202201 Math 122 Assignment 1 Solution ideas

1. Let d be "*I have a million dollars*", g be "*I would buy you a green dress*", and m be "*I would buy you a monkey*". Then, in symbols, the statement "*If I had a million dollars, I'd buy you a green dress and a monkey*" is $d \to (g \wedge m)$.

   The statement "*If I buy you neither a green dress nor a monkey, then I don't have a million dollars'*" is $(\neg g \wedge \neg m) \to \neg d$. It is not logically equivalent to the given statement. If d is true, g is true and m is false then the given statement is false while this statement is true.

   The statement "*I don't have a million dollars, or I'd buy you a green dress and a monkey*" is $\neg d \vee (g \wedge m)$. It is logically equivalent to the given statement since $p \to q \Leftrightarrow \neg p \vee q$.

   The statement "*To buy you a green dress or a monkey, I need to have a million dollars*" is $(g \vee m) \to d$. It is not logically equivalent to the given statement. If d is true and both g and m are false, then the given statement is false while this statement is true.

2. Let:

   "q: you do well on the Math 122 quizzes",

   "n: you have gone through the notes at least three times", and

   "t: you have done at least three of the old tests".

   Then the first statement is $q \to (n \wedge t)$ and the second statement is $(\neg n \wedge \neg t) \to \neg q$.

   The two statements are not logically equivalent. If q and n are true and t is false, then the first statement is false and the second one is true.

   (iii) Unclear/ can't tell. Suppose you have done at least three of the old quizzes but not gone through the notes at all. Then, the first statement is true if q is false and false if q is true, while the second statement is always true. Thus the two statements can give conflicting information.

3. Let s_1, s_2, s_3 be statements. Suppose $s_1 \to s_2$ and $s_2 \to s_3$ and $s_3 \to s_1$ are tautologies.

   (a) Suppose s_1 is true. Since $s_1 \to s_2$ is always true, s_2 is true.

   (b) Suppose s_1 is false. Since $s_3 \to s_1$ is always true, s_3 is false. By the same argument, since $s_2 \to s_3$ is always true, s_2 is false.

   (c) Yes s_1 and s_2 logically equivalent. By (a) and (b) the statement $s_1 \leftrightarrow s_2$ is a tautology.

   (d) Yes. The argument is identical to the argument in (c) except for the subscripts.

4. (a) We have $p \vee q \Leftrightarrow \neg p \to q$ and $p \wedge q \Leftrightarrow \neg\neg(p \wedge q) \Leftrightarrow \neg(p \to \neg q)$.

   (b) We know every statement has a representation using $\wedge, \vee$ and $\neg$. By (a), statements involving $\wedge$ of $\vee$ have a representation using $\to$ and $\neg$. Thus, every statement has a representation using only the logical connective $\to$ and $\neg$.

   (c) We have $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow \neg[(p \to q) \to \neg(q \to p)]$ by (a).

5. (a) $(\neg a \to b) \wedge [\neg b \vee \neg(a \wedge b)]$

$$
\begin{array}{llr}
\Leftrightarrow & (a \vee b) \wedge [\neg b \vee (\neg a \vee \neg b)] & \text{Known L.E., DeMorgan} \\
\Leftrightarrow & (a \vee b) \wedge [(\neg b \vee \neg b) \vee \neg a)] & \text{Commutative, Associative} \\
\Leftrightarrow & (a \vee b) \wedge [\neg b \vee \neg a] & \text{Idempotent} \\
\Leftrightarrow & [(a \vee b) \wedge \neg b) \vee [(a \vee b) \wedge \neg a] & \text{Distributive} \\
\Leftrightarrow & [(a \wedge \neg b) \vee (b \wedge \neg b)] & \\
 & \quad \vee [(a \wedge \neg a) \vee (b \wedge \neg a)] & \text{Distributive} \\
\Leftrightarrow & [(a \wedge \neg b) \vee \mathbf{0}] \vee [\mathbf{0} \vee (b \wedge \neg a)] & \text{Known Contradictions} \\
\Leftrightarrow & (a \wedge \neg b) \vee (b \wedge \neg a) & \text{Identity} \\
\Leftrightarrow & \neg(a \to b) \vee \neg(b \to a) & \text{Known L.E.} \\
\Leftrightarrow & \neg[(a \to b) \wedge (b \to a)] & \text{DeMorgan} \\
\Leftrightarrow & \neg(a \leftrightarrow b) & \text{Known L.E.}
\end{array}
$$

   (b) $p \wedge [(\neg q \leftrightarrow p) \wedge q]$

$$
\begin{array}{llr}
\Leftrightarrow & p \wedge ([(q \vee p) \wedge (\neg p \vee \neg q)] \wedge q) & \text{Known L.E.} \\
\Leftrightarrow & (p \wedge q) \wedge [(q \vee p) \wedge \neg(p \wedge q)] & \text{Commutative, Associative, DeMorgan} \\
\Leftrightarrow & [(p \wedge q) \wedge \neg(p \wedge q)] \wedge (q \vee p) & \text{Commutative, Associative} \\
\Leftrightarrow & \mathbf{0} \wedge (q \vee p) & \text{Known Contradiction} \\
\Leftrightarrow & \mathbf{0} & \text{Dominance}
\end{array}
$$

   Therefore the given statement is a contradiction.