CSC 225 FALL 2022
ALGORITHMS AND DATA STRUCTURES I
ASSIGNMENT 1 - SOLUTIONS
UNIVERSITY OF VICTORIA

1. a) $7! = 5040$

   b) $4! \cdot 3! = 24 \cdot 6 = 144$

   c) $5! \cdot 3! = 120 \cdot 6 = 720$

   d) $4! \cdot 3! \cdot 2 = 24 \cdot 6 \cdot 2 = 144 \cdot 2 = 288$

2. a) $\binom{13}{3}\binom{39}{2} = \frac{13!}{3!10!} \cdot \frac{39!}{2!37!} = \frac{13\cdot12\cdot11}{3\cdot2} \cdot \frac{39\cdot38}{2} = 286 \cdot 741 = 211{,}926$

   b) $\binom{13}{3}\binom{39}{2} + \binom{13}{2}\binom{39}{3} + \binom{13}{1}\binom{39}{4} + \binom{13}{0}\binom{39}{5} = 2{,}569{,}788$
   or $\binom{52}{5} - \binom{13}{4}\binom{39}{1} - \binom{13}{5}\binom{39}{0} = 2{,}569{,}788$

   c) $\binom{13}{2}\binom{13}{3} = 78 \cdot 286 = 22{,}308$

   d) $\binom{12}{1}\binom{12}{2}\binom{2}{2} + \binom{1}{1}\binom{12}{2}\binom{2}{1}\binom{24}{1} + \binom{12}{1}\binom{1}{1}\binom{12}{1}\binom{2}{1}\binom{24}{1} + \binom{1}{1}\binom{1}{1}\binom{12}{1}\binom{24}{2} =$
   $14{,}184$

3. Let n be a positive integer with $n > 2$, then

$$\begin{aligned}
\binom{n}{2} + \binom{n-1}{2} &= \frac{n!}{2!\,(n-2)!} + \frac{(n-1)!}{2!\,(n-3)!} \\
&= \frac{n!}{2(n-2)!} + \frac{(n-2)(n-1)!}{(n-2)2(n-3)!} \\
&= \frac{n! + (n-2)(n-1)!}{2(n-2)!} \\
&= \frac{\big(n + (n-2)\big)(n-1)!}{2(n-2)!} \\
&= \frac{2(n-1)(n-1)!}{2(n-2)!} \\
&= \frac{(n-1)(n-1)(n-2)!}{(n-2)!} \\
&= (n-1)^2
\end{aligned}$$

which is a perfect square for all $n>2$.

4. Consider $x_1 + x_2 + x_3 + x_4 = 20$,

a) Let $x_i \geq 0,\ 1 \leq i \leq 4$. Here, we have $n = 4$ distinct objects and we are choosing $r = 20$ of them, with repetition. So,

$$\binom{n+r-1}{r} = \binom{4+20-1}{20} = \binom{23}{20} = 1771$$

b) Now let $x_1, x_2 \geq 2, x_3, x_4 \geq 1$. Here we start with $x_1, x_2 = 2$ and $x_3, x_4 = 1$. We now only need to distribute the remaining $r = 20 - 6 = 14$ integers among the $n = 4$ variables.

$$\binom{4+14-1}{14} = \binom{17}{14} = 680$$

c) Let $x_i \geq -1$, $1 \leq i \leq 4$. This is the same as solving $x_1 + x_2 + x_3 + x_4 = 24$ where each $x_i \geq 0$, $1 \leq i \leq 4$. So, $n = 4$ and $r = 24$ and

$$\binom{4+24-1}{24} = \binom{27}{24} = 2925$$

Or, a) plus number of ways for one $x_i = -1$ plus number of ways for two $x_i = -1$ plus number of ways for three $x_i = -1$. So,

$$\binom{4+20-1}{20} + 4 \cdot \binom{3+21-1}{21} + 6 \cdot \binom{2+22-1}{22} + 4 \cdot \binom{1+23-1}{23}$$
$$= \binom{23}{20} + 4 \cdot \binom{23}{21} + 6 \cdot \binom{23}{22} + 4 \cdot \binom{23}{23} = 1771 + 1012 + 138 + 4 = 2925$$

d) Finally, let $x_i \geq 0,\ 1 \leq i \leq 3,$ and $2 \leq x_4 \leq 7$. Here we have 6 cases, when $x_4 = 2$, $x_4 = 3$, …, $x_4 = 7$.

$$\binom{3+18-1}{18} + \binom{3+17-1}{17} + \binom{3+16-1}{16} + \binom{3+15-1}{15} + \binom{3+14-1}{14} + \binom{3+13-1}{13}$$
$$= \binom{20}{18} + \binom{19}{17} + \binom{18}{16} + \binom{17}{15} + \binom{16}{14} + \binom{15}{13} = 190 + 171 + 153 + 136 + 120 + 105 = 875$$

5. Let x_i be the total number of resumes sent by the end of day i, for $1 \leq i \leq 42$. By the original assumptions, we have

$$1 \leq x_1 < x_2 < \cdots < x_{42} \leq 60$$

Here, we have 42 distinct integers in the interval [1,60]. Now, we want to show that there is a period of consecutive days where exactly 23 resumes are sent out. In other words, show that there exists $i > j$ such that $x_i - x_j = 23$ (or $x_i = x_j + 23$.)
So, let's add 23 to each of the above values to get another 42 distinct values. Thus,

$$24 \leq x_1 + 23 < x_2 + 23 < \cdots < x_{42} + 23 \leq 83$$

Now, all told we have 84 integers in the range [1,83]. By the Pigeonhole Principle, at least two of the 84 integers are equal. But, each group of 42 integers are distinct, thus one of the x_i in the first 42 must be equal to one of the $x_j + 23$ in the second group.