Student: Arfaz Hossain **Date:** 02/28/22	**Instructor:** Muhammad Awais **Course:** Math 101 A04 Spring 2022	**Assignment:** HW-5 [Sections 10.1, 10.2 & 10.3]

Does the series $\sum_{n=1}^{\infty} \frac{5\,\mathbf{tan}^{-1} n}{1+n^2}$ converge or diverge?

The given series $\sum_{n=1}^{\infty} \frac{5\,\mathbf{tan}^{-1} n}{1+n^2}$ is not a geometric series, a telescoping series, a harmonic series, or a p-series.

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). The integral test states that the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

The integral $\int_1^{\infty} \frac{5\,\mathbf{tan}^{-1} x}{1+x^2}dx$ can be used to determine the convergence of $\sum_{n=1}^{\infty} \frac{5\,\mathbf{tan}^{-1} n}{1+n^2}$.

Evaluate the integral using substitution method with $u = \mathbf{tan}^{-1} x$. Taking the differentials on both sides of the equation, $du = \frac{1}{1+x^2}\,dx$.

Rewrite the integral with the appropriate substitution of the variable.

$$\int_1^N \frac{5\,\mathbf{tan}^{-1} x}{1+x^2}\,dx = \int_{x=1}^{x=N} 5u\,du$$

Evalute the integral with respect to u.

$$\int_{x=1}^{x=N} 5u\,du = \left[\frac{5u^2}{2}\right]_{x=1}^{x=N}$$

Substitute $u = \mathbf{tan}^{-1} x$ back into the expression.

$$\left[\frac{5u^2}{2}\right]_{x=1}^{x=N} = \left[5\left(\frac{\left(\mathbf{tan}^{-1}(x)\right)^2}{2}\right)\right]_1^N$$

Evaluate the integral at the limits and simplify.

$$\left[5\left(\frac{\left(\mathbf{tan}^{-1}(x)\right)^2}{2}\right)\right]_1^N = 5\left(\frac{\left(\mathbf{tan}^{-1}(N)\right)^2}{2}\right) - 5\left(\frac{\pi^2}{32}\right)$$

Take the limit as $N \to \infty$. The function $\mathbf{tan}^{-1} x$ is the only term containing N. The limit of this function as $N \to \infty$ is $\frac{\pi}{2}$.

Therefore, the integral converges to a finite number.

$$\lim_{N\to\infty} 5\left(\frac{\left(\mathbf{tan}^{-1}(N)\right)^2}{2}\right) - 5\left(\frac{\pi^2}{32}\right) = \frac{5}{2}\left(\frac{\pi}{2}\right)^2 - 5\left(\frac{\pi^2}{32}\right)$$

Thus, the series $\sum_{n=1}^{\infty} \frac{5\,\mathbf{tan}^{-1} n}{1+n^2}$ converges by the Integral Test.