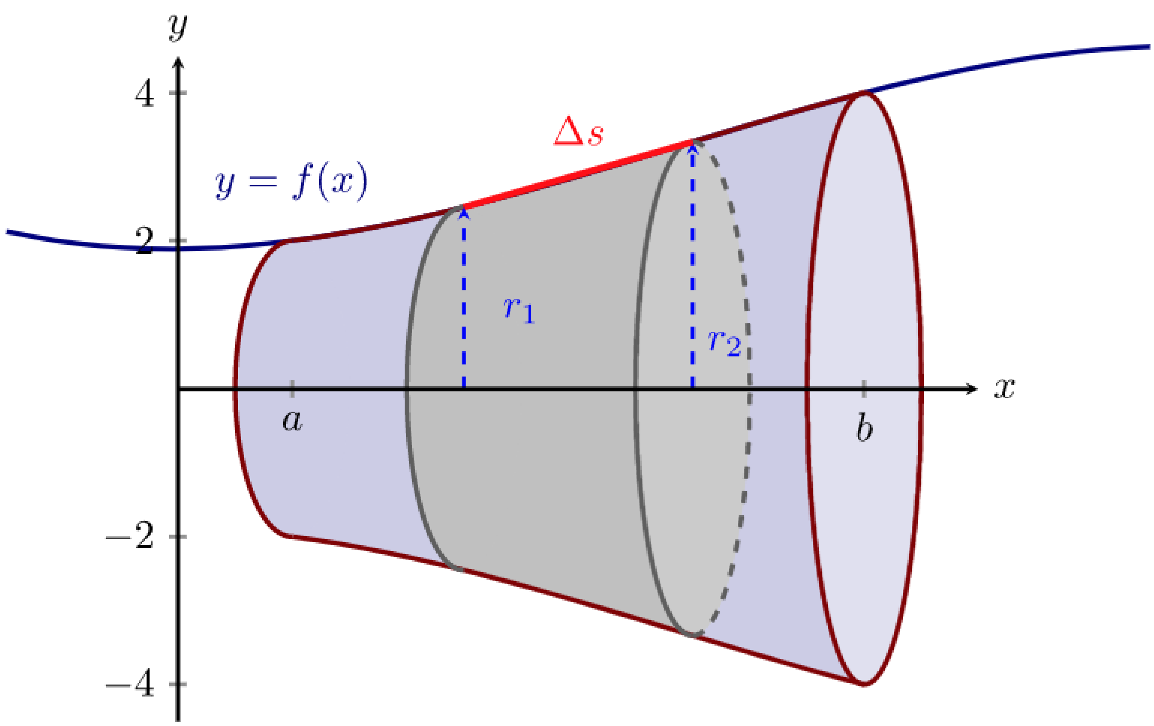
Chapter 1

Integration Techniques and Applications



1.1 Brief Review of Integration

The first month focuses heavily on picking up where Calculus I left off...integration.

There are two types of integration in Calculus I,

- Definite Integration: $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$
- Indefinite Integration: $\displaystyle\int f(x)dx = F(x) + C$ where $F'(x) = f(x)$.

<u>Indefinite Integration</u>

This is when we try to find the antiderivative of a function. Usually we do this to apply it to definite integration. That is,

If $F'(x) = f(x)$ then $\displaystyle\int f(x)dx = F(x) + C$

We call $F(x)$ an antiderivative of $f(x)$ and $F(x) + C$ the most generalized antiderivative of $f(x)$.

Note: Do not forget the constant of integration!

<u>Definite Integration</u>

This is defined by the use of Riemann sums,

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$. This geometrically represents the area between the graphs of $y = f(x)$ and $y = 0$ over the region $a \le x \le b$.

Another, and in my personal opinion, a better way to think of this integral is that it adds up the area of a bunch of rectangles with area $f(x)dx$ which we represent as $f(a + k\Delta x)\Delta x$ for $\Delta x = \dfrac{b-a}{n}$ of infinitesimal size. The reason why to view it as a summation of terms $f(x)dx$ rather than area is because in the future you might replace these terms with other quantities that shouldn't be interpreted geometrically.

Example: Consider a metal rod where we measure along only it's length from the bottom $x = 0$ to $x = L$ if it is L units in length.

Let $\rho(x_0)$ represent the density of the bar at the point $x = x_0$. As dx is thought of as an infinitesimal of Δx, a volume portion of the bar, then $\rho(x)dx$ is the product of density and volume in a small region of the bar yielding the mass in that region. To get the total mass of the bar we sum up everything to obtain

$$\text{Mass} = \int_0^L \rho(x)dx$$

Note: This example illustrates that like the derivative, there are two interpretations of integration. One physical and one geometric. Do not pick favourites on how you like to interpret integration, you need the flexibility of both to solve problems.

By the second part of the **Fundamental Theorem of Calculus** there is a relationship between antiderivatives and definite integration. This is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$.

Example: Find the area between $y = x$ and $y = x^2 - 2x$.

Example: Compute $\displaystyle\int_0^{\pi/6}(2\sec(\theta)+\sec(\theta)\tan(\theta))d\theta$

There are basic rules to integration:

- $\displaystyle\int(f(x)+g(x))dx=\int f(x)dx+\int g(x)dx$
- $\displaystyle\int kf(x)dx=k\int f(x)dx$

Integration (indefinite) undoes differentiation. Undoing something is hard! Unlike differentiation, we are very limited in ways to find an antiderivative. I like to compare differentiation and antidifferentiation to a plate. Differentiation would be smashing the plate and antidifferentiation would be gluing it back together.

Substitution (undoing the chain rule)

This dealt with integrals of the form

$$\int_a^b f(g(x))g'(x)dx$$

By letting $u = g(x)$ we have $du = g'(x)dx$. Moreover, when $x = a$ then $u = g(a)$ and when $x = b$ then $u = g(b)$. So this becomes

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

This type of substitution letting $u = g(x)$ is called a **push-forward** substitution. There is another type of substitution called a **pull-back** substitution that we cover when we discuss trigonometric substitution.

Example: Compute $\displaystyle\int_0^1 xe^{-x^2}dx$

There are two special types of substitution that we do which don't seem as obvious to fit into the form above. Consider the following integral

$$\int_a^b f(x+c)dx$$

By letting $u = x + c$ we obtain $du = dx$. The bounds (if present) change as $u = a + c$ and $u = b + c$ when $x = a$ and $x = b$ respectively. This gives us

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

I like to call these **shifts**.

Example: Compute $\displaystyle\int \frac{x}{(x+1)^{2020}}dx$

Now consider the integral below,

$$\int_a^b f(kx)dx$$

where $k \neq 0$. We may use the substitution $u = kx$ to obtain $du = kdx$. The bounds change as well to integrate from $u = ka$ to $u = kb$. Now we have a problem with the above, we don't have a kdx term in the integral. We get around this issue as follows:

$$\int_a^b f(kx)dx = \frac{1}{k}\int_a^b f(kx)(kdx) = \frac{1}{k}\int_{ka}^{kb} f(u)du$$

where we just use the fact that $1 = k/k$ to introduce the k term we need. I like to call these **scalings**.

Example: Compute $\int_0^{\pi/15} \sin(5x)dx$.

1.2 (Section 8.1) Some Integration Techniques and Tricks

Most integral aren't immediately clear on what the antiderivative is. Often algebraic tricks are required or some clever use of identities.

1.2.1 Completing the Square

Very useful! This technique is use to turn two terms where a variable appears into a single term. For terms like $x^2 + ax$ it turns them into shifts, which you can sub for. A reminder of how to complete the square:

Example: Compute $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$.

1.2.2 Using Trigonometric Identities

We will learn more about using trig identities as the course develops. Currently a very helpful identity to know if $\cos^2(x) + \sin^2(x) = 1$. The example below is an example similar to the idea of the previous example, to turn two terms into a single term that is easier to control.

Example: Compute $\displaystyle\int_0^{\pi/2} \sqrt{1 - \cos(\theta)} \; d\theta$.

1.2.3 Long Division

This is for rational expressions where the degree of the numerator is higher than or equal to the degree of the polynomial on the denominator. A reminder that a rational function is of the form:

$$\frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. Fome examples are as follows:

To evaluate these integrals we use long division to put it into a more favorable form. This doesn't always work, but we will learn a another technique to pass such a roadblock when we study partial fractions in this chapter.

Example: Compute $\displaystyle\int \frac{x^3+x}{x-1}dx$.

(Continued...)

1.2.4 Rewriting (a matter of perception)

Look to rewrite an expression. Sometimes a good technique is to look for common factors. This is a little tricky at first. When we say common factors, we don't only mean common factors of a polynomial.

Example: Compute $\int (27e^{9x} + e^{12x})^{1/3} dx$.

1.2.5 Dumb Luck

I'm sorry to say but integrals, as you have probably seen at this point, are not as straight-forward to compute as derivatives. We know really only one technique so far to compute integrals, substitution.

Sometimes we can make a substitution that changes the integral into something we can evaluate without a good reason why it would have worked initially. Math is often preached as a logical step from one to the next, but it honestly isn't always that case. Some things are solved by trial and error. This isn't a fault in math, this is a fault in the nature of the universe and it's reality.

Note: Saying an answer just came to you without a logical reason when there **IS** *a logical reason does not justify the result. It just comes off as a lack of work.*

Example: Compute $\int \frac{\sqrt{x}}{1+x^3} dx$.

Some ‘dumb luck’ methods are a bit more clear on where to start. Sometimes it’s because we want to reduce the complexity of the number of terms. Below we do an example of using a substitution to ‘eliminate’ a constant to avoid expansion.

Example: Compute $\displaystyle\int 27e^{-3x}(1+3e^{-x})^4 dx$.

1.2.6 Mixing Techniques

It is possible you may have to mix techniques to complete a problem!

Example: Compute $\int \frac{x-6}{\sqrt{8x-x^2}}dx$.

1.2.7 Organizing Definite Integral Substitution

One last note is when evaluating integrals is to be as organized as possible. Usually integrals require several steps and if another person, whether it be a colleague or a professor, sees work splattered randomly across the page or is broken into several obscure segments...they will just give up on trying to follow your reasoning. Best way to learn not to do this is to observe how people professionally present information and to mimic it into your technique. Aesthetic and presentation is crucial. You also want to ease the amount of work you have to do by avoiding several side calculations that don't remain consistent with the flow of your solution.

One such issue I commonly see students do is to not substitute for the bounds in definite integrals when solving something by substitution. Many extra steps are taken and it loses focus of the reader. While this is minor at this stage, it does make a difference when we study trigonometric substitution. It cuts down on time drastically when we substitute for the bounds in this section.

Example: Given that $f(t)$ is a continuous function on $\mathbb{R}$ such that $\displaystyle\int_1^4 f(t)dt = 3$ compute $\displaystyle\int_{-2}^{-1/2} f(2t+5)dt$.

1.3 (Section 7.1) Logarithms and Exponentials

The textbook goes through extensive measures to demonstrate how to reconstruct the exponential and all properties of the logarithm if we initially define the natural logarithm by

$$\ln(x) = \int_1^x \frac{1}{t}dt, \qquad x > 0$$

the process is long and worth a read. What we will do is give examples of computing logarithmic and exponential type integrals.

1.3.1 Integrals Involving the Natural Logarithm

It is defined as $\ln(x) = \displaystyle\int_1^x \frac{1}{t}dt$ or as the inverse function of $f(x) = e^x$. Since it is the inverse of $f(x) = e^x$ then

$$e^{\ln(x)} = x \qquad \text{and} \qquad \ln(e^x) = x$$

The logarithm is amazing in that it handles exponents and multiplication operations extremely well.

Properties

- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a/b) = \ln(a) - \ln(b)$
- $\ln(a^b) = b\ln(a)$

Note

The notation $\ln a^b$ is ambiguous. For example, $\ln(x^2) = \ln(x \cdot x)$ but $(\ln(x))^2 = (\ln(x)) \cdot (\ln(x))$. In general notice that $\ln(a^b) \neq (\ln(a))^b$ so you must always specify where to place the parenthesis unless an author explicitly states what the notation means.

Since $\ln|x| = \displaystyle\int \frac{dx}{x} + C$ it is commonly found in dealing with integrals of the form

$$\int_a^b \frac{f'(x)}{f(x)}dx$$

since $u = f(x)$ gives $du = f'(x)dx$ and thus

$$\int_a^b \frac{f'(x)}{f(x)}dx = \int_{f(a)}^{f(b)} \frac{du}{u} = \ln|u|\Big|_{f(a)}^{f(b)}$$

Example: Compute $\displaystyle\int_{-1}^0 \frac{3dx}{3x-2}$

Example: Compute $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}dx$.

Example: Compute $\displaystyle\int \frac{1}{\arctan(4x)(1+16x^2)}dx$.

1.3.2 Integrals Involving the Exponential Function

The counterpart to the logarithm. It is denoted either $f(x) = e^x$ or $f(x) = \exp(x)$ and can be defined in several different ways. It was first defined by the following limit

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

or you may define it as the inverse of $\ln(x) = \displaystyle\int_1^x \frac{dt}{t}$. We will learn later that another common definition some use is

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

which will be discussed in detail in chapter 10.

It has the amazing property that

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \text{and} \qquad \int e^x dx = e^x + C$$

Because it is what some might call a "fixed function" in calculus, it appears very commonly when solving most problems.

Properties

- $e^{a+b} = e^a e^b$
- $\dfrac{e^a}{e^b} = e^{a-b}$
- $e^{-a} = \dfrac{1}{e^a}$
- $(e^a)^b = e^{ab}$

Note

These are just regular exponent laws. It is because, surprisingly, the way the exponential is defined as above (using limits) you would not expect it to be a function with these properties! It turns out that $\exp(x)$ shares all the properties that exponential laws give, thus we use an alternate notation $\exp(x) = e^x$ to make things easier for us. We **represent** the exponential function as a number to the variable because nothing changes if we do this.

Example: Compute $\int_0^{\sqrt{\ln(\pi)}} 2xe^{x^2} \cos(e^{x^2}) dx$.

1.3.3 Other Logarithmic and Exponential Integrals

We have the logarithms and exponentials of other bases

$$f(x) = a^x \qquad \text{and} \qquad g(x) = \log_a(x)$$

They are inverses and thus related by

$$a^{\log_a(x)}x \qquad \text{and} \qquad \log_a(a^x) = x$$

with the integration and differentiation rules

$$\frac{d}{dx}\left[a^x\right] = (\ln(a))a^x \qquad\qquad \int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)} \qquad\qquad \int \log_a(x)dx = \underline{\text{LEARN THIS LATER!}}$$

Example: Compute $\displaystyle\int_1^e \frac{2\ln(10)\log_{10}(x)}{x}dx$.

Example: Compute $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan(t)} \sec^2(t)dt$.

1.4 (Section 7.2) Separable Differential Equations and Modeling

1.4.1 Separable Differential Equations

Definition

A differential equation is an equation involving an unknown function and its derivatives.

Definition

A separable differential equation (SDE) is one of the form $y' = f(x)g(y)$. $y' = y^2 x$

Example: Which of the following are SDE's?

$\frac{dy}{dx} = e^x e^y \Leftarrow \quad \frac{dy}{dx} = e^{x+y} \qquad \frac{dy}{dx} = \sin(x+y) \qquad \frac{dy}{dx} = \underbrace{x^2 \log_2(x)}_{g(x)} \underbrace{\ln(y)}_{f(y)} \qquad \frac{dy}{dx} = xy + x = x(y+1)$

Procedure for Solving an SDE

1. Provided $g(y) \neq 0$, divide both sides by $g(y)$ to obtain

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$u = x^2$

$\Rightarrow du = 2x\,dx$

$du = \frac{du}{dx} dx$

2. Integrate both sides with respect to x to obtain

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx$$

3. Use the substitution rule for integrals $dy = \frac{dy}{dx} dx$ to obtain

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

4. Integrate the previous expression

Note that solutions to SDE's are usually implicit and can seldomly be solved for explicitly.

Example: Solve explicitly for $y(x)$ if $xy' = 1 + y^2$. $\Rightarrow \frac{dy}{dx} = \underbrace{\frac{1}{x}}_{f(x)} \underbrace{(1+y^2)}_{g(y)}$

1. $\frac{1}{1+y^2} \frac{dy}{dx} = \frac{1}{x}$

2. $\int \frac{1}{1+y^2} \frac{dy}{dx} dx = \int \frac{1}{x} dx$

3. $\int \frac{1}{1+y^2} dy = \int \frac{1}{x} dx$

4. $\arctan(y) = \ln|x| + C$

Take tan of both sides

$\Rightarrow y = \tan(\ln|x| + C)$ //

1.4.2 Unlimited Population Growth Model (Malthus Model)

Malthus Model

A population growth with no limitations imposed is modeled by

$$\begin{cases} \dfrac{dy}{dt} = ky & \text{(Differential Equation)} \\ y(0) = y_0 & \text{(Initial Condition)} \end{cases}$$

where k is a constant and t is time. It has the unique solution

$$y = y_0 e^{kt}$$

Proof: Following the procedure of solving an SDE we have

$$\Rightarrow \frac{1}{y}\frac{dy}{dt} = k$$

$$\Rightarrow \int \frac{1}{y}\frac{dy}{dt}dt = \int kdt$$

$$\Rightarrow \int \frac{1}{y}dy = \int kdt$$

$$\Rightarrow \ln|y| = kt + C$$

By taking the exponential of both sides we obtain

$$e^{\ln|y|} = e^{kt+C} \Rightarrow |y| = e^C e^{kt}$$

As an additional assumption, we are using this to model the growth of a positive quantity of something (e.g. population). As a result, we may assume that $y(t) > 0$ and so the above simplifies to

$$y(t) = e^C e^{kt}$$

Now we invoke the condition $y_0 = y(0) = e^C e^0 = e^C(1) = e^C$ and so the solution is given by

$$y(t) = y_0 e^{kt}$$

■

Example: The biomass of a yeast culture in an experiment is 29g. After 30 min the mass is 37g. Assuming the equation for unlimited population growth models this, how long will it take for the mass to double from it's initial population size?

Malthus model $\Rightarrow$ $\underline{y(t) = y_0 e^{kt}}$

Initial population is $y(0) = 29 = y_0$

$\underline{y(t) = 29e^{kt}}$

Given: $\underbrace{y(30) = 37}_{\text{determines } k}$ Want to solve: $\underline{y(t) = 2 \times 29 = 58}$

$t = 30, y = 37$

$\longrightarrow$ $37 = 29e^{30k}$

$\Rightarrow \frac{37}{29} = e^{30k} \Rightarrow \ln(37/29) = 30k \Rightarrow \underline{k = \frac{1}{30}\ln(37/29)}$

So $y(t) = 29e^{kt}$ where

Now solve $y(t) = 58$

$\Rightarrow 58 = 29e^{kt}$

$\Rightarrow 2 = e^{kt} \Rightarrow \ln(2) = kt$

$\Rightarrow t = \frac{\ln(2)}{k} = \frac{\ln(2)}{\frac{1}{30}\ln(37/29)}$

$= \frac{30\ln(2)}{\ln(37/29)}$

≈ 85.36 minutes

$\approx$ 1 hour 25 min

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1.4.3 Radioactive Decay

Radioactive Decay Model

A substance undergoing radioactive decay is modeled by

$$\begin{cases} \dfrac{dy}{dt} = -ky & \text{(Differential Equation)} \\ y(0) = y_0 & \text{(Initial Condition)} \end{cases}$$

where $k > 0$. This has the unique solution

$$y(t) = y_0 e^{-kt}$$

which is obtained from the previous model by replacing k with $-k$. The constant k is usually obtained by knowing the half-life of a substance. It satisfies the equation

$$\text{Half-Life} = \frac{\ln(2)}{k}$$

Example: The half-life of carbon-14 is 5730 years. Find the age of a sample in which 10% of the radioactive substance has decayed.

Half-life $\Rightarrow$ $5730 = \dfrac{\ln(2)}{k} \Rightarrow k = \dfrac{\ln(2)}{5730}$ // use the model $y(t) = y_0 e^{-kt}$

Let $t=0$ be the initial age of the object.

at the current time $y(t^*) = 0.9 y_0$

$$\Rightarrow 0.9 \cancel{y_0} = \cancel{y_0} e^{-kt^*}$$

$$\Rightarrow 0.9 = e^{-kt^*}$$

$$\Rightarrow \ln(0.9) = -kt^*$$

$$\Rightarrow t^* = -\frac{\ln(0.9)}{k} = -\frac{\ln(0.9)}{\ln(2)/5730}$$

$$= -\frac{5730 \ln(0.9)}{\ln(2)}$$

$$\approx 870.98 \text{ years}$$

1.4.4 Heat Transfer: Newton's Law of Cooling

Newton's Law of Cooling

If $H(t)$ is the temperature of an object at time t and H_s is the constant surrounding temperature the model is given by

$$\begin{cases} \dfrac{dH}{dt} = -k(H - H_s) & \text{(Differential Equation)} \\ H(0) = H_0 & \text{(Initial Condition)} \end{cases}$$

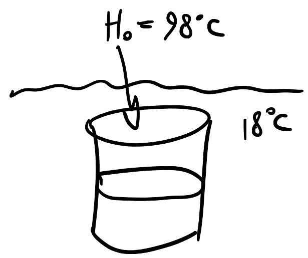
where k is a constant. You can obtain the unique solution by acknowledging the fact that because H_s is a constant then

$$\frac{dH}{dt} = \frac{d}{dt}[H - H_s]$$

thus the above differential equation becomes $\dfrac{d}{dt}[H - H_s] = -k(H - H_s)$. By the previous model with $y = H - H_s$ we obtain the solution $H - H_s = (H_0 - H_s)e^{-kt}$. Solving for H gives us

$$H(t) = H_s + (H_0 - H_s)e^{-kt}$$

Example: In Breaking Bad, Walter and Jesse cook "soup". Accidentally the soup is heated to 98^oC by Jesse screwing up again. After Walter yells at Jesse and tells him '**I am the one who cooks**' he tries to cool the soup by immersing it in a container surrounded by 18^oC water. After 5 minutes the temperature of the soup is 38^oC. The batch will be ruined if it doesn't reach 20^oC within 10 minutes. Is this batch spoiled?



$$H(t) = 18 + (98 - 18)e^{-kt}$$
$$= 18 + 80e^{-kt}$$

Given $H(5) = 38 \Rightarrow 38 = 18 + 80e^{-5k}$

$\Rightarrow 20 = 80e^{-5k}$

$\Rightarrow \frac{1}{4} = e^{-5k}$

$\Rightarrow \ln(1/4) = -5k$

$\Rightarrow k = \dfrac{\ln(1/4)}{-5}$

Solve $H(t) = 20$

$\Rightarrow 18 + 80e^{-kt} = 20$

$\Rightarrow 80e^{-kt} = 2$

$\Rightarrow e^{-kt} = \frac{1}{40}$

$\Rightarrow -kt = \ln(1/40) \Rightarrow t = -\dfrac{\ln(1/40)}{k} = \cancel{+}\dfrac{\ln(1/40)}{\cancel{-}\frac{1}{5}\ln(1/4)} = \dfrac{5\ln(1/40)}{\ln(1/4)}$

≈ 13.305

It's spoiled! $\leftarrow$ or about 13 min

1.5 (Section 8.2) Integration By Parts (IBP)

1.5.1 Establishing Integration by Parts and LIPET

Integration by parts is an integration rule that deals with undoing the product rule. Let $u(x)$ and $v(x)$ be differentiable functions. Then by the product rule,

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x over $[a, b]$ yields

$$\int_a^b \frac{d}{dx}[uv]\, dx = \int_a^b u\frac{dv}{dx}dx + \int_a^b v\frac{du}{dx}dx$$

$$\Rightarrow uv\Big|_a^b = \int_a^b u\frac{dv}{dx}dx + \int_a^b v\frac{du}{dx}dx$$

Then rewrite this as

$$\int_a^b u\frac{dv}{dx}dx = uv\Big|_a^b - \int_a^b v\frac{du}{dx}dx$$

This is also commonly represented as the following.

Integration by Parts

Let $u(x)$ and $v(x)$ be differentiable functions on an interval (a, b) and continuous on $[a, b]$. Then...

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

The idea is that you manipulate the product rule to obtain a way to express an integral product in the form of another integral product. The hope is that this rearrangement simplifies the integration. Notice that when you start with u on the left hand side you differentiate it in the right hand side. When you start with $\frac{dv}{dx}$ on the left hand side you find the anti-derivative v on the right hand side. In short, you often make one term in the product simpler while you make the other term more complicated. The hope is that the simplification of one term outdoes complicating the other to make the integral doable.

Example: Compute $\displaystyle\int x\cos(x)dx$ using $u = x$ and $dv = \cos(x)dx$.

Be careful to make good choices! Selecting which term to differentiate and which one to integrate matters! Let's revisit the previous example where we make the other choice of assigning u and v.

Example: Compute one iteration of IBP for $\int x\cos(x)dx$ using $u = \cos(x)$ and $dv = xdx$.

You can see the previous example did not work out well after a single iteration. The purpose is to simplify the expression! Not make it even more complicated! Thankfully there is an acronym called **LIPET** which helps us determine which term to set as our u-term in some cases.

LIPET

Consider an integral of the form $\int f(x)g(x)dx$ where $f(x)$ and $g(x)$ are either a **L**ogarithmic function, **I**nverse trigonometric function, **P**olynomial function, **E**xponential function, or **T**rigonometric function. **LIPET** is an acronym that tells you that you set your u term as the first function that is present on the list:

- **L** = Logarithmic function
- **I** = Inverse trigonometric function
- **P** = Polynomial function
- **E** = Exponential function
- **T** = Trigonometric function

Example: The above LIPET system does not hold for the following integrals! Can you explain why? Ability to find an antiderivative is **NOT** the reason!

$$\int x\sqrt{1-x^2}dx \qquad \int \ln(y)e^y\tan(y)dy \qquad \int \frac{1-t^2}{1+t^2}e^t dt$$

1.5.2 Dominant Logarithms and Inverse Functions in LIPET

Logarithmic and inverse functions are dramatically more complicated that their derivatives. For this reason, they often become a good first choice as what to set your u-term to be when performing integration by parts! The derivative of these expressions are usually some rational type function.

Example: Compute $\int_1^e x^3 \ln(x) dx$.

1.5.3 Dominant Polynomials in LIPET

When we encounter polynomials as the first thing on the list in LIPET, it's easy to see how the expression reduce to a simpler form at each step! Specifically, these are integrals that are the product of a polynomial with either an exponential or trigonometric function.

Example: Compute $\int (x^2 + 2x)\cos(2x)dx$.

Example: Compute $\int x \sec^2(x)dx$.

1.5.4 Dominant Exponential or Trigonometric Functions in LIPET

These are integrals where both terms in the product cycle back to a multiple of their original form after differentiation and integration. For example look at the chain of $\cos(x)$ after differentiating multiple times.

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \to \frac{d}{dx}[-\sin(x)] = -\cos(x)$$

and so after differentiating twice we are back to a multiple of $\cos(x)$. Similarly differentiating e^{5x} once gives $5e^{5x}$, which is a multiple of itself. For these you use IBP by selecting one term in the product to keep being differentiated and the other term to keep being integrated until we wind up with a multiple of our original integral. Then you solve the equation for the integral.

Example: Compute $\displaystyle\int e^{3x}\sin(2x)dx$.

1.5.5 Substitutions Leading to IBP

This is the integration by parts form of "Dumb Luck". You make a substitution that happens to lead to IBP.

Example: Compute $\int_0^{\pi^2} \sin(\sqrt{x})dx$.

1.5.6 Integrals of Lonely Logarithms and Inverse Functions (Ninja's)

Products might be hidden in plain sight when you encounter integrals. Such is the case for lonely integrands like

$$\int \arctan(x)dx \qquad \int \log_3(x)dx \qquad \int (\ln(x))^2 dx$$

where a product that may be exploited by IBP is actually hidden. If $f(x)$ is a logarithmic or inverse function then you may rewrite the integral as

$$\int f(x)dx = \int f(x) \cdot 1dx$$

to which you set $u = f(x)$ and $dv = 1dx$. This doesn't necessarily fall under LIPET, since we don't have "constants" on that list. I would just remember it as a special exception.

Example: Compute $\displaystyle\int \ln(x)dx$.

Example: Compute $\displaystyle\int \arccos(x)dx$.

1.5.7 The Tabular Method

Sometimes you may have to use IBP several times. Fortunately enough there is a pattern you can use in a table to obtain the result. Take for instance the following,

$$\int uv'''dx = uv'' - \int u'v''dx \tag{1.1}$$

$$= uv'' - \left(u'v' - \int u''v'dx\right) = uv'' - u'v' + \int u''v'dx \tag{1.2}$$

$$= uv'' - u'v' + u''v - \int u'''vdx \tag{1.3}$$

Notice how one term keeps being applied derivatives to while the other term keeps being integrated. The product is taken between the two and the sign alternates. We can record this in a table quite conveniently.

Example: Use the tabular method on the previous example $\int e^{3x}\sin(2x)dx$.

Example: Use the tabular method to compute $\int (x^4 + 3x^2)3^x dx$.

Note

Under LIPET, the tabular method works best on polynomial, exponential, or trigonometric dominant integrals. It does not work well on logarithmic or inverse trigonometric dominant integrals.

1.6 (Section 8.3) Trigonometric Integrals

We know how to integrate some basic trigonometric functions.

- $\displaystyle\int \sin(x)dx = -\cos(x) + C$
- $\displaystyle\int \csc^2(x)dx = -\cot(x) + C$
- $\displaystyle\int \tan(x)dx = \ln|\sec(x)| + C$

and so forth. But what about more complicated terms like

$$\int \sin^{2020}(x)\cos^3(x)dx \qquad \text{and} \qquad \int \tan^2(x)\sec^2(x)dx?$$

The textbook Thomas' Calculus really doesn't give you justice on the techniques to solve these problems. It avoids categorizing techniques to not add to the density of material, but it's better to know it than attack these problems half blind!

1.6.1 Products of Sines and Cosines of Different Composition

This is to deal with products of sines and cosines, **not raised to a power**, where the inputs of both differ. For example integrals of the form

$$\int \cos(5x)\sin(2x)dx$$

To deal with these, due to the fact that both terms in the product, do cycle, you can use "repetitive in all terms" IBP. However, a quicker trick is to use the product to sum identities.

Procedure for Integrals Consisting of a Products of Sine and/or Cosine with Different Inputs

1. Use one of the following three appropriate product to sum formulas:
   - $\cos(mx)\cos(nx) = \dfrac{1}{2}\left(\cos((n+m)x) + \cos((n-m)x)\right)$
   - $\sin(mx)\sin(nx) = \dfrac{1}{2}\left(\cos((n-m)x) - \cos((n+m)x)\right)$
   - $\sin(nx)\cos(mx) = \dfrac{1}{2}\left(\sin((n+m)x) + \sin((n-m)x)\right)$
2. Integrate the remaining basic integral.

These are the so called "product to sum" formulas. For example,

$$\sin(3x)\sin(2x) = \frac{1}{2}\left(\cos((3-2)x) - \cos((3+2)x)\right) = \frac{1}{2}\left(\cos(x) - \cos(5x)\right)$$

which turns the product into something easily, and quickly, integrable.

Example: Compute $\int \cos(2x)\sin(4x)dx$.

1.6.2 Products of Sines and Cosines, Same Input, Raised to a Power

These are integrals of the form

$$\int_a^b \sin^m(x)\cos^n(x)dx$$

where the input might be scaled or shifted. These divide into two cases:

Procedure for Computing $\int_a^b \sin^m(x)\cos^n(x)dx$

- m and/or n is odd:
  1. Take off a single term from the odd power and collect it with your dx term to form a du.
  2. Now that the remaining terms are raised to an even power, convert them to the other trigonometric function using $\sin^2(x) + \cos^2(x) = 1$.
  3. Complete the u-substitution and integrate with the du term formed in the first step.

- m and n are even:
  1. Convert all terms using the identities

     $$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

  2. If all your terms are not basic integrals, expand everything.
  3. If you encounter all even terms repeat the first step on those integrand terms. Else, your integrand terms are either of the case where you have a basic integral or m and/or n are odd. Use the appropriate procedure on those.

Example: Compute $\int \sin^3(x)\cos^3(x)dx$.

Example: Compute $\int \cos^7(x)dx$.

Example: Compute $\int \sin^2(x)\cos^2(x)dx$.

1.6.3 Eliminating Roots

Procedure for Computing Integrals Containing $\sqrt{1 \pm \cos(mx)}$ where m is Even

1. Use the double angle identity $1+\cos(2x) = 2\cos^2(x)$ or $1-\cos(2x) = 2\sin^2(x)$ to eliminate the square root.
2. Use appropriate integration techniques afterwards.

Example: Compute $\int_0^{\pi/4} \sqrt{1-\cos(4x)}dx$.

Procedure for Computing Integrals Containing $\sqrt{1 \pm \sin(nx)}$

1. Multiply the integrand by the conjugate $\dfrac{\sqrt{1 \mp \sin(nx)}}{\sqrt{1 \mp \sin(nx)}}$
2. Expand the terms and use the identity $1 - \sin^2(nx) = \cos^2(nx)$ to eliminate the square root.
3. Use appropriate integration techniques afterwards.

Example: Compute $\displaystyle\int_0^{\pi/4} \sqrt{1 - \sin(2x)}dx$.

1.6.4 Lonely Powers of Tangent or Secant

This is for integrals of the form

$$\int \tan^m(x)dx \qquad \text{or} \int \sec^n(x)dx$$

Unlike the case with powers of sine and cosine, they require their own categorization.

Procedure for Computing $\int_a^b \tan^m(x)dx$ or $\int_a^b \sec^n(x)dx$

- $n \geq 3$:

  1. Pull of a $\sec^2(x)$ term and perform IBP with $dv = \sec^2(x)dx$. Your integral will reduce as the following reduction formula:

  $$\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$$

  2. Complete the integration if you are at a basic integral of $\sec^2(x)$ or $\sec(x)$. Otherwise, repeat the first step on the appropriate integral.

- $m \geq 3$ is odd:

  1. Pull off a single tangent term to and multiply by $\dfrac{\sec(x)}{\sec(x)}$ to form

  $$\int \tan^m(x)dx = \int \tan^{m-1}(x)\frac{\sec(x)\tan(x)}{\sec(x)}dx$$

  2. Construct $du = \sec(x)\tan(x)dx$ and convert the remaining tangent terms to secants using $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete your u-substitution using $u = \sec(x)$ and integrate.

- $m \geq 2$ is even:

  1. Convert all your tangent terms into secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. If the result is a basic integral, integrate it. Otherwise, expand the result to construct a sum of powers of secants.
  3. Integrate the basic integral terms of $\sec^2(x)$, $\sec(x)$ and constants, then use the procedure of integrating higher powers of secants mentioned above.

Example: Compute $\int \tan^2(x)dx$.

Example: Compute $\int \sec^4(x)dx$.

Example: Compute $\int \sec^3(x)dx$.

1.6.5 Powers of Tangent and Secant with Same Input

This is for integrals of the form

$$\int \tan^m(x)\sec^n(x)dx$$

where the inputs may be shifted or scaled. The strategies for power products of cosecant and cotangent are identical. In this, it is assumed that $m, n \geq 1$ (not a lonely integrand).

Procedure for Computing a Non-Lonely $\int_a^b \tan^m(x)\sec^n(x)dx$

- m is odd:
  1. Pull off a single $\tan(x)$ and $\sec(x)$ term and construct $du = \sec(x)\tan(x)dx$.
  2. Convert the remaining tangent terms using the identity $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete the u-substitution with $u = \sec(x)$ and integrate.
- n is even:
  1. Pull of a $\sec^2(x)$ term and construct $du = \sec^2(x)dx$.
  2. Convert the remaining secant terms using the identity $\sec^2(x) = \tan^2(x) + 1$.
  3. Complete the u-substitution with $u = \tan(x)$ and integrate.
- m is even and n is odd:
  1. Convert all the tangent terms to secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. Expand your integrand to get a sum of powers of secants.
  3. Use the appropriate integration techniques for integrating lonely powers of secants.

Example: Compute $\int \sec(x)\tan^3(x)dx$.

(Continued...)

At Home Exercise: Use the appropriate procedure to compute $\int \tan^2(x)\sec^3(x)dx$

1.7 (Section 8.4) Trigonometric Substitution

1.7.1 Forms of Trigonometric Subsitution

These are used to deal with integrals containing the terms

$$a^2 - u^2 \qquad \text{or} \qquad a^2 + u^2 \qquad \text{or} \qquad u^2 - a^2$$

where a is a constant and u is the variable of integration. This is where we use a substitution method called a **pullback** substitution. The two substitution methods **pullback** and **push-forward** are as follows:

- Pullback: Substitutions of the form $x = g(u)$.
- Push-forward: Substitutions of the form $u = h(x)$.

The above terms resemble trigonometric identities

- $a^2 - u^2$ resembles $1 - \sin^2(\theta) = \cos^2(\theta)$
- $a^2 + u^2$ resembles $1 + \tan^2(\theta) = \sec^2(\theta)$
- $u^2 - a^2$ resembles $\sec^2(\theta) - 1 = \tan^2(\theta)$

Example: Convert the following integral

$$\int \frac{x^4}{9 + 4x^2} dx$$

to a trigonometric integral using the substitution $2x = 3\tan(\theta)$. Do not complete the integration.

Procedure for Computing Integrals Containing $u^2 \pm a^2$ and $a^2 - u^2$

The procedure for all the following is essentially the same. Use the substitution $u = a \times$ (Appropriate Trig Function), use techniques of trigonometric integrals to complete the integration, then convert back.

- Containing $a^2 - u^2$:

  1. Let $u = a\sin(\theta)$ and compute $du = a\cos(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arcsin(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sin(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $a^2 + u^2$:

  1. Let $u = a\tan(\theta)$ and compute $du = a\sec^2(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arctan(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\tan(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $u^2 - a^2$:

  1. Let $u = a\sec(\theta)$ and compute $du = a\sec(\theta)\tan(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \text{arcsec}(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sec(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

1.7.2 Sine Substitutions

In these integrals we always have a domain restriction of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Example: Compute $\int \frac{x^3}{\sqrt{5-4x^2}} dx$.

1.7.3 Tangent Substitutions $a^2 + u^2 = u^2 + a^2$

In this we always have a domain restriction of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ as well.

Example: Compute $\int \frac{1}{(4x^2+9)^2} dx$.

$u^2 + a^2 \iff 4x^2 + 9$
$= (2x)^2 + 3^2$

$$= \int \frac{1}{(9\tan^2(\theta) + 9)^2} \left(\frac{3}{2}\sec^2(\theta) d\theta\right)$$

Set $u = a\,\text{trig}(\theta)$

$\Rightarrow 2x = 3\tan(\theta)$

$\Rightarrow x = \frac{3}{2}\tan(\theta)$

$dx = \frac{3}{2}\sec^2(\theta) d\theta$

$$= \frac{3}{2} \int \frac{\sec^2(\theta)}{81(\tan^2(\theta)+1)^2} d\theta$$

$$= \frac{3}{2} \int \frac{\sec^2(\theta)}{81(\sec^2(\theta))^2} d\theta = \frac{1}{54} \int \frac{\sec^2(\theta)}{\sec^4(\theta)} d\theta$$

$$= \frac{1}{54} \int \frac{1}{\sec^2(\theta)} d\theta$$

$$= \frac{1}{54} \int \cos^2(\theta) d\theta$$

Halfangle

$$= \frac{1}{54} \int \left(\frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{1}{108} \int (1 + \cos(2\theta)) d\theta$$

use
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$= \frac{1}{108}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$

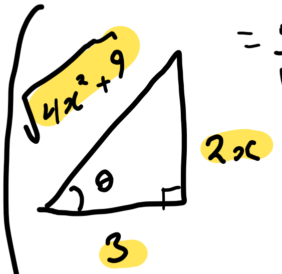
$\sin(\arctan(\frac{2x}{3}))$ X

$$= \frac{1}{108}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

original sub

$\tan(\theta) = \frac{2x}{3}$

$= \frac{opp}{hyp}$



$\theta = \arctan\left(\frac{2x}{3}\right)$

$$= \frac{1}{108}\left(\arctan\left(\frac{2x}{3}\right) + \frac{2x}{\sqrt{4x^2+9}} \cdot \frac{3}{\sqrt{4x^2+9}}\right) + C$$

1.7.4 Secant Substitutions

$$\int \frac{dx}{\sqrt{u^2-4}}$$

For terms of the form $u^2 - a^2$ you must have a restriction of $|u| > a$ for secant substitution to make any sense. If $u > a$ then $0 < \theta < \frac{\pi}{2}$ and if $u < -a$ then $\frac{\pi}{2} < \theta < \pi$.

Example: Compute $\int \frac{\sqrt{x^2-25}}{x} dx$, $x > 5$.

$u^2 - a^2 \iff x^2 - 25$

$\Rightarrow x^2 - 5^2$

$\Rightarrow u^2 - a^2$ where $u = x$ & $a = 5$.

$u = a\,\text{trig}(\theta)$

$\Rightarrow x = 5\sec(\theta)$

$dx = 5\sec(\theta)\tan(\theta)\,d\theta$

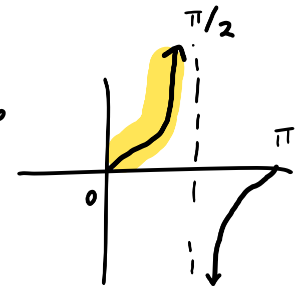
$$= \int \frac{\sqrt{25\sec^2(\theta) - 25}}{\cancel{5}\cancel{\sec}(\theta)} \cancel{5}\cancel{\sec}(\theta)\tan(\theta)\,d\theta$$

$$= \int \sqrt{25(\sec^2(\theta) - 1)}\ \tan(\theta)\,d\theta$$

$$= \int \sqrt{25\tan^2(\theta)}\ \tan(\theta)\,d\theta$$

$$= 5\int |\tan(\theta)|\ \tan(\theta)\,d\theta$$

$\tan(x) > 0$



$$= 5\int \tan^2(\theta)\,d\theta$$

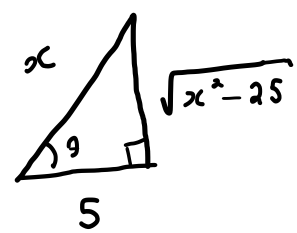
$$= 5\int (\sec^2(\theta) - 1)\,d\theta$$

$$= 5(\tan\theta - \theta) + C$$

$$= 5\left(\frac{\sqrt{x^2-25}}{5} - \text{arcsec}\left(\frac{x}{5}\right)\right) + C \ //$$

$\frac{x}{5} = \sec(\theta) \Rightarrow$

$= \frac{hyp}{adj}$



1.7.5 More Obscure u Terms in Trigonometric Substitution

Example: Compute $\displaystyle\int_{\ln(3)}^{\ln(3\sqrt{3})} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt.$

online

$u^2 + a^2$ $\qquad u = e^t \qquad a = 3$

$u = a\tan(\theta)$

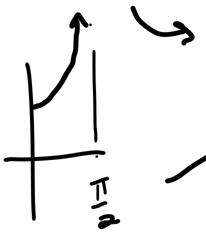
$\Rightarrow e^t = 3\tan(\theta)$

$e^t dt = 3\sec^2(\theta)\,d\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3\sec^2(\theta)\,d\theta}{\sqrt{9\tan^2(\theta)+9}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(\theta)\,d\theta}{\sqrt{\sec^2(\theta)}}$$

$\sec(\theta) > 0$



$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(\theta)}{|\sec(\theta)|}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta)+\tan(\theta)|\,\Big|_{\pi/4}^{\pi/3}$$

$$= \ln\left|\sec\left(\tfrac{\pi}{3}\right)+\tan\left(\tfrac{\pi}{3}\right)\right| - \ln\left|\sec\left(\tfrac{\pi}{4}\right)+\tan\left(\tfrac{\pi}{4}\right)\right|$$

$$= \ln|2+\sqrt{3}| - \ln|\sqrt{2}+1|$$

$$= \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right) \,//$$

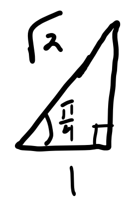
Bounds

- $t = \ln(3) \Rightarrow e^{\ln(3)} = 3\tan(\theta)$

  $\Rightarrow 3 = 3\tan(\theta)$

  $\Rightarrow \tan(\theta) = 1$

  $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow \theta = \pi/4$

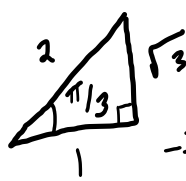


- $t = \ln(3\sqrt{3}) \Rightarrow e^{\ln(3\sqrt{3})} = 3\tan(\theta)$

  $\Rightarrow 3\sqrt{3} = 3\tan(\theta)$

  $\Rightarrow \sqrt{3} = \tan(\theta)$

  $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow \theta = \pi/3$



Example: Compute $\int \sqrt{x}\sqrt{1-x}\,dx$.

$u = \sqrt{x}$ and $a = 1$

$\Rightarrow a^2 - u^2 \iff 1^2 - (\sqrt{x})^2$

$= 1 - x$

So set $u = a\sin(\theta)$

$\Rightarrow \sqrt{x} = \sin(\theta)$

$\Rightarrow \frac{1}{2\sqrt{x}}\,dx = \cos(\theta)\,d\theta$

$\Rightarrow dx = 2\sqrt{x}\cos(\theta)\,d\theta$

$= 2\sin(\theta)\cos(\theta)\,d\theta$

$$= \int \sin(\theta)\sqrt{1-\sin^2(\theta)}\cdot 2\sin(\theta)\cos(\theta)\,d\theta$$

$$= 2\int \sin^2(\theta)\sqrt{\cos^2(\theta)}\cos(\theta)\,d\theta$$

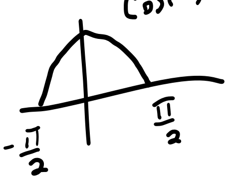
$$= 2\int \sin^2(\theta)\,|\cos(\theta)|\cos(\theta)\,d\theta$$

$$= 2\int \sin^2(\theta)\cos^2(\theta)\,d\theta$$

sine sub

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\cos(\theta) > 0$



we already computed this in 8.3 on page 46. copy and paste the procedure to obtain

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right) + C$$

← use double angle identity

$\sin(2A) = 2\sin(A)\cos(A)$

$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\cos(2\theta)\right) + C$$

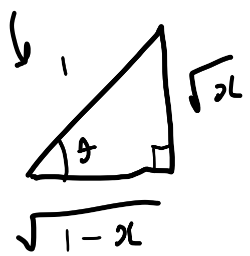
$$= \frac{1}{4}\left(\theta - \sin(\theta)\cos(\theta)\left(\cos^2(\theta) - \sin^2(\theta)\right)\right) + C$$

use

$\sin(2A) = 2\sin(\theta)\cos(\theta)$

$\cos(2A) = \cos^2(A) - \sin^2(A)$

$\sin(\theta) = \frac{\sqrt{x}}{1}$



$$= \frac{1}{4}\left(\arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x}\left((1-x) - x\right)\right) + C$$

$$= \frac{1}{4}\left(\arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x}\,(1-2x)\right) + C \quad //$$

unsimplifying

1.8 (Section 8.5) ~~Partial~~ Fractions

This is, quite bleakly stated, as the process of 'unsimplifying' a fraction to integrate it.

Example: Since $\dfrac{3x+11}{x^2-x-6} = \dfrac{4}{x-3} - \dfrac{1}{x+2}$ then

$(x-3)(x+2)$

$$\int \frac{3x+11}{x^2-x-6}dx = \int \frac{4}{x-3}dx - \int \frac{1}{x+2}dx = 4\ln|x-3| - \ln|x+2| + C$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, a rational function with $\deg(P(x)) < \deg(Q(x))$ where $P(x)$ and $Q(x)$ are polynomials you start by factoring $Q(x)$. We want to express, like above, this rational in the form

$$f(x) = \frac{p_1(x)}{q_1^{s_1}(x)} + \frac{p_2(x)}{q_2^{s_2}(x)} + \cdots + \frac{p_n(x)}{q_n^{s_n}(x)}$$

where $\deg(p_i(x))$ $=\deg(q_i(x)) - 1$ (this allows us to represent it in the form of a logarithmic integral). This form is called the **partial fraction decomposition** of $f(x)$. The forms we suggest in the decomposition depend on how $Q(x)$ factors.

1.8.1 Linear Terms Present

Suppose $Q(x)$ is factored, the following terms present means we suggest the following form.

$$\int \frac{3x+11}{(x-3)(x+2)}dx$$

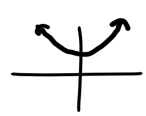
Term in $Q(x)$	Suggested Term(s) in Decomposition to Add
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$

Example: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x+4}{(2x+1)^2(x-1)}$.

Suggest $f(x) = \dfrac{A}{x-1} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$

$x^2 - x - 6 = (x-3)(x+2)$

x^2+1 can't be factored



Discriminant?

1.8.2 Irreducible Quadratic Terms Present

Definition

A quadratic of the form $ax^2 + bx + c$ is called **irreducible** if $b^2 - 4ac < 0$. That is, it cannot be factored into linear factors over the real numbers.

Suppose $Q(x)$ is factored, the following terms present means we suggest the following form.

Irreducible term in $Q(x)$	Suggested Term(s) in Decomposition to Add
$ax^2 + bx + c$	$\dfrac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\dfrac{A_1x + B_1}{ax^2 + bx + c} + \dfrac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \dfrac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Example: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x^2 + 4x}{(x^2 + 1)^2(x - 1)^2(x - 3)}$. //

$\underline{x}^2 + \underline{0}x + \underline{1}$

$b^2 - 4ac = 0^2 - 4(1)(1)$
$= -4 < 0$

Suggest

$$f(x) = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{E}{x - 1}$$
$$+ \frac{F}{(x - 1)^2} + \frac{G}{x - 3}$$

$\Rightarrow$ Page 63 $\Leftarrow$

Example: Construct the partial fraction decomposition $f(x) = \frac{x+1}{x^4+x^2}$.

$$f(x) = \frac{x+1}{x^2(x^2+1)} = \frac{x+1}{(x-0)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$a=1,\ b=0,\ c=1$

$b^2-4ac = 0-4<0$

want to find A, B, C, D.

multiply both sides by $Q(x) = x^2(x^2+1)$

$$\Rightarrow \quad x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)(x^2)$$

$$\Rightarrow \quad x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$= (A+C)x^3 + (B+D)x^2 + Ax + B$$

$0x^3 + 0x^2 + x + 1$

$$\Rightarrow \begin{cases} A+C=0 & (1) \\ B+D=0 & (2) \\ A=1 & (3) \\ B=1 & (4) \end{cases}$$

$A=1$

$(1) \Rightarrow C = -A = -1$

$(2) \overset{B=1}{\Rightarrow} D = -B = -1$

$A=1,\ B=1,\ C=-1,\ D=-1$

$$\frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{1}{x} + \frac{1}{x^2} + \frac{-x-1}{x^2+1}$$

$$= \frac{1}{x} + \frac{1}{x^2} - \frac{x}{x^2+1} - \frac{1}{x^2+1}$$

$$\int f(x)\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

Let $u = x^2+1$

$du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

$$= \ln|x| - \frac{1}{x} - \int \frac{x}{x^2+1}\,dx - \arctan(x) + C$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\int \frac{du}{u}\,du - \arctan(x) + C$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|u| - \arctan(x) + C$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$$

1.8.3 Special Case Where All Terms Are Linear

This is the **BEST** case scenario. You can solve for all the constants in the easiest manner possible. It's best to see this "trick" by example.

Example: Compute the partial fraction decomposition of $f(x) = \dfrac{3x^2+1}{(x-3)(x-4)(x-2)}$.

$$\frac{3x^2+1}{(x-3)(x-4)(x-2)} = \frac{A}{x-3} + \frac{B}{x-4} + \frac{C}{x-2}$$

$$\Rightarrow \quad 3x^2+1 = A(x-4)(x-2) + B(x-3)(x-2) + C(x-3)(x-4)$$

plug in $x=3$:

$$3\cdot 9 + 1 = A(-1)(1) + \emptyset + \emptyset \Rightarrow 28 = -A$$

$$\Rightarrow A = -28$$

plug in $x=4$:

$$3\cdot 16 + 1 = \emptyset + B(1)(+2) + \emptyset \Rightarrow 49 = +2B$$

$$\Rightarrow B = +\frac{49}{2}$$

plug in $x=2$:

$$3\cdot 4 + 1 = \emptyset + \emptyset + C(-1)(-2) \Rightarrow 13 = 2C$$

$$\Rightarrow C = \frac{13}{2}$$

So

$$f(x) = -\frac{28}{x-3} + \frac{49}{2}\cdot\frac{1}{x-4} + \frac{13}{2}\cdot\frac{1}{x-2}$$

$$\int f(x)\,dx = -28\ln|x-3| + \frac{49}{2}\ln|x-4| + \frac{13}{2}\ln|x-2| + C \;//$$

1.8.4 Example of Computation an Integral by Partial Fractions

Example: Compute $\int \frac{x^2 - x + 2}{x^3 - 1} dx$.

online

By a difference of cubes

$$x^3 - 1^3 = (x-1)(x^2+x+1)$$

Note x^2+x+1 is irreducible as $b^2 - 4ac = 1^2 - 4(1)(1) = -3 < 0$

So $\frac{x^2 - x + 2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$\Rightarrow x^2 - x + 2 = A(x^2+x+1) + (Bx+C)(x-1)$

If $x = 1 \Rightarrow 1 - 1 + 2 = A(1+1+1) + \phi$

$\Rightarrow 2 = 3A \Rightarrow A = 2/3$ //

$\Rightarrow 3x^2 - 3x + 6 = 2(x^2+x+1) + 3(Bx+C)(x-1)$

If $x = 0 \Rightarrow 0 \cdot 0 + 6 = 2(0+0+1) + 3(0+C)(-1)$

$\Rightarrow 6 = 2 - 3C \Rightarrow 4 = -3C$

$\Rightarrow C = -4/3$ //

$\Rightarrow 3x^2 - 3x + 6 = 2(x^2+x+1) + 3(Bx - \frac{4}{3})(x-1)$

Pick any $x \neq 0, 1$ now to get B, say $x = -2$

$\Rightarrow 3 \cdot 4 + 6 + 6 = 2(4 - 2 + 1) + 3(-2B - \frac{4}{3})(-3)$

$\Rightarrow 24 = 6 - 6B + 12$

$\Rightarrow 6 = -6B \Rightarrow B = -1$ //

So $\int f(x)\,dx = \int \left(\frac{2}{3} \cdot \frac{1}{x-1} + \frac{-x - 4/3}{x^2+x+1} \right) dx$

$= \frac{1}{3} \int \left(\frac{2}{x-1} - \frac{3x+4}{x^2+x+1} \right) dx$

Need to complete square

(Continued...)

$$x^2+x+1 = x^2+x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+1$$
$$= \left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}+1$$
$$= \left(x+\tfrac{1}{2}\right)^2+3/4$$

$x = u - 1/2$

$$= \frac{1}{3}\int\left(\frac{2}{x-1} - \frac{3x+4}{(x+\frac{1}{2})^2+3/4}\right)dx$$

$u = x + 1/2$

$du = dx$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{3}\int \frac{3(u-1/2)+4}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

$$= \frac{2}{3}\ln|x-1| - \int\frac{u}{u^2+\frac{3}{4}}\,du - \frac{7}{6}\int\frac{du}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$\int \frac{f'(x)}{f(x)}\,dx$

let

$v = u^2+\frac{3}{4}$

$\frac{1}{2}dv = u\,du$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{2}\int\frac{dv}{v} - \frac{7}{6}\cdot\frac{1}{\sqrt{3}/2}\arctan\left(\frac{2x}{\sqrt{3}}\right)$$

Basic integral

$\int\frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{2}\ln|v| - \frac{7}{3\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right)+C$$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{2}\ln|u^2+3/4| - \frac{7}{3\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right)+C$$

$$= \frac{2}{3}\ln|x-1| - \frac{1}{2}\ln\left(\left(x+\tfrac{1}{2}\right)^2+3/4\right) - \frac{7}{3\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right)+C \quad //$$

Note

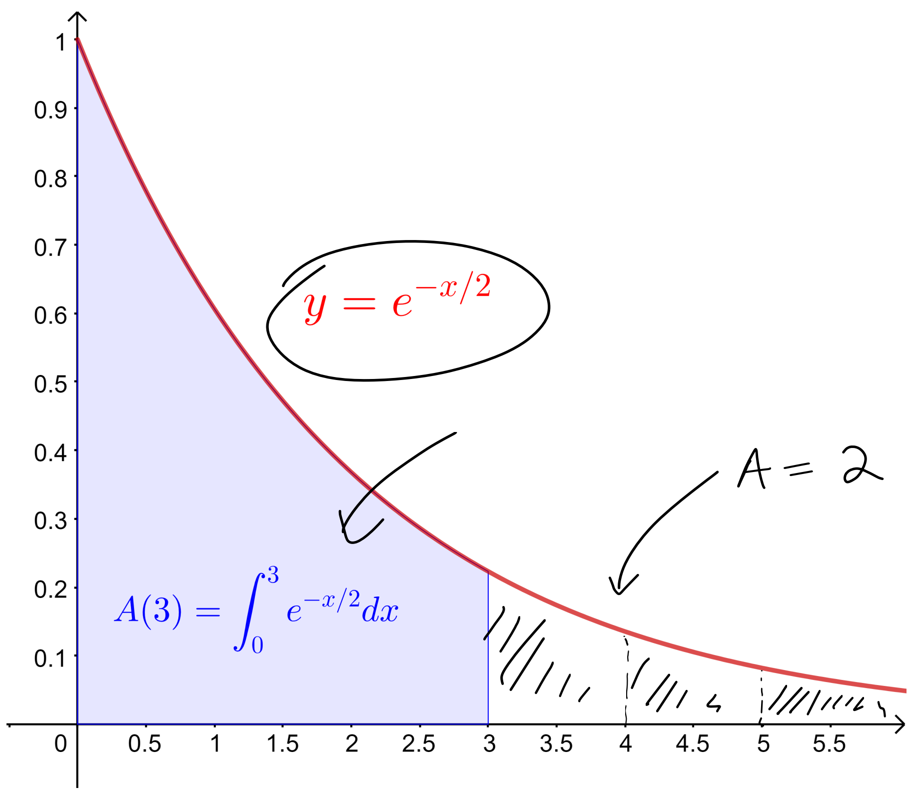
If the degree of the denominator is higher than the numerator, you must use long division first to re-express the integrand! Note also that some of the integrals which may be solved by partial fractions may also be solved by trigonometric substitution.

1.9 (Section 8.8) Improper Integrals

1.9.1 Defining Improper Integrals and Convergence

This section truthfully makes more sense upon the introduction of sequences, but we make do for the present time. These are integrals over a region of space that, in a sense, are infinite in size. You will find that some results might be counter intuitive at first, although the results are entirely based upon the way that things are defined.

Consider the function $y = e^{-x/2}$. As this function is entirely positive it makes sense to talk about the area between it and the x-axis over the region $[0, b]$ for some value b. This defines the area function $A(b) = \int_0^b e^{-x/2}dx$.



For each value of b the area is finite. Indeed, we may compute

$$A(b) = \int_0^b e^{-x/2}dx = -2e^{-x/2}\Big|_0^b = -2(e^{-b/2} - 1)$$

For each increased value of b we extend the area. It is possible that the area settles down to a finite value even though the region where we take it over is infinite. That is, by taking the limit as $b \to \infty$ we obtain

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} -2(e^{-b/2} - 1) = -2(0 - 1) = 2$$

$$e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

This is an illustration of an improper integral. There are two types of improper integrals, one where the region of integration is infinite and the other where there is a vertical asymptote in the region of integration.

Definition

An integral with infinite limits (i.e. integral over an infinite region) is called a **Type I Integral**. It is defined as:

1. $\displaystyle\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\left(\int_a^b f(x)dx\right)$
2. $\displaystyle\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\left(\int_a^b f(x)dx\right)$
3. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$, where c is any real number and the two integrals are defined as in the previous two points.

All these definitions assume that $f(x)$ is continuous over the region of integration.

Often we care little about the **value** of the integral. We often care about the behaviour of the integral. There are two options, either an integral results in concrete finite value or it does not.

Example: Determine whether the following integral $\displaystyle\int_1^{\infty}\frac{1}{x}dx$ is a finite value or not.

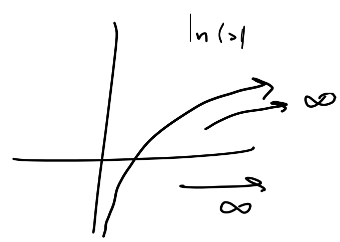
$$= \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}dx\right)$$

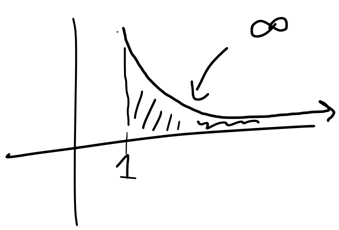
$$= \lim_{b\to\infty}\left(\ln|x|\Big|_1^b\right)$$

$$= \lim_{b\to\infty} \ln|b| - \ln|1|$$

$$= \infty - 0$$

$$= \infty$$





Not finite

Definition

is convergent

If an integral results in a finite value we say the integral **converges**. If an integral is not convergent we say it is **divergent**. The present tense verbal forms of these are that the integral **converges** or it **diverges**.

1.9.2 p-Integrals and the Integral Comparison Tests

Often we determine convergence or divergence of an integral by comparing it (in some manner) to a simpler and more well known integral that we understand the behaviour of. The simplest integrals we understand the convergent/divergent behaviour of are the p-Integrals.

Definition

A Type I **p-integral** is an integral of the form $\int_1^\infty \frac{dx}{x^p}$.

$\int_1^\infty \frac{dx}{\sqrt{x}}$ $p = 1/2$ divirge

convergent $\int_1^\infty \frac{dx}{x^\pi}$

Theorem

Consider the Type I p-integral. If $p > 1$ the p-Integral converges. If $p \leq 1$ the p-Integral diverges.

Proof: If $p = 1$ then by the previous example we saw that the integral diverges. If $p > 1$ then by the power rule we obtain the following

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p}\Big|_1^\infty = \frac{1}{1-p}\left(\lim_{b\to\infty} b^{1-p} - 1\right)$$

We can see that if $p > 1$ then the exponent of b^{1-p} is negative and thus

$$p < 1 \Rightarrow \lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$$

On the other hand, if $p < 1$ then the exponent of b^{1-p} is positive and thus

$$p > 1 \Rightarrow \lim_{b\to\infty} b^{1-p} = \infty$$

thus establishing the above theorem as desired ■.

Note

Many people casually call every integral of the form $\int_a^b \frac{dx}{x^p}$ a p-integral. There is no problem with this usually because the context is often clear. Usually, an integral of the form $\int_a^\infty \frac{dx}{x^p}$ where $a > 0$ is colloquially called a *Type I p-integral*. On the other hand, integrals of the form $\int_0^b \frac{dx}{x^p}$ where $b > 0$ is colloquially called a *Type II p-integral*.

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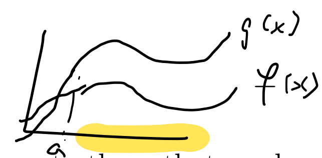


Below is the theorem we use to compare unknown integrals of Type I Improper to those that are known.

Direct Comparison Test

Let f and g be continuous functions on $[a,\infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then...

1. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges.
2. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

$$0 \leq \int_a^\infty f(x)\,dx \leq \underbrace{\int_a^\infty g(x)\,dx}_{=M}$$

Example: Determine whether or not the integral $\int_1^\infty \frac{dx}{1+x^4}$ converges or diverges.

Let $g(x) = \frac{1}{x^4}$ and $f(x) = \frac{1}{x^4+1}$. Note that $g(x), f(x) \geq 0$ on $x \geq 1$. Also

$$\frac{1}{x^4+1} \leq \frac{1}{x^4}$$

because the denominator is smaller on $x \geq 1$.

Since $\int_1^\infty \frac{dx}{x^4}$ is a p-integral with $p=4>1$ so it converges. Since this bounds our original integral above, by the DCT our original integral converges.

Example: Determine whether or not the integral $\int_6^\infty \frac{dx}{6\sqrt{x^2-25}}$ converges or diverges. $\sim \int_6^\infty \frac{dx}{\sqrt{x^2}} \sim \int_6^\infty \frac{dx}{6x} = \frac{1}{6}\int_6^\infty \frac{dx}{x}$

Let $f(x) = \frac{1}{6x}$ and $g(x) = \frac{1}{6\sqrt{x^2-25}}$ on $x \geq 6$. Note $f(x), g(x) \geq 0$ on $x \geq 6$. Note

$$\frac{1}{6x} = \frac{1}{6\sqrt{x^2-0}} \leq \frac{1}{6\sqrt{x^2-25}}$$

Because the denominator is smaller in $\frac{1}{6\sqrt{x^2-0}}$ on $x \geq 6$.

Since $\int_6^\infty \frac{dx}{6x}$ is a p-integral with $p=1 \leq 1$ it diverges.

As our original integral is bounded below by $\int_6 \frac{dx}{6x}$ then by DCT, it diverges.

We have one more test for Type I Integrals.

Limit Comparison Test

Let f and g be continuous and positive functions on $[a, \infty)$ with

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is positive and finite, i.e. $0 < L < \infty$. Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge together or diverge together.

Example: Determine whether or not the integral $\int_1^\infty \frac{1-e^{-x}}{x} dx$ converges or diverges.

$$\boxed{\frac{1-e^{-x}}{x} \overset{\text{large } x}{\sim} \frac{1-0}{x} \sim \frac{1}{x}}$$

Let $f(x) = \frac{1-e^{-x}}{x}$ and $g(x) = \frac{1}{x}$. These are continuous and positive on $x \geq 1$. Form

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1-e^{-x}}{x} \div \frac{1}{x}$$

$$= \lim_{x\to\infty} \frac{1-e^{-x}}{\cancel{x}} \times \cancel{x}$$

$$= \lim_{x\to\infty} 1 - e^{-x} = 1 - 0 = 1$$

Since $0 < L < \infty$ and $\int_1^\infty \frac{dx}{x}$ diverges because it is a p-integral with $p = 1 \leq 1$ then by LCT, $\int_1^\infty \frac{1-e^{-x}}{x} dx$ diverges as well ☒

1.9.3 Vertical Asymptotes

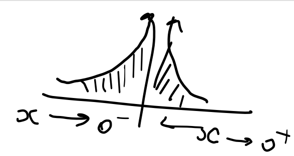
You can only invoke the Fundamental Theorem of Calculus if the function is continuous over the region of integration. Thus the Fundamental Theorem of Calculus does not apply to the integral

$$\int_{-1}^{1} \frac{dx}{x^2}$$

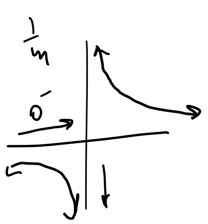
Finding an antiderivative and computing $F(x)\Big|_{-1}^{1}$ will give you the incorrect answer. This is because there is a vertical asymptote disrupting the continuity over this region of integration. We define such functions as below.

> Definition
>
> If the integrand is discontinuous (and has a vertical asymptote) in the region of integration we call it a **Type II Integral**. If $f(x)$ is continuous over $(a,c)\cup(c,b)$ (i.e. only has a vertical asymptote at $x=c$) then we define
>
> $$\int_a^b f(x)dx = \lim_{m\to c^-}\left(\int_a^m f(x)dx\right) + \lim_{n\to c^+}\left(\int_n^b f(x)dx\right)$$

Example: Determine whether or not the integral $\int_{-1}^{1}\frac{1}{x^2}dx$ converges or diverges.



$$= \lim_{m\to 0^-} \int_{-1}^{m} \frac{dx}{x^2} + \lim_{n\to 0^+} \int_n^1 \frac{dx}{x^2}$$



$$= \lim_{m\to 0^-} \left(-\frac{1}{x}\right)\Big|_{-1}^{m} + \lim_{n\to 0^+} \left(-\frac{1}{x}\right)\Big|_n^1$$

$$= \lim_{m\to 0^-} \left(-\frac{1}{m} - \left(-\frac{1}{-1}\right)\right) + \lim_{n\to 0^+} \left(-\frac{1}{1} - \left(-\frac{1}{n}\right)\right)$$

Note $\lim_{m\to 0^-} \frac{1}{m} = -\infty$ so as $\lim_{m\to 0^-}\int_{-1}^{m}\frac{dx}{x^2}$ diverges

then $\int_{-1}^{1}\frac{dx}{x^2}$ diverges ∎

Definition

A Type II **p-integral** is an integral of the form $\displaystyle\int_0^1 \frac{dx}{x^p}$.

Theorem

Consider the Type II p-integral. If $p \geq 1$ the p-integral diverges. If $p < 1$ the p-integral converges.

Note

This is almost opposite of the result for Type I p-integrals.

The Type II improper integrals has their own comparison theorem. We will omit the limit comparison version and just mention the direct comparison one to keep things simple.

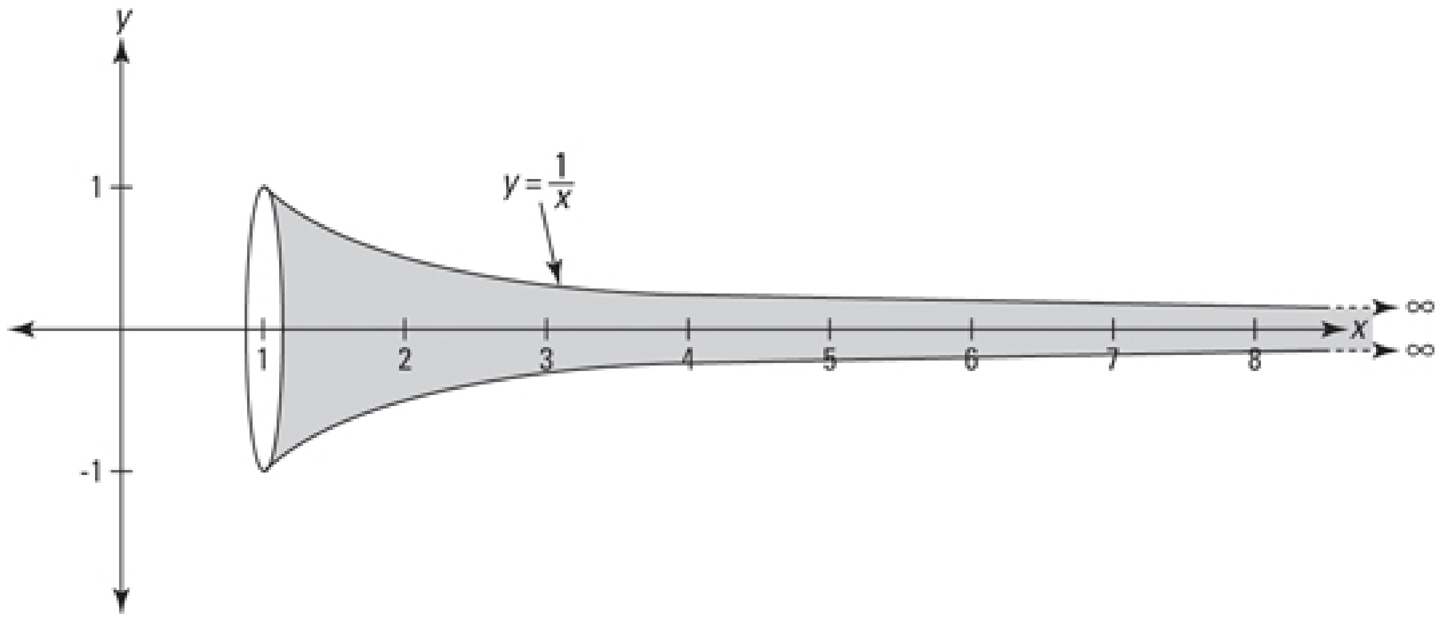
Direct Comparison Test for Type II Integrals

Let f and g be continuous functions on $I = (a, c) \cup (c, b)$ with $0 \leq f(x) \leq g(x)$ on I. Then...

1. $\displaystyle\int_a^b f(x)dx$ converges if $\displaystyle\int_a^b g(x)dx$ converges.
2. $\displaystyle\int_a^b g(x)dx$ diverges if $\displaystyle\int_a^b f(x)dx$ diverges.

1.9.4 Intuition is Lost in the Land of Infinity

Gabriel's horn is a vuvuzela like surface obtained by rotating the curve $f(x) = 1/x$ around the x-axis where $x \geq 1$.



It has a volume given by $V = \pi \int_1^\infty \frac{1}{x^2} dx$ and surface area $SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} dx$.

Example: Show that the surface area of Gabrielle's Horn is infinite, yet the volume is finite.

The volume is finite as $\pi \int_1^\infty \frac{dx}{x^2}$ is a p-integral with $p = 2 > 1$ and thus converges.

Next, Consider $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}$ which are both positive and continuous on $x \geq 1$.

Note

$$\frac{1}{x} \leq \frac{1}{x}\sqrt{1+0} \leq \frac{1}{x}\sqrt{1+\frac{1}{x^4}}$$

as $\frac{1}{x^4} \geq 0$ on $x \geq 1$

Since $\int_1^\infty \frac{1}{x} dx$ is a p-integral with $p = 1 \leq 1$, it diverges.

Since our original integral is bounded below by a divergent one then by DCT it diverges as well. Since $SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}} dx$ then the surface area is infinite (since the integrand is positive) as desired! ∎

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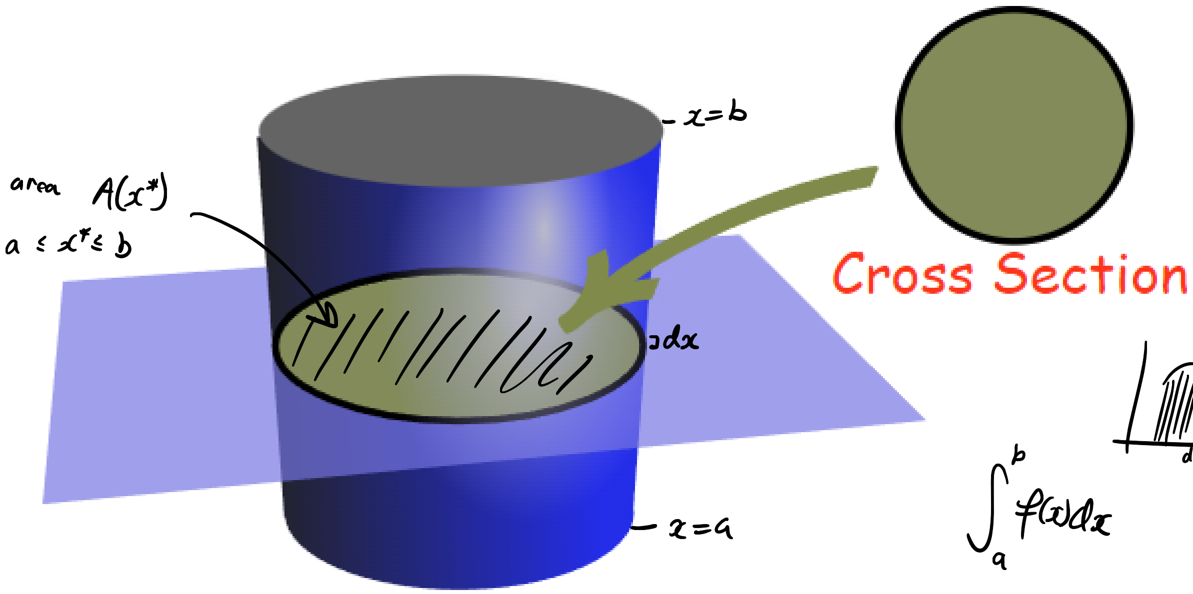
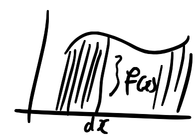
1.10 (Section 6.1) Volumes Using Cross Sections

1.10.1 Defining Volumes by Cross Sections

We approach computing the volume of a solid by use of **cross-sections**. Cross-sections are region formed by intersecting a solid with a plane.





$$\int_a^b f(x)dx$$

The idea is to find the area of each cross section and then add them up, through means of integration, to get the volume. Thus in this chapter you should not think of integration as area under a curve but rather the more physical interpretation of summing a quantity over a region.

Definition

small volume

The volume of a solid with cross sectional area $A(x)$ from $x = a$ to $x = b$ is $V = \int_a^b A(x)dx$

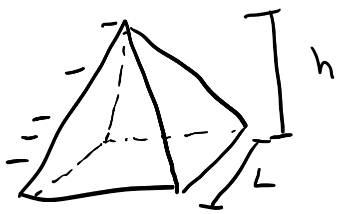
Procedure

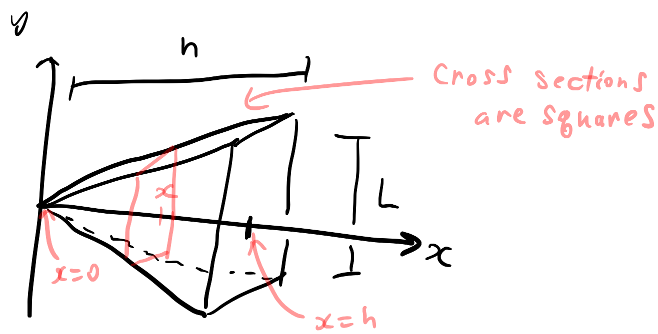
1. ⋆ **SKETCH THE SOLID AND A TYPICAL CROSS-SECTION** ⋆
2. Find a formula for $A(x)$
3. Find the limits of integration
4. Compute $V = \displaystyle\int_a^b A(x)dx$

Cavalieri's Principle

If two solids have the same height and cross-sectional area at every point along that height, then they have the same volume.

Example: Find the volume of a pyramid of height h whose base is square with sides of length L.

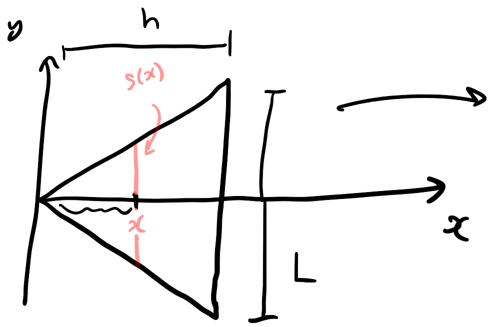


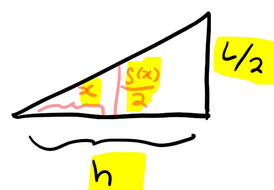


so

$A(x) = (s(x))^2$ where $s(x)$ is the side length of the cross section at x

pick a good angle to look from





By similar triangles

$$\frac{L/2}{h} = \frac{s(x)/2}{x}$$

$$\Rightarrow s(x) = \frac{L}{\cancel{2}h}(\cancel{2}x) = \frac{L}{h}x$$

Form $A(x) = (s(x))^2 = \left(\frac{L}{h}x\right)^2$

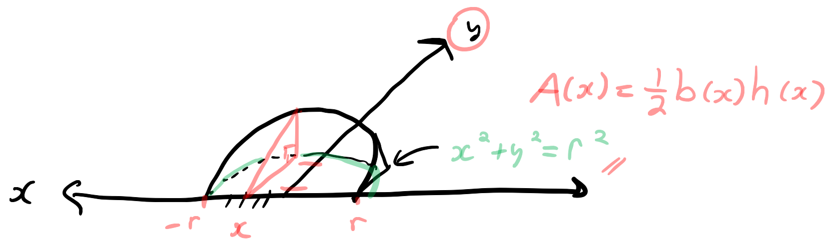
$$= \frac{L^2}{h^2}x^2 \quad \text{in } 0 \leq x \leq h$$

so $V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx$

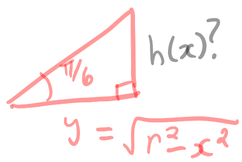
$$= \frac{L^2}{h^2} \cdot \frac{1}{3}x^3 \Big|_0^h$$

$$= \frac{L^2}{h^2} \cdot \frac{1}{3}(h^3 - 0^3) = \frac{L^2 h^3}{3h^2} = \frac{1}{3}L^2 h$$

Example: Find the volume of a cheese wedge cut from a circular cylinder of radius r if the angle between the top and bottom is $\pi/6$ radians.

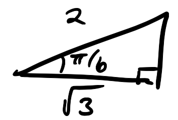


Zoom in at a good angle



$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$

$$\Rightarrow \tan(\pi/6) = \frac{h(x)}{\sqrt{r^2 - x^2}}$$



$$\Rightarrow h(x) = \sqrt{r^2 - x^2} \cdot \frac{1}{\sqrt{3}}$$

Now form $A(x) = \frac{1}{2} b(x) h(x)$

$$= \frac{1}{2}\sqrt{r^2 - x^2}\left(\frac{1}{\sqrt{3}}\sqrt{r^2 - x^2}\right) = \frac{1}{2\sqrt{3}}(r^2 - x^2)$$

So $V = \int_{-r}^{r} A(x)\,dx = \frac{1}{2\sqrt{3}}\int_{-r}^{r}(r^2 - x^2)\,dx$

$$= \frac{1}{2\sqrt{3}}\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{-r}^{r}$$

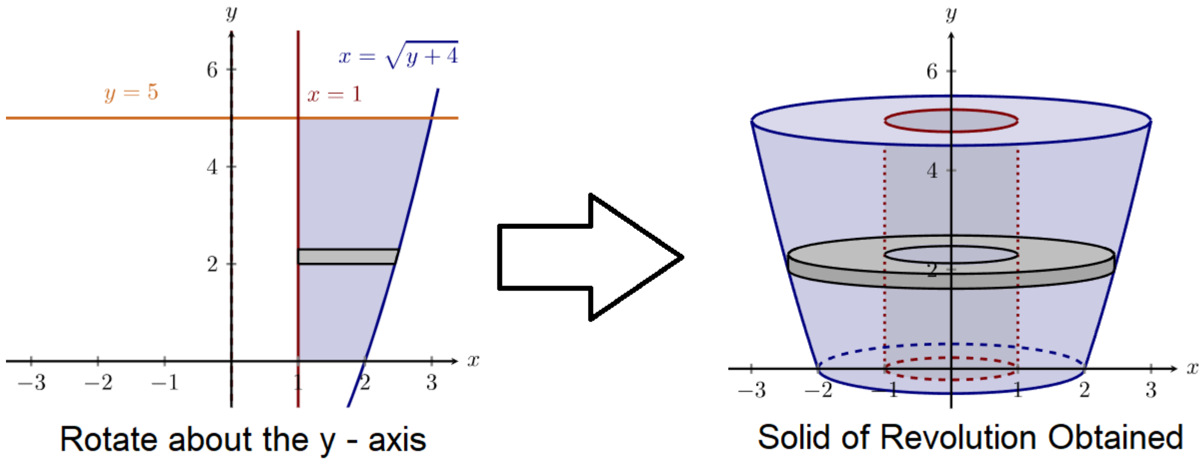
$$= \frac{1}{2\sqrt{3}}\left\{\left(r^2(r) - \frac{1}{3}r^3\right) - \left(r^2(-r) - \frac{1}{3}(-r)^3\right)\right\}$$

$$= \frac{1}{2\sqrt{3}}\left\{\underbrace{r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3}_{= 2r^3 - \frac{2}{3}r^3 = \frac{4r^3}{3}}\right\}$$

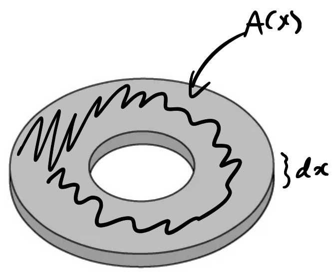
$$= \frac{2r^3}{3\sqrt{3}}$$

1.10.2 Solids of Revolution and their Volume by Washers

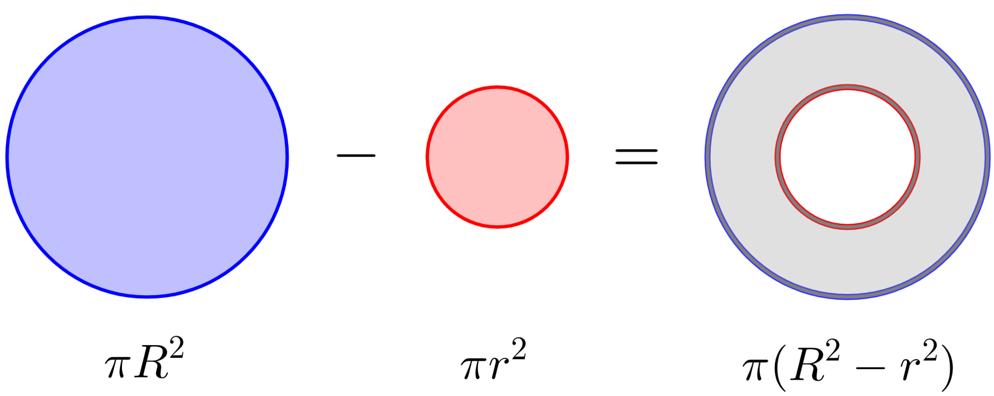
A solid of revolution is a solid obtained by rotating a curve around a line parallel (or equal) to an axis.



One may notice that the cross-sections are “so-called” **washers** (flat donut shapes).



To find the area of a washer we use the fact that it’s area is the difference of two circles.



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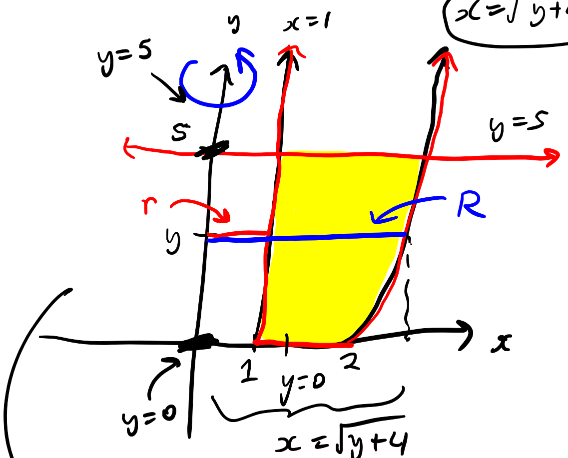
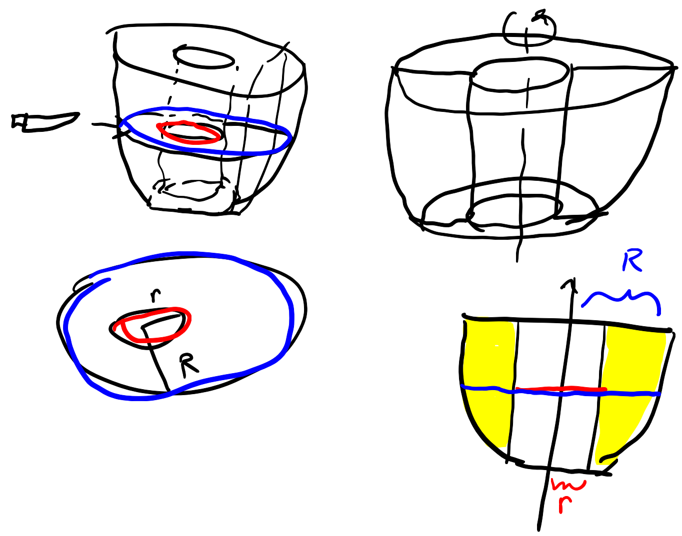
Example: ~~Continue with the example above~~. Consider the region bounded by $x = \sqrt{y+4}$, $x = 1$, $y = 0$ and $y = 5$ and form the solid of revolution by revolving the region about the y-axis. ~~Construct the cross sectional area $A(y)$ at each moment $0 \le y \le 5$ and use this to~~ construct the volume of the solid.

$x = \sqrt{y+4} \Rightarrow x^2 = y + 4 \Rightarrow y = x^2 - 4$





By the picture $r = 1$

and $R = \sqrt{y+4}$

Thus

$$V = \int_0^5 (\pi R^2 - \pi r^2)\,dy = \int_0^5 \pi\left((\sqrt{y+4})^2 - (1)^2\right)dy$$

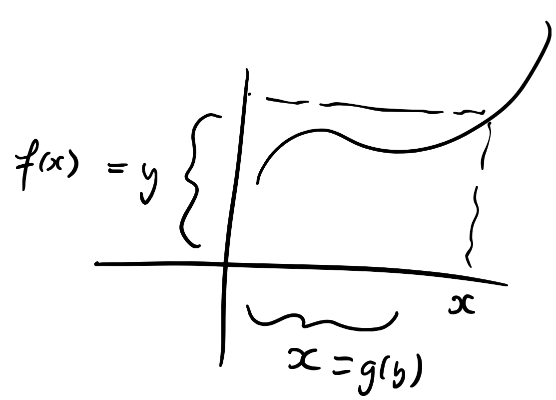
$$= \pi \int_0^5 (y + 4 - 1)\,dy$$

$$= \pi \int_0^5 (y+3)\,dy$$

$$= \pi\left(\tfrac{1}{2}y^2 + 3y\right)\Big|_0^5$$

$$= \pi\left(\frac{25}{2} + 15 - 0 - 0\right)$$

$$= \frac{55\pi}{2} \text{ units}^3$$



The Washer Method for Solids of Revolution

Let R be a region in the xy-plane that is revolved about an axis. If it is revolved around a line of the form $y = C$ (including the x-axis) then your volume is given by

$$V = \pi \int_a^b (R^2 - r^2)dx$$

where $A(x) = R(x)^2 - r(x)^2$ represents the cross sectional area as the difference of the bigger and smaller radii over the region $[a, b]$. If it is revolved around a line of the form $x = C$ (including the y-axis) then your volume is given by

$$V = \pi \int_c^d (R^2 - r^2)dy$$

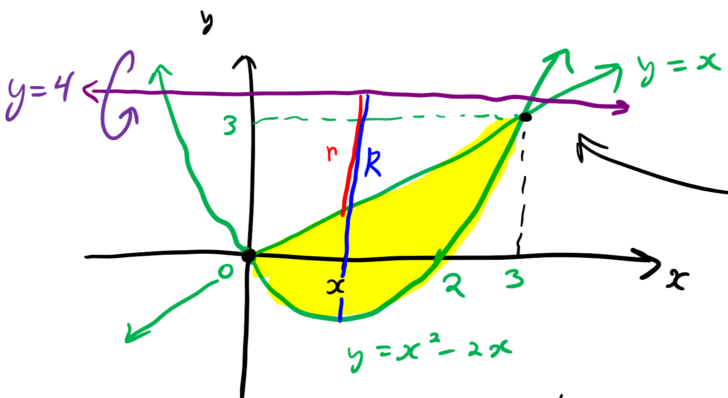
where $A(y) = R(y)^2 - r(y)^2$ represents the cross sectional area as the difference of the bigger and smaller radii over the region $[c, d]$.

In either case, if $r = 0$ we call this **The Disk Method** instead.

Note

The picture is what tells you how to set up your R and r. There is no formula for constructing it and is situation specific depending on your picture. Don't try asking for one because if anyone has a specific formula for you that covers "all cases", I can construct a counterexample where it fails.

Example: Find the volume of the solid of revolution formed by the region bounded by $y = x^2 - 2x$ and $y = x$ and revolving it about the line $y = 4$.

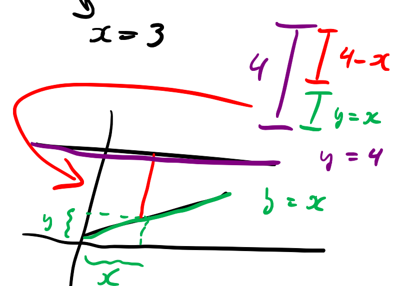


$y = x^2 - 2x$
$= x(x-2)$

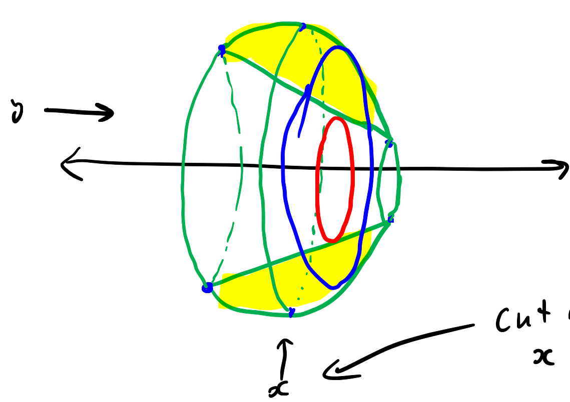
$y = x$

$x^2 - 2x = x$
$\Rightarrow x^2 - 3x = 0$
$\Rightarrow x(x-3) = 0$

$x = 0$ $\quad$ $x = 3$

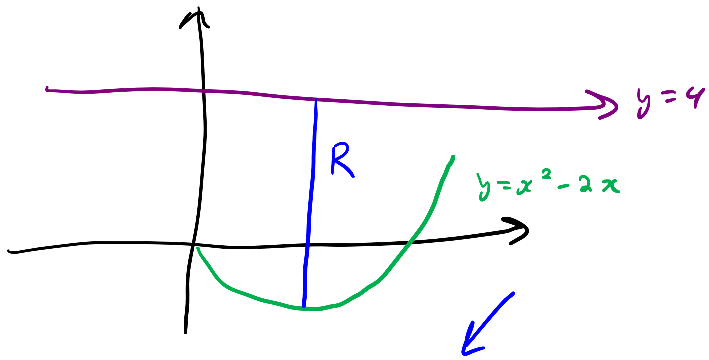


So $r = 4 - x$



Cut across
x to make washers

(Continued...)



4 $R = 4-(x^2-2x)$ $y = x^2 - 2x$

So $R = 4 - x^2 + 2x$

Thus

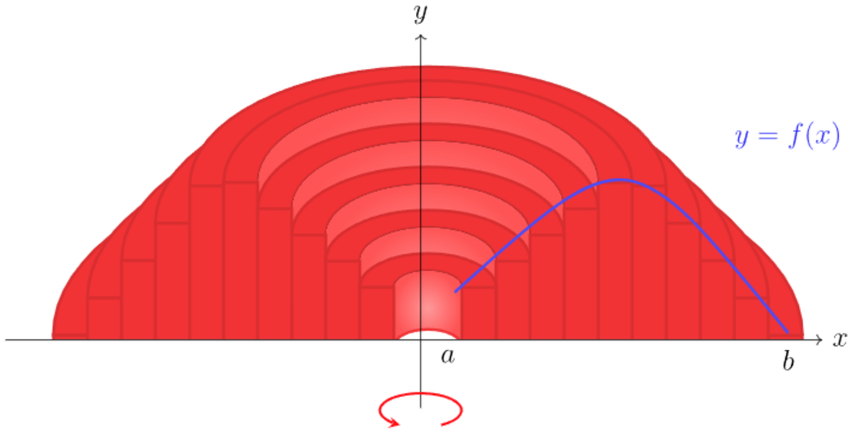
$\pi R^2 - \pi r^2$

$$V = \int_0^3 \pi\left((4-x^2+2x)^2 - (4-x)^2\right)dx$$

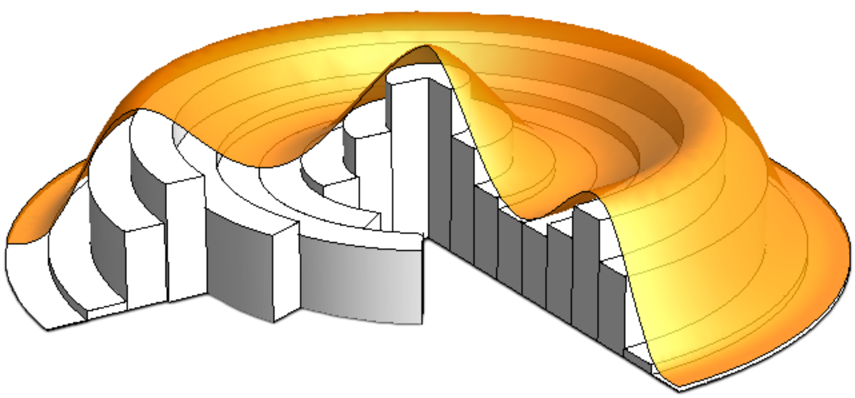
$$= \underline{\underline{\text{Maff}}}$$

1.11 (Section 6.2) The Shell Method

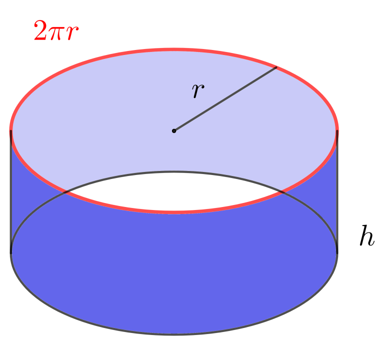
This section covers an alternate method to compute the volume of a solid of revolution. This is the method by looking at a solid more like a matryoshka (Russian) doll. For these, the idea is to find the surface area of a "shell" and add them up. If you want to think of the washer method as deconstructing a cake into its pancake like layers, then the shell method is like deconstructing an onion into its spherical like layers (as opposed to rings).



In this formulation of the solid, we see that the three-dimensional object is layered like a Russian doll where each layer is a tube-like cylinder.



If we imagine that the radius of a cylinder is r and the height is h, then the surface are of this object is $A = 2\pi rh$.



Surface Area $= 2\pi rh$

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Once again you integrate to add them up, obtaining the volume.

The Shell Method for Solids of Revolution

Let R be a region in the xy-plane that is revolved about an axis. If it is revolved around a line of the form $y = C$ (including the x-axis) then your volume is given by

$$V = 2\pi \int_c^d rh dy$$

where $A(y) = 2\pi r(y)h(y)$ represents the surface area of a typical shell where the domain of the radius is $[c, d]$. If it is revolved around a line of the form $x = C$ (including the y-axis) then your volume is given by

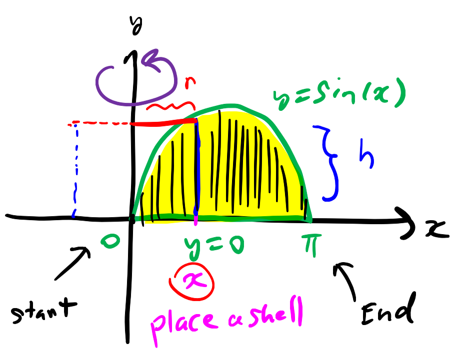
$$V = 2\pi \int_a^b rh dx$$

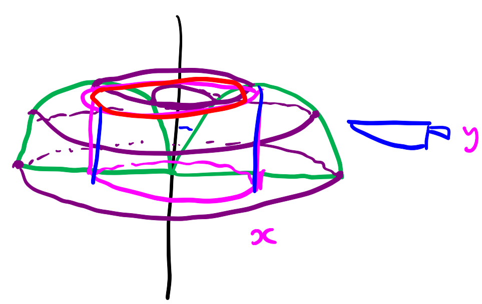
where $A(x) = 2\pi r(x)h(x)$ represents the surface area of a typical shell where the domain of the radius is $[a, b]$.

Note

The variable you integrate will be opposite of that for washers! This becomes very evident when drawing a picture!

Example: Consider the region bounded by the curve $y = \sin(x)$ and the x-axis over $[0, \pi]$. Find the volume of the solid of revolution obtained by revolving this region about the y-axis.





From the picture $r = x$

Also $h = \sin(x) - 0 = \sin(x)$

$$V = \int_0^\pi 2\pi r h \, dx = \int_0^\pi 2\pi x \sin(x) \, dx$$

$$= 2\pi \int_0^\pi \underset{P}{x} \underset{T}{\sin(x)} dx$$ ← use IBP

LIPET

$u = x$ $du = dx$

$dv = \sin(x)dx$ $v = -\cos(x)$

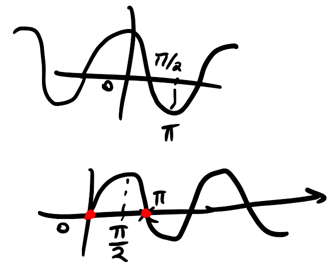
$$= 2\pi \left\{ -x\cos(x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos(x))\,dx \right\}$$

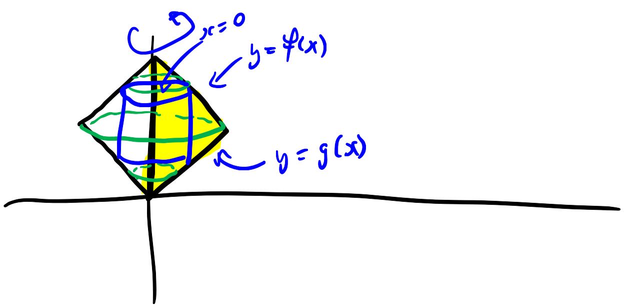
(Continued...)

$$= 2\pi \left\{ -\pi\cos(\pi) - 0 + \sin(x)\Big|_0^{\pi} \right\}$$

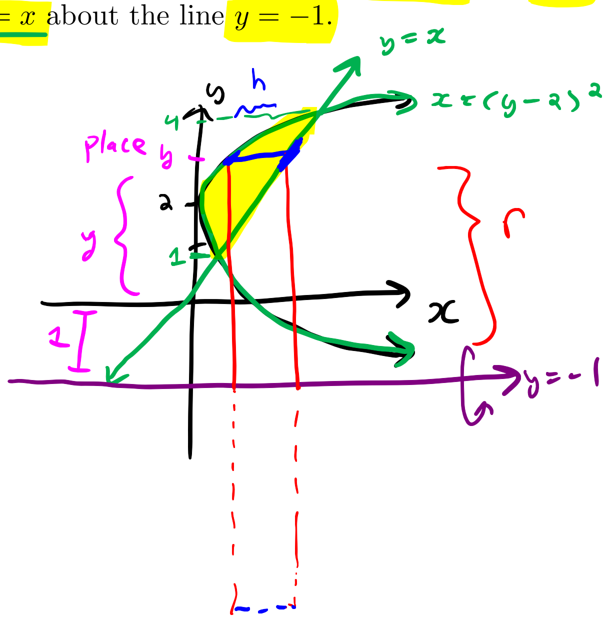
$$= \underline{2\pi} \left\{ -\underline{\pi(-1)} + (\cancel{\sin(\pi)}^{0} - \cancel{\sin(0)}^{0}) \right\}$$

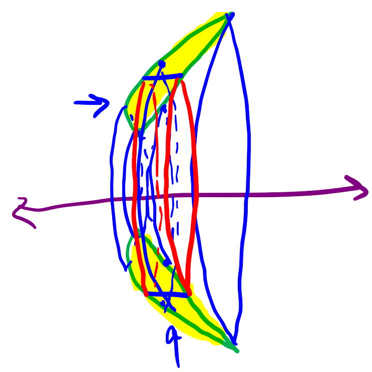
$$= 2\pi^2 \text{ units}^3$$





Example: Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.





The height is $h = \overset{\text{line}}{y} - \overset{\text{parabola}}{(y-2)^2}$

horizontal distance

The radius is $r = y+1$

Bounds are given by intersection

$(y-2)^2 = y \Rightarrow y^2 - 4y + 4 = y \Rightarrow y^2 - 5y + 4 = 0$

$\Rightarrow (y-4)(y-1) = 0$

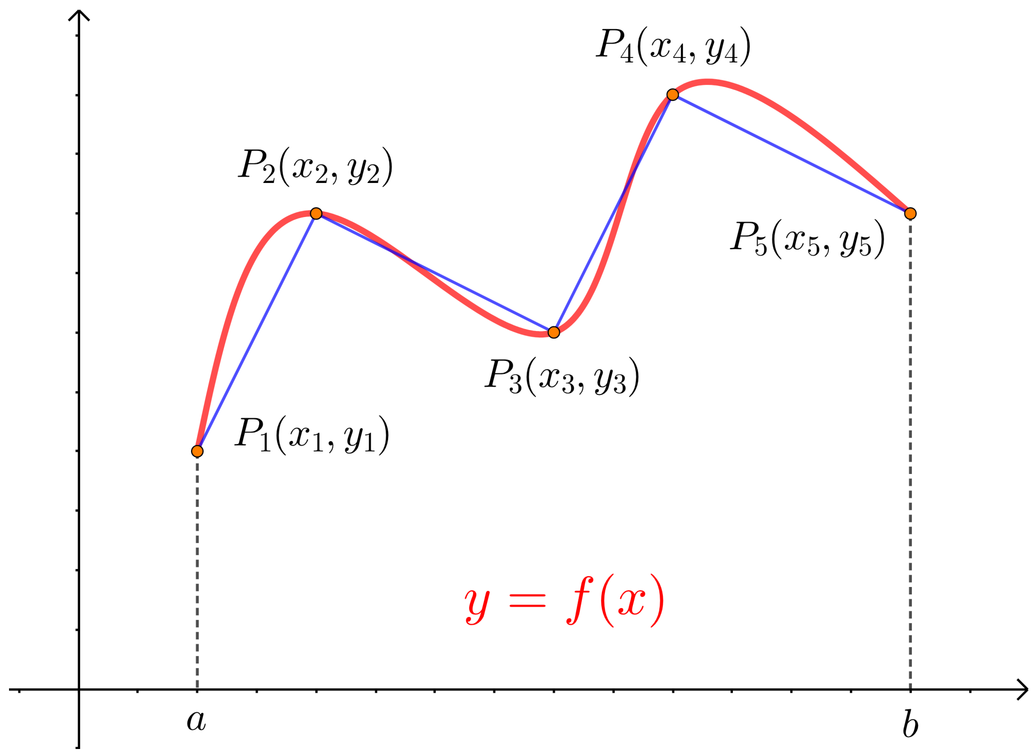
(Continued...)

so $V = 2\pi \int_1^4 \overbrace{(y+1)}^{r} \left(\overbrace{y - (y-2)^2}^{h} \right) dy$

1.12 (Section 6.3) Arc-Length

1.12.1 Formulating the Arc-Length

The purpose of this section is to both construct a good definition for the length of a curve and give some examples. We start with a curve $y = f(x)$ and consider approximating the length by computing the lengths of the segments through a series of consecutive points $P_1, ..., P_n$ as in the image below.



If we let $|P_{i-1}P_i|$ represent the length of the line segment connecting point $P_{i-1}(x_{i-1}, y_{i-1})$ to point $P_i(x_i, y_i)$ then we form an approximation given by

$$\text{Length} \approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

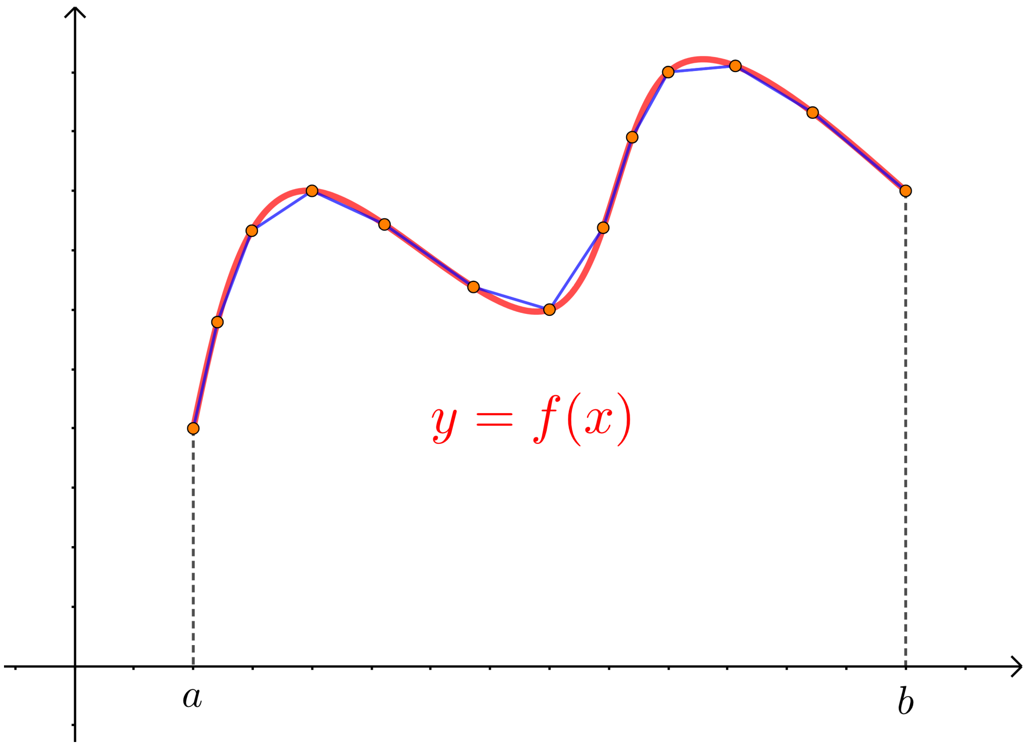
For the sake of argument, lets assume the points are evenly spaced so that $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$. Then, we obtain the approximation

$$\text{Length} \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\,\Delta x$$

whereby we use the **Mean Value Theorem** (MVT) to say that the secant slopes can be represented in terms of the derivative for an intermediary value in the interval (x_i, x_{i-1}). That is, $(f(x_i) - f(x_{i-1}))/(x_i - x_{i-1}) = f'(x_i^*)$ for some point x_i^* in (x_i, x_{i-1}). Thus we obtain

$$\text{Length} \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\,\Delta x$$

Now, as we take the limit as $n \to \infty$ our approximation gets better.



But as we take this limit we have, by definition, a formulation of the definite integral.

$$\text{Length} \overset{DEF}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x = \int_a^b \sqrt{1 + f'(x)^2}dx$$

and similar formulations may be made for curves expressed in the form of $x = g(y)$.

Definition

Let $y = f(x)$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). The **arc-length** of the graph of its representing curve over $[a, b]$ is

$$L = \int_a^b \underbrace{\sqrt{1 + f'(x)^2}dx}_{ds}$$

Similarly, let $x = g(y)$ be a function that is continuous on $[c, d]$ and differentiable on (c, d). The **arc-length** of the graph of its representing curve over $[c, d]$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}dy$$

In either case, we commonly abbreviate an Arc-Length integral as $\displaystyle\int ds$.

1.12.2 Computation of Arc-Length

Example: Find the length of $y = \frac{x^2}{2} - \frac{\ln(x)}{4}$ over [1, 3].

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$f(x) = \frac{x^2}{2} - \frac{\ln(x)}{4}$$

$$\Rightarrow f'(x) = \frac{2x}{2} - \frac{1}{4x} = x - \frac{1}{4x}$$

$$= \int_1^3 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{1+x^2 - 2(\cancel{x})\left(\frac{1}{4\cancel{x}}\right) + \frac{1}{16x^2}}\,dx = \int_1^3 \sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}\,dx$$

$(a-b)^2 = a^2 - 2ab + b^2$ //

$(a+b)^2 = a^2 + 2ab + b^2$ //

$$= \int_1^3 \sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}\,dx$$

$$= \int_1^3 \sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx$$

$4 + \frac{1}{4}\ln(3)$

$$= \int_1^3 \left|x+\frac{1}{4x}\right|dx = \int_1^3 \left(x+\frac{1}{4x}\right)dx = \frac{1}{2}x^2 + \frac{1}{4}\ln|x|\Big|_1^3 \quad //$$

$$= \frac{1}{2}\cdot 9 + \frac{1}{4}\ln(3) - \frac{1}{2}(1)^2 - \frac{1}{4}\emptyset$$

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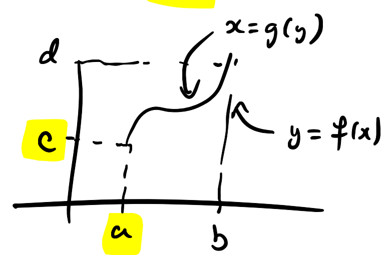
Example: Show that the arc-length of the function $y = \left(\frac{3x}{2}\right)^{2/3} + 1$ over $0 \le x \le \frac{2}{3}3^{3/2}$ is much more difficult to compute in terms of x instead of in terms of y. Compute the length.

$$f(x) = \left(\frac{3x}{2}\right)^{2/3}+1 \Rightarrow f'(x) = \frac{2}{3}\left(\frac{3x}{2}\right)^{-1/3}\frac{d}{dx}\left[\frac{3x}{2}\right] + \emptyset$$

$$= \frac{2}{3}\left(\frac{3x}{2}\right)^{-1/3}\cdot(3/2) = \left(\frac{3x}{2}\right)^{-1/3}$$

So $L = \int_0^{\frac{2}{3}\cdot 3^{3/2}} \sqrt{1+\left(\frac{3x}{2}\right)^{-2/3}}\,dx$

Hard!! Try switching variables?



$$y = \left(\frac{3x}{2}\right)^{2/3}+1 \Rightarrow y-1 = \left(\frac{3x}{2}\right)^{2/3} \Rightarrow (y-1)^{3/2} = \frac{3x}{2} \Rightarrow \frac{2}{3}(y-1)^{3/2} = x$$

Bounds (not a u-sub)

$x = 0 \Rightarrow y = \left(\frac{3(0)}{2}\right)^{2/3}+1 = 1$

$x = \frac{2}{3}\cdot 3^{2/3} \Rightarrow y = \left(\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot 3^{3/2}\right)^{2/3}+1$

$= 3 + 1 = 4$

$$\Rightarrow L = \int_1^4 \sqrt{1+g'(y)^2}\,dy \qquad g'(y) = \frac{\cancel{2}}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{2}}\right)(y-1)^{1/2}(1-0) = \sqrt{y-1}$$

$$= \int_1^4 \sqrt{1+(y-1)}\,dy$$

$$= \int_1^4 \sqrt{y}\,dy = \frac{2}{3}y^{3/2}\Big|_1^4$$

$$= \frac{2}{3}\left\{ \underbrace{4^{3/2} - 1^{3/2}}_{=8-1} \right\}$$

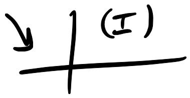
$$= \frac{14}{3}$$

Example: Determine the length of $y = \ln(\sec(x))$ over the interval $[0, \pi/4]$

$$f'(x) = \frac{1}{\cancel{\sec(x)}} \cancel{\sec(x)}\tan(x) = \tan(x)$$

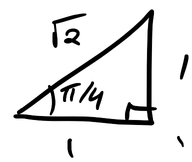
$$L = \int_0^{\pi/4} \sqrt{1+\tan^2(x)}\, dx = \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx$$

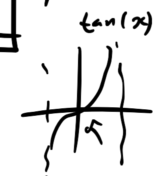
$\sec(x) > 0$



$$= \int_0^{\pi/4} |\sec(x)|\, dx$$

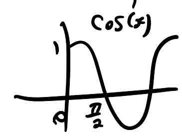
$$= \int_0^{\pi/4} \sec(x)\, dx$$





$$= \ln|\sec(x)+\tan(x)|\Big|_0^{\pi/4}$$

$$= \ln|\sqrt{2}+1| - \ln|\underbrace{\sec(0)}_{=\frac{1}{1}} + \underbrace{\tan(0)}_{=0}|$$



$$= \ln(\sqrt{2}+1) - \overset{0}{\cancel{\ln}(1)}$$

$$= \ln(\sqrt{2}+1)$$

/omit

1.12.3 The Surface Area of a Solid of Revolution

Definition

Consider the solid of revolution obtained by revolving the function $y = f(x)$ over the interval $[a, b]$ about the x-axis. We define the surface area as

$$\text{SA} = \int_a^b 2\pi f(x) ds = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} dx$$

Similarly, if we revolved a function $x = g(y)$ over the interval $[c, d]$ about the y-axis then we define the surface area of the resulting solid of revolution to be

$$\text{SA} = \int_c^d 2\pi g(y) ds = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2} dy$$

Example: Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$ in $[-2, 2]$ about the x-axis.