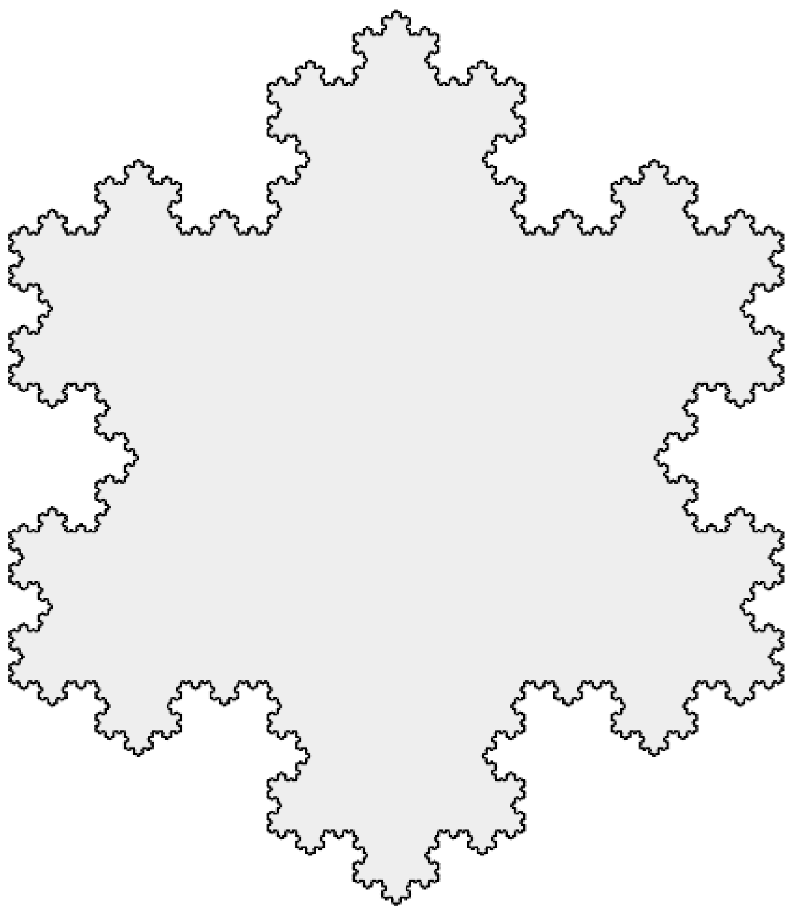
Chapter 2

Series



2.1 (Section 10.1) Introduction to Sequences

Definition

A **sequence** is an ordered list $a_1, a_2, a_3, \ldots$ and may be abbreviated as $\{a_i\}$ or $\{a_i\}_{i=1}^{\infty}$

These sequences may literally be any list of objects. In mathematics we always imply it to be infinite if no context is given although people use the term sequence colloquially to describe lists of finite objects.

Example: You can have a sequence of animals: $\{$*Dogs, Birds, Quoakkas, Lizards, Picasso's Cats, ...*$\}$

Example: You can have a sequence of dates: $\{$*June 3rd, July 12th, August 20th, June 3rd, September 10th,...*$\}$

Hence some sequences can repeat elements in them but their order is important.

2.1.1 Ways to Describe a Sequence

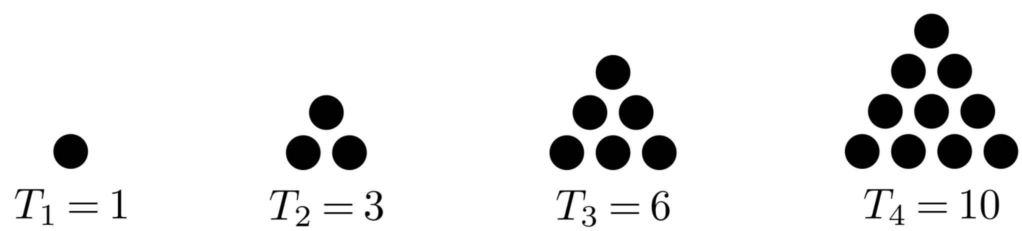
There are, in essence, three common ways to describe a sequence.

Explicit Description of a Sequence

These are sequences of the form $a_n = f(n)$ for some function where n is a sequence of integers starting from some initial value $n \geq k$.

Example: Triangular numbers are given by the sequence $T_n = \dfrac{n(n+1)}{2}$ where $n \geq 1$. We compute

$$T_1 = \frac{1(2)}{2} = 1 \qquad T_2 = \frac{2(3)}{2} = 3 \qquad T_3 = \frac{3(4)}{2} = 6 \qquad T_4 = \frac{4(5)}{2} = 10$$



Example: Arithmetic Sequences are given by $a_n = a + (n-1)d$ for $n \geq 1$ where a is some *initial value* and d is a called the *common difference*. Here you start at some value a and add the number d to obtain the next number in the sequence. For example if $a_n = 10 + 5(n-1)$ for $n \geq 1$ the first few terms of the sequence are...

$$a_1 = 10 + 5(0) = 10 \qquad a_2 = 10 + 5(1) = 15 \qquad a_3 = 10 + 5(2) = 20 \qquad a_4 = 10 + 5(3) = 25$$

so you start at $a_1 = 10$ and add 5 each time to get the next number.

Recursive Definition of a Sequence

These are sequences of the form $a_n = f(a_1, a_2, ..., a_{n-1})$ for $n \geq k$ where initial values a_1, a_2, ..., a_{k-1} are given. You generate the new values from the previous values.

Example: The most famous sequence is undeniably the Fibonacci sequence and is given by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ and $f_0 = 1$, $f_1 = 1$. Give the next few numbers in the Fibonacci sequence.

$f_0, f_1, f_2, f_3, f_4, f_5, f_6, \ldots$

$\hookrightarrow 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$f_0 = 1$ Given

$f_1 = 1$ Given

$f_2 = f_{2-1} + f_{2-2} = f_1 + f_0 = 1 + 1 = 2$

$f_3 = f_{3-1} + f_{3-2} = f_2 + f_1 = 2 + 1 = 3$

$f_4 = f_{4-1} + f_{4-2} = f_3 + f_2 = 3 + 2 = 5$

$\vdots$

Example: The logistic recursive sequence is given by $x_{n+1} = rx_n(1 - x_n)$ for $n \geq 0$ where x_0 is some specified initial value in $[0, 1]$. For example if $r = \frac{3}{2}$ and $x_0 = \frac{1}{2}$ then

$$x_1 = \frac{3}{2}\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{3}{8}$$

$$x_2 = \frac{3}{2}\left(\frac{3}{8}\right)\left(1 - \frac{3}{8}\right) = \frac{45}{128}$$

$$x_3 = \frac{3}{2}\left(\frac{45}{128}\right)\left(1 - \frac{45}{128}\right) = \frac{11205}{32768}$$

and so forth. This sequence is best calculated using a computer. It exhibits a very interesting behaviour as you vary the parameter r. I recommend reading about the relation of the Logistic Map and Chaos Theory. Below is an excellent video provided by Veritasium that described the Logistic Map:

==Link to Veritasium Video==

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The Description of a Sequence by a Mathematical Anarchist

This is where you specify a sequence by random rules described in English (Well, English for this class).

Example: The look and say sequence is the sequence as follows

$$1, 11, 21, 1211, 111221, 312211, ...$$

Determine the rule for generating elements of this sequence and generate the next two numbers of the sequence.

$$\ldots, 13112221, 1113213211, \ldots$$

2.1.2 Convergence and Divergence

Sequences (like we've seen with improper integrals) either approach a single finite value or they do not. That is, they either converge or diverge.

Example: Consider the sequence $a_n = \dfrac{n^2+1}{3n^2-2}$ where $n \geq 1$. Determine the limit as $n \to \infty$. Does it converge or diverge?

$$a_1, a_2, a_3, \ldots, \Rightarrow \frac{1^2+1}{3(1)^2-2}, \frac{2^2+1}{3(2)^2-2}, \frac{3^2+1}{3(3^2)-2}, \ldots$$

$$\Rightarrow \frac{2}{1}, \frac{5}{10}, \frac{10}{25}, \ldots$$

The limit is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1 \div \frac{1}{n^2}}{3n^2-2 \div \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{3-\frac{2}{n^2}} = \frac{1+0}{3-0} = \frac{1}{3}$$

Example: All previous limit laws hold. Squeeze, l'Hôpital, etc. For example consider $a_n = \dfrac{\cos(n)}{n}$ where $n \geq 1$. Determine the limit as $n \to \infty$. Does it converge or diverge?

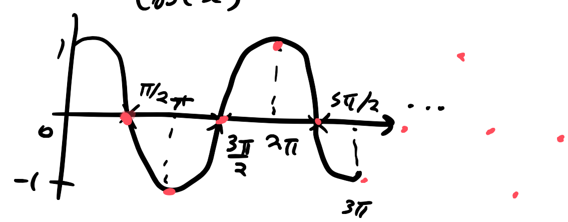
Since $-1 \leq \cos(n) \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ holds for all $n \geq 1$.

Since $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0$

Then by Sandwich Theorem (squeeze lemma)

$$\lim_{n\to\infty} \frac{\cos(n)}{n} = 0.$$

Example: Consider the sequence $a_n = \cos\left(\frac{n\pi}{2}\right)$ where $n \geq 0$. Determine the limit as $n \to \infty$. Does it converge or diverge?



$a_1 = \cos\left(\frac{\pi}{2}\right) = 0$

$a_2 = \cos\left(\frac{2\pi}{2}\right) = -1$ $\quad \frac{2\pi}{2} \nearrow \pi$

$a_3 = \cos\left(\frac{3\pi}{2}\right) = 0$

$a_4 = \cos\left(\frac{4\pi}{2}\right) = 1$ $\quad \frac{4\pi}{2} \nearrow 2\pi$

$a_5 = \cos\left(\frac{5\pi}{2}\right) = 0$

$a_6 = \cos\left(\frac{6\pi}{2}\right) = -1$ $\quad \frac{6\pi}{2} \nearrow 3\pi$

This oscillates indefinitely

$\therefore$ Diverges.

$a_7 = 0$

$a_8 = 1$

$a_9 = 0$

$a_{10} = -1$

$a_{11} = 0$

$\vdots$

Example: Given that the recursive sequence $a_{n+1} = \sqrt{2 + a_n}$, for $n \geq 1$ with $a_1 = \sqrt{2}$ converges, determine it's limit.

Given $\lim_{n\to\infty} a_n = L$ for some L.

Take $n \to \infty$ of both sides of the recursive formula

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$$

$$= \sqrt{2 + \lim_{n\to\infty} a_n}$$

equation for L

$$\Rightarrow \quad L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L$$

$$\Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow (L-2)(L+1) = 0$$

$L = 2$ or $L = -1$ which one?

Every element in seq. is pos.

$\therefore L = \lim_{n\to\infty} a_n = 2$

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Example: Provided a sequence converges you can determine the limit of a recursive equation by setting up an equation you may solve for. For example consider the sequence obtained by successive ratio's of the Fibonacci sequence $R_n = f_{n+1}/f_n$ where $n \geq 0$.

Start with $f_{n+1} = f_n + f_{n-1}$ where $n \geq 1$ and $f_0 = f_1 = 1$

$$\Rightarrow \frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n}$$

$$\Rightarrow \frac{f_{n+1}}{f_n} = 1 + \frac{1}{\frac{f_n}{f_{n-1}}} \Rightarrow R_n = 1 + \frac{1}{R_{n-1}} \quad \text{for } n \geq 1$$

where $R_0 = \frac{f_1}{f_0} = \frac{1}{1} = 1$,

with $R_n = 1 + \frac{1}{R_{n-1}}$, $n \geq 1$ and $R_0 = 1$.

Take $n \to \infty$ and Let $L = \lim_{n\to\infty} R_n$. Then

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(1 + \frac{1}{R_{n-1}}\right)$$

$$\Rightarrow L = 1 + \frac{1}{L} \Rightarrow L^2 = L + 1$$

$$\Rightarrow L^2 - L - 1 = 0$$

Need to use quadratic formula

$$\Rightarrow L = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

so $L = \frac{1+\sqrt{5}}{2}$ OR $L = \frac{1-\sqrt{5}}{2}$. which one?

well, $R_n > 0$ for all n but $\frac{1-\sqrt{5}}{2} < 0$ so it must be

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$$

The quantity $\frac{1+\sqrt{5}}{2}$ is called the Golden Ratio and is abbreviated by $\frac{1+\sqrt{5}}{2} = \varphi$.

2.1.3 Useful Results

Theorem

If $f(x)$ is continuous and $\{a_n\}$ is a sequence in the domain of f then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

Example: Consider $f(x) = \ln(x)$ and $a_n = \dfrac{n}{n+1}$ where $n \geq 1$. Determine $\lim_{n\to\infty} f(a_n)$.

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}\right) = \ln\left(\frac{1}{1+0}\right) = \ln(1)$$

$$= \emptyset //$$

Theorem

These are useful limits to memorize. As $n \to \infty$,

- $\ln(n)/n \to 0$
- $n^{1/n} \to 1$
- $(\ln(n))^{1/n} \to 1$
- $a^{1/n} \to 1$ if $a > 0$
- $a^n \to 0$ if $|a| < 1$
- $\left(1 + \frac{a}{n}\right)^n \to e^a$
- $a^n/n! \to 0$

Example: Consider $a_n = \left(\dfrac{n}{n+1}\right)^n$ for $n \geq 1$. Determine the limit as $n \to \infty$.

$$2 = (2^{-1})^{-1} = \left(\tfrac{1}{2}\right)^{-1}$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

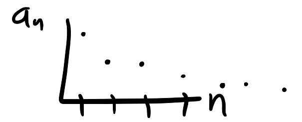
$$a^{bc} = (a^b)^c$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

#6 with $a=1$

$$= (e^1)^{-1}$$

$$= e^{-1} //$$

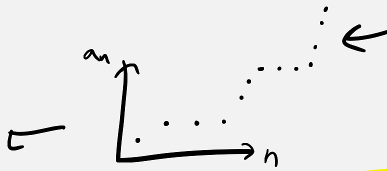


2.1.4 Monotonic Sequences and the Monotonic Convergence Theorem

Definition

A sequence $\{a_n\}$ is...

- non-increasing if $a_{n+1} \leq a_n$ for all n.
- non-decreasing if $a_{n+1} \geq a_n$ for all n.

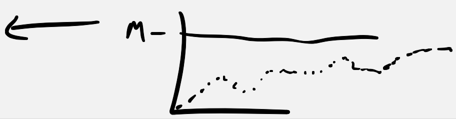


A sequence that is either non-increasing or non-decreasing is also called monotonic.

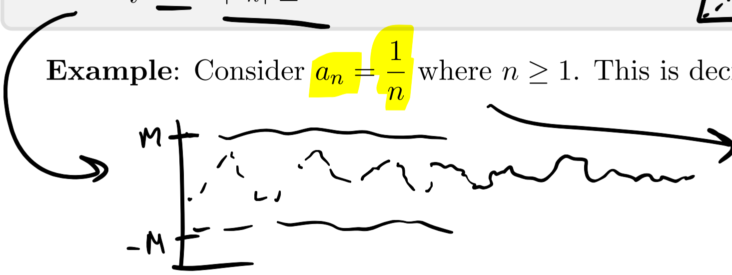
Definition

A sequence $\{a_n\}$ is bounded...

- below by M if $a_n \geq M$ for all n.
- above by M if $a_n \leq M$ for all n.
- by M if $|a_n| \leq M$ for all n.



Example: Consider $a_n = \dfrac{1}{n}$ where $n \geq 1$. This is decreasing and bounded below by $M = -2$. Explain why.



$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Since every $a_n > -2$ then $M = -2$ is a lower bound.

Monotonic Convergence Theorem

If $\{a_n\}$ is either bounded below and non-increasing or bounded above and non-decreasing then the sequence converges.

Example: Argue that the sequence $a_n = \dfrac{a_{n-1}}{n}$ where $n \geq 1$ converges without solving for it explicitly.

$a_0 = 1$

$a_1 = \dfrac{a_0}{1} = \dfrac{1}{1} > 0$

$a_2 = \dfrac{a_1}{2} = \dfrac{1}{2} > 0$

$a_3 = \dfrac{a_2}{3} = \dfrac{1/2}{3} = 1/6 > 0$

$\vdots$

all $a_n \geq 0$ so it is bounded below.

Then we see

$a_1 > a_2 > a_3 > a_4 > \cdots$

So it is decreasing.

Thus by MTC the sequence converges.

2.2 (Section 10.2) Infinite Series

2.2.1 Defining an Infinite Sum

Intuitive Description: An infinite series (or commonly just called a series) is the sum of an infinite sequence of numbers. That is, if $\{a_k\}_{k=1}^{\infty}$ is a sequence then a series whose terms are this sequence is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots.$$

The problem with this definition is that it doesn't give context of what it means to take an infinite sum. We understand limits and can work from there. It should illustrate the fact that as we taking more values to add, we are approaching some value. Thus we need to somehow incorporate this into the definition.

(Actual) Definition

Given a sequence $\{a_k\}$ as above we define the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

We say that the series

$\Rightarrow$ Page 100b

$$\sum_{k=1}^{\infty} a_k$$

converges to L if $\lim_{n\to\infty} S_n = L$. If the Sequence $\{S_n\}$ diverges we say the series diverges.

Now, most of the time you will never be able to find this limit L. It is often impossible and just calculated using very abstract techniques or approximated by computers. Mostly we care about convergence or divergence. This section focuses on two series you can actually find the value of while the rest of this chapter focuses on just determining convergence or divergence without ever being able to find this value.

Example: Consider the series

$$\sum_{k=1}^{\infty} (-1)^k = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + \cdots$$

We have that the partial sums are $S_n = \sum_{k=1}^{n} (-1)^k$. We compute a few terms to find that

$\{S_n\} \sim -1, 0, -1, 0, \ldots$

$$S_1 = (-1)^1 = -1$$
$$S_2 = (-1)^1 + (-1)^2 = -1 + 1 = 0$$
$$S_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1$$
$$S_4 = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = -1 + 1 - 1 + 1 = 0$$

At this point you can probably tell that $\{S_n\}_{n=1}^{\infty} = \{-1, 0, -1, 0, -1, 0, -1, \ldots\}$. We see that for this sequence $\lim_{n\to\infty} S_n$ does not exist. It oscillates between two values. Therefore the above series diverges.

$4\left(\frac{1}{2}\right)^0 + 4\left(\frac{1}{2}\right)^1 + 4\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \cdots$

$a = 4 \quad r = 1/2$

2.2.2 Geometric Series

Definition

A geometric series is a series of the form

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

Theorem

If $|r| < 1$ then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

else if $|r| \geq 1$ the series diverges.

Proof: Consider the partial sum $S_n = \sum_{k=1}^{n} ar^k = a + ar + ar^2 + \cdots + ar^{n-1}$. We may derive an explicit formula for the partial sums of this particular series as follows. Scale partial sum by r to obtain

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Then form the difference $S_n - rS_n$ to obtain

$$(1-r)S_n = S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^n) = a - ar^n$$

due to many of the terms canceling out. Now, solve for the partial sum to obtain

$$S_n = \frac{a(1-r^n)}{1-r}$$

We observe that this sequence converges provided that $|r| < 1$ and diverges otherwise. In particular, if $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$ and thus

$$\sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty} S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

■

$$\sum_{k=1}^{\infty} ar^{k-1} = ar^{1-1} + ar^{2-1} + ar^{3-1} + \cdots = a + ar + ar^2 + ar^3 + \cdots$$

$$\sum_{k=0}^{\infty} ar^{k} = ar^{0} + ar^{1} + ar^{2} + \cdots = a + ar + ar^2 + ar^3 + \cdots$$

Example: Determine if $\displaystyle\sum_{k=0}^{\infty}\left(\frac{e}{\pi}\right)^k$ converges or diverges. If it converges find it's sum.

$$= \left(\frac{e}{\pi}\right)^0 + \left(\frac{e}{\pi}\right)^1 + \left(\frac{e}{\pi}\right)^2 + \left(\frac{e}{\pi}\right)^3 + \cdots$$

$$= 1 + 1\left(\frac{e}{\pi}\right) + 1\left(\frac{e}{\pi}\right)^2 + 1\left(\frac{e}{\pi}\right)^3 + \cdots$$

$a = 1$

$r = e/\pi$

Check with $r = e/\pi$

$\Rightarrow |r| = \left|\frac{e}{\pi}\right| \approx \left|\frac{2.7}{3.1}\right| < 1$

So the Geo series Converges

$$= \frac{1}{1 - e/\pi} = \frac{\pi}{\pi - e}$$

$\left(\frac{1}{3}\right)^k$

Example: Determine if $\displaystyle\sum_{k=1}^{\infty}\ln\left(\frac{1}{3^k}\right)$ converges or diverges. If it converges find it's sum.

The partial sums are

$\ln(a^b) = b \cdot \ln(a)$

$$S_n = \sum_{k=1}^{n} \ln\left(\left(\tfrac{1}{3}\right)^k\right)$$

$$= \sum_{k=1}^{n} k \cdot \ln(1/3) = 1\cdot\ln(1/3) + 2\cdot\ln(1/3) + \cdots$$

$$= \ln(1/3)\,(1 + 2 + \cdots)$$

Not a geometric series

$$= \ln(1/3)\cdot\sum_{k=1}^{n} k$$

$$= \ln(1/3)\cdot\frac{n(n+1)}{2}$$

$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Note $\displaystyle\lim_{n\to\infty} S_n = \ln(1/3)\cdot\lim_{n\to\infty}\frac{n(n+1)}{2} \sim \frac{\infty}{2}$

$$= \underbrace{\ln(1/3)}_{<0}\cdot\infty$$

$$= -\infty$$

$\therefore$ It diverges

$(-1)^0 \quad (-1)^1 \quad (-1)^2 \quad (-1)^3 \quad (-1)^4$

Example: Determine if the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$ converges or diverges. If it converges find it's sum.

$a^k b^k = (ab)^k$

$$= \sum_{k=0}^{\infty} 1 \cdot (-1)^k \left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} 1 \cdot \left(-\frac{1}{2}\right)^k$$

Geometric series

$a = 1 \quad r = -1/2$

and check that

$|r| = |-\frac{1}{2}| = \frac{1}{2} < 1$

$\therefore$ The geo series converges

$$= \frac{a}{1-r} = \frac{1}{1-(-1/2)} = \frac{1}{1+1/2} = \frac{1}{3/2} = 2/3$$

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Example: Express the number 1.414414414414... as the ratio of two integers.

2.2.3 Telescoping Series

Definition

A telescoping series is a series of the form

$$\sum_{k=1}^{\infty}(a_{k-m}-a_{k-l})$$

You can find the sum (or determine convergence/divergence) of these series by investigating and finding a closed formula for the partial sums.

Example: Determine whether the series $\displaystyle\sum_{k=1}^{\infty}\frac{40\cancel{k}}{(2k-1)(2k+1)}$ converges or diverges. If it converges find it's sum.

$$= \frac{40}{(2(1)-1)(2\cdot 1+1)} + \frac{40\cdot 2}{(2\cdot 2-1)(2\cdot 2+1)} + \cdots$$

use partial fractions on

$$\frac{40}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1} = \frac{20}{2k-1} - \frac{20}{2k+1}$$

$$\Rightarrow 40 = A(2k+1) + B(2k-1)$$

Let $k = 1/2 \Rightarrow 40 = A(2) + B\cancel{0} \Rightarrow A = 20$

$k = -1/2 \Rightarrow 40 = A\cancel{0} + B(-2) \Rightarrow B = -20$

$$\text{So } S_n = \sum_{k=1}^{n}\frac{40}{(2k-1)(2k+1)} = \sum_{k=1}^{n}\left(\frac{20}{2k-1} - \frac{20}{2k+1}\right)$$

$$S_1 = 20\left\{\underbrace{\frac{1}{2-1} - \frac{1}{2+1}}_{k=1}\right\} = 20\left\{1 - \frac{1}{3}\right\}$$

$$S_2 = 20\left\{\underbrace{\left(1-\frac{1}{3}\right)}_{k=1} + \underbrace{\left(\frac{1}{4-1} - \frac{1}{4+1}\right)}_{k=2}\right\} = 20\left\{1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{5}\right\}$$

$$= 20\left\{1 - \frac{1}{5}\right\}$$

$$S_3 = 20\left\{\underbrace{\left(1-\frac{1}{5}\right)}_{k=1,2} + \underbrace{\left(\frac{1}{6-1} - \frac{1}{6+1}\right)}_{k=3}\right\} = 20\left\{1 - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{7}\right\} = 20\left\{1 - \frac{1}{7}\right\}$$

$\vdots$

$$S_n = 20\left(1 - \frac{1}{2n+1}\right) \quad \text{So } \sum_{n=1}^{\infty}\frac{40}{(2k-1)(2k+1)} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} 20\left(1 - \frac{1}{2n+1}\right)$$

$$= 20(1-0) = 20$$

$$\ln\left(\left(\frac{n+1}{n}\right)^{1/2}\right)$$

Example: Determine if the series $\sum_{n=1}^{\infty} \ln\left(\sqrt{\frac{n+1}{n}}\right)$ converges or diverges. If it converges find it's sum.

$$= \sum_{n=1}^{\infty} \frac{1}{2} \cdot \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\ln(n+1) - \ln(n)\right)$$

$$S_N = \sum_{n=1}^{N} \frac{1}{2} \cdot \left(\ln(n+1) - \ln(n)\right)$$

$$S_1 = \frac{1}{2}\left(\ln(2) - \cancel{\ln(1)}^{0}\right) = \frac{1}{2}\ln(2)$$

$$S_2 = \frac{1}{2}\left(\underbrace{\cancel{\ln(2)}}_{n=1} + \underbrace{(\ln(3) - \cancel{\ln(2)})}_{n=2}\right) = \frac{1}{2}\ln(3)$$

$$S_3 = \frac{1}{2}\left(\underbrace{\cancel{\ln(3)}}_{n=1,2} + (\ln(4) - \cancel{\ln(3)})\right) = \frac{1}{2}\ln(4)$$

$$\vdots$$

$$S_N = \frac{1}{2}\ln(N+1)$$

$\therefore$ The diveages as the partial sums limit is

$$\lim_{N\to\infty} \frac{1}{2}\ln(N+1) = \infty \;//$$

2.2.4 The n-th Term Divergence Test

If you are **given** the fact that a series $\sum_{n=1}^{\infty} a_n$ converges what can you say about the convergence of the sequence in the sum, $\{a_n\}$? Notice that we may single out elements of the sequence using partial sums by the following

$$S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = a_n$$

Then if we are told $\sum_{n=1}^{\infty} a_n$ converges, that implies that $\lim_{n\to\infty} S_n = L$ for some number L. Thus we obtain

$$\lim_{n\to\infty} S_n = L \quad \text{and} \quad \lim_{n\to\infty} S_{n-1} = L$$

and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = L - L = 0$$

$\{\frac{1}{n}\}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\{\frac{1}{n^2}\}$

This gives us a result but nothing that is useful.

(Mostly Useless But Still Should Know This) Theorem

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

⇒ Page 106 ⇐

We can reword this theorem to become useful but it requires a little bit of logic.

Definition and Result

Given a statement "If p then q" the *contrapositive* of this statement is "If NOT q then NOT p". The contrapositive of a statement has the same logical implication.

Example: Consider the statement "If it is a crow then it is black" (where we are talking about the common crow before any biology people start talking about blue jays). So if you see a crow you can say it is black. This also implies that the following statement is also true, "If it is not black then it is not a crow".

With this we can form the useful theorem.

n-th Term Divergence Test

If it is NOT TRUE that $\lim_{n\to\infty} a_n = 0$ then it is NOT TRUE that $\sum_{n=1}^{\infty} a_n$ converges. That is, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. The statement $\lim_{n\to\infty} a_n \neq 0$ is still satisfied if a_n diverges.

Example: Consider the series $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right)$. $= \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{2\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) + \cdots$

Sequence terms are $a_n = \cos\left(\frac{n\pi}{2}\right)$. In an earlier example we saw that

$\{a_n\} \sim 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots$

Thus $\{a_n\}$ oscillates so $\lim_{n\to\infty} a_n \neq 0$

$\therefore$ By the divergence test

$\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right)$

$f(n)^n$

$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$

Example: Consider the series $\sum_{n=1}^{\infty} \left(1 - \frac{\pi}{n}\right)^{-n} = \left(1-\frac{\pi}{1}\right)^{-1} + \left(1-\frac{\pi}{2}\right)^{-2} + \left(1-\frac{\pi}{3}\right)^{-3} + \cdots$

$a_n = \left(1-\frac{\pi}{n}\right)^{-n}$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\left(1 + \frac{-\pi}{n}\right)^n\right)^{-1}$, $a = -\pi$

$= \left(e^{-\pi}\right)^{-1} = e^{\pi} \neq 0$

$\therefore$ By the divergence test

$\sum_{n=1}^{\infty} \left(1-\frac{\pi}{n}\right)^{-n}$

$$\lim_{k\to\infty} \frac{4k^2-k^4}{10+2k^4} = \text{maff}$$
$$= -\frac{1}{2}$$

Example: A friend has determined that for the series $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2-k^4}{10+2k^4}$ that it diverges because $\displaystyle\lim_{k\to\infty} a_k = -\frac{1}{2} \neq 0$. However, they wrote down

$$\sum_{k=1}^{\infty} \frac{4k^2-k^4}{10+2k^4} = -\frac{1}{2}$$

Explain why the student is incorrect.

$\sum a_n \neq$ any number because it diverges

The quantity of $\sum a_n$ is not related to the quantity $\lim a_n$ in any way

Example: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We see that $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$. Is there anything you can say about the value convergence or divergence of the series? Can you conclude anything about the value of the series?

NO! You'd be assuming the hypothesis!

No! The sequence limit is not related in quantity to the series quantity.

Definition

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **Harmonic Series**.

2.2.5 Properties of Series

Theorem

Let $\sum a_n$ and $\sum b_n$ be two convergent series and let k be a constant. Then...

1. $\sum(a_n + b_n) = \sum a_n + \sum b_n$
2. $\sum(a_n - b_n) = \sum a_n - \sum b_n$
3. $\sum k a_n = k \sum a_n$

and consequently all the above expressions are convergent series.

There are also corollaries that occur as a consequence of the above theorem.

Theorem

1. Every non-zero constant multiple of a divergent series is also divergent.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then both $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ diverge.

Example: Explain why the series $\sum_{k=1}^{\infty} \left(\left(\frac{2}{3}\right)^k - \frac{k}{k+1} \right)$ diverges.

$a_k = \left(\frac{2}{3}\right)^k$ $\qquad$ $b_k = \frac{k}{k+1}$

Note $\sum a_k = \sum \left(\frac{2}{3}\right)^k$ ← Geometric series with $r = 2/3$

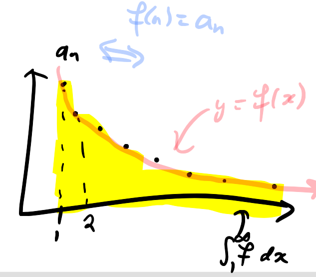
as $|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$

$\Rightarrow \sum a_k$ converges

Note $\sum b_k = \sum \frac{k}{k+1}$ diverges by the divergence test

as $\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+1/k} = \frac{1}{1+0} = 1 \neq 0$

$\therefore \sum(a_k - b_k)$ diverges.



2.3 (Section 10.3) Integral Test

2.3.1 Constructing the Integral Test and Examples

Formulating the Integral Test

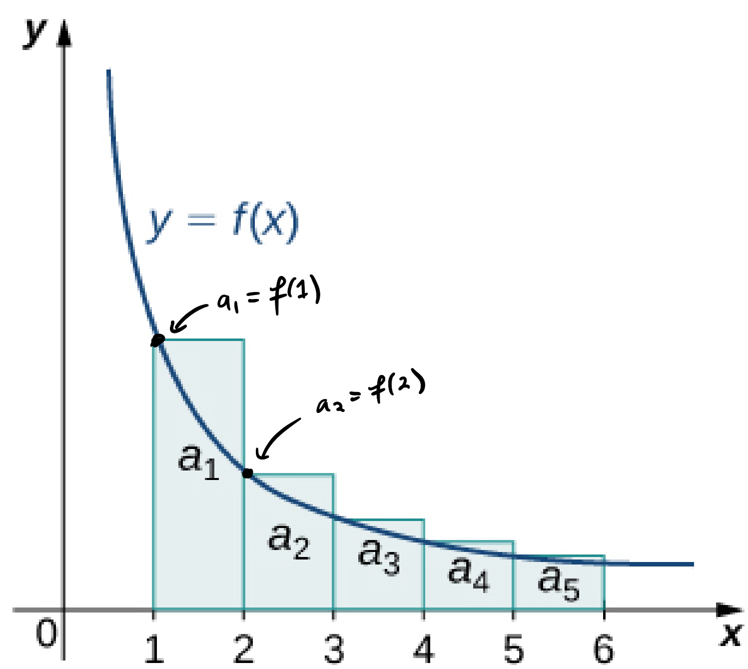
Suppose that f is a continuous, positive and decreasing function such that $f(n) = a_n$ on the interval $[1, \infty)$. Then...

$$\int_1^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} f(x)dx$$

To demonstrate this result, we consider rewriting the series as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ as

$$\sum_{n=1}^{\infty} a_n = a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots$$

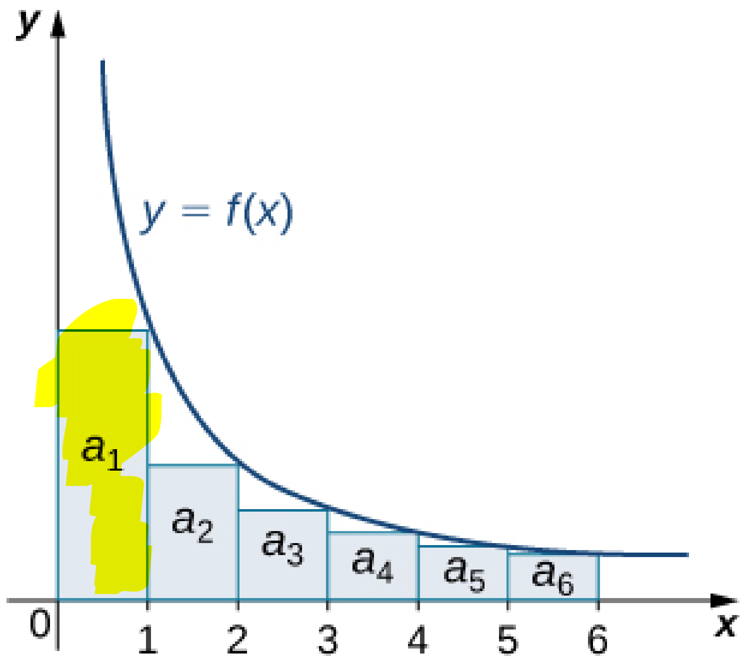
which we will interpret each term $a_i \cdot 1$ to geometrically represent the area of a rectangle of height $H = a_i$ and width $W = 1$. Now, we may fit these rectangles to the curve of $y = f(x)$ as follows to obtain



which holds graphically since $f(1) = a_1$, $f(2) = a_2$, ... etc. Based on this, we see that the sum of the rectangles forms an over approximation and we obtain the lower bound of

$$\int_1^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n$$

For the upper bound, we take the above picture and move all the rectangles one unit over to the left. This still is allowed since by the symmetry of the rectangle we still have $f(1) = a_1$, $f(2) = a_2$, ... etc.



However, we are taking the area for $n \geq 1$, and thus we obtain

$$\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)dx$$

to which we add a_1 to both sides of the inequality to obtain

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} f(x)dx$$

as desired. From this, we can determine the behaviour of certain series by a representing integral and vice versa (essentially a comparison test).

The Integral Test

I II III Construct

Let $f(x)$ be continuous, positive, and decreasing on $[k, \infty)$ and that $f(n) = a_n$ then

$$\int_k^{\infty} f(x)dx \qquad \text{and} \qquad \sum_{n=k}^{\infty} a_n$$

either both converge or both diverge. ← conclude Determine with theorem

Note

Since $\int_1^{\infty} \frac{dx}{x^p}$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ both converge or diverge together the nature of convergence for p-integrals is the same as the p-series form.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi+1}}$ Converges as $p = \pi + 1 > 1$. and is a p-series.

> **Note**
>
> A COMMON mistake that students make is that they assume that the integral is equal to the series! THIS IS NOT THE CASE! They just have the same convergent behaviour! That is,
>
> $$\int_k^\infty f(x)dx \neq \sum_{n=k}^{\infty} f(n)$$
>
> In this course, you should assume that unless you have a Geometric Series, Telescoping Series, or (a yet to be mentioned) Taylor Series then all hope of finding an exact value for a series is lost. Very advanced techniques are required.

Example: Determine the nature of convergence for $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. $= \frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1} + \cdots$

We have $a_n = \frac{1}{n^2+1}$ as a sequence.

Fit $f(x) = \frac{1}{x^2+1}$ for $x \geq 1$

I) Continuous on $x \geq 1$? Yes! Rational with no division by zero.

II) Positive on $x \geq 1$? Yes! $x^2 \geq 0 \Rightarrow \frac{1}{x^2+1} > 0$.

III) Decreasing on $x \geq 1$? Yes as $f'(x) = -\frac{1}{(x^2+1)^2}(2x) = -\frac{2x}{(x^2+1)^2} < 0$

By the integral test $\int_1^\infty \frac{dx}{x^2+1}$ and $\sum_{n=1}^\infty \frac{1}{n^2+1}$ both converge or diverge together

As $\int_1^\infty \frac{dx}{x^2+1} = \arctan(x)\Big|_1^\infty = \arctan(\infty) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ Converges

then $\sum_{n=1}^\infty \frac{1}{n^2+1}$ also converges ■

Example: Determine the nature of convergence for $\sum_{n=2}^{\infty} ne^{-n}$.

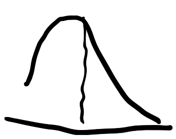
⇒ Page 112 ⇐

Let $f(x) = xe^{-x}$ for $x \geq 2$

I) positive? Yes, as $x \geq 2$ so $x > 0$. Also $e^{-x} > 0$ so $xe^{-x} > 0$.

II) Continuous? Yes, it is the product of a polynomial and an exponential.

III) Decreasing? $f'(x) = e^{-x} - xe^{-x} = (1-x)\underbrace{e^{-x}}_{>0}$ then as $x \geq 2$



(Eventually)

LIPET

we have $1-x<0$ so $f'(x)<0$ and thus $f(x)$ is decreasing.

Determine the nature of $\int_2^\infty xe^{-x}\,dx$

$u = x$, $du = dx$

$dv = e^{-x}dx$, $v = -e^{-x}$

$\sim \frac{\infty}{\infty}$

$= -xe^{-x}\Big|_2^\infty - \int_2^\infty -e^{-x}dx = \lim_{b\to\infty}\left(-\frac{b}{e^b} + \frac{2}{e^2}\right) + \int_2^\infty e^{-x}dx$

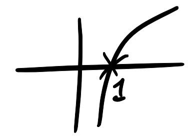
$\overset{L'H}{=} \lim_{b\to\infty}\left(-\frac{1}{e^b} + \frac{2}{e^2}\right) - e^{-x}\Big|_2^\infty$

$= \left(-\frac{1}{e^\infty} + \frac{2}{e^2}\right) - \left(\frac{1}{e^\infty} - \frac{1}{e^2}\right)$

$= 0 + \frac{2}{e^2} - 0 + \frac{1}{e^2}$ Finite!

$\therefore$ as $\int_2^\infty xe^{-x}\,dx$ converges then $\sum_{n=2}^{\infty} ne^{-n}$ converges by the integral test.

Example: Determine the nature of convergence for $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$.



$$f(x) = \frac{1}{x\ln(x)} \quad \text{for } x \geq 2.$$

I) positive? Yes as $x \geq 2$ and $\ln(x) > 0$ for $x > 1$ thus $\frac{1}{x\ln(x)} > 0$ on $x \geq 2$.

II) Continuous? Yes as $\ln(x)$ is continuous and x is cont so $\frac{1}{x\ln(x)}$ is cont as $x\ln(x) \neq 0$ on $x \geq 2$.

III) Decreasing?

$$f'(x) = \frac{d}{dx}\left[(x\ln(x))^{-1}\right] = -(x\ln(x))^{-2}\frac{d}{dx}\left[x\ln(x)\right]$$

$$= -\frac{1}{(x\ln(x))^2}\left(\ln(x) + x\cdot\frac{1}{x}\right)$$

$$= \underbrace{-}_{<0}\frac{1}{\underbrace{(x\ln(x))^2}_{>0}}\underbrace{(\ln(x)+1)}_{>0 \text{ on } x\geq 2} < 0$$

Thus $f(x)$ is decreasing.

Investigate $\displaystyle\int_2^\infty \frac{dx}{x\ln(x)}$

Let $u = \ln(x)$, $du = \frac{1}{x}dx$

Bounds

$x = 2 \Rightarrow u = \ln(2)$

$x \to \infty \Rightarrow u \to \infty$

$$= \int_{\ln(2)}^{\infty} \frac{du}{u^1} \longleftarrow \text{p-integral with } p = 1 \leq 1 \;\therefore \int_2^\infty f(x)\,dx \text{ diverges}$$

$\therefore$ By the integral test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges.

> Note
>
> We will be discussing other series tests so that we have a wide range of tools at our disposal for determining the behaviour of series. A **BIG** indication that you might want to start with an integral test is if a logarithm is present.

At Home Exercise: Show that $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k \neq \int_1^\infty \left(\frac{2}{3}\right)^x dx.$

omit

2.3.2 Error Estimation

We only know how to find the sum of geometric and telescoping series, but we can approximate others by partial sums

$$\sum_{n=k}^{N} a_n \approx \sum_{n=k}^{\infty} a_n$$

for very very large N. Let $S = \sum_{n=k}^{\infty} a_n$ and $S_N = \sum_{n=k}^{N} a_n$. We can form the error in the estimation (summing up to the term $n = N$),

$$R_N = S - S_N$$

Suppose that S converges under the integral test. If $f(n) = a_n$ under the required conditions earlier then one can show the error is bounded by

$$\int_{N+1}^{\infty} f(x)dx \le R_N \le \int_{N}^{\infty} f(x)dx$$

by rearranging the inequality formed at the beginning of this section.

Example: Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by bounding it with $N = 10$ terms.

2.4 (Section 10.4) Comparison Tests

2.4.1 The Direct Comparison Test for Series

Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $0 \leq a_n \leq b_n$ for all n. Then

- If $\sum b_n$ converges then $\sum a_n$ converges
- If $\sum a_n$ diverges then $\sum b_n$ diverges

The logic behind this is identical to that for improper integrals.

Example: Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges or diverges.

Let $a_n = \frac{1}{n^3+n}$ for $n \geq 1$. Note

$$\frac{1}{n^3+n} \leq \underbrace{\frac{1}{n^3}}_{=b_n} \text{ by decreasing the denominator}$$

also note $a_n, b_n \geq 0$ on $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (it is a p-series with $p = 3 > 1$) then by the DCT $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

Example: Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges or diverges.

Consider $a_n = \frac{2^n+3^n}{3^n+4^n}$ on $n \geq 1$.

Note

$$\frac{2^n+3^n}{3^n+4^n} \leq \frac{2^n+3^n}{4^n} = \frac{2^n}{4^n} + \frac{3^n}{4^n} = \left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n$$
$$= \underbrace{\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n}_{=b_n}$$

Note $a_n, b_n \geq 0$ on $n \geq 1$.

Investigate $\displaystyle\sum_{n=1}^{\infty} \left\{ \left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n \right\}$ $\leftarrow r_1 = \frac{1}{2}$, $r_2 = \frac{3}{4}$ note $|r_1| = \frac{1}{2} < 1$, $|r_2| = \frac{3}{4} < 1$ so $\sum r_1^n$ and $\sum r_2^n$ converge

$\sum \left(\frac{1}{2}\right)^n$ $\sum \left(\frac{3}{4}\right)^n$ Geometric

$\therefore$ So $\sum b_n$ converges

$\therefore$ By the DCT $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges $\blacksquare$

2.4.2 The Limit Comparison Test for Series

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ and let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then provided

- $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or diverge
- extra $L = 0$, then if $\sum b_n$ converges then $\sum a_n$ converges
- $L = \infty$, then if $\sum b_n$ diverges then $\sum a_n$ diverges

Think about these results intuitively!

Example: Determine whether or not the series $\sum_{n=3}^{\infty} \sqrt{\frac{n^5+3n}{2n^2+4}}$ converges or diverges.

$a_n = \sqrt{\frac{n^5+3n}{2n^2+4}}$ for $n \geq 3$. Large n $\sim \sqrt{\frac{n^5}{2n^2}} = \sqrt{\frac{n^3}{2}} = \frac{n^{3/2}}{\sqrt{2}} = b_n$

Form $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{\frac{n^5+3n}{2n^2+4}} \div \sqrt{\frac{n^3}{2}}$

$= \lim_{n\to\infty} \sqrt{\frac{n^5+3n}{2n^2+4}} \times \sqrt{\frac{2}{n^3}}$

$= \lim_{n\to\infty} \sqrt{\frac{2n^5+6n}{2n^5+4n^3}}$

$= \lim_{n\to\infty} \sqrt{\frac{2+6/n^4}{2+4/n^2}} = \sqrt{\frac{2+0}{2+0}} = \sqrt{\frac{2}{2}} = \sqrt{1} = 1$

Thus $0 < L < \infty$. We investigate $\sum b_n = \sum_{n=3}^{\infty} \frac{n^{3/2}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{n=3}^{\infty} n^{3/2}$

$= \frac{1}{\sqrt{2}} \sum_{n=3}^{\infty} \frac{1}{n^{-3/2}}$

p-series with $p = -3/2 \leq 1$ and thus diverges.

$\therefore$ By the LCT $\sum_{n=3}^{\infty} a_n$ also diverges

$$\overset{\text{Large}}{\sim} \frac{n\ln(n)}{n^2} \sim \frac{\ln(n)}{n} = b_n$$

Example: Determine whether or not the series $\displaystyle\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+5}$ converges or diverges.

Let $a_n = \dfrac{1+n\ln(n)}{n^2+5}$ and $b_n = \dfrac{\ln(n)}{n}$ on $n \geq 2$. Note $a_n, b_n > 0$.

Form $\displaystyle\lim_{n\to\infty} a_n \div b_n = \lim_{n\to\infty} \frac{1+n\ln(n)}{n^2+5} \div \frac{\ln(n)}{n}$

$$= \lim_{n\to\infty} \frac{1+n\ln(n)}{n^2+5} \times \frac{n}{\ln(n)}$$

$$= \lim_{n\to\infty} \frac{n+n^2\ln(n)}{n^2\ln(n)+5\ln(n)} \times \frac{\frac{1}{n^2\ln(n)}}{\frac{1}{n^2\ln(n)}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n\ln(n)} + 1}{1+\frac{5}{n^2}} = \frac{0+1}{1+0} = 1 = L$$

and thus $0 < L < \infty$. Investigate $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. use the integral test

$f(x) = \dfrac{\ln(x)}{x}$ on $x \geq 2$.

I)
II) Online
III)

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx \qquad \begin{aligned} u &= \ln(x) \\ du &= \tfrac{1}{x}\,dx \end{aligned}$$

$= \displaystyle\int_{\ln(2)}^{\infty} u\,du = \frac{1}{2}u^2\Big|_{\ln(2)}^{\infty} = \infty$ thus by the integral test $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges.

Going back $\displaystyle\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+5}$ diverges by the LCT $\blacksquare$

On line

Example: Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}e^{n}}$ converges or diverges.

$\Rightarrow$ Page 119 $\Leftarrow$

2.5 (Section 10.5) Absolute Convergence, Ratio and Root Tests

2.5.1 The Absolute Convergence Test

Definition

An alternating series is a series of the form

$$\sum_{n=k}^{\infty}(-1)^n a_n$$

where $a_n \geq 0$ for all $n \geq k$.

e.g. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

$= \frac{(-1)^1}{1} + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \cdots$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

Example: Determine if the alternating series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{4^{n+1}}{5^n}$ converges or diverges.

$$= \sum_{n=0}^{\infty}(-1)(-1)^n \frac{4^n \cdot 4}{5^n}$$

$$= \sum_{n=0}^{\infty}(-4)\cdot\left(-\frac{4}{5}\right)^n \quad \text{Geometric Series}$$

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r|<1$

$r = -\frac{4}{5} \Rightarrow |r| = \frac{4}{5} < 1$ so

$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{4^{n+1}}{5^n}$ converges $\blacksquare$

We can handle them for geometric series. Other tests will be required for other series.

Absolute Convergence Test

If $\sum |a_n|$ converges then $\sum a_n$ converges.

Proof: Notice that $|a_n|$ is either a_n or $-a_n$ by definition of the absolute value (depending on the sign of a_n). Thus we can say

$$0 \leq a_n + |a_n| \leq |a_n| + |a_n| = 2|a_n|$$

Since we are assuming $\sum |a_n|$ is convergence then $\sum 2|a_n|$ is also convergent. As $a_n + |a_n|$ and $2|a_n|$ are non-negative them by a comparison test $\sum(a_n + |a_n|)$ is convergent as $\sum 2|a_n|$ is. Then we may write

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

and as $\sum a_n$ is the difference of two convergent series, it is also convergent. $\blacksquare$

Note

This is currently our only test mentioned that handles series with negative terms in determining convergence. The divergence test handles series with negative terms but only determines divergence. Be careful on how to interpret this theorem. There is NO COMPARISON happening here. The comparison tests previously mentioned only apply to series whose terms are non-negative. Comparison does not apply to series that have negative terms. Also, if $\sum |a_n|$ diverges there is nothing you can conclude!

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$.

$$= \frac{(-1)^3}{1^2} + \frac{(-1)^4}{2^2} + \frac{(-1)^5}{3^2} + \frac{(-1)^6}{4^2} + \cdots = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$$

Investigate $\displaystyle\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+2}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|(-1)^{n+2}|}{|n^2|} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

This is a p-series with $p = 2 > 1$

So it converges

$\therefore$ By the ACT as $\sum |a_n|$ converge then $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$ Converges ∎

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$.

Investigate $\displaystyle\sum |a_n| = \sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{|n^3|} = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$

Note that for $|a_n| = \dfrac{|\sin(n)|}{n^3}$ for $n \geq 1$ that

$$\frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3} \quad \text{as} \quad |\sin(n)| \leq 1 \quad (\because -1 \leq \sin(n) \leq 1)$$

Letting $b_n = \dfrac{1}{n^3}$ for $n \geq 1$ we note that $|a_n|, b_n > 0$.

By the DCT, Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as it is a p-series with $p = 3 > 1$

then $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ converges.

$\therefore$ By the ACT, Since $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3}$ converges then $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ converges ∎

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \underline{(-1)^n} \underline{\left(1 + \frac{3}{n}\right)^n}$.

Try using ACT and investigate $\sum |a_n| = \sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^n$ Reminder $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$

we see by the divergence test that $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3 \neq 0$

and thus $\sum |a_n|$ diverges. OUR ATTEMPT FAILS.

By deduction (in our brain) the only option left to try is divergence test.

For $a_n = (-1)^n \left(1+\frac{3}{n}\right)^n$ when $n \geq 1$ we have $\lim_{n\to\infty} \underbrace{(-1)^n}_{\pm 1} \underbrace{\left(1+\frac{3}{n}\right)^n}_{e^3}$ e^3, $-e^3$ oscillate between

which is non-zero. Thus $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{3}{n}\right)^n$

diverges by the divergence test $\boxtimes$

> **Definition**
>
> If $\sum |a_n|$ is convergent we say $\sum a_n$ is absolutely convergent.

Thus all but the last example mentioned are series that are absolutely convergent. Absolute convergence is a level of convergence that is VERY strong. There are many things you can do with series that is absolutely convergent when manipulating them... while for series that are convergent (but not absolutely convergent) you are very restricted.

2.5.2 The Ratio Test

This is everybody's favourite test.

The Ratio Test

Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Then if...

- $L < 1$ the series converges absolutely.
- $L > 1$ the series diverges.
- $L = 1$ then the test is inconclusive. You must apply a different test as this one does not work.

Note

This test is useful for **FACTORIALS** (especially!!!), polynomials, and simple exponents.

Example: Determine the nature of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{3^n(n+1)}{n!}$.

$n! = n(n-1)(n-2)\cdots(2)(1)$

$4! = 4\cdot 3\cdot 2\cdot 1$

$9! = 9\cdot 8\cdot 7\cdot \cdots \cdot 3\cdot 2\cdot 1$

$0! \overset{def}{=} 1$

Consider $a_n = \dfrac{3^n(n+1)}{n!}$ for $n \geq 0$.

Form $L = \lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1+1)}{(n+1)!} \div \frac{3^n(n+1)}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{n+1}(n+2)}{(n+1)!} \times \frac{n!}{3^n(n+1)}\right| = \lim_{n\to\infty}\left|\frac{\cancel{3^{n+1}}}{\cancel{3^n}}\cdot\frac{n+2}{n+1}\cdot\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|3\cdot\frac{n+2}{n+1}\cdot\frac{\cancel{n(n-1)(n-2)\cdots(2)(1)}}{(n+1)\cancel{(n)(n-1)\cdots(2)(1)}}\right|$$

$$= \lim_{n\to\infty}\left|3\cdot\frac{n+2}{n+1}\cdot\frac{1}{n+1}\right|$$

$$= \lim_{n\to\infty}\left|3\cdot\frac{1+2/n}{1+1/n}\cdot\frac{1}{n+1}\right|$$

$$= \left|3\cdot\frac{1+0}{1+0}\cdot 0\right| = 0$$

$$\frac{3!}{4!} = \frac{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \frac{1}{4}$$

$$\frac{7!}{8!} = \frac{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{8\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \frac{1}{8}$$

Since $L = 0 < 1$ then $\displaystyle\sum_{n=0}^{\infty}\frac{3^n(n+1)}{n!}$ converges by the Ratio Test.

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Consider $a_n = \dfrac{n!\,n!}{(2n)!}$ for $n \geq 1$.

Form
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(n+1)!}{(2(n+1))!} \div \frac{n!\,n!}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!\,n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{n}\cdot\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}} \cdot \frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}} \cdot \frac{\cancel{(2n)}\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{(2)}\cancel{(1)}}{(2n+2)(2n+1)\cancel{(2n)}\cancel{(2n-1)}\cdots\cancel{(2)}\cancel{(1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \right| \overset{\text{ratl}}{=} \frac{1}{4}$$

Since $L = \frac{1}{4} < 1$ then $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges by the ratio test.

$\Rightarrow$ page 124 $\Leftarrow$

2.5.3 The Root Test

The Root Test

Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

then if...

- $L < 1$ then the series converges absolutely.
- $L > 1$ then the series diverges.
- $L = 1$ then the test is inconclusive. You must use a different test as this one does not work.

Note

This is useful for bad exponents

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$.

Form $a_n = \sin^n\left(\frac{1}{\sqrt{n}}\right) = \left(\sin\left(\frac{1}{\sqrt{n}}\right)\right)^n$ for $n \geq 1$.

Construct

$$L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\left(\sin\left(\tfrac{1}{\sqrt{n}}\right)\right)^n\right|^{1/n}$$

$$= \lim_{n\to\infty} \left(\left|\sin\left(\tfrac{1}{\sqrt{n}}\right)\right|^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \left|\sin\left(\tfrac{1}{\sqrt{n}}\right)\right|^{n/n}$$

$$= \lim_{n\to\infty} \left|\sin\left(\tfrac{1}{\sqrt{n}}\right)\right|$$

$$= \left|\lim_{n\to\infty} \sin\left(\tfrac{1}{\sqrt{n}}\right)\right| \quad \because |x| \text{ is cont}$$

$$= \left|\sin\left(\lim_{n\to\infty} \tfrac{1}{\sqrt{n}}\right)\right| \quad \because \sin(x) \text{ is cont}$$

$$= |\sin(0)| = |0| = 0$$

Thus $L = 0 < 1$ and so $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$ converges by the root test.

$$(1+\tfrac{a}{n})^n \xrightarrow[n\to\infty]{} e^a$$

$$(n^{1/n}) \xrightarrow[n\to\infty]{} 1$$

Example: Determine the nature of convergence of $\displaystyle\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^2}$.

Form $a_n = \dfrac{2^{n^2}}{n^2}$ for $n \geq 3$

Construct

$$L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{2^{n^2}}{n^2} \right|^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{2^{n^2}}{n^2} \right)^{1/n}$$

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$

$$= \lim_{n\to\infty} \frac{2^{n^2/n}}{(n^2)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{2^n}{(n^{1/n})^2} = \frac{2^\infty}{(1)^2}$$

$$= \frac{\infty}{1} = \infty$$

$\because \lim_{n\to\infty} n^{1/n} = 1$ by limits list in 10.1.

Thus $L \to \infty > 1$ and thus $\displaystyle\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^2}$ diverges by the root test.

Example: Determine the nature of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$.

Form $a_n = \dfrac{n!}{(-n)^n}$ for $n \geq 1$.

Construct $L = \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{(-(n+1))^{n+1}} \div \frac{n!}{(-n)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{(-(n+1))^{n+1}} \times \frac{(-n)^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \times \frac{(-n)^n}{(-(n+1))^{n+1}} \right|$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}$$

$$= \lim_{n\to\infty} \left| (n+1) \cdot \frac{(-1)^n n^n}{(-1)^{n+1}(n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{n+1}}{-1} \cdot \frac{1}{\cancel{(n+1)}} \cdot \frac{n^n}{(n+1)^n} \right| = \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^n \right|$$

absolute value

$$\lim_{n\to\infty} (1+\tfrac{a}{n})^n = e^a$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-1}$$

$\therefore$ Since $L = \frac{1}{e} < 1$ then $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$ converges by the ratio test ■ $\leftarrow$ $= (e^1)^{-1} = e^{-1} = \frac{1}{e} < 1$

2.6 (Section 10.6) Alternating Series Test

2.6.1 Conditional Convergence and the Alternating Series Test

Definition

If a series $\sum b_n$ converges but $\sum |b_n|$ diverges then series $\sum b_n$ is called conditionally convergent.

Note

Conditionally convergent series DO converge, but they have a "lower level" of convergence. The level of convergence is rather slow and weak. When working with conditionally convergent series you are rather limited to what you can do with them algebraically.

The Alternating Series Test (AST)

The series $\sum (-1)^{n+1} a_n$ converges if all the following are satisfied:

- All $a_n \geq 0$.
- All terms in $\{a_n\}$ are eventually all non-increasing.
- $\lim_{n\to\infty} a_n = 0$.

Proof: Let's examine even partial sums. Note that as we assume $a_{n+1} \leq a_n$ (non-increasing) then $a_n - a_{n+1} \geq 0$. We have

$$S_2 = a_1 - a_2$$

$$S_4 = a_1 - a_2 + a_3 - a_4 = S_2 + a_3 - a_4 \geq S_2 + 0 = S_2$$

$$S_6 = S_4 + a_5 - a_6 \geq S_4 + 0 = S_4$$

So $\{S_{2n}\}$ is increasing. Also $S_{2n} \leq a_1$ since

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots a_{2n-1} - a_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

as $a_n - a_{n+1} \geq 0$. Thus by monotonic convergence $\{S_{2n}\}$ converges (i.e. $S_{2n} \to L$). If N is odd then for S_N we have $\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty}(S_{2m} + a_{2m+1}) = L + 0 = L$. ■

Note

To show a series is absolutely convergent you need to show $\sum |a_n|$ converges and THAT'S IT. To show a series is conditionally convergent you need to show $\sum a_n$ converges under the alternating series test AND show that $\sum |a_n|$ diverges by some other test.

$\sum a_n$:

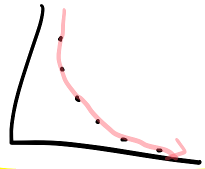
Absolute Convergence

$\sum |a_n|$ converge

$\Rightarrow$ imply $\sum a_n$ converges strongly by ACT

Conditional Convergence

$\sum |a_n|$ diverges but $\sum a_n$ converges



Example: Determine the nature of convergence of the alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Is it conditionally or absolutely convergent?

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Try AST with $a_n = \frac{1}{n}$ for $n \geq 1$.

- Since $n \geq 1$ then $\frac{1}{n} > 0$ so $a_n > 0$.
- Fit $f(x) = \frac{1}{x}$ about $a_n = \frac{1}{n}$. then as $f'(x) = -\frac{1}{x^2} < 0$ for $x \geq 1$ then a_n lies on a dec curve, so $\{a_n\}$ decreases.
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By the AST, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Investigate $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$. This diverges as it is a p-series with $p = 1 \leq 1$ (or you can say it's the Harmonic series).

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Example: Determine the nature of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$.

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If you try AST, it will not work as

$$\lim_{n\to\infty} \frac{n^2}{n^2+5} = \lim_{n\to\infty} \frac{1}{1+5/n^2} = \frac{1}{1+0} = 1 \neq 0$$

In this circumstance where $\lim_{n\to\infty} a_n \neq 0$ the divergence test immediately applies

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} \begin{matrix} \nearrow 1 \\ \text{oscillates} \\ \searrow -1 \end{matrix} \neq 0$$

and thus by the divergence test $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$

$$\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \begin{array}{l} \nearrow \underline{\underline{\text{Converge}}} \text{ if } p>0 \\ \searrow \text{diverse if } p \le 0 \end{array} \right)$$

$$\sim \sum \frac{\sqrt{n}}{n+4} \sim \sum \frac{\sqrt{n}}{n} \sim \sum \frac{1}{\sqrt{n}} \quad p = 1/2 \le 1$$

diverge

Example: Determine the nature of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$. Is it conditionally or absolutely convergent?

$$\sim \sum \frac{(-1)^n \sqrt{n}}{n} \sim \sum \frac{(-1)^n}{\sqrt{n}} \quad p = 1/2 > 0$$

Gut tells me it is conditionally convergent.

Try AST, set $a_n = \frac{\sqrt{n}}{n+4}$ for $n \geq 0$

- $a_n \geq 0$ as $n \geq 0 \Rightarrow \sqrt{n} \geq 0$ and $n+4 > 0 \Rightarrow \frac{\sqrt{n}}{n+4} > 0$ ✓
- Fit $f(x) = \frac{\sqrt{x}}{x+4}$ for $x \geq 0$ to a_n.

$$\Rightarrow f'(x) = \frac{(x+4)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1+0)}{(x+4)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{(x+4) - 2x}{2\sqrt{x}(x+4)^2} ✓$$

$$= \frac{4-x}{2\sqrt{x}(x+4)^2} < 0$$

for $x \geq 5$

So $f(x)$ is eventually dec $\Rightarrow \{a_n\}$ is as well

- $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+4/n} = \frac{0}{1+0} = 0$ ✓

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n}}{n+4}$ is convergent by the AST.

Investigate $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n+4} = \sum |a_n| \overset{\text{large}}{\sim} \sum \frac{\sqrt{n}}{n} \sim \sum \frac{1}{n^{1/2}}$ (gut says it diverges)

Set $b_n = \frac{1}{n^{1/2}}$ for $n \geq 1$. write $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n+4} = \frac{\sqrt{0}}{0+4} + \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$

Compare with $a_n = \frac{\sqrt{n}}{n+4}$ for $n \geq 1$. Note $a_n, b_n > 0$.

Construct $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} \div \frac{1}{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} \times \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{n+4}$$

$$= \lim_{n\to\infty} \frac{1}{1+4/n} = \frac{1}{1+0} = 1$$

Thus as $0 < L = 1 < \infty$ then $\sum \frac{\sqrt{n}}{n+4}$ and $\sum \frac{1}{\sqrt{n}}$ both diverge or converge by LCT. Since $\sum \frac{1}{\sqrt{n}}$ diverges as it is a p-series with $p = \frac{1}{2} \le 1$ then $\sum \frac{\sqrt{n}}{n+4}$ diverges.

$\therefore$ As $\sum |a_n|$ diverge but $\sum a_n$ converges then $\sum a_n$ is conditionally convergent ■

omit

2.6.2 Error Estimation of Series that Converge by the AST

The sum $\sum(-1)^{n+1}a_n = S$ always lies between two successive partial sums S_N and S_{N+1}. Furthermore the error (remainder) is bounded by

$$R_N = |S - S_N| \le a_{n+1}$$

Example: Find the error in approximating $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ using 15 terms.

2.7 (Section 10.7) Power Series

2.7.1 Power Series Functions and Their Domain

Sometimes we can define a function by a series. When the terms in the sum are simple polynomials we call it a "power series".

Definition

A **power series** about $x = a$ (called centered at a) is a function of the form

$$f(x) = \sum_{n=k}^{\infty} c_n(x-a)^n = c_k(x-a)^k + c_{k+1}(x-a)^{k+1} + \cdots$$

where all c_n are constant.

Example: Find a power series form of $f(x) = \dfrac{1}{1-x}$ and find the domain of the series form.

Reminder that for a geometric series $a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n$

$= \dfrac{a}{1-r}$ if $|r|<1$

So $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} (1)x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$

$r = x$ and $a = 1$

for $|x| < 1$.

Think of it like this, the partial sums $S_N(x) = \sum_{n=1}^{N} x^{n-1} = 1+x+x^2+\cdots+x^N$ are polynomials that approximate $f(x) = \dfrac{1}{1-x}$. The approximation gets better as $N \to \infty$.

This is an example I like as it illustrates that a region of convergence describes the domain of such a function.

Example: For what values of x does

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

converge? (i.e. what is the domain of $f(x)$?)

Always use Ratio or Root test in some way!

Set $a_n = (-1)^{n+1} \dfrac{x^n}{n}$ for $n \geq 1$ where x is fixed.

Construct

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{n+1}}{n+1} \div \frac{(-1)^n x^n}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{n+1}}{n+1} \times \frac{n}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| 1 \cdot x \cdot \frac{1}{1+1/n} \right| = \left| 1 \cdot x \cdot \frac{1}{1+0} \right|$$

$$= |x|$$

If $L = |x| < 1$ then it converges (absolutely)

$\Rightarrow -1 < x < 1.$

But the above is inconclusive for $L = 1 \Rightarrow |x| = 1 \Rightarrow x = \pm 1.$

A different test needs to be used!

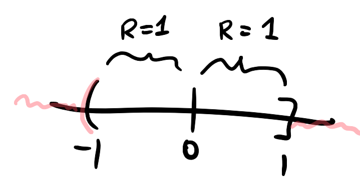
Investigate

$\checkmark$ $f(1) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ this is an alternating Harmonic and thus converges.

$\times$ $f(-1) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1) \cdot \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} -\frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ ← Harmonic series and thus diverges.

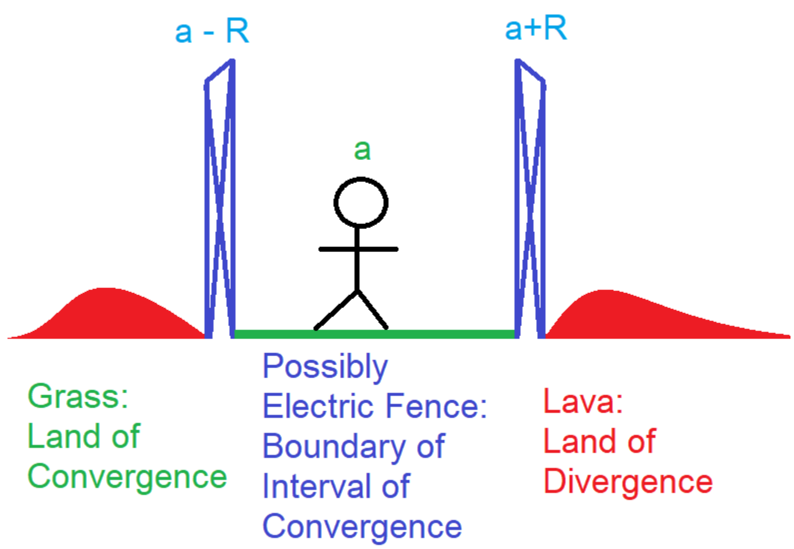
$\therefore$ $f(x)$ converges for $\underline{\underline{-1 < x \leq 1}}$.

$\text{Dom}(f) = (-1, 1]$



⇒ Page 132 ⇐

So the ratio test (or less commonly the root test) gives a fence between convergence and divergence, then you check to see if the fence itself is dangerous (divergent) or not (convergent).



Theorem

A series will either converge absolutely at a point $x = a$, in an interval $|x - a| < R$ or everywhere. The region of convergence is called the **interval of convergence** (check boundary). The number R is called the **radius of convergence**.

Example: Determine the interval and radius of convergence of $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Identify $a_n = \dfrac{x^n}{n!}$ for $n \geq 0$.

Form for fixed x ⇒ $L = \displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{\cancel{n}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}}{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{(2)}\cancel{(1)}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

Thus $L = 0 < 1$ (always) and thus $f(x)$ converges for all x.

I.o.C. ⇒ $\mathrm{Dom}(f) = (-\infty, \infty)$ and $R = \infty$

Example: Determine the interval and radius of convergence of $f(x) = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n (x-2)^n$ and be sure to check the endpoints.

$a = 2$

use the Root Test

Form $L = \lim_{n\to\infty} |a_n|^{1/n}$

$= \lim_{n\to\infty} \left| \underline{\left(1+\tfrac{1}{n}\right)^n} (x-2)^n \right|^{1/n}$

$= \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^{n/n} |x-2|^{n/n}$

$= \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)|x-2|$ $\quad \tfrac{1}{n} \to 0$

$= (1+0)|x-2|$

$= |x-2|$

Thus $f(x)$ converges absolutely for $L = |x-2| < 1$

$\Rightarrow -1 < x - 2 < 1$ $\quad (+2)$

$\Rightarrow 1 < x < 3$

Check the endpoints $\Rightarrow$ $a_n = \left(1+\tfrac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n = e^1 \neq 0$

$f(1) = \sum_{n=1}^{\infty} \left(1+\tfrac{1}{n}\right)^n (-1)^n \rightarrow$ Take $\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n (-1)^n$ $\to e$, $\to -e$ oscillates $\neq 0$

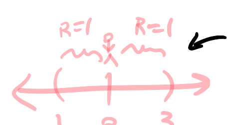
$\therefore$ Diverges by the divergence test

$f(3) = \sum_{n=1}^{\infty} \left(1+\tfrac{1}{n}\right)^n (1)^n = \sum_{n=1}^{\infty} \left(1+\tfrac{1}{n}\right)^n \rightarrow$ Take $\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n = e \neq 0$

$\therefore$ Diverges by the divergence test

Thus the I.O.C. $\Rightarrow$ Dom$(f) = (1, 3)$ //

and $R = \frac{b-a}{2} = \frac{3-1}{2} = 1$ //



$R = \frac{b-a}{2}$

$-1<x\leq 1$ $\quad$ all x

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}+\sum_{n=1}^{\infty}\frac{x^n}{n!}=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}+\frac{1}{n!}\right)x^n$$

for $-1<x<1$

2.7.2 Operations of Power Series

Addition and Scalar Multiplication

If $\sum a_nx^n$ and $\sum b_nx^n$ converge for $|x|<R$ then

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)+\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

and

$$\sum_{n=0}^{\infty}ka_nx^n=k\sum_{n=0}^{\infty}a_nx^n$$

converge for $|x|<R$ where k is a real number.

Multiplication

If $\sum a_nx^n$ and $\sum b_nx^n$ converge for $|x|<R$ then

Convolution

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n$$

converges for $|x|<R$.

Substitution

$f(x)=\sum a_nx^n \Rightarrow f(g(x))=\sum a_n(g(x))^n$

If $\sum a_nx^n$ converges for $|x|<R$ and $g(x)$ is continuous then $\sum a_n(g(x))^n$ converges for all values x such that $|g(x)|<R$.

Example: Consider $f(x)=\dfrac{1}{4+x^2}$. Find an expression for this function in power series form and determine the interval of convergence.

Turn $\frac{1}{4+x^2}$ into $\frac{a}{1-r}$

$$=\frac{1}{4}\cdot\frac{1}{1+\frac{x^2}{4}}=\frac{1}{4}\cdot\frac{1}{1-\left(-\frac{x^2}{4}\right)}=\frac{1/4}{1-\left(-\frac{x^2}{4}\right)}\sim\frac{a}{1-r}\quad \text{w}\quad a=\frac{1}{4},\ r=-\frac{x^2}{4}$$

$$=\sum_{n=0}^{\infty}\frac{1}{4}\left(-\frac{x^2}{4}\right)^n=\sum_{n=0}^{\infty}\frac{1}{4}\frac{(-1)^n}{4^n}\cdot x^{2n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n}$$

regularly this converges for $|r|<1$

$\Rightarrow \left|-\frac{x^2}{4}\right|<1$

$\Rightarrow \frac{x^2}{4}<1 \Rightarrow x^2<4$

$\Rightarrow -2<x<2$ //

Differentiation

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ then

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-(k+1))c_n(x-a)^{n-k}$$

all converge over $|x-a| < R$.

Example: Consider the power series representation of the function $f(x) = \dfrac{1}{1-x}$ given by $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ which converges over $|x| < 1$. Use this to construct power series representations of its derivatives.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{on } -1 < x < 1$$

$$\overset{\frac{d}{dx}}{\Rightarrow} -\frac{1}{(1-x)^2}(0-1) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{on } -1 < x < 1$$

$$-\frac{1}{(1-x)^2}(0-1) = \frac{1}{(1-x)^2}$$

$$\overset{\frac{d^2}{dx^2}}{\Rightarrow} -\frac{2}{(1-x)^3}(0-1) = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2 + 6x + 12x^2 + \cdots \quad \text{on } -1 < x < 1$$

$$-\frac{2}{(1-x)^3}(0-1) = \frac{2}{(1-x)^3}$$

$$\vdots$$

Integration

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges over $|x-a| < R$ then

$$\int f(x)dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right) + C$$

converges over $|x-a| < R$.

Example: Consider $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ which converges over $|x| < 1$. Use this to construct a power series representation of $f(x) = \ln(1+x)$ and determine the interval of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{over} \quad -1 < x < 1$$

↳ sub in $-x$ for x

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{over } -1 < -x < 1 \Leftrightarrow -1 < x < 1$$

$\int (\,)dx$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$

$$\int \frac{1}{1+x}dx = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \ldots)dx$$

$$\Rightarrow \ln|1+x| = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots + C$$

also this holds on $-1 < x < 1$.

Determine C by plugging in $x = 0$

$$\Rightarrow \underbrace{\ln(1)}_{=0} = \underbrace{0 - \frac{1}{2}0^2 + \frac{1}{3}0^3}_{=0} - \ldots + \cancel{C} \quad C = 0$$

$$\text{so } \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad // \quad \text{on } -1 < x < 1.$$

$x+1=e^{ax}$

2.8 (Section 10.8) Taylor and MacLaurin Series

A Taylor series (and Maclaurin) is a polynomial series that (potentially) approximates a known function $f(x)$. Let's assume we are GIVEN that a known function $f(x)$ has a power series representation (e.g. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$) and consider it's representation

$f(a) \quad f'(a) \quad \frac{f''(a)}{2} \quad \ldots$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

> Note
>
> We are ASSUMING a KNOWN function $f(x)$ has a power series form. This is a really massive assumption!!

Then we shall attempt to solve for all terms in the sequence $\{a_n\}$. We see that

$$f(a) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots \Rightarrow f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots = a_1$$

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + \cdots \rightarrow f''(a) = 2a_2 + 3 \cdot 2a_3(0) + \cdots = 2a_2$$

$$f'''(x) = 3 \cdot 2a_3 + \cdots \Rightarrow f'''(a) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2(0) + \cdots = 3 \cdot 2a_3$$

$$\vdots$$

$$f^{(k)}(x) = k \cdot (k-1) \cdots (2)(1)a_k + (k+1)(k)(k-1) \cdots (2)a_{k+1}(x-a) + \cdots \Rightarrow f^{(k)}(a) = k \cdot (k-1) \cdots (2)(1)a_k = k!a_k$$

and thus $a_k = \frac{f^{(k)}(a)}{k!}$. Thus if we are TOLD a specific function has a power series representation and we want to compute it, we can use this to determine the representation of it in power series form without clever algebraic tricks or relating it to a geometric series.

> Definition
>
> Let f be a function with derivatives of all orders about an interval containing $x = a$. Then the **Taylor series** about $x = a$ of $f(x)$ is the series
>
> $$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
>
> If $a = 0$ it is called a **Maclaurin series** (just to give him some credit)

Nah : (Read for interest).

2.8.1 The Unreasonable Assumption of Being a Taylor Series

There is still one question, for each value of x within an appropriate interval is it true that $P(x) = f(x)$? It required a massive assumption, that the function $f(x)$ COULD BE expressed as a power series. Knowing whether or not you can do this requires tools we don't have. You can always compute the Taylor series of any function but you might not get equality of $P(x)$ and $f(x)$. Let's have a brief chat about how functions are classified in calculus.

Definition

Let I be an open interval and denote $C^0(I)$ as the collection of functions that are continuous on I (called class zero). We also further denote $C^k(I)$ as the collection of functions whose derivatives all the way up to order k are continuous on I (called class k) and denote $C^\infty(I)$ as the collection of function who have derivatives of ALL orders and are continuous on I (called **smooth functions**). Lastly, we also represent functions who have a power series representation on I as $\mathfrak{A}(I)$ (called **analytic functions**).

Note

To illustrate this, $f(x) = |x|$ is a class zero function on $\mathbb{R}$ (the collection of all real numbers) as it is continuous but it's derivative is not continuous. The function $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ is an analytic function on $(-1, 1)$ as it has a power series representation on this interval.

Here's how the structuring in calculus works...

$$C^0(I) \supset C^1(I) \supset C^2(I) \supset \cdots C^\infty(I) \supset \mathfrak{A}(I)$$

where $A \supset B$ means the collection of all objects in B can be found in A. By this chain, analytic is the ULTIMATE form of differentiability. You can have functions with derivatives of ALL orders and are continuous but still don't have a power series representation! Making your way down the chain is harder at each step. Functions further down the chain are much smoother and nicer than functions near the start of the chain.

Example Show the function

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

is strictly a class one function on $\mathbb{R}$ (i.e. lies in the collection $C^1(\mathbb{R})$ but not $C^2(\mathbb{R})$).

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2.8.2 Examples of Constructing a Taylor Series Directly

Example: Compute the Taylor series of $f(x) = \sin(x)$ about $x = 0$. $\rightarrow a = 0$

so $P(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ where $a_n = \boxed{\frac{f^{(n)}(a)}{n!}}$

$a=0$ in this case

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\sin(x)$	$\sin(0) = 0$
1	$\cos(x)$	$\cos(0) = 1$
2	$-\sin(x)$	$-\sin(0) = 0$
3	$-\cos(x)$	$-\cos(0) = -1$
4	$\sin(x)$	$\sin(0) = 0$ repeats
5	.	1
6	.	0
7	|	-1
8	|	0
9	|	1
10		0
11		-1
12	.	⋮

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= \cancel{\frac{0}{0!}} + \frac{1}{1!}x + \cancel{\frac{0}{2!}}x^2 + \frac{-1}{3!}x^3 + \cancel{\frac{0}{4!}}x^4$$

$$+ \frac{1}{5!}x^5 + \cancel{\frac{0}{6!}}x^6 + \frac{-1}{7!}x^7 + \cdots$$

Reminder
$0! = 1$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9$$

$$- \frac{1}{11!}x^{11} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$n=0 \Rightarrow x$
$n=1 \Rightarrow x^3$
$n=2 \Rightarrow x^5$
$n=3 \Rightarrow x^7$
⋮

Example: Find the Taylor series of $f(x) = e^x$ about $x = 0$.

$a=0$

$a=0$

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	e^x	$e^0 = 1$
1	e^x	$e^0 = 1$
2	e^x	$e^0 = 1$
$\vdots$	$\vdots$	$\vdots$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

so $$P(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

2.8.3 Taylor Polynomials of Order N

Definition

The **Taylor polynomial of order** N to $f(x)$ at $x = a$ is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Definition

If $P(x)$ is the Taylor polynomial of $f(x)$ we call $f(x)$ the **generating function** of $P(x)$.

Two questions remain in this theory of wanting to replace $f(x)$ with $P(x)$ (because polynomials are nicer to work with):

- When can we expect $f(x) = P(x)$ for each x? That is, when can we expect that $\lim_{N\to\infty} P_N(x) = f(x)$?
- If the previous condition is satisfied (yay!) how accurate are the finite order approximations $P_N(x)$? (Because we might want to understand the function using a computer and computers don't do infinity so using some smaller terms to approximate values might be good enough).

Example: We can't always expect that $P(x) = f(x)$. With great difficulty one can show that for

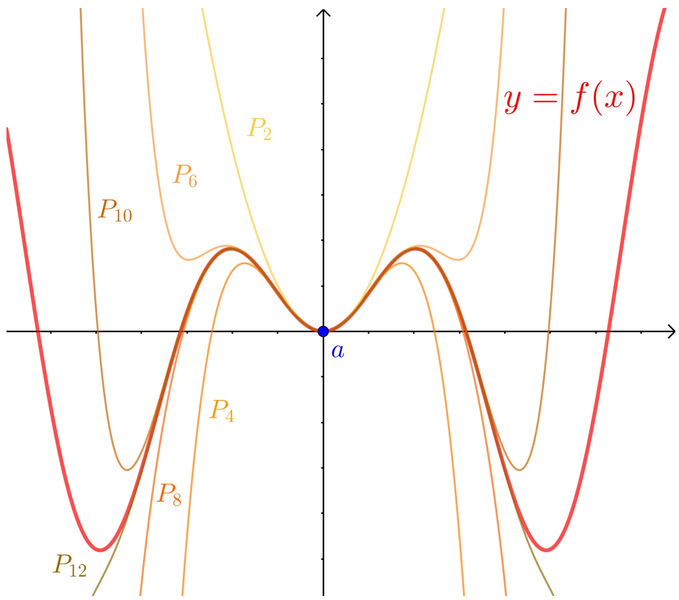
$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$$

That derivatives of all orders exist and are continuous with the further fact that $f^{(n)}(0) = 0$. Thus for a Taylor series centered at $x = 0$ we have $P(x) = 0$. However, clearly $f(x)$ is not identically zero! Thus they are not equal. This function is an example of a function that is $C^\infty(\mathbb{R})$ but not $\mathfrak{A}(\mathbb{R})$!

2.9 (Section 10.9) Convergence of Taylor Series

2.9.1 Taylor's Theorem and Approximating Functions

Alright, it's time to answer the big question. When does the Taylor series $P(x)$ equal the generating function $f(x)$? When does $P_N(x) \to f(x)$ as $N \to \infty$?



Do the polynomials approach $f(x)$ as $N \to \infty$? By the mean value theorem we have the following theorem to help us out.

Taylor's Theorem

If f is of class C^N on an interval containing a and b then there exists a number c between a and b such that

$$f(b) = P_N(b) + \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1} = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(b-a)^n\right) + \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1}$$

This theorem extends (in a complicated way) so that in an interval I where f is of class C^∞ then for all x in I containing a,

$$f(x) = P_N(x) + R_N(x)$$

for all N where $R_N(x) = \dfrac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$ for some c between x and a.

Alright, this might seem complicated. We're working VERY HARD to approximate the function $f(x)$ with the Taylor polynomials. Let's break down what's happening, we have said that if the function is smooth (nice enough) then we can represent it as the sum of two functions

$$f(x) = \overset{\to P(x)}{P_N(x)} + \overset{\to 0}{R_N(x)}$$

Now here's the part we have to work on. Since $P_N(x) \to P(x)$ as $N \to \infty$ then we need to only show that this other weird function $R_N(x)$ satisfies $R_N(x) \to 0$ as $N \to \infty$. This will allow us to conclude that

$$f(x) = P_N(x) + R_N(x) \Rightarrow \lim_{N\to\infty} f(x) = \lim_{N\to\infty} (P_N(x) + R_N(x)) \Rightarrow f(x) = P(x) + 0 = P(x)$$

Often one can estimate $R_N(x)$ without ever knowing c.

Example: Show the function $f(x) = e^x$ is equal to it's Taylor series.

$$e^x = P_N(x) + R_N(x) \quad \text{where} \quad P_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!} \quad \text{(earlier)}$$

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

Some unknown 'c'

Since all derivatives of e^x are e^x then

$$R_N(x) = \frac{e^c}{(N+1)!} x^{N+1} \quad \text{for some } c.$$

For any fixed $x \Rightarrow \lim_{N\to\infty} R_N(x) = \lim_{N\to\infty} \frac{e^c x^{N+1}}{(N+1)!} = 0$

exponential is so much smaller

so $e^x = \lim_{N\to\infty} (P_N(x) + R_N(x)) = P(x) + 0$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{over } I = (-\infty, \infty)$$

2.9.2 Taylor Series with Bounded Derivatives

Theorem

If there is a constant M such that $|f^{(N+1)}(t)| \leq M$ for all t between x and a then $R_N(x)$ satisfies

$$|R_N(x)| \leq M\frac{|x-a|^{N+1}}{(N+1)!}$$

If this holds for every N and the Taylor conditions are satisfied then $P(x) = f(x)$.

Example: Show that $f(x) = \sin(x)$ is equal to it's Taylor series.

By Taylor's Theorem $\quad \sin(x) = P_N(x) + R_N(x)$

$$= \sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_N(x)$$

where $\quad R_N(x) = \dfrac{f^{(2N+2)}(c)}{(2N+2)!} x^{2N+2}$

$f(c) = \sin(c)$

$f'(c) = \cos(c)$

$f''(c) = -\sin(c)$

$f'''(c) = -\cos(c)$

$f^{(4)}(c) = \sin(c)$

$\vdots$

$$f^{(2N+2)}(c) = \begin{cases} \sin(c) \\ \cos(c) \\ -\sin(c) \\ \text{OR} \\ -\cos(c) \end{cases} \quad \text{depending}$$

$\because |\cos(c)|, |\sin(c)| \leq 1$

Note then $\quad |R_N(x)| = \dfrac{|f^{(2N+2)}(c)|}{(2N+2)!}|x|^{2N+2} \leq \dfrac{1}{(2N+2)!}|x|^{2N+2}$

$$\text{so } \lim_{N\to\infty} |R_N(x)| = \lim_{N\to\infty} \frac{|x|^{2N+2}}{(2N+2)!} = 0$$

$$\Rightarrow \lim_{N\to\infty} R_N(x) = 0$$

$$\therefore \sin(x) = \lim_{N\to\infty}\left(P_N(x) + \overset{0}{\cancel{R_N(x)}}\right)$$

$$= P(x) + 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Can show I.o.c. $\Rightarrow (-\infty, \infty)$.

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2.9.3 Important Analytic Functions

The following is a short list of some analytic functions and their Taylor series form.

Important Taylor Series Expression

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in $I = \mathbb{R}$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ in $I = \mathbb{R}$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ in $I = \mathbb{R}$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ in $I = (-1, 1)$
- $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ in $I = (-1, 1)$

Note

Ahem, for future calculus courses (and this one) you should have these MEMORIZED. No if's and's or but's. Many expressions for other functions are obtained using these.

Example: Use the above to indirectly construct a Taylor Series for the integral function

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{on } (-\infty, \infty)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \, dt$$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \, dt$$

$$\frac{2}{\sqrt{\pi}} \left\{ x - \frac{1}{3}x^3 + \frac{x^5}{2! \cdot 5} - \frac{x^7}{3! \cdot 7} + \cdots \right\}$$

$$= \frac{2}{\sqrt{\pi}} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n!(2n+1)} \right\} \Bigg|_{t=0}^{t=x}$$

bunch of zeroes

$$= \frac{2}{\sqrt{\pi}} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} - \sum_{n=0}^{\infty} \cancel{\frac{(-1)^n \cdot 0^{2n+1}}{n!(2n+1)}} \right\} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

on $(-\infty, \infty)$

Note

The above integral function is the famous "Error Function".

2.9.4 Approximations Using Taylor Series

Example: Find $\boxed{P_4(x)}$ of $f(x) = e^x \cos(x)$.

"oh five" means terms deg 5 or bigger

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(5)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(5)$$

on $(-\infty, \infty)$

Form

$$e^x \cos(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(5)\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(5)\right)$$

$$= 1 - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} + O(5) + x - \frac{x^3}{2!} + O(5) + O(5) + \cancel{\frac{x^2}{2!}} - \frac{x^4}{2!2!} + O(5)$$
$$+ O(5) + \frac{x^3}{3!} + O(5) + O(5) + O(5) + \frac{x^4}{4!} + O(5) + O(5) + O(5) + O(5)$$

$$= 1 + x + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{2}{4!} - \frac{1}{2!2!}\right)x^4 + O(5)$$

$$P_4(x) = 1 + x + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{2}{4!} - \frac{1}{2!2!}\right)x^4$$

In an alternating series you can see how well $P_N(x)$ approximates $f(x)$.

omit

Example: For what values of x will $P_3(x) = x - \dfrac{x^3}{3!}$ approximate $\sin(x)$ with an error no bigger than 3×10^{-4}?

2.10 (Section 10.10) Binomial Series and Applications of Taylor Series

In this section we discuss a bit of practicality of using and developing Taylor series. Before we do so, we discuss Binomial series.

2.10.1 Binomial Series

These are the series representation of $(1+x)^m$. Before you get all "just expand it and it becomes a polynomial. Polynomials are their own Taylor series" we consider the case where m is not a positive integer as well.

We defin the binomial coefficient

"m choose k"

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

$$\binom{m}{4} = \frac{m(m-1)(m-2)(m-3)}{4!}$$

for $k \geq 1$ where k is an integer and we set

$$\binom{m}{1} = m \qquad\qquad \binom{m}{2} = \frac{m(m-1)}{2!}$$

We compute for $f(x) = (1+x)^m$ at $x = 0$,

$(1+x)^{1/2} =$

$P(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$

$$f(x) = (1+x)^m \Rightarrow f(0) = 1$$

$$f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^m$	$(1+0)^m = 1$
1	$m(1+x)^{m-1}$	$m(1+0)^{m-1} = m$
2	$m(m-1)(1+x)^{m-2}$	$m(m-1)(1+0)^{m-2} = m(m-1)$
$\vdots$	$\vdots$	$\vdots$

and so forth. Hence

$$f^{(k)}(0) = m(m-1)(m-2)\cdots(m-k+1)$$

and thus the series is

$$P(x) = 1 + \sum_{k=1}^{\infty} \underbrace{\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}}_{f^{(k)}(0)} x^k = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

One can show that by the ratio test and inspecting the endpoints it converges (absolutely) for $|x| < 1$. Showing $P(x) = (1+x)^m$ is a little tricky. The book shows a trick and guides you in one of the exercises of this section.

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k \quad \text{for } -1 < x < 1.$$

Example: Form a series representation of $\sqrt{5}$.

$m = 1/2$

Consider $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

use Binomial series

$$(1+x)^{1/2} = 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} x^k$$

Say we expand to degree 3.

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

$$= 1 + \binom{1/2}{1}x^1 + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \cdots$$

$$= 1 + \frac{1/2}{1!}x + \frac{1/2(1/2-1)}{2!}x^2 + \frac{1/2(1/2-1)(1/2-2)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{-1/4}{2\cdot 1}x^2 + \frac{\cancel{3}/8}{\cancel{3}\cdot 2\cdot 1}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \quad \text{on } (-1,1)$$

pick $x = 1/4$ in $(-1,1)$.

$$\sqrt{1+1/4} = \sqrt{\frac{4+1}{4}} = \frac{\sqrt{5}}{2} = 1 + \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{8}\left(\frac{1}{16}\right) + \frac{1}{16}\left(\frac{1}{64}\right) + \cdots$$

$$\Rightarrow \sqrt{5} = 2 + \left(\frac{1}{4}\right) - \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{8}\left(\frac{1}{64}\right) + \cdots$$

2.10.2 Applications of Taylor Series

One should not underestimate the "power" of power series. Their complexity is worth the effort. They are the foundation of much theory.

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = i^2 \cdot i = -i$
$i^4 = -i^2 = 1$
$\vdots$

Complex Variables: We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Let $x = i\theta$ then

$i = \sqrt{-1}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

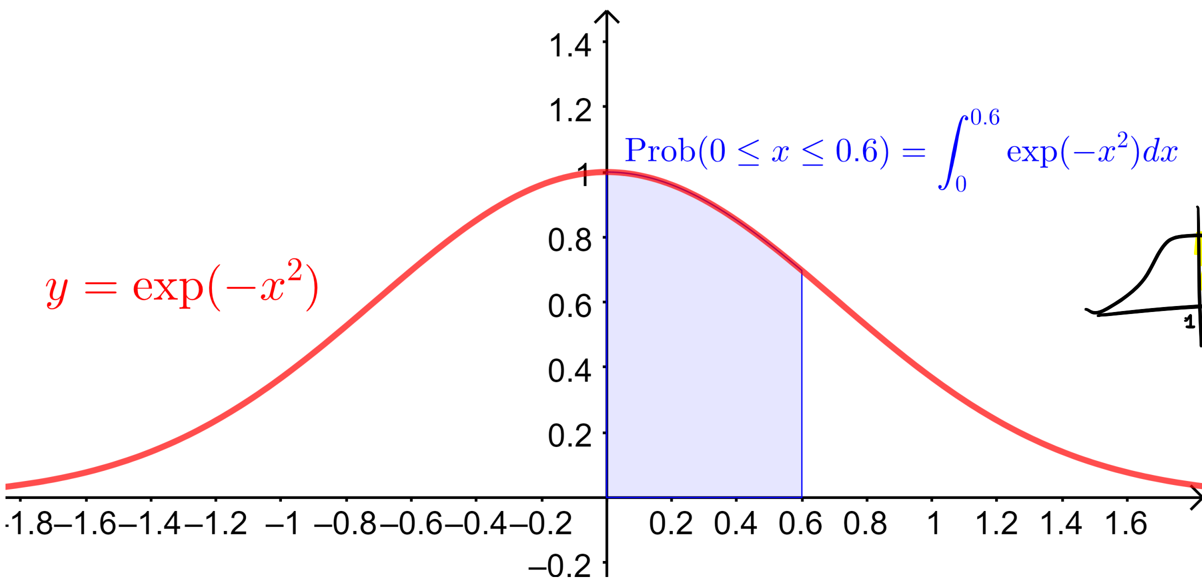
since $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ...and the pattern repeats. Thus

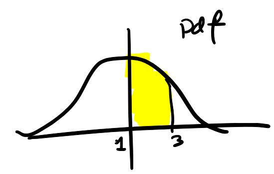
$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) = \cos(\theta) + i\sin(\theta)$$

Ta-da! $\square$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Probability Theory: One of (if not the most) important distribution is the bell curve!





$\int e^{-x^2} dx$

Many natural occurrences follow a bell curve distribution. Ideally, in a class grades you want them to follow a bell (normal) distribution. The probability of something occuring is given by the area between two points under the curve. In quantum physics, for example, the probability of a particle existing in a location can follow a bell curve.

Example: In the ground state of the harmonic oscillator, there is a non-zero probability of finding a particle outside the classically allowed region. The probability of finding the particle outside of the region $[-a, a]$ is given by

$$\frac{1}{\sqrt{\pi}}\sqrt{A} \qquad\qquad A$$

$$F(a) = 1 - \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-a}^{a} \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx \qquad F(a) = 1 - B\int_{-a}^{a} e^{(-Ax^2)} dx$$

Determine the probability of finding the particle outside the region where $a = \sqrt{\dfrac{\hbar}{m\omega}}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$A = \frac{m\omega}{\hbar}$$

$$e^{-Ax^2} = \sum_{n=0}^{\infty} \frac{(-Ax^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n A^n x^{2n}}{n!}$$

$$= 1 - Ax^2 + \frac{A^2x^4}{2!} - \frac{A^3x^6}{3!} + \cdots$$

Sub into $F(a)$

$$\int e^{\cos(x)} dx$$

$$F(a) = 1 - \frac{1}{\sqrt{\pi}}\sqrt{A}\int_{-a}^{a} e^{-Ax^2} dx$$

$$= 1 - \frac{1}{\sqrt{\pi}}\sqrt{A}\int_{-a}^{a} \left(\sum_{n=0}^{\infty} \frac{(-1)^n A^n x^{2n}}{n!}\right) dx$$

$$= 1 - \frac{1}{\sqrt{\pi}}\sqrt{A}\left(\sum_{n=0}^{\infty} \frac{(-1)^n A^n x^{2n+1}}{n!\,(2n+1)}\right)\Bigg|_{x=-a}^{x=a}$$

$$(-1)^n \qquad (-1)^{2n+1}$$

$$= 1 - \frac{1}{\sqrt{\pi}}\sqrt{A}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n A^n a^{2n+1}}{n!\,(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n A^n (-a)^{2n+1}}{n!\,(2n+1)}\right\}$$

$$= 1 - \frac{1}{\sqrt{\pi}}\sqrt{A}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n A^n a^{2n+1}}{n!\,(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^{3n+1} A^n a^{2n+1}}{n!\,(2n+1)}\right\}$$

$$= 1 - \frac{2}{\sqrt{\pi}}\sqrt{A} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n A^n a^{2n+1}}{n!\,(2n+1)}$$

Approximating Values: One can show using

$\frac{a}{1-r}$, $a=1$, $r=-x^2$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

is convergent for $|x| < 1$. Upon integrating and solving to obtain $C = 0$,

$$\arctan(x) = \left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\right) + C$$

$$\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

holds for $-1 < x \leq 1$ (after checking the endpoints). Thus we may compute

$$\frac{\pi}{4} \underset{\arctan}{\overset{\tan}{\rightleftarrows}} 1$$

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\Rightarrow \pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

however it does converge very slowly. You'll need several terms to for a good approximation (the reason it's slow is due to conditional convergence).

Evaluating Limits: One can use Taylor series to compute various limits.

Example: Compute $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2} = \lim_{x\to 0}\frac{\left(\cancel{1}+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-\left(\cancel{1}+\cancel{x}\right)}{x^2}$

$$= \lim_{x\to 0}\frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{1}{2!}+\cancelto{0}{\frac{x}{3!}}+\cancelto{0}{\frac{x^2}{4!}}+\cancelto{0}{\cdots}\right)$$

$$= \frac{1}{2!} = \frac{1}{2}$$