Math 110 - Homework 6
Topic: Subspaces, dimension, linear transformations

Due at 6:00pm (Pacific) on Friday, October 22, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 3.4 and 4.1 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

This week's material is mostly about establishing a theoretical framework. After unravelling the definitions you will find that the computations that need to be done are the same as the ones from previous weeks, so there are no new MATLAB commands for this week.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let S be the collection of vectors in $\mathbb{R}^3$ where $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in S if and only if $x + y \leq z$. Determine whether or not S is a subspace of $\mathbb{R}^3$. If it is a subspace, prove that it is. If not, explain why not.

> **Solution:** S is not a subspace of $\mathbb{R}^3$, because S is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is in S (because $1 + 1 \leq 3$), but $(-1)\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is not in S (because $(-1) + (-1) > -3$).

2. Let $\overrightarrow{w_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{w_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Let S be the collection of vectors in $\mathbb{R}^3$ where $\overrightarrow{v}$ is in S if and only if $\mathrm{proj}_{\overrightarrow{w_1}}(\overrightarrow{v}) + \mathrm{proj}_{\overrightarrow{w_2}}(\overrightarrow{v}) = \overrightarrow{0}$. Determine whether or not S is a subspace of $\mathbb{R}^3$. If it is a subspace, prove that it is. If not, explain why not.

> **Solution:** We will prove that S is a subspace of $\mathbb{R}^3$.
>
> First, observe that $\mathrm{proj}_{\overrightarrow{w_1}}(\overrightarrow{0}) + \mathrm{proj}_{\overrightarrow{w_2}}(\overrightarrow{0}) = \overrightarrow{0} + \overrightarrow{0} = \overrightarrow{0}$, so $\overrightarrow{0}$ is in S (and in particular, S is not empty).
>
> Suppose that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are in S. Then:
>
> $$\begin{aligned} \mathrm{proj}_{\overrightarrow{w_1}}(\overrightarrow{v_1} + \overrightarrow{v_2}) + \mathrm{proj}_{\overrightarrow{w_2}}(\overrightarrow{v_1} + \overrightarrow{v_2}) &= \left(\frac{\overrightarrow{w_1} \cdot (\overrightarrow{v_1} + \overrightarrow{v_2})}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2} \cdot (\overrightarrow{v_1} + \overrightarrow{v_2})}{\overrightarrow{w_1} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} \\ &= \left(\frac{\overrightarrow{w_1} \cdot \overrightarrow{v_1} + \overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2} \cdot \overrightarrow{v_1} + \overrightarrow{w_2} \cdot \overrightarrow{v_2}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} \\ &= \left(\frac{\overrightarrow{w_1} \cdot \overrightarrow{v_1}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2} \cdot \overrightarrow{v_1}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} + \left(\frac{\overrightarrow{w_2} \cdot \overrightarrow{v_2}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} \\ &= \left(\frac{\overrightarrow{w_1} \cdot \overrightarrow{v_1}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2} \cdot \overrightarrow{v_1}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} + \left(\frac{\overrightarrow{w_1} \cdot \overrightarrow{v_2}}{\overrightarrow{w_1} \cdot \overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2} \cdot \overrightarrow{v_2}}{\overrightarrow{w_2} \cdot \overrightarrow{w_2}}\right)\overrightarrow{w_2} \\ &= (\mathrm{proj}_{\overrightarrow{w_1}}(\overrightarrow{v_1}) + \mathrm{proj}_{\overrightarrow{w_2}}(\overrightarrow{v_1})) + (\mathrm{proj}_{\overrightarrow{w_1}}(\overrightarrow{v_1}) + \mathrm{proj}_{\overrightarrow{w_2}}(\overrightarrow{v_2})) \\ &= \overrightarrow{0} + \overrightarrow{0} \\ &= \overrightarrow{0} \end{aligned}$$
>
> This shows that $\overrightarrow{v_1} + \overrightarrow{v_2}$ is in S.

Finally, suppose $\overrightarrow{v}$ is in S and c is a scalar. Then:

$$\begin{aligned}
\text{proj}_{\overrightarrow{w_1}}(c\overrightarrow{v}) + \text{proj}_{\overrightarrow{w_2}}(c\overrightarrow{v_1}) &= \left(\frac{\overrightarrow{w_1}\cdot(c\overrightarrow{v})}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\right)\overrightarrow{w_1} + \left(\frac{\overrightarrow{w_2}\cdot(c\overrightarrow{v})}{\overrightarrow{w_2}\cdot\overrightarrow{w_2}}\right)\overrightarrow{w_2} \\
&= c\left(\frac{\overrightarrow{w_1}\cdot\overrightarrow{v}}{\overrightarrow{w_1}\cdot\overrightarrow{w_1}}\right)\overrightarrow{w_1} + c\left(\frac{\overrightarrow{w_2}\cdot\overrightarrow{v}}{\overrightarrow{w_2}\cdot\overrightarrow{w_2}}\right)\overrightarrow{w_2} \\
&= c\,\text{proj}_{\overrightarrow{w_1}}(\overrightarrow{v}) + c\,\text{proj}_{\overrightarrow{w_2}}(\overrightarrow{v}) \\
&= c(\text{proj}_{\overrightarrow{w_1}}(\overrightarrow{v}) + \text{proj}_{\overrightarrow{w_2}}(\overrightarrow{v})) \\
&= c\,\overrightarrow{0} \\
&= \overrightarrow{0}
\end{aligned}$$

This shows that $c\overrightarrow{v}$ is in S, and completes the proof that S is a subspace.

3. Suppose that S is a subspace of $\mathbb{R}^2$, and that every vector of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$ is in S. Show that $S = \mathbb{R}^2$.

Aside: We have already seen in class that a line that doesn't go through the origin is not a subspace of $\mathbb{R}^2$; this question shows that the only subspace of $\mathbb{R}^2$ that contains the line $y = x + 1$ is the whole of $\mathbb{R}^2$.

Solution: Since S contains all vectors of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$ it contains, in particular, the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Now since S is a subspace it also contains every linear combination of those two vectors. For any $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = (-x)\begin{bmatrix} -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore every vector in $\mathbb{R}^2$ is a linear combination of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and hence every vector in $\mathbb{R}^2$ is in S.

4. Let S be the collection of vectors in $\mathbb{R}^4$ where $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is in S if and only if $x + 2y = z + 2w$. You may assume, without proving it, that S is a subspace of $\mathbb{R}^4$. Calculate $\dim(S)$.

Solution: Rearranging the equation used to define S, we have that a vector is in S if and only if it satisfies $x = -2y + z + 2w$. That is, the vectors in S have the following form:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2y+z+2w \\ y \\ z \\ w \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$S = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

To find the dimension of S we need a basis for S. We already have a spanning set, so all we need to do is remove any unnecessary vectors until it is linearly independent. We start by checking if it is already independent:

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the vectors in our spanning set are already independent, and so we have shown that they form a basis for S. There are three vectors in our basis, so $\dim(S) = 3$.

5. Let $\overrightarrow{w}$ be a vector in $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\overrightarrow{v}) = \operatorname{perp}_{\overrightarrow{w}}(\overrightarrow{v})$. Prove that T is a linear transformation.

Solution: For any vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, we have:

$$\begin{aligned}
\operatorname{perp}_{\overrightarrow{w}}(\overrightarrow{v_1} + \overrightarrow{v_2}) &= \overrightarrow{v_1} + \overrightarrow{v_2} - \operatorname{proj}_{\overrightarrow{w}}(\overrightarrow{v_1} + \overrightarrow{v_2}) \\
&= \overrightarrow{v_1} + \overrightarrow{v_2} - \left(\frac{\overrightarrow{w} \cdot (\overrightarrow{v_1} + \overrightarrow{v_2})}{\overrightarrow{w} \cdot \overrightarrow{w}} \right) \overrightarrow{w} \\
&= \overrightarrow{v_1} + \overrightarrow{v_2} - \left(\frac{\overrightarrow{w} \cdot \overrightarrow{v_1}}{\overrightarrow{w} \cdot \overrightarrow{w}} \right) \overrightarrow{w} - \left(\frac{\overrightarrow{w} \cdot \overrightarrow{v_2}}{\overrightarrow{w} \cdot \overrightarrow{w}} \right) \overrightarrow{w} \\
&= (\overrightarrow{v_1} - \operatorname{proj}_{\overrightarrow{w}}(\overrightarrow{v_1})) + (\overrightarrow{v_2} - \operatorname{proj}_{\overrightarrow{w}}(\overrightarrow{v_2})) \\
&= \operatorname{perp}_{\overrightarrow{w}}(\overrightarrow{v_1}) + \operatorname{perp}_{\overrightarrow{w}}(\overrightarrow{v_2})
\end{aligned}$$

For any vector $\overrightarrow{v}$ and any scalar c:

$$\begin{aligned}
\operatorname{perp}_{\overrightarrow{w}}(c\overrightarrow{v}) &= c\overrightarrow{v} - \operatorname{proj}_{\overrightarrow{w}}(c\overrightarrow{v}) \\
&= c\overrightarrow{v} - \left(\frac{\overrightarrow{w} \cdot (c\overrightarrow{v})}{\overrightarrow{w} \cdot \overrightarrow{v}} \right) \overrightarrow{w} \\
&= c\overrightarrow{v} - c\left(\frac{\overrightarrow{w} \cdot \overrightarrow{w}}{\overrightarrow{w} \cdot \overrightarrow{v}} \right) \overrightarrow{w} \\
&= c\left(\overrightarrow{v} - \left(\frac{\overrightarrow{w} \cdot \overrightarrow{w}}{\overrightarrow{w} \cdot \overrightarrow{v}} \right) \overrightarrow{w} \right) \\
&= c \operatorname{perp}_{\overrightarrow{w}}(\overrightarrow{v})
\end{aligned}$$

We have verified both properties required for a function to be a linear transformation, so the proof is complete.

Part II: Concepts and connections

As this week did not introduce any new MATLAB commands, we have chosen to keep the numbers relatively pleasant, and put all of the questions in Part I.