CSC 225

Algorithms and Data Structures: I

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ECS 516

Building up a heap bottom-up

- n standard insert-operations for a heap result in $O(n \log(n))$ time.
- Can we build up a heap for n given elements faster? Is $O(n)$ possible?

Bottom-Up Heap (pgs. 176-178)

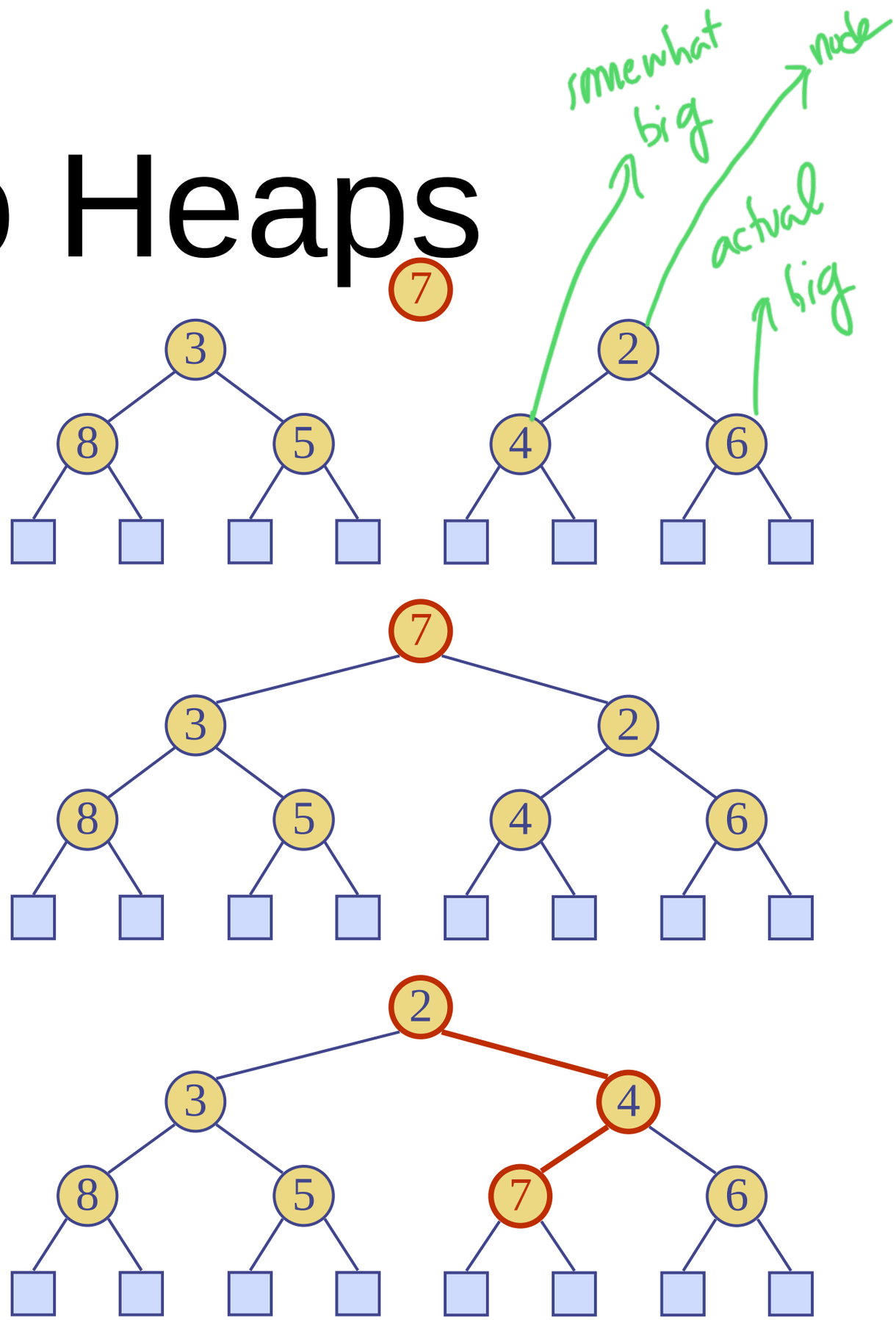
```
Algorithm BottomUpHeap(S):
    Input: A list S storing m keys
    Output: A heap T storing the m keys

    if S is empty then
        return external node
    remove the first key, k, from S
    split S in half, lists S_1 and S_2
    T_1 ← BottomUpHeap(S_1)
    T_2 ← BottomUpHeap(S_2)
    T ← merge(k,T_1,T_2)
    DownHeap(T,root)
    return T
```

Merging Two Heaps

- We are given two heaps and a key *k*

- We create a new heap with the root node storing *k* and with the two heaps as subtrees

- We perform downheap to restore the heap-order property



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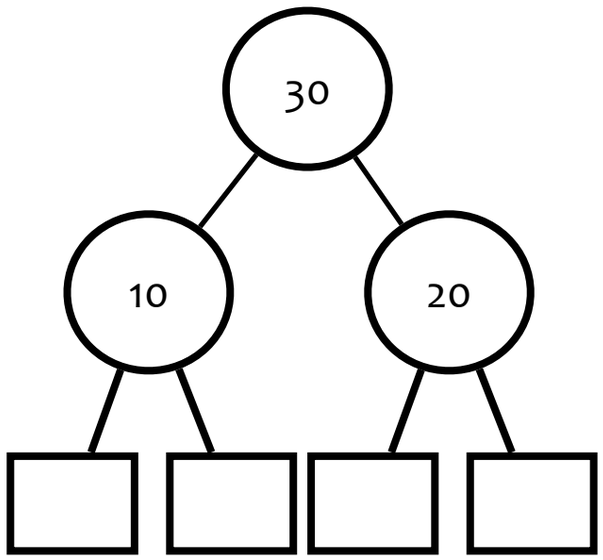

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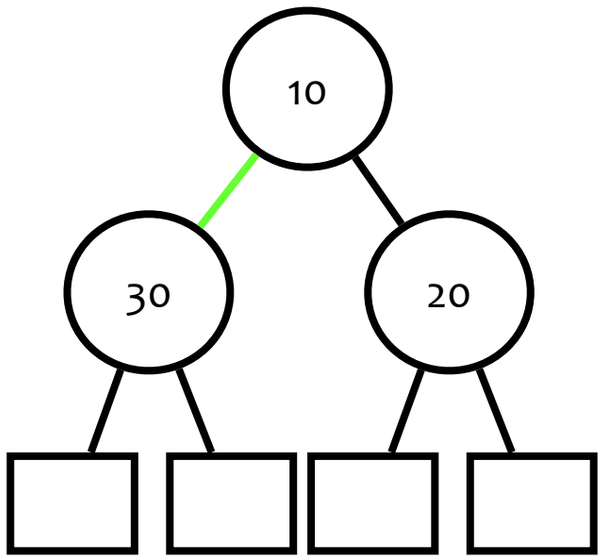

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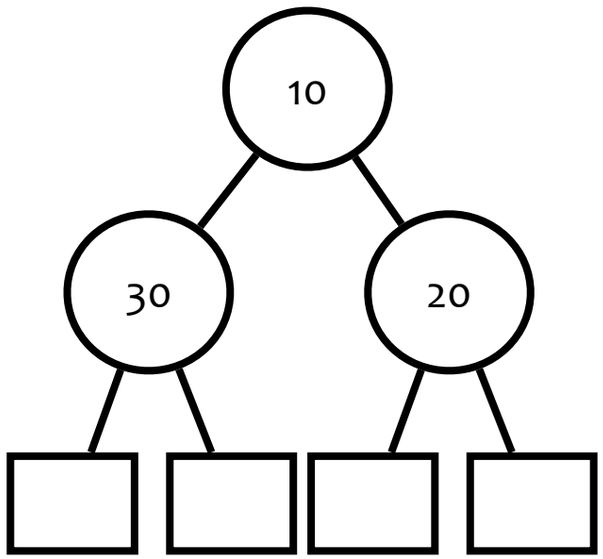

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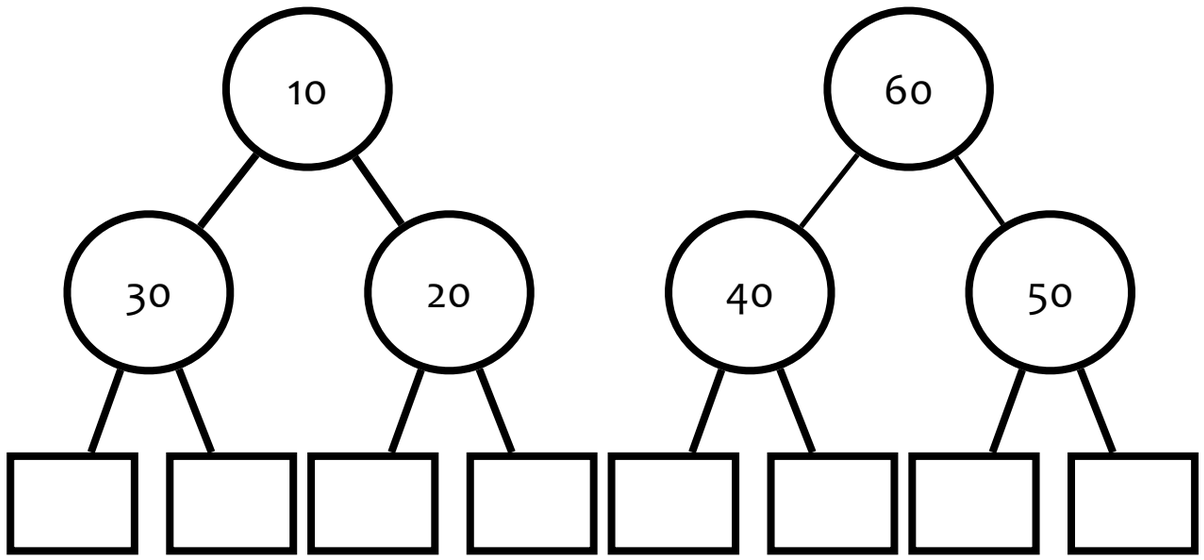

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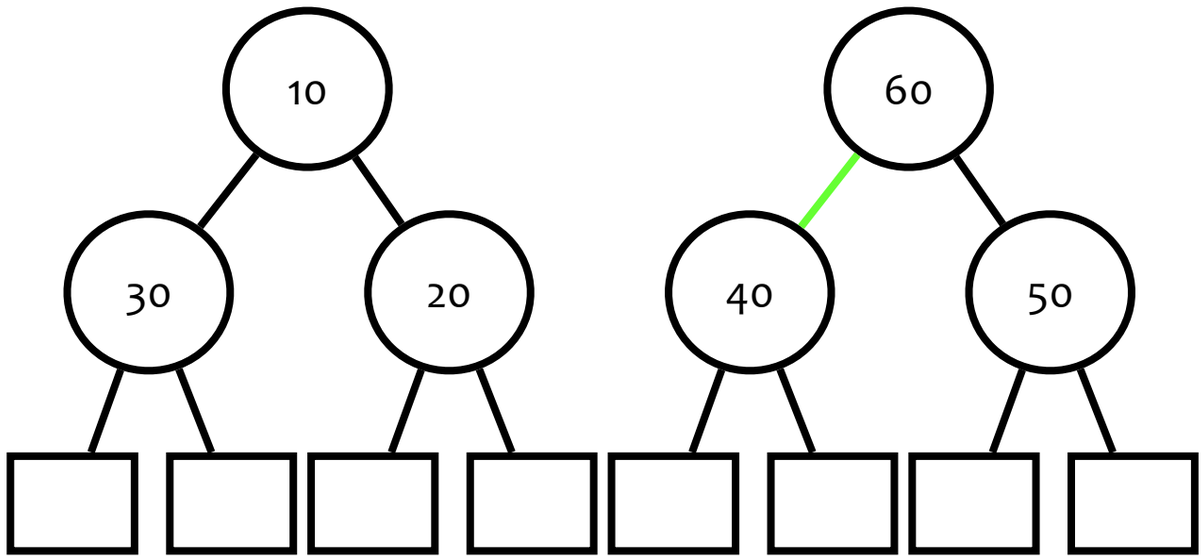

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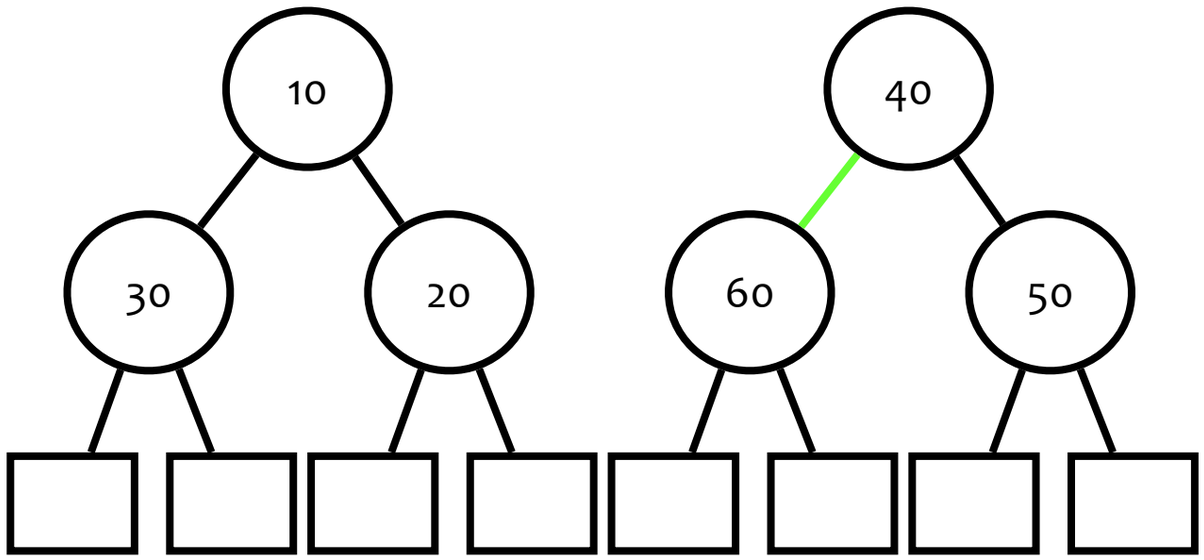

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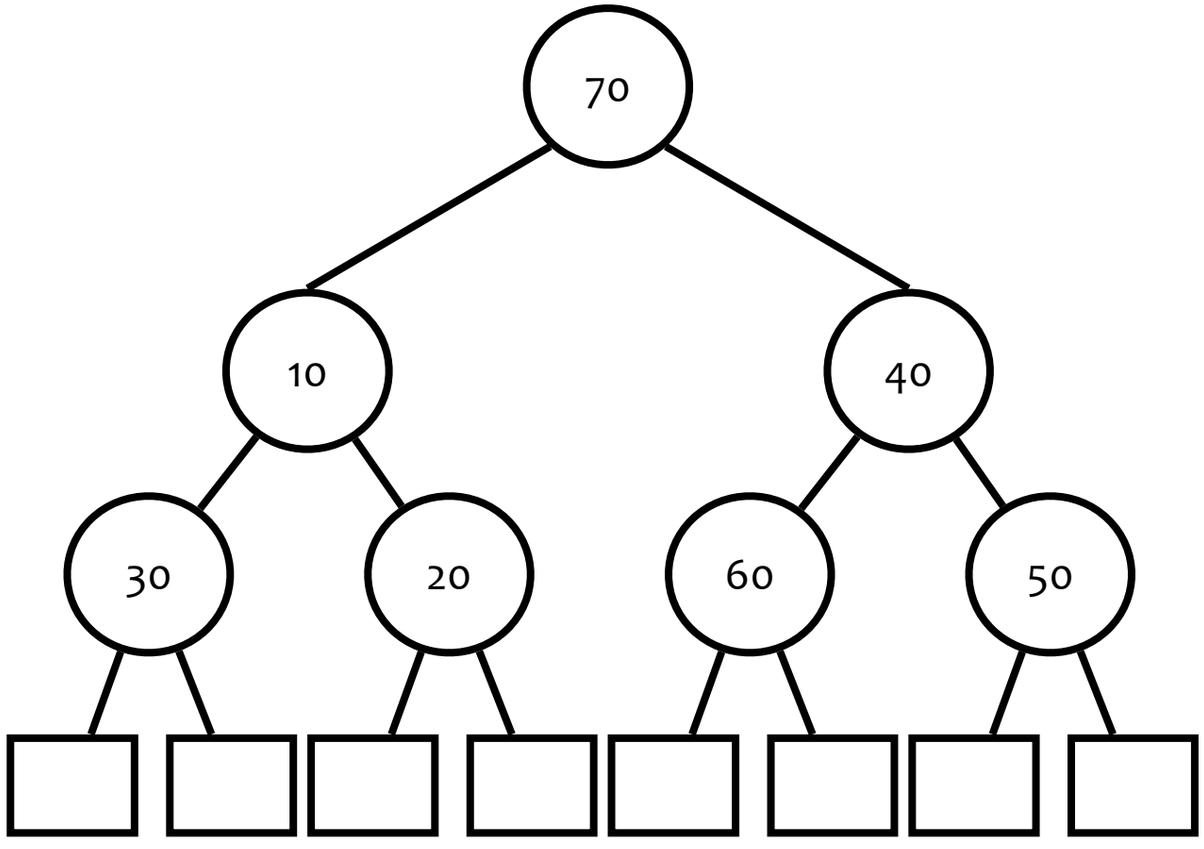

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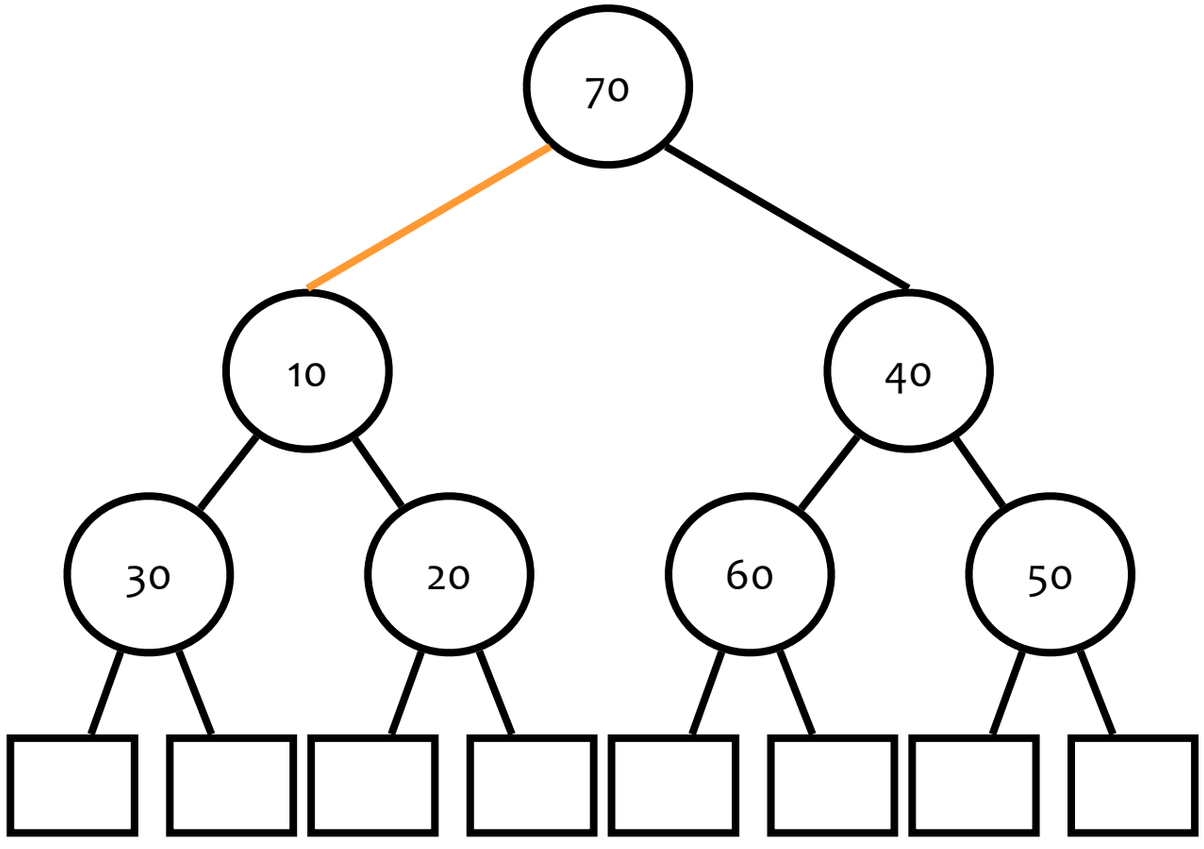

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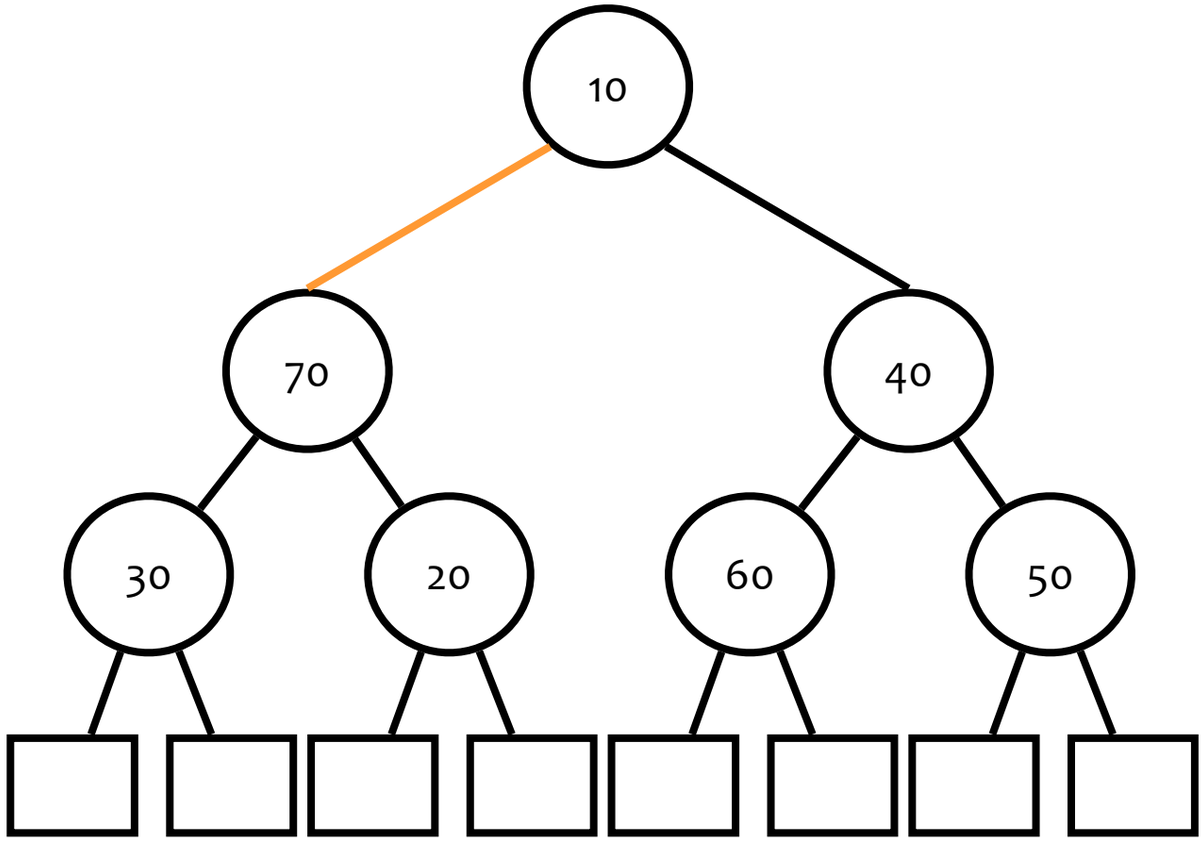

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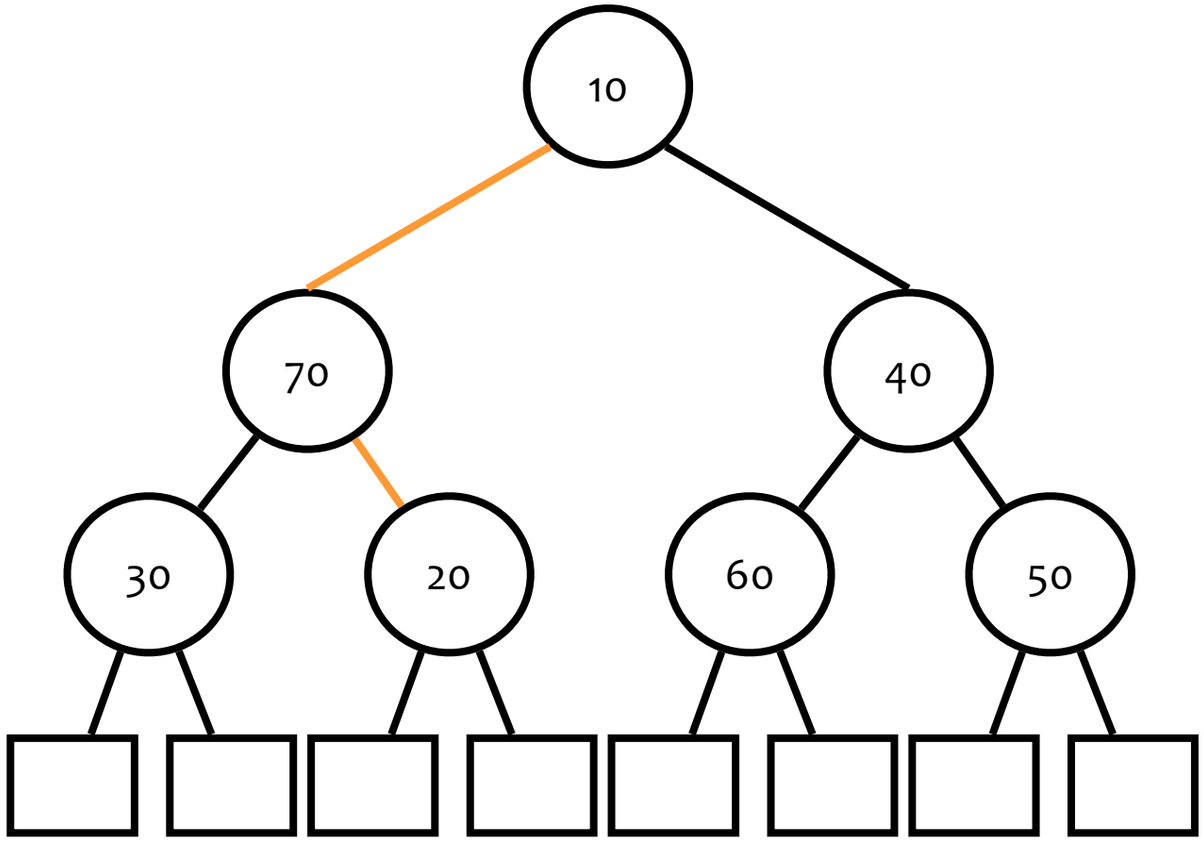

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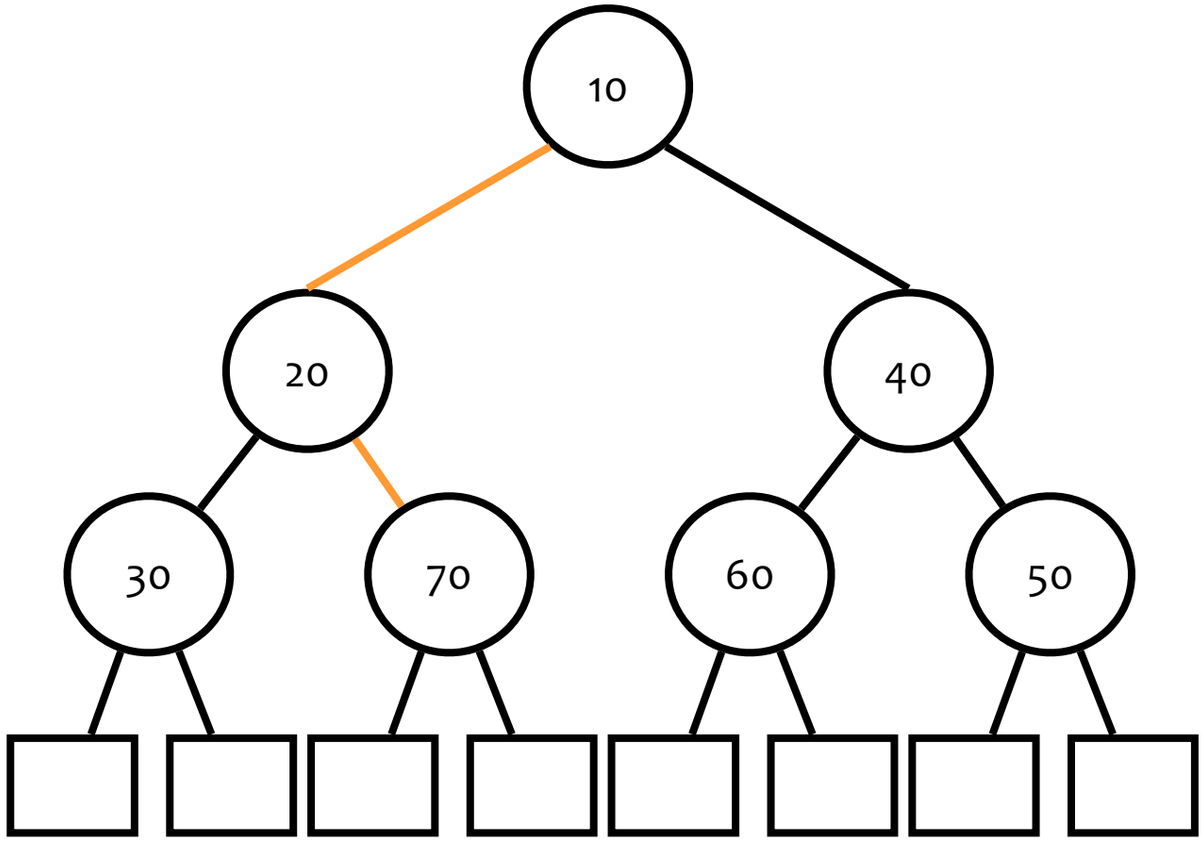

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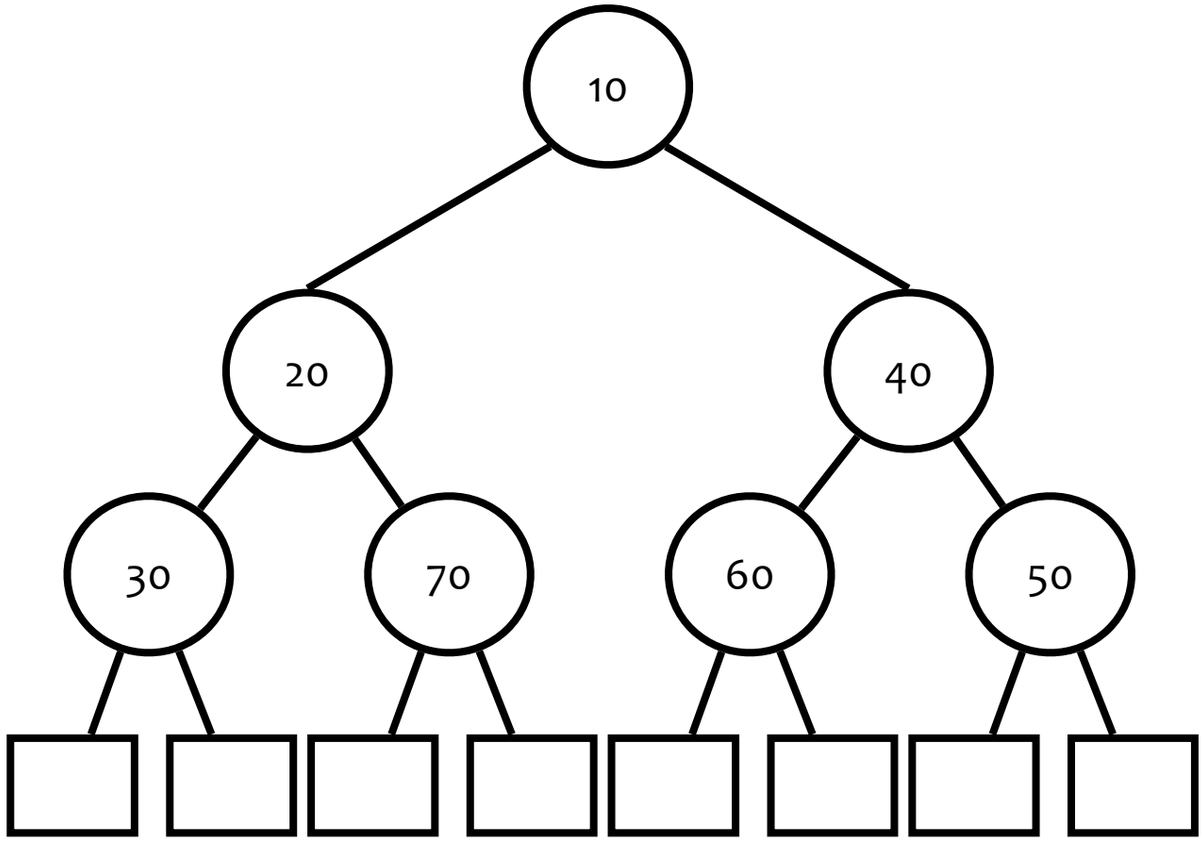

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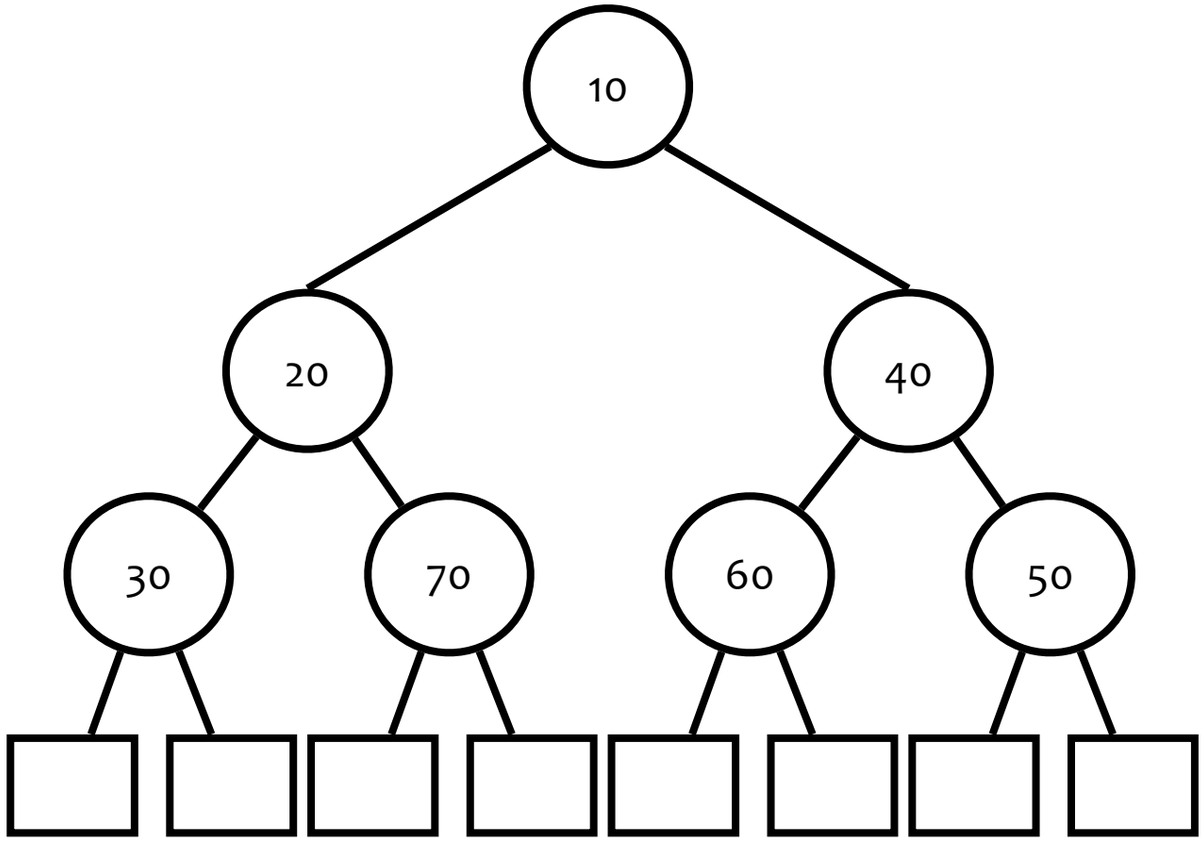

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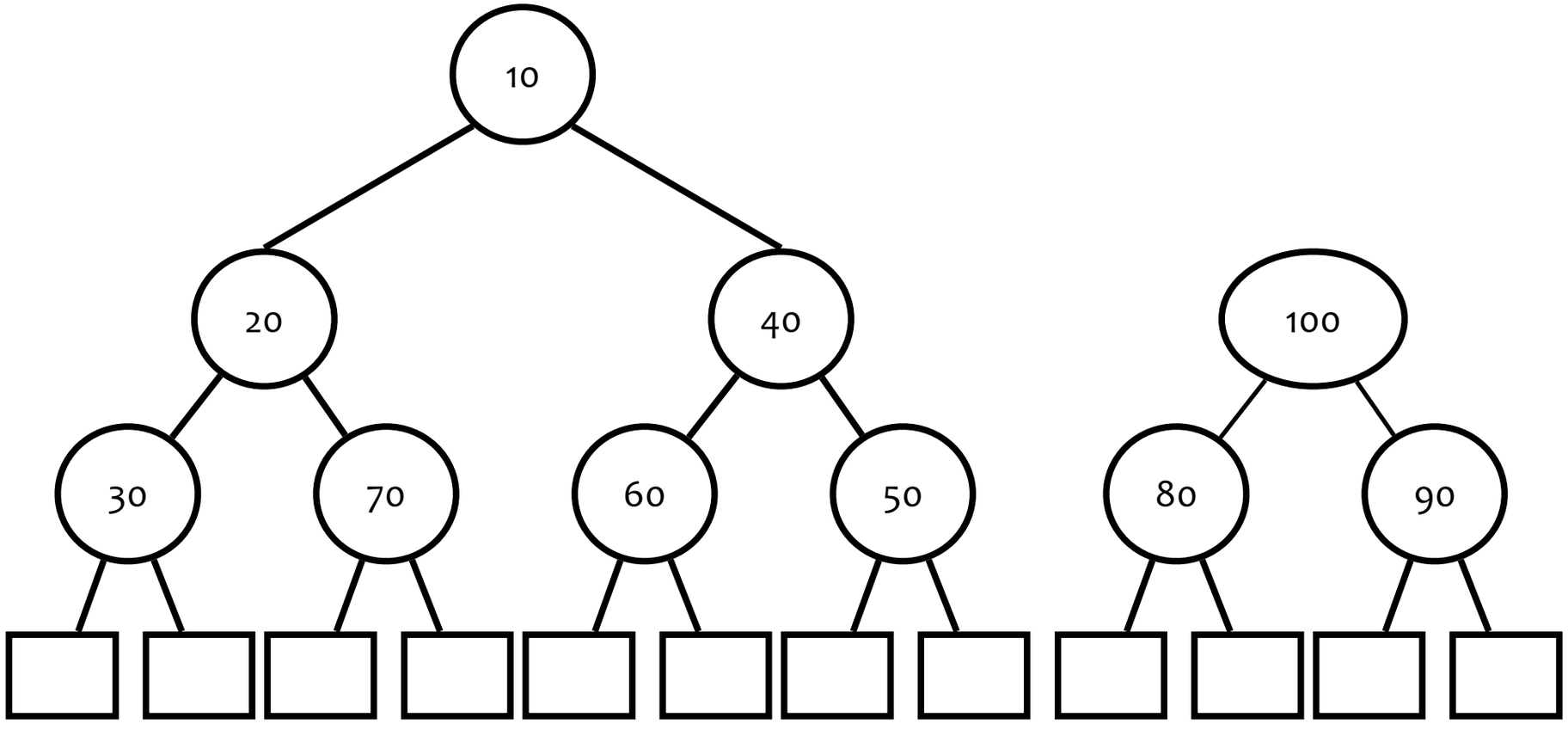

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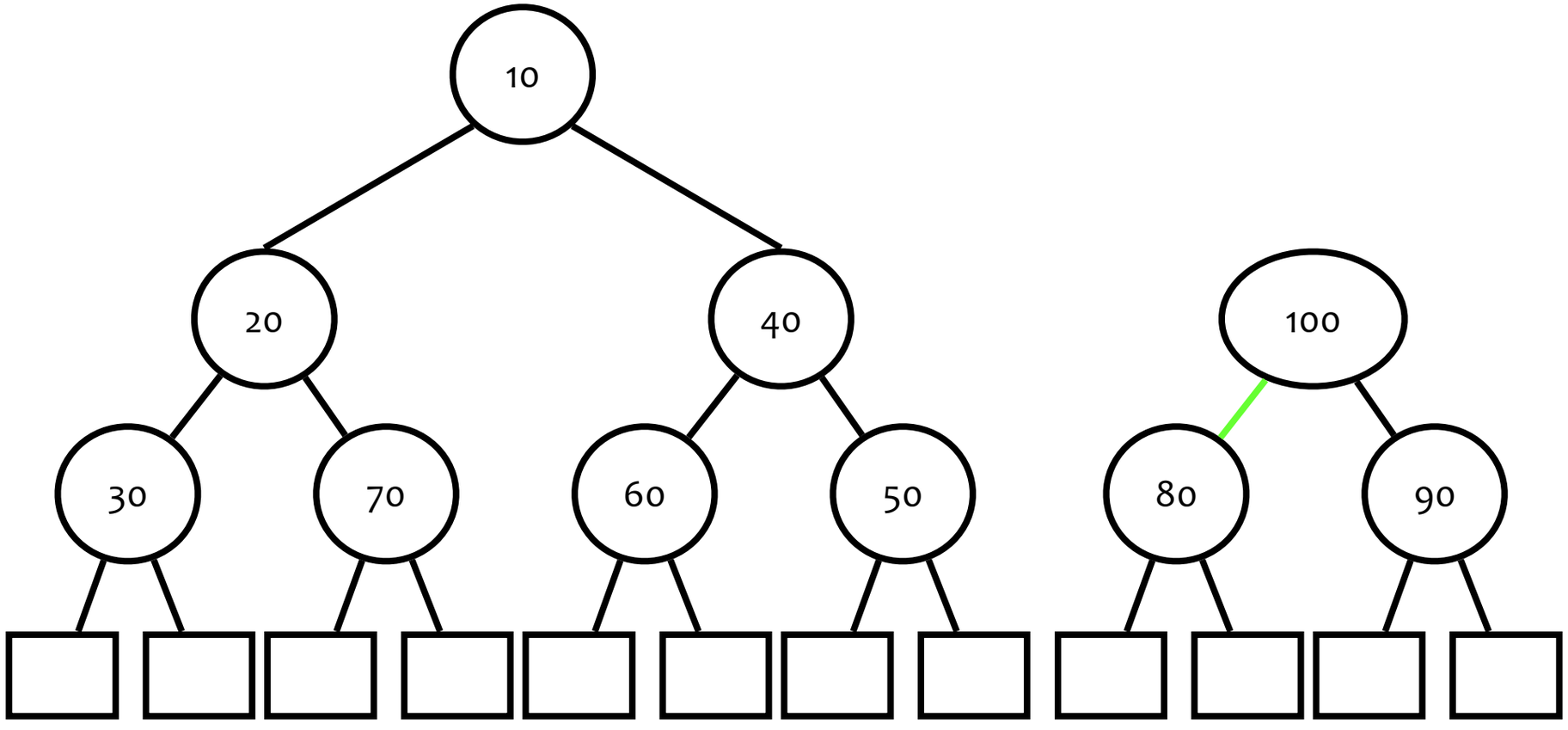

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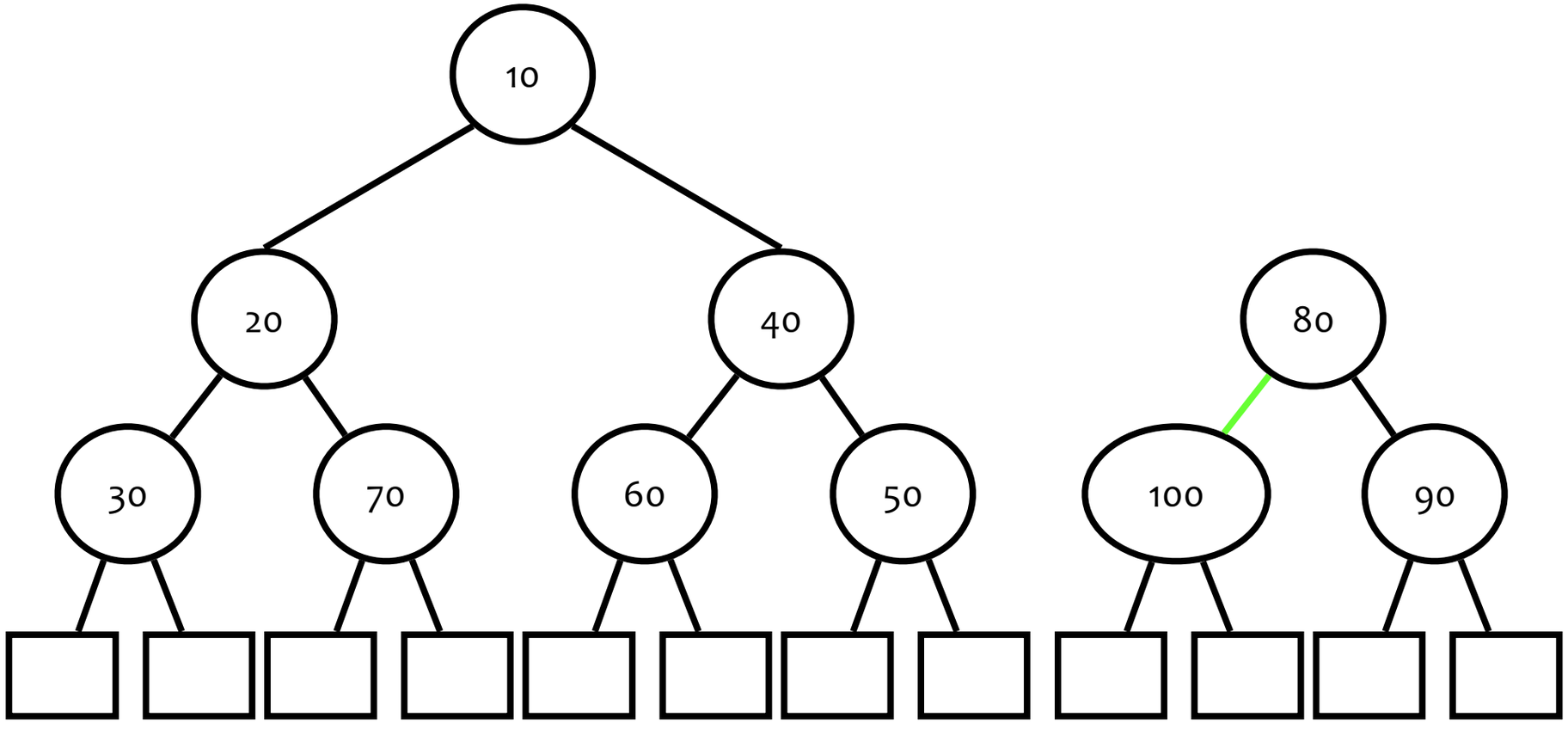

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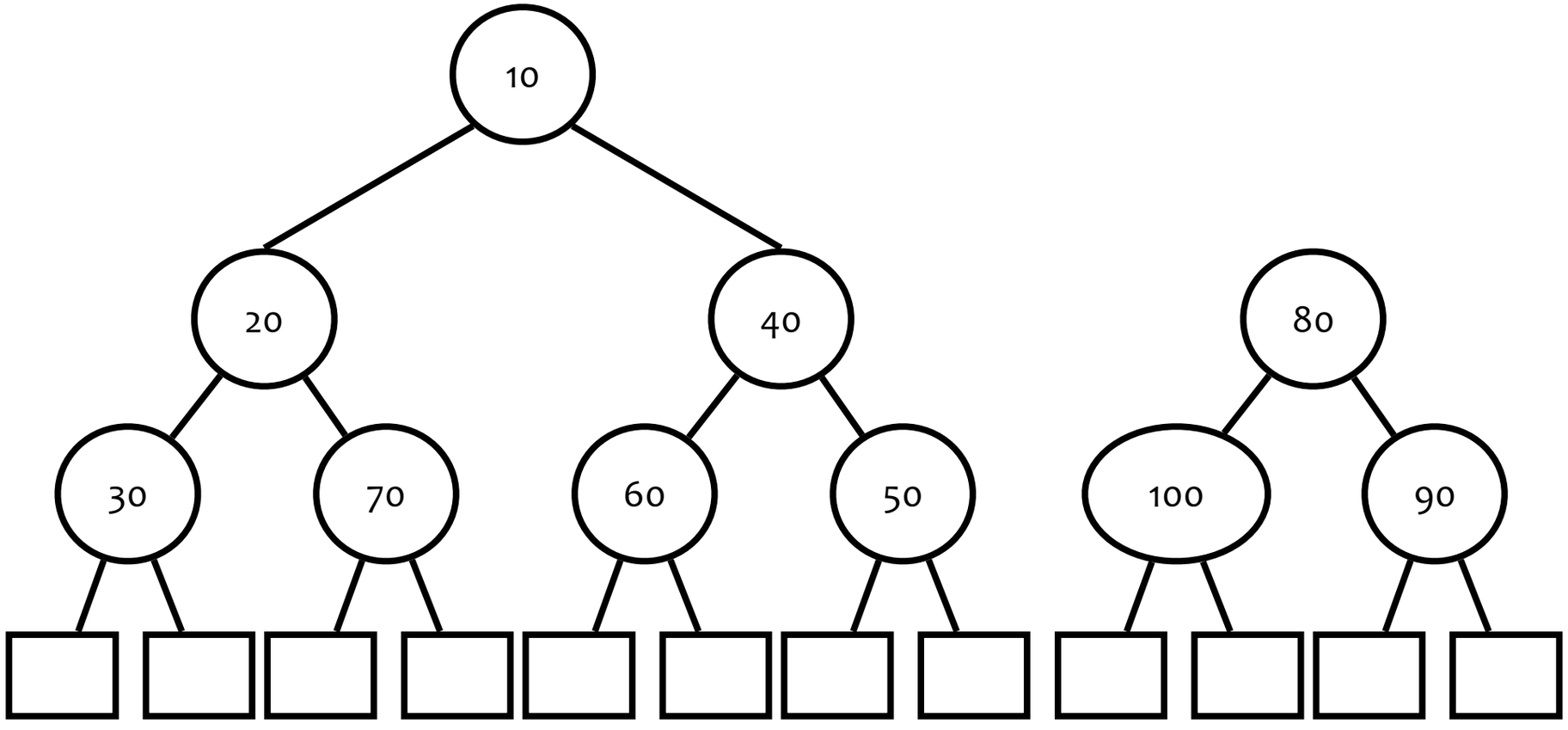

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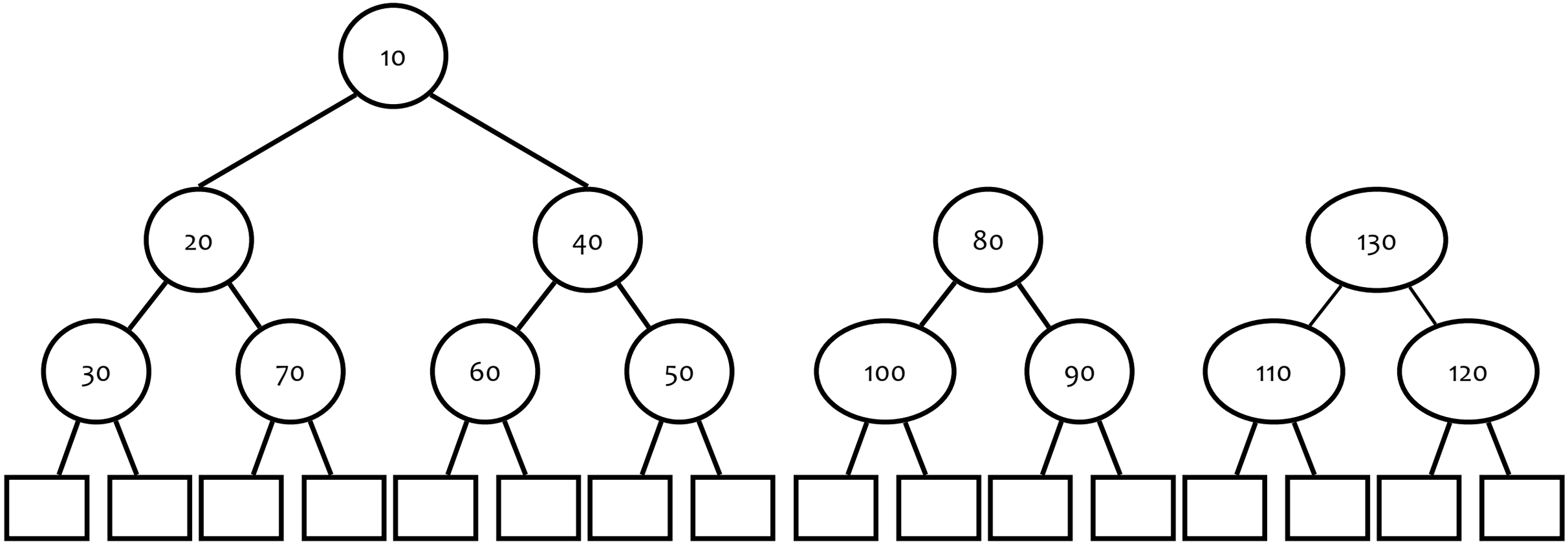

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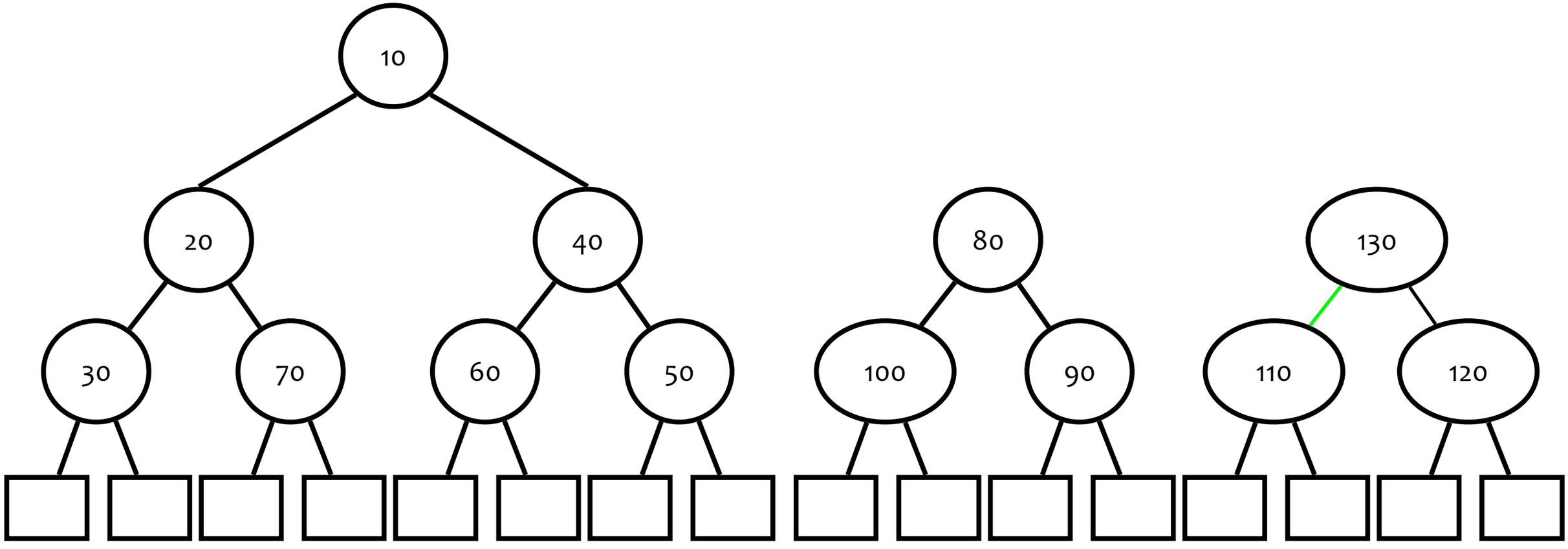

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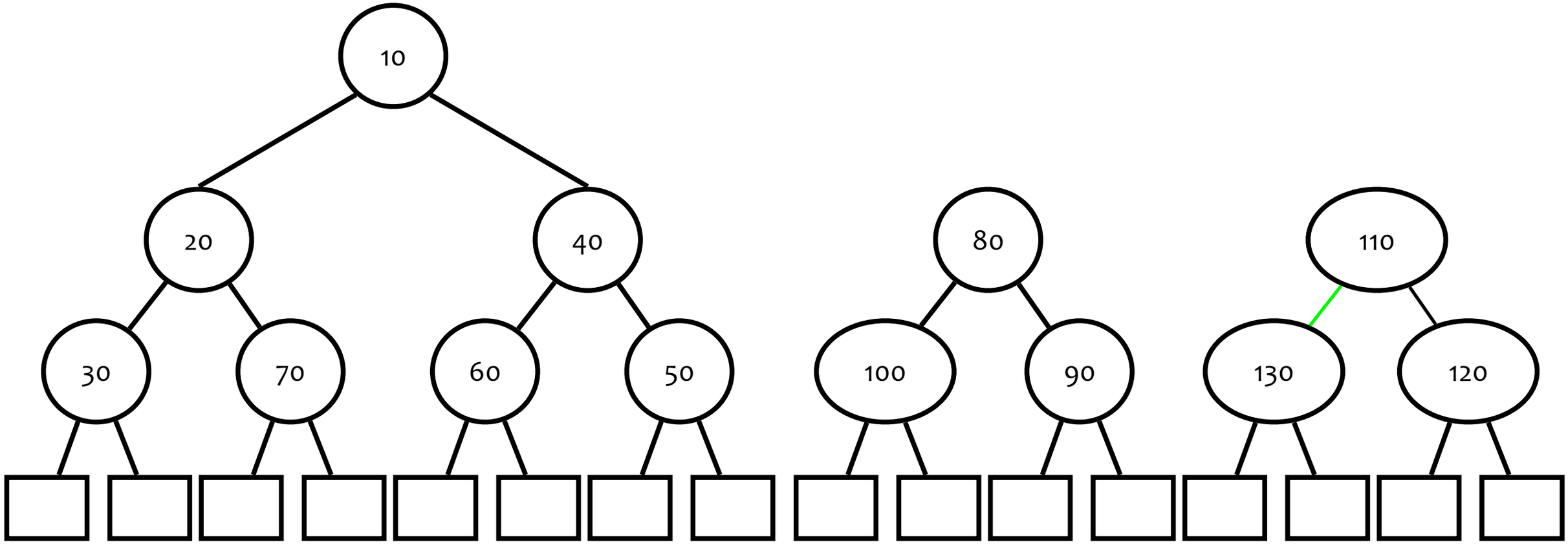

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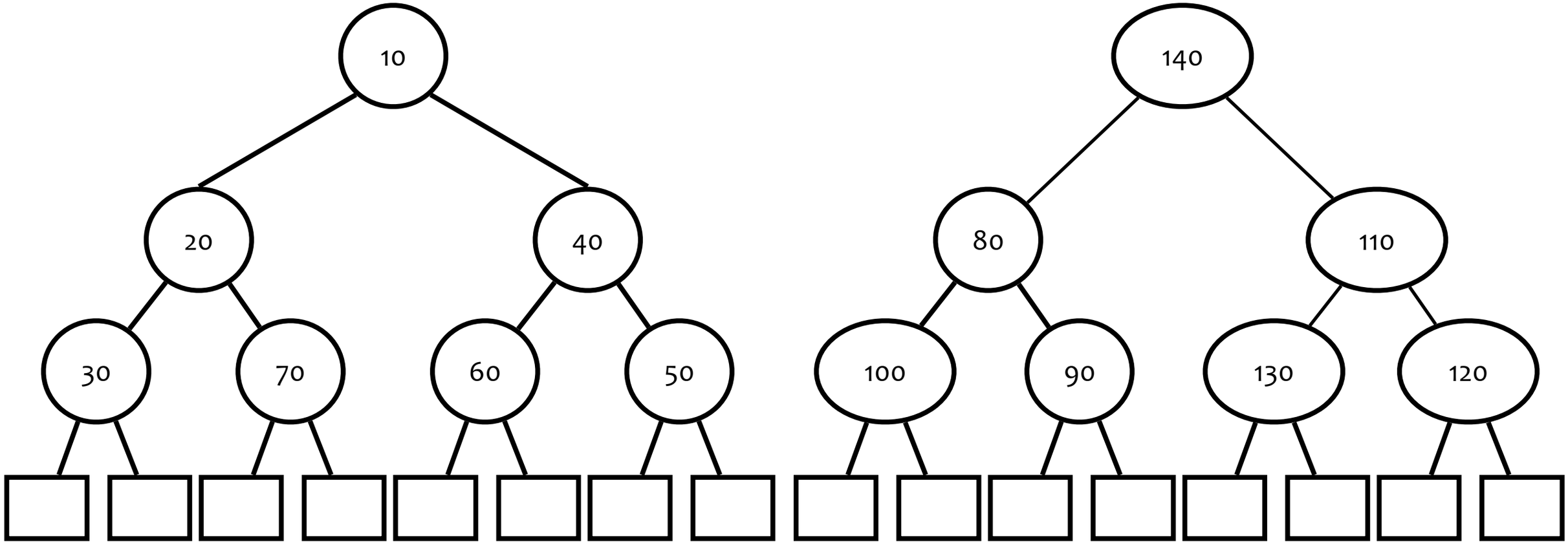

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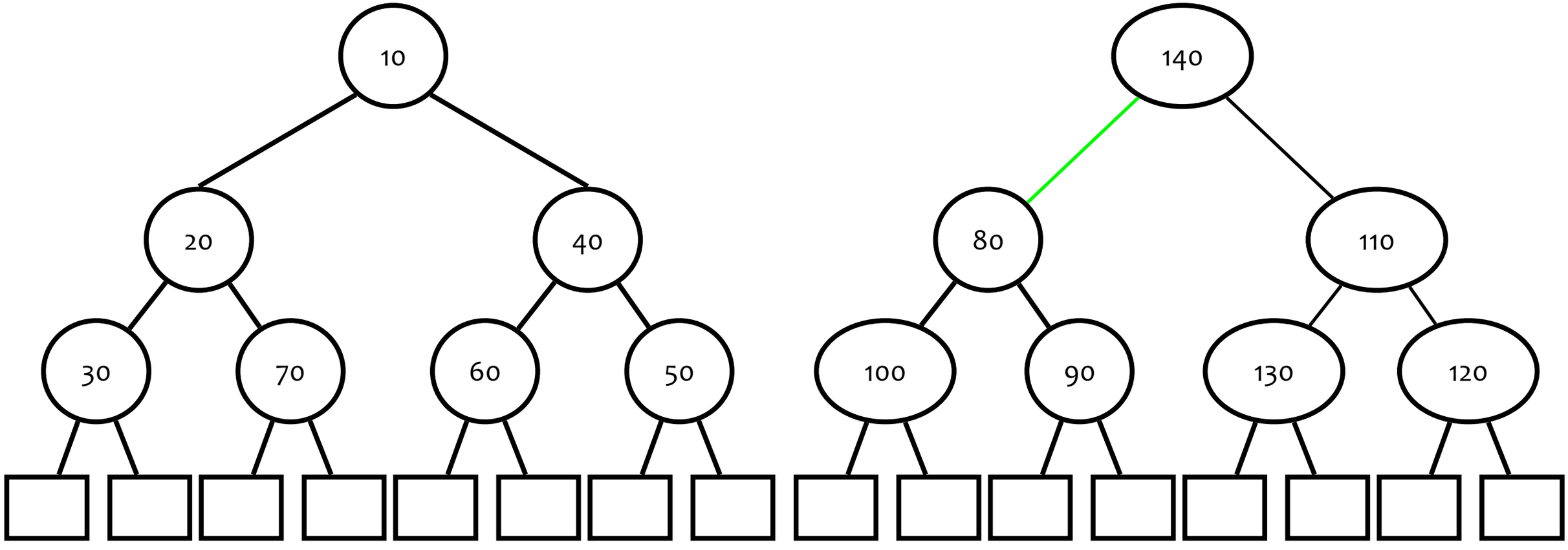

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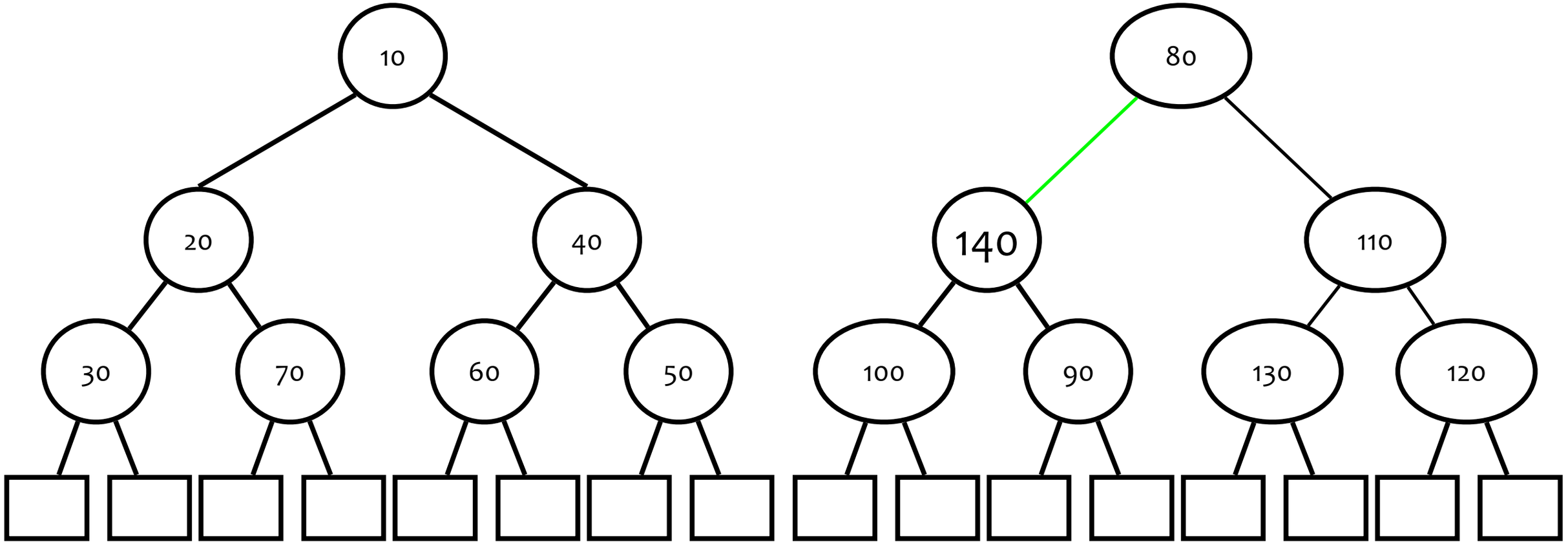

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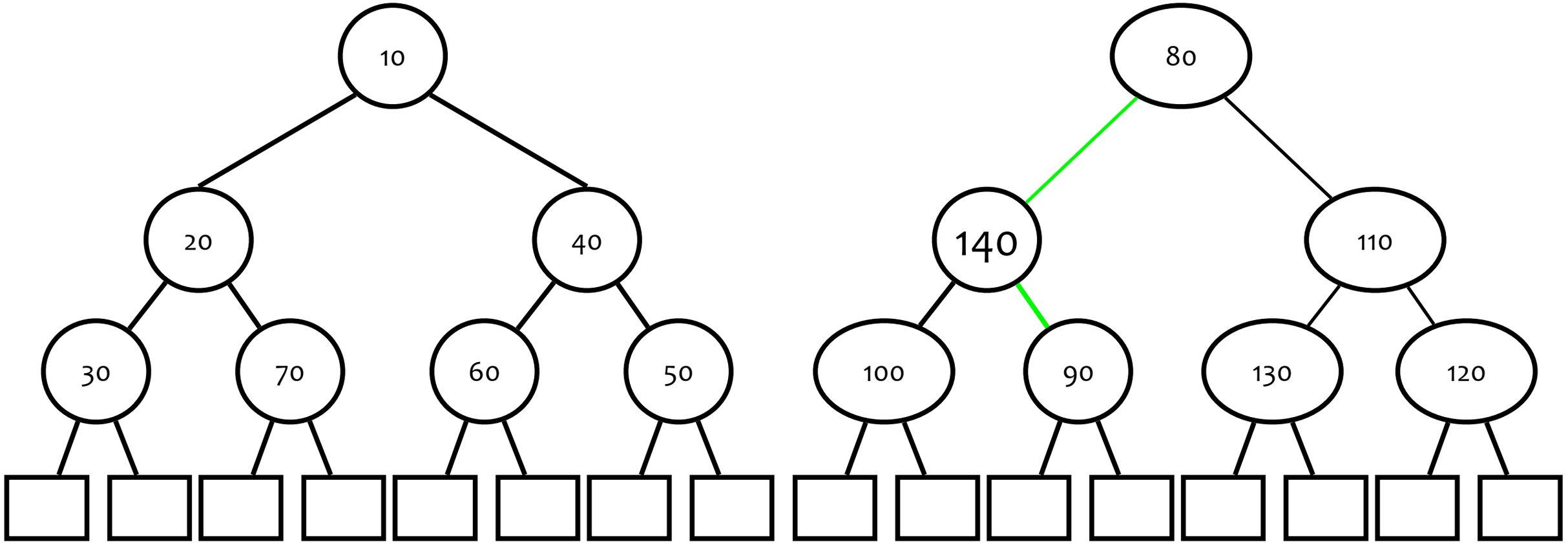

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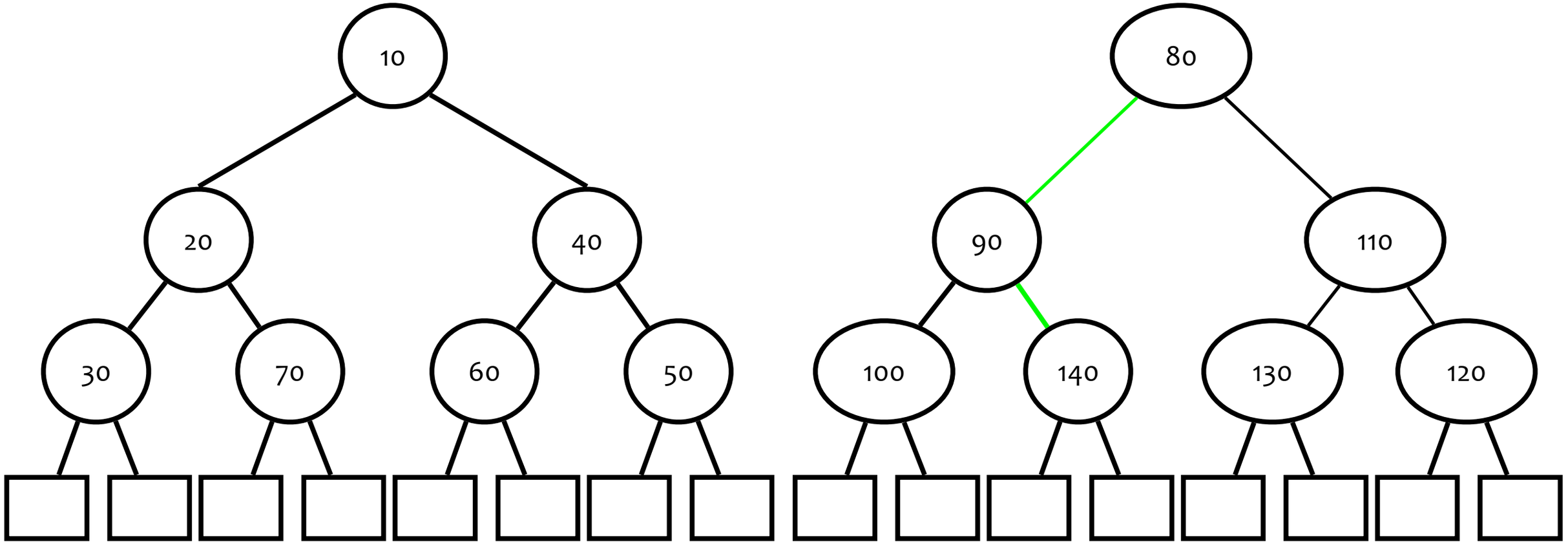

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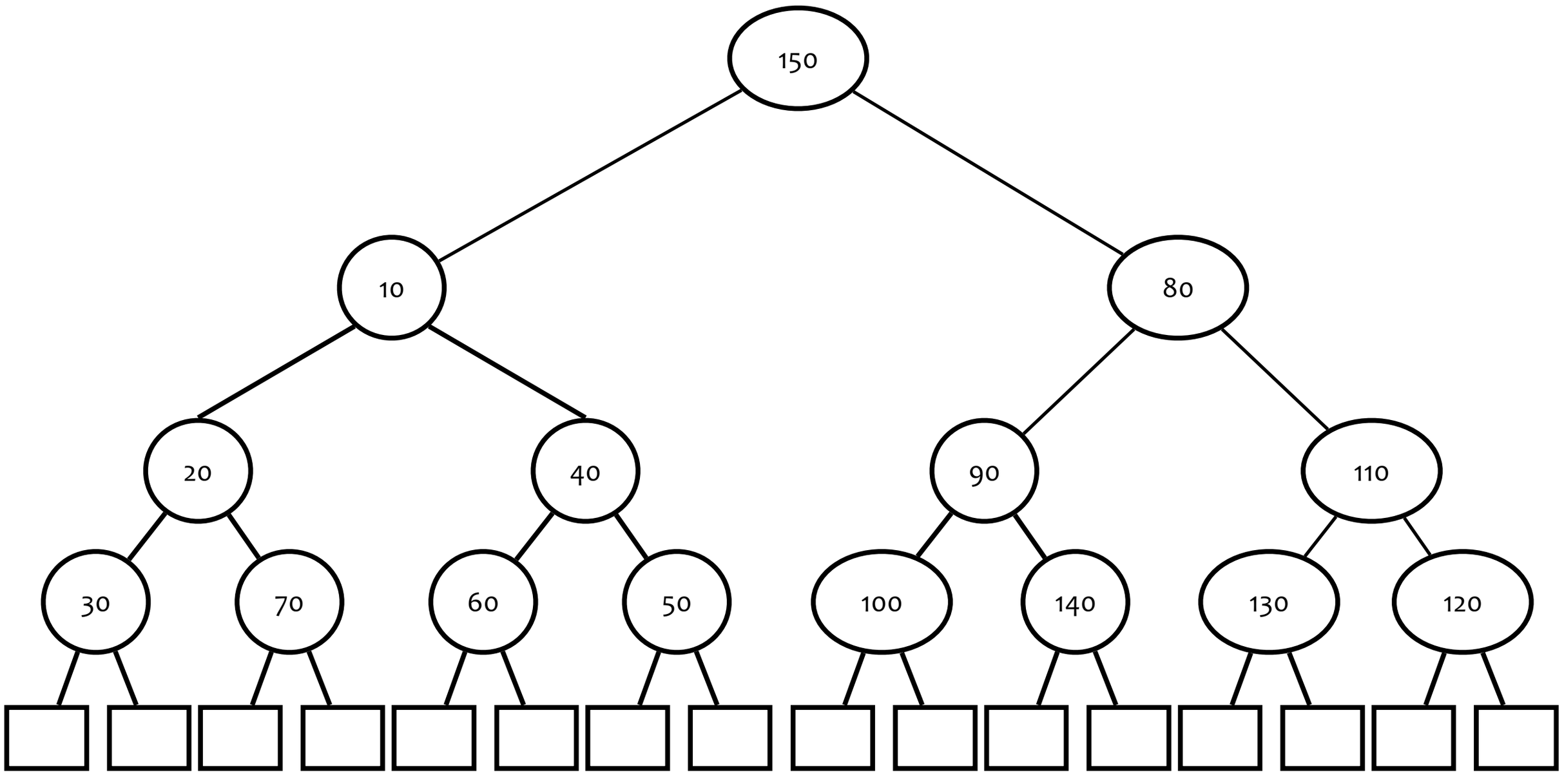

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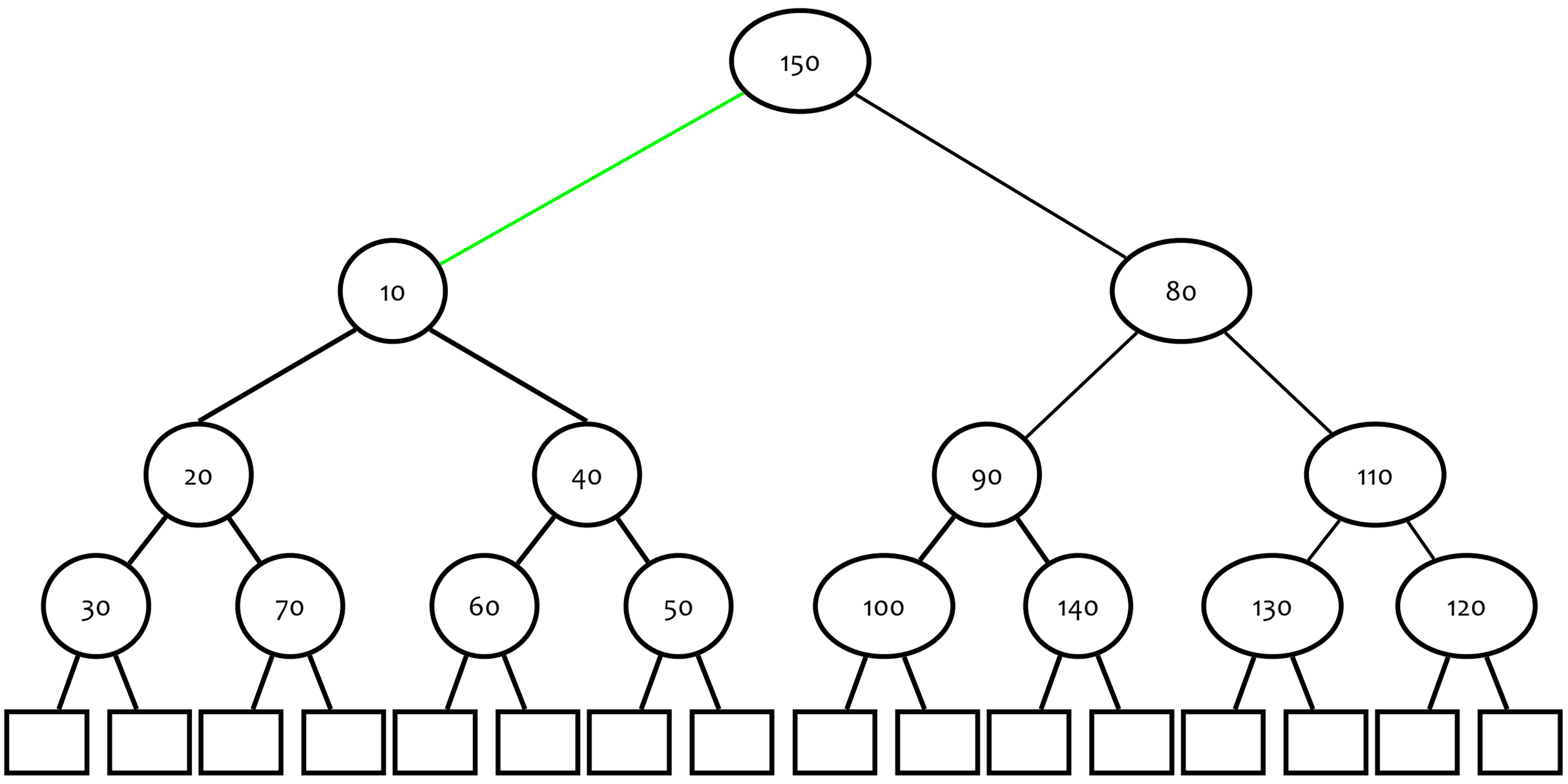

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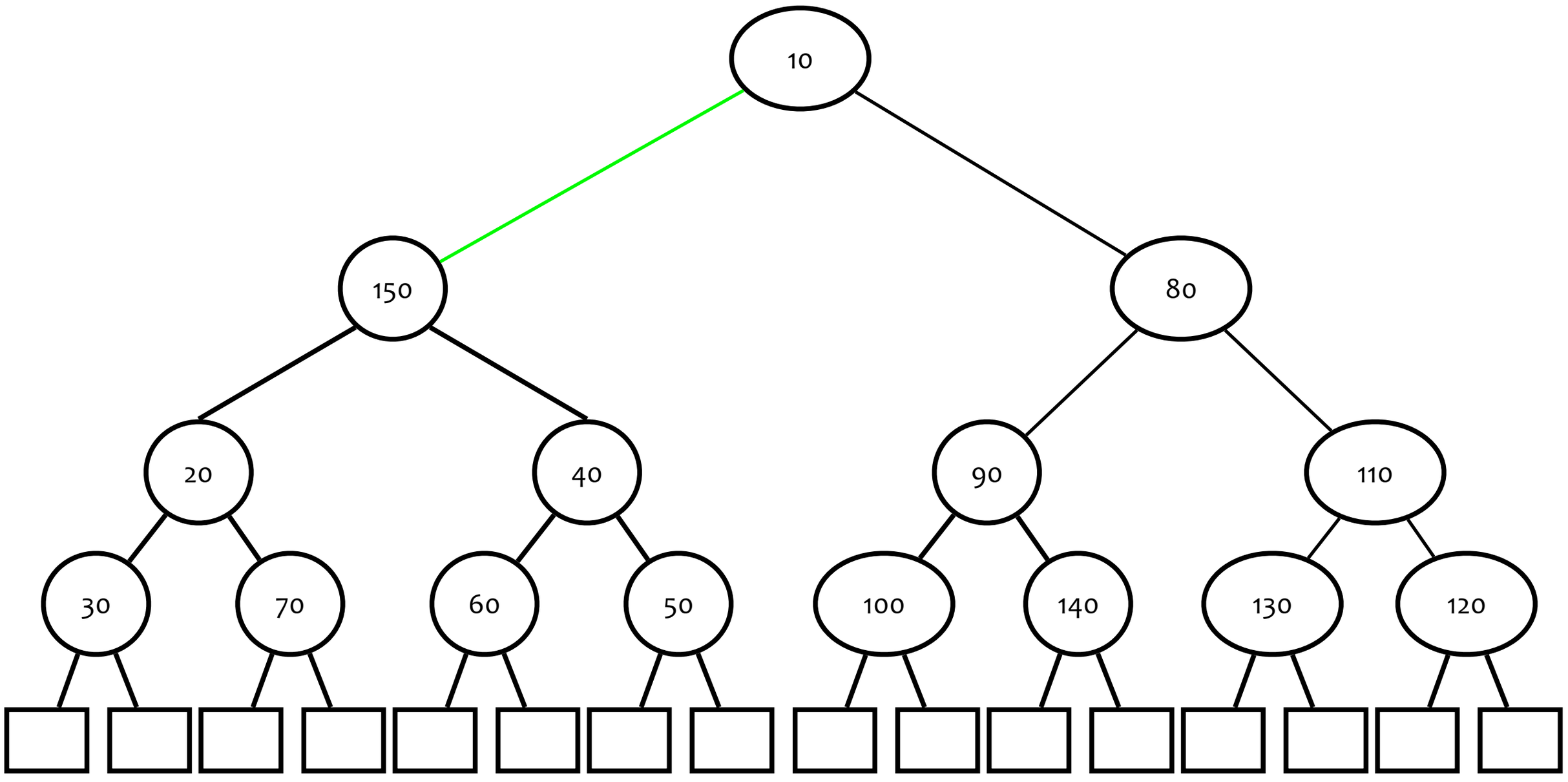

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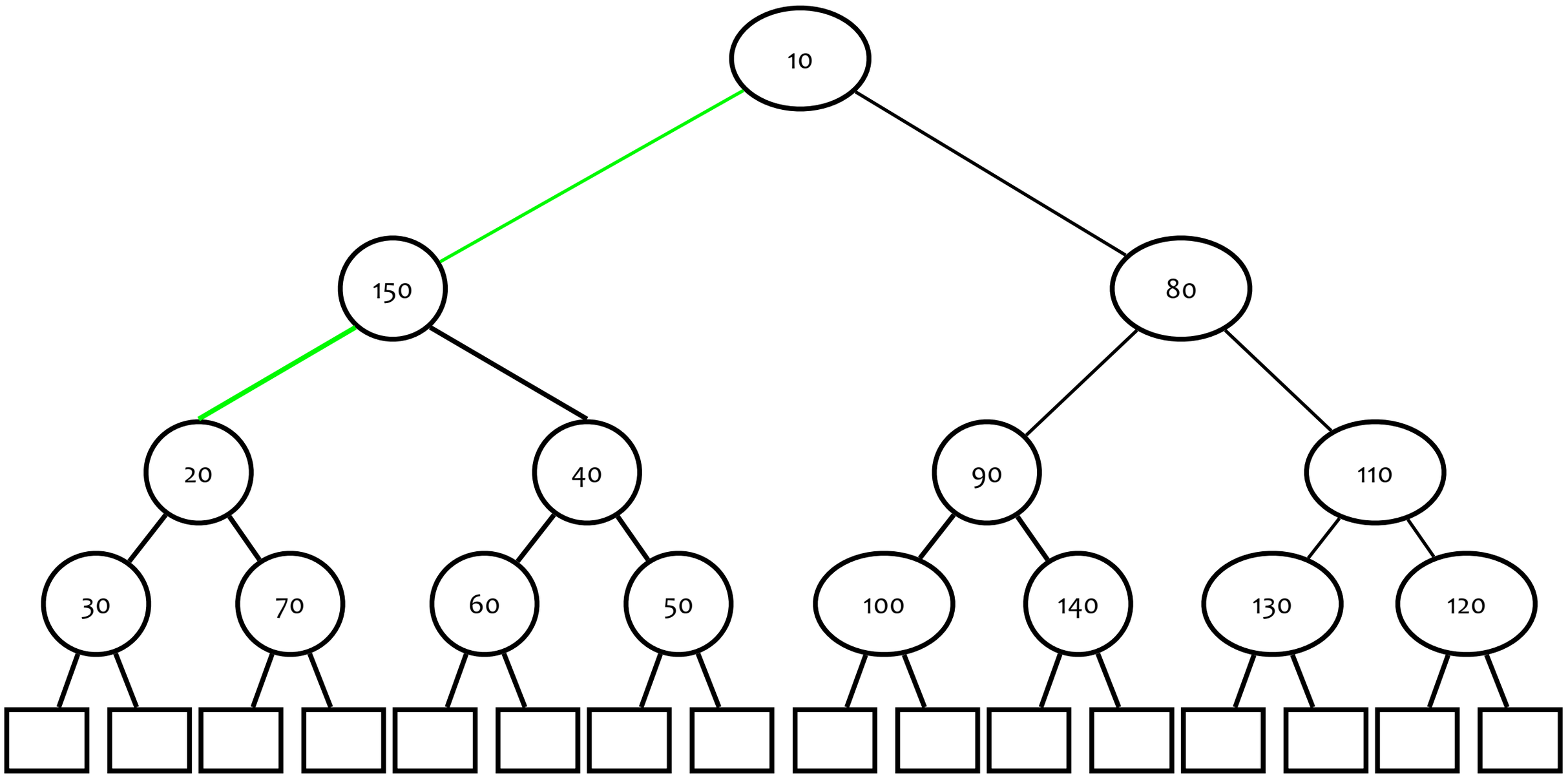

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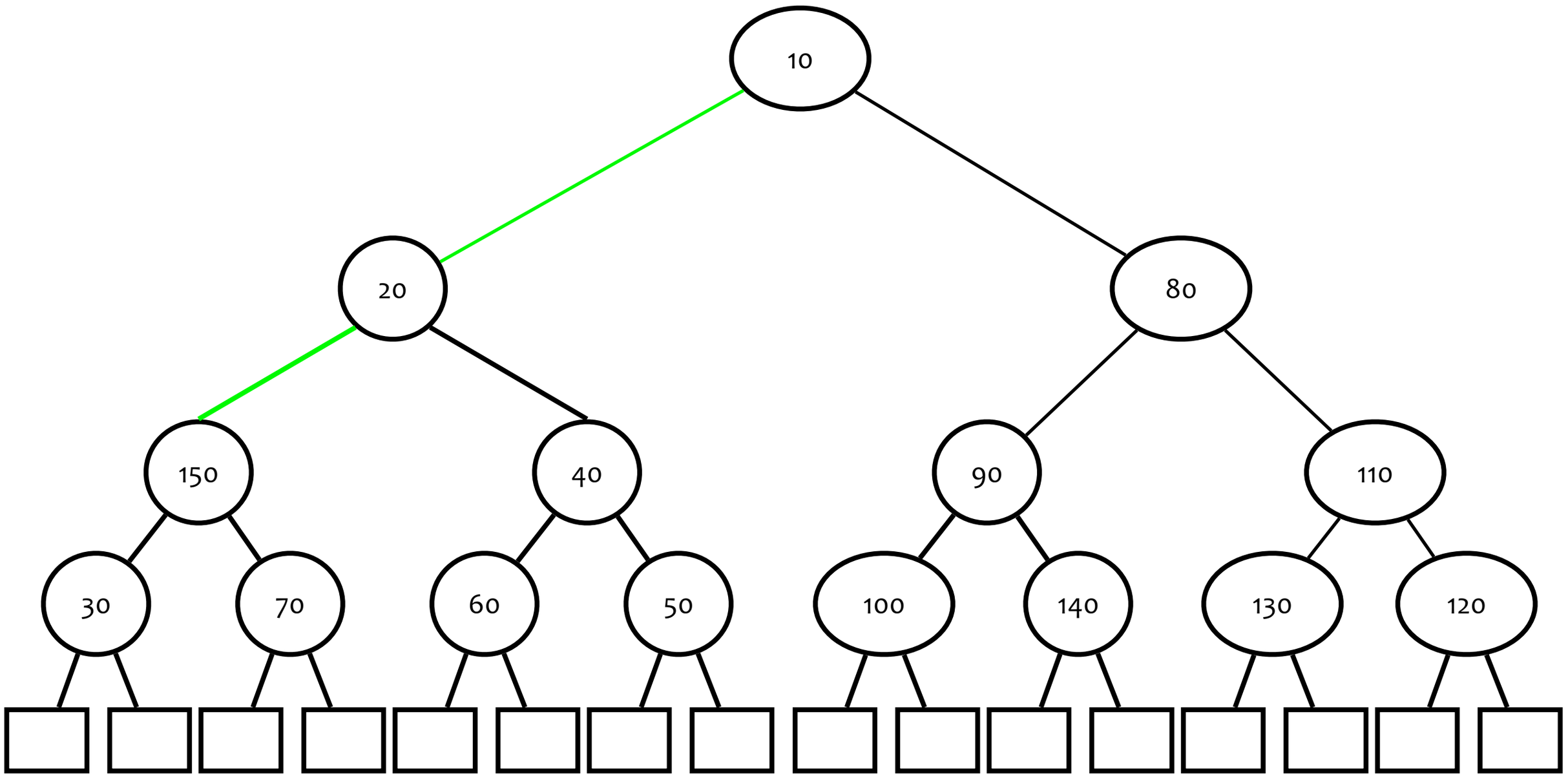

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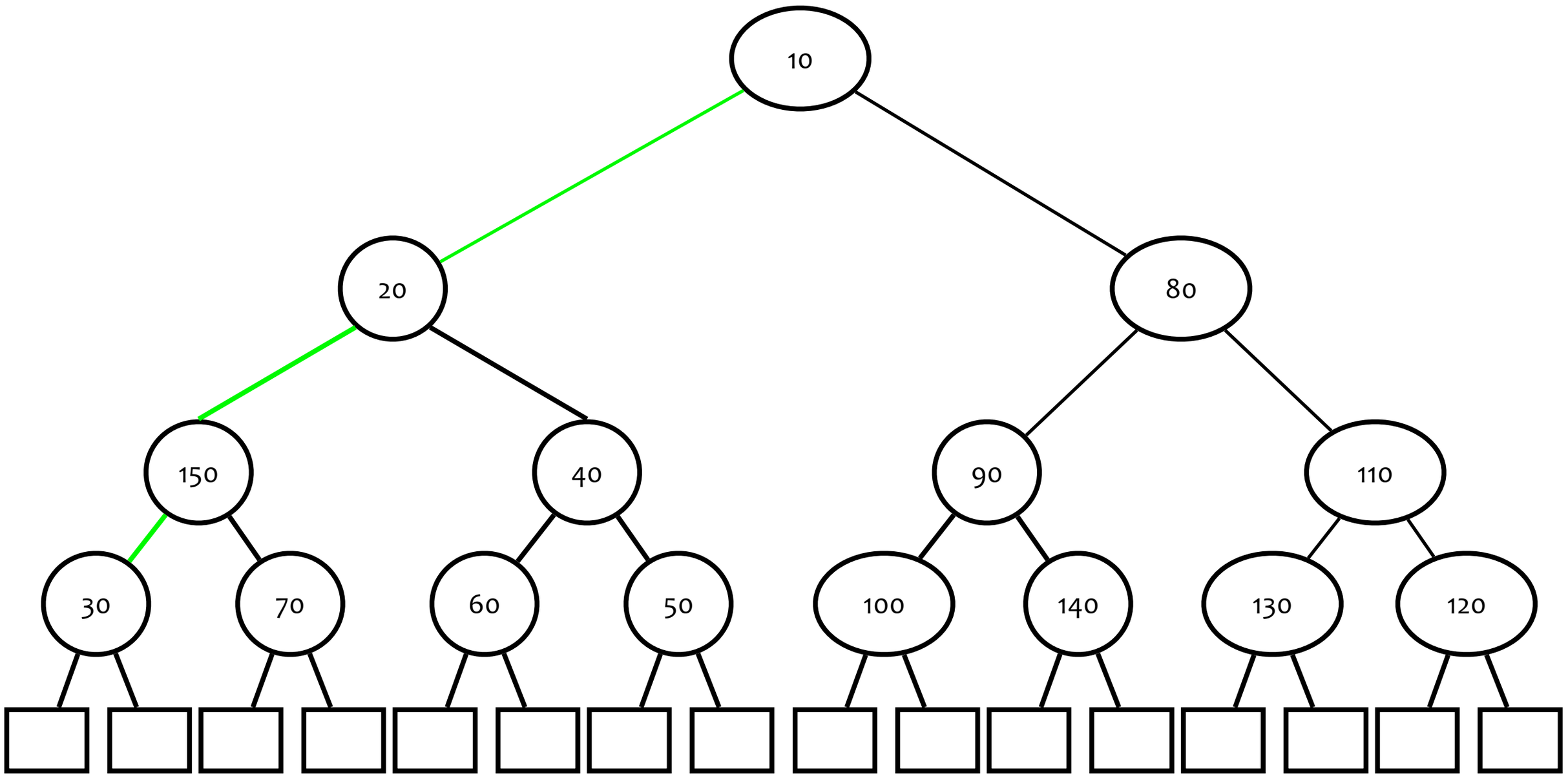

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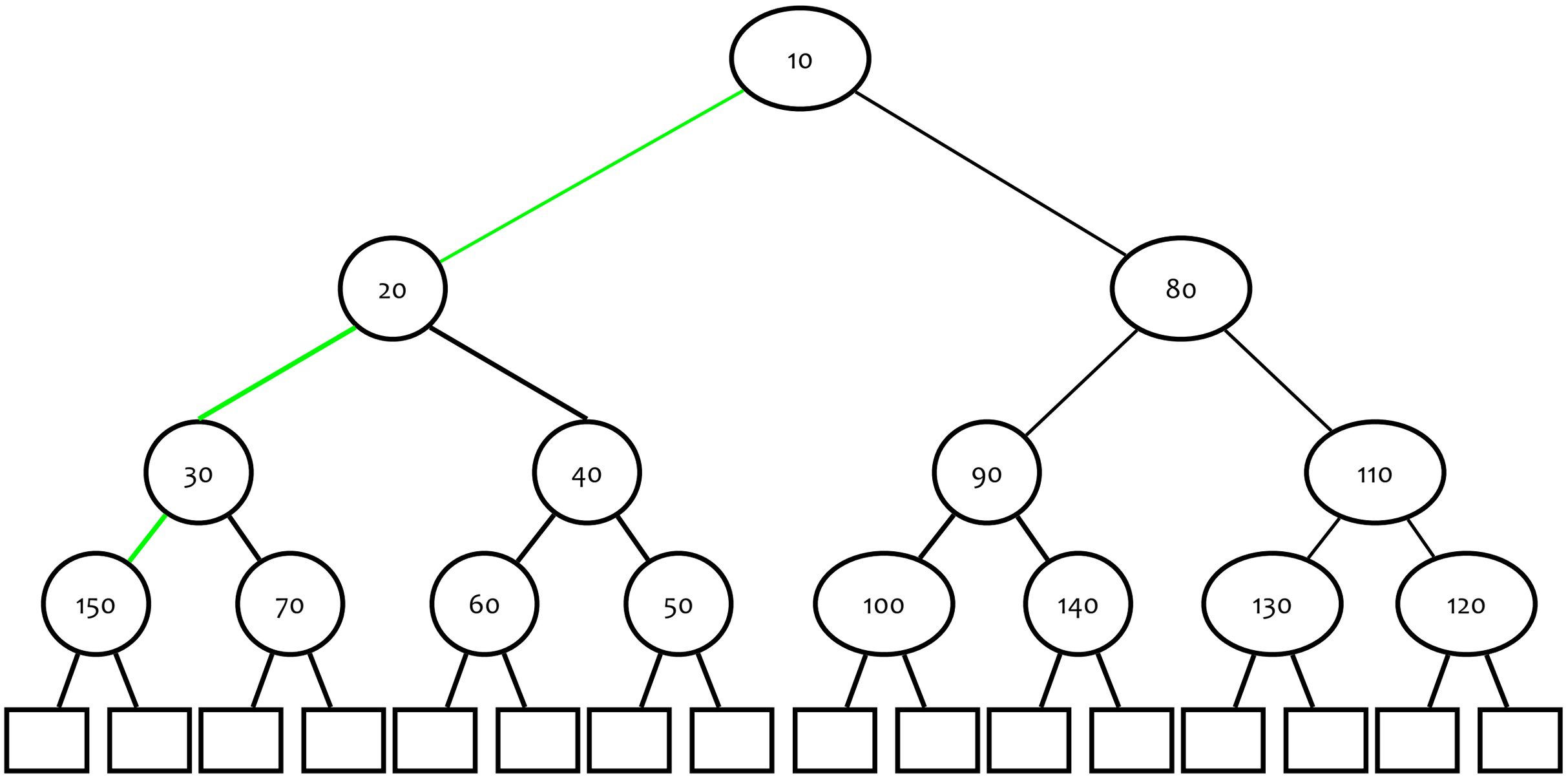

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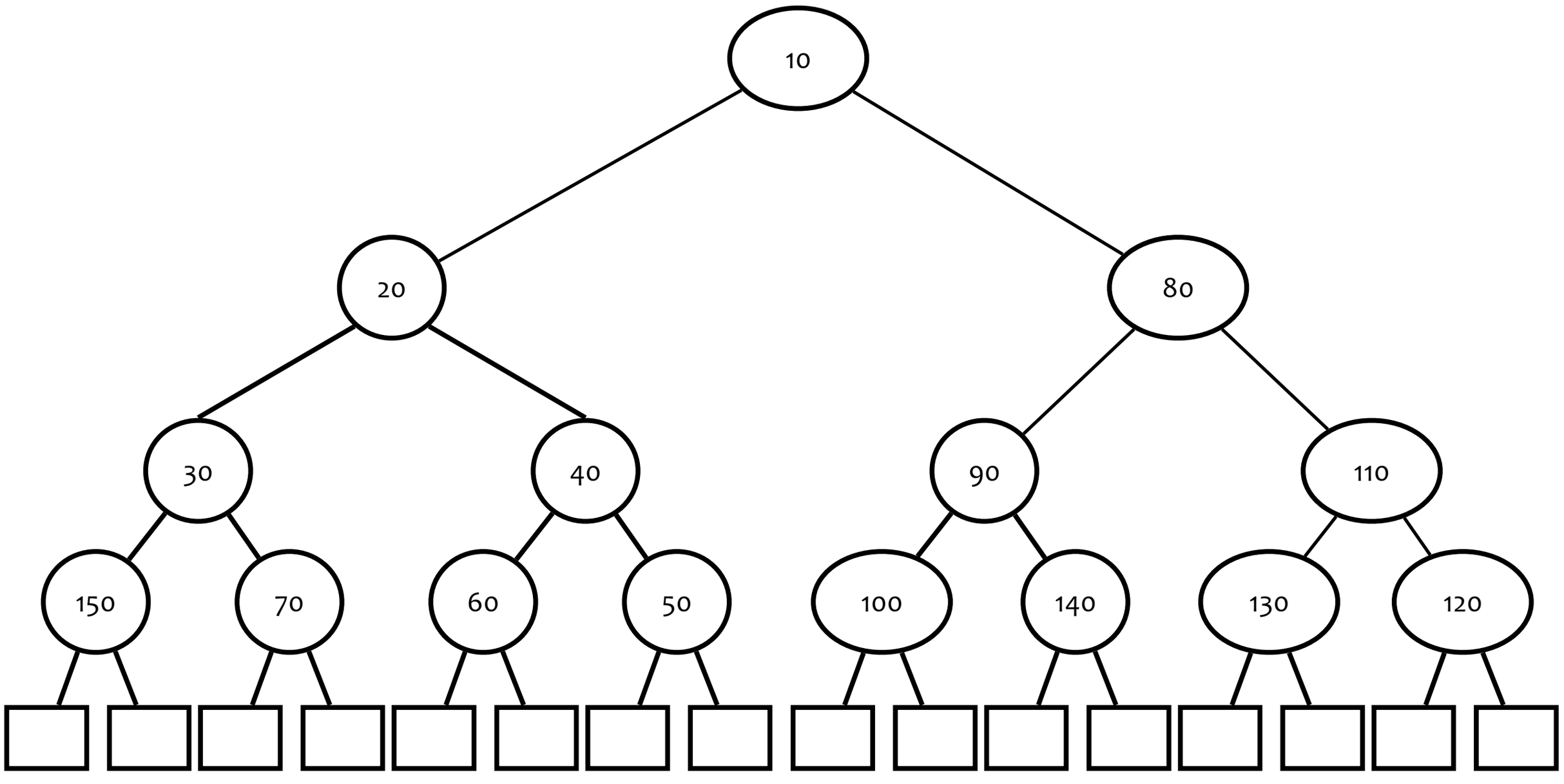
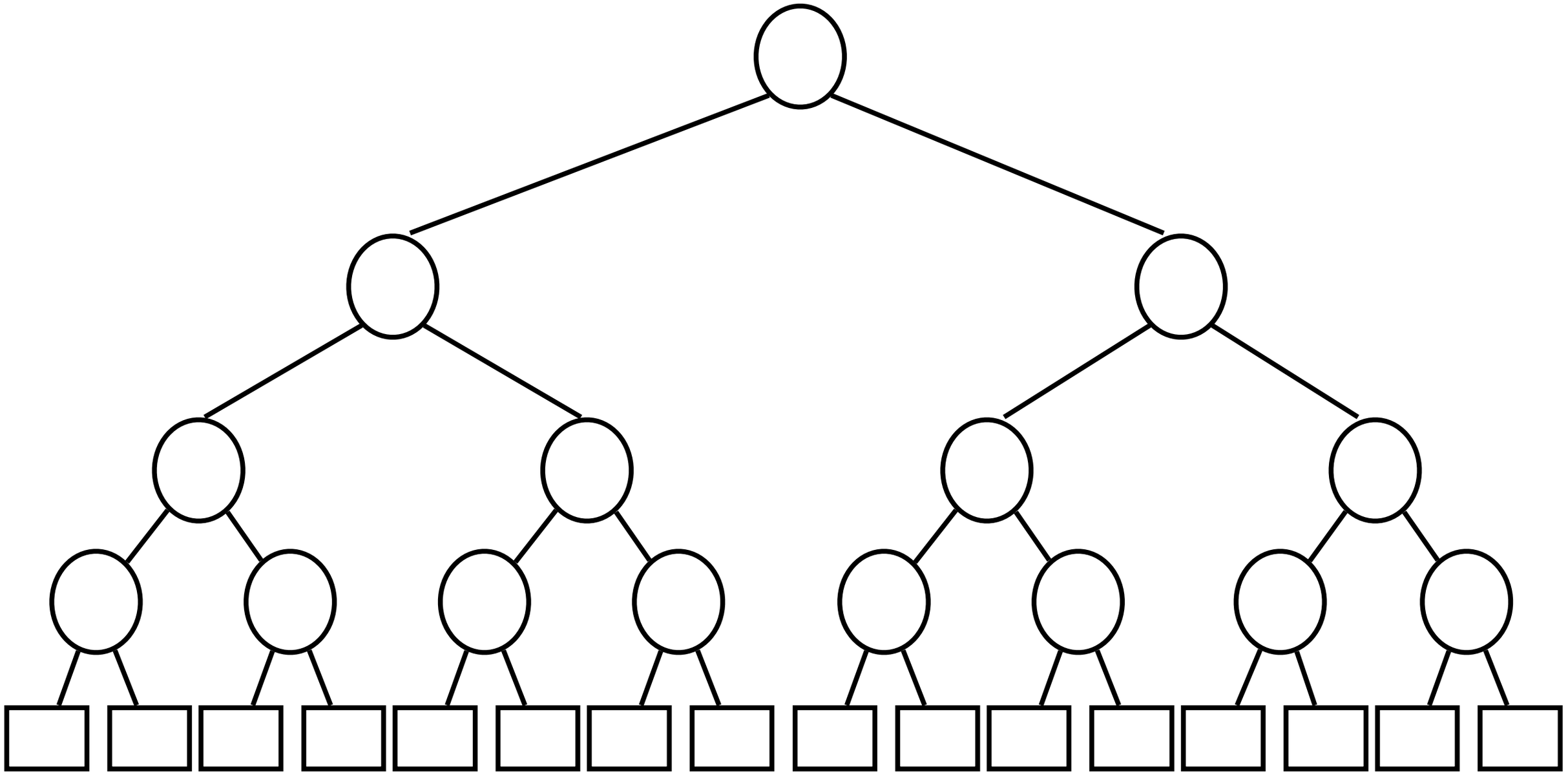
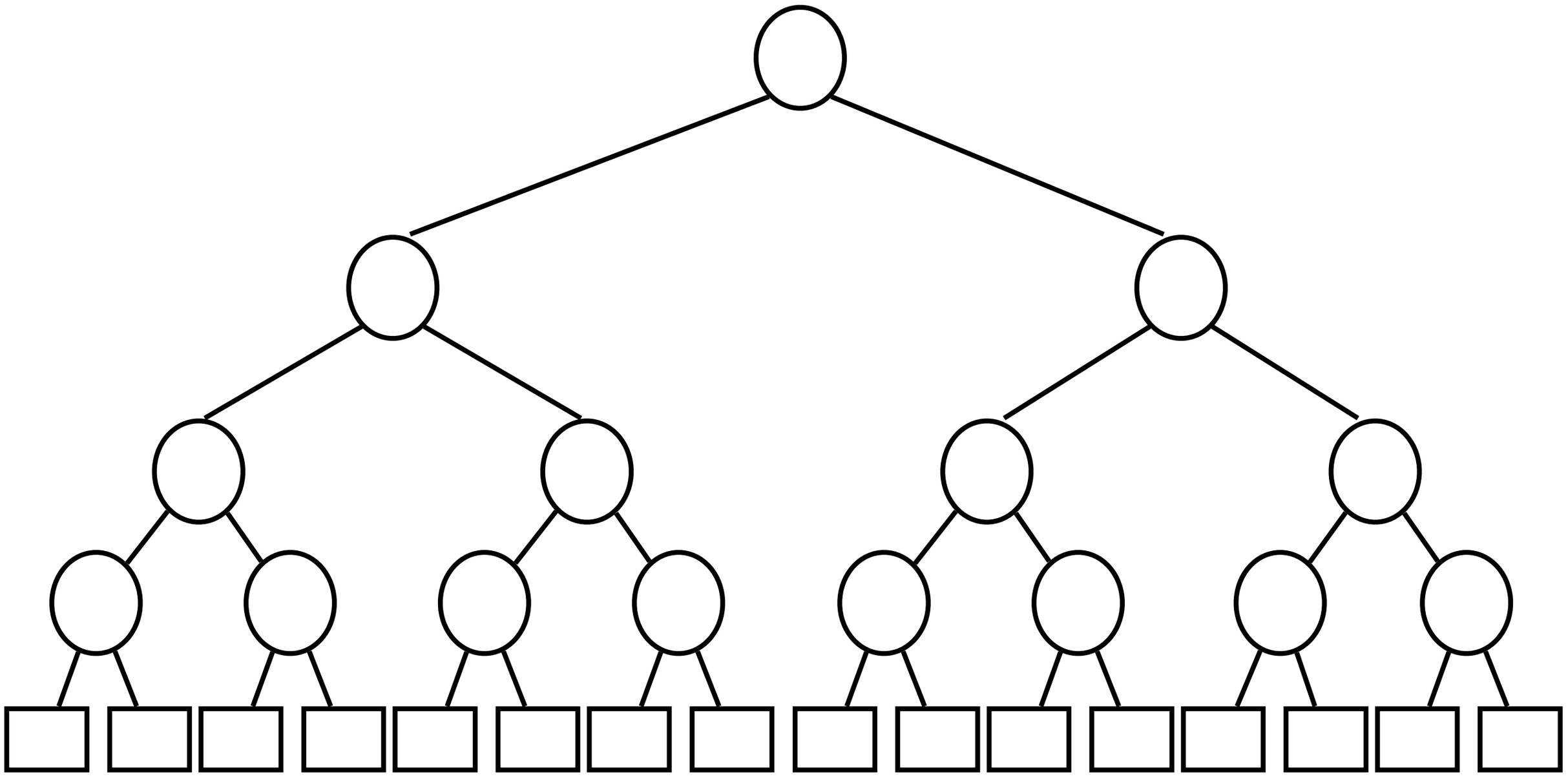


Did we really insert all n elements in O(n) time??



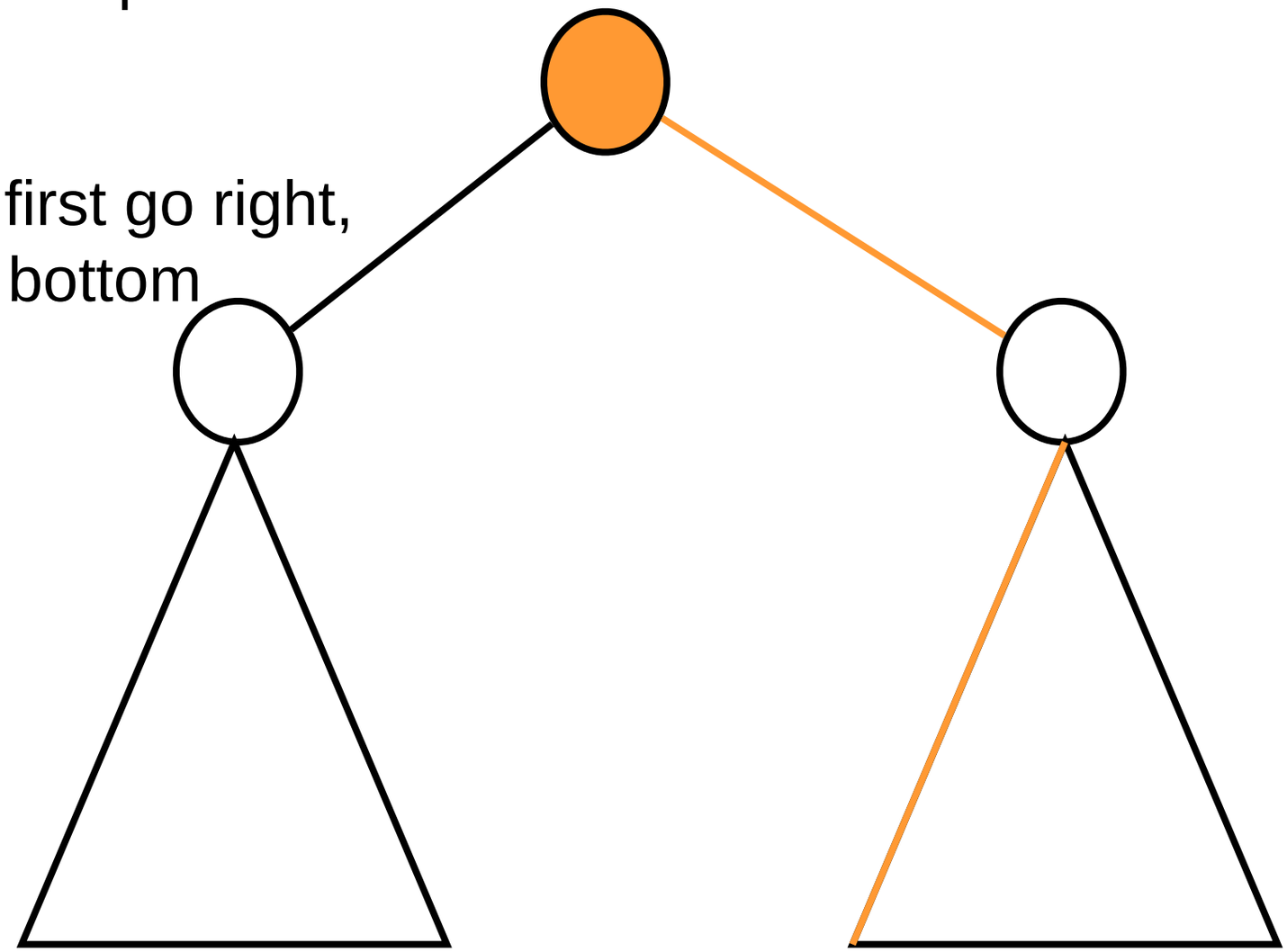
We show: max# bubble down ops < # heap edges



Proof idea

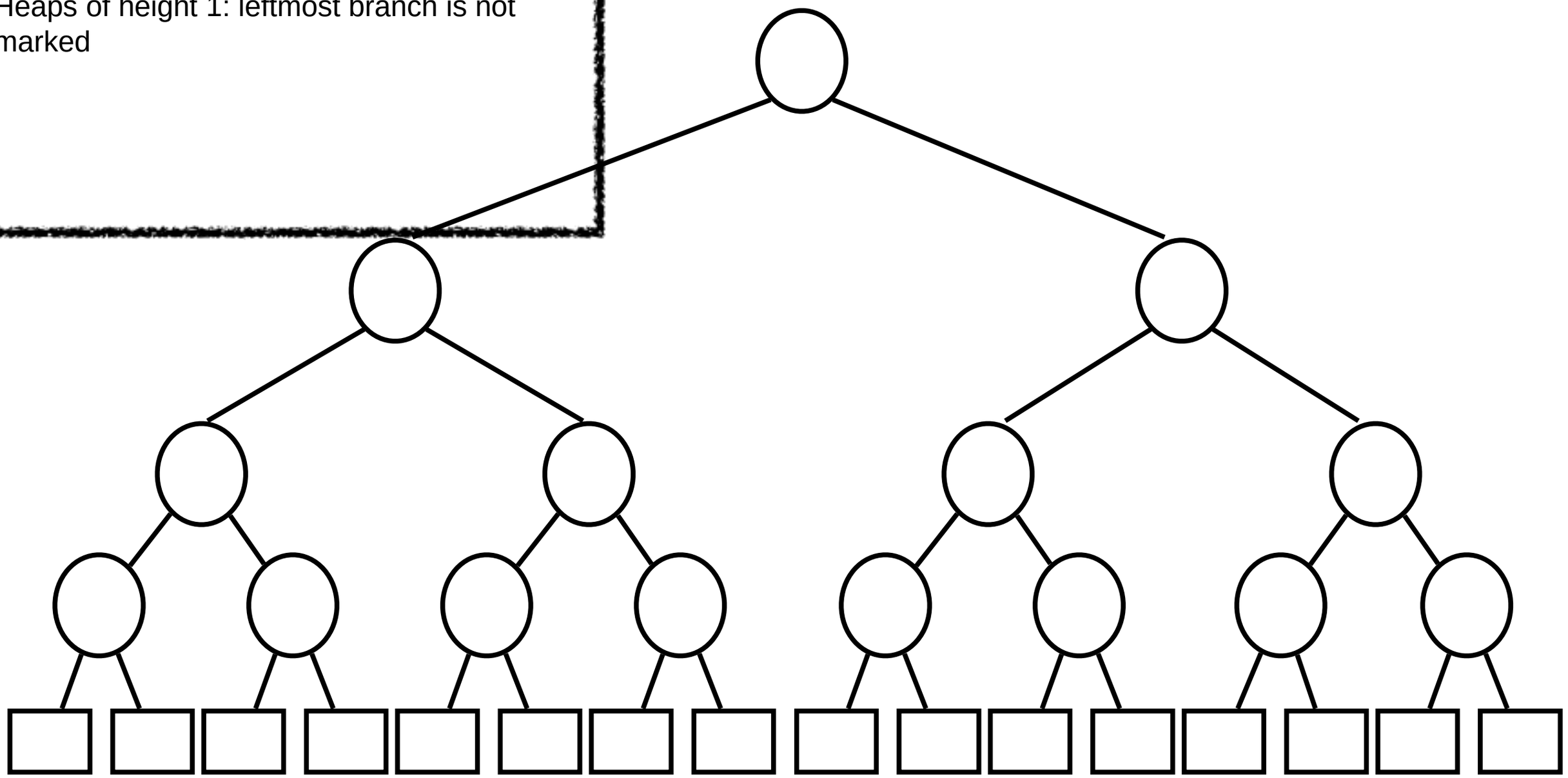
For each new node joining two heaps: mark path of maximum number of bubble-down operations

For marking: first go right, then left until bottom

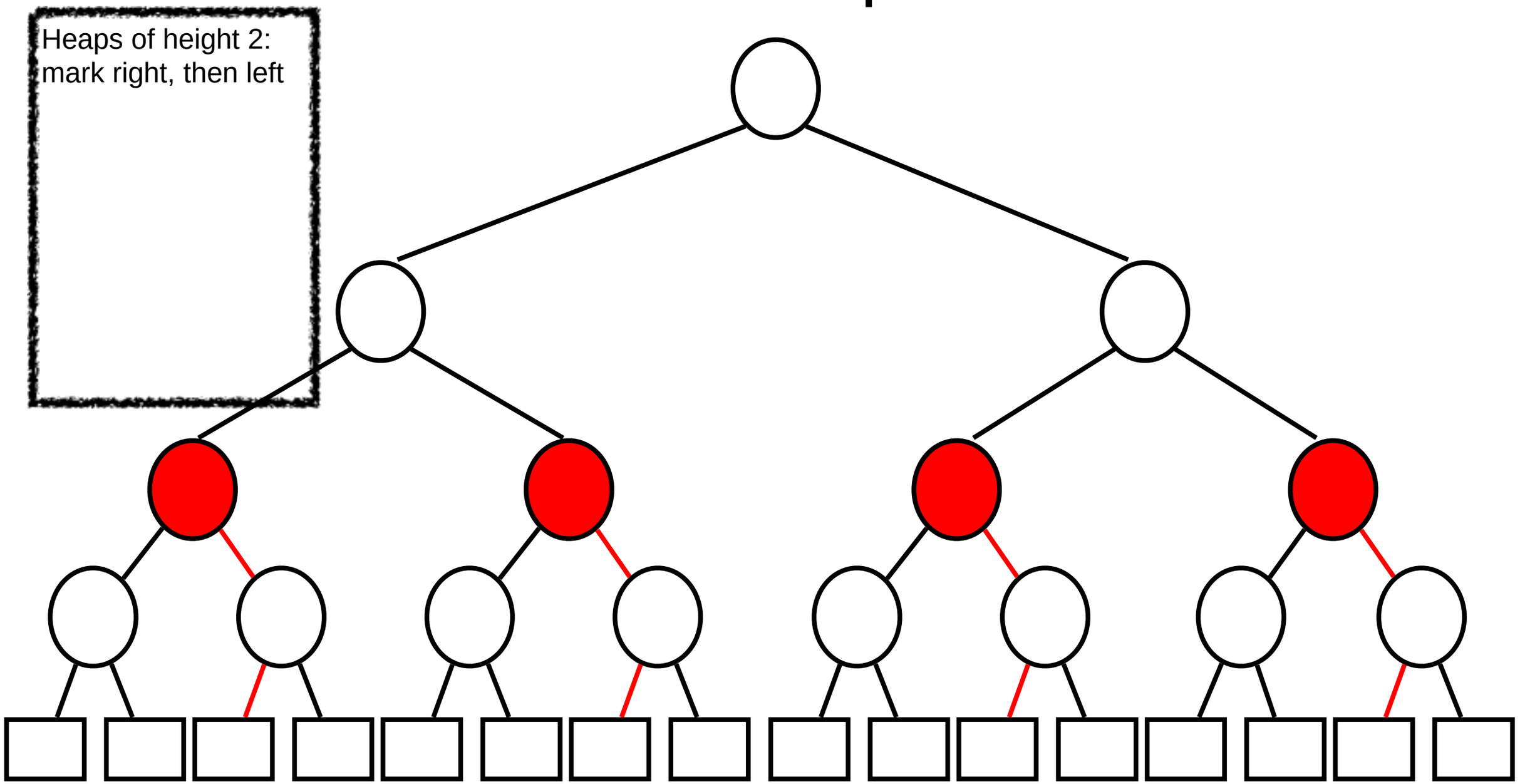


For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 1: leftmost branch is not marked



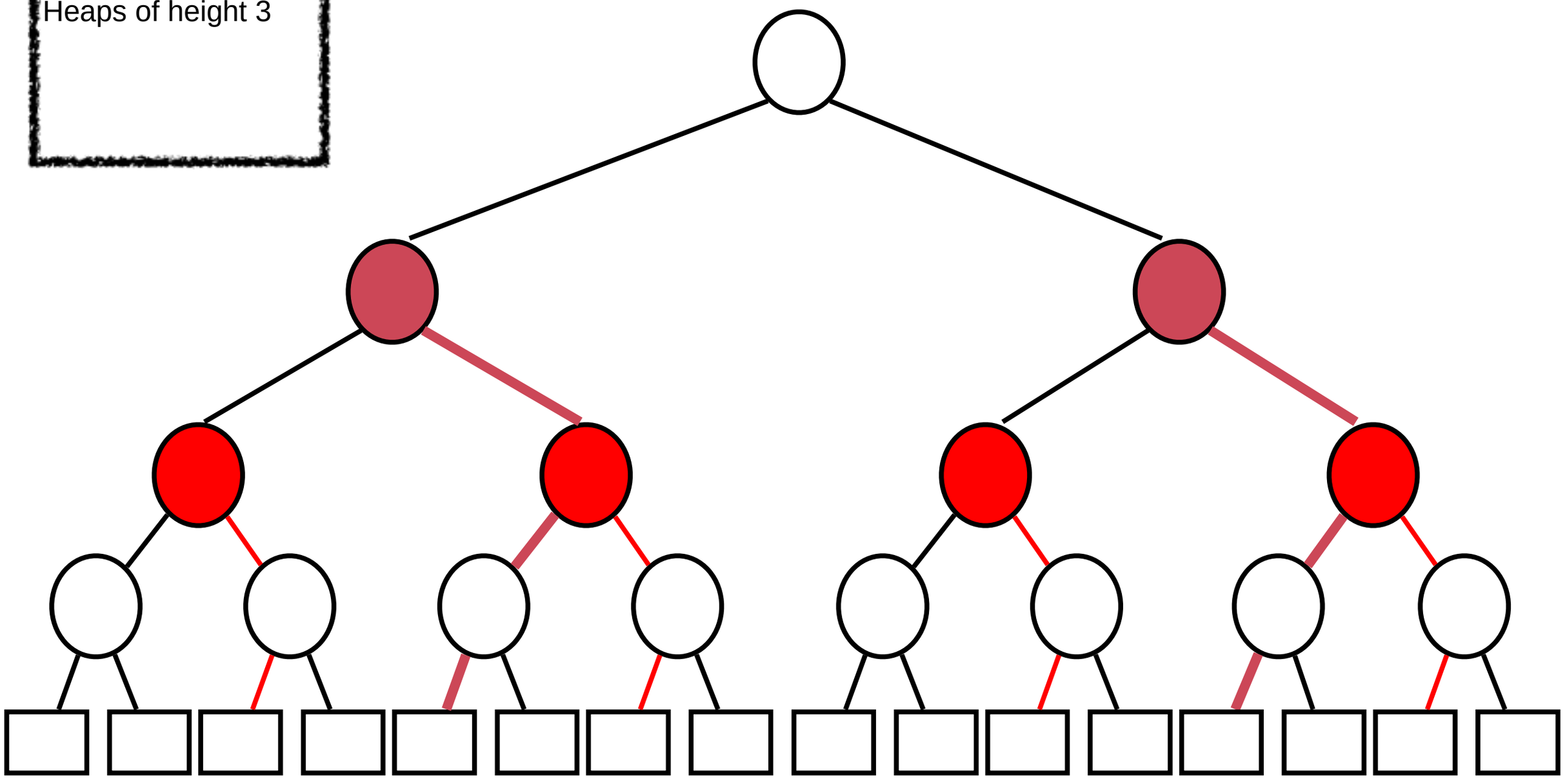
For each new node joining two heaps: mark path with maximum number of bubble-down operations



For each height-2 heap, leftmost branch not marked

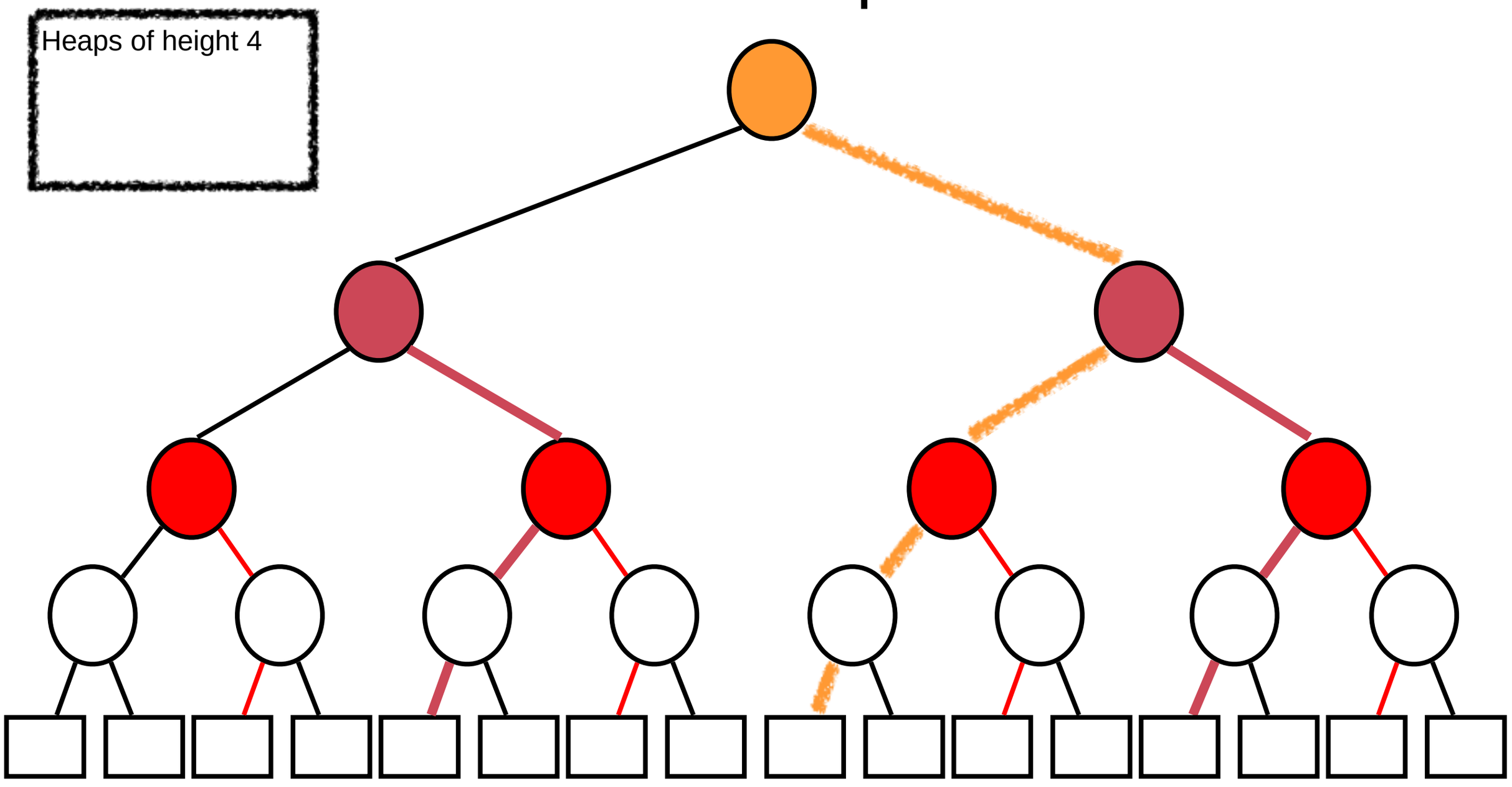
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 3



For each height-3 heap, leftmost branch not marked

For each new node joining two heaps: mark path with maximum number of bubble-down operations



For height-4 heap, leftmost branch not marked

Inductive argument: marking procedure will never mark all edges in heap, since the leftmost branch is never marked

- Note: leftmost branch in height-h heap: not marked
- When joining 2 heaps of height h to heap of height $h + 1$: new edges to be marked are
  - edge joining new node and right heap of height h, and
  - edges on left path in the right heap of height h
- We conclude: leftmost branch in height $(h+1)$ heap is not marked

Build Heap In-place

Algorithm buildHeap(A,n):
    for $i \leftarrow \lfloor n/2 \rfloor$ **to** 1 **do**
        downHeap(A,i)

Algorithm downHeap(A,i):
    $l \leftarrow 2i$
    $r \leftarrow 2i + 1$
    if $l \leq n \wedge A[l] < A[i]$ **then**
        $min \leftarrow l$
    else
        $min \leftarrow i$
    if $r \leq n \wedge A[r] < A[min]$ **then**
        $min \leftarrow r$
    if $i \neq min$ **then**
        swap(i,min)
        downHeap(A,min)

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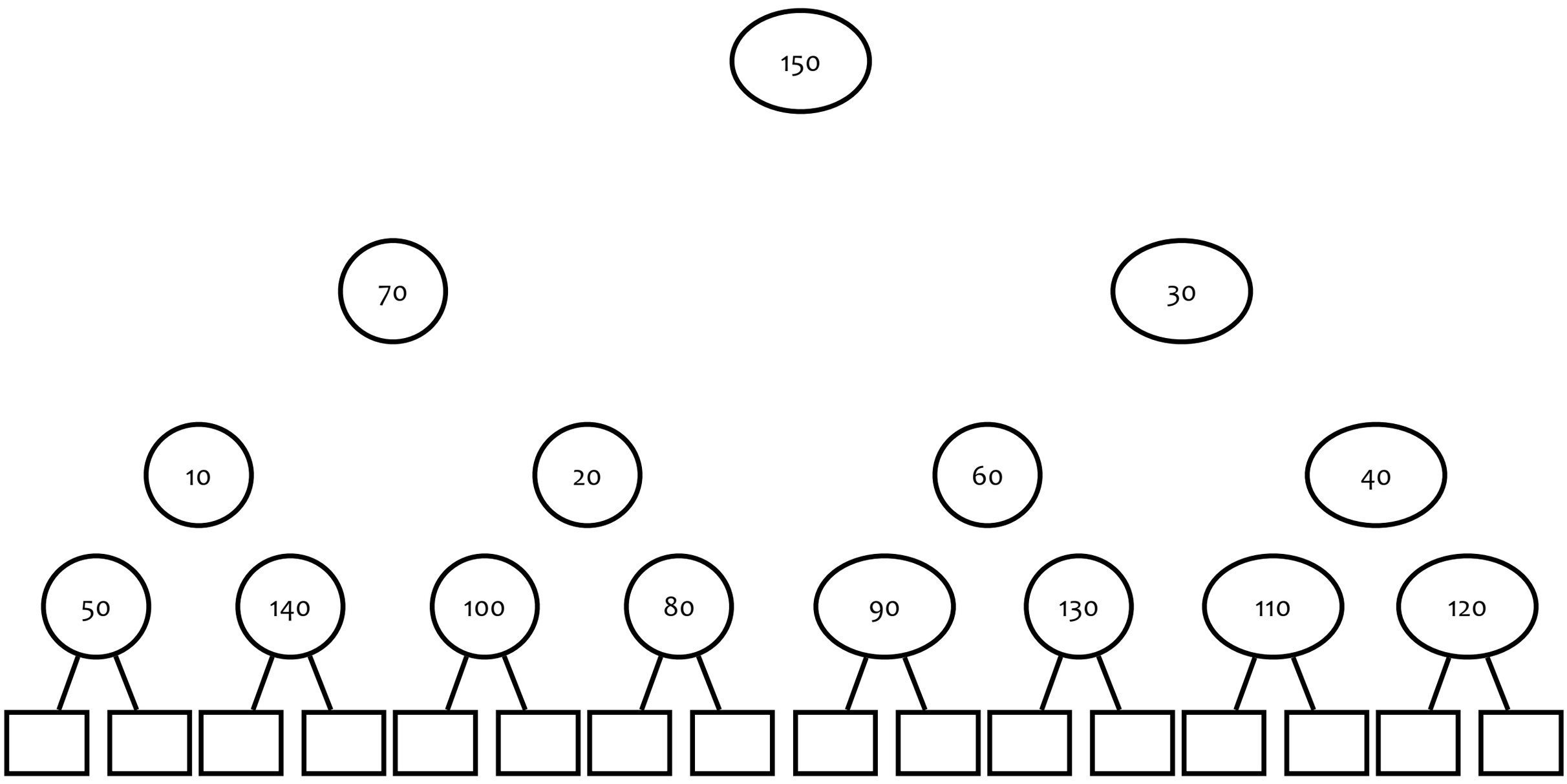

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