CHAPTER 3

Solutions to Chapter-End Problems

A. Key Concepts

Recognizing Cash Flows:

3.1 Boarding: an annuity paid monthly, at the end of the months May to September, brought to present worth to the beginning of May. Monthly estimates obtained by historical measurements.

Breeding: annuity due, with a six-month period over 10 years

Heating, water and sewage: monthly annuity, estimated from the average monthly costs over the last few years

Food: annuity due over 10 years, estimated by the amount paid for food in previous years

3.2 Rent: monthly annuity for eight periods. Amount known from rental agreement.

Bus: weekly annuity. Amount estimated as 5 return trips, or if on monthly bus pass, an annuity due as the cost of the pass.

Groceries: weekly annuity, amount estimated or observed weekly average

Lunch: weekly annuity, amount estimated or observed weekly average

Printing and copying: two single payments, one at the end of four months and one at the end of eight months, or an annuity with a four-month period. Amount estimated or observed from previous years.

Christmas presents: single payment, amount as budgeted

Christmas extra cash: single payment, amount as estimated

Single Disbursements or Receipts:

3.3 P = 100
i = 8%
N = 15

Using the formula: $F = P(1 + i)^N = 100(1 + 0.08)^{15} = 317.22$

Or using the compound interest tables:
$F = P(F/P, 8\%, 15) = 100(3.1722) = 317.22$

About $317 will be in the bank account.

3.4 $F = 1\ 000\ 000$
$i = 12\%$
$N = 30$

Using the formula: $P = F/(1 + i)^N = 1\ 000\ 000/(1 + 0.12)^{30} = 33\ 377.92$

Using the tables produces a slightly different result due to the number of significant digits in the table:

$P = F(P/F, 12\%, 30) = 1\ 000\ 000(0.0334) = 33\ 400$

You should invest about R 33 400.

3.5 $1725(F/P, i, 5) = 3450$
$(F/P, i, 5) = 2$

$(F/P, 14\%, 5) = 1.9254$
$(F/P, 15\%, 5) = 2.0114$

Solve for i using linear interpolation:
$i = 0.14 + (0.15 - 0.14)[(2 - 1.9254)/(2.0114 - 1.9254)] = 0.1487 = 14.87\%$

In 10 years, you will have:
$F = 1725(F/P, 14.87\%, 10) = 1725(1 + 0.1487)^{10} = \$6900$

Annuities:

3.6
$$\begin{aligned} P &= 500(P/A, 0.5\%, 12\times20) + 5000 \\ &= 500[(1 + 0.005)^{240} - 1]/[0.005(1 + 0.005)^{240}] + 5000 \\ &= 69\ 790 + 5000 \\ &= 74\ 790 \end{aligned}$$

Morris purchased the house for £74 790.

3.7 Using the capital recovery formula:

$$\begin{aligned} A &= (P - S)(A/P, i, N) + Si \\ &= (45\ 000 - 25\ 000)(A/P, 0.15, 5) + 25\ 000(0.15) \\ &= 20\ 000(0.29832) + 3\ 750 \end{aligned}$$

= 9 716.40

The juicer would have to save a little over $18 400 per year.

3.8 F = 10 000
i = 0.06/12 = 0.005 per month
N = 24

A = 10 000(A/F, 0.5%, 24) = 10 000(0.03932) = 393.20

Fred has to save a little over $393 per month.

Arithmetic Gradient Series:

3.9 **(a)** F = 40(F/A, 1%, 24) = 40(26.969) = 1079

(b) F = [30 + 1(A/G, 1%, 24)](F/A, 1%, 24)
= [30 + 1(11.010)](26.969) = 1106

3.10 First find the annuity value of the prize by converting the gradient into an annuity:

A = A' + G(A/G, i, N)
= 1000 + 1000(A/G, 15%, 20)
= 1000 + 1000(5.3651) = 6365.1

Converting the annuity into a present value gives:

P = A(P/A, 15%, 20) = 6365.1(6.2593) = 39 841

Therefore, the winning ticket has a present worth of about £39 800. Since 10 000 tickets are to be sold, on average each ticket is worth £39 800/10 000 = £3.98.

Geometric Gradient Series:

3.11 First find the growth adjusted interest rate:

$i^\circ = (1 + i)/(1 + g) - 1 = 1.2/1.1 - 1 = 0.0909 = 9.09\%$

We then make use of the geometric series to present worth conversion factor with A = 10 000, g = 10%, $i^\circ = 0.0909$, and N = 10. Since the growth adjusted interest rate is positive, we can make use of the present worth conversion factor.

$P = A(P/A, g, i, N) = A(P/A, i^\circ, N)/(1 + g)$

$= 10\,000(P/A, 9.09\%, 10)/(1 + 0.1)$
$= 10\,000(6.3923/1.1) = 58\,110$

We then use the present worth factor to calculate the worth in 10 years:

$F = 58\,110(F/P, 20\%, 10) = 58\,110(6.1917) = 359\,810$

The savings would have accumulated to about $360 000.

3.12 First find the growth adjusted interest rate:

$i^\circ = (1 + i)/(1 + g) - 1 = 1.1/1.2 - 1 = -0.0833 = -8.333\%$

We then make use of the geometric series to present worth conversion factor with A = 10 000, g = 20%, $i^\circ = -0.0833$, and N = 10. Since the growth adjusted interest rate is negative, we cannot make use of the present worth conversion factor.

$P = A(P/A, g, i, N) = A[(1 + i^\circ)^N - 1]/[i^\circ(1 + i^\circ)^N][1/(1+g)]$
$= 10\,000[(1 - 0.08333)^{10} - 1]/[-0.08333(1 - 0.08333)^{10}][1/(1 + 0.2)]$
$= 138\,710$

We then use the present worth factor to calculate the worth in 10 years:

$F = 138\,710(F/P, 10\%, 10) = 138\,710(2.5937) = 359\,790.$

The savings would have accumulated to about $360 000.

Non-standard Annuities and Gradients:

3.13 *Method 1*: Consider the annuities as separate future payments.

Year	**Present Worth**		
7	10 000(P/F, 9%, 7) = 10 000(0.54703)		= 5470.3
12	10 000(P/F, 9%, 12) = 10 000(0.35553)		= 3555.3
17	10 000(P/F, 9%, 17) = 10 000(0.23107)		= 2310.7
22	10 000(P/F, 9%, 22) = 10 000(0.15018)		= 1501.8
		Total	= 12 838.1

The investment is worth about $12 838 today.

Method 2: Convert the compounding period from yearly, to every five years. This can be done with the effective interest rate formula:

$i_e = (1 + 0.09)^5 - 1 = 53.86\%$

The present value at the end of period 2 is then:

P_2 = 10 000(P/A, 53.86%, 4)

and so the present worth today is:

P_0 = 10 000(P/A, 53.86%, 4)(P/F, 9%, 2)
= 10 000(1.5253)(0.84168)
= \$12 838

Method 3: Convert the annuity to an equivalent yearly annuity. This can be done by considering only the first payment as a future value, and finding the equivalent annuity over the five-year period using the sinking fund factor:

A = 10 000(A/F, 9%, 5)

This yearly annuity is brought to present worth at the end of period 2:

P_2 = 10 000(A/F, 9%, 5)(P/A, 9%, 20)

and so the present worth today is:

P_0 = 10 000(A/F, 9%, 5)(P/A, 9%, 20)(P/F, 9%, 2)
= 10 000(0.16709)(9.1285)(0.84168)
= \$12 838

Note that each method produces the same amount, allowing for rounding. When you have a choice in methods as in this example, your choice will depend on what you find convenient, or what is the most efficient computationally.

3.14 Amount owed in one year:
P(1-year) = 2000(F/P, 0.5%, 12) = 2000(1.0616) = 2123.2

Convert this to an annuity:
A = P(1-year)(A/P, 0.5%, 24) = 2123.2(0.04433) = 94.13

Payments will be \$94.13 per month.

3.15 Find the effective annual interest rate first in order to determine the effective interest rate for the four-month period:

$i_e = (1+.12/12)^{12} - 1 = 0.1268$

$0.1268 = (1 + i_{4\text{-months}})^3 - 1$
$i_{4\text{-months}} = (1.1268)^{1/3} - 1 = 0.040604$

Using this interest rate,

$$P = (P/A, 4.0806\%, 20)[400 + 100(A/G, 4.0806\%, 20)]$$
$$= 13.5179[400 + 100(8.1904)]$$
$$= 16\ 478.91$$

The present worth is $16 479.

When n Goes to Infinity:

3.16 The present worth computations for the full capacity tunnel can be found as follows:

First, the $100 000 paid at the end of 10 years can be thought of as a future amount which has an equivalent annuity:

$$A = 100\ 000(A/F, 8\%, 10) = 100\ 000(0.06903) = 6903$$

Thus, at 8% interest, $100 000 every 10 years is equivalent to $6903 every year.

Since the tunnel will have (approximately) an infinite life, the present worth of the lining repairs can be found using the capitalized cost formula. Added to the initial cost, the total present worth is thus:

$$P(\text{total}) = P(\text{initial cost}) + P(\text{lining}) = 3\ 000\ 000 + 6903/0.08 = 3\ 086\ 288$$

Alternatively, convert to a 10-year annuity period, changing the interest rate instead of the payment amount:

$$i = (1 + 0.08)^{10} - 1 = 1.1589$$

$$P = 3\ 000\ 000 + 100\ 000/1.1589 = \$3\ 086\ 287$$

3.17 $P = A/i = 18\ 000/0.07 = 257\ 143$

GA must donate a little over $257 000.

3.18 Sum the present worth of the $350 annuity and the 10 000 future value at 5% per period over $20\times2 = 40$ periods.

$$P = 350(P/A, 5\%, 40) + 10\ 000(P/F, 5\%, 40)$$
$$= 350(17.158) + 10\ 000(0.14205)$$
$$= 7425.80$$

I should pay no more than (approximately) \$7426 for the bond.

3.19 $P = 10\,000(P/F, 4\%, 18) + (10\,000 \times 0.09/2)(P/A, 4\%, 18)$
$= 10\,000(0.49363) + 450(12.659) = 10\,632$

The bond is worth about \$10 632 today.

Estimating Unknowns:

3.20 Using linear interpolation:

$$X^* = 100 + (200 - 100) \times [(300\,000\,000 - 200\,000\,000)/(360\,000\,000 - 200\,000\,000)]$$
$$= 100 + 100(100\,000\,000/160\,000\,000)$$
$$= 100 + 100(0.625) = 162.5$$

Trenny should be able to afford a 162.5 MW plant.

3.21 From basic principles:

$F = P(1 + i)^N$
$5000 = 2300(1 + i)^7$
$1 + i = (5000/2300)^{1/7}$
$i = (5000/2300)^{1/7} - 1 = 0.1173 = 11.73\%$

Or from the tables:

$F = P(F/P, i, N)$
$5000 = 2300(F/P, i, 7)$
$(F/P, i, 7) = 5000/2300 = 2.1739$

$(F/P, 11\%, 7) = 2.0761$
$(F/P, 12\%, 7) = 2.2106$

By linearly interpolating between the two, we get:

$i = 11 + (12 - 11)[(2.1739 - 2.0761)/(2.2106 - 2.0761)] = 11.73\%$

3.22 You could use either the capital recovery factor or the series present worth factor.

Method 1: Using series present worth factor

$2000 = 40(P/A, 1\%, N)$

We can solve using the formula:

$2000 = 40[(1 + 0.01)^N – 1]/[0.01(1 + 0.01)^N]$
$0.5 = (1.01^N – 1)/1.01^N$
$1 = 1.01^N – 0.5(1.01^N)$
$2 = 1.01^N$
$\ln(2) = N[\ln(1.01)]$
$N = \ln(2)/\ln(1.01) = 69.66$

It will take her about 70 months or about 5.8 years.

Method 2: Using linear interpolation

2000 = 40(P/A, 1%, N)
(P/A, 1%, N) = 50

(P/A, 1%, 70) = 50.169
(P/A, 1%, 65) = 47.627

Linearly interpolating:

N = 65 + 5[(50 – 47.627)/(50.169 – 47.627)] = 69.97

The small difference is the error in linearly interpolating.

B. Applications

3.23 $5000 = (7\times 20\ 000)(P/F, i, 18) = 140\ 000/(1 + i)^{18}$
$(1 + i)^{18} = 140\ 000/5000$
$i = 28^{1/18} – 1 = 0.20336$

The annual rate of return must be 20.3%.

3.24 This problem requires the solution for N in the Sinking Fund Factor. Starting with the definition, the explicit formula for N can be obtained through some manipulation:

$A = F(A/F, i, N) = Fi/[(1 + i)^N – 1]$

which leads to:

$(1 + i)^N = (iF + A)/A$

Taking the logarithm of both sides and rearranging gives:

N = ln[(iF + A)/A]/ln(1 + i)
= ln[(0.10×50 000 + 7000)/7000]/ln(1 + 0.1)
= 5.655

It will take about 5.7 years for the members to save the $50 000.

Note that this problem can also be effectively solved by trial and error calculations.

3.25 **(a)** 2000 + 350 + 210 = 100(P/A, 3%, N)
(P/A, 3%, N) = 2560/100 = 25.6

Solve for N using linear interpolation:
N = 45 + (50 – 45)(25.6 – 24.519)/(25.730 – 24.519) = 49.4632

It will take about 50 months to complete her payment.

(b) A = 2560(A/P, 3%, 24) = 2560(0.05905) = 151.168

Yoko's monthly payment will have to be $151.

3.26 **(a)** Deposit of $15 per week:
F = (15×4)(F/A, 1.5%, 36) = 60[(1.015^{36} – 1)/0.015] = $2836.56

Deposit of $20 per week:
F = (20×4)(F/A, 1.5%, 36) = 80 × (1.015^{36} – 1)/0.015 = $3782.08

(b) A = 5000(A/F, 1.5%, 36) = 5000[0.015/(1.015^{36} – 1)] = 105.76

Rinku needs to deposit $105.76/4 = $26.44 per week.

3.27 A = (P – S)(A/P, i, N) + Si
= (140 000 – 37 000)(A/P, 14%, 5) + 37 000(0.14)
= 133 000(0.2918) + 37 000(0.14)
= 43 989.4

The investment would have to save about $43 989 per year over its 5-year life.

3.28 Using linear interpolation:

X* = 500 + (800 – 500)[(15 000 – 11 350)/(18 950 – 11 350)]
= 500 + 300(3650/7600)

= 500 + 300(0.48026) = 644.08

Enrique would have to invest about $644 per month.

3.29 The present worth of the mortgage is 260 000 at the end of August, last year.

i = 0.12/12 = 0.01 or 1% per month
Length of mortgage = 12×20 = 240 months

Payments are: A = P(A/P, i, N) =260 000(A/P, 1%, 240) = 2862.82

Method 1: Find the future worth of 260 000 at the end of 5 months.

FW(mortgage, end of 5 months) = 260 000(F/P, 1%, 5) = 273 262.61

Future worth of 5 payments at end of 5 months:
FW(5 payments) = 2862.82(F/A, 1%, 5) = 14603.28

Amount owing at end of January
= FW(mortgage) – FW(5 payments)
= 273262.61 – 14603.28 = 258659.33

They still owe $258 659.

Method 2: Find the worth of 5 payments as of end of August.

PW(5 payments) = 2862.82(P/A, 1%, 5) = 13894.52

PW(debt at end of Aug) = 260 000
PW(amount owed) = 2600 000 – 13894.52 = 246105.48

Amount owing at end of Jan = 246105.48(F/P, 1%, 5) = 2586.59

They still owe $258 659.

3.30 A' = 10 000
G = 1000
i = 0.15
N = 6 for gradient to annuity
6 for annuity to present value
2 for future value to present value

P = [10 000 + 1000(A/G, 15%, 6)](P/A, 15%, 6)(P/F, 15%, 2)
= [10 000 + 1000(2.0971)](3.7844)(0.75614)
= 34 616.29

The software is worth \$34 616 today.

3.31 One way to solve this problem is to sum the future worth of each individual cash flow:

Balance:
F1 = 2400(F/P, 1%, 24) = 2400(1.2697) = 3047.28

Salary:
F2 = 120(F/A, 1%, 24) = 120(26.973) = 3236.76

Dividends:
F3 = 200(F/P, 1%, 20) = 200(1.2202) = 244.04
F4 = 200(F/P, 1%, 16) = 200(1.1726) = 234.52
F5 = 200(F/P, 1%, 12) = 200(1.1268) = 225.36
F6 = 200(F/P, 1%, 8) = 200(1.0829) = 216.58
F7 = 200(F/P, 1%, 4) = 200(1.0406) = 208.12
F8 = 200

Fee:
F9 = [10+1(A/G, 1%, 24)](F/A, 1%, 24) = [10+1(11.024)](26.973) = 567.08

Total future value = F1 + F2 + … + F8 – F9 = 7045.74

Clem will have saved about \$7046.

3.32 P = –30 000 – 1600(P/A, 0.5%, 24) + 40 000(P/F, 0.5%, 24)
+ 2000(P/A, 0.5%, 12) + 2400(P/A, 0.5%, 12)(P/F, 0.5%, 12)
– [400 + 40(A/G, 0.5%, 18)](P/A, 0.5%, 18)(P/F, 0.5%, 6)
= –30 000 – 1600(22.558) + 40 000(0.88721) + 2000(11.616)
+ 2400(11.616)(0.94192) – [400 + 40(8.3198)](17.168)(0.97052)
= –30 000 – 36 092 + 35 548 + 13 232 + 26 260 – 12 210 = 6678

The present worth of this investment is \$6678. Yogajothi should buy the house.

3.33 P = 15 000(P/A, 12%, 8) = 15 000(4.9676) = 74 514

Barnaby Circuit Boards could afford to spend up to about \$75 000 on the wave soldering machine.

3.34 This is an arithmetic gradient with

A' = 100 000
G = 10 000 per year

N = 30 years
i = 0.08 per year

A = A' + G(A/G, 8%, 30)
= 100 000 + 10 000(9.1897) = 100 000 + 91 897 = 191 897

P = A(P/A, i, N) = 191 897(P/A, 8%, 30) = 191 897(11.258) = 2 160 376

Yes, it is a good deal.

3.35 A = 100 000
g = 10% per year
i = 8% per year
N = 30 years

$i^\circ = (1 + i)/(1 + g) - 1 = 1.08/1.1 - 1 = -0.01818$

For a negative interest rate, we cannot use the Series Present Worth Factor from tables, so we must use the full formula:

$$P = 100\,000[(1 - 0.01818)^{30} - 1]/[-0.01818(1 - 0.01818)^{30}][1/(1 + 0.1)]$$
$$= 3\,670\,261$$

No, Leon would have to pay more than $1 000 000 more.

3.36 Amount available for annuity = 20 000 – 1200 = 18 800

P = 18 800
i = 0.07/12 per month
N = 55×12 = 660 months

$$A = P(A/P, i, N) = P[i(1 + i)^N]/[(1 + i)^N - 1]$$
$$= 18\,800[(0.07/12)(1 + 0.07/12)^{660}]/[(1 + 0.07/12)^{660} - 1]$$
$$= 112.10$$

The approximate daily amount can be calculated from:

Daily amount = (112.10×12)/365 = $3.68

No, Tina would not have enough money to retire. She would have about $3.68 available to spend each day.

3.37 Construction costs (in $millions):

P_1 = 20(P/A, 6%, 5)(P/F, 6%, 4) = 20(4.2124)(0.79209) = 66.728

Maintenance and Repair costs (in $millions):

$i° = (1 + 0.06)/(1 + 0.01) – 1 = 0.0495 \cong 5.0\%$

$P_2 = 2[(P/A, 5\%, 35)/(1 + 0.01)](P/F, 6\%, 9)$
$= 2(16.374/1.01)(0.5919) = 19.1914$

$P = P_1 + P_2 = 85.9193$

The present cost of the water supply project is about $86 million.

3.38 P = 10 000(P/A, 9%, 5) + 20 000(P/A, 9%, 10)(P/F, 9%, 5)
= 10 000(3.8896) + 20 000(6.4176)(0.649 93)
= 122 316

The present worth is €122 316.

3.39 The actual amount loaned is 500 – 45 = $455

45 = 455(A/P, i, N)
(A/P, i, 12) = 45/455 = 0.0989

(A/P, 2.5%, 12) = 0.09749
(A/P, 3%, 12) = 0.10046

By linear interpolation:

i = 2.5 + 0.5[(0.0989 – 0.09749)/(0.10046 – 0.09749)] = 2.73% per month

The effective interest rate is:
$i_e = (1 + 0.02737)^{12} – 1 = 38.3\%$ per year

3.40 g = 0.01
i = 0.015 (monthly)
$i_e = (1 + 0.015)^2 – 1 = 0.03022$ (bimonthly)
$i° = (1 + i_e)/(1 + g) – 1 = 1.0302/1.01 – 1 = 0.02$

We assume that Shamsir's monthly profit remains the same as the previous month when there is no increase. For example, his cash flows for months 1, 2, 3, and 4 would be $10 000, $10 100, $10 100, and $10 201.

Noting that there are actually two sets of geometric gradient series, identical in all aspects but different only in timing of the first cash flow, we can determine the present value as follows:

$P = 10\,000(P/A, g, i_e, 12) + 1000(P/A, g, i_e, 12)(F/P, i, 1)$
$= 10\,000[(P/A, i^\circ, 12)/(1 + g)][1 + (F/P, 0.015, 1)]$
$= 10\,000[(P/A, 0.02, 12)/1.01](1 + 1.0150)$
$= 10\,000(10.575/1.01)(2.015)$
$= 210\,976$

The present value of all his profit over the next 2 years is about $211 000.

3.41 The 6-month interest rate can be calculated from:

$300\% = 3 = (1 + i)^2 - 1$
$(1 + i)^2 = 4$
$1 + i = 2$
$i = 1 = 100\%$

Then:

$P = 5000(P/F, 100\%, 10) + 500(P/A, 100\%, 10)$
$= 5000(0.00098) + 500(0.99902) = 504.41$

I should pay no more than $504 for the bond now.

3.42 $A = 5000(0.015) = 75$
$i = 0.08/4 = 2\%$

$P = 75 + 75(P/A, 2\%, 25) + 5000(P/F, 2\%, 25)$
$= 75 + 75(19.523) + 5000(0.6095) = 4586.88$

You would be willing to pay up to $4587 for the bond.

C. More Challenging Problems

3.43 **(a)** It is not necessary to know how long the mortgages will last, as long as they are equal.

For the old plan: Using X to represent the quarterly payments, the present value of one three-month payment cycle is calculated.

$P(\text{quarterly payment}) = X/(1 + i)^1 = X/1.06 = 0.943X$

For the new plan: Each payment is now 0.3X. The present value of three months of payments is

$P = 0.3X(P/A, 2\%, 3) = 0.3X(2.8839) = 0.865X$

The value of three months of the old plan is greater than that of the new plan. As you prefer to minimize the value of the payments, you prefer the new plan.

(b) $i_e = (1 + i_s)^m - 1 = (1 + 0.06)^4 - 1 = 26.24\%$ for the old plan
$i_e = (1 + i_s)^m - 1 = (1 + 0.02)^{12} - 1 = 26.82\%$ for the new plan

The new plan has a higher effective yearly interest rate.

3.44 The cash flow series consists of G at the end of period 2, 2G at the end of period 3 and so on up to (N – 1)G at the end of the Nth period.

The first step is to convert each period's gradient amount to its future value:

$F = G(1 + i)^{N-2} + 2G(1 + i)^{N-3} + ... + (N - 2)G(1 + i) + (N - 1)G$

Multiplying by (1 + i) and subtracting F gives:

$F(1 + i) = G(1 + i)^{N-1} + 2G(1 + i)^{N-2} + ... + (N - 2)G(1 + i)^2 + (N - 1)G(1 + i)$
$F(1 + i) - F = G(1 + i)^{N-1} + G(1 + i)^{N-2} + ... + G(1 + i) - (N - 1)G$
$Fi = G[(1 + i)^{N-1} + (1 + i)^{N-2} + (1 + i)^{N-3} + ... + (1 + i) + 1] - NG$

Noting that the amount in the square brackets is the Series Compound Amount Factor:

$Fi = G(F/A, i, N) - NG$

Multiplying both sides by the Sinking Fund Factor:

$Fi(A/F, i , N) = G - NG(A/F, i, N)$

$Ai = G - NG(A/F, i, N) = G[1 - N(A/F, i, N)]$

$$A = G\left(\frac{1}{i} - \frac{Ni}{i[(1+i)^N - 1]}\right) = G\left(\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right)$$

3.45 The present worth of a geometric series is:

$$P = \frac{A}{1+i} + \frac{A(1+g)}{(1+i)^2} + ... + \frac{A(1+g)^{N-1}}{(1+i)^N}$$

If we divide and multiply each term in the present worth expression by (1 + g) and then simplify, we get

$$P = \frac{A}{1+g}\left(\frac{1+g}{1+i} + \frac{(1+g)^2}{(1+i)^2} + \ldots + \frac{(1+g)^N}{(1+i)^N}\right)$$

Then, substitute the growth adjusted interest rate, i°:

$$i^{\circ} = \frac{1+i}{1+g} - 1 \text{ so that } \frac{1}{1+i^{\circ}} = \frac{1+g}{1+i}$$

into the present worth expression and we get:

$$P = \frac{A}{1+g}\left(\frac{1}{1+i^{\circ}} + \frac{1}{(1+i^{\circ})^2} + \ldots + \frac{1}{(1+i^{\circ})^N}\right)$$

The right hand side is simply the present worth of an annuity where the constant cash flow each period is A/(1 + g) and the interest rate is i°. We can write this as

$$P = A\left(\frac{\frac{1}{1+i^{\circ}} + \frac{1}{(1+i^{\circ})^2} + \ldots + \frac{1}{(1+i^{\circ})^N}}{1+g}\right) = A\,\frac{(P/A, i^{\circ}, N)}{1+g}$$

so that: $$(P/A,g,i,N) = \frac{(P/A, i^{\circ}, N)}{1+g} = \left(\frac{(1+i^{\circ})^N - 1}{i^{\circ}(1+i^{\circ})^N}\right)\frac{1}{1+g}$$

3.46 i° = (1 + 0.05)/(1 + 0.5) – 1 = – 0.3 = –30%

PW_{sales} = 1 456 988(P/A, –30%, 5)/(1.5)
= 1 456 988[$(0.7^5 – 1)/(–0.3\times0.7^5)$]/(1.5)
= 16 026 550

Selling price = 16 026 550/2 = 8 013 275

Ruby should sell the company for just over 8 million dollars.

3.47 The worth of the loan at month 52 is:
F(loan) = 80 000(F/P, 1%, 52) = 134 215.11

The future worth of the payments at month 52 is:
F(payments) = 2000(F/A, 1%, 51)(F/P, 1%, 1) = 133 537.78

The difference is the amount of the last payment:
Last Payment = F(loan) – F(payments) = 677.33

Clarence's last payment will be about $677.

3.48 Only the spreadsheet for part (a) is provided.
(a)

Capital Amount		$50 000		
Annual Interest Rate		8.00%		
No. of years to repay		15		
Payment Periods	**Annual Payment**	**Interest Received**	**Recovered Capital**	**Unrecovered Capital**
0				50000
1	5841	4000	1841	48159
2	5841	3853	1989	46170
3	5841	3694	2148	44022
4	5841	3522	2320	41702
5	5841	3336	2505	39197
6	5841	3136	2706	36491
7	5841	2919	2922	33569
8	5841	2686	3156	30413
9	5841	2433	3408	27004
10	5841	2160	3681	23323
11	5841	1866	3976	19348
12	5841	1548	4294	15054
13	5841	1204	4637	10417
14	5841	833	5008	5409
15	5841	433	5409	0
Total			50000	

3.49 Once the project is started, the savings effectively reduce the costs by $50 000 to $100 000 per month.

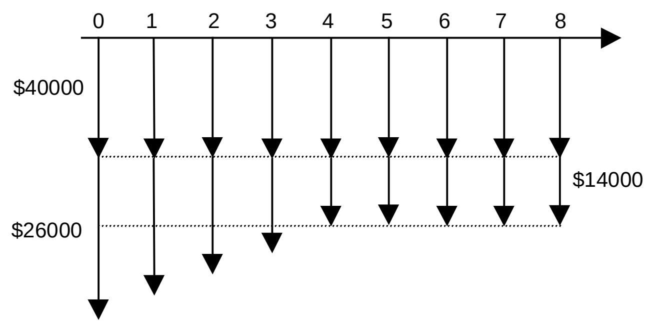
Let X = start of project.

250 000(F/P, 1.5%, X) + 50 000(F/A, 1.5%, X) = 150 000(P/A, 1.5%, 24)
5(F/P, 1.5%, X) + (F/A, 1.5%, X) = 3(20.030) = 60.09

At X = 36: LHS = 5(1.7091) + 47.276 = 60.09
At X = 40: LHS = 5(1.814) + 54.2679 = 60.09

The project can start about 38 months from now.

3.50 **(a)** Cash flow diagram



(b) There will be a total of 9 payments. The payments can be considered in two portions:

1) The salary costs and the constant portion of the other costs are a 9-period annuity due. The equipment and facility cost declines from $26 000 to a constant $14 000.

At the end of eight years, the payments are worth:
F_A = A (F/A, 7%, 9) = (40 000 + 14 000)(11.978) = 646 812

This includes the first payment, and is equivalent to treating the initial payment separate from a standard 8 period annuity:

F_A= 54 000(F/P, 7%, 8) + 54 000(F/A, 7%, 8)

2) The declining costs can be modelled as an annuity due less an arithmetic gradient, after four years:

F_{B1} = (26 000 – 14 000)(F/A, 7%, 5) = 12 000(5.7507) = 69 008

The gradient’s worth after the same four years is

F_{B2} = 3000(A/G, 7%, 5)(F/A, 7% 5) = 3000(1.8650)(5.7507) = 32 175

For both F_{B1} and F_{B2}, N = 5 to conform with the standard form for an annuity and a gradient (the first cash flow for an annuity is after one year, while the first cash flow for a gradient is after two years).

The value of both cash flows after the eight years is the annuity's value minus the gradient's value, taken forward to the eighth year:

F_B = (69 008 – 32 175)(F/P, 7%, 4) = (36 833)(1.3108) = 48 281

The total worth of the project at the end of the eight years is:

$F_A + F_B$ = 646 812 + 48 281 = \$695 093

3.51 g = 0.05 per month
i = 0.01 per month
i° = (1 + i)/(1 + g) – 1 = 1.01/1.05 – 1 = –0.0380

P(expenses)
= 15 000(P/A, g, i, 12)
= 15 000[(P/A, i°, 12)/(1 + g)]
= 15 000 $[(1.038^{12} - 1)/(0.038 \times 1.038^{12})]/1.05$ = 135 642

Let A be the amount of the monthly instalment:

P(grant) = A(P/A, 1%, 6)(P/F, 1%, 12) = A(5.7955)(0.88745) = 5.1432A

By letting P(expenses) = P(grant), solve for A:

5.1432A = 135 642
A = \$26 373

The amount of the monthly instalment must be 26 373 元.

3.52 The present worth computations for the concrete pool are:

P = 1 500 000 + 200 000(A/F, 5%, 10)/0.05
= 1 500 000 + 200 000(0.07951)/0.05
= \$1 818 040

3.53 a) $i^0 > 0$; $P = \lim_{N \to \infty} A(P/A, i^0, N) = \frac{A}{i^0}$

b) $P = \frac{A}{1+i} + \frac{A(1+g)}{(1+i)^2} + ... + \frac{A(1+g)^{n-1}}{(1+i)^n} + ...$

noting that $\frac{A(1+g)^{n-1}}{(1+i)^n} = \frac{A(1+g)^{n-2}}{(1+i)^{n-1}} \frac{(1+g)}{(1+i)}$

and that since $g > 1$, $\frac{(1+g)}{(1+i)} > 1$

then $\frac{A(1+g)^{n-1}}{(1+i)^n} > \frac{A(1+g)^{n-2}}{(1+i)^{n-1}}$

Consequently *P* is the sum of an infinite number of monotonically increasing amounts, and takes on an infinite value.

c) $P = \lim_{N \to \infty} N\left(\frac{A}{1+g}\right) = \infty$

d) $i^0 > 0$; $P = \lim_{N \to \infty} A(P/A, i^0, N) = \frac{A}{i^0}$

Notes for Case-in-Point 3.1

1) No right answer – both are to blame.

2) On the one hand, large infrastructure projects need to be financed. On the other hand, there is a moral hazard for politicians.

3) It is probably not reasonable for political leaders to understand the time value of money. But it is reasonable for them to have competent advisors who do.

4) About 340 months or just over 28 years.

Notes for Mini-Case 3.1

Oil industry projects tend to be in the range of 100's of millions to tens of billions of dollars of investment. As one might expect, very capable people think through all aspects of a project – technical, financial, environmental, political – very carefully. Oil companies take planning seriously, and their engineers are very experienced. The $90 billion of cancelled projects in 2009 does not represent incompetence, but rather something fundamental, which is that no matter how careful the planning process might be, one can never fully predict how the future will turn out. There are three key issues that have strongly affected the Canadian oil sands.

The first is that crude oil is a commodity, meaning that there are so many sources that no one can easily predict or control its price. Even if the cost of production is absolutely certain, the profitability and hence project viability depends very much on the world price, which itself is subject to many random forces. No-one predicted that the price of oil would rise to a peak of US$145 per barrel in 2008, or drop the same year to below US$50. Both are extremes outside of the planning window imagined earlier in the same decade. For conventional oil production, costs can be controlled to a degree. Exploration efforts can be reduced, for example, when prices are low. But oil sands project costs are much more difficult to cut back on once they are started, with very large capital costs to recover, and with high direct costs per barrel of oil produced. Investment decisions are thus more vulnerable to low oil prices.

The second confounding issue with the oil sands concerns social, environmental and political forces at play. Mining oil sands is very dirty work, damaging the physical sites and producing vast quantities of carbon dioxide. Early oil sands projects sometimes completely failed to recognize the costs of managing their environmental impact, and ended up with unexpected costs for administration, legal bills, remediation and meeting regulations. Even in recent years where such costs are recognized and taken into account in the planning process, they are very unpredictable because of the inherent nature of social and political intervention. Otherwise benign activities can suddenly become difficult and expensive if they somehow gain media attention. This has happened in several cases in the oil sands. For example, in 2011 international media attention was given to the “Rethink Alberta” movement, which was started by several environmental groups to dissuade tourists from visiting Alberta unless new oil sands developments were stopped.

A third problem is a confluence of global trends, which has been compounded by the oil sand’s remote location. Between 2003 and 2008, for example, a key global force was the industrialization of China. China’s industrialization led to an increase in demand for oil and gas worldwide, causing oil companies to attempt to increase supply. This in turn caused a global proliferation of oil and gas developments, creating a tremendous demand for manufacturing capacity, technical expertise and specialized equipment. This demand significantly increased the costs of acquiring the resources necessary to develop the oil sands. In particular, with a tight labour pool and an isolated location, costs of labour in Alberta skyrocketed – to about double previously estimated costs, far beyond even the most pessimistic expectations. Although some trends can be predicted, others defy prediction.

Even with the best planning and with the most reliable process and product, the future remains unpredictable. Economic analyses still need to be done, however, despite an uncertain future. There are sophisticated ways to deal with uncertainty about future cash flows, some of which are discussed in Chapter 12. In most cases it makes sense to carry out the economic analysis with a range of possible values for future cash flows. But, as the oil sands example shows, there are situations when even this approach does not account for what the future actually holds.

Solutions to All Additional Problems

Note: Solutions to odd-numbered problems are provided on the Student CD-ROM.

3S.1

Mr Dashwood must find the least value of N for which
$1500 < 100(P/A,5\%,N)$
which is equivalent to finding N such that
$(P/A,5\%,N) > 15$

Consulting Appendix A, this first occurs when N = 29 years.

If the interest rate is 10%, we come to the end of the table in Appendix A before finding a value of N for which (P/A,*10%*,N) > 15. We therefore go on to look at the capitalized value of the annuity, which is 100 / 0.1, or £1000. Thus, if he can invest his money at 10%, Mr Dashwood can afford to support his widowed half-sister indefinitely for a cost equivalent to a single present payment of £1000.

3S.2

There are several ways of tackling this. One way is to convert the bi-annuity to an equivalent annuity:

$A1 = A\ (A/F,i,2)$

where ¥$A1$ is the annual payment equivalent to getting a final payment of ¥A at the end of two years. Then we can convert ¥$A1$ to its present worth using a formula we already know:

P = A1(P/A,i,2N) = A(A/F,i,2)(P/A,i,2N)

3S.3

Having drawn a cash-flow diagram, we write down the PW equivalent of each of the costs:

PW = 32 000 − 16 000(P/F,15%,8) + 20 000 ($P/\overline{A}$,15%,8) + 600 + 550(P/A,15%,7) – 50(P/G,15%,7)

(represent the decreasing insurance costs as an annuity plus a negative arithmetic gradient)

= €122 322.

Common mistakes are to treat the insurance payments as coming at year's end (this is what you assume if you calculate the insurance as 600(P/A,15%,7) − 50(P/G,15%,7)) and to treat the labour costs as occurring at year's end instead of continuously. (It is rare for a company to wait till the end of the year before paying the year's wages.)

3S.4

The present worth for each of the two cases can be calculated as follows:

Option 1:

PW = 230 000 (P/A,0.12,10) = 230 000(5.65) = 元1 299 500

Option 2 :

PW = 1 100 000 + 500 000 (P/F,0.12,3) + 10 000 (P/A,0.12,3)
+ 20 000 (P/A,0.12,7)(P/F,0.12,3) − 500 000 (P/F,0.12,10)
=元1 383 890

So the first option is better.

It is also possible to do the comparison in terms of equivalent uniform annual cost; this has the advantage that no calculation is needed for Option 1, though it makes the Option 2 calculations a bit harder.

A possible variant is to assume that in Option 1, the company pays rent at the beginning of each year rather than at the end. This makes quite a difference—the present Option 1 cost goes up to 元1 455 000, so the second option is then better.

3S.5

The money accumulating in the company's interest-bearing account is a discrete cash flow, continuously compounded. It is convenient to consider a time interval of six months; for this interval, the nominal interest rate is 2%, and the study period is ten six-monthly intervals.

We first convert the arithmetic increase in income to an equivalent annuity, A':
A' = 10 000(A/G,0.02,10) = 10 000(4.3351) = 43 451

So the incoming cash flow is equivalent to the sum of this and the base annuity, ¥100 000.

The future worth of the total equivalent annuity is therefore:
F = 143 451(F/A,0.02,10) = 143 451(10.9598) = 1 572 194

So if each of the students were to stay in the group, each would have a one-fifth share of this, which is ¥314 439. The student who leaves the group will instead get a one-fifth share of the present value of the total equivalent annuity. Its present value is
P = 143 451(P/A,0.02,10) = 143 451(8.97313) = 1 287 204

And a one-fifth share of this is ¥257 441.

The student who left the group deposits this at 3% annual interest, so at the end of five years she will have 257 441 (F/P,0.03,5) = 257 441(1.1593) = ¥298 238. Thus she has lost ¥314 439 − 298 238, which is ¥16 201.

3S.6

B. (As the MARR increases, the importance of future paybacks diminishes the further in the future they are.)

3S.7

In solving this, the essential thing is to get a fair basis of comparison between the three alternatives. If all three alternatives have a cash flow in common, you do not need to represent that cash flow explicitly.

So, on all three options you have the BMW and you do not have the old car. (You have either sold it or traded it in.) In Option A, you have $1000 from selling the old car; in Option B, you have the $10 000 you saved, plus $1000 from selling the old car; in Option C, you have nothing. (You can add or subtract the same quantity from each of the Year Zero figures without affecting the solution.)

We write down the annual cash flows for each option:

Dealer A:
Year 0: You sell the old car for $1000 salvage value.

Year 1: You pay $4000 principal plus 20 000(0.05) interest.

Year 2: You pay $4000 principal plus 16 000(0.05) interest.

Year 3: You pay $4000 principal plus 12 000(0.05) interest.

Year 4: You pay $4000 principal plus 8000(0.05) interest.

Year 5: You pay $4000 principal plus 4000(0.05) interest, and you are done.

Dealer B:
Year 0: You sell the old car for $1000 salvage value and keep your $10 000 in the bank.

Year 1: You pay $5000 principal plus 30 000(0.02) interest.

Year 2: You pay $5000 principal plus 25 000(0.02) interest.

Year 3: You pay $5000 principal plus 20 000(0.02) interest.

Year 4: You pay $5000 principal plus 15 000(0.02) interest.

Year 5: You pay $5000 principal plus 10 000(0.02) interest.

Year 6: You pay $5000 principal plus 5000(0.02) interest, and you are done.

Dealer C:
Year 0: You get nothing.

Year 4: You pay $15 000 (F/P, 0.1,4) and you are done.

Summarizing in a table:

Option/Year	*0*	*1*	*2*	*3*	*4*	*5*	*6*	***Present Value***
A	1000	−5000	−4800	−4600	−4400	−4200	0	−14 606
B	11 000	−5600	−5500	−5400	−5300	−5200	−5100	−9399
C	0	0	0	0	−21 962	0	0	−12 557

You find the present worth of each of these sums, using an interest rate of 15%, and choose the smallest. This is done in the accompanying spreadsheet, **3S_7.xls.** We see from the spreadsheet results that Option B is by far the best.