Student: Arfaz Hossain **Date:** 02/28/22	**Instructor:** Muhammad Awais **Course:** Math 101 A04 Spring 2022	**Assignment:** HW-5 [Sections 10.1, 10.2 & 10.3]

Use the Integral Test to determine if the series shown below converges or diverges. Be sure to check that the conditions of the Integral Test are satisfied.

$$\sum_{n=1}^{\infty} \frac{5}{n^2+121}$$

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the Integral Test states that the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

Check to see whether the Integral Test can be applied to this series. First identify f(n).

$$f(n) = \frac{5}{n^2+121}$$

Since the first index in the sum is n = 1, see if the conditions hold for N = 1. Since f(x) is a rational function whose denominator has no real zeros, it is continuous for all $x \geq 1$.

Since $x^2 \geq 0$, the denominator $x^2 + 121 \geq 0$ for all x. Also, the numerator is a constant, positive value. So, f(x) is positive for all $x \geq 1$.

Since the denominator of f(x), $x^2 + 121$, is increasing, and the numerator is constant, f(x) is decreasing for all $x \geq 1$.

Therefore, f(x) is continuous, positive, and decreasing for all $x \geq 1$, so the conditions of the Integral Test are satisfied with N = 1. Now determine whether the integral $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{5}{x^2+121}dx$ converges or diverges.

Use a basic integration formula to evaluate the integral. To evaluate $\int_1^{\infty} \frac{5}{x^2+121}dx$, use the formula

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\,\mathbf{tan}^{-1}\left(\frac{x}{a}\right) + C.$$

Apply the formula $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\,\mathbf{tan}^{-1}\left(\frac{x}{a}\right) + C$ with a = 11 to evaluate the indefinite integral $\int \frac{5}{x^2+121}dx$.

$$\int \frac{5}{x^2+121}dx = \frac{5}{11}\,\mathbf{tan}^{-1}\left(\frac{x}{11}\right) + C$$

Apply the Fundamental Theorem of Calculus to evaluate the integral $\int_1^{\infty} \frac{5}{x^2+121}dx$.

$$\begin{aligned}\int_1^{\infty} \frac{5}{x^2+121}dx &= \lim_{b\to\infty}\left[\frac{5}{11}\,\mathbf{tan}^{-1}\left(\frac{x}{11}\right)\right]_1^b \\ &= \lim_{b\to\infty}\left[\frac{5}{11}\,\mathbf{tan}^{-1}\left(\frac{b}{11}\right) - \frac{5}{11}\,\mathbf{tan}^{-1}\left(\frac{1}{11}\right)\right] \\ &= \frac{5}{11}\lim_{b\to\infty}\mathbf{tan}^{-1}\left(\frac{b}{11}\right) - \frac{5}{11}\,\mathbf{tan}^{-1}\left(\frac{1}{11}\right)\end{aligned}$$

Since $\lim_{b\to\infty} \textbf{tan}^{-1}\left(\frac{b}{11}\right) = \frac{\pi}{2}$, then $\frac{5}{11}\lim_{b\to\infty} \textbf{tan}^{-1}\left(\frac{b}{11}\right) = \frac{5\pi}{22}$.

Use the value of the limit to finish evaluating the integral.

$$\begin{aligned}\int_1^\infty \frac{5}{x^2+121}dx &= \frac{5\pi}{22} - \frac{5}{11}\textbf{tan}^{-1}\left(\frac{1}{11}\right)\\ &= \frac{5}{11}\left(\frac{\pi}{2} - \textbf{tan}^{-1}\left(\frac{1}{11}\right)\right)\end{aligned}$$

Thus, the integral $\int_1^\infty \frac{5}{x^2+121}dx$ converges. Since the integral converges, the series $\sum_{n=1}^{\infty} \frac{5}{n^2+121}$ also converges by the Integral Test.