Assignment 2 Solution Ideas

1. Let m : I go motorcycling
   r : it rains
   w : it is warm
   g : I wear rain gear
   d : I decide to get wet
   p : my friends phone me to go riding

Then the argument is

$$\begin{array}{l} m \rightarrow r \\ (r \wedge w) \rightarrow (g \vee d) \\ p \rightarrow (m \wedge w) \\ \neg d \\ \hline \therefore\ p \rightarrow g \end{array}$$

The argument is valid. Following Section 1.13, since the conclusion is an implication, it suffices to take p as an extra premise and obtain g as the conclusion

1.	$m \to r$	Premise
2.	$(r \wedge w) \to (g \vee d)$	"
3.	$p \to (m \wedge w)$	"
4.	$\neg d$	"
5.	p	extra Premise to prove implic'n
6.	$\therefore m \wedge w$	3, 5 M.P.
7.	m	6 Conj. Simp.
8.	$\therefore r$	1, 7 M.P.
9.	w	6 Conj Simp
10.	$r \wedge w$	8, 9 Conjunction
11.	$g \vee d$	2, 10 M.P.
12.	$\neg d \to g$	11, L.E.
13	$\therefore g$	4, 12 M.P.

2a Every positive real number has exactly two square roots

True. If $x > 0$ then $(-\sqrt{x})^2 = \sqrt{x}^2 = x$, $-\sqrt{x} \neq \sqrt{x}$, and no other number squared equals x.

b. $\mathcal{U} = \mathbb{Z}$.

$\forall x, \exists y, y < x$

c. $\neg \forall n, \exists m, m \cdot n = n$

$\Leftrightarrow \exists n, \neg \exists m, m \cdot n = n$

$\Leftrightarrow \exists n, \forall m, mn \neq n$

$\Leftrightarrow \exists n, \forall m, (mn < m) \vee (mn > n)$

3. a Suppose S_2 is true.
Then $\forall n\ p(n)$ is true or $\forall n\ q(n)$ is true.
In either case $p(n) \vee q(n)$ is true for every n.
That is $\forall n, p(n) \vee q(n)$ is true
(i.e. S_1 is true)

b. No. Let $U = \{1, 2\}$
$p(n)$ be "$n=1$" and $q(n)$ be "$n=2$".
Then $\forall n, p(n) \vee q(n)$ is true.
But neither $\forall n, p(n)$ nor $\forall n, q(n)$ is true, so
$\forall n, p(n) \vee \forall n, q(n)$ is false. $\therefore S_1$ & S_2 are not L.E.

c. Yes. Suppose S_1 is true.

$\therefore$ $p(n) \vee q(n)$ is true for some n.

$\therefore$ $p(n)$ is true for some n, or $q(n)$ is true for some n.

$\therefore$ $\exists n, p(n)$ is true, or $\exists n, q(n)$ is true.

$\therefore$ $\exists n, p(n) \vee \exists n, q(n)$ is true.

Suppose S_2 is true. Then $\exists n, p(n)$ is true, or $\exists n, q(n)$ is true.

$\therefore$ $p(n)$ is true for some n, or $q(n)$ is true for some n.

$\therefore$ $\exists n, p(n)$ is true, or $\exists n, q(n)$ is true

$\therefore$ $\exists n, p(n) \vee \exists n, q(n)$ is true

$\therefore$ S_1 & S_2 are logically equivalent.

4.a We prove the contrapositive: if n is not a multiple of 3, then n^4 is not a multiple of 3.

Suppose n is not a multiple of 3. Then the remainder when n is divided by 3 equals 1 or 2. We consider each case in turn.

<u>Case 1</u> The remainder when n is divided by 3 equals 1.

$\therefore$ There is an integer k so that $n = 3k+1$.

$$\therefore\ n^4 = (3k+1)^4$$
$$= 81k^4 + 27k^3 + 9k^2 + 3k + 1$$
$$= 3(27k^4 + 9k^3 + 3k^2 + k) + 1$$

Since $27k^2 + 9k^3 + 3k^2 + k$ is an integer, n^4 is not a multiple of 3.

Case 2 The remainder when n is divided by 3 equals 2

$\therefore$ There is an integer l so that $n = 3l+2$

$$\begin{aligned}
\therefore \quad n^4 &= (3l+2)^4 \\
&= 81l^4 + 54l^3 + 36l^2 + 24l + 16 \\
&= 81l^4 + 54l^3 + 36l^2 + 24l + 15 + 1 \\
&= 3(27l^4 + 18l^3 + 12l^2 + 8l + 5) + 1
\end{aligned}$$

Since $27l^4 + 18l^3 + 12l^2 + 8l + 5$ is an integer, n^4 is not a multiple of 3.

The desired conclusion holds in both cases.

$\therefore$ If n^4 is a multiple of 3, then n is a multiple of 3.

b. The proof is by contradiction. Suppose $\sqrt{8}$ is rational. Then there exist integers a & b so that

$$\sqrt{8} = 2\sqrt{2} = \frac{a}{b}$$

$\therefore \sqrt{2} = a/2b$, a contradiction.

$\therefore \sqrt{8}$ is irrational

c. The proof is by contradiction. Suppose $\sqrt{2^{2m+1}}$ is rational. Then there exist integers a & b so that

$$\sqrt{2^{2m+1}} = 2^m\sqrt{2} = a/b$$

$\therefore \sqrt{2} = a/2^m b$, a contradiction

$\therefore \sqrt{2^{2m+1}}$ is irrational for any positive integer m.