MATHEMATICS 101 (Sections A01-A05),
Midterm # 1, February 08, 2020.
Time: 90 minutes

Last name: ______________________________ StudentID: $V00$______________

First name: ______________________________ Tutorial section number: # A04 T35

Problem #	1	2	3	4	5	6	7	8	9	10	11	12	TOTAL
Points (max)	2	2	4	2	2	4	3	2	4	2	3	3	33
Score	1	2	1	0	1	0	∅	1.5	∅.5	0	1.5	1.5	10

- Only calculators which start with Sharp EL-510R are allowed.
- This test consists of 12 questions and has 14 pages (including this cover and a **Formula sheet** on the back of the first page).
  - All Questions are long-answer questions and can have partial marks.
  - Write your full answer in this booklet in the provided space for every question. You need to show your work for all answers, as we may disallow any answer which is not properly justified.
- Before starting your test enter your Name (Last, First), student ID, and tutorial section number (T01 - T35) on this page.
- If you complet the exam more than 10 minutes before the end of the examination, you are free to leave the room after submitting your paper. Please **remain seated** for the remaining 10 minutes of the examination. It is important to minimize the disruptions in the room.
- At the end of 90-minute test, turn-in this booklet.
- **Do not remove** any pages from this booklet, including the formula sheet.
- No cell phones, smart watches or external papers are allowed to be brought to the table while you are in the examination room. If you have any of the unpermitted items with you at the table, raize your hand now and give the item(s) to one of the invigilators.
- Use the back of each page as your draft paper.
- This is version B of the Midterm #1.

MATHEMATICS 101 (Sections A01 - A05, Spring 2020)
Formula sheet
Midterm #1

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, (u < a)$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C, (u > a)$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$$\cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

$$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$$

1. (2 points) Evaluate $\int \tan(2x)\, dx$

$= \int \frac{1}{2} \cdot \tan(u)\, du$

$u = 2x$

$du = 2dx$

$= \frac{1}{2} \int \tan(u)\, du$

$\frac{du}{2} = dx$

$= \frac{1}{2} \int \frac{\sin(u)}{\cos(u)}\, du$

$\tan = \frac{\sin}{\cos}$

$= \frac{1}{2} \int \sin(u) \cdot \sec(u)\, du$

$\int \sin \quad -\cos$

$\ln \cos$

$= \frac{1}{2}(-\cos(u) \cdot \ln|\sec u + \tan u|) + c$

$= -\frac{1}{2} \cos(2x) \cdot \ln|\sec(2x) + \tan(2x)| + c$

1

ANSWER: $-\frac{1}{2} \cos(2x) \cdot \ln|\sec(2x) + \tan(2x)| + c$

2. (2 points) Evaluate $\int xe^{3x}\,dx$

$\int u\,dv = uv - \int v\,du$

$u = x$ $\quad v = \frac{1}{3}e^{3x}$

$du = 1dx$ $\quad dv = e^{3x}dx$

$\int xe^{3x}dx = x \cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$

$= \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx$

$u = 3x$

$du = 3dx$

$= \quad - \frac{1}{3}\cdot\frac{1}{3}\int e^{u}du$

$= \frac{xe^{3x}}{3} - \frac{1}{9}e^{u} + C$

$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$

2

ANSWER: $\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$

3. (4 points) A thermometer is removed from a room where the temperature is 90°F and is taken outside, where the air temperature is 10°F. After half minute the thermometer reads 45°F. What is the reading of the thermometer at $t = 1$ min?

Show your **complete work in deriving solution** to the differential equation.

Newton's Law of Cooling: $\frac{dH}{dt} = -k\,(H - H_s)$ or $\frac{dT}{dt} = -k\,(T - T_s)$.

$$\frac{dT}{(T - T_s)} = -k\,dt$$

$$\int \frac{dT}{T - T_s} = \int -k\,dt$$

$$\ln|T - T_s| + c = -kt + c$$

$$\ln|90 - 45| = -k \cdot 0.5$$

$$\ln|45 - T_s| = -k \cdot 0.5$$

$$\frac{\ln|90 - 45|}{\ln|45 - T_s|} = 1$$

1

ANSWER:

4. (2 points) During the first examination in Calculus I, the following question was given:

Calculate the definite integral:

$$\int_{-1}^{5} \sqrt{x^2 - 4x + 4}\, dx.$$

A student provided the following answer:

$$12.5 - 10 - 0.5 + 2$$

$$\int_{-1}^{5} \sqrt{x^2 - 4x + 4}\, dx = \int_{-1}^{5} \sqrt{(x-2)^2}\, dx = \int_{-1}^{5} (x-2)dx = \left(\frac{x^2}{2} - 2x\right)\Bigg|_{-1}^{5} = \underbrace{\frac{25}{2} - 10 - \left(\frac{1}{2} + 2\right)}_{\text{this is not zero lol}} = 0.$$

Provide detailed correct solution to this question.

Write your numerical answer in the ANSWER box below.

$$\int_{-1}^{5} \sqrt{x^2-4x+4}\ dx \qquad x^2-4x+4 = (x-2)^2$$

$$= \int_{-1}^{5} \sqrt{(x-2)^2}\ dx \qquad u = x-2 \qquad u = 5-2 = 3$$

$$du = 1dx \qquad u = -1-2 = -3$$

$$= \int_{-3}^{3} \sqrt{u^2}\ du$$

$$= \int_{-3}^{3} |u| du$$

$$= \left[\frac{u^2}{2}\right]_{-3}^{3} = \frac{3^2}{2} - \frac{(-3)^2}{2} = 0$$

ANSWER: 0 ✗

5. (2 points) Evaluate $\int_{-\infty}^{0} e^{-|3x|}\,dx$.

indefinite integral

$$\int_{-\infty}^{0} e^{-|3x|}\,dx = \int_{-\infty}^{-1} e^{-|3x|}\,dx + \int_{-1}^{0} e^{-|3x|}\,dx$$

$$= \lim_{b\to-\infty} \int_{b}^{-1} e^{-|3x|}\,dx + \int_{-1}^{0} e^{-|3x|}\,dx$$

bad one

'normal' integral

$$= \lim_{b\to-\infty}\left[-\frac{1}{3}e^{-|3x|}\right]_b^{-1} + \left[-\frac{1}{3}e^{-|3x|}\right]_{-1}^{0}$$

$$\int e^{-|3x|}\,dx \qquad u = |3x| = 3x$$

$$du = 3\,dx$$

x always negative on this interval

$$\frac{1}{3}\int e^{-u}\,du$$

$$= -\frac{1}{3}e^{-u}$$

$$= -\frac{1}{3}e^{-|3x|}$$

$$= \lim_{b\to-\infty}\left(-\frac{1}{3}e^{-|-1\cdot 3|} + \frac{1}{3}e^{-|3b|}\right) - \frac{1}{3}\cdot e^{-3}$$

$$= -\frac{1}{3}e^{-3} - \frac{1}{3}e^{-3}$$

$$= -\frac{2}{3}e^{-3}$$

as $b \to -\infty$,

$e^{-x} \to 0 \qquad e^{-x} = \frac{1}{e^x}$

ANSWER: $-\frac{2}{3e^3}$ ✗

6. (4 points) Evaluate $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

$$\frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+C,\quad u<a$$

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

a^2-x^2 $\qquad x = a\sin\theta,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Let $x = 2\sin\theta$

$$= \int x^2 \cdot (4-x^2)^{-1/2}\,dx$$

well i'm lost

this sub doesn't seem to help

$$= (4-x^2)^{-1/2} \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot x(4-x^2)^{-3/2}$$

that is not any better

$u = (4-x^2)^{-1/2}$ $\qquad v = \frac{1}{3}x^3$

$dv = x^2$

$du = -\frac{1}{2}(4-x^2)^{-3/2} \cdot -2x$

int. by parts doesn't turn out well

$du = x(4-x^2)^{-3/2}$

$\int u\,dv = uv - \int v\,du$ X

ANSWER:

7. (3 points) Evaluate $\int \frac{\ln(\sin(x))}{\sec(x)}\,dx$

$\emptyset$

$= \int \ln(\sin x) \cdot \cos(x)\,dx$

$\int \ln(x) =$

$= \int e^{\ln(\sin x)} \cdot e^{\cos(x)}\,dx$

$= \int \sin(x) \cdot e^{\cos(x)}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= -\int e^u\,du$

$= -e^u$

$= -e^{\cos x}$

$= -\cos x$

ANSWER: $-\cos x$ X

8. (2 points) Evaluate $\int \sec^6 x \, dx$

1.5

tilg sub?

$$\int \sec^6 x \, dx = \int (\sec^2 x)^2 \cdot \sec^2 x \, dx$$

$\sec^2\theta = \tan^2\theta - 1$

$$= \int (\tan^2 x + 1)^2 \cdot \sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int (u^2 - 1)^2 \, du$$

$$= \int (u^4 - 2u^2 + 1) \, du$$

$$= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C$$

$$= \frac{1}{5}\tan^5 x - \frac{2}{3}\tan^3 x + \tan x + C$$

ANSWER: $\frac{1}{5}\tan^5 x - \frac{2}{3}\tan^3 x + \tan x + C$

9. (4 points) Evaluate $\int_{-1}^{0} \frac{dx}{(x-1)(x^2+1)}$

partial fractions

0.5

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x^2+1)}$$

$$1 = A(x^2-1) + B(x-1)$$

$$1 = Ax^2 - A + Bx - B$$

$$1 = Ax^2 + Bx - (A+B)$$

$$1 = -(A+B)$$

$$-1 = A+B$$

$$A = -1, \quad B = 0$$

$$= -\frac{1}{x-1}$$

$$\int_{-1}^{0} -\frac{dx}{x-1} = -\int_{-1}^{0} \frac{dx}{x-1} = -\ln|x-1|\Big]_{-1}^{0}$$

$$= -\ln|0-1| + \ln|-1-1|$$

$$= 0 + \ln 2$$

ANSWER: $\ln(2)$

10. (2 points) Evaluate

$$\int \frac{x^2}{(1-x^3)^{5/2}}\,dx$$

$= \int x^2 \cdot (1-x^3)^{-\frac{5}{2}}\,dx$

$u = x^3$

$du = 3x^2\,dx$

$= \frac{1}{3}\int (1-u)^{-\frac{5}{2}}\,du$

reverse chain

$= \frac{1}{3}\cdot\left(-\frac{2}{3}(1-u)^{-\frac{3}{2}}\cdot\left(u-\frac{1}{2}u^2\right)\right)+c$

0

$= -\frac{2u-u^2}{9(1-u)^{3/2}}+c \quad = -\frac{2x^3-x^6}{9(1-x^3)^{3/2}}$

ANSWER: $-\frac{2x^3-x^6}{9(1-x^3)^{3/2}}$

11. (3 points) Evaluate $\int 4x\sec^2(2x)\,dx$.

$\int u\,dv = uv - \int v\,du$

$\int 4x\sec^2(2x)\,dx$

$u = 4x$ $\quad v = \frac{1}{2}\tan(2x)$

$du = 4dx$ $\quad dv = \sec^2(2x)\,dx$

$= 4x \cdot \frac{1}{2}\tan(2x) - \int \frac{1}{2}\tan(2x) \cdot 4dx$

$= 2x\tan(2x) - 2\int \tan(2x)\,dx$

see q1 for full solution

$= 2x\tan(2x) - 2\left(-\frac{1}{2}\cos(2x) \cdot \ln|\sec(2x) + \tan(2x)|\right) + C$

$= 2x\tan(2x) - \cos(2x)\ln|\sec(2x) + \tan(2x)| + C$

1.5

ANSWER: $2x\tan(2x) - \cos(2x)\ln|\sec(2x) + \tan(2x)| + C$

12. (3 points) Test integral on convergence, justifying each statement you make:

$$\int_2^\infty \frac{\sqrt{t^2+3}}{t^3}\,dt.$$

$\frac{1}{x^p}$ converges on $\int_1^\infty$ if $p \geq 1$

$$= \int_1^\infty \frac{\sqrt{t^2+3}}{t^3} - \int_1^2 \frac{\sqrt{t^2+3}}{t^3}$$

Looking at $1-\infty$ interval, $\sqrt{t^2+3}$ is always >1

$\sqrt{1^2+3} = \sqrt{4} = 2$

Therefore, $\frac{\sqrt{t^2+3}}{t^3} > \frac{1}{t^3}$

Using direct comparison test, we know that since $\frac{\sqrt{t^2+3}}{t^3}$ is greater than $\frac{1}{t^3}$, it will also converge.

Looking at interval $1-2$, $\frac{\sqrt{t^2+3}}{t^3}$ will always be >1

On this region, $\frac{1}{t^3}$ converges to $\frac{1}{8}$, so $\frac{\sqrt{t^2+3}}{t^3}$ will also converge.

1.5

ANSWER: Converges

—— END OF THE EXAMINATION ——