Math 110 - Homework 10
Topic: Eigenvalues and eigenvectors

Due at 6:00pm (Pacific) on Friday, November 26, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 5.1 of the online textbook, as well as the supplemental questions about complex numbers posted on Brightspace.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` is a square matrix then the command `eig(A)` returns a list of the eigenvalues of A, with each one repeated according to its algebraic multiplicity.
- If `A` is a square matrix and you run the command `[P,D] = eig(A)` then D will be a diagonal matrix whose entries are the eigenvalues of A (repeated according to algebraic multiplicity) and P will be a matrix whose columns are eigenvectors ordered so that the jth column of P is an eigenvector for the eigenvalue appearing in position (j, j) of D. If it is possible, P will be made to be invertible.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. Find all of the eigenvalues of A, as well as the algebraic and geometric multiplicities of each eigenvalue.

2. Let L be the line through the origin in $\mathbb{R}^2$ with slope 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto L, that is, $T(\overrightarrow{v}) = \text{proj}_{\overrightarrow{d}}(\overrightarrow{v})$, where $\overrightarrow{d}$ is a direction vector for L. You may use, without proof, the fact that T is a linear transformation.
   (a) Find the matrix $[T]$.
   (b) Find the eigenvalues of $[T]$.
   (c) Find a basis for each eigenspace of $[T]$.
   (d) Briefly explain what your results from (b) and (c) mean geometrically.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. In this question, as usual, $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ are the standard basis vectors for $\mathbb{R}^3$ (that is, $\overrightarrow{e_j}$ has a 1 in the jth position, and has 0 everywhere else).
   (a) Suppose that D is a 3×3 diagonal matrix. Show that $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ are eigenvectors of D.
   (b) Suppose that A is a 3×3 matrix, and that $\overrightarrow{e_1}, \overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are eigenvectors of A. Is it true that A must be a diagonal matrix? If so, explain why. If not, give a specific example of a non-diagonal matrix A for which $\overrightarrow{e_1}, \overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are eigenvectors.

2. Let $B = \begin{bmatrix} -2 & -14 & 0 & -6 & -12 & -6 \\ 21 & 17 & 25 & -8 & 16 & 8 \\ -1 & 3 & 5 & -2 & 4 & 2 \\ -8 & 4 & -20 & 4 & 12 & 16 \\ -26 & -2 & -80 & 8 & 4 & 2 \\ 4 & -2 & 40 & -12 & 4 & 12 \end{bmatrix}$.
   Explain why every eigenvector of B must have at least one entry that is not a real number.