Math 110 - Homework 5
Topic: Linear independence, span, rank

Due at 6:00pm (Pacific) on Friday, October 15, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 3.1-3.3 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` is a matrix then `rank(A)` calculates the rank of A.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $\vec{v_1} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ k \\ 4 \end{bmatrix}$.

   Find all values of k for which $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

Solution: We want to know when there is a unique solution (for a, b, c) to the equation:

$$a \begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ k \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We read this equation row-by-row, and thus produce a system of linear equations that we can solve (remember that student solutions should show the steps of the row reduction):

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -2 & 0 & 0 \\ 2 & -1 & k & 0 \\ -1 & 2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & k-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We could continue to RREF, but from this REF we already see that the system has a unique solution if and only if $k - 1 \neq 0$, while if $k - 1 = 0$ then c is a free variable and we get infinitely many solutions. Thus the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are independent if and only if $k \neq 1$.

2. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$, and $\vec{v_4} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

   Determine whether span$(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4})$ is a line, a plane, or all of $\mathbb{R}^3$. If it is a line or plane, describe the line or plane by giving general form equation(s).

Solution: We want to know for which x, y, z we can find a, b, c, d such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + d \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

Reading this as a system of equations, we get the following system, which we row-reduce (remember that student solutions should show the steps of the row reduction):

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & x \\ 3 & 1 & 5 & -2 & y \\ -2 & -1 & -3 & 1 & z \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & x \\ 0 & 1 & -1 & 1 & y - 3x \\ 0 & 0 & 0 & 0 & -x + y + z \end{array}\right].$$

From here we see that the system has a solution if and only if $-x+y+z = 0$. Thus span$(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4})$ is a plane in $\mathbb{R}^3$, and a general equation for that plane is $-x + y + z = 0$.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. Note that there may be questions that can be solved without doing any calculations, or where MATLAB is not helpful for the calculations you need to do; in such cases, do the calculations by hand. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Let $\vec{v_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 9 \\ 4 \end{bmatrix}$, and $\vec{v_4} = \begin{bmatrix} 4 \\ -6 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.
   Find one of these vectors that can be removed in such a way that the span of the other three is the same as $\text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4})$, or explain why doing so is impossible.

Solution: Recall that there is a vector that can be dropped if and only if the vectors are linearly dependent, and that in that case we can drop any of the vectors that is in the span of the remaining ones. We thus first check if the vectors are dependent or independent. We want to find a, b, c, d such that

$$a \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 5 \\ -7 \\ 0 \\ 9 \\ 4 \end{bmatrix} + d \begin{bmatrix} 4 \\ -6 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Converting to a system of linear equations and row reducing (either by hand, or by MATLAB), we get:

$$\left[\begin{array}{cccc|c} 2 & -1 & 5 & 4 & 0 \\ -3 & 1 & -7 & -6 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ 4 & -1 & 9 & 1 & 0 \\ 1 & -2 & 4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We thus see that in the above equation of vectors, we must have $d = 0$, $a = -2c$, and $b = c$, while c is a free variable. If we pick $c = 1$ we get the equation

$$-2 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 9 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -6 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Rearranging this a bit, we get

$$\vec{v_3} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 9 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \in \text{span}(\vec{v_1}, \vec{v_2}).$$

Since $\overrightarrow{v_3}$ is in the span of the other vectors, by a theorem from class we can remove $\overrightarrow{v_3}$ without affecting the span; that is,

$$\operatorname{span}(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}) = \operatorname{span}(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_4}).$$

We note that when we rearranged the equation above we could have been rearranged to allow us to remove $\overrightarrow{v_1}$ or $\overrightarrow{v_2}$ instead; any one, but only one, of these can be removed, and $\overrightarrow{v_4}$ cannot be removed.

2. Let $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Is there a 3×3 matrix B such that $\operatorname{rank}(B) = 2$ and A is row-equivalent to B? If so, find such a B. If not, explain why not.

Solution: We first calculate the rank of A, by row-reducing. We could either do this by hand, or by using MATLAB. Either way, we find

$$\operatorname{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From here we see that $\operatorname{rank}(A) = 3$. If B is a matrix that is row-equivalent to A then $\operatorname{RREF}(B) = \operatorname{RREF}(A)$, and so B also has rank 3. Thus there is no matrix B that is row-equivalent to A and has rank 2.