1 Big-Oh Analysis

Based on the definitions of Big-Oh prove the following.

a) $5n^2 + 6n + 12$ is $O(n^3)$

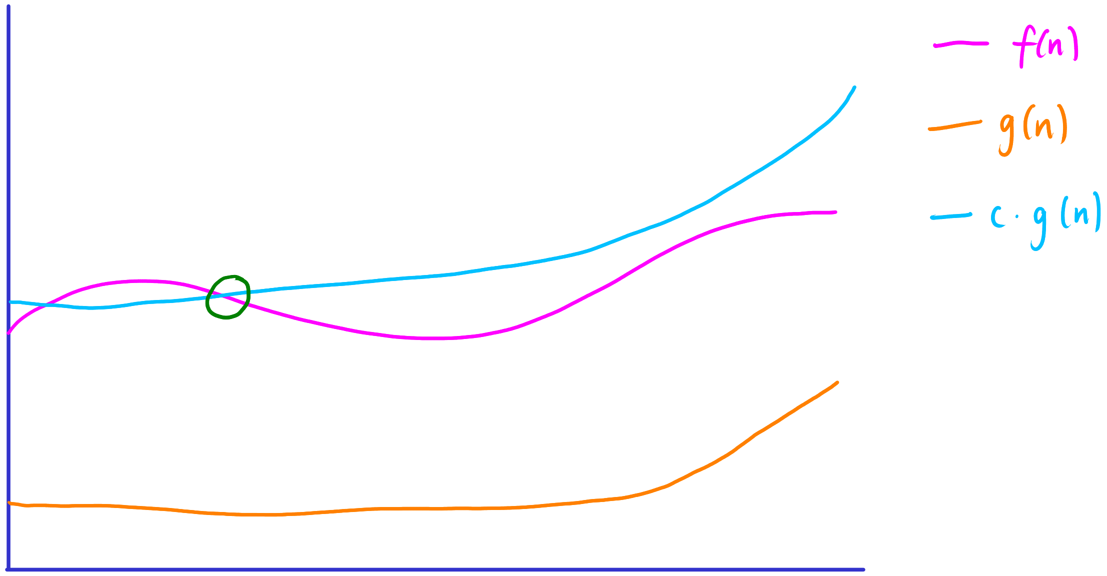
b) If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$, then $d(n)$ is $O(g(n))$

c) $\sum_{i=1}^{n} i^2$ is $O(n^3)$

We can prove $f(n)$ is $O(g(n))$ if we can find some values for c, n_0 such that

$$f(n) \leq c \cdot g(n) \quad \forall \ n \geq n_0 .$$

Visually, it looks like this:



We want to find the point where $g(n)$ overtakes $f(n)$, that is, where the line for $g(n)$ crosses the line for $f(n)$ and then they never cross again.

There should (eventually) exist such a point, but it might be far to the right or off the graph, or just difficult to compute.

disclaimer: multiply $g(n)$ by c actually stretches it vertically!!

To make our lives easier, we can "raise" $g(n)$ by multiplying it by some constant c, then we get the line $c \cdot g(n)$, which, for a good value of c, will overtake $f(n)$ at a much easier-to-compute spot. (Circled in green on the graph.)

a) here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^3$.

We want to compare $f(n)$ to some $c \cdot g(n)$, and the easiest way to do that is to turn all terms into n^3.

$$f(n) = 5n^2 + 6n + 12 \leq 5n^3 + 6n^3 + 12n^3 = 23n^3$$

here is our value for c

Now we just check when $23n^3$ overtakes $5n^2+6n+12$.
We actually just proved the inequality, so it must be that

$$23n^3 \geq 5n^2+6n+12 \quad \text{for any positive value of } n.$$

Thus, we can simply set $n_0 = 1$.

Final answer: $5n^2+6n+12 \leq 23n^3 \quad \forall\, n \geq 1$ (ie. $c = 23$, $n_0 = 1$)

Therefore $5n^2+6n+12$ is $O(n^3)$.

b) If $d(n)$ is $O(f(n))$, then $\exists\ c_1, n_1$ s.t.

$$d(n) \leq c_1 f(n) \quad \forall\ n \geq n_1$$

If $f(n)$ is $O(g(n))$, then $\exists\ c_2, n_2$ s.t.

$$f(n) \leq c_2 \cdot g(n) \quad \forall\ n \geq n_2$$

Thus,

$$d(n) \leq c_1 \cdot f(n) \leq c_1 c_2 \cdot g(n) \qquad \forall\ n \geq \max\{n_1, n_2\}$$

replaced

$\therefore$ $d(n)$ is also $O(g(n))$ (ie. $c = c_1 c_2$ and $n_0 = \max\{n_1, n_2\}$)

c) $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{(n^2+n)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\leq \frac{2n^3+3n^3+n^3}{6} = \frac{6n^3}{6} = n^3.$$

$\therefore$ $\sum_{i=1}^{n} i^2$ is $O(n^3)$ (ie. $c = 1$, $n_0 = 1$)

2 Big-Omega and Big-Theta Analysis

Prove the following:

a) $n^3 \log n$ is $\Omega(n^3)$

b) $5n^2 + 6n + 12$ is $\Theta(n^2)$

a) If $n^3 \log n$ grows faster than n^3, we would expect $\lim_{n\to\infty} \frac{n^3 \log n}{n^3} = \infty$ (or at least, > 0).

$$\lim_{n\to\infty} \frac{n^3 \log n}{n^3} = \lim_{n\to\infty} \log n = \infty > 0.$$

$\therefore$ $n^3 \log n$ is $\Omega(n^3)$.

b) If $f(n)$ and $g(n)$ grow at the same rate, we would expect $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ where $0 < c < \infty$.

$$\lim_{n\to\infty} \frac{5n^2 + 6n + 12}{n^2} = \lim_{n\to\infty} \left(5 + \frac{6}{n} + \frac{12}{n^2}\right) = 5.$$

(note $0 < 5 < \infty$).

$\therefore$ $5n^2 + 6n + 12$ is $\Theta(n^2)$.

3 Algorithm

An array A contains $n-1$ unique integers in the range $[0, n-1]$; that is, there is one number from this range not in A. Design an $O(n)$-time algorithm for finding the missing number that uses $O(1)$ extra space, i.e. you cannot make a copy of A, which would take $O(n)$ extra space.

$O(1)$ space, $O(n)$ time

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Algorithm missingNumber(A,n):
   Input: Array A containing n−1 unique integers in range
   [0,n−1]
   Output: Integer x ∈ [0,n−1] such that x is not in A.

   sum ← (n−1) ∗ n/2
   arraysum ← 0
   for i ← 0 to n−1 do
      arraysum ← arraysum + A[i]
   end
   return sum - arraysum
end
```