MATH 100, Fall, 2021
Tutorial #8
Story Problems and Newton's method

Q1 A printed page contains 24 in^2 of text with 1 in blank side margins and 1.5 inch top and bottom margins each. Find the dimensions of the page with smallest area that satisfies these constraints. In your group, make a sketch of the page and apply labels x and y to width and height of the printed area respectively. Convince yourself that the problem becomes: minimize $A = (x+2)(y+3)$ where $xy = 24$.

Assignment:

1. Formulate the problem as a single variable optimization problem (in terms of x or y, but not both) and then solve the problem via calculus to find the optimal page dimension. Explain how you know your solution is a minimum. State your conclusion in a grammatically correct sentence that tells the printer what size paper to order.

Q2 A business manufactures and sells $x \geq 0$ tons of a product each week with revenue $r(x)$ and operating costs $c(x)$ measured in thousands of \$. Its weekly profit at production level x is therefore $P(x) = r(x) - c(x)$. In your group, discuss the economist's rule-of-thumb that says the company's profit is maximized at output x_* when $r'(x_*) = c'(x_*)$ and interpret this in terms of a critical point for the profit function P.

Assignment:

1. Suppose that $r(x) = 50\sqrt{x}$ and that $c(x) = 10 + 6x^{3/2}$. Find the output x_* that maximizes profit for the firm. Summarize your findings in a grammatically correct sentence that can be understood by the president of the company.

Final thought: is there a way to be sure you have found a maximizer, and not a minimizer? The president might ask.

Q3 In your group, discuss how the function $g(x) = x^{12} - 2$ can be used to approximate the number $2^{\frac{1}{12}}$ by Newton's method.

Assignment:

1. If your current estimate of the root is x_n find Newton's formula for computing the next approximation x_{n+1}.
2. Set $x_0 = 1$, and derived the approximations $x_1, x_2 \ldots$ until your estimate agrees with the first 3 decimals of your calculator's estimate for $\sqrt[12]{2}$. Report back: calculator $= \ldots$; Newton after n steps$= \ldots$. Tell us what n you stopped at.

$\sqrt[12]{2}$ turns out to be an important number in music theory, so having a decimal approximation is actually a helpful thing to know: (Eg, see `https://en.wikipedia.org/wiki/Twelfth_root_of_two`)

Q4 As many of you have realized, Newton's method does not always work first time. Consider the function $f(x) = x^3 - 2x + 2$. Discuss in your group how you **know** that f has a root and set up a Newton iteration formula to compute its approximation x_{n+1} from the estimate x_n.

Assignment: Let $x_0 = 1$. Compute the Newton iteration x_1. Now compute x_2. What do you notice happening?

Q5 Consider the same function $f(x) = x^3 - 2x + 2$ from Q4. Despite your experience in Q4, discuss again why you are sure there must be a root for f. Can you find it?

Assignment: Set $x_0 = -2$. Compute x_1 and x_2 along with $f(x_1)$ and $f(x_2)$. Does the method now seem to be working?

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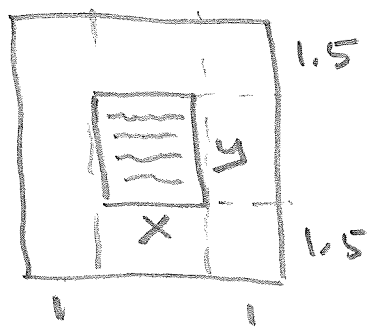
Tutorial Worksheet
Tutorial Section (T01, T02 etc)________
Tutorial Instructor Name:________________________
Question Number Attempted (Q1, Q2, etc) Q1

Your Name: Key
Your Student Number: V00____________
Today's Date:____________________



$A(x,y) = (x+2)(y+3)$

$xy = 24$ ← required

$0 < x < \infty$?

$y = \frac{24}{x}$. $\Rightarrow$ $A = A(x) = (x+2)\left(\frac{24}{x}+3\right)$

$= 24 + 3x + \frac{48}{x} + 6$

$= 3x + \frac{48}{x} + 30$

$A'(x) = 3 - \frac{48}{x^2} = 0 \Leftrightarrow x^2 = \frac{48}{3} = 16 \Leftrightarrow x = \pm 4$

Ignore $x=-4$ (obviously).

$A'(x) = \frac{1}{x^2}(3x^2 - 48)$.

sgn A': $\underset{0}{|}$ − − − − − − $\underset{4}{|}$ + + + + +

local min

The minimum area paper format is

width $= 4+2 = 6$ in
height $= 6+3 = 9$ in
} 6×9 format.

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Tutorial Worksheet
Tutorial Section (T01, T02 etc)________
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Question Number Attempted (Q1, Q2, etc) Q2

Your Name: Key
Your Student Number: V00__________
Today's Date:__________________

$$P(x) = 50\sqrt{x} - (10 + 6x^{3/2})$$

$$P'(x) = 50\,\frac{1}{2\sqrt{x}} - 18/2\, x^{1/2}$$

$$= \frac{25}{\sqrt{x}} - 9\sqrt{x} = 0 \iff \boxed{\frac{25}{9} = x}$$

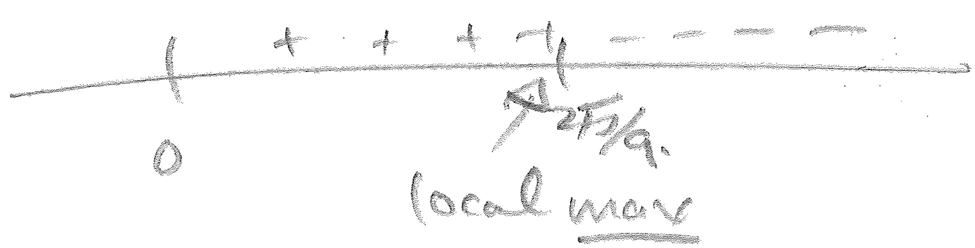
In order to maximize profit, your company should produce and sell $x_* = \frac{25}{9} \sim 2.77$ tons of product per week

If you do this, your profit will be \$45.555 thousands per week: \$45,555

Look at $p'(x) = \frac{1}{\sqrt{x}}(25 - 9x)$

this is $p'(x) > 0$ when $x < 25/9$.

$p'(x) < 0$ when $x > 25/9$.



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Question Number Attempted (Q1, Q2, etc) Q3

Looking for roots of $g(x) = x^{12} - 2$

Iteration $g'(x) = 12x^{11}$;

$$x_{n+1} = x_n - \frac{x_n^{12} - 2}{12x_n^{11}}$$

$$= x_n - \frac{1}{12}x_n + \frac{2}{12x_n^{11}}$$

$x_0 = 1$ $\boxed{= \frac{11}{12}x_n + \frac{1}{6x_n^{11}} = x_{n+1}}$ Iteration

$$x_1 = \frac{11}{12} + \frac{1}{6 \cdot 1^{11}} = \frac{13}{12} \approx 1.08333\ldots$$

$$x_2 = \frac{11}{12} \cdot \frac{13}{12} + \frac{1}{6 \cdot \left(\frac{13}{12}\right)^{11}} = \frac{143}{144} + \frac{12^{11}}{6 \cdot 13^{11}} \approx 1.06215\ldots$$

$$x_3 = \frac{11}{12} \cdot 1.06215 + \frac{1}{6 \cdot (1.06215)^{11}} \approx 1.05950\ldots$$

Calc $\sqrt[12]{2} \approx 1.0594\underline{6}$

Newton, 3 steps $= 1.059\underline{50}$

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Question Number Attempted (Q1, Q2, etc) Q4

$f(x) = x^3 - 2x + 2$

$f'(x) = 3x^2 - 2$

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}}$$

iteration rule

$x_0 = 1$

$$x_1 = 1 - \frac{1 - 2 + 2}{3 - 2} = 1 - 1 = 0$$

$x_1 = 0$

$$x_2 = 0 - \frac{2}{-2} = 1.$$

$x_2 = 1$

$x_3 = \cdots 0$ and so on.

$$x_n = \begin{cases} 0 & \text{if } n = 1, 3, 5 \cdots 2k+1 \\ 1 & \text{if } n = 0, 2, 4, \cdots 2k \end{cases}$$

No convergence Newton fails. etc.

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Today's Date:__________________

$$f(x) = x^3 - 2x + 2$$

$$f'(x) = 3x^2 - 2$$

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}$$

$$x_0 = -2$$

$$x_1 = -2 - \frac{-8+4+2}{12-2} = -2 + \frac{2}{10} = \boxed{-\frac{18}{10}}$$

$$f\left(-\frac{18}{10}\right) = (-1.8)^3 - 2(-1.8) + 2$$
$$\cong -0.232 \quad \text{(not so great)}$$

$$x_2 = -1.8 - \frac{(-1.8)^3 + 2(1.8) + 2}{3(1.8)^2 - 2} \approx -1.76995$$

$$f(-1.76995) \approx -0.0049 \quad \text{(better.)}$$

students to here

$$x_3 = \underbrace{-1.76995}_{\text{same}} - \frac{f(\text{same})}{f'(\text{same})} \approx -1.76929$$

$$f(-1.76929) \approx 0.00002 \quad \text{(good!)}$$