Table 5.7

Code Name of Cereal	Grams of Sugar per 1-oz Serving	% of RDA[a] of Vitamin A per 1-oz Serving	% of RDA of Vitamin C per 1-oz Serving	% of RDA of Protein per 1-oz Serving
U	1	25	25	6
V	7	25	2	4
W	12	25	2	4
X	0	60	40	20
Y	3	25	40	10
Z	2	25	40	10

[a]RDA = recommended daily allowance

Table 5.8

Vitamin Capsule	Vitamin Present in Capsule	Amount of Vitamin in Capsule in IU[a]	Dosage: Capsules / Day	No. of Capsules per Bottle
1	A	10,000	1	100
1	D	400	1	100
1	E	30	1	100
2	A	4,000	1	250
2	D	400	1	250
2	E	15	1	250

[a]IU = international units

5.5 The Pigeonhole Principle

A change of pace is in order as we introduce an interesting distribution principle. This principle may seem to have nothing in common with what we have been doing so far, but it will prove to be helpful nonetheless.

In mathematics one sometimes finds that an almost obvious idea, when applied in a rather subtle manner, is the key needed to solve a troublesome problem. On the list of such obvious ideas many would undoubtedly place the following rule, known as the *pigeonhole principle*.

The Pigeonhole Principle: **If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.**

One situation for 6 ($= m$) pigeons and 4 ($= n$) pigeonholes (actually birdhouses) is shown in Fig. 5.7. The general result readily follows by the method of proof by contradiction. If the result is not true, then each pigeonhole has at most one pigeon roosting in it — for a total of at most n ($< m$) pigeons. (Somewhere we have lost at least $m - n$ pigeons!)

But now what can pigeons roosting in pigeonholes have to do with mathematics — discrete, combinatorial, or otherwise? Actually, this principle can be applied in various problems in which we seek to establish whether a certain situation can actually occur. We

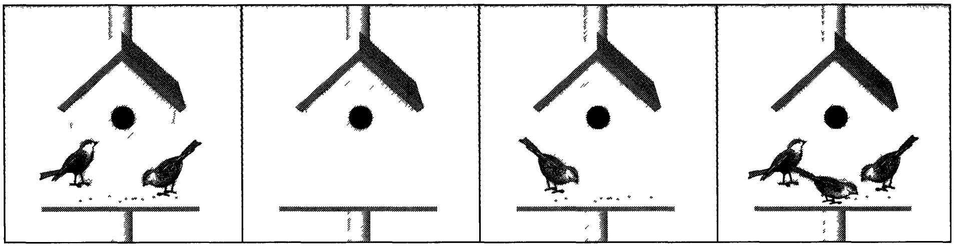

Figure 5.7

illustrate this principle in the following examples and shall find it useful in Section 5.6 and at other points in the text.

EXAMPLE 5.39

An office employs 13 file clerks, so at least two of them must have birthdays during the same month. Here we have 13 pigeons (the file clerks) and 12 pigeonholes (the months of the year).

Here is a second rather immediate application of our principle.

EXAMPLE 5.40

Larry returns from the laundromat with 12 pairs of socks (each pair a different color) in a laundry bag. Drawing the socks from the bag randomly, he'll have to draw at most 13 of them to get a matched pair.

From this point on, application of the pigeonhole principle may be more subtle.

EXAMPLE 5.41

Wilma operates a computer with a magnetic tape drive. One day she is given a tape that contains 500,000 "words" of four or fewer lowercase letters. (Consecutive words on the tape are separated by a blank character.) Can it be that the 500,000 words are all distinct?

From the rules of sum and product, the total number of different possible words, using four or fewer letters, is

$$26^4 + 26^3 + 26^2 + 26 = 475{,}254.$$

With these 475,254 words as the pigeonholes, and the 500,000 words on the tape as the pigeons, it follows that at least one word is repeated on the tape.

EXAMPLE 5.42

Let $S \subset \mathbf{Z}^+$, where $|S| = 37$. Then S contains two elements that have the same remainder upon division by 36.

Here the pigeons are the 37 positive integers in S. We know from the division algorithm (of Theorem 4.5) that when any positive integer n is divided by 36, there exists a unique quotient q and unique remainder r, where

$$n = 36q + r, \qquad 0 \leq r < 36.$$

The 36 possible values of r constitute the pigeonholes, and the result is now established by the pigeonhole principle.

EXAMPLE 5.43

Prove that if 101 integers are selected from the set $S = \{1, 2, 3, \ldots, 200\}$, then there are two integers such that one divides the other.

For each $x \in S$, we may write $x = 2^k y$, with $k \geq 0$, and $\gcd(2, y) = 1$. (This result follows from the Fundamental Theorem of Arithmetic.) Then y must be odd, so $y \in T = \{1, 3, 5, \ldots, 199\}$, where $|T| = 100$. Since 101 integers are selected from S, by the pigeonhole principle there are two distinct integers of the form $a = 2^m y$, $b = 2^n y$ for some (the same) $y \in T$. If $m < n$, then $a|b$; otherwise, we have $m > n$ and then $b|a$.

EXAMPLE 5.44

Any subset of size 6 from the set $S = \{1, 2, 3, \ldots, 9\}$ must contain two elements whose sum is 10.

Here the pigeons constitute a six-element subset of $\{1, 2, 3, \ldots, 9\}$, and the pigeonholes are the subsets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$. When the six pigeons go to their respective pigeonholes, they must fill at least one of the two-element subsets whose members sum to 10.

EXAMPLE 5.45

Triangle ACE is equilateral with $AC = 1$. If five points are selected from the interior of the triangle, there are at least two whose distance apart is less than $1/2$.

For the triangle in Fig. 5.8, the four smaller triangles are congruent equilateral triangles and $AB = 1/2$. We break up the interior of triangle ACE into the following four regions, which are mutually disjoint in pairs:

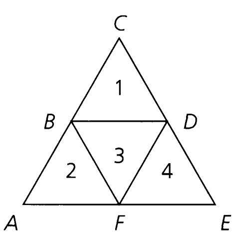

Figure 5.8

R_1: the interior of triangle BCD together with the points on the segment BD, excluding B and D.

R_2: the interior of triangle ABF.

R_3: the interior of triangle BDF together with the points on the segments BF and DF, excluding B, D, and F.

R_4: the interior of triangle FDE.

Now we apply the pigeonhole principle. Five points in the interior of triangle ACE must be such that at least two of them are in one of the four regions R_i, $1 \leq i \leq 4$, where any two points are separated by a distance less than $1/2$.

EXAMPLE 5.46

Let S be a set of six positive integers whose maximum is at most 14. Show that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

For each nonempty subset A of S, the sum of the elements in A, denoted s_A, satisfies $1 \leq s_A \leq 9 + 10 + \cdots + 14 = 69$, and there are $2^6 - 1 = 63$ nonempty subsets of S. We should like to draw the conclusion from the pigeonhole principle by letting the possible sums, from 1 to 69, be the pigeonholes, with the 63 nonempty subsets of S as the pigeons, but then we have too few pigeons.

So instead of considering all nonempty subsets of S, we cut back to those nonempty subsets A of S where $|A| \leq 5$. Then for each such subset A it follows that $1 \leq s_A \leq 10 + 11 + \cdots + 14 = 60$. There are 62 nonempty subsets A of S with $|A| \leq 5$ —namely, all the subsets of S except for $\emptyset$ and the set S itself. With 62 pigeons (the nonempty subsets A of S where $|A| \leq 5$) and 60 pigeonholes (the possible sums s_A), it follows by the pigeonhole principle that the elements of at least two of these 62 subsets must yield the same sum.

EXAMPLE 5.47

Let $m \in \mathbf{Z}^+$ with m odd. Prove that there exists a positive integer n such that m divides $2^n - 1$.

Consider the $m + 1$ positive integers $2^1 - 1, 2^2 - 1, 2^3 - 1, \ldots, 2^m - 1, 2^{m+1} - 1$. By the pigeonhole principle and the division algorithm there exist $s, t \in \mathbf{Z}^+$ with $1 \leq s < t \leq m + 1$, where $2^s - 1$ and $2^t - 1$ have the same remainder upon division by m. Hence $2^s - 1 = q_1 m + r$ and $2^t - 1 = q_2 m + r$, for $q_1, q_2 \in \mathbf{N}$, and $(2^t - 1) - (2^s - 1) = (q_2 m + r) - (q_1 m + r)$, so $2^t - 2^s = (q_2 - q_1)m$. But $2^t - 2^s = 2^s(2^{t-s} - 1)$; and since m is odd, we have $\gcd(2^s, m) = 1$. Hence $m|(2^{t-s} - 1)$, and the result follows with $n = t - s$.

EXAMPLE 5.48

While on a four-week vacation, Herbert will play at least one set of tennis each day, but he won't play more than 40 sets total during this time. Prove that no matter how he distributes his sets during the four weeks, there is a span of consecutive days during which he will play exactly 15 sets.

For $1 \leq i \leq 28$, let x_i be the total number of sets Herbert will play from the start of the vacation to the end of the ith day. Then $1 \leq x_1 < x_2 < \cdots < x_{28} \leq 40$, and $x_1 + 15 < \cdots < x_{28} + 15 \leq 55$. We now have the 28 distinct numbers $x_1, x_2, \ldots, x_{28}$ and the 28 distinct numbers $x_1 + 15, x_2 + 15, \ldots, x_{28} + 15$. These 56 numbers can take on only 55 different values, so at least two of them must be equal, and we conclude that there exist $1 \leq j < i \leq 28$ with $x_i = x_j + 15$. Hence, from the start of day $j + 1$ to the end of day i, Herbert will play exactly 15 sets of tennis.

Our last example for this section deals with a classic result that was first discovered in 1935 by Paul Erdös and George Szekeres.

EXAMPLE 5.49

Let us start by considering two particular examples:

1) Note how the sequence 6, 5, 8, 3, 7 (of length 5) contains the decreasing subsequence 6, 5, 3 (of length 3).
2) Now note how the sequence 11, 8, 7, 1, 9, 6, 5, 10, 3, 12 (of length 10) contains the increasing subsequence 8, 9, 10, 12 (of length 4).

These two instances demonstrate the general result: For each $n \in \mathbf{Z}^+$, a sequence of $n^2 + 1$ distinct real numbers contains a decreasing or increasing subsequence of length $n + 1$.

To verify this claim let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. For $1 \leq k \leq n^2 + 1$, let

x_k = the maximum length of a decreasing subsequence that ends with a_k, and

y_k = the maximum length of an increasing subsequence that ends with a_k.

For instance, our second particular example would provide

k	1	2	3	4	5	6	7	8	9	10
a_k	11	8	7	1	9	6	5	10	3	12
x_k	1	2	3	4	2	4	5	2	6	1
y_k	1	1	1	1	2	2	2	3	2	4

If, in general, there is no decreasing or increasing subsequence of length $n+1$, then $1 \leq x_k \leq n$ and $1 \leq y_k \leq n$ for all $1 \leq k \leq n^2+1$. Consequently, there are at most n^2 *distinct* ordered pairs (x_k, y_k). But we have n^2+1 ordered pairs (x_k, y_k), since $1 \leq k \leq n^2+1$. So the pigeonhole principle implies that there are two identical ordered pairs (x_i, y_i), (x_j, y_j), where $i \neq j$ — say $i < j$. Now the real numbers $a_1, a_2, \ldots, a_{n^2+1}$ are distinct, so if $a_i < a_j$ then $y_i < y_j$, while if $a_j < a_i$ then $x_j > x_i$. In either case we no longer have $(x_i, y_i) = (x_j, y_j)$. This contradiction tells us that $x_k = n+1$ or $y_k = n+1$ for some $n+1 \leq k \leq n^2+1$; the result then follows.

For an interesting application of this result, consider n^2+1 sumo wrestlers facing forward and standing shoulder to shoulder. (Here no two wrestlers have the same weight.) We can select $n+1$ of these wrestlers to take one step forward so that, as they are scanned from left to right, their successive weights either decrease or increase.

EXERCISES 5.5

1. In Example 5.40, what plays the roles of the pigeons and of the pigeonholes?

2. Show that if eight people are in a room, at least two of them have birthdays that occur on the same day of the week.

3. An auditorium has a seating capacity of 800. How many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first and last initials?

4. Let $S = \{3, 7, 11, 15, 19, \ldots, 95, 99, 103\}$. How many elements must we select from S to insure that there will be at least two whose sum is 110?

5. a) Prove that if 151 integers are selected from $\{1, 2, 3, \ldots, 300\}$, then the selection must include two integers x, y where $x|y$ or $y|x$.

b) Write a statement that generalizes the results of part (a) and Example 5.43.

6. Prove that if we select 101 integers from the set $S = \{1, 2, 3, \ldots, 200\}$, there exist m, n in the selection where $\gcd(m, n) = 1$.

7. a) Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$, there are at least two whose sum is 26.

b) Write a statement that generalizes the results of part (a) and Example 5.44.

8. a) If $S \subseteq \mathbf{Z}^+$ and $|S| \geq 3$, prove that there exist distinct $x, y \in S$ where $x + y$ is even.

b) Let $S \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$. Find the minimal value of $|S|$ that guarantees the existence of distinct ordered pairs $(x_1, x_2), (y_1, y_2) \in S$ such that $x_1 + y_1$ and $x_2 + y_2$ are both even.

c) Extending the ideas in parts (a) and (b), consider $S \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+ \times \mathbf{Z}^+$. What size must $|S|$ be to guarantee the existence of distinct ordered triples $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$ where $x_1 + y_1$, $x_2 + y_2$, and $x_3 + y_3$ are all even?

d) Generalize the results of parts (a), (b), and (c).

e) A point $P(x, y)$ in the Cartesian plane is called a *lattice point* if $x, y \in \mathbf{Z}$. Given distinct lattice points $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_n(x_n, y_n)$, determine the smallest value of n that guarantees the existence of $P_i(x_i, y_i), P_j(x_j, y_j)$, $1 \leq i < j \leq n$, such that the midpoint of the line segment connecting $P_i(x_i, y_i)$ and $P_j(x_j, y_j)$ is also a lattice point.

9. a) If 11 integers are selected from $\{1, 2, 3, \ldots, 100\}$, prove that there are at least two, say x and y, such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

b) Write a statement that generalizes the result of part (a).

10. Let triangle ABC be equilateral, with $AB = 1$. Show that if we select 10 points in the interior of this triangle, there must be at least two whose distance apart is less than $1/3$.

11. Let $ABCD$ be a square with $AB = 1$. Show that if we select five points in the interior of this square, there are at least two whose distance apart is less than $1/\sqrt{2}$.

12. Let $A \subseteq \{1, 2, 3, \ldots, 25\}$ where $|A| = 9$. For any subset B of A let s_B denote the sum of the elements in B. Prove that there are distinct subsets C, D of A such that $|C| = |D| = 5$ and $s_C = s_D$.

13. Let S be a set of five positive integers the maximum of which is at most 9. Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

14. During the first six weeks of his senior year in college, Brace sends out at least one resumé each day but no more than 60 resumés in total. Show that there is a period of consecutive days during which he sends out exactly 23 resumés.

15. Let $S \subset \mathbf{Z}^+$ with $|S| = 7$. For $\emptyset \neq A \subseteq S$, let s_A denote the sum of the elements in A. If m is the maximum element in S, find the possible values of m so that there will exist distinct subsets B, C of S with $s_B = s_C$.

16. Let $k \in \mathbf{Z}^+$. Prove that there exists a positive integer n such that $k|n$ and the only digits in n are 0's and 3's.

17. **a)** Find a sequence of four distinct real numbers with no decreasing or increasing subsequence of length 3.

b) Find a sequence of nine distinct real numbers with no decreasing or increasing subsequence of length 4.

c) Generalize the results in parts (a) and (b).

d) What do the preceding parts of this exercise tell us about Example 5.49?

18. The 50 members of Nardine's aerobics class line up to get their equipment. Assuming that no two of these people have the same height, show that eight of them (as the line is equipped from first to last) have successive heights that either decrease or increase.

19. For $k, n \in \mathbf{Z}^+$, prove that if $kn + 1$ pigeons occupy n pigeonholes, then at least one pigeonhole has $k + 1$ or more pigeons roosting in it.

20. How many times must we roll a single die in order to get the same score (a) at least twice? (b) at least three times? (c) at least n times, for $n \geq 4$?

21. **a)** Let $S \subset \mathbf{Z}^+$. What is the smallest value for $|S|$ that guarantees the existence of two elements $x, y \in S$ where x and y have the same remainder upon division by 1000?

b) What is the smallest value of n such that whenever $S \subseteq \mathbf{Z}^+$ and $|S| = n$, then there exist three elements $x, y, z \in S$ where all three have the same remainder upon division by 1000?

c) Write a statement that generalizes the results of parts (a) and (b) and Example 5.42.

22. For $m, n \in \mathbf{Z}^+$, prove that if m pigeons occupy n pigeonholes, then at least one pigeonhole has $\lfloor (m-1)/n \rfloor + 1$ or more pigeons roosting in it.

23. Let $p_1, p_2, \ldots, p_n \in \mathbf{Z}^+$. Prove that if $p_1 + p_2 + \cdots + p_n - n + 1$ pigeons occupy n pigeonholes, then either the first pigeonhole has p_1 or more pigeons roosting in it, or the second pigeonhole has p_2 or more pigeons roosting in it, ..., or the nth pigeonhole has p_n or more pigeons roosting in it.

24. Given 8 Perl books, 17 Visual BASIC† books, 6 Java books, 12 SQL books, and 20 C++ books, how many of these books must we select to insure that we have 10 books dealing with the same computer language?

5.6 Function Composition and Inverse Functions

When computing with the elements of $\mathbf{Z}$, we find that the (closed binary) operation of addition provides a method for combining two integers, say a and b, into a third integer, namely $a + b$. Furthermore, for each integer c there is a second integer d where $c + d = d + c = 0$, and we call d the additive *inverse* of c. (It is also true that c is the additive *inverse* of d.)

Turning to the elements of $\mathbf{R}$ and the (closed binary) operation of multiplication, we have a method for combining any $r, s \in \mathbf{R}$ into their product rs. And here, for each $t \in \mathbf{R}$, if $t \neq 0$, then there is a real number u such that $ut = tu = 1$. The real number u is called the multiplicative *inverse* of t. (The real number t is also the multiplicative *inverse* of u.)

In this section we first study a method for combining two functions into a single function. Then we develop the concept of the inverse (of a function) for functions with certain properties. To accomplish these objectives, we need the following preliminary ideas.

†Visual BASIC is a trademark of the Microsoft Corporation.