Q.1 (a) $\displaystyle\int_2^{\infty} \frac{x}{x^2+\sin x}\,dx$

Let $f(x) = \dfrac{x}{x^2+\sin x}$ , $g(x) = \dfrac{1}{x}$ ; both $f$ and $g$ are continuous on $[2,\infty)$

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{x^2}{x^2+\sin x} = \lim_{x\to\infty} \frac{1}{1+\frac{\sin x}{x^2}} = 1$$

Therefore, both $\displaystyle\int_2^{\infty} f(x)\,dx$ and $\displaystyle\int_2^{\infty} g(x)\,dx$ converge or both diverge. Since $\displaystyle\int_2^{\infty} \frac{1}{x}\,dx$ diverges, so does $\displaystyle\int_2^{\infty} \frac{x}{x^2+\sin x}\,dx$.

(b) $$\int_{-1}^{\infty} \frac{1}{\sqrt{x^4+1}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{x^4+1}}\,dx + \int_{1}^{\infty} \frac{1}{\sqrt{x^4+1}}\,dx = I_1 + I_2$$

Since $\frac{1}{\sqrt{x^4+1}}$ is positive, $I_1 = \int_{-1}^{1} \frac{1}{\sqrt{x^4+1}}\,dx$ is equal to some finite number.

$$x^4+1 > x^4$$

$$\Rightarrow \sqrt{x^4+1} > \sqrt{x^4}$$

$$\Rightarrow \frac{1}{\sqrt{x^4+1}} < \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$$

Convergence of $I_1$ and $I_2$ imply the convergence of $I = I_1 + I_2$

$$\Rightarrow I_2 = \int_{1}^{\infty} \frac{1}{\sqrt{x^4+1}}\,dx < \int_{1}^{\infty} \frac{1}{x^2}\,dx \;;$$

$\int_{1}^{\infty} \frac{1}{x^2}\,dx$ converges as its a p-integral with $p=2$.

Therefore, $I_2$ also converges by Direct Comparison.

(c) $\displaystyle\int_1^{\infty} \frac{1}{x+\cos^2 x}\,dx$

Taking $f(x) = \dfrac{1}{x+\cos^2 x}$, $g(x) = \dfrac{1}{x}$ ; $f$ and $g$ are continuous on $[1,\infty)$

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{1}{1+\frac{\cos^2 x}{x}} = 1$$

Therefore, both $\displaystyle\int_1^{\infty} f(x)\,dx$ and $\displaystyle\int_1^{\infty} g(x)\,dx$ converge or both diverge. Since $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges, so does the integral $\displaystyle\int_1^{\infty} \frac{1}{x+\cos^2 x}\,dx$.

(d). $$\int_0^\infty \frac{x\cos^2 x}{e^x}\,dx$$

Notice $\dfrac{x\cos^2 x}{e^x} \leq \dfrac{x}{e^x}$, and

$$\int_0^\infty \frac{x}{e^x}\,dx = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx\right]$$

$$= \lim_{b\to\infty}\left[-xe^{-x}\Big|_0^b - e^{-x}\Big|_0^b\right]$$

$$\Rightarrow \int_0^\infty \frac{x}{e^x}\,dx = 1$$

Therefore, since $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$ converges, so does $\displaystyle\int_0^\infty \frac{x\cos^2 x}{e^x}\,dx$.

(e) $\int_0^1 \tan(2\pi x)\, dx.$

let $u = 2\pi x$ when; $x = 0,\ u = 0$

$\Rightarrow \frac{du}{2\pi} = dx,$ $x = 1,\ 2\pi$

$$\Rightarrow \int_0^1 \tan(2\pi x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} \tan(u)\, du$$

Since $\tan(\theta)$ has VAs at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$,

the above integral does not converge.

(f). $$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} x e^{-x^2}\,dx + \int_{0}^{\infty} x e^{-x^2}\,dx$$

$$= \underbrace{\lim_{a\to\infty} \int_{-a}^{0} x e^{-x^2}\,dx}_{} + \lim_{b\to\infty} \int_{0}^{b} x e^{-x^2}\,dx$$

Let $u = -x$

when; $x = -a$, $u = a$

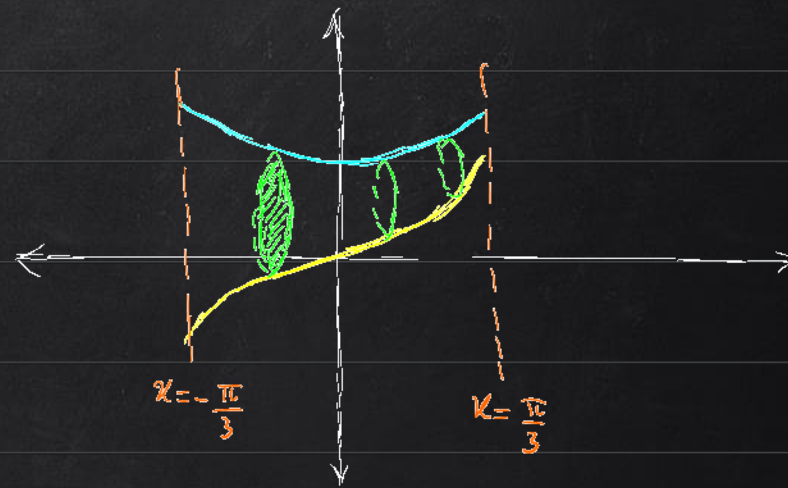
$x = 0$, $u = 0$

$$\Rightarrow \lim_{a\to\infty} \int_{-a}^{0} x e^{-x^2}\,dx = \lim_{a\to\infty} \int_{a}^{0} (-u)\, e^{-u^2} (-du) = \lim_{a\to\infty} \int_{a}^{0} u e^{-u^2}\,du$$

$$= -\lim_{a\to\infty} \int_{0}^{a} u e^{-u^2}\,du$$

Therefore, $$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = -\lim_{a\to\infty} \int_{0}^{a} u e^{-u^2}\,du + \lim_{b\to\infty} \int_{0}^{b} x e^{-x^2}\,dx = 0.$$

**Q. 2.** **(a).** Diameter of a typical cross-section $= \sec x - \tan x$.




$\Rightarrow$ Radius $= \dfrac{\sec x - \tan x}{2}$

$\Rightarrow$ Area of the disk $= \pi \left[\dfrac{\sec x - \tan x}{2}\right]^2$

$$= \frac{\pi}{4}\left[2\sec^2 x - 2\sec x \tan x - 1\right]$$

If $dx$ is the thickness of the disk, then

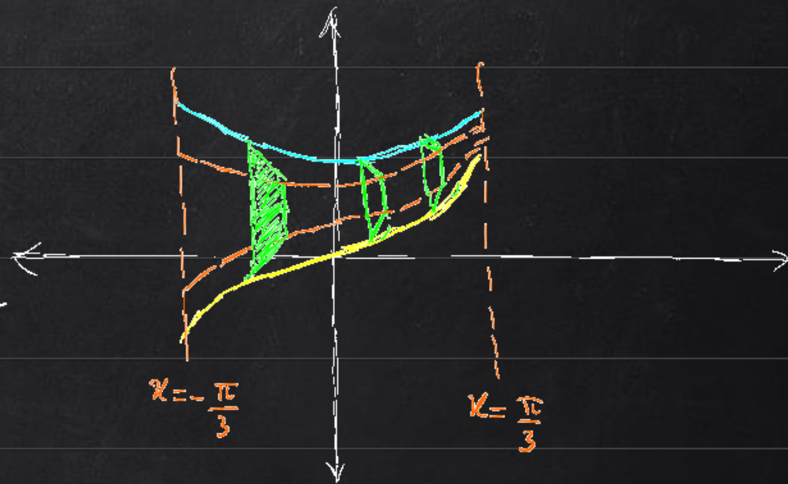
$$\text{Volume} = \frac{\pi}{4}\int_{-\pi/3}^{\pi/3} \left[2\sec^2 x - 2\sec x \tan x - 1\right] dx = \frac{\pi}{2}\left[\tan x - \sec x - \frac{x}{2}\right]_{-\pi/3}^{\pi/3}$$

$$= \sqrt{3}\,\pi - \frac{\pi^2}{6}$$

(b).

Length of base = $\sec x - \tan x$.

Area of the base = $(\sec x - \tan x)^2$




If $dx$ is the thickness;

Volum of cross-section $= (\sec x - \tan x)^2 dx = (2\sec^2 x - 2\sec x \tan x - 1)\, dx$

Therefore;

$$\text{Volume} = \int_{-\pi/3}^{\pi/3} [2\sec^2 x - 2\sec x \tan x - 1]\, dx = \Big[2\tan x - 2\sec x - x\Big]_{-\pi/3}^{\pi/3}$$

$$= 4\sqrt{3} - \frac{2\pi}{3}$$