Q.1 (a)

$$\int \frac{\sqrt{\sin^{-1}(x)}}{\sqrt{1-x^2}} \, dx$$

Let $u = \sin^{-1}(x)$

$$\Rightarrow \quad du = \frac{1}{\sqrt{1-x^2}} \, dx$$

$$\Rightarrow \int \frac{\sqrt{\sin^{-1}(x)}}{\sqrt{1-x^2}} \, dx = \int \sqrt{u} \, du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3} \left[\sin^{-1}(x)\right]^{3/2} + C$$

(b) $\int [x \ln(x)]^2\, dx = \int x^2 [\ln x]^2\, dx$

Let $t = \ln x \Rightarrow x = e^t$

$dt = \frac{1}{x} dx$ or $e^t\, dt = dx$

$\Rightarrow \int x^2 [\ln x]^2\, dx = \int t^2 e^{3t}\, dt$

Let $u = t^2$, $dv = e^{3t} dt$

$\Rightarrow du = 2t\, dt$, $v = \frac{e^{3t}}{3}$

$$= \frac{t^2 e^{3t}}{3} - \frac{2}{3}\int t e^{3t}\, dt$$

$$= \frac{t^2 e^{3t}}{3} - \frac{2}{3}\left[\frac{t e^{3t}}{3} - \frac{1}{3}\int e^{3t}\, dt\right];$$ By Part again

this finally gives:

$$\int x^2 [\ln x]^2\, dx = \frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}(\ln x) + \frac{2}{27}x^3 + C$$

(c). $$\int \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$

$$\Rightarrow \quad -du = (\sin x)\,dx$$

$$\Rightarrow \quad \int \frac{\sin x}{1+\cos^2 x}\,dx = -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1}(u) + C$$

$$= -\tan^{-1}[\cos x] + C$$

(d). $I = \int e^{x} \sin(2x)\, dx$

$$u = e^{x}, \quad dv = \sin(2x)\, dx$$

$$\Rightarrow du = e^{x} dx, \quad v = -\frac{\cos(2x)}{2}$$

$$\Rightarrow I = -\frac{e^{x}\cos(2x)}{2} + \frac{1}{2}\int e^{x}\cos(2x)\, dx$$

$$= -\frac{e^{x}\cos(2x)}{2} + \frac{1}{2}\left[\frac{e^{x}\sin(2x)}{2} - \frac{1}{2}\int e^{x}\sin(2x)\, dx\right]; \text{ By Parts again.}$$

$$= \frac{-e^{x}\cos(2x)}{2} + \frac{e^{x}\sin(2x)}{4} - \frac{1}{4} I$$

$$\Rightarrow \left(1 + \frac{1}{4}\right) I = -\frac{e^{x}\cos(2x)}{2} + \frac{e^{x}\sin(2x)}{4}$$

or $$I = \int e^{x}\sin(2x)\, dx = \frac{2e^{x}}{5}\left[-\cos(2x) + \frac{\sin(2x)}{2}\right] + C$$

(e). Doing a similar process as done in part (d), we get

$$\int e^{3x}\cos x \, dx = \frac{3e^{3x}}{10}\left[\cos x + \frac{\sin x}{3}\right] + C$$

(f) $\displaystyle\int \frac{\ln[\tan^{-1}(x)]}{1+x^2}\,dx$

Let $t = \tan^{-1}(x)$

$$\Rightarrow dt = \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \int \frac{\ln[\tan^{-1}(x)]}{1+x^2}\,dx = \int \ln(t)\,dt$$

$u = \ln(t)$, $dv = dt$

$$= t \cdot \ln(t) - t + C$$

$\Rightarrow du = \frac{1}{t}dt$, $v = t$

$$\Rightarrow \int \frac{\ln[\tan^{-1}(x)]}{1+x^2}\,dx = \tan^{-1}(x)\ln[\tan^{-1}(x)] - \tan^{-1}(x) + C$$

(g) $\int x \sin(\ln x)\, dx$

Let $t = \ln x \Rightarrow x = e^t$

$\Rightarrow dt = \frac{1}{x} dx$ or $e^t dt = dx$

$$\Rightarrow \int x \sin[\ln x]\, dx = \int e^{2t} \sin(t)\, dt$$

$$\overset{\text{By Parts}}{=} \frac{2}{5} e^{2t} \left[\sin t - \frac{\cos t}{2} \right] + C$$

$$\Rightarrow \int x \sin[\ln x]\, dx = \frac{2}{5} x^2 \left[\sin(\ln x) - \frac{\cos(\ln x)}{2} \right] + C$$

(h) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

let $u = e^x$

$\Rightarrow du = e^x\,dx$

$$\Rightarrow \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1}(u) + c$$

$$\Rightarrow \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \sin^{-1}[e^x] + c$$