ_x as the angle between the positive x-axis and the vector $\vec{A}$, the relationship is

$$\theta_x = \arccos\left(\frac{A_x}{\left|\vec{A}\right|}\right) \rightarrow A_x = \left|\vec{A}\right| \cos\theta_x. \tag{1.9}$$

There are similar relationships between A_y, A_z and the angles between $\vec{A}$ and the y- and z-axes respectively.

1.5 Scalar Product

Vectors can be multiplied together. However, unlike numbers, there are two ways to do this because each vector encodes direction as well as magnitude. One way to multiply vectors takes two vectors and results in a (dimensional) number. This is known as the scalar product – in mathematical literature this is often called the *inner product*, and in physics literature it is usually called the *dot product* because of the notation for this type of multiplication uses a 'dot'.

The definition of the scalar product between two vectors is that the result will be the product of the magnitudes of the two vectors and the cosine of the angle between them. Figure 1.7 shows an example with vectors, $\vec{A}$ and $\vec{B}$ and an angle, θ between the vectors. Symbolically, this is

$$\vec{A} \cdot \vec{B} \equiv \left|\vec{A}\right| \left|\vec{B}\right| \cos\theta, \tag{1.10}$$

where θ is the angle between vectors $\vec{A}$ and $\vec{B}$.

This definition implies that the components of the vector $\vec{A}$ can be determined by taking the scalar product of it with the appropriate unit vector. If θ_x is the angle between $\vec{A}$ and the x-axis, then

$$\begin{aligned} \vec{A} \cdot \hat{\imath} &= \left|\vec{A}\right| |\hat{\imath}| \cos\theta_x, \\ &= A_x. \end{aligned} \tag{1.11}$$

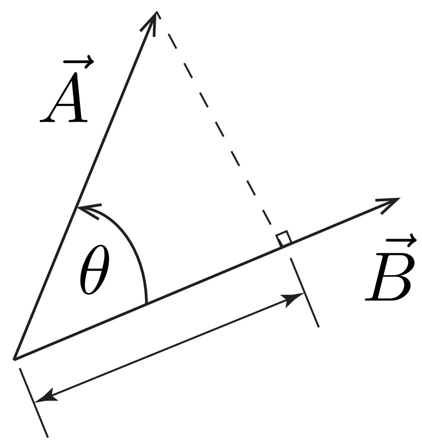

Figure 1.7: The angle θ between vectors $\vec{A}$ and $\vec{B}$.

Similarly,

$$\vec{A} \cdot \hat{\jmath} = A_y \quad \text{and} \quad \vec{A} \cdot \hat{k} = A_z. \tag{1.12}$$

The scalar product of two unit vectors can be calculated using the essential definition of scalar product. $\hat{\imath} \cdot \hat{\imath} = 1$, $\hat{\jmath} \cdot \hat{\jmath} = 1$, and $\hat{k} \cdot \hat{k} = 1$, because each of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ has magnitude 1, and each lines up with itself. As the vectors are orthogonal to each other, $\hat{\imath} \cdot \hat{\jmath} = 0$, $\hat{\imath} \cdot \hat{k} = 0$, and $\hat{\jmath} \cdot \hat{k} = 0$.

The scalar product of two vectors can be expressed in terms of their components using the known relations between the scalar product of the basis vectors and the distributive law.

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \cdot \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) \\
&= A_xB_x\hat{\imath} \cdot \hat{\imath} + A_xB_y\hat{\imath} \cdot \hat{\jmath} + A_xB_z\hat{\imath} \cdot \hat{k} \\
&\quad + A_yB_x\hat{\jmath} \cdot \hat{\imath} + A_yB_y\hat{\jmath} \cdot \hat{\jmath} + A_yB_z\hat{\jmath} \cdot \hat{k} \\
&\quad + A_zB_x\hat{k} \cdot \hat{\imath} + A_zB_y\hat{k} \cdot \hat{\jmath} + A_zB_z\hat{k} \cdot \hat{k} \\
&= A_xB_x + A_yB_y + A_zB_z
\end{aligned} \tag{1.13}$$

The relations which obtained the components of a vector through inner product with an appropriate unit vector can be readily obtained from this result. This also establishes that the inner product of a vector with itself is the sum of the square of its components; using the definition of the magnitude of a vector this means that

$$\left|\vec{A}\right| = \sqrt{\vec{A} \cdot \vec{A}}. \tag{1.14}$$

The scalar product is so called because it takes two vectors and produces a (dimensional) number without direction information – a scalar.

1.6 Vector Product

In addition to the scalar product, there is a second common way to multiply two vectors which produces another vector. In mathematical literature this is often called an *outer product*, whereas in physics it is usually called the *cross product* because of the symbol typically used to represent the operation. The cross product of two vectors is written as $\vec{A} \times \vec{B}$.

The magnitude of the vector product of two vectors is given by the product of the magnitudes of the two vectors and the sine of the angle between them. The direction of the cross product of two vectors is *perpendicular* to both of the individual vectors. This definition is unique except it does not specify direction; there are two unit vectors perpendicular to a plane, however one is the negative of the other. This ambiguity is resolved by the so-called *right hand rule*: the direction of the vector product is the direction that the thumb of a right hand would point if the fingers of the hand were aligned with the first vector while the palm of the hand was aligned towards the second vector. This direction is shown in figure 1.8. In a similar way to the definition of the scalar product, the vector products of the basis vectors are easy to compute:

$$\begin{array}{lll}
\hat{\imath} \times \hat{\imath} = 0 & \hat{\imath} \times \hat{\jmath} = \hat{k} & \hat{\imath} \times \hat{k} = -\hat{\jmath} \\
\hat{\jmath} \times \hat{\imath} = -\hat{k} & \hat{\jmath} \times \hat{\jmath} = 0 & \hat{\jmath} \times \hat{k} = \hat{\imath} \\
\hat{k} \times \hat{\imath} = \hat{\jmath} & \hat{k} \times \hat{\jmath} = -\hat{\imath} & \hat{k} \times \hat{k} = 0
\end{array}$$

The expression for the component version of the vector product can be

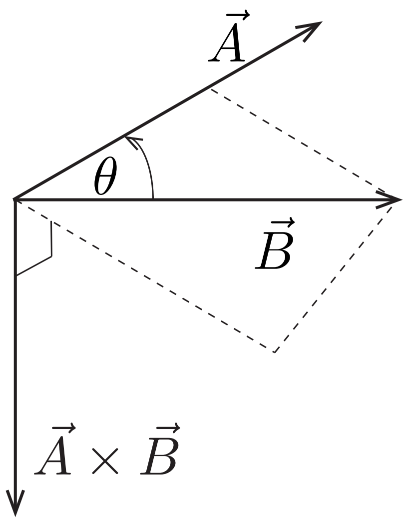

Figure 1.8: The vectors $\vec{A}$ and $\vec{B}$, together with $\vec{A} \times \vec{B}$.

similarly derived.

$$\begin{aligned}
\vec{A} \times \vec{B} = & \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \times \left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right) \\
= & A_xB_x\left(\hat{\imath}\times\hat{\imath}\right) + A_xB_y\left(\hat{\imath}\times\hat{\jmath}\right) + A_xB_z\left(\hat{\imath}\times\hat{k}\right) \\
& +A_yB_x\left(\hat{\jmath}\times\hat{\imath}\right) + A_yB_y\left(\hat{\jmath}\times\hat{\jmath}\right) + A_yB_z\left(\hat{\jmath}\times\hat{k}\right) \\
& +A_zB_x\left(\hat{k}\times\hat{\imath}\right) + A_zB_y\left(\hat{k}\times\hat{\jmath}\right) + A_zB_z\left(\hat{k}\times\hat{k}\right) \\
= & A_xB_y\hat{k} - A_xB_z\hat{\jmath} - A_yB_x\hat{k} + A_yB_z\hat{\imath} + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath} \\
= & \left(A_yB_z - A_zB_y\right)\hat{\imath} + \left(A_zB_x - A_xB_z\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}
\end{aligned} \tag{1.15}$$

The vector product is *antisymmetric* in its two arguments: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

The known relations for inner product make it obvious that $\vec{A}$ and $\vec{B}$ are both perpendicular to $\vec{A} \times \vec{B}$. Specifically, the scalar product of $\vec{A}$ with $\vec{A} \times \vec{B}$ is

$$\begin{aligned}
\vec{A}\cdot\left(\vec{A}\times\vec{B}\right) &= A_x\left(A_yB_z - A_zB_y\right) \\
&\quad +A_y\left(A_zB_x - A_xB_z\right) + A_z\left(A_xB_y - A_yB_x\right), \\
&= 0,
\end{aligned} \tag{1.16}$$

with a similar result for the scalar product of $\vec{B}$ and $\vec{A}\times\vec{B}$. Since in general neither $\left|\vec{A}\right|$ nor $\left|\vec{B}\right|$ vanish, this implies that the cosine of the angle between $\vec{A}$ and $\vec{A}\times\vec{B}$ is 0, which is equivalent to the vectors being perpendicular.

1.7 Advanced Topics

1.7.1 Vector Spaces

The vectors which are essential to the analysis done in introductory physics courses are a more general version of a mathematical structure known as a *vector space.*

The defining features of a vector space are

- There is a set of objects – called vectors – together with a commutative and associative addition rule, and a unique identity - a zero vector. This means that the sum of two vectors is another vector, that the order in which the vectors are summed does not matter, and that there is a 'zero' vector that can be added to anything without changing it.

- There is another set of objects – called a field – which can have both a multiplication and addition rule, together with a rule which allows an element of the field to be multiplied by a vector to get another vector. The typical example in Physics of this field is the set of real numbers.

- All vectors can be expressed as a *linear combination* of of a certain set of vectors known as a *basis.*

In this terminology, each set of vectors of the same dimensionality – $[m]$, $\left[\frac{m}{s}\right]$, and so on – forms a different vector space with the field of real numbers. This is because you cannot add vectors with different units. The set of all vectors expressed in the same coordinate system are *isomorphic* to each other, and they are unified by the fact that the scalar product is defined in such a way that the value of $\vec{A}\cdot\vec{B}$ can be calculated.

1.7.2 Tensors

There are many interesting quantities in Physics which cannot be appropriately expressed as either a scalar (a number with dimensions) or as a vector. These quantities typically occur when two (or more) directions are relevant. They are known as *tensors*; a tensor is described in terms of its *rank*, which corresponds to the number of directions that are relevant for the phenomenon it is describing. In this sense, a scalar is a 'rank 0' tensor and a vector is a 'rank 1' tensor.

Tensors themselves form a (mathematical) vector space as they can be added together, and multiplied by scalars. They are different in that the rules for multiplying a tensor by another tensor or for multiplying a tensor by a vector are more complicated to implement than those for vectors.

The approach taken in this text is that the chosen examples will be restricted in such a way that it is not necessary to use tensors to describe the aspect of physics that is under consideration. In those cases, there will be discussion which explains the simplification.

There are a few concrete examples of tensors that are worth enumerating:

- When a spring is compressed and bent, the force it exerts will not necessarily be directly back towards the equilibrium length. This can be expressed in terms of a 'stress-strain tensor', which will be simplified in this text to a single 'spring constant'.

- When a rigid object is rotating, the *angular momentum* will depend on the axis around which it rotates and on the mass distribution. While the axis can be expressed as a vector, the effect of the distribution of masses must be encoded in the 'moment of inertia tensor'. In this text only flat (plane) mass distributions will be considered, so only one component of the moment of inertia tensor is relevant.

- The vector (cross) product of two vectors is itself a rank 2 tensor because it depends on two directions. In this text, and conventionally in physics, this rank 2 tensor is treated as a vector because there are only three independent components.

Chapter 2

Translational Equilibrium

2.1 Introduction

The simplest example of physical interest which shows the importance of vector analysis is that of a *particle* which is in *equilibrium*. Newton's first law characterizes the net force on a object in equilibrium: if it is in equilibrium then it is subject to no net force. In the case of groups of interacting object Newton's third law relates the forces that two particular particles exert on each other. These two facts can be combined to analyze complex situations.

2.2 Particle Model

In Physics, as will be demonstrated throughout this book it is often useful to make *approximations*. These approximations are intended to simplify the analysis of a problem by removing extraneous information or quantities from consideration, while retaining enough of the key features to enable meaningful consideration. A pervasively common approximation throughout mechanics is the particle model.

The particle model is simply that all spatial extent of an object is completely ignored. The object is treated as though it only exists at a single point. All external forces which act on the object are considered to act at that location. The object is idealized to be perfectly rigid, so it does not deform or change shape in response to the forces to which it is subject. All rotation or twisting is also ignored.

An example of this idea in the case of a person would be to approximate the person as being located exactly at the tip of their nose. This clearly ignores the fact that the person exists with non-zero spatial extent, and that the person can be twisting or rotating. However, for the person, this is enough to determine the *overall* motion.

2.3 Newton's First Law

All objects are subject to Newton's first law, which states that *if* an object is subject to no *net* force *then* it will continue its uniform motion, neither changing its direction nor changing its speed. A direct application of this fact is that if an object is observed to be moving uniformly then it can be inferred that the net force on the object is zero; if an object is *at rest* and stationary then it similarly is known to be subject to no net force.

$$\sum_{external\ forces} \vec{F}_i = 0 \quad \text{uniform motion and equilibrium} \tag{2.1}$$

The net force on an object is the total of all forces *external* to itself that it is subject to. This is appropriate, in part, because of the particle approximation that is being used ignores any *internal* structure; a particle cannot exert a force on itself.

This inference, that an object moving uniformly is subject to no net force, is true in *inertial reference frames.* A reference frame is the combination of a set of basis vectors together with a point(the origin) from which locations are measured, and an inertial reference frame is one in which Newton's first law holds. An inertial reference frame has a constant set of basis vectors, and the origin is either stationary or moves uniformly.

Most reference frames which are naturally used in Physics are not actually inertial, but are approximately inertial over appropriately short timescales. The Earth is rotating so the coordinate system made of the basis vectors 'North', 'East', and 'Up' and an origin at a particular location on Earth's surface is not actually inertial. Over a time scale of a few minutes it can be usefully *approximated* as inertial, and in this text it will be assumed to be so, however over larger time and distance scales, such as hours or days, this is not true; it gives rise to effects such as the ones that make tropical cyclones spin a particular way. On a smaller scale, the coordinate system made of the basis vectors 'Forward', 'Right', and 'Up' and an origin defined by the location of a car's steering wheel is generically not going to be inertial.

2.4 Particles in Equilibrium

When analyzing an object in equilibrium the end goal is usually to *use* the fact that the particle is in equilibrium to *determine* something about one (or more) of the forces acting on it. The procedure employed is conceptually straight forward:

- Draw a diagram of the situation, carefully considering all external forces on the object.

- Abstract the diagram into a *free-body* diagram, which represents the object as a point particle and enumerates all forces the object experiences.

  - Represent the mass as a point.
  - Draw all the external force vectors acting on the mass.
  - Draw a coordinate system with the origin at the mass point.
  - Orient the coordinate system in a direction that will make the final equations involving the components of the forces easier to solve. Any orientation will work but some give simpler components to work with than others. There is no rule for this, the process requires the application of both trial and error and experience.

- Use the known facts about the motion of the particle to *infer* something about the net force. If the particle is stationary and not changing how it moves, then Newton's first law requires that the total force on the particle is zero.

- Manipulate the expression

  $$\vec{F}_{net} = 0 \tag{2.2}$$

  to express the unknown quantities in terms of known quantities. This can be typically done by considering the components of the net force in an appropriate set of basis vectors.

- Solve the resulting expressions to determine the needed quantities.

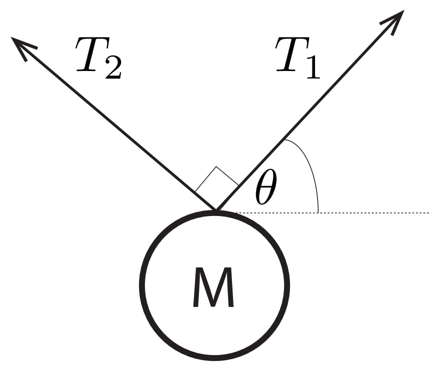

Figure 2.1: A sketch depicting a ball of mass m supported by two ropes.

2.4.1 Equilibrium Example

To illustrate the technique, consider the case of a ball of mass m being supported by two ropes. The first rope exerts a force T_1 up and to the right at an angle θ above the horizontal, and the second rope exerts a force T_2 up and to the left at an angle of 90° with respect to the first rope. A sketch of this situation is in figure 2.1. If θ is known the values of T_1 and T_2 can be determined, if T_2 is known the values of θ and T_1 can be determined, and conversly if T_1 is known the values of T_2 and θ can be determined. The free-body diagram (figure 2.2) for this situation enumerates the forces on the ball. They are the downward force of gravity $\vec{F}_g$, the force from the first rope $\vec{F}_1$, and the force from the second rope $\vec{F}_2$. The forces are such that $\left|\vec{F}_1\right| = T_1$ and $\left|\vec{F}_2\right| = T_2$. The force of gravity has magnitude mg and acts downward. In the diagram the forces are illustrated with appropriate directions, and all forces, including the downward force of gravity, are included. Since the ball is in equilibrium, Newton's first law can be expressed as

$$\begin{aligned} \vec{F}_{net} &= 0 \\ \text{and } \vec{F}_{net} &= \vec{F}_g + \vec{F}_1 + \vec{F}_2 \\ \text{so } 0 &= \vec{F}_g + \vec{F}_1 + \vec{F}_2 \end{aligned} \tag{2.3}$$

One of the purposes of the free-body diagram is to make it easier to *visualize* how to decompose vectors into their components.

Now that the conditions for equilibrium have been expressed, the vector expression in equation 2.3 is converted into a number of expressions – one for

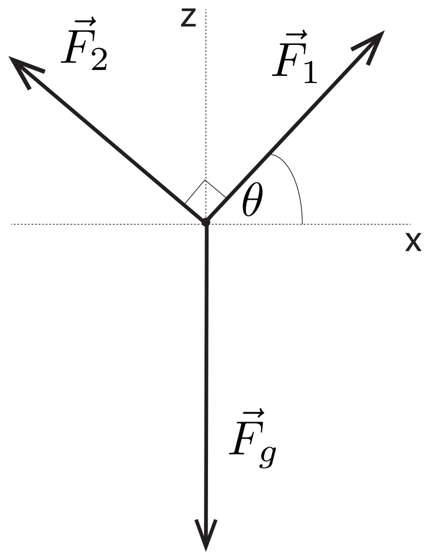

Figure 2.2: A free body diagram illustrating the forces acting on the ball.

each *component* of the relevant vectors. There are two completely equivalent ways to derive these expressions:

- Express each vector in the form $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, then add the vectors together using the rules for vector addition. Since a vector is only zero if all its components are zero, then the resulting coefficients of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are each independently zero.

- Take the vector expression $0 = \vec{F}_{net}$ and find the components by taking the inner product of each side with the appropriate unit vector. Calculating $0 \cdot \hat{\imath} = \vec{F}_{net} \cdot \hat{\imath}$ will give the expression generated by considering the x-component of the net force.

These two methods are equivalent because the inner products is both distributive and associative, and addition of vectors is linear: the x-component of the sum of a number of vectors is the same as the sum of the x-components of the vectors.

In the example under consideration, expressing the vectors in components gives

$$\begin{aligned}\vec{F}_g &= -mg\hat{k} \\ \vec{F}_1 &= T_1 \cos\theta \hat{\imath} + T_1 \sin\theta \hat{k} \\ \vec{F}_2 &= -T_2 \sin\theta \hat{\imath} + T_2 \cos\theta \hat{k}\end{aligned} \tag{2.4}$$

so the net force obtained by adding them together is

$$\vec{F}_{net} = (T_1 \cos\theta - T_2 \sin\theta)\,\hat{\imath} + (-mg + T_1 \sin\theta + T_2 \cos\theta)\,\hat{k} \tag{2.5}$$

Considering that $\vec{F}_{net} = 0$ so both the x-component and z-component of the net force have to be zero gives two relations:

$$\begin{aligned}0 &= T_1 \cos\theta - T_2 \sin\theta \quad \mathrm{x-component} \\ 0 &= -mg + T_1 \sin\theta + T_2 \cos\theta \quad \mathrm{z-component}\end{aligned} \tag{2.6}$$

The alternative method, taking the expression $\vec{F}_{net} = 0$ and calculating the inner product with both sides yields the same results.

The expressions in equation 2.6 are a set of two equations in a total of four variables. If m and θ are assumed to be known they can be solved to result in $T_1 = mg \sin\theta$ and $T_2 = mg \cos\theta$; if m and T_1 are known then $\theta = \sin^{-1}\left(\frac{T_1}{mg}\right)$ and $T_2 = mg\sqrt{1 - \left(\frac{T_1}{mg}\right)^2}$. Other combinations of known variables can be solved for the appropriate unknown quantities.

It is important to note that the final (resulting) set of relations *does not* depend on the set of basis vectors used. Instead of using the x, y, and z coordinate axes with the corresponding basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, it is possible to use a set of basis vectors which are *rotated* relative to the x-axis. As shown in figure 2.3, the $\hat{n}$ and $\hat{p}$ vectors are rotated by θ counterclockwise from the $\hat{\imath}$ and $\hat{k}$ vectors. Starting with the vector expression for equilibrium

$$0 = \vec{F}_g + \vec{F}_1 + \vec{F}_2 \tag{2.7}$$

it is possible to get two component equations by taking the inner product of both sides with $\hat{n}$ and $\hat{p}$ respectively. Since $\hat{n}$ is at 90° to $\vec{F}_2$ and $\hat{p}$ is at 90° to $\vec{F}_1$, both inner products can be calculated directly.

$$\begin{aligned}0 \cdot \hat{n} &= \left(\vec{F}_g + \vec{F}_1 + \vec{F}_2\right) \cdot \hat{n} \\ 0 &= mg \cos(90 + \theta) + T_1 + 0\end{aligned} \tag{2.8}$$

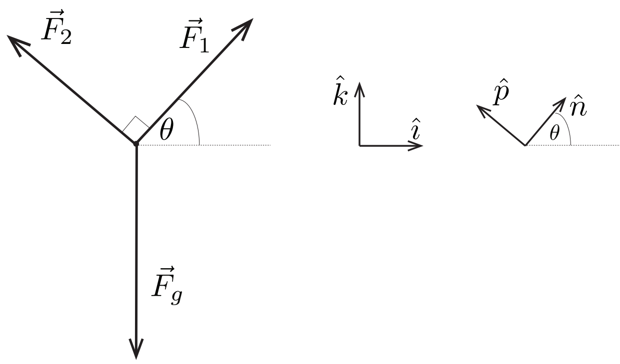

Figure 2.3: The $\hat{n}$ and $\hat{p}$ coordinate system compared with the $\hat{\imath}$ and $\hat{k}$ basis vectors.

and using trigonometric identities[1] yields $T_1 = mg\sin\theta$ and $T_2 = mg\cos\theta$ exactly as calculated earlier.

2.5 Newton's Third Law

It is very common for two objects to *interact*. This term has a special meaning in Physics: the statement that two objects interact indicates that they exert forces on each other. There is a particular relationship between those forces, expressed in Newton's Third Law. When two objects interact the force that the first object exerts on the second object is equal in magnitude but opposite in direction to the force that the second object exerts on the first. This is often colloquially referred to as 'the forces are equal and opposite', but it is more conveniently expressed mathematically using vectors

$$\vec{F}_{1\ on\ 2} = -\vec{F}_{2\ on\ 1} \tag{2.10}$$

This is *always* true, whether the interacting objects are large or small, moving or stationary, accelerating or in uniform motion. It can be a powerful

[1]

$$\begin{aligned} \cos(\theta \pm 90) &= \mp\sin\theta \\ \sin(\theta \pm 90) &= +\cos\theta \end{aligned} \tag{2.9}$$

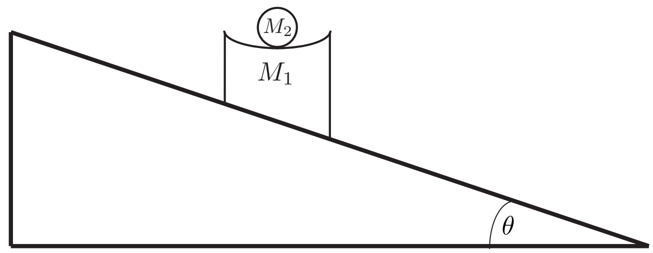

Figure 2.4: An object of mass m_1 on a slope with a second object of mass m_2 resting on top of it.

tool in analysing complex combinations of particles because it is sometimes easier to determine the force one object is subject to or exerts than another.

When analysing a *system* of particles which is in equilibrium the analytical procedure is very similar to what is done for a single particle. One draws a free-body diagram for each individual particle, being careful to note the interaction forces. Those interaction forces are generally unknown, but are related to each other via Newton's Third Law.

2.5.1 Two masses interacting

Consider an object of mass m_1 sitting on a rough sloped surface, with a second object of mass m_2 resting on top of it. The slope makes an angle of θ with the horizontal. The two objects are subject to the downward force of gravity of magnitudes m_1g and m_2g repectively, the object on the slope is subject to a 'normal' force at 90° to the surface on which it is resting, and to a 'friction' force which runs up along the slope. This situation is illustrated in figure 2.4. Supposing that θ and the masses were known, the problem is to determine the magnitudes of the 'normal' and 'friction' forces. Object with mass m_2 is a ball resting in a bowl shaped depression on top of the object with mass m_1. The exact direction of the normal force on the ball may not be immediately obvious. However, it will become clear by analyzing the free body diagrams. Each of the two objects is treated individually because each is, itself, in equilibrium. Each has its own free-body diagram, where it is treated as though it is a point particle. Since the particles are interacting, the forces that each exerts on the other are included in the free body diagrams in figure 2.5. At this stage in the problem, all that is known about the forces that each mass exerts on the other is that they must obey

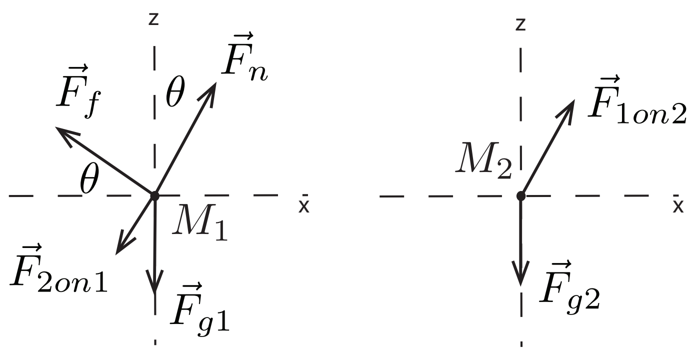

Figure 2.5: Free body diagrams for m_1 and m_2. The forces that the two masses exert on each other are drawn in a way that obviously respects Newton's third law, but is deliberately uncertain about the direction of either force.

Newton's laws. Noticing that each mass is in equilibrium gives the following set of expressions:

$$\begin{aligned} 0 &= \vec{F}_{g1} + \vec{F}_f + \vec{F}_n + \vec{F}_{2\ on\ 1} && \text{First law on first mass} \\ 0 &= \vec{F}_{g2} + \vec{F}_{1\ on\ 2} && \text{First law on second mass} \\ \vec{F}_{1\ on\ 2} &= -\vec{F}_{2\ on\ 1} && \text{Thirdlaw} \end{aligned} \tag{2.11}$$

It is possible to solve these *directly* by *assuming* a direction for the force that the first mass exerts on the second and breaking everything into components as in the previous question. This is not the most efficient way, however, to deal with such problems: using vectors we can immediately write that $\vec{F}_{1\ on\ 2} = -\vec{F}_{g2}$ so $\vec{F}_{2\ on\ 1} = \vec{F}_{g2}$. This can be substituted into the first expression, and solving as previously gives that

$$\begin{aligned} \left|\vec{F}_f\right| &= (m_1 + m_2)\, g \sin\theta \\ \left|\vec{F}_n\right| &= (m_1 + m_2)\, g \cos\theta \end{aligned} \tag{2.12}$$

In this particular example the end result looks as though the two objects were *effectively* a single object with total mass $m_1 + m_2$. Taking the point of view that the two objects form a *single* componsite object, the forces $\vec{F}_{1\ on\ 2}$ and $\vec{F}_{2\ on\ 1}$ both can be thought of as *internal* forces. If the free body diagram for the composite object were drawn it would look essentially identical to that in figure 2.3; the internal forces are omitted.

2.6 Forces

Many of the forces which are discussed in Physics are not *fundamental* in the sense that they come from underlying universal physical laws. They are conventional in the sense that the forces are *assumed* to work in the way described.

2.6.1 Gravity near the Earth's surface

Newton's law of gravitation can be used to determine the gravitational force that a particular object exerts on another one. For object of the human scale which are near the surface of the Earth the overwhelmingly dominant contribution to the force they experience is due to the gravitational force of the Earth itself. This gravitational force is *proportional* to the mass of the affected object, and it is directed directly down:

$$\vec{F}_g = -mg\hat{k} \tag{2.13}$$

This is an approximation which ignores changes in the gravitational force due to changes in height, variations in the Earth's density, and other massive objects. As long as distances being considered do not greatly exceed $1000m$ and the calculation precision sought is at most tenths of a percent, this approximation can be used without concern.

2.6.2 Force provided by a rope

When physicists refer to a 'rope' or 'string' they typically mean an object with two important properties: The magnitude of the force exerted by a rope is known as the 'tension' of the rope, and the direction of the force that a rope exerts is along the length of the rope away from whatever object the force is exerted on. The mathematically precise way of expressing this is to say that the direction of the force exerted by the rope is *tangent* to the rope; the text will provide a mathematical definition of the tangent line in terms of derivatives when calculus is discussed.

Ropes are usually assumed to be massless; in this case the tension in the rope is constant throughout it. In the case that a rope is not massless the tension will vary along the length of the rope. Ropes are also often assumed to be able to exert an arbitrarily large force; this is another approximation. It assumes that the rope will not stretch or break, and can be thought of as equivalent to the assumption that the rope is a rigid object.

2.6.3 Normal Force

When two objects are in contact they typically exert a force one on another. One component of this force is exerted perpendicular to the plane where the objects are touching. This is known as the *Normal Force*, and it is called this because it acts perpendicular (or 'normal') to the surface. This force is usually denoted as $\vec{F}_n$.

The normal force is almost always calculated based on the assumption that the objects involved are perfectly rigid. This means that the force will be *whatever is necessary* to prevent them from moving into each other. In the case of a box resting on a surface the normal force the surface is assumed to exert whatever is necessary to hold the box in equilibrium along a direction which is perpendicular to the surface. The normal force is a constraining force, and is not nessarily related in any way to the force exerted by gravity on an object.

2.6.4 Friction

Two objects in contact may exert a force on each other which is directed along their plane of contact. Contact forces of this type are known as *frictional forces*. The friction force depend on the materials which are in contact. There are two different cases where friction is relevant: when the two objects are moving relative to each other, and when the two objects are stationary relative to each other.

In the case where the two objects move relative to each other, the friction force is known as *kinetic* friction. An example where kinetic friction is relevant is the case where a box slides across a horizontal surface. The magnitude of the force of kinetic friction is approximated as

$$\left|\vec{F}_k\right| = \mu_k \left|\vec{F}_n\right| \tag{2.14}$$

The direction of $\vec{F}_k$ is *opposite* to the relative motion of the objects: if a box slides to the left on the floor then the floor exerts a force of kinetic friction to the right on the box and the box exerts a force to the left on the floor. In the expression for the magnitude of the frictional force the quantity $\left|\vec{F}_n\right|$ is the magnitude of the normal force that one of the two objects exerts on the other, and μ_k is a quantity known as the 'coefficient of kinetic friction'. μ_k must, in principle, be measured for any pair of surfaces, and is approximately constant; it conveys all information about how smooth or rough the two surfaces are. The value of μ_k for a steel surface on smooth ice, in a case like skating, is relatively small, and the similar value for sandpaper on concrete is quite high.

In the case where two objects are stationary relative to each other, the friction force is known as *static* friction. The force of static friction is qualitatively different than kinetic friction because static friction can vary. To understand why consider a large mass (such as a refrigerator) sitting on a horizontal surface: if there are no forces on the mass other than the downwards force of gravity and the upwards normal force from the surface then it can be in equilibrium. Suppose that the refrigerator is pushed horizontally by a small force. Unless the surface it is on is extraordinarily slippery the refrigerator is expected to remain at rest and in equilibrium. The friction force must then be enough to keep the refrigerator at rest, and it will remain at rest in response to progressively larger applied forces until the applied force is so large that the refrigerator starts to slip; the fact that the object starts to slip implies that there is a maximum force of friction that can be exerted. This is expressed as

$$\left|\vec{F}_s\right| \leq \mu_s \left|\vec{F}_n\right| \tag{2.15}$$

The direction of the static friction force is along the plane between the two objects in whatever direction is necessary to maintain equilibrium. It is important to note that the force of static friction, like the force exerted by a rope and like the normal force is a constraining force. It can, in principle, be anything up to the limit described above.

2.6.5 Friction example

A box is being pulled by a rope which goes up and to the right at an angle of θ with respect to the x-axis, it has a mass m and rests on a horizontal rough surface with which it has a coefficient of static friction μ_s. Determine the minimum tension in the rope at which the box will start to slide. This situation is illustrated in figure 2.6. What this problem probes is the boundary between two behaviours for a physical system. If the tension in the rope is small the box will remain at rest and in equilibrium, and if the tension in the rope is large enough the box will start to slide because the force of static friction cannot supply a large enough force to keep the box at rest. The problem of *exactly* how the box will move when it is subject to a non-zero total force will be dealt with in a subsequent chapter – for the purposes of this discussion it is enough to note that in that case the box is not in equilibrium.

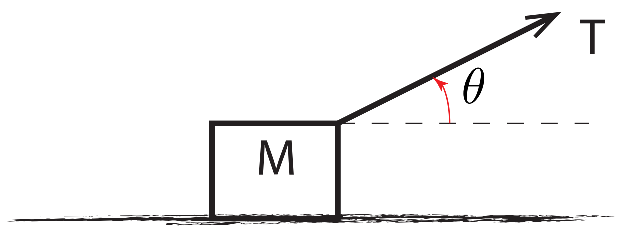

Figure 2.6: A mass resting on a horizontal surface pulled up an to the right by a rope.

There are two conceptual steps to this problem. The first is to *assume* equilibrium and then calculate the appropriate relations between the known forces (the rope, and gravity) and the derivable forces (the normal and friction force). The second step is to combine the results from the first step into a relation which describes which values of the tension in the rope are consistent with equilibrium.

As before, the first step is to enumerate the forces on the box via a free-body diagram. The forces are the downward force of gravity, the upwards normal force of the surface on the box, the pulling force from the rope up and to the right, and the force of static friction to the left. The free body diagram is in figure 2.7. The first step is to assume that the tension in the rope is denoted T, and that the box is in equilibrium. The application of Newton's law for equilibrium is

$$\begin{aligned}
\vec{F}_{net} &= 0 && \text{condition for equilibrium} \\
\vec{F}_{net} &= \vec{F}_g + \vec{F}_n + \vec{F}_s + \vec{F}_r && \text{enumerating forces} \\
0 &= \vec{F}_g + \vec{F}_n + \vec{F}_s + \vec{F}_r && (2.16)
\end{aligned}$$

As before, taking the horizontal and vertical components gives the following pair of linear equations

$$\begin{aligned}
\text{x component} \quad 0 &= -\left|\vec{F}_s\right| + T\cos\theta \\
\text{z component} \quad 0 &= \left|\vec{F}_n\right| - mg + T\sin\theta \qquad (2.17)
\end{aligned}$$

If T and m are known, these relations are enough to determine $\vec{F}_s$ and $\vec{F}_n$.

To combine the relations in 2.17 in order to make a relation which determines what values of T are consistent with equilibrium requires an ad-

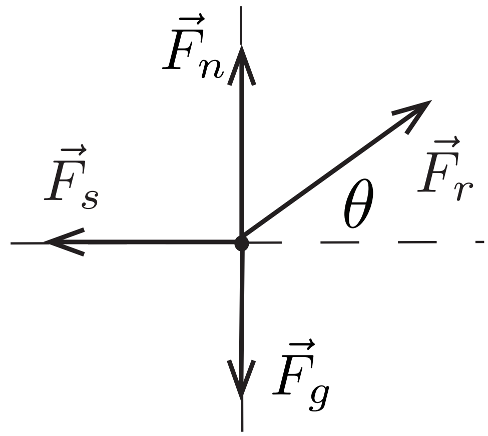

Figure 2.7: A free body diagram for the mass resting on a horizontal surface pulled up an to the right by a rope.

ditional piece of Physics: the relation between the normal force and the force of static friction on an object. Since 2.17 gives that $\left|\vec{F}_s\right| = T\cos\theta$ and $\left|\vec{F}_n\right| = mg - T\sin\theta$, the relation between normal force and friction force becomes

$$\begin{aligned} \left|\vec{F}_s\right| &\leq \mu_s \left|\vec{F}_n\right| \quad \text{friction} \\ T\cos\theta &\leq \mu_s\left(mg - T\sin\theta\right) \quad \text{substituting} \end{aligned} \tag{2.18}$$

which gives an overall relation for T as

$$T \leq \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta} \tag{2.19}$$

The interpretation of this is when T satisfies this inequality there will be an equilibrium situation whereas when T does not satisfy the inequality there cannot be equilibrium.

Chapter 3

Rotational Equilibrium

3.1 Introduction

A second example of extensive practical interest which shows the importance of vector analysis is that of a *rigid object* which is in equilibrium. As discussed in the previous chapter, Newton's first law characterizes the net force of an object in equilibrium. For rigid objects, there are additional considerations depending on the location at which the forces are applied. These can be used to find additional relationships between the external forces on a rigid object.

3.2 Rigid Body Model

The rigid body model is an *extension* of the point particle model, designed to account for the fact that modelling an object as a point particle does not capture all known information about forces which act on an extended object. Calling an extended object 'rigid' is an approximation which neglects any possibility that the object may bend or flex, and also neglects any possibility that it will break.

More specifically, a rigid object is modelled as a large number of individual particles, each of which can exert a force on its neighbours. The forces that they exert can be *arbitrarily* large and are assumed to be whatever is needed to maintain their position relative to the other pieces of the rigid body. When an object bends or breaks, it is a sign that this assumption is incorrect: that the pieces of the object cannot exert enough forces on each other to remain fixed. An alternative way of making the same statement is that treating a body as rigid is an approximation; the approximation is

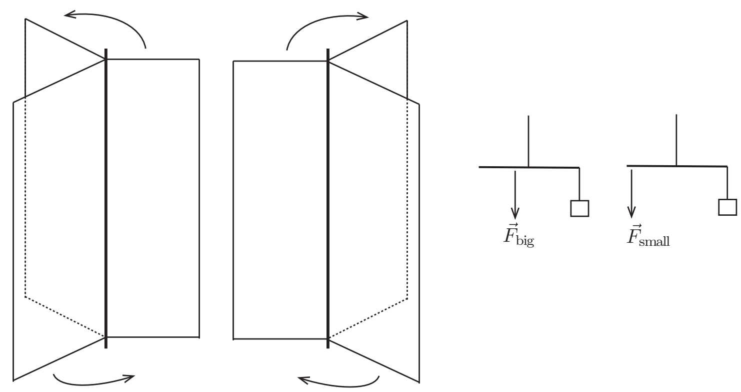

Figure 3.1: A revolving door and a scale.

valid in the case when the applied forces on the body are *small* compared to the possible internal forces.

3.3 Conditions for Rotational Equilibrium

Consider an object such as a revolving door; if the door is subject to zero *net* force then in the particle model the door will remain in equilibrium; it will not move. However common experience suggests that the door may *rotate* in a manner that depends on where it is pushed. Similarly, consider a situation where an object is held suspended from one end a horizontal bar which is pivoted in the middle. To hold the bar level a force must be exerted on the other side of the pivot, and the required force varies with distance from the pivot – as the location the force is exerted moves away from the pivot point, the magnitude of the required force decreases. These situations are illustrated in figure 3.1 These situations for extended objects illustrate that to describe a rigid object it is necessary to consider more than simply the net forces. There must be a quantity that also accounts for *where* the force is being exerted.

The quantity *torque* serves this purpose. The torque $\vec{\tau}$ exerted by force $\vec{F}$ about (or 'around') point P is defined to be

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{3.1}$$

where $\vec{r}$ is the vector from the 'pivot point' P to the location where the force is exerted. Torque is a *vector* quantity, as it is calculated from the cross product (vector product) of two other vector quantities. It is important to note that torque depends on the location P that is chosen as the pivot, or origin. The condition for a rigid object to be in *rotational equilibrium*, that is for the rigid object to not be changing how it is rotating, is that the *net* torque must vanish.

$$0 = \sum_i \vec{\tau}_i \text{ for rotational equilibrium} \tag{3.2}$$

An object which is stationary and not changing how it rotates is in both translational and rotational equilibrium. In that case, the object is described by both Newton's first law *and* the condition that net torque is zero.

$$\begin{aligned} 0 = \vec{F}_{net} &= \sum_i \vec{F}_i \\ 0 = \vec{\tau}_{net} &= \sum_i \vec{r}_i \times \vec{F}_i \end{aligned} \tag{3.3}$$

In this expression, there are assumed to be a number of different forces, $\vec{F}_1, \vec{F}_2, \ldots$, and each force is exerted on an object at the locations $\vec{r}_1, \vec{r}_2, \ldots$ respectively measured from origin P. This provides a set of equations which are linear in the forces $\vec{F}_i$ which can be used to determine or predict an unknown force.

3.3.1 Independence of torque on origin in the case of net zero force

A fact which can be used to simplify many problems involving torque is that as long as the net external force on a rigid object is zero then the torque does not depend on the location chosen as the origin or pivot point. This can often be used to eliminate unknown quantities from the set of equations generated by equilibrium considerations - it is very important to note that this does not change the *physics* content of the problem, it simply may make the problem more straight forward to solve.

Consider an extended and rigid object which is subject to n external forces $\vec{F}_i$ (for $i = 1, \ldots, n$). Each of these forces acts at a location $\vec{r}_i$ measured relative to the centre of the object. This is illustrated in figure 3.2. The condition that the net torque vanishes when measured about the centre of the object can be expressed as

$$0 = \vec{\tau}_{net} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 \tag{3.4}$$

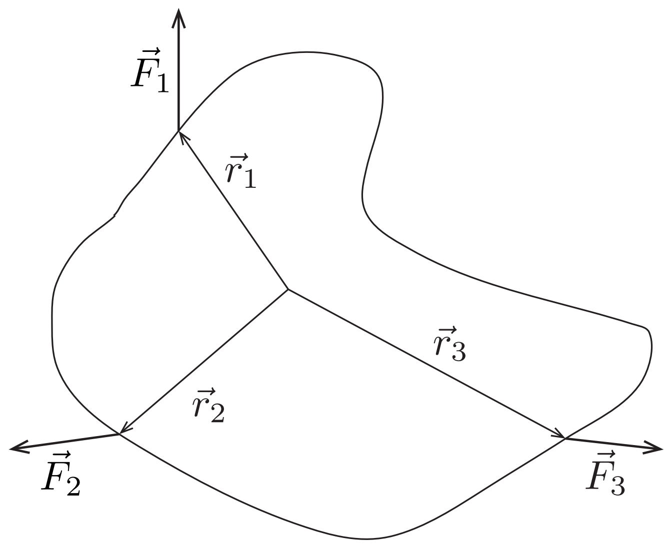

Figure 3.2: Three forces acting on a rigid object.

The torque measured about a different point is calculated by appropriately modifying the $\vec{r}$ vectors appropriately. In figure 3.3 the situation is the same as that illustrated in figure 3.2 except that the pivot is taken to be point P; the centre of the rigid object is at location $\vec{r}_0$ relative to P. In this basis, the total torque will be based on the $\vec{r}_i'$ vectors which are measured from P: $\vec{r}_1' = \vec{r}_0 + \vec{r}_1$ and $\vec{r}_2'$ and $\vec{r}_3'$ are similarly calculated. To calculate the net torque around P, we find

$$\begin{aligned}\vec{\tau}_{around\ P} &= \vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2 + \vec{r}_3' \times \vec{F}_3 \\ &= (\vec{r}_0 + \vec{r}_1) \times \vec{F}_1 + (\vec{r}_0 + \vec{r}_2) \times \vec{F}_2 + (\vec{r}_0 + \vec{r}_3) \times \vec{F}_3 \\ &= \vec{r}_0 \times \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) + \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 \qquad (3.5)\end{aligned}$$

Since the net force is zero ($\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$), this is the *same* as that calculated for the torque about the centre of the object. This demonstrates that in the case of no net force the total torque is independent of origin.

3.3.2 Example of rotational equilibrium

As a specific example of applying the conditions of rotational equilibrium, consider the case of a child of mass m_1 sitting on one end of a massless, rigid

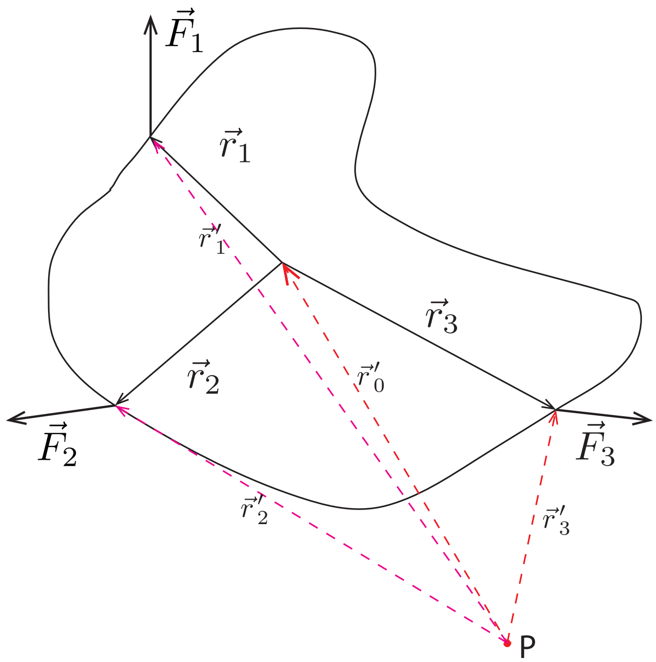

Figure 3.3: Three forces acting on a rigid object which is located at a distance $\vec{r}_0$ from the origin.

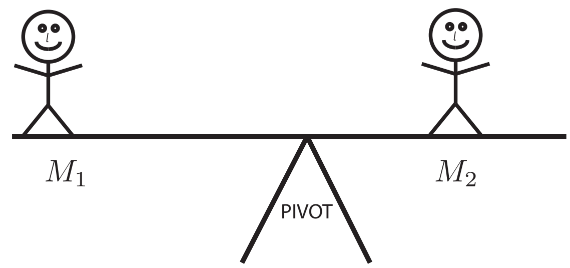

Figure 3.4: A child and an adult sit on a seesaw.

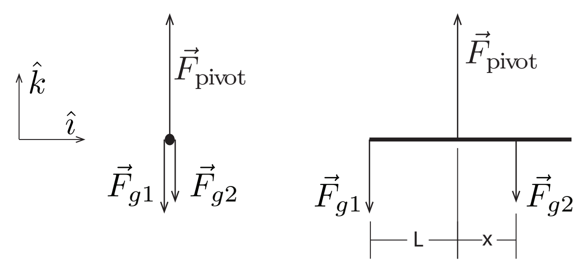

Figure 3.5: A free-body diagram for the system with the child and adult on the seesaw.

rod of length $2L$. The rod is balanced on a pivot at its geometric centre, and an adult of mass m_2 sits on the rod on the other side of the pivot from the child. At what distance x from the pivot must the adult sit for the rod to be in equilibrium? This situation is illustrated in figure 3.4.

The first step in analyzing a problem like this is to draw both a free-body diagram and a sketch which allows the position ($\vec{r}_i$) that each force is exerted to be determined. It is also important to be very specific about *which* object is being considered; for this example, the object of interest is the massless and rigid rod. The appropriate free-body diagram is shown in figure 3.5. It should be noted that since the people are subject to only the forces from gravity and the contact force from the rod, and since they are in equilibrium, the force that each person exerts on the rod has already been determined to be equal to the downward force gravity exerts on each person.

Taking the origin as the left end of the rod, the positions each force is exerted, and the expression of each force in components are

$$\begin{array}{rcc} \text{First person} & \vec{r}_1 = 0 & \vec{F}_{g1} = -m_1 g\hat{k} \\ \text{Second person} & \vec{r}_2 = (L+x)\,\hat{\imath} & \vec{F}_{g2} = -m_2 g\hat{k} \\ \text{Pivot} & \vec{r}_p = L\hat{\imath} & \vec{F}_{pivot} = F_p\hat{k} \end{array} \tag{3.6}$$

The net force is the sum of the three forces on the rod, and since the rod is in *translational* equilibrium, it must be zero. When expressed in components, this is

$$\begin{aligned} 0 = \vec{F}_{net} &= \vec{F}_{g1} + \vec{F}_{g2} + \vec{F}_{pivot} \\ 0 &= -m_1 g - m_2 g + F_p \quad \text{z component} \end{aligned} \tag{3.7}$$

Recalling the method for calculation of cross product, the fact that the rod is in *rotational* equilibrium means that it is subject to zero net torque.

$$\begin{aligned} 0 = \vec{\tau}_{net} &= \vec{r}_1 \times \vec{F}_{g1} + \vec{r}_2 \times \vec{F}_{g2} + \vec{r}_p \times \vec{F}_{pivot} \\ 0 &= \vec{0} \times \left(-m_1 g\hat{k}\right) + [(L+x)\hat{\imath}] \times \left(-m_2 g\hat{k}\right) + [L\hat{\imath}] \times \left(F_p\hat{k}\right) \\ &= (L+x)(m_2 g)\hat{\jmath} - LF_p\hat{\jmath} \\ 0 &= m_2 g(L+x) - LF_p \quad \text{y component} \end{aligned} \tag{3.8}$$

In expressions 3.7 and 3.8 there are *two* unknowns: x and F_p. Solving for the unknowns it can be determined that

$$\begin{aligned} F_p &= m_1 g + m_2 g \\ \text{so } 0 &= m_2 g(L+x) - LF_p \\ \text{gives } x &= \frac{m_1}{m_2} L \end{aligned} \tag{3.9}$$

This same problem could be done using a different origin, and it would lead to the same result. Taking the origin at the pivot point, the position each force is exerted and the expressions of each force in components are

$$\begin{array}{rcc} \text{First person} & \vec{r}_1 = -L\hat{\imath} & \vec{F}_{g1} = -m_1 g\hat{k} \\ \text{Second person} & \vec{r}_2 = x\hat{\imath} & \vec{F}_{g2} = -m_2 g\hat{k} \\ \text{Pivot} & \vec{r}_p = 0 & \vec{F}_{pivot} = F_p\hat{k} \end{array} \tag{3.10}$$

The condition of *translational* equilibrium can be analyzed the same way as above, and again give the relationship that

$$F_p = m_1 g + m_2 g \tag{3.11}$$

The fact that the rod is in *rotational* equilibrium, and hence experiences no torque is expressed as

$$\begin{aligned} 0 = \vec{\tau}_{net} &= \vec{r}_1 \times \vec{F}_{g1} + \vec{r}_2 \times \vec{F}_{g2} + \vec{r}_p \times \vec{F}_{pivot} \\ 0 &= [-L\hat{\imath}] \times \left(-m_1 g\hat{k}\right) + [x\hat{\imath}] \times \left(-m_2 g\hat{k}\right) + 0 \times \left(F_p\hat{k}\right) \\ &= -m_1 gL\hat{\jmath} + m_2 gx\hat{\jmath} \\ \text{y component} \quad 0 &= m_2 gx - m_1 gL \end{aligned} \tag{3.12}$$

The value of x obtained in either method was the same; this will always be true. Very often in analyzing Physics problems there are several correct ways to obtain the desired information; the only difference is that some times there is one or another which offers a more direct way of performing the calculation.

3.4 Centres of Mass

The example discussed above dealt with an *massless* rigid rod which had two people sitting on it. This problem could have been analyzed more simply using the concept of 'Centre of Mass'.

The definition of the centre of mass of a collection of objects is the average location of the collection of objects *weighted* by mass. If there are n objects with mass m_i at locations $\vec{r}_i$, then the location of the centre of mass is

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \tag{3.13}$$

For objects which have spatial extent such as a rigid rod, there is a similar rule: The object is analysed as though it was made of a very large number of small, identical pieces each of mass dm, and each with location $\vec{r}$:

$$\vec{r}_{CM} = \frac{\int_{object} \vec{r} dm}{\int_{object} dm} \tag{3.14}$$

This equations means that extended objects have their centres of mass calculated in the same way as collections of point particles; integration is a mathematical technique to add together a large number of very small quantities. For a uniform object – one with a constant density everywhere – the centre of mass is located at the geometric centre.

The centre of mass of an object is sometimes colloquially called the 'centre of gravity'. The reason for this name is that if a rigid object is subject to the uniform gravitational force near the surface of the earth it can be treated as though the force of gravity *only* acts at its center of mass. For *translational* equilibrium the object is modelled as a point particle, and so the total gravitational force comes from simply adding up the gravitational force on each piece. When *rotational* equilibrium is a consideration, the location of each component is important. For a rigid object composed of n individual pieces, each of mass m_i at location $\vec{r}_i$, the total torque due to gravity (around the origin) is

$$\begin{aligned}
\vec{\tau}_g &= \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{g,i} \\
&= \sum_{i=1}^{n} \vec{r}_i \times \left(-m_i g \hat{k}\right) \\
&= \sum_{i=1}^{n} \left(\frac{m_i \vec{r}_i}{m_1 + m_2 + \ldots}\right) \times \left(-(m_1 + m_2 + \ldots) g \hat{k}\right) \\
&= \vec{r}_{CM} \times \vec{F}_{g,total} \qquad (3.15)
\end{aligned}$$

where the total mass is given by $\sum_{i=1}^{n} m_i$.

This demonstrates that, for the purposes of analyzing torque, the force of gravity (near the Earth's surface) can be treated as though it all occurs at the centre of mass.

3.4.1 Revisiting rotational equilibrium with centre of mass

The example illustrated in 3.4 can be re-analyzed using the concept of centre of mass. Considering the two people (and the massless rod) as a single system it is possible to find the location of their centre of mass.

In the coordinate system where the smaller person is at the origin, m_1 is at $\vec{r}_1 = 0$, and m_2 is at $\vec{r}_2 = (L + x)\,\hat{\imath}$. The centre of mass of the two people is at

$$\vec{r}_{CM} = \frac{m_1 \vec{0} + m_2 (L + x)\,\hat{\imath}}{m_1 + m_2} \qquad (3.16)$$

The condition on x for rotational equilibrium was that $x = \frac{m_1}{m_2} L$; applying that gives that

$$\vec{r}_{CM} = L\hat{\imath} \qquad (3.17)$$

which reframes the original question as 'where does the larger person need to be so that the centre of mass is above the pivot?'

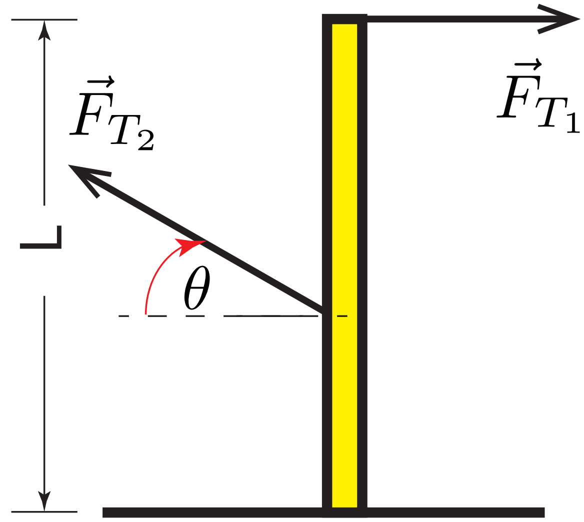

Figure 3.6: A vertical rod attached to two ropes.

3.5 Two examples

3.5.1 A vertical rod

Consider a rigid and uniform rod of length L and mass M, which is balanced vertically on a surface with which it has coefficient of static friction μ, and is tethered to two ropes. One rope is attached to the top end and is horizontally pulling to the right with a force T_1 while the second rope is attached at the mid-point of the rod and pulls with a force of magnitude T_2 up an to the left at an angle of θ above the horizontal. This situation is illustrated in figure 3.6. What is the maximum magnitude of T_1 that the rod can remain in equilibrium?

To address this problem, there are two considerations: the condition of translational equilibrium, and that of rotational equilibrium. As always the starting point is a free-body diagram, and a diagram which indicates where the forces are exerted. This diagram is shown in figure 3.7. The forces exerted on the rod are, taking the bottom of the rod as the origin:

- The downward force of gravity $\vec{F}_g = -Mg\hat{k}$, exerted at the geometric centre of the rod: $\vec{r}_g = \frac{L}{2}\hat{\imath}$.

- The force from rope at the top of the rod. $\vec{F}_1 = T_1\hat{\imath}$ exerted at $\vec{r}_1 = L\hat{k}$

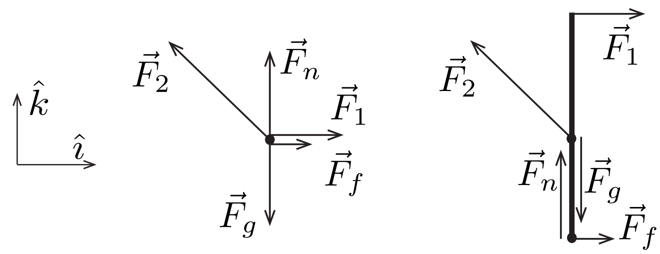

Figure 3.7: A free body diagram showing a vertical rod attached to two ropes.

- The force from the second rope. $\vec{F}_2 = -T_2 \cos\theta \hat{\imath} + T_2 \sin\theta \hat{k}$.
- The normal force exerted by the ground. $\vec{F}_n = F_n \hat{k}$ exerted at $\vec{r}_n = \vec{0}$.
- The friction force exerted by the ground. $\vec{F}_f = F_f \hat{\imath}$ exerted at $\vec{r}_f = \vec{0}$. The friction force will be along the ground, hence in the x-direction, but at this stage it is not certain whether F_f will be positive or negative, i.e. whether the force will point to the right or to the left.

Applying the condition that $\vec{F}_{net} = 0$ and $\vec{\tau}_{net} = 0$, together with the $\vec{F}$s and $\vec{r}$s listed above gives the following three expressions:

$$\begin{aligned} \text{Force x component}: \quad 0 &= T_1 - T_2 \cos\theta + F_f \\ \text{Force z component}: \quad 0 &= -Mg + T_2 \sin\theta + F_n \\ \text{Torque y component}: \quad 0 &= LT_1 - \frac{L}{2} T_2 \cos\theta \end{aligned} \tag{3.18}$$

From this set of equations all of F_f, F_n, and T_2 can be expressed in terms of the known quantities (M and θ) and one unknown quantity (T_1) as follows:

$$\begin{aligned} T_2 &= \frac{2}{\cos\theta} T_1 \\ F_n &= Mg - 2T_1 \tan\theta \\ F_f &= T_1 \end{aligned} \tag{3.19}$$

These are the conditions for equilibrium; to determine if equilibrium is *possible* for a given value of T_1 the next step is to use the fact known about

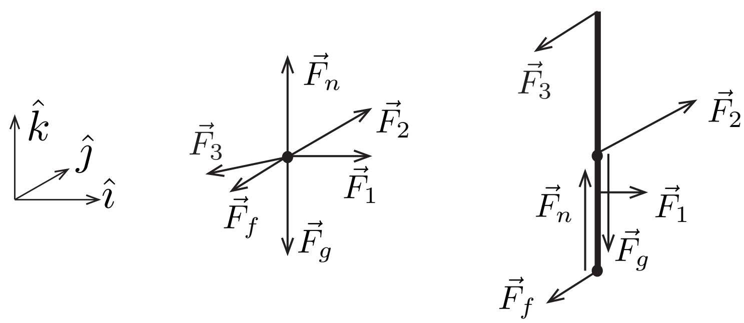

Figure 3.8: A vertical rod attached to three ropes pulling horizontally.

friction, that $\left|\vec{F}_f\right| \leq \mu \left|\vec{F}_n\right|$. Applying this indicates that

$$\begin{aligned} F_f &\leq \mu F_n \\ T_1 &\leq \mu \left(Mg - 2T_1 \tan\theta\right) \\ T_1 &\leq \frac{\mu Mg}{1 + 2\tan\theta} \end{aligned} \tag{3.20}$$

What this indicates is that any time $T_1 > \frac{\mu Mg}{1+2\tan\theta}$ for this scenario there cannot be equilibrium.

3.5.2 Torque considering three dimensions

Consider a rod of mass m and length L which is oriented vertically ($\hat{k}$ direction) on a horizontal rough surface and is subject to horizontal forces from three ropes. One rope is fixed a distance of $\frac{L}{4}$ from the ground and exerts a force $\vec{F}_1 = F\hat{\imath}$. A second rope is fixed a distance $\frac{L}{2}$ from the ground and exerts a force $\vec{F}_2 = F\hat{\jmath}$. A third rope exerts a horizontal force with components $\vec{F}_3 = F_{3,x}\hat{\imath} + F_{3,y}\hat{\jmath}$. What are the components of $\vec{F}_3$, and what is the force of friction exerted on the bottom of the rod?

The free-body diagram for this situation is illustrated in figure 3.8. When these are analyzed in the same pattern as above, the resulting relations are:

$$\begin{aligned}
\text{x component Force}: \quad 0 &= F + F_{3,x} + F_{F,x} \\
\text{y component Force}: \quad 0 &= F + F_{3,y} + F_{f,y} \\
\text{z component Force}: \quad 0 &= -mg + F_n \\
\text{x component Torque}: \quad 0 &= -\frac{L}{2}F - LF_{3,y} \\
\text{y component Torque}: \quad 0 &= \frac{L}{4}F + LF_{3,x}
\end{aligned} \tag{3.21}$$

which results in $\vec{F}_3 = -\frac{1}{4}F\hat{\imath} - \frac{1}{2}F\hat{\jmath}$ and $\vec{F}_f = -\frac{3}{4}F\hat{\imath} - \frac{1}{2}F\hat{\jmath}$.

Chapter 4

Differential Calculus

4.1 Introduction

The text so far has discussed the use of *vectors* to describe position, forces, and other quantities. All the problems considered so far have been *static* in nature; the fact that an object moved was, at best, incidental to the analysis done. Physics is more than the study of static objects. Physics studies how an object moves: how the *position vector* of the object under consideration *changes in time*, and how that motion changes in response to applied forces.

To study motion it is necessary to have a precise and mathematical expression which quantifies the colloquial term 'change'. Just as it was useful to develop vector analysis to describe position, the idea of change requires the development of *differential calculus*

4.2 Graphical Representations of functions

A tool that is often used in analysis in both Physics and Mathematics is the idea of a graphical representation of a function. In the case of physics, these are normally used to guide intuition about how a particular object of quantity will behave - in this sense, they are an adjunct, rather than a critical part of the problem.

When illustrating a function of a single variable, the convention in mathematics is typically to draw a pair of axes, label the horizontal axis x, label the vertical axis y, and to describe a line drawn on that plot by the *function* $y = f(x)$. What this means is that *given* a particular value of x it is possible to uniquely determine a corresponding y (using the rule that $y = f(x)$) and the point (x, y) appears on the plot. As there are many pos-

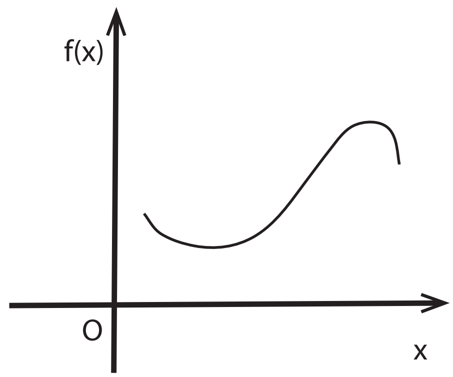

Figure 4.1: A sketch of a function $y = f(x)$.

sible values of x that could be chosen the line drawn is the set of *all* points generated by that rule. In this sense, the graph and the statement of the function $f(x)$ are equivalent expressions of the same information. A critical piece of nomenclature is that the x is typically known in mathematics as the 'independent variable' while the y (calculated by $f(x)$) is the 'dependent variable'. The sense of this is that *knowing* the independent variable allows the *determination* of the dependent variable.

The usage in Physics is somewhat different than in Mathematics. The *main* difference is that there are different independent variable depending on context. In the case of motion, the *position* (a vector) of an object varies depending on the independent variable time (t): knowing the time allows the determination of the position of an object. A second example is that of quantities such as electric potential, or potential energy in general, which depend on position: in this case location is the independent variable and potential can be determined depending on location.

4.3 Derivatives and Slopes

The fundamental question that differential calculus attempts to solve is the following: *Given a small change in the independent variable, what is the corresponding change in the dependent variable?* Another way of expressing the same idea in a physics context is to ask *how much does an object move in a very small amount of time δt?*

The development of these ideas will be done using the convention where

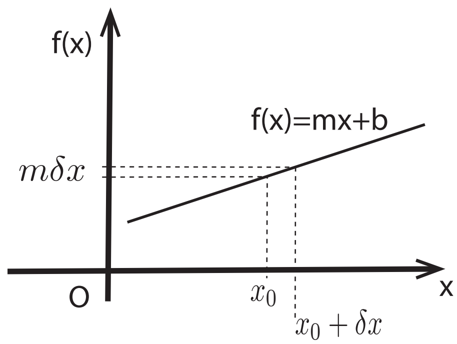

Figure 4.2: The linear function $f(x) = mx + b$. The change in $f(x)$ as x changes from x_0 to $x_0 + \delta x$ is shown.

the independent variable is x and the dependent variable is $f(x)$. The resulting expressions can be easily translated to contexts where the independent variable is differently named through appropriate substitution of variable names.

The initial instructive case is that of a linear function: one given by $f(x) = mx + b$ where m and b are constants. Suppose that the value of $f(x)$ is known at a particular value of the dependent variable, specifically the location $x = x_0$. If x is increased by a small amount δx the new value of x is $x_0 + \delta x$, and the new value of $f(x)$ is $f(x_0 + \delta x) = m(x_0 + \delta x) + b$. The amount that the function $f(x)$ changed as its argument (the independent variable) changed from x_0 to $x_0 + \delta x$ is

$$\begin{aligned} \text{change in } f(x) &= f(x_0 + \delta x) - f(x_0) \\ &= m(x_0 + \delta x) + b - [mx_0 + b] = m\delta x \end{aligned} \tag{4.1}$$

This is illustrated in figure 4.2. The change in $f(x)$ as the argument increases by δx is $m\delta x$. For this linear function, the change did not depend on what x_0 was, and it only depended linearly on δx.

A second instructive case is that of a simple quadratic function: one given by $f(x) = x^2$. Following a similar procedure to the linear case, the change in $f(x)$ as x is increased from x_0 to $x_0 + \delta x$ is calculated.

$$\begin{aligned} \text{change in } f(x) &= f(x_0 + \delta x) - f(x_0) \\ &= (x_0 + \delta x)^2 + (x_0)^2 \\ &= 2x_0\delta x + (\delta x)^2 \end{aligned} \tag{4.2}$$

There are two differences between this and the linear function: the change *depends* on x_0, and there are higher order terms (in this case, quadratic) terms in δx. It is important to remember that the motivating question was what the change in the function was for a *small* change in the independent variable. In other words, for a small value of δx. If δx is small, then $(\delta x)^2$ (or any positive power of δx) will be very small; for small enough δxs all higher powers of δx are so small they can be ignored. In this case

$$\text{change in } f(x) \quad \approx \quad 2x_0\delta x \tag{4.3}$$

This illustrates the general point that for an arbitrary $f(x)$ there will be a related function, called $f'(x)$ which satisfies

$$\begin{aligned} \text{change in } f(x) &\approx f'(x_0)\delta x \\ \text{or } f(x_0+\delta x) &\approx f(x_0) + f'(x_0)\delta x \end{aligned} \tag{4.4}$$

The algorithm for calculating this function $f'(x)$ which does not make use of an imprecise claim that 'higher powers of δx all get very small' is the following:

$$f'(x_0) \quad = \quad \lim_{\delta x \to 0} \frac{f(x_0+\delta x) - f(x_0)}{\delta x} \tag{4.5}$$

This limiting process ensures that all powers of δx which are higher than linear will evaluate to zero, but the linear term remains. The commonly used symbol for $f'(x)$, this function which is linearly proportional to the change of the dependent variable is

$$\frac{d}{dx}f(x) = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \tag{4.6}$$

This is known as the *derivative* of $f(x)$. The derivative can be thought of graphically as the *slope* of the tangent line to the function $y = f(x)$, however in Physics it is typically more useful to think of the derivative as the quantity that gives a linear approximation to the change of a function:

$$f(x+\delta x) \approx f(x) + \left(\frac{d}{dx}f(x)\right)\delta x \text{ for small } \delta x \tag{4.7}$$

4.4 Calculating Derivatives

Many Physics problems can be addressed by knowing how to differentiate a relatively small number of functions. These functions, and general rules for calculating other derivatives are discussed.

4.4.1 Polynomials

The derivative of a constant is 0. This can be understood because derivatives measure the change of a function (a dependent variable) based on a small change in the argument (the independent variable).

Suppose that $f(x) = c$ (with c a constant). Then $f(x + \delta x) = c$ as well, and using the definition of derivative:

$$\begin{aligned} \frac{d}{dx} f(x) &= \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ \frac{d}{dx} c &= \lim_{\delta x \to 0} \frac{c - c}{\delta x} = 0 \end{aligned} \tag{4.8}$$

The derivative of a power of x can be calculated in a similar way. Suppose that $f(x) = x^n$ where n is a fixed number. In that case, the derivative, using the definition above, can be calculated to be

$$\frac{d}{dx} x^n = n x^{n-1} \tag{4.9}$$

As an example of that, suppose that $n = 3$, so we have $f(x) = x^3$, then using the definition of derivative:

$$\begin{aligned} \frac{d}{dx} f(x) &= \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ \frac{d}{dx} x^3 &= \lim_{\delta x \to 0} \frac{(x + \delta x)^3 - x^3}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{x^3 + 3x^2 \delta x + 3x(\delta x)^2 + (\delta x)^3 - x^3}{\delta x} \\ &= 3x^2 \end{aligned} \tag{4.10}$$

This example should be suggestive that the rule described works for all positive integer ns; it is, in fact, correct for all values of n.

The derivative of the sum of two functions is the sum of the derivatives:

$$\frac{d}{dx} \left(f(x) + g(x) \right) = \left(\frac{d}{dx} f(x) \right) + \left(\frac{d}{dx} g(x) \right) \tag{4.11}$$

This statement is the same as the statement that the derivative operator is a *linear*. The reason the derivative of the sum of two functions is the sum of the derivatives is that since the total function is the sum, any change will be due to a change in one function, the other function, or both. The implication of this is that there is a simple rule for the derivative of the product of a number and a function. If c is any constant and $f(x)$ is any function, then

$$\frac{d}{dx} c f(x) = c \frac{d}{dx} f(x) \tag{4.12}$$

This can be motivated by considering $c = 2$ and $f(x) = x^2$. Based on the rule above,

$$\begin{aligned} \frac{d}{dx}\left(2x^2\right) &= \frac{d}{dx}\left(x^2 + x^2\right) \\ &= \left(\frac{d}{dx}x^2\right) + \left(\frac{d}{dx}x^2\right) \\ &= 2\left(\frac{d}{dx}x^2\right) = 2\left(2x\right) \end{aligned} \tag{4.13}$$

This motivates the result for any positive integer, but the result is valid for any value of c.

4.4.2 Product rule and Chain rule

If two functions are multiplied by each other, there is a simple expression for their total derivative. Assuming that $f(x)$ and $g(x)$ are both arbitrary functions of x, and $f'(x)$ and $g'(x)$ are their respective derivatives, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \tag{4.14}$$

This relation is known as the product rule. The reason for this form of the rule is that as x changes by a small amount to $x + \delta x$ both $f(x)$ and $g(x)$ change a bit, each proportional to δx – in fact the new value of

$$\begin{aligned} f(x+\delta x)g(x+\delta x) &\approx \left(f(x) + f'(x)\delta x\right)\left(g(x) + g'(x)\delta x\right) \\ &= f(x)g(x) + f'(x)g(x)\delta x + f(x)g'(x)\delta x + f'(x)g'(x)\delta x^2 \end{aligned} \tag{4.15}$$

The piece that is linear in δx is the derivative of $f(x)g(x)$.

This can be illustrated using the known facts about polynomials. Consider the function $h(x) = x^3 + 4x^2$. It's derivative can be calculated easily:

$$\frac{d}{dx}h(x) = \frac{d}{dx}\left(x^3 + 4x^2\right) = 3x^2 + 8x \tag{4.16}$$

However, $h(x) = x^2\,(x+4)$, so with $f(x) = x^2$ and $g(x) = x+4$ (and hence $f'(x) = 2x$ and $g'(x) = 1$) it is possible to use the product rule to determine the derivative:

$$\begin{aligned}\frac{d}{dx}h(x) &= \frac{d}{dx}\left(f(x)g(x)\right) \\ &= f'(x)g(x) + f(x)g'(x) \\ &= 2x\,(x+4) + x^2\,(1) = 3x^2 + 8x \qquad (4.17)\end{aligned}$$

This is not a *proof*, but rather a suggestive argument. Note that this also confirms a previous rule, that $\frac{d}{dx}cf(x) = cf'(x)$: this is consistent with the product rule and the observation that if $g(x) = c$ then $g'(x) = 0$.

When taking the derivative of a function that does not have just the independent variable, but rather a *function* of that independent variable as its argument, the chain rule is used. If $f(x)$ and $g(x)$ are both functions then

$$\frac{d}{dx}f\left(g(x)\right) = f'\left(g(x)\right)g'(x) \qquad (4.18)$$

The motivation for this rule is that as x changes, $g(x)$ changes by a little (proportional to $g'(x)$ and δx). However the change in $g(x)$ means that the *argument* of the function f has changed by the change in $g(x)$, not just by the change in x, so the effective change is $g'(x)\delta x$ however the derivative measures the constant of proportionality of the change of f to δx.

The chain rule can be illustrated by considering the function $h(x) = \left(x^3 + 3x\right)^2$. The function can be expanded so $h(x) = x^6 + 6x^4 + 9x^2$ which makes it obvious that $h'(x) = 6x^5 + 24x^3 + 18x$. The function can also be written as $h(x) = f(g(x))$ where $g(x) = x^3 + 3x$ and $f(x) = x^2$; Since then $f'(x) = 2x$ and $g'(x) = 3x^2 + 3$, we have

$$\begin{aligned}\frac{d}{dx}h(x) &= \frac{d}{dx}f\left(g(x)\right) \\ &= f'\left(g(x)\right)g'(x) \\ &= 2\left(x^3 + 3x\right)\left[3x^2 + 3\right] \\ &= 6x^5 + 24x^3 + 18x \qquad (4.19)\end{aligned}$$

as expected. Again, this example does not prove that the rule is generally true, however it *suggests* that the rule is true because the rule reproduces a result which can be obtained another way.

The so-called quotient rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2} \qquad (4.20)$$

is a straightforward application of the product rule and the chain rule.

4.5 The exponential function and the logarithm

There is a *special* function that has the unique property that the derivative of the function has the same value as the function itself. In other words, that

$$\frac{d}{dx}f(x) = f(x) \tag{4.21}$$

This function is known as the *exponential* function, and is denoted e^x. This means that

$$\begin{aligned} \frac{d}{dx}e^x &= e^x \quad \text{and} \\ \frac{d}{dx}e^{ax} &= ae^{ax} \end{aligned} \tag{4.22}$$

The second equality comes from the application of the chain rule and the quantity a being a constant.

It is also possible to deduce the rule for multiplying two exponential functions from this and the product rule. Consider $h(x) = e^x e^{2x}$. Applying the product rule (and the chain rule) the derivative is

$$\frac{d}{dx}h(x) = e^x e^{2x} + e^x\left(2e^{2x}\right) = 3e^x e^{2x} \tag{4.23}$$

and since the derivative of $e^x e^{2x} = 3e^x e^{2x}$ the definition of the exponential function is such that it must be e^{3x}. In general, this means that $e^{f(x)}e^{g(x)} = e^{f(x)+g(x)}$.

The *natural* logarithm is the function that is the *inverse* of the exponential function:

$$\ln\left(e^x\right) = x \quad \text{and} \quad e^{\ln x} = x \tag{4.24}$$

The chain rule can be used to determine the derivative of the logarithm. Consider $h(x) = e^{\ln x} = x$; clearly $\frac{d}{dx}h(x) = 1$, but using the chain rule, with $f(x) = e^x$ and $g(x) = \ln x$ we have

$$\begin{aligned} \frac{d}{dx}e^{\ln x} &= e^{\ln x}\left(\frac{d}{dx}\ln x\right) \\ &= x\left(\frac{d}{dx}\ln x\right) \end{aligned} \tag{4.25}$$

In order for this to be 1, as required, it must be that $\frac{d}{dx}\ln x = \frac{1}{x}$.

This requirement can be used to determine how the logarithm of a product behaves: Consider $h(x) = \ln(f(x)g(x))$. Then

$$\begin{aligned}
\frac{d}{dx}h(x) &= \frac{d}{dx}\ln(f(x)g(x)) \\
&= \frac{1}{f(x)g(x)}\frac{d}{dx}(f(x)g(x)) \quad \text{by chain rule} \\
&= \frac{1}{f(x)g(x)}\left(f'(x)g(x) + f(x)g'(x)\right) \quad \text{by product rule} \\
&= \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad (4.26)
\end{aligned}$$

which suggests that $\ln(f(x)g(x)) = \ln f(x) + \ln g(x)$.

4.6 Interlude: complex numbers

An exceptionally easy application of algebra is the complex number. The complex numbers are a combination of regular ('real') numbers and multiples of i. The quantity i has the property that $i^2 = -1$.

One situation where imaginary numbers can be very useful is in the solution of quadratic equations: In the expression for the solution of a quadratic equation there is a square root, and the argument of the square root can be either positive or negative. In the case where the quantity inside the square root is *negative* an i can be extracted, and the possible solutions have an imaginary component. A wonderful fact is that all polynomials of nth order have a total of n complex numbers for which the polynomial is zero.

A different useful fact, and one that is closely related to the study of vectors earlier is that a complex number can be thought of as a geometrical object: specifically a two-dimensional vector. Figure 4.3 shows the point z on a cartesian coordinate system where the axes are x and iy.

Using the notation $z = x + iy$ for a complex number it can be shown that

$$\begin{aligned}
z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\
&= (x_1 + x_2) + i(y_1 + y_2) \qquad (4.27)
\end{aligned}$$

This is exactly the addition rule for two vectors in a regular two-dimensional vector space. Using this intuition, there is a relation between the *angle* θ between the positive real (x) axis and the line from the origin ($z = 0$) to the point $z = x + iy$. Using notation developed for vectors relation is that

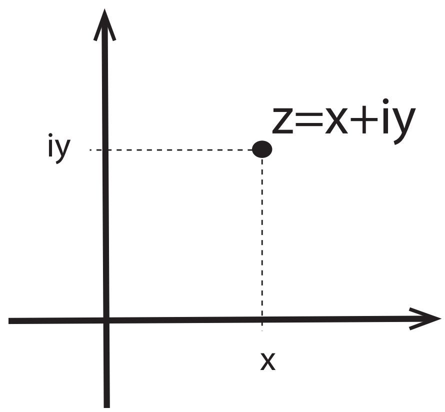

Figure 4.3: The point $z = x + iy$.

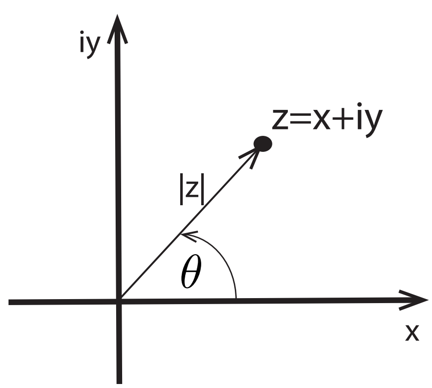

Figure 4.4: The point $z = x + iy$ with the angle θ shown.

$x = |z|\cos\theta$ and $y = |z|\sin\theta$. In this, as expected, $|z| = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$ as is illustrated in figure 4.4.

There is also an obvious multiplication rule for two complex numbers:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - r_1r_2\sin\theta_1\sin\theta_2) \\ &+ ir_1r_2(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \end{aligned} \tag{4.28}$$

This was obtained by a straightforward application of the distributive rule.

Using the identities that $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\sin(A+B) = \sin A\cos B + \cos A\sin B$, it is apparent that the angle that the *product* of z_1 and z_2 makes with the real axis is the *sum* of the angles that z_1 and z_2 make. The magnitude of the result is clearly the product of the magnitudes of z_1 and z_2.

These two facts give rise to a second representation of complex numbers: For a complex number defined by a magnitude r and an angle θ with the real axis, we can write

$$z = r\cos\theta + ir\sin\theta \tag{4.29}$$

and the addition rule just derived for angles means that the combination $\cos\theta + i\sin\theta$ can be written

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{4.30}$$

This is a relationship of fundamental importance that is used often in many physics applications. For example, the product of two complex numbers can be written as,

$$z_1z_2 = r_1e^{\theta_1}r_2e^{\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} \tag{4.31}$$

4.7 Sine and Cosine

Using Euler's identity, that

$$e^{ix} = \cos x + i\sin x \tag{4.32}$$

together with the chain rule can be used to determine the derivatives of $\sin x$ and $\cos x$.

Consider

$$\begin{aligned}
\frac{d}{dx}e^{ix} &= \frac{d}{dx}\left(\cos x + i\sin x\right) \\
ie^{ix} &= \frac{d}{dx}\cos x + i\frac{d}{dx}\sin x \\
i\left(\cos x + i\sin x\right) &= \frac{d}{dx}\cos x + i\frac{d}{dx}\sin x \\
-\sin x + i\cos x &= \frac{d}{dx}\cos x + i\frac{d}{dx}\sin x \qquad (4.33)
\end{aligned}$$

Equating the real and imaginary parts of this expression gives

$$\begin{aligned}
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\sin x &= \cos x \qquad (4.34)
\end{aligned}$$

Derivatives of other functions such as $\tan x$ can be obtained using known values for the derivatives of $\sin x$ and $\cos x$ together with the product and chain rules.

4.8 Advanced topic: Taylor polynomials

It is possible to take many derivatives of a function, and most functions in Physics are continuous and well-behaved, so that their derivatives are also well-behaved. This allows the function to be well-approximated near a point by knowing its value at that particular point and its derivatives.

The general mnemonic is to say that

$$\begin{aligned}
f(x_0+\delta x) &= \sum_{n=0}^{\infty}\frac{1}{n!}f^n(x_0)\left(\delta x\right)^n \\
&= f(x_0) + f'(x_0)\delta x + \frac{1}{2}f''(x_0)\left(\delta x\right)^2 + \frac{1}{6}f'''(x_0)\left(\delta x\right)^3 + \ldots \qquad (4.35)
\end{aligned}$$

For most purposes in Physics, when a function is being approximated it suffices to know the first few terms of this Taylor expansion. Note that the additional terms are *modifications* to the linear approximation that is the 'soul' of the derivative.

Some of the most commonly used expansions are around $x = 0$. In terms of those:

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\
\cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\
\sin x &= x - \frac{1}{3}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \qquad (4.36)
\end{aligned}$$

The expansion for e^x can be verified by direct differentiation. The expansions for $\sin x$ and $\cos x$ can be checked by differentiating each other. They can also be checked using the Euler expression $e^{ix} = \cos x + i \sin x$ and the expansion for e^x together with the fact that $i^3 = -i,\ i^5 = i, \dots$. They can also be checked by taking the nth derivatives and evaluating at $x = 0$ as in the defining relation for the Taylor polynomials. Note that the derivative of the expansion for $\ln(1+x)$ gives a geometric series which converges (for $|x| < 1$) to $\frac{1}{1+x}$ as expected.

Chapter 5

Kinematics

5.1 Introduction

The physics of moving objects is called **kinematics**. It is remarkable subject for its enormous breadth of applications as well as its mathematical beauty. The combination of vectors and calculus allows us the describe the motion of any particle. A three-dimensional approach is developed first and then motion in one dimension is treated as a special case. In many ways the three-dimensional treatment is less ambiguous and more straight forward then one dimensional kinematics.

5.2 Displacement

The remarkable aspect of kinematics is that it can all be derived and understood by starting with one idea that most people already have a good intuitive understanding of. The distance and direction between two places or points in space is all that is needed, along with the mathematics of vectors and calculus. In physics, this quantity is call the **displacement**.

Consider two points, an initial point, P_i, and a final point, P_f. An example of two points and their coordinates in a cartesian coordinate system is illustrated in figure 5.1. The coordinates of the points are given by (x_i, y_i, z_i) and (x_f, y_f, z_f), respectively. **Position vectors** are drawn from the origin to each point. The displacement is defined as the following vector quantity with units of meters:

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i. \tag{5.1}$$

The Δ indicates a difference is being taken between two quantities; in this case, two position vectors. When the Δ is used, it is always the **final - initial**

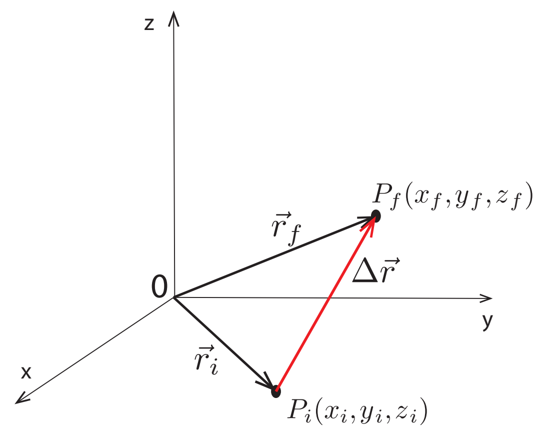

Figure 5.1: Vector description of displacement.

quantity. Following the rules for vector subtraction given in chapter 1, one gets,

$$\begin{aligned}
\vec{r}_i &= x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}, \\
\vec{r}_f &= x_f\hat{\imath} + y_f\hat{\jmath} + z_f\hat{k}, \\
\Delta\vec{r} &= (x_f\hat{\imath} + y_f\hat{\jmath} + z_f\hat{k}) - (x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}), \\
&= (x_f - x_i)\hat{\imath} + (y_f - y_i)\hat{\jmath} + (z_f - z_i)\hat{k}, \\
&= \Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}.
\end{aligned} \tag{5.2}$$

This is an equation given in terms of the x, y and z values.

The next step is to imagine that a particle moves from the initial point to the final point. The position can be written as position vector that depends on time,

$$\vec{r}_i = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \tag{5.3}$$

where $x(t)$ is a function of time and so on for $y(t)$ and $z(t)$. Here the importance of cartesian coordinates with orthogonal axes comes into play. Even though the trajectory of the particle could describe a possibly very complicated function in space, it can always be described by knowing the coordinates, individually, as a function of time.

The initial position can now be written as the position vector at an initial time, t_i, and similarly the final position is given by the position at a final time, t_f. This is illustrated in figure 5.2. The equation for the position vectors at the initial and final times are given by,

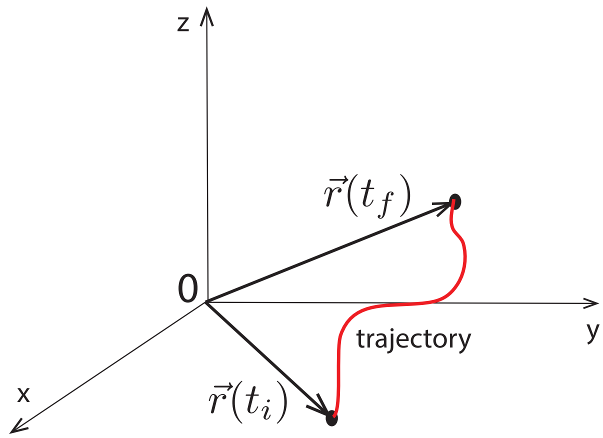

Figure 5.2: Vector description of position as a function of time

$$\begin{aligned}
\vec{r}(t_i) &= x(t_i)\hat{\imath} + y(t_i)\hat{\jmath} + z(t_i)\hat{k}, \\
\vec{r}(t_f) &= x(t_f)\hat{\imath} + y(t_f)\hat{\jmath} + z(t_f)\hat{k}.
\end{aligned} \tag{5.4}$$

Consider what the displacement is between these points.

$$\begin{aligned}
\Delta\vec{r} &= [x(t_f)\hat{\imath} + y(t_f)\hat{\jmath} + z(t_f)\hat{k}] - [x(t_i)\hat{\imath} + y(t_i)\hat{\jmath} + z(t_i)\hat{k}], \\
&= [x(t_f) - x(t_i)]\hat{\imath} + [y(t_f) - y(t_i)]\hat{\jmath} + [z(t_f) - z(t_i)]\hat{k}.
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
\Delta x = x(t_f) - x(t_f), \\
\Delta y = y(t_f) - y(t_f), \\
\Delta z = z(t_f) - z(t_f).
\end{aligned} \tag{5.6}$$

It is exactly the same vector as in figure 5.1. it is not the trajectory shown in figure 5.2. The displacement vector gives the straight line distance and direction from the initial point to the final point.

5.3 Space and Time

In order to set up a theory that can describe the motion of an object through space as a function of time, a closer look at what is meant by space, time and an object is required. Newtonian mechanics is based on the idea of absolute space. Imagine a set of three orthogonal infinitely long rulers. A position along the ruler is absolute. It does not depend on time, the other rulers or any objects. Space is uniform and isotropic.

Time is also taken to be absolute. It does not depend on the space position or any objects. Time passes at a uniform unchanging rate.

Objects are taken as particles that exist at a point and the kinematics described in this chapter are those of a particle. The justification for this model and how it is related to extended objects will be discussed in future chapters.

5.3.1 Average Velocity

The displacement defined in equation 5.5 can be used to define an average velocity,

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}, \tag{5.7}$$

where $\Delta t = t_f - t_i$. The average velocity is a vector with units of meters/second $[m/s]$. It is quite different from what one would refer to as average speed, say from a car journey. There, the distance travelled along the trajectory (i.e. along the road) divided by the time elapsed is what is colloquially referred to as average speed. However, the average velocity defined in equation 5.7 is the more useful physical quantity even though the displacement gives the straight line distance.

5.4 Instantaneous Velocity

When moving along a path that is not straight the direction and magnitude of the motion changes continually. The average velocity is a physical quantity with direction and magnitude but is fixed by the initial and final positions used in the calculation. At this, point the tools of differential calculus are used to describe a vector velocity that is given as a function of time.

This could be quite a difficult problem. The trajectory of the particle through space seems like the natural place to start. An example of a charged particle, starting with a initial velocity and moving under the influence of a magnetic and electric field is shown in figure 5.3. The particle starts at $x = 1$[m], $y = 0$[m] and $z = 0$[m] at an initial time, $t = 0$[s], and proceeds in the direction of the arrow until the plot ends at $t = 15$[s]. No closed form equation of the type, $f(x(t), y(t), z(t)) = c$ where c is a constant exists for this trajectory but particles certainly do follow this path.

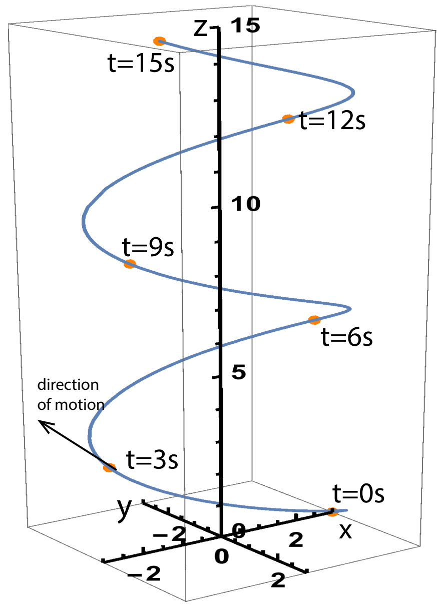

Figure 5.3: A charged particle moving in a spiral-like trajectory under the influence of magnetic and electric fields.

However, there is a simpler way to look at trajectories and, moreover, it is just equation 5.3. All the points at the end of the vector $\vec{r}(t)$ as the parameter t, time, changes make up the trajectory. The three functions $x(t)$, $y(t)$ and $z(t)$ that give the components of $\vec{r}(t)$ together describe the trajectory.

Equation 5.3 can be modified to find the velocity at each instant by taking the limit as $\Delta t \to 0$ and finding the displacement between $\vec{r}(t)$ and $\vec{r}(t + \Delta t)$,

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}(t)}{dt}. \tag{5.8}$$

This looks like and in fact is a derivative of a vector. It can be written as,

$$\vec{v}(t) = \frac{d}{dt}[x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}] = \frac{dx(t)}{dt}\hat{\imath} + \frac{dy(t)}{dt}\hat{\jmath} + \frac{dz(t)}{dt}\hat{k}. \tag{5.9}$$

The derivative gives the slope of the trajectory at a give time, t. This is illustrated in figure 5.4 See chapter 4 for an explanation of the derivatives used in equations 5.8 and 5.9.

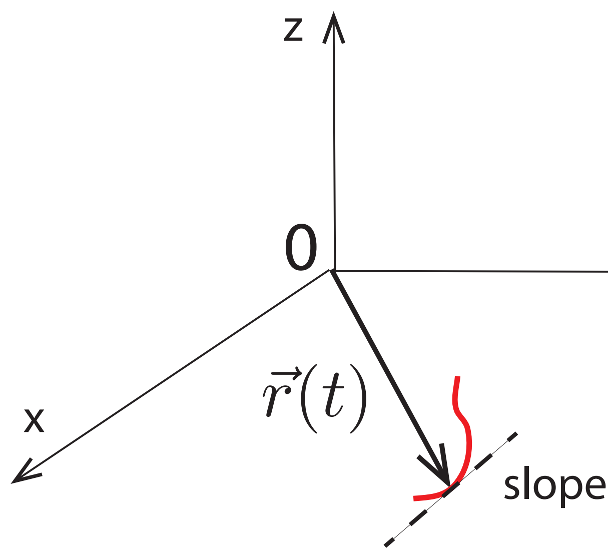

Figure 5.4: The position vector at a time t is shown along with a segment of a particle trajectory. The slope of the trajectory is shown as a dashed line. The slope is given by the derivative of the position vector as explained in the text.

Note that $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are constant and therefore, the derivative of these cartesian unit vectors is zero. The velocity vector can be written in terms of velocity components, $\vec{v}(t) = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath} + v_z(t)\hat{k}$ and therefore the components are given by,

$$\begin{aligned} v_x(t) &= \frac{dx(t)}{dt}, \\ v_y(t) &= \frac{dy(t)}{dt}, \\ v_z(t) &= \frac{dz(t)}{dt}. \end{aligned} \tag{5.10}$$

Therefore, if we know these functions, we know the velocity. The magnitude of the velocity $|\vec{v}(t)| = \sqrt{v_x(t)^2 + v_y(t)^2 + v_z(t)^2}$ is known as the speed.

5.5 Acceleration

Acceleration is the change of velocity with time. The analogous equations that were derived for velocity are derived for acceleration. The average acceleration is a vector with units [m/s^{-1}] given by,

$$\vec{v}_{avg} = \frac{\Delta \vec{v}}{\Delta t}. \tag{5.11}$$

The change in velocity from the initial time to the final time is illustrated

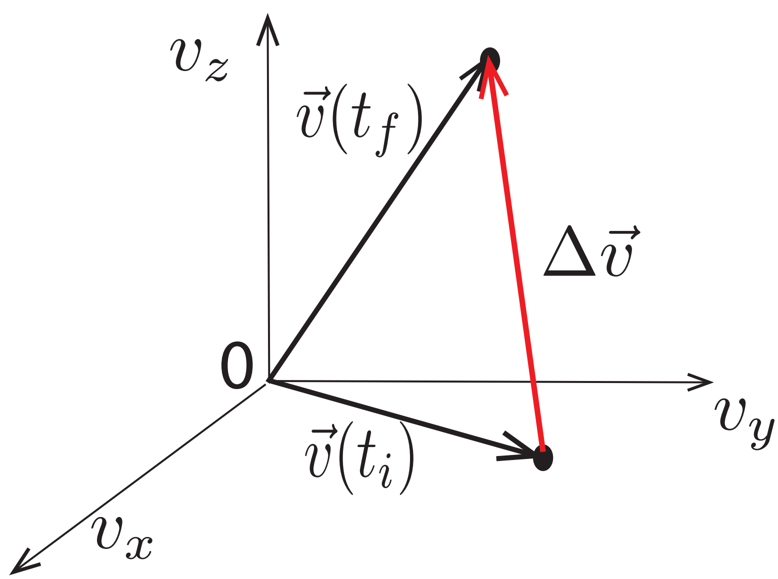

Figure 5.5: The change in velocity is the difference between the initial and final velocity vectors. Note that the axes are now velocity components.

in figure 5.5. The instantaneous acceleration is found by differentiation. Imagine the final and initial times approaching each other until they are only separated by an infinitesimal time. The limit as $\Delta t \to 0$ will give the derivative,

$$\vec{a} = \frac{d\vec{v}(t)}{dt} \tag{5.12}$$

As with velocity, each of the acceleration components is given by the derivatives,

$$\begin{aligned} a_x(t) &= \frac{dv_x(t)}{dt}, \\ a_y(t) &= \frac{dv_y(t)}{dt}, \\ a_z(t) &= \frac{dv_z(t)}{dt}. \end{aligned} \tag{5.13}$$

The acceleration components in equation 5.13 are given in terms of the derivatives of the velocity components. Recall, the velocity components are given in terms of the position components in equation 5.10. Therefore, substitution gives,

$$\begin{aligned} a_x(t) &= \frac{dv_x(t)}{dt} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}, \\ a_y(t) &= \frac{dv_y(t)}{dt} = \frac{d}{dt}\left(\frac{dy(t)}{dt}\right) = \frac{d^2y(t)}{dt^2}, \\ a_z(t) &= \frac{dv_z(t)}{dt} = \frac{d}{dt}\left(\frac{dz(t)}{dt}\right) = \frac{d^2z(t)}{dt^2}. \end{aligned} \tag{5.14}$$

and therefore,

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}. \tag{5.15}$$

5.6 Motion in One Dimension

Consider an object moving in one direction, for example a train on a single straight track. The position vector can be written as $\vec{r}(t) = x(t)\hat{\imath}$. This form emphasizes that x(t) is the component of the position vector in the $\hat{\imath}$ direction. A component can be positive, negative or zero. Figure 5.6 shows an example that illustrates some important points.

First, the particle starts at $x = 0$ at $t = 0$, i.e. $x(0) = 0$. It does not have to start at $x = 0$ but often the coordinates can be chosen so that it does. Next, the particle moves with constant velocity because the line in the x versus t graph has a constant slope. The velocity component is positive and the particle is moving to larger values of x.

At some later time, t_1, the particle velocity component changes from positive to negative and reaches the maximum value of x. Eventually the particle travels backwards and returns to $x = 0$ at $t = t_2$ and eventually to a maximum negative value of x at t_3. Finally, the velocity component changes to positive again and the particle returns to $x = 0$ at $t = t_4$.

Visualize the whole trip. A particle heads off, turns around, passes its starting point, turns around again and returns to the starting point.

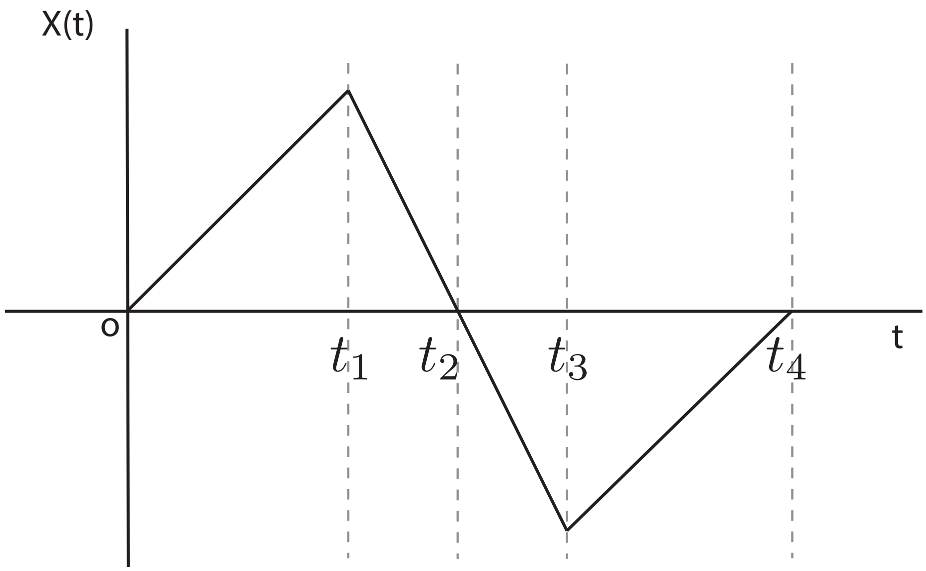

Figure 5.6: The graph shows position as a function of time.

An important special case is constant acceleration in one dimension. Starting with the following expression for position as a function of time,

$$\vec{x}(t) = (x_0 + v_0 t + \frac{1}{2} a t^2)\hat{\imath}, \tag{5.16}$$

where x_0, v_0 and a are constants with names that reflect the physical meaning that will ultimately be given to them, the velocity is calculated by taking the derivative of the position vector with respect to time. It is

$$\vec{v}(t) = \frac{d}{dt}\vec{x}(t) = (v_0 + at)\hat{\imath}. \tag{5.17}$$

It only has an x direction and therefore the x component of the velocity is $v_x(t) = v_0 + at$.

The acceleration is given by the derivative of the velocity with respect to time,

$$\vec{a}(t) = a\hat{\imath}, \tag{5.18}$$

demonstrating that the acceleration is a constant. It has a constant direction, $\hat{\imath}$, the x direction,and a constant magnitude, $a_x = a$.

At this point a couple of observations can be made. First, the position coordinate was labelled x but could have been y or z. In three dimensions, it is traditional to use z as up, in other words away from the centre of the earth or opposite to the pull of gravity. The horizontal coordinates are usually labelled x and y. in the two dimensional case, motion in a plane, up is often labelled as y. This is the convention used in this text.

Second, the fact that the acceleration is constant is what defines this kinematic situation. It is therefore, perhaps more satisfactory to start with acceleration and somehow derive the velocity and position equations. This is possible using integration in the sense that it is like an inverse derivative.

5.6.1 Projectile Motion with Gravity

A projectile launched straight upwards will have an acceleration $\vec{a} = -g\hat{\jmath} = -9.8\hat{\jmath}[\mathrm{m/s^2}]$, where $\hat{\jmath}$, the y direction, is taken to be upwards. Therefore $-\hat{\jmath}$ is the downward direction and the magnitude of gravity near the surface of the earth is denoted as g and is approximately constant with the value $9.8[\mathrm{m/s^2}]$. The equations for position and velocity are the same as equations 5.16 and 5.17 but with $a = -g$

$$\begin{aligned} y(t) &= y_0 + v_0 t - \frac{1}{2}gt^2, \qquad (5.19)\\ v_y(t) &= v_0 - gt,\\ a_y &= -g. \end{aligned}$$

Note that, a projectile launched straight downwards would mean that the initial velocity component is negative, $v_0 < 0$. If the object were dropped from rest, $v_0 = 0$. The quantity y_0 indicates the initial height of the object. This could be positive, zero or negative.

5.7 Projectile Motion

A projectile can be launched in a direction other than just straight up or down. In figure 5.7, an example trajectory is plotted on the X-Y plane. The initial launch angle is the slope of the curve at $x = y = 0$ when $t = 0$. This is the direction of the initial velocity. The velocity is calculated by taking the derivative of the position vector with respect to time.

The position vector, velocity vector and acceleration vector can all be written completely generally as,

$$\vec{r}(t) = (x_0 + v_{0x}t)\hat{\imath} + (y_0 + v_{0y}t - \frac{1}{2}gt^2)\hat{\jmath}, \tag{5.20}$$
$$\vec{v}(t) = v_{0x}\hat{\imath} + (v_0y - gt)\hat{\jmath},$$
$$\vec{a}(t) = o\hat{\imath} - g\hat{\jmath},$$

where the quantities, x_0, y_0, v_{0x}, v_{0y} and $-g$ are constants that are needed in order to calculate a specific example. Note, there are two sets of equations,

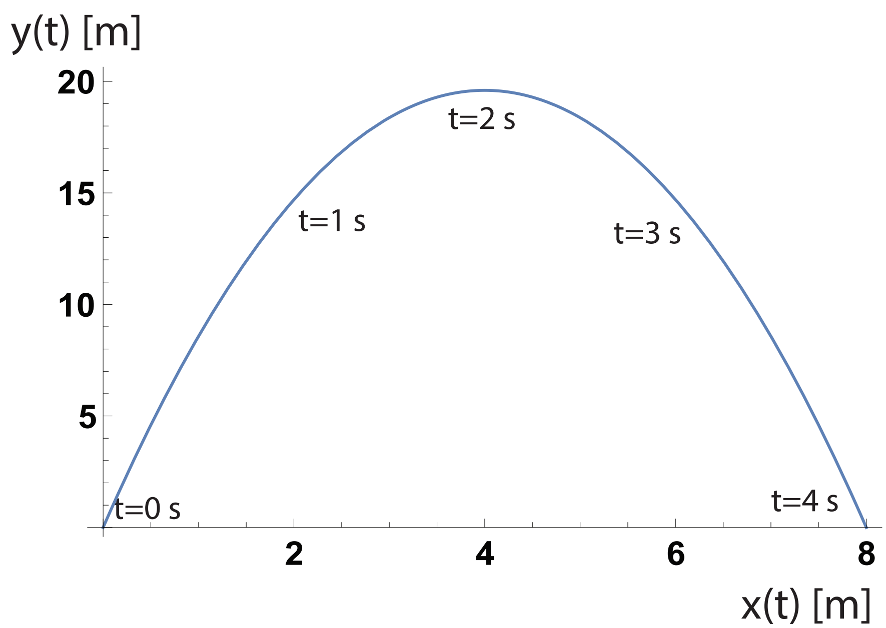

Figure 5.7: The position of a particle is plotted in x and y. Where the particle is at different times is indicated by markers and time stamps. The vector position plotted is given by $\vec{r}(t) = (0+2t)\hat{\imath}+(0+19.6t-1/2gt^2)\hat{\jmath}$. This corresponds to $x_0 = y_0 = 0$[m], $v_{0x} = 2$[m/s], $v_{0y} = 19.6$[m/s], $a_x = 0$[m/s^2] and $a_y = -g = -9.8$[m/s^2].

the equations for the x components and the ones for the y components. The x and y motion are independent of each other. This is a consequence of using cartesian coordinates.

Figure 5.8 shows the x and y positions as a function of time. The derivatives of these functions, i.e. the x and y components of the velocity are shown

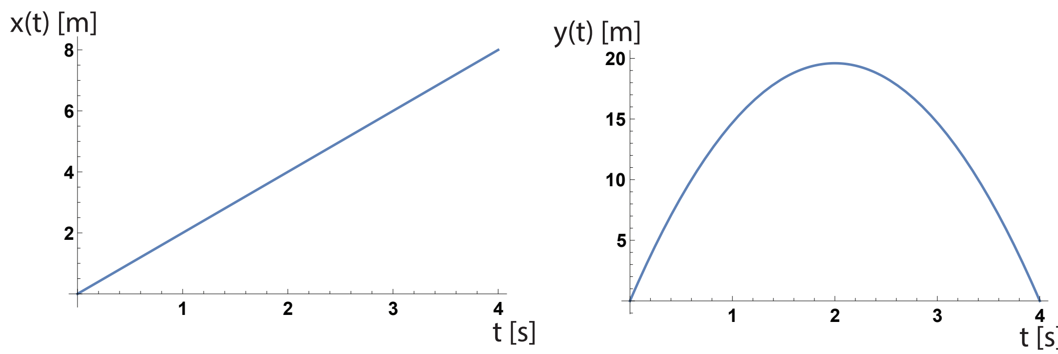

Figure 5.8: The left side of the figure shows the x position of a particle plotted against time. Notice that the x position has a constant slope corresponding to a constant x component of the velocity, v_x. The right side of the figure shows the y position of a particle plotted against time. The parabolic shape indicates a constant acceleration in the y direction, $a_y = -g$.

in figure 5.9. The velocity in the x direction is constant and the velocity in y direction is a line with negative slope that corresponds to the y component of the acceleration, $a_y = -g$.

The motion has a constant velocity in the x direction and a constant acceleration in the y direction. In chapter 2, inertial frames (also called reference frames) were introduced. An observer could be an inertial frame that moves along the x direction with the same velocity as the projectile. The observer would see the projectile go straight up and down. The y direction is more complicated because the object is undergoing acceleration in this direction.

5.8 Motion in a Circle

A particle moving in a circle, has a constant radius and hence magnitude but its direction changes with time. The position vector for the particle in figure 5.10 can be written as

$$\vec{r}(t) = R\cos\theta(t)\hat{\imath} + R\sin\theta(t)\hat{\jmath}. \tag{5.21}$$

The magnitude of the position vector is $|\vec{r}(t)| = R$. The velocity is calculated by taking the derivative. Using $\frac{d\cos\theta}{d\theta} = -\sin\theta$, $\frac{d\sin\theta}{d\theta} = \cos\theta$ and the

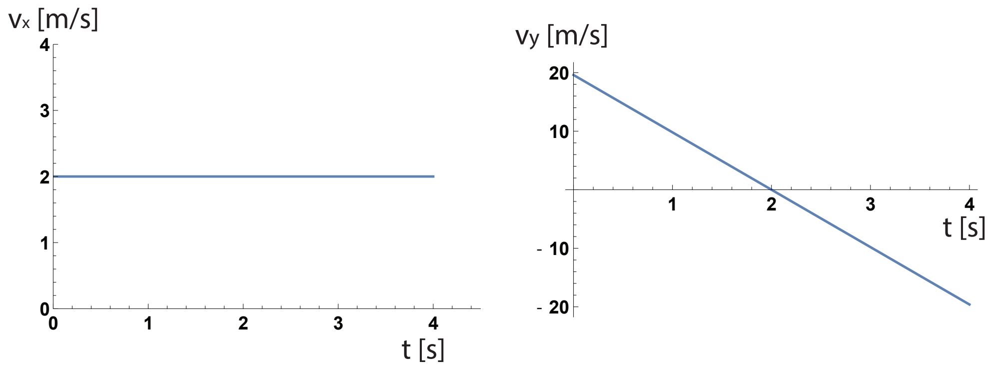

Figure 5.9: The left side of the figure shows the x component of the velocity, v_x, of a particle plotted against time. Notice that $v_x = v_{0x}$ is constant and hence $a_x = 0$. The right side of the figure shows the y component of the velocity, v_y, of a particle plotted against time. The linear shape indicates a constant negative acceleration in the y direction, $a_y = -g$.

chain rule, $\frac{df(\theta(t)}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt}$, gives.

$$\vec{v}(t) = -R\frac{d\theta(t)}{dt}\sin\theta(t)\hat{\imath} + R\frac{d\theta(t)}{dt}\cos\theta(t)\hat{\jmath}. \tag{5.22}$$

The speed the particle goes around the circle is the magnitude of the velocity, $|\vec{v}(t)| = R\frac{d\theta(t)}{dt}$. The simplest case is a constant speed. Given, $\theta(t) = \omega t$, where ω, the Greek letter omega in lower case, is a constant. Thus $|\vec{v}(t)| = R\frac{d\theta(t)}{dt} = r\omega$, also a constant. The quantity ω is called the angular velocity. In this case the velocity becomes,

$$\vec{v}(t) = -R\omega\sin\omega t\hat{\imath} + R\omega\cos\omega t\hat{\jmath}. \tag{5.23}$$

Note the speed is a constant but the velocity is not because the direction of the velocity is continually changing. This implies there is an acceleration.

First, the direction of the velocity with respect to the position vector can be calculated by taking the dot product,

$$\vec{r}\cdot\vec{v} = r_x v_x + r_y v_y = 0 \tag{5.24}$$

and therefore the angle between the velocity of the position vector (radius of the circle) is 90°. This shows the velocity is tangent to the circle and is, in

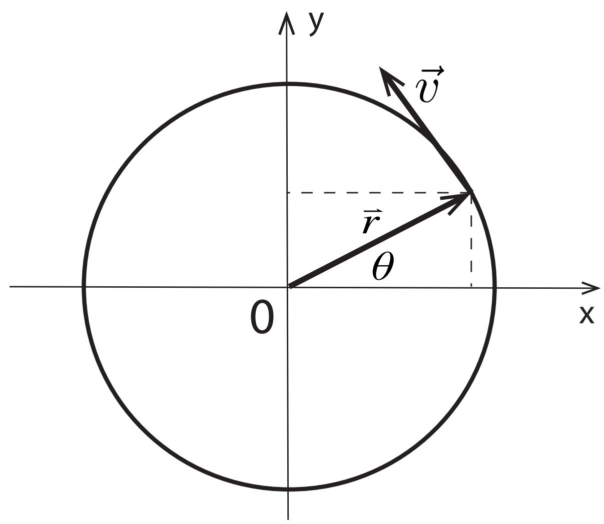

Figure 5.10: The position of a particle moving in a circle is shown. The position vector $\vec{r}$ has a magnitude R and at a given instant in time the angle from the x-axis ($\hat{\imath}$ direction) is given by θ.

fact, a general result for a circle regardless of whether the speed is constant or not.

The acceleration is calculated by differentiating the velocity. In the case of constant angular velocity, it is given by,

$$\vec{a}(t) = -R\omega^2 \cos\theta(t)\hat{\imath} - R\omega^2 \sin\theta(t)\hat{\jmath} = -\omega^2\vec{r}(t). \tag{5.25}$$

Calculating the magnitude of the acceleration gives the important equation for a particle undergoing circular motion with a constant angular speed,

$$|\vec{a}| = R\omega^2, \tag{5.26}$$

and the acceleration is pointing in the direction $-\vec{r}(t)$. in other words, towards the centre of the circle. Note that, the time it takes to go around the circle once, is called the period and denoted T. Since $\theta = \omega t$ and the angle once around a circle is 2π, then $2\pi = \omega T$ or

$$\omega = \frac{2\pi}{T}. \tag{5.27}$$

The circumference of the circle is given by $2\pi R$ and hence the speed is given by $v = |\vec{v}| = \frac{2\pi R}{T}$ and therefore the speed v is related to ω by

$$v = R\omega. \tag{5.28}$$

Substituting the equation for angular velocity 5.28 into the equation for the centripetal acceleration 5.26 gives the expression,

$$a = \frac{v^2}{R}. \tag{5.29}$$

It is essential to keep in mind that the equations after equation 5.22 are for **Uniform Circular Motion**, in other words, for constant speed v.

A more general expression for the acceleration can be derived by differentiating the completely general expression for the velocity of a particle moving in a circle that is given in equation 5.22. The result has a component towards the centre of the circle and a radial component that represents the change in the speed of the particle.

Chapter 6

Newton's Second Law

6.1 Introduction

Newton's second law combines a number of themes which have been developed so far in the text. In Chapter 2 the central idea was that if there was no *net* force on an object it would remain in equilibrium; this idea can be expressed using the nomenclature of Chapter **??** with the phrase that if there is no net force on an object it will not *accelerate*. The implication was also the other way: if an object was observed to *not* be accelerating then it could be inferred to be subject to zero net force.

The open question of explainng how an object responds to a non-zero net force is the topic of this chapter.

6.2 Statement of the Second Law

Newton's second law relates the net force on an object to the object's mass and the object's acceleration. In words, the second law is that: *An object which is subject to a non-zero net force will experience an acceleration. This acceleration will be proportional to the net force on the object and inversely proportional to the object's mass.* The second law is often conveniently expressed in mathematical form:

$$\vec{a} = \frac{\vec{F}_{net}}{m} \,. \tag{6.1}$$

Notice that this expression has not been re-written as $\vec{F}_{net} = m\vec{a}$, even though it could obviously be rearranged that way. The reason is to emphasize the logical steps behind the statement. In particular, it is to emphasize

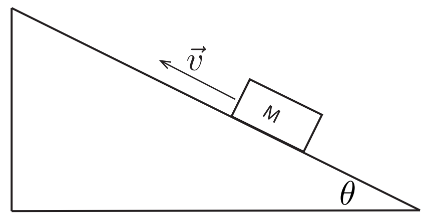

Figure 6.1: A mass moving up a slope while subject to kinetic friction.

that the acceleration is *derived* from two other quantities which are, in principle, known.

The principle application of this rule comes in two forms:

- Given the net force on an object the acceleration (and hence, given appropriate information, the position as a function of time) can be determined.
- Given the acceleration of the object, the *required* net force can be determined, and knowledge of the net force will allow determination (as appropriate) of one of the forces that the object was subject to.

6.3 Examples of application of Newton's Second Law

The following examples are chosen to convey the underlying idea and procedures for applying Newton's second law in the special cases of both rectilinear (constant acceleration) and curvilinear motion.

6.3.1 A block sliding up a rough slope

Consider a block of mass M that is sliding along a slope which makes an angle with the horizontal of θ. The block has an initial velocity of $\vec{v}$ in a direction up the slope, and is subject to a friction force from the surface which is characterized by a coefficient of kinetic friction μ_k. This situation is illustrated in figure 6.1.

Knowing that the magnitude of the initial velocity is $|\vec{v}|$ and that the initial velocity was measured at time $t = 0s$ when the mass was at the origin, how long does it take until the mass comes to rest, and how far up the slope does it go?

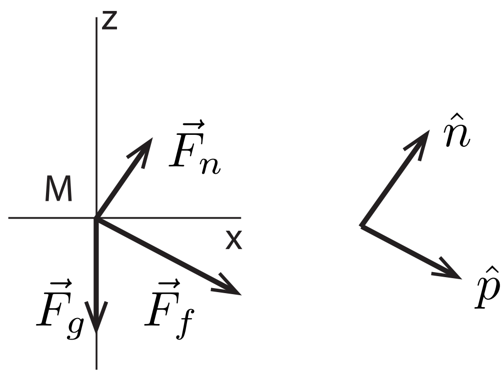

Figure 6.2: A free-body diagram mass moving up a slope while subject to kinetic friction.

In order to appropriately address this question, the first step is to conceptualize the question: The mass is known to be on a slope, and from consideration of its motion it should be possible to determine all forces on it. Knowing the forces allows determination of the net force, and hence the acceleration, and the problem then becomes one of applying knowledge about constant acceleration motion.

There are three forces acting on the mass: the downwards force of gravity, the normal force, and the force of kinetic friction which is in the *opposite direction* of motion. The free-body diagram is in figure 6.2. It is convenient, although not necessary to express the forces in terms of the unit vectors $\hat{n}$ and $\hat{p}$:

$$\begin{aligned} \vec{F}_n &= \left|\vec{F}_n\right| \hat{n} \\ \vec{F}_f &= \left|\vec{F}_f\right| \hat{p} \\ \vec{F}_g &= -Mg\cos\theta\hat{n} + Mg\sin\theta\hat{p}. \end{aligned} \tag{6.2}$$

Applying Newton's second law gives that

$$\begin{aligned} \vec{a} &= \frac{\vec{F}_{net}}{M} \\ &= \left(\frac{\left|\vec{F}_n\right|}{M} - g\cos\theta\right)\hat{n} + \left(\frac{\left|\vec{F}_f\right|}{M} + g\sin\theta\right)\hat{p}. \end{aligned} \tag{6.3}$$

Now, apply the consideration that the mass is not going to come off the slope. This means that the $\hat{n}$ component of the acceleration is zero ($\hat{n} \cdot \vec{a} = 0$), which gives that the magnitude of the normal force is $\left|\vec{F}_n\right| = Mg\cos\theta$, which allows the determination of the magnitude fo the friction force: $\left|\vec{F}_f\right| = \mu_k Mg\cos\theta$ so then the acceleration is

$$\vec{a} = (\mu_k \cos\theta + \sin\theta)\, g\hat{p}. \tag{6.4}$$

Now, since the acceleration and initial velocity are known, and since the initial position is defined to be at the origin, the general rule that

$$\vec{r}(t) = r_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \tag{6.5}$$

can be applied, since this is a case where the acceleration is constant and the time at which the position and velocity were measured are $t = 0s$. This means that

$$\begin{aligned} \vec{r}(t) &= \vec{0} + (-\left|\vec{v}\right|\hat{p})\, t + \frac{1}{2}\left[(\mu_k \cos\theta + \sin\theta)\, g\hat{p}\right] t^2 \\ \text{so } \vec{v}(t) &= (-\left|\vec{v}\right|\hat{p}) + \left[(\mu_k \cos\theta + \sin\theta)\, g\hat{p}\right] t \end{aligned} \tag{6.6}$$

This means that at time $t_{stop} = \frac{|\vec{v}|}{g(\mu_k \cos\theta + \sin\theta)t}$ the velocity with be $\vec{v} = 0$ – the mass will have stopped. This means that the mass will have undergone a displacement, so

$$\vec{r}(t_{stop}) = -\frac{1}{2}\frac{\left(|\vec{v}|^2\right)}{g\left(\mu_k \cos\theta + \sin\theta\right)}\hat{p}. \tag{6.7}$$

Note that the component along $\hat{p}$ is negative because the mass has moved in the negative $\hat{p}$ direction.

6.3.2 Masses in contact

Three objects with masses m_1, m_2 and m_3 respectively are on a horizontal frictionless surface. The masses are touching each other and the one to the left is subject to a horizontal pushing force of magnitude P. Determine the force exerted by the second mass (the one with the mass m_2) on the first mass (the one with mass m_1). This situation is illustrated in figure 6.3.

As always, the first step is to create appropriate free-body diagrams for each mass. These are illustrated in figure 6.4. Using these diagrams and

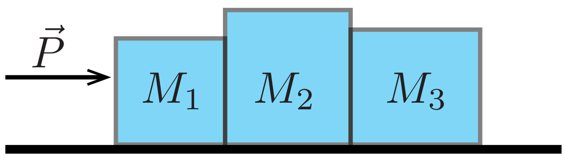

Figure 6.3: A sketch of three masses being pushed horizontally on a frictionless surface.

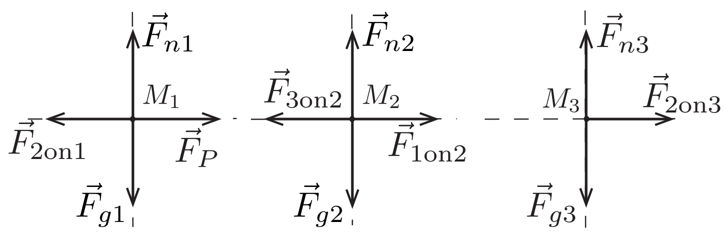

Figure 6.4: Three free-body diagrams for the three masses being pushed horizontally on a frictionless surface.

Newton's second law gives that

$$
\begin{aligned}
\vec{a}_1 &= \frac{1}{m_1}\left[\vec{F}_{g1} + \vec{F}_{N1} + \vec{F}_p + \vec{F}_{2on1}\right] \\
\vec{a}_2 &= \frac{1}{m_2}\left[\vec{F}_{g2} + \vec{F}_{N2} + \vec{F}_{1on2} + \vec{F}_{3on2}\right] \\
\vec{a}_3 &= \frac{1}{m_3}\left[\vec{F}_{g3} + \vec{F}_{N3} + \vec{F}_{2on3}\right] \qquad (6.8)
\end{aligned}
$$

The next step is to use insight about the physical situation to derive some facts about the acceleration of each mass. In this case, the masses are not going to come off the horizontal plane on which they sit, so a_{1z}, a_{2z}, and $a_{3z} = 0$. In addition, the masses are going to move *together*, so that $\vec{a}_1 = \vec{a}_2 = \vec{a}_3$. This means that the acceleeration of each can be written $\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = a\hat{\imath}$ where a is a number that needs to be determined.

Substituting this *parametrization* into expression 6.8 together with the observation that the downwards force of gravity is counteracted by the upwards normal force results in the following system of equations:

$$\begin{aligned} a\hat{\imath} &= \frac{1}{m_1}\left[P\hat{\imath} + \vec{F}_{2on1}\right] \\ a\hat{\imath} &= \frac{1}{m_2}\left[\vec{F}_{3on2} + \vec{F}_{1on2}\right] \\ a\hat{\imath} &= \frac{1}{m_3}\vec{F}_{2on3} \end{aligned} \tag{6.9}$$

Together with Newton's third law (which relates $\vec{F}_{2on1} = -\vec{F}_{1on2}$ and $\vec{F}_{3on2} = -\vec{F}_{2on3}$) this is a system of linear equations which can be easily solved. The final results are that

$$\begin{aligned} \vec{a}_1 = \vec{a}_2 = \vec{a}_3 &= \frac{P}{m_1 + m_2 + m_3}\hat{\imath} \\ \vec{F}_{3on2} &= -\frac{m_3}{m_1 + m_2 + m_3}P\hat{\imath} \\ \vec{F}_{2on1} &= -\frac{m_2 + m_3}{m_1 + m_2 + m_3}P\hat{\imath} \end{aligned} \tag{6.10}$$

Note that this result shows that the net force on any individual part of this system also appropriate for that acceleration.

6.3.3 Atwood machine

Consider an 'Atwood machine', which consists of two objects (of mass m_1 and m_2) which are suspended via a massless and inextensible rope over a massless and frictionless pulley. This system is shown in figure 6.5. Determine the magnitude of the acceleration of the object of mass m_1, and determine the tension in the rope.

There is some physics knowledge hidden in the statement of the problem: the fact that the rope is assumed to be massless means that it is not necessary to worry about how much rope is on either side of the pulley. The fact that the pulley is massless and frictionless means that the tension is constant throughout the rope, that is on both sides of the pulley; the reason for this will be explained in more detail in Chapter 10. The appropriate free body diagrams are shown in figure 6.6.

Using the free-body diagrams and Newton's second law, the accelerations of the two masses can be written:

$$\begin{aligned} \vec{a}_1 &= \frac{1}{m_1}\left[\vec{F}_{T1} + \vec{F}_{g1}\right] \\ \vec{a}_2 &= \frac{1}{m_2}\left[\vec{F}_{T2} + \vec{F}_{g2}\right] \end{aligned} \tag{6.11}$$

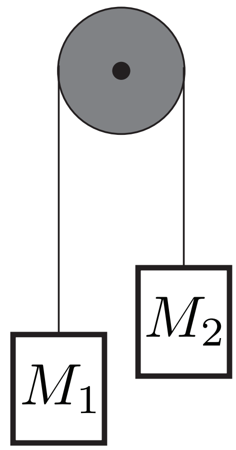

Figure 6.5: An Atwood machine consisting of two masses suspended by an inextensible rope over a massless and frictionless pulley.

The force due to tension in each case is $T\hat{k}$, and the downwards force of gravity is $-mg\hat{k}$. Based on this, each mass will *only* accelerate in the vertical direction.

As in previous problems the next step is to determine the relations between the various accelerations. Since the rope is inextensible the amount that one mass moves up will be exactly counteracted by the other mass moving down. This can be expressed mathematically as $\vec{a}_1 = -\vec{a}_2$ Applying this to expression 6.11 gives

$$\frac{1}{m_1}\left[T\hat{k} - m_1 g\hat{k}\right] = -\frac{1}{m_2}\left[T\hat{k} - m_2 g\hat{k}\right] \tag{6.12}$$

This can be rearranged and solved for the tension, giving that $T = \frac{2m_1 m_2}{m_1+m_2}g$, and then this can be substituted into the expression for either acceleration and so it can be found that

$$\begin{aligned} \vec{a}_1 &= \frac{m_2 - m_1}{m_2 + m_1} g\hat{k} \\ \vec{a}_2 &= \frac{m_1 - m_2}{m_1 + m_2} g\hat{k} \end{aligned} \tag{6.13}$$

As outlined, the two accelerations are opposite in direction but equal in magnitude.

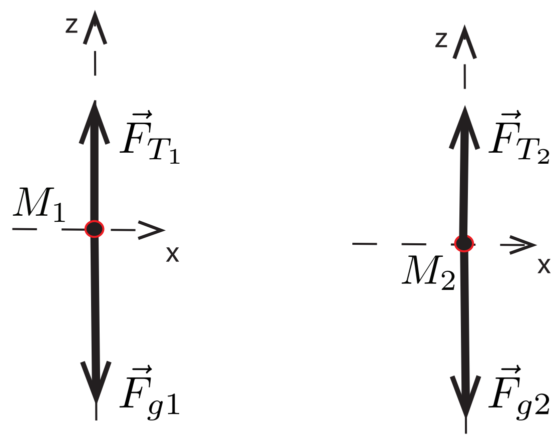

Figure 6.6: The two free-body diagrams for an Atwood machine consisting of two masses suspended by an inextensible rope over a massless and frictionless pulley.

6.3.4 Conical pendulum

A mass is attached to a string of length L and allowed to swing in a horizontal circle. What is the relation between the angle θ that the string makes with the vertical and the speed of the mass? The situation is depicted in figure 6.7.

There are two and only two force acting on the mass: the tension from the string and the force of gravity. This is shown in the free body diagram in figure 6.8. Newton's second law determines that for this mass

$$\begin{aligned} \vec{a} &= \frac{1}{m}\left[\vec{F}_T + \vec{F}_g\right] \\ &= \frac{1}{m}\left[T\sin\theta\hat{c} + T\cos\theta\hat{k} - mg\hat{k}\right] \end{aligned} \tag{6.14}$$

In the above, $\hat{c}$ is the unit vector towards the center of the circle.

Since it is known that the mass is travelling in a uniform horizontal circle, the acceleration can be expressed as

$$\vec{a} = \frac{|\vec{v}|^2}{r}\hat{c} \tag{6.15}$$

Comparing this to Newton's second law, there are two relationships which

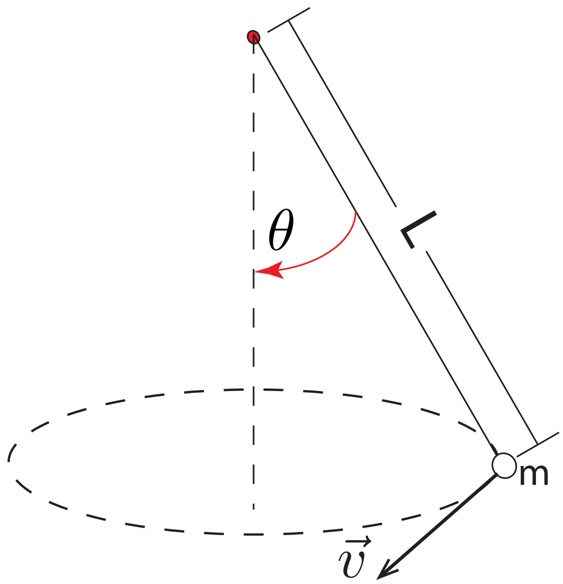

Figure 6.7: A conical pendulum .

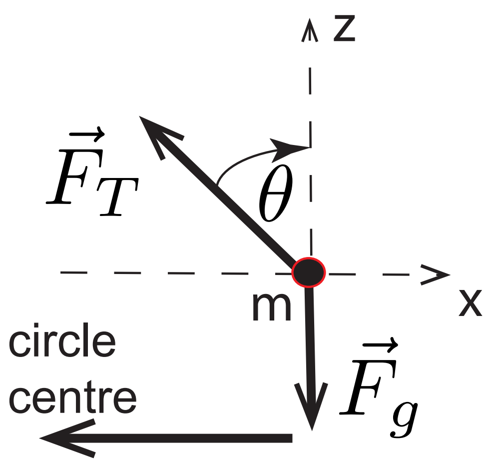

Figure 6.8: Free body diagram for a conical pendulum .

become apparent:

$$\begin{aligned} z \text{ component } & 0 = T\cos\theta - mg \\ c \text{ component } & \frac{|\vec{v}|^2}{r} = T\sin\theta \end{aligned} \tag{6.16}$$

Solving for T using the first the first equation gives $T = \frac{mg}{\cos\theta}$ and substituting that into the second gives $\frac{|\vec{v}|^2}{r} = mg\frac{\sin\theta}{\cos\theta}$. The last piece of geometry is that the radius of the circle, r, is obviously $r = L\sin\theta$, giving in total that

$$|\vec{v}|^2 = gL\frac{\sin^2\theta}{\cos\theta}. \tag{6.17}$$

Chapter 7

Forces

7.1 Introduction

The text so far has discussed *how* forces affect objects, but it has not treated forces as fundamental quantities that can be derived from properties of objects such as their mass, position, or charge. In this chapter a number of commonly encountered forces are introduced; these include fundamental forces such as gravity, electric, and magnetic forces. There are also some *effective* forces which are explainable in terms of the fundamental forces discussed, but also they are amenable to simple mathematical treatment.

7.2 Gravity

Any two massive objects will exert an attractive force on each other. Through careful measurement and experimentation it was determined that this force is *directly* proportional to the mass of each object, and *inversely* proportional to the square of the separation distance between the objects.

If two objects, Object 1 and Object 2, have masses m_1 and m_2 respectively, and are located at positions $\vec{r}_1$ and $\vec{r}_2$, then the force *on* Object 2 *by* Object 1 can be expressed mathematically as

$$\vec{F}_{1\ on\ 2} = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}. \tag{7.1}$$

In this, G is a constant with a numerical value of approximately $G = 6.67 \times 10^{-11}\left[N\frac{m^2}{kg^2}\right]$, and the quantity $\frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|}$ is the *unit* vector *from* Object 1 *to* Object 2. This situation is illustrated in figure 7.1. Writing the expression with an overall minus sign may seem to be an odd choice. It will be seen

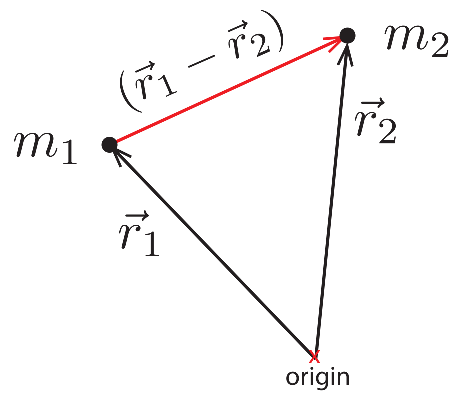

Figure 7.1: Object 1 has mass m_1 and it located at $\vec{r}_1$; Object 2 has mass m_2 and is located at $\vec{r}_2$.

later that using a minus sign with attractive forces and a plus sign with repulsive forces will simplify calculations.

This expression is strictly an approximation which is only perfectly accurate for *point particles* or particles that are perfectly spherically symmetric, and the position referred to is the exact centre. In other cases the force derived from this expression is approximately accurate when treating the total force; there are, however, observable effects of the part of the force neglected by this approximation. For example, the fact that the Moon is 'tidally locked' to the Earth, which is the same as saying that the same face of the moon is always towards the Earth, is due to the action, over many years, of the forces which are neglected in this approximation.

It is instructive to inquire why gravity can be reasonably approximated by $\vec{F}_g = -mg\hat{k}$ near the surface of the Earth. Consider two people on the surface of the earth separated by a distance d, and suppose that one has mass m_1 and the other has mass m_2. This is illustrated in figure 7.2. First, examine the direction that each will experience the force of gravity due to the approximately symmetric Earth: The two directions from the people towards the centre of the Earth are not the same. In fact, the two vectors have an angle θ between them, and as the people are separated by a distance d as measured over the surface of the Earth, from the diagram it can be seen that the angle $\theta = \frac{d}{R_E}$. On the scales where the approximation of gravity as 'straight down' is valid, d will be at most a few kilometers. The radius of Earth is about $6000km$, so $\theta \approx 10^{-3}$ at most. Using the Taylor expansions

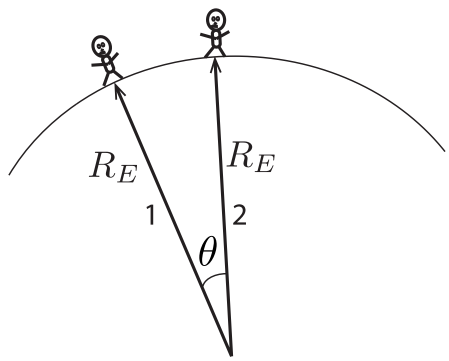

Figure 7.2: Two people a distance d apart on the surface of the Earth.

for sin and cos, it is apparent that component of the force of gravity on person 2 which is perpendicular to the force of gravity on person 1 is at most a thousandth of the component which is in the same direction. This establishes that the two forces are in approximately the same direction.

The next thing to consider is the magnitude: for each, $\left|\vec{F}_g\right| = G\frac{m_E m_i}{R_E^2}$ where m_i is m_1 or m_2 as appropriate, and R_E is the radius of the Earth. This distance is the same for both people. Now, notice that the coefficient of the person's mass is the same in both cases: $G\frac{m_E}{R_E^2}$: this is the quantity which is normally written as g. These two things establish that the Newtonian expression for the force of gravity is a generalization of the simple 'gravity pulls down' model discussed earlier.

7.2.1 Star orbiting another star

Suppose that there are two stars which are together in a double-star system with masses m_1 and m_2 respectively. If each star is moving in a circular orbit, what is the relation between the star's separation and their orbital period?

The first step in this problem is to understand the geometry: the two stars will orbit their common centre of mass, as illustrated in figure 7.3. The speeds at which the stars orbit are $|\vec{v}_1|$ and $|\vec{v}_2|$ respectively, and since this is uniform circular motion the stars have constant speeds. The key observation is that the stars move around their common center of mass, and that this center of mass is located a distance r_1 from star 1 and r_2 from star 2. As discussed in chapter 3 the relation between r_1 and $_2$ can be derived by

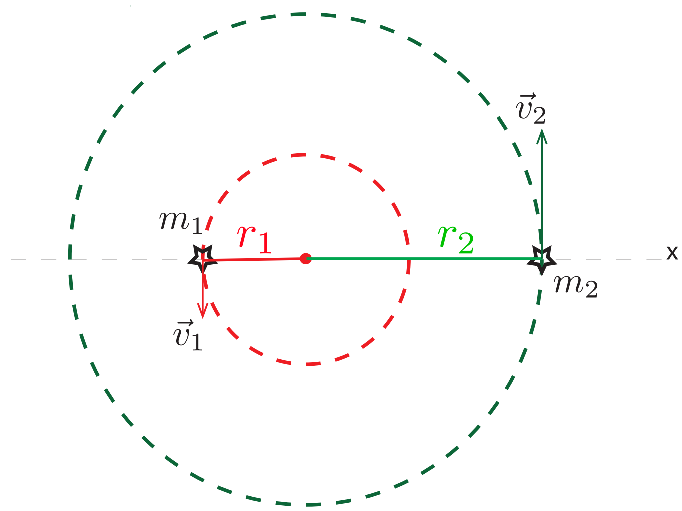

Figure 7.3: Two stars orbiting each other.

assuming that the center of mass is the origin and then using the definition of the center of mass. This results in $m_1 r_1 = m_2 r_2$.

Now, consider star 2. At the time illustrated in the figure, where the stars are lined up along the x-axis, it is subject to a force

$$\vec{F}_{on\ star\ 2} = -G\frac{m_1 m_2}{(r_1 + r_2)^2}\hat{\imath} \tag{7.2}$$

The reason for the direction is that the vector from star 1 to star 2 (the unit vector from equation 7.1) is along the x-axis. Now, the star is moving is a circle of radius r_2 at a constant speed, so

$$\vec{a}_{of\ star\ 2} = -\frac{|\vec{v}_2|^2}{r_2}\hat{\imath} \tag{7.3}$$

The reason for the $-\hat{\imath}$ direction is that it is the direction towards the centre of the circle in which the star was moving. Now, Newton's second law can be used to relate the two facts:

$$\begin{aligned} \vec{a}_{of\ star\ 2} &= \frac{1}{m_2}\vec{F}_{on\ star\ 2} \\ \frac{|\vec{v}_2|^2}{r_2} &= G\frac{m_1}{(r_1 + r_2)^2} \end{aligned} \tag{7.4}$$

The speed of star 2 is related to its orbital period T as $2\pi r_2 = |\vec{v}_2| T$, and r_2 can be related to the total separation as $r_2 = \frac{r_1+r_2}{\frac{m_2}{m_1}+1}$ which can be derived from,

$$\begin{aligned} m_1 r_1 &= m_2 r_2 \\ r_1 + r_2 &= \frac{m_2 r_2}{m_1} + r_2 \\ &= r_2 \left(\frac{m_2}{m_1} + 1 \right). \end{aligned} \tag{7.5}$$

These two facts allow substitution into the relation derived from Newton's second law as in expression 7.4, giving

$$\begin{aligned} \left(\frac{2\pi r_2}{T} \right)^2 \frac{1}{r_2} &= G \frac{m_1}{(r_1 + r_2)^2} \\ r_2 \left(r_1 + r_2 \right)^2 &= \frac{G}{4\pi^2} m_1 T^2 \\ \left(r_1 + r_2 \right)^3 &= \frac{G}{4\pi^2} \left(m_1 + m_2 \right) T^2 \end{aligned} \tag{7.6}$$

In the case that one of the stars is much more massive than the other, so that it is essentially at the centre of mass of the two star system, this reduces to $r^3 = \frac{GM}{4\pi^2} T^2$ where M is the mass of the larger star and r is the orbital radius of the smaller star. This relation can be used to measure the mass of planets with satellites orbiting them, stars with planets orbiting them, or the mass of black holes which have stars orbiting them.

7.2.2 Lagrange Points

Given the previous discussion, it would be easy to expect that there is no configuration where two objects orbiting the same star at different radii would be able to remain in a particular stable alignment. However, there are a few locations where this is possible because one is the subject of forces from both the orbited star and the associated planet. Real-world examples of this phenomenon include the Trojan asteroids in Jupiter's orbit and the placement of some research satellites near Earth's orbit.

Suppose that a planet of mass m_p is orbiting a star of mass m_s at a radius R. At what distance from the centre of the star should a satellite be placed so that it remains between the planet and star, as shown in figure 7.4. Note that this is equivalent to the orbital periods of the planet and satellite being the same. Find the distance d at which the satellite orbits

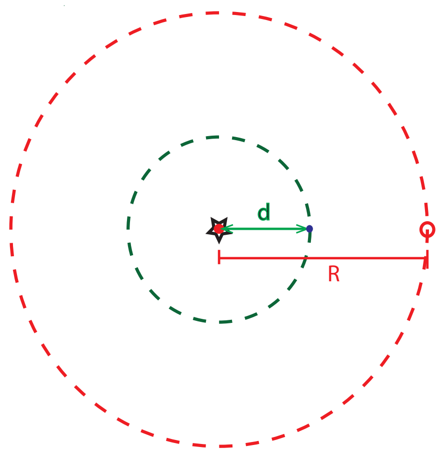

Figure 7.4: A planet orbiting a star with a satellite between them.

the star as a function of R, m_p and m_s.

The key part of this question is an approximation based on relative size. The planet is assumed to be much smaller than the star, so it orbits as described above. The satellite is assumed to be much smaller than either the planet or the star, so there is no meaningful force on the planet or star due to the satellite. This approximation can be represented as $m_{sat} \ll m_p \ll m_s$.

In this case the net force on the satellite can be immediately written (for the situation where the star is the origin and the star-planet line is the x-axis) as

$$\vec{F}_{net,\ sat} = -G\frac{m_s m_{sat}}{d^2}\hat{\imath} + G\frac{m_p m_{sat}}{(R-d)^2}\hat{\imath} \tag{7.7}$$

and the fact that the the satellite is going in a circle with the star at the center says that

$$\vec{a}_{sat} = -\frac{|\vec{v}_{sat}|^2}{d}\hat{\imath} \tag{7.8}$$

Using Newton's second law to equate the *required* acceleration with the *derived* net force gives

$$\begin{aligned} G\frac{m_s}{d^2} - G\frac{m_p}{(R-d)^2} &= \frac{|\vec{v}_{sat}|^2}{d} \\ Gm_s (R-d)^2 - Gm_p d^2 &= |\vec{v}_{sat}|^2 d (R-d)^2 \\ &= d^3 (R-d)^2 \left(\frac{2\pi}{T}\right)^2 \end{aligned} \tag{7.9}$$

In this, the orbital period T of the satellite has been explicitly pulled out, but in the previous example it was determined that for a planet orbiting a star ($m_p \ll m_s$) the relationship between the period of the planet and radius was $\frac{(2\pi)^2}{T^2} = \frac{Gm_s}{R^3}$, so substituting this gives

$$\begin{aligned} m_s\,(R-d)^2 - m_p d^2 &= m_s\left(\frac{d}{R}\right)^3 (R-d)^2 \\ m_s\left(1-\frac{d}{R}\right)^2 - m_p\left(\frac{d}{R}\right)^2 &= m_s\left(\frac{d}{R}\right)^3\left(1-\frac{d}{R}\right)^2 \qquad (7.10) \end{aligned}$$

This is now a polynomial expression which must be solved for d to answer the original question.

Unfortunately it is impossible to solve arbitrary quintic equations. In order to determine the quantity d, an approximation is used. Since the planet is much smaller than the star it is assumed that the value of d will be very close to R, in other words $d = R(1-\delta)$ where δ is a small parameter. In terms of δ the expression that must be solved for the position of this stable orbital point is

$$\begin{aligned} m_s\,(1-(1-\delta))^2 - m_p\,(1-\delta)^2 &= m_s\,(1-\delta)^3\,(1-(1-\delta))^2 \\ -m_p\left(1-2\delta+\delta^2\right) &= m_s\left(1-3\delta+3\delta^2-\delta^3\right)\delta^2 - m_s\delta^2 \\ m_p &\approx 3m_s\delta^3 \qquad (7.11) \end{aligned}$$

This was obtained by ignoring all higher terms in delta on each side of the expression, and is analogous to taking the first terms in the Taylor expansion as in chapter 4. So, the radius d of this stable orbit is $d \approx R\left(1-\left(\frac{m_p}{3m_s}\right)^{\frac{1}{3}}\right)$.

7.3 Electric Force

Particles which are *charged* exert force on each other in proportion to the amount of each charge and inversely proportional to the square of their separation. This is very similar to the force of gravity, but there is an important difference related to charge: Charge can be in two kinds (normally called 'positive' and 'negative') and the force between two similar charges is repulsive rather than attractive. The force between two charges of opposite sign is attractive.

If there is a charge q_1 at location $\vec{r}_1$ and there is another charge q_2 at location $\vec{r}_2$ then the force *on* the second charge *due to* the first charge is

$$\vec{F}_{1\ on\ 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_2-\vec{r}_1|^2}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|} \qquad (7.12)$$

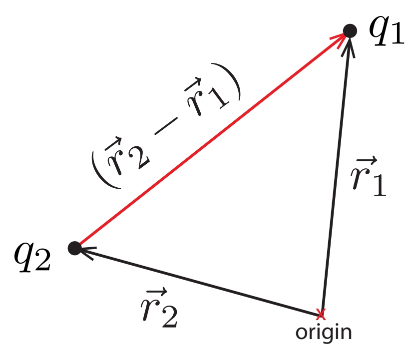

Figure 7.5: A charge q_1 at location $\vec{r}_1$ and a charge q_2 at location $\vec{r}_2$.

In this expression the quantity $\frac{1}{4\pi\epsilon_0}$ is a constant with an approximate value of $9.00 \times 10^9 \left[N\frac{m^2}{C^2}\right]$ and the charge is expressed in units called 'Coulombs' and abbreviated C. The term $\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|}$ is the unit vector from the first charge to the second charge, and if the two charges are both positive or both negative then the force on the second charge will be *away* from the first. This situation is illustrated in figure 7.5. There is an overall plus sign in front of the expression rather than a minus sign as there was for gravity. When the charges have opposite signs and the force is attractive $q_1q_2 = -|q_1||q_2|$ which gives an overall minus sign.

As in the case of gravity, this expression is strictly true for *point charges* and *spherically symmetric* charge configurations; in the case of extended charge configurations the relatively large strength of the electric force is the reason that the effects of many charges are usually not treated as simply point charges.

7.3.1 Electric Field

For many cases it is useful to express the force on a *particular* charge due to the effect of *all other* nearby charges. A charge q at point $\vec{r}$ experiences a force which is the net force due to all the other charges. This is related to the quantity called the 'Electric Field' which is defined as

$$\vec{F}_{net,\ elec} = q\vec{E}(\vec{r}). \tag{7.13}$$

The expression $\vec{E}(\vec{r})$ emphasizes that the electric field depends on the location at which it is measured. A positive charge will feel a force in the direction of the electric field, while a negative charge will experience a force in the opposite direction of the electric field. The electric field encodes the magnitude and location of the nearby charges. The units of electric field are Newtons per Coulomb, $[\frac{N}{C}]$.

7.4 Lorentz Force

The electric force describes the effect of *stationary* charges on a particular charge. If the charges are moving, there is another effect. A moving charge in a *Magnetic field* experiences a force due to this field of

$$\vec{F} = q\vec{v} \times \vec{B}(\vec{r}) \tag{7.14}$$

where q is the charge of the particle in question, $\vec{v}$ is the particle's velocity, and $\vec{B}(\vec{r})$ is the magnetic field at the location of the particle. The units of magnetic field are Newton-seconds per meter per Coulomb: $\left[\frac{N\ s}{m\ C}\right]$, which is known as a 'Tesla'.

The magnetic field is produced by other moving charges, and, similar to the electric field, accounts for the position, velocity, and charge of those other charges. The determination of the magnetic field for a few moving charges is relatively complicated compared to that of the electric field. It is normally treated in detail in a senior-level undergraduate Physics course. Since the calculation of a magnetic field from first principles takes more calculus than is presumed in this text, the field will be, if necessary, specified.

Note that the force exerted by the magnetic field on a moving charge is *perpendicular* to the direction that the charge moves. A net force which is perpendicular to the direction of motion will change the direction of motion but will not change the speed of the moving object.

7.4.1 Circular motion in a magnetic field

At the instant $t = 0s$ particle of charge q and mass m is moving with velocity $\vec{v} = v_0\hat{\imath}$ and is at the origin in a region where the magnetic field is $\vec{B} = B_0\hat{k}$. When will it have velocity $-v_0\hat{\imath}$, and where will it be then?

The net force on the particle is

$$\begin{aligned}\vec{F}_{net} &= q\vec{v} \times \vec{B} \\ &= q\left(v_0\hat{\imath}\right) \times \left(B_0\hat{k}\right) \\ &= -qv_0B_0\hat{\jmath}\end{aligned} \tag{7.15}$$

If the charge is positive it will feel a force in the *negative* y-direction, and if the charge is negative it will feel a force in the *positive* y-direction.

As discussed, the Lorentz force always acts perpendicular to the velocity, which means that the particle changes its velocity without changing its speed. In the case where the velocity is perpendicular to the magnetic field the particle will move in uniform circular motion. As the particle is moving in circular motion the acceleration is known to be

$$|\vec{a}| = \frac{|\vec{v}|^2}{R} \tag{7.16}$$

where R is the (currently unknown) radius of the circular motion.

Comparing the known magnitude of the acceleration with the known force, it can be seen that

$$\frac{|q|\, v_0 B_0}{m} = \frac{v_0^2}{R} \rightarrow R = \frac{mv_0}{qB_0} \tag{7.17}$$

which answers the question *where* will the particle be when the velocity is $-v_0\hat{\imath}$: assuming it started at the origin then if the charge was positive it will be at $-2\frac{mv_0}{qB_0}\hat{\jmath}$ and if the charge was negative it would be at $2\frac{mv_0}{qB_0}\hat{\jmath}$. The direction is determined by considering the direction of the force at the initial time.

The amount of time taken can also be determined, because the speed $v_0 = \frac{2\pi R}{T}$ where T is the period of the circular motion. By substitution, T can be seen to be $\frac{m}{|q|B_0}$, which means that at time $t = \frac{m}{2|q|B_0}$ the charge will be halfway through its circle. This situation is shown in figure 7.6.

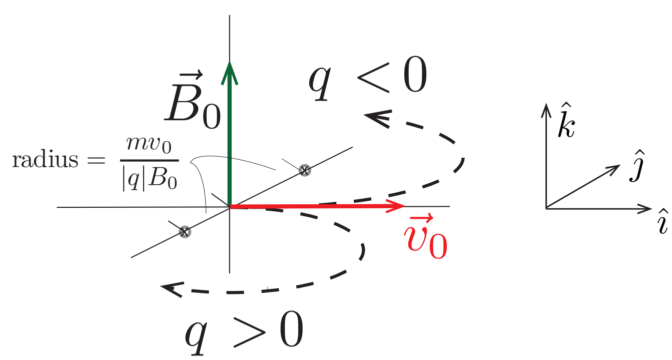

Figure 7.6: A charge q starting at the origin subject to the Lorentz force.

7.4.2 Velocity Selector

Suppose that a charged particle is moving in a region where there is both a constant electric field $\vec{E}$ and a constant magnetic field $\vec{B}$, where $\left|\vec{E}\cdot\vec{B}\right| < \left|\vec{E}\right|\left|\vec{B}\right|$. What is the condition on the velocity of the charge such that it is subject to no net force?

The key idea is that the fact that $\left|\vec{E}\cdot\vec{B}\right| \neq \left|\vec{E}\right|\left|\vec{B}\right|$ means that $\vec{E}$ and $\vec{B}$ are not in the same (or exactly opposite) directions. The net force that the charged particle is subject to is

$$\vec{F}_{net} = q\vec{E} + q\vec{v}\times\vec{B}. \tag{7.18}$$

For the net force to be zero the relation that must be satisfied is

$$\vec{E} = -\vec{v}\times\vec{B}. \tag{7.19}$$

In other words, $\vec{E}$, $\vec{B}$ and the selected component of $\vec{v}$ are all mutually perpendicular to each other. Assume that $\vec{E} = E_x\hat{\imath}$; this assumption does not affect generality at all, because $\vec{E}$ *must* point in some direction, and the coordinate system for the problem is just chosen so that it is along the x-axis.

Further assume that $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$; again this is general because this defines the component of $\vec{B}$ which is *not* along $\vec{E}$ as the y-axis, but it could have been called anything. Finally, with $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$, the required relation between $\vec{E}$, $\vec{v}$, and $\vec{B}$ is

$$\begin{aligned}
\text{x component}: \ E_x &= v_zB_y \\
\text{y component}: \ 0 &= -v_zB_x \\
\text{z component}: \ 0 &= v_yB_x - v_xB_y
\end{aligned} \tag{7.20}$$

Consistency of the y-component of this set of relations requires either $v_z = 0$ or $B_x = 0$. Of those two options, $v_z = 0$ is discarded because that would mean that $E_x = 0$, meaning that there is no electric field. So, it has been determined that $B_x = 0$, so for consistency of the z-component of the set of relations then either $v_x = 0$ or $B_y = 0$. The option that $B_y = 0$ is discarded because that would mean there was no magnetic field, so $v_x = 0$. Since $B_y \neq 0$, consistency of the x-component requires that $v_z = \frac{E_x}{B_y}$. Note that v_y has not been constrained.

The import of this is that the conditions for no force on a moving charged particle are as follows:

- $\vec{B}$ is perpendicular to $\vec{E}$
- The component of velocity perpendicular to both $\vec{E}$ and $\vec{B}$ (oriented in direction $\vec{E} \times \vec{B}$) is $\frac{|\vec{E}|}{|\vec{B}|}$
- The component of velocity along $\vec{E}$ is zero
- The component of velocity along $\vec{B}$ could be anything.

7.5 Springs

A non-fundamental, but none-the-less very useful type of force to understand is that of the idealized spring. The observation that some objects will *stretch* if pulled, that the amount that the object stretches is proportional to the applied force, that some objects will *compress* if pushed at both ends, and the amount they compress is proportional to the magnitude of the applied force can be mathematically expressed. For most of these objects the amount of stretch or compression is *linear* in the applied force.

This fact is normally expressed in a form known as Hooke's Law:

$$\left|\vec{F}\right| = k\,|\Delta\vec{x}| \tag{7.21}$$

In this, $\left|\vec{F}\right|$ is the magnitude of the force applied to either end of the object, $|\Delta\vec{x}|$ is the magnitude of the change in relative positions of the two ends, and k is a number which depends on the material and its configuration, normally known as the 'spring constant'. k is a quantity which is measured experimentally. Using Newton's third law, the force *exerted* by the spring can be determined by the same relationship. The direction of the force exerted by the spring is in such a direction as to take $\Delta\vec{x}$ to zero – it is a 'restoring' force.

It is important to note two idealizations in this: the first is that the springs described by Hooke's law are assumed not to ever break or stretch so far that they are irretrievably deformed. This assumption is analogous to assuming that objects are rigid - a useful approximation as long as the objects are not subject to forces which are much bigger than what they can sustain. The second idealization is that springs are assumed to only act along their axis of symmetry. To truely express the force exerted by the end of an object if it is pushed some amount away from its equilibrium position would require relating the exerted force to the displacement of the end via a tensor because it is not always clear that the force would exactly be back towards the unstretched configuration.

Consider the idealized spring that is not over stretched, twisted or pulled in more than one direction. Taking the direction the spring is compressed or stretched in to be the x axis then the spring has a force law given by,

$$\vec{F}(x) = -kx\hat{\imath}, \tag{7.22}$$

where $F_x(x) = -kx$ is the x-component of the force. An example with $k = 5$ [N]/[m] is given in figure 7.7.

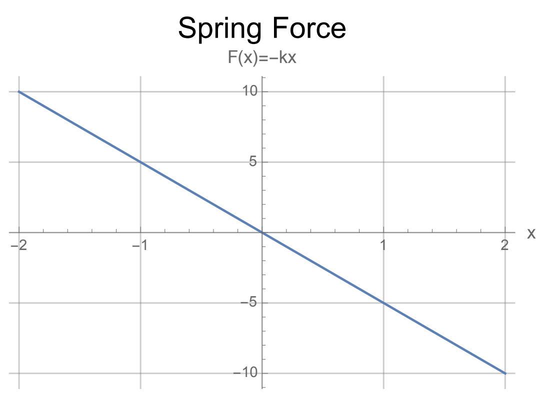

Figure 7.7: The spring force is a linear function with a negative slope, $-k$.

7.6 Pressure and Buoyancy

For most of the text the applied forces have been considered to act at a specific point, but in many cases the forces are *distributed* or exerted over an area. One important example of such a force is pressure, particularly that exerted by a fluid. The underlying *mechanical* explanation for why a fluid exerts a force on something touching or containing it is that the fluid (such as air or water) consists of a large number of molecules which are individually moving. When the molecules hit the edge of the thing that contains them they bounce back, having changed their velocity. As their velocity changed, they must have accelerated, and therefore experienced a net force. By Newton's third law, the fact that the molecules experienced a force means they must have exerted a force on whatever contained them. Since there are many individual particles, describing the force each one exerts is not particularly useful, however describing the overall effect of their interaction is.

The force exerted by a substance *spread over an area* is defined by

$$d\vec{F} = Pd\vec{A} \tag{7.23}$$

In this, the $d\vec{A}$ is the area of a small region over which the small force $d\vec{F}$ is exerted. The magnitude of $d\vec{A}$ is the area, and the direction is perpendicularly outward from the fluid which is exerting the force. This defines pressure as a constant of proportionality between the area on which the force is exerted and the force: for the same total force a larger area implies a smaller pressure.

For many fluids the pressure needs to be measured and determined based on the thermodynamic properties it exhibits, but for a class of (idealized) fluids known as 'incompressible' fluids it is possible to determine the pressure as long as the pressure is known at one point. Incompressible fluids are an idealization, just as perfectly rigid bodies are an idealization: the idea is that the fluid will not change its density no matter what force it is subject to. Water is actually a good example of an approximately incompressible fluid.

The pressure in an incompressible fluid can be determined by considering a small cylinder-shaped piece of fluid, as shown in figure 7.8. If the upper surface and the bottom surface both have area A, then the net force on the fluid within the 'cylinder' has the following z-component:

$$\begin{aligned} F_{net,z} &= F_{pressure,\ above,z} + F_{pressure,\ below,\ z} + F_{g,z} \\ &= -P(z+\delta z)A + P(z)A - mg \\ &= -\left(P(z+\delta z) - P(z)\right)A - \rho A \delta z g \end{aligned} \tag{7.24}$$

where $\rho = m/V$ is the density of the fluid.

Now, assuming that the cylinder of fluid is in equilibrium (and since it was arbitrary it should be), the net z-component of force should be zero. This can be then rearranges as

$$\begin{aligned} \frac{P(z+\delta z) - P(z)}{\delta z} &= -\rho g \\ \text{as } \delta z \to 0 \text{ becomes } \frac{d}{dz}P(z) &= -\rho g \end{aligned} \tag{7.25}$$

When ρ is constant, as it is for incompressible fluids, this *differential equation* has the solution

$$P(z) = P_0 - \rho g z \tag{7.26}$$

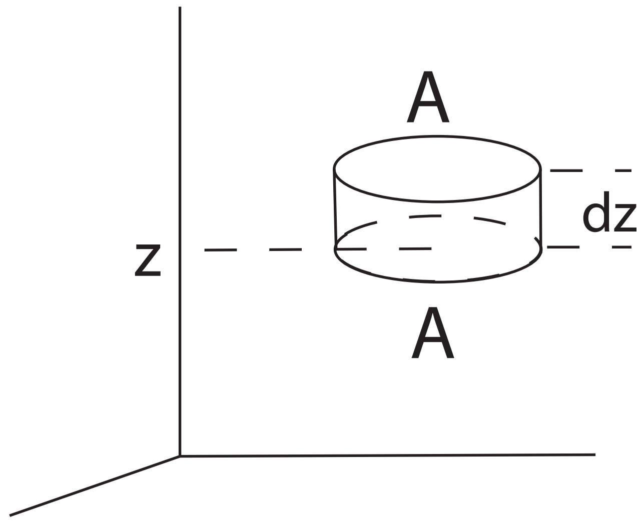

Figure 7.8: A small disc of an incompressible fluid.

where P_0 is the pressure in the fluid at *any point* as long as z is taken to be zero there. In particular, the pressure is equal to atmospheric pressure at the point where a fluid meets the air.

7.6.1 Bouyancy

The pressure difference between different levels in an incompressible fluid are the cause of a well-known phenomenon: the fact that objects float. Consider a box-shaped object which is floating in a liquid of density ρ_l. The object is uniform, has density ρ, and has side lengths L_x, L_y, and L_z as shown in figure 7.9. Determine how high above the surface of the liquid the top of the box is.

To do this the idea of equilibrium is employed. The box is in equilibrium, which means that the net force on it is zero, and in particular the z-component of the net force is zero. The only forces available are that due to gravity near the earth's surface and pressure on each side. Since the pressure force is exerted perpendicular to the surface, only the pressure force on

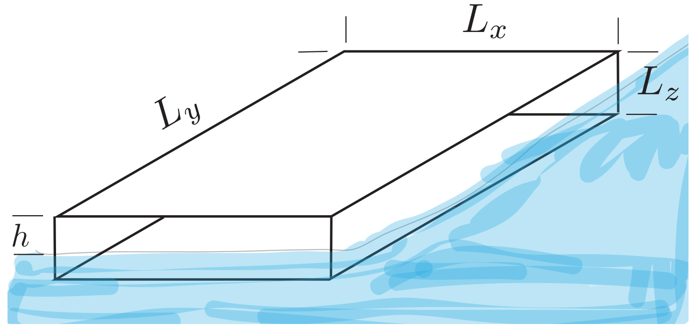

Figure 7.9: A uniform object of density ρ floats in a liquid of density ρ_l.

the top and bottom contribute.

$$\begin{aligned} F_{net,z} &= -Mg + F_{p,bottom} + F_{p,top} \\ 0 &= -\left(\rho L_x L_y L_z\right) g + \left(L_x L_y\right) P(bottom) - \left(L_x L_y\right) P(top) \end{aligned} \tag{7.27}$$

Now, the pressure at the top of the object is atmospheric pressure (P_{atm}). The pressure at the top of the liquid is also atmospheric pressure. The pressure at the bottom of the object can be determined by the assumption that the liquid is incompressible and the expression derived earlier that $P(z) = P_0 - \rho_l g z$. In this the reference level ($z = 0$) is the top of the liquid, so at that level $P_0 = P_{atm}$. The location of the bottom of the object is at $z = h - L_z$ (note that this is negative). Therefore, the pressure at the bottom of the object is $P_{atm} - \rho_l g\left(h - L_z\right)$.

Substituting in, it can be determined that

$$\begin{aligned} 0 &= -\rho L_z g + \left(P_{atm} + \rho_l g L_z - \rho_l g h\right) - P_{atm} \\ h &= L_z\left(1 - \frac{\rho}{\rho_l}\right). \end{aligned} \tag{7.28}$$

So the amount of the object above the liquid depends on the relative density between it and the liquid; the lower the average density the higher it floats.

This example also shows a general principle: that the upwards force on an object which is (either partially or fully) submerged in a liquid due to the presence of the liquid is proportional to the *weight* of the displaced liquid. In this particular example the (net) magnitude of the upwards force due to pressure in the liquid was $\rho_l\left(L_z - h\right) L_x L_y g$, or Mg where M is the mass of the displaced liquid. This is also true for more complicated shapes: any slope can be modelled as a sum of very small horizontal and vertical steps: the horizontal steps will present vertical surfaces on which pressure forces are exerted upwards.

Chapter 8

Integral Calculus

8.1 Introduction

Consider a person holding a spherical ball with the fingers of their two hands. The force that the person exerts is spread over the many individual points of contact: the finger-tips. The net force exerted by the person is the sum of the force exerted by each finger. In contrast, imagine a person holding the same ball on the palms of their hands. There are likely to be extended regions of contact between the palms and the ball, and the direction of the exerted force will change depending on exactly where on the hand it is, so it appears less obvious how to determine the *total* force exerted.

This problem, and similar problems where a quantity is somehow *continuously* distributed need a new mathematical technique to encode this idea of summing a quantity which continuously varies. The technique is called *integration*

8.2 Enclosed area and Riemann sums

The subject of integration is best understood starting from the following concrete problem. Suppose that one knows a function given by $f(x)$, with the restriction that $f(x) > 0$ for all x, and that one has chosen two numbers a and b subject to $a < b$. Further suppose that one wishes to determine the area of the region in the xy-plane bounded by the lines $y = 0$, $y = f(x)$, $x = a$ and $x = b$, as illustrated in figure 8.1.

It is possible to *estimate* the total area with the following procedure: choose an integer n which is bigger than 1. Divide the region between $x = a$ and $x = b$ into n equal-sized strips each of width $\delta x = \frac{b-a}{n}$. Determine the

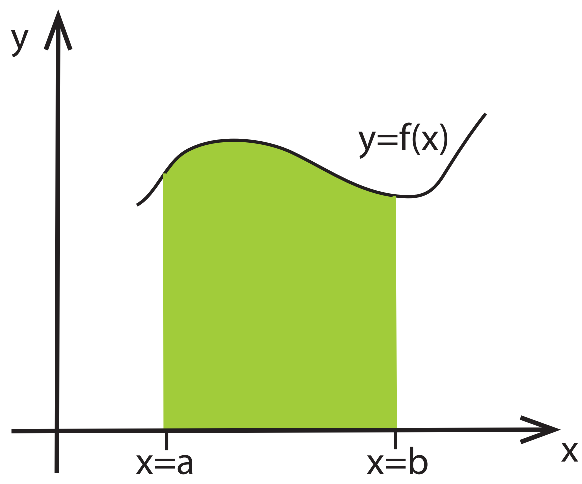

Figure 8.1: The area in the xy plane surrounded by the lines $y = 0$, $y = f(x)$, $x = a$, and $x = b$.

value of $f(x)$ at the left-hand end of each of the intervals. For each strip, its area can be approximated as 'the value of $f(x)$ at the left-hand end of the strip' multiplied by the width of the strip. Summing the estimated areas of each strip gives an approximation of the total area:

$$\text{Area} \approx \sum_{i=1}^{n} f\left(a + (i-1)\delta x\right) \delta x \tag{8.1}$$

This procedure is illustrated in figure 8.2. Note that in some cases the area estimated for the strip underestimates the true area, and in some cases the area overestimates the true area.

In order to make a more accurate approximation the value of n could chosen to be larger, and it should seem intuitively reasonable that as $n \to \infty$ (corresponding to $\delta x \to 0$) that the approximation of the area should become the actual area. This is modelled in figure 8.3. Approximating the area of the strip by a rectangle either underestimates the area of the strip or overestimates it. The area omitted, or included in error, is approximately a triangle, as shown in figure 8.4. The height of the triangle is approximately determined by the slope of the function $f'(x)$ at the left side of the strip and by the width of the strip, so the triangle's area is approximately

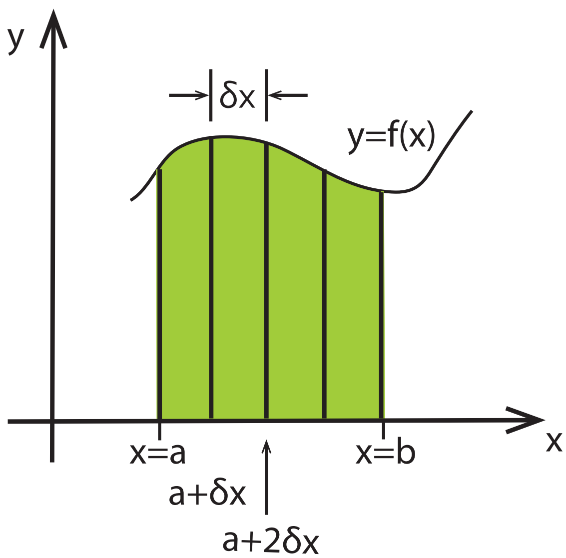

Figure 8.2: The area in the xy plane surrounded by the lines $y = 0$, $y = f(x)$, $x = a$, and $x = b$ is approximated by $n = 4$ rectangles.

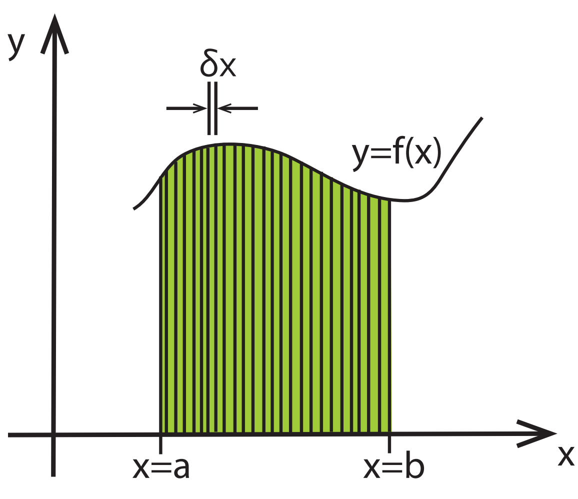

Figure 8.3: The area in the xy plane surrounded by the lines $y = 0$, $y = f(x)$, $x = a$, and $x = b$ is approximated by a large number rectangles.

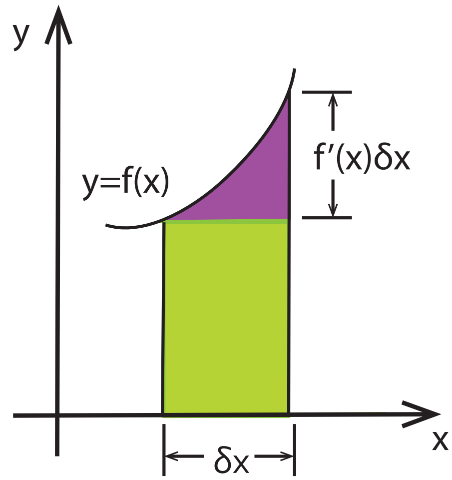

Figure 8.4: A strip of width δx is illustrated. The region inside the strip (black outline) consists of the approximating rectangle (green shading) and an error piece. The error piece is roughly a triangle of width δx and height $f'(x)\delta x$ (violet shading).

$\frac{1}{2}\delta x\,[f'(x)\delta x] = \frac{\delta x^2}{2}f'(x)$. The ratio of the estimation error to the estimated area will get smaller as δx gets smaller, which substantiates the intuition that as δx gets smaller the approximation of the area becomes more accurate.

The name for this type of procedure, adding a very large number of small elements together to get a total, is an *integral* or a *Riemann sum*. The definition and symbol for this procedure is

$$\text{Area} = \lim_{\delta x \to 0} \sum_{i=1}^{n} f\left(a + (i-1)\delta x\right)\delta x \equiv \int_a^b f(x)dx \tag{8.2}$$

The integral symbol $\int$ should be thought of as a variation of a sum, the term $f(x)$ should be thought of as the height of each 'little' rectangle, and the dx as the width of each little rectangle.

8.3 Fundamental Theorem of Calculus

The preceding discussion establishes that an integral is a sum of a very large number of small contributions. This fact can be used to establish a

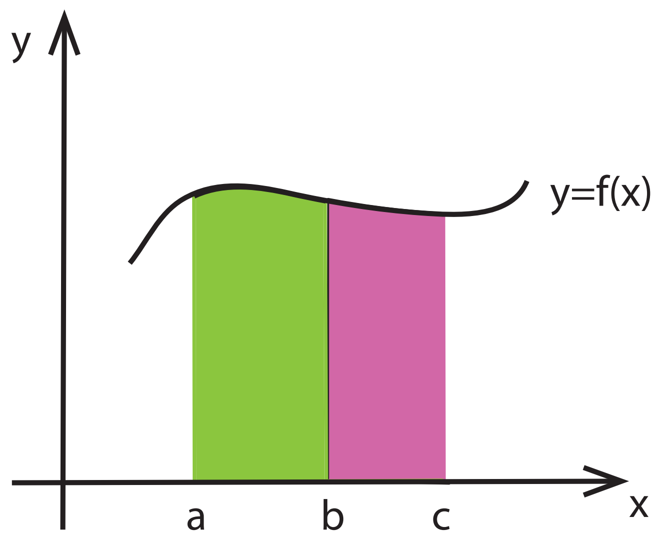

Figure 8.5: In this figure the region bounded by the lines $y = 0$, $y = f(x)$, $x = a$, and $x = b$ is shaded in red. The region bounded by the lines $y = 0$, $y = f(x)$, $x = b$, and $x = c$ is shaded in green. The area region bounded by the lines $y = 0$, $y = f(x)$, $x = a$, and $x = c$ is the sum of the areas of those two regions.

simpler way of evaluating integrals than calculating the limit of a sum of infinitesimals. This simpler way is the *Fundamental Theorem of Calculus.*

The fundamental theorem can be constructed from simple observations. The first is that integration is additive. The area enclosed by the lines $y = 0$, $y = f(x)$, $x = a$ and $x = c$ is the same as the sum of the areas enclosed by the same y bounds, from $x = a$ to $x = b$, and from $x = b$ to $x = c$. When illustrated, as in figure 8.5 this is obvious. The formal expression of this observation is

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{8.3}$$

The result of an integral depends on three things. These are the function $f(x)$ under consideration, and the two end-points of the integral a and b. The same graphical analysis as above shows that the change to the integral coming from any change to the upper limit of integration is *independent* of the lower limit of integration. This is illustrated in figure 8.6. This, together

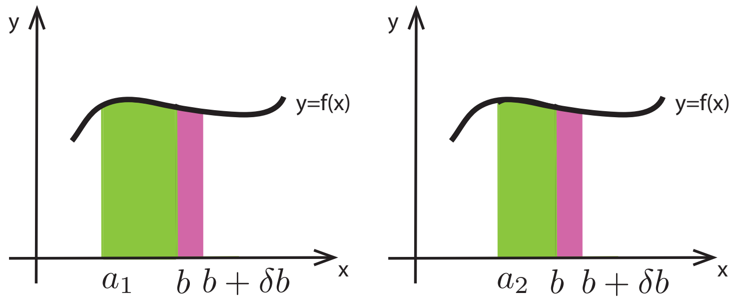

Figure 8.6: The areas indicated by the integrals $\int_{a_1}^{b} f(x)dx$ and $\int_{a_2}^{b} f(x)dx$ are shown. In each case, if b was increased to $b+\delta b$ the increase in the value of the integral would be the same.

with equation 8.3, establishes that an integral must be given by

$$\int_a^b f(x)dx = F(b) - F(a) \tag{8.4}$$

where the function $F(x)$ in the result of the integral is related to the function $f(x)$ which appears in the integral.

In order to determine the relation between $F(x)$ and $f(x)$ the natural thing to do is to use the techniques from chapter 4. To determine how much an integral changes by varying the end-point, consider the difference:

$$\begin{aligned}
\int_a^{b+\delta b} f(x)dx - \int_a^b f(x)dx &= [F(b+\delta b) - F(a)] - [F(b) - F(a)] \\
&= F(b+\delta b) - F(b) \qquad (8.5) \\
&= \int_b^{b+\delta b} f(x)dx.
\end{aligned}$$

Since δb is presumed to be small, the integral can be approximated by a single 'slice' giving

$$\begin{aligned}
F(b+\delta b) - F(b) &\approx f(b)\delta b \\
f(b) &\approx \frac{F(b+\delta b) - F(b)}{\delta b} \qquad (8.6)
\end{aligned}$$

In the limit that δb is small, this means that the derivative of $F(x)$ is $f(x)$.

This fact gives us the fundamental theorem of calculus: Assuming that $f(x)$ and $F(x)$ are related as $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{8.7}$$

Since the integral was shown to be related, in a simple case, to the area under a curve, the implication is that there are three equivalent ways of calculating an integral:

- Calculate, from geometric considerations, the area enclosed by the appropriate graph.
- Estimate the integral by taking the sum of a large number of small rectangles.
- Given the integrand $f(x)$, identify a function $F(x)$ which has $f(x)$ as its derivative, and then calculate $F(b) - F(a)$.

8.4 Examples applying the Fundamental Theorem to simple functions

The preceding development gives three different ways of calculating the value of an integral. Depending on the function being considered the 'simplest' method may be different. In particular, the meaning of the 'enclosed area' can be obscure in the case of the integrated function becoming negative, it is sometimes not straightforward to identify the antiderivative $F(x)$ which is associated with $f(x)$, and the Riemann sum method has not (in this text) been shown to always converge. The following examples show how the three methods can be applied to a number of different functions.

8.4.1 Constant functions

Consider the function $f(x) = 2$, and evaluate the integral $\int_1^4 f(x)dx$. This integral is shown in figure 8.7

The first method, finding the enclosed area, is straightforward: the area enclosed is a rectangle three units long and two units high, so the area (hence result of the integral) is 6.

The second method, splitting the area into a number of approximate rectangles, is similarly straightforward. Assuming that the area is split into

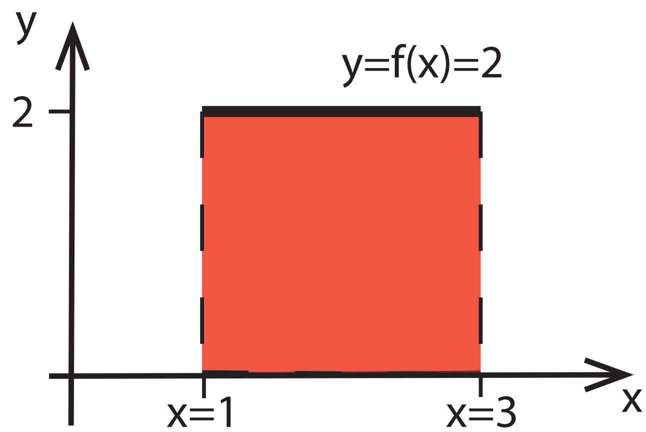

Figure 8.7: The function $f(x) = 2$. The area coloured red is the integral to be calculated.

10 rectangles the width of each would be 0.3, and then the integral would be

$$\begin{aligned}
\int_1^4 f(x)dx &\approx f(1.0)0.3 + f(1.3)0.3 + \ldots + f(3.7)0.3 \\
&= 0.3\left(f(1.0) + f(1.3) + \ldots + f(3.7)\right) \\
&= 0.3\left(2 + 2 + \ldots + 2\right)\ \text{10total} \\
&= 0.3 \cdot 2 \cdot 10 = 6 \qquad (8.8)
\end{aligned}$$

The third method, identifying an antiderivative, is also straightforward. Since $f(x) = 2$ it is necessary to find a function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$. In this case, such an appropriate function is $F(x) = 2x$. Then,

$$\begin{aligned}
\int_1^4 f(x)dx &= \int_1^4 2dx \\
&= F(4) - F(1) \\
&= 2 \cdot 4 - 2 \cdot 1 = 6. \qquad (8.9)
\end{aligned}$$

. This result is the same as those of the previous two methods.

8.4.2 Linear functions

Now, consider the function $f(x) = 3x - 5$, and the integral $\int_2^3 f(x)dx$. This integral is shown in figure 8.8.

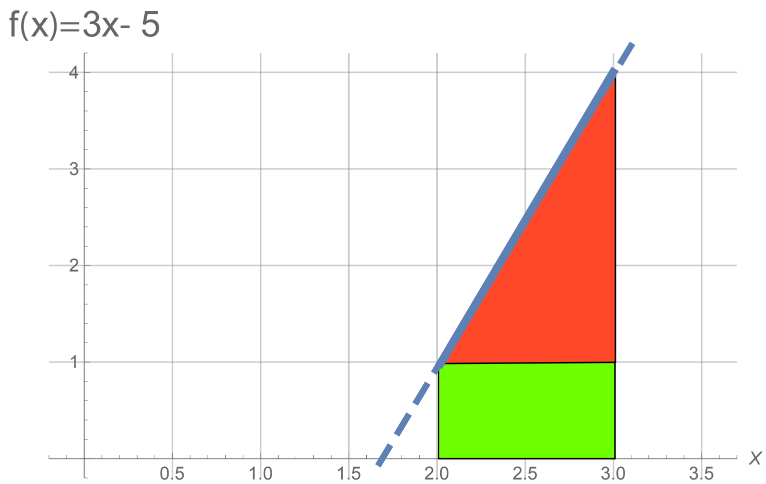

Figure 8.8: The function $f(x) = 3x - 5$. The function is represented by a solid black line between $x = 2$ and $x = 3$, and a dashed line elsewhere. The area bounded by this function is divided into a green rectangle and a red triangle.

The first method, where the integral is determined by geometrical considerations, can be resolved by considering the area confined to be the sum of a rectangle one unit wide and one unit high, and a right triangle which has base one unit and height three units. The total area of these is 2.5.

The second method, approximating the area by adding the areas of successively narrower rectangles, is somewhat more involved than it was when applied to the constant function. This is because the rectangles do not perfectly approximate the area, but 'miss' some, so the number of rectangles must be increased. Supposing that the region between $x = 2$ and $x = 3$ were broken into 20 pieces, each would have width 0.05, and then the integral would be approximated as:

$$\begin{aligned} \int_2^3 f(x)dx &\approx f(2.0)0.05 + f(2.05)0.05 + \dots f(2.95)0.05 \\ &= 0.05\,(1.0 + 1.15 + \dots 3.85) = 2.425 \end{aligned} \tag{8.10}$$

The estimate is somewhat close to what was obtained by the area method, but the results are not the same. Repeating the procedure with the region divided into 100 pieces of width 0.01 results in an approximation of $\int_2^3 f(x)dx \approx 2.485$, while 500 pieces with a width of 0.002 results in the integral approximated as 2.497. Dividing the region into 10,000 pieces results in an approximation of 2.49985. The larger the number of rectangles used in the approximation, the closer the approximation comes to 2.5. Notice that the approximation is always marginally less than that value; the reason for this is that the function $f(x)$ has a positive first derivative, so approximating the height of the strip rectangle by the height at the left end always underestimates the area.

The third method, finding an antiderivative, is relatively straightforward as in the first case. The function being integrated is $f(x) = 3x-5$; a function $F(x)$ which has $f(x)$ as its derivative is $F(x) = \frac{3}{2}x^2 - 5x$. Based on this, the integral is

$$\begin{aligned} \int_2^3 f(x)dx &= \int_2^3 (3x-5)\,dx \\ &= F(3) - F(2) \\ &= \left[\frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{3}{2}(2)^2 - 5(2)\right] = 2.5 \end{aligned} \tag{8.11}$$

Notice that $F(x) = \frac{3}{2}x^2 - 5x + C$ where C is an arbitrary constant would have served just as well, since in the evaluation of the integral the C would have been added to the $F(3)$ term, but would then have been subtracted off because of its inclusion in the $F(2)$ term.

8.4.3 Quadratic Function

Consider the function $f(x) = 3x^2 - 1$ and evaluate the integral $\int_{-1}^2 f(x)dx$. This function is shown in figure 8.9

The 'area' method is not very instructive in this case, because there is not a well-known expression for the area enclosed by a parabola, and the fact that the function $f(x)$ is negative for some of the x-values considered in the integral makes it uncertain, from an area perspective, how to account for those.

The approximation method, which divides the integral and evaluates it as a *sum* gives a definite answer. Dividing the region into 30 rectangles,

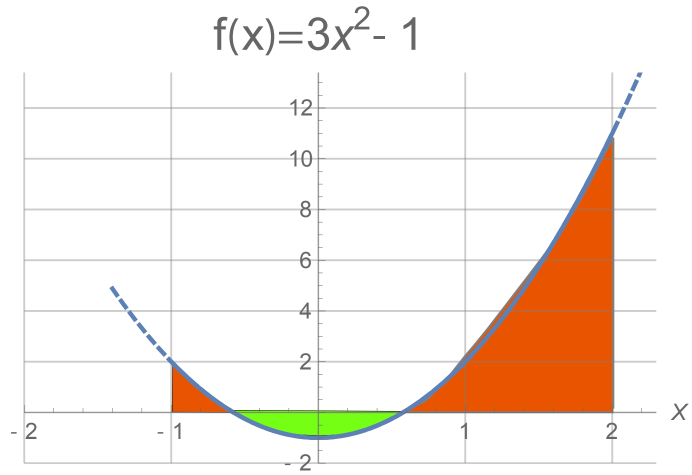

Figure 8.9: The function $f(x) = 3x^2 - 1$. The area coloured red is the region between $x = -1$ and $x = -\frac{1}{\sqrt{3}}$ and between $x = \frac{1}{\sqrt{3}}$ and $x = 2$; the region where $f(x)$ is positive. The area coloured green is the region between $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$; the region where $f(x)$ is negative.

each of width 0.1 approximates the integral as,

$$
\begin{aligned}
\int_{-1}^{2} f(x)dx &\approx f(-1.0)0.1 + f(-0.9)0.1 + \ldots f(-0.5)0.1 \\
&+ \ldots f(0.5)0.1 + \ldots f(1.9)0.1 \\
&= 0.1\,(2.0 + 1.43 + \ldots + (-0.25) \ldots + -0.25 + \ldots 9.83) \\
&= 5.565
\end{aligned}
\tag{8.12}
$$

Repeating the exercise with 300 rectangles estimates the integral to be 5.95515, while 3000 rectangles give an estimate of 5.9955, and 300,000 rectangles give an estimate of 5.999955. As the number of divisions increases the estimate gets closer and closer to 6.

Using the fundamental theorem of calculus confirms this result. The function $f(x) = 3x^2 - 1$, and for this function the $F(x)$ such that $\frac{d}{dx}F(x) =$

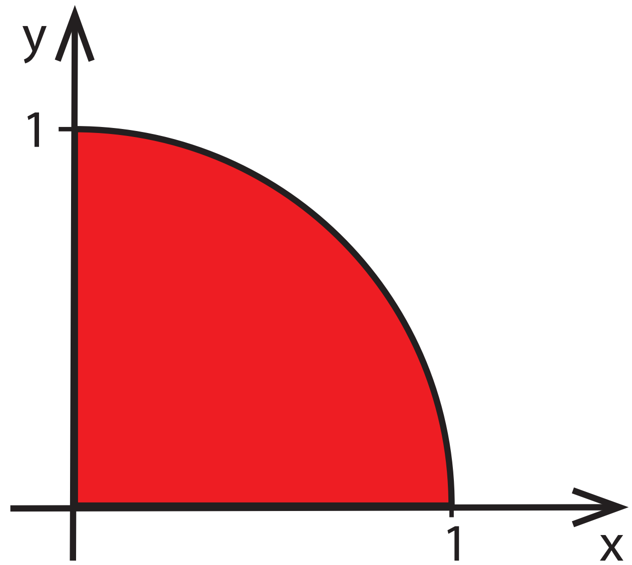

Figure 8.10: The function $f(x) = \sqrt{1-x^2}$ in the region between $x = 0$ and $x = 1$.

$f(x)$ is $F(x) = x^3 - x$. This means that

$$\begin{aligned} \int_{-1}^{2} f(x)dx &= \int_{-1}^{2} \left(3x^2 - 1\right) dx \\ &= F(2) - F(-1) \\ &= \left[(2)^3 - 1\,(2)\right] - \left[(-1)^3 - 1\,(-1)\right] = 6 \end{aligned} \tag{8.13}$$

The key observation from this example is that the fundamental theorem of calculus and the approximation and Riemann sum method coincide even if the integrand is negative sometimes.

8.4.4 Portion of a circle

Consider the function $f(x) = \sqrt{1-x^2}$ and evaluate the integral $\int_0^1 f(x)dx$. This function is shown in figure 8.10. The 'enclosed area' method is helpful in this case because the area in question is one quarter of a circle of radius 1, and therefore the enclosed area is $\frac{\pi}{4}$.

The approximation method will, as in previous cases, give progressively closer to the actual value of the integral as the number of divisions increases.

Dividing the region from 0 to 1 into 50 sections, each of width 0.02, gives the following expression for the integral:

$$\begin{aligned}\int_0^1 f(x)dx &\approx f(0.0)0.02 + f(0.02)0.02 + \ldots + f(0.98)0.02 \\ &= 0.02\,(1.0 + 0.9998 + \ldots + 0.1990)\,0.02 = 0.79457 \qquad (8.14)\end{aligned}$$

Similarly, for a division into 500 regions this method gives 0.78637, for 5000 regions the method gives 0.78550, and for 500,000 regions the method gives 0.785399, which is approaching the numerical value of $\frac{\pi}{4} \approx 0.785398$.

The antiderivative method in this case requires knowledge of more sophisticated integral techniques. The function to be integrated is $f(x) = \sqrt{1-x^2}$. A function $F(x)$ with this as its derivative is $F(x) = \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x$. This can be verified by taking the derivative, noticing that $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$. This means that

$$\begin{aligned}\int_0^1 f(x)dx &= \int_0^1 \sqrt{1-x^2}dx \\ &= F(1) - F(0) \\ &= \left[\frac{(1)\sqrt{1-(1)^2}}{2} + \frac{1}{2}\sin^{-1}(1)\right] - \left[\frac{0\sqrt{1-0^2}}{2} + \frac{1}{2}\sin^{-1} 0\right] \\ &= \frac{1}{2}\sin^{-1} 1 = \frac{\pi}{4} \qquad (8.15)\end{aligned}$$

8.5 Application of Integral Calculus to Physics problems

Integration will be a central technique in the development and exposition in the following four chapters, covering Momentum, Angular Momentum, the Work-Energy theorem, and Kinetic and Potential energy. The problems that will be solved fall, broadly, into two categories: The first is finding the total effect on the motion of an object in a time- or position-dependent force, and the second is finding the total value of some quantity which is distributed in space. In each case the technique employed is conceptually the same: the quantity to be calculated is *originally* written as a sum of a large number of small quantities – in the spirit of the 'approximation' method of the previous section – and then this sum is interpreted as a definite integral and evaluated using the fundamental theorem of calculus. Two examples will make this clearer.

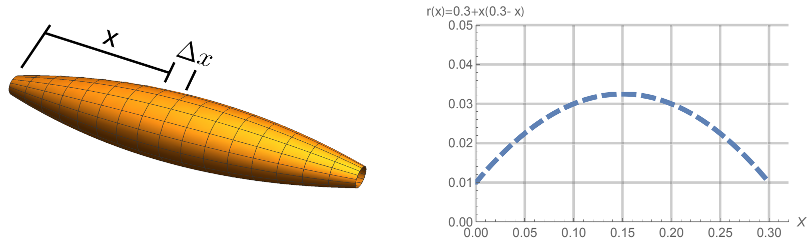

Figure 8.11: The cylindrical piece of wood of circular cross-section and varying radius. Part (a) shows the wood in perspective and illustrates the location of a circular piece. Part (b) shows the radius as a function of x, the distance from one end.

8.5.1 Mass of a cylindrical solid

Suppose that a piece of wood of density $\rho = 800\frac{kg}{m^3}$ is shaped into a rod of circular cross-section which is $0.3m$ long, whose radius, as a function of distance (x) from one end is given by $r(x) = 0.01[m] + 1[m^{-1}]x(0.3[m] - x)$ where the units of the numerical constants are marked by square brackets. This is illustrated in figure 8.11. What is the total mass of the piece of wood?

The mass of an object of constant density is given by $M = \rho V$ where V is the volume. Since there is not an *obvious* formula describing the volume of this piece of wood described it will be necessary to calculate it on its own. To do this, imagine that the piece of wood is split into a large number of circular pieces, each of width Δx, and assume that Δx is small enough that the radius of the circular piece is *approximately* constant. For the ith, piece which is a distance x_i from the reference end, the area of the circle is $\pi\left(r(x_i)\right)^2$, so the volume of the piece is $\pi\left(r(x_i)\right)^2\Delta x$. This means that the volume is approximately

$$\begin{aligned} V &= \sum_{all\ end\ points} \pi\left(r(x_i)\right)^2\Delta x \\ &= \sum_{all\ end\ points} \pi\left(0.0001[m^2] + 0.02[m]x_i(0.3[m] - x_i) + x_i^2(0.3[m] - x_i)^2\right)\Delta x \end{aligned} \tag{8.16}$$

Note that the units of numerical constants are enclosed in square brackets.

The volume has the form $\sum f(x_i)\Delta x$, which is the form that our approximations took in the previous section. Since the sums from the previous section gave the same answers as the value derived from the fundamental theorem of calculus, it is reasonable to calculate somethign by either method, so

$$\begin{aligned}
V &= \sum_{all\ end\ points} \pi\left(0.0001[m^2] + 0.02[m]x_i(0.3[m]-x_i) + 1[m^{-2}]x_i^2(0.3[m]-x_i)^2\right)\Delta x \\
&= \int_{x=0[m]}^{3[m]} \left(\pi\left(0.0001[m^2] + 0.02[m]x(0.3[m]-x) + 1[m^{-2}]x^2(0.3[m]-x)^2\right)\right) dx \\
&= \int_{x=0[m]}^{3[m]} \left(0.0001\pi m^2 + 0.006\pi m x + 0.07\pi x^2 - 1\pi m^{-2}x^4\right) dx \qquad (8.17)
\end{aligned}$$

A function $F(x)$ which satisfies

$$\frac{d}{dx}F(x) = 0.0001\pi[m^2] + 0.006\pi[m]x + 0.07\pi x^2 - 1\pi[m^{-2}]x^4 \qquad (8.18)$$

is

$$F(x) = 0.0001\pi[m^2]x + 0.003[m]\pi x^2 + \frac{0.07\pi}{3}x^3 - \frac{\pi}{5}[m^{-2}]x^5 \qquad (8.19)$$

which gives that

$$V = F(0.3[m]) - F(0[m]) = 4.45\times 10^{-3}[m^3] - 0[m^3] \qquad (8.20)$$

Given the volume of $4.45 \times 10^{-3}[m^3]$ the mass will be $3.56[kg]$.

8.5.2 Displacement given non-constant velocity

Suppose that a particle moves subject to a non-constant velocity $\vec{v}(t) = 4\left[\frac{m}{s}\right]\hat{\imath} + 2\left[\frac{m}{s^2}\right]\hat{\jmath}t$. What is its displacement between the times $t = -1[s]$ and $t = 2[s]$?

To analyse this problem we use two separate lines of reasoning. The first is that, by definition, velocity is the time derivative of position. This means that, by the fundamental theorem of calculus,

$$\Delta\vec{r} = \vec{r}(2[s]) - \vec{r}(-1[s]) = \int_{-1[s]}^{2[s]} \frac{d}{dt}r(t)dt = \int_{-1[s]}^{2[s]} \vec{v}(t)dt \qquad (8.21)$$

so the desired quantity is an integral, however it is the integral of a vector function.

To determine how to evaluate that integral, we return to the primitive definition of velocity. We know that for a *constant* velocity the displacement travelled in a time Δt is $\Delta\vec{r} = \vec{v}\Delta t$. Applying this to our problem, we divide the time between $t = -1[s]$ and $t = 2[s]$ into a large number of small segments each of length Δt, and then approximate:

$$\begin{aligned}
\Delta\vec{r} &= \sum_{all\ t_i's} \vec{v}(t_i)\Delta t \\
&= \sum_{all\ t_i's} \left(4\left[\frac{m}{s}\right]\right)\hat{\imath} + 2\left(\left[\frac{m}{s^2}\right]\hat{\jmath}t_i\right)\Delta t \\
&= \left(\sum_{all\ t_i's} 4\left[\frac{m}{s}\right]\Delta t\right)\hat{\imath} + \left(\sum_{all\ t_i's} 2\left[\frac{m}{s^2}\right]t_i\Delta t\right)\hat{\jmath} \qquad (8.22)
\end{aligned}$$

Each of the prefactors multiplying a unit vector can be converted into an integral, so

$$\begin{aligned}
\Delta\vec{r} &= \left(\int_{-1[s]}^{2[s]} 4\left[\frac{m}{s}\right]dt\right)\hat{\imath} + \left(\int_{-1[s]}^{2[s]} 2\left[\frac{m}{s^2}\right]tdt\right)\hat{\jmath} \\
&= \left(4\left[\frac{m}{s}\right](2[s]) - 4\left[\frac{m}{s}\right](-1[s])\right)\hat{\imath} + \left(1\left[\frac{m}{s^2}\right](2[s])^2 - 1\frac{m}{s^2}(-1s)^2\right)\hat{\imath} \\
&= 12[m]\hat{\imath} + 3[m]\hat{\jmath} \qquad (8.23)
\end{aligned}$$

The key lesson from this example are that the integration of a *vector* function can be treated as a number of *individual* integrals, one for each component of the relevant vector.

Chapter 9

Momentum

9.1 Introduction

If the nature of all the forces acting on a body are known, Newton's laws and the technique of free body diagrams can be used to solve the equations of motion for the acceleration. Given the acceleration and starting conditions, all other kinematic quantities can also be calculated. However, often the exact nature of a force is not known. Yet, it is still possible to determine a great deal of physics by using conserved quantities. Moreover, this technique can be used to deduce the actual nature of the force.

Momentum is one of three conserved physical quantities that are found in mechanics. In classical mechanics, momentum is given by,

$$\vec{p} = m\vec{v} \tag{9.1}$$

where $\vec{p}$ is a vector quantity proportional to the velocity and the constant of proportionality is the mass. There is no special unit for momentum. It is just referred to as $\frac{[kg\ m]}{[s]}$.

9.2 Force

The momentum of a system can change with time. The derivative of the momentum with respect to time is given by

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}m\vec{v} = m\vec{a} = \vec{F}, \tag{9.2}$$

for the case of constant mass, m. We see that Newton's Second Law has been rewritten as

$$\vec{F} = \frac{d}{dt}\vec{p} \tag{9.3}$$

If the mass of the object is changing with respect to time then the chain rule gives

$$\frac{d}{dt}\vec{p} = m\vec{a} + \vec{v}\frac{d}{dt}m(t). \tag{9.4}$$

The new term $\vec{v}\frac{d}{dt}m(t)$ is often called thrust. Consider a person with a tube of water. When the water is squirted out of the tube with a velocity $\vec{v}$ then the person experiences a force. This is the way rockets work.

The formulation of Newton's second law using momentum is therefore more general than the $F = ma$ formula. Newton actually wrote the second law using momentum.

9.3 Conservation of Momentum

Momentum is more important than just a more general way of writing the Second Law. If there are no net external forces on a body then the total net force is given by

$$\sum \vec{F} = 0 \tag{9.5}$$

and therefore

$$\begin{aligned} \frac{d}{dt}\vec{p} &= 0 \\ \vec{p} &= \text{constant} \end{aligned} \tag{9.6}$$

the momentum of the body is constant. In physics, we say that the momentum is conserved; meaning unchanged.

In order to use momentum as a conserved quantity, the concept of a system is introduced as a closed physical situation that may include multiple bodies. In particular, the interest is in systems with constant total momentum, $\vec{p}_{total} = \sum \vec{p}_i =$ constant. An example is given in figure 9.1 which illustrated in the upper portion a body with mass, M_1, moving with velocity, $\vec{v}_1$ towards a body at rest with mass M_2. The system in this case is both bodies, M_1 and M_2. The collision involves a force between the two bodies but Newton's Third Law gives,

$$\vec{F}_{1 \text{ on } 2} = -\vec{F}_{2 \text{ on } 1}, \tag{9.7}$$

and therefore the sum of these forces is zero. If there are no additional external forces acting on the system, i.e. both bodies, then the total net force on the system is zero and the total momentum is conserved. The total momentum is the sum of the momentum of both bodies,

$$\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 = M_1\vec{v}_i, \tag{9.8}$$

where the momentum of the target particle is zero because in this example

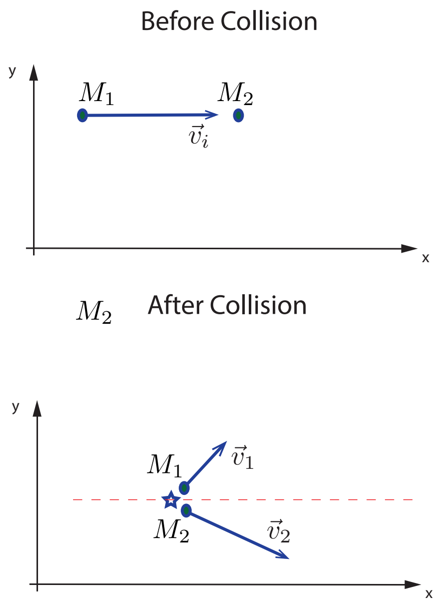

Figure 9.1: A body collides with another body at rest.

the target mass is at rest.

The upper part of figure 9.1 is labelled 'Before Collision'. In the lower part of the figure the situation after the collision is illustrated. The total momentum after the collision is the same as before the collision because it is conserved. However, now the total momentum can be written as

$$\vec{p}_{total} = M_1\vec{v}_1 + M_2\vec{v}_2. \tag{9.9}$$

and equating the total momentum before and after the collision gives,

$$\begin{gathered} \vec{p}_{\text{total init}} = \vec{p}_{\text{total final}}, \\ M_1\vec{v}_i = \vec{p}_1 + \vec{p}_2 = M_1\vec{v}_1 + M_2\vec{v}_2. \end{gathered} \tag{9.10}$$

Note, equations 9.8 and 9.9 are vector equations and therefore can be written in terms of their vector components,

$$\begin{aligned} M_1 v_{i,x} &= M_1 v_{1,x} + M_2 v_{2,x}, \\ 0 &= M_1 v_{1,y} + M_2 v_{2,y}, \end{aligned} \tag{9.11}$$

where the vectors are given in terms of the x and y components. Collision problems like this will be further discussed in Chapter 11.

Another example is a body that explodes into several parts. This is illustrated in figure 9.2. In this case, before the explosion the body is at

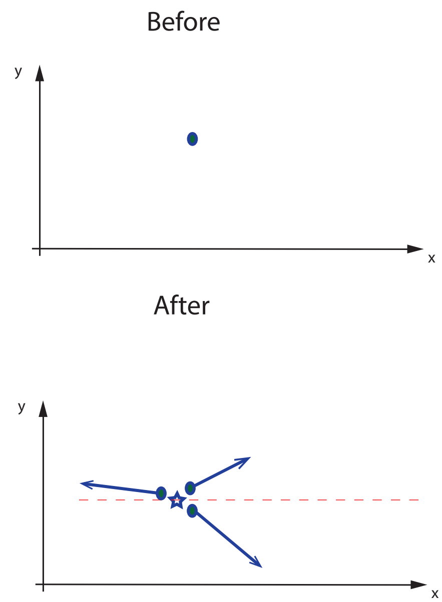

Figure 9.2: A body at rest explodes.

rest and therefore the total momentum is zero. After the explosion, the fragments fly off in many directions with many velocities. The conservation of momentum implies that when the momenta of all the fragments is summed as vectors, the total momentum must be zero.

9.4 Impulse

In the example of the colliding bodies, the exact nature of the force between them is not known. Other examples are a ball hit by a bat or a molecule of gas bouncing off the wall of the container that holds it. Some information about the nature of the force can be deduced from the change in momentum of the interacting body. Newton's law, $\vec{F} = \frac{d\vec{p}}{dt}$ can be rewritten as,

$$d\vec{p} = \vec{F}dt. \tag{9.12}$$

This is a vector equation that represents a separate equation for each component. Integrating both sides gives,

$$\int d\vec{p} \;=\; \int \vec{F}(t)\,dt. \tag{9.13}$$

where again there is a separate equation for each component. The integral with respect time of the force is called the Impulse.

In order, to make the calculus easier to follow, rewrite equation 9.13 for the x component,

$$\int_{p_{x,\text{init}}}^{p_{x,\text{final}}} p_x(t) \;=\; \int_{t_\text{init}}^{t_\text{final}} F_x(t)\,dt \tag{9.14}$$

and explicitly show the limits on the integrals. The initial momentum is the value of the momentum at an initial time, $p_{x,\text{init}} = p_x(t_\text{init})$ and the final momentum is $p_{x,\text{final}} = p_x(t_\text{final})$. Refer to the chapter on integral calculus for details but recall these are Riemann sums or the area under the function. In the case of $\int dp$, the function is a constant set to the number 1. Therefore the area, is just $A = 1 \times (p_{x,\text{final}} - p_{x,\text{init}})$ which is Δp_x.

Note, if the mass of the bodies is constant, the change in momentum can be calculated as,

$$\Delta\vec{p} = m\vec{v}_\text{final} - m\vec{v}_\text{init}. \tag{9.15}$$

Generally the exact nature of the force is unknown. An arbitrary shaped force as a function of time is illustrated in figure 9.3. If the time the force acts on the body is known, $\Delta t = t_f - t_i.$, then the actual force can be replaced by an average force. The integral of the force as a function of time gives an area. The same area can be made by taking an average force, shown as the constant line in the figure and labelled as $< F_x(t) >$ multiplied by the time, Δt, the force acts on the body. In other words,

$$\Delta p_x = \int_{t_\text{i}}^{t_\text{f}} F_x(t)\,dt =< F_x > \Delta t. \tag{9.16}$$

Consider a rubber ball being thrown at a fence. The incoming ball velocity and out going ball velocity are known. Therefore Δp is known. If the interaction of the ball is measured to take a time T, then the average force the ball exerts on the wall is given by $< F >= \frac{\Delta p}{T}$. The time, T, is how long it takes the ball to compress and then expand back. If the force is not too large it may be give by the spring law. Steel has a much bigger spring constant than rubber and therefore T would be smaller. Thus, the same Δp would result in a bigger average force because the time T would be smaller.

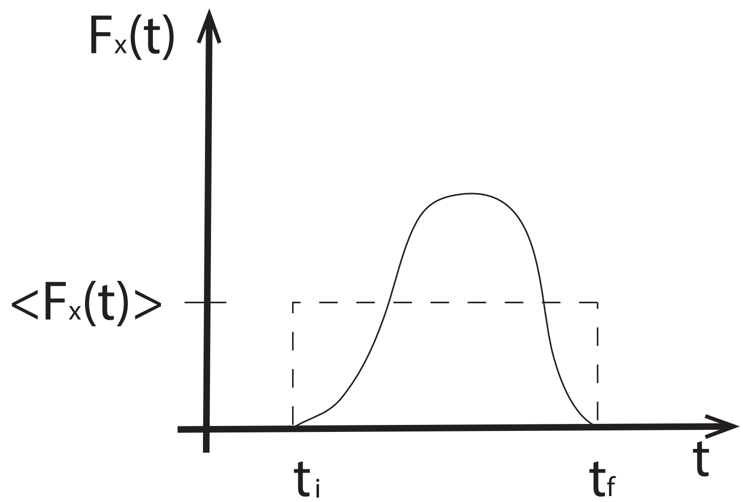

Figure 9.3: The area under the function is the same as the area inside the rectangle. The height of the rectangle represents the average force.

What if a steady stream of balls hit the wall, for example, from a jet of water. Let $\mu = \frac{\Delta m}{\Delta t}$ amount of water per unit time passing a point along the jet. When the jet bounces off the wall, the change in momentum per unit time will be $\mu(v_{xf} - v_{xi})$ and will be equal to the average force on the wall. A stream of water can knock over a wall.

9.5 Centre of Mass Frame

Applying Newton's Laws to determine the acceleration of an object requires a knowledge of the total net force force on an object in an inertial frame. In other words, an observer must be at rest or have a constant velocity with respect the body being studied and all of the external forces on the body have to be taken into account. Implicit in this is the fact that different observers in different inertial frames will measure the same forces and the same accelerations.

In Newtonian mechanics, the phrase, " observers in different inertial frames", comes down to observers with a constant relative velocity. A mathematical treatment of this phenomenon can be found in the Advanced section at the end of this chapter under the heading of Galilean Transformations.

9.5.1 Zero Momentum Frame

Momentum of a system is conserved when the total external forces on the system add to zero. It is always possible to find an inertial frame where the conserved momentum is zero. The case of the exploding body shown in figure 9.2 is a simple example where the total momentum is zero right from the start. Figure 9.1 is a case where the total momentum is not zero. In this situation, one object with momentum is incident on another that has zero momentum. If an observer moves along with the incident object, $\vec{v_{obs}} = \vec{v_i}$, that observer would see mass, M_1, at rest and hence have zero momentum. Object, M_2 has a velocity that is the negative of the observer velocity, $\vec{v}_{target} = -\vec{v_{obs}} = -\vec{v_i}$.

A velocity for an observer of a zero momentum frame for figure 9.1 can be calculated starting from equation 9.8 and adding in an observer velocity, $\vec{u}$ moving to the right. The particle velocity seen by the moving observer is given by $\vec{v} - \vec{u}$ and hence,

$$\begin{aligned} \vec{p}_{total} &= \vec{p}_1 + \vec{p}_2, \\ \vec{p}_{total} &= M_1(\vec{v}_i - \vec{u}) - M_2\vec{u} = 0. \end{aligned} \tag{9.17}$$

resulting in $\vec{u} = \frac{M_1\vec{v}_i}{M_1+M_2}$ the velocity of the observer necessary to make the total momentum zero. This is a special case where an incident object was directed on a target at rest. In general, a set of particles will have a total momentum,

$$\vec{p}_{total} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\vec{v}_i, \tag{9.18}$$

An observer moving with velocity $\vec{u}$ will measure a total momentum given by

$$\sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i(\vec{v}_i - \vec{u}). \tag{9.19}$$

Setting this equal to zero and solving for $\vec{u}$ yields,

$$\vec{u} = \frac{\sum_{i=1}^{n} m_i\vec{v}_i}{\sum_{i=1}^{n} m_i}. \tag{9.20}$$

The limits on the sum are generally assumed to run from $i = 1, \ldots n$ and therefore the expression is usually written as $\vec{u} = \frac{\sum m_i\vec{v}_i}{\sum m_i}$.

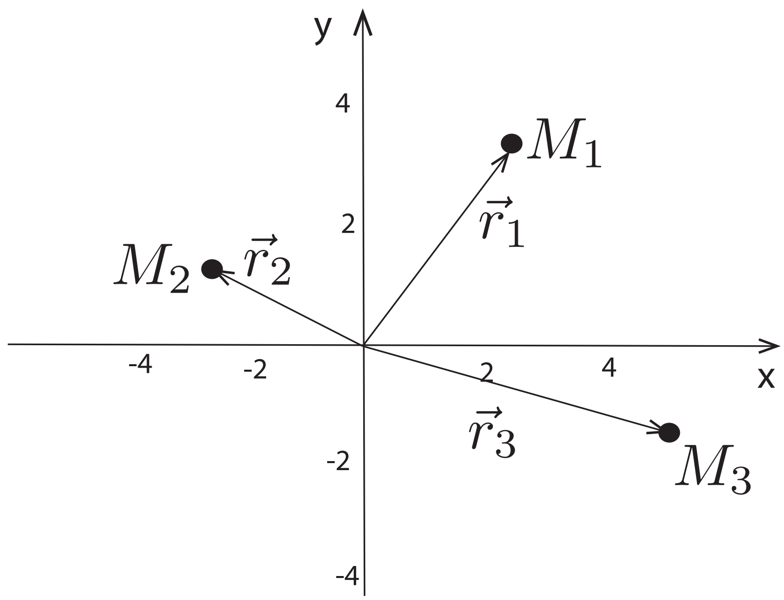

Figure 9.4: Three bodies

9.5.2 Centre of Mass Position

The zero momentum frame is known as the centre of mass frame. First seen in chapter 3, centre of mass is a concept with many applications. Recall, the centre of mass position is defined as

$$\vec{r}_{\text{CM}} = \frac{\sum m_i \vec{r}_i}{\sum m_i}, \tag{9.21}$$

where $\sum m_i$ is the total mass in the system and $\sum m_i \vec{r}_i$ is the mass weighted vector sum of all of the position vectors. Figure 9.4 shows an example where there are three masses with position vectors. The centre of mass position, $\vec{r}_{\text{CM}}$, is an average position [1] weighted by the size of the masses. Figure 9.6 shows an example. Two masses, each mass m, are located at $x = 1$ and $x = 3$ respectively. The centre of mass position has an x component only

[1] Given a set of measurements of a quantity called, x, the average is $\frac{1}{n_{tot}} \sum x_i$ where n_{tot} is the number of samples. A weighted average is given by $\frac{1}{\sum w_i} \sum w_i x_i$, where w_i are the weights.

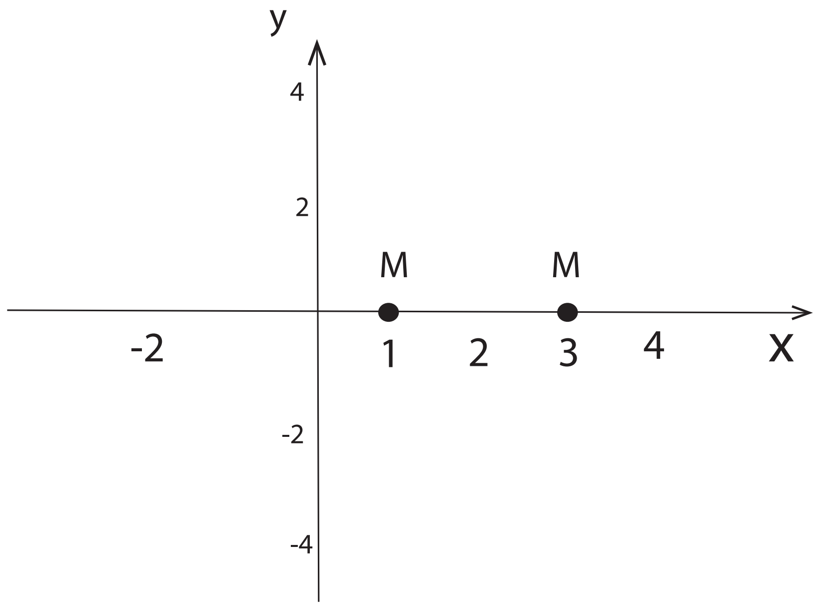

Figure 9.5: Two bodies

and is given by

$$\vec{r}_{\rm CM} = \frac{m \times 1 + m \times 3}{m+m}\hat{\imath} = \frac{4m}{2m}\hat{\imath} = 2\hat{\imath}, \tag{9.22}$$

which is a point exactly in between the two masses. If the mass at $x = 1$ is heavier than the mass at $x = 3$ then the centre of mass will move towards the mass at $x = 1$.

Some of the power of this approach can be seen if we consider the situation where the masses are moving. Their velocities are calculated by differentiating the position vectors with respect to time. The same applies to the centre of mass vector which is just a weighted sum of postion vectors,

$$\begin{aligned} \vec{v}_{\rm CM} &= \frac{\mathrm{d}\vec{r}_{\rm CM}}{\mathrm{d}t}, \\ &= \frac{1}{\sum m_i}\sum m_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}, \\ &= \frac{\sum m_i \vec{v}_i}{\sum m_i}, \end{aligned} \tag{9.23}$$

where $\vec{v}_{\text{CM}}$ is the velocity of the centre of mass. This is the velocity of the observer. The sum over the masses in the denominator just gives the total mass. The sum in the numerator of the masses times the velocities gives the total momentum. Compare equation 9.20 with equation 9.23. One sees that the centre of mass velocity is the velocity of the inertial frame where the total conserved momentum equal to zero if there are no external forces acting on the system.

9.5.3 Nonzero Acceleration

The centre of mass is useful even if the system being considered has a nonzero net force acting on it. The full treatment is a subject for a more advanced text. However, one important result is derived by differentiating the centre of mass velocity with respect to time. The result is the acceleration of the centre of mass position,

$$\begin{aligned}
\vec{a}_{\text{CM}} &= \frac{\mathrm{d}\vec{v}_{\text{CM}}}{\mathrm{d}t}, \\
&= \frac{1}{\sum m_i} \sum m_i \frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t}, \\
&= \frac{\sum m_i \vec{a}_i}{\sum m_i},
\end{aligned} \tag{9.24}$$

where $\vec{a}_i$ are the acceleration of the individual masses. Rearranging this expression gives,

$$\begin{aligned}
\sum m_i \; \vec{a}_{\text{CM}} &= \sum m_i \vec{a}_i \\
M_{\text{total}} \vec{a}_{\text{CM}} &= \sum m_i \vec{a}_i, \\
M_{\text{total}} \vec{a}_{\text{CM}} &= \sum \vec{F}_{\text{total}}.
\end{aligned} \tag{9.25}$$

Remarkably, the total force applied to the system is equal to the total mass times the acceleration of the centre of mass point. In other words, a system of particles acts as if the sum of its mass was at the centre of mass position; and the sum of all the forces on the particles is equal to a total force acting on the total mass at the centre of mass position.

The earlier chapters on forces used the technique of free body diagrams where the forces acted at a point. This result proves that treating the body as a point with the total mass of the body located there is a legitimate approach. A body can be treated as a point particle when calculating the effect of forces.

9.6 Advanced

9.6.1 Galilean Transformation

Consider two observers, each has an inertial reference frame associated with them, S labelled the unprimed frame and S' labelled the primed frame, respectively, because of the kinematic quantities in the frames will be labelled with or without primes.

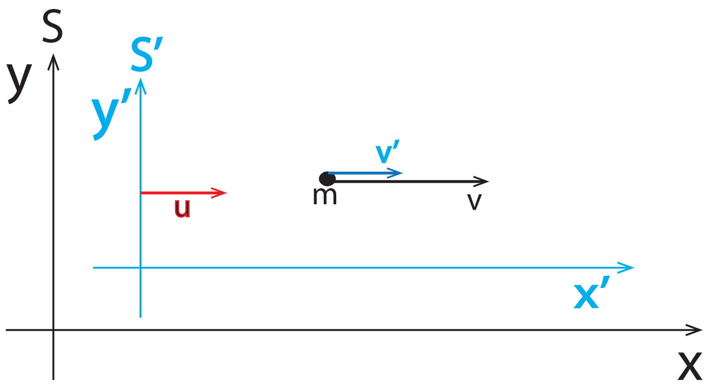

Figure 9.6: A body seen by two observers. In the unprimed frame, S, the body is moving with velocity, v. The primed frame, S', is indicated in blue and is moving in the x direction with velocity u. The body has a different velocity in the primed frame, v', given by $v' = v - u$. with velocity u.

A body in the unprimed frame, S, indicated by a mass m, has a position given as x and y. The primed frame, S' is moving with velocity u in the x direction with respect to S. The body in the S' frame will have coordinates, x' and y'. They related by,

$$
\begin{aligned}
x' &= x - ut, \qquad (9.26)\\
y' &= y,
\end{aligned}
$$

where it is assumed that at $t = 0$ the two reference frames overlap at $x = x' = 0$.

Differentiating the position coordinates in equation 9.28, gives,

$$
\begin{aligned}
\frac{\mathrm{d}x'}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - u, \qquad (9.27)\\
\frac{\mathrm{d}y'}{\mathrm{d}t} &= \frac{\mathrm{d}y'}{\mathrm{d}t},
\end{aligned}
$$

which is equivalent to

$$v' = v - ut, \tag{9.28}$$
$$v'_y = v_y.$$

This is called the Galilean addition of velocities. It has been illustrated here for a frame velocity in the x direction. It generalizes to any direction,

$$\vec{x}' = \vec{x} - \vec{u}. \tag{9.29}$$

.

Note that the velocity, $\vec{u}$ of the frame S' with respect to the frame S is a constant. Therefore, differentiating gives,

$$\frac{\mathrm{d}\vec{v}'}{\mathrm{d}t} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \frac{\mathrm{d}\vec{u}}{\mathrm{d}t}, \tag{9.30}$$
$$\vec{a}' = \vec{a}.$$

In other words the accelerations are identical in both reference frames and therefore the forces are identical as well.

Chapter 10

Angular Momentum

10.1 Introduction

Angular momentum is a second conserved physical quantity that is found in mechanics. In classical mechanics, angular momentum is given by,

$$\vec{L} = \vec{r} \times \vec{p}. \tag{10.1}$$

where $\vec{p}$ is the vector momentum and $\vec{r}$ is a position vector from an arbitrary point to the particle. The cross product requires $\vec{L}$ to be perpendicular to both the $\vec{r}$ and $\vec{p}$ vectors.

There is no special unit for angular momentum. It is just referred to as $\frac{[kg \cdot m^2]}{[s]}$. While momentum was related to the translation of particles through space. Angular momentum is related to the rotation of particles in space.

10.2 Torque

The angular momentum of a system can change with time. The derivative of the angular momentum with respect for time is given by

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \left(\frac{d}{dt}\vec{r}\right) \times \vec{p} + \vec{r} \times \frac{d}{dt}\vec{p}. \tag{10.2}$$

The deriviative of $\vec{r}$ gives the velocity $\vec{v}$. The velocity is parallel to the momentum $\vec{p} = m\vec{v}$ and the cross product of parallel vectors is identically zero. The derivative of the momentum in the second term gives $\frac{d}{dt}\vec{p} = \vec{F}$. Therefore, the derivative of angular momentum with respect to time reduces to

$$\frac{d}{dt}\vec{L} = \vec{r} \times \vec{F} = \vec{\tau}, \tag{10.3}$$

where $\vec{\tau}$ is the torque. This is analogous to the situation with momentum:

$$\begin{aligned} \vec{F} &= \frac{d}{dt}\vec{p}, \\ \vec{\tau} &= \frac{d}{dt}\vec{L}. \end{aligned} \tag{10.4}$$

At this point, when discussing momentum, the connection to $\vec{F} = m\vec{a}$ was made. There is an analogous situation wth angular moment but the analogue to mass for rotating objects will be treated below in the section on moment of inertia.

10.3 Conservation of Angular Momentum

Angular Momentum is more important than just a more general way of writing the formula for torque. If there are no net external torques on a body then the total net torque is given by

$$\vec{\tau} = 0, \tag{10.5}$$

and therefore

$$\begin{aligned} \frac{d}{dt}\vec{L} &= 0, \\ \vec{L} &= \text{constant}, \end{aligned} \tag{10.6}$$

the angular momentum of the body is conserved, i.e. constant. In order to use angular momentum as a conserved quantity, the concept of a system as a closed physical situation that may include multiple bodies is used again. In particular, the interest is in systems with constant total momentum, $\vec{L}_{total} = \sum \vec{L}_i = \text{constant}$.

An example is given in figure 10.1. Initially, a body is moving with constant circular motion. The initial point of the position vector can be chosen to make the mathematics clearer; for example as the origin of a coordinate system or at pivot point that an object is rotating around. In the figure, the position vector $\vec{r}_c$ goes from the centre of the circle to the body moving round the circle. The velocity of the body, $\vec{v}$ is perpendicular to the position vector. For a body of mass, m, the momentum is given by $m\vec{v}$. The orthogonality of the position and momentum vectors makes the cross product straight forward,

$$\vec{L} = \vec{r}_c \times m\vec{v}_c \tag{10.7}$$

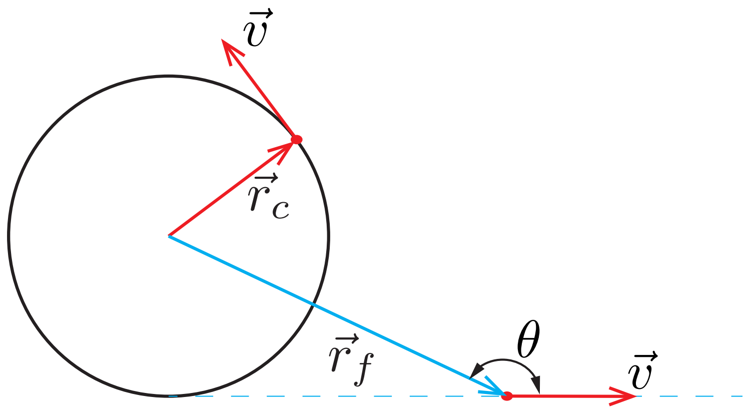

Figure 10.1: A body is rotating around a central pivot in uniform circular motion. At the bottom of the circle the body is released and moves in a straight line to the right.

where the angular momentum is perpendicular to the circle and using the right-hand rule is outwards from the page. The magnitude of the angular momentum is $|\vec{L}| = r_c m v_c$.

Angular momentum is conserved if there are no net torques acting on the body. Torque is given by $\vec{r}_c \times \vec{F}$ where the force is along the direction of the acceleration and in the case of uniform circular motion is towards the centre of the circle. In other words, along the $-\vec{r}_c$ direction. The cross product of parallel or antiparallel vectors always gives zero. Therefore the zero torque condition applies.

Next the string or force holding the mass in a circle is cut. The mass flies off in a straight line. Again, there are no forces and hence no torques acting on the mass. The angular momentum is given by,

$$\vec{L} = \vec{r}_f \times m\vec{v}_f. \tag{10.8}$$

The angular momentum vector is still pointing upwards from the page but the magnitude is now $|\vec{L}| = r_f m v_f sin(\theta)$.

The angular momentum in the circle and for the free mass have to be the same. This can be verified as follows. The magnitude of the velocity of the mass in the circle, v_c must be same as the magnitude of the velocity of the mass when it is released. Hence, $v_c = v_f = v$. The quantity $r_f sin(\theta) = r_c$

because it is a right angle triangle. Therefore,

$$\begin{aligned} r_c m v_c &= r_c m v, \\ r_f m v_f &= r_c m v, \end{aligned} \tag{10.9}$$

and we see that the angular momentum of the mass in uniform circular motion and the angular momentum of the mass after it is free are identical as expected.

Consider what happens if the string holding the mass in a circle were shortened or increased. If the string is pulled along the direction of the radius vector there is no torque. Starting with a mass moving in a circle with radius r_1 and with constant speed v_1 and changing the radius to r_2, gives from the conservation of angular momentum,

$$r_1 m v_1 = r_2 m v_2. \tag{10.10}$$

Therefore the velocity after the string length is changed becomes $v_2 = \frac{r_1 v_1}{r_2}$. This is related to why figure skaters spin faster as they pull there arms in closer to their bodies.

10.4 Rigid Bodies

An object with finite extent that does not deform is called a rigid body. Virtually any object will deform if enough force is applied to it but it is often a good approximation to treat a body as rigid. Figure 10.2 shows an arbitrary shaped rigid body rotating around an axis located at the origin of the coordinate system superimposed on the body. Three elements of mass have been singled out in the figure. The mass of the entire object is the sum of all the mass elements that make up the body. Note that each mass element rotates with the same angular velocity, ω while the radial velocity of an element depends on how far it is away from the axis of rotation, $v = \omega r$.

The angular momentum for each mass element of the object can be calculated. The sum over all the elements will give the total angular momentum

$$\begin{aligned} \vec{L}_i &= \vec{r}_i \times m_i \vec{v}_i \\ L_{z,i} &= r_i m_i v_i \\ &= m_i r_i^2 \omega, \end{aligned} \tag{10.11}$$

summing to get the total angular momentum gives,

$$\begin{aligned} L_{\text{total}} &= \sum L_{z,i}, \\ L_{\text{total}} &= \omega \sum m_i r_i^2, \end{aligned} \tag{10.12}$$

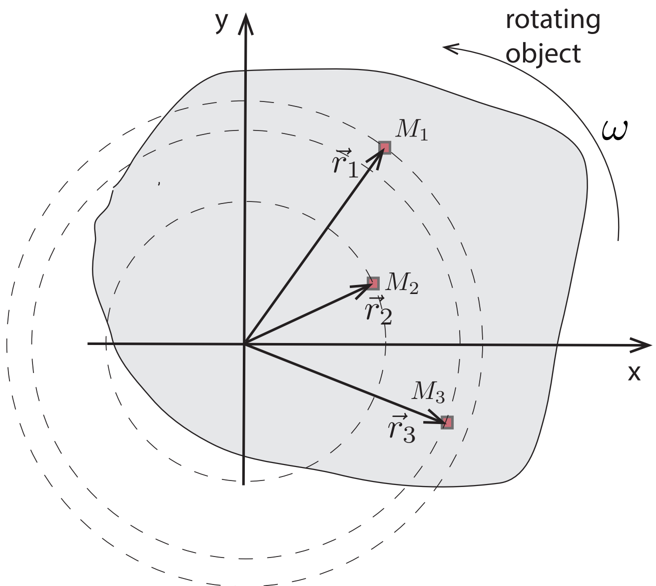

Figure 10.2: A rigid body is rotating around an axis located at the centre of the superimposed coordinate system. Each point in the body rotates about the axis with an identical value of ω.

where the term multiplying the angular velocity, ω is defined as the moment of inertia,

$$I = \sum m_i r_i^2 \tag{10.13}$$

and hence the total angular momentum is given by $\vec{L} = I\omega_z$. The angular velocity ω_z is actually the component of a vector. However, vector analysis of torques will be deferred to a more advanced treatment and only one dimensional angular velocities will be considered here.

The angular velocity of the rigid body does not have to be constant. The body could be spinning up, i.e. going around more quickly with time or vice-versa spinning down. In other words, ω could be a function of time. This corresponds to a rotational acceleration in one dimension, $\alpha_z = \frac{\mathrm{d}\omega_z}{\mathrm{d}t}$. This would correspond to a torque,

$$\tau_z = \frac{\mathrm{d}L_z}{\mathrm{d}t} = \frac{\mathrm{d}I\omega_z}{\mathrm{d}t} = I\alpha_z \tag{10.14}$$

For a rigid body the sum is replaced by an integral. The object is considered to be made up of infinitesimal pixels that have a mass $\mathrm{d}m$. This gives,

$$I = \int r^2 dm. \tag{10.15}$$

The amount of mass at a distance r depends on the shape and density of the object. If the density, ρ, is known as a function of position, then

$$m = \rho(x, y, z)V \tag{10.16}$$

where V is the volume. Therefore $dm = \rho dV$ and the moment of inertia can be written as $I = \int \rho(r)dV$. See table for a list of common shapes and their corresponding moments of inertia. It is important to note that the moment of inertia is a scalar quantity. Therefore they can be added together in the same way masses can be summed.

Description of the geometry	Moment of Inertia
A solid cylinder of radius, R, and mass, m, around the axis of the cylinder	$\frac{mR^2}{2}$
A thin shell cylinder of radius, R, and mass, m around the axis of the cylinder	mR^2
A solid sphere of radius, R, and mass, m around an axis of the sphere	$\frac{mR^2}{5}$
A hollow sphere of radius, R, and mass, m around an axis of the sphere	$\frac{mR^2}{3}$
A cylindrical rod with radius, R, length, L and mass, m around the centre of the rod	$\frac{mL^2}{12}$
A thin rectangular plate with sides, a and, b and mass, m around the centre of the plate	$\frac{m(a^2+b^2)}{12}$

Table 10.1: The moment of inertia is calculated around an axis that goes through the centre of mass

10.5 One dimensional rotational kinematics

A rigid body rotating around a fixed axis is a special case albeit of considerable practical importance. The angular velocity will always be along the rotation axis in this case. In general that is not true. Having a fixed axis also is equivalent to have a fixed plane that the rotation occurs in.

Taking the axis of rotation to be z axis of a coordinate system. A point on the object a distance r from the axis that rotates through an angle θ (note–always in radians) will travel along an arc with length $s = r\theta$. The speed is given by $v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega$.

If there is a constant rate of rotation is given by $\omega_{0z} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$ and the initial starting angle is θ_0 then the angle the object rotates through an angle given by

$$\theta(t) = \theta_0 + \omega_{0z} t \tag{10.17}$$

If there is a constant angular acceleration from a constant net torque, it can be written as

$$\begin{aligned} \alpha_z &= \frac{\mathrm{d}\omega_z(t)}{\mathrm{d}t} = \text{constant} \\ &= \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}, \end{aligned} \tag{10.18}$$

and therefore

$$a = r\alpha_z. \tag{10.19}$$

Solutions for $\omega_z(t)$ and $\theta(t)$ are given by

$$\begin{aligned} \omega_z(t) &= \omega_{0z} + \alpha_z t \\ \theta(t) &= \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 \end{aligned} \tag{10.20}$$

These equations are completely analogous with the equations developed for one dimensional translational motion with constant acceleration.

10.6 Examples - Atwood machine with massive pulley

Earlier, the Atwood machine with a massless pulley was studied. Now the massive pulley case can be understood. Figures 10.3 and 10.4 show the Atwood machine and the free body diagrams for each mass. The masses are held up by ropes with tensions, $T_1\hat{k}$ and $T_2\hat{k}$ for masses M_1 and M_2. The pulley is supported on a central axis. There are only forces along the z direction and therefore the equations of motion are given only for the z component of the net forces. The motion of the masses are connected such that $\vec{a}_1 = -\vec{a}_2$. It is convenient to assume $M_1 > M_2$ and therefore $\vec{a}_1 = -a\hat{k}$,

$$\begin{aligned} \Sigma F_{1z} &= T_1 - M_1 g &&= -M_1 a, \\ \Sigma F_{2z} &= T_2 - M_1 g &&= M_2 a, \\ \Sigma F_{Pz} &= N_P - M_P g &&= 0. \end{aligned} \tag{10.21}$$

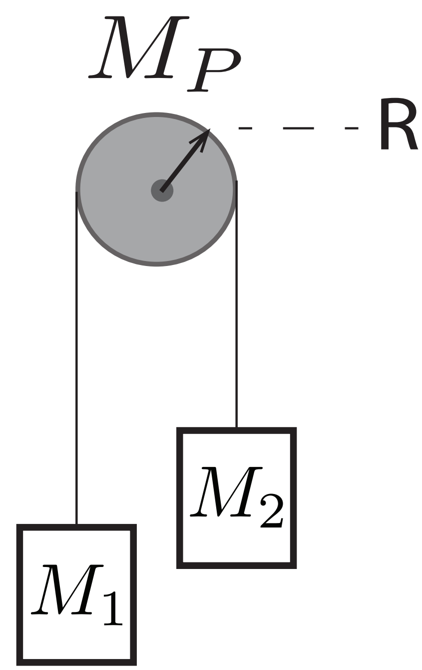

Figure 10.3: Two masses are supported by a rope and a massive pulley of radius R and mass M_P.

all of the torques are along the y axis and therefore the torque equation is given by,

$$\Sigma\tau_y = RT_1 - RT_2 = I_P\alpha_y, \tag{10.22}$$

where α_y is the angular acceleration around the y axis and I_P is the moment of inertia of the pulley.

Rearrange equation 10.22 to give an expression for $T_1 - T_2$. Rearrange equations 10.22 to give expressions for T_1 and T_2 and then substitute them in to the expression for the difference in tensions. This yields with some algebra,

$$(M_1 - M_2)g - (M_1 + M_2)a = \frac{I_P\alpha_y}{R}. \tag{10.23}$$

The rotational acceleration is related to the translational acceleration by

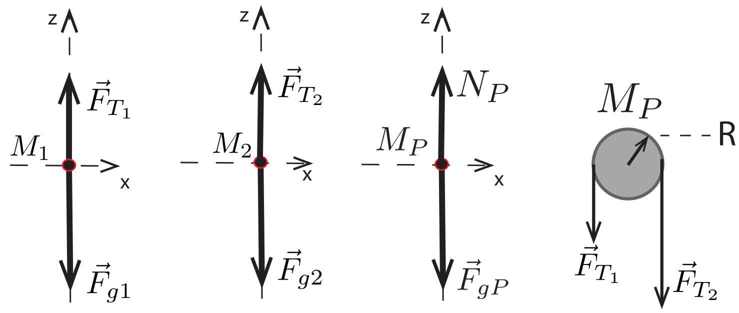

Figure 10.4: The free body diagram for the forces and torques.

$|\alpha_y| = \frac{|a|}{R}$. Substituting in this expression and collecting terms gives,

$$\begin{aligned} [(M_1 + M_2) + \frac{I_P}{R^2}]a &= (M_1 - M_2)g \qquad (10.24) \\ a &= \frac{(M_1 - M_2)g}{[(M_1 + M_2) + \frac{I_P}{R^2}]}. \end{aligned}$$

The expression for the angular acceleration was written with absolute value signs. The value of α_y can be positive or negative if the rotation is clockwise or anticlockwise, respectively. In this case, it is clockwise and α_y is positive.

Notice that if $I_P = 0$ the equation reverts to the solution of the Atwoods machine with massless frictionless pulley. The moment of inertia for a solid circular disk or cylinder is $M_P R^2/2$. Using this expression for the moment of inertia of the pulley then the acceleration becomes,

$$a = \frac{(M_1 - M_2)g}{[(M_1 + M_2) + \frac{M_P}{2}]}. \qquad (10.25)$$

Including a massive pulley reduces the acceleration. More inertia has been added to the system of masses and the pulley.

10.7 Parallel Axis Theorem

Note that in table 10.1 the moments of inertia are given for axes that are along a symmetry axis. In other words, the rotation axis goes through the centre of mass of the object. Figure 10.5 shows rotation around the point of contact between the wheel and the surface it is rolling on. The moment

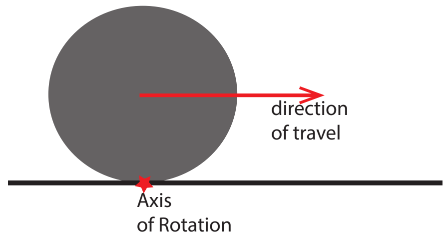

Figure 10.5: A wheel rolls along a level surface. the rotation axis is where the wheel contacts the surface.

of inertia around the contact point can be calculated using the parallel axis theorem give the moment of inertia for an axis that goes through he centre of mass. Figure 10.6 shows an object with mass, m and with a known moment

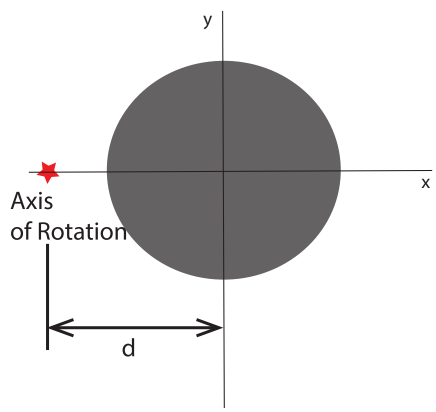

Figure 10.6: An axis of rotation isa distance d from the centre of mass of an object with a moment of inertia, I_{cm}, about the axis that goes through the centre of mass.

of inertia, I_{cm} calculated about an axis that goes through the centre of mass of the object. The moment of inertia, I_{rot} about a parallel axis a distance d away is given by,

$$I_{rot} = I_{cm} + md^2 \tag{10.26}$$

10.8 Rolling Objects with constant speed

When a circular object is rolling and not slipping then there is a fixed relationship between the angular velocity and the distance the linear distance the object moves. As the circle rotates it moves along the surface the same distance as the arc length it has rotated through. This is illustrated in figure 10.7. The point of contact of the wheel moves along the rim through

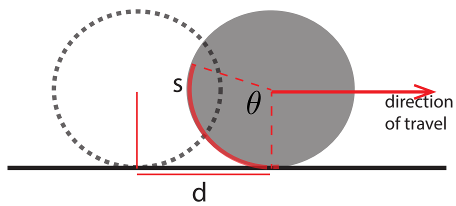

Figure 10.7: A wheel rolls along a level surface. the rotation axis is where the wheel contacts the surface.

an arc length $s = r\theta$. At the same time the wheel advances by the same distance $d = s = r\theta$ along the surface. If the angular velocity of the wheel is $\omega = \frac{\Delta\theta}{t}$ then the rim of the wheel moves with $v = r\omega$ and this is also the speed of the wheel moving over the surface.

Chapter 11

Work and Energy

11.1 Introduction

The concepts of mechanical work and kinetic energy are the basic starting point for a more advanced discussion of energy in general. One of the fascinating aspects of energy is its ability to take on different forms. For example, electrical energy, acoustic energy and electromagnetic energy like light are just a few common forms of energy. Moreover, it is possible to change one type of energy into another. Mechanical work and energy are clearly defined and make a good starting point for any discussion of energy. It is also a useful physical concept for solving problems.

11.2 Mechanical Work with constant Forces

The simplest case of mechanical work is a constant force acting on a mass, it could be pushing or pulling in the direction of motion, and moving that mass a distance $|\Delta\vec{r}|$ as is illustrated in figure 11.1. The work done, W, is defined as,

$$W = |\vec{F}||\Delta\vec{R}|, \tag{11.1}$$

the product of the magnitude of the force applied to the mass and the magnitude of the displacement along the direction of the force. Notice that work is a scalar quantity with units of joules, $[J] = [N][m]$.

In figure 11.2, the force is still constant but at an angle to the displacement. It is the component of the force along the direction of the displacement, $\vec{F}\cos\theta$ that contributes to the work. The work is the dot product of

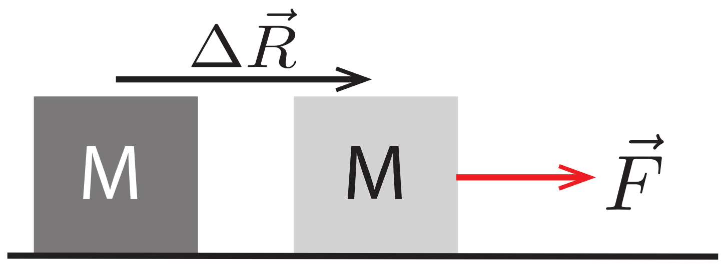

Figure 11.1: A constant force acts on a body to move it a distance $|\Delta\vec{R}|$ along the same direction as the force.

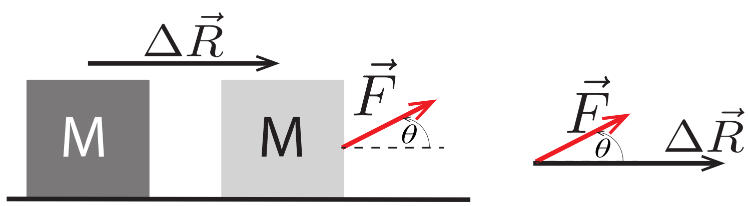

Figure 11.2: A constant force acts on a body to move it a distance $|\Delta\vec{R}|$ at an angle θ to the force direction.

the force and displacement vectors.

$$W = \vec{F} \cdot \Delta\vec{R} = |\vec{F}||\Delta\vec{R}| \cos\theta. \tag{11.2}$$

The component of the force along the displacement is automatically taken care of by the dot product. This equation reveals a lot more information about work. If the force is along the displacement it gives the simplest case, equation 11.1. if the angle is 90°, then the work done by the force is identically zero. Similarly, if the force is acting in the opposite direction of the displacement then the work is negative. For example, a mass sliding on a frictionless surface has a force acting to slow it down. The work done by this force is negative.

A block being pulled along a surface with friction illustrates all three cases as seen in figure 11.3 and the free body diagram as seen in figure 11.4. The object is being displaced to the right along the surface. The gravitational force and the normal force are perpendicular to the displacement.

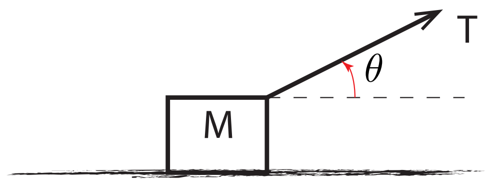

Figure 11.3: A constant force acts on a body to move it a distance $|\Delta\vec{R}|$ at an angle θ to the force direction.

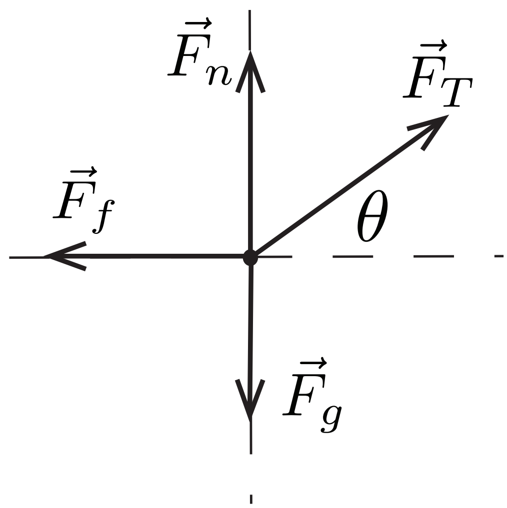

Figure 11.4: The free body diagram for the case of a constant force acting on a body to move it a distance $|\Delta\vec{R}|$ at an angle θ to the force direction.

Therefore their dot product with the displacement is zero and the work done by these forces is zero. The friction force is exactly opposite to the displacement. The angle between the force and the displacement vectors is $\theta = 180°$ and therefore $\cos\theta = -1$. This makes the work done by the friction equal to,

$$W_f = \vec{F}_f \cdot \Delta\vec{R} = -|\vec{F}_f||\Delta\vec{R}|. \tag{11.3}$$

The work done by the force, $\vec{F}_T$ is given by,

$$W_T = \vec{F}_T \cdot \Delta\vec{R} = |\vec{F}_T||\Delta\vec{R}|\cos\theta. \tag{11.4}$$

Notice that when discussing the work done by forces, it is essential to specify the force and the displacement of the mass.

In the case of the mass being pulled along the surface with friction, there are two forces doing work on the mass, M. The total work is just the scalar sum of the two values of the work.

$$Wtot = W_f + W_T. \tag{11.5}$$

11.3 Forces that vary with position

Spring forces, electric and gravitational forces vary with position. The spring force depends on the extension of the spring, the electric force depends on the separation of the charges and gravity depends on the distance between masses. These forces can do work on a mass by displacing it. However the formula for the constant forces is not applicable. The general case of a force that varies with position and changes direction with respect to the direction of the displacement is beyond the scope of this text. However, the case of a force that varies in strength along the direction of the displacement can be calculated by considering infinitesimal intervals. Take the direction of the displacement to be the x-axis. Consider an infinitesimal displacement $\mathrm{d}x$. The work done to move a mass by $\mathrm{d}x$ is given by,

$$\mathrm{d}W = F_x(x)\mathrm{d}x. \tag{11.6}$$

The total work is found by summing all of the infinitesimal quantities of work. This is given by the integral,

$$W = \int_{x_{init}}^{x_{final}} F_x(x)\mathrm{d}x, \tag{11.7}$$

where x_{init} and x_{final} are the initial and final values of the displacement along the x-axis, i.e. $\Delta x = x_{final} - x_{init}$. The integral can be calculated as long as the force $F_x(x)$ is known.

Recall the spring force is given by $F(x) = -kx$ as shown in figure 11.5. Zero is the point where the spring is in equilibrium. The region of positive x corresponds to the spring being stretched while in the negative x region the spring is compressed. The shaded areas on the graph correspond to the area under the function. Recall, if the area is below the x axis then it is considered a negative area. the following examples illustrates how this works.

A spring has been pulled pulled by an external force from an x position, $x = x_0$ to a final position x_f. The displacement is $\Delta\vec{r} = (x_f - x_0)\hat{\imath}$. The x component of the displacement is $x_f - x_0$. The external force will do positive work on the mass attached to the spring. The Direction of the spring force (towards the left) is in the opposite direction to the displacement. The spring does negative work on the mass. A calculation demonstrates this,

$$\begin{aligned} \Delta x &= (x_f - x_0 < 0) \\ F_x(x) &= -kx \\ W_S &= \int_{x_0}^{x_f} -kx \mathrm{d}x \\ &= -\frac{1}{2}kx^2|_{x_0}^{x_f} \\ &= -\frac{1}{2}k(x_f^2 - x_0^2) \\ &= \frac{1}{2}k(x_0^2 - x_f^2). \end{aligned} \tag{11.8}$$

This is shown as the negative area on the graph in figure 11.5. The amount of work done by the spring on the mass attached to the spring depends on the specific values of the starting position of the spring and the final position where it ends up.

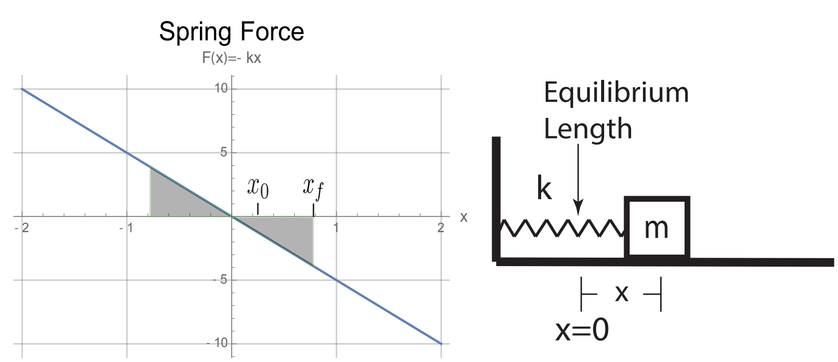

Figure 11.5: The function $f(x) = -kx$ where $k = 5$ is plotted. The shaded areas represent the area under the curve given by the function $f(x)$. This is equivalent to an integral.

11.4 Kinetic Energy

In the previous example with the spring, it was left in a stretched state that was being maintained by an external force acting on the mass attached to the spring. If the spring is released, in other words the external force is removed, the spring will pull the mass back to the equilibrium position where the mass will have a nonzero velocity. The mass will continue past the equilibrium position. The motion of the mass is called *kinetic energy.*

Kinetic energy is connected to work as can be understood from the first example in this chapter, illustrated in figure 11.1 . A mass sits at rest on a frictionless surface. A constant force acts on the mass and the mass moves through a displacement. The force acting on the mass will accelerate it in the horizontal direction. At the end of the displacement, the acceleration of the mass will have changed the velocity of the mass from its initial value. In this case, the result is a nonzero velocity.

Making use of Newton's Second Law for force, $F_x = ma_x$, and the kinematics of acceleration, $a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}$, the velocity and kinetic energy can be calculated as follows,

$$\begin{aligned} \mathrm{d}W &= F_x(x)\mathrm{d}x, \\ &= ma_x(x)\mathrm{d}x, \\ &= m\frac{\mathrm{d}v_x(x)}{\mathrm{d}t}\mathrm{d}x. \end{aligned} \tag{11.9}$$

The x position is a function of time, $x = x(t)$ and therefore the the velocity is a function of a function, $v_x(x) = v_x(x(t))$. Using the chain rule of calculus, $\frac{\mathrm{d}v_x(x)}{\mathrm{d}t} = \frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ and the kinematics of velocity, $\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(t)$ gives the following expression,

$$\begin{aligned} \mathrm{d}W &= mv_x(x)\frac{\mathrm{d}v_x(x)}{\mathrm{d}x}\mathrm{d}x, \\ &= mv_x(x)\mathrm{d}v_x, \end{aligned} \tag{11.10}$$

where in the last step the $\mathrm{d}x$ on top is cancelled with the one on the bottom. Now, the infinitesimal elements of work, $\mathrm{d}W$, can be summed in an integral. The integration is over $\mathrm{d}v_x$. When $x = x_0$, $v_x(x) = v_x(x_0)$ and when $x = x_f$,

$v_x(x) = v_x(x_f)$. This gives,

$$\begin{aligned}
\int_0^W \mathrm{d}W &= m\int_{v_x(x_0)}^{v_x(x_f)}, \quad (11.11)\\
W &= \frac{1}{2}mv_x^2|_{v_x(x_0)}^{v_x(x_f)},\\
&= \frac{1}{2}m[v_x(x_f)]^2 - \frac{1}{2}[v_x(x_0)]^2,\\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}v_0^2.
\end{aligned}$$

Defining kinetic energy as, $K = 1/2mv^2$, gives the *Work-Energy Theorem*,

$$W = K_{final} - K_{initial}. \quad (11.12)$$

The change in kinetic energy of the mass sliding over the frictionless surface is given by the work done on the mass by the applied force. The work can be calculated from equation 11.7. For example, in the case of the spring, the work done between two positions, x_0 and x_f, was calculated in equations 11.8. Using the Work-Energy Theorem the change in kinetic energy can be calculated. If the velocity at one of the positions is known, then the magnitude of the other can be calculated. For example if the mass on the spring is held at rest at x_0, i.e. $v_0 = 0$, then the magnitude of the velocity at x_f can be calculated.

11.5 Energy and work for a rotating rigid body

A body that is rotating has kinetic energy associated with it. Consider a rigid body with a moment of inertia, I, that is rotating with angular velocity, ω. The kinetic energy is given by,

$$K = \frac{1}{2}I\omega^2. \quad (11.13)$$

If a torque $\tau(\theta)$ is applied to a rigid body and causes it to rotate from an initial position given by θ_i to a final position θ_f, an amount of work is done given by,

$$W = \int_{\theta_i}^{\theta_f} \tau(\theta)\mathrm{d}\theta \quad (11.14)$$

where the torque may be a function of the angular position of the rigid body.

11.6 Power

Power is defined as the rate of change of work with respect to time,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{11.15}$$

The units are watts which is joules per second, [W]=[J]/[s].

For a constant force, the work is given by $W = \vec{F} \cdot \Delta\vec{R}$. Substituting this expression into equation 11.15 and using the product rule gives

$$P = \frac{\mathrm{d}\vec{F}}{\mathrm{d}t} \cdot \Delta\vec{R} + \vec{F} \cdot \frac{\mathrm{d}\Delta\vec{R}}{\mathrm{d}t}. \tag{11.16}$$

The derivative of the constant force is zero and recalling that the derivative of the displacement with respect to times is the velocity gives

$$P = \vec{F} \cdot \vec{v}. \tag{11.17}$$

11.7 Collisions and Scattering

A collision or scattering event was discussed in chapter 9 on momentum where it was determined that momentum was conserved if there were no net external forces acting on the particles. In a scattering process, the kinetic energy also can be used to analyze the collision. There are three cases,

- **elastic scattering** where the total kinetic energy is the same before and after the collision;
- **totally inelastic scattering** where there is a maximum difference in the total kinetic energy before and after the collision;
- **inelastic scattering** where the kinetic energy is different before and after the collision but not necessarily by the maximum amount.

11.7.1 Elastic Scattering

Figure 11.6 shows two masses colliding. Panel A at the top left shows what is often called the laboratory frame of reference. Implicitly an observer is considered to be at rest with respect to the coordinates. The colliding particles have momenta $\vec{P}_1 = m_1\vec{v}_1$ and $\vec{P}_2 = m_2\vec{v}_2$. The example shows a specific case where m_2 is at rest and the x-axis of the coordinate system is aligned with $\vec{P}_1$. While this appears to be a specific case, it is in fact

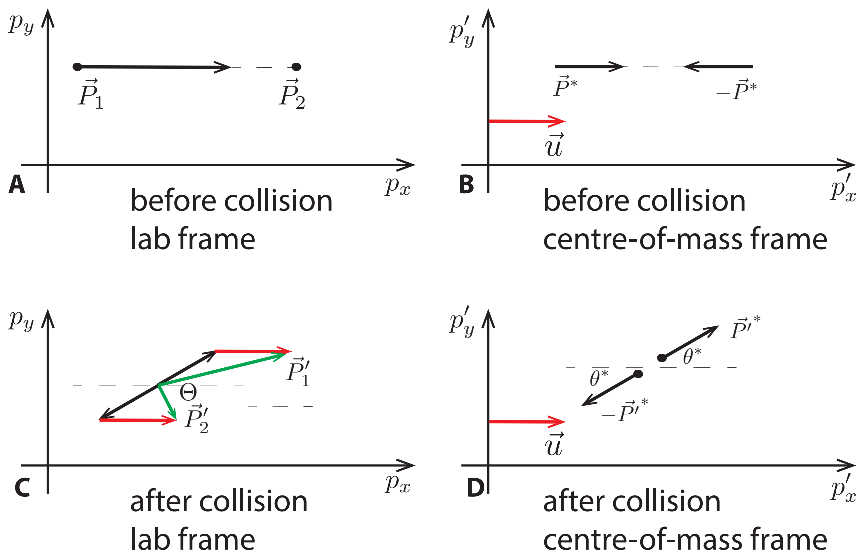

Figure 11.6: Elastic scattering example. Two masses collide with momenta $\vec{P}_1$ and $\vec{P}_2$.

possible to arrange any two body collision to look like this by choosing the appropriate velocity of the observer.

After the masses collide they move apart as illustrated in panel C at the bottom left side of figure 11.6 with momenta that are now indicated with a prime, $\vec{P}_1'$ and $\vec{P}_2'$ and green coloured arrows. The vectors drawn in black and red help calculate the final momenta through the method of the transforming reference frames.

It is possible to solve the momentum conservation and kinetic energy conservation equations directly:

$$\begin{aligned} \vec{P}_1 + \vec{P}_2 &= \vec{P}_1' + \vec{P}_2', \\ \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 &= \frac{1}{2}m_1 {v'}_1^2 + \frac{1}{2}m_2 {v'}_2^2. \end{aligned} \tag{11.18}$$

The first equation involves vectors and therefore is really three equations; one for each of the x, y and z components of the momenta. The kinetic energy equation involves the square of the magnitude of the velocity vector. These equations can be solved but the algebra is difficult because of the squares.

It is easier to use the centre-of-mass frame, also known as the zero momentum frame, discussed in chapter refchap:Momentum on momentum. In panel B on the top right hand side of figure 11.6, the momentum vectors have been transformed to a reference frame where the total momentum is zero. It is easy to see what is needed to accomplish this when the masses are equal, $m_1 = m_2$. This is the situation when billiard balls collide. If an observer moves to the right with one half of $\vec{v}_1$, i.e.$\vec{u} = \vec{v}_1/2$, then two things happen. Mass m_2 is now moving to the left with velocity $\vec{v}_2' = -\vec{v}_1/2$ and mass m_1 is moving to the right with half of its original mass, $\vec{v}_1/2$. in other words they collide head on with equal and opposite momentum.

The general case for $m_1 \neq m_2$ uses equations 9.17. Since the momentum is conserved, it does not matter what the velocity of the observer frame is. Start with the conservation of momentum and subtract an observer velocity, $\vec{u}$.

$$\begin{aligned} \vec{P}_1 + \vec{P}_2 &= \vec{P}_1' + \vec{P}_2', \qquad (11.19)\\ m_1\vec{v}_1 + m_2\vec{v}_2 &= m_1\vec{v}_1' + m_2\vec{v}_2', \\ m_1(\vec{v}_1 - \vec{u}) + m_2(\vec{v}_2 - \vec{u}) &= m_1(\vec{v}_1' - \vec{u}) + m_2(\vec{v}_2' - \vec{u}). \end{aligned}$$

The quantity $(m_1 + m_2)\vec{u}$ has been subtracted from both sides of the equation. Therefore the equation remains unchanged and momentum is still conserved. The centre-of-mass frame is where $(m_1 + m_2)\vec{u}$ is exactly right to make the total momentum zero before and after the collision. setting the third equation of 11.19 to zero gives

$$\begin{aligned} (m_1 + m_2)\vec{u} &= m_1\vec{v}_1 + m_2\vec{v}_2 \qquad (11.20)\\ (m_1 + m_2)\vec{u} &= m_1\vec{v}_1' + m_2\vec{v}_2' \end{aligned}$$

resulting in $\vec{u} = \frac{m_1\vec{v}_i + m_2\vec{v}_1}{m_1+m_2} = \frac{m_1\vec{v}_1' + m_2\vec{v}_2'}{m_1+M_2}$, the velocity of the observer necessary to make the total momentum zero.

Recall that in general,

$$\vec{p}_{total} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\vec{v}_i, \qquad (11.21)$$

An observer moving with velocity $\vec{u}$ will measure a total momentum given by

$$\sum_{i=1}^{n} \vec{p}_i^{\,*} = \sum_{i=1}^{n} m_i(\vec{v}_i - \vec{u}) = \sum_{i=1}^{n} \vec{p}_i - \vec{u}\sum_{i=1}^{n} m_i, \qquad (11.22)$$

where $\vec{p}_i^{\,*}$ is the vector-momentum of each particle in the centre of mass frame before the collision. Sometimes the last term of equation 11.22 is called the centre of mass momentum'

$$\vec{P}_{\rm cm} = \vec{u}\sum_{i=1}^{n} m_i. \tag{11.23}$$

Solving for $\vec{u}$ yields,

$$\vec{u} = \frac{\sum_{i=1}^{n} m_i\vec{v}_i}{\sum_{i=1}^{n} m_i}. \tag{11.24}$$

The limits on the sum are generally assumed to run over all of the particles and therefore the expression is usually written as $\vec{u} = \frac{\sum m_i\vec{v}_i}{\sum m_i}$.

Applying equation 11.24 to the case where $m_1 = m_2 = m$, $\vec{v}_1 = v_1\hat{\imath}$ and $\vec{v}_2 = 0$, one gets $\vec{u} = v_1\hat{\imath}/2$, exactly expected from the arguments made earlier. Using equation 11.22, gives $\vec{p}_1^{\,*} = \vec{v}_1/2$ labelled as $\vec{P}^{\,*}$ in the upper right hand panel B of figure 11.6 and $\vec{p}_2^{\,*} = -m\vec{v}_1/2$ labelled as $-\vec{P}^{\,*}$. The centre-of-mass momentum, $\vec{P}_{\rm cm} = \frac{(2m)\vec{v}_1}{2}\hat{\imath} = m\vec{v}_1\hat{\imath}$.

All this work now pays off. The lower right panel D of figure 11.6 shows the situation in the centre-of-mass frame after the collision. The momentum vectors have the same magnitude before and after the collision for the case of elastic scattering because the kinetic energy is conserved in addition to the momentum. Only their direction changes by an angle, θ^*. They remain back to back because the total momentum is conserved and therefore remains zero.

Again, for the case where $m_1 = m_2 = m$, $\vec{v}_1 = v_1\hat{\imath}$ and $\vec{v}_2 = 0$, the momenta after scattering are given by,

$$\begin{aligned} \vec{P'}^{*} &= mv_1[\cos\theta^*\hat{\imath} + \sin\theta^*\hat{\jmath}], \qquad (11.25)\\ -\vec{P'}^{*} &= mv_1[\cos(\theta^* + \pi)\hat{\imath} + \sin(\theta^* + \pi)\hat{\jmath}]\\ &= -mv_1\left[\cos\theta^*\hat{\imath} + \sin\theta^*\hat{\jmath}\right]. \end{aligned}$$

The real payoff can be seen in the lower left panel C of figure 11.6. The back to back CM frame vectors are shown in black. In order to get to the laboratory frame the vector $\vec{P}_{\rm cm}$ shown in red is added to the CM vectors. This is a straight-forward operation.

Once again, for the case where $m_1 = m_2$, $\vec{v}_1 = v_1\hat{\imath}$ and $\vec{v}_2 = 0$, the momenta after scattering in the laboratory frame are given by,

$$\begin{aligned}
\vec{P}_1' &= \vec{P'}^* + \vec{P}_{\rm cm}, \\
&= (mv_1 \cos\theta^* + mv_1)\hat{\imath} + mv_1 \sin\theta^* \hat{\jmath}, \\
&= mv_1(1 + \cos\theta^*)\hat{\imath} + mv_1 \sin\theta^* \hat{\jmath}, \\
\vec{P}_2' &= -\ P'^* + \vec{P}_{\rm cm}, \\
&= (-mv_1 \cos\theta^* + mv_1)\hat{\imath} - mv_1 \sin\theta^* \hat{\jmath}, \\
&= mv_1(1 - \cos\theta^*)\hat{\imath} - mv_1 \sin\theta^* \hat{\jmath}
\end{aligned} \tag{11.26}$$

The angle between the outgoing momenta in the laboratory frame is labelled as Θ. It can be calculated by taking the dot product between the two outgoing momenta. This gives,

$$\vec{P}_1' \cdot \vec{P}_2' = m^2 v_1^2 (1 - \cos^2\theta^*) - m^2 v_1^2 \sin^2\theta^* = 0. \tag{11.27}$$

A dot product equal to zero means the angle $\Theta = \pi/2$. The direction of the two equal mass balls always makes a right angle.

11.7.2 Totally Inelastic Scattering

Total inelastic scattering is characterized by the maximum loss of kinetic energy. This is easiest to understand in the centre-of-mass frame. By definition the momentum adds to zero. The kinetic energy of the incident particles will add up to something nonzero. After the collision, the kinetic energy of all the particles will be zero. This is illustrated in figure **??**. The upper two panels look the same as for elastic scattering but the lower right panel, D, is different. In the centre-of-mass frame after scattering the kinetic energy is zero. It has been dissipated as heat or some other form of energy that is lost. The effect in the laboratory frame is that the final state particles have the centre-of-mass momentum only. The illustration shows a mass moving along the x-axis and colliding with a mass at rest. This situation is easier to visualize than the general case. To be explicit, if mass m_1 has velocity $v_1\hat{\imath}$ and if mass m_2 has velocity zero, then

$$\begin{aligned}
m_1 v_1 \hat{\imath} + 0 &= \vec{P}_{\rm cm}, \\
\vec{v}' &= \frac{m_1 v_1}{m_1 + m_2}\hat{\imath}
\end{aligned} \tag{11.28}$$

However, the general case with multiple moving masses going in different directions is now straight forward to understand. The sum of all the incident momenta will define the centre-of-mass momentum and this will be the momentum of the final state.

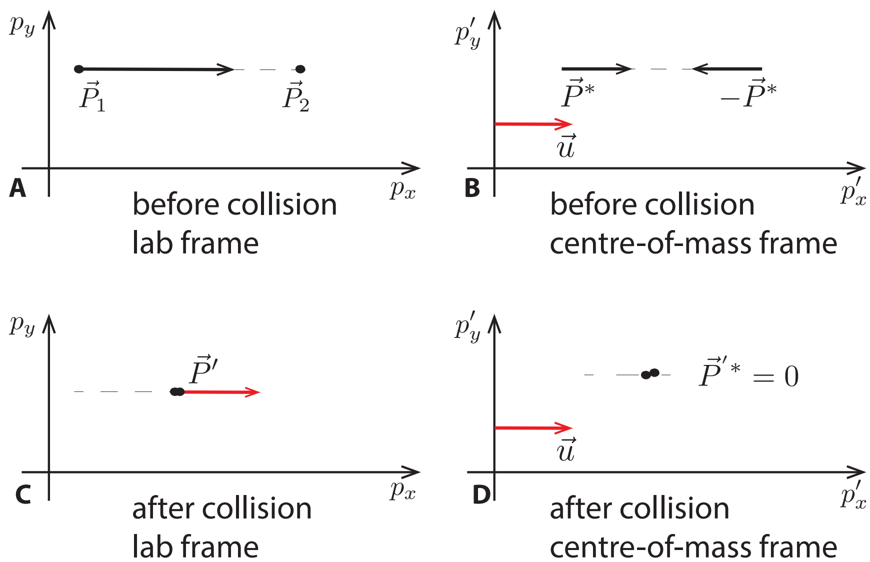

Figure 11.7: Totally Inelastic scattering example. Two masses collide with momenta $\vec{P}_1$ and $\vec{P}_2$. They lose the maximum possible kinetic energy.

11.7.3 Inelastic Scattering

Inelastic scattering covers all of the other possibilities. The kinetic energy is not conserved but also it is not zero in the centre-of-mass frame. In the centre-of-mass frame it is nonzero and can be less than the kinetic energy of the initial before scattering state. In this case some kinetic energy has been lost. Or it can be more than the kinetic energy in the initial state. In this case, some internal energy carried by the particles is released. for example, a proton can collide with an antiproton and make a positive and negative pion. The pions are lighter than the proton and will have more kinetic energy. The source of this energy is an important result from special relativity.

Chapter 12

Potential Energy

12.1 Introduction

The concepts of work and energy are important as they describe changes in the motion, either translation or rotation, of an object. However, it is not always easy to calculate the work done by a force on an object. If the path taken by the object is complicated, or if the force changes depending on the position or velocity of the object it may be challenging to calculate the work done. However, for *some* forces, the work done by the force does not depend on the path taken by the object, but rather only on the beginning and ending points. For this kind of force the work done can be encoded in a quantity called the 'Potential Energy', which simplifies some problems which would otherwise require more advanced mathematics.

12.2 Work-Energy Theorem Revisited

As was discussed in chapter 11 the work-energy theorem links the change in kinetic energy of an object to the work done on it. That relation is

$$\Delta KE = W_{net} = \int_{start}^{end} \vec{F} \cdot d\vec{r} \tag{12.1}$$

and understanding 'Potential Energy' requires understanding how the work for a varying force is calculated.

12.2.1 Work done by a varying force - procedure

There is a standard procedure for calculating the work done by a variable force. The steps are to

- Write an expression for the position of the object at a general point along the path it will follow. This expression, or function, will depend upon a single parameter which describes how far along the path the object has gone. This is known as 'parametrizing' the path taken, and results in an expression such as

$$\vec{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath} + z(s)\hat{k} \tag{12.2}$$

  In this example the single parameter is s; in kinematic problems covered in chapter 5 the usual parameter used was the time t.

- Based on the position of the object the vector describing the 'small step' that the object takes as the parameter changes from s to $s + ds$. The expression for this is

$$d\vec{r}(s) = \frac{d\vec{r}}{ds}ds = \left(\frac{dx(s)}{ds}\hat{\imath} + \frac{dy(s)}{ds}\hat{\jmath} + \frac{dz(s)}{ds}\hat{k}\right) ds \tag{12.3}$$

- Based on the position of the object, calculate the force on the object at that instant. This generates an expression for $\vec{F}$ which depends on s, the parameter.

- Calculate the quantity $\vec{F}(s) \cdot d\vec{r}$. This is a quantity that depends on the parameter s and what it represents is the amount of work done during the step which takes the object on a displacement $d\vec{r}$. What this *effectively* does is model each small step along the object's path as being subject to a constant force.

- Calculate the total work by evaluating

$$W = \int_{smallest\ s}^{biggest\ s} \vec{F}(s) \cdot d\vec{r} \tag{12.4}$$

  The quantity calculated is a function solely of s, so it is a single variable integral.

12.2.2 Work done by varying force - example

Consider the following position-dependent force described in the xy plane:

$$\vec{F} = C\left(\frac{-y}{x^2 + y^2}\hat{\imath} + \frac{x}{x^2 + y^2}\hat{\jmath}\right) \tag{12.5}$$

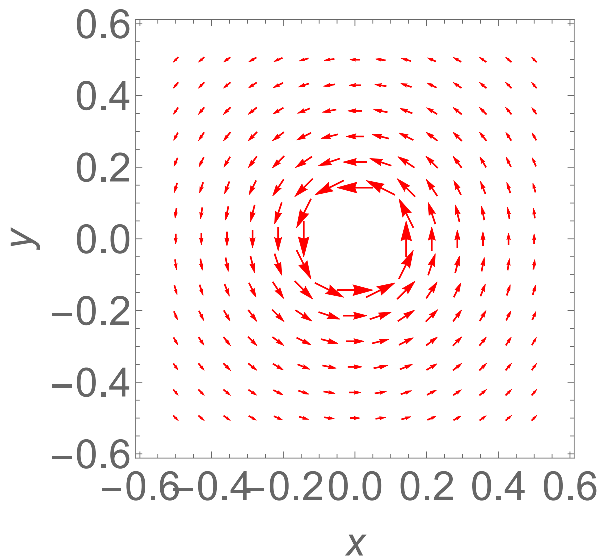

Figure 12.1: An illustration of the position-dependent force $\vec{F} = C\left(\frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right)$

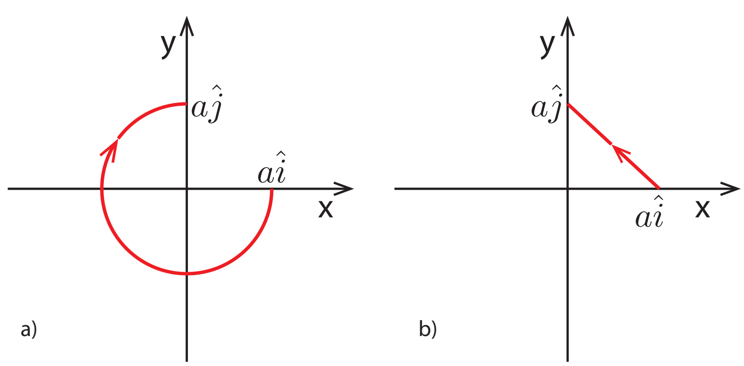

Figure 12.2: Part a is an illustration of the clockwise path from $a\hat{\imath}$ to $a\hat{\jmath}$ along the line $|\vec{r}| = a$. Part b is an illustration of a straight line path from $a\hat{\imath}$ to $a\hat{\jmath}$.

This force, drawn in space, appears sort of like a whirlpool, curling around the origin. This is illustrated in figure 12.1 and suppose that a particle travels from $\vec{r}_i = a\hat{\imath}$ to $\vec{r}_f = a\hat{\jmath}$ along a curved path with $|\vec{r}| = a$ clockwise, as drawn. This is illustrated in part a of figure 12.2

To determine the work done along this path, follow the steps given in the previous section.

- Imagining the parameter s to be like the angle, write

$$\vec{r}(s) = a\cos s\hat{\imath} - a\sin s\hat{\jmath} \tag{12.6}$$

  In this parametrization the value $s = 0$ corresponds to the initial point, and the value $s = \frac{3\pi}{2}$ corresponds to the final point since $\vec{r}(0) = a\hat{\imath}$ and $\vec{r}(\frac{3\pi}{2}) = a\hat{\jmath}$.

- Using this parametrization the step is

$$d\vec{r} = (-a\sin s\hat{\imath} - a\cos s\hat{\jmath})\,ds \tag{12.7}$$

- Knowing the value of $\vec{F}$, when the particle is at the location given by the parameter s it can be determined that

$$\begin{aligned}
\vec{F}(s) &= C\left(\frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right) \\
&= C\left(\frac{-(-a\sin s)}{(a\cos s)^2 + (-a\sin s)^2}\hat{\imath} + \frac{a\cos s}{(a\cos s)^2 + (-a\sin s)^2}\hat{\jmath}\right) \\
&= \frac{C}{a}(\sin s\hat{\imath} + \cos s\hat{\jmath}) \qquad (12.8)
\end{aligned}$$

- Now, to calculate the work done in a short step, the expression is

$$\begin{aligned}
dW &= \vec{F}(s)\cdot d\vec{r} \\
&= \frac{C}{a}(\sin s\hat{\imath} + \cos s\hat{\jmath})\cdot(-a\sin s\hat{\imath} - a\cos s\hat{\jmath})\,ds \\
&= -C ds \qquad (12.9)
\end{aligned}$$

- Finally, this is integrated, giving

$$\begin{aligned}
W &= \int_{start}^{stop} \vec{F}\cdot d\vec{r} \\
&= \int_0^{\frac{3\pi}{2}} (-C ds) = -\frac{3\pi}{2}C \qquad (12.10)
\end{aligned}$$

The procedure shows that for the specified force and the specified path, the total work done is $-\frac{3\pi}{2}C$.

Now, consider the slightly different path shown in part b of figure 12.2. In this case, the path is the straight line between $a\hat{\imath}$ and $a\hat{\jmath}$. To calculate the work done by the same force on the object as it moves along this path the same procedure is followed.

- Parametrize the path. In this case for values of the parameter s between 0 and 1 the position is given by

$$\vec{r}(s) = a\,(1-s)\,\hat{\imath} + as\hat{\jmath} \tag{12.11}$$

Note that in this case $\vec{r}(0) = a\hat{\imath}$ and $\vec{r}(1) = a\hat{\jmath}$, and for intermediate values $\vec{r}(s)$ obviously falls along the straight line between those points.

- Calculating $d\vec{r}$ gives

$$d\vec{r}(s) = (-a\hat{\imath} + a\hat{\jmath})\,ds \tag{12.12}$$

Notice that the vector $d\vec{r}$ points along the line from the starting to the ending points.

- For the values of the position listed the force is

$$\begin{aligned}\vec{F}(s) &= C\left(\frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}\right)\\ &= C\left(\frac{-as}{(a(1-s))^2+(as)^2}\hat{\imath} + \frac{as}{(a(1-s))^2+(as)^2}\hat{\jmath}\right)\\ &= \frac{C}{a}\frac{s\hat{\imath}+(1-s)\hat{\jmath}}{2s^2-2s+1}\end{aligned} \tag{12.13}$$

- From this, the work done in a little step along $d\vec{r}$ is

$$\begin{aligned}dW &= \vec{F}(s)\cdot d\vec{r}\\ &= \frac{C}{a}\frac{s\hat{\imath}+(1-s)\hat{\jmath}}{2s^2-2s+1}\cdot(-a\hat{\imath}+a\hat{\jmath})\,ds\\ &= C\frac{1}{1-2s+2s^2}ds\end{aligned} \tag{12.14}$$

- Knowing dW, it is possible to determine the total work done as

$$\begin{aligned} W &= \int_{start}^{stop} \vec{F} \cdot d\vec{r} \\ &= \int_0^1 C \frac{1}{1-2s+2s^2} ds \\ &= C - \tan^{-1}(1-2s)\Big|_0^1 = C\frac{\pi}{2} \end{aligned} \tag{12.15}$$

In this pair of examples the work done by the force depends upon the path taken from the initial location to the final location. In cases such as this there is nothing to do if one wishes to calculate the total work done by a force except to *calculate it* using the method outlined above. However, there is a class of forces for which the work done by it is *independent* of the path an object takes. These forces are known as *conservative* forces and are the focus of this chapter.

12.2.3 Conservative and Non-Conservative forces

As explained above a conservative force is one where the work done on the object is independent of the path taken by the object subject to this force. For this to be true, the fundamental theorem of calculus requires that the force must be expressible in a particular way. The force cannot be velocity-dependent - as is the case for the Lorentz force which depends on both speed and direction of motion or frictional forces which depend on direction of motion - and it cannot be a 'circulating' force like the one specified in the example above. In fact, the force must be expressible as a derivative of a function of only position, as described below.

Suppose that there were a a conservative force $\vec{F}$ which acted on an object which follows some path between locations $\vec{r}_1$ and $\vec{r}_2$. The work done is then

$$W(\vec{r}_1 \ to \ \vec{r}_2) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{12.16}$$

Now suppose that one wishes to determine the work done by the force on a path between $\vec{r}_1$ and $\vec{r}_2 + \delta\vec{r}$ where $\delta\vec{r}$ is small. Then, the work done is

$$\begin{aligned} W(\vec{r}_1 \ to \ \vec{r}_2 + \delta\vec{r}) &= \int_{\vec{r}_1}^{\vec{r}_2+\delta\vec{r}} \vec{F} \cdot d\vec{r} \\ &\approx \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} + \vec{F}(\vec{r}_2) \cdot \delta\vec{r} \\ &= W(\vec{r}_1 \ to \ \vec{r}_2) + \vec{F}(\vec{r}_2) \cdot \delta\vec{r} \end{aligned} \tag{12.17}$$

The approximation in the second step comes from the definition of integration, and the force is the force at the location from which the 'last step' starts.

Notice that this means that

$$\vec{F}(\vec{r}_2) \cdot \delta\vec{r} = W\left(\vec{r}_1 \; to \; \vec{r}_2 + \delta\vec{r}\right) - W\left(\vec{r}_1 \; to \; \vec{r}_2\right) \tag{12.18}$$

Now, suppose that $\delta\vec{r} = \delta x\hat{\imath}$, then the x-component of the force $\vec{F}$ is given as

$$F_x(\vec{r}_2) = \frac{W\left(\vec{r}_1 \; to \; \vec{r}_2 + \delta\vec{r}\right) - W\left(\vec{r}_1 \; to \; \vec{r}_2\right)}{\delta x} \tag{12.19}$$

and similarly if $\delta\vec{r}$ were $\delta y\hat{\jmath}$ or $\delta z\hat{k}$ it would be found that

$$\begin{aligned} F_y(\vec{r}_2) &= \frac{W\left(\vec{r}_1 \; to \; \vec{r}_2 + \delta\vec{r}\right) - W\left(\vec{r}_1 \; to \; \vec{r}_2\right)}{\delta y} \\ F_z(\vec{r}_2) &= \frac{W\left(\vec{r}_1 \; to \; \vec{r}_2 + \delta\vec{r}\right) - W\left(\vec{r}_1 \; to \; \vec{r}_2\right)}{\delta z} \end{aligned} \tag{12.20}$$

respectively. Because of the assumption that the force is conservative, or equivalently that the work done does not depend on the path taken, the argument of section 8.3 implies that this result is entirely independent of $\vec{r}_1$. In other words $W\left(\vec{r}_1 \; to \; \vec{r}_2\right) = G(\vec{r}_2) - G(\vec{r}_1)$ for some as yet undetermined function $G(\vec{r})$.

In the limit that $\delta\vec{r} \to 0$, this shows that the components of the conservative force are related to the derivatives of this function:

$$\vec{F}_x = \frac{d}{dx}G(\vec{r}) \tag{12.21}$$

with similar expressions for the y and z components.

12.2.4 Work-Energy theorem and potential energy

Returning now to the work-energy theorem, it is possible to categorize the forces that the object experiences are either those of conservative forces or non-conservative forces. The work-energy theorem then becomes

$$\begin{aligned} \Delta KE &= W_{net} \\ \Delta KE &= W_{conservative} + W_{non-conservative} \\ W_{non-conservative} &= \Delta KE - W_{conservative} \\ W_{non-conservative} &= \Delta KE + \Delta PE \end{aligned} \tag{12.22}$$

where the last equality is a definition: the change in potential energy is *defined* as being the negative of the amount of work that is done by conservative forces. In other words,

$$\Delta PE = -W_{conservative}. \tag{12.23}$$

This means that the potential energy is a *function* of position only, and that the conservative force is defined by the relationship that

$$F_x(\vec{r}) = -\frac{d}{dx}PE(\vec{r}) \tag{12.24}$$

There are similar expressions for the y- and z-components of the force. In general this can be written as

$$\vec{F}(\vec{r}) = -\vec{\nabla}PE(\vec{r}) = -\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)PE(\vec{r}) \tag{12.25}$$

The meaning of the ∂ symbol is that it is a *partial* derivative. These are simply the same as regular derivatives but with all other variables held constant. The ∇ symbol is defined in 12.25 and is used to make a compact notation and to emphasize that the result is a vector.

Conservative forces are, as their name implies, related to the conservation of mechanical systems. Consider the case where only conservatives forces are acting on a body. Using the last equation in **??** and setting $W_{non-conservative} = 0$ gives,

$$\begin{aligned} 0 &= \Delta KE + \Delta PE, \\ 0 &= (K_f - K_i) + (PE_f - PE_i), \end{aligned} \tag{12.26}$$

which can be rewritten as

$$\begin{aligned} K_i + PE_i &= K_f + PE_f, \\ E_i &= E_f, \end{aligned} \tag{12.27}$$

where a new quantity $E = K + PE$ has been defined. It is the total mechanical energy for a conserved system. It is important to note that there may be multiple conserved forces in a problem. Their potential energies simply add,

$$E = K + \sum_{j=1}^{n} PEj \tag{12.28}$$

where n is the number of conserved forces in the problem, For example, there could be gravity and a spring force.

The advantage of the potential energy formulation is that many for problems the only quantity of interest is the speed of an object. This has the effect of converting problems which would need to be solved with calculus to algebraic problems. In the following sections, some types of forces and their associated potential energies are cataloged.

12.3 Constant forces

Suppose that an object is subject to a *constant* force, $\vec{F}_c$. If the object undergoes a displacement of $\Delta\vec{r}$, the change in potential energy will be, as outlined above

$$\begin{aligned} \Delta PE &= -W_{conservative} \\ &= -\int_{\vec{r}_i}^{\vec{r}_i+\Delta\vec{r}} \vec{F}_c \cdot d\vec{r} \\ &= -\vec{F}_c \cdot \Delta\vec{r} \end{aligned} \tag{12.29}$$

In this, we have used the fact that the force is *constant* and performed the integral.

12.3.1 Gravity near the Earth's surface

An application of formula 12.29 is the (approximately) constant force of gravity on a mass m near the Earth's surface. In this, the force is $\vec{F} = -mg\hat{k}$. Supposing that $\delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}$, this gives that

$$\Delta PE = -\left(-mg\hat{k}\right) \cdot \left(\Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}\right) = mg\Delta z \tag{12.30}$$

This expression is consistent with a definition of potential energy as

$$PE_{gravity}(\vec{r}) = mgz + C \tag{12.31}$$

where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, so z is the z-component of the position, and C is an arbitrary constant. The constant C is chosen so that the potential energy is 0 at some reference level. The value of C is arbitrary, and does not change the value of the force. This can be thought of as like the arbitrary constant that accompanies an indefinite integral.

Notice that this expression for gravitational potential energy is consistent with the discussion of the relation between force and potential energy. Evaluating the force using equation 12.25 gives

$$\begin{aligned}
\vec{F}(\vec{r}) &= -\vec{\nabla}PE\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)\\
&= -\vec{\nabla}\left(mgz+C\right)\\
&= -\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(mgz+C\right)\\
&= -\hat{k}mg = -mg\hat{k} \qquad (12.32)
\end{aligned}$$

exactly as expected.

12.3.2 Constant Electric Field

A second application of the formula 12.29 is to the motion of a particle of charge q in a constant electric field. In this example, the force is $\vec{F} = q\vec{E}$, and if the charged particle has moved a displacement $\delta\vec{r}$ then the change in potential energy is

$$\Delta PE = -q\vec{E}\cdot\Delta\vec{r} \qquad (12.33)$$

If we choose a coordinate system where the positive x-axis is lined up with the direction of the electric field, this is consistent with a potential energy of

$$PE(\vec{r}) = -q\left|\vec{E}\right|x + C \qquad (12.34)$$

Since the x-axis is aligned with $\vec{E}$, this expression says that the potential energy *decreases* as the (positively charged) particle moves in the positive x-direction, along the direction of the electric field. This is exactly the same as in the case of the constant gravitational field – in that case the potential energy decreases as the mass moves along the direction of the gravitational field (which is *down*). The additive constant is, as in the constant gravitational field, simply like the constant in indefinite integrals. Changing C is *equivalent* to changing the position that is the origin ($\vec{r} = 0$).

Conversely if the charge is negative, the potential energy *increases* as the (positively charged) particle moves in the positive x-direction, along the direction of the electric field.

12.3.3 Example

Suppose that a ball of mass m is attached by a light, inextensible string to a disk of radius R and moment of inertia I which is free to rotate around

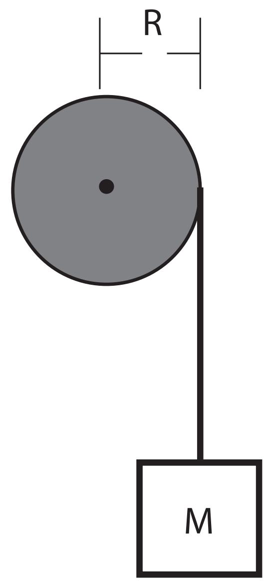

Figure 12.3: An illustration of a mass attached to a pulley by a rope.

its center of mass on a frictionless pulley. This is shown in figure 12.3. If the ball is allowed to drop from rest, what is the speed of the ball once it has dropped a distance d?

This *could* be solved using the methods of chapter 10. In that case the plan would be to write out a system of equations for each object using Newton's law, then to solve those to determine the unknown acceleration of the mass, and finally to use that acceleration together with kinematics to determine the speed after it has moved down by d.

Instead, we solve this using energy considerations: the initial state is when the mass and disk are stationary. The final state is when the mass has moved down by a distance d. We recall that the kinetic energy of a moving mass is $KE = \frac{1}{2}m\left|\vec{v}\right|^2$, that the kinetic energy of a rotating object is $KE = \frac{1}{2}I\left|\frac{d\theta}{dt}\right|^2$, and that the relation between the rotational speed of the disk and the speed of the mass is $\left|\vec{v}\right| = R\left|\frac{d\theta}{dt}\right|$ because the string does not slip.

Using the work-energy theorem in its potential energy form, and noting that there is no friction, so no non-conservative force at work, we have

$$\begin{aligned} W_{non-conservative} &= \Delta KE + \Delta PE \\ 0 &= \left(\frac{1}{2}m\left|\vec{v}\right|^2 - 0 + \frac{1}{2}I\left|\frac{d\theta}{dt}\right|^2\right) - (0 + mg\Delta z) \\ &= \frac{1}{2}m\left|\vec{v}\right|^2 + \frac{1}{2}\frac{I}{R^2}\left|\vec{v}\right|^2 + mg\left(-d\right) \\ \left|\vec{v}\right| &= \sqrt{\frac{2mgd}{m + \frac{I}{R^2}}} \end{aligned} \tag{12.35}$$

12.4 Springs

As outlined in sections 7.5 and 11.3, the force exerted by a spring *along its axis of symmetry* is $\vec{F}_s = -k\delta\vec{r}$ where $\delta\vec{r}$ is the amount that the springs ends have been displaced from each other relative to their unstretched and uncompressed position. Springs exert a conservative force to the extent that they can be well-described by Hooke's law.

To find the expression for the potential energy of a spring, consider the following situation: The spring is aligned with the x-axis, and the left end is held fixed. The right end is free to move, and the coordinate system has its origin at the point where the right end of the spring naturally sits. This is shown in figure 12.4. Now, imagine that the right end of the spring is

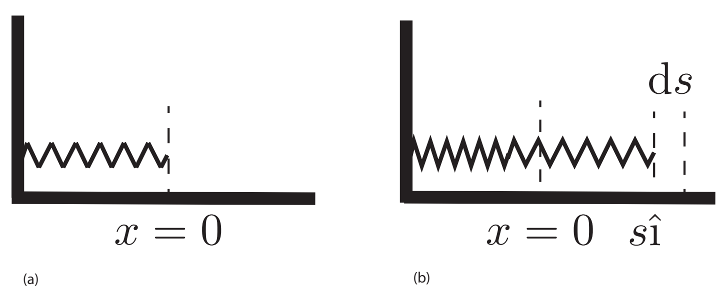

Figure 12.4: A spring being stretched by an amount s along the x-axis.

moved (staying along the x-axis) from $\vec{r}_i = x_i\hat{\imath}$ to $\vec{r}_f = x_f\hat{\imath}$. To determine the amount of work done (hence the change in potential energy) we follow the procedure outlined earlier in the chapter.

- The location of the end of the spring is $\vec{r} = s\hat{\imath}$ for s between x_i and x_f. Note that this encodes the assumption that the end of the spring only moves along the x-axis.
- The small step taken by the end of the spring is $d\vec{r} = ds\hat{\imath}$.
- The force that the end of the spring exerts when it is at $s\hat{\imath}$ is $\vec{F}_{spring} = -ks\hat{\imath}$.
- The amount of work done during the step $d\vec{r}$ is $dW = \vec{F}_{spring} \cdot d\vec{r}$, and is $dW = -ksds$.
- Integrating to get the the work done gives

$$\begin{aligned} W_{spring} &= \int \vec{F}_{spring} \cdot d\vec{r} \\ &= \int_{x_i}^{x_f} -ksds \\ &= -k\frac{s^2}{2}\bigg|_{x_i}^{x_f} \\ &= -k\frac{x_f^2}{2} + k\frac{x_i^2}{2} \end{aligned} \tag{12.36}$$

Using this result, together with the fact that $\Delta PE = -W_{conservative}$ gives that the potential energy of a spring, stretched or compressed by x from its equilibrium length along its axis of symmetry and with a spring constant k is

$$PE_{spring} = \frac{k}{2}x^2 + C \tag{12.37}$$

12.4.1 Example - Spring and friction

A block of mass m is moving at velocity $v\hat{\imath}$ over a horizontal and frictionless surface. It then hits one end of a spring that is held fixed at the other end. The spring has spring constant of k, and the ground under the extended spring would have a coefficient of kinetic friction of μ_k. By how much does the spring compress as the block comes to a stop?

This situation is depicted in figure 12.5 To solve this we use the work-energy theorem in the potential energy form, recognizing that there is a non-conservative force (friction) acting. The work-energy theorem gives that

$$\begin{aligned} W_{non-conservative} &= \Delta KE + \Delta PE \\ &= \frac{1}{2}m\left|\vec{v}_f\right|^2 - \frac{1}{2}m\left|\vec{v}_i\right|^2 + \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \end{aligned} \tag{12.38}$$

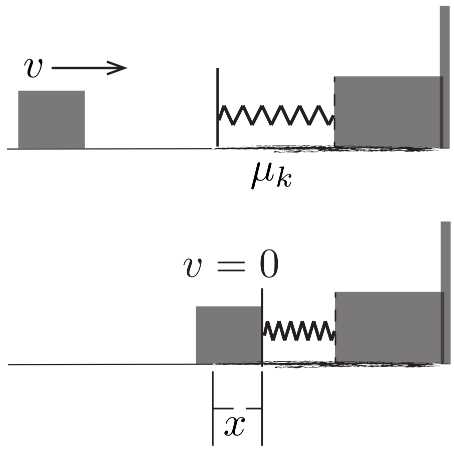

Figure 12.5: A mass hitting a spring

The quantity we wish is x_f (the final compression of the spring) and the spring starts uncompressed so $x_i = 0$. The final velocity is 0 since the mass is stopped.

The unknown quantity is the work done by the non-conservative force. To calculate the work done, we follow the procedure above. We notice that the normal force on the block has magnitude mg since it is not accelerating vertically. We parametrize the position of the block as $s\hat{\imath}$ for s between 0 and x_f. The step length is $ds\hat{\imath}$. The force of friction is $-\mu_k mg\hat{\imath}$, so the work is

$$\begin{aligned} W_{non-conservative} &= \int \vec{F}_{friction} \cdot d\vec{r} \\ &= \int_0^{x_f} (-\mu_k mg\hat{\imath}) \cdot ds\hat{\imath} \\ &= -\mu_k mg \int_0^{x_f} ds = -\mu_k mg x_f \end{aligned} \tag{12.39}$$

This means that our previous expression gives

$$\begin{aligned} W_{non-conservative} &= \frac{1}{2}m\,|\vec{v}_f|^2 - \frac{1}{2}m\,|\vec{v}_i|^2 + \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \\ -\mu_k mg x_f &= -\frac{1}{2}mv^2 + \frac{1}{2}kx_f^2 \\ x_f &= \sqrt{\left(\mu_k g\frac{m}{k}\right)^2 + \frac{m}{k}v^2} - \mu_k g\frac{m}{k} \end{aligned} \tag{12.40}$$

Vinitial is 0; (1/2)k(xi) is 0

Determining the value for x_f used the quadratic formula and the observation that $x_f < 0$ is not a viable answer.

12.5 Gravity and Electrostatic forces

The expression for the force of gravity on m_2 at $\vec{r}_2$ due to m_1 at $\vec{r}_1$ is

$$\vec{F}_{1\ on\ 2} = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{12.41}$$

while the expression for the Coulomb force on charge q_2 at $\vec{r}_2$ due to q_1 at $\vec{r}_1$ is

$$\vec{F}_{1\ on\ 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{12.42}$$

Both of these, aside from constants, are proportional to $\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}$. Because of this common proportionality the expressions for the potential energy for both gravity and the Coulomb force will be very similar.

12.5.1 The integral of $\frac{1}{r^2}\hat{r}$

As a step in the determination of the potential energies for gravity and the Coulomb force we will need to determine the work done by those forces as an object moves in their field. The work, in either case, will be proportional to the integral of $\frac{1}{r^2}\hat{r}$, so it is necessary to determine what that integral would be for arbitrary starting and ending points.

We start by making a coordinate choice: we assume that the object (either mass or charge) that is *creating* the force is placed at the origin, so $\vec{r}_1 = 0$, and the object that *experiences* the force starts at $\vec{r}_2 = x_i\hat{\imath}$ and moves to a final position of $x_f\hat{\imath}$ along the x-axis. This amounts to a choice of which axis we describe as the x-axis. If the final position we actually wish for the second object is not along the line we instead imagine, because the force was conservative so work is not dependent on path, that it moved directly to its final distance, and then moved at a constant radius (so no work was done on it) to a final location.

So, following the procedure for calculating the work by a variable force above:

- Object 2 is at $\vec{r}_2 = s\hat{\imath}$ with values of s between x_i and x_f.
- The object moves in a step $d\vec{r} = ds\hat{\imath}$.

- The value of the integrand $\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|^3}$ at the location of object 2 is $\frac{1}{s^2}\hat{\imath}$ since $\vec{r}_1 = 0$.

- The integral to be calculated is then

$$\begin{aligned}\int_{x_i}^{x_f}\left(\frac{1}{s^2}\hat{\imath}\right)\cdot ds\hat{\imath} &= \int_{x_i}^{x_f}\frac{1}{s^2}ds \\ &= -\frac{1}{s}\bigg|_{x_i}^{x_f} = -\frac{1}{x_f}+\frac{1}{x_i} \end{aligned} \tag{12.43}$$

where the x-axis was chosen along the radius.

This integral is the basis of how the potential energy for either the Coulomb force or gravity are calculated. Since x_i and x_f are the initial and final separation of the two objects considered, it is to be expected that either potential energy will depend on the inverse of the separation.

The generalization of the result above is that for two objects at positions $\vec{r}_1 = x_1\hat{\imath}+y_1\hat{\jmath}+z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath}+y_2\hat{\jmath}+z_2\hat{k}$ the potential energy will be proportional to $\frac{1}{|\vec{r}_2-\vec{r}_1|}$. This means that

$$\begin{aligned} PE &\propto \frac{1}{|\vec{r}_2-\vec{r}_1|} \\ &= \frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \end{aligned} \tag{12.44}$$

Since the *force* is proportional to the derivative of the potential energy with respect to the position of the object feeling that force, the x-component of force on object 2 is proportional to

$$\begin{aligned} F_{x\ (on\ 2)} &\propto \frac{\partial}{\partial x_2}\frac{1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \\ &= -\frac{1}{2}\frac{1}{((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2)^{3/2}}2(x_2-x_1) \\ &= -\frac{1}{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \end{aligned} \tag{12.45}$$

which is exactly the x-component of $\frac{1}{r^2}\hat{r}$. The y- and z-components are similar.

12.5.2 Potential Energy for the Coulomb force

As noted above the expression for the Coulomb force on charge q_2 at $\vec{r}_2$ due to q_1 at $\vec{r}_1$ is

$$\vec{F}_{1\ on\ 2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{12.46}$$

Supposing charge 1 to be at the origin and charge 2 to start at $x_i\hat{\imath}$ and move to $x_f\hat{\imath}$ in the manner described in the previous section, the work on charge 2 is

$$\begin{aligned} W &= \int \vec{F}\cdot d\vec{r} \\ &= \int_{x_i}^{x_f} \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{s^2} ds \\ &= -\frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{x_f} + \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{x_i} \end{aligned} \tag{12.47}$$

as above. Using the relationship that $W_c = -\Delta PE$, the potential energy for the Coulomb force becomes

$$PE = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|} + C \tag{12.48}$$

The integration constant C is normally chosen so that the potential energy vanishes as $\vec{r}_2$ moves far away from $\vec{r}_1$, so conventionally $C = 0$. i.e. as $r \to \infty$ then $PE \to 0$.

12.5.3 Potential Energy for Newtonian Gravity

As similarly noted above, the expression for the force of gravity on m_2 at $\vec{r}_2$ due to m_1 at $\vec{r}_1$ is

$$\vec{F}_{1\ on\ 2} = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \tag{12.49}$$

As before, assuming that mass 1 is at the origin and mass 2 is to start at $x_i\hat{\imath}$ and move to $x_f\hat{\imath}$ in the manner described in the previous section, the work on mass 2 is

$$\begin{aligned} W &= \int \vec{F}\cdot d\vec{r} \\ &= \int_{x_i}^{x_f} \left(-G\frac{m_1 m_2}{s^2} ds\right) \\ &= G\frac{m_1 m_2}{x_f} - G\frac{m_1 m_2}{x_i} \end{aligned} \tag{12.50}$$

Using the relationship that $W_c = -\Delta PE$ this means that the gravitational potential energy for two masses m_1 and m_2 respectively at positions $\vec{r}_1$ and $\vec{r}_2$ is

$$PE = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} + C \tag{12.51}$$

The integration constant C is also typically chosen so that the potential energy vanishes when the masses are separated by a long distance. In that case, when closer the potential energy is *negative* because as masses move towards each other gravity does positive work speeding them up.

It is somewhat noteworthy that the expression for gravitational potential energy in this case is very diffeerent from that for the constant gravitational field near the Earth's surface. To see that they are *approximately* the same, in the region where the constant gravitational force is a good approximation, it is important to put them in similar context: The opposite of the 'up' direction is down towards the center of the earth, and using Earth's center as the origin of a coordinate system (so $\vec{r}_1 = 0$ and $m_1 = M$, the mass of the Earth) and an object of mass $m_2 = m$ a distance of $R_E + z$ from the origin (so a distance z above the surface of the Earth). Applying the result of equation 12.51 to this situation we have

$$PE = -G\frac{Mm}{R_E + z} \tag{12.52}$$

Now, assume that $z \ll R_E$ and Taylor expand around $z = 0$ to linear order, which gives

$$PE \approx -G\frac{Mm}{R_E} + G\frac{Mm}{R_E^2}z \tag{12.53}$$

With the identification that $g = G\frac{M}{R_E^2}$, this is exactly $PE = mgz + C$, so the two expressions are consistent with each other.

12.5.4 Example - Escape Speed and Orbital Energy

Suppose that a rocket, mass m, is at rest on the surface of a planet of mass M and radius R. What is the minimum speed it would need to attain in order to be able to completely escape from the planet? This speed is known as the 'escape speed'.

Neglecting the effects of any atmospheric resistance, this can be calculated directly using the work-energy theorem. The minimum speed would be such that when the rocket is very far from the planet the rocket had minimal kinetic energy (and hence negligible velocity). The rocket starts with $|\vec{r}_2 - \vec{r}_1| = R$ and ends up with $|\vec{r}_2 - \vec{r}_1| \approx \infty$ This means that

$$\begin{aligned}
W_{non-conservative} &= \Delta KE + \Delta PE \\
0 &= 0 - \frac{1}{2}mv^2 + \left(-G\frac{Mm}{\infty}\right) - \left(-G\frac{Mm}{R}\right) \\
v &= \sqrt{\frac{2GM}{R}} \qquad (12.54)
\end{aligned}$$

so the escape speed from a planet depends on the planet's mass and radius, not on the mass of the rocket trying to escape.

If the rocket is in a circular orbit at height, R_O, then it would be a distance $R + R_O$ from the centre of the earth. If it is moving with a speed v in orbit then,

$$\frac{GMm}{(R + R_O)^2} = \frac{mv^2}{(R + R_O)} \qquad (12.55)$$

and therefore

$$v^2 = \frac{GM}{R + R_O}. \qquad (12.56)$$

The mechanical energy of the orbit is given by

$$\begin{aligned}
E &= KE + PE \qquad (12.57) \\
&= \frac{mv^2}{2} - \frac{GMm}{R + R_O} \\
&= \frac{1}{2}\frac{GMm}{R + R_O} - \frac{GMm}{R + R_O} \\
&= -\frac{1}{2}\frac{GMm}{R + R_O}
\end{aligned}$$