UNIVERSITY OF VICTORIA

PHYS110 Lab Manual

Authors:
Travis MARTIN
Mark LAIDLAW

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Introduction to Phys 110 Distance Learning Labs

Intent

Since you aren't able to perform labs in a normal lab room with scientific equipment, we felt that it was valuable to show you how you can perform simple experiments and use the scientific method in your own homes. One of the most important skills we want you to take away from this experience is the method used for analysis, and how it lends to our understanding of the reliability of the experimental results.

We are not expecting students to get values that perfectly match known quantities/constants. In fact, we would be highly surprised if you did! We realize that there are many inaccuracies, confounding influences, and unmodelled environmental effects that will make the experiments very imprecise. Not only is that okay, it is exactly what we want. You are expected to put your best effort into overcoming these issues by using techniques to introduce consistency and reliability into your labs, and you should always include details of such efforts in your lab reports. But you won't lose marks simply for getting an unexpected result. How you deal with, analyze and respond to the unexpected result will be what you are graded on, not the value that you get.

In this lab manual, you will find descriptions of the lab projects and a series of appendices. Overall, there are four lab Projects, which are broken down into smaller Parts that are meant to be completed every week. The goal here is to not overwhelm you with large activities that are due every other week, but instead provide a steady, maintainable workload. Within each Part, you will see a description of the activity/experiment you are expected to perform, a description of how you will be expected to analyze the results of your activity, a description of the writeup you are expected to submit, and a rubric that shows you how you will be marked.

One thing you will not find in this lab manual is a Problem Statement, or Objective, or other such description of your activity. Instead, you will find descriptions of the learning objectives – what we expect you to get out of the experience – and background information motivating the experiment you will be performing. You will be expected to include in your writeup a valid introduction to the material and a purpose statement written in paragraph form. You will be, in essence, writing a scientific paper rather than a laboratory report on activities.

Remember, the learning goals of the labs are very different from the scientific purpose of the experiment.

Materials

For the lab component of the course, you will need to obtain a number of materials or use materials that you have readily available to you. The hope is that you should have access to materials that will work without having to spend much money.

Materials:

- A long, straight, rigid object such as a wooden board, that you can use as a ramp to slide things down;
- A means to hold one end of the wooden board up to produce the ramp;
- A tape measure or other measuring device capable of measuring at least 2m;
- A means to measure time remotely (such as a stop watch, a cell phone or a laptop) with sub-second precision;
- A series of objects that are flat on at least one side, such as coins, bottle cap lids;
- A series of objects that are smoothly cylindrical or spherical, with different cross sections, such as a tape roll or a roll of coins;
- Other things like string, tin foil, tape, paper, pens, and things like that will be helpful.

Project 1: Measurements and Statistics

1.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix A: Statistics
- Appendix B: Spreadsheets
- Appendix C: Hypothesis Testing
- Appendix D: Graphical Analysis

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

1.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to the statistical nature of measurements.
- Introduce students to the mathematical techniques that are used to analyze the the statistics arising from measurements.
- Introduce students to how to communicate measured quantities.
- Introduce students to how to assess and communicate the quality of the measurements.

1.3 Background

Imagine flipping a coin, or better yet find a coin and flip it. Consider the physics involved – you apply a force to the coin for a short period of time, the coin flips in the air and lands on one of its faces. Some of the applied force results in the translational motion of the coin, and some of the force results in the rotational motion of the coin.

Now imagine flipping the coin again, or repeat the coin flip. Do you expect the coin to land in the same location? Do you expect the coin to land on the same face? Even subtle variations in the applied forces can result in potentially significant changes to the outcome.

With regards to the side of the coin that is upwards when the coin lands, most people would judge that there is a 50% chance of it landing on Heads versus Tails. Most people have also never checked. By examining Fig. 1.1, it is obvious that the two sides are not the same. Thus, the coin lacks the symmetry by which it could be argued that the coin is fair – meaning an equal chance to land on Heads versus Tails.

FIGURE 1.1: Image of a Canadian quarter, showing both sides. The sides do not demonstrate any significant amount of symmetry.

Naturally, a scientific approach to examining the coin is to perform an experiment. Flip the coin, say, 100 times and record the results. If it comes up with 56 times Heads and 44 times Tails, you might be inclined to say that the coin is biased towards Heads. What if you repeated the experiment and found 48 times Heads and 52 times Tails? Is it reasonable to assume that the bias changed between experiments?

Perhaps you flip the coin one million times, and the results are 51.3% heads and 48.7% tails. You are lead to believe that the coin is not perfectly unbiased. How do you properly communicate your results to others? How do you communicate how confident you are that your results are accurate and reproducible? These are all important questions to a scientist.

This is the nature of statistical data and one of the most challenging things to account for in the sciences and Engineering fields. As it turns out, there are statistical fluctuations in absolutely every measurement that can be made, and thus in every physical constant (e.g. gravitational constant) and every phenomenological constant (e.g. coefficient of friction, heat capacity of a material, conductivity of a wire, etc). For example, if you purchase 100 metre-sticks, there will likely be a 1mm (or more) variation among the actual physical lengths of the sticks – materials change due to temperature, pressure, humidity and other factors. If you have 100 metal rulers, there may be more consistency in the lengths, but the distance indicators/ticks have some physical dimension to them and different people looking at the same measurement might differ in their judgment on the length by 0.2mm or more.

Suffice to say that the statistical nature of measurements cannot be completely eliminated, and thus must be accounted for when taking measurements, performing experiments and designing technology. This first laboratory project will introduce you to some of the statistical calculations that are involved when calculating quantities. You are introduced to the idea by assuming that your aim towards a target is a biased quantity, and you will want to numerically characterize and communicate your aim.

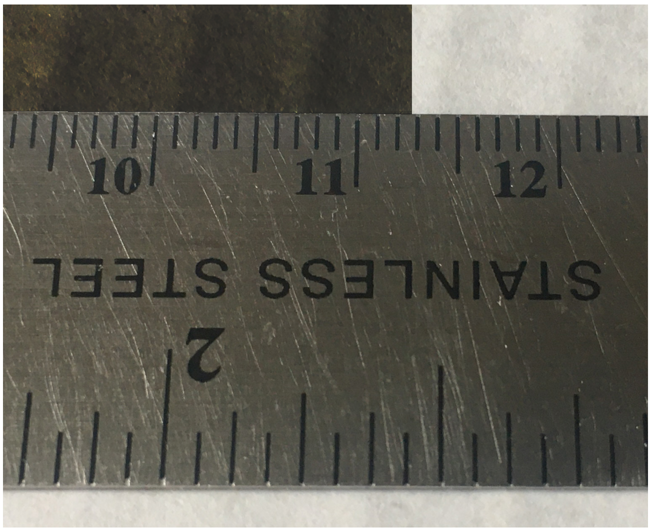


FIGURE 1.2: Close up of a ruler, showing the finite thickness of the distance marks. The thickness of the marks and the spaces between marks require personal judgment when using the device, resulting in variations between people performing the measurements. In this image, any value between 11.25cm and 11.30cm would be considered a reasonable measurement. Any attempt to claim a number more precise than 0.01cm with this ruler should be immediately recognized as unscientific.

1.4 Part 1

1.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ruler/measuring tape
- Masking tape (or other kind of tape)
- A coin or other small, non-rolling object

1.4.2 Activity Instructions

Using a rectangular item with perpendicular sides (e.g. a large book, a baking tray), lay the item down on the floor. Lay strips of tape following the edges of the rectangular item so that you have a large + sign on the ground. This will form your xy axes for measuring against. Indicate in some way which direction is positive and negative on each of your axes.

From some distance away, approximately 1m, toss your coin/small object towards the origin of the axes. Using your ruler/measuring tape, identify the (x, y) position of where your coin lands. To measure the x component of position, hold the ruler parallel to the x axis and measure the distance from the centre of the y axis to the centre of the object. Similarly, measure the y component of position.

Repeat this process until you have a total of 80 (x, y) positions. You will be grouping them into 8 groups of 10.

1.4.3 Analysis & Submission

Your goal is to analyze the data to convey the precision and accuracy of your aim. Remember: precision refers to how tightly packed the data points are, and accuracy refers to how close the points are to an expected value. Precision is measured with the *standard deviation of the mean*, δ, and you will performing an accuracy measurement in the next Part of this Project. The mean value for the positions represents the approximate location of the aim. There are many reasons why this value will not be and should not be the actual target on the paper, so do not be concerned if it is further from the target than you want it to be. To understand the equations needed for this analysis, you will need to review Appendix A.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo of your setup that shows a location of your coin/small object and your ruler/measuring tape in place for measuring the location.

2. Include a table of your values with columns for grouping/set (1-10), trial (1-10), x and y positions.

3. Determine the mean value of the x and y locations for one of the groups/sets of trials. Show your work.

4. Determine the standard deviation (σ) of the x and y locations for one of the sets of data. Show your work.

5. Estimate the *standard deviation of the mean* (δ) of the x and y location. Show your work.

6. Respond to the following questions/instructions:
   (a) How could you improve the precision of this experiment using reasonably priced/accessible equipment?
   (b) How could you improve the accuracy of this experiment using reasonably priced/accessible equipment?
   (c) Why is the mean value used to represent the aim?
   (d) Sometimes a percentage precision (e.g. $\delta_x/\bar{x}$) is quoted in scientific publications. Why would this not be a good method for quoting the precision of the aim? (Hint: Consider what happens as the aim gets worse.)

1.5 Part 2

1.5.1 Required Equipment

No equipment is needed.

1.5.2 Activity Instructions

All of the data has already been collected in Part 1. You will be doing analysis for this part of the project.

1.5.3 Analysis & Submission

For each set of the data from Part 1, you will need to calculate the mean and standard deviation. Summarize the results for each set in a table and produce a scatter plot of your means.

To estimate the reliability of the data, you will need to treat each of the mean values as a separate experiment. A calculation of the mean value of this set of mean values will give a more precise measurement of the location of the aim, and a standard deviation calculation of the set of mean values will give you a measure of the replicability of the measurement.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Create a table for each set of data from Part 1 with columns for trial, x, y. At the bottom, add two extra rows. In the first of these rows, type the label **Mean** in the trial column, and in the second of the rows type the label **Std Dev**. In the x and y column, include the corresponding value.

2. Using a spreadsheet program, estimate the standard deviation of the mean (δ) for each of the x and y locations of the aim trials.

3. Create a table summarizing your results. You should have seven columns: set number, $\bar{x}$, σ_x, δ_x, $\bar{y}$, σ_y, δ_y.

4. Use Plot.ly or Excel to create a scatter plot of your aim values for each trial, with the standard deviation of the mean (δ_x, δ_y) as your error bars.

5. Calculate the overall mean ($\bar{X}$, $\bar{Y}$) and standard deviation of the means ($\sigma_{\bar{x}}$, $\sigma_{\bar{y}}$) from the mean values of the aim trials.

6. Perform a statistical test on the x and y aim, as compared to the ideal values.

7. Respond to the following questions/instructions:
   (a) Was the estimate of the precision of the experiment comparable to the value found from replication? Justify/support your answer.
   (b) What numerical value represents the accuracy of the aim in the experiment? Is the aim in the experiment consistent with being “on target”?
   (c) By repeating the experiment a thousand times, instead of twenty, what should happen to your estimations of your accuracy and precision?
   (d) Hypothetically, in one version of the experiment darts are used to hit a target on a wall, while in another version coins are tossed at a target on the floor. How should the use of coins affect the experiment versus the darts?

Project 2: Determining Phenomenological Constants

2.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix A: Statistics
- Appendix D: Graphical Analysis
- Appendix E: Using Logger Pro

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

2.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to determining physical quantities via experiment.
- Introduce students to the application of statistics to determining physical quantities.
- Expand on students' communication of measurements and experiments.

2.3 Background

Every constant used in theoretical calculations must be measured experimentally. Some of these include universal constants, such as the gravitational constant, G. These constants can be measured in many different ways and are confirmed by many different experiments. Other constants are known as phenomenological constants – values that represent a specific situation and do not have the same broad applicability as the universal constants. Some examples include the density of a given material, the heat capacity of a material, and the coefficient of friction between two materials. You may notice a trend – phenomenological constants are typically constants that apply to specific materials or specific situations.

Consider car tires as an excellent example of a material with phenomenological properties. While all tires are made of rubber, not all rubber is produced identically. The shape and design of the treads can have a large impact on the effectiveness of the tire, and the tire is expected to drive on many different surfaces – concrete, pavement, wet pavement, snow/ice, gravel, etc. When a company produces a tire, they need to test the friction on many different surfaces and report a quantity that reflects the effectiveness of the tires under different conditions.

In a scientific setting, friction is typically parameterized in terms of a coefficient of friction, μ, with different values in static and kinetic settings. The frictional force is modelled differently in the static and kinetic settings, with the equations:

$$|\vec{F}_{static}| \leq \mu_s |\vec{F}_N| \tag{2.1}$$

$$|\vec{F}_{kinetic}| = \mu_k |\vec{F}_N| \tag{2.2}$$

where the force $\vec{F}_N$ represents the contact force that points perpendicular to the surface of contact. On a horizontal surface with no other applied forces in the vertical direction, $\vec{F}_N = mg\hat{z}$. However, on sloped surfaces, or where other forces are present, the normal force is simply the force necessary to maintain equilibrium in the direction perpendicular to the surface of contact.

The coefficient of kinetic friction is well understood to have a smaller value than the coefficient of static friction. If we conceptualize friction as due to roughness and imperfections in the surface, as in Figure 2.1, we then we can understand that this difference in coefficients is due to the different ways in which the surfaces interact in the two settings. However, this is only one conceptualization that doesn't account for other contributing factors.

FIGURE 2.1: Two surfaces illustrate one conceptualization of the origin of friction, which helps explain why static and kinetic friction are different. In static friction, the imperfections in the surfaces result in mechanical forces that must be overcome to start motion. In kinetic friction, the surfaces reach a steady-state, where only the tips of the imperfections align, reducing the frictional force.

In this project, you will be designing your own experimental setup to test static and kinetic friction coefficients, and the equations that are used to model them. You will be using what you previously learned about statistics and applying these techniques to characterize your results. Keep in mind that while scientists and engineers perform these types of measurements to greater precision, they can never avoid the variance/uncertainty in their measurements. Throughout these experiments, you should try to remember that there isn't a correct value that you are trying to achieve. Instead, you are simply trying to characterize materials.

In the next Project, you will be exploring measurement uncertainties arising from the devices that are used to perform the measurements. For now, we will assume that the ruler and/or angle measuring device(s) you use are free from measurement uncertainty and will just focus on the statistical uncertainty associated with your methodology.

2.4 Part 1

2.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ruler/measuring tape
- Video camera
- Long, flat surface for sliding things on

2.4.2 Activity Instructions

Note: It is more difficult to measure the coefficient of kinetic friction than static friction. For kinetic friction, you will be taking timing measurements of the object sliding down the slope, which is why it is helpful to find a **long** enough slope, and objects with small enough coefficients of static friction that they slide at a low angle. Both of these will increase the time over which the object slides and allow a more precise reading of the time. Since you need to use the same objects for both, you should take care now to find a good slope.

Measuring Coefficient of Static Friction

Construct a surface at a slope relative to the ground in such a way that you can adjust and measure the angle of the slope readily. There are several ways that you can determine the angle of the slope:

- Use a protractor or other angle measuring tool to directly measure the angle – be sure to align the origin of the device to the vertex of your slope;
- Measure the length of your sloped surface and the height of the end of your surface and use trigonometry to calculate the angle;
- Take a side-view photo parallel to the ground and use an art program to measure the angle;
- Use an angle measuring app on your phone.

There are other possible ways to determine the angle of the slope. You are free to use any method you wish, but you will need to justify the method you use in your report. If you use an app or an art program, you will need to take screenshots of your method so that the marker can assess your technique. If you use a protractor or measure lengths, you should take and include a photo that shows the method, as well as including sample calculations if necessary. In other words, you need to justify your technique to your marker sufficiently so that the marker can assess that you were careful and accurate enough.

With your variable slope constructed, test it out with objects you have available to you. Find small objects and place one at the top of the slope surface while it is horizontal, as in Figure 2.2. Now increase the angle of the slope until the object begins to slide down the surface. Do this with a variety of objects that appear to begin sliding down at different angles. Some options could include coins or bottle caps, or other small objects. Note: Do not use anything that will roll.

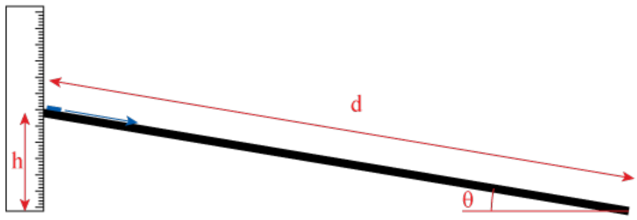


FIGURE 2.2: This diagram illustrates the setup you are expected to produce. You should be able to either measure the angle directly or calculate the angle using distance measurements.

Once you have found two very different objects (different masses, shapes, sizes, and materials), choose one and increase the angle from zero until the object begins to slide down the slope. Record that angle. Return the slope surface to horizontal, place the object at the top again, and increase the angle slowly until the object slides down again. Record your value. Repeat this process until you have 5 measurements that provide you with the angle at which the objects begins to slide. Note: It is critically important that you return your slope to horizontal before taking subsequent measurements. Failure to do so is a failure to perform the experiment correctly.

Now repeat the entire process again for the other object. Your final data should be two different sets of 5 measurements of the slope angle at which the object just starts to slip down the slope.

Measuring Coefficient of Kinetic Friction

You will now be sliding an object down your slope and recording the amount of time it takes to slide.

Mark a specific spot near the top of your sloped surface from which you will release the object (the *start* point of the motion), and measure the distance from this marked spot to the bottom of the slope (the *end* point of the motion). This is the sliding distance, and it should be the same for both objects, for all trials.

To measure the time, you will be recording a video of the event and using Logger Pro to analyze the amount of time it takes for the object to slide down the slope. See Appendix E for more information on how to do this. When recording your video, you will need to clearly see the start point and the end point of the slide, however it is not critically important for your camera to be in a fixed location or be steady (it would be if you were doing position measurements, however). Either set up your camera in a location that can clearly see the start and end of the motion, or hold your camera in such a way that the start and end of the motion is captured.

For each object, start with the slope raised to an angle slightly greater than the angle you found in for the static friction and record the angle measurement(s). Start filming. Place the object at the marked starting location on the slope and release it.

Increase the slope again, determine the angle, and record the object sliding down. Do this until you have 6 different slides recorded for 6 different angles, for each object.

Note: You should increase the angle in increments of approximately 3 (and no more than 5) degrees each time.

2.4.3 Analysis & Submission

In your analysis, you should draw a free body diagram of the setup of the device, and solve the system of forces in a way that allows you to determine the coefficient of static friction from the angle at which the object begins to slide. If you need to use the acceleration of gravity, $g = 9.81\text{m/s}^2$ will work fine for this Project.

Using these equations, you can determine the coefficient of friction from the measurement of the angle. Since you measured the angle multiple times, you will have multiple measurements of the coefficient of friction (some of which may be the same as others, but not all should be identical). Based on this, you can determine a quantity that represents your best estimate of the coefficient of static friction along with an uncertainty in its value.

Use LoggerPro to analyze the video for each of the slides. Observe the time of the first frame before you see the object begin to move, then the time of the last frame when the object reaches the end of the slope.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Describe your experimental setup, including photos of the setup.

2. Describe in detail the method you used for determining the angle of the slope. If you calculated the angle from other measurements, show a sample calculation.

3. Describe your experimental procedure for measuring the slide time of an object. Include images of the first video frame used for timing for one of your trials and the final frame for the same trial.

4. Draw a free body diagram (FBD) of the object resting on the slope, with complete and thorough labeling.

5. Use Newton's Laws and your FBD to show that the coefficient of static friction μ depends on the maximum angle before slippage θ via $\mu = \tan\theta$.

6. Create a table for your static friction data. You should have columns to indicate the object, for trial number, and for the measurements of the angle (distance measurements if used, and the calculated angle value), and the coefficient of static friction. If you used distances to calculate the angle, include a sample calculation of how you did it.

7. Determine the mean coefficient of static friction, as well as the uncertainty in the coefficient of static friction from your table of values. Show your work.

8. Create a table for the kinetic friction data. You should have columns to indicate the object, for the angle measurements, and for the slide-time measurement (from Logger Pro).

9. Respond to the following questions/instructions:
   (a) Why did you choose the objects you chose?
   (b) How might the object's size, shape and mass affect the results?
   (c) How might the object's size, shape and mass affect the uncertainty in the result?

(d) Describe the steps you took to ensure that the results of your experiment are accurate.

(e) Describe the steps you took to reduce the measurement uncertainty in your experiment.

(f) Describe the steps you took to reduce the statistical uncertainty in your experiment.

2.5 Part 2

2.5.1 Required Equipment

No equipment will be needed.

2.5.2 Activity Instructions

All of the data collection occurred in Part 1. You will be analyzing the data for Part 2.

2.5.3 Analysis & Submission

Using Newton's 2nd Law, determine a relationship between the coefficient of friction and the acceleration of the object down the slope and the angle of the slope. You can re-use your free-body diagram from Part 1, and modify your analysis to account for the acceleration of the object.

Use your measurement of the times, Δt, to determine the acceleration your object must have experienced, assuming a constant acceleration model, for each of the slides you did in Part 1. Using Newton's 1st and 2nd Laws, you can determine a relationship between the coefficient of friction, the acceleration of the object, and the angle of the slope. From this, you can calculate the coefficient of friction for each of the trials, at each of the 5 angles used. Using formulas in a spreadsheet will make this very simple.

Now you need to graph your data using either Plot.ly or Excel. Information on how to graph using Plot.ly is included in Appendix D.

Use the least squares method to calculate the slope and intercept (and their uncertainties) for each of the graphs. Note: You must calculate the slope/intercept/uncertainty yourself, rather than using premade functions in spreadsheet programs. However, you can still use the spreadsheet program to make this a lot easier. Add columns to your table for x^2, xy, and y^2, then find the averages $\bar{x}$, $\bar{y}$, $\bar{x^2}$, $\bar{xy}$, $\bar{y^2}$. There are formulas in the appendix that will allow you to quickly calculate the slope/intercept/uncertainty using these averages, and the number of data points you have, N. See Appendix D for more information on how to do this. Remember, x is the value of whatever goes on the x-axis of the graph you will make, and y is the value of whatever goes on the y axis of the graph you will make.

The theoretical version of this graph is shown in Fig. 2.3, however yours may differ in the kinetic region, depending on a variety of influences. Some things to consider:

- if your five data points don't form a straight line, and are scattered, this is likely statistical uncertainty;
- if your five data points show a linearly increasing or decreasing trend to them, this is a linear influence - examine your equations to find which quantity might be affecting this;

- if your five data points show a non-linear (i.e. curved) trend to them, then this is a non-linear influence - examine your equations to find which quantity might be affecting this.

You will not lose marks if your graph does not show a nearly horizontal line in the kinetic region – remember that it is an academic integrity violation to modify your lab data to fit an expected outcome. If your data does not behave as expected, you will simply be tasked with explaining the origin of the behaviour in the Discussion section of your write up.

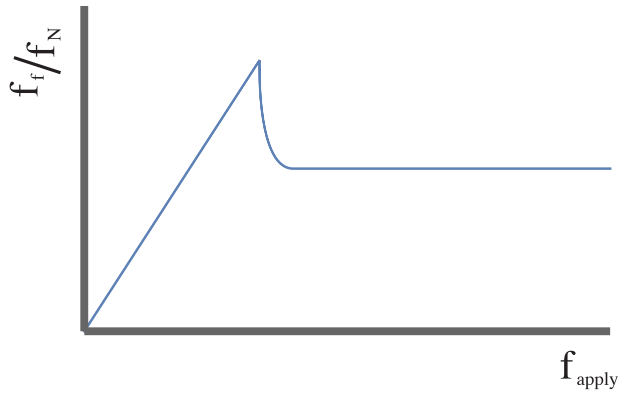


FIGURE 2.3: This graph represents the relationship between the force of friction and the applied force. The sloped region on the left shows that the frictional force is not a constant value when in the static regime. The horizontal region on the right shows that the frictional force is a constant value regardless of the applied force in the kinetic regime.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Draw a free body diagram of the object resting on the slope, with complete and thorough labeling. Include a kinetic diagram showing $m\vec{a}$.

2. Use kinematics to show that the relationship between the acceleration of the object a, the slide distance d, and the slide time Δt is $a = 2d/\Delta t^2$.

3. Use Newton's Laws to solve the system of forces based on the free-body diagram you drew, to show that the relationship between the coefficient of kinetic friction μ_k, the angle of the slope θ, and the acceleration down the slope a is $\mu_k = \frac{g \sin\theta}{g \cos\theta} - \frac{a}{g \cos\theta}$.

4. Copy your table of values for the kinetic scenario from Part 1, and add columns for the acceleration a, the specific force down the slope ($g \sin\theta$), and the coefficient of kinetic friction μ_k.

5. Assuming you are going to plot of the coefficient of friction on the y axis versus the specific force down the slope, $g \sin\theta$, on the x axis, calculate $\bar{x}$, $\bar{x^2}$, $\bar{y}$, $\bar{y^2}$, $\bar{xy}$ for each object.

6. Using least squares analysis, find the slope and intercept of your six data points in the kinetic region for each object from your mean values above. You should use the mean values you calculated above to simplify your calculation. You should write out each equation that you used, but you can use your spreadsheet to perform the calculation.

7. Find the uncertainty in the slope and intercept of your data points in the kinetic region using the mean values calculated. You should write out each equation that you used, but you can use your spreadsheet to perform the calculation.

8. Using Plot.ly or Microsoft Excel, make a scatter plot that includes both the static and kinetic region for each of your objects. Error bars are not needed for this plot.

9. Perform a statistical test to compare the slope results of your least squares analysis with the theoretical value you expected to get.

10. Respond to the following questions/instructions:
    (a) How precise were you able to perform measurements of the time? Would greater precision on the timing measurement have improved the results of this experiment? Justify your answer.
    (b) Is the coefficient of static friction you found larger or smaller than the coefficient of kinetic friction? Is this what you expected? Explain your answer.
    (c) Is your coefficient of kinetic friction a constant value, regardless of the normal force? If not, describe, using equations if necessary, what might have caused the value to have a non-constant trend.
    (d) In hindsight, what factors might have influenced your results but were not taken into account in the analysis?
    (e) How could you have performed the experiment differently with changes to the methodology or with reasonably available equipment in order to account for these influences?

Project 3: Scientific Error Analysis

3.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix F: Uncertainty Analysis
- Appendix D: Graphical Analysis
- Appendix E: Using Logger Pro

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

3.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to error analysis.
- Introduce students to confirming universal constants.
- Introduce students to comparisons to known values.
- Expand upon student understanding of graphical analysis techniques.

3.3 Background

In the previous project, you examined methods for determining a phenomenological constant. In this project, you will be determining a universal constant – a value that is well defined to high precision due to many past experiments. In such cases, the experiment is less about seeing if the universal constant is wrong, but more about testing your experimental setup. In a sense, this is validation and calibration.

Validation and calibration are critical parts of the scientific process. Until recently, the distance we take for granted to be 1m was determined by a rod of metal that was stored in very precise temperature, pressure and humidity conditions so that it's length was not affected by any environmental factors. Any company that wanted to produce a high precision distance measuring device would have to verify their device against this standard. Similarly, an object existed that represented 1kg. There have been similar standards for other units, as well.

Modern calibration approaches recognize that many devices measure distances that are much smaller or larger than 1m, or masses that are much smaller or larger than 1kg, or times that are much smaller or larger than 1s, etcetera. Thus, the definition of these units is starting to be shifted to be based on universal constants in a way that allows people to validate their measurement devices with physics instead of against an arbitrary object locked away in a vault.

Now imagine you are tasked with building a complicated device that can be used to measure some property. How would you check to see if this device is working correctly? Let's use, for our example, a mass scale. Small mass scales are mass-produced (pun intended), but large ones must be custom built. You could apply theoretical calculations to determine how the device *should* work, but reality often interferes with theoretical calculations. For example, do the theoretical calculations account for flexing or deformation of the materials used? Are all of the parts manufactured to precisely the right dimensions or are some of them off by some small amount? Are the materials used exactly what the specifications call for, or are there imperfections in their manufacturing?

There are far too many factors to account for to rely on theoretical calculations to determine how the device should respond precisely to a given mass. Instead, we rely on the designs to ensure that the measurements are possible, and then calibrate the device to a known quantity. This requires us to know that known quantity to a precision that is greater than the precision which our device can measure.

In this laboratory project, you aren't going to be constructing a measuring device, because that is too complicated for a home-laboratory experiment. Instead, you are going to be using measurements to determine a known quantity, the acceleration due to gravity, and seeing how close you can get. You are also going to explore possible ways to improve both the accuracy and the precision of your value.

However, during your experiment, you will also be applying statistics in a different way – through error analysis and graphical analysis. In science, the term *error* does not imply a mistake. Instead, it refers to the possibility that the outcome of the experiment is not repeatable. This manifests, as you have seen, as quoting measured values along with an uncertainty/precision for the measurement. So what happens when you calculate a theoretical quantity using measured values? Well, you must use error/uncertainty analysis (also known as error propagation) to determine how the uncertainties on the individual components of the calculation affect the final value. Details of this procedure is covered in Appendix F.

In this experiment, you will be using projectile motion to measure the acceleration due to gravity, commonly referred to as g. The value of g depends on a person's location and elevation:

$$g(\phi, h) = \left(9.7803253359\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(\frac{1+0.00193185\sin^2\phi}{\sqrt{1-0.00669438\sin^2\phi}}\right) - \left(3.086\times10^{-6}\frac{1}{\mathrm{s}^2}\right)h \tag{3.1}$$

where h is the elevation above sea level at the location and ϕ is the latitude. You can perform an internet search for your latitude and elevation in your city. If you are interested, you can read more about how this equation is produced by exploring the Department of Defence World Geodetic System 1984 (found here). For Victoria, BC, the value is 9.8093m/s^2.

3.4 Part 1

3.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ruler/measuring tape
- Long, flat surface to slide things on
- Tape
- Some kind of markable surface
- Plumb bob (e.g. a weight at the end of a string)

3.4.2 Activity Instructions

Your ultimate goal is to modify the slope that you used for the last project to launch projectiles horizontally off of a raised platform. This is easier than it might sound, but some precautions need to be incorporated, because precision is important.

For your raised platform, you should use something moderately high, like a table or a countertop. The next thing you will need to do is create a curve for the end of the slope so that your object smoothly transitions from the diagonal motion to horizontal motion. This can be accomplished with tape and either paper or tinfoil, but you are free to try to create your own solution. An illustration of the design you are aiming for is shown in Figure 3.1.

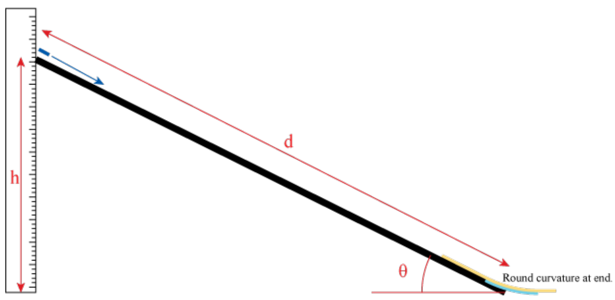


FIGURE 3.1: Your structure should be arranged like this, close to the edge of a table or countertop so that your chosen small object slides down the ramp and is launched horizontally off of the edge. This will make the calculations much simpler for you than launching the object at an angle relative to horizontal, but it does introduce some uncertainty into the analysis.

For this experiment, you won't need to know the angle of the slope precisely – but you will need to be able to take distance and time measurements as precisely as possible. You should modify your slope to add a series of marks/indicators every

$\sim$ 4cm from the top of the slope until you have five indicators. The spacing is up to you, but you will be using each of these points. If you place the indicators along the path of your object, you will affect the slide of the object, so ensure that these marks do not affect the slide of the object. You should also measure and record the height above ground of the countertop/table/surface that you are using.

For the distance the object travels, you will first need to locate the point on the ground directly beneath where the object enters into projectile motion. This can be accomplished by attaching a small mass to the end of a string (string, thread, dental floss, anything similar should work) and using gravity to indicate the point directly below the counter. To measure the location of impact, you should use something that will indicate the location – one idea is to spread a powder on the ground that will be disturbed when the object hits, or using a sheet of tinfoil that will be dented when the object impacts. At this point, your setup should look something like Figure 3.2.

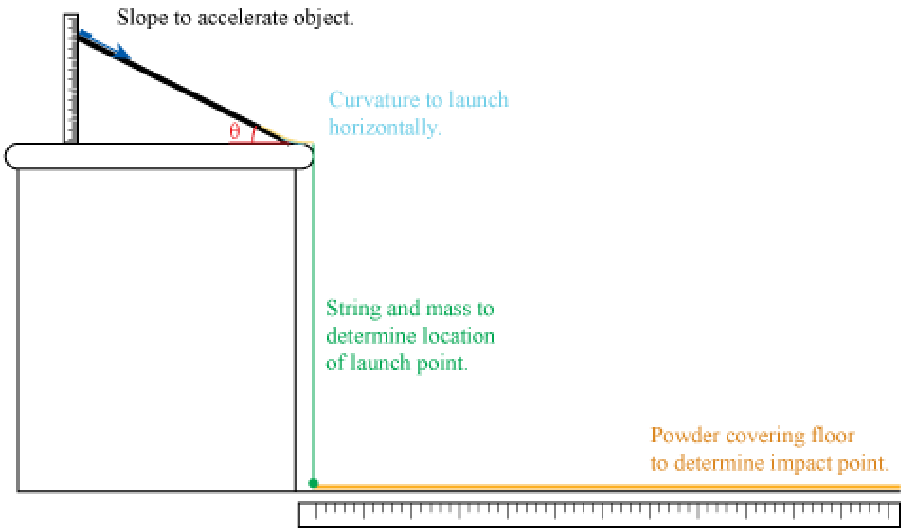


FIGURE 3.2: Your full setup should be something similar to this. You are free to make other choices in the arrangement so long as the outcome achieves the same goal. No matter what you choose to do, you will need to include pictures in your report showing how it works.

To record your data, place your projectile object on one of the marks and release it from rest while video recording the motion. Measure the location on the ground where the object impacted, and the release location along your slope. Repeat this process four times (total of 5 data points), using the exact same starting point for your object. Then repeat this at each of the other starting-line indicators.

When you are done taking measurements, use Logger Pro to analyze the video and determine the amount of time it took for the object to slide down the slope.

3.4.3 Analysis

The first step will be in determining the measured quantities and their uncertainties. For the diagonal distance that the object travels along the slope, ℓ, the measurement should be the same for each trial from the same starting point, and the uncertainty will be the inherent measurement uncertainty of the device you used (typically 1 of the smallest division of the device). For the distance that the object travels while in projectile motion, d, you have multiple measurements and can calculate a statistical uncertainty from these measurements. Similarly for the time values for the object sliding, you have multiple measurements and can calculate a statistical uncertainty.

Your object follows a path that must be described in a piece-wise fashion. When the object is on the ramp, the forces affecting the object have a different resultant (i.e. the $m\vec{a}$ part of Newton's 2nd Law) than when the object is in projectile motion. Thus, we need to solve the kinematics separately for these two situations.

While you might now have an understanding of the frictional forces involved in the motion of your object, we don't actually need to know them to solve this problem. It is sufficient to say that the acceleration of your object along the ramp is constant – none of the forces are dependent on the velocity or position of your object on the ramp. Thus, we get the following kinematic equations:

$$\ell = \ell_o + v_o T + \frac{1}{2} a T^2 \tag{3.2}$$

$$v = \frac{d\ell}{dt} = v_o + aT \tag{3.3}$$

where ℓ is the location of the object along the slope at time $t = T$, v_o is the speed it is traveling at time $t = 0$, and ℓ_o is the position the object is located at time $t = 0$. This assumes that the object is undergoing constant acceleration, a. Thus, the distance that the object travels along the ramp ($d = |\ell - \ell_o|$) and the time taken to travel along the ramp (T) are enough to get an approximation of the velocity of the object as it enters into projectile motion.

To analyze the uncertainty of the velocity, you are going to use the statistics in a different way than in the previous two Projects. From the timing values from your video, calculate the average time taken to slide down the slope and its standard deviation of the mean. Then use this quantity, the uncertainty on the distance traveled, and proper uncertainty propagation to determine the uncertainty on the velocity of the object as it is launched off the slope.

For the portion of the motion where the object is a projectile, you can neglect the effects of drag forces as being small due to the low speed and short flight time of the object. Thus, you have a classic horizontal projectile motion problem, with an initial speed determined from our T and d measurements above. The object then lands at position x after descending a distance h from the surface top. This results in the equations:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \begin{bmatrix} v_{xo} \\ v_{yo} \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} a_x \\ a_y \end{bmatrix} t^2 \tag{3.4}$$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix} + \begin{bmatrix} v_{xo} \\ 0 \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 0 \\ -g \end{bmatrix} t^2 \tag{3.5}$$

Since you are not measuring the time of flight, you will need to use the $\hat{x}$ and $\hat{y}$ components of this equation to eliminate t from your equation. This is also why the equations use a T for the time on the ramp, to distinguish between the two.

You will need to combine all of these equations to get a relationship between the projectile distance, x, the launch velocity of the object, v_{xo}, the height of the launch point, h, and the acceleration due to gravity, g. Solve for g to determine the acceleration due to gravity, and use the equation you derive to perform uncertainty propagation to include all sources of uncertainty from all measurements.

Note: When making tables, you should feel free to rename column headings with symbols. For example, "slide distance" could have a column heading of "d", which makes the table look much less cramped.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Describe your experimental setup, including photos of the setup.

2. Starting with the basic kinematics equations provided, show that the relationship between the launch velocity of the object and the slide distance/time is $v_{xo} = 2d/T$.

3. Starting with the basic kinematics equations provided, show that the relationship between gravitational acceleration, horizontal launch speed, initial height, and distance traveled is $h = \frac{1}{2}g\frac{x^2}{v_{xo}^2}$.

4. Let your projectile slide from each starting point five times and create a table of your readings with the following columns: Trial Number, Slide Distance, Slide Time, Height of Fall, Projectile Distance.

5. Create a new table that summarizes your values from the above table. You should have columns with slide distance (with uncertainty), mean slide time, uncertainty in slide time, height of fall (with uncertainty), mean projectile distance, uncertainty in projectile distance. (Slide distance and height of fall have the same uncertainty, so this can go in the header for that column.)

6. Respond to the following questions/instructions:

   (a) Describe the steps you took to ensure that the results of your experiment are accurate.

   (b) Describe the steps you took to reduce the measurement uncertainty in your experiment.

   (c) Describe the steps you took to reduce the statistical uncertainty in your experiment.

3.5 Part 2

3.5.1 Required Equipment

No equipment will be needed.

3.5.2 Activity Instructions

You have already collected all of your data. You will be doing analysis in Part 2.

3.5.3 Analysis

You should have six data points now corresponding to different locations along your slope, and different landing points. However, instead of calculating g six times, you are going to graph and perform a linear fit of the data. Using the equations you developed in Part 1, determine the value and uncertainty of the launch velocity and the landing distance from your data.

Plot all six data points on a graph, and perform a least-squares analysis of the slope and intercept of your graph. You will also need to calculate the uncertainty in your slope and intercept.

Write the equation that relates the distance the object travels, x, to the slope distance and velocity, and show how this can be written in the form of a linear line, $y = Ax + B$. Then identify the slope of this line in terms of the corresponding physical quantities. Use this relationship to calculate g and an uncertainty for g.

In your analysis, you should perform a statistical comparison for the value of g to the known value, which will give you a measure of your accuracy. You should also perform a comparison of your intercept value to the expected value for the intercept of your graph line.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Using uncertainty propagation, show that the equation for the uncertainty in the launch velocity is $\delta_v = v\sqrt{\left(\frac{\delta_d}{d}\right)^2 + \left(\frac{\delta_T}{T}\right)^2}$.

2. Take one set of slides for one starting location from Part 1 and show the full calculation of determining the launch velocity from the mean slide time & slide distance, along with the uncertainty.

3. Create a table with columns for mean projectile distance x, uncertainty in mean projectile distance δ_x, mean velocity, mean slide time, uncertainty in mean slide time, launch velocity, uncertainty in launch velocity. Most of the table can be copied from Part 1, but you will need to add the velocity values.

4. Assuming you are going to plot of the distance traveled on the y axis versus the launch velocity on the x axis, calculate $\bar{x}$, $\bar{x^2}$, $\bar{y}$, $\bar{y^2}$, $\bar{xy}$ for each object.

5. Perform a least squares analysis on the distance traveled versus the launch velocity and calculate the slope and intercept from the mean values above.

6. Calculate the uncertainty in the slope and intercept from the mean values above.

7. Using Plot.ly or Microsoft Excel, create a graph of the distance traveled versus the launch velocity. nclude the appropriate error bars in your plot. Add a linear line to your plot that has the same slope and intercept as you found in the least squares analysis.

8. By comparing your equation for the projectile distance as a function of the launch velocity to the equation for a straight line, show that $g = 2h/slope^2$. Calculate your value of g from your slope.

9. Use uncertainty propagation to calculate the uncertainty on g from the uncertainty on the slope and the other quantities that it depends on.

10. Perform a statistical test to compare the value of g you found with the accepted value.

11. Respond to the following questions/instructions:
    (a) What is the largest contribution to the uncertainty in your data points on the plot? Is it statistical or a measurement uncertainty? Explain how you came to this determination.
    (b) Was your value for the intercept consistent with the value you expected from your theoretical calculations? Discuss the implications and possible causes why the intercept value was not exactly equal to the theoretical value.
    (c) How precise were you able to perform a determination of g?
    (d) How accurate was your determination of g?
    (e) Why might a graphical analysis be used for g instead of using a single value for g? What could cause these to be significantly different?
    (f) In hindsight, what factors might have influenced your results but were not taken into account in the analysis?
    (g) How could you have performed the experiment differently with changes to the methodology or with reasonably available equipment in order to account for these external influences?
    (h) In hindsight, how might you have improved the precision of your measurements? (Hint: If the statistical uncertainty is larger than the measurement uncertainty, more precise equipment is not the answer.)

Project 4: Hypothesis Testing

4.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix C: Hypothesis Testing
- All other appendices

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

4.2 Learning Objectives

The goals of this laboratory project are as follows:

- Summarize students' knowledge of all aspects of the scientific process.
- Introduce students to hypothesis testing.

4.3 Background

4.3.1 Modeling

The way an object behaves when affected by forces is highly dependent on the properties of the object. The mass obviously affects the translational acceleration of the object, according to Newton's 2nd Law, but also the distribution of that mass. A force acting at a point on an object may produce both a translational effect as well as a rotational effect. To predict the behaviour of objects when subjected to forces, we produce models that are intended to represent the object. These models have parameters – quantities that can be measured by observing the behaviour of the object, and that can be calculated based on theoretical representations of the behaviour.

When developing theoretical equations to determine the model parameters, there needs to be a way to confirm whether the equation is accurate. An experiment that depends on the model parameter must be performed, so that a direct measure of the parameter is found. Then a calculation is performed to determine the parameter from the theory. Finally, a statistical comparison, such as a t-test, is performed to determine whether the measured and theoretical parameters are consistent with each other. The purpose is to disprove or rule out incorrect theories – theories that determine parameters that do not match the observed/measured values. This is the most fundamental process in science: science does not prove theories, but instead disproves bad hypotheses.

To put this into more concrete terms, consider rotation. When an object experiences an unbalanced torque, $\vec{\tau}$, it will begin to rotate. The model that we use to describe this is based on Newton's Laws for rotation: $\sum \vec{\tau} = I\vec{\alpha}$. The angular acceleration, $\vec{\alpha}$, is a quantity that describes how the object is moving, but isn't a model parameter. However, the moment of inertia, I, is a model parameter. Moments of inertia depend on the distribution of the mass of the object about the centre of rotation. Some example theoretical formulas for moments of inertia are given in Figure 4.1.

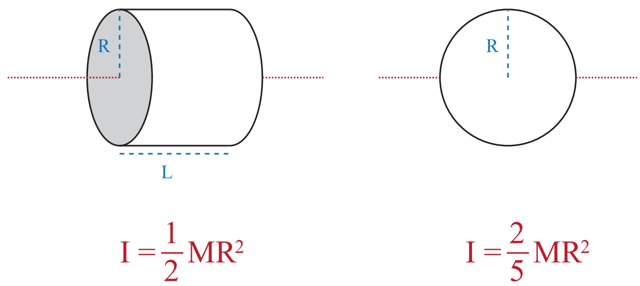


FIGURE 4.1: The moment of inertia of a solid cylinder and a solid sphere depend on the radius of the circular cross section perpendicular to the axis of rotation. To construct the moment of inertia from a compound shape, moments of inertia add in superposition.

If the density of an object at location $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is $\rho(x, y, z)$, then the moment of inertia is given by:

$$I = \int_{volume} \rho(x, y, z)|\Delta\vec{r}|^2 dV \tag{4.1}$$

where $\Delta\vec{r}$ is the shortest distance from the material located at (x, y, z) to the axis of rotation.

So if we want to find the moment of inertia about the long axis of a solid cylinder of uniform density, mass M, length L, and radius R, we first find the density:

$$V = \pi R^2 L \tag{4.2}$$

$$\rho = M/V = \frac{M}{\pi(R^2)L} \tag{4.3}$$

For a cylinder, the volumetric integral is $dV = 2\pi r dr dz$, where we have assumed that the central/rotational axis of the cylinder is along the z axis. Thus, our integration to determine the moment of inertia is:

$$I = \int_0^L \int_0^R \rho r^2 2\pi r dr dz = 2\pi L \rho \int_0^R r^3 dr \quad (4.4)$$

$$= 2\pi L \frac{M}{\pi R^2 L} \frac{R^4}{4} = \frac{1}{2} M R^2 \quad (4.5)$$

Note that in this case, the length of the cylinder is irrelevant to the overall moment of inertia. Moments of inertia typically depend only on how the mass is distributed along the dimensions perpendicular to the axis of rotation. Expressions for the moment of inertia of simple objects is readily available via internet search.

The formula for the moment of inertia of an object is a theoretically derived formula. Given the formula for I, and measurements of the mass and dimensions of the object, a theoretical value (and corresponding uncertainty) can be determined. If an experiment is performed that allows us to indirectly measure the moment of inertia, then the measured value can be compared to the theoretical value. If the theoretical and measured values disagree with each other, then there is a problem with the theoretical calculation (or perhaps a problem with the experimental design).

In this case, an experiment such as measuring the rotational acceleration of the object for a known applied torque would provide the data needed to measure the moment of inertia. When compared to the theoretical value, statistical agreement (such as with a t-test) means that the theory cannot cannot be disproved. However, if there isn't statistical agreement, then the theoretical value that was calculated was incorrect and the theory is disproved. (Again, this assumes that there aren't problems with the experiment that result in poor measurements, or other such issues.) This process is one of the critical challenges for experimental scientists.

4.3.2 Rolling without slipping

Wheels are ubiquitous in our society – automobiles, airplanes, bicycles, skateboards, inline skates, and many other technology incorporate the wheel to facilitate controlled travel along the ground. However, non-physicists might be surprised to realize that wheels operate in the *static* friction regime under normal conditions, despite their constant motion. When a wheel is rolling, the contact with the ground is static, even though the centroid of the wheel is in motion. This is illustrated in Figure 4.2.

FIGURE 4.2: A wheel rotating while its central axis is stationary has the same tangential speed at both the top and the bottom, as illustrated on the left side. A wheel rotating while it is rolling along the ground moves without slipping. Thus the tangential speed at the top is necessarily twice the linear speed of the central axis, as shown on the right, and the tangential speed of the point touching the ground is zero.

The position of a point on a circle located at angle ϕ in cartesian coordinates relative to the centre of the circle is

$$\vec{r}(\phi) = R\cos(\phi)\hat{x} + R\sin(\phi)\hat{y}. \quad (4.6)$$

If the wheel is rotating at an angular rate of ω in the clockwise (negative rotation) direction, then the position of the same point at some time t is

$$\vec{r}(\phi, t) = R\cos(-\omega t + \phi)\hat{x} + R\sin(-\omega t + \phi)\hat{y}. \quad (4.7)$$

The velocity of this point can be found by taking the time derivative:

$$\vec{v}(\phi, t) = \omega R\sin(-\omega t + \phi)\hat{x} - \omega R\cos(-\omega t + \phi)\hat{y}. \quad (4.8)$$

And the acceleration can be found by taking a second time derivative:

$$\vec{a}(\phi, t) = -\omega^2 R\sin(-\omega t + \phi)\hat{x} - \omega^2 R\cos(-\omega t + \phi)\hat{y}. \quad (4.9)$$

The position of the point directly below the centre of the circle will always be $-\omega t + \phi = 3\pi/2$, so plugging this into our velocity equation, we find that the velocity of the point below the centre is $\vec{v} = -\omega R\hat{x}$. Similarly, the velocity of the point directly above the centre of the circle ($-\omega t + \phi = \pi/2$) is $\vec{v} = \omega R\hat{x}$.

Now if we consider the tangential velocity of the centroid of the wheel to be $v_\circ\hat{x}$ relative to the ground, then we add this to the velocity of each point when we change frames of reference from that of the stationary wheel to that of the stationary ground. For a wheel moving horizontally at speed $v_\circ$, the velocity of the point below the centroid is $\vec{v} = (v_\circ - \omega R)\hat{x}$ and the velocity of the point above the centroid is $\vec{v} = (v_\circ + \omega R)\hat{x}$. Since we know the point below the centroid is locally and instantaneously stationary, this tells us that $v_\circ - \omega R = 0$ or $\omega = v_\circ/R$, and it also tells us that the velocity of the point above the centroid is $\vec{v} = 2\omega R\hat{x}$.

This entire derivation is simply to prove that the translational velocity, v, of the central axis is linearly proportional to the angular velocity of the wheel, ω, such that $v = \omega R$. This is something that we showed in the stationary reference frame of the wheel, but had to prove that it also applied when switching to the reference frame of the ground. Thus, we can switch between linear and rotational kinematics as necessary. For the position of the centroid x of a rolling object of outer radius R, we have:

$$x = x_\circ + v_\circ t + \frac{1}{2}at^2 \quad (4.10)$$

while for a point along the surface of the object, we have:

$$\phi = \phi_\circ + \omega_o t + \frac{1}{2}\alpha t^2 \quad (4.11)$$

$$(4.12)$$

where these are connected by the relationships $x = \phi R$, $v = \omega R$ and $a = \alpha R$.

For an object rolling down a slope without sliding, the static friction involved does not result in a loss of energy, as the kinetic friction did in Project 2. Instead, the static friction facilitates the rotation and translation. Thus, we do not need to use forces to calculate quantities, and can instead use energy. To understand the behaviour of an object rolling down a hill, we use conservation of energy and incorporate rotational kinetic energy.

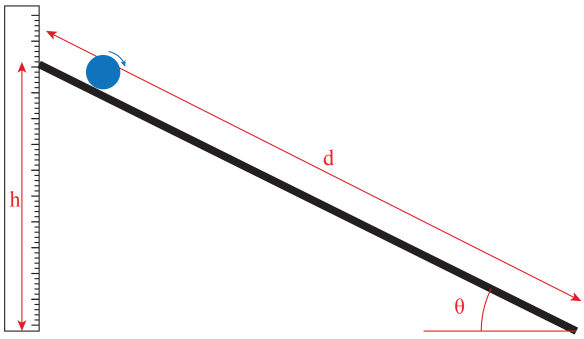


FIGURE 4.3: As an object rolls down a slope, it converts gravitational potential energy to linear and rotational kinetic energy.

Since the object rolls along a surface close to the surface of the Earth, we can use the simplified potential energy formula:

$$\Delta U = mg\Delta z \tag{4.13}$$

Similarly, the linear kinetic energy formula is straight forward as well:

$$\Delta K_{lin} = \frac{1}{2}m(v_f^2 - v_i^2) \tag{4.14}$$

The rotational kinetic energy formula is very similar to linear kinetic energy:

$$\Delta K_{rot} = \frac{1}{2}I\omega^2 \tag{4.15}$$

where I is the moment of inertia for the object and ω is the angular velocity of the object.

Conservation of energy then tells us that:

$$\Delta U + \Delta K_{lin} + \Delta K_{rot} = 0 \tag{4.16}$$

4.4 Part 1

4.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ruler/measuring tape
- Long, flat surface to roll things on

4.4.2 Activity Instructions

You will be reusing your slope surface that you have been using in the last two projects. However, you will need new objects that move down the slope – you will need to find objects that roll in a straight line. Your first task will be to find two objects with different distributions of mass along their cross section that you can roll down the surface.

Whatever objects you pick, ensure that your two objects have different forms. You will likely find only objects that are either solid cylinders, hollow cylinders, or solid spheres. Some ideas for object that you might have lying around:

- Roll of coins (solid cylinder)
- Roll of tape (hollow cylinder)
- Hockey puck (solid cylinder)
- Golf Ball (solid sphere)
- Other type of ball (often hollow sphere)

It is acceptable for your object to be comprised of multiple different materials, so don't worry too much. (Hint: Keep this in mind when you are doing the final write up for this Project!)

Your slope should still have marks on it indicating known distances along it from Project 3. Starting at one of them, roll one of your objects down the slope while video recording, and repeat the process until you have enough measurements to estimate the statistical uncertainty on the timing measurement. Choose four other locations along your slope and repeat this procedure. Thus, you will have time and distance measurements for five different points along your slope.

Repeat this procedure and all data collection for your second object as well.

4.4.3 Analysis

The focus of your analysis for this part will be on analyzing the graphs of the measurements you took of the time and distances for the objects rolling down the slope.

Determine the mean value of your timings for each distance value and for each object, along with the uncertainty, and make a graph of time versus distance for each object. You should notice that the graphs are not linear, and so will need to be linearized. You can linearize the graph by taking the square of the time values – don't forget to calculate the new uncertainties for the square of the time values, following proper uncertainty propagation.

Now calculate the slope and intercept for each of these linearized graphs using least squares methods. Plot a line of best-fit for each graph overtop of the existing data points.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so

in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Describe your experimental setup, including photos of the setup. Show a close up of the end of the slope and how you solved the transition to horizontal motion.

2. Roll your objects down the slope five times from five different distances and record the time it takes to complete the roll. Make a table of your results that indicate the object, the trial number, the distance it travels, and the time taken to roll the distance. Include measurement uncertainties for all quantities that have measurement uncertainties.

3. Create a new table that summarizes your results for each roll distance. You will need a column for the distance traveled and a mean value for the time, along with a combined statistical and measurement uncertainty for the time. Add another column that includes the square of the mean value of the time, along with the uncertainty on the square of the time.

4. Create a scatter plot graph of the time versus the roll distance, including uncertainties as error bars.

5. Create a scatter plot graph of the square of the time versus the roll distance, including uncertainties.

6. Determine the slope and intercept using least squares analysis.

7. Determine the uncertainty in the slope and intercept using least squares analysis.

8. Respond to the following questions/instructions:

   (a) Describe the steps you took to ensure that the results of your experiment are accurate.

   (b) Describe the steps you took to reduce the measurement uncertainty in your experiment.

   (c) Describe the steps you took to reduce the statistical uncertainty in your experiment.

   (d) Comment on whether the time versus distance graph looks linear, and if not what type of graph it looks like.

(e) Comment on whether the time squared versus distance graph looks linear, and if not what type of graph it looks like.

4.5 Part 2

4.5.1 Required Equipment

No equipment will be needed.

4.5.2 Activity Instructions

This part of Project 4 is entirely based on analysis. You will not be collecting any more data.

4.5.3 Analysis

In Part 1, you collected and graphed your data and determined a slope and intercept for a linearized version of your data. In Part 2, you will be using the slope to calculate an experimental value of the specific moment of inertia (I/M) of your objects, and then calculating a theoretical value for the specific moment of inertia from measurements of the object's dimensions, and then comparing these.

Using rotational kinematics and Newton's Laws, you will need to solve for the acceleration a of an arbitrary object with mass m, outer radius R, and moment of inertia I rolling down a slope of angle θ without slipping. From this acceleration, and using 1D kinematics equations, you will need to find an equation relating the time t it takes for an object with acceleration a to roll a distance d, starting from rest. Since you graphed t^2 versus distance d, you should rearrange your formula to fit a linear line and identify how the slope you calculated relates to the parameters you used in your solution (d, θ, I, m, R).

You should find that your result depends on the ratio of the moment of inertia to the mass I/m. This is important, since you did not measure the mass of your object. Theoretically, the overall mass is irrelevant to the acceleration, the same as it is for objects falling (all objects fall at the same rate, g, regardless of mass), and only the distribution of the mass matters. Use your calculated slope, and this equation, to determine a theoretical value for your specific moment of inertia, I/m. You will be expected to perform uncertainty propagation to connect the uncertainty in your slope to the uncertainty in your specific moment of inertia.

Next you will need to use your measurements of the dimensions of the objects to calculate theoretical values for the specific moment of inertia, along with their associated uncertainties. Lastly, you should perform a two-sample t-test to determine whether there is agreement between the experimental and theoretical values.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Measure the dimensions of your objects. If they are cylinders, you will need a diameter and a radius. If they are spherical, you will just need a diameter. Use your measurements to calculate the specific moment of inertia (I/m) based on the theoretical formula for the shape of the objects you are using.

2. Write down the uncertainty in your measurements of your object dimensions. Use these and uncertainty propagation to determine the uncertainty in your specific moment of inertia using the derivative form of the uncertainty derivation (i.e. do not use templates).

3. Use rotational kinematics and conservation of energy to show that the distance the object rolls down the slope d is related to the square of the time taken to roll down the slope T via the equation: $d = \dfrac{g \sin\theta}{2(1 + I/mR^2)} T^2$.

4. Comparing the above relationship to the equation for a straight line of d vs T^2, show that the specific moment of inertia I/m is related to the slope A as $I/m = \left(\dfrac{g \sin\theta}{2A} - 1\right) R^2$. Use this to solve for the experimental value of the specific moment of inertia.

5. Use uncertainty propagation to solve for the uncertainty in the experimental value of the specific moment of inertia.

6. Compare the theoretical value to the experimental value.

7. Respond to the following questions/instructions:

   (a) Did the two values agree? If not, what does this imply about the theoretical value as compared to the experimental value? If they do, comment on whether this is a reliable experiment or caused by some other issue - quantify your justification.

   (b) Did you make any simplifying assumptions or approximations in calculating the theoretical moment of inertia? If so, what were they and what information would be needed to avoid making these assumptions/approximations?

   (c) Given the precision of the measurements, can you exclude drastically different geometries with the same mass and radius? (e.g. if you use a solid disk template, can you exclude a sphere of same radius and mass?) Justify your answer with math by calculating I/MR^2 and its uncertainty.

(d) What is the largest contribution to the uncertainty in your data points on the plot from Part 1? Is it statistical or a measurement uncertainty? Explain how you came to this determination.

(e) What is the largest contribution to the uncertainty in your slope? Explain how you came to this determination.

(f) In hindsight, what factors might have influenced your results but were not taken into account in the analysis?

(g) How could you have performed the experiment differently with changes to the methodology or with reasonably available equipment in order to account for these external influences?

(h) In hindsight, how might you have improved the precision of your measurements? (Hint: If the statistical uncertainty is larger than the measurement uncertainty, more precise equipment is not the answer.)

(i) Explain, referencing your largest sources of uncertainty above, how you could have increased both the accuracy and precision of your determination of g.

Appendix A: Statistics

A.1 Mean and Standard Deviation

Statistics is not incorporated commonly into the theory/lecture portion of physics courses, yet is one of the most important tools in scientific research. At the very least, when measurements are performed, there will always be some amount of statistical variation in the results such that the results will be slightly different if the experiment were performed again. Reducing the amount of this variation is one of the primary challenges of experimental scientists.

As a simple example, consider measuring the time it takes an object to fall down a tube. Even if we assume the timing device is perfectly accurate, there are subtle variations in the atmosphere and other environmental factors that can result in slight differences in the time it takes the object to fall. Subtle differences in the start and stop triggers for the timing device usually make up a larger component of the variation, resulting in a seemingly identically performed measurement but producing a slightly different value. When human judgment is involved, such as a human lining up a ruler, or starting/stopping a timer, or in some way reacting to a situation, the amount of statistical variation always increases.

In this course, we are going to assume that all of the random variation that occurs in an experiment follows a Normal or Gaussian distribution. A Gaussian distribution is readily characterized by a mean value (μ) and a standard deviation (σ), as shown in Figure A.1. The mean value represents the **the expected value of the measurement**, while the standard deviation is a **measure of how precise is the measurement** or how wide is the distribution. A smaller standard deviation is thus a more precise measurement, and a narrower curve. This is illustrated in Figure A.2.

In practice, it is not feasible to perform a measurement enough times to verify that it does follow a Gaussian distribution. Instead, scientists typically perform enough measurements to make an estimation of the mean value and the standard deviation. In this course, $N = 20$ measurements will be considered enough to estimate these values.

The mean value of a measurement is determined as the sum of all of the measurements, divided by the total number of measurements. If each measurement is identified as an x_i with the subscript indicating which measurement is being referred

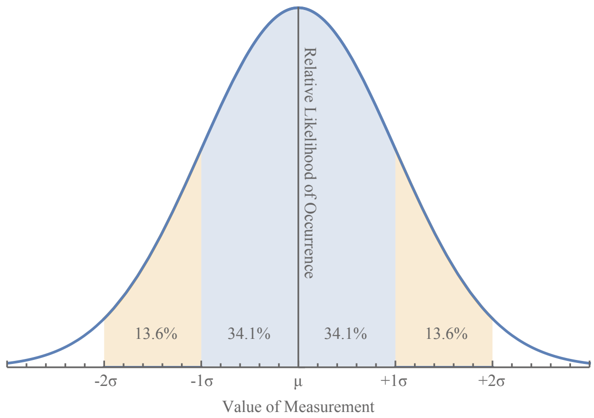


FIGURE A.1: Gaussian or Normal probability distribution. When applied to a measurement in an experiment, the x-axis represents the value of the measurement, while the y-axis represents the relative probability of measuring that value. Approximately 64% of all measurements will fall within one standard deviation ($\pm 1\sigma$) of the mean, while approximately 95% of all measurements will fall within two standard deviations.

to, then the formula would be:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ...}{N} \tag{A.1}$$

$$= \frac{1}{N}\sum_{i=1}^{N} x_i \tag{A.2}$$

The standard deviation is calculated by finding the mean square deviation and then taking the square root:

$$\sigma_x = \sqrt{\frac{\sum\limits_{i=1}^{N}(x_i - \bar{x})^2}{N-1}} \tag{A.3}$$

$$= \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + ...}{N-1}} \tag{A.4}$$

You may have seen an equation for the standard deviation before that only has an N in the denominator under the square root. This is an approximation that is used for cases where N is really large, where $N - 1 \approx N$. However, the $N - 1$ is important in situations where the number of values used to calculate the mean is small.

Important note: Calculating the standard deviation from a small set of numbers is unreliable, and often under-estimates the value. As a result, if there are fewer than 10 values usable for calculating a standard deviation, then the the standard deviation should be treated as a quarter of the distance between the maximum and minimum values. In the form of an equation: $\sigma_x \approx \frac{x_{max} - x_{min}}{4}$.

Here is an example that you can reproduce if you want to check your understanding of these equations:

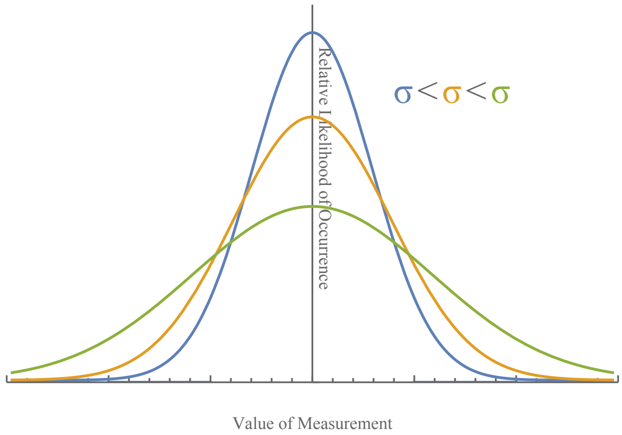


FIGURE A.2: Illustration of Gaussian distributions with different standard deviation (σ) values. The largest standard deviation corresponds to the green curve, which is the broadest of the three curves. In order for 64% of all measurements to fall within one standard deviation, no matter what is the standard deviation value, the curves are naturally shorter for a larger standard deviation.

EXAMPLE:
Calculate the mean and standard deviation of the following numbers:
18.8, 25.4, 19.5, 20.9, 16.3, 16.6, 20.8, 22.4, 16.5, 21.7, 19.1, 23.4, 21.1, 18.4, 22.4, 19.1, 18.8, 15.9, 25.0, 20.4

$$\begin{aligned}
\bar{x} &= (18.8 + 25.4 + 19.5 + 20.9 + 16.3 + 16.6 + 20.8 + 22.4 + 16.5 + 21.7 + 19.1 \\
&\quad +23.4 + 21.1 + 18.4 + 22.4 + 19.1 + 18.8 + 15.9 + 25.0 + 20.4)/20 \\
&= 20.125
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= ((18.8 - 20.125)^2 + (25.4 - 20.125)^2 + (19.5 - 20.125)^2 + (20.9 - 20.125)^2 \\
&\quad +(16.3 - 20.125)^2 + (16.6 - 20.125)^2 + (20.8 - 20.125)^2 + (22.4 - 20.125)^2 \\
&\quad +(16.5 - 20.125)^2 + (21.7 - 20.125)^2 + (19.1 - 20.125)^2 + (23.4 - 20.125)^2 \\
&\quad +(21.1 - 20.125)^2 + (18.4 - 20.125)^2 + (22.4 - 20.125)^2 + (19.1 - 20.125)^2 \\
&\quad +(18.8 - 20.125)^2 + (15.9 - 20.125)^2 + (25.0 - 20.125)^2 \\
&\quad +(20.4 - 20.125)^2)/19 \\
&= 7.5925 \\
\sigma_x &= 2.755
\end{aligned}$$

The mean value of this set of numbers is $\bar{x} = 20.125$ and the standard deviation is $\sigma_x = 2.755$.

A.2 Significant Digits, Uncertainty, Standard Deviation of the Mean

In math and science classes, it is common for students to be held to a standard when communicating the results of their calculations. This standard is commonly called **significant digits** or **significant figures**. Students are taught to round their answers to the same precision of the least-precise number in the calculation. For example, when adding $2.467 + 1.23 + 3.1$, the full answer is 6.797, but students are expected to write down 6.8 because the least precise number in the set has two significant digits, so the answer should have two significant digits.

On the surface, this looks like an arbitrary set of rules, or at least rules that are designed around simplifying the answer for easier marking. For example, $\sqrt{2.000} = 1.41421356237309505....$ For a marker's purposes, 1.414 would be sufficient to indicate that the student calculated the number correctly, so all the extra digits just take up space on the paper.

However, the origin of significant figures is actually in the sciences and has to do with a part of statistics and experimentation known as **Uncertainty** or **Error**. More information about uncertainty and error calculations is in Appendix F. However, to understand that material, it is critical to first understand the **standard deviation of the mean** ($\sigma_{\bar{x}}$).

When performing an experiment, it is impossible to take every possible measurement. For example, there are too many dogs to measure the length or mass of all dogs in order to accurately describe the average length or mass of all dogs. While less obvious, measuring the length of an object can be done multiple times in multiple different ways with multiple different measuring devices – performing all of these measurements is not feasible. Instead, we accept that the best that can be done is to take a *sample* of measurements out of the entire *population* of possible measurements.

The standard deviation value is a measure representing the likelihood of new individual measurements being in proximity to the mean value of our sample. The standard deviation *of the mean* is a measure representing how close we might get to our original value if we were to repeat all of the measurements over again – how likely two subsequent experiments are to produce the same result. The standard deviation of the mean is also referred to as the **uncertainty** or **error** value. The use of the term *error* does not refer to a mistake, but rather the likelihood that the value is not representative of the population.

The standard deviation of the mean can be determined in two ways. The obvious way would be to repeat the experiment many times and then calculate the standard deviation of the mean values from all of the different experiments. This is long and laborious. A simpler way is to *estimate* the standard deviation of the mean by using the standard deviation value from a single experiment using the formula:

$$\sigma_{\bar{x}} \approx \sigma_x / \sqrt{N} \tag{A.5}$$

This estimation equation is not meant to replace performing experiments multiple times – replication of scientific studies is important to the scientific method. Instead, it is used to assess the precision of the mean value that was determined in the experiment.

This might make more sense with an example:

EXAMPLE:
Continuing our example from the previous section, here are the results from collecting multiple sets of random numbers that follow the same distribution.

Set 1: 23.3, 14.9, 24.9, 20.8, 16.3, 14.5, 19., 22.6, 18.2, 20.9, 23.9, 16.9, 18.1, 11.4, 21.9, 23.1, 20.7, 24.2, 16.4, 17.1

Set 2: 18.6, 27.5, 18.5, 15.9, 23.7, 18.4, 19.2, 18.3, 22.8, 19.1, 20.1, 17.3, 25.6, 19.3, 22.1, 20.8, 13.8, 22.0, 17.6, 25.2

Set 3: 17.5, 17.2, 15.7, 24.3, 19.1, 18.8, 19.0, 25.7, 17.0, 18.6, 24.2, 18.0, 18.1, 20.2, 19.5, 22.4, 25.6, 18.6, 17.3, 24.4

Set 4: 16.6, 18.8, 18.0, 19.9, 22.4, 18.2, 21.0, 24.6, 20.1, 22.9, 19.2, 23.2, 21.6, 19.9, 17.1, 19.2, 16.3, 15.0, 20.4, 18.0

Set 5: 19.2, 20.7, 20.9, 16.4, 15.9, 22.7, 20.2, 20.3, 17.6, 20.0, 22.5, 20.7, 21.8, 19.0, 17.8, 19.0, 21.5, 17.2, 16.6, 18.2

Set 6: 20.7, 17.8, 21.5, 22.5, 18.7, 21.3, 12.4, 21.9, 21.6, 23.8, 15.0, 21.1, 20.2, 21.1, 22.1, 19.7, 14.3, 20.2, 22.0, 16.9

Set 7: 21.2, 20.0, 19.5, 23.4, 17.3, 20.7, 19.7, 21.9, 24.4, 23.6, 20.8, 21.6, 14.8, 21.8, 20.3, 22.9, 17.6, 21.6, 25.6, 18.1

Set 8: 18.8, 21.1, 21.3, 26.1, 23.7, 14.6, 20.7, 14.0, 21.4, 26.4, 17.6, 19.9, 18.1, 18.8, 15.8, 12.6, 19.0, 24.5, 20.9, 20.4

Set 9: 18.6, 23.6, 20.9, 22.8, 19.5, 21.8, 23.7, 23.3, 17.7, 13.7, 15.5, 22.5, 22.9, 23.5, 20.4, 14.1, 16.1, 18.7, 18.8, 22.6

Set 10: 16.0, 21.5, 13.2, 18.1, 16.3, 25.9, 22.4, 19.1, 26.0, 23.0, 17.0, 24.5, 22.0, 18.5, 18.7, 15.0, 18.1, 19.0, 19.3, 17.8

The mean values for each of these sets is: 19.945, 20.485, 18.960, 18.500, 21.450, 20.475, 19.145, 20.030, 20.140, 19.045, respectively. The average value of this set of mean values is 19.8775.

The standard deviation of these mean values is:

$$\begin{aligned}
\sigma_{\bar{x}} \quad = \quad & \big((19.945 - 19.8775)^2 + (20.485 - 19.8775)^2 + (18.960 - 19.8775)^2 \\
& +(18.500 - 19.8775)^2 + (21.450 - 19.8775)^2 + (20.475 - 19.8775)^2 + (19.145 - 19.8775)^2 \\
& +(20.030 - 19.8775)^2 + (20.140 - 19.8775)^2 + (19.045 - 19.8775)^2\big)\ /9 \\
& = 0.68767 \approx 0.69
\end{aligned}$$

If we use our estimation formula on the dataset from the previous example, we get:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \frac{2.75545}{\sqrt{20}} = 0.61614 \approx 0.62$$

Note: This technique only works because all of the sets of numbers were drawn from the same population.

As you can see in the example, the two methods to determine the standard deviation of the mean produce very similar values. Remember that both of these are *estimations* of the theorized *true* value, so neither one of them should be treated as the correct value to compare against. Since both calculations return similar results, it is reasonable that the simpler one can be used for practical purposes.

Significant figures arise when communicating a scientifically determined quantity. In the above example, the standard deviation of the mean was 0.61614 – but remember that this was simply an estimate. The first significant digit was the same between our long method and our short method, so the first digit is the only digit that is important. Thus, we truncate our standard deviation of the mean to the **first significant digit**. In some cases, for advanced and highly precise science, two significant digits of the standard deviation of the mean can be used. We will only use one in this class.

With this value truncated, we then report the mean value rounded to the same precision as the standard deviation of the mean/uncertainty/error (all three terms refer to the same quantity). If $\sigma_{\bar{x}} = 0.6$, then we need to round the mean value (20.125) to one decimal place: $\bar{x} = 20.1$. The last thing we do is report the value using the following format:

$$\bar{x} = 20.1 \pm 0.6$$

This makes the statement that the mean value was determined to be 20.1, and it is believed that there is a $\sim 64\%$ chance that the true mean value of the population is within the range $20.1 - 0.6$ and $20.1 + 0.6$.

Thus, significant figures tell us about how precisely we know the value of an object. Including more significant figures in the value than the leading digit(s) of the uncertainty would make a claim about knowledge that is not supportable by evidence. In math and physics classes, the use of significant figures is meant to prepare students for their application in engineering and the sciences. While it is true that we can know the value of radicals and fractions to infinite precision, the use of that precision is irrelevant in practice – typically only the precision of the uncertainty matters.

EXAMPLE:
Given the following information on mean values, standard deviations and sample sizes, write the mean values and the errors in the correct format.

a) $\bar{x} = 0.4512858123\text{m}$, $\sigma_x = 0.0155123\text{m}$, $N = 10000$

b) $\bar{t} = 5.6235123 \times 10^5\text{s}$, $\sigma_t = 45.9725\text{s}$, $N = 50$

c) $\bar{m} = 1.2459001 \times 10^{-2}\text{g}$, $\sigma_m = 4.578 \times 10^{-4}\text{g}$, $N = 1000$

Solutions:

a) $\sigma_{\bar{x}} = 0.0155123/\sqrt{10000}\ \text{m} = 0.000155123\ \text{m} \approx 0.0002\ \text{m}$

$$\bar{x} = (0.4513 \pm 0.0002)\ \text{m}$$

b) $\sigma_{\bar{t}} = 45.9725/\sqrt{50}\ \text{s} = 6.50149\ \text{s} \approx 7\ \text{s}$

$$\bar{t} = (5.62351 \pm 0.00007) \times 10^5\ \text{s}$$

c) $\sigma_{\bar{m}} = 4.578 \times 10^{-4}/\sqrt{1000}\ \text{g} = 0.0000144769\ \text{g} \approx 1 \times 10^{-5}\text{g}$

$$\bar{m} = (1.246 \pm 0.001) \times 10^{-2}\ \text{g}$$

Appendix B: Using Spreadsheets

Spreadsheet programs are fairly ubiquitous software – the major ones are Microsoft Excel, Apple Numbers and Google Sheets. All of them have the same basic functionality, with some slight differences between them that are either cosmetic or require significantly advanced knowledge to use. For this reason, only Google Sheets will be described in this appendix – Google Sheets is free and available on all platforms, and so is the most widely usable one by students. However, there will be no consequences for choosing to use a different one.

The view of a spreadsheet looks like Figure B.1. It is effectively a table with columns labeled with letters and the rows labeled with numbers. A particular cell within the spreadsheet is thus identified as a combination of a letter with a number. For example, the upper left most cell is referenced as $A1$. Cells can have text within them, numbers, or they can perform mathematical operations based on the values in other cells.

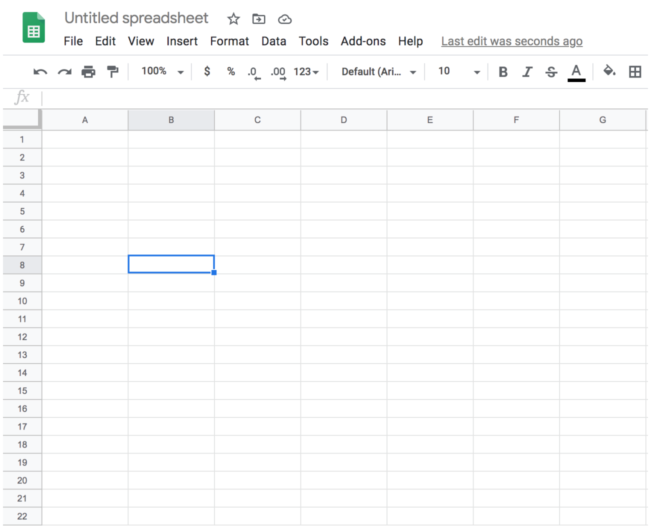


FIGURE B.1: Image of a spreadsheet showing the first few rows and columns. The cell $B8$ is selected in this case.

B.1 Creating Tables

Spreadsheets can be used to easily create tables of data that can be copied and pasted into a document. Figure B.2 shows a simple array of values with column headings.

By pressing the Borders button, shown in the bottom left of Figure B.3, it is possible to add structural lines around cells to help with organization. Typically, these properties will copy over to any document for displaying data.

	A	B
1	x-values	y-values
2	1	0.3
3	2	0.1
4	3	1.5
5	4	2.3
6	5	2.1
7	6	5.5
8	7	1.2
9	8	4.4
10	9	6.6
11	10	1.2

FIGURE B.2: A simple spreadsheet showing multiple columns of data with multiple rows. Column headings are included at the top of each column.

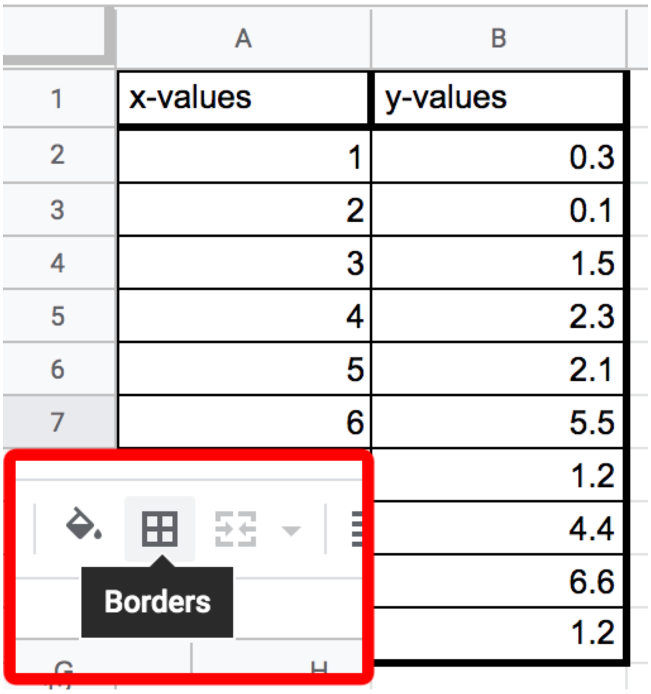


FIGURE B.3: The same simple spreadsheet, but now with thick lines around it and around the column headings, with thin lines separating the data cells. This is just one way to improve visibility of the data. There is no fixed style that must be used. However, some kind of organizational style is helpful to improve readability.

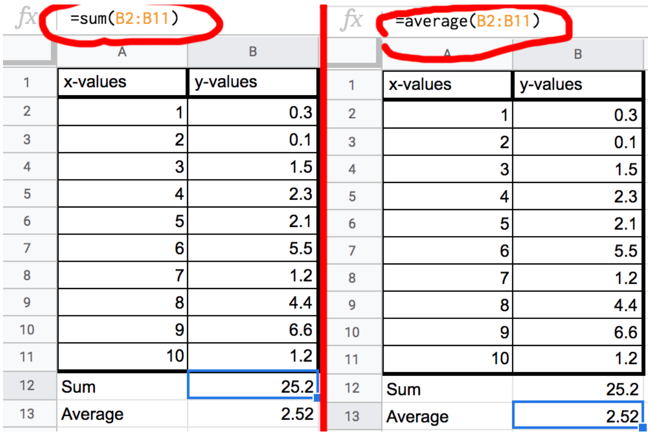
B.2 Formula Basics

A cell can contain text/numbers, or it can contain the results of formulas. To use a formula, the first character in the cell must be an equal sign, =. Common mathematical functions include:

Name	Symbol	Example	Approx. Result
Addition	$+$	$= 3.2 + 2.3$	5.5
Subtraction	$-$	$= 3.2 - 2.3$	0.9
Multiplication	$*$	$= 3.2 * 2.3$	7.36
Division	$/$	$= 3.2/2.3$	1.391
Exponents	^	$= 3.2$^2.3	14.52
Square Root	sqrt()	$=$ sqrt(4)	2
Powers of Euler's Number	exp()	$=$ exp(3.2)	24.53
Log Base 10	log()	$=$ log(3.2)	0.505
Natural Log	ln()	$=$ ln(3.2)	1.16
Cosine (radians)	cos()	$=$ cos(3.2)	-0.998
Sine (radians)	sin()	$=$ sin(3.2)	-0.058
Tangent (radians)	tan()	$=$ tan(3.2)	0.058
Arccos (radians)	acos()	$=$ acos(0.3)	1.266
Arcsin (radians)	asin()	$=$ asin(0.3)	0.305
Arctan (radians)	atan()	$=$ atan(0.3)	0.291
Round	round()	$=$ round(0.39, 1)	0.4

The strength of using a spreadsheet is not in performing manually entered formulas, but in performing mathematical calculations based on the values entered into specific cells. All of the above formulas can be combined with data entered in cells. For example, if cell $A1$ has contents 3.2 and cell $B1$ has contents 2.3, then the addition example would be just as easily accomplished with $= A1 + B1$.

Some functions require ranges of cells as input. A range of cells is accomplished with a semi-colon between the upper left cell and the lower right cell. For example $A1 : B4$ would be a selection of all cells between $A1$ and $B4$ – a total of 8 cells. Two common functions that you will find useful for your lab reports, summation and averaging, are shown in Figure B.4.



=sum(B2:B11)

	A	B
1	x-values	y-values
2	1	0.3
3	2	0.1
4	3	1.5
5	4	2.3
6	5	2.1
7	6	5.5
8	7	1.2
9	8	4.4
10	9	6.6
11	10	1.2
12	Sum	25.2
13	Average	2.52

=average(B2:B11)

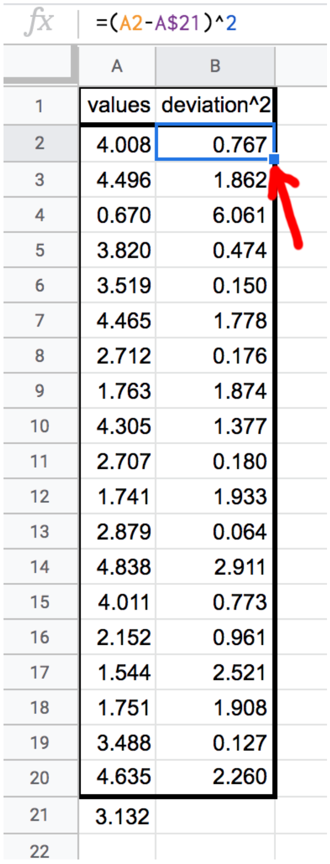
	A	B
1	x-values	y-values
2	1	0.3
3	2	0.1
4	3	1.5
5	4	2.3
6	5	2.1
7	6	5.5
8	7	1.2
9	8	4.4
10	9	6.6
11	10	1.2
12	Sum	25.2
13	Average	2.52

FIGURE B.4: Summation and addition are very commonly used in spreadsheets. These will be helpful when calculating averages and standard deviations.

EXAMPLE:
Use a spreadsheet to calculate the Standard Deviation of the following series of values:
4.008, 4.496, 0.670, 3.820, 3.519, 4.465, 2.712, 1.763, 4.305, 2.707, 1.741, 2.879, 4.838, 4.011, 2.152, 1.544, 1.751, 3.488, 4.635

To do this, we need to create a column of these values. The top-most cell in the column, $A1$, will be the label for the data. Then all of the values will be entered beneath that label heading in cells $A2 : A20$. Below the list of values, calculate the average for that column by entering the formula $= Average(A2 : A20)$.

The next thing to do is create a second column beside the one with our values, to contain the square of the deviation ($(x - \bar{x})^2$). This is accomplished by entering the formula $= (A2 - A21)\hat{\ }2$ into cell $B2$. This needs to be repeated for cell $B3$ through $B20$. There is an easier way to do this, however, by autocompleting the entering of values in the cells below $B2$. Change the formula to be $= (A2 - A\$21)\hat{\ }2$, where the \$ will lock the location of cell $A21$ into the formula, but allow other cells to change. Then click and drag the blue square in the bottom right corner of cell $B2$ down to the bottom. This is shown in Figure B.5.



fx =(A2-A$21)^2

	A	B
1	values	deviation^2
2	4.008	0.767
3	4.496	1.862
4	0.670	6.061
5	3.820	0.474
6	3.519	0.150
7	4.465	1.778
8	2.712	0.176
9	1.763	1.874
10	4.305	1.377
11	2.707	0.180
12	1.741	1.933
13	2.879	0.064
14	4.838	2.911
15	4.011	0.773
16	2.152	0.961
17	1.544	2.521
18	1.751	1.908
19	3.488	0.127
20	4.635	2.260
21	3.132	
22		

FIGURE B.5: The values are entered into column A, and then the deviation squared is calculated in column B. The red arrow shows where the blue box is that can be used to auto-fill the cells beneath.

The formula for the standard deviation is:

$$\sigma_x = \sqrt{\frac{(\sum(x-\bar{x})^2}{N-1}}$$

Column B contains the individual square deviations. Thus, we want to add up all of the values in column B, divide by $N-1$, and then take the square root. This can be accomplished by entering the formula $= sqrt(sum(B2:B20)/18)$ into cell $B21$. This is shown in Figure B.6.

B.3 Plotting

While Microsoft Excel will do plotting well enough for the purposes of these labs, Google Sheets will not. As a result, it is recommended that you use Plotly. It is free to use by visiting https://chart-studio.plotly.com/create/#/. It also has some spreadsheet capabilities, however it does not perform calculations. Thus, the calculations will need to be done in Excel, Numbers or Sheets, and then the plotting can be done in Plotly.

Plotting is much more easily explained by example. Thus, all explanations below will be based on the following data, that is assumed to be entered into a spreadsheet fx =sqrt(sum(B2:B20)/18)

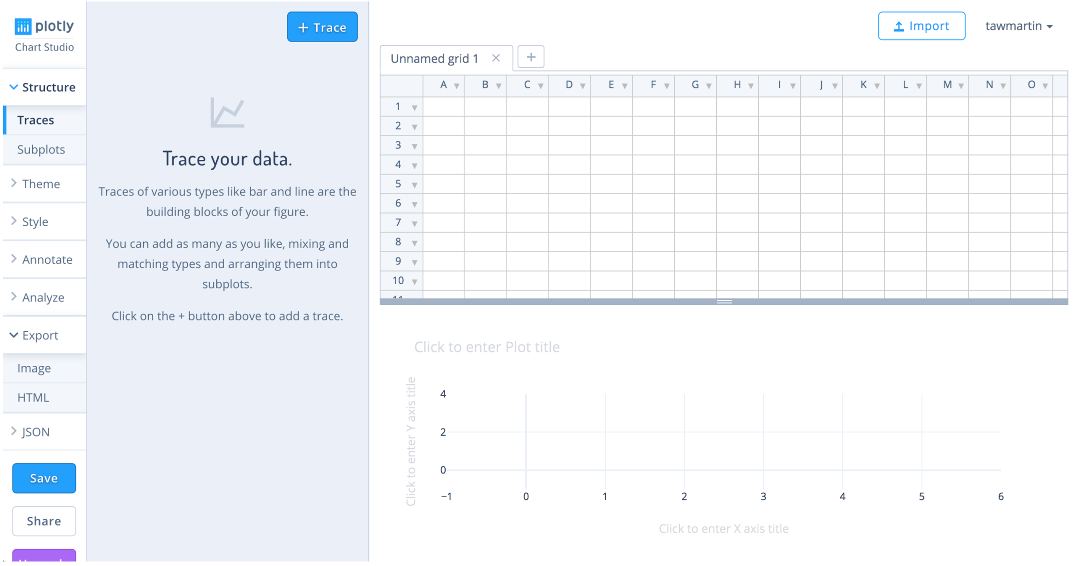
	A	B	C
3	4.496	1.862	
4	0.670	6.061	
5	3.820	0.474	
6	3.519	0.150	
7	4.465	1.778	
8	2.712	0.176	
9	1.763	1.874	
10	4.305	1.377	
11	2.707	0.180	
12	1.741	1.933	
13	2.879	0.064	
14	4.838	2.911	
15	4.011	0.773	
16	2.152	0.961	
17	1.544	2.521	
18	1.751	1.908	
19	3.488	0.127	
20	4.635	2.260	
21	3.132	1.251	
22			

FIGURE B.6: The values are entered into column A, and then the deviation squared is calculated in column B. The red arrow shows where the blue box is that can be used to auto-fill the cells beneath.

like Google Sheets:

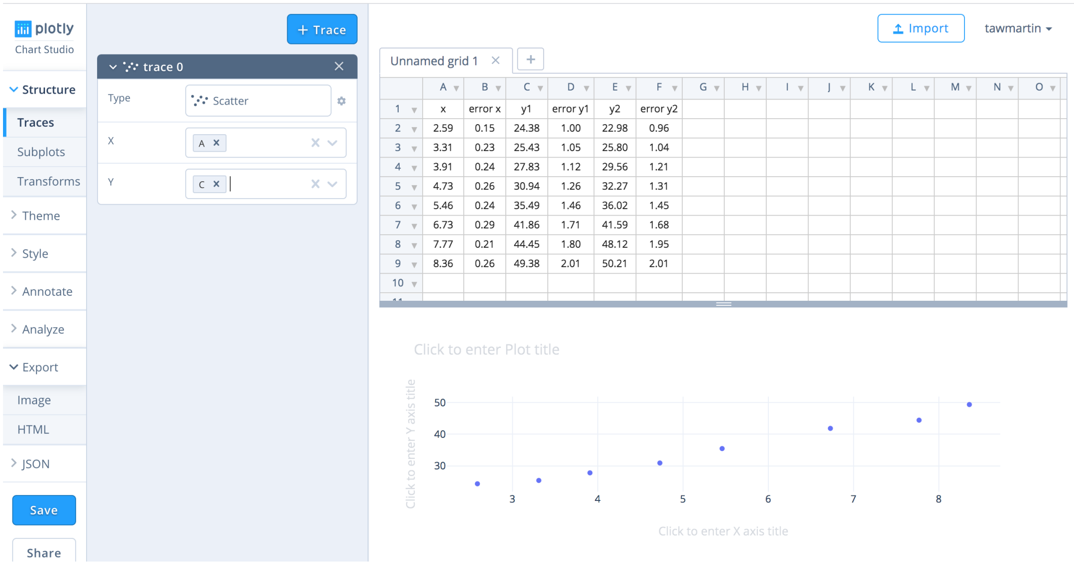
	A	B	C	D	E	F
1	x	error x	y1	error y1	y2	error y2
2	2.59	0.15	24.38	1.00	22.98	0.96
3	3.31	0.23	25.43	1.05	25.80	1.04
4	3.91	0.24	27.83	1.12	29.56	1.21
5	4.73	0.26	30.94	1.26	32.27	1.31
6	5.46	0.24	35.49	1.46	36.02	1.45
7	6.73	0.29	41.86	1.71	41.59	1.68
8	7.77	0.21	44.45	1.80	48.12	1.95
9	8.36	0.26	49.38	2.01	50.21	2.01

When you first open Plotly, you will see a screen that looks similar to:



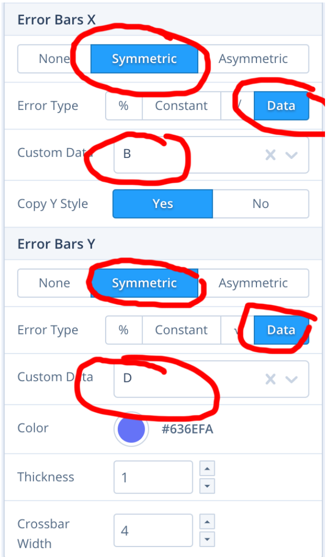
Copy your data from your spreadsheet and paste it into the spreadsheet area of Plotly.

To create a graph, click on the *Structure* button on the left, and then *Traces*, then on the blue *+Trace* button. For the x values, select column A, and for the y values select column C. The result should look something like this:



Repeat this same procedure but for column A and column E to create a second graph.

To add error bars, click on the *Style* button on the left, and then *Traces*, which appears underneath it. Scroll down in the middle menu until you find the *Error Bars X* and *Error Bars Y* entries. Choose *Symmetric*, then *Data*, and then enter the column of your x error bars for the *Error Bars X* entry. Repeat for *Error Bars Y*. Continue scrolling to find the same entries for your other trace(s). This should look like:



The last few things to finish up the plot will be to add labels to the x and y axes, as well as labels to the datasets in the legends.

Also, it is possible to use Plotly to automatically determine the line-of-best-fit and include it on the plot. To do so, choose the *Analyze* and then *Curve Fitting* option, and click on the *Run* button for the dataset that you want to produce a line for. Note that Plotly will give a value for the slope and the intercept, however you will be expected to calculate these yourself and show your calculation in your report. You can use Plotly's values to check if you did it correctly, though. Plotly does not give estimates of the uncertainty in the line parameters (slope and intercept), which you will be expected to include in your reports.

A completed plot might look like Figure B.7.

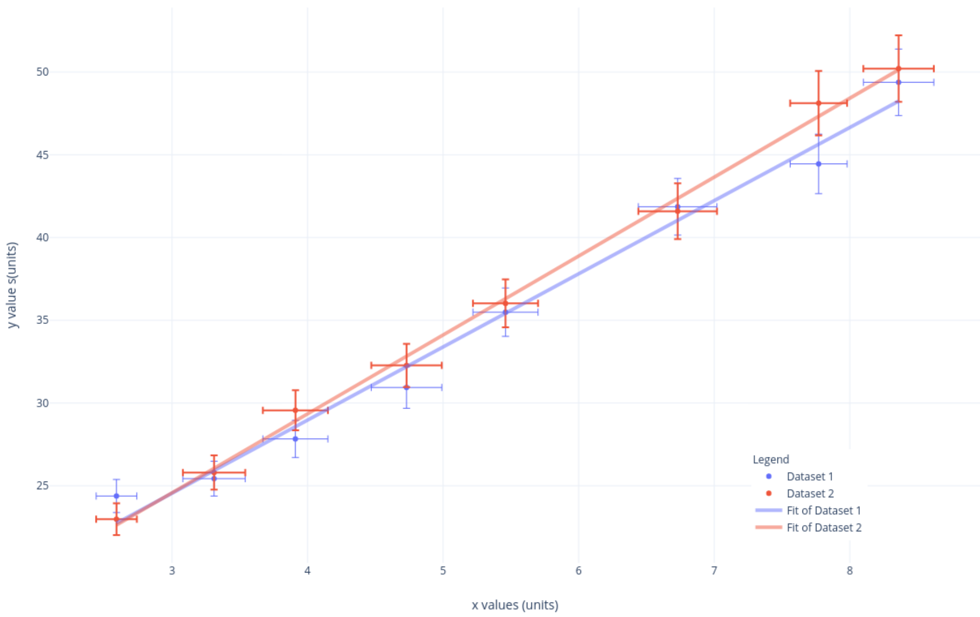


FIGURE B.7: Example of a completed plot produced using Plotly. Plot includes two different datasets, best fit lines, a legend, and proper labeling of the axes. The equation for the line of best fit can be included on the graph or in the caption.

Appendix C: Hypothesis Testing

A central feature of the scientific method is that hypotheses and theories are never proven. This is contrary to much of what you might read in the media, where words like *prove* and *cause* are used frequently to describe the results of a scientific experiment. The use of these terms misrepresent the goal of science altogether – the scientific method is used to find predictive patterns among natural phenomena, and that is all.

Consider the advancement of astronomy over the last several hundred years. For thousands of years, our understanding of the "heavens" was dominated by geocentric theory – where Earth is the centre of the universe – and the motions of celestial objects was predicted by the Ptolomaic model. The predictions of the Ptolomaic model were accurate over short periods of time, to the precision of the instruments that were available during that era, but advancements in technology changed human interpretation of the observed data of the stars. After geocentric theory came heliocentric theory, dominated by the contributions of Kepler and Galileo. Heliocentric theory posited the Sun at the centre of the universe, and allowed for orbits to be elliptical rather than just circular (as in geocentric theory). When Newton's Laws were developed, it displaced heliocentric theory, providing an explanation that the centre of orbits was the centre-of-mass of the system of orbiting objects, and no longer required that the Sun was the centre of the universe. Later observations of the movement of the stars and Mercury showed that Newton's Laws were also insufficient/inaccurate at predicting the future positions of objects, and the Theory of Relativity became dominant as our understanding of the motion of objects.

In each case, an existing theory was disproven, and a new theory was adopted. The reason the new theory was adopted was because the predictions of the new theory more accurately matched the observations. In turn, each new theory was disproven by later, more precise observations.

Scientific knowledge, when developed correctly, is the body of knowledge that makes useful predictions about future phenomena, and has not yet been disproven. For many, this understanding is disconcerting and they misuse the term *theory* to mean something other than how it is used in science. To be clear: a hypothesis is an idea about nature that has not yet been tested, or has been insufficiently tested – this is the word that should be used by people who deny scientific knowledge. A theory is an idea about nature that is worded in such a way as to produce predictions (hypotheses) about nature that, when tested, would conclusively disprove the idea if the predictions did not match reality, and that has been tested multiple times and in multiple different ways in attempts to disprove it.

Developing a theory is very difficult, and well beyond the activities of undergraduate students. However, testing a hypothesis – the first step in developing a theory, and the most important step – is a useful activity in which to participate. Each person should be empowered to test hypotheses, rather than simply trust the words of others. This appendix will provide an introduction into performing such tests.

Important Note: Not every experiment focuses on hypothesis testing. Scientific endeavours are often broken down into smaller subsections. Sometimes experiments are performed to measure an experimental or phenomenological constant, for example, where there is no known value. In these studies, typically the first study testing a hypothesis, there is nothing to compare against, and so hypothesis testing is not necessarily an intrinsic activity done. However, measures of precision are still critical in these studies, even if accuracy is not.

C.1 Accuracy and Precision

Intrinsically related to the concept of hypothesis testing is the idea of accuracy and precision. These terms are often used interchangeably in society, but they should not be – they have very different scientific meanings:
Precision – a measure of the reproducibility of a result of a measurement or experiment. **Accuracy** – a measure of how close a measured value is to another version of that value (measured or theoretical).

C.1.1 Precision

In practice, the term precision describes the uncertainty of a value. For example, $x = 42.3 \pm 0.3$cm has an uncertainty of $\delta_x = 0.3$cm. This is a description of the precision of the value of x. Sometimes, the precision is quoted as a percentage. In the previous example, the percent precision would be $0.3/42.3 = 0.7\%$, and the number could just as correctly be quoted as $x = 42.3\,\text{cm} \pm 0.7\%$.

However, using percentage quotes of precision can give the wrong impression, and so the two methods are not completely interchangeable. This is best explained by example: if a metre-stick has a measurement uncertainty of ± 0.1cm, then measuring a distance of 10cm versus 80cm would have the exact same measurement uncertainty, and yet a different percentage uncertainty. This gives the false impression that the distance of 80cm is more accurately measured, when it is, in fact, measured to the same precision.

To that end, measurement uncertainties should never be quoted as a percentage. Percentage uncertainties should only be used for values that are calculated from multiple different sources, and error propagation has been used to determine the overall uncertainty on the value. In such cases, either method of communicating precision is valid. Alternatively, percentage values can be used for statistical uncertainties. In all cases, using the direct value (the non-percentage one) is always valid.

C.1.2 Accuracy

In practice, the term accuracy describes how similar two distinct, independently measured/calculated values are to each other. For example, if the gravitational constant, G, is measured in two completely distinct experiments, accuracy would be a descriptor of how similar the two values are. Thus, accuracy takes into account both the difference in the two values, and the uncertainty.

For two separate values, $x_1 \pm \delta_{x_1}$ and $x_2 \pm \delta_{x_2}$, that follow a Gaussian probability distribution, a numerical measure of the accuracy would be:

$$t = \frac{x_1 - x_2}{\sqrt{\delta_{x_1}^2 + \delta_{x_2}^2}} \tag{C.1}$$

A larger value of t indicates a worse accuracy – improving accuracy means decreasing the value of t. This can be done in two ways: a smaller difference between x_1 and x_2 will produce a smaller t, but larger uncertainty values can also result in a smaller value of t. Obviously, the first one is more ideal. However, this tells an important truth about science – the larger the uncertainty (the lower the precision), the more likely two things are to agree but the less useful is the result.

In the case that one of the two values has a very high precision, the denominator of the t value will be dominated by the larger of the two uncertainties. In such cases, it is reasonable to approximate the t value as:

$$t \approx \frac{x_1 - x_2}{\delta_{x_1}} \tag{C.2}$$

assuming $\delta_{x_1} \gg \delta_{x_2}$. This is particularly used when calibrating, or when verifying an experimental apparatus/technique, as these are cases when the experimental value is viewed as less reliable than an accepted or target value.

A graphical illustration of the differences between precision and accuracy are shown in Figure C.1.

C.2 Disproving Hypotheses - Statistical Testing

In science, when testing a new hypothesis, there must always be a *null hypothesis* as an alternative. The *null hypothesis* is typically simply the statement that the proposed idea does not match nature. Null hypotheses is more complicated than what is necessary for this course. Instead, we will simply focus on basic hypothesis testing.

Note: There are a wide variety of statistical tests that are used for the multitude of difference types of scenarios that can arise. The one you will be doing in this class is related to the Student's T test, which is a comparison of mean values for samples. If you take a more advanced statistics course, you may be doing Z tests, p-value tests, and a variety of others.

The basic hypothesis we will be using is that the measure of accuracy discussed above, t, should follow a Gaussian probability distribution with a mean value of 0 and a standard deviation of 1. What this means in practice is that in a comparison of two values, $x_1 \pm \delta_{x_1}$ and $x_2 \pm \delta_{x_2}$, there is a $\sim 5\%$ chance that x_1 and x_2 are distinct

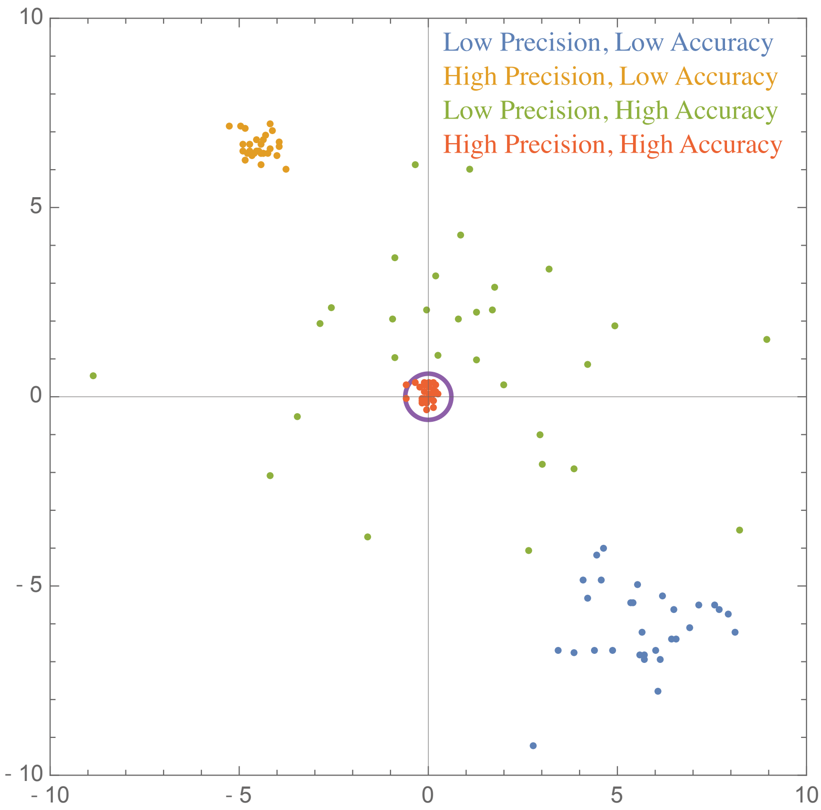


FIGURE C.1: This graph illustrates precision and accuracy. Points clustered close together (orange and yellow) represent higher precision than points that have a wide distribution (green and blue). Points whose mean value is closer to the centre of the purple circle (green and orange) represent higher accuracy than points that have a central value further from the purple circle (blue and yellow).

values (do not *agree* with each other) if $|t| > 2$. Thus, if $|t| < 2$, then we can say that x_1 and x_2 are both measuring the same quantity to the precision of the experiment(s).

In short, if $|t| > 2$, there is only a $\sim 5\%$ chance that the two mean values "agree with each other". Thus, we say that the two measurements "do not agree" with each other (with 95% probabilistic confidence).

Remember, if t is small because the precision is poor, that is not a useful or helpful outcome when trying to push the boundaries of science. However, for home experiments, such as in this laboratory, it is an indicator that we aren't neglecting anything in the analysis.

Typically, if the theory/model is sound and applied correctly, then $|t| > 2$ should only occur when there is something that isn't properly accounted for. For example, if we try to model an object sliding down a slope without including the coefficient of friction, the acceleration will be wrong and our prediction of time taken to travel some distance will be wrong. Or if we assume that friction is a fixed value and independent of the speed of the object, then our model might similarly fail to predict the results.

Science is more interesting when hypotheses are disproven (e.g. $|t| > 2$), because that is when the investigation into why the model/theory was not accurate at predicting results occurs. This is the fecund nature of science – it must always push precision and accuracy boundaries to improve predictive power. In your lab reports, if you find that you get an unexpected result, you will be tasked with trying to explain why your result was unexpected and proposing ways to fix the problem for future experimentation.

Appendix D: Graphical Analysis

When taking measurements, there are many things that can impact the results – some are random in nature and some are biased in some way. For example, imagine measuring the speed of an object by measuring the time it takes to travel some distance. If your timing method is consistently late in starting the timer but not stopping the timer, then the speed will consistently be measured too high. It doesn't matter how many times you repeat the experiment, you will get the same results. However, through experimental design it is possible to eliminate these factors. This is illustrated in Figure D.1.

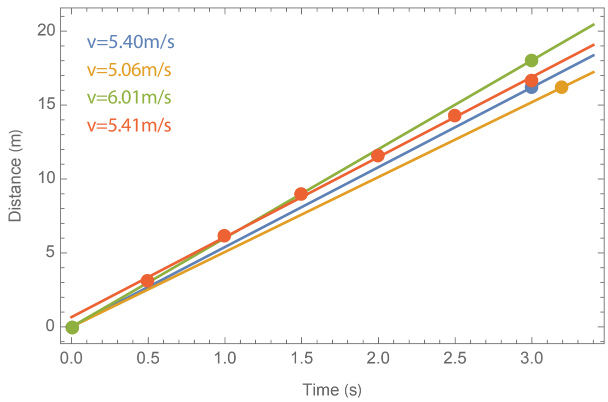


FIGURE D.1: This illustrates four different measurements of distance and time – the blue represents a perfect measurement scenario, with no random fluctuations or errors in the experimental process. The green and yellow lines represent the effect of some type of influence on the single measurement of distance and time. The red represents the effect of measuring multiple time/distance pairs. Speeds determined from this data are shown in the upper left of the figure.

In Figure D.1, there is one critical mistake that is incorporated into the experimental design for all but the red line: the assumption that there is a fixed origin point for $t = 0$s and $d = 0$m. The origin in an experiment is not actually a measured value, it is an assumed value. It is impossible to make a measurement of a length of time of 0s, nor a distance measurement of 0m. Thus, it is critically important to never assume that the origin is a value in your graph. This is the first lesson of graphical analysis. In the figure, it is also possible to see that the red data points have some statistical fluctuations to them, and yet these effects are averaged out over the multiple data points to give a good measure of the speed. This is the same effect as taking multiple measurements of the same quantity. This is the second lesson of graphical analysis.

This approach to analyzing data is part of a larger technique called *model fitting*, and there are a wide variety of mathematical approaches that can be used for very complicated mathematical models. In this course, we will be using a *least squares fit* on linearized data. That means we will simply be looking at determining the slope of a straight line using a technique that chooses the slope by minimizing the square of the deviation of all the data points from the line.

D.1 Plotting Data

When plotting data, it isn't always obvious which is the *controlled variable* and which is the *responding variable*. In this type of analysis, it isn't critically important which is put on the x-axis and which is put on the y-axis. The thing to keep in mind is the results of the analysis – when performing a linear line fit to data, you are determining the slope: $\frac{dy}{dx}$. This will tie in to the section on Interpreting the Results.

Each data point that is measured has a corresponding uncertainty to it. These uncertainties must be incorporated into the graph in a particular way. If a measurement gives a value of $y \pm \delta_y$, then there are three points of interest that need to be indicated on the graph: y, $y + \delta_y$ and $y - \delta_y$. The same is true for the x value, where there is a known uncertainty. The $\pm\delta$ is shown with *error bars* that illustrate the confidence range of the location of the value. The way this is drawn for a single data point is shown in Figure D.2. This is effectively making the statement "The best estimate of the location of this data point is at the point shown, but I am confident that the true value lies somewhere within the range shown."

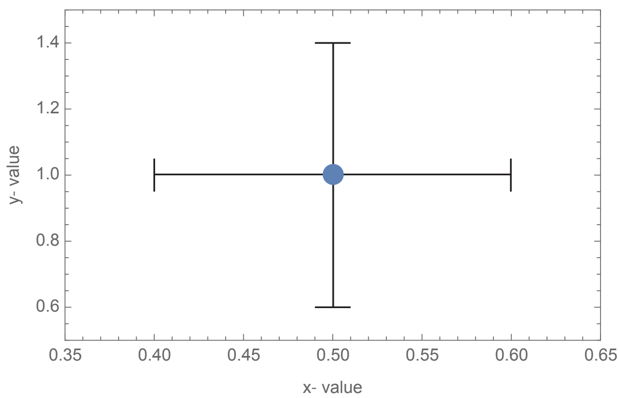


FIGURE D.2: A single data point with the corresponding error bars. This data point illustrates $\delta_x = 0.1$ and $\delta_y = 0.2$. Units are not included because this is an arbitrary example illustrating the graphical approach to error bars.

When put together with all data points, the graph should look like Figure D.3. For some experiments, the precision of some measurements may be very high. For example, when using highly accurate timing equipment, the uncertainty might be $\delta_t = 0.01$s while the data points are plotted on the range 0s to 10s. In such cases, the uncertainty bars parallel to the time axis would be too small to show, and thus it is reasonable to be neglect them. This should only be done if the error bars themselves are smaller than the width of the marker that is used to illustrate the point on the figure, or are otherwise visually unreasonable. Some judgment in the scientist is allowed when making this determination.

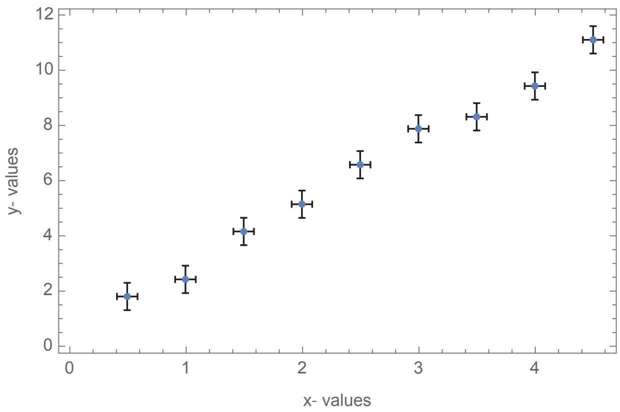


FIGURE D.3: This graph illustrates plotting an entire dataset along with uncertainties in the x and y axes. The uncertainties displayed here are $\delta_x = 0.1$ and $\delta_y = 0.5$. Uncertainties of $\delta_x < 0.05$ or $\delta_y < 0.2$ might be reasonable to leave off on this scale.

D.2 Least Squares Analysis

Least squares analysis is performed by minimizing the square of the deviation. That is a statement which is best explained mathematically and visually. A linear fit makes the prediction that the data follows a linear line of the formula: $y = Ax + B$. The parameter A is the slope and the parameter B is the intercept. The *deviation* is the vertical distance between the data point (x_i, y_i) and the value predicted by the linear line, $y(x_i) = Ax_i + B$:

$$\Delta y_i = y_i - y(x_i) = y_i - (Ax_i + B) \tag{D.1}$$

This is illustrated in Figure D.4. Keep in mind that before any calculations are done, the values of A and B are unknown – so at this point, we assume that there must exist some *best-fit* values of A and B, and our goal is to find them.

If A and B were known values, the deviation could be calculated for every single data point (x_i, y_i). The total deviation square is the quantity that we want to minimize:

$$\Delta_{total} = \sum_{i=1}^{N} (\Delta y_i)^2 = \sum_{i=1}^{N} (y_i - (Ax_i + B))^2 = (y_1 - (Ax_1 + B))^2 + (y_2 - (Ax_2 + B))^2 + ... \tag{D.2}$$

For poorly fitting values of A and B, the square of the deviation will always increase. For example, imagine the correct slope but an incorrect intercept – the deviation will be larger for all values. Alternatively, if the slope is incorrect, the deviation will at best be small for one or two points where the line passes near the data points, but will be larger for all other values. This is illustrated in Figure D.5. Thus, given the equation for the total deviation square, it is clear that it is a concave up parabola in both A and B. Thus, to find the minimum, we simply need to find the value of A

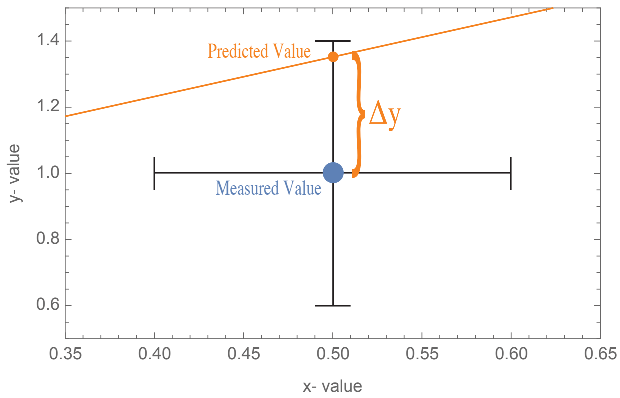


FIGURE D.4: Data point and hypothetical best-fit line illustrating what is meant by the term *deviation*. The deviation, Δy is shown on the graph.

and B that satisfies:

$$\frac{d\Delta_{total}}{dA} = 0 \qquad \frac{d\Delta_{total}}{dB} = 0 \tag{D.3}$$

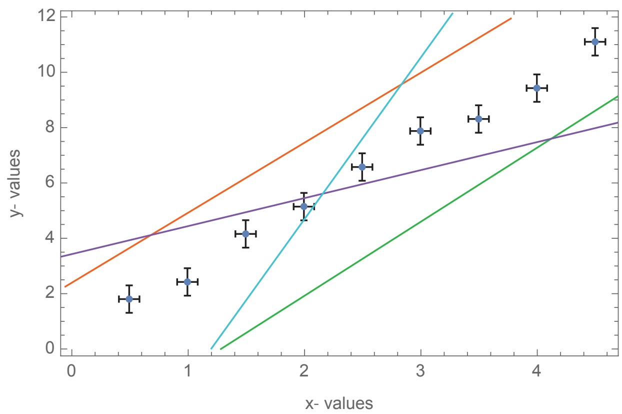


FIGURE D.5: This graph illustrates poor choices of lines to fit the data. When the slope and/or intercept are off, the sum of the square of the deviations increases.

Expanding out the terms in the sum, we get a different expression for the total deviation square:

$$\begin{aligned} \Delta_{total} &= \sum_{i=1}^{N}(y_i^2 + A^2x_i^2 + B^2 - 2Ay_ix_i - 2By_i + 2ABx_i) && \text{(D.4)} \\ &= \sum_{i=1}^{N} y_i^2 + A^2\sum_{i=1}^{N} x_i^2 + NB^2 - 2A\sum_{i=1}^{N} x_iy_i - 2B\sum_{i=1}^{N} y_i + 2AB\sum_{i=1}^{N} x_i && \text{(D.5)} \\ &= N(\overline{y^2} + A^2\overline{x^2} + B^2 - 2A\overline{xy} - 2B\overline{y} + 2AB\overline{x}) && \text{(D.6)} \end{aligned}$$

Thus, the derivatives are:

$$\frac{d\Delta_{total}}{dA} = 2A\sum_{i=1}^{N} x_i^2 - 2\sum_{i=1}^{N} x_i y_i + 2B\sum_{i=1}^{N} x_i = 0 \tag{D.7}$$

$$\frac{d\Delta_{total}}{dB} = 2NB - 2\sum_{i=1}^{N} y_i + 2A\sum_{i=1}^{N} x_i = 0 \tag{D.8}$$

This is two equations with two unknowns – we want to solve these equations simultaneously for A and B. The simplest way to do this would be to solve one of them for A in terms of B, then substitute that into the other equation to eliminate any factors of A and produce an equation that depends only on B. Solving for B then allows solving for A in the original equation. This results in the following:

$$A = \frac{N\sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \tag{D.9}$$

$$B = \frac{\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i\right) - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i y_i\right)}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \tag{D.10}$$

Dividing the numerator and denominator simultaneously by N^2 gives a way of writing these equations in terms of the averages $\overline{x}$, $\overline{y}$, $\overline{xy}$ and $\overline{x^2}$:

$$A = \frac{(\overline{xy}) - (\overline{x})(\overline{y})}{(\overline{x^2}) - (\overline{x})^2} \tag{D.11}$$

$$B = \frac{(\overline{x^2})(\overline{y}) - (\overline{x})(\overline{xy})}{(\overline{x^2}) - (\overline{x})^2} \tag{D.12}$$

These calculations would be tedious if done by hand, but are simple when done with the aid of computing. For these labs, you will be expected to use a spreadsheet program such as Microsoft Excel, Apple Numbers, or Google Sheets. Information on how to do this is included in Appendix B. As a side note, it turns out that the point $(\bar{x}, \bar{y})$ is a point that satisfies the equation $y = Ax + B$ for the best fit values of A and B. Thus, the *best-fit line* has slope A, intercept B, and passes through the point $(\bar{x}, \bar{y})$.

A careful observer might be wondering why there is little attention paid to the error bars when performing this analysis. Incorporating uncertainties in calculating A and B is more complicated than necessary for this course. However, the error bars are still important. For the purposes of these labs, a visual inspection of the graph and best-fit line should be done to ensure that none of the data points are more than two error bars distances away from the best fit line. The likelihood of this happening purely randomly is less than 3%, and such a situation is more likely caused by a bias or mistake in measurement or calculation. If you find that any of your points lie further than two error bar distances away from the best fit line, the data point should either be re-measured or re-calculated, depending on where the mistake lies, and the analysis should be repeated. This should emphasize the importance of proper data collection and calculations to minimize the effort involved in performing the analysis.

This does not mean that there is no uncertainty at all in the values of A and B. Instead, the uncertainty is calculated in a different manner. The derivation of the uncertainties in A and B can be found in uncertainty texts and will not be reproduced here. These uncertainties depend on the uncertainty in the value of Δ_{total}:

$$\delta_\Delta = \sqrt{\frac{\Delta_{total}}{N-2}} \tag{D.13}$$

With this value, the uncertainties in A and B are given by:

$$\delta_A = \delta_\Delta \sqrt{\frac{N}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}} = \delta_\Delta \sqrt{\frac{1}{N(\overline{x^2} - \overline{x}^2)}} \tag{D.14}$$

$$\delta_B = \delta_\Delta \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}} = \delta_\Delta \sqrt{\frac{\overline{x^2}}{N(\overline{x^2} - \overline{x}^2)}} \tag{D.15}$$

A completed graph illustrating the results is shown in Figure D.6.

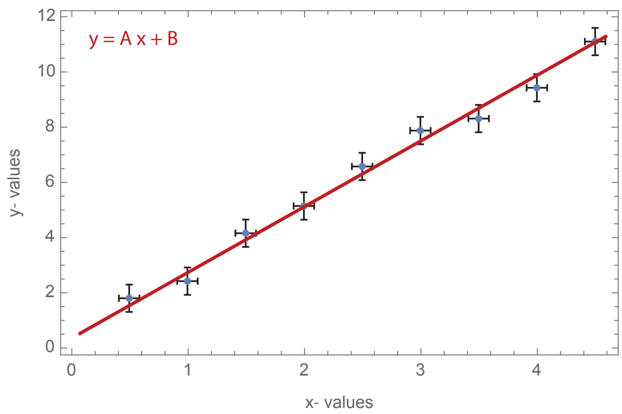


FIGURE D.6: This graph illustrates all of the necessary components of graphically displaying your data, including error bars, the best-fit line, and the equation for the best fit line. If there are multiple lines shown on the graph, the equation for the best-fit line should be kept separate from the graph.

D.3 Interpreting the Results

In graphing an objects distance from an origin versus time, it should be obvious that the slope of the graph is the speed of the object and the intercept is the distance from the origin at $t = 0$s. The interpretation of the slope and intercept is more difficult for more complicated experiments.

Consider Newton's Law of Gravitation, $\vec{F} = \frac{Gm_1m_2}{r^2}\hat{r}$. The letter G represents the universal gravitational constant, but all other components of the equation could potentially be measurable. If we measure the magnitude of the force F and vary only one of the masses while keeping the other mass and the distance, r, constant, then we have the following relationship:

$$F = \left(\frac{Gm_2}{r^2}\right) m_1$$

Keeping m_2 and r constant, and varying m_1 allows us to identify the y-axis with the force, F, and the x-axis with the mass, m_1. In other words:

$$\begin{array}{ccccccc} F & = & \left(\frac{Gm_2}{r^2}\right) & m_1 & + & ? \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ y & = & A & x & + & B \end{array}$$

Thus, a graph of F versus m_1 would produce a slope that can be equated with Gm_2/r^2. If the whole point of the experiment is to determine the gravitational constant, G, then our slope could be used as:

$$\begin{aligned} A &= \frac{Gm_2}{r^2} \\ G &= \frac{Ar^2}{m_2} \end{aligned}$$

The uncertainty of A is determined from the graph data, as shown in the previous section. Given measurement uncertainties on r and m_2, the uncertainty of the constant G can be determined using uncertainty propagation.

The likelihood of the data producing an intercept that is zero for a model that predicts a zero intercept is effectively negligibly small. Thus, it is important to consider the meaning of an intercept where the model (the physics equation) being used does not include one. In the situation proposed above, determining G, a small intercept is likely due to random fluctuations of the data, but a larger intercept is possible without ruining the experiment altogether. A negative intercept could be caused by a minimum force needed to produce a measurement in the force-meter. A positive intercept could be caused by a force-meter that is not properly calibrated (the zero point is not set correctly). So long as the force varies correctly with the mass, the presence of such an intercept has no impact on the results. This is one of the reasons why graphical analysis is so useful – a single measurement of the force and mass would not account for these kinds of biased influences, and would give a drastically incorrect value for G.

There are times when the model has non-linear dependencies. Again returning to the situation above, F is linearly proportional to the masses m_1 and m_2, but it is proportional to square inverse of distance. Instead of using F and m_1 to determine G, it would be possible to fix m_1 and m_2 and vary r and still use the least square technique. In order to do this, we do not plot F versus r, but rather F versus r^{-2}. Thus, we see:

$$\begin{array}{ccccccc} F & = & (Gm_2m_1) & \frac{1}{r^2} & + & ? \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ y & = & A & x & + & B \end{array}$$

and

$$\begin{aligned} A &= Gm_1m_2 \\ G &= \frac{A}{m_1m_2} \end{aligned}$$

This is known as *linearizing* the graph. When doing this kind of analysis, it is important to remember that the uncertainties on the linearized data need to be properly propagated. The uncertainty for r^{-2} is not the same as δ_r (the uncertainty on r). Instead:

$$\delta_{\frac{1}{r^2}} = \left| \frac{2\delta_r}{r^3} \right|$$

These values would need to be calculated for each value of r that is measured, and used in plotting the data.

Appendix E: Using Logger Pro

Logger Pro software is available for download and use by students during the Pandemic. Otherwise, there is a free 6 month trial. This software can be used to record timing information from a video, as well as many other advanced data logging and analysis activities. For the purpose of this lab, you will only be using Logger Pro to more accurately record the timing of objects moving on a video.

E.1 Video Timing Analysis

To analyze your video, first transfer the video file from your camera (e.g. cell phone) to your computer and move that file to a known location. If you are using your webcam on your computer to record the video, then save the video file to a known location. Ideally, use the location where you are keeping your write up of your lab report.

Next, open Logger Pro. The default view for a new analysis should look something like Fig. E.1. It is not absolutely necessary, but it may be helpful to move the spreadsheet and graph frames around to look like Fig. E.2 if you choose to learn/use some of the more advanced features of Logger Pro.

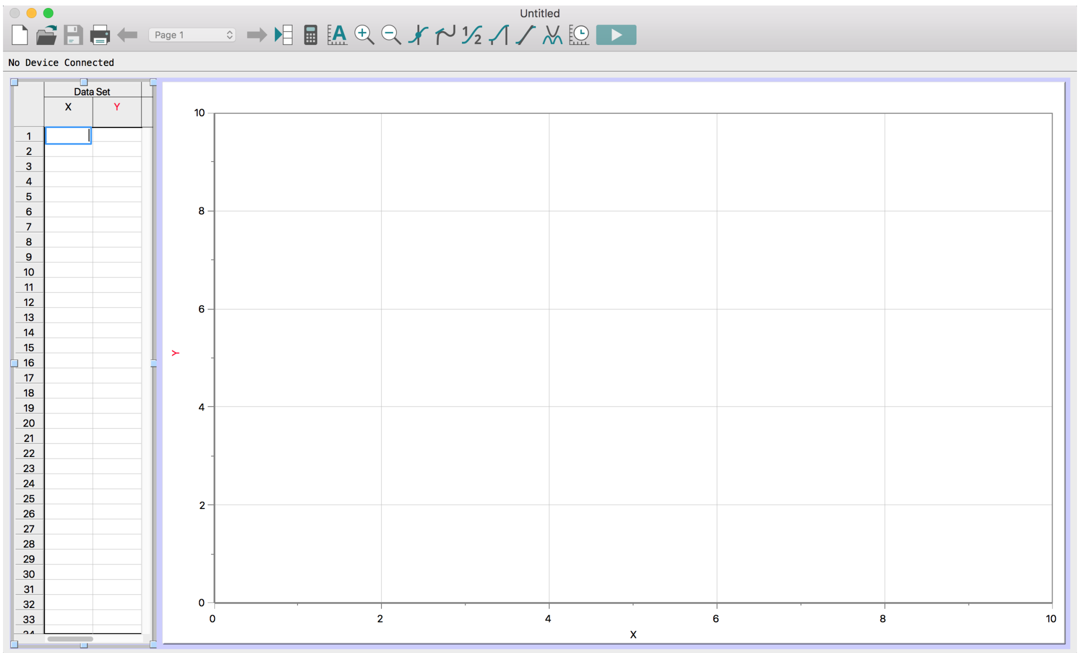


FIGURE E.1: Default view for a new Logger Pro data analysis.

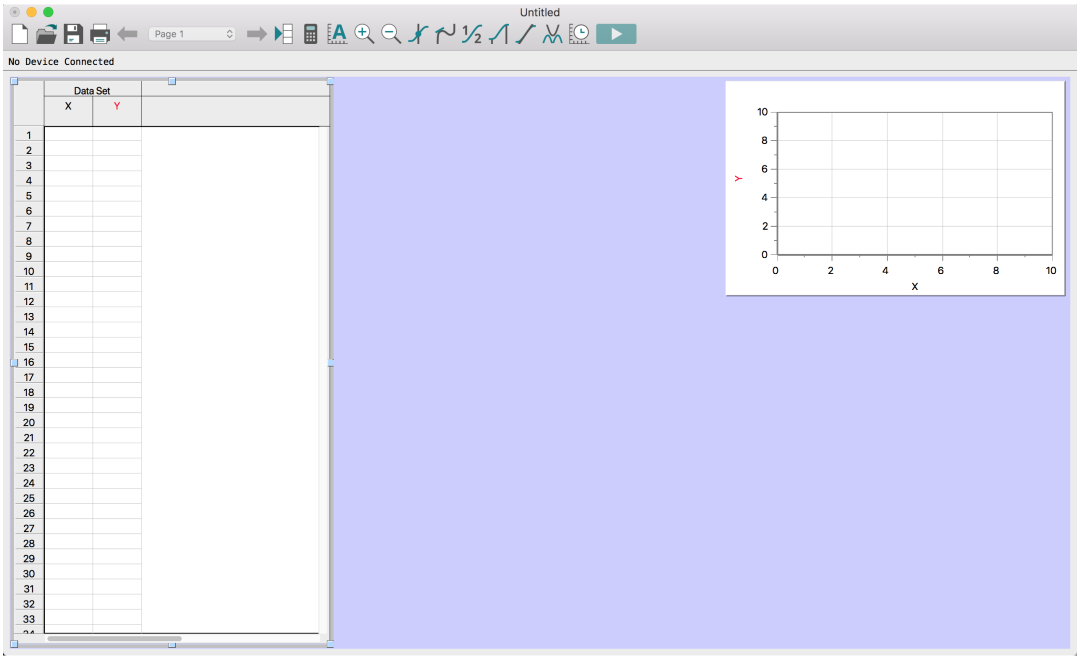


FIGURE E.2: Shifted graph and spreadsheet frames for a new analysis.

Next, click on the **Insert** menu, followed by the **Movie** option. A file location window will appear – locate the video file you saved and import it to your analysis. Your screen should look similar to Fig. E.3, where the video in the example images here is of a finger moving in front of a computer monitor. Press the button in the bottom right of the video, circled in red on Fig. E.3, to access the video analysis tools.

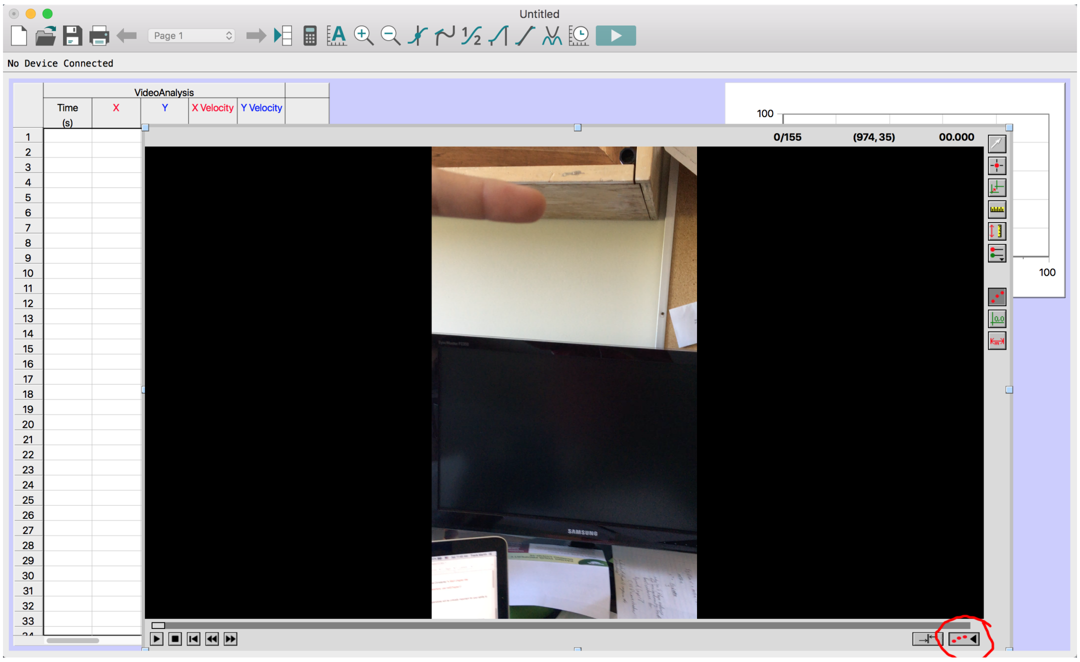


FIGURE E.3: Logger Pro after a video has been imported.

At the bottom of the video is a slider bar that controls the timing of the video, which is circled in orange. In Fig. E.4, the video has been advanced until the finger passes in front of the top of the monitor. In the top right corner of the video file is a readout of the time of the selected frame of the video, circled in orange in Fig. E.4. In this case, the selected frame is at 2.201s. If you press the double triangle button on the bottom left, you can advance the video by 1 frame. This is your uncertainty in the timing of the frame. The next frame in this example was at 2.235s, thus the time of the frame should be recorded as 2.201 $\pm$ 0.034s.

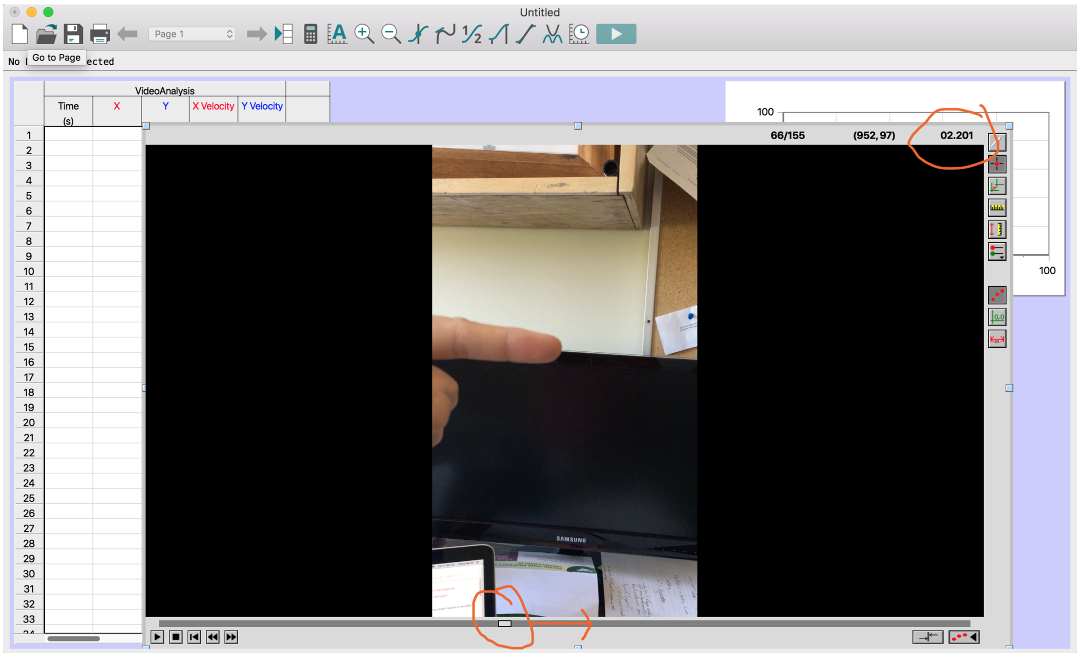


FIGURE E.4: View showing the advancement of the video and the recording of the timing for a frame.

Advancing the video until the finger moves to the bottom of the monitor gives a readout that looks like Fig. E.5. As you can see, in the top right corner the new reading is 4.436s, which would be recorded as 4.436 $\pm$ 0.034s, including the uncertainty in the value.

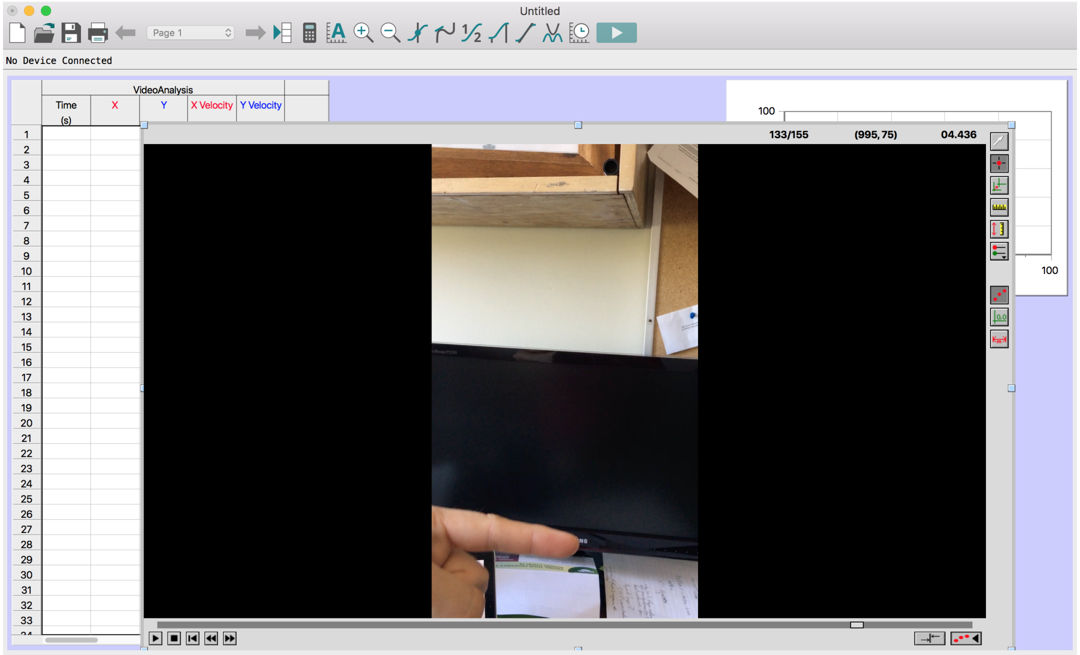


FIGURE E.5: Advancing the video until the next significant frame of interest results in this screenshot.

Using this, you can now answer the question: How long did it take for the finger to move infront of the monitor? (A trivial question, but you will be asking the similar question – "How long did it take for an object to move down a slope?") The answer is $4.436\text{s} - 2.201\text{s} = 2.235\text{s}$, with an uncertainty of $\sqrt{2}(0.035\text{s}) = 0.049\text{s}$. Thus, our final answer, including proper significant figures is: $\Delta t = 2.24 \pm 0.05\text{s}$.

E.2 Download and Installation Information

Detailed Instructions:
For more details on how to download and install Logger Pro, see https://www.vernier.com/til/2069/

Windows 10
Link: http://www.vernier.com/d/ihg4n

macOS 10.15, 10.14, 10.13
Link: http://www.vernier.com/d/gxzsu

For Windows and Mac computers that are no longer receiving updates, you will need an older version of Logger Pro. If your version is not listed below, please contact support@vernier.com.

Windows 8.1, 7
Link: https://www.vernier.com/d/otayn
Windows 8
Link: https://www.vernier.com/d/rcsq5

Password: conservation

Windows XP and Vista
Link: https://www.vernier.com/d/7tjc0
Password: climate

Mac OS X 10.12, 10.11, 10.10
Link: https://www.vernier.com/d/bstnw

Mac OS X 10.9
Link: https://www.vernier.com/d/oewti
Password: experiment

Mac OS X 10.8
Link: https://www.vernier.com/d/pdwat
Password: exploration

Mac OS X 10.7
Link: https://www.vernier.com/d/dpen3
Password: experiment

Appendix F: Uncertainty Analysis

In many texts, the topic of uncertainty is referred to as *error*. For many students, this conveys an incorrect understanding. The term *error* is historically applied to this analysis as a reference to the likelihood that the experimentally determined value is inaccurate or incorrect, or that the theory that the experiment is testing is incorrect. The use of the term does not imply any error on the part of the experiment or the experimentor: any errors or mistakes should be corrected, and are in no way admissible by the use of uncertainty calculations.

A very brief overview of the origin of uncertainty in the context of Gaussian statistics was given in Appendix A. This appendix will focus on the application of uncertainty in more complicated systems. It is important to note that the study of uncertainty has great depth that is not fully explored or used in this class. These labs and this lab manual are meant as an introduction to the important concepts only.

Instead of the more laborious notation of standard deviation of the mean, $\sigma_{\bar{x}}$, and to acknowledge that we are approximating standard deviations of the mean using the formula $\sigma_{\bar{x}} \approx \sigma_x/\sqrt{N}$, we will be using a slightly different notation to refer to the *uncertainty* of a quantity:

$$\sigma_{\bar{x}} = \delta_x \tag{F.1}$$

Thus a δ_x will refer to the uncertainty in quantity x and σ_x will refer to the standard deviation of a series of measurements of x. The overbar on the x in a subscript can be both hard to write and hard to read.

In cases where there is both a statistical uncertainty and a measurement uncertainty, these should be combined as:

$$\delta_{combined} = \sqrt{\delta^2_{statistical} + \delta^2_{measurement}} \tag{F.2}$$

For example, if the distance an object travels in projectile motion is measured multiple times, under apparent identical conditions, then the location of impact has multiple measurements of it, from which to calculate the statistical uncertainty. However, the measuring device probably has an uncertainty on the order of 0.5-1mm. Thus, these two components of uncertainty are independent of each other and should be combined.

If $\delta_{statistical} > 4\delta_{measurement}$, the measurement uncertainty can be neglected.

F.1 Origin of Uncertainty Propagation

There are few things which can be measured directly – distance, time, mass are three of them. Other types of quantities, such as forces and energies, must be calculated from measurements of more fundamental quantities. Even devices that purport to measure these quantities directly are, in fact, using more fundamental measurements and translating the fundamental measurement into the desired quantity using a calibrated conversion. In the Appendix A, the origin of uncertainty in measured quantities was discussed, and now we want to explore how these uncertainties propagate through as we use the measurements in contexts like calibrated conversions, and determining calculated values.

The formula for propagation of uncertainty has as its origin the Taylor Series for approximating a function using a polynomial. For those that are unfamiliar with Taylor Series, a brief description is included in this lab manual. Given a function of one variable, $f(x)$, for which we know the value of the function and all of its derivatives at location $x = a$, we can approximate the value for $f(x)$ at values of x that are very close to a with the following formula:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + ... \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k \end{aligned} \tag{F.3}$$

The accuracy of this approximation is shown in Figure F.1. Clearly the more terms included in the expression, the more similar the polynomial is to the original function.

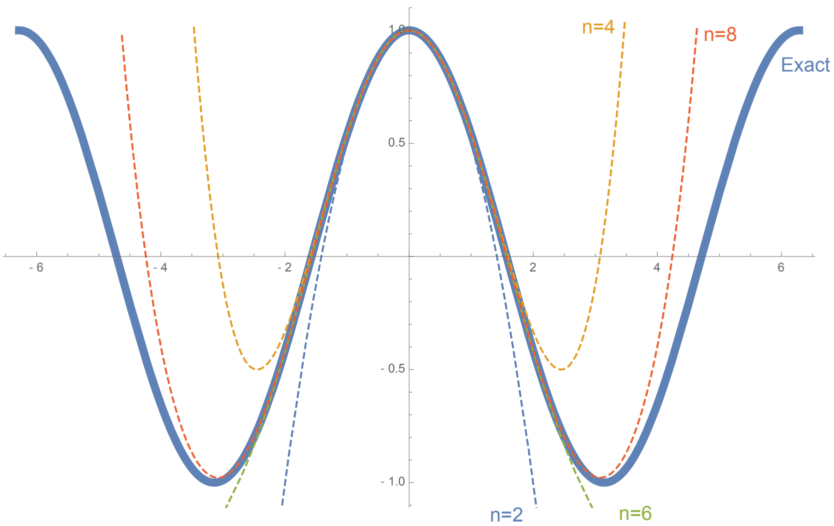


FIGURE F.1: The function $\cos x$ has a Taylor series about the point $x = 0$ of $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + ...$ In this figure, the thick blue line is the exact value for $\cos x$ over the range $[-2\pi, 2\pi]$. The blue dashed line is the Taylor series truncated at the x^2 term, the yellow dashed line is the series truncated at the x^4 term, the green dashed line is the series truncated at the x^6 term, and the red dashed line is the series truncated at the x^8 term.

If we change notation slightly, its application to uncertainties becomes more obvious: let $\delta_f = f(x) - f(a)$ and $\delta_x = x - a$, the differences between the approximated values and the hypothetically true values. Rearranging the equation, we then get:

$$\delta_f \approx f'(a)\delta_x + \frac{1}{2}f''(a)(\delta_x)^2 + \frac{1}{6}f'''(a)(\delta_x)^3 + ... \tag{F.4}$$

If δ_x is a small value (say $\delta_x = 0.01$) then $(\delta_x)^2$ is a very small number ($(\delta_x)^2 = 0.0001$). Thus, the terms in the sum have a smaller and smaller effect as the power of the δ_x factor increases.

We can perform a *linear approximation* by truncating the terms with $(\delta_x)^k$ for $k \geq 2$ to get much more simple formula:

$$\delta_f \approx \frac{df}{dx}\delta_x \tag{F.5}$$

where the derivative of f with respect to x is evaluated at the measured value. This is the most basic formula that can be applied to uncertainties – the uncertainty in a *function* of x depends on the value of the derivative of that function and the value of the uncertainty of x. Since uncertainty applies equally on either side of the central value, a negative value for the slope is irrelevant: $-1 \times (\pm\delta_x) = \mp\delta_x$, which has the same effect. Thus, we typically write the derivative factor in absolute values:

$$\delta_f \approx \left|\frac{df}{dx}\right|\delta_x \tag{F.6}$$

EXAMPLE:
The surface of sand on the beach looks like a cosine function with a wavelength of 12.3cm and an amplitude of 3.1cm, and you want to test this hypothesis. At a horizontal distance of 4.0 ± 0.1cm from one of the peaks, the height of the sand has dropped 4.4 ± 0.1cm relative to the nearest peak. If the surface of the sand really does follow a cosine function, how much should the height of the sand drop at a point 4.0 ± 0.1cm from the nearest peak? (i.e. Does $y_{measured}(4.0\text{cm}) = -4.4\text{cm}$ match our prediction that $y_{theory}(x) = 3.1\,\text{cm}(\cos(2\pi x/12.3\text{cm}) - 1)$?)

The modelled height of the sand relative to the nearest peak is $y_{theory}(x) = 3.1\,\text{cm}(\cos(2\pi x/12.3\,\text{cm}) - 1)\text{cm}$, while the measured value at $x = 4.0 \pm 0.1\text{cm}$ is $y = -4.4 \pm 0.1\text{cm}$. The goal of the calculation is to determine what the height *should* be if it followed the cosine model. We know what is height the sand at that point, so we want to test against what it should be.

$$y_{theory}(4.0) = 3.1(\cos(2\pi 4.0\,\text{cm}/12.3\,\text{cm}) - 1)\,\text{cm} = -4.5109\,\text{cm}$$

We know the value of the height of the sand, but we also need to know the uncertainty.

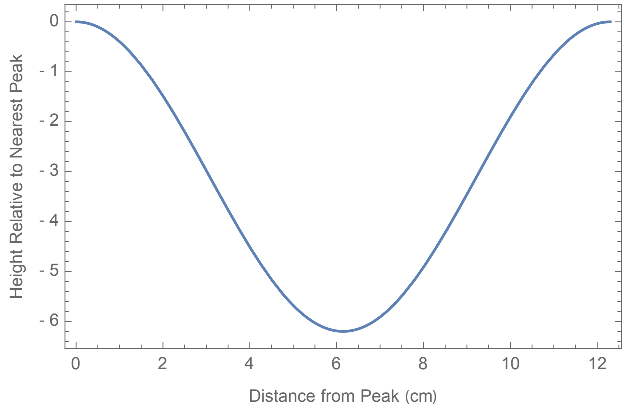


FIGURE F.2: The height of the sand relative to the nearest peak follows a $\cos(x) - 1$ type function. Given the wavelength and amplitude, the equation for the height of the sand should be $y_{theory}(x) = 3.1(\cos(2\pi x/12.3\,\text{cm}) - 1)$cm. At $x = 4.0$cm, the height of the sand was measured to be $y = -4.4$cm.

$$\begin{aligned}
\frac{df}{dx} &= -\frac{2\pi}{12.3\,\text{cm}} 3.1\text{cm}\sin(2\pi x/12.3\,\text{cm}) \\
\left.\frac{df}{dx}\right|_{x=4.0\,\text{cm}} &= -\frac{2\pi}{12.3\,\text{cm}} 3.1\,\text{cm}\sin(2\pi 4.0\,\text{cm}/12.3\,\text{cm}) = -1.41005 \\
\delta_f &= \left|\left.\frac{df}{dx}\right|_{x=a}\right| \delta_x = (1.41005)(0.1\,\text{cm}) = 0.141005\,\text{cm}
\end{aligned}$$

Thus, the height of the sand should be -4.5 ± 0.1cm according to the cosine model. The height that was measured was -4.4 ± 0.1cm. As shown in Figure F.3, there is significant similarity between these two values. Without performing a calculation to see if these two values agree with each other, it is reasonable to have intuition that the measured value supports the modelled equation. Another way to say this is, "We cannot exclude the possibility that the theoretical model is accurate based on the measurements taken and their precision." Mathematical comparisons of these results will be discussed in more detail in Appendix C.

F.2 Multivariate Uncertainty Propagation

The Taylor expansion explanation for uncertainty analysis is simple for a single variable. Taylor expanding for multiple variables is more complicated. However, it is easy to write up to the linear term. In one dimension, the expansion was:

$$f(x) \approx f(a) + f'(a)(x-a) + ... \tag{F.7}$$

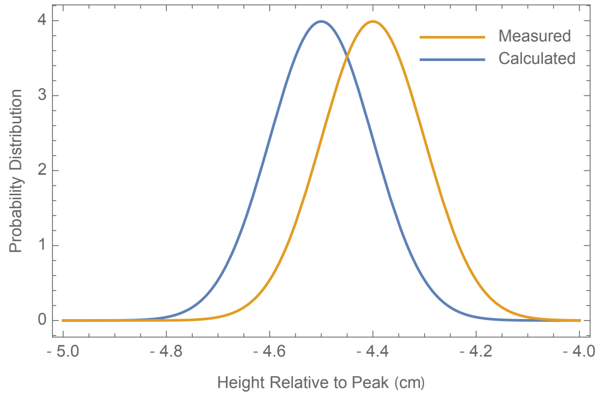


FIGURE F.3: The probability distributions for the measured and modelled values.

For multiple variables, we just add on extra terms for each of the other variables:

$$f(x,y,z,...) = f(a,b,c,...) + \frac{df}{dx}(x-a) + \frac{df}{dy}(y-b) + \frac{df}{dz}(z-c) + ... \tag{F.8}$$

Shifting the notation to include the δ's, it becomes:

$$\delta_f = \frac{df}{dx}\delta_x + \frac{df}{dy}\delta_y + \frac{df}{dz}\delta_z + ... \tag{F.9}$$

Note: In the derivatives, the variables x, y, z, etc are evaluated at the measured values.

However, this isn't the end of the explanation. In the previous section, we simply took the absolute value of the derivative, because the uncertainty is symmetric – the Gaussian curve behaves identically on either side of the mean. Now we have to account for the possibility that some uncertainties could overlap. For example, if we measure distance and time to determine velocity, then longer distances will also correspond to longer times. Using the formula above would miscount these overlapping effects. Thus, the uncertainties between these two have a component that is related to each other in some way.

To do this, we first square each side of the equation and expand:

$$\delta_f^2 = (\frac{df}{dx}\delta_x)^2 + (\frac{df}{dy}\delta_y)^2 + (\frac{df}{dz}\delta_z)^2 + 2\frac{df}{dx}\frac{df}{dy}\delta_x\delta_y + 2\frac{df}{dx}\frac{df}{dz}\delta_x\delta_z + 2\frac{df}{dy}\frac{df}{dz}\delta_y\delta_z + ... \tag{F.10}$$

This equation now represents the uncertainty in f, δ_f, if the measurements are completely positively correlated (e.g. any increase in x results in a proportionally identical increase in y). A negative sign in front of the factors of 2 in the cross terms would be if the measurements were completely anti-correlated (e.g. any increase in x results in a proportionally identical decrease in y). And a factor of 0 instead of 2 would indicate that the uncertainties are completely uncorrelated. In general, we can replace the factors of 2 with a number ρ_{ij} to represent the correlation between factor i and factor j. However, in this course, we are going to assume that the uncertainties are uncorrelated and leave correlation factors for more advanced courses.

Thus, the equation to use for calculating the uncertainty of a multivariate theoretical equation is:

$$\delta_f = \sqrt{\left(\frac{df}{dx}\delta_x\right)^2 + \left(\frac{df}{dy}\delta_y\right)^2 + \left(\frac{df}{dz}\delta_z\right)^2 + ...} \tag{F.11}$$

This is the most general expression that can always be used for uncorrelated uncertainties. However, physics has many formulas that are very similar in mathematical structure. Thus, it is sometimes easier to use the template method to calculate the uncertainties. This is covered in the next section.

EXAMPLE:
You want to determine the density of a uniform rectangular prism. The density formula is $\rho = m/V$. The mass is measured to be 22.3 ± 0.4kg, the length is measured to be 0.42 ± 0.02m, the width is measured to be 0.12 ± 0.02m, and the height is measured to be 0.23 ± 0.02m. What is the density and uncertainty in the density?

The volume of the prism is given by the formula $V = \ell wh$, where ℓ is the length, w is the width and h is the height. Thus, the combined expression for density is:

$$\rho = \frac{m}{\ell wh} = 22.3\,\text{kg}/0.42\,\text{m} \cdot 0.12\,\text{m} \cdot 0.23\,\text{m} = 1923.74\,\text{kg}\,\text{m}^{-3}$$

We first need to take a series of derivatives:

$$\begin{aligned}
\frac{d\rho}{dm} &= \frac{1}{\ell wh} = \frac{1}{0.42\,\text{m} \cdot 0.12\,\text{m} \cdot 0.23\,\text{m}} = 86.27\,\text{m}^{-3} \\
\frac{d\rho}{d\ell} &= -\frac{m}{\ell^2 wh} = \frac{22.3\,\text{kg}}{0.12\,\text{m} \cdot 0.23\,\text{m}} = 807.97\,\text{kg}\,\text{m}^{-2} \\
\frac{d\rho}{dw} &= -\frac{m}{\ell w^2 h} = \frac{22.3\,\text{kg}}{0.42\,\text{m} \cdot 0.23\,\text{m}} = 230.85\,\text{kg}\,\text{m}^{-2} \\
\frac{d\rho}{dh} &= -\frac{m}{\ell wh^2} = \frac{22.3\,\text{kg}}{0.42\,\text{m} \cdot 0.12\,\text{m}} = 442.46\,\text{kg}\,\text{m}^{-2}
\end{aligned}$$

Putting this together in the uncertainty formula, we find:

$$\begin{aligned}
\delta_f &= \sqrt{\left(\frac{d\rho}{dm}\delta_m\right)^2 + \left(\frac{d\rho}{d\ell}\delta_\ell\right)^2 + \left(\frac{d\rho}{dw}\delta_w\right)^2 + \left(\frac{d\rho}{dh}\delta_h\right)^2} \\
&= \big((86.27\,\text{m}^{-3} \cdot 0.4\,\text{kg})^2 + (807.97\,\text{kg}\,\text{m}^{-2} \cdot 0.02\,\text{m})^2 + (230.85\,\text{kg}\,\text{m}^{-2} \cdot 0.02\,\text{m})^2 \\
&\quad + (442.46\,\text{kg}\,\text{m}^{-2} \cdot 0.02\,\text{m})^2\big)^{1/2} \\
&= 39.39\,\text{kg}\,\text{m}^{-3}
\end{aligned}$$

Thus, rounding our value to the same precision as the leading digit of the uncertainty gives us:

$$\rho = 1920 \pm 40\,\text{kg}\,\text{m}^{-3}$$

F.3 Uncertainty Propagation Templates

In physics, there are a number of very common types of equations that are used. In particular, summations occur frequently (e.g. in superposition), as well as polynomials, and trig functions and exponentials are also common. To simplify your analysis, the uncertainties for these types of template functions are shown below. If you don't want to take derivatives and use the full formula – which always works – you can use these templates. The risk is in whether you have used the template correctly.

Type	Function	Uncertainty
Scalar Multiplication	$f(x) = Ax$	$\delta_f = \lvert A\rvert \delta_x$
Addition/ Subtraction	$f(x, y, ...) = \pm Ax \pm By \pm ...$	$\delta_f = \sqrt{(A\delta_x)^2 + (B\delta_y)^2 + ...}$
Multiplication	$f(x, y, ...) = Axy...$	$\delta_f = \lvert f\rvert \sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2 + ...}$
Division	$f(x, y, ...) = Axy^{-1}...$	$\delta_f = \lvert f\rvert \sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2 + ...}$
Exponents	$f(x) = Ax^n$	$\delta_f = \lvert n \cdot f\rvert \left\lvert \frac{\delta_x}{x} \right\rvert$
Exponents w/ Multiplication	$f(x, y, ...) = Ax^n y^m ...$	$\delta_f = \lvert f\rvert \sqrt{\left(n\frac{\delta_x}{x}\right)^2 + \left(m\frac{\delta_y}{y}\right)^2 + ...}$
Exponentials	$f(x) = Ae^{Bx}$	$\delta_f = \lvert fB\rvert \delta_x$
Logarithms	$f(x) = A\log(Bx)$	$\delta_f = \left\lvert A\frac{\delta_x}{x}\right\rvert$
Sine	$f(x) = A\sin(Bx)$	$\delta_f = \lvert AB\cos(Bx)\rvert \delta_x$
Cosine	$f(x) = A\cos(Bx)$	$\delta_f = \lvert AB\sin(Bx)\rvert \delta_x$

Note: This table assumes that only x, y and z are values that have uncertainties. The constants A, B and n should be taken as values without uncertainty, such as π or integers.