Q:1. We have

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \quad e^{-\ln(2)} = 1 - \ln(2) + \frac{[\ln(2)]^2}{2!} - \frac{[\ln(2)]^3}{3!} + \cdots$$

Therefore, the sum of the series $= e^{-\ln(2)}$

$$= \frac{1}{e^{\ln(2)}}$$

$$= \frac{1}{2}$$

Q: 2. Using the series for $\sin(3x)$:

$$\lim_{x \to 0} \left[\frac{\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \cdots\right)}{x^3} + \frac{a}{x^2} + b \right] = 0$$

$$\Rightarrow \quad \lim_{x \to 0} \left[\frac{3}{x^2} - \frac{9}{2} + \frac{3^5}{5!} x^2 - \cdots + \frac{a}{x^2} + b \right] = 0$$

This is only possible if;

$$\frac{3}{x^2} + \frac{a}{x^2} = 0 \quad \text{and} \quad b - \frac{9}{2} = 0$$

$$\Rightarrow \quad a = -3 \quad \text{and} \quad b = \frac{9}{2}$$

Q:3. (a)

$$\lim_{t\to 0} \frac{1-\cos t}{1+t-e^t} = \lim_{t\to 0} \frac{\cancel{1} - \left(\cancel{1} - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right)}{\cancel{(1+t)} - \left(\cancel{1} + \cancel{t} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)}$$

$$= \lim_{t\to 0} \frac{\frac{1}{2!} - \frac{t^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{t}{3!} + \cdots}$$

$$= \frac{1/2!}{-1/2!}$$

$$= -1$$

(b)

$$\lim_{s\to 0} \frac{\sin(s) + \frac{1}{6}s^3 - s}{s^5} = \lim_{s\to 0} \frac{\left(\cancel{s} - \cancel{\frac{s^3}{3!}} + \frac{s^5}{5!} - \cdots\right) + \cancel{\frac{s^3}{3!}} - \cancel{s}}{s^5}$$

$$= \lim_{s\to 0} \frac{\frac{s^5}{5!} - \frac{s^7}{7!} + \cdots}{s^5}$$

$$= \lim_{s\to 0}\left[\frac{1}{5!}+\frac{s^2}{7!}-\cdots\right]$$

$$= \frac{1}{5!}$$

(c). $$\lim_{r\to 0}\frac{1-\cos^2 r}{\ln(1-r)+\sin r} = \lim_{r\to 0}\frac{1-\left[1-r^2+r^4/3-\cdots\right]}{\left[-r-\frac{r^2}{2}-\frac{r^3}{3}-\cdots\right]+\left[r-\frac{r^3}{3!}+\frac{r^5}{5!}-\cdots\right]}$$

$$= \lim_{r\to 0}\frac{r^2-r^4/3+\cdots}{-\frac{r^2}{2}-\frac{2r^3}{3}-\cdots}$$

$$= \lim_{r\to 0}\frac{1-\frac{r^2}{3}+\cdots}{-\frac{1}{2}-\frac{2}{3}r-\cdots} = -2$$

Q. 4.

$$\int \frac{e^t - 1}{t}\, dt = \int \frac{\left(1 + t + t^2/2! + t^3/3! + \cdots\right) - 1}{t}\, dt$$

$$= \int \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right) dt$$

$$= \left(t + \frac{t^2}{2 \cdot 2!} + \frac{t^3}{3 \cdot 3!} + \cdots\right) + C$$

$$= \sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} + C$$

Q: 5. $\displaystyle\int_0^t \cos(x^3)\,dx$

We have $\displaystyle\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$\Rightarrow \int_0^t \cos(x^3)\,dx = \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(6n+1)(2n)!} \Bigg|_{x=0}^{x=t}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+1}}{(6n+1)(2n)!}$$

Q: 6. We can show that the series representation of $\tan^{-1}(x)$ is given by:

$$\tan^{-1}(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \qquad -1 \le z \le 1$$

Taking $z = 1$

$$\Rightarrow \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

or $$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$