| **Student:** Arfaz Hossain | **Instructor:** Muhammad Awais | **Assignment:** HW-7 [Sections 10.7 & 10.8] |
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Find the Maclaurin series of the function.

$f(x) = 8\cos 6\pi x$

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Let f be a function with derivatives of all orders throughout some interval containing 0 as an interior point. The Maclaurin series of f, shown below, is the Taylor series generated by f at x = 0.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots$$

Take the first few derivatives of $8\cos 6\pi x$ to find a pattern.

$$\begin{aligned} f(x) &= 8\cos 6\pi x \\ f'(x) &= -48\pi \sin 6\pi x \\ f''(x) &= -288\pi^2 \cos 6\pi x \\ f'''(x) &= 1728\pi^3 \sin 6\pi x \end{aligned}$$

Generalize the function and its derivatives. Notice that the even numbered derivatives all contain cosine, while the odd numbered derivatives all contain sine. Thus, when looking for a pattern, first find two separate formulas, one for $f^{(2n)}$ and one for $f^{(2n+1)}$.

Start with the even numbered derivatives. The fourth derivative has been added to the list to make it easier to see a pattern. Notice that the sign of the derivatives alternates. Write an alternating factor that represents this pattern.

$(-1)^n$

| n | $f^{(2n)}$ |
|---|---|
| n = 0 | $f(x) = 8\cos 6\pi x$ |
| n = 1 | $f''(x) = -288\pi^2 \cos 6\pi x$ |
| n = 2 | $f^{(4)} = 10{,}368\pi^4 \cos 6\pi x$ |

Now determine the coefficient, not including the sign. The pattern for the coefficient is often easier to see in a factored form. Write the formula to describe the coefficient.

$8 \cdot 6^{2n} \cdot \pi^{2n}$

| n | coefficient |
|---|---|
| n = 0 | 8 |
| n = 1 | $8 \cdot 36 \cdot \pi^2$ |
| n = 2 | $8 \cdot 1296 \cdot \pi^4$ |

Therefore, $f^{(2n)}(x) = (-1)^n \cdot 8 \cdot 6^{2n} \cdot \pi^{2n} \cos 6x$.

Now examine the odd numbered derivatives. The fifth derivative has been added to the list to make it easier to see a pattern. Write the formula for $f^{(2n+1)}(x)$.

$f^{(2n+1)}(x) = (-1)^{n+1} \cdot 8 \cdot 6^{2n+1} \cdot \pi^{2n+1} \sin 6x$

| n | $f^{(2n+1)}(x)$ |
|---|---|
| n = 0 | $f'(x) = -48\pi \sin 6\pi x$ |
| n = 1 | $f'''(x) = 1728\pi^3 \sin 6\pi x$ |
| n = 2 | $f^{(5)}(x) = -62{,}208\pi^5 \sin 6\pi x$ |

Notice that at x = 0, $\cos x = 1$ and $\sin x = 0$. Now recall that in the formula for a Maclaurin series, the derivatives are evaluated at x = 0. Simplify the expressions for $f^{(2n)}(0)$ and $f^{(2n+1)}(0)$.

$f^{(2n)}(0) = (-1)^n \cdot 8 \cdot 6^{2n} \cdot \pi^{2n}$ and $f^{(2n+1)}(0) = 0$

Since $-288\pi^2 \cos 6\pi x$ for all n, only even powers of x occur in the Maclaurin series for $f(x) = 8\cos 6\pi x$. Complete the first two nonzero terms of the Maclaurin series by evaluating $f^{(2n)}(0)$ for n = 0 to obtain the first term of the sequence, f(0) = 48. Then evaluate $f^{(2n)}(0)$ for n = 1 to obtain $f''(0) = -288\pi^2$. Dividing by 2! gives the coefficient for the $x^2$-term.

$$\begin{aligned} & f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots \\ & = 8 + 0x + \left(-144\pi^2x^2\right) + \frac{0x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots \end{aligned}$$

Finally, write the Maclaurin series for f in summation notation. Only the non-zero terms are shown in the expansion below.

Find the Maclaurin series of the function by substituting $(-1)^n \cdot 8 \cdot 6^{2n} \cdot \pi^{2n}$ for the general term $\frac{f^{(2n)}(0)x^{(2n)}}{(2n)!}$.

$$8 - 144\pi^2x^2 + \cdots + \frac{f^{(2n)}(0)x^{(2n)}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 8 \cdot 6^{2n} \cdot \pi^{2n} \cdot x^{2n}}{(2n)!}$$