Math 110, Fall 2021, Test 1A Sample Answers

Note: Students were required to show their steps in solving the problems. For brevity these solutions omit intermediate steps of row reductions.

Instructions:

- You may use a calculator on this test, but the only permitted calculators are SHARP brand calculators with model numbers beginning EL-510R. No other electronic devices are permitted.
- No notes, textbooks, or other outside materials or aids are permitted.
- For questions with numerical answers, either give your answer in exact form or give it as a decimal to two decimal places.
- For **all** questions you must show your work to be given credit, even if your answer is correct.
- For questions 1–3, show your work and then enter your final answer in the box provided.
- This test is printed double-sided - be sure not to miss the questions on the back of the first page! For the long-answer questions the backs of the pages are additional space for your solution.

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(1 point) 1. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. Calculate $\vec{v} + 2\vec{w} - 2(\vec{w} - \vec{v})$.

Solution:

$$\vec{v} + 2\vec{w} - 2(\vec{w} - \vec{v}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} - 2 \left(\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}.$$

Answer: $\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$

(1 point) 2. Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Find $\cos(\theta)$.

Solution:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{-2}{\sqrt{5}\sqrt{2}}.$$

Answer: $\dfrac{-2}{\sqrt{5}\sqrt{2}}$

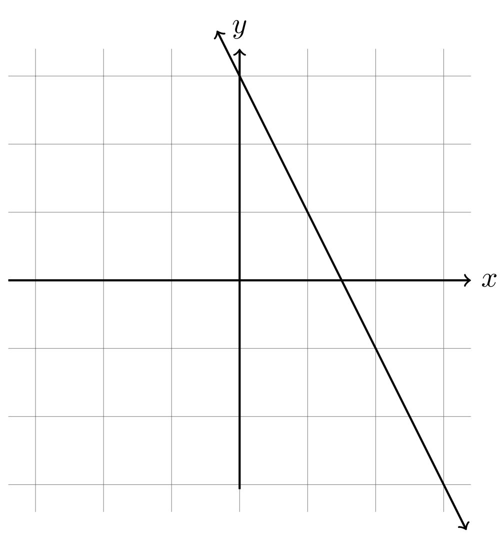
(1 point) 3. Find all values of x such that $\begin{bmatrix} 2x \\ 0 \\ x-1 \end{bmatrix}$ has length 1.

Solution: We have $\left\| \begin{bmatrix} 2x \\ 0 \\ x-1 \end{bmatrix} \right\| = \sqrt{(2x)^2 + 0^2 + (x-1)^2} = \sqrt{5x^2 - 2x + 1}$. To make the length 1 we therefore require $5x^2 - 2x + 1 = 1$, meaning $x(5x-2) = 0$, so $x = 0$ or $x = 2/5$.

Answer: $x = 0$ and $x = 2/5$

(1 point) 4. Sketch the line in $\mathbb{R}^2$ that has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Very briefly (one sentence is enough) tell us how you decided where to draw the line.

Solution: By plugging in $t = 0$ and $t = 1$ we find that the line passes through the points $(1, 1)$ and $(0, 3)$, so we draw the only line that passes through both of those points.



(4 points) 5. Determine whether the following system of linear equations in variables x_1, x_2, x_3, x_4 has no solution, exactly one solution, or infinitely many solutions.

$$x_1 - 2x_2 + x_3 = -2$$

$$2x_1 - x_2 + 3x_4 = 0$$

$$x_2 + x_3 + x_4 = 1$$

Solution: We set up an augmented matrix and row reduce.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & -2 \\ 2 & -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3/5 \\ 0 & 1 & 0 & 1 & 6/5 \\ 0 & 0 & 1 & 0 & -1/5 \end{array}\right].$$

We see from the reduced row echelon form that x_4 is a free variable, and therefore there are infinitely many solutions.

(4 points) 6. Find all values of k for which $\begin{bmatrix} 1 \\ k \\ k \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Solution: We want to know for which k there are a, b such that

$$a \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k \end{bmatrix}.$$

We treat this vector equation as a system of linear equations in variables a and b, and row reduce:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -3 & 2 & k \\ 2 & 1 & k \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -k-2 \\ 0 & 1 & -k-3 \\ 0 & 0 & 4k+7 \end{array}\right].$$

This system has a solution if and only if $4k+7=0$, that is, if and only if $k=-7/4$. Therefore the only k for which $\begin{bmatrix} 1 \\ k \\ k \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is $k=-7/4$.

(4 points) 7. Let L be the line in $\mathbb{R}^3$ that passes through the points $(4, 6, 0)$ and $(1, 1, 1)$. Let P be the plane in $\mathbb{R}^3$ that is orthogonal to L and passes through the point $(2, -1, -2)$. Find, with justification, a vector equation for P.

Solution: Since P is orthogonal to L, a direction vector for L will be a normal vector to P. Such a vector is $\vec{n} = \begin{bmatrix} 4-1 \\ 6-1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$. We know that P passes through $(2, -1, -2)$, so in normal form the equation for P is

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

Expanding the dot products we obtain the general form

$$3x + 5y - z = 3.$$

We rearrange this equation to say

$$z = -3 + 3x + 5y,$$

and then by substituting we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -3 + 3x + 5y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

The equation above is a vector equation for P (it is not the only possible correct answer).

(4 points) 8. Suppose that $\vec{v_1}, \vec{v_2}, \vec{w}$ are vectors in $\mathbb{R}^n$, and that $\vec{v_1} \perp \vec{w}$ and $\vec{v_2} \perp \vec{w}$. Show that every linear combination of $\vec{v_1}$ and $\vec{v_2}$ is orthogonal to $\vec{w}$.

Note: In this question we want you to write a general argument, so you should not choose specific numbers for any of the objects in the question.

Solution: The fact that $\vec{v_1} \perp \vec{w}$ and $\vec{v_2} \perp \vec{w}$ means that $\vec{w} \cdot \vec{v_1} = 0$ and $\vec{w} \cdot \vec{v_2} = 0$.

A linear combination of $\vec{v_1}$ and $\vec{v_2}$ has the form $a\vec{v_1} + b\vec{v_2}$ for some scalars a and b. We then use properties of the dot product to calculate:

$$\begin{aligned} \vec{w} \cdot (a\vec{v_1} + b\vec{v_2}) &= \vec{w} \cdot (a\vec{v_1}) + \vec{w} \cdot (b\vec{v_2}) \\ &= a(\vec{w} \cdot \vec{v_1}) + b(\vec{w} \cdot \vec{v_1}) \\ &= a(0) + b(0) \\ &= 0 \end{aligned}$$

Therefore $\vec{w} \perp (a\vec{v_1} + b\vec{w_2})$, as required.