Math 110, Fall 2021, Test 1C Sample Answers

Note: Students were required to show their steps in solving the problems. For brevity these solutions omit intermediate steps of row reductions.

Instructions:

- You may use a calculator on this test, but the only permitted calculators are SHARP brand calculators with model numbers beginning EL-510R. No other electronic devices are permitted.
- No notes, textbooks, or other outside materials or aids are permitted.
- For questions with numerical answers, either give your answer in exact form or give it as a decimal to two decimal places.
- For **all** questions you must show your work to be given credit, even if your answer is correct.
- For questions 1–3, show your work and then enter your final answer in the box provided.
- This test is printed double-sided - be sure not to miss the questions on the back of the first page! For the long-answer questions the backs of the pages are additional space for your solution.

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(1 point) 1. Let $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Calculate $\vec{v} + 3\vec{w} - 3(\vec{w} + \vec{v})$.

Solution:

$$\vec{v} + 3\vec{w} - 3(\vec{w} + \vec{v}) = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 3 \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -6 \\ 2 \end{bmatrix}$$

Answer: $\begin{bmatrix} 0 \\ -6 \\ 2 \end{bmatrix}$

(1 point) 2. Let $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Find $\cos(\theta)$.

Solution:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{-1}{\sqrt{6}\sqrt{6}} = \frac{-1}{6}.$$

The last step of simplification was not required.

Answer: $\frac{-1}{6}$

(1 point) 3. Find all values of x such that $\begin{bmatrix} x \\ x+1 \\ x \end{bmatrix}$ has length 1.

Solution: We have

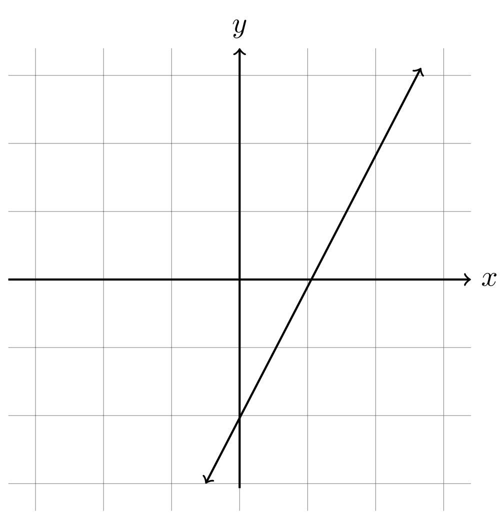
$$\left\| \begin{bmatrix} x \\ x+1 \\ x \end{bmatrix} \right\| = \sqrt{x^2 + (x+1)^2 + x^2} = \sqrt{3x^2 + 2x + 1}.$$

To have length 1 we need $3x^2 + 2x + 1 = 1$, so $x(3x+2) = 0$. Thus $x = 0$ or $x = -2/3$.

Answer: $x = 0$ and $x = -2/3$

(1 point) 4. Sketch the line in $\mathbb{R}^2$ that has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Very briefly (one sentence is enough) tell us how you decided where to draw the line.

Solution: By setting $t = 0$ and $t = 1$ we find that $(1, 0)$ and $(0, -2)$ are on the line, so we draw the only line containing those two points.



(4 points) 5. Determine whether the following system of linear equations in variables x_1, x_2, x_3, x_4 has no solution, exactly one solution, or infinitely many solutions.

$$x_1 + 2x_2 - x_3 = 3$$

$$-2x_1 - 3x_2 + 2x_4 = 0$$

$$x_2 + x_3 + x_4 = -2$$

Solution: We write an augmented matrix and row reduce.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 3 \\ -2 & -3 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & -1 \\ 0 & 1 & 0 & 4/3 & 2/3 \\ 0 & 0 & 1 & -1/3 & -8/3 \end{array}\right].$$

We see from the reduced row echelon form that x_4 is a free variable, and therefore there are infinitely many solutions.

(4 points) 6. Find all values of k for which $\begin{bmatrix} k \\ 2 \\ k \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Solution: We want to know for which k there are a, b such that

$$a \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 2 \\ k \end{bmatrix}.$$

Treating this vector equation as a system of linear equations in variables a and b, we solve:

$$\left[\begin{array}{cc|c} 2 & 1 & k \\ 1 & -1 & 2 \\ -3 & 1 & k \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & k+2 \end{array}\right].$$

This system has a solution if and only if $k = -2$. Therefore the only k for which $\begin{bmatrix} k \\ 2 \\ k \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is $k = -2$.

(4 points) 7. Let L be the line in $\mathbb{R}^3$ that passes through the points $(-1, 3, 1)$ and $(1, 0, -1)$. Let P be the plane in $\mathbb{R}^3$ that is orthogonal to L and passes through the point $(0, 2, 1)$. Find, with justification, a vector equation for P.

Solution: Since P is orthogonal to L, a direction vector for L will be a normal vector to P. Such a vector is $\vec{n} = \begin{bmatrix} -1-1 \\ 3-0 \\ 1-(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$. We know that P passes through $(0, 2, 1)$, so in normal form the equation for P is

$$\begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Expanding the dot products we obtain the general form

$$-2x + 3y + 2z = 8.$$

We rearrange this equation to say

$$z = 4 + x - \frac{3}{2}y,$$

and then by substituting we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 4 + x - \frac{3}{2}y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -3/2 \end{bmatrix}.$$

The equation above is a vector equation for P (it is not the only possible correct answer).

(4 points) 8. Suppose that $\vec{v_1}$, $\vec{v_2}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and that c is a scalar. Show that

$$\text{proj}_{\vec{w}}(\vec{v_1}+\vec{v_2}) = \text{proj}_{\vec{w}}(\vec{v_1}) + \text{proj}_{\vec{w}}(\vec{v_2}).$$

Note: In this question we want you to write a general argument, so you should not choose specific numbers for any of the objects in the question.

Solution: We use properties of the dot product to calculate:

$$\begin{aligned}
\text{proj}_{\vec{w}}(\vec{v_1}+\vec{v_2}) &= \left(\frac{\vec{w}\cdot(\vec{v_1}+\vec{v_2})}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= \left(\frac{\vec{w}\cdot\vec{v_1}+\vec{w}\cdot\vec{v_2}}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= \left(\frac{\vec{w}\cdot\vec{v_1}}{\vec{w}\cdot\vec{w}}\right)\vec{w} + \left(\frac{\vec{w}\cdot\vec{v_2}}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= \text{proj}_{\vec{w}}(\vec{v_1}) + \text{proj}_{\vec{w}}(\vec{v_2})
\end{aligned}$$