MATH 100 / MATH 109

Calculus I / Introduction to Calculus

2020 - 2021

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Contents

Introduction

Welcome to MATH 100 (Calculus I) or MATH 109 (Introduction to Calculus), 2020 - 2021. This Course Handbook is written to be compatible with both courses. In this course handbook, you will find:

- A pre-calculus review chapter to help you assess your readiness.
- Worksheets that might be used in your MATH 100 or MATH 109 tutorials, or can be used for additional practice.
- Checklists of detailed learning objectives from each chapter.
- Practice problems (with answers or hints) for you to use to check your understanding as you learn the material.
- Chapter tests (and answers) for you to use when you review chapters, or at the end of the term as you study for the final exam.

Note:

You will sometimes see text displayed like this! These notes direct your attention to various features of the course handbook, or advise you about how to use them.

This course handbook does not replace your textbook, because it contains very little explanation of material. For explanations, you will need to attend your lectures and read the book. This course handbook does help you focus your attention on the relevant portions of the textbook, however. Some material from some sections of your textbook will not be covered in this course! Refer to the **Chapter Checklist** in each chapter to see which material you are responsible for. If you are ever in doubt, ask your instructor.

Note:

Ask your instructor about any discrepancies between the textbook, this course handbook, and your lecture!

How to use this course handbook

The Pre-Calculus chapter is organized differently from the main content of this course handbook. Read through this chapter to determine which topics you need to review; for best results, do this before classes begin.

The four main chapters are ℓ**Limits**, $\triangle$**Derivatives**, $\cup$**Using Derivatives**, and $\int$**Integrals**. They are organized to be used in the following way:

- **Before you begin:** read the pre-calculus material that is listed as prerequisite for this section, and make sure you are comfortable with it. Refer to the $\pm$**Pre-Calculus** chapter for more information.
- When included, any **Additional Material** is intended to be read with the textbook. This is material that your instructors would add to the textbook if they could.
- Read the **Chapter Checklist** before you read the textbook or attend lectures on the topics. This will help to direct your note-taking and focus your attention on what you need to learn.
- After reading the textbook and participating in lecture, try the **Check Your Understanding** problems for that chapter and make note of the problems you cannot yet solve.
- If after participating in all of the relevant lectures you still cannot solve all of the **Check Your Understanding** problems then you should seek additional help.
- After participating in all of the relevant tutorials you will know which **Worksheets** are not being used for your class this term. You can use the other worksheets for additional practice.
- If you wish to test your understanding of an entire chapter, make use of the practice tests in the **Review** chapter at the end of this course handbook.

Note:

Although we include answers and hints for all of the **Check Your Understanding** problems, there are no answers available for the **Worksheets**. Please ask your instructor for help if you get stuck.

$\pm$ Pre-Calculus

Note:

The material in this section corresponds in part to Chapter 1 in your textbook, but really includes a selection of material that is of particular importance to learning calculus. This means we might omit some topics from Chapter 1 and might also include topics you learned even before taking a 'precalculus' course.

Precalculus Skills

(FC) Functions: concepts

Central to Calculus is an understanding of functions and their properties.

1. Understand the definition of a function, and apply it in places where the "vertical line test" does not apply (such as x being a function of y).
2. Be able to determine the domain of any of the functions listed below (FE).
3. Be able to find the x-intercepts and y-intercept of any of the functions listed below (FE).
4. Understand functions as things that vary, including the notion of "$f(x)$ increases as x decreases" and "increasing" negative numbers.
5. Understand function composition, including notation, domain, range, and the definition of even and odd functions.
6. Understand what an inverse function is, and how to prove that a given function does or does not have an inverse.

(FE) Functions: examples

Some functions come up frequently, so it is important that you are comfortable working with them.

1. Basic functions: lines, parabolas, cubic functions, square roots.
2. More advanced functions: $|x|$, x^n, $x^{1/n}$, e^x, $\log_b(x)$, $\ln(x)$.
3. Trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\csc(x)$, $\sec(x)$, $\cot(x)$, $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$. (You should also know the sine, cosine, and tangent of the standard angles in radians.)

(N) Notation

In order to work with functions we need to be able to write about them. To do that, you need to be comfortable with:

1. Interval notation (for example, when describing the domain of a function).
2. Function composition notation (two forms: $f(g(x))$ is more common in calculus than $(f \circ g)(x)$).
3. The symbol "$=$" (never put it between two things that are not equal).
4. The symbol "∞" (which is not a number).

(A) Algebra

Once you understand what expressions and functions are and how to write about them with appropriate notation, you can manipulate them algebraically.

1. Understand how transformations affect the algebraic definition of a function.
2. Be able to compute the algebraic definition of a function that is formed by transforming a given function.
3. Be able to rationalize expressions like $\frac{1}{\sqrt{x}-10}$ with conjugate multiplication.
4. Be able to factor any polynomial of degree at most 2, perhaps by using the quadratic formula.
5. Be able to determine whether a is a root of a given polynomial.
6. Be able to use polynomial long division to express a rational function in quotient/remainder form.
7. Be able to explain why two expressions are or are not equal.
8. Be able to explain why two equations are or are not equivalent.
9. Be able to express the set of points represented by a graph as several equivalent equations. (Example: point/slope form of a line and slope/intercept for of the same line)
10. Be able to simplify an expression and identify resulting change of domain in the corresponding functions (for example x^2/x compared with x).

Before You Begin

Note:

Now that you know which topics are particularly important for this Calculus course, you can take the following True/False quiz to help identify areas you particularly need to review. Answers are in the next section, along with information about which topic each question relates to.

True or False?

For each of the following statements, determine whether it is True or False. If True, explain why. If False, provide a counterexample.

1. $f(x) = \pm\sqrt{x}$ is a function.
2. $g(x) = \sqrt{x}$ is a function.
3. The parabola $y = 2(x+4)^2 - 1$ has y-intercept -1.
4. The curve $y = e^x$ crosses the x-axis at the point $(1, 0)$.
5. The curve $y = \ln(x)$ crosses the x-axis at the point $(1, 0)$.
6. $2^a = e^{a\ln(2)}$
7. The function $\sqrt{x}$ increases as x increases.
8. $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$ for all x.
9. The function $y = \sin(x)$ is one-to-one and therefore has an inverse.
10. $\cot(x) = \cos(x)\csc(x)$.
11. The line $y - 4 = 2(x+1)$ is perpendicular to the line $y - 4 = \frac{1}{2}(x+1)$.
12. The line $y - 2 = 4(x+3)$ is perpendicular to the line $y = -\frac{1}{4}x$.
13. The domain of $\sqrt{4 - x^2}$ is $(-\infty, -2] \cup [2, \infty)$.
14. $\dfrac{1}{\sqrt{2x} - 3} = \dfrac{\sqrt{2x} + 3}{2x - 9}$.
15. The polynomial $3x^4 + x^3 - 4x^2 + 5x - 5$ has root $x = 1$.
16. If 2 is a root of the polynomial $p(x)$ then $(x+2)$ must be a factor of $p(x)$.
17. The functions $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x + 2$ are the same functions.
18. The curve $y = f(x+2) + 4$ is the curve $y = f(x)$ shifted up 4 units and to the right 2 units.
19. Every polynomial of degree 4 is an even function.
20. If $f(x)$ is not an even function then it is an odd function.
21. If $f(x)$ is an even function then it cannot also be an odd function.
22. If the domain of $f(x)$ is $(-\infty, \infty)$, then the functions $f(x)$ and $f(x+2)$ have the same range.

Before You Begin: Answers

Note:

To aid your review, the labels below each answer indicate which learning objective(s) that problem tested.

1. False: the input $x = 4$ has two outputs, 2 and -2.
   FC1, FE1
2. True: the function $\sqrt{x}$ outputs the positive square root of x for any $x \geq 0$.
   FC1, FE1
3. False: this parabola crosses the y-axis at the point $(0, 31)$.
   FC3, FE1
4. False: the curve $y = e^x$ never crosses the x-axis because $e^x > 0$ for all x.
   FC3, FE2
5. True: $\ln(1) = 0$ because $e^0 = 1$.
   FC3, FE2
6. True: this follows from logarithm and exponent rules.
   FE2, A7
7. True: if $a > b$ then $\sqrt{a} > \sqrt{b}$.
   FC4, FE1
8. False: if $x < 0$ then $\sqrt{x^2} = -x$. Rather, $\sqrt{x^2} = |x|$. Also, the domain of $(\sqrt{x})^2$ is only $[0, \infty)$ while the domain of x is $(-\infty, \infty)$.
   FC5, N2, A7
9. False: We define $\arcsin(x)$ to be the inverse of the function "$\sin(x)$ with domain $[-\pi/2, \pi/2]$".
   FC6, FE3
10. True, because $\csc(x) = \frac{1}{\sin(x)}$.
    FE3
11. False, but it is perpendicular to the line $y - 4 = -\frac{1}{2}(x + 1)$.
    FE1
12. True.
    FE1
13. False: the domain is $[-2, 2]$.
    FC2, FE5, FE2
14. True: the expressions are equal and as functions they have the same domain.
    A3, A7
15. True, because $3x^4 + x^3 - 4x^2 + 5x - 5 = 0$ if $x = 1$.
    A5
16. False, but $(x - 2)$ must be a factor of $p(x)$.
    A5
17. False: the function f has domain $(-\infty, 2) \cup (2, \infty)$ while the function g has domain $(-\infty, \infty)$.
    A10
18. False: it is shifted up 4 units and left 2 units.

A1

19. False: $f(x) = x^4 + x$ is not even.
    FC5
20. False: $f(x) = x^4 + x$ is neither even nor odd.
    FC5
21. False: $f(x) = 0$ is both even and odd – but it is the only such function!
    FC5
22. True: if y_0 is in the range of $f(x)$ then $f(x_0) = y_0$ for some x_0. This means that $f((x_0-2)+2) = y_0$, and so y_0 is also in the range of $f(x+2)$. Graphically speaking, $y = f(x+2)$ is obtained from $y = f(x)$ by shifting it left two units, which does not affect the range.
    A2, FC5

Note:

Now that you've identified areas you need to review, use your textbook and MyLab Math study plan to brush up those skills. Chapter 1 in your textbook reviews functions (including trigonometric functions, exponential functions, logarithmic functions, and inverse functions) and the "Getting Ready" chapter in your MyLab Math study plan reviews algebra.

Test Yourself

Note:

Now that you have reviewed the material, here are some problems to help you test your understanding of it.

1. Determine whether each of the following rules is a function.
   (a) Domain: all real numbers. Rule: assign to each real number the square of that number.
   (b) Domain: all people in Victoria. Rule: assign to each person that person's uncle.
   (c) Domain: all buildings at UVic. Rule: assign to each building the number of water fountains in the building.
2. Consider the function $g(x) = x^2 - 2x - 3$.
   (a) What is $g(0)$? For what values of x is $g(x) = 0$?
   (b) What is $g(2+h)$? Simplify your answer by multiplying out fully and collecting like terms.
   (c) What is $\dfrac{g(2+h) - g(2)}{h}$? Simplify your answer as much as possible.
   (d) What is $g\left(\dfrac{3}{q}\right)$? Express your answer as a single quotient.
3. Find the domain of each of the following composite functions. Write your answer in interval notation:
   (a) $f(x) = \sqrt{4 - x^2}$
   (b) $g(x) = \dfrac{1}{\sqrt[3]{x^2 - 3} - 2}$
   (c) $h(x) = a(b(x))$, where $a(x) = \dfrac{1}{x+2}$ and $b(x) = \dfrac{x}{x-3}$
4. Graph the function below.
$$f(x) = \begin{cases} 12 - x, & x \leq 2 \\ 5x, & x > 2 \end{cases}$$
5. A function $f(x)$ is *even* if $f(-x) = f(x)$ for every x and *odd* if $f(-x) = -f(x)$ for every x. Determine whether the given function is even, odd, or neither:
$$h(x) = \frac{-9x^3}{7x^2 + 6}$$
6. The function $\cos(x)$ is not one-to-one, and yet we often talk about its inverse function $\cos^{-1}(x)$ (also known as $\arccos(x)$).
   (a) Explain why $\cos(x)$ is not one-to-one by finding two real numbers x_1 and x_2 such that $x_1 \neq x_2$ and $\cos(x_1) = \cos(x_2)$.

(b) What is the domain of $\cos(x)$? What is the range of $\cos(x)$?

(c) Sketch $y = \cos(x)$ on the interval $-2\pi \leq x \leq 2\pi$. Illustrate your answer to part (3a) by drawing the appropriate horizontal line through your sketch.

(d) What is the largest number a for which $\cos(x)$ *is* one-to-one on the interval $[0, a]$?

(e) Sketch the inverse function, $\arccos(x)$, which has range $[0, a]$ (see part (3d)). What is the domain of $\arccos(x)$?

7. Let $f(x) = \frac{x}{x-7}$. Find a function $g(x)$ so that $f(g(x)) = x$.

8. Consider $h(x) = e^{-x} + 2$.

(a) What is the domain and range of $h(x)$?

(b) Is $h(x)$ one-to-one? If not, prove it. If so, find its inverse $h^{-1}(x)$. What is the domain and range of h^{-1} (if it exists)?

(c) Sketch $y = h(x)$. If $h(x)$ has an inverse, sketch $y = h^{-1}(x)$ on the same axes.

9. Even and Odd functions.

(a) Suppose that $f(x)$ is an even function with domain $(-\infty, \infty)$ and $g(x)$ is an odd function with domain $(-\infty, \infty)$. For each of the following functions, determine whether it is *even*, *odd*, or neither. Explain your answers.

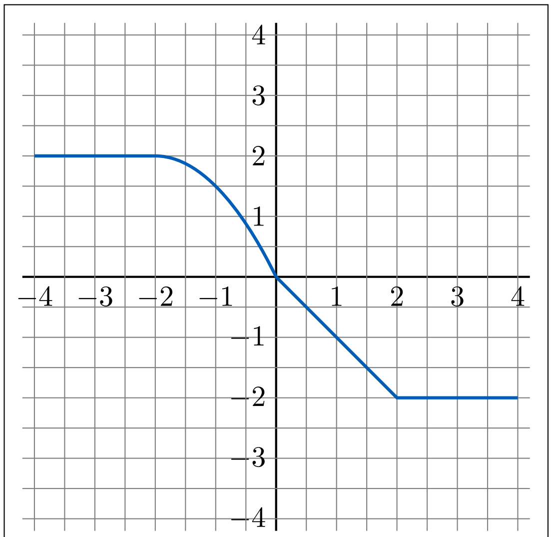
$$f(f(x)) \qquad g(g(x)) \qquad f(f(f(x))) \qquad g(g(g(x)))$$
$$f(g(x)) \qquad g(f(x)) \qquad f(x)g(x) \qquad f(x) + g(x)$$

(b) Is $g(x) = 2$ an even function, an odd function, or neither?

10. Find the domain of the following function. Express your answer in *interval* notation.

$$y = \frac{(x+2)(x+9)}{(x+9)\sqrt{x^2 - 25}}.$$

11. Given the sketch of $y = f(x)$ below, sketch $y = f(|x - 2|)$.



$y = f(x)$

12. Determine algebraically whether the following function is even, odd, or neither.

$$y = \frac{x^2+1}{10x}$$

13. Given $f(x) = \sqrt{x+2}$ and $g(x) = 4 - x$, determine the domain of each of the following functions:

    (a) $f(x) + g(x)$.
    (b) $\frac{f(x)}{g(x)}$.
    (c) $\frac{g(x)}{f(x)}$.
    (d) $f(g(x))$.
    (e) $g(f(x))$.

14. Explain why 5 is in the domain of $\sqrt{25-x^2}$ but is not in the domain of $\frac{x-5}{\sqrt{25-x^2}}$.

15. If $p(x) = ax^2 + bx + c$ and $a \neq 0$ then the roots of $p(x)$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, when defined. Using the method "completing the square" to solve the equation $0 = ax^2 + bx + c$, explain why.

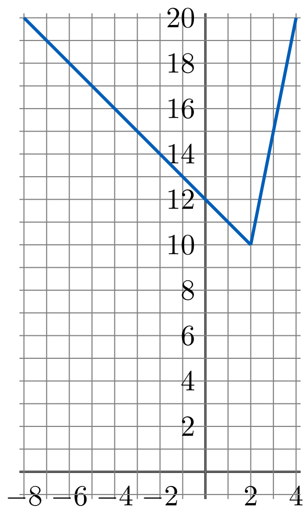
Test Yourself: Answers

1. The first is a function. The second is not, since people might have more than one uncle (the wording is poor, but that is deliberate). The third is a function.

2. This problem is about function notation primarily and order-of-operations secondarily. $g(0) = -3$, $g(-1) = 0$, $g(3) = 0$. The quotient simplifies to $h + 2$ and $g\left(\frac{3}{q}\right) = \frac{9-6q-3q^2}{q^2}$.

3. Domain of $f(x)$ is $[-2, 2]$. Domain of $g(x)$ is $(-\infty, -\sqrt{11}) \cup (-\sqrt{11}, \sqrt{11}) \cup (\sqrt{11}, \infty)$. Domain of $h(x)$ is $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

4. Graph:



5. Don't focus too much on the exponents of a function when determining if it is even or odd. Instead, you must actually evaluate $h(-x)$ and determine whether it is equal to $-h(x)$ or $h(x)$ (in this case, it is equal to $-h(x)$ and so the function is odd).

6. There are infinitely many such pairs x_1, x_2. One example is $\cos(\pi/2) = \cos(3\pi/2)$, so $\cos(x)$ is not one-to-one. The domain of $\cos(x)$ is $(-\infty, \infty)$ and its range is $[-1, 1]$. The largest a for which $\cos(x)$ is one-to-one on the interval $[0, a]$ is $a = \pi$, which means that the function $\arccos(x)$ has range $[0, \pi]$ (and domain $[-1, 1]$).

7. $g(x) = 7x/(x - 1)$

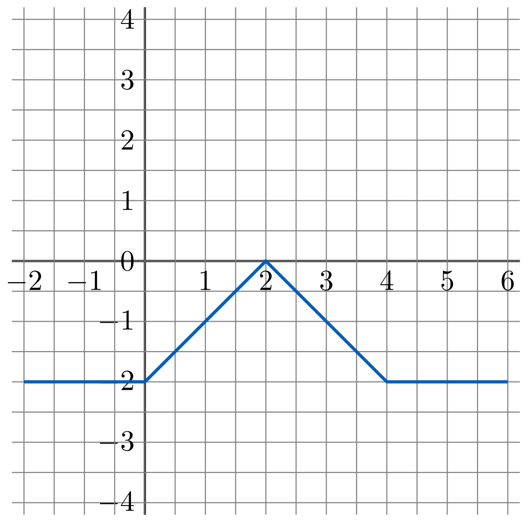
8. Domain is $(-\infty, \infty)$ and range is $(2, \infty)$. The function is indeed one-to-one and its inverse is $h^{-1}(x) = -\ln(x - 2)$, which some would prefer to write $\ln\left(\frac{1}{x-2}\right)$. Either is acceptable. A popular method to find an inverse function is to solve $x = e^{-y} + 2$ for y. Notice that this is exactly what we did in the previous problem, just replacing $g(x)$ with y.

9. $f(f(-x)) = f(f(x))$, EVEN. $g(g(-x)) = g(-g(x)) = -g(g(x))$, ODD. $f(f(f(x)))$ is EVEN. $g(g(g(x)))$ is ODD. You might be tempted to work this out "by example", which is helpful for brainstorming but would not be accepted as a solution. $f(g(-x)) = f(-g(x)) = f(g(x))$,

EVEN. $g(f(-x)) = g(f(x))$, EVEN. $f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x)$, ODD. $f(-x)+g(-x) = f(x) - g(x)$, NEITHER except in certain cases, such as if $f(x) = -1$, $g(x) = 1$. The function $g(x) = 2$ is even.

10. $(-\infty, -9) \cup (-9, -5) \cup (5, \infty)$.

11. This is $y = f(|x|)$, shifted two units to the right.



$y = f(|x - 2|)$

12. Odd: $\frac{(-x)^2+1}{10(-x)} = \frac{x^2+1}{-10x} = -\frac{x^2+1}{10x}$.

13. Given $f(x) = \sqrt{x+2}$ and $g(x) = 4 - x$, determine the domain of each of the following functions:

    (a) $f(x) + g(x)$ has domain $x \geq -2$.
    (b) $\frac{f(x)}{g(x)}$ has domain $[-2, 4) \cup (4, \infty)$.
    (c) $\frac{g(x)}{f(x)}$ has domain $x > -2$.
    (d) $f(g(x))$ has domain $x \leq 6$.
    (e) $g(f(x))$ has domain $x \geq -2$.

14. $\sqrt{25 - 5^2} = 0$, so 5 is in the domain of $\sqrt{25 - x^2}$ but $\frac{x-5}{\sqrt{25-x^2}}$ is undefined at $x = 5$.

15. If $p(x) = ax^2 + bx + c$ and $a \neq 0$ then

$$\begin{aligned} 0 &= ax^2 + bx + c \\ &= a\left(x^2 + \tfrac{b}{a}x + \tfrac{c}{a}\right). \end{aligned}$$

Now, in order to 'complete the square' we want to rewrite the expression $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ so it looks like $a(x+k)^2 + d$ for some constant k. Because $(x+k)^2 = x^2 + 2kx + k^2$, that means we should rewrite the above equation as

$$\begin{aligned}
0 &= a\left(x^2 + 2\left(\tfrac{b}{2a}\right)x + \left(\tfrac{b}{2a}\right)^2 - \left(\tfrac{b}{2a}\right)^2 + \tfrac{c}{a}\right) \\
&= a\left(x^2 + 2\left(\tfrac{b}{2a}\right)x + \left(\tfrac{b}{2a}\right)^2\right) - a\left(\left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a}\right) \\
&= a\left(x + \tfrac{b}{2a}\right)^2 - a\left(\left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a}\right).
\end{aligned}$$

Now we have an equation that we can solve for x:

$$\begin{aligned}
0 &= a\left(x + \tfrac{b}{2a}\right)^2 - a\left(\left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a}\right) \\
a\left(x + \tfrac{b}{2a}\right)^2 &= a\left(\left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a}\right) \\
\left(x + \tfrac{b}{2a}\right)^2 &= \left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a} \\
\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2}{4a^2} - \tfrac{c}{a} \\
\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a^2} \\
x + \tfrac{b}{2a} &= \pm\sqrt{\tfrac{b^2-4ac}{4a^2}} \\
x &= -\tfrac{b}{2a} \pm \sqrt{\tfrac{b^2-4ac}{4a^2}} \\
x &= -\tfrac{b}{2a} \pm \tfrac{\sqrt{b^2-4ac}}{2a} \\
x &= \tfrac{-b\pm\sqrt{b^2-4ac}}{2a}.
\end{aligned}$$

Do you really need to know how to re-derive the quadratic formula? No, memorizing it is enough – and a good idea. It is, however useful to be able to complete the square of a quadratic polynomial because it is easier to graph $y = a(x+k)^2 + d$ than it is to graph $y = ax^2 + bx + c$.

Note:

That concludes our Precalculus review; remember you can access more problems in Chapter 1 of your textbook as well as the “Getting Ready” and “Chapter 1” sections of your MyLab Math Study Guide. Still feeling unprepared? You might consider enrolling in MATH 120 for a comprehensive review of precalculus material. Talk to your instructor or an advisor if you are unsure what option is best for you.

ℓ Limits

In your textbook: *This material corresponds to Chapter 2, sections 2.1 - 2.2 & 2.4 - 2.6*

Before you begin: this chapter relies on the following pre-calculus material

- Transformations of functions (A1,A2)
- Domain of a function (FC2, FC5, A10)
- Standard functions (FE)
- Conjugate multiplication (A3)
- Polynomial factorization (A4, A5)
- Polynomial division (A6)
- Notation (N)

Additional Material

Note:

Sometimes we will deviate from or add to the material in your textbook. When that happens, there will be an "Additional Material" section in this course pack, like this one.

Your textbook gives you theorems and examples about how to evaluate limits or explain why a particular limit does not exist. Here are some more terms that your instructors are likely to use when talking about limits:

Definition 1. If $\lim\limits_{x\to a} f(x) = L$, then we say "$f(x)$ *converges to* L as x approaches a". If $\lim\limits_{x\to a} f(x)$ does not exist then we say "$f(x)$ *does not converge* or *diverges* as x approaches a".

Because we will be skipping the precise definition of a limit (section 2.3 in your textbook), the definitions in 2.4 of the textbook won't make sense to you! We will use the following definitions instead.

Definition 2. We say that $\lim\limits_{x\to c^-} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to, but less than, c.

Definition 3. We say that $\lim\limits_{x\to c^+} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close to, but greater than, c.

For the same reason, we need to use alternative definitions for section 2.6 as well:

Definition 4. We say that $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently large.

Definition 5. We say that $\lim_{x\to-\infty} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently small.

Notice that in Definition 5 "sufficiently small" refers to numbers that are very negative. For example, $-10,000,000 < -1,000,000$. If you prefer, you can replace the term "sufficiently small" with "sufficiently negative" or "negatively large" when you are thinking about this definition.

Definition 6. We say that $\lim_{x\to a} f(x) = \infty$ if for any positive number N we can make $f(x) \geq N$ by taking x sufficiently close to a.

Definition 7. We say that $\lim_{x\to a} f(x) = -\infty$ if for any negative number M we can make $f(x) \leq M$ by taking x sufficiently close to a.

Definition 8. We say that $\lim_{x\to\infty} f(x) = \infty$ if for any positive number N we can make $f(x) \geq N$ by taking x sufficiently large, and we say that $\lim_{x\to-\infty} f(x) = \infty$ if for any positive number N we can make $f(x) \geq N$ by taking x sufficiently small.

Definition 9. We say that $\lim_{x\to\infty} f(x) = -\infty$ if for any negative number M we can make $f(x) \leq M$ by taking x sufficiently large, and we say that $\lim_{x\to-\infty} f(x) = -\infty$ if for any negative number M we can make $f(x) \leq M$ by taking x sufficiently small.

Notice that $\lim_{x\to a} f(x) = \infty$ means that $\lim_{x\to a} f(x)$ *does not exist*, but fails to exist in a particular way. To remember this, get in the habit of pronouncing $\lim_{x\to a} f(x) = \infty$ as "$f(x)$ diverges to ∞ as x approaches a."
Finally, your book does not precisely define what an *oblique* (a.k.a. *slant*) asymptote is, although it does include a description of how to find a slant asymptote for the special case of a rational function. We will use the following definition for slant asymptotes in general:

Definition 10. Given real numbers a and b, where $a \neq 0$, we say that $y = ax + b$ is an *oblique asymptote* (also called a *slant asymptote*) of $y = f(x)$ if $f(x) \neq ax + b$ and

$$\lim_{x\to\infty} ((ax+b) - f(x)) = 0 \text{ or } \lim_{x\to-\infty} ((ax+b) - f(x)) = 0$$

For example, if $f(x) = 2x + 3 + \frac{1}{x}$ and $g(x) = 2x + 3$ we would say that $f(x)$ has a slant asymptote of $y = 2x + 3$ but $g(x)$ does not. Notice that (as your book says on page 107)if the degree of the numerator of a rational function is one more than the degree of its denominator then the rational function has a slant asymptote. However, other functions can also have slant asymptotes.

Chapter Checklist

Note:

In each chapter of this course handbook, we will include a summary of facts and concepts you are expected to know and understand from each section of your textbook. These are intended to help guide your reading, particularly since some chapters include material we won't be covering in this course. If you are ever in doubt about whether material in your textbook is required, ask your lecture instructor.

2.1: Rates of Change and Tangents to Curves

- You should know…
  - ☐ what the average rate of change of a function is.
  - ☐ what the instantaneous rate of change of a function is.
- You should understand…
  - ☐ how average and instantaneous rates of change relate to secant and tangent lines.
- You should be able to compute…
  - ☐ the average rate of change of a given function over a given interval.

2.2: Limits

- You should know…
  - ☐ the seven Limit Laws listed in your textbook.
  - ☐ the limit of a polynomial $p(x)$ as x approaches a constant.
  - ☐ the limit of a rational function $p(x)/q(x)$ as x approaches a constant c for which $q(c) \neq 0$.
  - ☐ the Sandwich Theorem (also known as the Squeeze Theorem).
- You should understand…
  - ☐ the meaning of $\lim_{x\to a} f(x) = L$.
  - ☐ how jump discontinuities and removable discontinuities relate to limits.
  - ☐ the difference between, for example, the functions $\frac{(x-1)(x+1)}{x-1}$ and $x+1$.
  - ☐ the Sandwich Theorem.
- You should be able to compute…
  - ☐ Limits of the form $\lim_{x\to a} c$, where c is a real number.
  - ☐ Limits of the form $\lim_{x\to a} p(x)$, where $p(x)$ is a polynomial.
  - ☐ Limits of the form $\lim_{x\to a} \frac{p(x)}{q(x)}$, where p and q are polynomials and after simplifying there is no remainder.
  - ☐ limits based on a graph of a function.
  - ☐ limits involving radicals that can be rationalized. **Example 9**
  - ☐ limits using the Sandwich Theorem. **Examples 10 and 11**

2.4: One-sided limits

- You should know...
  - ☐ the relationship between limits and one-sided limits.
  - ☐ that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (you do not need to be able to prove this).
- You should understand...
  - ☐ the meaning of a one-sided limit.
- You should be able to compute...
  - ☐ one-sided limits from a graph.
  - ☐ one-sided limits of a piecewise-defined function.
  - ☐ one-sided limits of anything for which you can compute a limit.

2.5: Continuity

- You should know...
  - ☐ the definition of "continuous at a point".
  - ☐ the definition of "continuous on a closed interval".
  - ☐ which common functions (such as polynomials, e^x, $\ln(x)$, trig functions, rational functions, etc.) are continuous on which intervals.
  - ☐ the Intermediate Value Theorem.
  - ☐ what a removable discontinuity is.
- You should understand...
  - ☐ what it means to be continuous.
  - ☐ when the Intermediate Value Theorem applies (and when it doesn't).
- You should be able to compute or prove...
  - ☐ that a function is (or is not) continuous at a point.
  - ☐ that a function is (or is not) continuous on a closed interval.
  - ☐ the "continuous extension to a point" of a function with a removable discontinuity.

2.6: Limits involving infinity

- You should know...
  - ☐ the "infinity versions" of the Limit Laws.
  - ☐ what a horizontal asymptote is.
  - ☐ what a vertical asymptote is.
  - ☐ what a slant (also known as oblique) asymptote is.
- You should understand...
  - ☐ the meaning of $\lim_{x\to a} f(x) = \infty$.
  - ☐ the meaning of $\lim_{x\to a} f(x) = -\infty$.
  - ☐ the meaning of $\lim_{x\to \infty} f(x) = L$.
  - ☐ the meaning of $\lim_{x\to -\infty} f(x) = M$.
  - ☐ the relationship between limits and asymptotes.
- You should be able to compute...

- ☐ any limit involving a rational function.
- ☐ limits involving rationalizing a numerator.
- ☐ the asymptotes of any rational function (vertical, horizontal, and slant).

Check Your Understanding: Limits

Note:

In each chapter, we provide some additional practice problems for you to think about. These are typically problems that don't translate too well to the online homework system, and are based on the entire chapter's material (and sometimes earlier material as well). **Answers and hints can be found at the end of the chapter**, and you are encouraged to ask for help in finding solutions.

True or False?

For each of the following statements, determine whether it is True or False. If True, explain why. If False, provide a counterexample.

Note:

Use the True or False questions to determine whether you **understand** definitions and theorems from the material. Some of them are tricky!

1. If $f(a)$ exists then $\lim\limits_{x\to a} f(x)$ exists.
2. If $\lim\limits_{x\to 0} f(x)g(x)$ exists then so do $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$.
3. If $\lim\limits_{x\to 0} f(x)$ exists then so do $\lim\limits_{x\to 0^-} f(x)$ and $\lim\limits_{x\to 0^+} f(x)$.
4. If $\lim\limits_{x\to 0^-} f(x)$ and $\lim\limits_{x\to 0^+} f(x)$ both exist then so does $\lim\limits_{x\to 0} f(x)$.
5. If $\lim\limits_{x\to 0} g(x) = \infty$ then $g(x)$ is discontinuous at $x = 0$.
6. If $y = h(x)$ has a slant asymptote at $y = 3x + 4$ then $\lim\limits_{x\to\infty} (h(x) - 3x) = 4$.

Find a Function:

Note:

Use the Find a Function problems to determine whether you understand how the concepts in this chapter **relate** to functions. These almost always have fairly simple solutions, so look for the least-complicated function you can find that has the required properties.

1. Find a function that has a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 2$.
2. Find a function that has a vertical asymptote at $x = -4$ and a slant asymptote at $y = 10 - x$.
3. Find a function that is continuous everywhere except at $x = 1$, where it has a removable discontinuity.
4. Find a function $f(x)$ that is discontinuous at $x = 2$ for which $f(2) = 0$ and $\lim_{x\to 2} f(x) = 2$.

Explain Your Answer:

Note:

Use the Explain Your Answer problems to determine whether you can **show your work** to support the sorts of computations and applications we expect. These are usually a mix of conceptual problems (which means they might be 'hard'), computational problems that we know students sometimes struggle to write solutions to, and computational problems that just don't work well in MyMathLab.

1. Using the fact that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, explain why $\lim_{x\to 0}\frac{\sin(7x)}{\sin(5x)} = \frac{7}{5}$.
2. Explain why the function $y = \sqrt{1+x^2}$ has slant asymptote $y = x$.
3. Use the Sandwich Theorem to explain why $\lim_{x\to\infty}\frac{\sin(x^2)}{x^2} = 0$.
4. If $\lim_{x\to 0^-} h(x) = 2$ and $\lim_{x\to 0^+} h(x) = 13$, explain why $\lim_{x\to 0} h(x)$ does not exist but $\lim_{x\to 0} h(x^2) = 13$.
5. Suppose that $y = g(x)$ is an odd function[a] and $g(x)$ is continuous on $(-\infty, \infty)$. Explain why $\lim_{x\to 0} g(x) = 0$.
6. Suppose that $y = f(x)$ is a continuous function, and we know that $f(x)$ is negative at $x = 2$ and positive at $x = 11$. Explain why the equation $f(x) = 0$ has at least one solution on the interval $[2, 11]$. Is it possible that $f(x) = 0$ has more than one solution on that interval? Illustrate your answer with a graph.
7. Explain why $\lim_{x\to 0}\frac{|x|}{x}$ does not exist. Illustrate your explanation with a sketch, but make sure your explanation does not rely on the sketch.
8. If $\lim_{x\to 7}\frac{f(x)-8}{x-7} = 0$, then $\lim_{x\to 7} f(x) = 8$. Explain why, using the limit laws carefully.

[a]Recall that a function $g(x)$ is *odd* if $g(-x) = -g(x)$ for every x in the function's domain; see $\pm$ Pre-Calculus, FC5.

Worksheets

Note:

Each chapter of this course pack comes with several worksheets. The worksheet actually used in tutorials will be one of these, or sometimes a combination, depending on student performance on the online homework and in lectures.

Getting Started with Limits

1. If $\lim_{x\to 3} f(x) = 14$ and $\lim_{x\to 3} g(x) = -2$, compute

$$\lim_{x\to 3}\left(3f(x)g(x) + \frac{g(x)}{(f(x))^2} - \sqrt{14f(x)} + \ln(3+g(x))\right).$$

2. Consider the function $g(x) = \sin(\frac{\pi}{x})$.

   (a) Evaluate $g\left(\frac{1}{10}\right)$, $g\left(\frac{1}{100}\right)$, and $g\left(\frac{1}{1000}\right)$. If you had only computed these values, what would you guess is the value of $\lim\limits_{x\to 0} g(x)$?

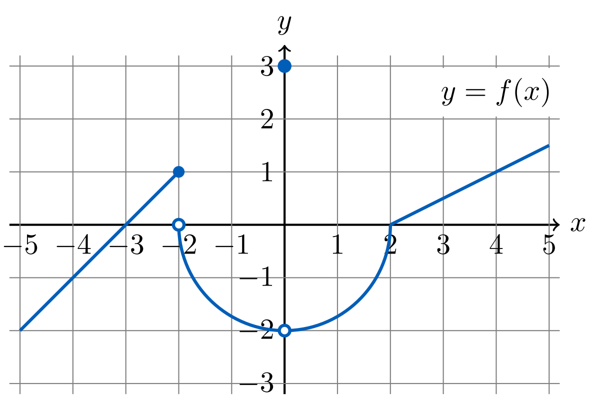
   (b) Evaluate $g\left(\frac{3}{10}\right)$, $g\left(\frac{3}{100}\right)$, and $g\left(\frac{3}{1000}\right)$. If you had only computed these values, what would you guess is the value of $\lim\limits_{x\to 0} g(x)$?

   (c) Explain why $\lim\limits_{x\to 0} g(x)$ does not exist.

   (d) Use the Sandwich Theorem to evaluate $\lim\limits_{x\to 0} x^2 g(x)$. *Hint: this limit does exist!*

3. Find functions $f(x)$ and $g(x)$ so that neither $\lim\limits_{x\to 2} f(x)$ nor $\lim\limits_{x\to 2} g(x)$ exists, but $\lim\limits_{x\to 2} \frac{f(x)}{g(x)}$ does exist.

4. Consider the function graphed below.



What is $\lim\limits_{x\to -2^-} f(x)$? What is $\lim\limits_{x\to -2^+} f(x)$?

What is $\lim\limits_{x\to 0^-} f(x)$? What is $\lim\limits_{x\to 0^+} f(x)$?

What is $\lim\limits_{x\to 0^-} f(f(x))$? What is $\lim\limits_{x\to 0^+} f(f(x))$?

Explain why $\lim\limits_{x\to -2} f(x)$ does not exist.

Explain why $\lim\limits_{x\to 0} f(f(x))$ *does* exist.

Limits and Graphs

1. (a) Using definition of "absolute value", graph function $f(x) = \dfrac{|6-2x|}{x-3}$.

   Clearly state the domain and the range of $f(x)$.

   (b) Using the graph in part (a) determine the following limits, or explain why a particular limit does not exist:

   $\lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3^-} f(x)$, $\lim\limits_{x\to 3} f(x)$.

   (c) Using definition of "least integer function" or "ceiling function" $y = \lceil x \rceil$ on p.5 of the text-book, graph function $g(x) = \lceil x \rceil + \lceil -x \rceil$.

   Clearly state the domain and the range of $g(x)$.

   (d) Using the graph in part (c) determine the following limits, or explain why a particular limit does not exist:

   $\lim\limits_{x\to 2^+} g(x)$, $\lim\limits_{x\to 2^-} g(x)$, $\lim\limits_{x\to 2} g(x)$.

2. Sketch the graph of an example of a function h that satisfies all of the given conditions:

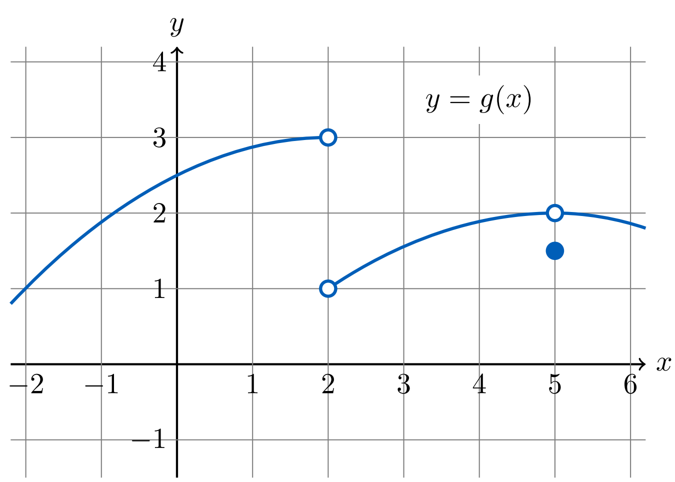
   $\lim\limits_{x\to 5^+} h(x) = 1$, $\lim\limits_{x\to 5^-} h(x) = 0$, $h(5) = 2$, $\lim\limits_{x\to -2} h(x) = 3$, $h(-2) = 1$.

3. Use the graph of a function g below to state the values (if they exist) of the following limits:

   (a) $\lim\limits_{x\to -2^-} (g(x) + 3g(-x))$ (b) $\lim\limits_{x\to -2^+} (g(x) + 3g(-x))$ (c) $\lim\limits_{x\to -2} (g(x) + 3g(-x))$

   (d) $\lim\limits_{x\to (\sqrt{2})^-} g(x^2)$ (e) $\lim\limits_{x\to 2} \sqrt{g(x+3)}$ (f) $\lim\limits_{x\to 5} g(g(x))$

   If a given limit does not exist, explain why.



Continued on next page...

Limits and Graphs, continued...

4. Victoria's new sewage treatment plant will spend C dollars a day to remove $p\%$ of the pollutants from the water before sending water into the ocean

   $C = \dfrac{25,000p}{100-p}$, $0 \leq p < 100$.

   (a) Find the cost of removing 20% of the pollutants.

   (b) Find the cost of removing 50% of the pollutants.

   (c) Find the cost of removing 90% of the pollutants.

   (d) Find the cost of removing 99.8% of the pollutants.

   (e) Calculate (and interpret) the limit of C as p approaches 100 from below.

   Disclamer: All the numbers and the cost formula in example 4 are not based on the expected costs for the actual sewage treatment plant.

Understanding Limits

1. If $\lim_{x\to 3} f(x) = 2$ and $\lim_{x\to 3} g(x) = -5$, compute

$$\lim_{x\to 3}\left(f(x)g(x) + \sqrt{f(x)^5} - e^{f(x)-g(x)+2} + \frac{f(x)}{2g(x)}\right).$$

2. Is "$\frac{0}{0}$" a number?

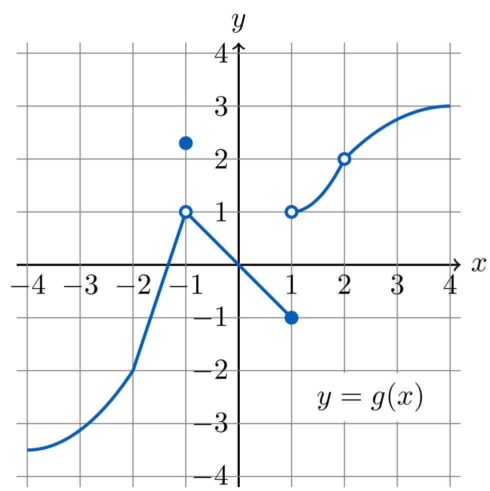
   (a) Notice that $\lim_{x\to 0}\frac{x^2}{x}$ looks like "$\frac{0}{0}$". Explain why $\lim_{x\to 0}\frac{x^2}{x} = 0$.

   (b) Notice that $\lim_{x\to 0}\frac{|x|}{x}$ looks like "$\frac{0}{0}$". Explain why $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

   (c) Notice that $\lim_{x\to 2}\frac{x-2}{\sqrt{x+14}-4}$ looks like "$\frac{0}{0}$". Explain why $\lim_{x\to 2}\frac{x-2}{\sqrt{x+14}-4} = 8$.

   (d) Find $f(x)$ and $g(x)$ so that $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$, and $\lim_{x\to 0}\frac{f(x)}{g(x)} = 63$.

3. The function $y = g(x)$ is graphed below. For each of the following limits, evaluate it or explain why it does not exist.



   (a) $\lim_{x\to -1^-} g(x)$, $\lim_{x\to -1^+} g(x)$, and $\lim_{x\to -1} g(x)$

   (b) $\lim_{x\to -1} g(x+3)$

   (c) $\lim_{x\to 0} 2g(x^2+1)$

   (d) $\lim_{x\to -1} g(g(x))$

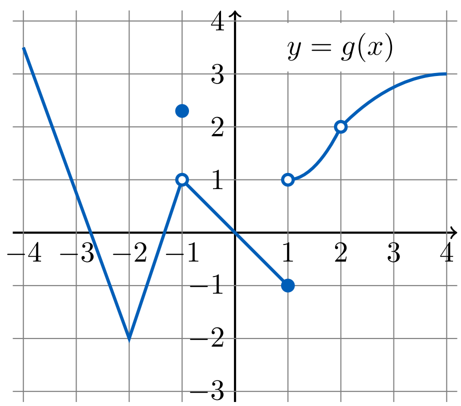
4. Find functions $f(x)$ and $g(x)$ so that neither $\lim_{x\to 1} f(x)$ nor $\lim_{x\to 1} g(x)$ exists, but $\lim_{x\to 1}(f(x)+g(x))$ does exist.

Crossing the road? Check both sides.

1. Evaluate the following limits.

$$\lim_{x\to 5^-} \frac{|5-x|}{x-5} \qquad \lim_{x\to 5^+} \frac{|5-x|}{x-5} \qquad \lim_{x\to 5} \frac{|5-x|}{x-5}$$

2. The function $y = g(x)$ is graphed below. Consider the function $f(x) = \dfrac{1}{g(x)}$. Find all points of discontinuity of the function $f(x)$. For each, explain why $f(x)$ is discontinuous there.

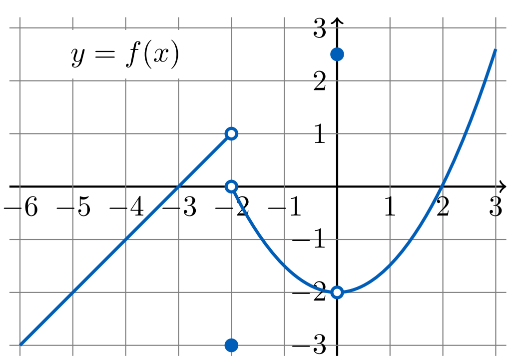


3. Each of the following solutions is incorrect. Explain what is incorrect about each of them, and find a correct solution. *Hint: the correct answer is* $4/5$.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{\sin(5x)} = \lim_{x\to 0} \frac{4x}{5x} = \lim_{x\to 0} \frac{4}{5} = \frac{4}{5}.$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{\sin(5x)} = \lim_{x\to 0} \frac{4\sin(x)}{5\sin(x)} = \lim_{x\to 0} \frac{4}{5} = \frac{4}{5}.$

4. Consider the function $y = f(x)$ graphed below.



What is $\displaystyle\lim_{x\to -2^-} f(x)$? What is $\displaystyle\lim_{x\to -2^+} f(x)$?

What is $\displaystyle\lim_{x\to 0^-} f(x)$? What is $\displaystyle\lim_{x\to 0^+} f(x)$?

What is $\displaystyle\lim_{x\to 0^-} f(f(x))$? What is $\displaystyle\lim_{x\to 0^+} f(f(x))$?

What is $\displaystyle\lim_{x\to -5^-} f(f(x))$? What is $\displaystyle\lim_{x\to -5^+} f(f(x))$?

Explain why $\displaystyle\lim_{x\to 0} f(f(x))$ *does* exist.

Does $\displaystyle\lim_{x\to -5} f(f(x))$ exist?

Tangents, slopes and asymptotes

1. The tangent line to the graph pf $y = h(x)$ at the point $(-2, 3)$ passes through the point $(4, -1)$. Find $h(-2)$ and $h'(-2)$.

2. Find equations of both tangent lines to the graph of $f(x) = x^2$ that pass through point $(x, y) = (-1, -3)$. Start by graphing the function f and drawing the correct tangent lines.

3. Calculate right-hand side and left-hand side derivatives of the function $g(x) = \lceil x \rceil$ at the point $x = 0$. Is function g differentiable at $x = 0$? List all reasons.

   If you are looking for the definition of "least integer function“ or "ceiling function“ $y = \lceil x \rceil$, you can find it on p.5 of the textbook.

4. Give an analytical formula of a function f that has three asymptotes: vertical asymptote $x = 0$, horizontal asymptote $y = 1$, and slant (or oblique) asymptote $y = 2x + 1$.

   Prove analytically that your function f has all three above mensioned asymptotes. Sketch your function f.

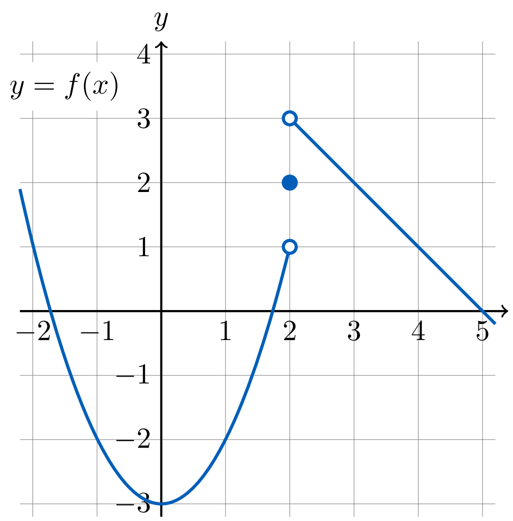
5. A park ranger on the West Coast Trail leaves his base at 7:00 am and takes his usual path on the trail, arriving to the campsite at 7:00 pm. The following morning, he starts at 7:00 am and takes the same trail back, arriving at the base at 7:00 pm. Use the Intermediate Value Theorem to show that there is a point on the trail that the park ranger will cross at exactly the same time of the day on both days.

   Notes: (1) The ranger can walk as fast or as slow as he wants at any moment in time, so do not make any assumptions about the ranger's speed on the way to and from the campsite.

   (2) The Intermediate Value Theorem talk about some function f. Let $f(t) = s(t) - r(t)$, where $s(t)$ and $r(t)$ are the position functions describing the walks to and from the campsite, respectively. What is the meaning of your function f? Can we apply the Intermediate Value Theorem to the function f?

Is there a limit?

1. **Calculate Limits** or show that limit does not exist:
   (a) $\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right)$
   (b) $\lim_{x\to+\infty}\frac{1+4x-x^4}{x+3x^2+2x^3}$
   (c) $\lim_{x\to+\infty}\left(x(\sqrt{x^2+4}-x)\right)$
   (d) $\lim_{x\to 3}\frac{\sqrt{x+13}-4}{x^2-9}$
   (e) $\lim_{x\to 1}\frac{\sin(2(x-1))}{x^2-7x+6}$
2. **Use the graph** of the function $y=f(x)$ below to determine the following limits:



$\lim_{x\to 1^-} f(x+1)$ $\quad\lim_{x\to 1^+} f(x+1)$ $\quad\lim_{x\to 1} f(x+1)$

$\lim_{x\to 5^-}\frac{f(x-2)}{f(x)}$ $\quad\lim_{x\to 5^+}\frac{f(x-2)}{f(x)}$ $\quad\lim_{x\to 5}\frac{f(x-2)}{f(x)}$

$\lim_{x\to 0}\frac{f(x+3)}{f(x)+3}$

3. **Magic of e** Use $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}=e\approx 2.71828$ to find
   $\lim_{t\to+\infty}\left(\frac{3t+2}{3t-1}\right)^{3t-1}$

See it From Both Sides

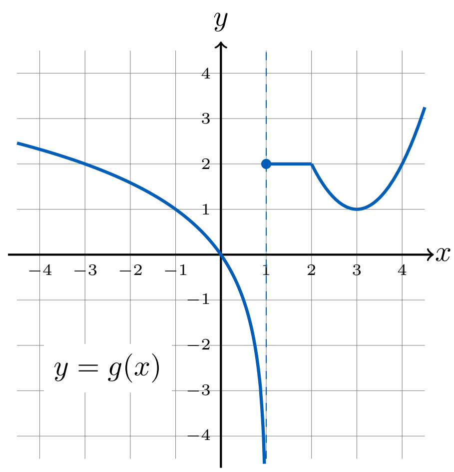
1. Let $f(x) = x^3 - 2x + 4$. *Without finding them*, explain why we know there must be values c_1, c_2, c_3 such that $f(c_1) = \pi$, $f(c_2) = -\sqrt{7}$, and $f(c_3) = 1,000,000$.

2. Each of the following solutions is incorrect. Explain what is incorrect about each of them, and find a correct solution. *Hint: the correct answer is* $5/2$.

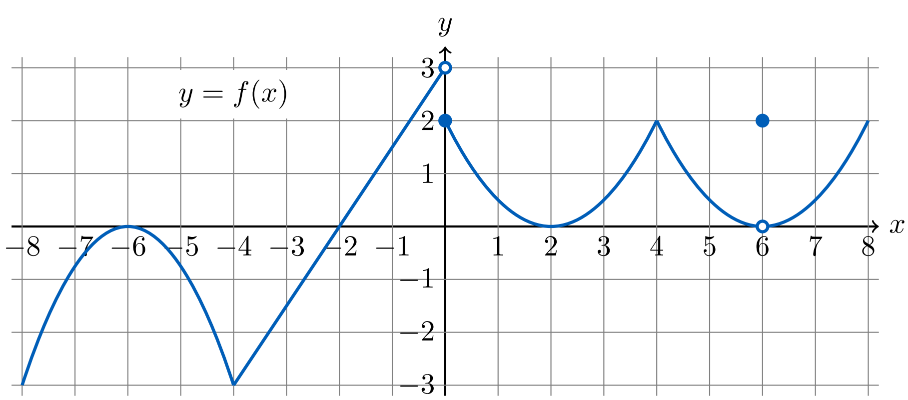
   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)} = \lim_{x\to 0} \frac{5x}{2x} = \lim_{x\to 0} \frac{5}{2} = \frac{5}{2}$.

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)} = \lim_{x\to 0} \frac{5\sin(x)}{2\sin(x)} = \lim_{x\to 0} \frac{5}{2} = \frac{5}{2}$.

3. Consider the function $y = g(x)$ graphed below. Explain why $g(g(x))$ *is* continuous at $x = 3$ but *is not* continuous at $x = -1$.



4. Consider the function $y = f(x)$ graphed below. At which of the following values is $f(f(x))$ *continuous*? $x = -6$, $x = 2$, $x = 6$. Explain.



Limits and Continuity

1. Sketch $y = \frac{|x|}{x}$ and $y = \frac{|x+3|}{x+3}$. On what open intervals are these functions continuous? Explain why.

2. Suppose that $g(x)$ has domain $(-\infty, \infty)$ and $\lim_{x\to 4^-} g(x) = 4$ and $\lim_{x\to 4^+} g(x) = -1$. Evaluate the following limits.

$$\lim_{x\to 4} g(x) \qquad \lim_{x\to 2} g\left(x^2\right) \qquad \lim_{x\to 4^+} g\left(20 - x^2\right)$$

3. *Without finding it*, explain how we know that there must be a number c such that $\cos(c) = \sqrt{c}$.

4. Consider $\displaystyle\lim_{x\to 0} \frac{1}{x^n}$.

   (a) For what integers n does $\displaystyle\lim_{x\to 0} \frac{1}{x^n} = \infty$?

   (b) For what integers n does $\displaystyle\lim_{x\to 0} \frac{1}{x^n} = -\infty$?

   (c) For what integers n does $\displaystyle\lim_{x\to 0} \frac{1}{x^n}$ not exist?

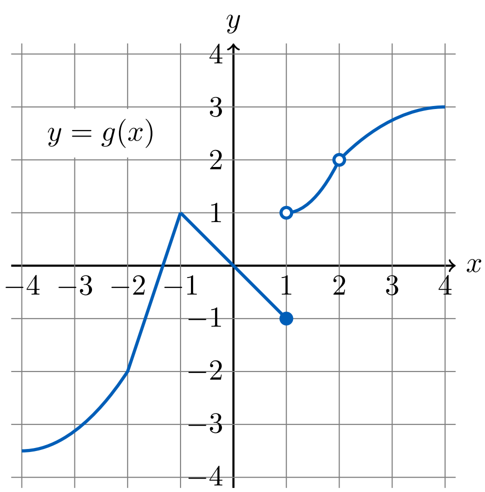
   (d) For what integers n does $\displaystyle\lim_{x\to 0} \frac{1}{x^n} = 0$?

5. Each of the following solutions is incorrect. Explain what is incorrect about each of them, and find a correct solution. *Hint: the correct answer is* $3/2$*.*

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3\sin(x)}{2\sin(x)} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{3x}{2x} = \lim_{x\to 0} 1 \cdot 1 \cdot \frac{3x}{2x} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$

6. For the function $y = g(x)$, graphed below, find all points of discontinuity. For each, explain why g is discontinuous there.



Sandwich Sines

Note:

All of the following limits involve the sine function; many of them also involve the Sandwich Theorem (but not all).

1. Evaluate $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.
2. Explain why $\lim\limits_{x\to\infty} \frac{\sin(x^2)}{x^2} = 0$.
3. Evaluate $\lim\limits_{x\to 0} \frac{\sin(x^2)}{x^2}$.
4. Evaluate $\lim\limits_{x\to 0^+} \cot(x)$.
5. Evaluate $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)$.
6. Explain why $\lim\limits_{x\to\infty} x \sin\left(\frac{1}{x}\right) = 1$.
7. Explain why $\lim\limits_{x\to\infty} \frac{x - \sin(x)}{x} = 1$.[b]

[b]Come back to this once you've learned l'Hospital's Rule and see what happens!

Rational Limits

1. Find a rational function $g(x)$ that has all of the following properties:

   $g(x)$ has a vertical asymptote at $x = 3$
   $g(x)$ has a slant asymptote of $y = x + 5$
   $g(x)$ is continuous on $(-\infty, 3) \cup (3, \infty)$.

   Write down an explicit expression for $g(x)$.

2. Suppose that $p(x)$ and $q(x)$ are polynomials. What sorts of asymptotes could the rational function $\frac{p(x)}{q(x)}$ have if...

   ... $p(x)$ has lower degree than $q(x)$?

   ... $p(x)$ has higher degree than $q(x)$?

   ... $p(x)$ has the same degree as $q(x)$?

3. Is it possible for a rational function to have both a slant asymptote and a horizontal asymptote? If so, provide an example. If not, explain why not.

4. Each of the following solutions is incorrect. Explain what is incorrect about each of them, and find a correct solution. *Hint: the correct answer is* $3/2$*.*

   (a) $\displaystyle \lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3x}{2x} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$

   (b) $\displaystyle \lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3\sin(x)}{2\sin(x)} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$

5. Consider the following function:

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

   (a) Sketch a rough graph of this function.
   Hint: for <u>any</u> two real numbers there is at least one rational number and at least one irrational number between them.

   (b) What is the domain of this function?

   (c) What is $\lim_{x\to 1} f(x)$?

   (d) What is $\lim_{x\to 0} f(x)$?

   (e) Is $f(x)$ continuous at 1?

   (f) Is $f(x)$ continuous at 0?

   (g) Find all of the points at which $f(x)$ is continuous.

“Check Your Understanding” Answers:

Note:

Each chapter includes answers to the "True or False" and "Find a Function" problems, as well as hints for "Explain Your Answer". If you are ever in doubt as to whether an answer is correct, or if you understand an answer but don’t think you could have worked it out yourself, seek help from your instructor.

True or False?

1. False.
2. False.
3. True.
4. False.
5. True.
6. True.

Find a Function:

1. One such function is $\frac{2x}{x-3}$.
2. One such function is $10 - x + \frac{1}{x+4}$.
3. One such function is $\frac{x-1}{x-1}$.
4. One such function is $f(x) = \begin{cases} x & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$.

Explain your answer:

1. Hint: notice that for $x \neq 0$
$$\frac{\sin(7x)}{\sin(5x)} = \frac{\sin(7x)}{7x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{7}{5}.$$
2. Hint: the function $y = \sqrt{1+x^2}$ has a slant asymptote $y = x$ if and only if at least one of the following is true:
$$\lim_{x\to-\infty}\left(\sqrt{1+x^2} - x\right) = 0 \text{ or } \lim_{x\to\infty}\left(\sqrt{1+x^2} - x\right) = 0.$$
Show that $\lim_{x\to\infty}\left(\sqrt{1+x^2} - x\right) = 0$ by rationalizing it.
Bonus: this function actually has *two* slant asymptotes! The other one is $y = -x$ – explain why.
3. Hint: notice that $-1 \leq \sin(x^2) \leq 1$. Use this to find an upper bound and a lower bound for $\frac{\sin(x^2)}{x^2}$.
4. Hint: If $x \neq 0$ then $x^2 > 0$.
5. Hint: If $g(x)$ is continuous then $\lim_{x\to0^-} g(x) = \lim_{x\to0^+} g(x)$. If $g(x)$ is odd then
$$\begin{aligned} \lim_{x\to0^-} g(x) &= \lim_{u\to0^+} g(-u) \\ &= \lim_{u\to0^+} -g(u). \end{aligned}$$
That gets you to within approximately two steps of a full solution.

6. Hint: Use the Intermediate Value Theorem.
7. Hint: $|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$.
8. Hint: You know $\lim_{x \to 7}(x-7)$. What is $\lim_{x \to 7}\left(\left(\frac{f(x)-8}{x-7}\right) \cdot (x-7)\right)$?

$\triangle$ Derivatives

In your textbook: *This material corresponds to Chapter 3, sections 3.1 - 3.9*

> **Before you begin:** this chapter relies on the following pre-calculus material
>
> - Equations of lines (FE1, A9)
> - Interval notation (N1)
> - Exponent and logarithm rules (FE2)
> - Standard functions, especially trigonometric functions (FE3)
> - Domain of a function (FC2, FC5, A10)
> - Function composition (FC5, N2)
> - Inverse functions (FC6, FE1, FE2, FE3)
> - Notation (N)

Additional Material

Your textbook introduces the concept of *vertical tangent line* in the exercises of section 3.1, but the definition given there is more restrictive than it needs to be. Here is our definition:

Definition 11. We say that the curve $y = f(x)$ has a *vertical tangent line* at $x = a$ if f is continuous at $x = a$ and $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \infty$ or $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = -\infty$.

In section 3.7, your textbook defines the *normal* line. We can define a vertical tangent line in terms of the normal line, too:

Definition 12. We say that the curve $y = f(x)$ has a *vertical tangent line* at $x = a$ if the line normal to $y = f(x)$ at $x = a$ is horizontal.

Chapter Checklist

3.1: Derivatives

- You should know…
  - ☐ the definition of *slope of the curve at a point*.
  - ☐ the definition of *tangent line to the curve at a point*.
  - ☐ the definition of *vertical tangent line*.

- ☐ the definition of *derivative of the function at a point*.

- You should understand...
  - ☐ the relationship between a derivative at a point and a tangent line at a point.
- You should be able to compute...
  - ☐ the slope of a curve at a point, given a function and a specified point.
  - ☐ the equation of a tangent line to a curve at a point, given a function and a specified point.
  - ☐ an *estimate* of the slope of a curve at a point, given a sketch of a function and a specified point.
  - ☐ an *estimate* of the tangent line to a curve at a point, given a sketch of a function and a specified point.

3.2: The derivative as a function

- You should know...
  - ☐ the definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.
  - ☐ the alternate definition $f'(x) = \lim_{z\to x} \dfrac{f(z)-f(x)}{z-x}$.
  - ☐ that if $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.
  - ☐ at least one function that is continuous but not differentiable at at least one point.
- You should understand...
  - ☐ the relationship between a function and its derivative in terms of slope.
  - ☐ the relationship between $f(x)$ and the roots of $f'(x)$.
- You should be able to compute...
  - ☐ the derivative of a function using the definition.

3.3: Differentiation rules

- You should know...
  - ☐ the following derivatives.
    $\dfrac{d}{dx}[c] \quad \dfrac{d}{dx}[x^n] \quad \dfrac{d}{dx}[cf(x)] \quad \dfrac{d}{dx}[f(x)g(x)] \quad \dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] \quad \dfrac{d}{dx}[e^x]$
- You should understand...
  - ☐ the relationship between the limit laws and the derivative rules.
- You should be able to compute...
  - ☐ the derivative of any function constructed from x, x^n, and e^x through addition, multiplication, and division.

3.4: The derivative as a rate of change

- You should know...
  - ☐ the definition of *velocity*.
  - ☐ the definition of *speed*.

- ☐ the definition of *acceleration*.
- ☐ the definition of *jerk*.

- You should understand...
  - ☐ the difference between velocity and speed.
  - ☐ the meaning of a *positive* or *negative* derivative of functions that represent distance, cost, and other models.

- You should be able to compute...
  - ☐ a rough sketch of $f'(x)$, given a sketch of $f(x)$.
  - ☐ a rough sketch of $f(x)$, given a sketch of $f'(x)$.

3.5: Derivatives of trigonometric functions

- You should know...
  - ☐ $\dfrac{d}{dx}[\sin(x)]$.
  - ☐ $\dfrac{d}{dx}[\cos(x)]$.
  - ☐ $\dfrac{d}{dx}[\tan(x)]$.

- You should understand...
  - ☐ the relationship between the horizontal tangent lines of $\sin(x)$ and the zeros of $\cos(x)$.
  - ☐ the relationship between the horizontal tangent lines of $\cos(x)$ and the zeros of $\sin(x)$.
  - ☐ how to use the quotient rule to find the derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$.
  - ☐ how to use the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ to compute the derivative of $\sin(x)$.
  - ☐ how to use the fact that $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$ to compute the derivative of $\cos(x)$.

- You should be able to compute...
  - ☐ $\dfrac{d}{dx}[\sec(x)]$
  - ☐ $\dfrac{d}{dx}[\csc(x)]$
  - ☐ $\dfrac{d}{dx}[\cot(x)]$
  - ☐ the derivative of functions consisting of products and quotients of polynomials, e^x, and the standard trig functions.

3.6: The chain rule

- You should know...
  - ☐ the Chain Rule.

- You should understand...
  - ☐ ...the Chain Rule.

- You should be able to compute...
  - ☐ the derivative of composite functions. **Example:** what is $\dfrac{d}{dx}\left[e^{x^2\cos\left(\frac{\pi}{2}\sqrt{1+x^4}\right)}\right]$?

3.7: Implicit differentiation

- You should know...
  - ☐ what an *implicitly defined* function is.
  - ☐ what a *normal line* is.
- You should understand...
  - ☐ the difference between an *explicit* relation and an *implicit* relation.
  - ☐ the relationship between implicit differentiation and the chain rule.
- You should be able to compute...
  - ☐ the tangent and normal lines of curves, including curves that are not functions (such as ellipses), by using implicit differentiation.

3.8: Derivatives of inverse function and logarithms

- You should know...
  - ☐ $\frac{d}{dx}[\ln|x|]$.
  - ☐ $\frac{d}{dx}[a^x]$, where a is a positive real number.
  - ☐ $\frac{d}{dx}[\log_a(x)]$, where a is a positive real number.
- You should understand...
  - ☐ how to derive the formula $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$.
  - ☐ the method of Logarithmic Differentiation (and when to use it).
- You should be able to compute...
  - ☐ $\frac{d}{dx}\left[f^{-1}(x)\right]$.
  - ☐ the derivative of a function of the form $h(x)^{g(x)}$ (for example, x^{3x}).

3.9: Inverse trigonometric functions

- You should know...
  - ☐ $\frac{d}{dx}[\arcsin(x)]$.
  - ☐ $\frac{d}{dx}[\arccos(x)]$.
  - ☐ $\frac{d}{dx}[\arctan(x)]$.
- You should understand...
  - ☐ how to find $\frac{d}{dx}\left[\cot^{-1}(x)\right]$, $\frac{d}{dx}\left[\csc^{-1}(x)\right]$, and $\frac{d}{dx}\left[\sec^{-1}(x)\right]$.
- You should be able to compute...
  - ☐ the derivative of functions involving inverse trig functions.

Check Your Understanding

True or False? For each of the following statements, determine whether it is True or False. If True, explain why. If False, provide a counterexample.

1. If $f'(a) = 3$ then the curve $y = f(x)$ has slope 3 at the point $x = a$.
2. If $g'(1) = 5$ and $g(1) = -4$ then the curve $y = g(x)$ has tangent line $y = 5x - 4$ at $x = 1$.
3. The derivative of $\sin(x)$ at $x = \pi$ is given by the following limit:
$$\lim_{x \to \pi} \frac{\sin(x)}{x - \pi}.$$
4. If $f(x)$ is differentiable on $(-\infty, \infty)$ and $f'(x)$ has a root at $x = a$ then $f(x)$ has a horizontal tangent line at $x = a$.
5. If $h(x)$ is differentiable at $x = b$ then $h(b)$ exists.
6. If $h(x)$ is continuous at $x = b$ then $h'(b)$ exists.
7. If an object's *speed* is positive then so is its *velocity*.
8. If f and g are differentiable functions then $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.
9. The derivative with respect to x of $\sin^2(x) + \cos^2(x)$ is 0.
10. Suppose water is pouring out of a tank, and let $h(t)$ represent the height of the water after t seconds. Then $h'(t) \leq 0$ for all $t \geq 0$.
11. If $f(x) = 8^x$ then $f'(x) = x8^{x-1}$.
12. If $y = 4x + 1$ is the tangent line to $y = f(x)$ at $x = 1$ then $y = -\frac{1}{4}x + \frac{21}{4}$ is the normal line to $y = f(x)$ at $x = 1$.
13. The curve defined implicitly by $(x - y)^2 = x + y - 1$ has a horizontal tangent line at the point $\left(1, \frac{1}{2}\right)$.
14. The curve defined implicitly by $(x - y)^2 = x + y - 1$ has a horizontal tangent line at the point $\left(\frac{7}{8}, \frac{3}{8}\right)$.
15. If $g(x) = \log_8(x)$ then $g'(x) = \frac{1}{\ln(8)x}$.
16. $\cos(\arcsin(x)) = \sqrt{1 - x^2}$.
17. $\tan^{-1}(x) = \frac{1}{\tan(x)}$.

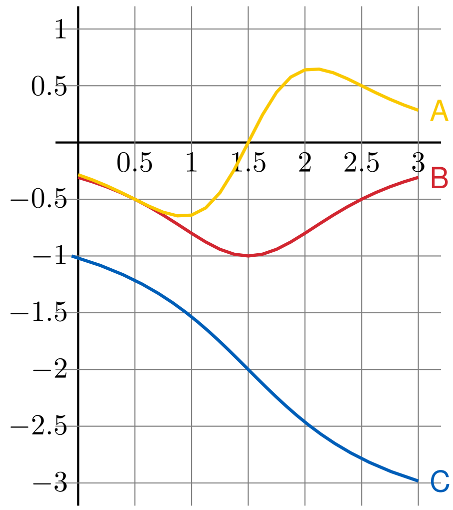
Find a Function:

1. Find a function that is continuous at $x = 0$ but not differentiable at $x = 0$.
2. Find a function $g(x)$ that is odd but whose derivative is even.
3. Find a function $h(x)$ that is even and whose derivative is also even.
4. Find a function $f(x)$ (a sketch suffices) such that $f(x)$ is continuous everywhere, $f'(x)$ has exactly one point of discontinuity, and $f''(x)$ has at least three points of discontinuity.

Explain your answer:

1. The function $f(x)$ defined below is continuous (you do not need to verify this fact). Explain why $f'(0) = 1$, using the limit definition of the derivative.
$$f(x) = \begin{cases} \frac{\sin(x^2)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

2. Three functions are graphed below. One is $f(x)$, one is $f'(x)$, and one is $f''(x)$. Explain how you know which is which.



3. Let $f(x) = x^3$. Using the limit definition of the derivative, explain why $f'(2) = 12$.
4. If $g(x) = \cos(x)$, explain why $g^{(2016)}(x) = \cos(x)$.
5. Explain why the function $\frac{1}{x^2}$ has no derivative at $x = 0$.
6. We say that a function $f(x)$ has a vertical tangent line at $x = a$ if f is continuous at $x = a$ and $\lim_{h\to0}\dfrac{f(a+h)-f(a)}{h} = \infty$ or $\lim_{h\to0}\dfrac{f(a+h)-f(a)}{h} = -\infty$. Explain why none of the following functions have a vertical tangent line at $x = 0$:
   - $f(x) = x^{4/5}$
   - $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$
   - $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$
7. Explain why $\dfrac{d}{dx}\left[(\cot(x)+\csc(x))(\cot(x)-\csc(x))\right] = 0$.
8. Use the product rule (not the chain rule) to explain why $\dfrac{d}{dx}\left[(1+e^x)^2\right] = 2e^x(e^x+1)$.
9. Suppose that an object thrown directly upward is at a height of $h(t)$ feet after t seconds, where $h(t) = -16(t-3)^2 + 150$. Explain how we know that the object's instantaneous velocity at $t = 3$ is 0. Why does this mean the object reaches its maximum height at $t = 3$?
10. Use the limit definition of derivative to explain why $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$.

    Hint: you will need to use the fact that $\lim_{h\to0}\dfrac{\cos(h)-1}{h} = 0$.
11. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then $f(g(x)) = x^2$ and $g(f(x)) = x^2$, both of which are differentiable at $x = 0$. On the other hand, $g(x)$ is *not* differentiable at $x = 0$. Explain why this does not contradict the Chain Rule.

12. The curve $x^2 + y^2 = 1$ is a circle with radius 1 centred at the point $(0, 0)$. Use implicit differentiation to explain why $\frac{dy}{dx} = -\frac{x}{y}$.
13. The curve $x^2 + y^2 = 1$ is a circle with radius 1 centred at the point $(0, 0)$. Using the fact that $\frac{dy}{dx} = -\frac{x}{y}$, explain why this curve has horizontal tangent lines at $(0, 1)$ and $(0, -1)$ and horizontal normal lines at $(-1, 0)$ and $(1, 0)$. Illustrate with a sketch.
14. The curve $x^2 + y^2 = 1$ is a circle with radius 1 centred at the point $(0, 0)$. Use implicit differentiation and the fact that $\frac{dy}{dx} = -\frac{x}{y}$ to explain why $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$.
15. Suppose that $f(x)$ is an invertible and differentiable function. Given the table of values below, explain why the slope of the tangent line to $y = f^{-1}(x)$ at $x = 2$ is $\frac{1}{12}$.

x	1	2	3	6
$f(x)$	5	6	2	1
$f'(x)$	10	11	12	13

16. Suppose that $g(x) = (4x + 3)^x$. Explain why $g'(x) = (4x + 3)^x \left(\ln(4x + 3) + \frac{4x}{4x+3}\right)$.
17. If $y = \sec^{-1}(x)$ then explain why $\sec(y)\tan(y)$ must be positive.
18. Use Theorem 3 from Section 3.8 to explain why

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2 - 1}} \text{ for } |x| > 1.$$

Hint: use your work for the problem above to explain the $|x|$ in the denominator. Why do we require $|x| > 1$?

Worksheets

Secants & Tangents

1. Warm up: equations of lines.
   - The line $y - \ell = m(x - k)$ has slope m. Verify that it contains the point (k, ℓ). What is the y-intercept of this line?
   - If $f(1) = 4$ and $f'(1) = -3$, what is an equation of the line tangent to $y = f(x)$ at $x = 1$? Express your answer in point/slope form.
   - The function $y = x^3 - 4x$ has slope $3a^2 - 4$ at $x = a$ (you do not need to verify this). What is an equation of the line tangent to $y = x^3 - 4x$ at $x = a$?

2. Let $f(x) = \sqrt{x+1}$.

   (a) *You will want a calculator for this*. Find the average rate of change of $f(x)$ on each of the following intervals: $[1, 1.5]$, $[1, 1.25]$, $[1, 1.1]$.

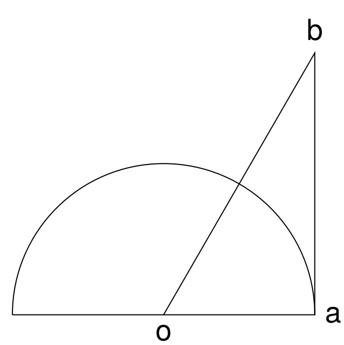
   (b) Evaluate $f'(1)$ using the definition $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

   (c) Evaluate $f'(1)$ using the definition $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

   (d) Which definition of $f'(1)$ do you think was easier to use? *No, there's no wrong answer!*

   (e) Does the average rate of change on the above intervals seem to be close to the instantaneous rate of change at $x = 1$?

3. The word "tangent" first appeared in *Geometria rotundi*, by Thomas Fincke (1583). As the title of the book suggests, Fincke was studying the geometry of circles – trigonometry. The diagram below partially replicates a figure from that book; the length of the line segment ab is exactly $\tan(\theta)$, where θ is the angle $\angle aob$ (this might not be obvious to you). Explain why it was reasonable to appropriate the word 'tangent' from this 'geometria rotundi' to apply it to what we call the 'tangent line' in calculus.[c]



[c]For the interested: *tangent* to mean "meeting at a point without intersecting" is from the Latin word *tangere* "to touch".

Derivatives and tangent lines

1. Suppose that u and v are differentiable functions such that $u(1) = 2$, $u'(1) = 0$, $v(1) = 7$, and $v'(1) = -3$.

   (a) Evaluate each of the following derivatives at $x = 1$:

   $$\frac{d}{dx}[u(x)v(x)] \quad \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] \quad \frac{d}{dx}\left[\frac{v(x)}{u(x)}\right] \quad \frac{d}{dx}[3u(x) - 7v(x)]$$

   (b) Find an equation for the line tangent to $y = u(x)v(x)$ at $x = 1$.

2. The function $|x|$ is not differentiable at $x = 0$...

   (a) Sketch $y = |x|$ and determine the slope of the function on $(-\infty, 0)$ and on $(0, \infty)$. Is $|x|$ continuous at $x = 0$?

   (b) Use the definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ to prove that $f(x) = |x|$ is not differentiable at $x = 0$.

   (c) Use any method you wish to prove that $f(x) = |x|$ *is* differentiable at $x = a$ as long as $a \neq 0$.

3. There are two distinct straight lines that pass through the point $(1, 0)$ and are tangent to the curve $y = x^3$ (note that $(1, 0)$ is *not* a point on this curve!). Find the equations of both lines.

4. A function may be differentiable but not have a second derivative. Consider the function

   $$g(x) = \begin{cases} x^2 & x \leq 0, \\ 0 & x > 0. \end{cases}$$

   (a) Use the limit definition to calculate $g'(x)$.

   (b) Explain why $g''(0)$ does not exist.

5. A function may be differentiable but not have a continuous derivative. Consider the function

   $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   (a) Use the limit definition to calculate $f'(x)$.

   (b) Explain why $f'(0)$ does exist, but $f'(x)$ is not continuous at $x = 0$.

6. Find all values of k so that the curves $y = kx^2$ and $y = k(x-4)^2$ intersect orthogonally (two curves are said to intersect orthogonally at a point (x, y) if they intersect at (x, y) and their tangent lines at (x, y) are perpendicular).

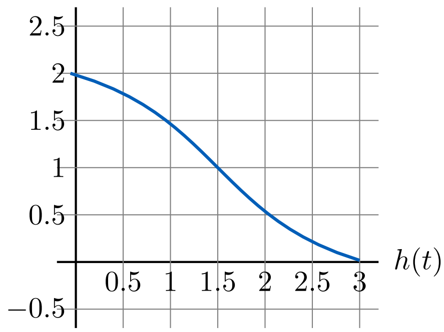
It's about "change"

1. Suppose that f and g are differentiable functions and

$$f(1) = 2 \qquad g'(1) = 4 \qquad g(1) = 2 \qquad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = 3 \text{ at } x = 1.$$

   What is $f'(1)$?

2. Suppose that the distance between a particle and a fixed point after t seconds is given by the function $h(t)$ sketched below.



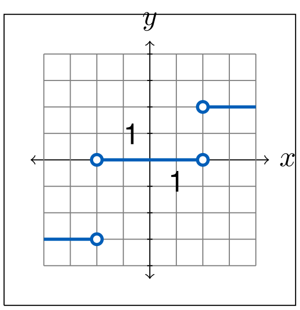
   (a) Sketch a graph of the velocity of this particle.
   (b) Sketch a graph of the speed of this particle.
   (c) Sketch a graph of the acceleration of this particle.

3. Is there a value of b that will make the following function differentiable at $x = 0$?

$$g(x) = \begin{cases} bx, & \text{if } x < 0 \\ -\sin(x), & \text{if } x \geq 0 \end{cases}$$

4. Two points of light are moving along the straight segment of the line OY, with the distance of the points from the origin O at the time t described by two functions: $f(t) = \dfrac{t^3}{3} - 4$ and $g(t) = \dfrac{7}{2}t^2 - 12t + 3$. At what moment(s) in time are the velocities of the two points of light equal?

5. The function $y = g'(x)$ is graphed below. If $g(x)$ is continuous and $g(2) = 3$, sketch a graph of $y = g(x)$ and $y = g''(x)$.



Ready to differentiate?

1. **Definition of Derivative**:
   (a) Find the formula for calculating derivative of function $y = \frac{3x}{2x+1}$ at every point of the domain using definition of derivative.
   (b) Confirm your formula derived in part (a) by calculating derivative of the same function $y = \frac{3x}{2x+1}$ using appropriate differentiation rules.
2. **Tangent and Normal lines**: Derive equations of the two lines "tangent" and "normal" to the curve $y = x^2 + 2x - 2$ at the point on the curve with x coordinate equal to 1.
3. **Differentiation Rules**: Find derivatives of the following functions (do not need to simplify):
   (a) $y = 2x^3 + 4\sqrt[3]{x^4} - 5/x^2$
   (b) $y = x^2(\cos x) \cdot \ln x$
   (c) $y = (\log_2(5x)) \cdot e^{3x}$
   (d) $y = 2^{-\cos^2(3x)}$
4. Find $\frac{dy}{dx}$ if
   (a) $x^3 + y^3 - 3xy = 0$
   (b) $2^x + 2^y = 2^{x+y}$

Derivatives Rule

1. Let $f(x) = \frac{x^6}{100}$.
   (a) What is $f'(x)$?
   (b) What is $f''(x)$?
   (c) What is $f'''(x)$?
   (d) What is $f^{(4)}(x)$, $f^{(5)}(x)$, $f^{(6)}(x)$, $f^{(7)}(x)$, $f^{(8)}(x)$, etc.?

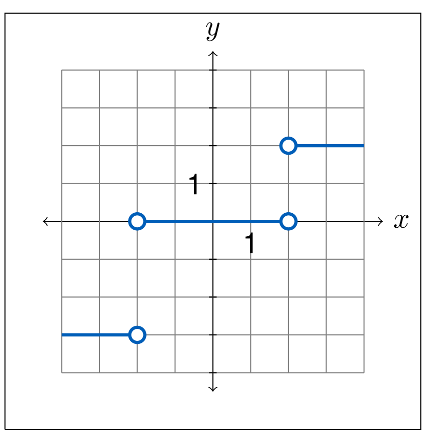
2. Suppose that u and v are differentiable functions and:
$$u(1) = 2 \qquad u'(1) = 0 \qquad v(1) = 7 \qquad v'(1) = -3$$
   (a) Evaluate each of the following derivatives at $x = 1$:
$$\frac{d}{dx}[u(x)v(x)] \quad \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] \quad \frac{d}{dx}\left[\frac{v(x)}{u(x)}\right] \quad \frac{d}{dx}[3u(x) - 7v(x)]$$
   (b) Find an equation for the line tangent to $y = u(x)v(x)$ at $x = 1$.

3. Suppose that a rock is dropped on a distant planet, and it is observed that the rock's distance from the surface of the planet after t seconds is $t_0 - 10t^2$ meters, where t_0 is the height from which the rock was dropped. What is the acceleration due to gravity on that planet? If the planet is the same size as Earth, does it have greater or lesser mass than Earth?

4. Without using the chain rule, explain why $\frac{d}{dx}[e^{nx}] = ne^{nx}$ for any positive integer n.

5. Consider the function
$$g(x) = \begin{cases} ax, & x \leq 0 \\ x^2 - 3x, & x > 0 \end{cases}$$
   (a) For what value of a is $g(x)$ differentiable everywhere?
   (b) Sketch $y = g(x)$ and $y = g'(x)$ for this value of a.

6. The function $y = g'(x)$ is graphed below. If $g(x)$ is continuous and $g(0) = 0$, sketch a graph of $y = g(x)$ and $y = g''(x)$.



Trigonometry

1. For each of the following functions, find all x in the interval $[0, 2\pi]$ such that $f(x)$ has a horizontal tangent line at x.
   (a) $f(x) = x - \sin(x) + 1$
   (b) $f(x) = \frac{1}{2}\sin^2(x)$

2. Is there a value of b that will make the following function differentiable at $x = 0$?
$$g(x) = \begin{cases} x + b, & \text{if } x < 0 \\ \cos(x), & \text{if } x \geq 0 \end{cases}$$

3. Consider the function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ c & x = 0 \end{cases}$.
   (a) Find all values of c that make $f(x)$ continuous at 0.
   (b) For each value of c that you found in (a), determine whether $f(x)$ is differentiable at $x = 0$.
   (c) For each value of c for which $f(x)$ is differentiable at 0, determine whether or not $f''(0)$ exists.

4. The function $\cos(x)$ is not one-to-one, and yet we often talk about its inverse function $\cos^{-1}(x)$ (also known as $\arccos(x)$).
   (a) Explain why $\cos(x)$ is not one-to-one by finding two real numbers x_1 and x_2 such that $x_1 \neq x_2$ and $\cos(x_1) = \cos(x_2)$.
   (b) What is the domain of $\cos(x)$? What is the range of $\cos(x)$?
   (c) Sketch $y = \cos(x)$ on the interval $-2\pi \leq x \leq 2\pi$. Illustrate your answer to part (4a) by drawing the appropriate horizontal line through your sketch.
   (d) What is the largest number a for which $\cos(x)$ *is* one-to-one on the interval $[0, a]$?
   (e) Sketch the inverse function, $\arccos(x)$, which has range $[0, a]$ (see part (4d)). What is the domain of $\arccos(x)$?

5. Our convention is that the function $\sin(x)$ always has x measured in radians. Let us define two new functions, $\sin_d(x)$ and $\cos_d(x)$, to be the sine and cosine functions when x is measured in degrees. Thus, for example, $\sin_d(30) = \sin(\pi/6) = 1/2$.
   (a) Find a formula that relates $\sin_d$ to $\sin$ and/or $\cos$. Do the same for $\cos_d$.
   (b) Show that the derivative of $\sin_d(x)$ is *not* $\cos_d(x)$.

"Chain" rules!

1. Draw a function f that safisfies all four of the following statements:

   function f is continuous and invertable, derivatives at the point (x, y) are equal to: at $\left(-1, \frac{3}{2}\right)$ is -2, at $\left(\frac{1}{2}, 1\right)$ is 0, at $(3, -2)$ is $-\frac{1}{4}$.

   List all known slopes of the tangent lines to the function f^{-1}, including the exact coordinates of the points (x, y) where the slope of the tangent lines to the function f^{-1} is known. Justify.

2. Find the derivative of the function:

   (a) $g(x) = \dfrac{\arccos x}{x+1}$ (b) $f(t) = \sin(\arccos 6t)$

   (c) $y = x\arctan(2x) - \dfrac{1}{4}\ln(1+4x^2)$

   You need to be able to quickly derive the formula for the derivatives of the given inverse trigonometric functions.

3. Find $\dfrac{dy}{dx}$ using Chain rule, implicit differentiation or logarithmic differentiation, when appropriate:

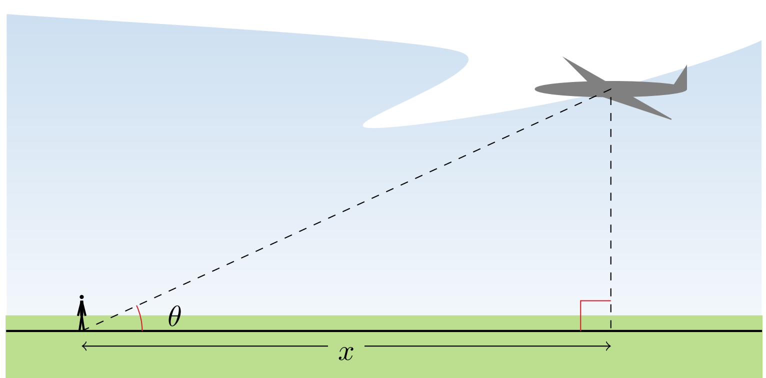
   (a) $\arctan xy = \arcsin(x+y)$ (b) $y = \dfrac{(2x+1)^3(x^2-1)^2}{x+3}$

   (c) $x\sin y = y\cos x$ (d) $y\ln(2x+1) = x+2$

4. An airplane flies parallel to the ground at an altitude of 3 miles towards a point directly above the observer on the ground. The speed of the plane is 400 miles per hour. As the plane gets closer to the observer, the angle θ (see the picture below) is increasing.

   (a) Write the formula relating θ and x.

   (b) Find $\dfrac{d\theta}{dt}$ at two specific moments: when $x = 10$ miles and when $x = 2$ miles.



Chains

1. Given the information in the table below, compute the following derivatives with respect to x:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

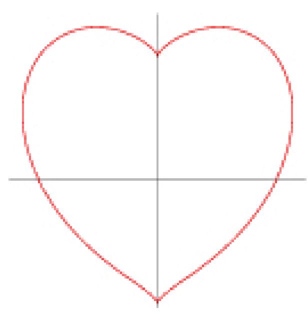
(a) $2f(x)$ at $x = 2$ (b) $f(x) + g(x)$ at $x = 3$
(c) $f(g(x))$ at $x = 2$ (d) $\sqrt{(f(x))^2 + (g(x))^2}$ at $x = 2$

2. Suppose that a is a positive real number. Evaluate each of the following derivatives:

$$\frac{d}{dx}[x^a] \qquad \frac{d}{dx}[a^x] \qquad \frac{d}{dx}[\log_a(x)] \qquad \frac{d}{dx}[(ax)^x] \qquad \frac{d}{dx}[x^{ax}]$$

3. Our convention is that the function $\sin(x)$ always has x measured in radians. Let us define two new functions, $\sin_d(x)$ and $\cos_d(x)$, to be the sine and cosine functions when x is measured in degrees. Thus, for example, $\sin_d(30) = \sin(\pi/6) = 1/2$.
   (a) Find a formula that relates $\sin_d$ to $\sin$ and/or $\cos$. Do the same for $\cos_d$.
   (b) Show that the derivative of $\sin_d(x)$ is *not* $\cos_d(x)$.

4. Consider the curve defined by $(x^2 + y^2 - 1)^3 = x^2y^3$. A sketch of this graph is given below.[d] What is $\dfrac{dy}{dx}$?



[d]Image from Wolfram Mathworld, `http://mathworld.wolfram.com/HeartCurve.html`. This curve is a cross-section of the so-called *heart surface*.

Inverses and inverse trigonometry

1. Let $f(x) = \sqrt{2x+1}$, defined for x in the interval $(-1/2, \infty)$.
   (a) Show that $f(x)$ is one-to-one on the interval $(-1/2, \infty)$.
   (b) Find a formula for $f^{-1}(x)$.
   (c) Sketch $f(x)$, $f^{-1}(x)$, and the line $y = x$ on the same grid.
2. Let $f(x) = \dfrac{ax+b}{cx+d}$, where a, b, c, d are constants.
   (a) Suppose that $ad - bc = 0$. Show that $f(x)$ is not one-to-one. *Hint*: Consider the case $c = 0$ separately. If $c \neq 0$, multiply the formula for $f(x)$ by $\frac{c}{c}$
   (b) Now suppose that $ad - bc \neq 0$. Show that f is one-to-one, and find a formula for f^{-1}.
   (c) Suppose again that $ad - bc \neq 0$. Find a formula for the derivative of f^{-1}.
3. The function $\cos(x)$ is not one-to-one, and yet we often talk about its inverse function $\cos^{-1}(x)$ (also known as $\arccos(x)$).
   (a) Explain why $\cos(x)$ is not one-to-one by finding two real numbers x_1 and x_2 such that $x_1 \neq x_2$ and $\cos(x_1) = \cos(x_2)$.
   (b) What is the domain of $\cos(x)$? What is the range of $\cos(x)$?
   (c) Sketch $y = \cos(x)$ on the interval $-2\pi \leq x \leq 2\pi$. Illustrate your answer to part (3a) by drawing the appropriate horizontal line through your sketch.
   (d) What is the largest number a for which $\cos(x)$ *is* one-to-one on the interval $[0, a]$?
   (e) Sketch the inverse function, $\arccos(x)$, which has range $[0, a]$ (see part (3d)). What is the domain of $\arccos(x)$?
4. Recall that $y = \arctan(x)$ if and only if $\tan(y) = x$ and y is in the interval $(-\pi/2, \pi/2)$. Use implicit differentiation to prove that

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}.$$

   Then verify your answer by using Theorem 3 from Section 3.8, which states that

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

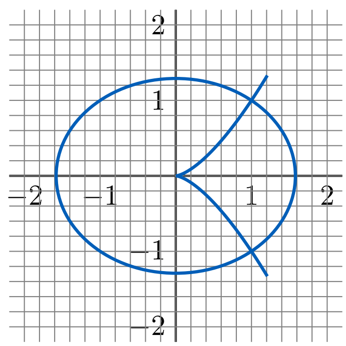
Implicit Differentiation (It's Just the Chain Rule)

1. Given the information in the table below, compute the following derivatives with respect to x:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	3	$1/3$	-2
3	3	$-\pi/3$	$\pi/2$	5

   (a) $2f(x)$ at $x=2$
   (b) $f(x)+g(x)$ at $x=3$
   (c) $f(g(x))$ at $x=2$
   (d) $\sin(f(x)g(x))$ at $x=3$

2. Consider the curves $y^2=x^3$ and $2x^2+3y^2=5$, graphed below.



   (a) Does $2x^2+3y^2=5$ have any horizontal tangent lines? Any horizontal normal lines?
   (b) Verify that these two curves intersect at the points $(1,1)$ and at $(1,-1)$.
   (c) Find the equation for the line tangent to $2x^2+3y^2=5$ at $(1,1)$ and at $(1,-1)$.
   (d) Find the equation for the line tangent to $y^2=x^3$ at $(1,1)$ and at $(1,-1)$. What is the relationship between the tangent lines for these two curves at these two points?

3. An ellipse in the x,y-plane has the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are positive real numbers that determine the length of the major and minor axes. Use implicit differentiation to determine at what points an ellipse has horizontal tangent lines and at what points an ellipse has horizontal normal lines. Illustrate your answer with a sketch.

Two Ways About It

1. Compute the derivative of $\cot(x)$ in two ways:
   (a) by using the Quotient Rule, because $\cot(x) = \frac{\cos(x)}{\sin(x)}$.
   (b) by using the Chain Rule and the derivative of $\tan(x)$, because $\cot(x) = (\tan(x))^{-1}$ (where both exist).
   (c) Verify that your two answers are equal.

2. Notice that $y = \arctan(x)$ if and only if $\tan(y) = x$. Use implicit differentiation to prove that

$$\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}.$$

   Then verify your answer by using Theorem 3 from Section 3.8, which states that

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}.$$

3. Compute the derivative of $\ln\left(\frac{3x^2+x}{x^4+2}\right)$ in two ways:
   (a) by using the Chain Rule and the Quotient Rule.
   (b) by using logarithm rules, and then using the Chain Rule and the Sum Rule.
   (c) Verify that your two answers are equal.

4. Compute the derivative of $\frac{3x^2+x}{x^4+2}$ in two ways:
   (a) by using the Quotient Rule.
   (b) by using Logarithmic Differentiation.
   (c) Verify that your two answers are equal.

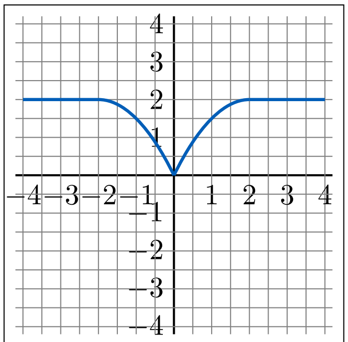
"Check Your Understanding" Answers

True or False?

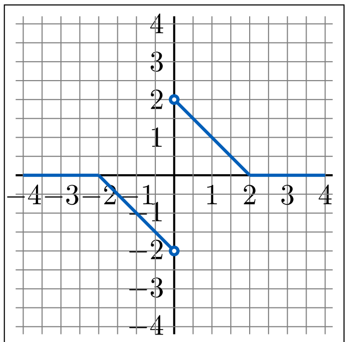
1. True.
2. False.
3. True.
4. True.
5. True.
6. False.
7. False.
8. False.
9. True.
10. True.
11. False.
12. True.
13. False. The point $(1, \frac{1}{2})$ is not a point on this curve.
14. True.
15. True.
16. True.
17. False.

Find a Function:

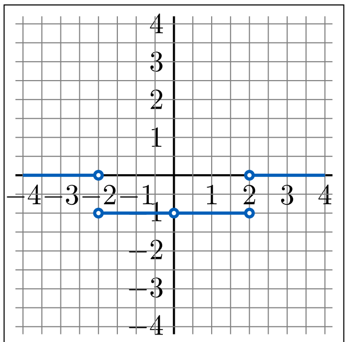
1. $|x|$ is one such function.
2. $g(x) = x$ is one such function.
3. $f(x) = c$, for any constant c, is such a function.
4. Here is one such function:



$y = h(x)$



$y = h'(x)$



$y = h''(x)$

Note:

Find a Function #4 was a particularly tricky problem! If you understand the above solution then you are following this material. If you *came up with your own solution* then you are good at manipulating functions.

Explain your answer:

1. Hint: It is easiest to use the definition of $f'(0)$, rather than the definition of $f'(x)$.
$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{\sin(x^2)}{x} - 0}{x-0} = \dots$$
2. Hint: It would be helpful to chart out on what intervals each function is positive/negative and increasing/decreasing. Then look for correspondences.
3. Hint: This is a straightforward computation of
$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}.$$
4. Hint: Compute $g'(x)$, $g''(x)$, $g'''(x)$, and $g^{(4)}(x)$ to see the pattern.
5. Hint: If a function is differentiable at $x=0$ then it is also continuous at $x=0$.
6. Hint: For each of the given functions, evaluate the one-sided limits as $h \to 0$ and see what happens.
7. Hint: $\cot^2(x) - \csc^2(x) = -1$.
8. Hint: First rewrite as $(1+e^x)(1+e^x)$.
9. Hint: Find $h'(3)$.
10. Hint: Already given.
11. Hint: The Chain Rule only applies to $f(g(x))$ if $g'(x)$ is differentiable at the point. If the Chain Rule does not apply, then anything is possible.
12. Hint: Straightforward implicit differentiation computation.
13. Hint: Remember that the normal line has slope $-1/(\frac{dy}{dx})$.
14. Hint: You already know that $\frac{dy}{dx} = -\frac{x}{y}$, so take another derivative with respect to x and then replace any $\frac{dy}{dx}$ you get with $-\frac{x}{y}$.
15. Hint: At some point you will need to figure out why $f^{-1}(2) = 3$.
16. Hint: Use Logarithmic Differentiation.
17. Hint: Remember that we must restrict the domain of $\sec(x)$ in order for it to be a one-to-one (and therefore invertible) function. Therefore, $y = \sec^{-1}(x)$ implies that $0 \le y \le \pi$.
18. Hint: Already given. Notice that if $|x| \le 1$ then $x^2 - 1 \le 0$.

$\cup$ Using Derivatives

In your textbook: *This material corresponds to Chapter 3, sections 3.10 & 3.11 as well as Chapter 4, sections 4.1 - 4.7*

Before you begin: this chapter relies on the following pre-calculus material

- Standard functions, especially trigonometric functions (FE3)
- Factoring polynomials (A4, A5)
- Finding intercepts of lines (FC3)
- Notation (N)

Chapter Checklist

3.10: Related rates

- You should know…
  - ☐ the six-step process for dealing with a Related Rates problem (or any other systematic approach that works for you).
- You should understand…
  - ☐ how to use formulas for volumes to set up and solve Related Rates problems.
- You should be able to compute…
  - ☐ the rate of change of a given variable, given enough partial information. (Yes, that's quite vague. That's because 'compute' is too weak a word for what is involved with solving problems in section 3.10 – take a look at the assigned homework to see the sort of thing we are expecting.)

3.11: Linearization and Differentials

- You should know…
  - ☐ what the *linearization* of a function $f(x)$ at the point $x = a$ is.
  - ☐ what the *differential* dy is.
- You should understand…
  - ☐ the relationship between linearization and tangent lines.
  - ☐ how to use linearization to approximate values such as $\sqrt{4.01}$ and $\sin(3)$.

- ☐ how to use differentials to approximate values.

- You should be able to compute...
  - ☐ approximations of things like $8.1^{1/3}$, $\sin(\pi/2 + 0.01)$, etc. using linearization or differentials.

4.1: Extreme Values of Functions

- You should know...
  - ☐ what an absolute maximum is.
  - ☐ what an absolute minimum is.
  - ☐ what a local maximum is.
  - ☐ what a local minimum is.
  - ☐ what a critical number (a.k.a critical point) is.
- You should understand...
  - ☐ the Extreme Value Theorem.
  - ☐ the First Derivative Theorem.
- You should be able to compute...
  - ☐ absolute extrema of a function on a closed interval by using the Extreme Value Theorem and critical numbers.

4.2: The Mean Value Theorem

- You should know...
  - ☐ the statement of *Rolle's Theorem*.
  - ☐ the statement of the *Mean Value Theorem*.
- You should understand...
  - ☐ how to use Rolle's Theorem to prove that horizontal tangent lines exist in an interval.
  - ☐ the relationship between average rate of change and instantaneous rate of change.
  - ☐ how to prove that a function has exactly one zero on a given interval.
- You should be able to compute...
  - ☐ the average rate of change of a function on a given interval.
  - ☐ (sometimes) the value of c that is promised by the Mean Value Theorem.‘

4.3: The First Derivative Test

- You should know...
  - ☐ the first derivative test for local extrema.
- You should understand...
  - ☐ the first derivative test for local extrema.
- You should be able to compute...
  - ☐ the local extrema of a function on a closed, open, or unbounded interval.

4.4: Concavity and Curve Sketching

- You should know…
  - ☐ what a point of inflection is.
  - ☐ the second derivative test for local extrema.
- You should understand…
  - ☐ the relationship between $f''(x)$ and the concavity of $f(x)$.
  - ☐ how to use $f'(x)$ and $f''(x)$ to sketch the curve $y = f(x)$.
- You should be able to compute…
  - ☐ intervals on which a function is increasing and decreasing.
  - ☐ intervals on which a function is concave up and concave down.

4.5: Indeterminate forms and l'Hospital's Rule

- You should know…
  - ☐ when l'Hospital's Rule can (and cannot) be used.
  - ☐ what the indeterminate forms are.
- You should understand…
  - ☐ how to use l'Hospital's Rule (and how to explain why it can be used).
  - ☐ how to use logarithms to deal with limits that involve indeterminate powers.
- You should be able to compute…
  - ☐ limits involving indeterminate forms by using l'Hospital's Rule (when appropriate).

Note:

Guillaume de l'Hospital published what is now called "l'Hospital's Rule" in 1696. Johann Bernoulli is the one who discovered the rule, however. He was tutor to the Marquis de l'Hospital, and signed a contract allowing the Marquis to publish his work. In the 17th and 18th centuries, the spelling was *l'Hospital*, and that is how Guillaume himself spelled his name. Since then, the French spelling rules have changed and his name is now spelled *l'Hôpital*.

4.6: Applied Optimization

In the first few sections of Chapter 4, we learned how to find local and global extrema of functions. Now we will apply this skill to modelling problems.

- You should know…
  - ☐ how to find the maximum or minimum of a function.
- You should understand…
  - ☐ the relationship between section 4.6 and sections 4.2 - 4.4.
- You should be able to compute…
  - ☐ the optimal value of an application problem.

- It is a good idea to review all of the word problems from this section in your textbook. For problems that are not assigned, at least set up the relevant figures and equations.

4.7: Newton's Method

- You should know...
  - ☐ Newton's Method.
- You should understand...
  - ☐ the relationship between Newton's Method and roots of a tangent line.
  - ☐ the relationship between Newton's Method and Linearization.
- You should be able to compute...
  - ☐ approximations of roots, intersections of curves, and numbers like π, $\sqrt{2}$, etc. using Newton's Method.

Check Your Understanding

Note:

Remember that answers and hints can be found at the end of the chapter.

True or False?

For each of the following statements, determine whether it is True or False. If True, explain why. If False, provide a counterexample.

1. If the sides of a cube are growing at a rate of 3 cm per second then the volume of the cube is growing at a rate of 27 cm^3 per second when the sides are each 1 cm.
2. Suppose a circle has radius r, where r is growing at a rate of 3 cm per second. Then the area of the circle is growing twice as quickly when $r = 16$ as when $r = 8$.
3. If $f(x)$ is a function and $L(x)$ is the linearization centred at $x = a$, then $L(b)$ is always close to $f(b)$.
4. If $y = f(x)$ then $dy = f'(x)dx$.
5. If $y = f(x)$ then $\Delta y = f'(x)\Delta x$.
6. The formula used in Newton's Method is obtained by finding the roots of tangent lines.
7. Every function has an absolute maximum on the closed interval $[-1, 1]$.
8. If a function has an absolute maximum on $(-\infty, \infty)$ then it must also have an absolute minimum on $(-\infty, \infty)$.
9. Let $f(x)$ be a continuous function. If $f(a)$ is a local minimum then $f'(a) = 0$.
10. Let $f(x)$ be a continuous and differentiable function. If $f'(a) = 0$ then $f(a)$ is either a local minimum or a local maximum (or perhaps both, for example if $f(x)$ is a constant function).
11. If $f(x)$ is a twice-differentiable function $f''(2) = 0$ then $f(x)$ has a point of inflection at $(2, f(2))$.
12. If $f(x)$ is a twice-differentiable function with $f'(2) = 0$ and $f''(2) = 10$ then $f(x)$ has a local minimum at the point $(2, f(2))$.
13. If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} g(x) = 0$ then we may apply l'Hospital's Rule to $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$.
14. l'Hospital's Rule was proven by Guillaume de l'Hospital, the Marquis de l'Hospital.

Find a Function:

1. Find a function $f(x)$ such that if $y = f(x)$ then the differential Δy is given by $\Delta y = 3\cos(2x)$.
2. Which of the following functions satisfy the hypothesis of the Mean Value Theorem on the given interval, and which do not?
   (a) $f(x) = x^{2/3}$ on $[-3, 5]$
   (b) $f(x) = x^{4/3}$ on $[-3, 5]$
   (c) $f(x) = x^{2/3}$ on $[2, 10]$
   (d) $f(x) = x^{4/3}$ on $[2, 10]$
   (e) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ -1, & x = 1 \end{cases}$ on $[0, 1]$

3. For each of the following functions, determine a value of the constant that would make it be continuous.
   (a) $f(x) = \begin{cases} 0^x, & x \neq 0 \\ a, & x = 0 \end{cases}$
   (b) $g(x) = \begin{cases} x^0, & x \neq 0 \\ a, & x = 0 \end{cases}$
   (c) $h(x) = \begin{cases} x^x, & x \neq 0 \\ a, & x = 0 \end{cases}$
4. Find a pair of functions, $f(x)$ and $g(x)$, such that $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$, and
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 42.$$
5. Find a function $h(x)$ such that $h(x) = 0$ when $x = \sqrt{5}$.
6. Find a function $g(x)$ such that $g(x) = 0$ when $y = \cos(x)$ and $y = x$ intersect.

Explain your answer:

1. In some parts of Northwest British Columbia, speed limits are enforced by aircraft. Suppose that the pilot of a police aircraft is flying at a constant rate of 195 kilometres per hour at a steady height of 5 km above a level, straight road. The police officer sees a car approaching and with radar measures that when the line-of-sight distance between the aircraft and the car is exactly 13 kilometres, the line-of-sight distance is decreasing at a rate of 312 kilometres per hour. Explain why this means the car was at that instant driving at a speed of 143 kilometres per hour. Be sure to include a well-labeled figure.
2. Water is emptying out of a conical funnel (point down) at a rate of 50 cm^3/min. The funnel has height 12cm and radius 6cm at the top. Explain why we know that the radius of the surface of the water is shrinking at a rate of $\frac{1}{\pi}$ cm/min when the water's depth is 10cm. Be sure to include a well-labeled figure.
3. Suppose we are shining a laser pointer at a vertical wall from a point on the ground 4 feet away from the wall, and the light of the laser is moving up a straight line on the wall. (A strange thing to do, perhaps, but notice that it results in a nice right triangle with base 4 and variable height). At the moment when the angle between the laser beam and the ground is $\pi/3$, the light is moving up the wall at a rate of 2 feet per second. Explain why this means the angle between the laser beam and the ground is growing at a rate of $1/8$ radians per second at that moment. Be sure to include a well-labeled figure.
4. The surface area of a cube with side length x is $S = 6x^2$. Use differentials to explain why if the side length of the cube changes by 0.1 the surface area of the cube changes by approximately $1.2x$.
5. The diameter of a tree was measured to be 22in on January 1st, 2015. On January 1st, 2016 the circumference of the tree's trunk was measured to have increased by 1 inch. Use differentials to explain why this means the diameter of the tree had changed by approximately $1/\pi$ inches.

6. Suppose we want to use a Linearization to approximate $\cot(9/4)$. Explain why we should centre our Linearization at $x = 3\pi/4$, and why the resulting approximation would be $3\pi/2 - 11/2$.
7. Show that if $h \neq 0$ then applying Newton's Method to
$$f(x) = \begin{cases} \sqrt{x}, & \text{if} \quad x \geq 0 \\ \sqrt{-x}, & \text{if} \quad x < 0 \end{cases}$$
with initial guess $x_0 = h$ will fail. Draw a picture to illustrate the situation.
Hint: To get started, compute x_1 and x_2 in terms of h. You should see a pattern.
8. If $f'(x_0) = 0$ then Newton's Method with initial guess x_0 will fail to find a root of $y = f(x)$. Explain why, and illustrate your explanation with a sketch.
9. In some parts of Northwest British Columbia, speed limits are enforced by aircraft. This is usually done by painting white lines on the roadway and measuring how long it takes a car to get from one line to another. Suppose that a red car is observed to cross the first line at 10:15am and the second line, which is 300 meters away, 10 seconds later. Explain why this means the red car was definitely exceeding the speed limit of 100 km/h at some time during those 10 seconds.
10. In some parts of Northwest British Columbia, speed limits are enforced by aircraft. This is usually done by painting white lines on the roadway and measuring how long it takes a car to get from one line to another. Explain why it is possible for a car to be exceeding the speed limit but still not be caught by this method.
11. A runner completes a 6.2 mile race in 33 minutes. Use the Mean Value Theorem to explain why she must have been running at *exactly* 11 miles per hour at least **two times** during the race. Assume that her instantaneous velocity at the start of the race and at the end of the race is 0 miles per hour.
12. Suppose that $g'(x) = (x-2)^2(3-x)(x+5)$. Explain why we know that $g(x)$ is increasing on the interval $(-5, 3)$.
13. Suppose that $g'(x) = (x-2)^2(3-x)(x+5)$. Explain why we know that $g(x)$ has a local minimum at $x = -5$ and a local maximum at $x = 3$.
14. Explain why l'Hospital's Rule *can* be applied to the following limit but will *not* help you evaluate it:
$$\lim_{x\to\infty} \frac{2^x - 3^x}{3^x + 4^x}$$
15. The function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x$ has a local maximum at $x = -3$. Explain how to prove this with the *second derivative test*.
16. Explain why it is impossible for a function $g(x)$ to satisfy all of the following properties at the same time:
$$g'(x) > 0 \text{ for } x \text{ in } (-\infty, 2),\ g'(x) < 0 \text{ for } x \text{ in } (2, \infty),\ g''(2) > 0.$$
17. Use calculus to explain why the largest area of a rectangle with perimeter 8m is 4m^2. Be sure to explain why this area is *largest* and not *smallest*.
18. Explain why the *lightest* open-top right circular cylindrical can made from aluminium that can hold $1000m^3$ has radius $\frac{10}{\sqrt[3]{\pi}}m$.

Worksheets

Related Rates of Change

1. A garbage compactor presses a cube of waste into a smaller cube in such a way that the volume of the cube decreases at 4 cubic metres per minute. How fast is the surface area of the cube changing at the moment when the volume of the cube is 64 cubic metres?

2. A kite is flying at a height of 40 metres above the ground, and the wind is causing the kite to move horizontally at a rate of 5 metres per second away from the person holding the string on the ground. Assuming the kite string is always a straight line, how fast is the length of kite string increasing when 50 metres of string is already out?

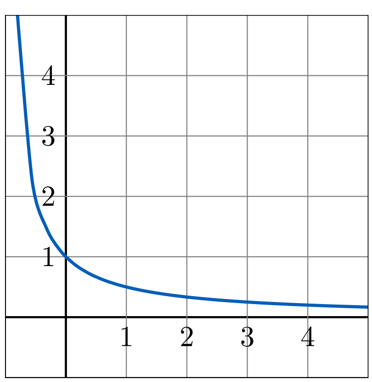
3. According to the special theory of relativity, a particle whose mass at rest is M moving with velocity v will have mass $\dfrac{M}{\sqrt{1-v^2/c^2}}$, where c is the (constant) speed of light. Suppose that at a particular moment a particle is travelling with velocity $v = \frac{1}{4}c$ and has acceleration $a = \frac{1}{10}c$. How fast is the mass of the particle changing at that moment? Express your answer in terms of c.

4. Suppose that a point is moving in the plane so that at time t (in seconds) the point is located at the point in the first quadrant where the curves $xy = t$ and $y = x + t$ intersect, where x and y are measured in centimetres. How fast is the distance from the point to the origin changing when $t = 1$? *Note*: Your final answer should be in exact form (i.e., no decimal approximations), but you do not need to simplify.

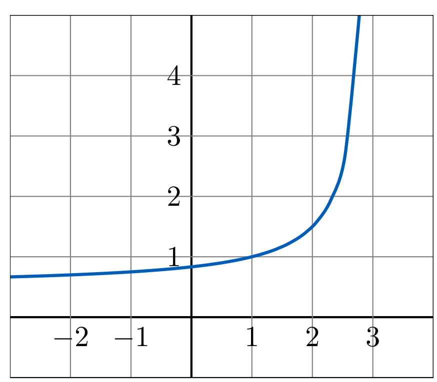
5. Using the graphs of $y = f(x)$ and $y = g(x)$ given below, use the appropriate formula that relates $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, and $\dfrac{dy}{dx}$ to answer the following questions.

   (a) If $y = f(x)$ and $\dfrac{dx}{dt}$ is negative, is $\dfrac{dy}{dt}$ is positive or negative?

   (b) If $y = g(x)$ and $\dfrac{dy}{dt}$ is positive, is $\dfrac{dx}{dt}$ is positive or negative?



$y = f(x)$



$y = g(x)$

Relating Rates

1. Unit conversions:

   (a) There are 12 inches in a foot and there are 60 seconds in a minute. If the area of an object is increasing at a rate of 36 in^2 per minute, at what rate is it changing in ft^2 per second?

   (b) There are 360 degrees in a circle, and 2π radians in a circle. If a wheel is rotating at a rate of 90 degrees per minute, at what rate is it rotating in radians per second?

2. A rectangle has side lengths x and y, where both are varying with respect to time. At the instant that $x = 9$ and $y = 3$, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -2$.

   (a) At what rate is the area of the rectangle changing at that moment? Is the area growing or shrinking?

   (b) At what rate is the perimeter of the rectangle changing at that moment? Is the perimeter growing or shrinking?

3. The radius of a sphere is decreasing at a rate of 2 in/sec.

   (a) When the radius is exactly 4 inches, at what rate does the sphere's **volume**[e] change?

   (b) When the radius is exactly 4 inches, at what rate does the sphere's **surface area**[f] change?

4. Sand is falling into a conical pile in such a way that the diameter of the base of the cone is always exactly equal to its height (this seems pretty unlikely, but it makes the problem work out nicely). Suppose the sand is being added to the pile at a rate of 150 m^3 per minute. When the height of the sand pile is 10m, at what rate is the radius changing? Is the radius growing or shrinking? *Hint: the volume of a cone with radius r and height h is $\pi r^2 h/3$.*

5. A 13-ft ladder is leaning against a house when its base starts to slide away from the wall. At the instant when the base is 12 ft from the house, the base is moving at a rate of 5ft/sec. *Note: you may not assume that the base is moving at a constant rate.*

   (a) Sketch a picture of this situation, labeling it well. Which lengths in your figure are constant and which are varying with time?

   (b) How quickly is the top of the ladder sliding when the base is 12 ft from the house?

   (c) Interpret your answer in terms of positive rate of change or negative rate of change: is the top of the ladder sliding down or up?

   (d) Let θ be the angle between the ladder and the ground. At what rate is θ changing when the base is 12 ft from the house? Is this angle getting larger or smaller?

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[e] Volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

[f] Surface area of a sphere with radius r is $4\pi r^2$.

Curve Sketching with Calculus

1. Sketch a function $y = g(x)$ such that:
   - $g(0) = 1$,
   - $\lim_{x\to\infty} g(x) = 1$ and $\lim_{x\to-\infty} g(x) = -1$
   - $g'(x)$ is positive on $(-\infty, 2) \cup (2, 5)$,
   - $g'(x)$ is negative on $(5, \infty)$,
   - $g''(x)$ is positive on $(-\infty, 2) \cup (3, 5) \cup (5, \infty)$, and
   - $g''(x)$ is negative on $(2, 3)$.

2. Sketch $y = g(x)$ **without using a calculator** given that $g(0) = 2$ and $g'(x) = x(x^2 - 9)$. Be sure to determine on what intervals $g(x)$ is concave up or concave down. *You do not need to find a formula for $g(x)$.*

3. Consider the function $f(x) = \sqrt[3]{x^2 - 4}$.

   (a) What are the x-intercepts and y-intercepts of $y = f(x)$?

   (b) On what intervals is $f(x)$ increasing? decreasing? You may use the fact that

   $$f'(x) = \frac{2x}{3(x^2 - 4)^{2/3}}.$$

   (c) According to the First Derivative Test, at what values of x does $f(x)$ have local extrema? Which are maxima and which are minima?

   (d) On what intervals is $f(x)$ concave up? concave down? You may use the fact that

   $$f''(x) = \frac{-2(x^2 + 12)}{9(x^2 - 4)^{5/3}}.$$

   (e) What are the points of inflection (if any) of $f(x)$?

   (f) According to the Second Derivative Test, at what values of x does $f(x)$ have local extrema? Which are maxima and which are minima? If your answer to this question does not agree with your answer to (3c) then you have made an error somewhere – find and correct it!

   (g) Use your answers to sketch a graph of $y = f(x)$.

Sketchy

1. Sketch $y = g(x)$ **without using a calculator** given that $g(1) = 3$ and $g'(x) = (x-1)(x^2-25)$. Your sketch should include the features of the graph that you can find *without* finding a formula for $g(x)$. Be sure to determine where $g(x)$ is concave up or concave down.

2. Sketch a function $y = h(x)$ such that:

   - $h(0) = 1$,
   - $\lim_{x\to\infty} h(x) = 1$ and $\lim_{x\to-\infty} h(x) = -1$
   - $h'(x)$ is positive on $(-\infty, 2) \cup (2, 5)$,
   - $h'(x)$ is negative on $(5, \infty)$,
   - $h''(x)$ is positive on $(-\infty, 2) \cup (3, 5) \cup (5, \infty)$, and
   - $h''(x)$ is negative on $(2, 3)$.

3. Sketch a function $y = f(x)$ such that $f(-3) = 5$ and $f(1) = 5$ but $f'(x)$ is never 0. Explain why this does not contradict Rolle's Theorem.

4. Sketch a function $y = r(x)$ such that $r(-2) = 3$ and $r(2) = 5$, and $r(x)$ is differentiable on the interval $(-3, 3)$. What does the Mean Value Theorem tell you about $r'(x)$? Indicate in your sketch the location(s) of the point(s) guaranteed to exist by the Mean Value Theorem.

5. Sketch a function $y = s(x)$ that is continuous on $[-4, 4]$, has a local maximum at $x = 0$, but does not have a horizontal tangent line at $x = 0$.

6. Sketch a function that is continuous and differentiable on $[-4, 4]$ and has an absolute maximum at $x = 4$ and a local maximum at $x = 2$.

Approximations

1. **Linearizations**

   (a) If $f(x)$ is a function, what is the *linearization*[9] of $y = f(x)$ at the point $x = a$?

   (b) The linearization, $L(x)$, can be used to approximate $f(x)$ for values of x that are close to a. Use a linearization at $x = 8$ to approximate $f(8.1)$, where $f(x) = \sqrt[3]{x}$.

2. **Newton's Method**

   (a) If $f(x)$ is a function, and x_n is our current estimate of a root of $f(x)$, what does *Newton's Method* give us as our next estimate, x_{n+1}?

   (b) Newton's Method can be used to approximate the roots of functions. Use Newton's Method with initial guess $x_0 = 2$ to approximate the root of $f(x) = x^3 - 8.1$ (just find x_1).

3. **Differentials**

   (a) If $y = f(x)$, what is the differential dy?

   (b) Differentials can be used to approximate change in y caused by change in x, because we assume that $\Delta y \approx dy$ and $\Delta x = dx$. Use differentials to approximate the difference between $\sqrt[3]{8.1}$ and $\sqrt[3]{8}$ by letting $y = \sqrt[3]{x}$. Use this to approximate $\sqrt[3]{8.1}$.

4. **Connections**

   (a) Explain why both linearization and Newton's Method resulted in an approximation of the number $\sqrt[3]{8.1}$.

   (b) Explain why the linearization and differentials methods both resulted in exactly the same approximation. It might help to draw a picture.

   (c) Newton's Method is an iterative process, unlike linearization or differentials. Improve your approximation of $\sqrt[3]{8.1}$ by performing another iteration of Newton's Method to find x_2.

5. Approximate each of the following numbers by first using an appropriate linearization and then by using Newton's Method on an appropriate function.

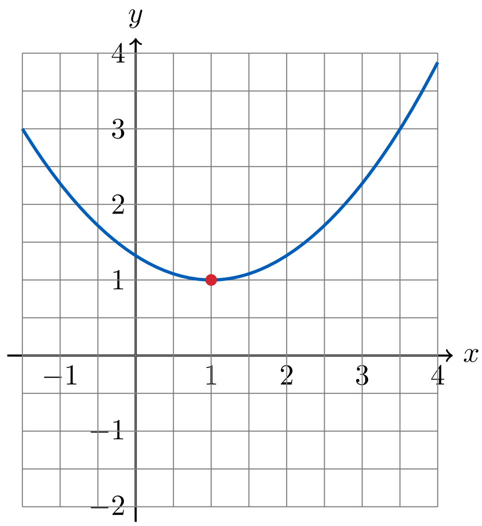
   (a) $\sqrt[4]{81.1}$

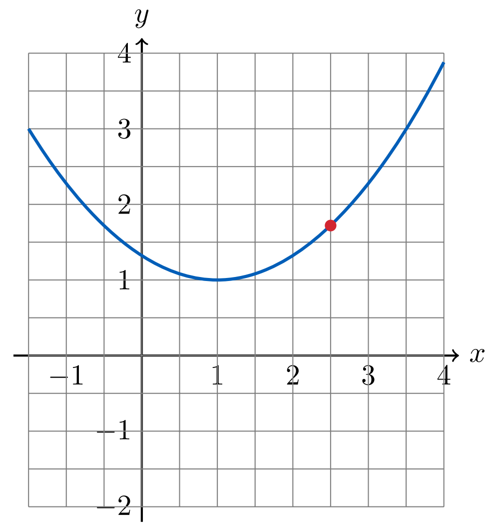
   (b) $e^{1.01}$

Continued on next page . . .

[9]The word "linearization" comes from the word "linear" – why does that make sense?

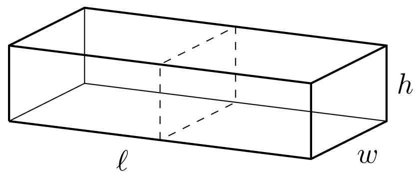
6. Consider the curve $y = f(x)$ below. Sketch the linearization at $x = 1$. For what values of x is the linearization within 0.5 of $f(x)$? Now sketch the linearization at $x = 2.5$. For what values of x is *that* linearization within 0.5 of $f(x)$?





Optimizing and approximating

1. You are going to ship a rectangular box **with a square end** (so $h = w$ in the figure below). The 'length' of the box is the length of its longest side, ℓ. The 'girth' of the box is the distance around it (dashed line in figure below):



   (a) The U.S. Postal Service will not accept any "mailpiece" whose length plus girth is more than 108 inches. What are the dimensions of the largest-volume box you can ship this way?[h]

   (b) Canada Post considers a parcel to be a "*Regular Parcel* item" if its length is at most 2m and the sum of its length and girth is at most 3m. What are the dimensions of the largest-volume box you can ship this way?[i]

   (c) Given that 1m$=3.28084$ft$=39.3701$in, which of these boxes has the largest volume?

2. You want to make a can in the shape of a cylinder with a top and bottom to hold a volume V of liquid. Show that the can with minimum surface area has height equal to twice its radius.

3. What point on the curve $y = \sqrt{x}$ is closest to the point $(3, 0)$?

4. Approximate $\ln(1.01)$ by using an appropriate linearization, and by using one iteration of Newton's method on an appropriate function.

5. Explain why we cannot use Newton's method to find a root of $f(x) = x^3 - 2x + 2$ with initial guess $x_0 = 1$. Illustrate your explanation with a sketch. *Note*: To see the problem you will need to do more than one iteration of Newton's method.

[h] `https://pe.usps.com/businessmail101?ViewName=Parcels` You might find this problem easier to work with in feet (there are 12 inches in a foot).

[i] `https://www.canadapost-postescanada.ca/tools/pg/4_Preparing/Prep_parcels-e.pdf`

It shall not derive me!

1. **It's all about the functions**:

   (a) Find all asymptotes of the curve $y = \dfrac{x^3}{2(x+1)^2}$, or prove that they do not exist.

   (b) Analyse the function $y = \arctan(x) - x$ to find:

      i. All points of the inflection;

      ii. All the intervals where the function is concave upward

      iii. All the intervals where the function is concave downward.

   (c) Find the maximum and the minimum values of the function $y = x + 3\sqrt[3]{x}$ on the interval $[-1, 1]$.

   (d) Prove algebraically that $x > \ln(1+x)$ for all $x > 0$. (Hint: it might help to graph both functions $f(x) = x$ and $g(x) = \ln(1+x)$ and observe the behaviour of those functions).

2. **Detailed graph**: Complete the study of the function $y = -\ln(x^2 - 4x + 5)$ and graph this function. (Hint: Did you find and label points of the inflection, local and global extremum, all asymptotes, etc?)

3. **Need for Speed**: A body is moving along the straight line OX based on the following law:
   $x = \dfrac{1}{3}t^3 - \dfrac{7}{2}t^2 + 10t - 16$.
   Determine the “velocity" and “acceleration" of this body. At what moment(s) in time does the direction of the motion of the body change?

l'Hospitalization

Note:

Remember always to indicate whether you are using l'Hospital's Rule and (if you are) why it is applicable.

1. Do **not** evaluate any of the following limits:[j]

$\lim_{x\to\infty} \frac{x^2+13}{13x-1}$	$\lim_{x\to 1} \frac{\ln(x)}{x}$	$\lim_{x\to 0^+} \frac{e^x}{11x}$	$\lim_{x\to 2} \frac{2x^2-8}{e^x-e^2}$
$\lim_{x\to\infty}(x^2+1)e^{-x}$	$\lim_{x\to 1}(x-1)\ln(x)$	$\lim_{x\to 0^+} \ln(x)\sin(x)$	$\lim_{x\to\infty} e^x \ln(x)$
$\lim_{x\to\infty} x^x$	$\lim_{x\to 1}(x-1)^{\ln(x)}$	$\lim_{x\to 0^+} \ln(x)^{x^2+13}$	$\lim_{x\to 2}(x-1)^{1-\frac{2}{x}}$

   (a) Which of the above limits are in indeterminate form? (Indicate type).

   (b) Which are in a correct form for using l'Hospital's Rule?

2. Let's examine "0^0"...

   (a) John says that $0^0 = 0$, because $0^x = 0$ for any x. Convince John that he is wrong by evaluating $\lim_{x\to 0} x^0$, which should equal 0^0 if 0^0 is a number.

   (b) Tracey says that $0^0 = 1$, because $x^0 = 1$ for any x. Convince Tracey that she is wrong by evaluating $\lim_{x\to 0} 0^x$, which should equal 0^0 if 0^0 is a number.

   (c) Pat tries to compromise between John and Tracey by setting $L = \lim_{x\to 0^+} x^x$ and then evaluating $\ln(L) = \lim_{x\to 0^+} \ln(x^x)$. This will require l'Hospital's rule – what does Pat discover?[k]

3. The *hyperbolic tangent* function is defined by

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

   In this problem we will use the hyperbolic tangent function to see that l'Hospital's rule is not always the best way to evaluate limits.

   (a) Explain why the limit $\lim\limits_{x\to\infty} \tanh(x) = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is in a suitable form for applying l'Hospital's rule.

   (b) Attempt to evaluate $\lim\limits_{x\to\infty} \tanh(x)$ by applying l'Hospital's rule. What went wrong? (You will need to use l'Hospital's rule more than once to see the problem).

   (c) Without using l'Hospital's rule, show that $\lim\limits_{x\to\infty} \tanh(x) = 1$.

[j] Unless you really want to and have finished the rest of the worksheet.

[k] This doesn't prove that Tracey is correct, by the way – because we have found two limits of the form "0^0" that have different values, we have demonstrated that "0^0" is an indeterminate form.

Choose Your Own Adventure

l'Hospitalization

Remember always to indicate whether you are using l'Hospital's Rule and (if you are) why it is applicable.

1. Do **not** evaluate any of the following limits:[1]

$$\begin{array}{llll} \lim_{x\to\infty} \frac{x^2+13}{13x-1} & \lim_{x\to 1} \frac{\ln(x)}{x} & \lim_{x\to 0^+} \frac{e^x}{11x} & \lim_{x\to 2} \frac{2x^2-8}{e^x-e^2} \\ \lim_{x\to\infty}(x^2+1)e^{-x} & \lim_{x\to 1}(x-1)\ln(x) & \lim_{x\to 0^+} \ln(x)\sin(x) & \lim_{x\to\infty} e^x \ln(x) \\ \lim_{x\to\infty} x^x & \lim_{x\to 1}(x-1)^{\ln(x)} & \lim_{x\to 0^+} \ln(x)^{x^2+13} & \lim_{x\to 2}(x-1)^{1-\frac{2}{x}} \end{array}$$

   (a) Which of the above limits are in indeterminate form? (Indicate type).

   (b) Which are in a correct form for using l'Hospital's Rule?

2. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x^n}$, where n is *any* positive integer.

   (b) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n}$, where n is *any* positive integer.
   Hint: It might help to first consider $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}, \lim_{x\to\infty} \frac{e^x}{x^2}, \lim_{x\to\infty} \frac{e^x}{x^3}, \ldots$.

3. In this problem we will explore why "1^∞" is an indeterminate form.

   (a) Prove that $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x^2}\right)^x = 1$.

   (b) Prove that $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

   (c) Prove that $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{\ln(x)}\right)^x = \infty$.

Continued on the next page . . .

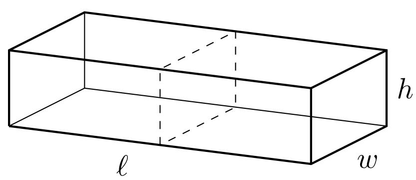
[1]Unless you really want to and have finished the rest of the worksheet – including the second page!

Choose Your Own Adventure, cont.

Optimization

1. You are going to make an open-top rectangular box by taking an 8cm-by-15cm piece of cardboard, cutting identically-sized squares from each of the corners, and folding up the resulting sides. What sized squares should you cut out in order to make the largest-volume box this way?

2. You are going to ship a rectangular box **with a square end** (so $h = w$ in the figure below). The 'length' of the box is the length of its longest side, ℓ. The 'girth' of the box is the distance around it (dashed line in figure below):



   (a) The U.S. Postal Service will not accept any "mailpiece" whose length plus girth is more than 108 inches. What are the dimensions of the largest-volume box you can ship this way?[m]

   (b) Canada Post considers a parcel to be a "*Regular Parcel* item" if its length is at most 2m and the sum of its length and girth is at most 3m. What are the dimensions of the largest-volume box you can ship this way?[n]

   (c) Given that 1m$=3.28084$ft$=39.3701$in, which of these boxes has the largest volume?

3. What point on the curve $y = \sqrt{x}$ is closest to the point $(3, 0)$?

[m] `https://pe.usps.com/businessmail101?ViewName=Parcels` You might find this problem easier to work with in feet (there are 12 inches in a foot).

[n] `https://www.canadapost-postescanada.ca/tools/pg/4_Preparing/Prep_parcels-e.pdf`

“Check Your Understanding” Answers

True or False?

1. False.
2. True.
3. False. The linearization is almost never a good approximation if b is far from a.
4. True.
5. False.
6. True.
7. False. Consider for example $1/x$.
8. False. Consider for example $-x^2$.
9. False. Consider for example $|x-a|$.
10. False. Consider for example $f(x) = x^3$.
11. False.
12. True.
13. False.
14. False: his tutor Johann Bernoulli discovered the rule.

Find a Function:

1. One such function is $f(x) = \frac{3}{2}\sin(2x)$. Another such function is $f(x) = \frac{3}{2}\sin(2x) + 42$ – we will discuss this further in the Integration chapter.
2. Which of the following functions satisfy the hypothesis of the Mean Value Theorem on the given interval, and which do not?
   (a) $f(x) = x^{2/3}$ on $[-3, 5]$ Not differentiable at 0, so MVT does not apply.
   (b) $f(x) = x^{4/3}$ on $[-3, 5]$ MVT applies.
   (c) $f(x) = x^{2/3}$ on $[2, 10]$ MVT applies.
   (d) $f(x) = x^{4/3}$ on $[2, 10]$ MVT applies.
   (e) $f(x) = \begin{cases} x, & 0 \le x < 1 \\ -1, & x = 1 \end{cases}$ Not continuous at $x = 1$, so MVT does not apply.
3. For f, $a = 0$. For g, $a = 1$. For h, $a = 1$. (This exercise demonstrates that 0^0 is an indeterminate form, by the way – can you explain why?)
4. One such pair is $f(x) = 42x$ and $g(x) = x$.
5. One such function is $h(x) = x^2 - 5$. (Another such function is $x - \sqrt{5}$, but that would not be as useful for using Newton’s Method to approximate $\sqrt{5}$.)
6. One such function is $g(x) = \cos(x) - x$.

Explain your answers:

1. Hint: use Related Rates. Don’t forget that the horizontal distance between the aircraft and the car is decreasing at a rate of $195 - x$ kilometres per hour, where x is the car’s speed.
2. Hint: use Related Rates. The volume of a cone with height h and radius r is $\frac{\pi}{3}r^2h$ – you need to find a relation between h and r.

3. Hint: use Related Rates. We need to relate the rate of change of one side of the right triangle to the rate of change of its opposite angle, which means you will need trigonometry. I recommend $\tan(\theta)$.
4. Hint: ΔS is approximately $dS = 12x dx$, and $dx = \Delta x$.
5. Hint: if D is the diameter of the tree and c is its circumference then $c = \pi D$. Use differentials to approximately related Δc and ΔD, then solve for ΔD.
6. Hint: linearizations work best when centred near the point of interest.
7. Hint: already given.
8. Hint: start with the sketch and remember that x_1 is the x-intercept of the tangent line at x_0.
9. Hint: this is actually how speed limits are enforced by aircraft, and it's much easier than using related rates as you did earlier. Use the Mean Value Theorem.
10. Hint: think about the Mean Value Theorem and its limitations. Disclaimer: this is not driving advice.
11. Hint: in addition to the Mean Value Theorem, you will need to use the Intermediate Value Theorem. Start by proving that the runner's speed was exactly $11.\overline{27}$ miles per hour at some moment during the race.
12. Hint: remember that $g(x)$ is increasing if $g'(x) > 0$.
13. Hint: First Derivative Test.
14. Hint: Try l'Hospital's Rule and see if it helps.
15. Hint: Second Derivative Test.
16. Hint: Compare the results of the First Derivative Test with the results of the Second Derivative Test at $x = 2$.
17. Hint: Optimization.
18. Hint: Optimization. An open-top cylinder with height h and radius r has volume $\pi r^2 h$ and surface area $\pi r^2 + 2\pi r h$.

$\int$ Integrals

In your textbook: *This material corresponds to Chapter 4, section 4.8, and Chapter 5.*

Before you begin: this chapter relies on the following pre-calculus material.

- Standard functions (FE)
- Notation (N)

Chapter Checklist

4.8: Antiderivatives

- You should know…
  - ☐ the definition of antiderivative.
  - ☐ the definitions of indefinite integral, integrand, and variable of integration.
- You should understand…
  - ☐ the relationship between a derivative and an antiderivative.
- You should be able to compute…
  - ☐ the most general antiderivative of many functions, including polynomials, $\sin(x)$, $\cos(x)$, $\sec^2(x)$, and some types of rational functions.

5.1: Estimating area with finite sums

This section seems unrelated to the previous chapter, but the area under the curve will turn out to be related to antiderivatives.

- You should know…
  - ☐ the definition of *average value*.
  - ☐ what is meant by *upper sum* and *lower sum*.
- You should understand…
  - ☐ the meaning of the area under a curve that represents, for example, position.
  - ☐ the difference between *displacement* and *distance traveled*.
- You should be able to compute…
  - ☐ estimates of the area under a curve by using a fixed number of rectangles.

5.2: Sigma notation & Limits of finite sums

Improving the estimates from the previous section requires more and more rectangles, which brings us back to the notion of a limit.

- You should know...
  - ☐ what the notation $\sum_{k=1}^{n} a_k$ means.
  - ☐ algebra rules for finite sums.
  - ☐ that $\sum_{k=1}^{n} c = nc$ and $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
  - ☐ the definition of (and notation for) a Riemann Sum.
- You should understand...
  - ☐ the relationship between Riemann Sums and the area under a curve.
- You should be able to compute...
  - ☐ $\sum_{k=1}^{n} a_k$ given a_k and n. For example, $\sum_{k=1}^{4}(2k+1) = 3+5+7+9 = 24$.
  - ☐ the limit $\lim_{n\to\infty} \sum_{k=1}^{n}(\frac{c}{n} + \frac{dk}{n^2})$ for any real numbers c and d.
  - ☐ the area under the curve $y = cx + d$ on the interval $[a, b]$ by using a Riemann Sum and a limit.

5.3: The definite integral

A definite integral measures the signed area under a curve.

- You should know...
  - ☐ what a definite integral is.
  - ☐ the algebraic properties of definite integrals.
- You should understand...
  - ☐ the relationship between Riemann Sums and definite integrals.
  - ☐ the relationship between definite integrals and area under a curve.
  - ☐ when the area under a curve is negative and when it is positive.
- You should be able to compute...
  - ☐ a definite integral by using known area formulas.
  - ☐ a definite integral by combining other, known, definite integrals.
  - ☐ the average value of a function using definite integrals.
  - ☐ the sign (negative or positive) of a definite integral.

5.4: The Fundamental Theorem of Calculus

As their names suggest, the theorems in this section are fundamental to (integral) calculus.

- You should know...
  - ☐ the Fundamental Theorem of Calculus, Part 1.
  - ☐ the Fundamental Theorem of Calculus, Part 2.
  - ☐ the Net Change Theorem
- You should understand...
  - ☐ how the FToC, Part 2, is related to the FToC, Part 1.

- ☐ how the Net Change Theorem is related to the FToC, Part 2.

- You should be able to compute...
  - ☐ definite integrals using the Fundamental Theorem of Calculus, Part 2.
  - ☐ the derivative of definite integrals by using the Fundamental Theorem of Calculus, Part 1. Examples: $\int_a^x f(t)dt$ or $\int_a^{2e^x} f(t)dt$ or $\int_{\ln(x)}^{x^2} f(t)dt$

5.5: Substitution
This section is all about undoing the Chain Rule. In MATH 101 you will learn more methods for finding the antiderivative of complicated functions.

- You should understand...
  - ☐ the relationship between the Chain Rule and the method of Substitution.
- You should be able to compute...
  - ☐ indefinite integrals of the form $\int f(g(x))g'(x)dx$.

5.6: Area between curves
This section actually has two parts: using the method of substitution with definite integrals, and computing the area between curves.

- You should know...
  - ☐ the difference between signed area and total area.
  - ☐ what the area of the region between two curves is.
- You should understand...
  - ☐ the time-saving tricks you can use for definite integrals of the form $\int_{-a}^{a} f(x)dx$ when $f(x)$ is an even function or $f(x)$ is an odd function.
  - ☐ when it is easier to compute the area of a region by using y as the variable of integration rather than x.
- You should be able to compute...
  - ☐ definite integrals using the substitution method.
  - ☐ the area under a curve or between two curves, sometimes with y as the variable of integration.

8.7: Numerical Integration
Some integrals cannot be directly computed. This section describes two methods of approximation that are better than approximating by rectangles.

- You should understand...
  - ☐ how to approximate area under a curve with a finite number of trapezoids.
  - ☐ Simpson's Rule.
- You should be able to compute...
  - ☐ an approximation of the area under a curve with 4, 5, or 6 trapezoids.

Check Your Understanding

Fill in the blanks:

$$\sum_{k=1}^{n} c = \qquad \sum_{k=1}^{n} k = \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

True or False?

For each of the following statements, determine whether it is True or False. If True, explain why. If False, provide a counterexample.

1. $\arctan(2x) + 13$ is an antiderivative of $\frac{2}{1+4x^2}$.
2. If $f(x)$ is an antiderivative of $g(x)$ then so is $f(x) + \pi$.
3. The most general antiderivative of $\sec^4(x)$ is $\tan^2(x) + C$.
4. If $s(t)$ represents the amount of money deposited into Bill's bank account on day t of March then the area under the curve $y = s(t)$ on the interval $[0, 10]$ represents the total amount of money Bill's account increased by (could be negative) over the first ten days of March.
5. $\displaystyle\sum_{k=1}^{n} 2k = n(n+1)$.
6. $\displaystyle\sum_{k=4}^{6} \frac{k}{2k+1} = \sum_{k=1}^{6} \frac{k}{2k+1} - \sum_{k=1}^{3} \frac{k}{2k+1}$.
7. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{54k^2}{n^3} - \frac{9k}{n^2} \right) = \int_0^3 \left(2x^2 - x\right)\, dx$.
8. If $\int_1^5 f(x)dx = 11$ and $\int_1^7 f(x)dx = 8$ then $\int_5^7 f(x)dx = -3$.
9. $\int_4^x f(t)dt$ is a function of x.
10. If $F(x)$ is an antiderivative of $f(x)$ and $G(x) = F(x) + 13$ then $\int_a^b f(x)dx = G(b) - G(a)$.
11. $\int_0^4 \sin(2x)dx = \int \frac{1}{2}\sin(u)du$.
12. $\int_0^4 \sin(2x)dx = \int_0^4 \frac{1}{2}\sin(u)du$.
13. The area between two curves is sometimes negative.
14. If $v(t)$ represents the velocity of a car at time t then the signed area under the curve $y = v(t)$ on $[0, 10]$ represents the *total distance* the car traveled on $[0, 10]$.
15. If $f(x)$ is an odd function then $\int_{-5}^0 f(x)dx = \int_5^0 f(x)dx$.

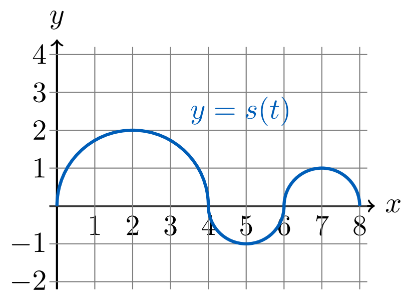
Find a Function:

1. Find a function whose derivative is $\cos^4(x)\sin(x)$.

2. Find a function $f(x)$ such that $f'(x) = 3x^2 + 4x - 12$ and $f(1) = -6$.

3. Find a definite integral equal to $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{8k^2}{n^3} + \frac{2}{n} \right)$. (Bonus: evaluate it using sum rules and limits.)

4. Suppose the curve below represents a particle's distance from a fixed point as a function of time. Find (but do not evaluate) an integral or sum of integrals that represent the *total distance* the particle traveled over the time interval $[0, 8]$. Find (but do not evaluate) an integral or sum of integrals that represent the particle's *displacement* over the time interval $[0, 8]$.



5. Find a function $g(t)$ such that the derivative of $\int_{-5}^{x^2} g(t)dt$ is $2x(1 + x^6)^2$.

Explain your answers:

1. Explain why there are infinitely many functions $g(x)$ such that $g''(x) = x$ and $g'(1) = 2$, and express this infinite family of functions algebraically.

2. Consider the curve $y = \cos(x)$ on the interval $[0, \pi/2]$. If we estimate the area under this curve using a finite number of rectangles and the *midpoint rule* then we will get an *overestimate* of the area. Explain why.
Hint: to explain this rigorously you will need to use both the first and second derivative.

3. Use geometry to explain why the area under the curve $\sqrt{16 - x^2}$ on the interval $[0, 4]$ is exactly 4π. Illustrate with a well-labeled sketch.

4. You have already explained why the area under the curve $\sqrt{16 - x^2}$ on the interval $[0, 4]$ is 4π. Explain why this means that the average value of the curve on that interval is π, and illustrate by drawing the appropriate horizontal line on your sketch above.

5. Explain why $\sum_{k=1}^{100} \sqrt{k} = \sum_{k=5}^{104} \sqrt{k-4}$ without evaluating either sum.

6. Suppose that the average value of $f(x)$ on the interval $[a, b]$ is C. Explain why $\int_a^b C dx = \int_a^b f(x)dx$.

7. Suppose that $f(x) = \int_0^{\tan(x)} \frac{1}{1+t^2} dt$. Explain why $f'(x) = 1$ for all x in the domain of $f(x)$. What does that tell you about $f(x)$?

8. Suppose that a runner's velocity at time t and acceleration at time t are given by $v(t)$ and $a(t)$, respectively. If the runner starts to run at $t = 0$ and comes to a stop at $t = 8$, explain why $\int_0^8 a(t)dt = 0$.

9. Consider the curve $y = (x+1)^2$ on the interval $[0, 4]$ and the curve $y = x^2$ on the interval $[1, 5]$.

   (a) Explain why the areas under these two curves on these two intervals should be the same, without evaluating either, and illustrate with a sketch.

   (b) The Riemann sum below can represent either the area under the curve $y = (x+1)^2$ on the interval $[0, 4]$ or the area under the curve $y = x^2$ on the interval $[1, 5]$. Verify this claim by finding Δ_k, c_k, and the Riemann sum S_n for each.

$$\sum_{k=1}^{n} \left(\left(\frac{4k}{n} + 1 \right)^2 \frac{4}{n} \right)$$

   (c) Now use the Fundamental Theorem of Calculus, Part 2, to evaluate the two areas explicitly and verify that they are both equal to $124/3$.

10. Suppose that f is continuous on $(-\infty, \infty)$ and g is differentiable on $(-\infty, \infty)$. Explain why for any constant a

$$\frac{d}{dx} \left[\int_a^{g(x)} f(t)dt \right] = f(g(x))g'(x).$$

11. Explain why the total area of the region between $x = y^3$ and $x = y^2$ is $1/12$. Do you think this integral is easier to compute in "dy" form or in "dx" form?

12. Explain why the total area of the region between $y = x^2 + x + 10$ and $y = -x^2 + x + 15$ is $2 \displaystyle\int_0^{\sqrt{5/2}} (5 - 2x^2)dx$.

13. *Without using the Evaluation Theorem* (a.k.a. the Fundamental Theorem of Calculus, Part 2) explain why $\displaystyle\int_{\ln(0.5)}^{\ln(2)} \frac{10(x^2+3)}{\sqrt[3]{x}} dx = 0$.

14. *Without using the Evaluation Theorem* (a.k.a. the Fundamental Theorem of Calculus, Part 2) explain why $\displaystyle\int_{\ln(1)}^{0} \sqrt{1 + \sin^2(x)} dx = 0$.

15. The area of a trapezoid with base lengths b_1 and b_2 and height h is $\frac{1}{2}(b_1 + b_2)h$. Use this fact to explain the trapezoid rule:

> Suppose that $[a, b]$ is partitioned into n equally-sized subintervals with endpoints $a = x_0$, x_1, x_2, $\ldots x_{n-1}$, $x_n = b$ and $f(x)$ is integrable over $[a, b]$. Then
>
> $$\int_a^b f(x)dx \approx \frac{b-a}{2n} \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right).$$

Worksheets

Deer John

1. Find the most general antiderivative of each of the following functions.

   (a) $x^3 + 2x^2 + 3x + \frac{4}{x}$

   (b) $\sin(3x)$

   (c) e^{2x}

   (d) 2^x

2. Evaluate the following finite sums.

   (a) $\sum_{k=1}^{100} 1$

   (b) $\sum_{k=1}^{100}(2k+3)$

   (c) $\sum_{k=25}^{100}(2k+3)$

   (d) $\sum_{k=1}^{n} nk$

3. John claims that the most general antiderivative of $\frac{5}{\sqrt{1-x^2}}$ is

$$10\sqrt{1-x^2} + C.$$

   Check his answer by using differentiation. Is he correct? If not, what is the correct antiderivative?

4. John is driving near campus one evening. At time $t = 0$ he sees a deer in the road in front of him. Perception time plus reaction time ranges between 0.5 seconds at 1.25 seconds before most people are able to slam on the brakes. The following table records John's speed at various times during the tense seconds that follow, in meters per second:

t:	0	0.5	1.0	1.5	2.0	2.5	3
speed:	11.11	11.11	8.86	6.61	2.36	2.11	0

   If the deer was 23 m away from John's car's front bumper at $t = 0$, was he able to stop in time? What if the deer had been 20 m away at $t = 0$?

5. Approximate the area under the curve $y = \sqrt{x^2 - 1}$ on the interval $[1, 4]$ in the following three ways.

   (a) Using three rectangles whose height is determined by the right endpoints.

   (b) Using three rectangles whose height is determined by the left endpoints.

   (c) Using three rectangles whose height is determined by the midpoints.

   Which of the above estimates are *overestimates* and which are *underestimates*? Explain.

Finite sums

1. Consider the function $f(x) = x - 2$.
   (a) Estimate the area under $f(x)$ from $x = -1$ to $x = 2$ by using a Riemann sum with 3 equally sized subintervals and right endpoints.
   (b) Sketch $f(x)$ from $x = -1$ to $x = 2$, along with the rectangles used in the estimate from (1a).
   (c) Is the estimate from (1a) an overestimate, an underestimate, or equal to the exact area? Explain your answer with reference to your sketch.
2. Repeat Question 1 for the function $f(x) = x^3 - 7x^2 + 7x + 55$.
3. Repeat Question 1 for the function $f(x) = |\cos(\pi x)|$.
4. Once we have studied the Fundamental Theorem of Calculus, you will be able to show that $4\int_0^1 \frac{1}{1+x^2}\,dx = \pi$. Use the following methods to estimate $\int_0^1 \frac{1}{1+x^2}\,dx$, then use the result to estimate π.
   (a) A Riemann sum with $n = 4$ equally sized subintervals and right endpoints.
   (b) The trapezoid rule with $n = 4$.

Summing Up

1. Let $f(x) = 2x + 3$. We will construct a Riemann sum for this function on the interval $[a, b]$, where $a < b$, using equally-sized subintervals and right-hand-endpoints.

   (a) Find a formula for Δ_k.

   (b) Find a formula for c_k.

   (c) Use your formulas above to find the Riemann sum S_n.

   (d) Evaluate $\lim_{n\to\infty} S_n$ to calculate the area under the curve.

   (e) Check your answer by using geometry.

2. Thinking backwards: Each sum below is a Riemann sum for a function on the interval $[1, 3]$ with right-hand-endpoints. Figure out what the function was for each.

$$\sum_{k=1}^{n} \frac{2}{n}\sqrt{1-\left(1+\frac{2k}{n}\right)^2} \qquad \sum_{k=1}^{n} \frac{2}{n}\sin\left(1+\frac{2k}{n}\right) \qquad \sum_{k=1}^{n}\left(\frac{8k}{n^2}+\frac{6}{n}\right)$$

3. Consider the limit $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{4}{n}\sqrt{16-\frac{16k^2}{n^2}}$.

   (a) By recognizing $\sum_{k=1}^{n}\frac{4}{n}\sqrt{16-\frac{16k^2}{n^2}}$ as a Riemann sum, express this limit as a definite integral.

   (b) Use geometry to evaluate that definite integral.

4. Suppose that $\int_1^5 f(x)dx = 11$ and $\int_1^5 g(x)dx = 3$ and $\int_3^5 g(x)dx = 6$. Compute each of the following definite integrals:

$$\int_1^5 (3f(x) - 2g(x))\,dx \qquad \int_5^1 f(x)dx \qquad \int_1^3 g(x)dx$$

$$\int_3^5 2g(x)dx \qquad \int_1^5 0.5f(x)dx - \int_5^3 2g(x)dx \qquad \int_3^3 f(x)dx$$

It's Fundamental

Note:

Tutorials will have ended for the term before this worksheet can be used, so you won't be able to get guidance from your tutorial TA on this worksheet. To make up for it, there are some hints on the next page – and you can as always seek additional assistance on CourseSpaces, in the Math & Stats Assistance Centre, and in your instructors' office hours.

1. What is the average value of $\tan^2(x)\sec^2(x)$ on the interval $[0, \pi/4]$?
2. Suppose that $f(x)$ is differentiable and $f'(x) > 0$ on $[a, b]$. Which is larger, $f(a)(b-a)$ or $\int_a^b f(x)dx$? Or are they equal? Explain your answer.[o]
3. Compute the derivative of each of the following:
   (a) $\displaystyle\int_0^x \sin^4(t)dt$
   (b) $\displaystyle\int_{-1}^{x^2} (1+t^4)^2 dt$
   (c) $\displaystyle\int_{\sqrt{x}}^{x^2} (1+3t^2+5t^4)dt$
   (d) $\displaystyle\int_0^{12} \sqrt{1+12t^2+5t^4}dt$
4. Evaluate each of the following integrals, where $a, b, c,$ and m are constants:
   (a) $\int_\pi^0 \sin(x)dx$
   (b) $\int_a^b c dx$
   (c) $\int_a^b (mx+c)dx$
   (d) $\int \sin(3x)dx$
   (e) $\int \sqrt{1+t^2}tdt$
   (f) $\int xe^{x^2}dx$
   (g) $\int (1+t)^{2016}tdt$
5. If $f(x) = \int_1^{e^x} \sqrt{16-r^2}dr$, what is $f(0)$?

[o]The MVT for integrals might help.

Hints to "It's Fundamental"

1. To evaluate $\frac{1}{\pi/4}\int_0^{\pi/4}\tan^2(x)\sec^2(x)dx$ you will need to use a substitution. Your only choices are $u=\tan(x)$ or $u=\sec(x)$ – which one will work?

2. Use the Mean Value Theorem for Integrals to say something about $f(c)(b-a)$ for some special value of c. If we know that $f'(x)>0$ on $[a,b]$, we can say something about the relationship between $f(c)$ and $f(a)$.

3. These are listed in order of increasing difficulty. You do not need to evaluate any of these integrals! Instead, use the Fundamental Theorem of Calculus, Part 1, which states that for any constant a and integrable function f:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right]=f(x).$$

From the FToC1 and the Chain Rule, you can conclude that:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right]=f(g(x))g'(x).$$

From the algebraic properties of definite integrals, you can conclude that:

$$\int_{h(x)}^{g(x)} f(t)dt=-\int_0^{h(x)} f(t)dt+\int_0^{g(x)} f(t)dt.$$

Those three facts are all you need to finish this problem.

4. The first few require the Fundamental Theorem of Calculus, part II. The others require you to use a substitution:

   (a) Hint: notice that the lower bound is larger than the upper bound.
   (b) You know this should be $(b-a)c$ because of geometry.
   (c) $(m/2)(b^2-a^2)+c(b-a)dx$.
   (d) Use $u=3x$.
   (e) Use $u=1+t^2$.
   (f) Use $u=x^2$.
   (g) Use $u=1+t$, and then don't forget to replace the extra t with something...

5. Notice that $f(0)=\int_1^1\sqrt{16-r^2}dr$.

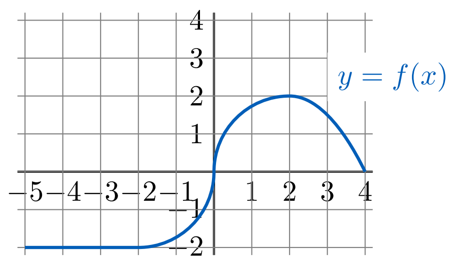
The Space Between

Note:

Tutorials will have ended for the term before this worksheet can be used, so you won't be able to get guidance from your tutorial TA on this worksheet. To make up for it, there are some hints on the next page – and you can as always seek additional assistance on CourseSpaces, in the Math & Stats Assistance Centre, and in your instructors' office hours.

1. Given the sketch of $y = f(x)$ below, put the following definite integrals in order from smallest to largest without attempting to evaluate them:

$$\int_{-4}^{0} f(x)dx, \int_{0}^{-4} f(x)dx, \int_{0}^{-2} f(x)dx, \int_{0}^{2} f(x)dx, \int_{0}^{4} f(x)dx$$



2. What is the area between the parabola $y = -x^2$ and the line $y = 1 + x$ on the interval $[-1, 0]$?
3. Consider the region enclosed by the parabola $y = x^2 + 2$ and the lines $y = 2x + 2$ and $y = 4$.
   (a) Sketch the region, and find the points of intersection between the three curves.
   (b) Express the area of this region as a sum of definite integrals with respect to x.
   (c) Express the area of this region as a single definite integral of the form $\int_0^4 g(y)dy$ and then evaluate it.
   (d) Check your answer to (3c) by evaluating the sum of integrals in (3b).
4. What is the area under the curve $y = \sin(x^3)$ on the interval $[-\pi, \pi]$?
5. Evaluate $\int_{-1}^{0} t(t + 1)^{16}dt$.

Hints to "The Space Between"

1. Hint: it might be helpful to approximate each of these definite integrals by thinking of them as signed areas.
2. Hint: the area is $\int_{-1}^{0}(1+x+x^2)dx$ because the line is above the parabola.
3. Hint: when working with "dy" then you should think of the parabola as $x = \sqrt{y-2}$, one line as $x = \frac{1}{2}y - 2$, and the interval as y in $[0, 4]$.
4. Hint: you do not need to find an antiderivative in order to evaluate $\int_{-\pi}^{\pi}\sin(x^3)dx$.
5. Hint: use the substitution $u = t + 1$.

Statistically

Note:

If the probability density function of some continuous random variable is given by $f(x)$ then the probability that the random variable is between a and b is

$$\int_a^b f(x)dx.$$

What does that mean? It means that statisticians need calculus! There's one particularly important probability density function: the *Standard Normal Distribution* (sometimes called the *bell curve*) has probability density function $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. For more information, take a statistics course – for now all you need to know is that in order to actually use the normal distribution we need to work with the definite integral

$$\int_{-a}^{a} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

1. Estimate the area under the curve $y = e^{-x^2}$ on the interval $[-1, 1]$ by using 4 rectangles and the right-hand-rule.
2. Estimate the area under the curve $y = e^{-x^2}$ on the interval $[-1, 1]$ by using 4 subintervals and the trapezoid rule.
3. Estimate the area under the curve $y = e^{-x^2}$ on the interval $[-1, 1]$ by using 4 subintervals and Simpson's Rule:

   Simpson's Rule:

   $$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

   where $y_0, y_1, \ldots, y_n$ are the values of f at the endpoints of the n subintervals, n is even, and $\Delta x = (b - a)/n$.

4. It turns out that $\int_{-1}^{1} e^{-x^2} dx$ is approximately 1.49365. Which of your estimates above was the closest?

Note:

There is no algebraic expression for $\int_{-1}^{1} e^{-x^2} dx$, which is why statisticians need to use *statistics tables* when working with the normal distribution.

Solutions to “Statistically”

Note:

Because this worksheet is so computational, here are full solutions.

1. The right endpoints are -0.5, 0, 0.5, and 1, so the heights of the rectangles are $e^{-0.25}$, 1, $e^{-0.25}$, and e^{-1}. The width of each rectangle is 0.5, so the total area of the four rectangles is $0.5\left(2e^{-0.25}+1+e^{-1}\right)$, which simplifies to $0.5+e^{-0.25}+0.5e^{-1}\approx 1.46274$.

2. The subintervals are $[-1,-0.5]$ and $[-0.5,0]$ and $[0,0.5]$ and $[0.5,1]$. The areas of the resulting four trapezoids are:

Interval:	$[-1,-0.5]$	$[-0.5,0]$	$[0.0.5]$	$[0.5,1]$
Trapezoid's area:	$\frac{\left(e^{-1}+e^{-0.25}\right)}{2}\cdot\frac{1}{2}$	$\frac{\left(e^{-0.25}+1\right)}{2}\cdot\frac{1}{2}$	$\frac{\left(1+e^{-0.25}\right)}{2}\cdot\frac{1}{2}$	$\frac{\left(e^{-0.25}+e^{-1}\right)}{2}\cdot\frac{1}{2}$

so the total area of the four trapezoids is

$$\begin{aligned}\tfrac{1}{4}\left(e^{-1}+2e^{-0.25}+2+2e^{-0.25}+e^{-1}\right) &= \tfrac{1}{4}\left(2e^{-1}+4e^{-0.25}+2\right)\\ &= \tfrac{1}{2}e^{-1}+e^{-0.25}+\tfrac{1}{2}\\ &\approx 1.46274.\end{aligned}$$

That is exactly the same area we got with rectangles and the right-hand rule! Why? Basically because the function is even and we have an even number of subintervals.

3. Simpson's Rule says that we can approximate $\int_{-1}^{1} e^{-x^2}\,dx$ by

$$\frac{0.5}{3}\left(e^{-1}+4e^{-0.25}+2e^{0}+4e^{-0.25}+e^{-1}\right),$$

which is approximately 1.49436.

4. Simpson's Rule yielded the best result.

“Check Your Understanding” Answers

True or False?

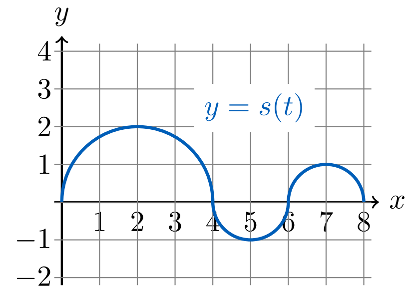
1. True.
2. True.
3. False.
4. True.
5. True.
6. True.
7. True.
8. True.
9. True.
10. True.
11. False.
12. False.
13. False.
14. False – it represents the car’s displacement.
15. True.

Find a Function:

1. All such functions have the form $-\frac{1}{5}\cos^5(x)+C$, where C is any real number
2. $f(x) = x^3+2x^2-12x+C$, so $f(1) = 1+2-12+C = -9+C$, so $C = 3$ and $f(x) = x^3+2x^2-12x+3$. There is no other such function.
3. One such definite integral is $\int_0^2 (x^2+1)dx$. Problems of this type can be tricky, because you have to think backwards, so here is a full solution:
   The simplest Riemann sum formula involves right-endpoints and an interval of the form $[0,b]$ because then $\Delta_k = \frac{b}{n}$ and $f(c_k) = f\left(\frac{bk}{n}\right)$. We therefore need:
   $$\sum_{k=1}^{n} \frac{b}{n}\cdot f\left(\frac{b}{n}\right) = \sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{2}{n}\right).$$
   Now notice that we can factor $2/n$ out in the summand:
   $$\sum_{k=1}^{n} \frac{b}{n}\cdot f\left(\frac{b}{n}\right) = \sum_{k=1}^{n}\frac{2}{n}\left(\frac{4k^2}{n^2}+1\right).$$
   It’s starting to look like $b/n = 2/n$, which means we are looking for some function $f(x)$ such that $f(2k/n) = 4k^2/n^2+1 = (2k/n)^2+1$. The integral we are looking for is therefore $\int_0^2 (x^2+1)dx$.
   Of course, there are infinitely many other correct answers, such as $\int_{-1}^{1}((x+1)^2+1)dx$.
   Bonus:
   $$\begin{aligned}\textstyle\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{2}{n}\right) &= \textstyle\sum_{k=1}^{n}\frac{8k^2}{n^3}+\sum_{k=1}^{n}\frac{2}{n}\\ &= \textstyle\frac{8}{n^3}\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}\frac{2}{n}\\ &= \textstyle\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}+n\cdot\frac{2}{n}\\ &= \textstyle\frac{8n(n+1)(2n+1)}{6n^3}+2\end{aligned}$$

So $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{2}{n}\right)=\lim_{n\to\infty}\frac{8n(n+1)(2n+1)}{6n^3}+2=\frac{16}{6}+2$.

4. Total distance is $\int_0^4 s(t)dt-\int_4^6 s(t)dt+\int_6^8 s(t)dt$. Displacement is $\int_0^8 s(t)dt$.



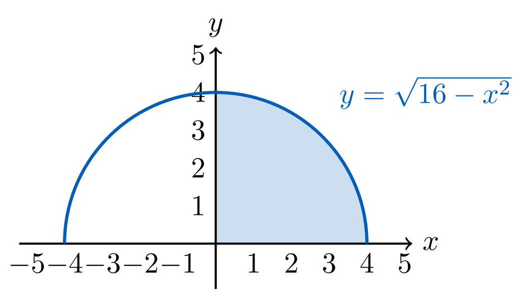
5. $g(t)=(1+t^3)^2$.

Explain your answer:

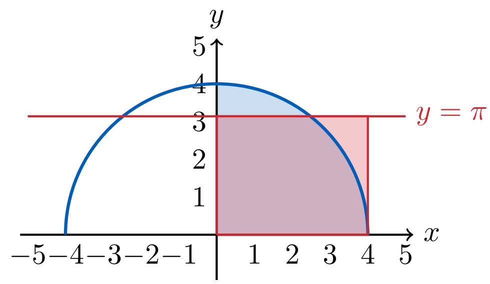
Note:

Because much of this material will be on the final exam but not on any midterm, we have included solutions rather than only hints for some of these problems.

1. If $g''(x)=x$ then $g'(x)=\frac{1}{2}x^2+C$ for some constant C. Because $g'(1)=2$, this means that $\frac{1}{2}+C=2$, so $C=\frac{3}{2}$ and so $g'(x)=\frac{1}{2}x^2+\frac{3}{2}$. This in turn means that $g(x)=\frac{1}{6}x^3+\frac{3}{2}x+D$ for some constant D. Any function of this form satisfies the requirements, so the infinite family can be expressed as "any function of the form $\frac{1}{6}x^3+\frac{3}{2}x+D$, where D is a real number."
2. Notice that $\frac{dy}{dx}=-\sin(x)$, and for x in $[0,\pi/2]$ we know that $-\sin(x)\le 0$. This means that the function is decreasing, so the right half of each rectangle is above the curve while the left half is below the curve. We want to argue that it is decreasing more quickly as x grows from 0 to $\pi/2$ (because that would mean that the right half of the rectangle adds more extra area than the left half misses). To see this, notice that $\frac{d^2y}{dx^2}=-\cos(x)\le 0$ on $[0,\pi/2]$. This means that the derivative is decreasing (becoming more and more negative) as x grows from 0 to $\pi/2$, just as we hoped.
3. The curve $y=\sqrt{16-x^2}$ is the upper half of the circle with radius 4 centred at $(0,0)$, and the area under this curve on the interval $[0,4]$ is the area of one half of this half circle, so we just need to compute $\frac{1}{4}\pi(4)^2$, which is 4π.



4. The average value is found by dividing the area by the length of the interval, and $4\pi/(4-0)=\pi$. The line $y=\pi$ is drawn below: you can see that the resulting rectangle seems to have the same area as the shaded quarter-circle region.



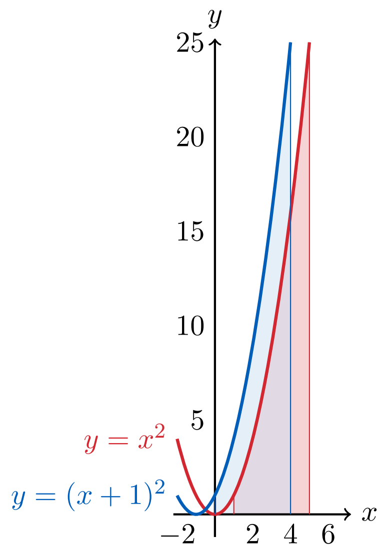
5. Let $m = k - 4$. Then $\sum_{k=5}^{104} \sqrt{k-4} = \sum_{m=1}^{100} \sqrt{m}$. Since the variable of summation doesn't matter, this is the same as $\sum_{k=1}^{100} \sqrt{k}$.
6. By definition, $C = \frac{1}{b-a} \int_a^b f(x)dx$, so $C(b-a) = \int_a^b f(x)dx$. On the other hand, $\int_a^b C dx = C(b-a)$ because the integral represents the area of the rectangle with width $b-a$ and height C (where either value can be negative).
7. The function $f(x)$ has exactly the same domain as $\tan(x)$, which is

$$\cdots \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \cdots$$

On each of these intervals, we can use the Fundamental Theorem of Calculus, Part 1: $f'(x) = \frac{1}{1+\tan^2(x)} \cdot \sec^2(x) = 1$.
This tells us that (again, on each of the intervals) $f(x) = x + C$ for some constant C, because all functions with derivative 1 take that form. We could work out what C is for each interval. For example, $f(0) = \int_0^{\tan(0)} \frac{1}{1+t^2} dt = \int_0^0 \frac{1}{1+t^2} dt = 0$, and so $f(x) = x$ for $-\pi/2 < x < \pi/2$. On the other hand, $f(\pi) = \int_0^{\tan(\pi)} \frac{1}{1+t^2} dt = \int_0^0 \frac{1}{1+t^2} dt = 0$, and so $f(\pi) = \pi + C = 0$ implies that $f(x) = x - \pi$ for $\pi/2 < 3\pi/2$.
You might be tempted to use the Fundamental Theorem of Calculus, Part 2:

$$f(x) = \arctan(t)|_0^{\tan(x)} = \arctan(\tan(x)) - \arctan(0) = x,$$

but that only works for $-\pi/2 < x < \pi/2$ because $\arctan(\tan(x)) = x$ only for $-\pi/2 < x < \pi/2$.
8. Recall that $a(t) = v'(t)$, so $\int_0^8 a(t)dt = v(8) - v(0) = 0 - 0 = 0$.
9. (a) These areas should be the same because $y = (x+1)^2$ is obtained from $y = x^2$ by shifting it left by one, and the interval $[0, 4]$ is obtained from the interval $[1, 5]$ by shifting it left by one as well.



(b) First consider the curve $y = (x+1)^2$ on the interval $[0,4]$. Notice that $\Delta_k = 4/n$, and if we use the right endpoint of each interval we find that $c_k = 0 + 4k/n = 4k/n$. The Riemann sum is therefore $S_n = \sum_{k=1}^{n}\left(\left(\frac{4k}{n}+1\right)^2\frac{4}{n}\right)$. Now consider the curve $y = x^2$ on the interval $[1,5]$. Again, $\Delta_k = 4/n$, but this time $c_k = 1 + 4k/n$. The Riemann sum is therefore $S_n = \sum_{k=1}^{n}\left(\left(1+\frac{4k}{n}\right)^2\frac{4}{n}\right) = \sum_{k=1}^{n}\left(\left(\frac{4k}{n}+1\right)^2\frac{4}{n}\right)$.

(c) We want to show that $\int_0^4 (x+1)^2 dx = \int_1^5 x^2 dx = 124/3$.

$$\begin{aligned}\int_0^4 (x^2+2x+1)dx &= \tfrac{1}{3}x^3 + x^2 + x\Big]_0^4 \\ &= \tfrac{1}{3}4^3 + 4^2 + 4 \\ &= 124/3.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_1^5 x^2 dx &= \tfrac{1}{3}x^3\Big]_1^5 \\ &= \left(\tfrac{1}{3}5^3\right) - \left(\tfrac{1}{3}1^3\right) \\ &= 124/3.\end{aligned}$$

10. Let $F(x) = \int_a^x f(t)dt$. Then by the Fundamental Theorem of Calculus, Part 1, $F'(x) = f(x)$. Now, $\int_a^{g(x)} f(t)dt = F(g(x))$ and so the Chain Rule tells us that

$$\begin{aligned}\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] &= \frac{d}{dx}\left[F(g(x))\right] \\ &= F'(g(x))g'(x) \\ &= f(g(x))g'(x),\end{aligned}$$

as desired.

11. The two curves intersect at the points $(0,0)$ and $(1,1)$. Let's consider a "dy" integral in order to avoid $\sqrt{x}$ and $\sqrt[3]{x}$. For $0 \le y \le 1$, $y^3 < y^2$ and so the integral that represents the area is:

$$\int_0^1 (y^2 - y^3)dy = \frac{1}{3}y^3 - \frac{1}{4}y^4\Big]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) - (0-0) = \frac{1}{12}.$$

Note:

If you had gotten a negative answer here, you'd realize you had made a mistake, because total area is always positive. The most common cause for that would be incorrectly guessing which curve is "above" the other.

12. The two curves intersect at $x = -\sqrt{5/2}$ and $x = \sqrt{5/2}$. The integral that represents the area is then
$$\int_{-\sqrt{5/2}}^{\sqrt{5/2}}((-x^2+x+15)-(x^2+x+10))dx = \int_{-\sqrt{5/2}}^{\sqrt{5/2}}(5-2x^2)dx.$$
Because $2x^2-5$ is an even function,
$$\int_{-\sqrt{5/2}}^{\sqrt{5/2}}(2x^2-5)dx = 2\int_0^{\sqrt{5/2}}(2x^2-5)dx.$$
13. $\ln(0.5) = -\ln(2)$ and the integrand is an odd function.
14. $\ln(1) = 0$.
15. The 'height' of each trapezoid is $\frac{b-a}{n}$ and the 'base' lengths of the kth trapezoid are $f(c_{k-1})$ and $f(c_k)$ – the trapezoid is sideways, so 'base' and 'height' are odd words to use. This means that the area of the kth trapezoid is $\frac{1}{2}(x_{k-1}+x_k)\frac{b-a}{n}$. Summing up we get:
$$\int_a^b f(x)dx \approx \sum_{k=1}^{n}\left(\frac{b-a}{2n}\left(f(x_{k-1})+f(x_{k-2})\right)\right),$$
which is
$$\frac{b-a}{2n}\left(f(x_0)+f(x_1)\right)+\frac{b-a}{2n}\left(f(x_1)+f(x_2)\right)+\cdots+\frac{b-a}{2n}\left(f(x_{n-2})+f(x_{n-1})\right)+\frac{b-a}{2n}\left(f(x_{n-1})+f(x_n)\right),$$
which simplifies to
$$\frac{b-a}{2n}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right).$$

Review

In this chapter we include two practice tests for each of the chapters in this course handbook. You can use these to check your mastery of course material as you go. Your class's tests might have a very different format, and your tests might not correspond exactly to the chapters in this handbook. Make sure to follow your instructor's guidance when preparing for your real tests.

Note:

To get the most out of these practice tests, you should write them in a quiet and timed environment (the time limit is indicated on each test). Do not let yourself use any tools that you would not have available to you on a midterm – be particularly careful to use only an approved calculator.

Answers are available at the end of each section. For best results, do not look at them until you have completed the practice test.

Review #1: Limits

Practice Test ℓA (50 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

1. Determine $\lim_{x\to 2} \dfrac{x\sqrt{4x+1}}{x+1}$.

   (A) -6 (B) -3 (C) -2 (D) 0 (E) 2 (F) 3 (G) 6 (H) $-\infty$ (I) ∞
   (J) The limit does not exist and is not infinite.

2. Determine $\lim_{x\to 1} \dfrac{x^2-1}{x^2+2x-3}$.

   (A) -3 (B) $-1/2$ (C) $-1/3$ (D) 0 (E) $1/3$ (F) $1/2$ (G) 3 (H) $-\infty$ (I) ∞
   (J) The limit does not exist and is not infinite.

3. Determine $\lim_{x\to \infty} \dfrac{3x^2-5x-19}{\sqrt{4x^4+7x^2-12}}$.

   (A) -3 (B) $-3/2$ (C) $-3/4$ (D) 0 (E) $3/4$ (F) $3/2$ (G) 3 (H) $-\infty$ (I) ∞
   (J) The limit does not exist and is not infinite.

4. What value of a makes the function below continuous everywhere?

$$f(x) = \begin{cases} 3x^2-2 & \text{if } x \leq -2 \\ ax+2 & \text{if } x > -2 \end{cases}$$

   (A) -16 (B) -14 (C) -8 (D) -4 (E) 0 (F) 4 (G) 8 (H) 14 (I) 16 (J) None

5. Determine $\lim_{x\to 2^+} \dfrac{5x}{2-x}$.

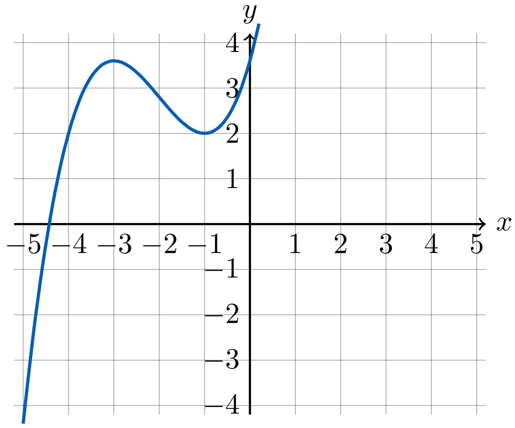
   (A) -10 (B) -5 (C) $-5/2$ (D) 0 (E) $5/2$ (F) 5 (G) 10 (H) $-\infty$ (I) ∞
   (J) The limit does not exist and is not infinite.

6. Determine $\lim_{x\to 11} \dfrac{\sqrt{x-2}-3}{x-11}$.

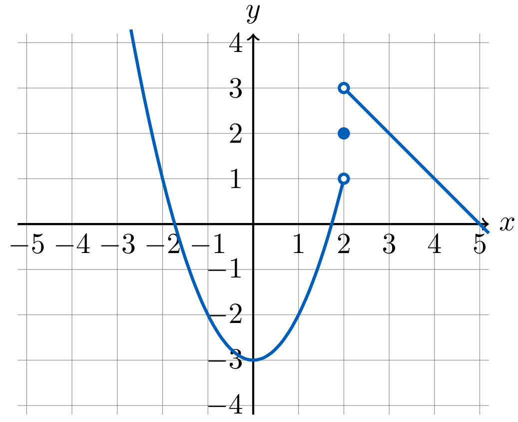
   (A) -11 (B) -6 (C) $-1/6$ (D) 0 (E) $1/6$ (F) 6 (G) 11 (H) $-\infty$ (I) ∞
   (J) The limit does not exist and is not infinite.

7. Use the graphs of $f(x)$ and $g(x)$ below to determine each of the following limits or explain why the limit does not exist:

$$\lim_{x\to 0} g(x) \qquad \lim_{x\to -3} f(x-1) \qquad \lim_{x\to 2^-} (g(x)+4) \qquad \lim_{x\to -1} g(f(x))$$



$y = f(x)$



$y = g(x)$

8. Use the sandwich theorem to calculate $\displaystyle\lim_{x\to-\infty} \frac{4\sin(2x-3)}{(x+1)^2}$.

9. Determine $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x(1+\cos(3x))}$.

10. In each of the problems below, give an example of a function that satisfies all the necessary requirements or explain why no such function exists. You do not need to prove that your example works and if you use any important theorems from class be sure to clearly indicate which theorem you are using and why it applies.

    (a) Give an explicit formula of a function $f(x)$ that is continuous at every real number except at $x = -3$ and has slant asymptote (a.k.a. oblique asymptote) at $y = 2x - 5$, or explain why no such function exists.

    (b) Give an explicit formula of a function $g(x)$ that is continuous at every real number, has $g(-1) = 5$, $g(2) = -3$, but such that $g(x) = 0$ has no solutions, or explain why no such function exists.

End of Practice Test ℓA

Practice Test ℓB (120 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

1. Find $\lim_{h\to 0} \dfrac{5}{\sqrt{5h+1}+1}$.

   (A) -2.0 (B) -1.5 (C) -1 (D) -0.5 (E) 0.5 (F) 1 (G) 1.5 (H) 2.0 (I) 2.5 (J) 3.0

2. Find $\lim_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$.

   (A) -3.0 (B) -1.0 (C) -0.6 (D) -0.3 (E) 0.0 (F) 0.3 (G) 0.6 (H) 1.0 (I) 2.0 (J) 3.0

3. Find $\lim_{t\to 4^-} (t - \lfloor t \rfloor)$.

   (A) -4.0 (B) -3.0 (C) -2.0 (D) -1.0 (E) 0.0 (F) 1.0 (G) 2.0 (H) 3.0 (I) 4.0 (J) 5.0

4. Find $\lim_{t\to 0^-} \dfrac{2t}{\tan t}$.

   (A) -3.0 (B) -2.0 (C) -1.0 (D) -0.5 (E) 0.0 (F) 0.5 (G) 1.0 (H) 2.0 (I) 3.0 (J) 4.0

5. Find $\lim_{x\to +\infty} \dfrac{\sqrt{x^2+2}}{2x+4}$.

   (A) $-\infty$ (B) -2.0 (C) -1.0 (D) -0.5 (E) 0.0 (F) 0.5 (G) 1.0 (H) 2.0 (I) 3.0 (J) $+\infty$

6. Find $\lim_{\theta\to -\infty} \dfrac{\cos\theta}{3\theta}$.

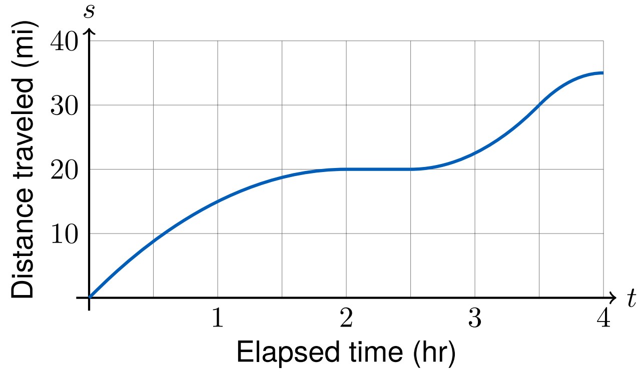
   (A) $-\infty$ (B) -3.0 (C) -1.0 (D) -0.3 (E) 0.0 (F) 0.3 (G) 1.0 (H) 2.0 (I) 3.0 (J) $+\infty$

7. Find $\lim_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$.

   (A) $-\dfrac{1}{a^2}$ (B) $-\dfrac{a^2}{2}$ (C) $-\dfrac{1}{2a^2}$ (D) $-2a^2$ (E) 0.0

   (F) $+2a^2$ (G) $+\dfrac{1}{2a^2}$ (H) $+\dfrac{a^2}{2}$ (I) $+\dfrac{1}{a^2}$ (J) Does not exist

8. The graph below shows the total distance s traveled by a bicyclist after t hours.



(a) Estimate the bicyclist's average speed over the time interval [1.0, 2.5].

(b) Estimate the bicyclist's instantaneous speed at the time $t = 0.5$.

9. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq 2$ and suppose that $\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5$.

   (a) Can we conclude anything about the values of f, g, and h at $x = 2$? Justify your answer.

   (b) Could $f(2) = 0$? Justify your answer.

   (c) Could $\lim_{x\to 2} f(x) = 0$? Justify your answer.

10. Give an example of a function $f(x)$ that is defined on the interval $[1, 5]$, and $f(x) \neq 0$ for all x on the interval $[1, 5]$, and $f(1) = -1$, $f(5) = 1$. You can define $y = f(x)$ using a graph or using a formula. Explain why your function $y = f(x)$ does not violate the Intermediate Value Theorem.

11. Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a nonremovable discontinuity. Explain how you know that $f(x)$ is discontinuous there and why the discontinuity is not removable.

12. Calculate $\lim_{x\to +\infty} (\sqrt{9x^2 - x} - 3x)$. Use your answer to determine an equation for the slant asymptote of the curve $y = \sqrt{9x^2 - x}$.

13. Consider the following piecewise-defined function:

$$g(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 - 2x - 1, & x < 0. \end{cases}$$

Determine the left-hand derivative and the right-hand derivative of $g(x)$ at the point $x_0 = 0$ using the definition of the one-sided derivative at a point. Then use your results to determine whether the function $g(x)$ is differentiable at the origin; justify your conclusion.

End of Practice Test ℓB

Review #1: Limits (Answers)

Practice Test ℓA

1. (E)
2. (F)
3. (F)
4. (D)
5. (H)
6. (E)
7. -3, 2, 5, 3
8. 0
9. $3/2$
10. (a) Many such functions; one is $f(x) = 2x - 5 + \frac{1}{x+3}$.
    (b) No such function, because of the Intermediate Value Theorem

Practice Test ℓB answers on the next page. For best results, take a break to review the material you missed on Practice Test ℓA and don't try Test ℓB for a day or two...

Practice Test ℓB

1. (I)
2. $1/3$, so the closest answer is (F)
3. (F)
4. (H)
5. (F)
6. (E)
7. (G)
8. Approximately $10/3$ miles per hour. A bit less than 15 miles per hour.
9. No, Yes, No (because the limit is -5).
10. There are many such functions; one is $f(x) = \begin{cases} -1 & x < 5 \\ 1 & x \geq 5 \end{cases}$, which does not violate the IVT because it is discontinuous at $x = 5$.
11. There are many such functions; one is $f(x) = \frac{1}{x-2}$, which has a vertical asymptote at $x = 2$.
12. The limit is $-1/6$, which means that $\left(\lim_{x\to\infty} \sqrt{9x^2 - x} - 3x + \frac{1}{6}\right) = \lim_{x\to\infty} \sqrt{9x^2 - x} - 3x + \lim_{x\to\infty} \frac{1}{6} = -\frac{1}{6} + \frac{1}{6} = 0$, which means that $y = 3x - \frac{1}{6}$ is a slant asymptote of $y = \sqrt{9x^2 - x}$.
13. 2 and -2, which means the function is not differentiable at the origin.

Review #2: Derivatives

Practice Test $\triangle$A (50 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

1. Given that $y = 3x^3 - x + 1$, find y' at $x = 2$.

   (A) 8 (B) 9 (C) 23 (D) 24 (E) 25
   (F) 30 (G) 34 (H) 35 (I) 36 (J) None of those

2. Given that $y = \log_2 x$, find y' at $x = e$.

   (A) $\frac{1}{e}$ (B) $\frac{e}{\ln 2}$ (C) $\frac{2}{\ln e}$ (D) $e \ln 2$ (E) 2
   (F) $\frac{2}{e}$ (G) $\frac{\ln 2}{e}$ (H) $2e$ (I) $\frac{1}{e \ln 2}$ (J) None of those

3. Given that $y = \frac{x+3}{1-x}$, find y' at $x = 2$.

   (A) -4 (B) -2 (C) -1 (D) $-1/2$ (E) Does not exist
   (F) $1/2$ (G) 1 (H) 2 (I) 4 (J) None of those

4. Given that $y = \frac{\cos(x)}{1+\sin(x)}$, find y' at $x = \frac{\pi}{6}$

   (A) -2 (B) $-3/2$ (C) -1 (D) $-2/3$ (E) Does not exist
   (F) 2 (G) $3/2$ (H) 1 (I) $2/3$ (J) None of those

5. From the options below select the correct equations of the asymptotes of the function. $y = x + 3 - \frac{2}{(x-1)^2}$.

   (a) Horizontal asymptote:

   (A) $x = -2$ (B) $y = x - 1$ (C) $x = -1$ (D) $y = 2$ (E) No horizontal asymptote
   (F) $x = 2$ (G) $y = x + 3$ (H) $x = 1$ (I) $y = 1$ (J) None of those

   (b) Vertical asymptote:

   (A) $x = -2$ (B) $y = x - 1$ (C) $x = -1$ (D) $y = 2$ (E) No vertical asymptote
   (F) $x = 2$ (G) $y = x + 3$ (H) $x = 1$ (I) $y = 1$ (J) None of those

   (c) Slant (or oblique) asymptote:

   (A) $x = -2$ (B) $y = x - 1$ (C) $x = -1$ (D) $y = 2$ (E) No slant asymptote
   (F) $x = 2$ (G) $y = x + 3$ (H) $x = 1$ (I) $y = 1$ (J) None of those

6. Given that $y = \pi x + \frac{1}{\cos^2(\pi x)}$, find y' at $x = \frac{1}{4}$.

   (A) -5π (B) -4π (C) $-\frac{3}{2}\pi$ (D) $-\pi$ (E) 0
   (F) π (G) $\frac{3}{2}\pi$ (H) 4π (I) 5π (J) None of those

7. Given that $y = \sec(\tan(x))$, find y'.

(A) $\sec^2(\tan(x))\tan(\tan(x))$ (B) $\sec(\tan(x))\tan(\tan(x))$ (C) $\sec^2(\tan(x))$
(D) $\dfrac{\cos^3(x) - 2\cos(x)\sin^2(x)}{\cos^4(x)}$ (E) $\sec(\tan(x))\tan(\tan(x))\sec^2(x)$ (F) None of those

8. Given that $y = \begin{cases} x^2 + \tan x & x \geq 0 \\ 3x & x < 0 \end{cases}$, find y' at $x = 0$.

(A) -3 (B) -2 (C) -1 (D) $-\frac{1}{3}$ (E) Does not exist
(F) 3 (G) 2 (H) $\frac{1}{3}$ (I) 0 (J) None of those

9. Given that $y = \sin(f(t))$, $f(0) = \frac{\pi}{3}$ and $f'(0) = 4$, find $\frac{dy}{dt}$ at $t = 0$.

(A) $-\frac{\sqrt{3}}{8}$ (B) $-\frac{1}{8}$ (C) -4 (D) -2 (E) Does not exist
(F) 2 (G) 4 (H) $\frac{1}{8}$ (I) $\frac{\sqrt{3}}{8}$ (J) None of those

10. Find $\frac{dy}{dx}$ at $(x, y) = (2, 1)$, given that $x^2y + xy^2 = 6$.

(A) $-\frac{10}{8}$ (B) $-\frac{5}{8}$ (C) $-\frac{1}{8}$ (D) -5 (E) Does not exist
(F) 5 (G) $\frac{1}{8}$ (H) $\frac{5}{8}$ (I) $\frac{10}{8}$ (J) None of those

End of Practice Test △A

Practice Test $\triangle$B (50 minutes)

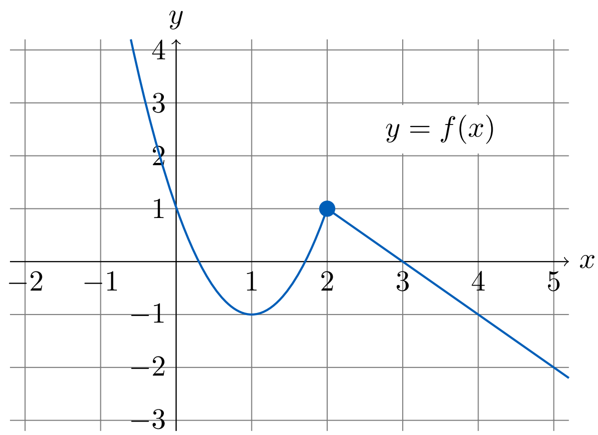
1. Consider the function $g(x) = 2x^4 - 3x^2 + 7x - 5$. Calculate $g'(1)$.

   (A) -5 (B) -3 (C) -1 (D) 0 (E) 1 (F) 3 (G) 5 (H) 7 (I) 9
   (J) Does not exist.

2. Consider the function $h(x) = \tan(3x)$. Determine $h'(\frac{\pi}{3})$.

   (A) -6 (B) -3 (C) -1 (D) 0 (E) 1 (F) 3 (G) 6 (H) 9 (I) 12
   (J) Does not exist.

3. Use the graph of $y = f(x)$ below to answer the following questions.



   (a) What is the **average** rate of change of $f(x)$ on $[0, 5]$?

   (A) $-5/3$ (B) $-3/5$ (C) $-1/5$ (D) 0 (E) $1/5$ (F) $3/5$ (G) $5/3$ (H) 3 (I) 5
   (J) Does not exist.

   (b) What is the **instantaneous** rate of change of $f(x)$ at $x = 2$?

   (A) -6 (B) -4 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 4 (I) 6
   (J) Does not exist.

   (c) Determine $\lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$.

   (A) -5 (B) -3 (C) -1 (D) $-1/2$ (E) 0 (F) 1 (G) 3 (H) 5 (I) 7
   (J) Does not exist.

4. Determine the slope of the line tangent to the curve $y = \ln\left(\dfrac{x^2 \cdot e^x}{(x+1)^3}\right)$ when $x = 2$.

   (A) -5 (B) -3 (C) -1 (D) $-1/2$ (E) 0 (F) $1/2$ (G) 1 (H) 3 (I) 5
   (J) Does not exist.

5. Calculate $\dfrac{d^2y}{dx^2}$ at $x = \pi$ for $y = \cos^2(x)$.

   (A) -2 (B) $-\sqrt{3}/2$ (C) -1 (D) $-1/2$ (E) 0 (F) $1/2$ (G) 1 (H) $\sqrt{3}/2$ (I) 2
   (J) Does not exist.

6. Consider the function $f(x) = \dfrac{x}{x+1}$.

   (a) Compute the derivative $f'(x)$ **using derivative rules**.

   (b) Compute the derivative $f'(x)$ **using the limit definition of derivative**.

7. Given the information in the table below, answer the following questions. Show enough work to justify your answers; any unsupported answers will receive no marks.

	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$x=-1$	$f(-1)=3$	$f'(-1)=0$	$g(-1)=-2$	$g'(-1)=2$
$x=0$	$f(0)=4$	$f'(0)=2$	$g(0)=1$	$g'(0)=-2$
$x=2$	$f(2)=5$	$f'(2)=-3$	$g(2)=4$	$g'(2)=-1$
$x=4$	$f(4)=2$	$f'(4)=-5$	$g(4)=7$	$g'(4)=6$

   (a) If $F(x) = 3f(x)$, what is $F'(0)$?

   (b) If $G(x) = (g(x))^3$, what is $G'(-1)$?

   (c) If $H(x) = f(x)g(x)$, what is $H'(2)$?

   (d) If $f(x)$ is one-to-one and $K(x) = f^{-1}(x)$, what is $K'(2)$?

8. Consider the curve $x^2 + 4y^2 - 4x + 16y + 12 = 0$.

   (a) Determine $\dfrac{dy}{dx}$.

   (b) Determine the x and y coordinates of any points on the curve where the tangent line is horizontal.

   (c) Determine the equation of the line tangent to the curve at the point $(4, -1)$.

End of Practice Test △B

Review #2: Derivatives (Answers)

Practice Test $\triangle$A

1. (H)
2. (I)
3. (I)
4. (D)
5. (E), (H), (G)
6. (I)
7. (E)
8. (E)
9. (F)
10. (B)

Practice Test $\triangle$B answers on the next page. For best results, take a break to review the material you missed on Practice Test $\triangle$A and don't try Test $\triangle$B for a day or two...

Practice Test $\triangle$B

1. (I)
2. (F)
3. (B), (J), (E)
4. (G)
5. (A)
6. $f'(x) = \frac{1}{(x+1)^2}$.
7. $F'(0) = 6$, $G'(-1) = 24$, $H'(2) = -17$, $K'(2) = -1/5$
8. (a) $\frac{dy}{dx} = \frac{2-x}{4y+8}$
   (b) $(2, -2-\sqrt{2})$ and $(2, -2+\sqrt{2})$
   (c) $y = -\frac{1}{2}(x-4) + 1$

Review #3: Using Derivatives

Practice Test ∪A (50 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

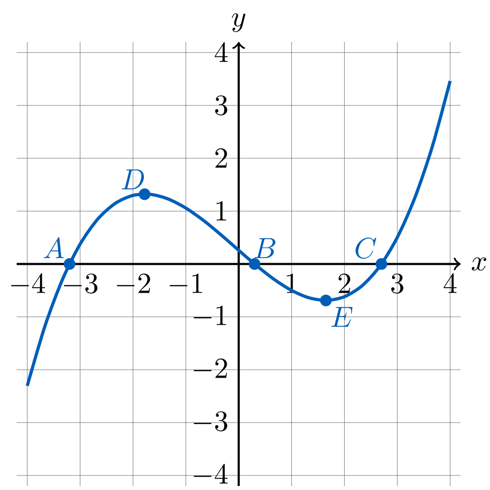
1. Let $f(x) = 3x^{\frac{2}{3}} - 4x$. What is the absolute **minimum** of f on the interval $[0, 8]$?

   (A) -20 (B) -10 (C) -5 (D) -1 (E) $-1/4$ (J) No Absolute Minimum
   (F) 0 (G) $1/4$ (H) 1 (I) 5

2. Determine the equation of the line tangent to $y = \arccos(x)$ at $x = 0$.

   (A) $y = x + \pi$ (B) $y = -x + \pi$ (C) $y = x$ (D) $y = -x$ (E) $y = -x + \frac{\pi}{2}$
   (F) $y = x + \frac{\pi}{2}$ (G) $y = -x - \frac{\pi}{2}$ (H) $y = -x + \frac{3\pi}{2}$ (I) $y = -1$ (J) $y = 1$

3. Use the graph of $y = f(x)$ below to answer the following questions.



$y = f(x)$

   (a) Suppose Newton's Method was performed with an initial guess of $x_0 = 2$. Which point of the function (if any) would it approximate?

   (A) A (B) B (C) C (D) D (E) E
   (F) A and D (G) B and E (H) C and E (I) None of these (J) Method Fails

   (b) At the point A on the graph above which of the following are true?

   (a) $f'(x) > 0$ (b) f is concave up (c) $f''(x) < 0$ (d) A is a critical point

   (A) (a) only (B) (b) only (C) (c) only (D) (d) only (E) (a) and (c)
   (F) (a) and (d) (G) (b) and (c) (H) (b), (c), and (d) (I) None (J) All

4. Suppose $f''(x) = (x+2)(x-1)^2(x-4)$. On what interval(s) is f **concave up**?

(A)	$(-\infty,-2)$ only	(B)	$(-2,1)$ only	(C)	$(1,4)$ only
(D)	$(4,\infty)$ only	(E)	$(-\infty,-2)\cup(1,4)$	(F)	$(-2,1)\cup(4,\infty)$
(G)	$(-\infty,-2)\cup(4,\infty)$	(H)	$(-2,1)\cup(1,4)$	(I)	Not enough information

5. Determine $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

6. Suppose you made a snowman during the last snowstorm and have managed to keep it cool enough that there is still some of the bottom sphere remaining. With the temperature warming up lately, it's been harder to keep your last bit of snowman. Suppose you know now that the volume of the sphere is shrinking at a rate of $32\text{cm}^3/\text{min}$. At what rate is the radius changing when the radius of the sphere is 8cm?
(It may be useful to know that the volume of a sphere is $\frac{4}{3}\pi r^3$)

7. Consider the function $f(x) = e^x + 2x^3$. Explain, using the Mean Value Theorem or Rolle's Theorem, why $f(x)$ cannot have two (different) roots.

8. Suppose you want to construct a closed rectangular box with volume $576\ cm^3$. The length along the base of the box must be twice the width along the base of the box. Find the dimensions of the box with the **minimum** total surface area.

9. Sketch the graph of a **continuous** function $f(x)$ that clearly satisfies all of the following characteristics. Your graph should clearly show the increasing, decreasing, and concave structure of f, as well as other features such as relative extrema, points of inflection, and asymptotes.

f has x-ints at $x=-8$ and 6 f has y-int at $y=5$
$f(5)=2$ $f(3)=7$ $f(1)=2$ $f(-4)=9$
$\lim_{x\to\infty} f(x) = -3$ $\lim_{x\to-\infty} f(x) = -\infty$
$f'(3)=0$ $f'(1)$ is undefined $f'(-4)=0$
$f''(1)$ is undefined $f''(5)=0$

Interval	$(-\infty,-4)$	$(-4,1)$	$(1,3)$	$(3,\infty)$
Sign of $f'(x)$	+	-	+	-

Interval	$(-\infty,1)$	$(1,5)$	$(5,\infty)$
Sign of $f''(x)$	-	-	+

End of Practice Test ∪A

Practice Test $\cup$B (50 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

1. Evaluate $\lim_{x\to 1} \frac{x^3-1}{\ln x}$.

   (A) -3 (B) -2 (C) -1 (D) $-\frac{1}{2}$ (E) Does not exist
   (F) $\frac{1}{2}$ (G) 1 (H) 2 (I) 3 (J) None of those

2. Evaluate $\lim_{t\to \pi} \frac{\sin t}{t^2-\pi^2}$.

   (A) $-\frac{1}{2\pi}$ (B) -2 (C) -1 (D) $-\frac{1}{2}$ (E) $\frac{4}{25}$
   (F) $\frac{1}{2\pi}$ (G) 2 (H) 1 (I) $\frac{1}{2}$ (J) None of those

3. Evaluate $\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$.

   (A) -3 (B) -2 (C) -1 (D) $-\frac{1}{2}$ (E) Does not exist
   (F) $\frac{1}{2}$ (G) 1 (H) 2 (I) 3 (J) None of those

4. Evaluate $\lim_{x\to \infty} \left(1+\sin\frac{1}{3x}\right)^x$.

   (A) $\frac{1}{3}$ (B) 1 (C) 2 (D) 0 (E) Does not exist
   (F) $e^{1/3}$ (G) e (H) e^2 (I) e^3 (J) None of those

5. Find the differential dy at $x=1$ when $y=x^5+37x$ and $dx=0.2$.

   (A) 0.2 (B) 1 (C) 3.8 (D) 4.2 (E) 8.4
   (F) 12.0 (G) 24.0 (H) 38.0 (I) 42.0 (J) None of those

6. For the function $f(x)=\frac{1}{x}$ on the interval $[a,b]=[1,2]$ find the value c that satisfies the Mean Value Theorem:
   $$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

   (A) $-\sqrt{2}$ (B) -1 (C) $-\frac{1}{2}$ (D) $-\frac{1}{4}$ (E) 0
   (F) $\frac{1}{4}$ (G) $\frac{1}{2}$ (H) 1 (I) $\sqrt{2}$ (J) None of those

7. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 360 m of wire at your disposal, what is the largest area you can enclose?

   (A) 90 (B) 100 (C) 180 (D) 270 (E) 400
   (F) $16,000$ (G) $16,200$ (H) $16,400$ (I) $16,600$ (J) None of those

8. Find a point of inflection $(c, f(c))$ for the function $f(x) = (x^2-1)^{2/3}$ with $c > 0$. The derivatives of this function are $f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$, $f''(x) = \dfrac{4(x^2-3)}{9(x^2-1)^{4/3}}$.

(A)	$(1,0)$	(B)	$\left(\sqrt{3}, 2^{2/3}\right)$	(C)	$\left(\sqrt{3}, 0\right)$	(D)	None of those
(E)	$(0,1)$	(F)	$\left(\sqrt{3}, -2^{2/3}\right)$	(G)	$(3,4)$	(H)	No inflection point with $c > 0$

9. Find an interval on which function $f(x) = xe^{-x^2}$ is increasing. The derivative of this function is $f'(x) = e^{-x^2}(1-2x^2)$.

(A)	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	(B)	$\left(-\frac{1}{\sqrt{2}}, +\infty\right)$	(C)	$\left(-\frac{1}{\sqrt{2}}, 0\right)$	(D)	$\left(-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)$
(E)	$\left(-\infty, +\frac{1}{\sqrt{2}}\right)$	(F)	$\left(+\frac{1}{\sqrt{2}}, +\infty\right)$	(G)	$\left(0, +\frac{1}{\sqrt{2}}\right)$	(H)	None of those

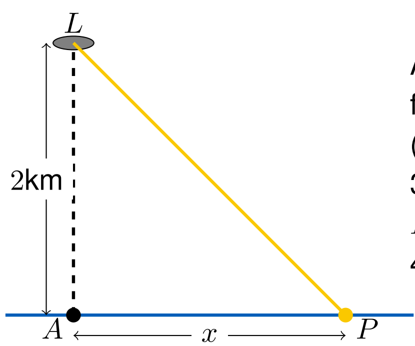
10. Find the global maximum of the function $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

(A)	0	(B)	1	(C)	2	(D)	e	(E)	e^2
(F)	10	(G)	$10e$	(H)	$10e^2$	(I)	20	(J)	None of those

11. Perform one iteration of the Newton's method to find the x coordinate of the point of intersection of the two curves $y = x^3$ and $y = x + 1$ starting from the guess $x_0 = 1.5$. Do not round your answer until the end, when you are selecting the multiple choice answer.

(A)	1.25	(B)	1.30	(C)	1.35	(D)	1.40	(E)	1.45
(F)	1.50	(G)	1.55	(H)	1.60	(I)	1.65	(J)	1.70

12.



A lighthouse L is located on a small island that is 2 km from the nearest point A on a long, straight shoreline (see figure to the left). If the lighthouse lamp rotates at 3 revolutions per minute, how fast is the illuminated spot P on the shoreline moving along the shoreline when it is 4 km from A?

You might find the following information useful: (1) one revolution corresponds to 2π radians, and (2) $1 + \tan^2 x = \sec^2 x$.

(A)	10	(B)	15	(C)	48	(D)	60	(E)	6π
(F)	10π	(G)	15π	(H)	48π	(I)	60π	(J)	None of those

End of Practice Test ∪B

Review #3: Using Derivatives (Answers)

Practice Test ∪A

1. (A)

2. (E)

3. (C), (E)

4. (G)

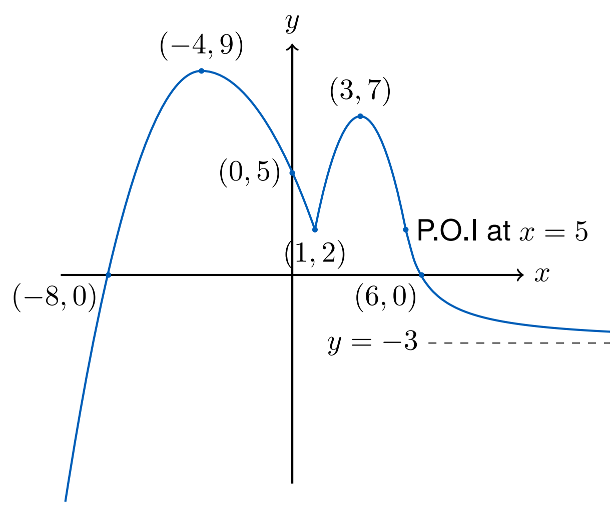
5. e^2

6. $-\frac{1}{8\pi}$cm/min

7. If there were numbers a, b with $a < b$ and $f(a) = f(b) = 0$ then (because $f(x)$ is continuous and differentiable everywhere) Rolle's Theorem would tell us that there is a number c between a and b such that $f'(c) = 0$. However, $f'(x) = e^x + 6x^2 > 0$, so there can be no such c.

8. 12cm$\times 6$cm$\times 8$cm

9. One possible function:



Practice Test ∪B answers on the next page. For best results, take a break to review the material you missed on Practice Test ∪A and don't try Test ∪B for a day or two...

Practice Test ∪B

1. (I)
2. (A)
3. (F)
4. (F)
5. (E)
6. (I)
7. (G)
8. (B)
9. (D)
10. (G)
11. Approximately 1.347826, so the closest choice is (C)
12. (I)

Review #4: Integrals

Practice Test $\int$A (50 minutes)

For multiple-choice questions requiring numerical answers, the correct answer might not be listed. Choose the value that is **nearest to your answer**; if your answer is equidistant from two nearest choices, choose the larger of these two choices.

1. Evaluate $\displaystyle\int_2^4 x dx$.

   (A) 0.0 (B) 1.0 (C) 2.0 (D) 3.0 (E) 4.0
   (F) 5.0 (G) 6.0 (H) 7.0 (I) 8.0 (J) 9.0

2. Evaluate $\displaystyle\sum_{k=1}^{4} \frac{k-1}{k}$.

   (A) 0 (B) 0.5 (C) 0.66 (D) 0.75 (E) 1.0
   (F) 1.5 (G) 2.0 (H) 2.5 (I) 3 (J) 4

3. Use three rectangles and the right-endpoint-rule to approximate the area under $y = \sqrt{x^3+1}$ on the interval $[-1, 2]$.

   (A) 1.0 (B) 1.5 (C) 2.0 (D) 2.5 (E) 3.0
   (F) 3.5 (G) 4.0 (H) 4.5 (I) 5.0 (J) 5.5

4. What is the most general antiderivative of $\dfrac{2x}{5+x^2}$?

   (A) $\dfrac{2}{2x} + C$ (B) $\dfrac{x^2}{5x + \frac{1}{3}x^3} + C$ (C) $\dfrac{-2(x^2-5)}{(x^2+5)^2} + C$

   (D) $\dfrac{2}{5+x^2} + C$ (E) $\dfrac{2}{5}\arctan(x) + C$ (F) $\dfrac{2x}{5}\arctan(x) + C$

   (G) $2x\ln(x^2+5) + C$ (H) $\ln(x^2+5) + C$ (I) $2\ln(x^2+5) + C$
   (J) None of these.

5. Suppose that $\int_0^{-1} f(x)\,dx = 3$, $\int_0^5 f(x)\,dx = -2$, and $\int_3^5 f(x)\,dx = 1$. Calculate $\int_{-1}^{3} f(x)\,dx$.

   (A) -8 (B) -6 (C) -4 (D) -2 (E) 0
   (F) 2 (G) 4 (H) 6 (I) 8 (J) Not enough information to answer.

6. Given $f(x) = \int_0^{x^2} \sqrt{t^4 + 16t^2}\,dt$, calculate $f'(3)$.

7. Evaluate $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$.
   Hint: Use the substitution $u = 1 + \sqrt{x}$.

8. What is the area of the region enclosed by the curves $y = -x^2 + 2x + 8$ and $y = 2x - 1$?

End of Practice Test $\int$A

Practice Test $\int$B (50 minutes)

1. Evaluate $\displaystyle\int_3^5 x dx$.

(A)	0.0	(B)	1.0	(C)	2.0	(D)	3.0	(E)	4.0
(F)	5.0	(G)	6.0	(H)	7.0	(I)	8.0	(J)	9.0

2. What is the **average value** of the function $y = 9x^2 + 10$ on the interval $[0, 3]$?

(A)	7	(B)	12	(C)	17	(D)	22	(E)	27
(F)	32	(G)	37	(H)	42	(I)	47	(J)	52

3. Compute $\displaystyle\int \sec^2(2x) + \frac{5}{1+x^2}\, dx$.

(A)	$\tan(2x) + 5\ln(1+x^2) + C$	(B)	$\tan^2(2x) + \frac{5}{2x}\ln(1+x^2) + C$
(C)	$\tan(2x) + \frac{5}{2x}\ln(1+x^2) + C$	(D)	$\frac{1}{2}\tan(2x) + 5\ln(1+x^2) + C$
(E)	$\frac{1}{\cos^2(2x)} + 5\ln(1+x^2) + C$	(F)	$\frac{1}{2}\tan(2x) + \frac{5}{2x}\ln(1+x^2) + C$
(G)	$\tan(2x) + 5\arctan(x^2) + C$	(H)	$\frac{1}{2}\tan(2x) + \arctan(5x) + C$
(I)	$\tan(2x) + \arctan(5x^2) + C$	(J)	$\frac{1}{2}\tan(2x) + 5\arctan(x) + C$

4. Find $s(2)$ given $\dfrac{ds}{dt} = 4t^3 - 5t$ and $s(0) = 3$.

(A)	-12	(B)	-9	(C)	-6	(D)	-4	(E)	0
(F)	4	(G)	6	(H)	9	(I)	12	(J)	Not enough information to answer.

5. If $f''(x) = 2$, $f(0) = 3$, and $f(1) = 2$, what is $f(x)$?

6. Evaluate $\int_{-1}^{0} t(1+t)^{19} dt$.

7. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{24k^2}{n^3} + \frac{4k}{n^2}\right)$.

   Hint: you do not need to use the fact that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$*, but you may.*

8. Calculate the area between $y = \sin(x)$ and the line tangent to $y = \sin(x)$ at $x = 0$, from $x = 0$ to $x = \pi$.

End of Practice Test $\int$B

Review #4: Integrals (Answers)

Practice Test $\int$A

1. (G)
2. $0 + 1/2 + 2/3 + 3/4$ is approximately 1.9167, so (G) is closest.
3. $4 + \sqrt{2}$, which is approximately 5.4142, so (J) is closest.
4. (H)
5. (B)
6. $f'(x) = \sqrt{x^8 + 16x^4} \cdot 2x$, so $f'(3) = 6\sqrt{7857}$ (which is approximately 531.8383).
7. $\int \frac{1}{1+\sqrt{x}} dx = 2 \int \left(1 - \frac{1}{u}\right) du$. The answer is $2(1 + \sqrt{x} - \ln(1 + \sqrt{x})) + C$.
8. 36

Practice Test $\int$B answers on the next page. For best results, take a break to review the material you missed on Practice Test $\int$A and don't try Test $\int$B for a day or two...

Practice Test $\int$B

1. (I)
2. (G)
3. (J)
4. (H)
5. $f(x) = x^2 - 2x + 3$.
6. Use the substitution $u = 1 + t$. The answer is $-1/420$, which is approximately -0.0024.
7. You could evaluate the sum and then the limit using the formula in the hint and remembering that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Or you could recognize this as the Riemann sum representing $\int_0^2 (3x^2 + x)dx$. The answer is 10.
8. The tangent line is $y = x$, so we are looking for the area between $y = x$ and $y = \sin(x)$ on the interval $[0, \pi]$. The answer is $\pi^2/2 - 2$, which is approximately 2.9348.

Content Checklist

Note:

All of these can be found in the relevant chapters above, but here they are in one place to aid your review as you prepare for the final exam.

(FC) Functions: concepts

Central to Calculus is an understanding of functions and their properties.

1. Understand the definition of a function, and apply it in places where the “vertical line test” does not apply (such as x being a function of y).
2. Be able to determine the domain of any of the functions listed below (FE).
3. Be able to find the x-intercepts and y-intercept of any of the functions listed below (FE).
4. Understand functions as things that vary, including the notion of “$f(x)$ increases as x decreases” and “increasing” negative numbers.
5. Understand function composition, including notation, domain, range, and the definition of even and odd functions.
6. Understand what an inverse function is, and how to prove that a given function does or does not have an inverse.

(FE) Functions: examples

Some functions come up frequently, so it is important that you are comfortable working with them.

1. Basic functions: lines, parabolas, cubic functions, square roots.
2. More advanced functions: $|x|$, x^n, $x^{1/n}$, e^x, $\log_b(x)$, $\ln(x)$.
3. Trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\csc(x)$, $\sec(x)$, $\cot(x)$, $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$. (You should also know the sine, cosine, and tangent of the standard angles in radians.)

(N) Notation

In order to work with functions we need to be able to write about them. To do that, you need to be comfortable with:

1. Interval notation (for example, when describing the domain of a function).
2. Function composition notation (two forms: $f(g(x))$ is more common in calculus than $(f \circ g)(x)$).
3. The symbol “$=$” (never put it between two things that are not equal).

4. The symbol “∞” (which is not a number).

(A) Algebra

Once you understand what expressions and functions are and how to write about them with appropriate notation, you can manipulate them algebraically.

1. Understand how transformations affect the algebraic definition of a function.
2. Be able to compute the algebraic definition of a function that is formed by transforming a given function.
3. Be able to rationalize expressions like $\frac{1}{\sqrt{x}-10}$ with conjugate multiplication.
4. Be able to factor any polynomial of degree at most 2, perhaps by using the quadratic formula.
5. Be able to determine whether a is a root of a given polynomial.
6. Be able to use polynomial long division to express a rational function in quotient/remainder form.
7. Be able to explain why two expressions are or are not equal.
8. Be able to explain why two equations are or are not equivalent.
9. Be able to express the set of points represented by a graph as several equivalent equations. (Example: point/slope form of a line and slope/intercept for of the same line)
10. Be able to simplify an expression and identify resulting change of domain in the corresponding functions (for example x^2/x compared with x).

2.1: Rates of Change and Tangents to Curves

- You should know…
  - ☐ what the average rate of change of a function is.
  - ☐ what the instantaneous rate of change of a function is.
- You should understand…
  - ☐ how average and instantaneous rates of change relate to secant and tangent lines.
- You should be able to compute…
  - ☐ the average rate of change of a given function over a given interval.

2.2: Limits

- You should know…
  - ☐ the seven Limit Laws listed in your textbook.
  - ☐ the limit of a polynomial $p(x)$ as x approaches a constant.
  - ☐ the limit of a rational function $p(x)/q(x)$ as x approaches a constant c for which $q(c) \neq 0$.
  - ☐ the Sandwich Theorem (also known as the Squeeze Theorem).
- You should understand…
  - ☐ the meaning of $\lim_{x \to a} f(x) = L$.
  - ☐ how jump discontinuities and removable discontinuities relate to limits.
  - ☐ the difference between, for example, the functions $\frac{(x-1)(x+1)}{x-1}$ and $x + 1$.
  - ☐ the Sandwich Theorem.

- You should be able to compute...
  - ☐ Limits of the form $\lim_{x\to a} c$, where c is a real number.
  - ☐ Limits of the form $\lim_{x\to a} p(x)$, where $p(x)$ is a polynomial.
  - ☐ Limits of the form $\lim_{x\to a} \frac{p(x)}{q(x)}$, where p and q are polynomials and after simplifying there is no remainder.
  - ☐ limits based on a graph of a function.
  - ☐ limits involving radicals that can be rationalized. **Example 9**
  - ☐ limits using the Sandwich Theorem. **Examples 10 and 11**

2.4: One-sided limits

- You should know...
  - ☐ the relationship between limits and one-sided limits.
  - ☐ that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (you do not need to be able to prove this).
- You should understand...
  - ☐ the meaning of a one-sided limit.
- You should be able to compute...
  - ☐ one-sided limits from a graph.
  - ☐ one-sided limits of a piecewise-defined function.
  - ☐ one-sided limits of anything for which you can compute a limit.

2.5: Continuity

- You should know...
  - ☐ the definition of “continuous at a point”.
  - ☐ the definition of “continuous on a closed interval”.
  - ☐ which common functions (such as polynomials, e^x, $\ln(x)$, trig functions, rational functions, etc.) are continuous on which intervals.
  - ☐ the Intermediate Value Theorem.
  - ☐ what a removable discontinuity is.
- You should understand...
  - ☐ what it means to be continuous.
  - ☐ when the Intermediate Value Theorem applies (and when it doesn’t).
- You should be able to compute or prove...
  - ☐ that a function is (or is not) continuous at a point.
  - ☐ that a function is (or is not) continuous on a closed interval.
  - ☐ the “continuous extension to a point” of a function with a removable discontinuity.

2.6: Limits involving infinity

- You should know...
  - ☐ the “infinity versions” of the Limit Laws.

- ☐ what a horizontal asymptote is.
- ☐ what a vertical asymptote is.
- ☐ what a slant (also known as oblique) asymptote is.

- You should understand…
  - ☐ the meaning of $\lim_{x\to a} f(x) = \infty$.
  - ☐ the meaning of $\lim_{x\to a} f(x) = -\infty$.
  - ☐ the meaning of $\lim_{x\to\infty} f(x) = L$.
  - ☐ the meaning of $\lim_{x\to-\infty} f(x) = M$.
  - ☐ the relationship between limits and asymptotes.
- You should be able to compute…
  - ☐ any limit involving a rational function.
  - ☐ limits involving rationalizing a numerator.
  - ☐ the asymptotes of any rational function (vertical, horizontal, and slant).

3.1: Derivatives

- You should know…
  - ☐ the definition of *slope of the curve at a point*.
  - ☐ the definition of *tangent line to the curve at a point*.
  - ☐ the definition of *vertical tangent line*.
  - ☐ the definition of *derivative of the function at a point*.
- You should understand…
  - ☐ the relationship between a derivative at a point and a tangent line at a point.
- You should be able to compute…
  - ☐ the slope of a curve at a point, given a function and a specified point.
  - ☐ the equation of a tangent line to a curve at a point, given a function and a specified point.
  - ☐ an *estimate* of the slope of a curve at a point, given a sketch of a function and a specified point.
  - ☐ an *estimate* of the tangent line to a curve at a point, given a sketch of a function and a specified point.

3.2: The derivative as a function

- You should know…
  - ☐ the definition $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.
  - ☐ the alternate definition $f'(x) = \lim\limits_{z\to x} \dfrac{f(z)-f(x)}{z-x}$.
  - ☐ that if $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.
  - ☐ at least one function that is continuous but not differentiable at at least one point.
- You should understand…
  - ☐ the relationship between a function and its derivative in terms of slope.
  - ☐ the relationship between $f(x)$ and the roots of $f'(x)$.

- You should be able to compute…
  - ☐ the derivative of a function using the definition.

3.3: Differentiation rules

- You should know…
  - ☐ the following derivatives.
    $\frac{d}{dx}[c]$ $\frac{d}{dx}[x^n]$ $\frac{d}{dx}[cf(x)]$ $\frac{d}{dx}[f(x)g(x)]$ $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ $\frac{d}{dx}[e^x]$
- You should understand…
  - ☐ the relationship between the limit laws and the derivative rules.
- You should be able to compute…
  - ☐ the derivative of any function constructed from x, x^n, and e^x through addition, multiplication, and division.

3.4: The derivative as a rate of change

- You should know…
  - ☐ the definition of *velocity*.
  - ☐ the definition of *speed*.
  - ☐ the definition of *acceleration*.
  - ☐ the definition of *jerk*.
- You should understand…
  - ☐ the difference between velocity and speed.
  - ☐ the meaning of a *positive* or *negative* derivative of functions that represent distance, cost, and other models.
- You should be able to compute…
  - ☐ a rough sketch of $f'(x)$, given a sketch of $f(x)$.
  - ☐ a rough sketch of $f(x)$, given a sketch of $f'(x)$.

3.5: Derivatives of trigonometric functions

- You should know…
  - ☐ $\frac{d}{dx}[\sin(x)]$.
  - ☐ $\frac{d}{dx}[\cos(x)]$.
  - ☐ $\frac{d}{dx}[\tan(x)]$.
- You should understand…
  - ☐ the relationship between the horizontal tangent lines of $\sin(x)$ and the zeros of $\cos(x)$.
  - ☐ the relationship between the horizontal tangent lines of $\cos(x)$ and the zeros of $\sin(x)$.
  - ☐ how to use the quotient rule to find the derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$.
  - ☐ how to use the fact that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ to compute the derivative of $\sin(x)$.

- ☐ how to use the fact that $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$ to compute the derivative of $\cos(x)$.

- You should be able to compute…
  - ☐ $\frac{d}{dx}[\sec(x)]$
  - ☐ $\frac{d}{dx}[\csc(x)]$
  - ☐ $\frac{d}{dx}[\cot(x)]$
  - ☐ the derivative of functions consisting of products and quotients of polynomials, e^x, and the standard trig functions.

3.6: The chain rule

- You should know…
  - ☐ the Chain Rule.
- You should understand…
  - ☐ …the Chain Rule.
- You should be able to compute…
  - ☐ the derivative of composite functions. **Example:** what is $\frac{d}{dx}\left[e^{x^2 \cos\left(\frac{\pi}{2}\sqrt{1+x^4}\right)}\right]$?

3.7: Implicit differentiation

- You should know…
  - ☐ what an *implicitly defined* function is.
  - ☐ what a *normal line* is.
- You should understand…
  - ☐ the difference between an *explicit* relation and an *implicit* relation.
  - ☐ the relationship between implicit differentiation and the chain rule.
- You should be able to compute…
  - ☐ the tangent and normal lines of curves, including curves that are not functions (such as ellipses), by using implicit differentiation.

3.8: Derivatives of inverse function and logarithms

- You should know…
  - ☐ $\frac{d}{dx}[\ln|x|]$.
  - ☐ $\frac{d}{dx}[a^x]$, where a is a positive real number.
  - ☐ $\frac{d}{dx}[\log_a(x)]$, where a is a positive real number.
- You should understand…
  - ☐ how to derive the formula $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$.
  - ☐ the method of Logarithmic Differentiation (and when to use it).

- You should be able to compute...
  - ☐ $\frac{d}{dx}\left[f^{-1}(x)\right]$.
  - ☐ the derivative of a function of the form $h(x)^{g(x)}$ (for example, x^{3x}).

3.9: Inverse trigonometric functions

- You should know...
  - ☐ $\frac{d}{dx}[\arcsin(x)]$.
  - ☐ $\frac{d}{dx}[\arccos(x)]$.
  - ☐ $\frac{d}{dx}[\arctan(x)]$.
- You should understand...
  - ☐ how to find $\frac{d}{dx}\left[\cot^{-1}(x)\right]$, $\frac{d}{dx}\left[\csc^{-1}(x)\right]$, and $\frac{d}{dx}\left[\sec^{-1}(x)\right]$.
- You should be able to compute...
  - ☐ the derivative of functions involving inverse trig functions.

3.10: Related rates

- You should know...
  - ☐ the six-step process for dealing with a Related Rates problem (or any other systematic approach that works for you).
- You should understand...
  - ☐ how to use formulas for volumes to set up and solve Related Rates problems.
- You should be able to compute...
  - ☐ the rate of change of a given variable, given enough partial information. (Yes, that's quite vague. That's because 'compute' is too weak a word for what is involved with solving problems in section 3.10 – take a look at the assigned homework to see the sort of thing we are expecting.)

3.11: Linearization and Differentials

- You should know...
  - ☐ what the *linearization* of a function $f(x)$ at the point $x = a$ is.
  - ☐ what the *differential* dy is.
- You should understand...
  - ☐ the relationship between linearization and tangent lines.
  - ☐ how to use linearization to approximate values such as $\sqrt{4.01}$ and $\sin(3)$.
  - ☐ how to use differentials to approximate values.
- You should be able to compute...
  - ☐ approximations of things like $8.1^{1/3}$, $\sin(\pi/2 + 0.01)$, etc. using linearization or differentials.

4.1: Extreme Values of Functions

- You should know…
  - ☐ what an absolute maximum is.
  - ☐ what an absolute minimum is.
  - ☐ what a local maximum is.
  - ☐ what a local minimum is.
  - ☐ what a critical number (a.k.a critical point) is.
- You should understand…
  - ☐ the Extreme Value Theorem.
  - ☐ the First Derivative Theorem.
- You should be able to compute…
  - ☐ absolute extrema of a function on a closed interval by using the Extreme Value Theorem and critical numbers.

4.2: The Mean Value Theorem

- You should know…
  - ☐ the statement of *Rolle's Theorem*.
  - ☐ the statement of the *Mean Value Theorem*.
- You should understand…
  - ☐ how to use Rolle's Theorem to prove that horizontal tangent lines exist in an interval.
  - ☐ the relationship between average rate of change and instantaneous rate of change.
  - ☐ how to prove that a function has exactly one zero on a given interval.
- You should be able to compute…
  - ☐ the average rate of change of a function on a given interval.
  - ☐ (sometimes) the value of c that is promised by the Mean Value Theorem.‘

4.3: The First Derivative Test

- You should know…
  - ☐ the first derivative test for local extrema.
- You should understand…
  - ☐ the first derivative test for local extrema.
- You should be able to compute…
  - ☐ the local extrema of a function on a closed, open, or unbounded interval.

4.4: Concavity and Curve Sketching

- You should know…
  - ☐ what a point of inflection is.
  - ☐ the second derivative test for local extrema.

- You should understand...
  - ☐ the relationship between $f''(x)$ and the concavity of $f(x)$.
  - ☐ how to use $f'(x)$ and $f''(x)$ to sketch the curve $y = f(x)$.
- You should be able to compute...
  - ☐ intervals on which a function is increasing and decreasing.
  - ☐ intervals on which a function is concave up and concave down.

4.5: Indeterminate forms and l'Hospital's Rule

- You should know...
  - ☐ when l'Hospital's Rule can (and cannot) be used.
  - ☐ what the indeterminate forms are.
- You should understand...
  - ☐ how to use l'Hospital's Rule (and how to explain why it can be used).
  - ☐ how to use logarithms to deal with limits that involve indeterminate powers.
- You should be able to compute...
  - ☐ limits involving indeterminate forms by using l'Hospital's Rule (when appropriate).

4.6: Applied Optimization
In the first few sections of Chapter 4, we learned how to find local and global extrema of functions. Now we will apply this skill to modelling problems.

- You should know...
  - ☐ how to find the maximum or minimum of a function.
- You should understand...
  - ☐ the relationship between section 4.6 and sections 4.2 - 4.4.
- You should be able to compute...
  - ☐ the optimal value of an application problem.
- It is a good idea to review all of the word problems from this section in your textbook. For problems that are not assigned, at least set up the relevant figures and equations.

4.7: Newton's Method

- You should know...
  - ☐ Newton's Method.
- You should understand...
  - ☐ the relationship between Newton's Method and roots of a tangent line.
  - ☐ the relationship between Newton's Method and Linearization.
- You should be able to compute...
  - ☐ approximations of roots, intersections of curves, and numbers like π, $\sqrt{2}$, etc. using Newton's Method.

4.8: Antiderivatives

- You should know…
  - ☐ the definition of antiderivative.
  - ☐ the definitions of indefinite integral, integrand, and variable of integration.
- You should understand…
  - ☐ the relationship between a derivative and an antiderivative.
- You should be able to compute…
  - ☐ the most general antiderivative of many functions, including polynomials, $\sin(x)$, $\cos(x)$, $\sec^2(x)$, and some types of rational functions.

5.1: Estimating area with finite sums
This section seems unrelated to the previous chapter, but the area under the curve will turn out to be related to antiderivatives.

- You should know…
  - ☐ the definition of *average value*.
  - ☐ what is meant by *upper sum* and *lower sum*.
- You should understand…
  - ☐ the meaning of the area under a curve that represents, for example, position.
  - ☐ the difference between *displacement* and *distance traveled*.
- You should be able to compute…
  - ☐ estimates of the area under a curve by using a fixed number of rectangles.

5.2: Sigma notation & Limits of finite sums
Improving the estimates from the previous section requires more and more rectangles, which brings us back to the notion of a limit.

- You should know…
  - ☐ what the notation $\sum_{k=1}^{n} a_k$ means.
  - ☐ algebra rules for finite sums.
  - ☐ that $\sum_{k=1}^{n} c = nc$ and $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
  - ☐ the definition of (and notation for) a Riemann Sum.
- You should understand…
  - ☐ the relationship between Riemann Sums and the area under a curve.
- You should be able to compute…
  - ☐ $\sum_{k=1}^{n} a_k$ given a_k and n. For example, $\sum_{k=1}^{4}(2k+1) = 3+5+7+9 = 24$.
  - ☐ the limit $\lim_{n\to\infty} \sum_{k=1}^{n}(\frac{c}{n} + \frac{dk}{n^2})$ for any real numbers c and d.
  - ☐ the area under the curve $y = cx + d$ on the interval $[a, b]$ by using a Riemann Sum and a limit.

5.3: The definite integral

A definite integral measures the signed area under a curve.

- You should know...
  - ☐ what a definite integral is.
  - ☐ the algebraic properties of definite integrals.
- You should understand...
  - ☐ the relationship between Riemann Sums and definite integrals.
  - ☐ the relationship between definite integrals and area under a curve.
  - ☐ when the area under a curve is negative and when it is positive.
- You should be able to compute...
  - ☐ a definite integral by using known area formulas.
  - ☐ a definite integral by combining other, known, definite integrals.
  - ☐ the average value of a function using definite integrals.
  - ☐ the sign (negative or positive) of a definite integral.

5.4: The Fundamental Theorem of Calculus

As their names suggest, the theorems in this section are fundamental to (integral) calculus.

- You should know...
  - ☐ the Fundamental Theorem of Calculus, Part 1.
  - ☐ the Fundamental Theorem of Calculus, Part 2.
  - ☐ the Net Change Theorem
- You should understand...
  - ☐ how the FToC, Part 2, is related to the FToC, Part 1.
  - ☐ how the Net Change Theorem is related to the FToC, Part 2.
- You should be able to compute...
  - ☐ definite integrals using the Fundamental Theorem of Calculus, Part 2.
  - ☐ the derivative of definite integrals by using the Fundamental Theorem of Calculus, Part 1. Examples: $\int_a^x f(t)dt$ or $\int_a^{2e^x} f(t)dt$ or $\int_{\ln(x)}^{x^2} f(t)dt$

5.5: Substitution

This section is all about undoing the Chain Rule. In MATH 101 you will learn more methods for finding the antiderivative of complicated functions.

- You should understand...
  - ☐ the relationship between the Chain Rule and the method of Substitution.
- You should be able to compute...
  - ☐ indefinite integrals of the form $\int f(g(x))g'(x)dx$.

5.6: Area between curves

This section actually has two parts: using the method of substitution with definite integrals, and computing the area between curves.

- You should know...
  - ☐ the difference between signed area and total area.
  - ☐ what the area of the region between two curves is.
- You should understand...
  - ☐ the time-saving tricks you can use for definite integrals of the form $\int_{-a}^{a} f(x)dx$ when $f(x)$ is an even function or $f(x)$ is an odd function.
  - ☐ when it is easier to compute the area of a region by using y as the variable of integration rather than x.
- You should be able to compute...
  - ☐ definite integrals using the substitution method.
  - ☐ the area under a curve or between two curves, sometimes with y as the variable of integration.

8.7: Numerical Integration

Some integrals cannot be directly computed. This section describes two methods of approximation that are better than approximating by rectangles.

- You should understand...
  - ☐ how to approximate area under a curve with a finite number of trapezoids.
  - ☐ Simpson's Rule.
- You should be able to compute...
  - ☐ an approximation of the area under a curve with 4, 5, or 6 trapezoids.