Math101 - Midterm I

Version B - February 12

Sections: [A01 - A05]

Last Name, First Name : ________________________________

Lecture Section : ______________________________________

Student ID : ____

- This examination has 15 problems [12 multiple choice and 3 long answer questions], worth a total of 30.
- It consists of 10 pages, including this one. Make sure your exam copy has the correct number of pages.
- You have 90 minutes to complete the exam.
- Answer each question in the appropriate space immediately below that question. Use the backsides as an extra space for rough work. Show all your work to get full marks. Unsupported correct answers may get zero marks. Also, you must fill the bubble sheet for multiple choice questions.
- No Textbooks, No Class Notes, and No Formula Sheets allowed.
- Only **Sharp EL-510R** or its variant calculator is allowed. No other calculator allowed.
- **No Electronic Communication Device of any sort (e.g. cell phones, laptops, iPods, translators, pagers) are allowed during the exam.**

Question	$1-12$[12]	14[6]	15[6]	16[6]	**Total**
Marks					

Q1. Let $f(x)$ be a continuous function whose integral values are given by the table below:

	$\int_2^5 f(x)dx$	$\int_2^4 f(x)dx$	$\int_5^{25} f(x)dx$	$\int_4^{25} f(x)dx$	$\int_2^{25} f(x)dx$	$\int_4^5 f(x)dx$
Value	3	4	8	7	4	1

Using this table, compute $\displaystyle\int_2^5 4tf(t^2)dt$.

(a) 2; (b) 14; (c) 8; (d) 4; (e) 12

Let $x = t^2$

$\Rightarrow dx = 2t\,dt$

when $t = 2, x = 4$

$t = 5, x = 25$

$\Rightarrow \displaystyle\int_2^5 4t f(t^2)\,dt = 2\int_4^{25} f(x)\,dx$

$= 2(7)$

$= 14$

Q2. During a test a student provided the incorrect answer to the integral

$$\int_1^2 2\sqrt{x^2 - 4x + 4}dx$$

given by

$$\int_1^2 2\sqrt{x^2-4x+4}dx = \int_1^2 2\sqrt{(x-2)^2}dx = \int_1^2 2(x-2)dx = \left(x^2 - 2x\right)\Big|_1^2 = 0-(-1) = 1$$

Determine the correct answer to the above integral by fixing the student's mistake.

(a) 0; (b) -1; (c) 1; (d) -2; (e) 3

$\displaystyle 2\int_1^2 \sqrt{(x-2)^2}\,dx = 2\int_1^2 |x-2|\,dx = -2\int_1^2 (x-2)\,dx$

$= -2\left(\frac{x^2}{2} - 2x\right)\Big|_1^2$

$= -2\left[\left(\frac{4}{2} - 4\right) - \left(\frac{1}{2} - 2\right)\right]$

$= -2\left[-2 + \frac{3}{2}\right]$

$= +1$

Q3. Suppose that $f(x)$ is a twice differentiable function whose values are given by the following table:

x	$f(x)$	$f'(x)$
0	6	5
2	9	8

Using this table, compute $\int_0^2 3xf''(x)dx$.

(a) 13; (b) 39; (c) 26; (d) 9; (e) 6

Let $u = 3x$, $dv = f''(x)\,dx$

$\Rightarrow$ $du = 3\,dx$, $v = f'(x)$

$$\Rightarrow \int_0^2 3xf''(x)\,dx = 3xf'(x)\Big|_0^2 - 3\int_0^2 f'(x)\,dx$$
$$= 6f'(2) - 3\left[f(x)\right]_0^2$$
$$= 48 - 3[f(2) - f(0)]$$
$$= 48 - 3[9-6] = 39$$

Q4. A certain substitution gives $\int \sin^3(x)\cos^n(x)dx = \int (u^7 - u^5)du$. Determine n.

(a) 1; (b) 3; (c) 4; (d) 5; (e) 7

$$\int \sin^3 x \cos^n x\,dx = \int \sin^2 x \cos^n x \sin x\,dx$$
$$= \int (1-\cos^2 x)\cos^n x\,(\sin x)\,dx$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\Rightarrow \int \sin^3 x \cos^n x\,dx = -\int (1-u^2)\,u^n\,du$$
$$= \int (u^{n+2} - u^n)\,du$$

n must be 5.

Q5. Evaluate $\int_0^{\pi} 8\cos^4(x)dx$.

(a) 3π; (b) π; (c) $\frac{3\pi}{2}$; (d) $\frac{5\pi}{2}$; (e) $\frac{\pi}{2}$

$$8\int_0^{\pi} \cos^2 x \cdot \cos^2 x \, dx = 8\int_0^{\pi} \frac{1+\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx$$

$$= 2\int_0^{\pi} [1+\cos(2x)]^2 \, dx$$

$$= 2\int_0^{\pi} 1+\cos^2(2x)+2\cos(2x) \, dx$$

$$= 2\int_0^{\pi} 1+\frac{1+\cos(4x)}{2}+2\cos(2x) \, dx$$

$$= 2\left[x+\frac{x}{2}+\frac{\sin(4x)}{8}+\sin(2x)\right]_0^{\pi}$$

$$= 3\pi$$

Q6. For what values of k does the integral $\int_e^{\infty} \frac{dx}{x(\ln(x))^{k-2}}$ converge?

(a) $k \geq -1$; (b) $k > -1$; (c) $k \geq 3$; (d) $k > 3$; (e) $k \geq 1$

$$= \lim_{b\to\infty} \int_e^{b} \frac{(\ln x)^{-(k-2)}}{x} \, dx$$

let $u = \ln x$, when $x = e$, $u = 1$

$\Rightarrow du = \frac{1}{x}dx$ $x \to \infty$, $u \to \infty$

$$= \lim_{b\to\infty} \int_1^{b} (u)^{-(k-2)} \, du \rightarrow (*)$$

$$= \lim_{b\to\infty} \left.\frac{u^{-k+3}}{-k+3}\right|_1^b = \lim_{b\to\infty} \frac{+1}{-k+3}\left[\frac{1}{b^{k-3}} - 1\right]$$

we must have $k-3>0$

$\Rightarrow k>3$

or, by using $(*)$ as a p-integral and taking $k+2>1$ for convergence.

Q7. If you split $\dfrac{6x^4 - 8x^3 + 7}{x^2(x^2+3)^3}$ into partial fractions, how many coefficients would you have to compute?

(a) 2; (b) 5; (c) 7; (d) 8; (e) none of these.

$$\frac{6x^4 - 8x^3 + 7}{x^2(x^2+3)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+3)} + \frac{Ex+F}{(x^2+3)^2} + \frac{Gx+H}{(x^2+3)^3}$$

8 Constants on the R.H.S

Q8. The indefinite integral of $f(x) = \dfrac{5x^2 + x + 5}{x^3 + x}$ contains, besides the integration constant,

(a) only logarithms; (b) logarithms and arctans; (c) arctans and rational functions; (d) only arctans; (e) only rational functions

$$\frac{5x^2+x+5}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad 5x^2+x+5 = A(x^2+1) + (Bx+C)x$$

$$= (A+B)x^2 + Cx + A$$

$$\Rightarrow \quad A = 5, \quad C = 1, \quad A+B = 5$$

$$\Rightarrow B = 0.$$

$$\Rightarrow \quad \frac{3x^2+x+3}{x(x^2+1)} = \frac{5}{x} + \frac{1}{x^2+1}$$; logs and arctans.

Q9. Compute the definite integral: $\displaystyle\int_{\ln(2)}^{\ln(3)} \frac{e^x dx}{e^{2x}+5e^x+6}$.

(a) $\ln(25/24)$; (b) $\ln(24/25)$; (c) $\ln(12/7)$; (d) $\ln(7/12)$; (e) None of these

Let $u = e^x$

$du = e^x dx$

$$\int_{\ln(2)}^{\ln(3)} \frac{e^x dx}{(e^x)^2+5e^x+6} = \int_2^3 \frac{1}{u^2+5u+6}\,du$$

$$= \int_2^3 \frac{1}{(u+3)(u+2)}\,du$$

$$\frac{1}{(u+3)(u+2)} = \frac{A}{u+3} + \frac{B}{u+2}$$

$\Rightarrow \quad 1 = (A+B)u + (2A+3B)$

$\Rightarrow \quad A+B=0, \quad 2A+3B=1$

these two give $A=-1, B=1$

$$= \int_2^3 \left(\frac{-1}{u+3} + \frac{1}{u+2}\right) du$$

$$= \Big[-\ln(u+3) + \ln(u+2)\Big]_2^3 = \left[\ln\left|\frac{u+2}{u+3}\right|\right]_2^3$$

$$= \ln\left(\frac{5}{6}\right) - \ln\left(\frac{4}{5}\right)$$

$$= \ln\left[\frac{5/6}{4/5}\right]$$

$$= \ln\left(\frac{25}{24}\right)$$

Q10. Which one is the value of $\displaystyle\int_1^e \frac{dx}{x(1+\ln(x))^6}$.

(a) 0; (b) $\frac{1}{5}$; (c) $\frac{31}{160}$; (d) $-\ln(31)$; (e) $-\frac{1}{5}(e^{-5}-1)$

Let $u = 1+\ln x$

$du = \frac{1}{x}\,dx$

$$\Rightarrow \quad \int_1^e \frac{dx}{x(1+\ln x)^6} = \int_1^2 \frac{du}{u^6}$$

$$= -\frac{1}{5u^5}\Big|_1^2$$

$$= -\frac{1}{5}\left[\frac{1}{32} - 1\right]$$

$$= \frac{31}{160}$$

Q11. Using the substitution $x + 1 = 3\sin\theta$, we can convert

$$\int 9(8 - 2x - x^2)^{-\frac{3}{2}}\, dx$$

to a trigonometric integral, given by:

(a) $\int \tan^2\theta d\theta$; (b) $\int \sec^3\theta d\theta$; (c) $\int \sin\theta d\theta$; (d) $\int \sec^2\theta d\theta$; (e) None of these.

$$\int 9\left[8 - (x^2+2x)\right]^{-\frac{3}{2}} dx = 9\int \left[8 - (x^2+2x+1-1)\right]^{-3/2} dx$$

$$= 9\int \left[8 - (x+1)^2 + 1\right]^{-3/2} dx$$

$x+1 = 3\sin\theta$

$\Rightarrow dx = 3\cos\theta\, d\theta$

$$= 9\int \left[9 - (x+1)^2\right]^{-3/2} dx$$

$$= 9.3\int \left(9 - 9\sin^2\theta\right)^{-3/2} \cos\theta\, d\theta$$

$$= 9.3\int 9^{-3/2}\left(1 - \sin^2\theta\right)^{-3/2} \cos\theta\, d\theta$$

$$= 27\int \frac{1}{27 \cdot \cos^3\theta} \cos\theta\, d\theta$$

$$= \int \sec^2\theta\, d\theta.$$

Q12. Find the arc-length of the curve whose equation is $y = \frac{2}{3}(x^2+1)^{3/2}$ for x in the interval $[0, 4]$. *Round your answer to two decimal places.*

(a) 8.73; (b) 10.01; (c) ~~40.66~~; 46.67 (d) -6.83 (e) 7.33

Here $y'(x) = \frac{2}{3} \cdot \frac{3}{2} (x^2+1)^{1/2}(2x)$

$\Rightarrow [y'(x)]^2 = (x^2+1)(4x^2)$

$\Rightarrow 1 + (y')^2 = (x^2+1)(4x^2) + 1 = 4x^4 + 4x^2 + 1 = (2x^2+1)^2$

Therefore Arc length $= \int_0^4 \sqrt{1+(y')^2}\, dx = \int_0^4 (2x^2+1)\, dx$

$$= \left[\frac{2x^3}{3} + x\right]_0^4$$

$$= \frac{128}{3} + 4$$

$$= \frac{140}{3} \simeq 46.67$$

Q13. Evaluate $\int_0^{\pi} 4x\cos^2(x)dx$.

Rewrite the above using the identity

$$\sin^2(x) = \frac{1+\cos(2x)}{2}.$$

Then the integral transforms as

$$\int_0^{\pi} 4x\left(\frac{1+\cos(2x)}{2}\right)dx = \int_0^{\pi}(2x+2x\cos(2x))dx = \int_0^{\pi} 2xdx + \int_0^{\pi} 2x\cos(2x)dx$$

The first integral is easy enough to compute. The second requires integrations by parts as

$$\begin{array}{ccc} u = 2x & & du = 2dx \\ & \searrow & \uparrow \\ dv = \cos(2x)dx & & v = \tfrac{1}{2}\sin(2x) \end{array}$$

yielding

$$\begin{aligned} I &= x^2\Big|_0^{\pi} + \left\{x\sin(2x)\Big|_0^{\pi} - \int_0^{\pi}\sin(2x)dx\right\} \\ &= \pi^2 - 0^2 + \left\{0 - 0 + \frac{1}{2}\cos(2x)\Big|_0^{\pi}\right\} \\ &= \pi^2 + \frac{1}{2}(\cos(2\pi) - \cos(0)) \\ &= \pi^2 + \frac{1}{2}(1-1) \\ &= \pi^2 \end{aligned}$$

Q14. Determine whether the following integral

$$\int_1^\infty \frac{2-\sin(5x)}{\sqrt{x}}dx$$

converges or diverges. Be sure to justify all inequalities and state all theorems. **No marks are given for the answer of "convergence" or "divergence". All marks are based on your argument leading to the correct answer**.

Cosine is a bounded function satisfying $-1 \leq \sin(u) \leq 1$ for any u. As such, we see that the integrand has a lower bound over $[1, \infty)$ given by

$$\frac{2-\sin(5x)}{\sqrt{x}} \geq \frac{2-1}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

where $x \geq 1$. However, the integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges as it is a type I p-integral with $p = 1/2 \leq 1$. Accordingly, by the direct comparison test the original integral diverges as its integrand is bounded below by a function whose integral diverges on those bounds.

Q15. Recall that a sphere of radius $a > 0$ has volume $V = \frac{4}{3}\pi a^3$. Verify this result in two different ways:

a) Using the method of Disks (and not using any other method), find the volume of the solid of revolution that is obtained by revolving the semi-circle $y = \sqrt{a^2 - x^2}$ around the x-axis:

$$V = \int_{-a}^{a} \pi \underbrace{(a^2 - x^2)}_{\text{disk radius}^2} dx = \pi \cdot \left(a^2 x - \frac{1}{3}x^3\right)\Bigg|_{x=-a}^{x=a} = \frac{4}{3}\pi a^3.$$

b) Using the method of Cylindrical Shells (and not using any other method), find the volume of the solid of revolution that is obtained by revolving the semi-circle $x^2 + y^2 = a^2$, $x \geq 0$, around the y-axis:

$$V = \int_0^a 2\pi \cdot \underbrace{x}_{\text{shell radius}} \cdot \underbrace{2\sqrt{a^2 - x^2}}_{\text{shell height}} dx = 4\pi \int_0^a x\sqrt{a^2 - x^2} dx$$

Here one hopes nobody tries IBP which makes this a bit of a mess either way one chooses the factors. Instead, one either subs the "obvious" $u = a^2 - x^2$ to get

$$= 4\pi \int_{a^2}^{0} u^{1/2} \frac{du}{-2} = -\frac{4\pi}{3} u^{3/2} \Bigg|_{u=a^2}^{u=0} = \frac{4\pi}{3} a^3$$

or a trig substitution $x = a\sin(\theta)$ makes it

$$= 4\pi \int_0^{\pi/2} a\sin(\theta) \cdot a\cos(\theta) \cdot a\cos(\theta) d\theta = 4\pi a^3 \int_0^{\pi/2} \cos^2(\theta)\sin(\theta) d\theta$$

$$= 4\pi a^3 \cdot \left(-\frac{1}{3}\cos^3(\theta)\right)\Bigg|_{\theta=0}^{\theta=\pi/2} = \frac{4\pi}{3} a^3$$

– Rough Work –