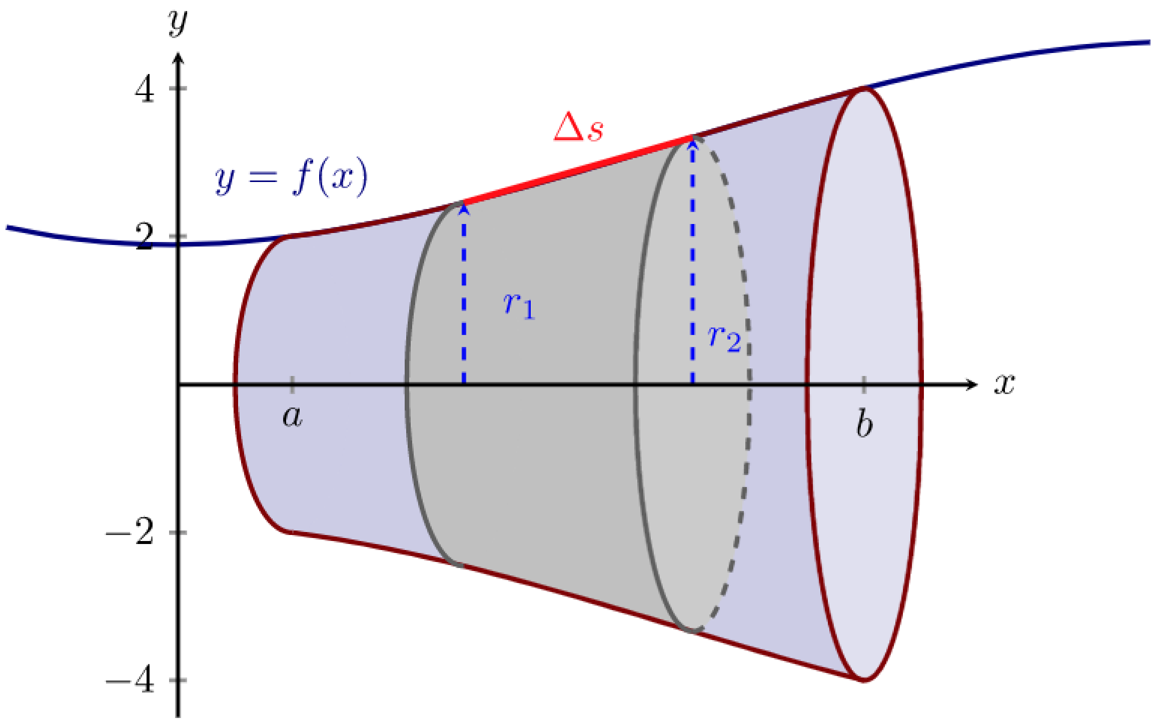
Chapter 1

Integration Techniques and Applications



1.1 Brief Review of Integration

The first month focuses heavily on picking up where Calculus I left off...integration.

There are two types of integration in Calculus I,

- Definite Integration: $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$
- Indefinite Integration: $\displaystyle\int f(x)dx = F(x) + C$ where $F'(x) = f(x)$.

Indefinite Integration

This is when we try to find the antiderivative of a function. Usually we do this to apply it to definite integration. That is,

If $F'(x) = f(x)$ then $\displaystyle\int f(x)dx = F(x) + C$

We call $F(x)$ an antiderivative of $f(x)$ and $F(x) + C$ the most generalized antiderivative of $f(x)$.

Note: Do not forget the constant of integration!

Definite Integration

This is defined by the use of Riemann sums,

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$. This geometrically represents the area between the graphs of $y = f(x)$ and $y = 0$ over the region $a \le x \le b$.

Another, and in my personal opinion, a better way to think of this integral is that it adds up the area of a bunch of rectangles with area $f(x)dx$ which we represent as $f(a + k\Delta x)\Delta x$ for $\Delta x = \dfrac{b-a}{n}$ of infinitesimal size. The reason why to view it as a summation of terms $f(x)dx$ rather than area is because in the future you might replace these terms with other quantities that shouldn't be interpreted geometrically.

Example: Consider a metal rod where we measure along only it's length from the bottom $x = 0$ to $x = L$ if it is L units in length.

Let $\rho(x_0)$ represent the density of the bar at the point $x = x_0$. As dx is thought of as an infinitesimal of Δx, a volume portion of the bar, then $\rho(x)dx$ is the product of density and volume in a small region of the bar yielding the mass in that region. To get the total mass of the bar we sum up everything to obtain

$$\text{Mass} = \int_0^L \rho(x)dx$$

Note: This example illustrates that like the derivative, there are two interpretations of integration. One physical and one geometric. Do not pick favourites on how you like to interpret integration, you need the flexibility of both to solve problems.

By the second part of the **Fundamental Theorem of Calculus** there is a relationship between antiderivatives and definite integration. This is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$.

Example: Find the area between $y = x$ and $y = x^2 - 2x$.

Example: Compute $\int_0^{\pi/6} (2\sec(\theta) + \sec(\theta)\tan(\theta))d\theta$

There are basic rules to integration:

- $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
- $\int kf(x)dx = k \int f(x)dx$

Integration (indefinite) undoes differentiation. Undoing something is hard! Unlike differentiation, we are very limited in ways to find an antiderivative. I like to compare differentiation and antidifferentiation to a plate. Differentiation would be smashing the plate and antidifferentiation would be gluing it back together.

Substitution (undoing the chain rule)

This dealt with integrals of the form

$$\int_a^b f(g(x))g'(x)dx$$

By letting $u = g(x)$ we have $du = g'(x)dx$. Moreover, when $x = a$ then $u = g(a)$ and when $x = b$ then $u = g(b)$. So this becomes

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

This type of substitution letting $u = g(x)$ is called a **push-forward** substitution. There is another type of substitution called a **pull-back** substitution that we cover when we discuss trigonometric substitution.

Example: Compute $\displaystyle\int_0^1 xe^{-x^2}dx$

There are two special types of substitution that we do which don't seem as obvious to fit into the form above. Consider the following integral

$$\int_a^b f(x+c)dx$$

By letting $u = x + c$ we obtain $du = dx$. The bounds (if present) change as $u = a + c$ and $u = b + c$ when $x = a$ and $x = b$ respectively. This gives us

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(u)du$$

I like to call these **shifts**.

Example: Compute $\displaystyle\int \frac{x}{(x+1)^{2020}}dx$

Now consider the integral below,

$$\int_a^b f(kx)dx$$

where $k \neq 0$. We may use the substitution $u = kx$ to obtain $du = kdx$. The bounds change as well to integrate from $u = ka$ to $u = kb$. Now we have a problem with the above, we don't have a kdx term in the integral. We get around this issue as follows:

$$\int_a^b f(kx)dx = \frac{1}{k}\int_a^b f(kx)(kdx) = \frac{1}{k}\int_{ka}^{kb} f(u)du$$

where we just use the fact that $1 = k/k$ to introduce the k term we need. I like to call these **scalings**.

Example: Compute $\displaystyle\int_0^{\pi/15} \sin(5x)dx$.

1.2 (Section 8.1) Some Integration Techniques and Tricks

Most integral aren't immediately clear on what the antiderivative is. Often algebraic tricks are required or some clever use of identities.

1.2.1 Completing the Square

Very useful! This technique is use to turn two terms where a variable appears into a single term. For terms like $x^2 + ax$ it turns them into shifts, which you can sub for. A reminder of how to complete the square:

$$x^2 + ax = x^2 + ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

Example: Compute $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$.

$$\begin{aligned} x^2 - 4x + 3 &= x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + 3 \\ &= x^2 - 4x + 4 - 4 + 3 \\ &= (x-2)^2 - 1 \end{aligned}$$

$$I = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$$

Let $u = x-2$

$du = (1-0)\,dx = dx$

$$= \int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$

$$= \operatorname{arcsec}|x-2| + C$$

1.2.2 Using Trigonometric Identities

We will learn more about using trig identities as the course develops. Currently a very helpful identity to know if $\cos^2(x) + \sin^2(x) = 1.$ The example below is an example similar to the idea of the previous example, to turn two terms into a single term that is easier to control.

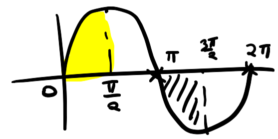
Example: Compute $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos(\theta)}\ \ d\theta.$

$$(a-b)(a+b) = a^2 - b^2$$

$$= \int_0^{\pi/2} \sqrt{1-\cos(\theta)} \cdot \frac{\sqrt{1+\cos(\theta)}}{\sqrt{1+\cos(\theta)}}\, d\theta$$

$$1-\cos^2(\theta) = \sin^2(\theta)$$

$$= \int_0^{\pi/2} \frac{\sqrt{1-\cos^2(\theta)}}{\sqrt{1+\cos(\theta)}}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin^2(\theta)}}{\sqrt{1+\cos(\theta)}}\, d\theta$$



$$= \int_0^{\pi/2} \frac{|\sin(\theta)|}{\sqrt{1+\cos(\theta)}}\, d\theta$$

$$\sin(\theta) > 0 \quad \text{in} \quad (0, \pi/2)$$

$$= \int_0^{\pi/2} \frac{\sin(\theta)}{\sqrt{1+\cos(\theta)}}\, d\theta$$

$$u = 1+\cos(\theta)$$
$$du = -\sin(\theta)\, d\theta$$
$$\Rightarrow -du = \sin(\theta)\, d\theta$$

$$= \int_2^1 \frac{-du}{\sqrt{u}}$$

$$\underline{\text{Bounds}} \qquad u = 1+\cos(\theta)$$
$$\theta = 0 \Rightarrow u = 1+\cos(0) = 1+1 = 2$$
$$\theta = \pi/2 \Rightarrow u = 1+\cos\left(\tfrac{\pi}{2}\right) = 1+0 = 1$$

$$= -\left.\frac{u^{-1/2+1}}{-1/2+1}\right|_2^1$$

$$= -\left.\frac{u^{1/2}}{1/2}\right|_2^1$$

$$= -2\sqrt{u}\,\Big|_2^1 = -2\sqrt{1} - (-2\sqrt{2})$$

$$= -2 + 2\sqrt{2}\ //$$

1.2.3 Long Division

This is for rational expressions where the degree of the numerator is higher than or equal to the degree of the polynomial on the denominator. A reminder that a rational function is of the form:

$$\frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials. Fome examples are as follows:

$$\frac{x^2+1}{x^3+3x} \qquad \frac{2x+4}{x+5} \qquad \cancel{\frac{\cos(x)}{1+x}} \qquad \cancel{\frac{\sqrt{x}+1}{1+x^2}}$$

Not Not

To evaluate these integrals we use long division to put it into a more favorable form. This doesn't always work, but we will learn a another technique to pass such a roadblock when we study partial fractions in this chapter.

Example: Compute $\int \frac{x^3+x}{x-1}dx$.

$$\begin{array}{r|l} & x^2+x+2 \quad \leftarrow \text{Quotient} \\ x-1 & x^3+0x^2+x+0 \\ & -(x^3-x^2) \\ \hline & 0+x^2+x \\ & -(x^2-x) \\ \hline & 0+2x+0 \\ & -(2x-2) \\ \hline & 0+2 \quad \leftarrow \text{Remainder} \end{array}$$

$$I = \int \frac{x^3+x}{x-1}dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+2x+2\ln|x-1|+C \;//$$

(Continued...)

1.2.4 Rewriting (a matter of perception)

Look to rewrite an expression. Sometimes a good technique is to look for common factors. This is a little tricky at first. When we say common factors, we don't only mean common factors of a polynomial.

Example: Compute $\int (27e^{9x} + e^{12x})^{1/3} dx$. $e^{12x} = e^{9x+3x} = e^{9x}e^{3x}$

$$= \int (27e^{9x} + e^{3x}e^{9x})^{1/3} dx$$

$$= \int (27 + e^{3x})^{1/3} (e^{9x})^{1/3} dx$$

$$= \int \underbrace{(27+e^{3x})}^{1/3} \underbrace{e^{3x} dx}$$

$u = 27 + e^{3x}$

$du = 3e^{3x} dx$

$\Rightarrow \frac{1}{3} du = e^{3x} dx$

$\frac{d}{dx}[e^{3x}] = e^{3x} \frac{d}{dx}[3x]$

$= 3e^{3x}$ //

$$= \int u^{1/3} \cdot \frac{1}{3} du$$

$$= \frac{1}{3} \cdot \frac{u^{1/3+1}}{1/3+1} + C$$

$$= \frac{1}{3} \cdot \frac{u^{4/3}}{4/3} + C = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{4} (27+e^{3x})^{4/3} + C$$

$$= \frac{1}{4} (27+e^{3x})^{4/3} + C \text{ //}$$

1.2.5 Dumb Luck

I'm sorry to say but integrals, as you have probably seen at this point, are not as straight-forward to compute as derivatives. We know really only one technique so far to compute integrals, substitution.

Sometimes we can make a substitution that changes the integral into something we can evaluate without a good reason why it would have worked initially. Math is often preached as a logical step from one to the next, but it honestly isn't always that case. Some things are solved by trial and error. This isn't a fault in math, this is a fault in the nature of the universe and it's reality.

Note: Saying an answer just came to you without a logical reason when there **IS** *a logical reason does not justify the result. It just comes off as a lack of work.*

Example: Compute $\int \frac{\sqrt{x}}{1+x^3}dx.$

Let $x^3 = t^2 \Rightarrow t = x^{3/2}$ taking $x > 0$.

$dt = \frac{3}{2}x^{1/2}dx$

$\Rightarrow \quad = \frac{3}{2}\sqrt{x}\,dx$

$\Rightarrow \frac{2}{3}dt = \sqrt{x}\,dx$

$dt = \frac{3}{2}\sqrt{x}\,dx$

$\frac{2}{3}dt = \sqrt{x}\,dx$

$= \frac{2}{3}\int \frac{dt}{1+t^2}$

$= \frac{2}{3}\arctan(t) + C$

$= \frac{2}{3}\arctan(x^{3/2}) + C$ //

Some 'dumb luck' methods are a bit more clear on where to start. Sometimes it's because we want to reduce the complexity of the number of terms. Below we do an example of using a substitution to 'eliminate' a constant to avoid expansion.

Example: Compute $\int 27e^{-3x}(1+3e^{-x})^4 dx$.

Let $u = 1+3e^{-x} \Rightarrow \frac{u-1}{3} = e^{-x} \Rightarrow \left(\frac{u-1}{3}\right)^2 = e^{-2x}$

$du = (0-3e^{-x})\,dx$

$\Rightarrow du = -3e^{-x}\,dx$

$\Rightarrow -\frac{1}{3}du = e^{-x}\,dx$

$$= \int 27 e^{-2x}(1+3e^{-x})^4 e^{-x}\,dx$$

$$= \int 27\left(\frac{u-1}{3}\right)^2 u^4 \cdot \left(-\frac{1}{3}du\right)$$

$$= \int -\frac{27}{27}(u^2-2u+1)u^4\,du$$

$$-\frac{\cancel{27}(u-1)^2u^4}{\cancel{3^2\cdot 3}} = -(u-1)^2u^4$$

$$= \int -(u^6-2u^5+u^4)\,du$$

$$= -\frac{1}{7}u^7 + \frac{2}{6}u^6 - \frac{1}{5}u^5 + C$$

$$= -\frac{1}{7}(1+3e^{-x})^7 + \frac{1}{3}(1+3e^{-x})^6 - \frac{1}{5}(1+3e^{-x})^5 + C \quad //$$

1.2.6 Mixing Techniques

It is possible you may have to mix techniques to complete a problem!

Example: Compute $\displaystyle\int \frac{x-6}{\sqrt{8x-x^2}}dx$.

First complete square

Technique #1: Completing the square

$$8x - x^2 = -(x^2-8x) = -\left(x^2-8x+\left(\tfrac{8}{2}\right)^2-\left(\tfrac{8}{2}\right)^2\right)$$
$$= -(x^2-8x+16-16)$$
$$= -((x-4)^2-16) = 16-(x-4)^2$$

$$\Rightarrow I = \int \frac{x-6}{\sqrt{16-(x-4)^2}}dx$$

Let $u = x-4 \Rightarrow du = dx$

Technique #2: u-Sub

Note: $x-6 = (x-4)-2$

$$= \int \frac{(x-4)-2}{\sqrt{16-(x-4)^2}}dx = \int \frac{u-2}{\sqrt{16-u^2}}du$$

Technique #3: Separating the integral

$$= \int \frac{u}{\sqrt{16-u^2}}du - 2\int \frac{du}{\sqrt{16-u^2}}$$

Let $v = 16-u^2$
$dv = -2u\,du$
$\Rightarrow -\frac{1}{2}dv = u\,du$

Technique #4: v-Sub

$$= -\frac{1}{2}\int \frac{dv}{\sqrt{v}} - 2\int \frac{du}{\sqrt{16-u^2}}$$

Technique #5: using algebra

$$= -\frac{1}{2}(2\sqrt{v}) - 2\int \frac{du}{4\sqrt{1-\left(\frac{u}{4}\right)^2}}$$

Let $w = \frac{u}{4}$
$\Rightarrow dw = \frac{1}{4}du$

$$= -\sqrt{v} - 2\int \frac{dw}{\sqrt{1-w^2}}$$

Technique #6: w-Sub

$$= -\sqrt{v} - 2\arcsin(w) + C$$
$$= -\sqrt{16-u^2} - 2\arcsin\left(\frac{u}{4}\right) + C$$
$$= -\sqrt{16-(x-4)^2} - 2\arcsin\left(\frac{x-4}{4}\right) + C \quad //$$

1.2.7 Organizing Definite Integral Substitution

One last note is when evaluating integrals is to be as organized as possible. Usually integrals require several steps and if another person, whether it be a colleague or a professor, sees work splattered randomly across the page or is broken into several obscure segments...they will just give up on trying to follow your reasoning. Best way to learn not to do this is to observe how people professionally present information and to mimic it into your technique. Aesthetic and presentation is crucial. You also want to ease the amount of work you have to do by avoiding several side calculations that don't remain consistent with the flow of your solution.

One such issue I commonly see students do is to not substitute for the bounds in definite integrals when solving something by substitution. Many extra steps are taken and it loses focus of the reader. While this is minor at this stage, it does make a difference when we study trigonometric substitution. It cuts down on time drastically when we substitute for the bounds in this section.

Example: Given that $f(t)$ is a continuous function on $\mathbb{R}$ such that $\int_1^4 f(t)dt = 3$ compute $\int_{-2}^{-1/2} f(2t+5)dt$.

$$\int_{-2}^{-1/2} f(2t+5)\,dt$$

$$= \int_1^4 f(u)\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\underbrace{\int_1^4 f(u)\,du}$$

$$= \frac{1}{2}\cdot 3 = 3/2$$

$u = 2t+5$

$du = 2dt$

$\Rightarrow \frac{1}{2}du = dt$

Bounds

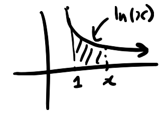
$t = -2 \Rightarrow u = 2(-2)+5 = 1$

$t = -1/2 \Rightarrow u = 2(-\frac{1}{2})+5 = 4$

1.3 (Section 7.1) Logarithms and Exponentials

The textbook goes through extensive measures to demonstrate how to reconstruct the exponential and all properties of the logarithm if we initially define the natural logarithm by

$f(x) = \frac{1}{x}$



$$\ln(x) = \int_1^x \frac{1}{t}dt, \qquad x > 0$$

the process is long and worth a read. What we will do is give examples of computing logarithmic and exponential type integrals.

1.3.1 Integrals Involving the Natural Logarithm

$(f \circ f^{-1})(x) = x$

It is defined as $\ln(x) = \int_1^x \frac{1}{t}dt$ or as the inverse function of $f(x) = e^x$. Since it is the inverse of $f(x) = e^x$ then

$$e^{\ln(x)} = x \qquad \text{and} \qquad \ln(e^x) = x$$

The logarithm is amazing in that it handles exponents and multiplication operations extremely well.

Properties

- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a/b) = \ln(a) - \ln(b)$
- $\ln(a^b) = b\ln(a)$

$\ln(x \cdot x)$ $\qquad$ $\ln(x) \cdot \ln(x)$

Note $\quad \ln x^2 \qquad \ln(x^2) \qquad (\ln(x))^2$

The notation $\ln a^b$ is ambiguous. For example, $\ln(x^2) = \ln(x \cdot x)$ but $(\ln(x))^2 = (\ln(x)) \cdot (\ln(x))$. In general notice that $\ln(a^b) \neq (\ln(a))^b$ so you must always specify where to place the parenthesis unless an author explicitly states what the notation means.

Since $\ln|x| = \int \frac{dx}{x} + C$ it is commonly found in dealing with integrals of the form

$$\int_a^b \frac{f'(x)}{f(x)}dx$$

since $u = f(x)$ gives $du = f'(x)dx$ and thus

$$\int_a^b \frac{f'(x)}{f(x)}dx = \int_{f(a)}^{f(b)} \frac{du}{u} = \ln|u|\Big|_{f(a)}^{f(b)}$$

Example: Compute $\int_{-1}^{0} \frac{3dx}{3x-2}$ $\qquad u = 3x - 2 \qquad du = 3dx$

Bounds

$x = -1 \Rightarrow u = 3(-1) - 2 = -5$

$x = 0 \Rightarrow u = 3(0) - 2 = -2$

$$= \int_{-5}^{-2} \frac{du}{u} = \ln|u|\Big|_{-5}^{-2} = \ln|-2| - \ln|-5|$$

$$= \ln(2) - \ln(5)$$

$$= \ln\left(\frac{2}{5}\right)$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Example: Compute $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx$.

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

$$= \int \frac{\ln(\ln(x))}{\ln(x)} \cdot \frac{1}{x}dx = \int \frac{\ln(u)}{u}du$$

Let $u = \ln(x)$

$du = \frac{1}{x}dx$

$$= \int \ln(u) \cdot \frac{1}{u}du$$

Let $v = \ln(u)$

$dv = \frac{1}{u}du$

$$= \int v\,dv$$

$$= \frac{1}{2}v^2 + C$$

$$= \frac{1}{2}(\ln(u))^2 + C$$

$$= \frac{1}{2}(\ln(\ln(x)))^2 + C$$

Example: Compute $\displaystyle\int \frac{1}{\arctan(4x)(1+16x^2)}\,dx$.

$$= \int \frac{\frac{1}{1+16x^2}}{\arctan(4x)}dx$$

$$\frac{a}{bc} = \frac{a/c}{b}$$

Let $u = \arctan(4x)$

$$du = \frac{1}{1+(4x)^2}\frac{d}{dx}[4x]\,dx$$

$$= \frac{4}{1+16x^2}dx$$

$$\Rightarrow \frac{1}{4}du = \frac{1}{1+16x^2}dx$$

$$= \int \frac{\frac{1}{4}du}{u}$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|\arctan(4x)| + C$$

1.3.2 Integrals Involving the Exponential Function

The counterpart to the logarithm. It is denoted either $f(x) = e^x$ or $f(x) = \exp(x)$ and can be defined in several different ways. It was first defined by the following limit

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

or you may define it as the inverse of $\ln(x) = \displaystyle\int_1^x \frac{dt}{t}$. We will learn later that another common definition some use is

$$e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$$

which will be discussed in detail in chapter 10.

It has the amazing property that

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \text{and} \qquad \int e^x dx = e^x + C$$

Because it is what some might call a "fixed function" in calculus, it appears very commonly when solving most problems.

Properties

- $e^{a+b} = e^a e^b$
- $\dfrac{e^a}{e^b} = e^{a-b}$
- $e^{-a} = \dfrac{1}{e^a}$
- $(e^a)^b = e^{ab}$

Note

These are just regular exponent laws. It is because, surprisingly, the way the exponential is defined as above (using limits) you would not expect it to be a function with these properties! It turns out that $\exp(x)$ shares all the properties that exponential laws give, thus we use an alternate notation $\exp(x) = e^x$ to make things easier for us. We **represent** the exponential function as a number to the variable because nothing changes if we do this.

Example: Compute $\int_0^{\sqrt{\ln(\pi)}} 2xe^{x^2}\cos(e^{x^2})\,dx.$

$u = x^2 \Rightarrow du = 2x\,dx$

Bounds

$x = 0 \Rightarrow u = 0^2 = 0$

$x = \sqrt{\ln(\pi)} \Rightarrow u = \left(\sqrt{\ln(\pi)}\right)^2 = \ln(\pi)$

$$= \int_0^{\ln(\pi)} e^u \cos(e^u)\,du$$

Let $v = e^u \Rightarrow dv = e^u\,du$

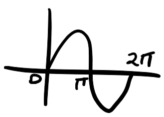
Bounds

$u = 0 \Rightarrow v = e^0 = 1$

$u = \ln(\pi) \Rightarrow v = e^{\ln(\pi)} = \pi$ $\because$ e and ln are inverses

$$= \int_1^{\pi} \cos(v)\,dv$$

$$= \sin(v)\Big|_1^{\pi}$$



$$= \sin(\pi) - \sin(1)$$

$$= 0 - \sin(1) = -\sin(1)$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

Bounds

$x = a \Rightarrow u = g(a)$

$x = b \Rightarrow u = g(b)$

1.3.3 Other Logarithmic and Exponential Integrals

We have the logarithms and exponentials of other bases

$$f(x) = a^x \qquad \text{and} \qquad g(x) = \log_a(x)$$

$$x \underset{\log_a(x)}{\overset{a^x}{\rightleftarrows}} y$$

They are inverses and thus related by

$$2^3 = 8 \qquad \log_2(8) = 3$$

$$a^{\log_a(x)} = x \qquad \text{and} \qquad \log_a(a^x) = x$$

with the integration and differentiation rules

$$\frac{d}{dx}[a^x] = (\ln(a))a^x \qquad \int a^x dx = \frac{a^x}{\ln(a)} + C$$

☆ Should be memorized

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)} \qquad \int \log_a(x)dx = \text{LEARN THIS LATER!}$$

☆ IBP in 8.2

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[a^x] = (\ln(a))a^x$$

Example: Compute $\displaystyle\int_1^e \frac{2\ln(10)\log_{10}(x)}{x}dx.$ Like $\sim \displaystyle\int_a^b C\frac{\ln(x)}{x}dx$

Let $u = \log_{10}(x) \Rightarrow du = \dfrac{1}{x\ln(10)}dx$

$\Rightarrow \ln(10)du = \frac{1}{x}dx$

Bounds

$x = 1 \Rightarrow u = \log_{10}(1) = 0$

$x = e \Rightarrow u = \log_{10}(e)$

$$= \int_0^{\log_{10}(e)} 2\ln(10)\cdot u\cdot \ln(10)du$$

$$= 2(\ln(10))^2\int_0^{\log_{10}(e)} u\,du$$

$$= \cancel{2}(\ln(10))^2\cdot\frac{1}{\cancel{2}}u^2\Big|_0^{\log_{10}(e)}$$

$$= (\ln(10))^2\left((\log_{10}(e))^2 - 0^2\right)$$

$$= (\ln(10))^2(\log_{10}(e))^2$$

Example: Compute $\displaystyle\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan(t)}\sec^2(t)\,dt.$

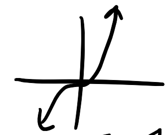
Let

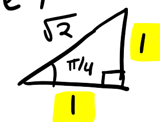
$u=\tan(t) \Rightarrow du=\sec^2(t)\,dt$

Bounds

$t=0 \Rightarrow u=\tan(0)=0$

$t=\frac{\pi}{4} \Rightarrow u=\tan(\frac{\pi}{4})=\frac{1}{1}=1$





$$=\int_0^1\left(\frac{1}{3}\right)^u du$$

$$=\left.\frac{(1/3)^u}{\ln(1/3)}\right|_0^1$$

$$=\frac{(1/3)^1}{\ln(1/3)}-\frac{(1/3)^0}{\ln(1/3)}=\frac{1/3}{\ln(1/3)}-\frac{1}{\ln(1/3)}$$

$$=-\frac{2}{3\ln(1/3)}=\frac{2}{3\ln(3)}$$

$\ln(1/3)=\ln(3^{-1})=-\ln(3).$

$$-\frac{2}{3\ln(1/3)}=-\frac{2}{3\ln(3^{-1})}=-(-1)\frac{2}{3\ln(3)}=\frac{2}{3\ln(3)}$$

$\ln(a^b)=b\cdot\ln(a)$

1.4 (Section 7.2) Separable Differential Equations and Modeling

1.4.1 Separable Differential Equations

Definition

A differential equation is an equation involving an unknown function and its derivatives.

Definition

A separable differential equation (SDE) is one of the form $y' = f(y)g(x)$.

Example: Which of the following are SDE's?

$$\frac{dy}{dx} = e^{x+y} \qquad \frac{dy}{dx} = \sin(x+y) \qquad \frac{dy}{dx} = x^2 \log_2(x)\ln(y) \qquad \frac{dy}{dx} = xy + x$$

Procedure for Solving an SDE

1. Provided $g(y) \neq 0$, divide both sides by $g(y)$ to obtain

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

2. Integrate both sides with respect to x to obtain

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$$

3. Use the substitution rule for integrals $dy = \dfrac{dy}{dx}dx$ to obtain

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

4. Integrate the previous expression

Note that solutions to SDE's are usually implicit and can seldomly be solved for explicitly.

Example: Solve explicitly for $y(x)$ if $xy' = 1 + y^2$.

1.4.2 Unlimited Population Growth Model (Malthus Model)

Malthus Model

A population growth with no limitations imposed is modeled by

$$\begin{cases} \dfrac{dy}{dt} = ky & \text{(Differential Equation)} \\ y(0) = y_0 & \text{(Initial Condition)} \end{cases}$$

where k is a constant and t is time. It has the unique solution

$$y = y_0 e^{kt}$$

Proof: Following the procedure of solving an SDE we have

$$\Rightarrow \frac{1}{y}\frac{dy}{dt} = k$$

$$\Rightarrow \int \frac{1}{y}\frac{dy}{dt}dt = \int kdt$$

$$\Rightarrow \int \frac{1}{y}dy = \int kdt$$

$$\Rightarrow \ln|y| = kt + C$$

By taking the exponential of both sides we obtain

$$e^{\ln|y|} = e^{kt+C} \Rightarrow |y| = e^C e^{kt}$$

As an additional assumption, we are using this to model the growth of a positive quantity of something (e.g. population). As a result, we may assume that $y(t) > 0$ and so the above simplifies to

$$y(t) = e^C e^{kt}$$

Now we invoke the condition $y_0 = y(0) = e^C e^0 = e^C(1) = e^C$ and so the solution is given by

$$y(t) = y_0 e^{kt}$$

■

Example: The biomass of a yeast culture in an experiment is 29g. After 30 min the mass is 37g. Assuming the equation for unlimited population growth models this, how long will it take for the mass to double from it's initial population size?

1.4.3 Radioactive Decay

Radioactive Decay Model

A substance undergoing radioactive decay is modeled by

$$\begin{cases} \dfrac{dy}{dt} = -ky & \text{(Differential Equation)} \\ y(0) = y_0 & \text{(Initial Condition)} \end{cases}$$

where $k > 0$. This has the unique solution

$$y(t) = y_0 e^{-kt}$$

which is obtained from the previous model by replacing k with $-k$. The constant k is usually obtained by knowing the half-life of a substance. It satisfies the equation

$$\text{Half-Life} = \frac{\ln(2)}{k}$$

Example: The half-life of carbon-14 is 5730 years. Find the age of a sample in which 10% of the radioactive substance has decayed.

1.4.4 Heat Transfer: Newton's Law of Cooling

Newton's Law of Cooling

If $H(t)$ is the temperature of an object at time t and H_s is the constant surrounding temperature the model is given by

$$\begin{cases} \dfrac{dH}{dt} = -k(H - H_s) & \text{(Differential Equation)} \\ H(0) = H_0 & \text{(Initial Condition)} \end{cases}$$

where k is a constant. You can obtain the unique solution by acknowledging the fact that because H_s is a constant then

$$\frac{dH}{dt} = \frac{d}{dt}\left[H - H_s\right]$$

thus the above differential equation becomes $\dfrac{d}{dt}\left[H - H_s\right] = -k(H - H_s)$. By the previous model with $y = H - H_s$ we obtain the solution $H - H_s = (H_0 - H_s)e^{-kt}$. Solving for H gives us

$$H(t) = H_s + (H_0 - H_s)e^{-kt}$$

Example: In Breaking Bad, Walter and Jesse cook "soup". Accidentally the soup is heated to 98^{o}C by Jesse screwing up again. After Walter yells at Jesse and tells him '**I am the one who cooks**' he tries to cool the soup by immersing it in a container surrounded by 18^{o}C water. After 5 minutes the temperature of the soup is 38^{o}C. The batch will be ruined if it doesn't reach 20^{o}C within 10 minutes. Is this batch spoiled?

1.5 (Section 8.2) Integration By Parts (IBP)

1.5.1 Establishing Integration by Parts and LIPET

Integration by parts is an integration rule that deals with undoing the product rule. Let $u(x)$ and $v(x)$ be differentiable functions. Then by the product rule,

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x over $[a, b]$ yields

$$\int_a^b \frac{d}{dx}[uv]\,dx = \int_a^b u\frac{dv}{dx}dx + \int_a^b v\frac{du}{dx}dx$$

$$\Rightarrow uv\Big|_a^b = \int_a^b u\frac{dv}{dx}dx + \int_a^b v\frac{du}{dx}dx$$

Then rewrite this as

$$\int_a^b u\frac{dv}{dx}dx = uv\Big|_a^b - \int_a^b v\frac{du}{dx}dx$$

This is also commonly represented as the following.

Integration by Parts (IBP)

Let $u(x)$ and $v(x)$ be differentiable functions on an interval (a, b) and continuous on $[a, b]$. Then...

$$\int_a^b \underline{udv} = \underline{uv}\Big|_a^b - \int_a^b \underline{vdu}$$

$u \to du$, $dv \to v$

The idea is that you manipulate the product rule to obtain a way to express an integral product in the form of another integral product. The hope is that this rearrangement simplifies the integration. Notice that when you start with u on the left hand side you differentiate it in the right hand side. When you start with $\frac{dv}{dx}$ on the left hand side you find the anti-derivative v on the right hand side. In short, you often make one term in the product simpler while you make the other term more complicated. The hope is that the simplification of one term outdoes complicating the other to make the integral doable.

Example: Compute $\int x\cos(x)dx$ using $u = x$ and $dv = \cos(x)dx$.

$\int u\,dv = uv - \int v\,du$

$u = x \qquad du = dx$

$dv = \cos(x)\,dx \qquad v = \sin(x)$

$= x\sin(x) - \int \sin(x)\,dx$

$= x\sin(x) - (-\cos(x)) + C$

$= x\sin(x) + \cos(x) + C$ //

$u = f(x) \qquad du =$

$dv = g(x)dx \qquad v =$

Be careful to make good choices! Selecting which term to differentiate and which one to integrate matters! Let's revisit the previous example where we make the other choice of assigning u and v.

Example: Compute one iteration of IBP for $\displaystyle\int x\cos(x)dx$ using $u=\cos(x)$ and $dv=xdx$.

$u=\cos(x)$ $\quad$ $du=-\sin(x)dx$

$dv=x\,dx$ $\quad$ $v=\frac{1}{2}x^2$

$$=\underbrace{\tfrac{1}{2}x^2\cos(x)}_{uv}\underbrace{-}_{-}\underbrace{\int \tfrac{1}{2}x^2(-\sin(x))dx}_{\int v\,du}$$

Way Harder!

You can see the previous example did not work out well after a single iteration. The purpose is to simplify the expression! Not make it even more complicated! Thankfully there is an acronym called **LIPET** which helps us determine which term to set as our u-term in some cases.

LIPET

Consider an integral of the form $\displaystyle\int f(x)g(x)dx$ where $f(x)$ and $g(x)$ are either a **L**ogarithmic function, **I**nverse trigonometric function, **P**olynomial function, **E**xponential function, or **T**rigonometric function. **LIPET** is an acronym that tells you that you set your u term as the first function that is present on the list:

- **L** = Logarithmic function
- **I** = Inverse trigonometric function
- **P** = Polynomial function
- **E** = Exponential function
- **T** = Trigonometric function

$\int x\ln(x)dx$ $\quad$ $u=\ln(x)$ $\quad$ $dv=x\,dx$

$\int x\tan(x)dx$ $\quad$ $u=x$ $\quad$ $dv=\tan(x)dx$

Example: The above LIPET system does not hold for the following integrals! Can you explain why? Ability to find an antiderivative is **NOT** the reason!

$$\int \underbrace{x}_{\text{Polynomial}}\underbrace{\sqrt{1-x^2}}_{\text{Algebraic}}dx \qquad \int \underbrace{\ln(y)e^y\tan(y)}_{\text{Product of 3 elementary functions. Not two!}}dy \qquad \int \underbrace{\frac{1-t^2}{1+t^2}}_{\text{Rational}}\underbrace{e^t}_{\text{Exponential}}dt$$

LIPET

1.5.2 Dominant Logarithms and Inverse Functions in LIPET

Logarithmic and inverse functions are dramatically more complicated that their derivatives. For this reason, they often become a good first choice as what to set your u-term to be when performing integration by parts! The derivative of these expressions are usually some rational type function.

Example: Compute $\int_1^e \underbrace{x^3}_{P} \underbrace{\ln(x)}_{L} dx$.

$u = \ln(x) \qquad du = \frac{1}{x}dx$

$dv = x^3 dx \qquad v = \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln(x) \Big|_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x} dx$

$= \frac{1}{4}e^4 \ln(e) - \frac{1}{4}(1)^4 \cancel{\ln(1)}^{0} - \frac{1}{4}\int_1^e x^3 dx$

$= \frac{1}{4}e^4 \cdot (1) - 0 - \frac{1}{4} \cdot \frac{1}{4}x^4 \Big|_1^e$

$= \frac{1}{4}e^4 - \frac{1}{16}(e^4 - 1^4)$

$= \frac{3}{16}e^4 + \frac{1}{16}$

1.5.3 Dominant Polynomials in LIPET

LIPET

When we encounter polynomials as the first thing on the list in LIPET, it's easy to see how the expression reduce to a simpler form at each step! Specifically, these are integrals that are the product of a polynomial with either an exponential or trigonometric function.

Example: Compute $\int \underbrace{(x^2+2x)}_{P} \underbrace{\cos(2x)}_{T} dx$.

$u = x^2+2x \qquad du = (2x+2)\,dx$

$dv = \cos(2x)\,dx \qquad v = \frac{1}{2}\sin(2x)$

$$= (x^2+2x)\cdot\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)(2x+2)\,dx$$

$$= \frac{1}{2}(x^2+2x)\sin(2x) - \int \underbrace{(x+1)}_{P}\underbrace{\sin(2x)}_{T}\,dx$$

Use IBP with LIPET again!

$u = x+1 \qquad du = dx$

$dv = \sin(2x)\,dx \qquad v = -\frac{1}{2}\cos(2x)$

$$= \frac{1}{2}(x^2+2x)\sin(2x) - \left\{(x+1)\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,dx\right\}$$

$$= \frac{1}{2}(x^2+2x)\sin(2x) + \frac{1}{2}(x+1)\cos(2x) - \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{1}{2}(x^2+2x)\sin(2x) + \frac{1}{2}(x+1)\cos(2x) - \frac{1}{4}\sin(2x) + C$$

Example: Compute $\int x\sec^2(x)dx$.

$u = x \qquad du = dx$

$dv = \sec^2(x)\,dx \qquad v = \tan(x)$

$$= x\tan(x) - \int \tan(x)\,dx$$

$$= x\tan(x) - \ln|\sec(x)| + C$$

LIPET ~ IPE

1.5.4 Dominant Exponential or Trigonometric Functions in LIPET

These are integrals where both terms in the product cycle back to a multiple of their original form after differentiation and integration. For example look at the chain of $\cos(x)$ after differentiating multiple times.

$$\frac{d}{dx}[\cos(x)] = -\sin(x) \rightarrow \frac{d}{dx}[-\sin(x)] = -\cos(x)$$

and so after differentiating twice we are back to a multiple of $\cos(x)$. Similarly differentiating e^{5x} once gives $5e^{5x}$, which is a multiple of itself. For these you use IBP by selecting one term in the product to keep being differentiated and the other term to keep being integrated until we wind up with a multiple of our original integral. Then you solve the equation for the integral.

Example: Compute $\int \underset{E}{e^{3x}} \underset{T}{\sin(2x)} dx$.

Let

$u = e^{3x}$ $\quad du = 3e^{3x}dx$

$dv = \sin(2x)dx \rightarrow v = -\frac{1}{2}\cos(2x)$

$$= -\frac{1}{2}e^{3x}\cos(2x) - \int -\frac{1}{2}\cos(2x)\cdot 3e^{3x}dx$$

$$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int \underset{E}{e^{3x}} \underset{T}{\cos(2x)}dx$$

$u = e^{3x}$ $\quad du = 3e^{3x}dx$

$dv = \cos(2x)dx \rightarrow v = \frac{1}{2}\sin(2x)$

$$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\left\{\frac{1}{2}e^{3x}\sin(2x) - \frac{3}{2}\int e^{3x}\sin(2x)dx\right\}$$

$$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) - \frac{9}{4}\underbrace{\int e^{3x}\sin(2x)dx}_{\text{OG integral!!}}$$

$$\text{Let } I = \int e^{3x}\sin(2x)dx$$

$$\Rightarrow I = \underbrace{-\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x)}_{} - \frac{9}{4}I$$

$$\Rightarrow \left(1+\frac{9}{4}\right)I = (\text{This})$$

$$\Rightarrow \frac{13}{4}I = (\text{This})$$

$$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x)\right) + C$$

1.5.5 Substitutions Leading to IBP

This is the integration by parts form of "Dumb Luck". You make a substitution that happens to lead to IBP.

Example: Compute $\int_0^{\pi^2} \sin(\sqrt{x})\,\underline{\underline{dx}}$.

u-sub

$$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$$
$$\Rightarrow 2\sqrt{x}\,du = dx$$
$$\Rightarrow 2u\,du = dx$$

Bounds

$$x = 0 \Rightarrow u = \sqrt{0} = 0$$
$$x = \pi^2 \Rightarrow u = \sqrt{\pi^2} = |\pi| = \pi$$

$$= \int_0^{\pi} \underset{\text{P}}{\underset{\uparrow}{2u}}\, \underset{\text{T}}{\underset{\uparrow}{\sin(u)}}\,du$$

use LIPET

$\underline{u} = 2u \qquad d\underline{u} = 2\,du$

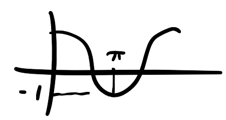
$d\underline{v} = \sin(u)\,du \qquad \underline{v} = -\cos(u)$

$$= -2u\cos(u)\Big|_0^{\pi} - \int_0^{\pi} -2\,\underline{\underline{\cos(u)}}\,du$$

$$= -2\pi\,\underline{\underline{\cos(\pi)}} - (-2(0)\cos(0)) + 2\sin(u)\Big|_0^{\pi}$$

$$= 2\pi - 0 + 2\left(\cancelto{0}{\sin(\pi)} - \cancelto{0}{\sin(0)}\right)$$

$$= 2\pi$$



1.5.6 Integrals of Lonely Logarithms and Inverse Functions (Ninja's)

Products might be hidden in plain sight when you encounter integrals. Such is the case for lonely integrands like

$$\int \arctan(x)dx \qquad \int \log_3(x)dx \qquad \int (\ln(x))^2 dx$$

where a product that may be exploited by IBP is actually hidden. If $f(x)$ is a logarithmic or inverse function then you may rewrite the integral as

$$\int f(x)dx = \int f(x) \cdot 1dx$$

to which you set $u = f(x)$ and $dv = 1dx$. This doesn't necessarily fall under LIPET, since we don't have "constants" on that list. I would just remember it as a special exception.

Example: Compute $\int \ln(x)dx$.

$u = \ln(x)$ $\quad du = \frac{1}{x}dx$

$dv = dx$ $\quad v = x$

$= x\ln(x) - \int x \cdot \frac{1}{x}dx$

$= x\ln(x) - \int 1dx$

$= x\ln(x) - x + C$ //

Example: Compute $\int \arccos(x)dx$.

$u = \arccos(x)$ $\quad du = -\frac{1}{\sqrt{1-x^2}}dx$

$dv = dx$ $\quad v = x$

$= x\arccos(x) - \int -\frac{x}{\sqrt{1-x^2}}dx$

u-sub

Let $u = 1-x^2$

$du = -2x\,dx$

$\Rightarrow -\frac{1}{2}du = x\,dx$

$= x\arccos(x) + \int -\frac{du}{2\sqrt{u}}$

$= x\arccos(x) - \frac{1}{2}(2\sqrt{u}) + C$

$= x\arccos(x) - \sqrt{1-x^2} + C$ //

1.5.7 The Tabular Method

Sometimes you may have to use IBP several times. Fortunately enough there is a pattern you can use in a table to obtain the result. Take for instance the following,

$$\int uv'''dx = uv'' - \int u'v''dx \tag{1.1}$$

$$= uv'' - \left(u'v' - \int u''v'dx\right) = uv'' - u'v' + \int u''v'dx \tag{1.2}$$

$$= uv'' - u'v' + u''v - \int u'''vdx \tag{1.3}$$

Notice how one term keeps being applied derivatives to while the other term keeps being integrated. The product is taken between the two and the sign alternates. We can record this in a table quite conveniently.

Example: Use the tabular method on the previous example $\int \underset{\underset{E}{\uparrow}}{e^{3x}} \underset{\underset{T}{\uparrow}}{\sin(2x)}dx$.

$+/-$	u	dv
$+$	e^{3x}	$\sin(2x)$
$-$	$3e^{3x}$	$-\frac{1}{2}\cos(2x)$
$+$	$9e^{3x}$	$-\frac{1}{4}\sin(2x)$

$$= (+1)e^{3x}\left(-\tfrac{1}{2}\cos(2x)\right) + (-1)(3e^{3x})\left(-\tfrac{1}{4}\sin(2x)\right) + \int (9e^{3x})\left(-\tfrac{1}{4}\sin(2x)\right)dx$$

$$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x) - \frac{9}{4}I$$

Same as last time $\Rightarrow$ $I = \dfrac{4}{13}\left(-\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{4}e^{3x}\sin(2x)\right) + C$

LIPET

Example: Use the tabular method to compute $\int \underbrace{(x^4+3x^2)}_{P}\underbrace{3^x}_{E}\,dx$.

+/-	u	dv
✓ $+$	x^4+3x^2	3^x
✓ $-$	$4x^3+6x$	$\frac{3^x}{\ln(3)}$
✓ $+$	$12x^2+6$	$\frac{3^x}{(\ln(3))^2}$
✓ $-$	$24x$	$\frac{3^x}{(\ln(3))^3}$
✓ $+$	24	$\frac{3^x}{(\ln(3))^4}$
$-$	0	$\frac{3^x}{(\ln(3))^5}$

$$= (+1)(x^4+3x^2)\frac{3^x}{\ln(3)}$$

$$+(-1)(4x^3+6x)\frac{3^x}{(\ln(3))^2}$$

$$+(+1)(12x^2+6)\frac{3^x}{(\ln(3))^3}$$

$$+(-1)(24x)\frac{3^x}{(\ln(3))^4}$$

$$+(+1)(24)\frac{3^x}{(\ln(3))^5}$$

$$+(-1)\int 0\cdot\frac{3^x}{(\ln(3))^5}\,dx \rightarrow \int 0\,dx = C$$

Note

Under LIPET, the tabular method works best on polynomial, exponential, or trigonometric dominant integrals. It does not work well on logarithmic or inverse trigonometric dominant integrals.

1.6 (Section 8.3) Trigonometric Integrals

We know how to integrate some basic trigonometric functions.

- $\displaystyle\int \sin(x)dx = -\cos(x) + C$
- $\displaystyle\int \csc^2(x)dx = -\cot(x) + C$
- $\displaystyle\int \tan(x)dx = \ln|\sec(x)| + C$

and so forth. But what about more complicated terms like

$$\int \sin^{2020}(x)\cos^3(x)dx \qquad \text{and} \qquad \int \tan^2(x)\sec^2(x)dx?$$

The textbook Thomas' Calculus really doesn't give you justice on the techniques to solve these problems. It avoids categorizing techniques to not add to the density of material, but it's better to know it than attack these problems half blind!

1.6.1 Products of Sines and Cosines of Different Composition

This is to deal with products of sines and cosines, **not raised to a power**, where the inputs of both differ. For example integrals of the form

$$\int \cos(5x)\sin(2x)dx$$

To deal with these, due to the fact that both terms in the product, do cycle, you can use "repetitive in all terms" IBP. However, a quicker trick is to use the product to sum identities.

Procedure for Integrals Consisting of a Products of Sine and/or Cosine with Different Inputs

1. Use one of the following three appropriate product to sum formulas:
   - $\cos(mx)\cos(nx) = \dfrac{1}{2}\left(\cos((n+m)x) + \cos((n-m)x)\right)$
   - $\sin(mx)\sin(nx) = \dfrac{1}{2}\left(\cos((n-m)x) - \cos((n+m)x)\right)$
   - $\sin(nx)\cos(mx) = \dfrac{1}{2}\left(\sin((n+m)x) + \sin((n-m)x)\right)$
2. Integrate the remaining basic integral.

These are the so called "product to sum" formulas. For example,

$$\sin(3x)\sin(2x) = \frac{1}{2}\left(\cos((3-2)x) - \cos((3+2)x)\right) = \frac{1}{2}\left(\cos(x) - \cos(5x)\right)$$

which turns the product into something easily, and quickly, integrable.

Example: Compute $\int \underline{\cos(2x)}\, \underline{\sin(4x)} dx$.

$$= \frac{1}{2} \int (\sin(4x - 2x) + \sin(4x + 2x))\, dx$$

$$= \frac{1}{2} \int (\sin(2x) + \sin(6x)) dx$$

$$= \frac{1}{2} \left\{ -\frac{1}{2}\cos(2x) - \frac{1}{6}\cos(6x) \right\} + C \;//$$

1.6.2 Products of Sines and Cosines, Same Input, Raised to a Power

These are integrals of the form

$$\int_a^b \sin^m(x)\cos^n(x)dx$$

where the input might be scaled or shifted. These divide into two cases:

Procedure for Computing $\int_a^b \sin^m(x)\cos^n(x)dx$

- m and/or n is odd:
  1. Take off a single term from the odd power and collect it with your dx term to form a du.
  2. Now that the remaining terms are raised to an even power, convert them to the other trigonometric function using $\sin^2(x) + \cos^2(x) = 1$.
  3. Complete the u-substitution and integrate with the du term formed in the first step.
- m and n are even:
  1. Convert all terms using the identities
  $$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \text{and} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$
  2. If all your terms are not basic integrals, expand everything.
  3. If you encounter all even terms repeat the first step on those integrand terms. Else, your integrand terms are either of the case where you have a basic integral or m and/or n are odd. Use the appropriate procedure on those.

Example: Compute $\int \sin^3(x)\cos^3(x)dx$.

want $u = \sin(x)$

$$= \int \sin^3(x)\cos^2(x)\underbrace{\cos(x)dx}_{du} \quad \#1$$

Let $u = \sin(x)$
$du = \cos(x)dx$

$$= \int \sin^3(x)(1-\sin^2(x))\cos(x)dx \quad \#2$$

$$= \int u^3(1-u^2)du \quad \#3$$

$$= \int (u^3 - u^5)du$$

$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

Example: Compute $\int \cos^7(x)dx$. $= \int \sin^0(x)\cos^7(x)dx$

$$= \int \cos^6(x) \underbrace{\cos(x)dx}_{du} \to u = \sin(x)$$

$\cos^2(x) + \sin^2(x) = 1$

$$= \int (\cos^2(x))^3 \cos(x)dx$$

$u^8 = (u^2)^4$

$$= \int (1-\sin^2(x))^3 \cos(x)\, dx$$

Let
$u = \sin(x)$
$du = \cos(x)dx$

$$= \int (1-u^2)^3 du$$

Binomial Expansion $(x+y)^n$

$$= \int (1 - 3u^2 + 3u^4 - u^6)\, du$$

$$= u - \frac{3}{3}u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \sin(x) - \sin^3(x) + \frac{3}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + C$$

$$\left(\sin^2(x)\right)^2 = \int\left(\frac{1-\cos(2x)}{2}\right)^2\left(\frac{1+\cos(2x)}{2}\right)dx$$

$$//$$

Example: Compute $\int \sin^2(x)\cos^2(x)dx$. $\longrightarrow \int \sin^4(x)\cos^2(x)dx$

$$= \int\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)dx$$

$$= \frac{1}{4}\int\left(1-\cancel{\cos(2x)}+\cancel{\cos(2x)}-\cos^2(2x)\right)dx$$

$$= \frac{1}{4}\int\left(1-\cos^2(2x)\right)dx$$

$$\cos^2(A) = \frac{1+\cos(2A)}{2}$$

$$= \frac{1}{4}\int 1\,dx - \frac{1}{4}\int \cos^2(2x)\,dx \qquad A = 2x$$

$$= \frac{1}{4}x - \frac{1}{4}\int\left(\frac{1+\cos(4x)}{2}\right)dx$$

$$= \frac{1}{4}x - \frac{1}{8}\int\left(1+\cos(4x)\right)dx \qquad \int 1\,dx = x + C.$$

$$= \frac{1}{4}x - \frac{1}{8}\left(x + \frac{1}{4}\sin(4x)\right) + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C \;//$$

1.6.3 Eliminating Roots

Procedure for Computing Integrals Containing $\sqrt{1 \pm \cos(mx)}$ where m is Even

1. Use the double angle identity $1+\cos(2x) = 2\cos^2(x)$ or $1-\cos(2x) = 2\sin^2(x)$ to eliminate the square root.
2. Use appropriate integration techniques afterwards.

Example: Compute $\boxed{\int_0^{\pi/4} \sqrt{1-\cos(4x)}dx}$.

$2\sin^2(A) = 1-\cos(2A)$ ← $A=2x$

$\Rightarrow 2\sin^2(2x) = 1-\cos(4x)$

$$= \int_0^{\pi/4} \sqrt{2\sin^2(2x)}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\sin(2x)|\, dx$$

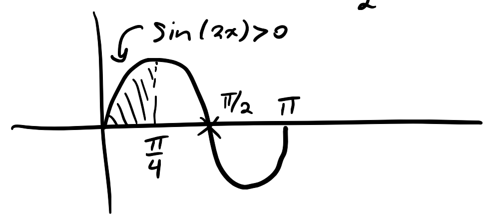
$$= \sqrt{2}\int_0^{\pi/4} \sin(2x)\, dx$$

$$= -\frac{\sqrt{2}}{2}\cos(2x)\Big|_0^{\pi/4}$$

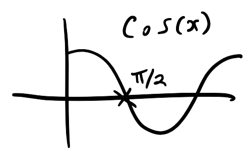
$$= -\frac{\sqrt{2}}{2}\left\{\cos(\pi/2) - \cos(0)\right\}$$

$$= -\frac{\sqrt{2}}{2}\{0-1\} = \frac{\sqrt{2}}{2}$$

$\frac{2\pi}{2} = \pi$



$\sin(Ax) \leftarrow \frac{2\pi}{|A|}$



Procedure for Computing Integrals Containing $\sqrt{1 \pm \sin(nx)}$

1. Multiply the integrand by the conjugate $\dfrac{\sqrt{1 \mp \sin(nx)}}{\sqrt{1 \mp \sin(nx)}}$
2. Expand the terms and use the identity $1 - \sin^2(nx) = \cos^2(nx)$ to eliminate the square root.
3. Use appropriate integration techniques afterwards.

Example: Compute $\displaystyle\int_0^{\pi/4} \sqrt{1 - \sin(2x)}\,dx$.

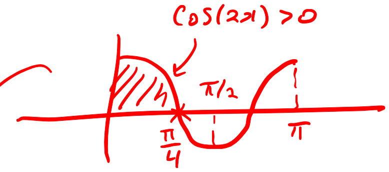
use conjugate like we did in 8.1

Stewart's Calculus

$$= \int_0^{\pi/4} \sqrt{1 - \sin(2x)} \cdot \frac{\sqrt{1 + \sin(2x)}}{\sqrt{1 + \sin(2x)}}\,dx$$

$$= \int_0^{\pi/4} \frac{\sqrt{1 - \sin^2(2x)}}{\sqrt{1 + \sin(2x)}}\,dx$$

$\cos(2x) > 0$



$$= \int_0^{\pi/4} \frac{\sqrt{\cos^2(2x)}}{\sqrt{1 + \sin(2x)}}\,dx = \int_0^{\pi/4} \frac{|\cos(2x)|}{\sqrt{1 + \sin(2x)}}\,dx$$

$$= \int_0^{\pi/4} \frac{\cos(2x)}{\sqrt{1 + \sin(2x)}}\,dx$$

Let $u = 1 + \sin(2x)$

$du = 2\cos(2x)\,dx$

$\Rightarrow \frac{1}{2}\,du = \cos(2x)\,dx$

Bounds

$x = 0 \Rightarrow u = 1 + \sin(0) = 1$

$x = \pi/4 \Rightarrow u = 1 + \sin(\frac{\pi}{2}) = 2$

$$= \int_1^2 \frac{1}{2} \cdot \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2}\sqrt{u}\right)\Big|_1^2$$

$$= \sqrt{2} - \sqrt{1} = \sqrt{2} - 1$$

1.6.4 Lonely Powers of Tangent or Secant

This is for integrals of the form

$$\int \tan^m(x)dx \qquad \text{or} \int \sec^n(x)dx$$

Unlike the case with powers of sine and cosine, they require their own categorization.

Procedure for Computing $\int_a^b \tan^m(x)dx$ or $\int_a^b \sec^n(x)dx$

- $n \geq 3$:
  1. Pull of a $\sec^2(x)$ term and perform IBP with $dv = \sec^2(x)dx$. Your integral will reduce as the following reduction formula:
  $$\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$$
  2. Complete the integration if you are at a basic integral of $\sec^2(x)$ or $\sec(x)$. Otherwise, repeat the first step on the appropriate integral.

- $m \geq 3$ is odd:
  1. Pull off a single tangent term to and multiply by $\frac{\sec(x)}{\sec(x)}$ to form
  $$\int \tan^m(x)dx = \int \tan^{m-1}(x)\frac{\sec(x)\tan(x)}{\sec(x)}dx$$
  2. Construct $du = \sec(x)\tan(x)dx$ and convert the remaining tangent terms to secants using $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete your u-substitution using $u = \sec(x)$ and integrate.

- $m \geq 2$ is even:
  ✓ 1. Convert all your tangent terms into secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. If the result is a basic integral, integrate it. Otherwise, expand the result to construct a sum of powers of secants.
  3. Integrate the basic integral terms of $\sec^2(x)$, $\sec(x)$ and constants, then use the procedure of integrating higher powers of secants mentioned above.

Example: Compute $\int \tan^2(x)dx$.

$$= \int (\sec^2(x) - 1)\,dx$$

$$= \tan(x) - x + C$$

$\tan^6(x)$
$= (\tan^2(x))^3$
$= (\sec^2(x) - 1)^3$

$\to \int \sec^2(x)dx = \tan(x) + C$

$\to \int \sec(x)dx$
$= \ln|\sec(x) + \tan(x)| + C$

$$\text{trig}^2(x) = (\text{trig}(x))^2$$

Example: Compute $\int \sec^4(x)dx$.

$$= \int \sec^2(x)\sec^2(x)\,dx$$

$u = \sec^2(x)$ $\qquad du = 2\sec(x)\sec(x)\tan(x)dx$

$dv = \sec^2(x)dx$ $\qquad v = \tan(x)$

$$= \sec^2(x)\tan(x) - 2\int \sec^2(x)\tan^2(x)\,dx$$

$$= \sec^2(x)\tan(x) - 2\int \sec^2(x)(\sec^2(x) - 1)\,dx$$

$$= \sec^2(x)\tan(x) - 2\int \sec^4(x)\,dx + 2\int \sec^2(x)\,dx$$

If $I = \int \sec^4(x)dx$

$$\Rightarrow I = \underbrace{\sec^2(x)\tan(x)}\ - 2I + \underbrace{2\int \sec^2(x)\,dx}$$

$\Rightarrow 3I =$ above

$$\Rightarrow I = \frac{1}{3}\left\{\sec^2(x)\tan(x) + 2\int \sec^2(x)\,dx\right\}$$

$$= \frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C$$

$$\int \sec^4(x)\,dx = \frac{\sec^2(x)\tan(x)}{3} + \frac{2}{3}\int \sec^{4-2}(x)\,dx$$

Example: Compute $\int \sec^3(x)dx$.

$$= \int \sec(x)\sec^2(x)\,dx$$

$u = \sec(x)$ → $du = \sec(x)\tan(x)\,dx$

$dv = \sec^2(x)\,dx$ → $v = \tan(x)$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

$$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)\,dx$$

$$= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx$$

$$= \sec(x)\tan(x) - I + \ln|\sec(x)+\tan(x)$$

Collect and solve I

$$\Rightarrow \quad I = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C \quad //$$

1.6.5 Powers of Tangent and Secant with Same Input

This is for integrals of the form

$$\int \tan^m(x)\sec^n(x)dx$$

where the inputs may be shifted or scaled. The strategies for power products of cosecant and cotangent are identical. In this, it is assumed that $m, n \geq 1$ (not a lonely integrand).

Procedure for Computing a Non-Lonely $\int_a^b \tan^m(x)\sec^n(x)dx$

- m is odd:
  1. Pull off a single $\tan(x)$ and $\sec(x)$ term and construct $du = \sec(x)\tan(x)dx$.
  2. Convert the remaining tangent terms using the identity $\tan^2(x) = \sec^2(x) - 1$.
  3. Complete the u-substitution with $u = \sec(x)$ and integrate.
- n is even:
  1. Pull of a $\sec^2(x)$ term and construct $du = \sec^2(x)dx$.
  2. Convert the remaining secant terms using the identity $\sec^2(x) = \tan^2(x) + 1$.
  3. Complete the u-substitution with $u = \tan(x)$ and integrate.
- m is even and n is odd: (Evil Case)
  1. Convert all the tangent terms to secants using the identity $\tan^2(x) = \sec^2(x) - 1$.
  2. Expand your integrand to get a sum of powers of secants.
  3. Use the appropriate integration techniques for integrating lonely powers of secants.

Example: Compute $\int \sec(x)\tan^3(x)dx$. m is odd n is odd

$$= \int \tan^2(x)\sec(x)\tan(x)dx$$

$$= \int (\sec^2(x) - 1)\sec(x)\tan(x)dx$$

Let $u = \sec(x)$ $du = \sec(x)\tan(x)dx$

$$= \int (u^2 - 1)du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\sec^3(x) - \sec(x) + C$$

(Continued...)

$$\int (\sec^2(x) - 1)\sec^3(x)\,dx = \int (\sec^5(x) - \sec^3(x))\,dx \;//$$

//

At Home Exercise: Use the appropriate procedure to compute $\int \tan^2(x)\sec^3(x)dx$

1.7 (Section 8.4) Trigonometric Substitution

1.7.1 Forms of Trigonometric Subsitution

$$\int \frac{1}{(2x)^2 - 3^2}\,dx$$

These are used to deal with integrals containing the terms

$$a^2 - u^2 \quad \text{or} \quad a^2 + u^2 \quad \text{or} \quad u^2 - a^2$$

where a is a constant and u is the variable of integration. This is where we use a substitution method called a **pullback** substitution. The two substitution methods **pullback** and **push-forward** are as follows:

- ✓ Pullback: Substitutions of the form $x = g(u)$. $\quad x = \sqrt{t} - 3$
- Push-forward: Substitutions of the form $u = h(x)$. $\quad u = x^2$

The above terms resemble trigonometric identities

$$\int \frac{1}{(2x)^2 - 3^2}\,dx$$

- $a^2 - u^2$ resembles $1 - \sin^2(\theta) = \cos^2(\theta)$?
- ✓ $a^2 + u^2$ resembles $\boxed{1 + \tan^2(\theta) = \sec^2(\theta)}$?
- $u^2 - a^2$ resembles $\sec^2(\theta) - 1 = \tan^2(\theta)$?

Example: Convert the following integral

$$\int \frac{x^4}{9 + 4x^2}\,dx$$

to a trigonometric integral using the substitution $\boxed{2x = 3\tan(\theta)}$. Do not complete the integration.

$$\int \frac{x^4}{3^2 + (2x)^2}\,dx$$

$u = 2x \qquad a = 3$

$a^2 + u^2$

Sub $u = a\tan(\theta)$

Let $\boxed{2x = 3\tan(\theta)} \Rightarrow x = \frac{3}{2}\tan(\theta)$

$\Rightarrow 2\,dx = 3\sec^2(\theta)\,d\theta$

$\Rightarrow dx = \frac{3}{2}\sec^2(\theta)\,d\theta$ ☆

$$= \int \frac{\left(\frac{3}{2}\tan(\theta)\right)^4}{9 + (3\tan(\theta))^2} \cdot \frac{3}{2}\sec^2(\theta)\,d\theta$$

$$= \int \frac{\frac{81}{16}\tan^4(\theta) \cdot \frac{3}{2}\sec^2(\theta)}{9 + 9\tan^2(\theta)}\,d\theta$$

$$= \int \frac{\frac{81}{16}\tan^4(\theta) \cdot \frac{3}{2}\sec^2(\theta)}{9(1 + \tan^2(\theta))}\,d\theta$$

$$= \int \frac{\frac{81}{16}\tan^4(\theta) \cdot \frac{3}{2}\cancel{\sec^2(\theta)}}{9(\cancel{\sec^2(\theta)})}\,d\theta = \frac{81}{16} \cdot \frac{3}{2} \cdot \frac{1}{9} \int \tan^4(\theta)\,d\theta$$

Procedure for Computing Integrals Containing $u^2 \pm a^2$ and $a^2 - u^2$

The procedure for all the following is essentially the same. Use the substitution $u = a \times$ (Appropriate Trig Function), use techniques of trigonometric integrals to complete the integration, then convert back.

- Containing $a^2 - u^2$:

  1. Let $u = a\sin(\theta)$ and compute $du = a\cos(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arcsin(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sin(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $a^2 + u^2$:

  1. Let $u = a\tan(\theta)$ and compute $du = a\sec^2(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \arctan(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\tan(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

- Containing $u^2 - a^2$:

  1. Let $u = a\sec(\theta)$ and compute $du = a\sec(\theta)\tan(\theta)d\theta$.
  2. Complete the substitution and use an appropriate integration technique to compute the resulting trigonometric integral.
  3. Convert all terms back. If you encounter a lonely θ use the identity $\theta = \operatorname{arcsec}(u/a)$. If you encounter a double angle, reduce to a single angle using a double angle identity. If you encounter a non-sine function, represent the identity $\sec(\theta) = \dfrac{u}{a}$ as a triangle and solve for the values of other trigonometric functions.

1.7.2 Sine Substitutions

In these integrals we always have a domain restriction of $\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$.

Example: Compute $\int \frac{x^3}{\sqrt{5-4x^2}}dx$. $= \int \frac{x^3}{\sqrt{(\sqrt{5})^2-(2x)^2}}dx$ Let $u = a\sin(\theta)$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\uparrow a^2 - u^2$

$2x = \sqrt{5}\sin(\theta)$

$2dx = \sqrt{5}\cos(\theta)d\theta$

$\Rightarrow dx = \frac{\sqrt{5}}{2}\cos(\theta)d\theta$

$$= \int \frac{\left(\frac{\sqrt{5}}{2}\sin(\theta)\right)^3}{\sqrt{5-(\sqrt{5}\sin(\theta))^2}}\frac{\sqrt{5}}{2}\cos(\theta)d\theta$$

$$= \int \frac{\frac{5^{3/2}}{2^3}\sin^3(\theta)\cdot\frac{5^{1/2}}{2}\cos(\theta)}{\sqrt{5-5\sin^2(\theta)}}d\theta = \frac{25}{16}\int\frac{\sin^3(\theta)\cos(\theta)}{\sqrt{5(1-\sin^2(\theta))}}d\theta$$

$$= \frac{25}{16}\int\frac{\sin^3(\theta)\cos(\theta)}{\sqrt{5}\sqrt{\cos^2(\theta)}}d\theta$$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \frac{25}{16\sqrt{5}}\int\frac{\sin^3(\theta)\cos(\theta)}{|\cos(\theta)|}d\theta$$

$$= \frac{25}{16\sqrt{5}}\int\frac{\sin^3(\theta)\cancel{\cos(\theta)}}{\cancel{\cos(\theta)}}d\theta$$

$$= \frac{25}{16\sqrt{5}}\int\sin^3(\theta)\,d\theta$$

Case of odd power in sines and cosines

$$= \frac{25}{16\sqrt{5}}\int\sin^2(\theta)\sin(\theta)d\theta$$

$$= \frac{25}{16\sqrt{5}}\int(1-\cos^2(\theta))\sin(\theta)d\theta$$

Let $u = \cos(\theta)$

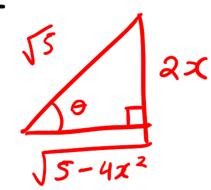
$-du = \sin(\theta)d\theta$

$$= \frac{25}{16\sqrt{5}}\int -(1-u^2)du$$

$$= \frac{25}{16\sqrt{5}}\left(-u + \frac{1}{3}u^3\right) + C = \frac{25}{16\sqrt{5}}\left(-\cos(\theta) + \frac{1}{3}\cos^3(\theta)\right) + C$$

$$= \frac{25}{16\sqrt{5}}\left(-\frac{\sqrt{5-4x^2}}{\sqrt{5}} + \frac{1}{3}\left(\frac{\sqrt{5-4x^2}}{\sqrt{5}}\right)^3\right) + C$$

Have $\sin(\theta) = \frac{2x}{\sqrt{5}} = \frac{\text{opp}}{\text{hyp}}$



1.7.3 Tangent Substitutions

In this we always have a domain restriction of $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ as well.

Example: Compute $\displaystyle\int \frac{1}{(4x^2+9)^2}dx$.

1.7.4 Secant Substitutions

For terms of the form $u^2 - a^2$ you must have a restriction of $|u| > a$ for secant substitution to make any sense. If $u > a$ then $0 < \theta < \dfrac{\pi}{2}$ and if $u < -a$ then $\dfrac{\pi}{2} < \theta < \pi$.

Example: Compute $\displaystyle\int \frac{\sqrt{x^2 - 25}}{x} dx$.

1.7.5 More Obscure u Terms in Trigonometric Substitution

Example: Compute $\displaystyle\int_{\ln(3)}^{\ln(3\sqrt{3})} \frac{e^t}{\sqrt{e^{2t}+9}}dt$.

Example: Compute $\int \sqrt{x}\sqrt{1-x}dx$.

1.8 (Section 8.5) Partial Fractions

This is, quite bleakly stated, as the process of 'unsimplifying' a fraction to integrate it.

Example: Since $\dfrac{3x+11}{x^2-x-6} = \dfrac{4}{x-3} - \dfrac{1}{x+2}$ then

$$\int \frac{3x+11}{x^2-x-6}dx = \int \frac{4}{x-3}dx - \int \frac{1}{x+2}dx = 4\ln|x-3| - \ln|x+2| + C$$

For $f(x) = \dfrac{P(x)}{Q(x)}$, a rational function with $\deg(P(x)) < \deg(Q(x))$ where $P(x)$ and $Q(x)$ are polynomials you start by factoring $Q(x)$. We want to express, like above, this rational in the form

$$f(x) = \frac{p_1(x)}{q_1^{s_1}(x)} + \frac{p_2(x)}{q_2^{s_2}(x)} + \cdots + \frac{p_n(x)}{q_n^{s_n}(x)}$$

where $\deg(p_i(x)) = \deg(q_i(x)) - 1$ (this allows us to represent it in the form of a logarithmic integral). This form is called the **partial fraction decomposition** of $f(x)$. The forms we suggest in the decomposition depend on how $Q(x)$ factors.

1.8.1 Linear Terms Present

Suppose $Q(x)$ is factored, the following terms present means we suggest the following form.

Term in $Q(x)$	Suggested Term(s) in Decomposition to Add
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$

Example: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x+4}{(2x+1)^2(x-1)}$.

1.8.2 Irreducible Quadratic Terms Present

Definition

A quadratic of the form $ax^2 + bx + c$ is called **irreducible** if $b^2 - 4ac < 0$. That is, it cannot be factored into linear factors over the real numbers.

Suppose $Q(x)$ is factored, the following terms present means we suggest the following form.

Irreducible term in $Q(x)$	Suggested Term(s) in Decomposition to Add
$ax^2 + bx + c$	$\dfrac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\dfrac{A_1x + B_1}{ax^2 + bx + c} + \dfrac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \dfrac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Example: Suggest a form for the partial fraction decomposition of $f(x) = \dfrac{3x^2 + 4x}{(x^2+1)^2(x-1)^2(x-3)}$.

Example: Construct the partial fraction decomposition $f(x) = \dfrac{x+1}{x^4+x^2}$.

1.8.3 Special Case Where All Terms Are Linear

This is the **BEST** case scenario. You can solve for all the constants in the easiest manner possible. It's best to see this "trick" by example.

Example: Compute the partial fraction decomposition of $f(x) = \dfrac{3x^2+1}{(x-3)(x-4)(x-2)}$.

1.8.4 Example of Computation an Integral by Partial Fractions

Example: Compute $\displaystyle\int \frac{x^2 - x + 2}{x^3 - 1} dx$.

(Continued...)

Note

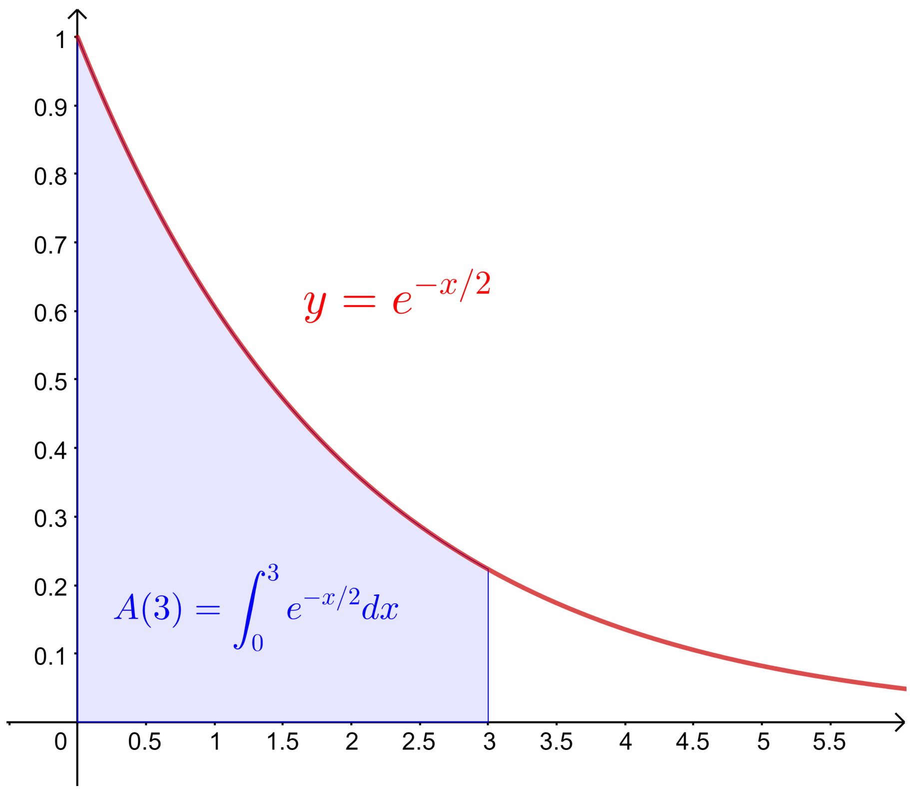
If the degree of the denominator is higher than the numerator, you must use long division first to re-express the integrand! Note also that some of the integrals which may be solved by partial fractions may also be solved by trigonometric substitution.

1.9 (Section 8.8) Improper Integrals

1.9.1 Defining Improper Integrals and Convergence

This section truthfully makes more sense upon the introduction of sequences, but we make do for the present time. These are integrals over a region of space that, in a sense, are infinite in size. You will find that some results might be counter intuitive at first, although the results are entirely based upon the way that things are defined.

Consider the function $y = e^{-x/2}$. As this function is entirely positive it makes sense to talke about the area between it and the x-axis over the region $[0, b]$ for some value b. This defines the area function $A(b) = \int_0^b e^{-x/2}dx$.



For each value of b the area is finite. Indeed, we may compute

$$A(b) = \int_0^b e^{-x/2}dx = -2e^{-x/2}\Big|_0^b = -2(e^{-b/2} - 1)$$

For each increased value of b we extend the area. It is possible that the area settles down to a finite value even though the region where we take it over is infinite. That is, by taking the limit as $b \to \infty$ we obtain

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} -2(e^{-b/2} - 1) = -2(0 - 1) = 2$$

This is an illustration of an improper integral. There are two types of improper integrals, one where the region of integration is infinite and the other where there is a vertical asymptote in the region of integration.

Definition

An integral with infinite limits (i.e. integral over an infinite region) is called a **Type I Integral**. It is defined as:

1. $\displaystyle\int_a^\infty f(x)dx = \lim_{b\to\infty}\left(\int_a^b f(x)dx\right)$
2. $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\left(\int_a^b f(x)dx\right)$
3. $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_x^\infty f(x)dx$, where c is any real number and the two integrals are defined as in the previous two points.

All these definitions assume that $f(x)$ is continuous over the region of integration.

Often we care little about the **value** of the integral. We often care about the behaviour of the integral. There are two options, either an integral results in concrete finite value or it does not.

Example: Determine whether the following integral $\displaystyle\int_1^\infty \frac{1}{x}dx$ is a finite value or not.

Definition

If an integral results in a finite value we say the integral **converges**. If an integral is not convergent we say it is **divergent**. The present tense verbal forms of these are that the integral **converges** or it **diverges**.

1.9.2 p-Integrals and the Integral Comparison Tests

Often we determine convergence or divergence of an integral by comparing it (in some manner) to a simpler and more well known integral that we understand the behaviour of. The simplest integrals we understand the convergent/divergent behaviour of are the p-Integrals.

Definition

A Type I p-**integral** is an integral of the form $\displaystyle\int_1^\infty \frac{dx}{x^p}$.

Theorem

Consider the Type I p-integral. If $p > 1$ the p-Integral converges. If $p \leq 1$ the p-Integral diverges.

Proof: If $p = 1$ then by the previous example we saw that the integral diverges. If $p > 1$ then by the power rule we obtain the following

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p}\Big|_1^\infty = \frac{1}{1-p}\left(\lim_{b\to\infty} b^{1-p} - 1\right)$$

We can see that if $p > 1$ then the exponent of b^{1-p} is negative and thus

$$p < 1 \Rightarrow \lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$$

On the other hand, if $p < 1$ then the exponent of b^{1-p} is positive and thus

$$p > 1 \Rightarrow \lim_{b\to\infty} b^{1-p} = \infty$$

thus establishing the above theorem as desired $\blacksquare$.

Note

Many people casually call every integral of the form $\displaystyle\int_a^b \frac{dx}{x^p}$ a p-integral. There is no problem with this usually because the context is often clear. Usually, an integral of the form $\displaystyle\int_a^\infty \frac{dx}{x^p}$ where $a > 0$ is colloquially called a *Type I p-integral*. On the other hand, integrals of the form $\displaystyle\int_0^b \frac{dx}{x^p}$ where $b > 0$ is colloquially called a *Type II p-integral*.

Below is the theorem we use to compare unknown integrals of Type I Improper to those that are known.

Direct Comparison Test

Let f and g be continuous functions on $[a,\infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then...

1. $\displaystyle\int_a^\infty f(x)dx$ converges if $\displaystyle\int_a^\infty g(x)dx$ converges.
2. $\displaystyle\int_a^\infty g(x)dx$ diverges if $\displaystyle\int_a^\infty f(x)dx$ diverges.

Example: Determine whether or not the integral $\displaystyle\int_1^\infty \frac{dx}{1+x^4}$ converges or diverges.

Example: Determine whether or not the integral $\displaystyle\int_6^\infty \frac{dx}{6\sqrt{x^2-25}}$ converges or diverges.

We have one more test for Type I Integrals.

Limit Comparison Test

Let f and g be continuous and positive functions on $[a, \infty)$ with

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is positive and finite, i.e. $0 < L < \infty$. Then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge together or diverge together.

Example: Determine whether or not the integral $\int_1^\infty \frac{1 - e^{-x}}{x} dx$ converges or diverges.

1.9.3 Vertical Asymptotes

You can only invoke the Fundamental Theorem of Calculus if the function is continuous over the region of integration. Thus the Fundamental Theorem of Calculus does not apply to the integral

$$\int_{-1}^{1} \frac{dx}{x^2}$$

Finding an antiderivative and computing $F(x)\Big|_{-1}^{1}$ will give you the incorrect answer. This is because there is a vertical asymptote disrupting the continuity over this region of integration. We define such functions as below.

> Definition
>
> If the integrand is discontinuous (and has a vertical asymptote) in the region of integration we call it a **Type II Integral**. If $f(x)$ is continuous over $(a, c) \cup (c, b)$ (i.e. only has a vertical asymptote at $x = c$) then we define
>
> $$\int_a^b f(x)dx = \lim_{m \to c^-} \left(\int_a^m f(x)dx \right) + \lim_{x \to c^+} \left(\int_n^b f(x)dx \right)$$

Example: Determine whether or not the integral $\int_{-1}^{1} \frac{1}{x^2} dx$ converges or diverges.

Definition

A Type II p**-integral** is an integral of the form $\displaystyle\int_0^1 \frac{dx}{x^p}$.

Theorem

Consider the Type II p-integral. If $p \geq 1$ the p-integral diverges. If $p < 1$ the p-integral converges.

Note

This is almost opposite of the result for Type I p-integrals.

The Type II improper integrals has their own comparison theorem. We will omit the limit comparison version and just mention the direct comparison one to keep things simple.

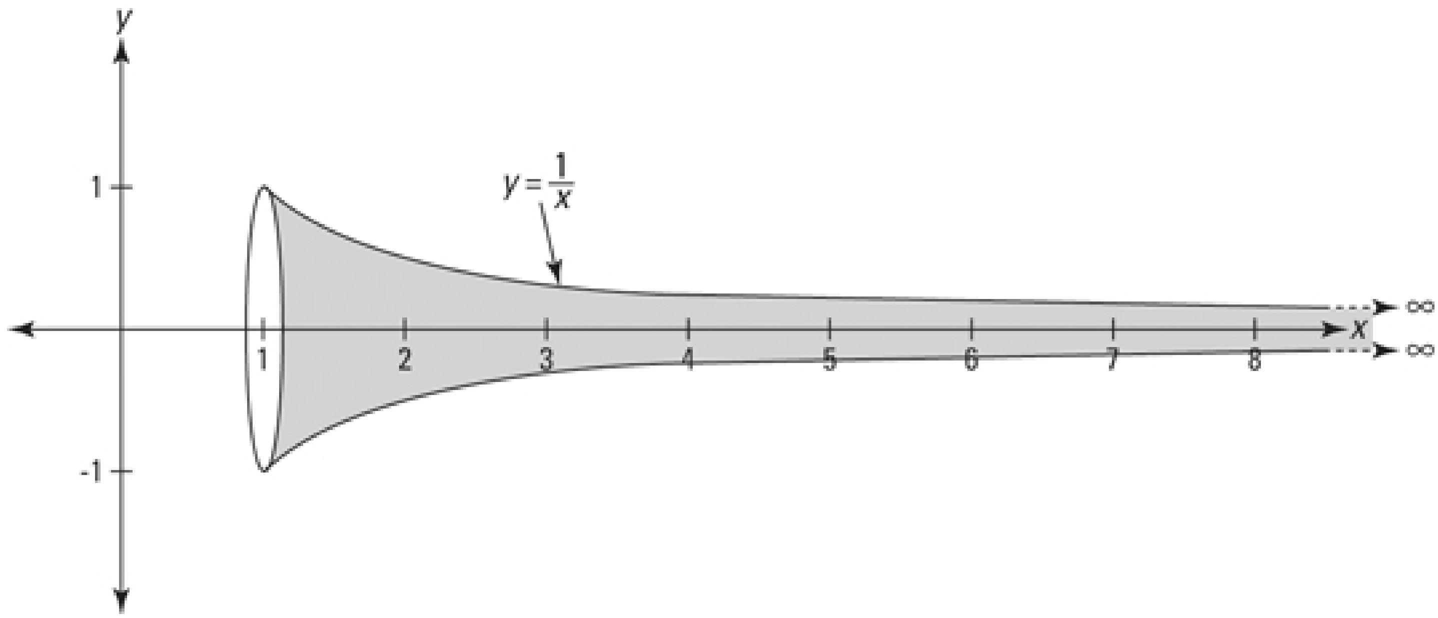
Direct Comparison Test for Type II Integrals

Let f and g be continuous functions on $I = (a, c) \cup (c, b)$ with $0 \leq f(x) \leq g(x)$ on I. Then...

1. $\displaystyle\int_a^b f(x)dx$ converges if $\displaystyle\int_a^b g(x)dx$ converges.
2. $\displaystyle\int_a^b g(x)dx$ diverges if $\displaystyle\int_a^b f(x)dx$ diverges.

1.9.4 Intuition is Lost in the Land of Infinity

Gabriel's horn is a vuvuzela like surface obtained by rotating the curve $f(x) = 1/x$ around the x-axis where $x \geq 1$.



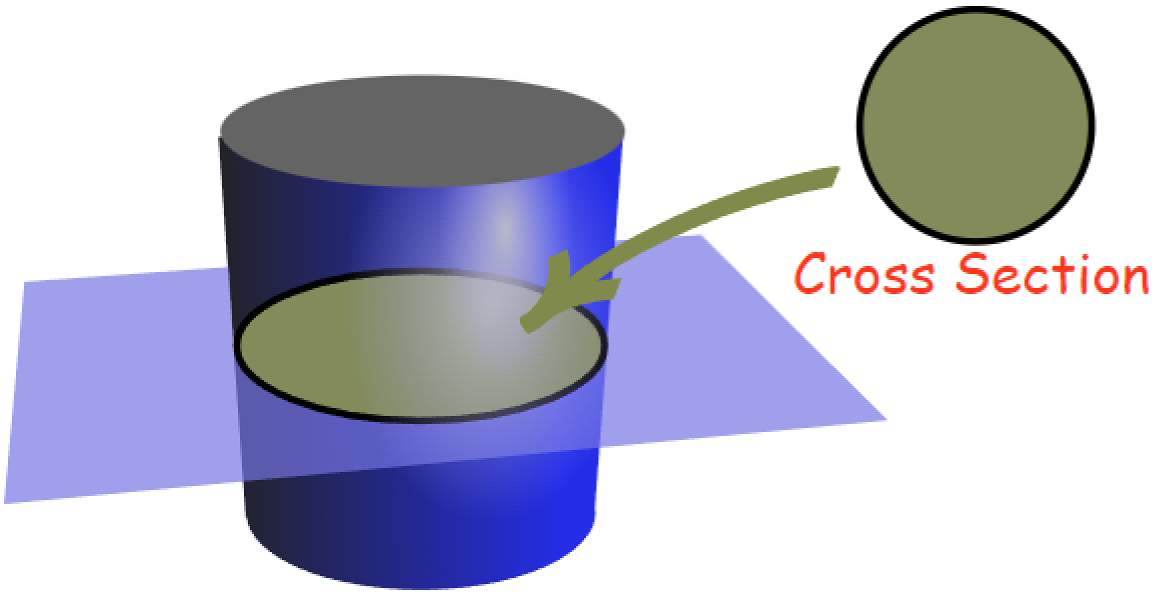
It has a volume given by $V = \pi \int_1^\infty \frac{1}{x^2} dx$ and surface area $SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} dx$.

Example: Show that the surface area of Gabrielle's Horn is infinite, yet the volume is finite.

1.10 (Section 6.1) Volumes Using Cross Sections

1.10.1 Defining Volumes by Cross Sections

We approach computing the volume of a solid by use of **cross-sections**. Cross-sections are region formed by intersecting a solid with a plane.



The idea is to find the area of each cross section and then add them up, through means of integration, to get the volume. Thus in this chapter you should not think of integration as area under a curve but rather the more physical interpretation of summing a quantity over a region.

Definition

The volume of a solid with cross sectional area $A(x)$ from $x = a$ to $x = b$ is $V = \int_a^b A(x)dx$

Procedure

1. ⋆ **SKETCH THE SOLID AND A TYPICAL CROSS-SECTION** ⋆
2. Find a formula for $A(x)$
3. Find the limits of integration
4. Compute $V = \int_a^b A(x)dx$

Cavalieri's Principle

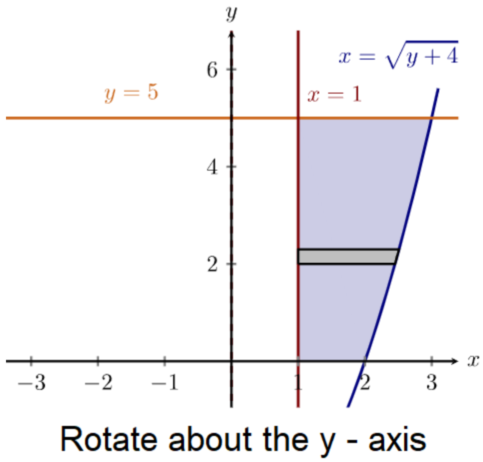
If two solids have the same height and cross-sectional area at every point along that height, then they have the same volume.

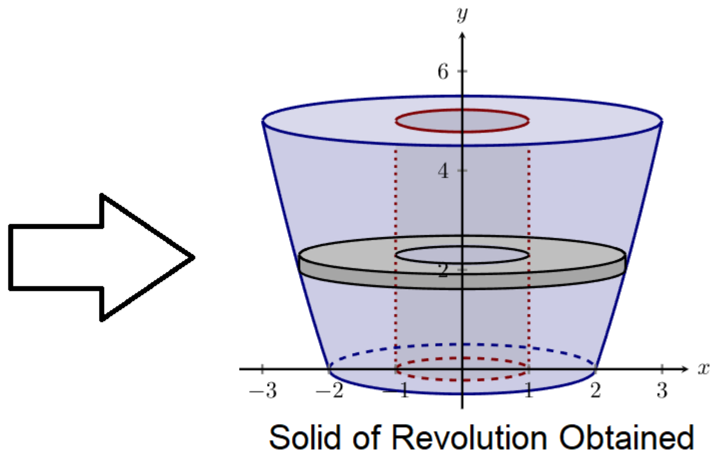
Example: Find the volume of a pyramid of height h whose base is square with sides of length L.

Example: Find the volume of a cheese wedge cute from a circular cylinder of radius r if the angle between the top and bottom is $\pi/6$ radians.

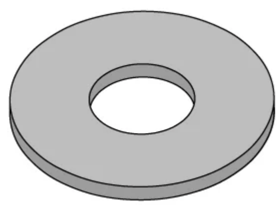
1.10.2 Solids of Revolution and their Volume by Washers

A solid of revolution is a solid obtained by rotating a curve around a line parallel (or equal) to an axis.

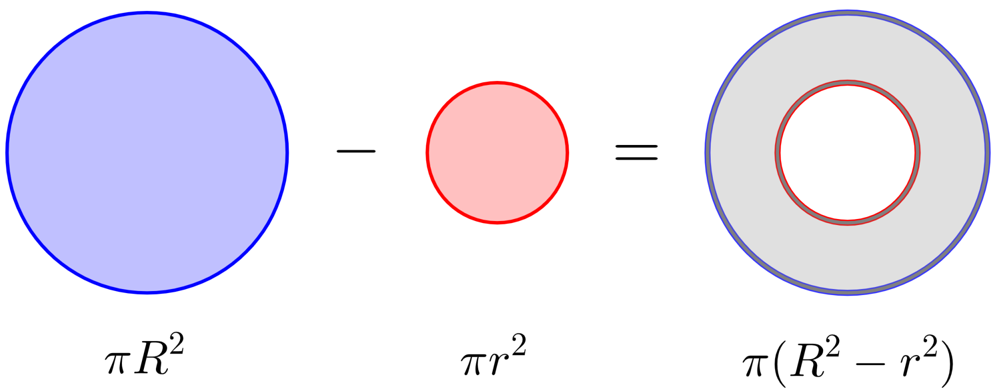




One may notice that the cross-sections are "so-called" **washers** (flat donut shapes).



To find the area of a washer we use the fact that it's area is the difference of two circles.



Example: Continue with the example above. Consider the region bounded by $x = \sqrt{y+4}$, $x = 1$, $y = 0$ and $y = 5$ and form the solid of revolution by revolving the region about the y-axis. Construct the cross-sectional area $A(y)$ at each moment $1 \leq y \leq 5$ and use this to construct the volume of the solid.

The Washer Method for Solids of Revolution

Let R be a region in the xy-plane that is revolved about an axis. If it is revolved around a line of the form $y = C$ (including the x-axis) then your volume is given by

$$V = \pi \int_a^b (R^2 - r^2)dx$$

where $A(x) = R(x)^2 - r(x)^2$ represents the cross sectional area as the difference of the bigger and smaller radii over the region $[a, b]$. If it is revolved around a line of the form $x = C$ (including the y-axis) then your volume is given by

$$V = \pi \int_c^d (R^2 - r^2)dy$$

where $A(y) = R(y)^2 - r(y)^2$ represents the cross sectional area as the difference of the bigger and smaller radii over the region $[c, d]$.

In either case, if $r = 0$ we call this **The Disk Method** instead.

Note

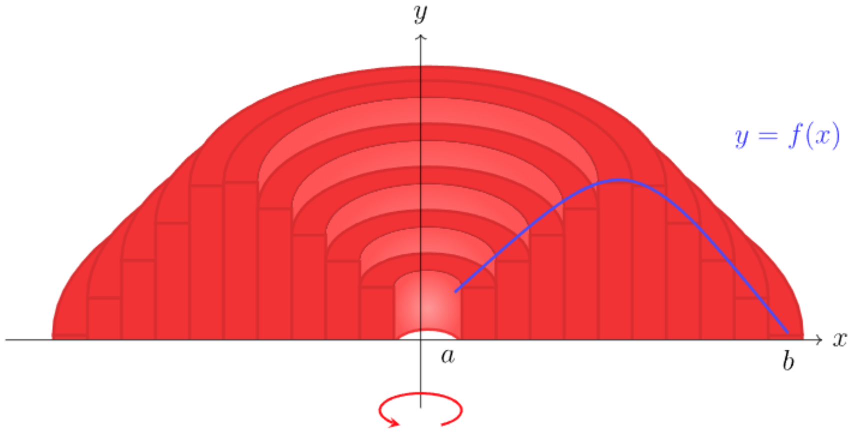
The picture is what tells you how to set up your R and r. There is no formula for constructing it and is situation specific depending on your picture. Don't try asking for one because if anyone has a specific formula for you that covers "all cases", I can construct a counterexample where it fails.

Example: Find the volume of the solid of revolution formed by the region bounded by $y = x^2 - 2x$ and $y = x$ and revolving it about the line $y = 4$.

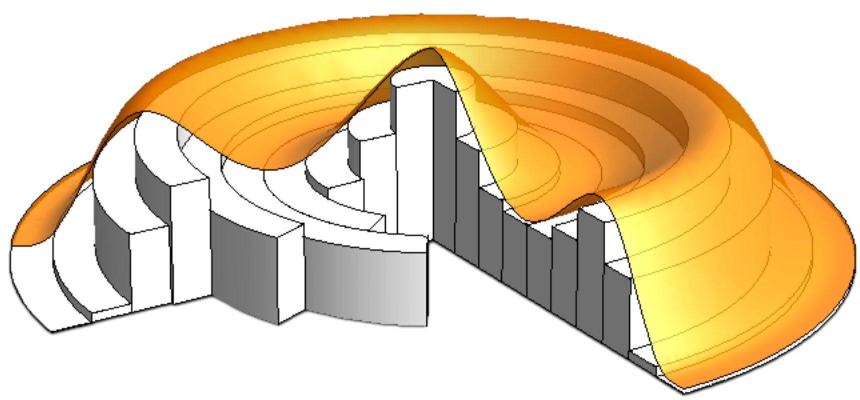
(Continued...)

1.11 (Section 6.2) The Shell Method

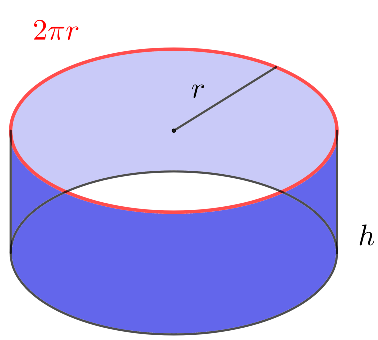
This section covers an alternate method to compute the volume of a solid of revolution. This is the method by looking at a solid more like a matryoshka (Russian) doll. For these, the idea is to find the surface area of a "shell" and add them up. If you want to think of the washer method as deconstructing a cake into its pancake like layers, then the shell method is like deconstructing an onion into its spherical like layers (as opposed to rings).



In this formulation of the solid, we see that the three-dimensional object is layered like a Russian doll where each layer is a tube-like cylinder.



If we imagine that the radius of a cylinder is r and the height is h, then the surface are of this object is $A = 2\pi rh$.



Surface Area $= 2\pi rh$

Once again you integrate to add them up, obtaining the volume.

The Shell Method for Solids of Revolution

Let R be a region in the xy-plane that is revolved about an axis. If it is revolved around a line of the form $y = C$ (including the x-axis) then your volume is given by

$$V = 2\pi \int_c^d rh dy$$

where $A(y) = 2\pi r(y)h(y)$ represents the surface area of a typical shell where the domain of the radius is $[c, d]$. If it is revolved around a line of the form $x = C$ (including the y-axis) then your volume is given by

$$V = 2\pi \int_a^b rh dx$$

where $A(x) = 2\pi r(x)h(x)$ represents the surface area of a typical shell where the domain of the radius is $[a, b]$.

Note

The variable you integrate will be opposite of that for washers! This becomes very evident when drawing a picture!

Example: Consider the region bounded by the curve $y = \sin(x)$ and the x-axis over $[0, \pi]$. Find the volume of the solid of revolution obtained by revolving this region about the y-axis.

(Continued...)

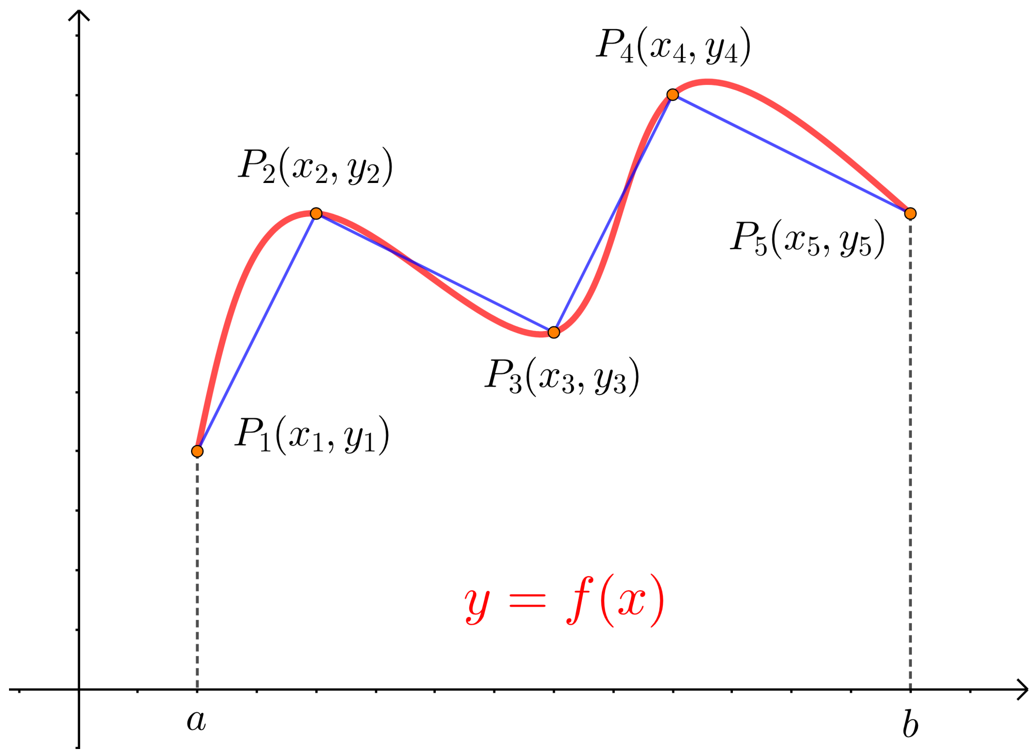
Example: Determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

(Continued...)

1.12 (Section 6.3) Arc-Length

1.12.1 Formulating the Arc-Length

The purpose of this section is to both construct a good definition for the length of a curve and give some examples. We start with a curve $y = f(x)$ and consider approximating the length by computing the lengths of the segments through a series of consecutive points $P_1, ..., P_n$ as in the image below.



If we let $|P_{i-1}P_i|$ represent the length of the line segment connecting point $P_{i-1}(x_{i-1}, y_{i-1})$ to point $P_i(x_i, y_i)$ then we form an approximation given by

$$\text{Length} \approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

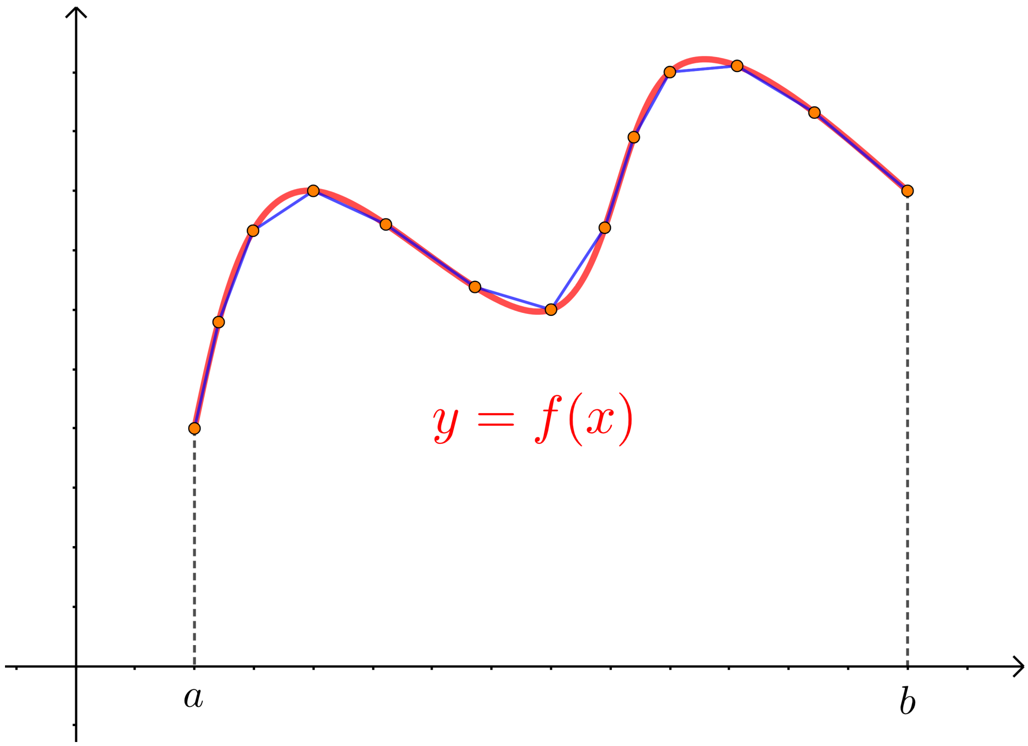
For the sake of argument, lets assume the points are evenly spaced so that $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$. Then, we obtain the approximation

$$\text{Length} \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\Delta x$$

whereby we use the **Mean Value Theorem** (MVT) to say that the secant slopes can be represented in terms of the derivative for an intermediary value in the interval (x_i, x_{i-1}). That is, $(f(x_i) - f(x_{i-1}))/(x_i - x_{i-1}) = f'(x_i^*)$ for some point x_i^* in (x_i, x_{i-1}). Thus we obtain

$$\text{Length} \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\Delta x$$

Now, as we take the limit as $n \to \infty$ our approximation gets better.



But as we take this limit we have, by definition, a formulation of the definite integral.

$$\text{Length} \stackrel{DEF}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x = \int_a^b \sqrt{1 + f'(x)^2}dx$$

and similar formulations may be made for curves expressed in the form of $x = g(y)$.

Definition

Let $y = f(x)$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). The **arc-length** of the graph of its representing curve over $[a, b]$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}dx$$

Similarly, let $x = g(y)$ be a function that is continuous on $[c, d]$ and differentiable on (c, d). The **arc-length** of the graph of its representing curve over $[c, d]$ is

$$L = \int_x^d \sqrt{1 + g'(y)^2}dy$$

In either case, we commonly abbreviate an Arc-Length integral as $\int ds$.

1.12.2 Computation of Arc-Length

Example: Find the length of $y = \dfrac{x^2}{2} - \dfrac{\ln(x)}{4}$ over $[1, 3]$.

Example: Show that the arc-length of the function $y = \left(\dfrac{3x}{2}\right)^{2/3} + 1$ over $0 \leq x \leq \dfrac{2}{3}3^{2/3}$ is much more difficult to compute in terms of x instead of in terms of y. Compute the length.

Example: Determine the length of $y = \ln(\sec(x))$ over the interval $[0, \pi/4]$

1.12.3 The Surface Area of a Solid of Revolution

Definition

Consider the solid of revolution obtained by revolving the function $y = f(x)$ over the interval $[a, b]$ about the x-axis. We define the surface area as

$$\text{SA} = \int_a^b 2\pi f(x) ds = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} dx$$

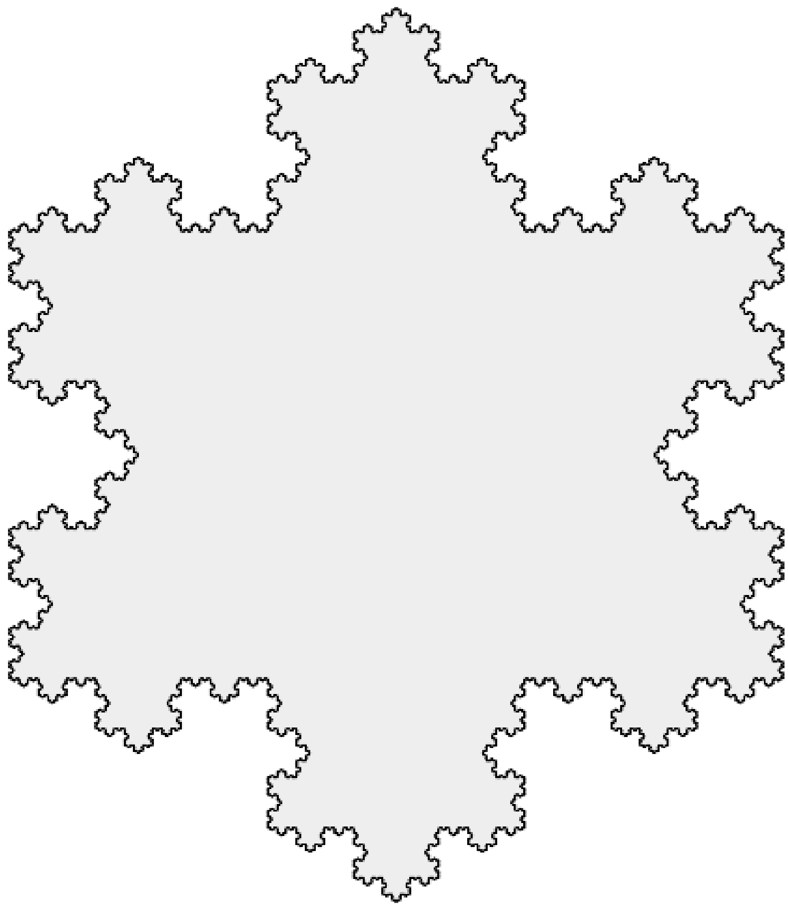
Similarly, if we revolved a function $x = g(y)$ over the interval $[c, d]$ about the y-axis then we define the surface area of the resulting solid of revolution to be

$$\text{SA} = \int_c^d 2\pi g(y) ds = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2} dy$$

Example: Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$ in $[-2, 2]$ about the x-axis.

Chapter 2

Series



2.1 (Section 10.1) Introduction to Sequences

Definition

A **sequence** is an ordered list $a_1, a_2, a_3, ...$ and may be abbreviated as $\{a_i\}$ or $\{a_i\}_{i=1}^{\infty}$

These sequences may literally be any list of objects. In mathematics we always imply it to be infinite if no context is given although people use the term sequence colloquially to describe lists of finite objects.

Example: You can have a sequence of animals: {*Dogs, Birds, Quoakkas, Lizards, Picasso's Cats, ...*}

Example: You can have a sequence of dates: {*June 3rd, July 12th, August 20th, June 3rd, September 10th,...*}

Hence some sequences can repeat elements in them but their order is important.

2.1.1 Ways to Describe a Sequence

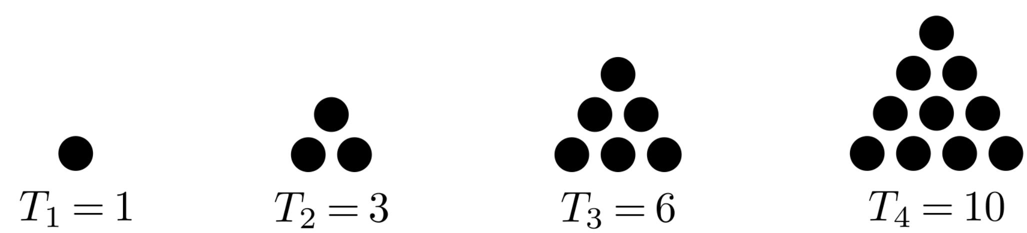
There are, in essence, three common ways to describe a sequence.

Explicit Description of a Sequence

These are sequences of the form $a_n = f(n)$ for some function where n is a sequence of integers starting from some initial value $n \geq k$.

Example: Triangular numbers are given by the sequence $T_n = \dfrac{n(n+1)}{2}$ where $n \geq 1$. We compute

$$T_1 = \frac{1(2)}{2} = 1 \qquad T_2 = \frac{2(3)}{2} = 3 \qquad T_3 = \frac{3(4)}{2} = 6 \qquad T_4 = \frac{4(5)}{2} = 10$$



Example: Arithmetic Sequences are given by $a_n = a + (n-1)d$ for $n \geq 1$ where a is some *initial value* and d is a called the *common difference*. Here you start at some value a and add the number d to obtain the next number in the sequence. For example if $a_n = 10 + 5(n-1)$ for $n \geq 1$ the first few terms of the sequence are...

$$a_1 = 10 + 5(0) = 10 \qquad a_2 = 10 + 5(1) = 15 \qquad a_3 = 10 + 5(2) = 20 \qquad a_4 = 10 + 5(3) = 25$$

so you start at $a_1 = 10$ and add 5 each time to get the next number.

Recursive Definition of a Sequence

These are sequences of the form $a_n = f(a_1, a_2, ..., a_{n-1})$ for $n \geq k$ where initial values a_1, a_2, ..., a_{k-1} are given. You generate the new values from the previous values.

Example: The most famous sequence is undeniably the Fibonacci sequence and is given by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ and $f_0 = 1$, $f_1 = 1$. Give the next few numbers in the Fibonacci sequence.

Example: The logistic recursive sequence is given by $x_{n+1} = rx_n(1 - x_n)$ for $n \geq 0$ where x_0 is some specified initial value in $[0, 1]$. For example if $k = \dfrac{3}{2}$ and $x_0 = \dfrac{1}{2}$ then

$$x_1 = \frac{3}{2}\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{3}{8}$$

$$x_2 = \frac{3}{2}\left(\frac{3}{8}\right)\left(1 - \frac{3}{8}\right) = \frac{45}{128}$$

$$x_3 = \frac{3}{2}\left(\frac{45}{128}\right)\left(1 - \frac{45}{128}\right) = \frac{11205}{32768}$$

and so forth. This sequence is best calculated using a computer. It exhibits a very interesting behaviour as you vary the parameter r. I recommend reading about the relation of the Logistic Map and Chaos Theory. Below is an excellent video provided by Veritasium that described the Logistic Map:

==Link to Veritasium Video==

The Description of a Sequence by a Mathematical Anarchist

This is where you specify a sequence by random rules described in English (Well, English for this class).

Example: The look and say sequence is the sequence as follows

$$1, 11, 21, 1211, 111221, 312211, ...$$

Determine the rule for generating elements of this sequence and generate the next two numbers of the sequence.

2.1.2 Convergence and Divergence

Sequences (like we've seen with improper integrals) either approach a single finite value or they do not. That is, they either converge or diverge.

Example: Consider the sequence $a_n = \dfrac{n^2+1}{3n^2+3}$ where $n \geq 1$. Determine the limit as $n \to \infty$. Does it converge or diverge?

Example: All previous limit laws hold. Squeeze, l'Hôpital, etc. For example consider $a_n = \dfrac{\cos(n)}{n}$ where $n \geq 1$. Determine the limit as $n \to \infty$. Does it converge or diverge?

Example: Consider the sequence $a_n = \cos\left(\frac{n\pi}{2}\right)$ where $n \geq 0$. Determine the limit as $n \to \infty$. Does it converge or diverge?

Example: Given that the recursive sequence $a_{n+1} = \sqrt{2 + a_n}$, for $n \geq 1$ with $a_1 = \sqrt{2}$ converges, determine it's limit.

Example: Provided a sequence converges you can determine the limit of a recursive equation by setting up an equation you may solve for. For example consider the sequence obtained by successive ratio's of the Fibonacci sequence $R_n = f_{n+1}/f_n$ where $n \geq 0$.

2.1.3 Useful Results

Theorem

If $f(x)$ is continuous and $\{a_n\}$ is a sequence in the domain of f then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

Example: Consider $f(x) = \ln(x)$ and $a_n = \dfrac{n}{n+1}$ where $n \geq 1$. Determine $\lim_{n\to\infty} f(a_n)$.

Theorem

These are useful limits to memorize. As $n \to \infty$,

- $\ln(n)/n \to 0$
- $n^{1/n} \to 1$
- $(\ln(n))^{1/n} \to 1$
- $a^{1/n} \to 1$ if $a > 0$
- $a^n \to 0$ if $|a| < 1$
- $\left(1 + \dfrac{a}{n}\right)^n \to e^a$
- $a^n/n! \to 0$

Example: Consider $a_n = \left(\dfrac{n}{n+1}\right)^n$ for $n \geq 1$. Determine the limit as $n \to \infty$.

2.1.4 Monotonic Sequences and the Monotonic Convergence Theorem

Definition

A sequence $\{a_n\}$ is...

- non-increasing if $a_{n+1} \leq a_n$ for all n.
- non-decreasing if $a_{n+1} \geq a_n$ for all n.

A sequence that is either non-increasing or non-decreasing is also called monotonic.

Definition

A sequence $\{a_n\}$ is bounded...

- below by M if $a_n \geq M$ for all n.
- above by M if $a_n \leq M$ for all n.
- by M if $|a_n| \leq M$ for all n.

Example: Consider $a_n = \dfrac{1}{n}$ where $n \geq 1$. This is decreasing and bounded below by $M = -2$. Explain why.

Monotonic Convergence Theorem

If $\{a_n\}$ is either bounded below and non-increasing or bounded above and non-decreasing then the sequence converges.

Example: Argue that the sequence $a_n = \dfrac{a_{n-1}}{n}$ where $n \geq 1$ converges without solving for it explicitly.

2.2 (Section 10.2) Infinite Series

2.2.1 Defining an Infinite Sum

Intuitive Description: An infinite series (or commonly just called a series) is the sum of an infinite sequence of numbers. That is, if $\{a_k\}_{k=1}^{\infty}$ is a sequence then a series whose terms are this sequence is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + ...$$

The problem with this definition is that it doesn't give context of what it means to take an infinite sum. We understand limits and can work from there. It should illustrate the fact that as we taking more values to add, we are approaching some value. Thus we need to somehow incorporate this into the definition.

(Actual) Definition

Given a sequence $\{a_k\}$ as above we define the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

We say that the series

$$\sum_{k=1}^{\infty} a_k$$

converges to L if $\lim_{n\to\infty} S_n = L$. If the Sequence $\{S_n\}$ diverges we say the series diverges.

Now, most of the time you will never be able to find this limit L. It is often impossible and just calculated using very abstract techniques or approximated by computers. Mostly we care about convergence or divergence. This section focuses on two series you can actually find the value of while the rest of this chapter focuses on just determining convergence or divergence without ever being able to find this value.

Example: Consider the series

$$\sum_{k=1}^{\infty} (-1)^k$$

We have that the partial sums are $S_n = \sum_{k=1}^{n} (-1)^k$. We compute a few terms to find that

$$S_1 = (-1)^1 = -1$$

$$S_2 = (-1)^1 + (-1)^2 = -1 + 1 = 0$$

$$S_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1$$

$$S_4 = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = -1 + 1 - 1 + 1 = 0$$

At this point you can probably tell that $\{S_n\}_{n=1}^{\infty} = \{-1, 0, -1, 0, -1, 0, -1, ...\}$. We see that for this sequence $\lim_{n\to\infty} S_n$ does not exist. It oscillates between two values. Therefore the above series diverges.

2.2.2 Geometric Series

Definition

A geometric series is a series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

Theorem

If $|r| < 1$ then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

else if $|r| \geq 1$ the series diverges.

Proof: Consider the partial sum $S_n = \sum_{k=1}^{n} ar^k = a + ar + ar^2 + \cdots + ar^{n-1}$. We may derive an explicit formula for the partial sums of this particular series as follows. Scale partial sum by r to obtain

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Then form the difference $S_n - rS_n$ to obtain

$$(1-r)S_n = S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^n) = a - ar^n$$

due to many of the terms canceling out. Now, solve for the partial sum to obtain

$$S_n = \frac{a(1-r^n)}{1-r}$$

We observe that this sequence converges provided that $|r| < 1$ and diverges otherwise. In particular, if $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$ and thus

$$\sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

$\blacksquare$

Example: Determine if $\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k$ converges or diverges. If it converges find it's sum.

Example: Determine if $\sum_{k=1}^{\infty} \ln\left(\frac{1}{3^k}\right)$ converges or diverges. If it converges find it's sum.

Example: Determine if the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ converges or diverges. If it converges find it's sum.

Example: Express the number 1.4144144144144... as the ratio of two integers.

2.2.3 Telescoping Series

Definition

A telescoping series is a series of the form

$$\sum_{k=1}^{\infty}(a_{k-m} - a_{k-l})$$

You can find the sum (or determine convergence/divergence) of these series by investigating and finding a closed formula for the partial sums.

Example: Determine whether the series $\displaystyle\sum_{k=1}^{\infty}\frac{40k}{(2k-1)(2k+1)}$ converges or diverges. If it converges find it's sum.

Example: Determine if the series $\sum_{n=1}^{\infty} \ln\left(\sqrt{\frac{n+1}{n}}\right)$ converges or diverges. If it converges find it's sum.

2.2.4 The n-th Term Divergence Test

If you are **given** the fact that a series $\sum_{n=1}^{\infty} a_n$ converges what can you say about the convergence of the sequence in the sum, $\{a_n\}$? Notice that we may single out elements of the sequence using partial sums by the following

$$S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = a_n$$

Then if we are told $\sum_{n=1}^{\infty} a_n$ converges, that implies that $\lim_{n\to\infty} S_n = L$ for some number L. Thus we obtain

$$\lim_{n\to\infty} S_n = L \qquad \text{and} \qquad \lim_{n\to\infty} S_{n-1} = L$$

and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = L - L = 0$$

This gives us a result but nothing that is useful.

(Mostly Useless But Still Should Know This) Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

We can reword this theorem to become useful but it requires a little bit of logic.

Definition and Result

Given a statement "If p then q" the *contrapositive* of this statement is "If NOT q then NOT p". The contrapositive of a statement has the same logical implication.

Example: Consider the statement "If it is a crow then it is black" (where we are talking about the common crow before any biology people start talking about blue jays). So if you see a crow you can say it is black. This also implies that the following statement is also true, "If it is not black then it is not a crow".

With this we can form the useful theorem.

n-th Term Divergence Test

If it is NOT TRUE that $\lim_{n\to\infty} a_n = 0$ then it is NOT TRUE that $\sum_{n=1}^{\infty} a_n$ converges. That is, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. The statement $\lim_{n\to\infty} a_n \neq 0$ is still satisfied if a_n diverges.

Example: Consider the series $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right)$.

Example: Consider the series $\sum_{n=1}^{\infty} \left(1 - \frac{\pi}{n}\right)^{-n}$.

Example: A friend has determined that for the series $\sum_{k=1} \frac{4k^2 - k^4}{10 + 2k^4}$ that it diverges because $\lim_{k\to\infty} a_k = -\frac{1}{2} \neq 0$. However, they wrote down

$$\sum_{k=1}^{\infty} \frac{4k^2 - k^4}{10 + 2k^4} = -\frac{1}{2}$$

Explain why the student is incorrect.

Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that $\lim_{n\to\infty} \frac{1}{n} = 0$. Is there anything you can say about the value convergence or divergence of the series? Can you conclude anything about the value of the series?

Definition

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **Harmonic Series**.

2.2.5 Properties of Series

Theorem

Let $\sum a_n$ and $\sum b_n$ be two convergent series and let k be a constant. Then...

1. $\sum(a_n + b_n) = \sum a_n + \sum b_n$
2. $\sum(a_n - b_n) = \sum a_n - \sum b_n$
3. $\sum k a_n = k \sum a_n$

and consequently all the above expressions are convergent series.

There are also corollaries that occur as a consequence of the above theorem.

Theorem

1. Every non-zero constant multiple of a divergent series is also divergent.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then both $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ diverge.

Example: Explain why the series $\displaystyle\sum_{k=1}^{\infty}\left(\left(\frac{2}{3}\right)^k - \frac{k}{k+1}\right)$ diverges.

2.3 (Section 10.3) Integral Test

2.3.1 Constructing the Integral Test and Examples

Formulating the Integral Test

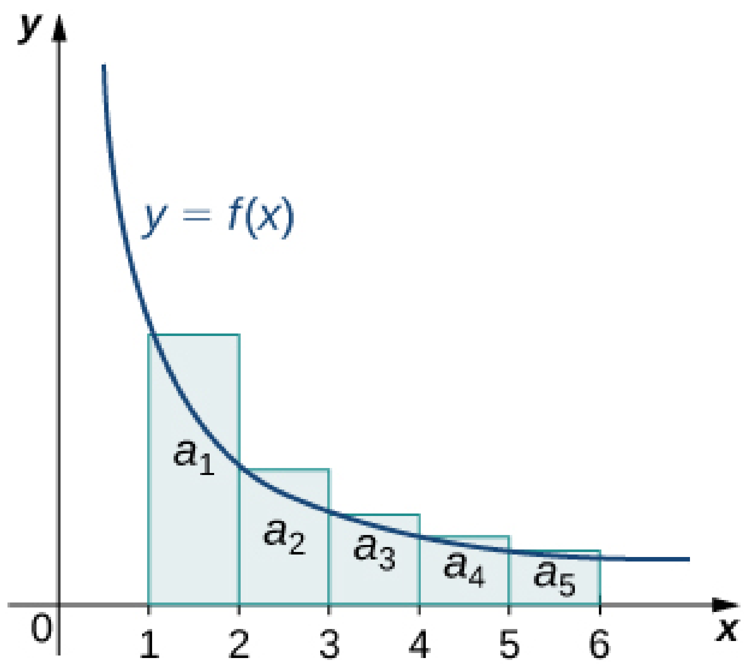
Suppose that f is a continuous, positive and decreasing function such that $f(n) = a_n$ on the interval $[1, \infty)$. Then...

$$\int_1^\infty f(x)dx < \sum_{n=1}^\infty a_n < a_1 + \int_1^\infty f(x)dx$$

To demonstrate this result, we consider rewriting the series as $\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \cdots$ as

$$\sum_{n=1}^\infty a_n = a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots$$

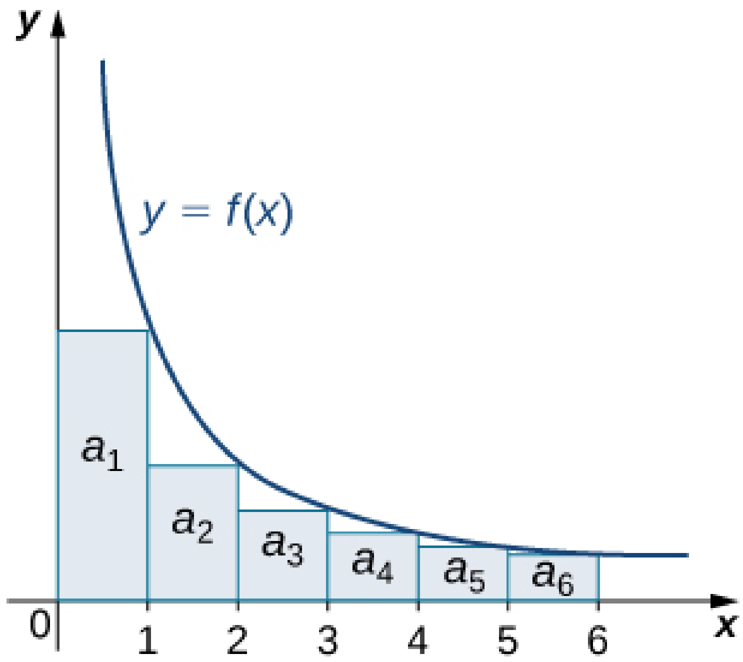
which we will interpret each term $a_i \cdot 1$ to geometrically represent the area of a rectangle of height $H = a_i$ and width $W = 1$. Now, we may fit these rectangles to the curve of $y = f(x)$ as follows to obtain



which holds graphically since $f(1) = a_1$, $f(2) = a_2$, ... etc. Based on this, we see that the sum of the rectangles forms an over approximation and we obtain the lower bound of

$$\int_1^\infty f(x)dx < \sum_{n=1}^\infty a_n$$

For the upper bound, we take the above picture and move all the rectangles one unit over to the left. This still is allowed since by the symmetry of the rectangle we still have $f(1) = a_1$, $f(2) = a_2$, ... etc.



However, we are taking the area for $n \geq 1$, and thus we obtain

$$\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)dx$$

to which we add a_1 to both sides of the inequality to obtain

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} f(x)dx$$

as desired. From this, we can determine the behaviour of certain series by a representing integral and vice versa (essentially a comparison test).

The Integral Test

Let $f(x)$ be continuous, positive, and decreasing on $[k, \infty)$ and that $f(n) = a_n$ then

$$\int_k^{\infty} f(x)dx \qquad \text{and} \qquad \sum_{n=k}^{\infty} a_n$$

either both converge or both diverge.

Note

Since $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ both converge or diverge together the nature of convergence for p-integrals is the same as the p-series form.

Note

A COMMON mistake that students make is that they assume that the integral is equal to the series! THIS IS NOT THE CASE! They just have the same convergent behaviour! That is,

$$\int_k^\infty f(x)dx \neq \sum_{n=k}^\infty f(n)$$

In this course, you should assume that unless you have a Geometric Series, Telescoping Series, or (a yet to be mentioned) Taylor Series then all hope of finding an exact value for a series is lost. Very advanced techniques are required.

Example: Determine the nature of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Example: Determine the nature of convergence for $\displaystyle\sum_{n=2}^{\infty} ne^{-n}$.

Example: Determine the nature of convergence for $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$.

Note

We will be discussing other series tests so that we have a wide range of tools at our disposal for determining the behaviour of series. A **BIG** indication that you might want to start with an integral test is if a logarithm is present.

At Home Exercise: Show that $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \neq \int_1^{\infty} \left(\frac{2}{3}\right)^x dx$.

2.3.2 Error Estimation

We only know how to find the sum of geometric and telescoping series, but we can approximate others by partial sums

$$\sum_{n=k}^{N} a_n \approx \sum_{n=k}^{\infty} a_n$$

for very very large N. Let $S = \sum_{n=k}^{\infty} a_n$ and $S_N = \sum_{n=k}^{N} a_n$. We can form the error in the estimation (summing up to the term $n = N$),

$$R_N = S - S_N$$

Suppose that S converges under the integral test. If $f(n) = a_n$ under the required conditions earlier then one can show the error is bounded by

$$\int_{N+1}^{\infty} f(x)dx \leq R_N \leq \int_{N}^{\infty} f(x)dx$$

by rearranging the inequality formed at the beginning of this section.

Example: Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by bounding it with $N = 10$ terms.

2.4 (Section 10.4) Comparison Tests

2.4.1 The Direct Comparison Test for Series

Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $0 \leq a_n \leq b_n$ for all n. Then

- If $\sum b_n$ converges then $\sum a_n$ converges
- If $\sum a_n$ diverges then $\sum b_n$ diverges

The logic behind this is identical to that for improper integrals.

Example: Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges or diverges.

Example: Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges or diverges.

2.4.2 The Limit Comparison Test for Series

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ and let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then provided

- $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or diverge
- $L = 0$, then if $\sum b_n$ converges then $\sum a_n$ converges
- $L = \infty$, then if $\sum b_n$ diverges then $\sum a_n$ diverges

Think about these results intuitively!

Example: Determine whether or not the series $\sum_{n=3}^{\infty} \sqrt{\frac{n^5 + 3n}{2n^2 + 4}}$ converges or diverges.

Example: Determine whether or not the series $\sum_{n=2}^{\infty} \frac{1 + n\ln(n)}{n^2 + 5}$ converges or diverges.

Example: Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}e^n}$ converges or diverges.

2.5 (Section 10.5) Absolute Convergence, Ratio and Root Tests

2.5.1 The Absolute Convergence Test

Definition

An alternating series is a series of the form

$$\sum_{n=k}^{\infty}(-1)^n a_n$$

where $a_n \geq 0$ for all $n \geq k$.

Example: Determine if the alternating series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{4^{n+1}}{5^n}$ converges or diverges.

We can handle them for geometric series. Other tests will be required for other series.

Absolute Convergence Test

If $\sum |a_n|$ converges then $\sum a_n$ converges.

Proof: Notice that $|a_n|$ is either a_n or $-a_n$ by definition of the absolute value (depending on the sign of a_n). Thus we can say

$$0 \leq a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n|$$

Since we are assuming $\sum |a_n|$ is convergence then $\sum 2|a_n|$ is also convergent. As $a_n + |a_n|$ and $2|a_n|$ are non-negative them by a comparison test $\sum(a_n + |a_n|)$ is convergent as $\sum 2|a_n|$ is. Then we may write

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

and as $\sum a_n$ is the difference of two convergent series, it is also convergent. ■

Note

This is currently our only test mentioned that handles series with negative terms in determining convergence. The divergence test handles series with negative terms but only determines divergence. Be careful on how to interpret this theorem. There is NO COMPARISON happening here. The comparison tests previously mentioned only apply to series whose terms are non-negative. Comparison does not apply to series that have negative terms. Also, if $\sum |a_n|$ diverges there is nothing you can conclude!

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$.

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$.

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{3}{n}\right)^n$.

Definition

If $\sum |a_n|$ is convergent we say $\sum a_n$ is absolutely convergent.

Thus all but the last example mentioned are series that are absolutely convergent. Absolute convergence is a level of convergence that is VERY strong. There are many things you can do with series that is absolutely convergent when manipulating them... while for series that are convergent (but not absolutely convergent) you are very restricted.

2.5.2 The Ratio Test

This is everybody's favourite test.

The Ratio Test

Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Then if...

- $L < 1$ the series converges absolutely.
- $L > 1$ the series diverges.
- $L = 1$ then the test is inconclusive. You must apply a different test as this one does not work.

Note

This test is useful for **FACTORIALS** (especially!!!), polynomials, and simple exponents.

Example: Determine the nature of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{3^n(n+1)}{n!}$.

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

2.5.3 The Root Test

The Root Test

Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

then if...

- $L < 1$ then the series converges absolutely.
- $L > 1$ then the series diverges.
- $L = 1$ then the test is inconclusive. You must use a different test as this one does not work.

Note

This is useful for bad exponents

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$.

Example: Determine the nature of convergence of $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^2}$.

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$.

2.6 (Section 10.6) Alternating Series Test

2.6.1 Conditional Convergence and the Alternating Series Test

Definition

If a series $\sum b_n$ converges but $\sum |b_n|$ diverges then series $\sum b_n$ is called conditionally convergent.

Note

Conditionally convergent series DO converge, but they have a "lower level" of convergence. The level of convergence is rather slow and weak. When working with conditionally convergent series you are rather limited to what you can do with them algebraically.

The Alternating Series Test

The series $\sum (-1)^{n+1} a_n$ converges if all the following are satisfied:

- All $a_n \geq 0$.
- All terms in $\{a_n\}$ are eventually all non-increasing.
- $\lim_{n\to\infty} a_n = 0$.

Proof: Let's examine even partial sums. Note that as we assume $a_{n+1} \leq a_n$ (non-increasing) then $a_n - a_{n+1} \geq 0$. We have

$$S_2 = a_1 - a_2$$

$$S_4 = a_1 - a_2 + a_3 - a_4 = S_2 + a_3 - a_4 \geq S_2 + 0 = S_2$$

$$S_6 = S_4 + a_5 - a_6 \geq S_4 + 0 = S_4$$

So $\{S_{2n}\}$ is increasing. Also $S_{2n} \leq a_1$ since

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots a_{2n-1} - a_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

as $a_n - a_{n+1} \geq 0$. Thus by monotonic convergence $\{S_{2n}\}$ converges (i.e. $S_{2n} \to L$). If N is odd then for S_N we have $\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} (S_{2m} + a_{2m+1}) = L + 0 = L$. ■

Note

To show a series is absolutely convergent you need to show $\sum |a_n|$ converges and THAT'S IT. To show a series is conditionally convergent you need to show $\sum a_n$ converges under the alternating series test AND show that $\sum |a_n|$ diverges by some other test.

Example: Determine the nature of convergence of the alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Is it conditionally or absolutely convergent?

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$.

Example: Determine the nature of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$. Is it conditionally or absolutely convergent?

2.6.2 Error Estimation of Series that Converge by the AST

The sum $\sum(-1)^{n+1}a_n = S$ always lies between two successive partial sums S_N and S_{N+1}. Furthermore the error (remainder) is bounded by

$$R_N = |S - S_N| \leq a_{n+1}$$

Example: Find the error in approximating $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ using 15 terms.

2.7 (Section 10.7) Power Series

2.7.1 Power Series Functions and Their Domain

Sometimes we can define a function by a series. When the terms in the sum are simple polynomials we call it a "power series".

Definition

A **power series** about $x = a$ (called centered at a) is a function of the form

$$f(x) = \sum_{n=k}^{\infty} c_n (x-a)^n = c_k(x-a)^k + c_{k+1}(x-a)^{k+1} + \cdots$$

where all c_n are constant.

Example: Find a power series form of $f(x) = \dfrac{1}{1-x}$ and find the domain of the series form.

Think of it like this, the partial sums $S_N(x) = \displaystyle\sum_{n=1}^{N} x^{n-1} = 1+x+x^2+\cdots+x^N$ are polynomials that approximate $f(x) = \dfrac{1}{1-x}$. The approximation gets better as $N \to \infty$.

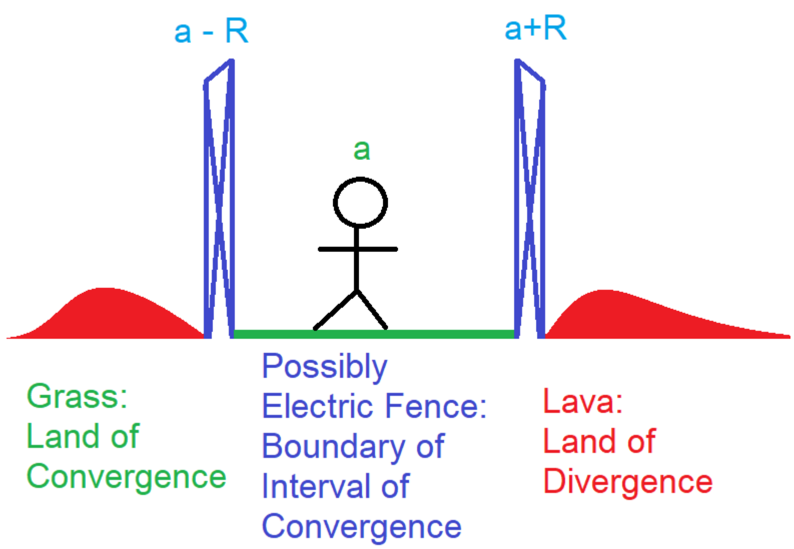
This is an example I like as it illustrates that a region of convergence describes the domain of such a function.

Example: For what values of x does

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

converge? (i.e. what is the domain of $f(x)$?)

So the ratio test (or less commonly the root test) gives a fence between convergence and divergence, then you check to see if the fence itself is dangerous (divergent) or not (convergent).



Theorem

A series will either converge absolutely at a point $x = a$, in an interval $|x - a| < R$ or everywhere. The region of convergence is called the **interval of convergence** (check boundary). The number R is called the **radius of convergence**.

Example: Determine the interval and radius of convergence of $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Example: Determine the interval and radius of convergence of $f(x) = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n (x-2)^n$ and be sure to check the endpoints.

2.7.2 Operations of Power Series

Addition and Scalar Multiplication

If $\sum a_n x^n$ and $\sum b_n x^n$ converge for $|x| < R$ then

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and

$$\sum_{n=0}^{\infty} k a_n x^n = k \sum_{n=0}^{\infty} a_n x^n$$

converge for $|x| < R$ where k is a real number.

Multiplication

If $\sum a_n x^n$ and $\sum b_n x^n$ converge for $|x| < R$ then

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

converges for $|x| < R$.

Substitution

If $\sum a_n x^n$ converges for $|x| < R$ and $g(x)$ is continuous then $\sum a_n (g(x))^n$ converges for all values x such that $|g(x)| < R$.

Example: Consider $f(x) = \dfrac{1}{4 + x^2}$. Find an expression for this function in power series form and determine the interval of convergence.

Differentiation

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ then

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-(k+1))c_n(x-a)^{n-k}$$

all converge over $|x-a| < R$.

Example: Consider the power series representation of the function $f(x) = \dfrac{1}{1-x}$ given by $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ which converges over $|x| < 1$. Use this to construct power series representations of its derivatives.

Integration

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges over $|x-a| < R$ then

$$\int f(x)dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \right) + C$$

converges over $|x-a| < R$.

Example: Consider $\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ which converges over $|x| < 1$. Use this to construct a power series representation of $f(x) = \ln(1+x)$ and determine the interval of convergence.

2.8 (Section 10.8) Taylor and MacLaurin Series

A Taylor series (and Maclaurin) is a polynomial series that (potentially) approximates a known function $f(x)$. Let's assume we are GIVEN that a known function $f(x)$ has a power series representation (e.g. $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$) and consider it's representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

Note

We are ASSUMING a KNOWN function $f(x)$ has a power series form. This is a really massive assumption!!

Then we shall attempt to solve for all terms in the sequence $\{a_n\}$. We see that

$$f(a) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots \Rightarrow f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots = a_1$$

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + \cdots \to f''(a) = 2a_2 + 3 \cdot 2a_3(0) + \cdots = 2a_2$$

$$f'''(x) = 3 \cdot 2a_3 + \cdots \Rightarrow f'''(a) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2(0) + \cdots = 3 \cdot 2a_3$$

$$\vdots$$

$$f^{(k)}(x) = k \cdot (k-1) \cdots (2)(1)a_k + (k+1)(k)(k-1) \cdots (2)a_{k+1}(x-a) + \cdots \Rightarrow f^{(k)}(a) = k \cdot (k-1) \cdots (2)(1)a_k = k!a_k$$

and thus $a_k = \dfrac{f^{(k)}(a)}{k!}$. Thus if we are TOLD a specific function has a power series representation and we want to compute it, we can use this to determine the representation of it in power series form without clever algebraic tricks or relating it to a geometric series.

Definition

Let f be a function with derivatives of all orders about an interval containing $x = a$. Then the **Taylor series** about $x = a$ of $f(x)$ is the series

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a = 0$ it is called a **Maclaurin series** (just to give him some credit)

2.8.1 The Unreasonable Assumption of Being a Taylor Series

There is still one question, for each value of x within an appropriate interval is it true that $P(x) = f(x)$? It required a massive assumption, that the function $f(x)$ COULD BE expressed as a power series. Knowing whether or not you can do this requires tools we don't have. You can always compute the Taylor series of any function but you might not get equality of $P(x)$ and $f(x)$. Let's have a brief chat about how functions are classified in calculus.

Definition

Let I be an open interval and denote $C^0(I)$ as the collection of functions that are continuous on I (called class zero). We also further denote $C^k(I)$ as the collection of functions whose derivatives all the way up to order k are continuous on I (called class k) and denote $C^\infty(I)$ as the collection of function who have derivatives of ALL orders and are continuous on I (called **smooth functions**). Lastly, we also represent functions who have a power series representation on I as $\mathfrak{A}(I)$ (called **analytic functions**).

Note

To illustrate this, $f(x) = |x|$ is a class zero function on $\mathbb{R}$ (the collection of all real numbers) as it is continuous but it's derivative is not continuous. The function $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ is an analytic function on $(-1, 1)$ as it has a power series representation on this interval.

Here's how the structuring in calculus works...

$$C^0(I) \supset C^1(I) \supset C^2(I) \supset \cdots C^\infty(I) \supset \mathfrak{A}(I)$$

where $A \supset B$ means the collection of all objects in B can be found in A. By this chain, analytic is the ULTIMATE form of differentiability. You can have functions with derivatives of ALL orders and are continuous but still don't have a power series representation! Making your way down the chain is harder at each step. Functions further down the chain are much smoother and nicer than functions near the start of the chain.

Example Show the function

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

is strictly a class one function on $\mathbb{R}$ (i.e. lies in the collection $C^1(\mathbb{R})$ but not $C^2(\mathbb{R})$).

2.8.2 Examples of Constructing a Taylor Series Directly

Example: Compute the Taylor series of $f(x) = \sin(x)$ about $x = 0$.

Example: Find the Taylor series of $f(x) = e^x$ about $x = 0$.

2.8.3 Taylor Polynomials of Order N

Definition

The **Taylor polynomial of order** N to $f(x)$ at $x = a$ is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Definition

If $P(x)$ is the Taylor polynomial of $f(x)$ we call $f(x)$ the **generating function** of $P(x)$.

Two questions remain in this theory of wanting to replace $f(x)$ with $P(x)$ (because polynomials are nicer to work with):

- When can we expect $f(x) = P(x)$ for each x? That is, when can we expect that $\lim_{N\to\infty} P_N(x) = f(x)$?
- If the previous condition is satisfied (yay!) how accurate are the finite order approximations $P_N(x)$? (Because we might want to understand the function using a computer and computers don't do infinity so using some smaller terms to approximate values might be good enough).

Example: We can't always expect that $P(x) = f(x)$. With great difficulty one can show that for

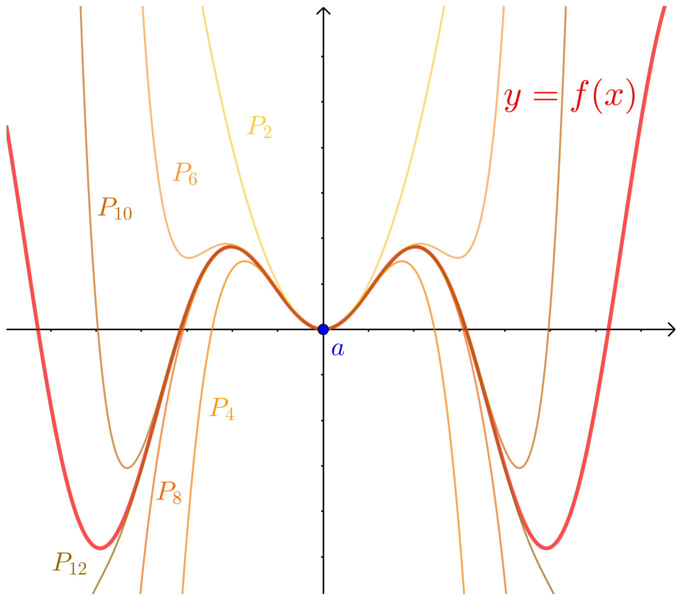
$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$$

That derivatives of all orders exist and are continuous with the further fact that $f^{(n)}(0) = 0$. Thus for a Taylor series centered at $x = 0$ we have $P(x) = 0$. However, clearly $f(x)$ is not identically zero! Thus they are not equal. This function is an example of a function that is $C^\infty(\mathbb{R})$ but not $\mathfrak{A}(\mathbb{R})$!

2.9 (Section 10.9) Convergence of Taylor Series

2.9.1 Taylor's Theorem and Approximating Functions

Alright, it's time to answer the big question. When does the Taylor series $P(x)$ equal the generating function $f(x)$? When does $P_N(x) \to f(x)$ as $N \to \infty$?



Do the polynomials approach $f(x)$ as $N \to \infty$? By the mean value theorem we have the following theorem to help us out.

Taylor's Theorem

If f is of class C^N on an interval containing a and b then there exists a number c between a and b such that

$$f(b) = P_N(b) + \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1} = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(b-a)^n\right) + \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1}$$

This theorem extends (in a complicated way) so that in an interval I where f is of class C^∞ then for all x in I containing a,

$$f(x) = P_N(x) + R_N(x)$$

for all N where $R_N(x) = \dfrac{f^{(N+1)}(x)}{(N+1)!}(x-a)^{N+1}$ for some c between x and a.

Alright, this might seem complicated. We're working VERY HARD to approximate the function $f(x)$ with the Taylor polynomials. Let's break down what's happening, we have said that if the function is smooth (nice enough) then we can represent it as the sum of two functions

$$f(x) = P_N(x) + R_N(x)$$

Now here's the part we have to work on. Since $P_N(x) \to P(x)$ as $N \to \infty$ then we need to only show that this other weird function $R_N(x)$ satisfies $R_N(x) \to 0$ as $N \to \infty$. This will allow us to conclude that

$$f(x) = P_N(x) + R_N(x) \Rightarrow \lim_{N\to\infty} f(x) = \lim_{N\to\infty} (P_N(x) + R_N(x)) \Rightarrow f(x) = P(x) + 0 = P(x)$$

Often one can estimate $R_N(x)$ without ever knowing c.

Example: Show the function $f(x) = e^x$ is equal to it's Taylor series.

2.9.2 Taylor Series with Bounded Derivatives

Theorem

If there is a constant M such that $|f^{(N+1)}(t)| \leq M$ for all t between x and a then $R_N(x)$ satisfies

$$|R_N(x)| \leq M \frac{|x-a|^{N+1}}{(N+1)!}$$

If this holds for every N and the Taylor conditions are satisfied then $P(x) = f(x)$.

Example: Show that $f(x) = \sin(x)$ is equal to it's Taylor series.

2.9.3 Important Analytic Functions

The following is a short list of some analytic functions and their Taylor series form.

Important Taylor Series Expression

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in $I = \mathbb{R}$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ in $I = \mathbb{R}$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ in $I = \mathbb{R}$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ in $I = (-1, 1)$
- $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ in $I = (-1, 1)$

Note

Ahem, for future calculus courses (and this one) you should have these MEMORIZED. No if's and's or but's. Many expressions for other functions are obtained using these.

Example: Use the above to indirectly construct a Taylor Series for the integral function

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

.

Note

The above integral function is the famous "Error Function".

2.9.4 Approximations Using Taylor Series

Example: Find $P_4(x)$ of $f(x) = e^x \cos(x)$.

In an alternating series you can see how well $P_N(x)$ approximates $f(x)$.

Example: For what values of x will $P_3(x) = x - \dfrac{x^3}{3!}$ approximate $\sin(x)$ with an error no bigger than 3×10^{-4}?

2.10 (Section 10.10) Binomial Series and Applications of Taylor Series

In this section we discuss a bit of practicality of using and developing Taylor series. Before we do so, we discuss Binomial series.

2.10.1 Binomial Series

These are the series representation of $(1+x)^m$. Before you get all "just expand it and it becomes a polynomial. Polynomials are their own Taylor series" we consider the case where m is not a positive integer as well.

We defin the binomial coefficient

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

for $k \geq 1$ where k is an integer and we set

$$\binom{m}{1} = m \qquad\qquad \binom{m}{2} = \frac{m(m-1)}{2!}$$

We compute for $f(x) = (1+x)^m$ at $x = 0$,

$$f(x) = (1+x)^m \Rightarrow f(0) = 1$$

$$f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$$

and so forth. Hence

$$f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1)$$

and thus the series is

$$P(x) = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} x^k = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

One can show that by the ratio test and inspecting the endpoints it converges (absolutely) for $|x| < 1$. Showing $P(x) = (1+x)^m$ is a little tricky. The book shows a trick and guides you in one of the exercises of this section.

Example: Form a series representation of $\sqrt{5}$.

2.10.2 Applications of Taylor Series

One should not underestimate the "power" of power series. Their complexity is worth the effort. They are the foundation of much theory.

Complex Variables: We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Let $x = i\theta$ then

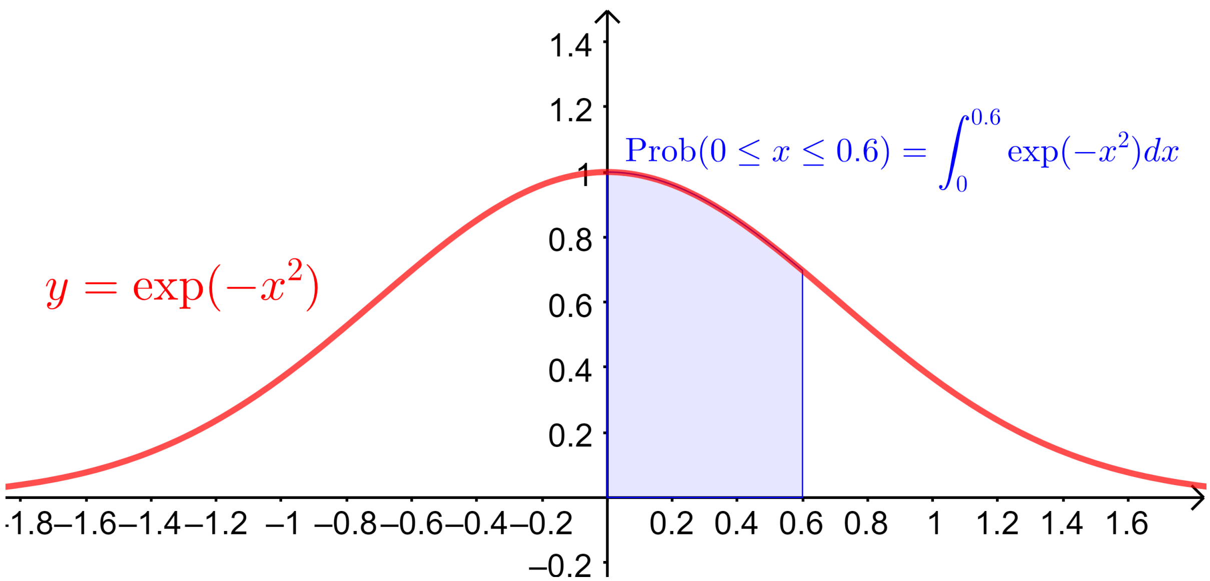
$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

since $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ...and the pattern repeats. Thus

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) = \cos(\theta) + i\sin(\theta)$$

Ta-da! □

Probability Theory: One of (if not the most) important distribution is the bell curve!



Many natural occurrences follow a bell curve distribution. Ideally, in a class grades you want them to follow a bell (normal) distribution. The probability of something occuring is given by the area between two points under the curve. In quantum physics, for example, the probability of a particle existing in a location can follow a bell curve.

Example: In the ground state of the harmonic oscillator, there is a non-zero probability of finding a particle outside the classically allowed region. The probability of finding the particle outside of the region $[-a, a]$ is given by

$$F(a) = 1 - \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-a}^{a} \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx$$

Determine the probability of finding the particle outside the region where $a = \sqrt{\frac{\hbar}{m\omega}}$.

Approximating Values: One can show using

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

is convergent for $|x| < 1$. Upon integrating and solving to obtain $C = 0$,

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

holds for $-1 < x \leq 1$ (after checking the endpoints). Thus we may compute

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\Rightarrow \pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

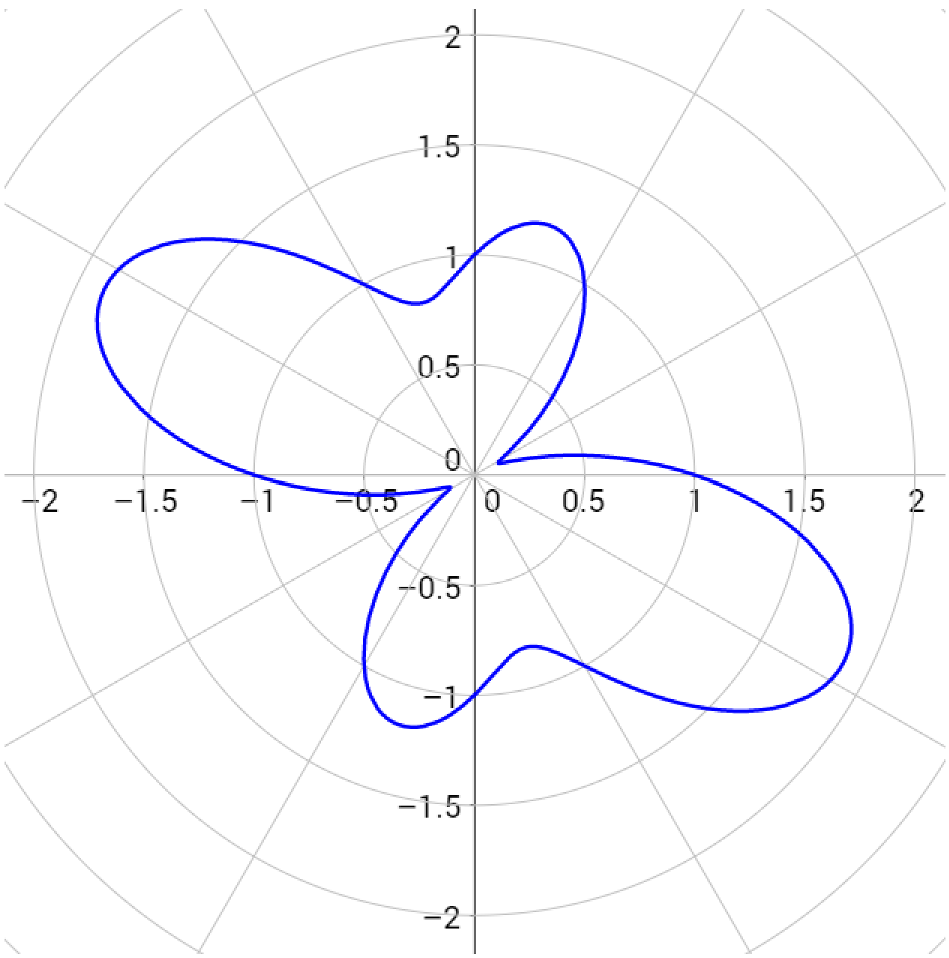
however it does converge very slowly. You'll need several terms to for a good approximation (the reason it's slow is due to conditional convergence).

Evaluating Limits: One can use Taylor series to compute various limits.

Example: Compute $\lim_{x \to 0} \dfrac{e^x - (1+x)}{x^2}$.

Chapter 3

Coordinate Systems



3.1 (Appendix A7) Complex Numbers

3.1.1 Introducing Complex Numbers and Algebraic Operations

Everything we've dealt with is over the collection of measurable quantities, the real numbers $\mathbb{R}$. The complex numbers $\mathbb{C}$ are an extension of the reals $\mathbb{R}$ such that we allow solutions to the equation

$$x^2 + 1 = 0$$

The "positive" solution $x = +\sqrt{-1} = i$ gives rise to "imaginary" parts of numbers. The space $\mathbb{C}$ consists of all numbers of the form

$$a + bi$$

where a and b are real.

Example: The equation $x^2 + 4x + 5 = 0$ has solutions

Addition and multiplication are defined as expected.

Example: Compute $(3 + 4i) - (-1 + 2i)$ and $(2 + i)(1 + 3i)$.

Definition

The **conjugate** of $z = a + bi$ is denoted and defined as $\bar{z} = a - bi$.

Definition

The **norm/modulus** of $z = a + bi$ is defined as

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}$$

Definition

We define **division** for $z = a + bi$, $w = c + di$ as

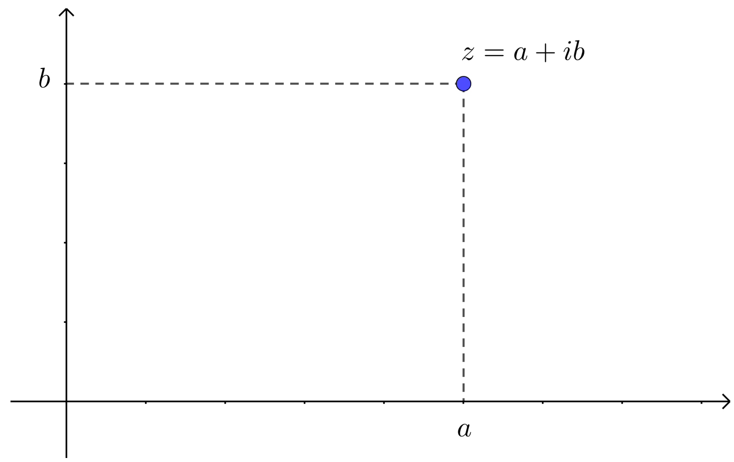
$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Essentially you compute it by rationalizing the denominator.

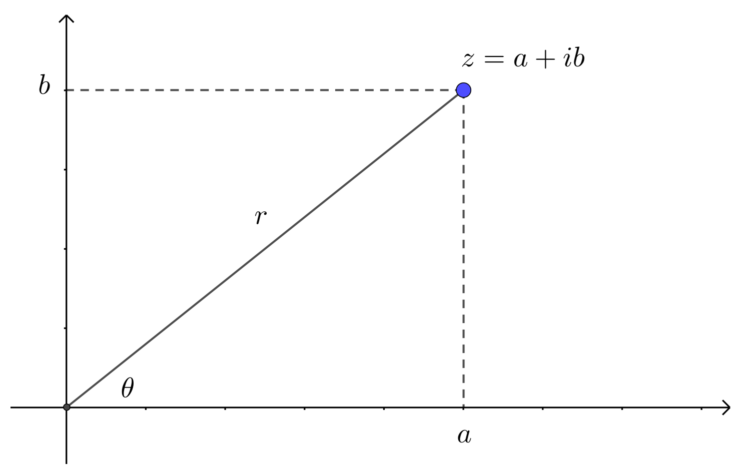
Example: Compute $\dfrac{2+3i}{1-7i}$.

3.1.2 Argand Diagrams and Polar Coordinates

There are two components to a complex number we need to measure instead of just the one component that real numbers have. Thus real numbers are represented by a single quantity while complex numbers are represented as a pair of numbers (thus a point). We call this representation an Argand diagram.



We can also represent them by "polar coordinates". The point makes an angle with the x-axis and a radius out from the origin.



From the diagram $a = r\cos(\theta)$ and $b = r\sin(\theta)$ where $r = |z|$.

Definition

Consider the complex number $z = a + ib$ and let θ be the angle between the segment connecting point representation of z with the origin and the positive x-axis. Provided that r represents the length of this segment, the **Polar Representation** of the point is

$$z = r(\cos(\theta) + i\sin(\theta))$$

Example: Express $1 - i$ in polar coordinates.

3.1.3 Euler's Formula

Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This allows us to write the polar form as

$$a + bi = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

This form allows us to exploit exponent properties.

Products: Taking the product gives $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Quotients: Taking the quotient gives $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Powers: Taking the power gives $(re^{i\theta})^n = r^n e^{i\theta n}$.

That last point is the most interesting because it gives us the following theorem.

De Moivre's Theorem

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Proof: Combine $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ with $(e^{i\theta})^n = e^{in\theta}$. ■

You can use this to generate identities for trigonometric functions!

Example: Use De Moivre's Theorem to generate identities for $\cos(2\theta)$ and $\sin(2\theta)$.

3.1.4 Roots of Complex Numbers

Every polynomial of degree n always has n-roots over $\mathbb{C}$ (called the Fundamental Theorem of Algebra). We can use the exponential form to 'easily' solve

$$z^n = C$$

for z where C is a constant (possibly a complex number $C = a + bi$).

Procedure for Finding All Roots of a Complex Number

Start with an equation of the form $z^n = C$.

1. Express C in exponential form as $C = re^{i\theta}$. Then

$$z^n = re^{i\theta}$$

2. Use the periodicity of the complex exponential (sine and cosine). Since sine and cosine are periodic with period 2π then

$$z^n = r\exp(i(\theta + 2\pi k))$$

for k any integer.

3. Take the n-th root of both sides,

$$z = r^{1/n}\exp\left(i\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, 3, ..., n-1$.

The reason it goes from k being all integers to just the first n values is because afterwards it repeats. So we can obtain the n roots from the first n values of k starting from zero.

Example: Find the all the fourth roots of -16.

3.2 (Section 11.1) Parametric Equations

3.2.1 Defining Parametric Curves

A curve C given by $f(x, y) = 0$ gives us a graph but it falls short on a few aspects.

> Definition
>
> If C is a curve and (x, y) is any point on C then provided there exists functions such that
>
> $$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$
>
> we call it a **parametric representation** of C.

Example: Consider $y^2 = x$. Then $y = t$ and $x = t^2$ is a parametric representation of this curve. All points can be mapped out by choosing values of t.

Why is this useful? For modeling the position of a particle by time.

Example: Suppose that the position of a particle follows the trajectory of the curve $y^2 = x^2 + 1$ given by the equations $y = \sec(\pi t)$ and $x = \tan(\pi t)$ where $-\pi/2 < t < \pi/2$. Determine the position of the particle after 1.25 seconds. Graph and Determine the orientation of the particle flow along the curve.

3.2.2 Graphing a Parametric Curve

Technique #1: Plotting Points

This is the worst way and I don't really condone it.

Example: Attempt to graph $x = 2\cos(t)$, $y = \sin(t)$ for $0 \le t \le 2\pi$ by plotting a few points.

If you do it this way, you might as well use GeoGebra.

Graphing Using GeoGebra

Using GeoGebra Classic 6, you can graph parametric curves given by $x = f(t)$ and $y = g(t)$ from $t = a$ to $t = b$ using the command

```
Curve(f(t),g(t),t,a,b)
```

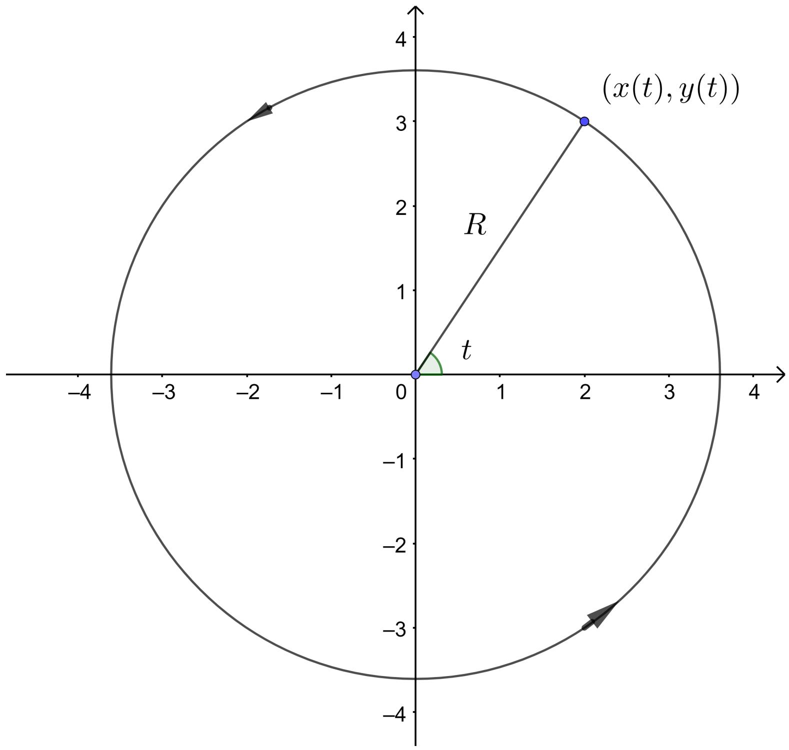
Technique #2: Finding the Curve C

This is usually done by solving one equation for t and plugging it in the other.

Example: Consider the curve given by $x = t^2 + t$, $y = 2t - 1$.

3.2.3 Standard Representation of a Circle

Consider the circle $x^2 + y^2 = R^2$.



We note $\cos(t) = \dfrac{x}{R}$, $\sin(t) = \dfrac{y}{R}$. So a parametric representation is

$$\begin{cases} x = R\cos(t) \\ y = R\sin(t) \end{cases}$$

for $0 \leq t < 2\pi$. The orientation is counter-clockwise as that represents the direction of the increasing angle (which is variable the circle is parametrized by).

3.2.4 Transformations

Reverse Orientation

Translation by a Point

3.3 (Section 11.2) Calculus of a Parameterized Curve

3.3.1 The Slope of a Parametric Curve

Let C be a curve given parametrically by $x = f(t)$, $y = g(t)$. We still want to discuss the slope of the tangent line $\dfrac{dy}{dx}$.

By the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Notation

People commonly use the following,

$$y' = \frac{dy}{dx} \qquad \dot{y} = \frac{dy}{dt}$$

to denote differentiation with respect to space and time respectively. Thus the above may be stated as $y' = \dfrac{\dot{y}}{\dot{x}}$

Theorem

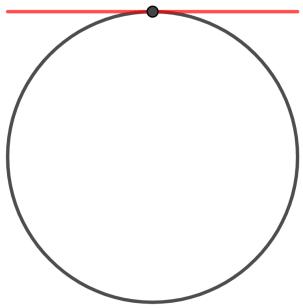
Provided $\dot{x} \neq 0$ we have $y' = \dfrac{\dot{y}}{\dot{x}}$. Provided $\dot{y} \neq 0$ we have $x' = \dfrac{\dot{x}}{\dot{y}}$.

Example: (Witch of Agnesi) Consider the curve given by $x = 2t$, $y = \dfrac{2}{1+t^2}$.

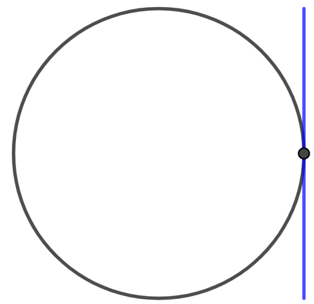
Example: Find the tangent line to $x = t^5 - 4t^3$, $y = t^2$ at $(0, 4)$.

3.3.2 Higher Order Derivatives and Vertical/Horizontal Tangency

Horizontal: $\dot{y} = 0$, (provided $\dot{x} \neq 0$)



Vertical: $\dot{x} = 0$, (provided $\dot{y} \neq 0$)



We can also talk about concavity. Since

$$\frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{\dot{x}}$$

then by the same procedure,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\dot{x}}$$

Theorem

Let $x(t)$ and $y(t)$ be parametric equations of a curve. Then

$$y'' = \frac{\frac{d}{dt}[y']}{\dot{x}}$$

Example: Consider $x = 1 - t^2$, $y = t^7 + t^5$.

3.3.3 Integration

Definition

The signed area between $y(x)$ and the x-axis over $[a, b]$ is $\int_a^b y dx$. The signed area between $x(y)$ and the y-axis over $[c, d]$ is $\int_c^d x dy$.

Example: Compute the area trapped between the curve $x = \cos^3(t)$, $y = \sin^3(t)$ and the x-axis over $0 \le t \le 2\pi$.

(Continued...)

Example: Find the area enclosed by the y-axis and the curve $x = t - t^2$, $y = 1 + e^{-t}$.

3.3.4 Arc-Length

We learned that the length of $y(x)$ over $[a, b]$ is

$$L = \int_a^b \sqrt{1 + y'^2} dx$$

Let's examine the integrand. Since $y' = \frac{\dot{y}}{\dot{x}}$ then

$$\sqrt{1 + y'^2} dx = \sqrt{1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2} dx = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}} dx$$

However $\frac{1}{\dot{x}} dx = t' dx = dt$. So the length of $x(t)$, $y(t)$ over $t = c$ to $t = d$ is the following.

Theorem

The arc length of a parametrized curve $x(t)$, $y(t)$ over $c \leq t \leq d$ is

$$L = \int_c^d \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

Example: Compute the length of a circle of radius R using the standard parametrization.

Example: Compute the length of the curve $x = \ln(\sec(t) + \tan(t)) - \sin(t)$, $y = \cos(t)$ over $0 \leq t \leq \pi/3$.

3.4 (Section 11.3) Polar Coordinates

3.4.1 Revisiting Polar Coordinates and Non-Uniqueness

We already talked a bit about polar form with complex numbers. The idea is that we measure points in the xy-plane using an angle θ and radius r.

Consider the point $(r, \theta) = (2, \pi/6)$.

The expression for a point is not necessarily unique! For example, technically $(2, \pi/6) = (2, \pi/6 + 2\pi)$.

In fact, $(2, \pi/6) = (2, \pi/6 + 2\pi n)$ for any integer n. Polar coordinates also allows negative distance (unless specified otherwise)!

For example consider the angle $\theta = \pi/6 + \pi = 7\pi/6$.

If we take the radius $r = -2$ then we measure the distance backwards, so $(2, \pi/6) = (-2, 7\pi/6) = (-2, 7\pi/6 + 2\pi n)$ for any integer n as well.

3.4.2 Polar to Cartesian (Points), Cartesian to Polar (Equations)

Expressing Cartesian Values in Terms of Polar Values

Consider a point (x, y) in Cartesian that makes an angle θ with the positive x axis and is a distance r from the origin. Then...

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

We can use this to convert equations to polar form. This is very useful for equation with the term $x^2 + y^2$. Given a specified r and θ this allows you to find the Cartesian equivalent of a point in polar coordinates.

Example: Find the Cartesian representation of the point $(r, \theta) = (2, \pi/6)$.

Example: Find the polar curve equation of $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$.

3.4.3 Cartesian to Polar (Points), Polar to Cartesian (Equations)

Expressing Polar Values in Terms of Cartesian Values

Solving the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ for r and θ gives

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan(\theta) = \dfrac{y}{x} \end{cases}$$

Where you solve for them explicitly depending on the "*Branch*" (this means what region the angle in the Cartesian plane: $0 \leq \theta < 2\pi$, $2\pi \leq \theta < 4\pi$, etc... and whether you use a positive or negative radius).

Example: Convert $(x, y) = (-4, -4)$ to polar coordinates.

Example: Convert the equation $r = -8\cos(\theta)$ to Cartesian.

3.4.4 Graphing Sinusoidal Curves

We will be talking about graphing polar curves from their equations in the next section. To do so, we will need to review how to graph sinusoidal curves.

<u>Review</u>:

$$r = A\sin(B\theta + C) + D \qquad \text{or} \qquad r = A\cos(B\theta + C) + D$$

- The amplitude is $|A|$ (height of the curve is $2|A|$).
- The period is $\dfrac{2\pi}{|B|}$ (length of one wave).
- The phase shift is $-C/B$ (horizontal shift).
- The vertical shift is D (vertical translation).

Example: Graph the curve $r = 2\sin(4\theta - 2) + 3$ in the $r\theta$-plane.

Example: Graph the curve $r = 1 + 2\sin(2\theta)$ in the $r\theta$-plane.

3.5 (Section 11.4) Graphing Polar Curves in the Cartesian Plane

3.5.1 Plotting Curves in the xy-Plane Based on Their $r\theta$-Graph

When graphing functions of the form $r = f(\theta)$ in the Cartesian plane, the best way is to follow along by graphing it initially in the $r\theta$-plane and then the xy-plane. This is best seen by example.

Example: Graph $r = 1 - \cos(\theta)$ in the Cartesian plane.

Example: Graph $r = 1 + \sin(\theta)$ in the Cartesian plane.

Example Graph $r = \cos(2\theta)$ in the Cartesian plane.

Example: Graph $r = 1 + 2\sin(\theta)$ in the Cartesian plane.

Example: Graph $r^2 = 4\cos(2\theta)$ in the Cartesian plane.

3.5.2 Graphing Polar Curves Using GeoGebra

Using Geogebra Classic 6, there is a way to plot polar curves of the form $r = f(\theta)$ with ease. Below is the command that you use to obtain this.

GeoGebra Polar Graphing

To graph the curve $r = f(\theta)$ over the region $a \leq \theta \leq b$ use the command

```
Curve[(f(t);t),t,a,b]
```

Note that the semicolon is important!

Example: Use GeoGebra Classic 6 to graph the polar curve $r(\theta) = \sin(\theta) + \sin^3(5\theta/2)$.

3.5.3 Calculus in Polar Coordinates

We can use $r = f(\theta)$ to describe a curve parametrically (in terms of a parameter θ) as

$$\begin{cases} x = r\cos(\theta) = f(\theta)\cos(\theta) \\ y = r\sin(\theta) = f(\theta)\sin(\theta) \end{cases}$$

From this use $y' = \dfrac{\dot{y}}{\dot{x}} = \dfrac{dy/d\theta}{dx/d\theta}$ to obtain the following formula for the slope.

Theorem

Let $r = f(\theta)$ be a differentiable curve in polar coordinates. Then the Cartesian slope of the curve at (r, θ) is given by the following.

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

You can also apply the same logic to finding the arc-length by creating a parametrization but it always simplifies to the following formula for a curve $r = f(\theta)$ and is given by the following.

Theorem

Let $r = f(\theta)$ be a differentiable curve in polar coordinates. Then the Arc-Length of the curve over $\alpha \le \theta \le \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Compute the slope of the tangent line to $r = 1 + 2\cos(\theta)$ at $\theta = \pi/4$.

Example: Find the length of the curve $r = 1 - \cos(\theta)$ for $0 \leq \theta \leq 2\pi$.

3.6 (Section 11.5) Area Trapped by Polar Curves

3.6.1 Formulating the Wedged Area Trapped by Polar Curves

Consider a curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$.

To find the area trapped above we take wedges instead of rectangles for simplicity.

We subdivide the angles into equal widths $\Delta\theta_n = \dfrac{\beta - \alpha}{n}$ each with a radius of $r_k = f(\theta_k)$ for some θ_k in each subregion. The formula for the area of each sector is

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_n = \frac{1}{2} f(\theta_k)^2 \Delta\theta_n$$

Then we take $n \to \infty$ and add them up to obtain

$$A = \frac{1}{2} \int_\alpha^\beta f(\theta)^2 d\theta$$

3.6.2 Examples of Computing Area

Example: Compute the area trapped inside of $r = 1 + \cos(\theta)$.

Example: Find the area trapped inside $r = 1$ and $r = 1 - \cos(\theta)$.

3.6.3 Splits in Integration

When integrating between two objects splits are a common occurence

The behaviour changes at $\theta = c$.

Example: Find the area common to the curves $r = 9\cos(\theta)$ and $r = 9\sin(\theta)$.