Solved problems on

Present Values, Future Values & Annuities

from Chapter 12 of John Ward's *The Young Mathematician's Guide* (1724)
Revised with modern notation, minor corrections and decimal currency
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Part I: Present and Future Values

Question I.1

What will \$256.50 amount to in seven years, at 6 percent per year compound interest?

Answer: We have a present value, and we want a future value, so we use (F/P,i,N). The relevant interest rate is 6% per year, and N = 7 years. \$256.50 x (F/P,6%,7) = \$385.68

Question I.2

What principal, or sum of money, must be put (or let) out to raise a stock of \$385.68 in seven years, at 6% per year compound interest?

Answer: We have a future value, and we want a present value, so we use (P/F,i,N). The relevant interest rate is 6% per year, and N = 7 years. \$385.68 x (P/F,6%,7) = \$256.50

Question I.3

In what time will \$256.50 raise a stock of (or amount to) \$385.68, allowing 6 percent per year compound interest?

Answer: We have a present value, a future value, and an interest rate. We want N. We'll have to use the full version of our expressions to solve for it.

\$256.50 x (F/P,6%,N) = \$385.68
$\$257.60 \times (1+6\%)^N = \$385.68$
$1.06^N = 385.68/256.50$
$N = \log_{1.06}(385.68/256.50)$
$N = \ln(385.68/256.50)/\ln(1.06)$
N = 7 (approx.)

Question I.4

If \$256.50 will amount to (or raise a stock of) \$385.68 in seven years' time, what must the rate of interest be, in percent per year?

Answer: We have a present value, a future value, and a time N. We want the interest rate, i. We'll have to use the full version of our expressions to solve for it.

$\$256.50 \times (F/P,i,7) = \$385.68$
$\$257.60 \times (1+i)^7 = \$385.68$
$(1+i)^7 = 385.68/256.50$
$i = (385.68/256.50)^{1/7} – 1 = 0.06 = 6\%$ (approx.)

Question I.5

What will \$375.50 amount to in nine years at interest of 6% per year?

Answer: We have a present value, an interest rate, and a time period. We want a future value. $F = \$375.50 \times (F/P,6\%,9) = \$634.40$

Question I.6

What principal (or sum) must be put to interest to raise a stock of \$634.40 in nine years' time, at interest of 6% per year?

Answer: We have a future value, an interest rate and a time period. We want a present value. $P = \$634.40 \times (P/F,6\%,9) = \$375.50$

Question I.7

In what time will \$375.50 raise a stock of (or amount to) \$634.40 at interest of 6% per year?

Answer: We have a present value, a future value and an interest rate. We want the time period, N. We'll have to use the full version of our expressions to solve for it.

$\$375.50 \times (F/P,6\%,N) = \$634.40$
$\$375.50 \times (1+6\%)^N = \$634.40$
$1.06^N = 634.40/375.50$
$N = \log_{1.06}(634.40/375.50)$
$N = \ln(634.40/375.50)/\ln(1.06)$
$N = 9$ (approx.)

Question I.8

In what time will \$563 amount to \$860 at 6% per year, compound interest?

Answer: We have a present value, a future value and an interest rate. We want the time period, N. We'll have to use the full version of our expressions to solve for it.

$\$563 \text{ x } (F/P,6\%,N) = \$860$
$\$563 \text{ x } (1+6\%)^N = \$860$
$1.06^N = 860/563$
$N = \log_{1.06}(860/563)$
$N = \ln(860/563)/\ln(1.06)$
$N = 7.2706487$ (approx.)

Multiplying the remainder by 365 days in a year: 0.2706487 x 365 = 98.79 (approx.) All together, the required time is 7 years and 99 days, to the nearest day.

Question I.9

Suppose it were required to find out what \$256 would amount to in fifteen years, at 8% per year compound interest. What is the answer?

Answer: We have a present value, and we want a future value. \$256 x (F/P,8%,15) = \$812.08[1]

Question I.10

What will \$365 amount to in seven years, at 4.5% interest per year?

Answer: We have a present value, and we want a future value. \$365 x (F/P,4.5%,7) = \$496.71.

[1] Ward's answer of \$811.46 appears to suffer from rounding error. That would correspond to a present value of \$255.81.

Question I.11

Suppose it were required to find the amount of \$375 for 210 days at interest of 6% per year.

Answer: The only difficulty here is that our time period is in days, and our interest rate is in terms of % per year. We can resolve this difficulty either by converting the days to years, or the interest rate per year to an equivalent interest rate per day.

Converting days to years:
Assume a non-leap year. Then 210 days = (210/365) years.
\$375 x (F/P,6%,(210/365)) = \$387.78

Converting interest per year to interest per day:
Assume a non-leap year. Then $(1 + 6\%) = (1 + i_{daily})^{365}$, and $i_{daily} = 1.06^{1/365} - 1$.
\$375 x (F/P,i_{daily},210) = \$387.78

Question I.12

Suppose it were required to find what \$265 would amount to in five years and 135 days, at 6 per cent interest per year. What is the answer?

Answer: Again, the difficulty is that we have days and years mixed up. We can either put everything in terms of years, or everything in terms of days. I'll ignore leap years, since we don't know whether there are one or two leap years in the mix.

Converting days to years:
Assuming no leap years, ever, 5 years + 135 days = 5 + 135/365 years.
\$265 x (F/P,6%,5+135/365) = \$362.36

Converting years to days:
Assuming no leap years, ever, 5 years + 135 days = (5 x 365) + 135 = 1960 days
From yearly to daily interest: $(1 + 6\%) = (1 + i_{daily})^{365}$, and $i_{daily} = 1.06^{1/365} - 1$
\$265 x (F/P,i_{daily},1960) = \$362.36

Question I.13

What principal will raise a stock of \$362.36 in 5 years and 135 days, at 6 per cent interest per year?

Answer: We have a desired future value, and need to find the present value that will give us that future value, given 6% interest per year and a period of 5 years and 135 days.

Again, the difficulty is that we have days and years mixed up. We can either put everything in terms of years, or everything in terms of days. I'll ignore leap years, since we don't know whether there are one or two leap years in the mix.

Converting days to years:
Assuming no leap years, ever, 5 years + 135 days = 5 + 135/365 years.
\$362.36 x (P/F,6%,5+135/365) = \$265

Converting years to days:
Assuming no leap years, ever, 5 years + 135 days = (5 x 365) + 135 = 1960 days
From yearly to daily interest: $(1 + 6\%) = (1 + i_{daily})^{365}$, and $i_{daily} = 1.06^{1/365} - 1$
\$362.36 x (P/F,i_{daily},1960) = \$265

Future value

Part II: Annuities

Question II.1

If \$30 yearly rent, or annuity, &c. be forlorn (viz. remain unpaid) nine years; what will it amount to, at 6% per year compound interest? (In modern terms, what is the future value, in nine years, of an annuity that pays \$30 per year for nine years, starting one year from now, if the interest rate is 6% per year?)

Answer: We have an annuity, A, and want an equivalent future value at the time of the last payment. This is what (F/A,i,N) is designed for:

A x (F/A,i,N) = \$30 x (F/A,6%,9) = \$344.74

We want the fund rising in the end from given Annuity, i, N

Since this is the first annuity question, I'll demonstrate this is the right answer by taking each payment to 'nine years from now' individually, then adding them all up.

Year 1 payment, forward 8 years: \$30 x $(1+6\%)^8$ = \$47.82
Year 2 payment, forward 7 years: \$30 x $(1+6\%)^7$ = \$45.11
Year 3 payment, forward 6 years: \$30 x $(1+6\%)^6$ = \$42.56
Year 4 payment, forward 5 years: \$30 x $(1+6\%)^5$ = \$40.15
Year 5 payment, forward 4 years: \$30 x $(1+6\%)^4$ = \$37.87
Year 6 payment, forward 3 years: \$30 x $(1+6\%)^3$ = \$35.73
Year 7 payment, forward 2 years: \$30 x $(1+6\%)^2$ = \$33.71
Year 8 payment, forward 1 years: \$30 x $(1+6\%)^1$ = \$31.80
Year 9 payment, forward 0 years: \$30 x $(1+6\%)^0$ = \$30.00

All together: \$(47.82 + 45.11 + 42.56 + 40.15 + 37.87 + 33.71 + 31.80 + 30.00) = \$344.74

Question II.2

Future Value

What yearly rent, or annuity, &c., being forlorn or unpaid nine years, will raise a stock of \$344.74 at interest of 6% per year? (In other words, if interest is 6% per year, what annuity, paid from Year 1 to Year 9, is equivalent to a single payment of \$344.74 in Year 9?)

Answer: We have a future value, and want an annuity so use (A/F,i,N).

F x (A/F,i,N) = \$344.74 x (A/F,6%,9) = \$30

- We have the fund, we want the annuity/ installment payments leading to the future fund.

Question II.3

In what time will \$30 yearly rent raise a stock or amount equal to \$344.74, allowing 6% interest per year? (Assume the first payment is one year from now.)

Answer: We have an annuity, a future value, and an interest rate. $\boxed{\text{We need N}}$. We'll have to expand our expression for this.

$30 \times (F/A, 6\%, N) = 344.74$
$30 \times (((1+6\%)^N - 1)/6\%) = 344.74$
$1.06^N = (.06 \times 344.74/30) + 1 = 1.68948$
$1.06^N = 1.68948$
$N = \log_{1.06}(1.68948)$
$N = \ln(1.68948)/\ln(1.06)$
$N = 9$ (approx.)

Future value of an amount

Annuity

N

F

Question II.4

If \$30 per year, being unpaid nine years, will amount to \$344.74, allowing compound interest for every payment as it becomes due, what must the rate of interest be in percent per year? (Assume the first payment is one year from now.)

Answer: We have an annuity, a future value, and a time period. We need an interest rate. We'll have to expand our expression for this.

$30 \times (F/A, i, 9) = 344.74$
$30 \times (((1 + i)^9 - 1)/i) = 344.74$
We can solve this using numerical methods, and find that i=6%...
But how did Ward solve it in 1724?

They used $A \times (F/A, i, N)$ Formula
So this is Annuity to Future Funding Formula.

we have Annuity (yearly), N (yearly) & Funding. We need Interest. By Formula we gonna get yearly interest

The process was as follows:
We have $F = 344.74$, and $A = 30$.
By Ward's calculations[2],
$F/A = 11.491317$, and $(F – A)/A = 10.491317$
Let $R = (1+i)$
$30 \times (((1 + i)^9 – 1)/i) = 344.74$
$A \times ((R^9 – 1)/(R – 1)) = F$
$AR^9 – AR = FR – F$
$(F/A)R – R^9 = (F – A)/A$
$11.491317R – R^9 = 10.491317$ (1)
Now, since $R = (1+i)$, we can approximate R9 by
$R^9 = 1^9 + 9 \times 1^8 \times i + 36i^2$
$1 + 9i + 36i^2 = R^9$ (2)
Also note that $(1+i) = R \rightarrow 11.491317 + 11.491317i = 11.491317R$ (3)
Subtract (2) from (3):
$10.491317 + 2.491317i – 36i^2 = 11.491317R – R^9$
But from (1), the right-hand-side is equal to 10.491317
$\rightarrow 10.491317 + 2.491317i – 36i^2 = 10.491317$
$36i^2 = 2.491317i$
$36i = 2.491317$
$i = 2.491317/36 = 6\%$ (according to Ward)
His numbers actually give $i = 0.0692$, which rounds up to 7%, but the discrepancy is due to rounding error.

$\rightarrow A_4$ $\rightarrow N_4 \rightarrow I_4$ A x (P\A,i,N)

Question II.5

What is \$30 yearly rent, to continue seven years, worth in ready money (present worth), allowing 6% compound interest per year to the purchaser? (Assume the first rent payment is one year from now.)

Answer: We have an annuity, an interest rate and a time period, and need to find a present value. Using the (P/A,i,N) factor, \$30 x (P/A,6%,7) = \$167.47

Question II.6

What annuity or yearly rent, to continue seven years, may be purchased for \$167.47 allowing 6% compound interest per year to the purchaser? (Assume the first annuity payment is one year from now.)

Answer: We have a present value, and want an annuity, so we use (A/P,i,N).
\$167.47 x (A/P,6%,7) = \$30

[2] Ward's numbers understandably have a bit of rounding error.

$A \times (P\backslash A,i,N) = F$

N4? A4 Funding (Present Value) I4

Question II.7

How long may one have a lease of \$30 yearly rent, for \$167.47, allowing 6% compound interest per year to the purchaser? (Assume the first rent payment is one year from now.)

Answer: We have an annuity, a present value and an interest rate, and need a time period. We need to expand our expressions for this.

$167.47 = $30 x (P/A,6%,N)
$(167.47/30) = (1.06^N – 1)/(.06 \times 1.06^N)$
$(167.47/30) \times (.06 \times 1.06^N) = (1.06^N – 1)$
$(.06 \times 167.47/30) \times 1.06^N = 1.06^N – 1$
$(1 – (.06 \times 167.47/30)) \times 1.06^N = 1$
$1.06^N = 1/(1 – (.06 \times 167.47/30))$
$N = \ln(1/(1 – (.06 \times 167.47/30)))/\ln(1.06)$
N = 7 (approx.)

Question II.8

Suppose one should give \$167.47 for the purchase of a pension, or annuity, of \$30 per year, to continue seven years (with the first payment being one year from now). At what rate of interest, in percent per year, would that purchase be made, allowing compound interest to the purchaser?

Answer: We have an annuity, a present value and a time period, and want an interest rate. We need to expand our expressions for this, and use numerical methods to solve (Ward refers to his solution for Question II.4.)

\$167.47 = \$30 x (P/A,i,7)
$(167.47/30) = ((1+i)^7 – 1)/(i \times (1+i)^7)$
Solving numerically, we find i = 6% per year (approx.)

Part III: Two-part questions

Question III.1

Suppose it were required to compute the present worth of $75 yearly rent, which is not to commence or be entered upon, until ten years from now, and then to continue seven years after that time, at 6% per year compound interest. What is the correct value? (Note: the first payment would be 11 years from now, or equivalently, at the very end of the tenth year.)

Answer: From the point of view of someone 10 years from now, the present value of that stream of seven payments of $75 is $75 x (P/A,6%,7) = $418.68.

From the point of view of someone today, the value of $418.68, ten years from now, is $418.68 x (P/F,6%,10) = $233.79.

Therefore, the present value of that proposed investment is $233.79.

Question III.2

What annuity or yearly rent to be entered upon ten years from now, and then to continue seven years after that time, may be purchased for $233.79 ready money (present worth), at 6% per year compound interest? (Note: the first payment would be 11 years from now, or if you prefer, or equivalently, at the very end of the tenth year.)

Answer: Let A be the annuity. From the point of view of someone 10 years from now, the present value of that stream of seven payments of A is A x (P/A,6%,7) = A x 5.58238. From the point of view of someone today, the value of A x 5.58238, ten years from now, is A x 5.582381 x (P/F,6%,10) = A x 3.11717. Setting $233.79 = A x 3.11717, A = $233.79/3.11717 = $75.

F = Future
P = Present
A = Annuity
i = Interest
N = Time

Useful Formulas

$$(F/P,i,N) = (1+i)^{N}$$

$$(P/F,i,N) = (1+i)^{-N}$$

$$(A/F,i,N) = \frac{i}{(1+i)^{N}-1}$$

$$(F/A,i,N) = \frac{[(1+i)^{N}-1]}{i}$$

$$(A/P,i,N) = \frac{i(1+i)^{N}}{[(1+i)^{N}-1]}$$

$$(P/A,i,N) = \frac{(1+i)^{N}-1}{[i(1+i)^{N}]}$$

Excel Equivalents

F x (P/F,i,N) =PV(i,N,,-F)
P x (F/P,i,N) =FV(i,N,,-P)
F x (A/F,i,N) =PMT(i,N,,-F)
A x (F/A,i,N) =FV(i,N,-A)
P x (A/P,i,N) =PMT(i,N,-A)
A x (P/A,i,N) =PV(i,N,-A)