ECON 180

Introduction to Principles of Microeconomics and Financial Project Evaluation

Lecture 15: The External Rate of Return

October 12, 2021

Version 1.05 (Nov. 16) – Fixed a typo on Slide 24.

Required Reading

- Biezma, M. V. & San Cristobal, J. R. (2006). Investment criteria for the selection of cogeneration plants – a state of the art review. *Applied Thermal Engineering, 26*(5-6), 583-588. Retrieved from https://doi-org.ezproxy.library.uvic.ca/10.1016/j.applthermaleng.2005.07.006
- **Summarizes and compares different project evaluation methods.**

Recommended Reading

- *Engineering Economics*, Chapter 5, sections 5.3.3, 5.3.4, 5.3.5 and 5.4
- Keane, S. M. (1979). The Internal Rate of Return and the Reinvestment Fallacy. *Abacus*, *15*(1), 48 – 55. Retrieved from https://doi-org.ezproxy.library.uvic.ca/10.1111/j.1467-6281.1979.tb00073.x
  - **There is no 'reinvestment assumption'. If you want the gory details, this short article will give them to you.**

Optional reading on IRR and reinvestment

- Hazen, G. B. (2003). A New Perspective on Multiple Internal Rates of Return. *The Engineering Economist, 48*(1), 31-51. Retrieved from https://doi-org.ezproxy.library.uvic.ca/10.1080/00137910308965050
  - **A suggestion on how to interpret multiple IRRs.**
- Kelleher, J. & MacCormack, J. (2004). Internal rate of return: A cautionary tale.[Web Page]. Retrieved from https://www.mckinsey.com/business-functions/strategy-and-corporate-finance/our-insights/internal-rate-of-return-a-cautionary-tale
  - **Explains the dangers of relying just on IRR. When reading, recall that MIRR is (more or less) another name for the ERR**.
- Magni, C. A. & Martin, J. D. (2017). The Reinvestment Rate Assumption Fallacy for IRR and NPV [MPRA Paper No. 83889]. Retrieved from https://mpra.ub.uni-muenchen.de/83889/
  - **More on the reinvestment assumption.**

Learning Objectives

- Understand the situations in which the IRR equation can have multiple roots.
- Know how to find the ERR/MIRR in cases where the IRR has multiple roots.
- Be able to calculate both approximate and exact ERR.

Relevant Solved Problems

- From Engineering Economics, 6th edition, Chapter 5
- Multiple IRR: Example 5.7, 5.35.b, 5.38.b
- ERR: Example 5.8, 5.17.a, 5.18.b, 5.22, 5.23, 5.26, 5.28, 5.35.c, 5.37, 5.41, 5.42
- Approximate ERR: Example 5.9, Review Problem 5.3, 5.17.b, 5.18.c

Notation Dictionary

(Not provided on quiz/final formula sheet)

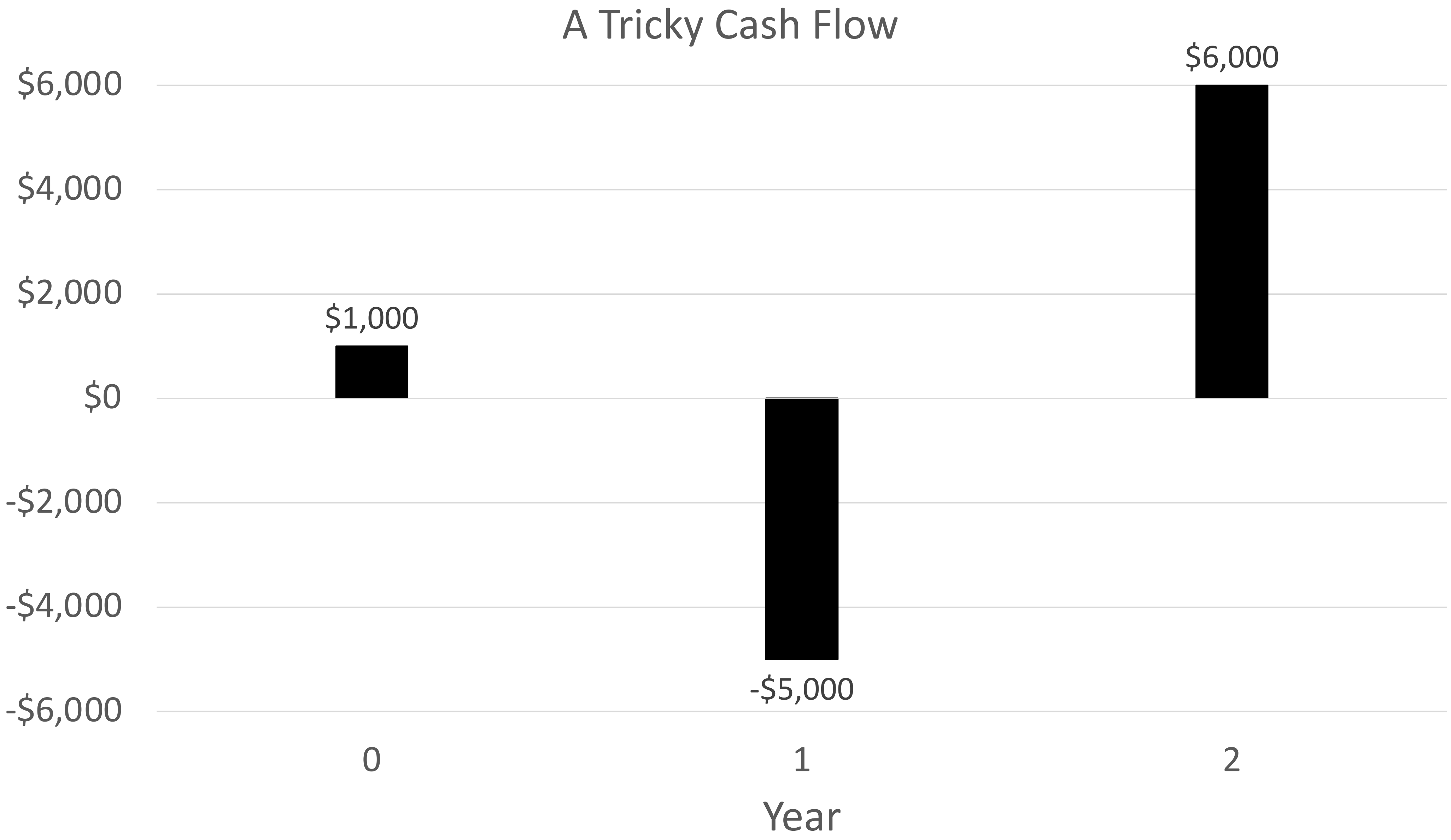
- A = Annuity
- F = Future Value
- IRR = Internal Rate of Return
- ERR = External Rate of Return
- MARR = Minimum Acceptable Rate of Return
- N = the N'th time period
- P = Present Value
- Green Text = Excel Formula

- Conversion factors are of the form
  (X/Y,z)
- Read as: X, given Y and z.
- X is the element we want.
- Y is the element we have.
- z represents additional parameters.
- e.g. (P/F,i,N)
- Present Value, given a Future Value at time N and interest rate i.

ESSENTIALS (11 slides)

Multiple IRRs?

- Not everything is rosy…
- There are as many possible IRRs as there are sign changes in a cash flow profile. That is, how many times $\sum C_t^j$ switches from negative to positive or vice versa.
- In a 'simple investment', where you pay up front then reap the benefits, there's only one sign change (cost to benefit).
- →A unique solution (if one exists)
- In more complicated cash flows… things can get tricky.


A Tricky Cash Flow
$6,000
$4,000
$2,000
$0
-$2,000
-$4,000
-$6,000
$1,000
-$5,000
$6,000
0
1
2
Year

Two paths, same destination

	IRR = 100%			IRR = 200%		
	YEAR 0	YEAR 1	YEAR 2	YEAR 0	YEAR 1	YEAR 2
NIO (Start of Year)	**$0**	-$2,000	$6,000	**$0**	-$3,000	$6,000
'Withdrawals'	**$1,000**	**-$5,000**	**$6,000**	**$1,000**	**-$5,000**	**$6,000**
Leaving…	-$1,000	$3,000	$0	-$1,000	$2,000	$0
Applying Interest	x (1 + 100%)	x (1 + 100%)	x (1 + 100%)	x (1 + 200%)	x (1 + 200%)	x (1 + 200%)
NIO (End of Year)	-$2,000	$6,000	**$0**	-$3,000	$6,000	**$0**

- Just like last lecture, we can think of the Net Initial Outlay (NIO) as an account we draw from in order to re-create the cash flows of the project.
- In this case, the NIO is $0, since the first cash flow in the project is income, not an expense – not a problem! We can 'borrow' at the IRR to recreate the first $1,000 cash flow, leaving us with a negative balance in the NIO…
- …which is fixed the following year, when the -$5,000 represents money going in to our (theoretical) 'bank account'. This deposit is listed on the table as a 'negative withdrawal'.
- As the table shows, **both r = 100% and r = 200% satisfy the conditions for being an IRR: allowing us to replicate the cash flows of the project, with nothing left over, from our net initial outlay (in this case $0).**

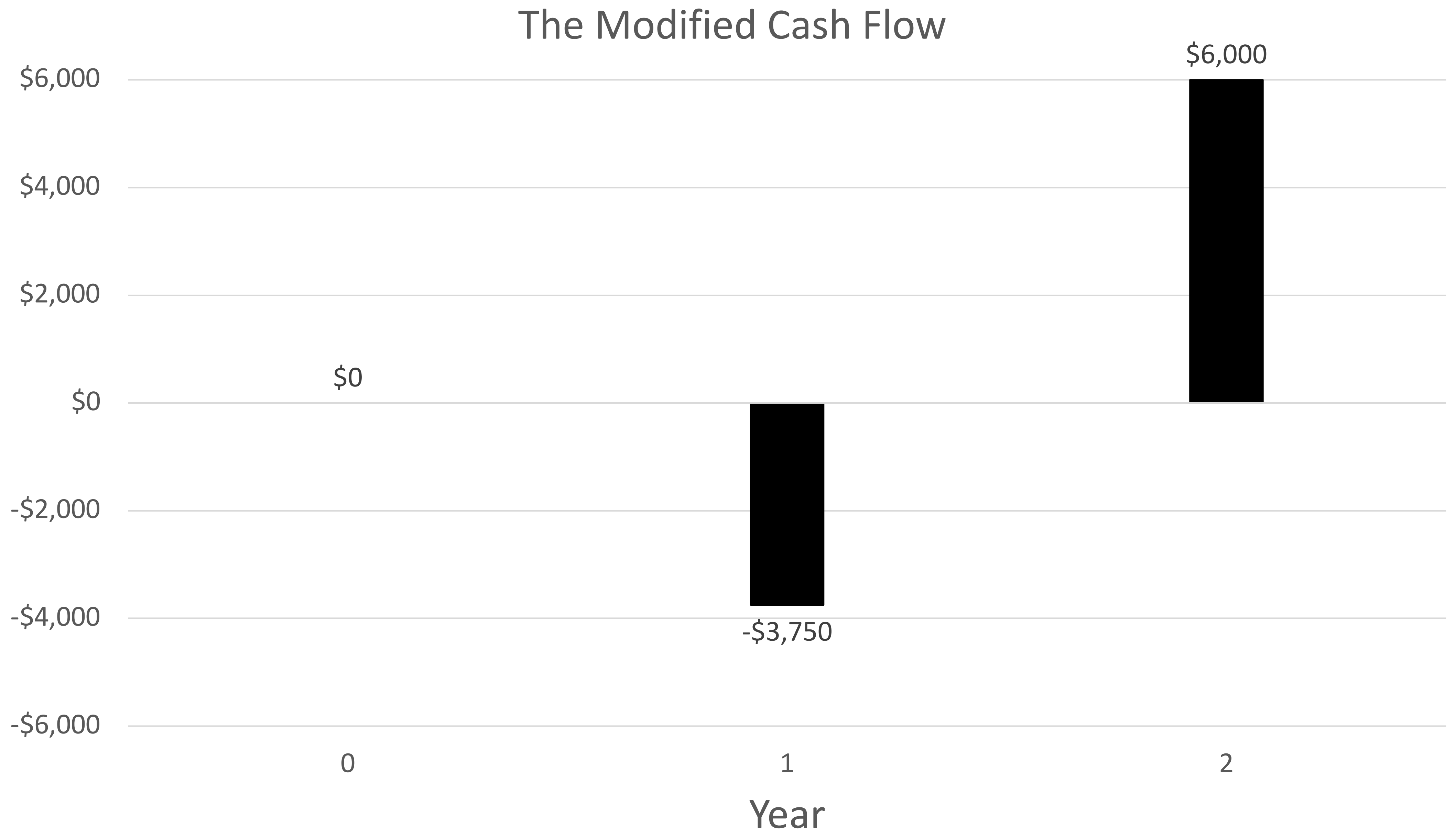
> Since "[t]he IRR of a project represents the maximum cost of capital a project can sustain" (Keane, 1979), this is a problem!

What’s going on?

- Recall from last lecture that the IRR and NPV measure different things: the IRR is a rate that lets you replicate the cash flows of the project starting from the net initial outlay.
- There are up to as many different real discount (interest) rates that will allow us to do this as there are sign changes in the cash flow profile (plus a bunch of complex solutions).
- They’re all ‘correct’, but we (still!) have no easy way of knowing which one gives the ranking we’re looking for - the one that can be used like we use NPV to choose between projects.
- **→It’d be nice if we had a procedure that got rid of the sign changes while leaving the NPV of the cash flow profile untouched**.
- This is the External Rate of Return (ERR), also called the Modified Internal Rate of Return (MIRR).
- (Note: Your textbook says the IRR procedure assumes reinvestment of interim cash flows at the IRR, and that this is what causes all the trouble. **This is a fallacy**, and has been known to be wrong for decades. There is no reinvestment assumption in the IRR. See optional readings for this lecture and the last for details.)
- Why worry? Perversely, this fallacy may lead project managers to reject projects with a high IRR, for fear of not being able to reinvest at that rate. (“The IRR is 30%, but our MARR is only 5%, so the project is unattainable.”) (Magni & Martin, 2017)

External (or Modified Internal) Rate of Return

- The ERR (or MIRR) avoids the multiple roots by assuming any unused funds attached to the project are invested at the MARR until they are needed.
- This gets rid of extra sign changes within the project's cash flows.
- Suppose, in our example, MARR = 25% per year.
- The project does not need cash until the first year, so we assume that there is no cash inflow in year 0: it is invested elsewhere at the MARR and arrives just in time in Year 1, augmented by its return.
- In the first year, the project now earns:
- 1,000(F/P,25%,1) – 5,000 = 1,250 – 5,000 = -$3750
- The modified cash flow now has only *one* sign change and a unique root to the IRR (well, ERR) equation. (It's 60% per year, by the way.)
- The ERR approach, properly applied, guarantees this.


The Modified Cash Flow
$6,000
$4,000
$2,000
$0
-$2,000
-$4,000
-$6,000
$0
-$3,750
$6,000
0
1
2
Year

It’s also rigged, and NOT the same as the IRR…

- The ERR **does** assume ‘extra’ cash is reinvested at the MARR…
- …BUT this is done ONLY to make the ‘many IRR’ problem go away…
- …since as discussed earlier, there’s no reinvestment assumption in the IRR itself: “Reinvestment is not implicit in the IRR … ; rather, it is a sufficient condition for solving ranking conflicts between NPV and IRR and the multiple-IRR problem.” (Magni & Martin, 2017)
- →The ERR has a <u>**different**</u> meaning than the IRR.
- It CAN’T be the ‘rate of return’ of the project (and only the project) because it takes the MARR into account – hence, ‘External’ rate.
- It’s the rate of return of a *portfolio* of projects that includes i) the project being considered, AND ii) other projects that earn the MARR.
- But, *applied appropriately*, it does give a single answer that will give project rankings consistent with the NPV…

‘Applied appropriately’?

- THAT’S the catch.
- Easy to describe: When the project is a net source of funds, assume they’re invested elsewhere at the MARR. When the project is a net user of funds, assume the ERR.
- Tricky to implement… Especially for complicated cash flows, it’s not always clear where to start or stop the MARR assumption.
- Thankfully… there’s an ‘approximate ERR’ calculation that we can make use of.

How to find the approximate ERR (MIRR)

- For all t such that $\sum C_t^j > 0$, bring $\sum C_t^j$ forward at the MARR to the time of the last cash flow, using (F/P,MARR,N) etc.
- For all t such that $\sum C_t^j < 0$, bring $\sum C_t^j$ forward at an unknown rate ERR, also to the time of the last cash flow, using (F/P,ERR,N) etc.
- Set the results of the first two steps equal to each other and solve for ERR. One equation, one unknown, one solution.
- This solution is the *approximate* ERR.
- Why isn't it exact? Because we're taking *all* negative net cash flows forward, not just the ones when the project balance is positive.

Intuitive Translation

- Pretend you're lazy, and leave all payments to the last minute.
- If money comes IN at time t (positive net cash flow), you redirect it to the firm's fallback project, where it earns the MARR.
- If money goes OUT at time t, you let it sit as a debt, accumulating interest at r%.
- Note: We're talking about the *net flows* at time t: if at time t=5 (say), there is a cash flow of -$5 and a cash flow of +$12, the *net* cash flow is $12 - $5 = $7, and we would bring that $7 forward to the last period of the project at the MARR.
- Finally, at the last time period of the project, N:
- IN flows at time t R_t will have become $R_t(1 + \mathrm{MARR})^{N-t}$
- OUT flows at time t D_t will have become $D_t(1 + r)^{N-t}$
- The rate r at which the accumulate earnings from the inflows can just cover the accumulated debt from the outflows is the approximate ERR.
- (R = 'Revenue', D = 'Dispositions' = 'Cash that's going out')

Good news: the approximate’s good enough

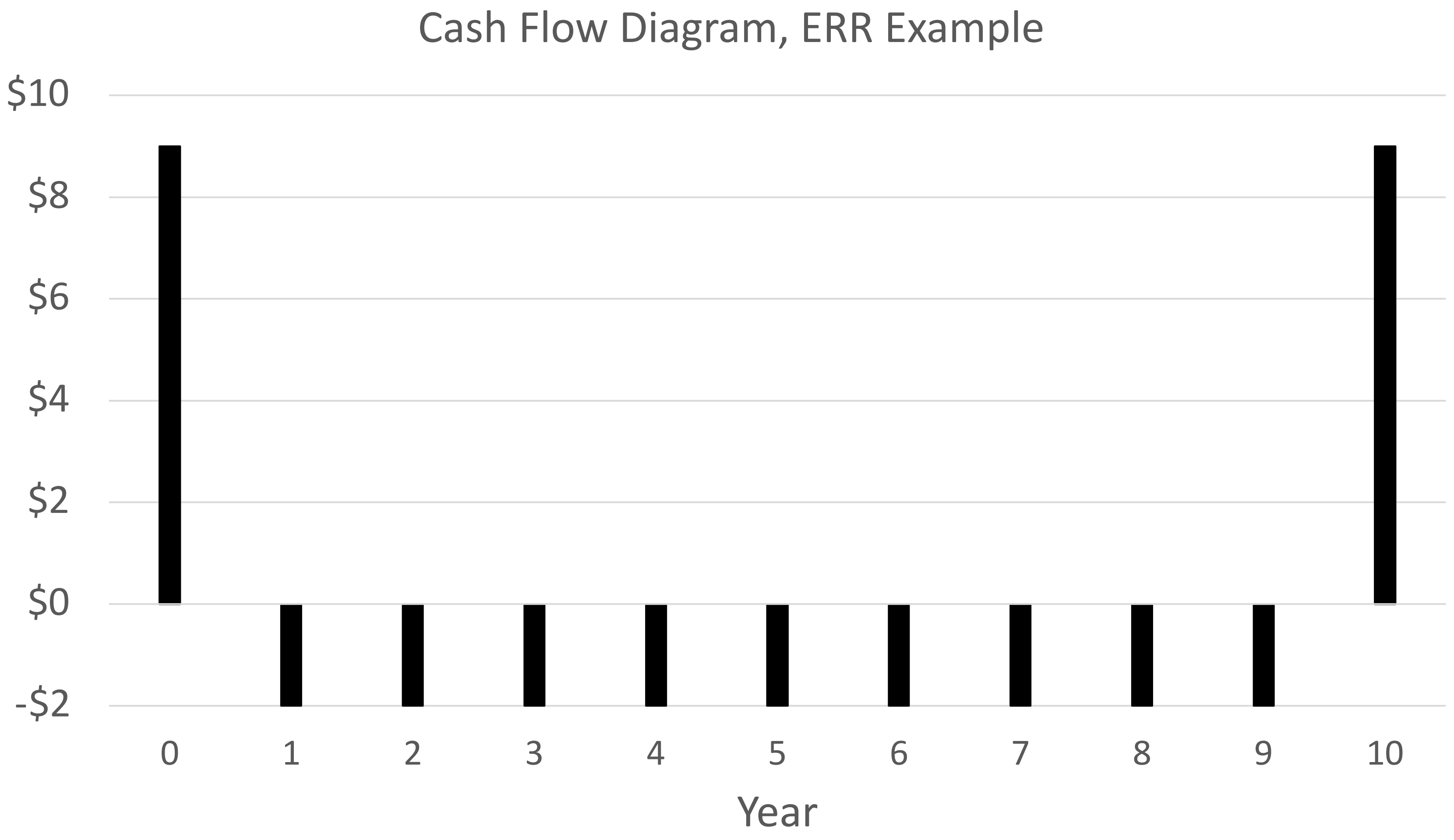
- …for most project evaluation purposes, anyway.
- It can be shown that the approximate ERR will always lie between the MARR and the true ERR.
- → If ERR <> MARR, the approximate ERR <> MARR.
- → The approximate ERR will still be a faithful guide as to whether an independent project is worthwhile….
- …and the approximate incremental ERR will work for mutually exclusive projects. (Example in Companion Spreadsheet)

AFTER HOURS

- Extended Example (5 slides)
- Mixed Time Scales (7 slides)

Example: Finding the (approximate) ERR

Year	Cash Flow
0	9
1	-2
2	-2
3	-2
4	-2
5	-2
6	-2
7	-2
8	-2
9	-2
10	9



Two sign changes, so two IRR solutions possible → Find ERR

<u>The steps we're going to take</u>:

- We need two items, which I'll call: <u>Negative</u> and <u>Positive</u>
- <u>Negative</u> = Sum of the future value of negative cash flows at N, evaluated at the approximate ERR.
- <u>Positive</u> = Sum of the future value of positive cash flows at N, evaluated at the MARR.
- At the correct value of the approximate ERR, <u>Negative + Positive</u> = 0.
- We'll first find the approximate ERR for our cash flow numerically, then we'll show how it could be done through linear interpolation.
- In this simple example it is, of course, possible to solve for the approximate ERR analytically. For more realistic examples or less regular cash flows (those that can't be described with just a few cash flow elements), this can get messy.

For completeness: The analytical solution

- Positive Cash Flows: 9 (Year 0), 9 (Year 10)
- Year 10 Values at the MARR = **Positive = 9(F/P,MARR,10) + 9**
- Negative Cash flows: An annuity with A = -2, N = 9.
- Year 10 Value at the ERR = **Negative = -2(P/A,ERR,9)(F/P,ERR,10)**
- At the approximate ERR, **Positive + Negative = 0**
- We can solve this analytically by expanding the cash flow elements.
- BUT this is messy: on the companion spreadsheet, Positive + Negative was initialized for a trial value of the ERR (10%), then Goalseek was used to find the ERR that set Positive + Negative to $0, through trial and error.

Solution (and Brute Force Check)

Year	Cash Flow	FV (Check)	
0	9	$23.34	At MARR
1	-2	-$5.35	At (Approx.) ERR
2	-2	-$4.80	At (Approx.) ERR
3	-2	-$4.30	At (Approx.) ERR
4	-2	-$3.85	At (Approx.) ERR
5	-2	-$3.45	At (Approx.) ERR
6	-2	-$3.10	At (Approx.) ERR
7	-2	-$2.78	At (Approx.) ERR
8	-2	-$2.49	At (Approx.) ERR
9	-2	-$2.23	At (Approx.) ERR
10	9	$9.00	Already FV

IRR	0%	Excel's IRR function

Total	$0.00

MARR	10%

Positive Cash Flows: 9 (Year 0), 9 (Year 10)

Year 10 Values at the MARR: 9(F/P,MARR,10) + 9

Call this 'Positive'

Positive	$32.34

Negative Cash flows: An annuity with A = -2, N = 9.

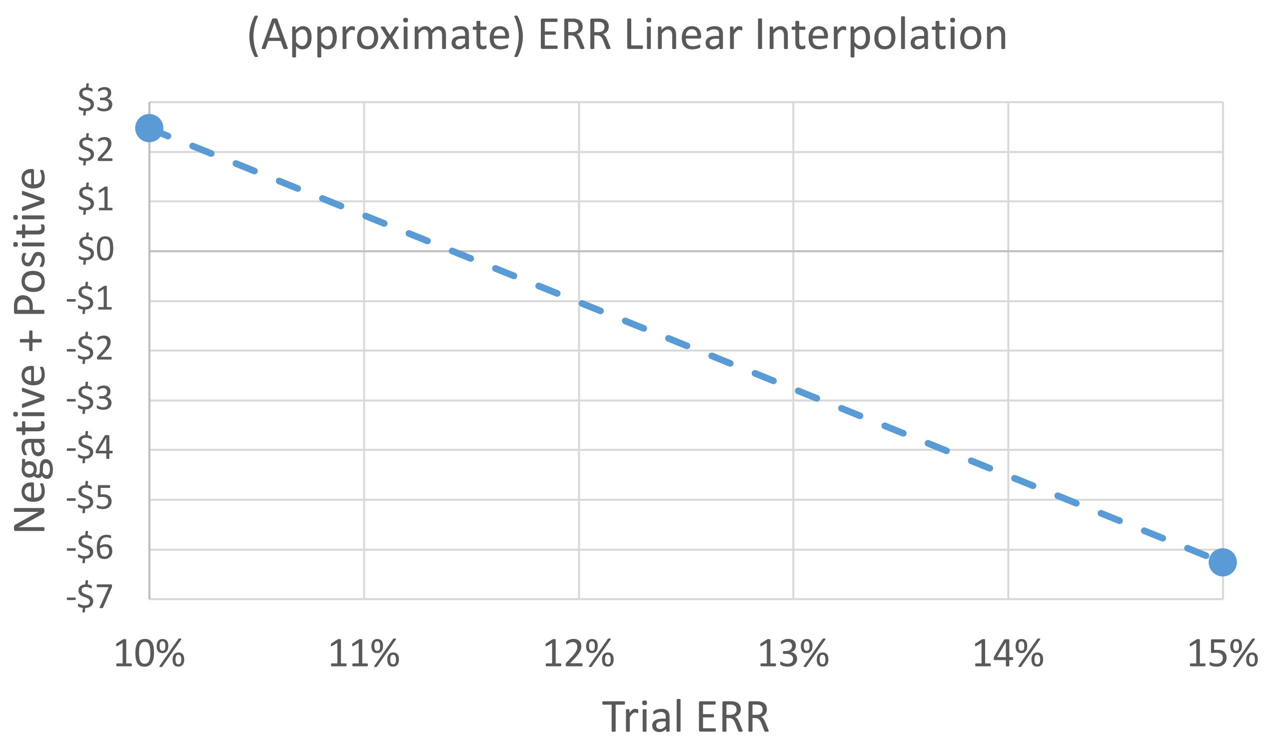
Year 10 Value at the ERR: -2(P/A,ERR,9)(F/P,ERR,10). Call this 'Negative'.

Negative + Positive	(Approx.) ERR
$0.00	**11.55%**

Sample solution using linear interpolation

Trial i	Negative + Positive
10%	$2.47
15%	-$6.26

Rise	-$8.73
Run	5%
Slope (m)	-$174.65



Slope = (y - y1)/(x - x1)

We want the x such that y = 0.

Solving, x = x1 - y1/m

x	11.41%	Using x1=10%, y1=$2.47
x	11.41%	Using x1=20%, y1=-$6.26

(Approx.) ERR	11.41%	Interpolation

What if you have mixed time scales?

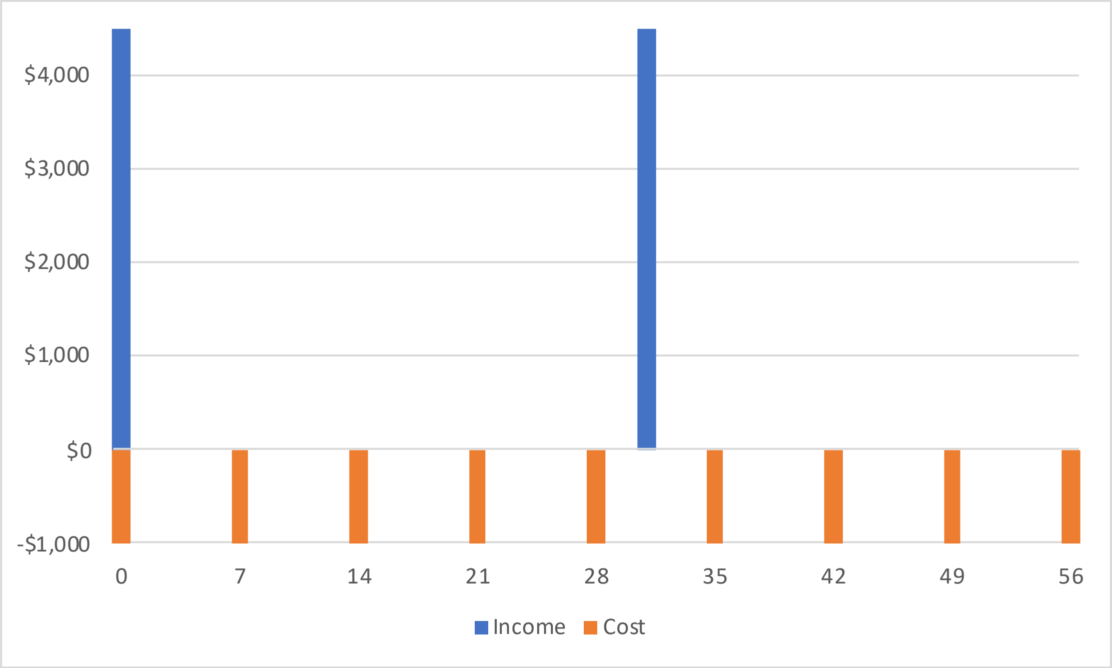
- Suppose your project takes up the months of March and April.
- Once a month, on the 1st of the month, you are paid \$4,500.
- Once a week, on the same day of the week, you incur \$1,000 in costs.
- The first weekly costs are incurred on the 1st of March.
- Your MARR is 10% per year.
- What is the approximate ERR?
- Remember that to find the approximate ERR, we take all positive *net* cash flows to the time period of the *last cash flow in the project*, *at the MARR*, and…
- …we take all negative *net* cash flows to the time period of the last cash flow in the project, at the ERR to be solved for.
- Then we set the sum of those amounts equal to zero, and solve for the ERR.

Setting things up

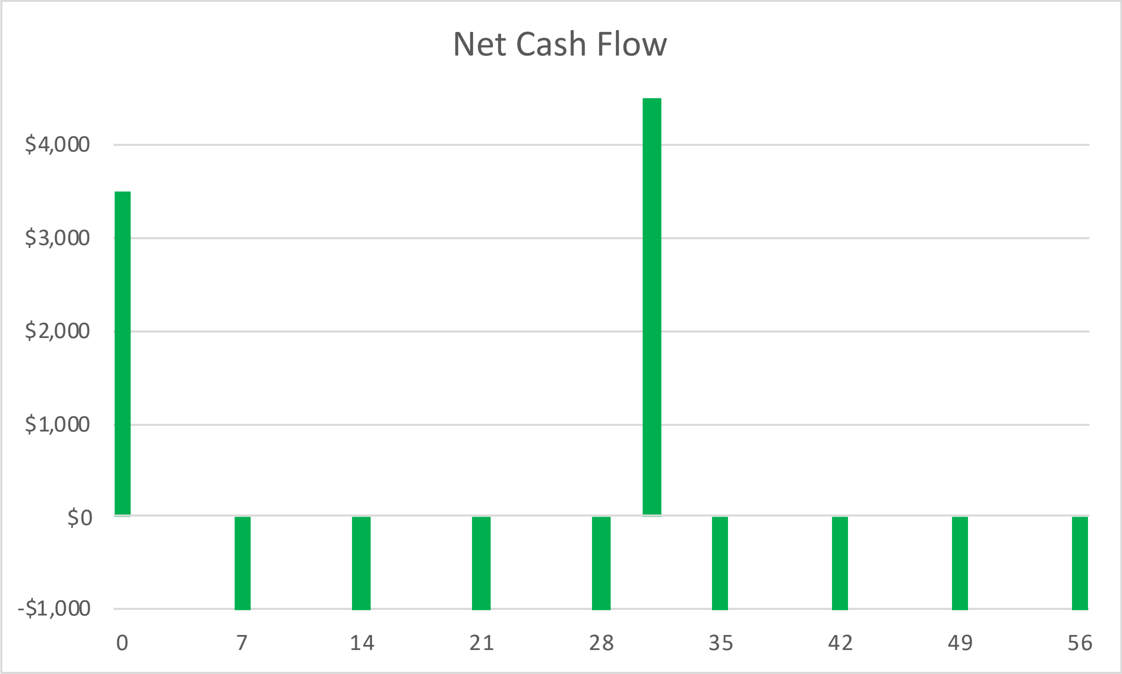
- March has 31 days and April have has 30 days.
- Call March 1 ‘Day 0’ for purposes of indexing. Then April 30 = Day 60.
- BUT ‘Day 60’ is *not* the date of our final cash flow.
- Costs are weekly, on the same day, and the first is on Day 0.
- →Remaining costs are on days with an index that’s a multiple of 7.
- The last cost is paid on Day 56 (= 7 x 8), since the next payment would be due in May, which is beyond the length of our project.
- →There are 9 payments of costs. (Day 0 + the other 8.)

Continuing setup

- Income is paid on the 1st of each month.
- The first payment is on Day 0, and since March has 31 days (Day 0 + Days 1 to 30), the last day of March is Day 30…
- →The second payment is on April 1 = Day 31.
- Day 0 is also a day on which costs must be paid.
- Since 31 is not divisible by 7, Day 31 is not a day on which costs are due.
- →Our two types of cash flow only overlap on Day 0.
- (We need to know this because we're performing operations on *net* cash flows.)



Cash Flows by Day Index



NET Cash Flows by Day Index

So, what do we need to bring forward?

- The last cash flow in our project is on Day 56 = Week 8.
- In terms of *net* positive cash flows, we have \$3,500 on Day 0, and \$4,500 on Day 31. It makes the most sense to use a *daily* rate and treat them as individual cash flows.
- We need to bring them to Day 56, so we need to bring the \$3,500 forward by (56 – 0) = 56 days, and the \$4,500 forward by (56 – 31) = 25 days.
- Since they're positive, we're bringing these forward at the MARR:
- \$3,500 x (F/P,$MARR_{daily}$,56) + \$4,500 x (F/P,$MARR_{daily}$,25)
- And since our MARR = 10%/year, $MARR_{daily} = (1+10\%)^{1/365} - 1$
- (Assuming it's not a leap year.)
- →Positive = \$8,081.03 after plugging in numbers.

What about the negatives?

- Our first cost payment, the one on Day 0, is already accounted for.
- It formed part of the *net* positive flow of $3,500 on Day 0.
- What's left? 8 cash flows, a week apart, of -$1,000.
- The final cost payment is also the last cash flow in the project.
- →We can use (F/A,ERR_{weekly},8) to calculate the Day 56 value.
- Remember that (F/A,i,N) takes N sequential payments of magnitude A and returns an equivalent single payment in the same time period as the last payment of the original sequence.
- That's exactly what we need right now.
- Negative = -$1,000 x (F/A,$ERR_{weekly}$,8)

Finishing up:

- Positive = \$8,081.03,
  Negative = -\$1,000 x (F/A,ERR_{weekly},8)
- \$8,081.03 -\$1,000 x (F/A,ERR_{weekly},8) = 0
- Solving (numerically) →
  ERR_{weekly} = 0.29% per week (approx.)
- ERR per year = $(1+ERR_{weekly})^{365/7}$
  = 16.16% per year (approx.)