Math 110 - Homework 11
Topic: Diagonalization and orthogonality

Due at 6:00pm (Pacific) on Friday, December 3, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 5.1 of the online textbook, as well as the supplemental questions about complex numbers posted on Brightspace.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

This week's material does not require any new MATLAB commands.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $A = \begin{bmatrix} 9 & -4 & -8 \\ 12 & -7 & -8 \\ 0 & 0 & -3 \end{bmatrix}$. Either find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ or explain why no such matrices exist. If you do find a P and D you do not need to calculate P^{-1}.

Solution: We begin by finding the eigenvalues (as usual, students needed to show steps in this question because it is in Part I).

$$\det(A - \lambda I) = \det \begin{bmatrix} 9-\lambda & -4 & -8 \\ 12 & -7-\lambda & -8 \\ 0 & 0 & -3-\lambda \end{bmatrix} = -(\lambda+3)^2(\lambda-5).$$

The eigenvalues are -3, with algebraic multiplicity 2, and 5, with algebraic multiplicity 1.

Next we look for a basis for each eigenspace. Starting with $\lambda = -3$ we have:

$$A - (-3)I = \begin{bmatrix} 12 & -4 & -8 \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus vectors in E_{-3} are of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x - \frac{1}{3}y - \frac{2}{3}z = 0$. That is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/3)y + (2/3)z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for E_{-3} is thus $\left\{ \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$. In particular,

$$\operatorname{geo}(-3) = \dim(E_{-3}) = 2 = \operatorname{alg}(-3),$$

so so far our matrix could still be diagonalizable. For future use we note that rescaling a basis still gives us a basis, so another (more pleasant looking) basis for E_{-3} is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Now we turn to the eigenvalue $\lambda = 5$.

$$A - 5I = \begin{bmatrix} 4 & -4 & -8 \\ 12 & -12 & -8 \\ 0 & 0 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Vectors in E_5 therefore have the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = y$ and $z = 0$. Thus a basis for E_5 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For this eigenvalue we have

$$\operatorname{geo}(5) = \dim(E_5) = 1 = \operatorname{alg}(5).$$

Every eigenvalue has the same algebraic as geometric multiplicity, so A is diagonalizable. To build D we put the eigenvalues on the diagonal:

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

and then to build P we put the basis vectors for the eigenspaces as the columns in an order corresponding to the order we chose for the eigenvalues in D:

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

Then by results from class, $A = PDP^{-1}$.

2. Let L be the line in $\mathbb{R}^3$ that passes through the points $(0,0,0)$ and $(3,3,-1)$.

   (a) Find a basis for $L^{\perp}$.

Solution: First we find the direction vector for L, which is $\vec{d} = \begin{bmatrix} 3-0 \\ 3-0 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$. Thus $L = \operatorname{span}\left(\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}\right)$. By a result from class, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $L^{\perp}$ if and only if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = 0$. That is, if and only if

$$3x + 3y - z = 0.$$

Rearranging this as $z = 3x + 3y$ we see that every vector in $L^{\perp}$ has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 3x+3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

As the two vectors appearing here are independent, this shows that a basis for $L^{\perp}$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}$.

   (b) Describe $L^{\perp}$ geometrically.

Solution: $L^{\perp}$ is the plane through the origin that is orthogonal to the line (i.e., it has normal vector $\vec{d}$).

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. For both parts of this question assume that the matrices are 3×3.

   (a) If A is an invertible matrix is it necessarily true that A is diagonalizable? If so, explain why. If not, give a specific example of a matrix A that is invertible but not diagonalizable (and explain why your example has these properties).

   Solution: Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A is triangular, so $\det(A)$ is the product of the diagonal entries, thus $\det(A) = 1$. In particular $\det(A) \neq 0$ so A is invertible. We next show that A is not diagonalizable. Again, since A is triangular we can just read the eigenvalues off the diagonal; the only eigenvalue is 1, and $\text{alg}(1) = 3$. Now we calculate:

   $$A - 1I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   From here we see that $\text{rank}(A) = 1$, so

   $$\text{geo}(1) = \dim(E_1) = \dim(\text{null}(A - 1I)) = 3 - \text{rank}(A) = 2.$$

   Thus $\text{geo}(1) \neq \text{alg}(1)$, so A is not diagonalizable.

   (b) If B is a diagonalizable matrix is it necessarily true that B is invertible? If so, explain why. If not, give a specific example of a matrix B that is diagonalizable but not invertible (and explain why your example has these properties).

   Solution: Let $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then B is a diagonal matrix, so it is certainly diagonalizable, and $\det(B) = 0$, so B is not invertible.

2. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vector $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$.

Solution: We start by finding a basis for $\mathbb{R}^4$ that contains $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$. To do that we need to come up with a total of four linearly independent vectors, one of which is the given one. As a guess, we consider $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. If we put these vectors as the columns of a matrix and ask

MATLAB to row reduce it we get I_4, so these four vectors are indeed linearly independent, and hence form a basis for $\mathbb{R}^4$. Define:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \qquad \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We now apply Gram-Schmidt to these vectors to transform them into an orthogonal basis. The first step is

$$\vec{w_1} = \vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}.$$

This step has already ensured that our orthogonal basis will contain the vector we wanted! For the remainder of the steps we have MATLAB carry out the necessary computations.

$$\vec{w_2} = \vec{v_2} - \mathrm{proj}_{\vec{w_1}}(\vec{v_2}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/15 \\ -2/15 \\ 1/5 \\ -1/15 \end{bmatrix}.$$

To make things look nicer we'll rescale this vector by 15 and in fact take

$$\vec{w_2} = \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}.$$

Next,

$$\begin{aligned} \vec{w_3} &= \vec{v_3} - \mathrm{proj}_{\vec{w_1}}(\vec{v_3}) - \mathrm{proj}_{\vec{w_2}}(\vec{v_3}) \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}} \right) \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 5/7 \\ 3/7 \\ -1/7 \end{bmatrix} \end{aligned}$$

Again, we rescale to make the numbers nicer, and choose

$$\vec{w_3} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix}.$$

Last, and most unpleasantly to look at, we have:

$$\begin{aligned}
\overrightarrow{w_4} &= \overrightarrow{v_4} - \operatorname{proj}_{\overrightarrow{w_1}}(\overrightarrow{v_4}) - \operatorname{proj}_{\overrightarrow{w_2}}(\overrightarrow{v_4}) - \operatorname{proj}_{\overrightarrow{w_3}}(\overrightarrow{v_4}) \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}} \right) \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix}} \right) \begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1/10 \\ 3/10 \end{bmatrix}
\end{aligned}$$

One last rescaling gives us

$$\overrightarrow{w_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Thus our orthogonal basis for $\mathbb{R}^4$ is

$$\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 5 \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Note that many, many other solutions are possible, depending on the vectors $\overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_3}$ chosen, how they are ordered, and whether or not the rescaling step is done. As long as you ended up with four orthogonal vectors and $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$ is one of them you have a correct final answer.