Math 110 (A01, A02, A03)
Test 2
Version: A
November 15, 2019
Time: 45 minutes

Student ID: V00 ____________ Family (Last) Name: ____________

Given (First) Name: ____________

Tutorial sections (check one):

- ☐ T01 (Jaimes Joschko, 2:30, CLE A127)
- ☐ T02 (MacKenzie Carr, 2:30, CLE A308)
- ☐ T03 (Jacob Nagrocki, 2:30, HHB 110)
- ☐ T04 (Jacob Nagrocki, 3:30, HHB 110)
- ☐ T05 (Jaimes Joschko, 3:30, CLE C112)
- ☐ T06 (MacKenzie Carr, 3:30, CLE A203)
- ☐ T07 (Jaimes Joschko, 4:30, CLE C112)
- ☐ T08 (Jacob Nagrocki, 4:30, HHB 110)
- ☐ T12 (MacKenzie Carr, 4:30, CLE A203)

Question(s)	Value	Score
Question 1	1	
Question 2	1	
Question 3	1	
Question 4	1	
Question 5	4	
Question 6	4	
Question 7	4	
Question 8	4	
Total	**20**	

Instructions:

1. Identifying information:
   (a) Enter your Student ID and name at the top of this page now.
   (b) Select your tutorial section above now.
2. Only the following materials are permitted:
   (a) Pens, pencils, erasers, and a ruler are permitted at your desk. If you have a pencil case it must be stored with your belongings in the front of the room.
   (b) You may use a Sharp calculator with a model number beginning with EL510-R. No other calculators are acceptable on this examination.
3. No notes, outside paper, or aid other than the ones listed above is permitted. You are responsible for ensuring that any unauthorized material is stored with your belongings at the front of the room.
4. Show all calculations on this paper for all problems. We may disallow any answer given without appropriate justification.
5. If you require extra space, use the backs of test pages or the blank page at the end of the test. Be sure to clearly indicate where you have done so. No outside paper is permitted.
6. If you need to leave the room during the test, raise your hand until an invigilator comes to you.
7. This test has 8 pages, including this cover and the blank page at the end.

For questions 1–4, enter your final answer in the box provided. You must show your work to be given credit, even if your answer is correct.
Leave all answers in exact form - do not give decimal approximations. If it is impossible to answer a question using the given information, write "NA" in the box.

(1 point) 1. Let $z = 3e^{i\pi/3}$ and $w = 5e^{i\pi/4}$. Suppose that zw is written in exponential form as $zw = re^{i\theta}$. Find the value of r.

Solution: $zw = (3e^{i\pi/3})(5e^{i\pi/4}) = (3 \cdot 5)e^{i(\pi/3+\pi/4)}$, so $r = 15$.

Answer: 15

(1 point) 2. Let $z = \frac{1}{1-i\sqrt{3}}$. Find the Cartesian form of $\overline{z}$.

Solution: We first calculate

$$z = \frac{1}{1-i\sqrt{3}}\left(\frac{1+i\sqrt{3}}{1+i\sqrt{3}}\right) = \frac{1+i\sqrt{3}}{4} = \frac{1}{4} + \frac{\sqrt{3}}{4}i.$$

Therefore $\overline{z} = \frac{1}{4} - \frac{\sqrt{3}}{4}i$.

Answer: $\frac{1}{4} - \frac{\sqrt{3}}{4}i$

(1 point) 3. Suppose that A is a 7×7 matrix, and $A^2 = A$. Find all possible values of $\det(A)$.

Solution: We have $\det(A^2) = \det(A)^2$, and also since $A^2 = A$ we have $\det(A^2) = \det(A)$. Thus we have $\det(A)^2 = \det(A)$, so $\det(A)$ is either 0 or 1.

Answer: $0, 1$

(1 point) 4. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and let $\mathbf{v} = \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix}$. Determine whether or not $\mathbf{v}$ is an eigenvector of A. If it is, write the corresponding eigenvalue in the box. If it is not, write "no" in the box.

Solution: We multiply:

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ -9 \end{bmatrix} = 3\mathbf{v}.$$

Therefore $\mathbf{v}$ is an eigenvector of A, with eigenvalue 3.

Answer: 3

(4 points) 5. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 0 & 10 \end{bmatrix}$. Find a basis for row$(A)$.

Solution: We row-reduce:

$$A \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for row(A) is the non-zero rows of the RREF, which means that in this case a basis is $\begin{bmatrix} 1 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & 2 & a & -1 \\ -1 & 0 & 0 & 1 \\ 2 & -3 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(3 points) (a) Calculate $\det(A)$.

Solution: We use cofactor expansion on the fourth row to obtain

$$\det(A) = (-3)\det\left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & -3 & 0 \end{bmatrix}\right).$$

Then we continue, now using cofactor expansion on the second row, to get:

$$\begin{aligned}
\det(A) &= (-3)\det\left(\begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & -3 & 0 \end{bmatrix}\right) \\
&= (-3)\left(\det\left(\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}\right)\right) \\
&= (-3)((0-3)-(-3-4)) \\
&= -12
\end{aligned}$$

(1 point) (b) For which values of a is A invertible? Justify your answer.

Solution: Since $\det(A) \neq 0$ no matter what a is, the matrix A is invertible for all values of a.

(4 points) 7. Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. It is a fact, which you do not need to prove, that -2 is an eigenvalue of A. Find a basis for the eigenspace $E_{-2}(A)$.

Solution: We need a basis for null$(A - (-2)I)$. So we row-reduce:

$$A - (-2)I = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Viewing this as a system of equations, we obtain $x - y + z = 0$, thus a vector is in $E_{-2}(A)$ if and only if

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore a basis for $E_{-2}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(4 points) 8. Let $\mathbf{w}$ be a vector in $\mathbb{R}^n$, and let S be the collection of vectors in $\mathbb{R}^n$ defined by saying that $\mathbf{v}$ is in S if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. Prove that S is a subspace of $\mathbb{R}^n$. No credit will be given for solutions that choose specific values of n or $\mathbf{w}$.

Solution: There are three conditions to be checked, so here we go:

1. Since $\mathbf{0} \cdot \mathbf{w} = 0$, we have $\mathbf{0}$ in S.
2. Suppose that $\mathbf{v}_1, \mathbf{v}_2$ are in S. Then $\mathbf{v}_1 \cdot \mathbf{w} = \mathbf{v}_2 \cdot \mathbf{w} = 0$. We calculate
$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w} + \mathbf{v}_2 \cdot \mathbf{w} = 0 + 0 = 0.$$
Therefore $\mathbf{v}_1 + \mathbf{v}_2$ is in S.
3. Suppose that $\mathbf{v}$ is in S, so $\mathbf{v} \cdot \mathbf{w} = 0$. Then for any scalar c,
$$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) = c0 = 0.$$
So $c\mathbf{v}$ is in S.

We've checked all three conditions, so we have proved that S is a subspace of $\mathbb{R}^n$.

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