Q.1 (a) $$\int \frac{\sqrt{\cos^{-1}(x)}}{\sqrt{1-x^2}}\,dx$$

Let $u = \cos^{-1}(x)$

$$\Rightarrow du = \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{\sqrt{\cos^{-1}(x)}}{\sqrt{1-x^2}}\,dx = -\int \sqrt{u}\,du$$

$$= -\frac{2u^{3/2}}{3} + C$$

$$= -\frac{2}{3}\left[\cos^{-1}(x)\right]^{3/2} + C$$

(b) $\displaystyle\int_0^{\pi/4} \sqrt{1-\cos(4\theta)}\, d\theta$

we have: $\sin(2\theta) = \sqrt{\dfrac{1-\cos(4\theta)}{2}}$

or $\sqrt{2}\sin(2\theta) = \sqrt{1-\cos(4\theta)}$,

$$\Rightarrow \int_0^{\pi/4} \sqrt{1-\cos(4\theta)}\, d\theta = \sqrt{2}\int_0^{\pi/4} \sin(2\theta)\, d\theta$$

$$= -\sqrt{2}\left[\frac{\cos(2\theta)}{2}\right]\Bigg|_{\theta=0}^{\theta=\pi/4}$$

$$= -\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$$

$$= \frac{1}{\sqrt{2}}$$

(c) $\displaystyle\int_0^{\pi/6} \sqrt{1+\cos(2x)}\ dx$

$$= \sqrt{2}\int_0^{\frac{\pi}{6}} \cos(x)\ dx$$

$$= \sqrt{2}\ \left[\sin(x)\right]\Big|_{x=0}^{x=\pi/6}$$

$$= \sqrt{2}\left[\sin(\pi/6) - \sin(0)\right]$$

$$= \sqrt{2}\left[\frac{1}{2} - 0\right]$$

$$= \frac{1}{\sqrt{2}}$$

(d) $\displaystyle\int \frac{x^2}{(x^2-1)^{5/2}}\,dx \quad , \quad x > 1$

Let $x = \sec u$

$\Rightarrow \quad dx = \sec u \tan u \, du$

$$\Rightarrow \int \frac{x^2}{(x^2-1)^{5/2}}\,dx = \int \frac{\sec^2 u}{(\sec^2 u - 1)^{5/2}} \sec u \tan u \, du$$

$$= \int \frac{\sec^3 u \tan u}{\tan^5 u}\,du$$

$$= \int \cot u \csc^3 u \, du$$

Let $\qquad w = \csc(u)$

$$dw = -\csc(u)\cot u \, du$$

$$= -\int w^2 \, dw$$

$$= -\frac{w^3}{3} + C$$

$$\csc^3\theta = \frac{\cos^3\theta}{\sin^3\theta} \cdot \frac{1}{\cos^3\theta}$$

$$= \frac{\sec^3\theta}{\tan^3\theta}$$

$$= \frac{\sec^3\theta}{(\sec^2\theta - 1)^{3/2}}$$

$$= -\frac{\csc^3 u}{3} + C$$

$$= -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

(e) $\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

We write $\dfrac{s^4+81}{s(s^2+9)^2} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+9} + \dfrac{Ds+E}{(s^2+9)^2}$

Solving this, we get

$A = 1, \quad B = 0, \quad C = 0, \quad D = -18, \quad E = 0$

$$\Rightarrow \int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{1}{s}\,ds - 18\int \frac{s}{(s^2+9)^2}\,ds$$

$$= \ln|s| + \frac{9}{s^2+9} + c$$

(f).

$$\int_0^1 \frac{dx}{(x+1)(x^2+1)}$$

using Partial fractions: $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

this gives $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{1}{2}$

$$\Rightarrow \int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx + \frac{1}{2}\int_0^1 \frac{1-x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1|\Big|_0^1 + \frac{1}{2}\tan^{-1}(x)\Big|_0^1 - \frac{1}{4}\ln(x^2+1)\Big|_0^1$$

$$= \frac{\pi}{8} + \frac{\ln(4)}{8}$$

(g) $$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \frac{1}{27}\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left[\left(\frac{2x}{3}\right)^2+1\right]^{3/2}}\,dx$$

Let $\frac{2x}{3} = \tan\theta \Rightarrow dx = \frac{3}{2}\sec^2\theta\,d\theta,$ $x=0 \Rightarrow \theta = 0$

$x = \frac{3\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\Rightarrow \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \frac{3}{16}\int_0^{\frac{\pi}{3}} \frac{\tan^3\theta\,\sec^2\theta}{(1+\tan^2\theta)^{3/2}}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^2\theta}{\sec^2\theta}(\sec\theta\tan\theta)\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{(\sec^2\theta-1)}{\sec^2\theta}\tan\theta\sec\theta\,d\theta$$

Let $w = \sec\theta,$ $dw = \sec\theta\tan\theta\,d\theta;$

$\theta = 0 \Rightarrow w = 1$

$\theta = \frac{\pi}{3} \Rightarrow w = 2$

$$= \frac{3}{16}\int_1^2 \frac{w^2-1}{w^2}\,dw$$

$$= \frac{3}{16}\left[w + \frac{1}{w}\right]\Bigg|_{w=1}^{w=2}$$

$$= \frac{3}{32}$$

(h) $$\int \frac{y^4 - 2y^2 + 4y + 1}{y^3 - y^2 - y + 1}\, dy$$

Using long division, $$\frac{y^4 - 2y^2 + 4y + 1}{y^3 - y^2 - y + 1} = (y+1) + \frac{4y}{y^3 - y^2 - y + 1}$$

$$= (y+1) + \frac{4y}{(y-1)^2(y+1)}$$

Further, $$\frac{4y}{(y-1)^2(y+1)} = \frac{A}{y-1} + \frac{B}{(y-1)^2} + \frac{C}{y+1}$$

gives $A = 1,\ B = 2,\ C = -1$

Putting this all together,

$$\int \frac{y^4 - 2y^2 + 4y + 1}{y^3 - y^2 - y + 1}\, dy = \int \left[(y+1) + \frac{1}{y-1} + \frac{2}{(y-1)^2} - \frac{1}{y+1} \right] dy$$

$$= \frac{y^2}{2} + y + \ln|y-1| - \ln|y+1| - \frac{2}{y-1} + C$$

(i) $$\int_0^1 \frac{x^2}{x^4-1}\,dx$$

$$\frac{x^2}{x^4-1} = \frac{Ax+B}{x^2-1} + \frac{Cx+D}{x^2+1}$$

this gives, $A=0,\ C=0,\quad B=1/2,\ D=1/2$

$$\Rightarrow \int_0^1 \frac{x^2}{x^4-1}\,dx = \frac{1}{2}\int_0^1 \underbrace{\frac{1}{x^2-1}}_{\text{Partial Fraction}}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{4}\int_0^1 \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx + \frac{1}{4}\tan^{-1}(x)\Big|_0^1$$

$$= \frac{1}{4}\left[\ln|x-1| - \ln|x+1|\right]\Big|_0^1 + \frac{1}{4}\tan^{-1}(x)\Big|_0^1$$

Does not converge