A3

Part 1

Q1 (a) $\vec{v_1}\cdot\vec{v_3}$

$$=\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}0\\0\end{bmatrix}$$

$$=(1\times0)+(2\times0)$$

$$=0+0$$

$$=0$$

(b) $\vec{v_3}\cdot\vec{v_4}$

$$=\begin{bmatrix}0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Hence the dot product of the two vectors doesn't make sence as they have different/unequal number of dimentions/elements.

(c) $\vec{v_2}\cdot(2\vec{v_1}-3\vec{v_2})$

$$2\vec{v_1}=2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

$$3\vec{v_2}=3\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}-6\\9\end{bmatrix}$$

$$2\vec{v_1}-3\vec{v_2}=\begin{bmatrix}2\\4\end{bmatrix}-\begin{bmatrix}-6\\9\end{bmatrix}$$

$$=\begin{bmatrix}2-(-6)\\4-(9)\end{bmatrix}=\begin{bmatrix}2+6\\4-9\end{bmatrix}=\begin{bmatrix}8\\-5\end{bmatrix}$$

$\vec{v_2}\cdot(2\vec{v_1}-3\vec{v_2})$

$$=\vec{v_2}\cdot\begin{bmatrix}8\\-5\end{bmatrix}$$

$$=\begin{bmatrix}-2\\3\end{bmatrix}\cdot\begin{bmatrix}8\\-5\end{bmatrix}$$

$$=(-2\times8)+(3\times-5)$$

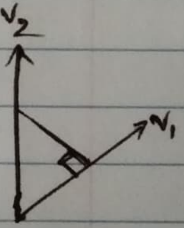
$$=-16-15=-31$$

(d) $\vec{v_2}\times(2\vec{v_1}-3\vec{v_2})$

$$2\vec{v_1}-3\vec{v_2}=\begin{bmatrix}8\\-5\end{bmatrix};\quad \vec{v_2}=\begin{bmatrix}-2\\3\end{bmatrix}$$

Since cross product of the two vectors will product a vector in a different direction (Third dimention) as such: $-14\hat{k}+0\hat{i}+0\hat{j}$, we can conclude that the given expression doesn't make sense.

(e) $\text{Proj}_{\vec{v_1}}(\vec{v_2})$



$$\text{proj}_{\vec{v_1}}(\vec{v_2})=\left(\frac{\vec{v_1}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\right)\vec{v_2}$$

$$=\left(\frac{4}{13}\right)\begin{bmatrix}-2\\3\end{bmatrix}$$

$$=\begin{bmatrix}-2\times\frac{4}{13}\\3\times\frac{4}{13}\end{bmatrix}$$

$$=\begin{bmatrix}-8/13\\12/13\end{bmatrix}$$

(Ans)

$$\vec{v_1}\cdot\vec{v_2}=\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}-2\\3\end{bmatrix}$$

$$=(1\times-2)+(2\times3)$$

$$=-2+6$$

$$=4$$

$$v_2\cdot v_2=\begin{bmatrix}-2\\3\end{bmatrix}\cdot\begin{bmatrix}-2\\3\end{bmatrix}$$

$$=(-2\cdot-2)+(3\cdot3)$$

$$=4+9=13$$

Q_2. Here, $P \equiv (3,-5)$

The general equation of the parallel line is $x - 3y = 5$

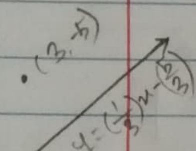
$$\Rightarrow x - 3y - 5 = 0$$

where, $a = 0$
$b = -3$
$c = -5$

$$\Rightarrow y = \left(\frac{1}{3}\right)x - \left(\frac{5}{3}\right)$$

The equation of the general line is:

$$y = \left(\frac{1}{3}\right)x + c$$ (where c can be any rational number)



The vector equation of the line is: $\vec{V} = \vec{P} + t\vec{d}$

Here, $\vec{V} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{3}x - \frac{5}{3} \end{bmatrix}$

$$= \begin{bmatrix} x \\ \frac{1}{3}x \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$$

Putting $x = t$, we'll get the vector form of the parallel line

$$\vec{V} = t\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$ where $\begin{pmatrix} a = 0 \\ b = -5/3 \end{pmatrix}$

Since we have the point of the line, we get the vector form of the line as:

$$\vec{V} = t\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$x = 3,\ y = -\frac{2}{3}$

$x = 4,\ y = -\frac{1}{3}$

$$d = \begin{bmatrix} 4-3 \\ -\frac{1}{3} - -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$$

Part 2

Q_1 We want to find a vector in R_6 which is orthogonal to $\vec{V_1}, \vec{V_2}$ and $\vec{V_3}$ and not equal to 0.

$$\vec{V_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix};\quad \vec{V_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix};\quad \vec{V_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

Let the vector be $\vec{V} = \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix}$

For finding that vector, it has to be equal to $\Big\}$ Dot products of orthogonal vectors are 0.

$$\vec{V_1} \cdot \vec{V} = \vec{V_2} \cdot \vec{V} = \vec{V_3} \cdot \vec{V} = 0$$

or,
$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = 0$$

or
$$\left.\begin{aligned} x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 0 \\ -x_1 + 2x_2 + x_3 - 4x_5 + 5x_6 &= 0 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 - x_6 &= 0 \end{aligned}\right\}$$

Putting it in augmented matrix form:
$$\left[\begin{array}{cccccc|c} 1 & 3 & 2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & -4 & 5 & 0 \\ 1 & 1 & 1 & 2 & 2 & -1 & 0 \end{array}\right]$$

Finding rref on MathLAB we get
$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 3 & -4 & 0 \\ 0 & 1 & 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 1 & 7 & -1 & 5 & 0 \end{array}\right]$$

or, $x_1 = x_4 - 3x_5 + 4x_6$ or $x_4 = x_1 + 3x_5 - 4x_6$

$x_2 = 4x_4 + 2x_6$ $\qquad x_4 = \dfrac{x_2 - 2x_6}{4}$

$x_3 = -7x_4 + x_5 - 5x_6$ $\qquad x_4 = \dfrac{x_5 - 5x_6 - x_3}{7}$

Its not possible to find the extreme values of the points, since there are more than one variables in this system of linear equation.

So, what we can do is take arbitrary values of $x_4\ x_5\ x_6$ in order to find

$\cancel{Q_2 : \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

the values of x_1, x_2 and x_3 such that

$$\vec{v_1} \cdot \vec{v} = \vec{v_2} \cdot \vec{v} = \vec{v_3} \cdot \vec{v} = 0$$

$x_4 = 0, \quad x_5 = 2 \quad x_6 = 1$

$\therefore x_1 = 0 - 3(2) + 4(1) = -6 + 4 = -2$

$\therefore x_2 = 4(0) + 2(1) = 0 + 2 = 2$

$\therefore x_3 = -7(0) + (2) - 5(1) = 0 + 2 - 5 = -3$

putting it, we get,

$$\vec{v_1} \cdot \vec{v} = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 3$$

$$\vec{v_2} \cdot \vec{v} = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix} = 0$$

$$\vec{v_3} \cdot \vec{v} = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = 0$$

Here, $\vec{v_1} \cdot \vec{v} \neq \vec{v_2} \cdot \vec{v} = \vec{v_3} \cdot \vec{v} = 0$

so, this vector doesn't fit the criteria. Let's look for another one!

Part 2 Question 1 (Part 2)

Let's try $x_4 = 1$ $\quad x_5 = 0$ $\quad x_6 = 0$

$$x_1 = x_4 - 3x_5 + 4x_6 = 1 - 3(0) + 4(0) = 1$$

$$x_2 = 4x_4 + 2x_6 = 4(1) + 2(0) = 4$$

$$x_3 = -7x_4 + x_5 - 5x_6 = -7(1) + (0) - 5(0) = -7$$

Now, $\vec{v}_1 \cdot \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$

$\vec{v}_2 \cdot \vec{v} = \begin{bmatrix} 1 \\ 4 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -4 \\ 5 \end{bmatrix} = 0$

$\vec{v}_3 \cdot \vec{v} = \begin{bmatrix} 1 \\ 4 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} = 0$

Here, $\vec{v}_1 \cdot \vec{v} = \vec{v}_2 \cdot \vec{v} = \vec{v}_3 \cdot \vec{v} = 0$

So, the vector that is orthogonal to the all three vectors is

$$\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -7 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Question 2

$\vec{V} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ Let $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix}$

Here,

$\vec{V} \cdot \vec{W} = \frac{1}{2}$

or, $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}$

or, $-2x + y = \frac{1}{2}$

or, $-4x + 2y = 1$

or, $y = \left(\frac{1+4x}{2}\right)$ —— (i)

$\|\vec{w}\| = 1$

or, $\sqrt{x^2 + y^2} = 1$

or, $x^2 + y^2 = 1$ —— (ii)

Putting (i) in (ii), we get

$x^2 + \left(\frac{1+4x}{2}\right)^2 = 1$

$\Rightarrow x^2 + \frac{1 + 16x^2 + 2(4x)}{4} = 1$

$\Rightarrow \frac{4x^2 + 1 + 16x^2 + 8x}{4} = 1$

$\Rightarrow 20x^2 + 1 - 4 + 8x = 0$

$\Rightarrow 20x^2 + 8x - 3 = 0$ —— (iii)

For a quadratic equation $(ax^2 + bx + c = 0)$, we have $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Putting it on (iii), we get, $x = \frac{-2 \pm \sqrt{19}}{10}$. Putting the vale of x in (i), we get,

$y = \left(\frac{1+4x}{2}\right) = \left(\frac{1 \pm 2\sqrt{19}}{10}\right)$

Putting the value of x and y in the vectore $\vec{w}$, we get,

$$\vec{w} = \begin{bmatrix} \frac{-2 \pm \sqrt{19}}{10} \\ \frac{1 \pm 2\sqrt{19}}{10} \end{bmatrix} = \begin{bmatrix} \frac{-2 + \sqrt{19}}{10} & \frac{-2 - \sqrt{19}}{10} \\ \frac{1 + 2\sqrt{19}}{10} & \frac{1 - 2\sqrt{19}}{10} \end{bmatrix}$$

$$\xrightarrow[R_2 \times 10]{R_1 \times 10} \begin{bmatrix} -2 + \sqrt{19} & -2 - \sqrt{19} \\ 1 + 2\sqrt{19} & 1 - 2\sqrt{19} \end{bmatrix}$$

(Ans)

Question 3

Given, $\vec{v}$ and $\vec{w}$ are two non-zero vectors in $\mathbb{R}^n$, and $\vec{v} \perp \vec{w}$

We have to show, $proj_{\vec{w}}\left(proj_{\vec{v}}(\vec{x})\right) = \vec{0}$

$$proj_{\vec{v}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|}\right)\vec{v}$$

$$proj_{\vec{w}}\left(proj_{\vec{v}}\vec{x}\right) = proj_{\vec{w}}\left(\left(\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|}\right)\vec{v}\right)$$

$$= \left(\frac{\left(\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|}\right)\vec{v}}{\|\vec{w}\|}\right)\vec{w}$$

let take $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ such that $\vec{v}\cdot\vec{w} = 0$

Let's take $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\vec{x}\cdot\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x(1) + y(0) = x$$

$$\|\vec{v}\| = \sqrt{0^2 + 1^2} = \sqrt{1^2} = 1$$

Now, $proj_{\vec{v}}\vec{x} = \left(\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|}\right)\vec{v} = \left(\frac{x}{1}\right)\vec{v} = x(\vec{v}) = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Now, $\text{proj}_{\vec{w}}(\text{proj}_{\vec{v}}\ \vec{x}) = \left(\dfrac{\left(\dfrac{x \cdot v}{\|\vec{v}\|}\right)\vec{v}}{\|\vec{w}\|}\right) \cdot \vec{w}$

Here $\|\vec{w}\| = \sqrt{0^2+1^2}$

$= \sqrt{1^2}$

$= 1$

$$= \left(\frac{\begin{bmatrix} x \\ 0 \end{bmatrix}}{\|\vec{w}\|}\right) \cdot \vec{w}$$

$$= \left(\begin{bmatrix} x \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x(0) + 0(1)$$

$$= 0 + 0 = 0$$

$\therefore\ \text{proj}_{\vec{w}}\left(\text{proj}_{\vec{v}}\ (\vec{x})\right) = \vec{0} = 0$

(Showed)