Math & Stats Assistance Centre

MATH 110 Exam Review Answer key

The following are brief answers to the MATH 110 Review Problems. If you have questions about how to solve any of the problems, please feel free to ask at the Math and Stats Assistance Centre.

Part 1 (True/False):

1. A linearly independent set of vectors in $\mathbb{R}^n$ has at most n vectors.

   Answer: True

   Notes: If S is a set of linearly independent vectors in $\mathbb{R}^n$ then the span of S is a subspace of $\mathbb{R}^n$ and therefore has dimension at most n. Also, S is a basis of that subspace (because the vectors are linearly independent). It follows that the dimension of the subspace is equal to the number of vectors in S.

2. If A is an $n \times n$ matrix and B is an $n \times p$ matrix such that $AB = \vec{0}$, then either A or B (or both) is the 0-matrix.

   Answer: False

   Notes: Many counterexamples possible. With this and the rest, start by looking for a small counterexample, which is usually possible. That is particularly important in a timed environment like a test, because larger examples are harder to work with.

   One possible counterexample is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

3. If A is an $n \times n$ matrix and rank$(A) = n$, then Det$(A) = 0$.

   Answer: False

   Notes: In fact, if A is an $n \times n$ matrix with rank$(A) = n$, then Det$(A) \neq 0$.

4. If A is a square matrix, then AA^t and A^tA are orthogonally diagonalizable.

   Answer: True; they're both symmetric.

5. For all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, $\vec{u} \cdot \vec{v} \geq 0$.

   Answer: False.

   Notes: One possible counterexample is $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

6. Any matrix can be transformed into reduced row echelon form by a finite sequence of elementary row operations.

   Answer: True.

7. If A is a 4×3 matrix and nullity of A^t is 2, then $\text{rank}(A) = 2$.

   Answer: True.

8. If B is a 3×5 matrix then the columns of B are linearly dependent as vectors.

   Answer: True.

9. If A and B are $n \times n$ matrices and B is obtained from A by a sequence of elementary row operations, then Det$(B) =$ Det(A).

   Answer: False.

10. If $\vec{v}$ is both in the row space and in the column space of a square matrix A, then $\vec{v} = \vec{0}$.

    Answer: False.

11. Every plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

    Answer: False.

12. If V and W are subspaces of $\mathbb{R}^n$, then the set of vectors that are in both V and W is also a subspace of $\mathbb{R}^n$.

    Answer: True.

13. If V and W are subspaces of $\mathbb{R}^n$, then the set of vectors that are in either V or W or both is also a subspace of $\mathbb{R}^n$.

    Answer: False.

14. If V and W are subspaces of $\mathbb{R}^n$ and S is the set of vectors of the form $\vec{w} - \vec{v}$, where $\vec{v} \in V$ and $\vec{w} \in W$, then S is a subspace of $\mathbb{R}^n$.

    Answer: True.

15. If matrices M and N have the same size, then $(M + N)^t = M^t + N^t$.

    Answer: True.

16. If matrices M and N are invertible, and MN exists, then MN is invertible and $(MN)^{-1} = N^{-1}M^{-1}$.

    Answer: True.

17. If matrix M is invertible, then M^t is invertible.

    Answer: True.

18. If matrices M, N, and U are square matrices such that $MN = MU$, then either M is the all-zero matrix or $N = U$.

    Answer: False.

    Notes: One possible counterexample: $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $N = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

19. If R is the reduced row echelon form of a matrix A, then $Col(A) = Col(R)$.

Notes: One possible counterexample: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ so that $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so $Col(A) = span\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ and $Col(R) = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

Answer: False.

20. If R is the reduced row echelon form of a matrix A, then $Row(A) = Row(R)$. **Answer:** True.

21. If $\vec{u}$ and $\vec{v}$ are nonzero, orthogonal vectors in $\mathbb{R}^2$, then they are independent.

    Answer: True.

22. If $\vec{u}$ and $\vec{v}$ are nonzero, independent vectors in $\mathbb{R}^2$, then they are orthogonal.

    Answer: False

    Notes: One counterexample is $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

23. If A and B are orthogonal $n \times n$ matrices, then so is AB.

    Answer: True.

    Notes: $(AB)^t AB = B^t A^t AB = I_r$, so we have shown that $(AB)^t(AB) = I_r$, which means AB is orthogonal.

Part 2:

1. Let $\vec{u} = \begin{bmatrix} 2 \\ 8 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 9 \end{bmatrix}$. Compute $(\vec{u} - \vec{v}) \cdot (3\vec{v})$.

   Answer: -240

2. Find the projection of the vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ onto the vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

   Answer: $proj_{\vec{v}}(\vec{u}) = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{6}{5} \end{bmatrix}$.

3. Find the distance from the point $Q(4, 2, 4)$ to the plane $2x_1 - x_2 + x_3 = 3$

   Answer: $\frac{7\sqrt{6}}{6}$

4. For what values of k is the solution to the following system of linear equations (a) a point, (b) a line, and (c) a plane?

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ x & + & ky & + & 3z & = & 1 \\ kx & + & y & + & 3z & = & -2 \end{array}$$

Answer:

(a) The solution is a point when $k \neq 1, 2$;

(b) The solution is a line when $k = 2$;

(c) There are no values of k for which the solution is a plane.

Note that the system of equations has **no** solution when $k = 1$.

5. If the vector $\vec{w}$ is a linear combination of the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$ with corresponding coefficients $c_1 = 3$, $c_2 = 2$, $c_3 = -1$, find vector $\vec{w}$.

Answer: $\vec{w} = \vec{v_3} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

6. Suppose A and B are 4×4 matrices with $\det A = 5$ and $\det B = 3$. Find $\det(AB^2A^t)$.

Answer: $\det(AB^2A^t) = 225$.

7. Find the inverse of $\begin{bmatrix} 1 & 0 & -3 \\ 6 & 1 & -13 \\ -1 & 4 & 25 \end{bmatrix}$, or explain why the inverse does not exist.

Answer: $\begin{bmatrix} \frac{77}{2} & -6 & \frac{3}{2} \\ -\frac{137}{2} & 11 & -\frac{5}{2} \\ \frac{25}{2} & -2 & \frac{1}{2} \end{bmatrix}$

Notes: Row reduce the augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 6 & 1 & -13 & 0 & 1 & 0 \\ -1 & 4 & 25 & 0 & 0 & 1 \end{array}\right]$ to (eventually) get $\left[\begin{array}{ccc|cc} 1 & 0 & 0 & \frac{77}{2} & -6 \\ 0 & 1 & 0 & -\frac{137}{2} & 11 \\ 0 & 0 & 1 & \frac{25}{2} & -2 \end{array}\right.$

8. Compute the determinant of the matrix $M = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$ by row-reducing the matrix.

Notes: This matrix is well-suited to cofactor expansion, but we want to practice using the other method! The purpose of this problem is to use the effects of row operations on a matrix's determinant. The following sequence of steps will work (of course, other sequences will too - there are many ways to row-reduce a matrix):

(a) Let $M_0 = M$ (so $det(M_0) = det(M)$)

(b) Obtain M_1 from M_0 by swapping rows 1 and 3 (so $det(M_0) = -det(M_1)$)

(c) Obtain M_2 from M_1 by clearing out column 1: $R_2 - 5R_1$, $R_3 - 2R_1$, and $R_4 + 2R_1$ (so $det(M_1) = det(M_2)$)

(d) Obtain M_3 from M_2 by swapping rows 2 and 3 (so $det(M_2) = -det(M_3)$)

(e) Obtain M_4 from M_3 by $-R_2$ (so $det(M_3) = -det(M_4)$)

(f) Obtain M_5 from M_4 by clearing out column 2: $R_1 + R_2$, $R_3 - 9R_2$, $R_4 + R_2$ (so $det(M_4) = det(M_5)$)

(g) Obtain M_6 from M_5 by $\frac{1}{2}R_3$ (so $det(M_5) = 2det(M_6)$)

(h) Obtain M_7 from M_6 by $\frac{1}{11}R_4$ (so $det(M_6) = 11det(M_7)$)

(i) Obtain I_4 by from M_7 by clearing out column 4 (so $(det(M_7) = det(I_4) = 1)$

Now, working backwards, $det(M) = -(-(-2 \cdot 11 \cdot 1)) = -22$.

Answer: $det(M) = -22$

9. Let W be the subspace of $\mathbb{R}^3$ spanned by vectors $w_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $w_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$. Find a basis for $W^\perp$

Answer: $\begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix}$

10. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 9 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 3 \\ 3 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 21 \\ 9 \\ -1 \end{bmatrix}$. Determine whether the systems $A\vec{x} = \vec{b}$ and $D\vec{x} = \vec{b}$ are consistent or inconsistent. If consistent, solve the system.

Answer:

- $A\vec{x} = \vec{b}$ is inconsistent.
- $D\vec{x} = \vec{b}$ is consistent and $\vec{x} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{2} \\ 7 \end{bmatrix}$

Notes: Row-reduce the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 21 \\ 2 & 5 & 3 & 9 \\ 1 & 0 & 9 & -1 \end{array}\right]$ to get $\left[\begin{array}{ccc|c} 1 & 0 & 9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$. Because the system corresponding to the row-reduced matrix is inconsistent, the system corresponding to $A\vec{x} = \vec{b}$ is also inconsistent.

Row-reduce the augmented matrix $\left[\begin{array}{ccc|c} 0 & 0 & 3 & 21 \\ 3 & 1 & 1 & 9 \\ 0 & 2 & 0 & -1 \end{array}\right]$ to get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/6 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 7 \end{array}\right]$, which has a unique solution.

11. If $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$, for which values of k is the vector $\vec{w} = \begin{bmatrix} k \\ 2 \\ 1 \end{bmatrix}$ in the $span$ of $\{\vec{u_1}, \vec{u_2}\}$? If W is the subspace of $\mathbb{R}^3$ spanned by vectors $\vec{u_1}, \vec{u_2}$, find a basis for $W^\perp$.

Answer:

- $k = \frac{1}{3}$
- $W^\perp = span \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Notes: There are at least three approaches you could use to find k:

- Notice that $\vec{w}$ is in the span of $\{\vec{u_1}, \vec{u_2}\}$ if and only if the matrix $\begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{w} \end{bmatrix}$ does not have full rank - so find the values of k for which that matrix has determinant 0.
- Notice that $\vec{w}$ is in the span of $\{\vec{u_1}, \vec{u_2}\}$ if and only if the matrix $\begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{w} \end{bmatrix}$ does not have full rank - so row-reduce that matrix and determine what values of k prevent you from reaching I_3 (you are looking for an all-zero row).
- Notice that if $\vec{w} = a\vec{u_1} + b\vec{u_2}$ then $b = 2$, and proceed from there to solve for a.

12. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ 5x + 3z \\ 2|z| \end{bmatrix}$. Is this transformation a linear transformation? If so, what is the standard matrix of T?

(b) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y + \sqrt{7}z \\ y - 5x + 3z \\ 2z + 0.1x \end{bmatrix}$. Is this transformation a linear transformation? If so, what is the standard matrix of S?

Answer: (a) Not a linear transformation. (b) S is a linear transformation with matrix $S \begin{bmatrix} 3 & -2 & \sqrt{7} \\ -5 & 1 & 3 \\ 0.1 & 0 & 2 \end{bmatrix}$

13. $B = \left\{ v_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis of $\mathbb{R}^3$. Find constants c_1, c_2, c_3 such that $w = c_1v_1 + c_2v_2 + c_3v_3$.

14. Find the standard matrix that performs the **clockwise** rotation of vectors in $\mathbb{R}^2$ about the origin by $\pi/4$ radians, then reflects resulting vectors over line $y=-x$, and then projects resulting vectors on $y-$axis. Is this linear transformation invertible?

Answer: $M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0-\frac{\sqrt{2}}{2} \end{bmatrix}$

Notes: Remember, if T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ then you can find its standard matrix by considering what effect T has on the standard basis vectors.

In this case, a clockwise rotation by $\pi/4$ takes the vector $\vec{e_1}$ to $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$, and takes the vector $\vec{e_2}$ to $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, so the matrix for that transformation is $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

Similarly, the reflection across the line $y=-x$ has matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, and the projection onto the y-axis has matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

It follows that the standard matrix for the whole transformation is the matrix product

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

This is not invertible, because (for example) the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ have the same image.

15. The vectors $\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x_2} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ form a basis for a subspace W of $\mathbb{R}^3$.

Apply the Gram-Schmidt Process to obtain an orthonormal basis for W.

Answer: An orthonormal basis for W is $\left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix} \right\}$

16. Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & -4 & 3 & -6 \\ 1 & 0 & -1 & 3 & -4 \\ 2 & 1 & -5 & 6 & -10 \\ -1 & -2 & 7 & -3 & 8 \end{bmatrix}$$

(a) Determine the rank and nullity of matrix A

Answer: The rank is 2, which means the nullity is 3.

Notes:

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 3 & -4 \\ 0 & 1 & -3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now it is easy to see that the rank is 2, and the rank-nullity theorem says that the nullity is $5 - 2 = 3$.

(b) Give a basis for the row space of A

Answer: $\left\{ \begin{bmatrix} 1 & 0 & -1 & 3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 & 0 & -2 \end{bmatrix} \right\}$

Notes: Remember that row-reduction does not affect the row space, so the basis of the row space of $RREF(A)$ is also a basis of the row space of A.

(c) Give a basis for the column space of A

Answer: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$

Notes: Remember that row-reduction *does* affect the column space, so you cannot directly read the column space out of the row-reduced matrix. You can, however, use the row-reduced matrix to identify which columns of the *original* matrix to keep (in this case, columns 1 and 2).

(d) Give a basis for the nullspace of A.

Answer: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Notes: Remember that row operations do not change the solution sets of the corresponding system of linear equations. That is, if $\vec{v}$ is a solution to $A\vec{v} = \vec{b}$ then $\vec{v}$ is also a solution to $(RREF(A))\vec{v} = \vec{b}$. In particular, $\vec{v}$ is in the null space of A if and only if $(RREF(A))\vec{v} = \vec{0}$, which means that we can find the null space of our row-reduced matrix instead. That is a much easier task than working directly with A - you are looking for three linearly-independent vectors whose product with $RREF(A)$ is $\vec{0}$.

17. For the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

(a) find an invertible matrix P and a diagonal matrix D such that $A = P^{-1}DP$.

Answer: $P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

Notes: First you want to find the eigenvalues, by calculating $det(A - \lambda I)$ to get the characteristic polynomial. In this case, the characteristic polynomial is $(7-\lambda)(1-\lambda)+8$, which has roots $\lambda = 5$ and $\lambda = 3$.

Now you want to find the eigenvectors. In this case, you will find that the eigenvectors for $\lambda = 5$ are the vectors of the form $\begin{bmatrix} t \\ -t \end{bmatrix}$, so choose $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. You will find that the eigenvectors for $\lambda = 3$ are those of the form $\begin{bmatrix} t \\ -2t \end{bmatrix}$, so choose $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Remember to be consistent with the placement between the eigenvalues and the eigenvectors when finding P to go with D.

(b) Find formulae for calculating $B = A^k$ for all values of $k \geq 1$.

Notes: Recall that $A = P^{-1}DP$ and so $A^k = P^{-1}D^kP$, and D^k is relatively easy to calculate - which is why we like to express matrices in this form when we can!

Answer: $A^k = \begin{bmatrix} -3^k + 2(5^k) & -3^k + 5^k \\ 2(3^k - 5^k) & 2(3^k) - 5^k \end{bmatrix}$

18. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A, the eigenvalues of A, and the eigenvector(s) corresponding to each eigenvalue.

Answer: Characteristic polynomial $\lambda(1-\lambda)(\lambda-3)$; $\lambda_1 = 0$ has eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 1$ has eigenvector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $\lambda_3 = 3$ has eigenvector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(b) Is A diagonalizable?

Answer: Yes, because the algebraic multiplicity equals the geometric multiplicity of each eigenvalue.

19. Let S be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that takes a vector $\vec{v}$ and rotates it $\pi/2$ radians. If $A\vec{x} = S(\vec{x})$, what are the real eigenvalues of A?

Answer: There are no real eigenvalues.

20. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that takes a vector $\vec{v}$ and projects it onto the xy-plane. If $B\vec{x} = T(\vec{x})$, what are the real eigenvalues of B?

Answer: $\lambda_1 = 1$ (which has algebraic multiplicity 2) and $\lambda_2 = 0$

Notes: Recall that if λ is an eigenvalue for the matrix B then $B\vec{v} = \lambda\vec{v}$. By considering the effect that T has on the standard basis vectors, find B - its characteristic polynomial turns out to be $(\lambda - 1)^2\lambda$, which is why $\lambda_1 = 1$ has algebraic multiplicity 2.

21. Find the polar form of the complex number $z = 1 + i$. Find z^7.

    Answer: $8\sqrt{2} + 8\sqrt{2}i$

    Notes: The radius is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the angle is $\pi/4$, so $z = \sqrt{2}e^{\pi i/4}$. Then $x^7 = \sqrt{2}^7 e^{7\pi i/4}$. Convert back to rectangular form.

22. What are the solutions to the equation $z^4 - 4i = 0$?

    Answer:

23. Find all real and complex roots of $-x^3 - 6x + 4x^2 + 24$.

    Answer: $-x^3 - 6x + 4x^2 + 24) = (x^2 + 6)(4 - x) = (x + 3i)(x - 3i)(4 - x)$ has roots $3i$, $-3i$, and 4.

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