Work-Energy theorem

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

parametrize path taken.

What if we used t as our parameter?

$$W_{net} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r}$$

$$\vec{r}(t) \rightsquigarrow d\vec{r} = \frac{d\vec{r}(t)}{dt} dt$$
$$= \vec{v}(t)\, dt$$

$$\vec{F}_{net}(t) \rightsquigarrow \vec{a}(t) = \frac{\vec{F}_{net}}{m}$$
$$\boxed{\vec{F}_{net} = m\vec{a}(t)}$$

$$W_{net} = \int_{t_i}^{t_f} (m\vec{a}(t)) \cdot (\vec{v}(t)\, dt)$$
$$= m \int_{t_i}^{t_f} \vec{a}(t) \cdot \vec{v}(t)\, dt$$

F.T.O.C

if $\frac{d}{dx}F(x) = g(x)$

then $\int_a^b g(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

$$\int_a^b \left(\frac{d}{dx}F(x)\right)dx = F(b) - F(a)$$

$$W_{net} = m\int_{t_i}^{t_f} \underbrace{\vec{a}(t)\cdot\vec{v}(t)}_{\text{is this } \frac{d}{dt} \text{ something?}}\,dt$$

What is

$$\frac{d}{dt}|\vec{v}(t)|^2 = \frac{d}{dt}\left(\vec{v}(t)\cdot\vec{v}(t)\right)$$

$$= \left(\frac{d}{dt}\vec{v}(t)\right)\cdot\vec{v}(t)$$

$$+ \vec{v}(t)\cdot\left(\frac{d}{dt}\vec{v}(t)\right)$$

$$= \vec{a}(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{a}(t)$$

$$= 2\,\vec{a}(t)\cdot\vec{v}(t)$$

$$W_{net} = \int_{t_i}^{t_f} \left(\frac{d}{dt} \frac{m}{2} |\vec{v}(t)|^2 \right) dt$$

$$= \underbrace{\frac{m}{2} |\vec{v}(t_f)|^2} - \frac{m}{2} |\vec{v}(t_i)|^2$$

Define for point particle

$$KE = \frac{1}{2} m |\vec{v}|^2$$

$$\boxed{W_{net} = KE_f - KE_i}$$

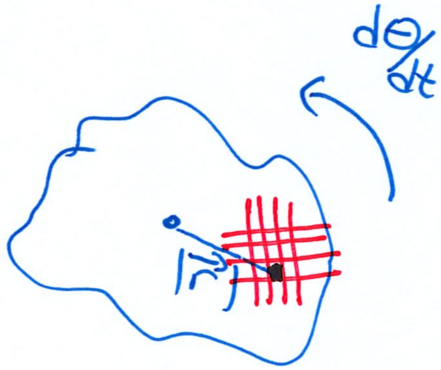
Work-Energy Theorem

Work-Energy

$$W_{net} = \Delta KE$$

$$KE = \frac{1}{2} m |\vec{v}(t)|^2 \quad \text{(point particle)}$$

What is KE for extended object?



plane lamina
rotating around
center of mass
($\vec{L}$ along axis of
rotation)

Think of object as made of
small pieces.
Find mass (dm) of each bit.
Find its speed
Find KE of each bit
Add them up.

Piece has mass dm

moving speed $|\vec{v}| = |\vec{r}| \, d\theta/dt$

KE of piece is

$$\frac{1}{2}(dm)|\vec{r}|^2\left(\frac{d\theta}{dt}\right)^2$$

Total KE

$$= \int \frac{1}{2} dm |\vec{r}|^2 \left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 \underbrace{\int dm |\vec{r}|^2}_{I\,!}$$

For rotating object $KE = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$

rotation rate

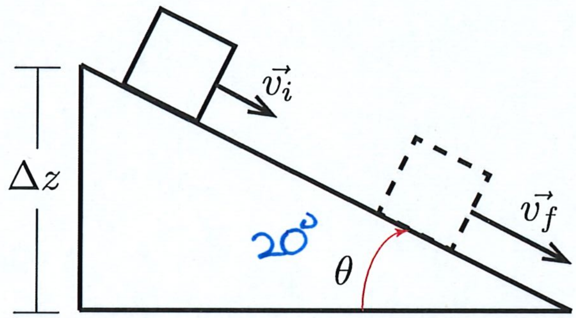
$$KE_{total} = \frac{1}{2} m |\vec{v}_{cm}|^2 + \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2$$

9-8-Example-WKE3

Work and Kinetic Energy - III

A $2kg$ block is on a rough slope. It is sliding down the slope, and has a coeffiecient of kinetic friction of 0.1 (μ_k) with the slope. The slope makes an angle of $\theta = 20°$ with the horizontal as shown.

The mass initially has a speed of $2\frac{m}{s}$. The vertical component of its displacement was $\Delta z = -1m$ (ie it went down by $1m$).



$W = \vec{F}\cdot\vec{\Delta r}$

- How much work is done by gravity?
- How much work is done by friction?
- How much work is done by the normal force?

$W_{net} = \Delta KE$

- What is the block's final kinetic energy?

$KE = \frac{1}{2}m v^2$

- What is the block's final speed?

What is $\vec{\Delta r}$

$\tan\theta = \frac{|\Delta z|}{\Delta x}$

$\vec{\Delta r} = \Delta x\,\hat{i} + \Delta z\,\hat{k}$

$$= \frac{|\Delta z|}{\tan\theta}\hat{i} + \Delta z\,\hat{k}$$

$$|\vec{\Delta r}| = \sqrt{\vec{\Delta r}\cdot\vec{\Delta r}} = \sqrt{\left(\frac{|\Delta z|}{\tan\theta}\right)^2 + (\Delta z)^2}$$

$$= |\Delta z|\sqrt{\frac{1}{\tan^2\theta} + 1}$$

$$\frac{1}{\left(\frac{\sin\theta}{\cos\theta}\right)^2}$$

$$\frac{\cos^2\theta}{\sin^2\theta}$$

$$= |\Delta z|\sqrt{\frac{1}{\sin^2\theta}}$$

$$= |\Delta z|\frac{1}{\sin\theta}$$

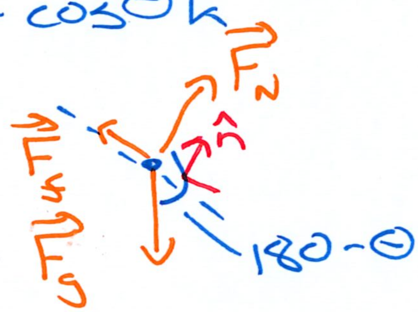
$$\vec{F_g} = -mg\hat{k}$$

$\vec{F_N}$: Unit vector

$$\cos(90-\theta)\hat{i} + \cos\theta\hat{k}$$

$$\sin\theta\hat{i} + \cos\theta\hat{k}$$

Magnitude:



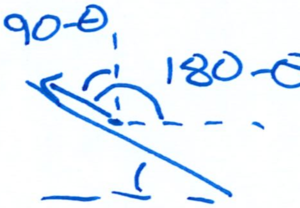
$$\hat{n}\cdot\vec{F}_{net} = 0$$

$$0 + |\vec{F_N}| + mg\cos(180-\theta) = 0$$

$$|\vec{F_N}| = mg\cos\theta$$

$$\vec{F_N} = mg\cos\theta(\sin\theta\hat{i} + \cos\theta\hat{k})$$

$\vec{F}_f$



unit vector

$$\cos(180-\theta)\hat{i} + \cos(90-\theta)\hat{k}$$

$$= -\cos\theta\hat{i} + \sin\theta\hat{k}$$

$$\vec{F}_f = \mu_k|\vec{F}_N|(-\cos\theta\hat{i} + \sin\theta\hat{k})$$

$$W_g = \vec{F}_g \cdot \Delta\vec{r} = (-mg\hat{k}) \cdot \left(\frac{|\Delta z|}{\tan\theta}\hat{i} + \Delta z\hat{k}\right)$$

$$= -mg\Delta z$$

$$= -(2\text{kg})\left(9.8\tfrac{\text{N}}{\text{kg}}\right)(-1\text{m})$$

$$= 19.6\text{J}$$

$$W_f = \vec{F}_f \cdot \Delta\vec{r} = \mu_k|\vec{F}_N|(-\cos\theta\hat{i} + \sin\theta\hat{k}) \cdot \left(\frac{|\Delta z|}{\tan\theta}\hat{i} + \Delta z\hat{k}\right)$$

$$= \mu_k|\vec{F}_N|\left(\frac{-\cos\theta|\Delta z|}{\tan\theta} + \sin\theta\Delta z\right)$$

$$= \mu_k(mg\cos\theta)\Delta z\left(\frac{\cos\theta}{\tan\theta} + \sin\theta\right)$$

$$\frac{\cos^2\theta}{\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)} + \frac{\sin^2\theta}{\sin\theta}$$

$$= \mu_k mg\cos\theta\frac{\Delta z}{\sin\theta}$$

$$= (0.1)(2\,kg)(9.8\tfrac{N}{kg})\frac{\cos 20^\circ}{\sin 20^\circ}(-1m)$$

$$= -5.39\,J$$

$$W_N = mg\cos\theta(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})\cdot\left(\frac{|\Delta z|}{\tan\theta}\hat{i} + \Delta z\,\hat{k}\right)$$

$$= mg\cos\theta\left(\frac{\sin\theta\,|\Delta z|}{\tan\theta} + \Delta z\cos\theta\right)$$

$$= mg\cos\theta\left(\cos\theta\,|\Delta z| + \Delta z\cos\theta\right)$$

$$= 0$$

$$W_{net} = W_g + W_f + W_N$$

$$= 19.6\,J + (-5.4\,J) + 0\,J$$

$$= 14.2\,J$$

$$W_{Net} = \Delta KE = KE_f - KE_i$$

$$KE_f = KE_i + W_{net}$$

$$= \frac{1}{2}m|\vec{v}|^2 + W_{net}$$

$$= \frac{1}{2}(2\,kg)(2\,m/s)^2 + 14.2\,J$$

$$KE_f = 18.2\,J$$

$$\frac{1}{2}m|\vec{v}_f|^2 = 18.2\,J$$

$$|\vec{v}_f| = 4.3\,m/s$$

Conservative & Non-Conservative forces

$$W = \int \vec{F} \cdot d\vec{r}$$

along path taken!

Does the path matter? Sometimes

Forces where path taken does matter (friction) Non-conservative

Forces where path doesn't matter (Gravity, Coulomb force, springs...)
Conservative (associated with a potential Energy)

9-10-Example-WKE4

Work and Kinetic Energy - IV

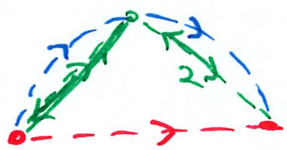
A block of mass $3kg$ is on a flat rough surface with which it has a coefficient of kinetic friction of 0.3.

The block starts at $-1m\hat{\imath}$ and moves to $1m\hat{\imath}$. How much work is done by friction if the path:

a • is the straight line from $-1m\hat{\imath}$ to $1m\hat{\imath}$?

b • is a triangle from $-1m\hat{\imath}$ to $1m\hat{\jmath}$ to $1m\hat{\imath}$?

c • is the half-circle of radius $1m$ centered at the origin going through $1m\hat{\jmath}$?



Purpose of example: illustrate W different

$W = \vec{F} \cdot \Delta\vec{r}$ $\qquad$ $W = \int \vec{F} \cdot d\vec{r}$

(a) Force constant

(b) Force changes, constant for the two steps

(c) Force changes; always opposite motion

(a) $\vec{F} = -\mu_k mg\,\hat{i}$

$\Delta\vec{r} = 1\text{m}\,\hat{i} - (-1\text{m}\,\hat{i}) = 2\text{m}\,\hat{i}$

$\vec{F}\cdot\Delta\vec{r} = (-\mu_k mg)(2\text{m})$

$= -17.6\,\text{J}$

(b) For step 1 $\quad \Delta\vec{r} = 1\text{m}\,\hat{j} - (-1\text{m}\,\hat{i})$

$= 1\text{m}\,\hat{i} + 1\text{m}\,\hat{j}$

$\vec{F} = \cancel{-}(\mu_k mg)\left[\frac{-1}{\sqrt{2}}\hat{i} + \frac{-1}{\sqrt{2}}\hat{j}\right]$

$\vec{F}\cdot\Delta\vec{r} = (\mu_k mg)\left(-\frac{2\text{m}}{\sqrt{2}}\right) = -12.5\,\text{J}$

For step 2 $\quad \Delta\vec{r} = 1\text{m}\,\hat{i} - 1\text{m}\,\hat{j}$

$\vec{F} = (\mu_k mg)\left(\frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$

$\vec{F}\cdot\Delta\vec{r} = -12.5\,\text{J}$

$W_{total} = -24.9\,\text{J}$

(c) Parametrize path

$\vec{r}(s) = -1\text{m}\cos s\,\hat{i} + 1\text{m}\sin s\,\hat{j}$

$0 \le s \le \pi$

$$d\vec{r} = \frac{d\vec{r}}{ds}ds = \left[(1m \sin s)\hat{i} + (1m \cos s\,\hat{j})\right]ds$$

$$\vec{F}_f = \mu_k mg(-\sin s\,\hat{i} - \cos s\,\hat{j})$$

$$\vec{F}_f \cdot d\vec{r} = \mu_k mg(-\sin s\,\hat{i} - \cos s\,\hat{j}) \cdot \left[1m \sin s\,\hat{i} + 1m \cos s\,\hat{j}\right]ds$$

$$= \mu_k mg(1m)ds(-\sin^2 s - \cos^2 s)$$

$$= -\mu_k mg(1m)ds$$

$$W = \int_0^{\pi} -\mu_k mg(1m)ds$$

$$= \int_0^{\pi} -8.82J\,ds \quad = -8.82J\,\pi = 27.7J$$