MATHEMATICS 101 (all sections), March 20, 2014,
Midterm # 3. Time: 2 hours

Cape Heather .... V00[redacted] .... A03 T01 ....

Name (Last, First) **Student ID** **Section**

Problems 2 - 8	..........13..........	1 point for each, max of 7 points
Problems 9 - 11	....	2 points for each, max of 6 points
Problem 12	..........1.5..........	3 points
Problem 13	..........2.5..........	4 points
Problem 14	..........4..........	4 points
Total:	..........21..........	24 points

- **As stated in the course outline, the only calculators allowed on any examination are the Sharp EL-510R, RN or RNB.**
- This test consists of 13 questions (numbered 2 through 14) and has 9 pages, including this cover. You need to **show your work** for all questions (2 through 14), as we may disallow any answer which is not properly justified.
- For questions with numerical answers, if the exact answer is not be among the options, select the numerical answer closest to yours. If the answer is equidistant from two nearest choices, select the largest of the two choices.
- Before starting your test fill out your name (Last, First), student number (drop V00, fill in last 6 digits), and the tutorial section number (T01 - T28) on the top of this exam paper and on the bubble sheet, using an HB or softer pencil.
- Enter "A" in the bubble sheet as your answer to Question 1 now.

1. Enter "A" in the bubble sheet as your answer to Question 1 now.

2. [1 point] Determine whether or not the sequence $\{a_n\}$, where $a_n = \ln\left(\frac{n+2}{4n}\right)$ converges, and find its limit L if it does converge.

$\ln(n+2) - \ln(4n)$

$\ln(3) + \ln 4 + \ln 5 + \ln 6$

$- \ln 4 + \ln 8 + \ln 12 + \ln \ldots$

$\ln \lim_{n\to\infty} \left(\frac{n+2}{4n}\right) = \frac{1}{4}$

$\lim_{n\to\infty} \ln(\frac{1}{4}) = -1.37$

(A) $L = -2.0$ (B) $L = -1.5$ (C) $L = -1.0$ (D) $L = -0.5$ (E) $L = 0.0$

(F) $L = 0.2$ (G) $L = 1.0$ (H) $L = 1.5$ (I) $L = 2.0$ (J) Diverges

3. [1 point] Calculate the norm (or modulus) of $z = -3 + 2i$

$\sqrt{-3^2 + 2^2}$

$\sqrt{13} = 3.60$

(A) ~~−4.0~~ (B) ~~−3.0~~ (C) ~~−2.0~~ (D) ~~−1.0~~ (E) ~~0.0~~

(F) ~~1.0~~ (G) ~~2.0~~ (H) ~~3.0~~ (I) 4.0 (J) ~~$|z|$ does not exist~~

4. [1 point] Determine whether or not the series $\sum_{n=0}^{\infty} -3\left(\frac{2}{5}\right)^n$ converges, and find the sum of the series if it converges.

$\frac{2}{5}$ $\quad \frac{-3}{1-2/5}$

$-3 \left[\left(\frac{2}{5}\right)^n\right]$

$\frac{1}{1-2/5}$ $\quad (\frac{5}{3}) \cdot 3$

(A) -5.0 (B) -4.0 (C) -3.0 (D) -2.0 (E) -1.0
(F) 1.0 (G) 2.0 (H) 3.0 (I) 4.0 (J) Diverges

$-3(0.4)$ $\quad -3(0.16)$ $\quad (0.064)-3$

$\ln\left(\frac{n+2}{4n}\right)$

$\ln(n+2) - \ln(4n)$

$\frac{n2}{i}$ derivative $\frac{1}{4}$

-0.29, -0.69 $\rightarrow -1.367 \rightarrow -1.38 \rightarrow -1.39$

5. [1 point] Determine whether or not the sequence $\{b_n\}$, where $b_n = \dfrac{2^n}{n!}$ converges, and find its limit L if it does converge.

$\frac{2^n}{n!}$ $\quad \frac{2^n}{n!}$ $\quad \dfrac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$ $\quad \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ $\quad n! > 2^n$ $\quad \frac{2(2^n)}{(n+1)}$

$\frac{2}{n+1}$ $\quad r < 1$ converges ✓

(A) Diverges (B) $L = -1.0$ (C) $L = -0.6$ (D) $L = -0.4$ (E) $L = -0.2$
(F) $L = 0.0$ (G) $L = 0.2$ (H) $L = 0.6$ (I) $L = 0.8$ (J) $L = 1.0$

6. [1 point] Determine the real part of the complex number $z = 10e^{\frac{11}{3}\pi i}$

$r e^{\frac{11\pi}{3} i}$ $\quad re^{i\theta} = r(\cos\theta + i\sin\theta)$

$10(\cos\theta$

$5 - 8.66i$

(A) -6.0 (B) -5.0 (C) -4.0 (D) -2.0 (E) -1.0
(F) 1.0 (G) 2.0 (H) 4.0 (I) 5.0 (J) 6.0

7. [1 point] Determine whether or not the series $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n+4}$ converges, and find the sum of the series if it converges.

$\frac{2}{n+4}$ $\quad \frac{2}{n+1} < \frac{2}{n}$ $\quad \frac{2}{7} + \frac{2}{8} + \frac{2}{9} + \frac{2}{10} + \frac{2}{11}$

$\frac{2}{n+4}$ $\quad \frac{1}{n+1} < \frac{1}{n}$ $\quad \frac{2}{7} + \frac{2}{8}$

$2\left[\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right]$

harmonic
diverge

(A) Diverges (B) 0.1 (C) 0.5 (D) 1.0 (E) 1.5
(F) 2.0 (G) 2.5 (H) 3.0 (I) 3.5 (J) 4.0

$$\frac{2^n}{n!}$$

$$\frac{1024}{3628800}$$

$$\frac{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}$$

$$\sum_{n3}^{\infty} \frac{2}{n+4} \qquad \lim \to 0$$

$$\frac{\frac{2}{(n+1)+4}}{\frac{2}{n+4}}$$

$$\frac{\cancel{2}}{n+5} \cdot \frac{n+4}{\cancel{2}}$$

$$\frac{n+4}{n+5} \qquad r < 1$$

$$r > 1$$

$2\left(\frac{1}{n+4}\right)$ contains part of harmonic

$$2\left[\frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11}\right] \cdots$$

$$\frac{n^2 \tan^{-1} n}{2-n^2} \qquad \frac{n^2 \tan^{-1} n}{2-n^2} \qquad \frac{\frac{2n}{1+n^2}}{2-n^2} \qquad \frac{2n}{1+n^2} \times \frac{2-n^2}{1}$$

$$\frac{4n-2n^3}{1+n^2} \to \frac{4-6n^2}{1+2n}$$

$$\frac{n^2 \tan^{-1} n}{2-n^2} \qquad \frac{n^2}{1+n^2} \cdot -2n$$

$$\frac{0-12n}{0+2} \quad 6$$

$$\frac{-2n^3}{1+n^2} \qquad \frac{-6n}{2n}$$

$\sin y = x$

$\sin y = x$

$\cos y \frac{dy}{dx} = 1$

$1 - \cos y$

$\frac{\sin}{\cos} + \frac{\cos}{\cos} = 1$

$\tan + 1 = \sec^2$

$\tan y = x$

$\sec^2 y = 1$

$\tan^2 + 1$

$\frac{n^2}{2-n^2}$ $\frac{2n}{-2n}$ $-1\left(\frac{\pi}{2}\right)$

$\pi/2$

8. [1 point] Determine whether or not the sequence $\{c_n\}$, where $c_n = \dfrac{n^2 \tan^{-1} n}{2-n^2}$ converges, and find its limit L if it does converge.

$0 \le \tan^{-1} n \le \frac{\pi}{2}$

$n^2 \tan^{-1} n$

$\lim_{n\to\infty} \frac{n^2}{2-n^2}$ $\lim_{n\to\infty} \frac{\tan^{-1} n}{2-n^2}$

$\frac{\frac{\pi}{2} n^2}{2-n^2}$

1

$-\frac{\frac{\pi}{2}}{2-n^2} \le \tan^{-1} n \le \frac{\pi}{2}$

$\frac{n^2}{1+n^2} \times \frac{1}{2-n^2}$

$0 \le \tan^{-1} n$

$\frac{n^2}{2+n^2+n^4}$ $\frac{2n}{2n+4n^3}$ $\frac{2}{12n^2}$

0

(A) Diverges	(B) -3.0	(C) -1.5	(D) -1.0	(E) -0.5
(F) 0.0	(G) 0.5	(H) 1.0	(I) 1.5	(J) 3.0 ?

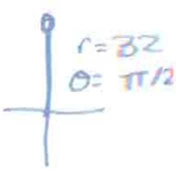
9. [2 points] Which of the ten complex numbers listed in the multiple choice is a solution of the equation $z^5 = 32i$, if any?

$n^{1/n} e^{i\alpha}$ $\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}$

$\sqrt[5]{32}$

$\sqrt[5]{32}\, e^{i\frac{\pi}{10}}$ $+\ e^{i\left(\frac{\pi}{10}+\frac{2\pi}{5}\right)}$ $\left(\frac{\pi}{10}+\frac{4\pi}{5}\right)$ $\left(\frac{\pi}{10}+\frac{6\pi}{10}\right)$

2



$\pi/2 / 5$

(A) $z = 2$ (B) $z = 2e^{i\pi/3}$ (C) $z = 2e^{i\pi/4}$ (D) $z = 2e^{i\pi/5}$ (E) $2e^{i\pi/6}$

(F) $z = 2e^{i\pi/7}$ (G) $z = 2e^{i\pi/8}$ (H) $z = 2e^{i\pi/9}$ (I) $z = 2e^{i\pi/10}$ (J) None of the above

$$\frac{n^2 \tan^{-1} n}{2-n^2} \qquad \frac{n^2}{2-n^2}$$

$$\lim_{n\to\infty} \frac{n^2}{2-n^2} \qquad \lim_{n\to\infty} \frac{\tan^{-1} n}{2-n^2}$$

$$\frac{2n}{-2n}$$

$$\boxed{\lim_{n\to\infty} -1} \qquad \lim_{n\to\infty} \frac{\frac{-\pi}{2}}{2-n^2} \le \frac{\tan^{-1} n}{2-n^2} \le \frac{\frac{\pi}{2}}{2-n^2}$$

$$0 \qquad\qquad 0$$

$$0$$

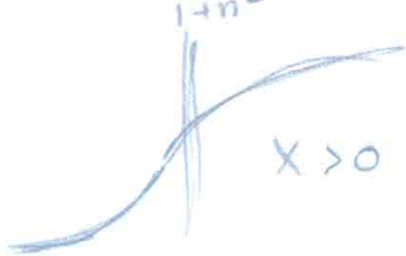
$$\lim_{n\to\infty} -1 \cdot \lim_{n\to\infty} 0 = 0.0$$

-1.5

$$\frac{2n\left(\frac{1}{1+n^2}\right)}{2-2n} \qquad \frac{\frac{2n}{1+n^2}}{-2n} \qquad \frac{2n}{1+n^2} \times \frac{1}{2n}$$

$$\frac{-1}{1+n^2}$$

$$\frac{n^2 \tan^{-1} n}{2-n^2}$$



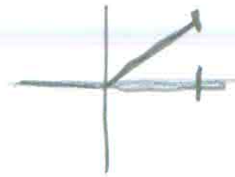
$X > 0$

$$\frac{2n}{2-2n} \qquad -1 \;($$

$$\frac{n^2}{2-n^2} \qquad \frac{\tan^{-1} n}{1}$$

$\Downarrow$

-1



$\pi/4$

10. [2 points] Find the 30th power of $z = 1 + i$

$z = (1+i)^{30}$

$r^{30}(\cos 30\theta + i \sin 30\theta)$

$r^{30}(\cos 30\theta + i\sin 30\theta)$

$2^{15}\left(\cos \frac{30\pi}{4} + i \sin \theta\right)$

$\sqrt{2}^{30}\left(\cos(\pi/4)30 + i\sin 30(\pi/4)\right)$

$2^{15}(-i\,1)$

$(0 \quad -1i)$

$-2^{15}i$

(A) $z^{30} = 2$ (B) $z^{30} = 2^{30}i$ (C) $z^{30} = 2^{15}i$ (D) $z^{30} = 2^{15} + i$ (E) $z^{30} = 0$

(F) $z^{30} = -2$ (G) $z^{30} = -2^{30}i$ (H) $z^{30} = -2^{15}i$ (I) $z^{30} = 2^{15} - i$ (J) None of the above

11. [2 points] Determine whether or not the series $\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{3}{2^n}$ converges, and find the sum of the series if it converges.

$\frac{3}{2^n}$ RATIO TEST

$3\left(\frac{1}{2}\right)^n$

$\frac{3^1}{2^n}$

$\frac{\frac{3}{2^{n+1}}}{\frac{3}{2^n}}$

$\frac{3}{(2)^n}$

$\frac{3}{2^{n+1}} \cdot \frac{2^n}{3}$

$\frac{3}{1} - \frac{3}{2} + \frac{3}{4} - \frac{3}{8}$

$r < 1$

converges absolutely

$3 + \frac{3}{4} + \frac{3}{2^4} + \frac{3}{2^6} = 4.0$

$-\left(\frac{3}{2^1} + \frac{3}{2^3} + \frac{3}{2^5}\right.$ 2

(A) -2.5 (B) -2.0 (C) -1.5 (D) -1.0 (E) 0.0

(F) 1.0 (G) 1.5 (H) 2.0 (I) 2.5 ~~(J) Diverges~~

$+\left[\frac{3}{2} + \frac{3}{8} + \frac{3}{2^5} + \frac{3}{2^7}\right. = 2$

$-\left[3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{2^6}\right. \sim 4$

converges

ab $|a_n|$ con — cond $|a_n|$ div.

1.5

12 [+ 2 = 3 points]

() Show that $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges.

$\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$

$\sum_{N=1}^{\infty} \ln(N+1) - \ln(N)$

$= +[\cancel{\ln(2)} + \cancel{\ln(3)} \dots \cancel{\ln(N)} + \ln(N+1)]$

$-[\ln(1) + \cancel{\ln(2)} \dots \cancel{\ln(N-1)} + \cancel{\ln(N)}]$

left with

$\ln(N+1) - \ln(1)$ as partial sum

$\ln(N+1) - 0$

$\lim_{n\to\infty} \ln(n+1) = \infty$

Limit of partial sum $= \infty$ $\therefore$ diverges

(b Show that $\sum_{n=1}^{\infty}(-1)^n \ln\left(\frac{n+1}{n}\right)$ converges conditionally.

0.5

Know that $\sum_{n=1}^{\infty} |a_n|$ diverges if $a_n = (-1)^n \ln\left(\frac{n+1}{n}\right)$

So $\sum_{n=1}^{\infty} a_n$ could converge conditionally or diverge.

To converge conditionally $(-1)^n \ln\left(\frac{n+1}{n}\right)$ must converge.

$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$

$\ln(1) = 0$

could converge or diverge

alternating series would be

$+[\ln(3/2) + \ln(5/4) + \ln(7/6) \dots]$ → getting smaller +

$-[\ln(2) + \ln(4/3) + \ln(6/5) \dots]$ → getting smaller −

will converge eventually

Ratio Test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ $\lim_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+1}\right) \to \infty}{\ln\left(\frac{n+1}{n}\right) \to \infty}$ ~~X~~ L'hop => $\frac{\ln(n+2) - \ln(n+1)}{\ln(n+1) - \ln(n)}$ L'hop => $\frac{\frac{1}{n+2} - \frac{1}{n+1}}{\frac{1}{n+1} - \frac{1}{n}}$ => $\frac{\frac{n+1-(n+2)}{(n+2)(n+1)}}{\frac{n-(n+1)}{n+1(n)}}$

Ratio test fails $1 <= \lim_{n\to\infty} \frac{n}{n+2} <=$ $\frac{-1}{(n+2)(n+1)} \times \frac{(n+1)(n)}{-1}$

$(-1)^n \quad \ln(n+1) - \ln(n)$

$-(\ln(2) - \ln(1)) \Rightarrow -\ln(2) + \ln(1)$

$+(\ln(3) - \ln(2) \qquad \ln(3) - \ln(2)$

$-(\ln(4) - \ln(3) \qquad -\ln(4) + \ln(3)$

$(-1) \quad \ln(n+1) - \ln(n)$

$-(\ln(2) - \ln(1))$

$+(\ln(3) - \ln(2)$

$-(\ln(4) - \ln(3))$

$(-1)^n \left(\ln(n+1) - \ln(n) \right)$

$-\ln(2) + \ln(1)$

$+ \quad \ln(3) - \ln(2)$

$+ \quad -\ln(4) + \ln(3)$

$(\ln($

$(-1)^n (\ln(n+1) - \ln(n))$

-0.405

$-\ln(2) + \ln(1)$

$+\ln(3) - \ln(2)$

$-\ln(4) + \ln(5)$

$\ln(1) + \ln(3) + \ln(3) + \ln(5) + \ln(5)$

$\ln(6) + \ln(10)$

$\ln\left(\frac{n+1}{n}\right) \qquad -\ln(2) + \ln(3) - \ln(4) + \ln(5)$

$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$

1.3 [1 + 3 = 4 points]

(a) Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges or diverges.

$\lim_{n\to\infty} \frac{1}{n^n} = 0$ Could conv or diverge

$\frac{1}{n^n} = \left(\frac{1}{n}\right)^n$

↓

Harmonic series which diverges

∴ diverges as well

X

0

$\ln y = n \ln\left(\frac{1}{n}\right)$

$\frac{\ln\left(\frac{1}{n}\right)}{\frac{1}{n}}$ LOP →

$\frac{n}{1/n^2}$

$\frac{n}{1} \cdot \frac{n^2}{1}$

$\lim_{n\to\infty} = 3n$

∞

(b) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ converges, and find the sum of the series if it converges.

$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} \Rightarrow \frac{2}{n(n+2)} \Rightarrow \frac{1}{n} - \frac{1}{n+2}$ TELESCOPING

Partial Fraction

$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{(n+2)}$

$An + 2A + Bn$

$An + Bn = 0$

$2A = 2$

$A = 1$

$B = -1$

$+\left[\frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \cdots \cancel{\frac{1}{N-1}}, \cancel{\frac{1}{N}}\right]$

$-\left[\cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \cdots \cancel{\frac{1}{N}} + \frac{1}{N+1} + \frac{1}{N+2}\right]$

$\underbrace{1 + \frac{1}{2}}_{\text{always } 1\frac{1}{2}} \ominus \frac{1}{N+1} \ominus \frac{1}{N+2}$ => Partial sum

always $1\frac{1}{2} + \lim_{n\to\infty} \frac{1}{n+1} + \frac{1}{n+2}$

both 0

Limit of partial sums = $1\frac{1}{2}$ = converging to

2.5/3

14. [2 + 2 = 4 points] Find the first three non-zero terms of the Taylor series:

(i) for $y = e^{3x}$ generated at the point $x = 0$

$y = e^{3x} \Rightarrow 1$

$y' = 3e^{3x} \Rightarrow 3$

$y'' = 9e^{3x} \Rightarrow 9$

$\frac{f^{K}(a)}{n!}(x-a)^{K}$

$\frac{1}{0!}(x-0)^{0} \Rightarrow 1$

$\frac{3}{1!}(x-0)^{1} \Rightarrow 3x$

$\frac{9}{2!}(x-0)^{2} \Rightarrow \frac{9x^2}{2!}$

(ii) for $y = \sqrt{x}$ generated at the point $x = 4$

$y = x^{1/2} \overset{f(4)}{\Rightarrow} 2$

$y' = \frac{1}{2}x^{-1/2} \Rightarrow 1/4$

$y'' = -\frac{1}{4}x^{-3/2} \Rightarrow -1/32$

$\frac{f^{K}(a)}{n!}(x-a)^{K}$

$\frac{2}{0!}(x-4)^{0} = 2$

$\frac{1/4}{1!}(x-4)^{1} = \frac{x-4}{4}$

$\frac{-1/32}{2!}(x-4)^{2} \quad \frac{\frac{-(x-4)^2}{32}}{2!} \Rightarrow \left(\frac{-(x-4)^2}{64}\right)$

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