202201 Math 122 [A01] Quiz #5

March 24th, 2022

Name: Solutions #V00: ____________

This test has 2 pages and 6 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except where explicitly noted, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

   F (a) For all real numbers x, $\lfloor x \rfloor + 1 = \lceil x \rceil$. $x=2$ $\lfloor 2 \rfloor + 1 = 3$ but $\lceil 2 \rceil = 2$

   T (b) $(2022)_7 = 702$ $2\times 7^3 + 2\times 7 + 2 = 702$

   T (c) The base 2 representation of 2^{122} has 123 digits.

   F (d) When -87 is divided by -5, the remainder is -2. No negative remainders.

2. [2] Consider the sequence $a_0, a_1, a_2, a_3, \ldots$ defined by the recursive defintion $a_0 = 6$ and $a_n = 4a_{n-1}+6$ for $n \geq 1$. Write the terms a_1 and a_2 in a way to help you see the pattern of how the terms are created. Then write a guess for the term a_n, and simplify to find an explicit formula for a_n (without $+\cdots+$). You do not need to prove your conjecture, but you do need to show work for how your guess for a_n was found and simplified.

$$a_1 = 4\cdot 6 + 6$$
$$a_2 = 4(4\cdot 6 + 6) + 6 = 4^2\cdot 6 + 4\cdot 6 + 6$$
$$a_n = 4^n\cdot 6 + 4^{n-1}\cdot 6 + \cdots + 4^2\cdot 6 + 4\cdot 6 + 6$$
$$= 6(4^n + 4^{n-1} + \cdots + 4^2 + 4 + 1)$$
$$= 6\left(\frac{4^{n+1}-1}{4-1}\right) = 2(4^{n+1}-1)$$

3. [2] Find the base 16 representation of 14891.

$$14891 = 16(930) + 11 \leftarrow B$$
$$930 = 16(58) + 2$$
$$58 = 16(3) + 10 \leftarrow A$$
$$3 = 16(0) + 3$$

$$14891 = (3A2B)_{16}$$

4. [3] Let $a, b, m, n \in \mathbb{Z}$. Prove that if $m|a$ and $n|b$, then $mn|ab$.

Suppose $m|a$ and $n|b$.
Then $\exists\, k \in \mathbb{Z}$ such that $a = mk$.
Also $\exists\, \ell \in \mathbb{Z}$ such that $b = n\ell$.
Now $ab = (mk)(n\ell) = mn(k\ell)$.
Since k and ℓ are integers, $k\ell$ is an integer,
so $mn \mid ab$. $\blacksquare$

5. [4] Use the Euclidean Algorithm to compute $d = \gcd(682, 165)$. Then use your work to find integers x and y such that $d = 682x + 165y$.

$682 = 165(4) + 22$
$165 = 22(7) + 11$
$22 = 11(2) + 0$

$\therefore \gcd(682, 165) = 11$

$22 = 682 - 165(4)$
$11 = 165 - 22(7)$

So $11 = 165 - 22(7)$
$11 = 165 - [682 - 165(4)](7)$
$11 = 165(29) + 682(-7)$

In $11 = 682x + 165y$ we have
$x = -7$ and $y = 29$.

6. [2] Indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

F (a) Consider $a, b, c \in \mathbb{Z}$. If $a|bc$, then $a|b$ and $a|c$. $\left\{\begin{array}{l}\text{Look at } a=8,\ b=6,\ c=4 \\ 8 \mid 24 \text{ but } 8 \nmid 6 \text{ and } 8 \nmid 4.\end{array}\right.$

F (b) There exist positive integers a and b such that $15^a = 14^b$. $\{$Not allowed by FTA, this would give two different prime power decompositions.

T (c) Let $a = 3^5 5^6 11^1$ and $b = 2^3 3^3 11^2$. Then $\gcd(a, b) = 3^3 11^1 = 297$.

T (d) For positive integers a and b, if $\gcd(a, b) = 4$, then $\mathrm{lcm}(a, b) = \dfrac{ab}{4}$.

$\gcd(a,b) \cdot \mathrm{lcm}(a,b) = ab.$