Student: Arfaz Hossain **Date:** 04/20/22	**Instructor:** Muhammad Awais **Course:** Math 101 A04 Spring 2022	**Assignment:** Practice Questions for Sections 11.4 & 11.5 [Not f

The equation $\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\,\textbf{sin}\,\theta + f(\theta)\,\textbf{cos}\,\theta}{f'(\theta)\,\textbf{cos}\,\theta - f(\theta)\,\textbf{sin}\,\theta}$ gives a formula for the derivative y' of a polar curve $r = f(\theta)$. The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$. Find the slope and concavity of the following curve at the given points.

$r = 8\theta$, $\theta = 2\pi, \frac{7\pi}{2}$

To find the slope, use the first derivative of the function.

So, start by finding $y' = \left.\frac{dy}{dx}\right|_{(r,\theta)}$ using the formula given in the problem statement.

The formula involves $f(\theta)$ and $f'(\theta)$. Identify these functions.

$f(\theta) = 8\theta$
$f'(\theta) = 8$

Substitute $f(\theta) = 8\theta$ and $f'(\theta) = 8$ into the formula for $y' = \left.\frac{dy}{dx}\right|_{(r,\theta)}$.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(r,\theta)} &= \frac{f'(\theta)\,\textbf{sin}\,\theta + f(\theta)\,\textbf{cos}\,\theta}{f'(\theta)\,\textbf{cos}\,\theta - f(\theta)\,\textbf{sin}\,\theta} \\ &= \frac{8\,\textbf{sin}\,\theta + 8\theta\,\textbf{cos}\,\theta}{8\,\textbf{cos}\,\theta - 8\theta\,\textbf{sin}\,\theta} \\ &= \frac{\textbf{sin}\,\theta + \theta\,\textbf{cos}\,\theta}{\textbf{cos}\,\theta - \theta\,\textbf{sin}\,\theta}\end{aligned}$$

Evaluate this at $\theta = 2\pi$.

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{(r,\theta)} &= \frac{\textbf{sin}\,\theta + \theta\,\textbf{cos}\,\theta}{\textbf{cos}\,\theta - \theta\,\textbf{sin}\,\theta} \\ \left.\frac{dy}{dx}\right|_{\theta = 2\pi} &= \frac{\textbf{sin}\,2\pi + 2\pi\,\textbf{cos}\,2\pi}{\textbf{cos}\,2\pi - 2\pi\,\textbf{sin}\,2\pi} \\ &= \frac{0 + 2\pi \bullet 1}{1 - 2\pi \bullet 0} \\ &= 2\pi\end{aligned}$$

This means the slope at $\theta = 2\pi$ is 2π.

To find the concavity, use the second derivative of the function.

In particular, the graph is concave up where $\frac{d^2y}{dx^2} > 0$ and is concave down where $\frac{d^2y}{dx^2} < 0$.

The formula for $\frac{d^2y}{dx^2}$ involves the derivative $\frac{dy'}{d\theta}$. Find this derivative.

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(y')$$
$$= \frac{d}{d\theta}\left(\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right)$$
$$= \frac{\theta^2 + 2}{(\cos\theta - \theta\sin\theta)^2}$$

The formula for $\frac{d^2y}{dx^2}$ also involves the derivative $\frac{dx}{d\theta}$. To find this derivative, first express x in terms of θ.

$$x = 8\theta\cos\theta$$

Find $\frac{dx}{d\theta}$.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(8\theta\cos\theta)$$
$$= 8\cos\theta - 8\theta\sin\theta$$

Substitute $\frac{dy'}{d\theta} = \frac{\theta^2 + 2}{(\cos\theta - \theta\sin\theta)^2}$ and $\frac{dx}{d\theta} = 8\cos\theta - 8\theta\sin\theta$ into the formula for $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$
$$= \frac{\dfrac{\theta^2 + 2}{(\cos\theta - \theta\sin\theta)^2}}{8\cos\theta - 8\theta\sin\theta}$$
$$= \frac{\theta^2 + 2}{8(\cos\theta - \theta\sin\theta)^3}$$

Evaluate this at $\theta = 2\pi$.

$$\left.\frac{d^2y}{dx^2}\right|_{(r,\theta)} = \frac{\theta^2 + 2}{8(\cos\theta - \theta\sin\theta)^3}$$
$$\left.\frac{d^2y}{dx^2}\right|_{\theta = 2\pi} = \frac{(2\pi)^2 + 2}{8(\cos 2\pi - 2\pi\sin 2\pi)^3}$$
$$= \frac{4\pi^2 + 2}{8(1 - 2\pi \cdot 0)^3}$$
$$= \frac{2\pi^2 + 1}{4}$$

Notice that $\frac{2\pi^2 + 1}{4} > 0$. This means that at $\theta = 2\pi$, the curve is concave up.

Now consider the second point, $\theta = \frac{7\pi}{2}$.

Find the slope at $\theta = \frac{7\pi}{2}$ using the formula for $y' = \left.\frac{dy}{dx}\right|_{(r,\theta)}$ found earlier.

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{7\pi}{2}} = \frac{\sin\frac{7\pi}{2} + \frac{7\pi}{2}\cos\frac{7\pi}{2}}{\cos\frac{7\pi}{2} - \frac{7\pi}{2}\sin\frac{7\pi}{2}}$$

$$= \frac{-1 + \frac{7\pi}{2}\bullet 0}{0 - \frac{7\pi}{2}\bullet(-1)}$$

$$= -\frac{2}{7\pi}$$

The slope at $\theta = \frac{7\pi}{2}$ is $-\frac{2}{7\pi}$.

To find the concavity at $\theta = \frac{7\pi}{2}$, first evaluate the formula for $\frac{d^2y}{dx^2}$ found earlier at $\theta = \frac{7\pi}{2}$.

$$\left.\frac{d^2y}{dx^2}\right|_{(r,\theta)} = \frac{\theta^2 + 2}{8(\cos\theta - \theta\sin\theta)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta = \frac{7\pi}{2}} = \frac{\left(\frac{7\pi}{2}\right)^2 + 2}{8\left(\cos\frac{7\pi}{2} - \frac{7\pi}{2}\sin\frac{7\pi}{2}\right)^3}$$

$$= \frac{\frac{49\pi^2}{4} + 2}{8\left(0 - \frac{7\pi}{2}\bullet(-1)\right)^3}$$

$$= \frac{49\pi^2 + 8}{1372\pi^3}$$

Notice that $\frac{49\pi^2 + 8}{1372\pi^3} > 0$. This means that at $\theta = \frac{7\pi}{2}$, the curve is concave up.