Math 110 - Homework 4
Topic: Lines and planes in $\mathbb{R}^3$

Due at 6:00pm (Pacific) on Friday, October 8, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 2.3 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

This week's material does not include new computations, just new ways of understanding computations we've already done. As a result, there are no new MATLAB commands needed this week.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Find a vector form equation for the plane in $\mathbb{R}^3$ with general form equation

$$3x - 5y + z = 2.$$

Solution: There are many ways to solve this problem. Perhaps the easiest is to re-write the general equation as $z = 2 - 3x + 5y$. Then for a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on the plane we must have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2 - 3x + 5y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

This is a vector equation for the plane. It is certainly not the only one!

2. Find general form equations for the line in $\mathbb{R}^3$ that passes through $(0, 2, 1)$ and $(1, -2, 2)$.

Solution: From the data we are given the easiest thing to find is the direction vector on the line, which is $\overrightarrow{d} = \begin{bmatrix} 1-0 \\ -2-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$. To get to general form we need two vectors, $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, each of which is orthogonal to $\overrightarrow{d}$. A vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that is orthogonal to $\overrightarrow{d}$ satisfies:

A = First Vector, B = Second Vector
V = [x; y; z]
n = normal vector, d = directional vector

1. For finding the Normal Vector Co-Ordinates,
   n.d1 = n.d2 = 0
2. For finding the General Form,
   V . n = A . n

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = a - 4b + c.$$

That is, we may choose any $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as long as $c = -a + 4b$. To get two such answers easily we'll take $a = 1, b = 0$ to get $\overrightarrow{n_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $a = 0, b = 1$ to get $\overrightarrow{n_2} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$. In normal form our line is therefore:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

Expanding these dot products gives us the general equations for the line:

$$x - z = -1$$

$$y + 4z = 6$$

This answer is not unique, as we could have made different choices for our normal vectors.

3. Let L_1 be the line in $\mathbb{R}^3$ defined by the general form equations:

$$x + y - z = 1$$

$$x + 4y = 2$$

Let L_2 be the line in $\mathbb{R}^3$ that contains the points $(1, 2, 3)$ and $(6, -1, 4)$. There is exactly one plane in $\mathbb{R}^3$ that contains both L_1 and L_2. Find a general form equation for this plane.

Solution: There are several ways to approach this question. From the given information it is probably easiest to find two direction vectors for the plane, and then convert from there to a normal vector, and then get to a general equation. One direction vector is fairly easy to find: Take the direction vector along L_2, which is $\overrightarrow{d_2} = \begin{bmatrix} 6-1 \\ -1-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

For the other direction vector we have a few options. One is to convert the description of L_1 into vector form and use the direction vector obtained that way. Another option is to find a point on L_1, and then use that point together with one of the points on L_2 to get another direction on the plane. This latter option is appealing, because we can find by inspection that $(2, 0, 1)$ satisfies the general equations of L_1. Using that point, together with $(1, 2, 3)$ we can take $\overrightarrow{d_1} = \begin{bmatrix} 1-2 \\ 2-0 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Now we want a normal vector for the plane, which will be a vector $\overrightarrow{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\overrightarrow{n} \cdot \overrightarrow{d_1} = 0 = \overrightarrow{n} \cdot \overrightarrow{d_2}$. That is, we need

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -a + 2b + 2c$$

$$0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = 5a - 3b + c$$

Solving this system, we find that $\overrightarrow{n} = \begin{bmatrix} 8 \\ 11 \\ -7 \end{bmatrix}$ (or any non-zero scalar multiple of this) works. Therefore, in normal form our plane is:

$$\begin{bmatrix} 8 \\ 11 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Finally, if we expand the dot products we get the general equation for the plane:

$$8x + 11y - 7z = 9.$$

Every correct answer is a non-zero multiple of this equation.

Part II: Concepts and connections

There are no Part II questions this week, as we aim to keep this assignment somewhat shorter than usual due to Wednesday's test.