Q.1 (a) $$\sum_n \frac{(2x-1)^n}{5^n\sqrt{n}}$$

$$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\cdot\frac{5^n\sqrt{n}}{(2x-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{|2x-1|}{5}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \frac{|2x-1|}{5}$$

For convergence, $\frac{|2x-1|}{5} < 1$

$\Rightarrow |2x-1| < 5 \Rightarrow -2 < x < 3$

Therefore, the interval is: $-2 \leqslant x < 3$.

At $x=3$:

$$\sum_n \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_n \frac{5^n}{5^n\sqrt{n}} = \sum_n \frac{1}{\sqrt{n}}$$

which is divergent.

At $x=-2$;

$$\sum_n \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_n \frac{(-5)^n}{5^n\sqrt{n}} = \sum_n \frac{(-1)^n}{\sqrt{n}},$$

This converges by Alternating Series test.

(b). $\sum_n \frac{(-1)^n (x-3)^n}{(2n+1)}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-3)^{n+1}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n (x-3)^n} \right|$$

$$= \lim_{n\to\infty} |x-3| \frac{2n+1}{2n+3}$$

$$= |x-3|$$

For convergence $|x-3| < 1$

$\Rightarrow -1 < x-3 < 1$

$\Rightarrow 2 < x < 4.$

Required interval: $2 < x \leq 4.$

At $x=2$,

$$\sum_n \frac{(-1)^n (x-3)^n}{(2n+1)} = \sum_n \frac{(-1)^{n+n}}{(2n+1)}$$

$$= \sum_n \frac{1}{2n+1} ;$$

divergent.

At $x=4$;

$$\sum_n \frac{(-1)^n (x-3)^n}{(2n+1)} = \sum_n \frac{(-1)^n}{2n+1}$$

Alternating series test, convergent

In a similar way:

(c) For $\sum_n \frac{(-1)^n x^n}{n^{1/3}}$, $-1 < x \leq 1$

(e) For $\sum_n \frac{x^n}{5^n n^5}$, $-5 \leq x \leq 5$

(f) For $\sum_n \frac{n}{4^n} (x+1)^n$, $-5 < x < 3$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(2x-1)^n}{2}$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x-1)^{n+1}}{2} \cdot \frac{2}{n!\,(2x-1)^n} \right|$$

$$= \lim_{n\to\infty} |2x-1|\,(n+1)$$

$$= \infty \quad \text{provided } x \neq \tfrac{1}{2}.$$

Therefore, $\displaystyle\sum_{n} \frac{n!\,(2x-1)^n}{2}$ Converges only at $x=\frac{1}{2}$.

Q.2. Suppose $f(x) = \sum_n a_n x^n$, then the second series is simply $f(x^2)$.

If the given series converges for $|x| < R$, then the second series will converge for $|x^2| < R$. Therefore the radius of convergence of $\sum_n a_n x^{2n}$ is $\sqrt{R}$.

Q.3. We have $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!}, \quad y''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{(2n-2)!}$$

let $p = n-1$

$\Rightarrow$ $p = 0$ to $p = \infty$

$$\Rightarrow \quad y''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{2(n-1)}}{[2(n-1)]!} = \sum_{p=0}^{\infty} \frac{(-1)^{p+1} \, x^{2p}}{(2p)!}$$

$$= - \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{(2p)!}$$

Therefore,

$$y'' + y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} - \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{(2p)!}$$

$$= 0 \qquad \boxtimes$$