UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2021 (Adapted)

MATH 122: Logic and Foundations

CRN: 12133 (A01), 12135 (A03), 12136 (A04), 12137 (A05)

Instructors: J. Noel (A01 & A03), K. Mynhardt (A04), & N. Morrison (A05)

Duration: 3 Hours.

Answers are to be written on the exam paper. Please write within the page borders.
No calculator is necessary, but a Sharp EL-510R (plus some letters) calculator is allowed.

This exam consists of 20 questions for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed. Question 1 consists of 28 true/false questions labelled TF 1 to TF 28 to be answered on the bubble sheet at the end of the booklet.

There are 12 numbered pages, not including covers, and one bubble sheet.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

Questions marked with an asterisk (*) have been adapted to suit the course coverage of the Spring 2022 semester.

Do your rough work here. Nothing written here will be marked.

1. [worth 14 points] Use the **bubble sheet** provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**. No justification is necessary.

[TF 1] $\neg p \wedge \neg q$ logically implies $p \to q$.

[TF 2] The negation of "*If I buy groceries and watch TV today, then I don't have enough time to study*" is "*If I don't buy groceries or I don't watch TV today, then I have enough time to study*".

[TF 3] The converse of "*The university will close when it it snowing or there is a heatwave*" is "*If the university is closed, then it is snowing or there is a heatwave*".

[TF 4] The contrapositive of "*All math majors must take a calculus course*" is "*Some people who take calculus courses are not math majors*".

In questions TF 5 to TF 8, let $S = \{1, 2, 3, \{\emptyset, 1, 2, 3\}\}$.

[TF 5] $\{1, 2, 3\} \in S$

[TF 6] $\{1, 2, 3\} \subseteq S$

[TF 7] $\emptyset \in S \cap \mathcal{P}(S)$

[TF 8] $|S| = 3$

[TF 9] For all sets A and B, if $B \subseteq A^c$, then $A \cap B = \emptyset$.

[TF 10] For all sets A, if $a \in A$, then $\{a\} \subseteq \mathcal{P}(A)$.

[TF 11] For all sets A, B and C, if $B \neq C$, then $A \cap B \neq A \cap C$.

[TF 12] There exist sets A, B and C such that $A \cap B \neq \emptyset$, $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$, but $A \cap B \cap C = \emptyset$.

[TF 13] Let $a, b \in \mathbb{Z}$. If p is prime and $p^2 \mid ab$, then $p \mid a$ and $p \mid b$.

[TF 14] If $n \in \mathbb{N}$, then there are $\left\lfloor \frac{n}{5} \right\rfloor$ positive integers less than or equal to n that are multiples of 5.

[TF 15] $\gcd(a, b) \mid \operatorname{lcm}(a, b)$.

[TF 16] If $\gcd(a,b) = 2$ and n is even, then there exist integers x and y such that $ax + by = n$.

[TF 17] $(222)_3 = (111)_6$

[TF 18] If $a^2 = b^3$, then every prime divisor of a^2 is a divisor of b.

[TF 19] The last digit of 33^{33} is 3.

[TF 20] $(d_2d_1d_0)_{10} \equiv d_2 + d_1 + d_0 \pmod{9}$.

[TF 21] There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $(1,1) \in f$ and $(1,2) \in f$.

[TF 22] If $f : A \to B$ is a function, then every element of the range of f has a pre-image.

[TF 23]* The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is onto.

[TF 24] The function $f : \{2,3\} \to \{-4,-9\}$ defined by $f(x) = -x^2$ has an inverse.

[TF 25] Any subset of $\mathbb{N} \times \mathbb{N}$ is countable.

[TF 26]* The set $A = \{2n + 1 : n \in \mathbb{Z}\}$ is countable.

[TF 27] Every uncountable set has a countable subset.

[TF 28]* There exists a one-to-one and onto function $f : \mathbb{Z} \to \mathbb{N}$.

2. [3] Use any method to determine whether $(p \leftrightarrow q) \rightarrow (p \wedge \neg q)$ is a tautology. Write a sentence that explains how your work shows your conclusion.

3. [4] Use known logical equivalences to show that $p \rightarrow \neg(q \wedge \neg r)$ is logically equivalent to $(p \wedge q) \rightarrow r$.

4. [2] Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\forall x, \exists y, xy \leq x+y$. Explain your reasoning.

5. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{ll} & \neg p \vee q \\ & q \rightarrow \neg r \\ & r \\ \hline \therefore & \neg p \end{array}$$

6. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{ll} & r \vee \neg s \\ & p \leftrightarrow \neg q \\ & s \rightarrow p \\ \hline \therefore & q \end{array}$$

7. [4] Let A and B be sets such that $A \subseteq B$. Prove that $A \oplus B \subseteq B$, using an argument that starts with "Take any $x \in A \oplus B$ ...". Then use the universe $\mathcal{U} = \{1, 2\}$ to demonstrate that $A \oplus B \subsetneq B$ is possible.

8. [4] Let A, B and C be sets. Show that $A \setminus (B \cup C) = (A \setminus B) \cap A \setminus C)$. Hint: there is a short argument that uses set-theoretic identities.

9. [3] A set A consisting of 40 integers contains 20 even numbers, four multiples of 10 and eight numbers that are relatively prime to both 2 and 5. How many odd numbers in A are multiples of 5?

10. [2] Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

11. [4] Use the Euclidean Algorithm to find $d = gcd(1350, 456)$ and then use your work to find integers x and y such that $1350x + 456y = d$.

12. [3] Find the base 7 representation of 2021.

13. Let k be an integer such that $k \equiv -1 \pmod 5$.

(a) [2] What is the remainder when $50k^{10} + 26k^3 - 17$ is divided by 5?

(b) [2] What is the last digit of the base 5 representation of k? Why?

14. [4] Use induction to prove that $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

15. [3] Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 3$, and $a_n = 4a_{n-1} + 3$ for $n \geq 2$. Express each of a_2, a_3, a_4, and a_5 as a sum of terms that involve the numbers 3, 4 and exponents. Then, use your work to guess a (correct) formula for a_n. You do not need to prove that your formula is correct. (Suggestion: first use your work to express a_n as a sum of n terms, as above, and then use that sum to obtain a formula.)

16. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 0$, $b_1 = 1$ and $b_n = 2b_{n-1} - b_{n-2} + 2$ for all $n \geq 2$. Use induction to prove that $b_n = n^2$ for all $n \geq 0$.

17. [4] Let $f : [0, \infty) \to \mathbb{N}$ be defined by $f(x) = \lfloor x + 1 \rfloor$. Show that f is onto, but not 1-1. (Note: onto means surjective and 1-1 means injective; also, $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$.)

18. [4]* Consider the relation $\mathcal{R}$ on the set $\mathbb{Z}$ defined by $(a, b) \in \mathcal{R}$ if and only if $a + b$ is an even integer. Answer each of the following and provide a proof or counterexample as an explanation.

    (a) Is $\mathcal{R}$ reflexive?

    (b) Is $\mathcal{R}$ symmetric?

    (c) Is $\mathcal{R}$ antisymmetric?

    (d) Is $\mathcal{R}$ transitive?

19. Let $A = \{1, 2, \ldots, 35\}$. Fill in the blanks. No justification is necessary.

    (a) [1] The number of functions $f : \{1, 2\} \to A$ is ______.

    (b) [1] The number of subsets of $A \times A$ is ______.

    (c) [1] The number of 1-1 functions $f : A \to \{1, 2, \ldots, 36\}$ is ______.

    (d) [1] The number of non-empty subsets of A that contain neither 2 nor 3 is ______.

20. [3] Use a diagonal sweeping argument to prove that $\mathbb{N} \times \mathbb{Z}$ is countable.

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