Supplemental material

- Topic: Complex numbers

Definition. A *complex number* is an expression of the form $a+bi$ where a and b are real numbers, and $i^2=-1$. The collection of all complex numbers is denoted $\mathbb{C}$.

We think of each real number as a complex number by identifying the real number a with the complex number $a+0i$.

Addition and subtraction

We add and subtract complex numbers in the natural way:

$$(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i.$$

Example.

$$(1+2i)+(3-\pi i)=4+(2-\pi)i,$$

and

$$(-1-i)-(\sqrt{2}+3i)=(-1-\sqrt{2})-4i.$$

Multiplication

To multiply complex numbers, do so in the most natural way (kind of like you would for polynomials), but remember that $i^2=-1$.

Example.

$$\begin{aligned}(2-4i)*(3+5i)&=2*3+(-4i)*3+2*(5i)+(-4i)*(5i)\\&=6-12i+10i-20i^2\\&=6-2i-20(-1)\\&=26-2i\end{aligned}$$

Division

To divide complex numbers, it is extremely helpful to notice that for any real numbers a and b,

$$(a+bi)(a-bi) = a^2 + b^2.$$

Definition. If $z = a + bi$ where a and b are real numbers, then we call $a - bi$ the *complex conjugate* of z, and write it as

$$\overline{z} = a - bi.$$

The *(complex) absolute value*, or *modulus*, of z is defined to be

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}.$$

Now notice that if z and w are complex numbers, then

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2},$$

and since $|w|^2$ is a real number, this reduces division to multiplication.

Example.

$$\begin{aligned}\frac{1+2i}{3-5i} &= \frac{(1+2i)(3+5i)}{(3-5i)(3+5i)} \\ &= \frac{-7+11i}{34} \\ &= -\frac{7}{34} + \frac{11}{34}i\end{aligned}$$

The take-away so far: Complex numbers behave very much like real numbers for arithmetic operations. We can add, subtract, multiply, and divide (except by $0 = 0 + 0i$), and nothing scary happens!

Here is the reason that complex numbers are better than real numbers for some purposes:

Theorem (Fundamental Theorem of Algebra)**.** *Suppose that we have a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where the coefficients $a_n, \ldots, a_1, a_0$ are complex numbers. Then there is a complex number x_0 such that $f(x_0) = 0$.*

Even better, if the degree of f is n then f has exactly n roots (counted with multiplicity) in $\mathbb{C}$, so f completely factors into linear terms over $\mathbb{C}$.

In fact, the quadratic formula still works for solving quadratics over $\mathbb{C}$.

Example. The quadratic formula gives the roots of $x^2 + x + 1 = 0$ as

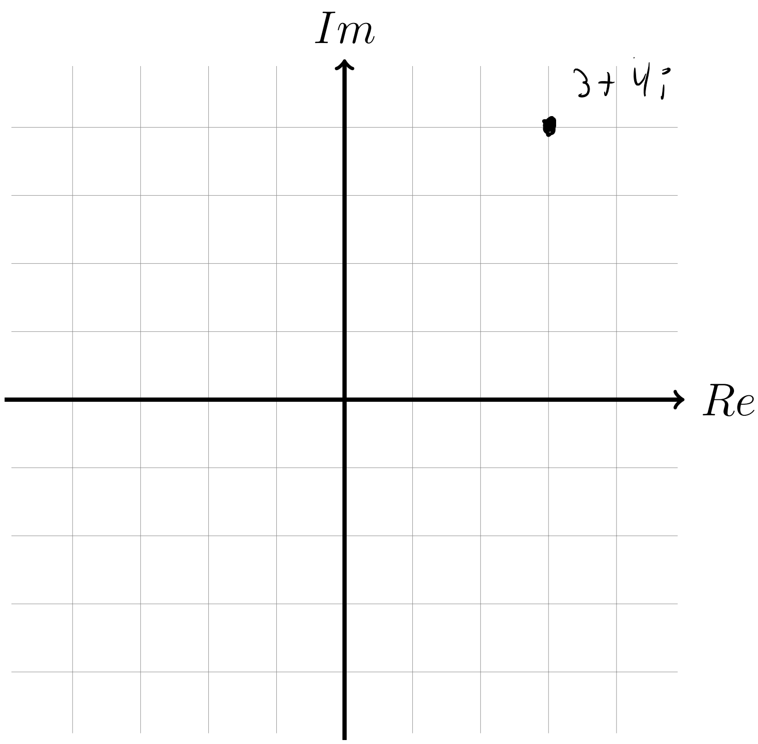
$$x = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

Representing $\mathbb{C}$ as $\mathbb{R}^2$

Definition. Suppose that a and b are real numbers, and $z = a + bi$. We call a the *real part* of z, written $\mathrm{Re}(z) = a$, and we call b the *imaginary part* of z, written $\mathrm{Im}(z) = b$.

It is sometimes useful to think of a complex number as being represented by a point in the plane, where we label the horizontal axis as the "real axis" and the vertical axis as the "imaginary axis".

Example. Let's draw the point $3 + 4i$.

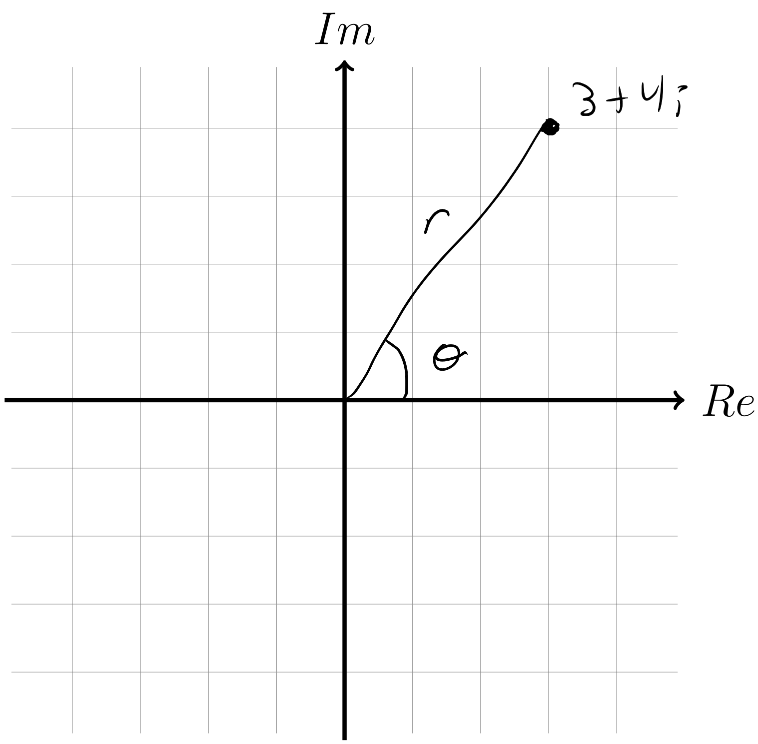


In this form it's pretty easy to check that addition, subtraction, and multiplication by real scalars are all just the same as the corresponding operations on vectors in $\mathbb{R}^2$, and that if $z = a + bi$ then $|z| = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|$.

Polar form

Definition. Let z be a complex number. The angle made by z to the positive real axis is the *argument*, written $\mathrm{Arg}(z)$. We choose the angle so that $-\pi < \mathrm{Arg}(z) \leq \pi$.
The *polar form* of z is $z = r(\cos(\theta) + i\sin(\theta))$, where $r = |z|$ and $\theta = \mathrm{Arg}(z)$.

Example. In our previous example $z = 3 + 4i$, we have:



If $z = a + bi = r(\cos(\theta) + i\sin(\theta))$ then the connections between a, b, r, θ are:

$$r = \sqrt{a^2 + b^2}$$

$$\tan(\theta) = b/a$$

$$a = r\cos(\theta)$$

$$b = r\sin(\theta)$$

Exponential form

It turns out that there is only one way to extend the function $f(x) = e^x$ to allow x to be a complex number, while still retaining all of the properties of exponentials that you expect. The formula for it is, for real numbers a and b,

$$e^{a+ib} = e^a(\cos(b) + i\sin(b)).$$

Thus

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta)),$$

so this is just polar form in disguise. But now notice:

$$(re^{i\theta})(se^{i\varphi}) = (rs)(e^{i\theta}e^{i\varphi}) = (rs)e^{i(\theta+\varphi)}.$$

So *multiplication of complex numbers multiplies lengths and adds angles*. We also get this lovely relation, relating many of the most important constants in mathematics:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1,$$

or

$$e^{i\pi} + 1 = 0.$$