UNIVERSITY OF VICTORIA

PHYS110 Lab Manual

Authors:
Travis MARTIN
Mark LAIDLAW

Prepared for the

Department of Physics and Astronomy

Contents

Introduction to PHYS110 Labs

In the classroom, you learn about the laws of physics and the behaviour of matter and energy. The equations that you use give you precise results, and you truncate that precision based on seemingly arbitrary "significant digits". In practice, this is a gross oversimplification of the practice of physics. These laboratory activities are meant to give you an overview of some of the challenging aspects of performing measurements and calculations with physics.

We are not expecting students to get values that perfectly match known quantities/constants. In fact, we would be highly surprised if you did! We realize that there are many inaccuracies, confounding influences, and unmodelled environmental effects that will make the experiments very imprecise. Not only is that okay, it is exactly what we want. You are expected to put your best effort into overcoming these issues by using techniques to introduce consistency and reliability into your labs, and you should always include details of such efforts in your lab reports. But you won't lose marks simply for getting an unexpected result. How you deal with, analyze and respond to the unexpected result will be what you are graded on, not the value that you get.

In this lab manual, you will find descriptions of the lab activities and a series of appendices. Overall, there are five main lab Projects, and one introductory lab Project. Each project includes a data taking activity that will require that you take and record measurements, often multiple times for the same activity. You will then have some analysis to transform your measurements into the quantities of interest for the Project, and answer some conceptual questions that are designed to encourage you to think about the greater picture and/or application of the material.

One thing you will not find in this lab manual is a Problem Statement, or Objective, or other such description of your activity. Instead, you will find descriptions of the learning objectives – what we expect you to get out of the experience – and background information motivating the experiment you will be performing.

Remember, the learning goals of the labs are very different from the scientific purpose of the experiment or activity.

Lab 0: Measurements and Statistics

1.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix A: Statistics
- Appendix C: Hypothesis Testing

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

1.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to the statistical nature of measurements.
- Introduce students to the mathematical techniques that are used to analyze the statistics arising from measurements.
- Introduce students to methods of using spreadsheets to analyze data.
- Introduce students to how to communicate measured and calculated quantities.
- Introduce students to how to assess and communicate the precision and accuracy of determined quantities.

1.3 Background

Imagine flipping a coin, or better yet find a coin and flip it. Consider the physics involved – you apply a force to the coin for a short period of time, the coin flips in the air and lands on one of its faces. Some of the applied force results in the translational motion of the coin, and some of the force results in the rotational motion of the coin.

Now imagine flipping the coin again, or repeat the coin flip. Do you expect the coin to land in the same location? Do you expect the coin to land on the same face? Even subtle variations in the applied forces can result in potentially significant changes to the outcome.

With regards to the side of the coin that is upwards when the coin lands, most people would judge that there is a 50% chance of it landing on Heads versus Tails. Most people have also never checked. By examining Fig. 1.1, it is obvious that the two sides are not the same. Thus, the coin lacks the symmetry by which it could be argued that the coin is fair – meaning an equal chance to land on Heads versus Tails.

FIGURE 1.1: Image of a Canadian quarter, showing both sides. The sides do not demonstrate any significant amount of symmetry.

Naturally, a scientific approach to examining the coin is to perform an experiment. Flip the coin, say, 100 times and record the results. If it comes up with 56 times Heads and 44 times Tails, you might be inclined to say that the coin is biased towards Heads. What if you repeated the experiment and found 48 times Heads and 52 times Tails? Is it reasonable to assume that the bias changed between experiments?

Perhaps you flip the coin one million times, and the results are 51.3% heads and 48.7% tails. You are lead to believe that the coin is not perfectly unbiased. How do you properly communicate your results to others? How do you communicate how confident you are that your results are accurate and reproducible? These are all important questions to a scientist.

This is the nature of statistical data and one of the most challenging things to account for in the sciences and engineering fields. As it turns out, there are statistical variations in absolutely every measurement that can be made, and thus in every physical constant (e.g. gravitational constant) and every phenomenological constant (e.g. coefficient of friction, heat capacity of a material, conductivity of a wire, etc). For example, if you purchase 100 metre-sticks, there will likely be a 1mm (or more) variation among the actual physical lengths of the sticks – materials change due to temperature, pressure, humidity and other factors. If you have 100 metal rulers, there may be more consistency in the lengths, but the distance indicators/ticks have some physical dimension to them and different people looking at the same measurement might differ in their judgment on the length by 0.2mm or more.

When measurements are performed scientifically, they are always repeated as many times as is reasonable or feasible, and the results are combined together. The mean value of all of the measurements represents the scientific estimate of the correct value. The standard deviation of all of the measurements is a measure of the most likely range of values in which any future single measurement will lie (i.e. approximately 67% of all future measurements should lie within $\pm 1\sigma$ of the mean). The standard deviation *of the mean* is a measure representing the range about the mean at which a new mean value determined from a repeat measurement should lie (i.e.

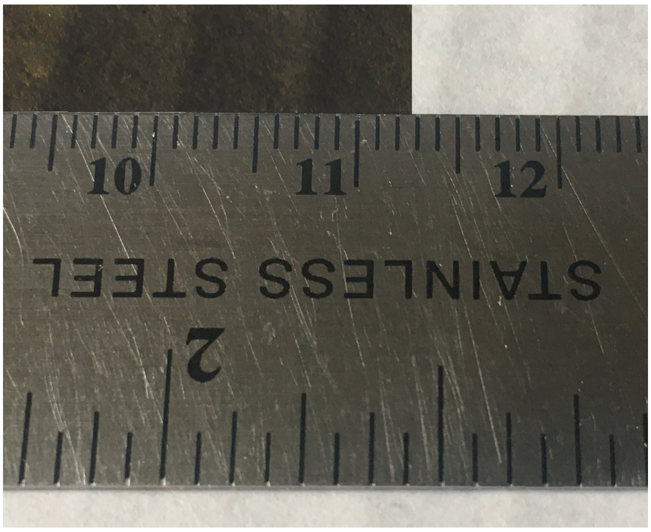


FIGURE 1.2: Close up of a ruler, showing the finite thickness of the distance marks. The thickness of the marks and the spaces between marks require personal judgment when using the device, resulting in variations between people performing the measurements. In this image, any value between 11.25cm and 11.30cm would be considered a reasonable measurement. Any attempt to claim a number more precise than 0.01cm with this ruler should be immediately recognized as unscientific.

repeating the experiment again should produce a mean value that has a 67% chance of being within $\pm 1\delta$ of the mean).

The precision of a scientific value is represented by the **uncertainty** value. For a value that is measured once, the measurement uncertainty represents the standard deviation *of the mean* of a series of measurements made by the manufacturer using the measuring device. It is typically 1/2 or 1 of the smallest division of the scale on the device. For example, a ruler may have the smallest measurement being mm - the measurement uncertainty of the device is either 0.5mm or 1mm. The choice of which to use would be up to the experimenter, determined by how accurately they honestly believe they can perform a measurement using the ruler. The measurement is then reported as the value plus or minus the uncertainty: e.g. 114.5 ± 0.5mm or 11.45 ± 0.05cm or 0.1145 ± 0.0005m.

For a value that is measured multiple times, the uncertainty is going to be a combination of the measurement uncertainty ($\delta_{measurement}$) and the statistical uncertainty (the standard deviation of the mean, $\delta_{statistical}$). The measurement uncertainty has to do with the reliability of the measuring device, and the statistical uncertainty has to do with the reliability of the experimental method.

Suffice to say that the statistical nature of measurements cannot be completely eliminated, and thus must be accounted for when taking measurements, performing experiments and designing technology. This first laboratory project will introduce you to some of the statistical calculations that are involved when calculating quantities. You are introduced to the idea by assuming that your aim towards a target is a biased quantity, and you will want to numerically characterize and communicate your aim.

1.4 Activity and Analysis

1.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Graph paper (2 pieces)
- Pencil
- Small metal washer

1.4.2 Activity Instructions

Take a piece of grid paper and lay it on top of a book or a stack of papers (this will help reduce bounce in the metal washer). Mark one of the intersections near the centre of the paper with your pencil – this is the spot you are aiming for. Lay the washer on your finger tip and align your eye and finger so that the washer is approximately above the marked point. Quickly pull your finger down and away so that the washer drops on to the paper. Ideally, the washer shouldn't bounce or move around once it lands if the technique is done correctly.

With the washer laying on the piece of paper, carefully hold it with your fingers and use your pencil to trace the inner edge of the washer. Do not trace the outer edge of the washer. Repeat this process 20 times. This will look like Fig. 1.3.

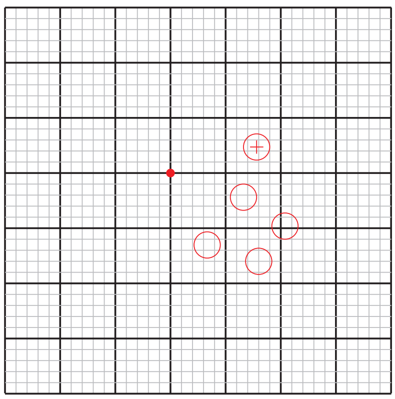


FIGURE 1.3: An example of what your graph paper should look like.

Get a second piece of paper and repeat the process but only with 10 marks.

Take each piece of graph paper and add an approximate centre for each circle. This is shown for one of the circles in Fig. 1.3. On each graph paper, use some of the open space to make a table listing the x and y locations of the centre of each circle relative to the dot that you drew. You do not need to use a ruler, just count the number of divisions of the graph paper (assume they are in mm).

Get your TA to sign your work. You may stay and continue to work until the end of the period, or you may leave at this point. Keep in mind that any work you do not finish during the lab period will need to be completed at home.

1.4.3 Analysis & Submission

Calculate the mean, standard deviation and standard deviation of the mean for the x and y data for your first dataset (with 20 points) and the mean value for the second data set. The formulas for this are included in Appendix A.

Calculate the number of standard deviations of the mean are between the mean values for the two data sets.

Note: The mean of your x and y values, along with the standard deviation of the mean, represents a measurement of "Your aim".

Answer any remaining questions in the worksheet.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo for each of your pieces of graph paper that clearly shows your data points and your TAs signature.

2. Calculate the mean value for the x and y data for the set of 20 trials. Show your work.

3. Calculate the standard deviation for x and y data for the set of 20 trials. Show your work.

4. Estimate the standard deviation of the mean for the x and y data for the set of 20 trials. Show your work.

5. Calculate the mean value for the x and y data for the set of 10 trials. Show your work.

6. Calculate the number of standard deviations of the mean between the mean values of the two data sets. This is known as a t value/score. (e.g. $t_x = \left|\frac{(\bar{x}_{20}-\bar{x}_{10})}{\delta_x}\right|$)

7. Respond to the following questions/instructions using complete sentences:
   (a) What is the precision of each individual measurement?
   (b) What is the precision of your aim?
   (c) Assuming the method to drop the washer does not change, how could you reduce the uncertainty in the location of your aim (that is, the mean location of your drops)?
   (d) What is the location of your aim, including the uncertainty? Use an appropriate number of significant figures following the scientific method for stating numbers.
   (e) Were the results of the second drop activity (10 drops) consistent with the results from the first (20 drops)? Explain your answer.
   (f) There are two ways to describe the uncertainty: as a value with a unit (e.g. (4.5 ± 0.2)cm), or as a percentage (e.g. 4.5cm $\pm$ 4%). Why might the percentage method not be an effective description with this type of measurement?

Lab 1: Measurements and Comparisons

2.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix B: Spreadsheets
- Appendix C: Hypothesis Testing
- Appendix E: Uncertainty Analysis

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

2.2 Learning Objectives

The goals of this laboratory project are as follows:

- Give students practice with measurements and uncertainties.
- Reinforce importance of repeating experiments under different conditions.
- Introduce students to uncertainty propagation.
- Give students practice with communicating measured and calculated quantities.
- Introduce students to more spreadsheet techniques.
- Introduce the concept of testing expectation versus experiment.
- Give students practice with statistical comparisons.

2.3 Background

When studying pulley systems in a classroom setting, there are two primary assumptions that are commonly made: 1. The tension is the same everywhere along the cable. 2. There is no friction in the pulley system. While introductory material also often assumes the pulleys are massless, this is only important for dynamic systems (systems in motion).

This makes pulley systems in a static situation excellent for exploring the accuracy and reliability of experiments. Accuracy and reliability are a central feature in all science, and one of the biggest challenges of performing scientific experiments. In this experiment, you will be arranging three masses on a two pulley system similar to Fig. 2.1, resolving the forces into components, and checking to see whether your measurements add up correctly.

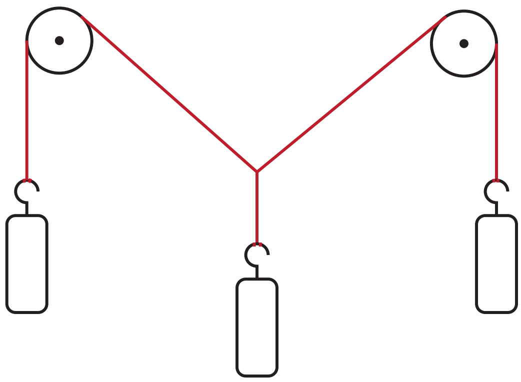


FIGURE 2.1: Arrangement of three masses on two pulleys, similar to the activity you will be performing.

2.3.1 Vectors

A basis is a description of fundamentally distinct but related quantities. We are taught the Euclidean basis at a young age without fully realizing what we are being taught – the Euclidean basis is just the three linear spatial dimensions, x, y, and z. The symbols that represent a single unit in one of the basis directions are a letter with a "hat" on it, such as $\hat{e}$ for arbitrary direction "e". In spatial directions, the two common symbol systems are $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. The "ijk" system is often used so that people learning these mathematical techniques do not get too caught up with the tendency to assume that x, y and z have intrinsic meaning to them – they are arbitrary designations of a coordinate system.

Simply put, a vector is a type of mathematical object which has a magnitude, and a direction in a basis. The magnitude indicates the scalar length of the vector and the unit (e.g. m, N, m/s^2), and the direction indicates the proportion of the vector in each of the available directions. The direction is typically described as a unit vector, which is indicated with a "hat" symbol (e.g. $\hat{r}$), and does have any reference to units. For example, $\vec{r} = R(e_x\hat{\imath} + e_y\hat{\jmath} + e_z\hat{k}) = 45\text{cm}(0.500\hat{\imath} + 0.500\hat{\jmath} + 0.707\hat{k})$ would represent a distance vector with a magnitude/length of 45cm, with 0.5 of that distance in the x-direction, 0.5 of that distance in the y-direction, and 0.707 of that distance in the z-direction. This might seem like the components add up to more than 1, but they do not: $0.5^2 + 0.5^2 + 0.707^2 = 0.25 + 0.25 + 0.5 = 1$. Thus, the direction part of the example vector does not change the length of the vector. This vector could also be written in component form as $\vec{r} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k} = 22.5\text{cm}\hat{\imath} + 22.5\text{cm}\hat{\jmath} + 31.8\text{cm}\hat{k}$.

It is often easier to measure the length/magnitude of a vector and angles relative to the basis directions than it is to know the components. Figure 2.2 illustrates breaking down a vector in different ways. In terms of components, the vector is $\vec{v} = v_x\hat{\imath} + v_z\hat{k}$, but it is more likely that, in practice, the magnitude, $|\vec{v}|$ and the angles are measured. To determine the components, you will need to understand scalar product rules:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta_{ab}) \tag{2.1}$$

The dot product math, shown above, can be used to determine the projection of one vector along another.

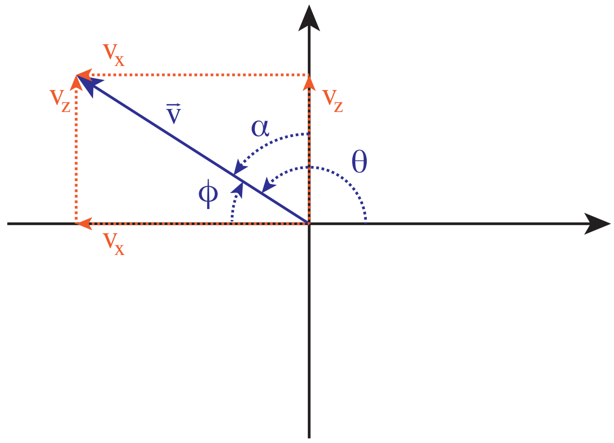


FIGURE 2.2: A two-dimensional vector $\vec{v}$, separated into components v_x and v_z, with angles measured relative to the positive x-axis (θ), the negative x-axis (ϕ) and the positive z-axis (α).

To find the projection of vector $\vec{v}$ in the x and z directions, you can take the dot product with the unit vector representing that direction (remember, unit vectors have a magnitude of 1, so $|\hat{i}| = 1$). Using the notation from Fig. 2.2, you can find the following:

$$\vec{v} \cdot \hat{i} = |\vec{v}||\hat{i}| \cos(\theta) = |\vec{v}| \cos(\theta) = v_x \tag{2.2}$$

Because $90 < \theta < 180$ in this example, v_x is going to be a negative number. Similarly, for the z-component, you would find:

$$\vec{v} \cdot \hat{k} = |\vec{v}||\hat{k}| \cos(\alpha) = |\vec{v}| \cos(\alpha) = v_z \tag{2.3}$$

Applying some trigonometric identities, and applying some physical understanding, this can be made significantly simpler. Only one angle needs to be measured. If you measure ϕ, for example, then the *negative* x-component of $\vec{v}$ would be determined with:

$$\vec{v} \cdot (-\hat{i}) = -|\vec{v}||\hat{i}| \cos(\phi) = -|\vec{v}| \cos(\phi) = v_x \tag{2.4}$$

And since $\alpha = 90 - \phi$, this means the z-component is simply:

$$|\vec{v}| \cos(\alpha) = |\vec{v}| \cos(90 - \phi) = |\vec{v}| \sin(\phi) = v_z \tag{2.5}$$

2.3.2 Addition of Vectors

A central tenet of physics is that vectors can be decomposed into components of an orthonormal (orthogonal/perpendicular, and normalized) basis, and each component can be combined independently. While angles and magnitudes/lengths may be easy to measure in practice, performing math with vectors is much harder to do with angles and magnitudes. This motivates switching to component form for easier manipulation.

When adding vectors, the unit vector tracks the component. The unit vector can be treated like a common factor when combining the components. For example, the addition of two vectors $\vec{a} = a_x\hat{\imath} + a_z\hat{k}$ and $\vec{b} = b_x\hat{\imath} + b_z\hat{k}$ can be combined as:

$$\vec{a} + \vec{b} = a_x\hat{\imath} + a_z\hat{k} + b_x\hat{\imath} + b_z\hat{k} = (a_x + b_x)\hat{\imath} + (a_z + b_z)\hat{k} \tag{2.6}$$

A similar thing applied when using vectored equations. Newton's First Law, $\sum \vec{F} = 0$, is a perfect example. The way it is written is just a very short form that implies a specific set of operations. For the addition of three forces, $\vec{F}_a$, $\vec{F}_b$ and $\vec{F}_c$, the components must each sum to zero:

$$\begin{aligned} \sum \vec{F} &= \vec{F}_a + \vec{F}_b + \vec{F}_c = 0 \\ &= (F_{ax} + F_{bx} + F_{cx})\hat{\imath} + (F_{ay} + F_{by} + F_{cy})\hat{\jmath} + (F_{az} + F_{bz} + F_{cz})\hat{k} \end{aligned}$$
$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

Or, in another way:

$$\begin{aligned} (F_{ax} + F_{bx} + F_{cx})\hat{\imath} = 0 \\ (F_{ay} + F_{by} + F_{cy})\hat{\jmath} = 0 \\ (F_{az} + F_{bz} + F_{cz})\hat{k} = 0 \end{aligned}$$

Subtraction of vectors is really just the addition of an inverted vector, and uses the same type of math. Thus, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$, where $(-\vec{b})$ is just the vector $\vec{b}$ with the signs of all components switched.

2.3.3 Propagating Uncertainties

This subsection will very briefly review some of the calculations you will need to perform for uncertainties. Much more detail is provided in Appendix E.

When two quantities are combined, their uncertainties must also be combined, but this is not a simple matter of adding the uncertainties together. Imagine using a ruler that is too short to measure the distance you want to determine. So you use the ruler to measure part way and make a mark, then measure from the mark to the other end. The location of the mark will have an uncertainty associated with it of, say, 1mm, and the distance that you measure from the mark to the other end will have an uncertainty of, say, 1mm, also. Naively, you might assume that the new uncertainty should just be 2mm. However, this doesn't account for the possibility that the mark underestimated the distance, and the final measurement overestimated the distance. Thus, uncertainties don't simply add together.

Uncertainties are combined using a procedure known as *propagation of uncertainties*.

Uncertainty analysis can be a very complicated process for high precision experiments. But for simple experiments, it doesn't need to be. For these labs, we will be adding uncertainties *in quadrature*. Given measurements of $x \pm \delta_x$, $y \pm \delta_y$, and constants without uncertainties A and B, the following three rules will be useful for this laboratory activity:

$$f(x, y) = Axy \quad \rightarrow \quad \delta_f = |f|\sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2} \tag{2.7}$$

$$f(x,y) = A\frac{x}{y} \quad \rightarrow \quad \delta_f = |f|\sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2} \tag{2.8}$$

$$f(x,y) = Ax + By \quad \rightarrow \quad \delta_f = \sqrt{(A\delta_x)^2 + (B\delta_y)^2}$$

2.4 Activity and Analysis

2.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ruler
- Square
- Hanging masses
- Drawing board with pulleys
- String
- Plain, white paper
- Mirror with hole
- Pencil/pen

2.4.2 Activity Instructions

The end product of your activity is to have a single piece of paper with three different sets of Y-shaped lines on it similar to Fig. 2.3. Each one will correspond to a different combination of masses.

Place three identical masses, one mass on each of the three loops of the string, and hang the string system from the pulleys similar to Fig. 2.1. Tape a piece of blank paper to the board behind the string so that the centre of the paper is approximately at the intersection point of the three parts of the string. The orientation and alignment of the paper do not need to be precise.

Use the mirror to identify three points along each string, and place a pencil mark in the centre of the hole in the mirror, then remove the mirror and draw a small circle around the dot to make it more visible. Remove the piece of paper and record the combination of masses you chose on it.

Now add another mass to the left side, such that the total mass on the left side is not more than the combined mass of the middle and right sides. Tape a new piece of paper on the board so that you can record the lines. Repeat the process to mark the lines. Remove the piece of paper and record the combination of masses you used.

Remove the extra mass from the left side and add a different mass to the right side, following the same criteria. Repeat the process of taping paper and recording your lines.

Using a ruler, draw lines through each set of 3 points that you marked, making three different Y shapes, similar to what is shown in Fig. 2.3. You may find that each set of three points do not perfectly align – do your best to draw a line through the middle of the three, minimizing the overall deviation between the line and the three points. Do NOT force the lines to meet at a central point, instead use the three points you drew as the only constraints for drawing the lines. Draw the central vertical line long enough, as shown in Fig. 2.3, and then draw the other two lines so that they cross the vertical. Note that your two branched lines may not meet at the same point along the vertical line - this is okay, and you should never force your data to meet your expectations. At the end of each branch of the Y shape, indicate the mass that was associated with that line.

The central vertical line will be your z-axis. Using your square, draw a line that is perpendicular to the z-axis on either side of it, through the point where each of your respective branches meet the z-axis. These will be your x-axes. Now pick a location along the two branching lines that is **not** one of the original points that you used to create the line. Use your ruler to measure the distance between the origin and the points you chose, and label this r. Next, use your square to draw lines perpendicular to the x and z axes up to the points you chose, label them x and z. Your final drawing should look something like Fig. 2.3.

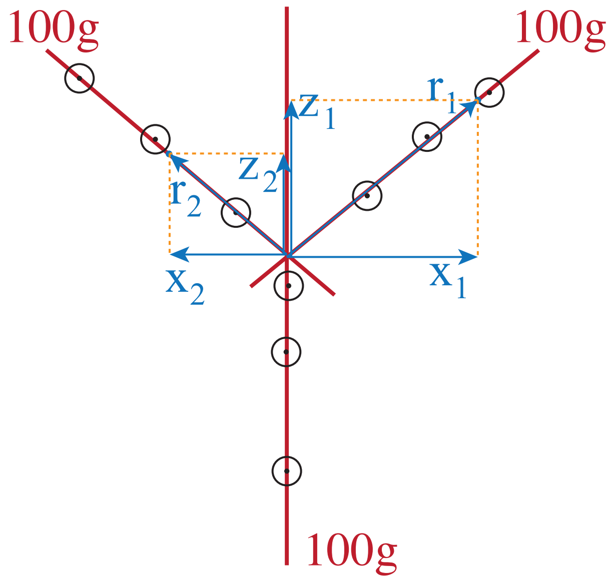


FIGURE 2.3: A depiction of all of the lines you will need to analyze your mass systems. **NOTE: Your three lines may not meet at the same point. Do NOT force them to meet at the same point. Draw the best fit line that you can for your three dots.**

You will need to do this for all three of the mass systems. Your marker should initial/sign off your lines and distances. Once you have your work signed off, you may leave and complete the rest of the work at home or you may stay and make the best of the scheduled time you have.

2.4.3 Analysis & Submission

Using Newton's First and Third Laws, it is clear that the tension in the string connected to each mass is due entirely to the gravitational force on the masses. However, there are two assumptions that must be made to solve this: 1. The string is massless. 2. There is no friction in the pulleys. Your ultimate goal in this lab is to test the validity of these assumptions. You are **not** trying to prove Newton's Laws, or test whether $F = mg$ or any other law or equation of physics.

If these assumptions hold true, then the tension forces of the string near the knot is determined by the gravitational force on the masses, and the sum of forces in the x and z directions should be 0.

Given that the tension is assumed to be $T = mg$ due to Newton's First and Third Laws, you will need to decompose these into x and z components. To do this, you could use a compass to measure angles, but it is just as easy to exploit similar triangles, as shown in Fig. 2.4. Since $\cos(\theta) = \frac{x}{r} = \frac{F_x}{T}$, it is easy to show that $F_x = T\frac{x}{r}$, and similarly $F_z = T\frac{z}{r}$.

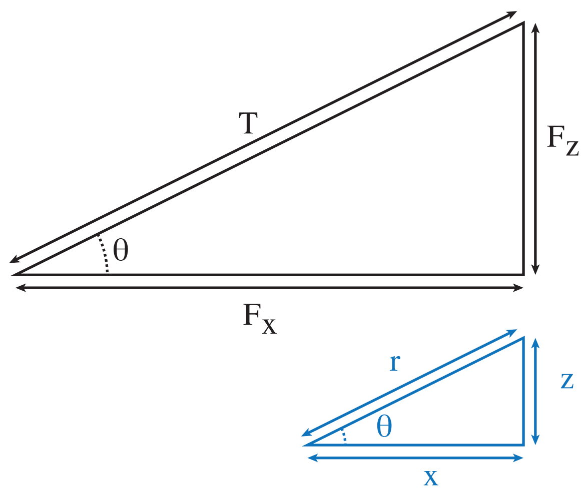


FIGURE 2.4: The similar triangles technique for determining components of the forces.

Once you have the components, you should sum them:

$$\begin{aligned} \sum F_x &= F_{x1} + F_{x2} + F_{x3} \\ \sum F_z &= F_{z1} + F_{z2} + F_{z3} \end{aligned}$$

In order to determine whether the assumptions are correct, you will need to perform a statistical comparison. Statistical comparisons are covered in Appendix C. A brief review of that appendix should make it clear that you will need to determine the uncertainty on $\sum F_x$ and $\sum F_z$. In future lab activities, you will be required to generate your own uncertainty formulas. For this one, while you are still learning, some of the details will be provided.

Each quantity you have measured should have an uncertainty associated with it: the uncertainty of the masses should be taken as 1% of the mass, and the uncertainty on the distances measured should be either 0.5mm or 1mm, depending on how accurate you feel you were able to measure using the ruler. The first step in determining the uncertainty of $\sum F$ is to find the uncertainties on the individual component forces. This can be done using the multiplication and division templates in the background section, or by reviewing the description in Appendix E. For example:

$$F_x = T\frac{x}{r} \quad \rightarrow \quad \delta_{Tx} = T_x\sqrt{\left(\frac{\delta_T}{T}\right)^2 + \left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_r}{r}\right)^2} \tag{2.9}$$

where $\delta_T = g\delta_m$. Once you have the uncertainties on each of the components of the summation, using the summation template for uncertainty will allow you to find the uncertainty on $\sum F$.

You will need to produce a table that looks similar to this for your lab writeup:

mass	r ($\pm\delta_r$ cm)	x ($\pm\delta_x$ cm)	z ($\pm\delta_z$ cm)	F_x (N)	δ_{Fx} (N)	F_z (N)	δ_{Fz} (N)
m_1							
m_2							
m_3							

For the distance column headings, r, x and z, you will be expected to include the measurement uncertainty in the column headings. This means you must replace the δ_r, for example, with the numerical value that is the uncertainty. Also note that **all** columns headings must include units. You may use millimetres, centimetres or metres if you wish.

With the value of $\sum F$ and $\delta_{\sum F}$, you can perform a statistical comparison. If your statistical comparison is small enough, then it is valid to say, "to the precision of this experiment, the assumptions made and physics analysis properly describe the equipment as it was used." However, if your statistical comparison is too large, then either: the equipment was used incorrectly, the assumptions are invalid, or Newton's Laws are not correct.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include photos of your papers showing all of the lines and measurements you made, along with your TAs signature. Note: If the quality of the photo is poor, you may not receive full marks.

2. Show all of formulas you used for determining each individual F_x and F_z from your measurements, along with the formulas for uncertainties. These formulas will be the ones you use in your spreadsheet to fill out your table of values.

3. Summarize your results in a table with columns for labeling the mass system and force, the x component of the force and its uncertainty, and the z component of the force and its uncertainty. (See the description in the Analysis and Writeup section.)

4. Show all of your calculations in summing the forces and determining the uncertainty in the sum of forces for each direction and for each combination of masses. (Do not do this in your spreadsheet.)

5. Explicitly state the expected value for the sum of forces in the x and z directions. Perform a statistical comparison of each sum of forces, as compared to the appropriate theoretical/expected value.

6. Respond to the following questions/instructions using complete sentences:
   (a) Why are three points used instead of two points for each branch of the force diagrams?
   (b) Why would putting all three points close together and making your Y shapes small be a bad idea for this experiment?
   (c) What does it mean if your statistical comparison is larger or smaller than 2? Why is a value of 2 used?
   (d) What is the largest component of the uncertainty in determining F_x and F_z?
   (e) How can you independently verify that your lines for determining the x and z components are square? Use this to test several of your lines and show your works.

Lab 2: Measuring Quantities In Multiple Ways – Quality Control

3.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix A: Statistics
- Appendix E: Uncertainty Analysis
- Appendix B: Using Spreadsheets
- Appendix C: Hypothesis Testing

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

3.2 Learning Objectives

The goals of this laboratory activity are as follows:

- Introduce students to methods of discretization.
- Introduce students to methods of determining quantities indirectly.
- Give students practice with using spreadsheet formulas.
- Give students practice with uncertainty propagation.
- Introduce students to comparisons against a known quantity.

3.3 Background

3.3.1 Measurements and Quality Control

In an experiment, it is often tempting to simplify the activity by measuring a quantity once and relying on the perceived accuracy of the measurement equipment and neglecting the impact of (unfounded) assumptions. There are many things that can go wrong when performing measurements – for example, equipment could be calibrated incorrectly, and thus returning incorrect measurements. Alternatively, assumptions are often made that may be invalid. Having a way to alternatively verify measurements, and then doing so regularly, is integral to quality control. Manufacturing and service companies typically pay quality control specialists quite well, as the consequence of poor quality production is more costly than the specialists!

There isn't a uniform way to perform quality control, and some ingenuity is required to meet the needs of the particular situation. For example, imagine you are monitoring the temperature of a substance. How can you be assured that the thermometer is correct? One way would be to get multiple thermometers that use different temperature sensors (e.g. thermal expansion thermometer vs thermocouple thermometer). If the measurements disagree between them, you know there is a problem with at least one of them. Using just two devices wouldn't tell you which device is working incorrectly (or maybe both are), but an array of thermometers might could be used and then the average temperature or the most commonly stated temperature could be used. Obviously, this amount of effort would only be used where there is little tolerance for variations in temperature.

An alternative approach to the thermometer example would be to re-calibrate the thermometer. This is done by performing an experiment in which the answer is already presumed, and has been established scientifically to great precision. The melting and boiling temperatures of many substances are well known, and the accuracy of the thermometer can be tested by measuring these quantities and comparing to their scientifically accepted values. In such a process, there is some reliance on the accuracy of the scientists who work diligently to provide high accuracy measurements of such quantities, but this is more reliable than, say, a company trying to sell a product.

In this lab, you will be performing quality control by measuring quantities in different ways and comparing them against each other. In each method, you will be making different assumptions. Your goal is to produce uniform circular motion using a pendulum, and then use that motion to calculate the acceleration due to gravity. One way to do this would be assume that the motion is circular, measure the average radius, and then use formulas for centripetal acceleration to calculate the acceleration due to gravity. Another way will be to approximate the motion as linear in small steps and then calculate the change in velocity, which will in turn give the acceleration. These methods will be discussed in greater detail below.

3.3.2 Circular Motion

For an object in uniform circular motion, the position of the object can be described mathematically by:

$$\vec{r}(t) = R\cos(\omega t + \phi)\hat{\imath} + R\sin(\omega t + \phi)\hat{\jmath} \tag{3.1}$$

where R is the radius, $\omega = 2\pi/T$ is the angular frequency as related to the time taken to complete one circle T, and ϕ is the phase constant that controls the starting location of the object relative to the coordinate system used. The velocity for the object is found by taking the time derivative:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -\omega R\sin(\omega t + \phi)\hat{\imath} + \omega R\cos(\omega t + \phi)\hat{\jmath} \tag{3.2}$$

and the acceleration can be found by taking the time derivative again:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -\omega^2 R\cos(\omega t + \phi)\hat{\imath} - \omega^2 R\sin(\omega t + \phi)\hat{\jmath} = -\omega^2\vec{r}(t) \tag{3.3}$$

While it is true that $\vec{a}(t) = -\omega^2\vec{r}(t)$, there is another relationship that is useful to consider: $|\vec{a}(t)| = |\vec{v}(t)|^2/|\vec{r}(t)|$, or, more more simply $a = v^2/R$. The centripetal acceleration is proportional to the square of the speed and the inverse of the radius of curvature. For this lab, you will be exploring this relationship in two different ways.

- For uniform circular motion, $\omega = 2\pi/T$ and $v = \omega R$, which gives the relationship $a = 4\pi^2 R/T^2$. Thus, measuring the radius of the circle and the period of motion is sufficient for determining the acceleration.
- Alternatively, you can discretize the motion and determine the change in velocity directly. This is shown below.

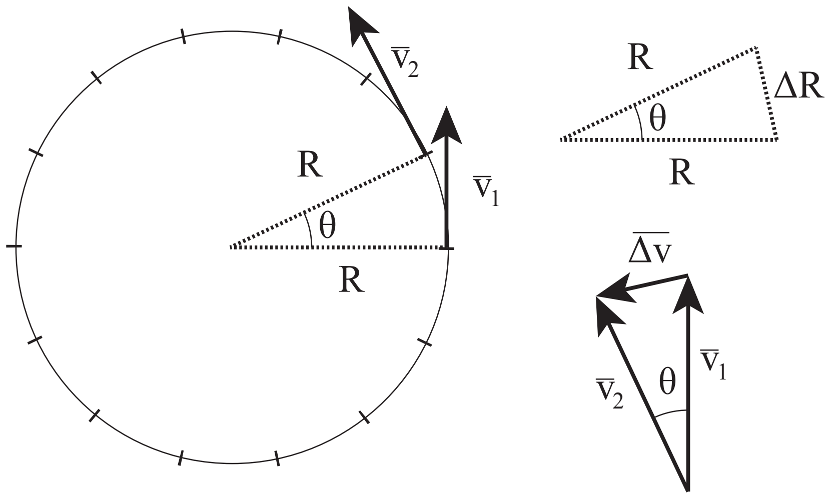


FIGURE 3.1: The velocity vectors shown have the same magnitude but differ in direction by some angle θ. The triangle formed by $\vec{v}_1$, $\vec{v}_2$ and $\vec{\Delta v}$ is similar to the triangle formed by the two radii and ΔR. Thus, measuring the distances in the ΔR triangle will allow you to calculate quantities in the $\vec{\Delta v}$ triangle.

Discretizing a path involves taking position measurements at regular time intervals, as shown in Fig. 3.1. For circular motion, the average of the two velocity vectors shown can be approximated as the arc-distance traveled divided by time, where the arc-distance is just the radius multiplied by the arc-angle, giving

$$v_{avg} \approx \frac{R\theta}{\Delta t} \tag{3.4}$$

And because the velocity vector and radial vector triangles share a common angle, they are similar triangles. This means that the ratios of similar sides should be equal, giving the relationship:

$$\frac{\Delta v}{v} = \frac{\Delta R}{R} \tag{3.5}$$

where we can use $v \approx v_{avg}$ for the velocity. This means that $a = v^2/R$ can be found for any given section of the circle by measuring θ, R and Δt:

$$a = \frac{\Delta v}{\Delta t} \approx \frac{R\theta^2}{\Delta t^2} \tag{3.6}$$

3.3.3 The Conical Pendulum

A conical pendulum is a pendulum that oscillates in both the x and y directions simultaneously, but with a $\pi/2$ phase difference between the two directions. The result is a pendulum bob that rotates in a circle, and the pendulum arm sweeps out a conical shape. Only two forces are affecting the pendulum: the tension in the pendulum arm and the force of gravity. Since the pendulum bob is moving in a circle, the pendulum bob should be in uniform circular motion, and thus experiencing centripetal acceleration.

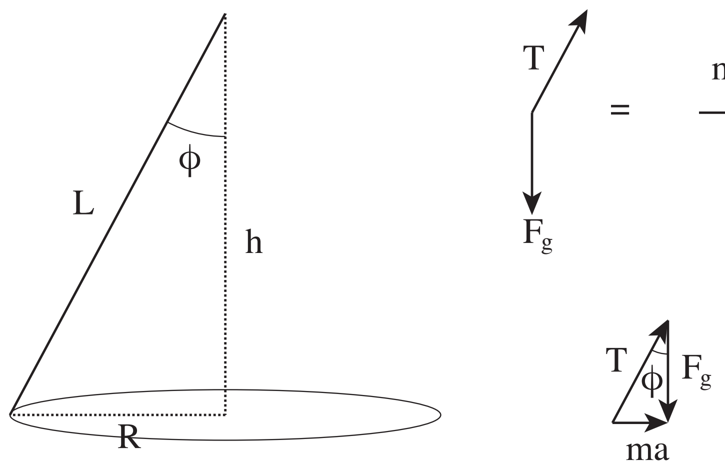


FIGURE 3.2: A conical pendulum is a pendulum that moves in a circle. The free-body diagram and kinetic diagram are shown in the upper right of the image, and the force triangle that is similar to the distance triangles is shown in the bottom right.

The sum of the tension and gravitational forces results in an inwards resultant and a centripetal acceleration: $\vec{T} + \vec{F}_g = m\vec{a}$. Because we know that the acceleration is perpendicular to the gravitational force, we can say that $|m\vec{a}|^2 + (mg)^2 = |\vec{T}|^2$ based purely on geometry. Similarly, we can see that $h^2 + R^2 = L^2$. The similar triangles allow us to determine the acceleration due to gravity, g, so long as we can measure the centripetal acceleration:

$$\begin{aligned} \tan\phi &= \frac{R}{h} = \frac{R}{\sqrt{L^2 - R^2}} \\ \tan\phi &= \frac{ma}{mg} = \frac{a}{g} \end{aligned}$$

Thus, you find:

$$g = a\frac{\sqrt{L^2 - R^2}}{R} \tag{3.7}$$

3.4 Activity and Analysis

3.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Spark timer
- Conical pendulum setup
- Metal plate with carbon paint
- White paper
- Ruler
- Compass
- Protractor
- Pencil and Pen

3.4.2 Activity Instructions

WARNING: An electrical shock from the spark timer apparatus can be very painful. Before activating the spark timer, ensure neither you nor your partner are touching any other part of the apparatus.

The conical pendulum setup should be arranged for you when you arrive, along with the spark timer. Ensure that your spark paper is placed with the labeled side upwards and set the spark timer to 10 sparks per second. Draw a cross ("+") on the paper, then use the centre of that cross and the compass to draw a $\approx 7\,\text{cm}$ circle on the paper. This will be your reference circle to help you determine if the pendulum is in circular motion, and when it is oscillating with the right radius.

Set your white paper beneath the pendulum while it is perfectly still so that the pendulum is as close to directly above the cross as possible, and activate the spark timer briefly so that the exact rest position of the pendulum is marked. To produce circular motion in the pendulum, apply a small force in a circular direction near the top of the pendulum wire. Using the circle as a guide, ensure that you are in circular motion with a radius larger than the reference circle you drew.

Over time, the pendulum will decrease in radius. When it is oscillating close to the radius of the reference circle, start the spark timer and keep it moving for one full revolution. Turn off the spark timer completely before removing your paper. The sparks will show on the underside of the paper.

Measure and record the length of the pendulum from the pivot point to the sparking tip. Then mark the central dot on your paper very clearly with a dark pen. Label each spark dot with a number, counting up from 1. Then measure and record the distance from the central dot to each spark dot. As you do so, draw a line connecting the central dot to each spark dot using your pencil. Using the lines you drew and your protractor, measure the angle between every other pair of lines, starting with 1-2. This will give you angles θ between points 1-2, 3-4, 5-6, 7-8, etc. If you have an odd number of lines, ignore the last one.

This should be enough to be signed off by your TA if you wish to leave and complete your analysis at home. However, you can stay and continue to work in the lab if you wish. You will need to have a photo of your paper showing your TA's signature to include in your lab submission.

3.4.3 Analysis & Submission

In this analysis, you will be using two different techniques to estimate g. At the end, you will have two (slightly) different values of g.

Determining g with the circumference

Given that the spark timer should be making sparks 20 times per second, that means the time between sparks should be $1/20\,\text{s}$. Using this information and the number of sparks you have, estimate the time for the pendulum to complete one full circle. You will need to estimate the uncertainty in the timing based on what you see and justify it in your submitted report. To do so, consider the precision of the spark timer settings and the pattern of sparks on your paper. There isn't one single correct answer to this, but there are wrong answers. You will be graded on the reasonability of your justification.

Using your radii measurements, calculate the average radius. If you had careful enough experimental technique, your radii should be close to uniform, but even if it looks perfectly uniform there is still measurement uncertainty involved that should result in some deviation. The uncertainty in the average radius will be the your standard deviation of the mean of the radii. Use this information along with the equation:

$$a = 4\pi^2 R/T^2 \tag{3.8}$$

to determine your centripetal acceleration. Then use Eq. 3.7 to estimate g. You should have uncertainties on R, T and L, which you will need to use uncertainty propagation to determine a formula for calculating uncertainty on a and then g from your measurement uncertainties.

Determine g with discretization

Copy your radii and angle measurements into a table that looks like this:

Section	R_{avg} ($\pm\delta_R$ cm)	θ_{ij} ($\pm\delta_\theta$ rad)	a_{ij} (m/s^2)
1-2			
3-4			
5-6			

You will fill in this table with the average radii of the two, $\frac{1}{2}(R_i + R_j)$, and calculate the centripetal acceleration from Eq. 3.6.

Once you have a list of centripetal accelerations, calculate the mean value of your a_{ij} values along with an estimate of the standard deviation of the mean. Your uncertainty will be the standard deviation of the mean, so you do not need to worry about calculating and propagating uncertainties to determine δ_a.

Finally, use Eq. 3.7 to estimate g. You should already have the formula to determine the uncertainty on g from previously.

Assessing your results

At this point, you should have two different determinations of g. You will then need to determine if either are statistically similar to the accepted value for g for Victoria, B.C. This is done with a simplified t-test (see Appendix C). The accepted value of g for Victoria, B.C. is $g = 9.8092\,\mathrm{m/s^2}$. You should also perform a comparison to see if the two values of g are comparable/compatible with each other.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo of your data showing your ticker marks, your radial lines, your labels and your angle/distance data.

2. Create a table for your data and angles, as described in the Analysis & Submission section of the lab manual.

3. Calculate (showing all steps) the mean radius and uncertainty, and the total time for one rotation of the pendulum, including uncertainty. Justify your uncertainty for the total time.

4. Derive an expression for the uncertainty in the centripetal acceleration in terms of the average radius and the rotational period.

5. Calculate the centripetal acceleration and its uncertainty using the radius and rotational period.

6. Calculate the mean centripetal acceleration from your table, along with its uncertainty, for the discretization data.

7. Derive an expression for the uncertainty in the gravitational acceleration as it depends on the length of the pendulum, radius of rotation, and the centripetal acceleration.

8. Calculate the gravitational acceleration and its uncertainty from your two values of centripetal acceleration.

9. Perform a statistical comparison between the g values determined via the two methods, and then comparing each of these to the accepted value of g. (Note: If the statistical tests all show agreement, this means your data is consistent and both techniques are valid. If not, ideally an experimenter would try to figure out why they did not agree. This is the proper process for performing a test with an expected result.)

10. Respond to the following questions/instructions using complete sentences:

(a) Identify as many assumptions that you can that were used in this lab design. This could include assumptions about physical effects that can be neglected, assumptions about validity of approximations, or anything else.

(b) Which method for determining g was more precise? Justify your answer.

(c) Which method for determining g was more accurate? Justify your answer.

(d) Which is the superior method for determining g - the single calculation of the total motion of the pendulum or the combination of calculations from discretization? Justify your answer.

(e) What was your largest source of uncertainty for each of the methods? Justify your answer.

(f) What could you do to improve (decrease) the uncertainty in this experiment?

Lab 3: Using Graphical Analysis

4.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix E: Uncertainty Analysis
- Appendix D: Graphical Analysis
- Appendix B: Using Spreadsheets
- Appendix C: Hypothesis Testing

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

4.2 Learning Objectives

The goals of this laboratory activity are as follows:

- Introduce students to video analysis and Logger Pro.
- Give students practice with discretization.
- Introduce students to methods for finding slopes of discretized curves.
- Introduce students to using least squares to determine the slope/intercept of a linear line.
- Give students practice with using spreadsheets, especially with calculations on columns of cells.
- Give students practice with communicating scientific quantities.

4.3 Background

4.3.1 Projectile Motion

The study of projectile motion covers the behaviour of objects that are in motion without thrust and subject to a gravitational force without a normal force. This seems like a complicated description, but that is because advanced projectile motion studies are quite complicated. At the most advanced levels, laminar and turbulent drag effects must be accounted for, along with the rotation of the Earth.

When speeds are low and the cross section of the projectile is small, drag effects are small enough that they can be neglected unless very high precision is necessary. When the time of flight is small, the rotation of the Earth can be neglected. These will be the assumptions that are used for this lab. Thus, the only force affecting the object is gravity, and constant acceleration kinematics can be used.

Constant acceleration kinematics occurs in a two dimensional plane - one dimension is defined by the direction of the acceleration, and the other dimension is the direction of the component of the initial velocity that is perpendicular to the acceleration. For the purposes of this lab, we will call the direction of gravity to be the $-\hat{k}$ direction, and the perpendicular direction of motion will be $\hat{\imath}$. In constant acceleration kinematics, the position of the object can be represented as as two dimensional vector:

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} + \begin{bmatrix} v_x(0) \\ v_z(0) \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} a_x \\ a_z \end{bmatrix} t^2 \tag{4.1}$$

In this parametrization, gravity points in the $-\hat{k}$ direction, so $a_x = 0$ and $a_z = -g = -9.8092\,\mathrm{m/s^2}$. The parameters $x(0)$ and $z(0)$ are the components of the initial position, and $v_x(0)$ and $v_z(0)$ are the components of the initial velocity. Often times the initial position can be chosen to be the origin, so $x(0) = 0$ and $z(0) = 0$, which simplifies the expression to:

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} v_x(0) \\ v_z(0) \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 0 \\ -g \end{bmatrix} t^2 \tag{4.2}$$

Taking the first time derivative of this equation gives the velocity as a function of time:

$$\begin{bmatrix} v_x(t) \\ v_z(t) \end{bmatrix} = \begin{bmatrix} v_x(0) \\ v_z(0) \end{bmatrix} + \begin{bmatrix} a_x \\ a_z \end{bmatrix} t \tag{4.3}$$

With the assumption that $a_x = 0$ and $a_z = -g$, this simplifies to:

$$\begin{bmatrix} v_x(t) \\ v_z(t) \end{bmatrix} = \begin{bmatrix} v_x(0) \\ v_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix} t \tag{4.4}$$

4.3.2 Estimating instantaneous velocity

Much of the mathematics of physics employs continuous variables, such as position and time. The continuous nature of these quantities is well established theoretically, however it is impractical to work with in an experiment. Typically measurements are made on a graduated scale, and therefore the measurements are discrete rather than continuous. For example, if you use a ruler to measure distances, the most accurate you can measure is approximately half of the smallest division on the ruler. Thus, when recording a series of distances, you can only record in integer steps of half of the smallest division on the ruler - this is not a continuous quantity in practice.

For the measurements themselves, this is often not a significant barrier, but this can have an impact on the calculation of quantities that depend on the measurements, especially when performing determining derivatives. Consider a situation where you have a series of distances measured at regular intervals in time. The first position is measured at t_1, $x(t_1)$, the second at t_2, $x(t_2)$, and the i'th measurement is measured at t_i, $x(t_i)$. The simplest method for calculating a derivative from data is to take either the forward or backward difference. The forward derivative would be:

$$\left.\frac{dx}{dt}\right|_{t=t_i} = \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} \tag{4.5}$$

while the backward derivative would be:

$$\left.\frac{dx}{dt}\right|_{t=t_i} = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \tag{4.6}$$

These are illustrated graphically in Fig. 4.1.

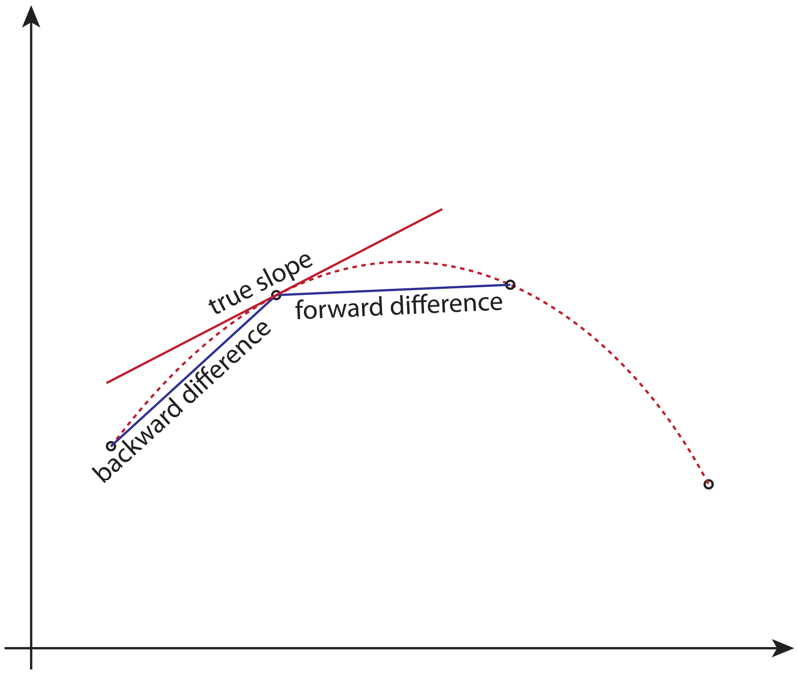


FIGURE 4.1: Visualization of forward difference and backward difference methods for estimating the slope from discrete data points.

There is also a central difference method which relies only on the next and previous positions, rather than the position of interest itself:

$$\left.\frac{dx}{dt}\right|_{t=t_i} = \frac{x(t_{i+1}) - x(t_{i-1})}{t_{i+1} - t_{i-1}} \tag{4.7}$$

Much more advanced is the five-point approximation, the derivation of which is not important for this lab:

$$\left.\frac{dx}{dt}\right|_{t=t_i} = \frac{-x(t_{i+2}) + 8x(t_{i+1}) - 8x(t_{i-1}) + x(t_{i-2})}{3(t_{i+2} - t_{i-2})} \tag{4.8}$$

You may notice that the slope for time t_i does not depend on the position at t_i in either of these two methods.

It is important to remember that each of these methods are approximations only. That means that there isn't a single ultimately superior method. While these are attempting to find the first derivative, the effectiveness of each of these approximation methods will depend on the higher order derivatives. For any curve with an acceleration term but minimal higher order derivatives, either the central difference method or the five-point method will be superior. It is up to you to choose which method you will use.

4.4 Activity and Analysis

4.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ball launch apparatus
- Camera and computer with LoggerPro

4.4.2 Activity Instructions

You will be launching a metal ball at an initial angle of 60-70° from the horizontal while recording the trajectory on video. Then you will be using LoggerPro to analyze the position of the projectile over time.

On the computer, start **Wmcap**, which will take some time until it shows a live image of the webcam. Under the **Options** menu, select **Video Capture Pin** and set the **Output Size** to **320 $\times$ 240**. Under the **Capture** menu, set the **Frame Rate** to **30 FPS**. Watch the live video while you launch the ball a few times. Ensure that the spring tension control and angle of launch are adjusted so that the entire trajectory of the ball fits within the camera's field of view. Once that is done, you should be ready to record your launch.

To record and analyze the video, follow these steps:

1. Under the **Capture** menu, select **Start Capture**.
2. Press **OK** in the dialogue box to start the recording and then fire the spring apparatus to launch the projectile. You will have 5 seconds by default from the time you start recording to when the recording stops. This is enough time to capture the projectile if you work quickly.
3. If something goes wrong with the launch (e.g. you were too slow at launching so it wasn't recorded completely), reset the projectile and the video recording, and capture another video.
4. Once you have a good recording, under the **File** menu, choose **Save Captured Video As ...**
5. Save the video with a distinctive name and a **.avi** extension into the **c: temp** folder.
6. Open **Logger Pro** from the Desktop, and under the **Insert** menu, select **Movie**.
7. Open your saved video file and press the **Enable Video Analysis** button in the lower right. The button shows three dots in a diagonal line, along with a triangle.

8. To set the scale of the video, you will need to use an object of known dimension that is the same distance from the camera as the projectile - this will tell you the relationship between the on-screen scale and the physical scale.
9. Going frame by frame in the video, use the **Add Point** selection tool and click on the position of the ball for every frame in the trajectory of the ball.
10. The data will appear in the data table, which includes the time stamp for each frame, as well as the x and z components of the position.

For each time stamp, subtract off the initial time of the launch of the projectile to find the *projectile time* you will need for your calculations in the analysis state. Copy these results down on a piece of paper in a table format.

Once you have copied down the table of projectile time and x and z position of your projectile, you will have enough to show your TA and get their signature to leave the lab room if you wish to complete your analysis at home.

4.4.3 Analysis

In your spreadsheet program, you will need to calculate the $v_x(t)$ and $v_z(t)$ using either the central-difference or five-point approximation methods. For either of these methods, you may not be able to calculate the speed components for the initial and final one or two points. This is expected and won't affect your results.

You should now have a table with five columns:

Time (s)	x (m)	z (m)	v_x (m/s)	v_z (m/s)

Using this table, produce four plots: x vs t, z vs t, v_x vs t and v_z vs t. You are expected to provide appropriate axis labels, a title and data ranges for your plots.

For each of the velocity columns, you will need to follow the procedure in Appendix D to perform a Least Squares analysis to determine the value and uncertainty of the slope and intercept. This will require you to make two more tables.

t	v_x	t^2	v_x^2	$v_x t$

t	v_z	t^2	z^2	$v_z t$

At the bottom of each column, you should calculate the average value of the values in the column above it. These two tables do not need to have units listed, as they are used for calculating the slope and intercept and their uncertainties. Note: Appendix D provides formulas to calculate slope (Eqs. D.11), intercept (Eq. D.12), and uncertainties (Eq. D.14, D.15) in terms of the x and y values of a graph. The x and y of the graphical analysis formulas represent whatever value is on the horizontal axis and vertical axis of the graph, respectively. For each of these tables, that means the **TIME** value is the horizontal (x) axis, and the vertical (y) axis is the other quantity.

The final step will be in analyzing the results of your calculations. From your slopes and intercepts, you need to interpret the following:

- Initial horizontal velocity
- Initial vertical velocity
- Acceleration in horizontal direction
- Acceleration in vertical direction

With this done, do your results match your expectations of the values? (Hint: Use a statistical comparison!)

Note: In this lab, measurement uncertainties were not recorded. This is because graphical analysis is an advanced method of using repeated measurements to determine quantities. Instead of using repeated measurements of the same quantity and then estimating the standard deviation of the mean, graphical analysis considers the deviations by comparing the data points to a hypothetical straight line.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo of your raw data with your TA's signature.

2. Create a table to show all of your data as described in the Analysis & Submission section of the lab manual. Clearly indicate which method was used for calculating the speeds and show a sample calculation.

3. Create the four graphs as described in the Analysis & Submission section of the lab manual.

4. Create a table for the last squares analysis of the v_x vs t data, as described in the Analysis & Submission section of the lab manual.

5. Determine the slope and intercept, along with their uncertainties. This should be done by hand using the average column values in the table above.

6. Create a table for the least squares analysis of the v_z vs t data, as described in the Analysis & Submission section of the lab manual.

7. Determine the slope and intercept, along with their uncertainties. This should be done by hand using the average column values in the table above.

8. State the initial horizontal and vertical components of velocity and acceleration of the projectile. (Simply state the quantities using proper significant figures.)

9. State the expected values of the components of the acceleration. Perform a statistical test for whether your measured acceleration component values agree with your expectations.

10. Respond to the following questions/instructions:
    (a) Were any assumptions or approximations involved in performing these calculations? List them and state how you think they might affect the results if they were not valid.
    (b) What do your statistical tests indicate? What are the implications of the results?

(c) What is the initial speed of your projectile? What is the initial angle from horizontal of the projectile?

Lab 4: Testing Predictions

5.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix B: Using Spreadsheets
- Appendix C: Hypothesis Testing
- Appendix E: Uncertainty Analysis

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

5.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to formulating and testing predictions.
- Give students practice with analysis and communication techniques established in the other lab activities.

5.3 Background

5.3.1 Hypothesis Testing

You have already used the mathematics of hypothesis testing. However, you haven't really been testing a hypothesis, as you haven't formed a hypothesis first before taking an action. This is an unfortunately common mistake that is made by scientists in every field - an hypothesis or prediction is written or modified after performing the measurement/experiment. The labs in this course are not called "Experiments" for the express purpose of not contributing to this trend, as you did not formulate a proper hypothesis and did not test that hypothesis.

In practice, scientists always form what is called the **null hypothesis**. The null hypothesis is simply the predicted outcome of an action/experiment under the assumption that the established theory is correct. Alternative hypotheses can also be formed, but are not necessary. The action/experiment is then performed, and a statistical comparison is performed between the outcome of the action/experiment and the prediction(s). If the two values (predicted and measured) are statistically similar to each other, then the hypothesis cannot be excluded.

This is why precision is so important. Imagine predicting the distance that something will travel based on Newton's Laws, and a measured initial velocity of, say, $v = (33 \pm 2)\,\mathrm{m/s}$ along a horizontal trajectory from a height of $h = (1.6 \pm 0.1)\,\mathrm{m}$. The predicted distance without accounting for any drag or other factors would be:

$$\begin{aligned} h &= \frac{1}{2}gt^2 \\ x &= vt = v\sqrt{\frac{2h}{g}} = 19\,\mathrm{m} \\ \delta_x &= x\sqrt{\left(\frac{0.5\delta_h}{h}\right)^2 + \left(\frac{\delta_v}{v}\right)^2} = 1\,\mathrm{m} \end{aligned}$$

This is an uncertainty of about 5%. If the object is measured to travel 17.5 m, a person might be inclined to use their intuition and say "17.5 m is not equal to 19 m, so clearly something is wrong." But a scientist would be forced to say, "Within the precision of the measurements, the null hypothesis (that Newton's Laws apply, that the measurements were correct, and that no drag or other effects are present) cannot be excluded." This is a complicated way of saying that 17.5 m and 19 m are effectively the same thing, since $t = \frac{19\,\mathrm{m} - 17.5\,\mathrm{m}}{1\,\mathrm{m}} < 2$. The problem here is the precision of the measurements. If the velocity and height could be known to a greater precision, then the outcome might be different.

Consider the same scenario but with 10 times greater precision: $v = 33.0 \pm 0.2\,\mathrm{m/s}$, $h = (1.60 \pm 0.01)\,\mathrm{m}$. The predicted distance becomes $x = (18.8 \pm 0.1)\,\mathrm{m}$ and the measurement of 17.5 m is clearly statistically significant from the predicted value. This would be sufficient to claim "There is a high probability that at least one of the components of the null hypothesis is incorrect." If an alternative hypothesis that accounted for drag using a newly created equation predicted a distance of $17.7 \pm 0.2\,\mathrm{m}$, then that hypothesis could not be excluded and thus the new drag equation could be claimed to be a better model of behaviour than excluding it.

5.3.2 Ballistic Pendulum

A ballistic pendulum is a device that traps a projectile and converts some of the kinetic energy of the projectile to gravitational potential energy. Typically, a ballistic pendulum is used to determine the energy/momentum/speed of a projectile that is difficult to measure in a different way. In this case, you won't be using it to determine the speed of the projectile but will instead be verifying the physics of the ballistic pendulum to test whether the theory, as applied, is sound. An example of a ballistic pendulum is illustrated in Fig. 5.1.

The simplest depiction of the physics of a ballistic pendulum starts with conservation of angular momentum. The pendulum will pivot about a fixed point, so we can use that as the centre-of-rotation. While the projectile is not initially in rotation, before the impact, we can still describe its motion in terms of the angular momentum about the pivot point. If the projectile has a mass of m and speed v and is traveling perfectly horizontally, then the angular momentum of the projectile about the pivot point prior to impact is:

$$L_i = mvZ \tag{5.1}$$

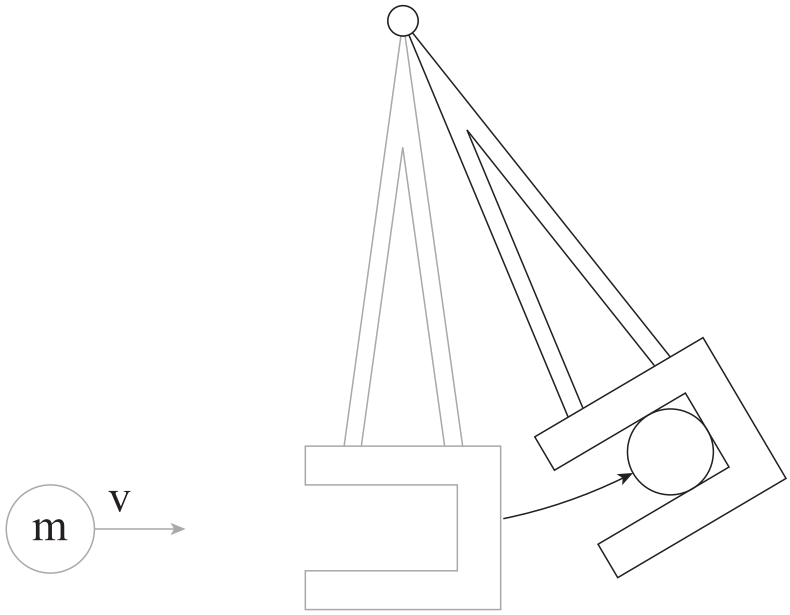


FIGURE 5.1: An example of the behaviour of a ballistic pendulum that captures a projectile and swings to some height. The height that the pendulum swings to indicates the amount of momentum that has been transferred to it by the projectile.

where Z is the distance between the pivot point and the centre of the projectile. The pendulum is initially at rest, and so it does not have any initial angular momentum. After the impact, the total angular momentum is conserved but it cannot be described as simply as for the projectile alone. The angular momentum after the impact is properly described as:

$$L_f = I\omega \tag{5.2}$$

where I is the moment of inertia of the combination of the pendulum and projectile, as measured about the pivot point, and ω is the angular velocity of the pendulum.

This is where the system needs to be modelled. Modelling is when assumptions and approximations are made to simplify a complicated system into something that can be solved practically. In this case, the simplest model for the moment of inertia of the pendulum is to assume that the pendulum is a simple pendulum, where the mass is not distributed over the entire object but instead concentrated at the centre-of-mass. You can verify this for yourself, but the centre of mass of the pendulum-plus-projectile happens to be very close to where the spark timer needle is located. Letting the distance from the pivot to the spark timer needle be R, this means the simplest model of the moment of inertia of the pendulum-plus-projectile is simply $I = (m + M)R^2$, where M is the mass of the pendulum alone. For clarity, the difference between R and Z is illustrated in Fig. 5.2.

Immediately after the collision, the pendulum will be rotating at the angular rate of ω, but will be slowing down as the energy is converted from kinetic into potential. When the pendulum reaches its highest point, all of the kinetic energy will have been converted into potential energy. In the absence of any dissipative forces (e.g. drag,

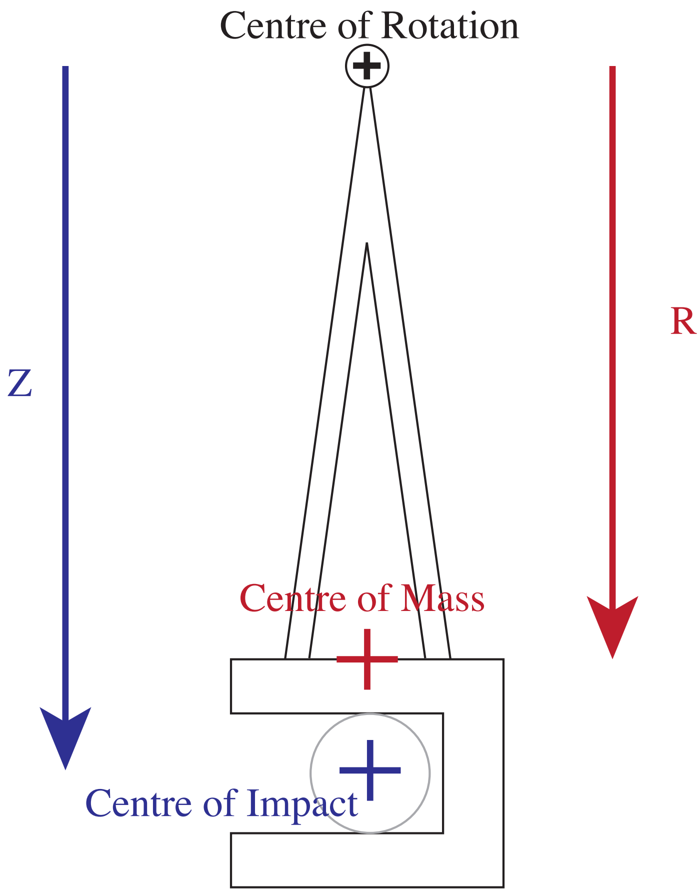


FIGURE 5.2: Illustration of the distances relevant for modelling the ballistic pendulum.

friction), the conservation of energy equation describes this adequately:

$$\Delta K = -\Delta P \quad \leftrightarrow \quad -\frac{1}{2}I\omega^2 = -(m+M)gh \tag{5.3}$$

where h is the maximum height that the centre of mass gains in the vertical direction.

Combining these equations and solving for the height in terms of the knowable/measurable quantities results in the relationship:

$$h = \frac{1}{2}\frac{m^2}{(m+M)^2}\frac{R^2}{Z^2}\frac{v^2}{g} \tag{5.4}$$

Assuming that g is known precisely enough that its uncertainty is not relevant, the remaining quantities all have uncertainties related to them. Solving for the uncertainty in h is best done using partial derivatives:

$$\delta_h = \sqrt{\left(\frac{\partial h}{\partial m}\delta_m\right)^2 + \left(\frac{\partial h}{\partial M}\delta_M\right)^2 + \left(\frac{\partial h}{\partial L}\delta_L\right)^2 + \left(\frac{\partial h}{\partial R}\delta_R\right)^2 + \left(\frac{\partial h}{\partial v}\delta_v\right)^2} \tag{5.5}$$

Remember, a partial derivative is just a derivative in which all other variables are treated as “constants” (so no need to use the chain rule on each one).

5.4 Activity & Analysis

5.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Ballistic pendulum apparatus
- Spark timer setup
- Ruler
- Mass scale
- Compass or square

5.4.2 Activity Instructions

Warning: Do not touch the aluminum plate or wire leads while the spark timer is on. Shocks from the system can burn small, deep and painful holes in your skin.

Remove the knurled nut to remove the pendulum from the apparatus. Measure the distance from the pivot point of the pendulum to the spark needle. Then, measure the mass of the pendulum and the projectile separately. Place the ball inside the pendulum and locate the centre of mass. The centre of mass should be approximately where the spark needle is. If you feel that it is not at that location, note in which direction you feel that it is biased and how that might affect your results.

Next, without replacing the pendulum, place the ball in the firing mechanism and load the spring to the first (lowest) setting. Fire the ball and watch for the approximate area it lands on the table. Place carbon paper in that area, and white paper on top so that the next time you fire, the impact on the carbon paper will mark the location where it lands. Reload the spring mechanism and fire the projectile again. Measure the distance from the spring mechanism to the location it landed and the height from which the ball was launched. Reload the device four more times and repeat the measurements so that you have five trials. Use constant acceleration kinematics to solve for the initial speed of the ball from the height and distance traveled, and then take the average speed as your estimate of the speed of the projectile.

Before continuing, perform all the necessary calculations for predicting the height that the pendulum will travel based on the speed of the ball that you just calculated, along with the properties of the pendulum that you measured. You MUST get this signed off by your TA before continuing.

Your uncertainty in the speed of the projectile will be statistical in nature, and you can propagate this uncertainty to determine the uncertainty in the predicted height the pendulum will travel. However, you do not need to have the uncertainty determined before you use the ballistic pendulum, so long as you have the value of the height prediction.

Replace the pendulum in the device with the knurled nut. Tape a piece of paper to the pendulum apparatus so that it covers the full range of swing of the pendulum. Use the pendulum as a guide to draw a vertical line along the pendulum when it is at rest, hanging vertically due to gravity. Adjust the spark pin so there is about a 2mm gap between the tip and the paper, and set the spark interval to 10ms. Load the projectile to the first setting and wait until the device is completely at rest before activating the spark timer and then firing the projectile. Turn off the spark timer.

Remove the paper and draw a circle around your starting spark and the highest spark, so that they are easy to see. Repeat the spark process four more times (so that you have five pieces of paper with sparks), including marking your lowest and highest spark from each run.

Use a square to draw a line that is perpendicular to the vertical and passes through your highest sparks for each run. Measure the height distance for each circled highest spark and write it on the piece of paper.

Once you have this done, you will have enough information to get your TAs signature on your papers and you may complete the rest of the lab analysis at home. You should have a total of six pieces of paper signed off by your TA, one with your calculations of your predictions for the maximum height, and five with spark timer arcs and height measurements.

5.4.3 Analysis

Create two tables, one for the distance information and velocity calculations, and one for the height measurements. Your tables should look something like this for the speed determination:

Trial	Distance ($\pm\delta_d$ m)	Speed (m/s)
1		
2		
3		

And this for the max height measurements:

Trial	Max Height ($\pm\delta_h$ m)
1	
2	
3	

From your data, calculate the average and uncertainty in the speed of the projectile. Then use this information to determine the predicted height of the pendulum, along with the uncertainty.

Find the average and uncertainty in the max height of the pendulum, and compare with your prediction.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo of your calculation of your prediction for the max height that you performed in the lab, showing your TA's signature. This should list all of the values you measured and their uncertainties.

2. Include photos of your pendulum arc papers, showing your TA's signature.

3. Create a table of your distance and height information, as describe in the Activity & Analysis section of the lab manual.

4. Calculate the average and uncertainty in the speed of the projectile. It is sufficient to show the formulas you use without showing intermediate steps of plugging in numbers. You do not need to perform uncertainty propagation, as the uncertainty in speed is statistical in nature.

5. Calculate your predicted height of the pendulum based on your measurements and the determined speed of the projectile.

6. Use uncertainty propagation to determine the uncertainty in the predicted max height value, as it depends on the uncertainty of your measurements.

7. Create a table of your max height information, as described in the Activity & Analysis section of the lab manual.

8. Calculate the average and uncertainty in the max height of the pendulum. It is sufficient to show the formulas you use without showing intermediate steps of plugging in numbers. You do not need to perform uncertainty propagation, as the uncertainty in height is statistical in nature.

9. Compare your two answers to determine if they are statistically compatible measurements.

10. Respond to the following questions/instructions:

    (a) What is the largest source of uncertainty in your prediction value?

(b) This experiment neglects many possible influences. List as many as you can and discuss how they might affect the results. Use equations to justify your answers.

(c) Assuming your results are not statistically compatible (whether they are or aren't), what is one aspect of the experiment/modelling would be good to verify/validate before claiming that the theory is incorrect? Explain how this could affect the results.

(d) What is another aspect of the experiment/modelling that would be good to verify/validate? Explain how this could affect the results.

Lab 5: Statistics and Histograms

6.1 Important Appendices

Understanding the information in the following appendices will be critically important for your ability to complete this laboratory project:

- Appendix A: Statistics
- Appendix B: Using Spreadsheets
- Appendix C: Hypothesis Testing

Before asking your lab instructor questions regarding this project, ensure that you have read these appendices. Any questions you ask that suggest you have not read these appendices will receive a response encouraging you to read the appropriate appendix first.

6.2 Learning Objectives

The goals of this laboratory project are as follows:

- Introduce students to histograms and probability distribution functions.

6.3 Background

All manufacturing processes have variations, and this is an unavoidable consequence of production. More care and effort can be put into the production to produce a higher quality, more precise product, but no two products produced by the same equipment will ever be completely identical. There will always be some variation.

Resistors are a perfect example of this. They are mass produced but have precisions of 10%, 5% or 1%, depending on the quality of the production method, and the design specifications. A precision of 5%, for example, means that the resistance value of any given resistor produced by that equipment should have a 67% chance of being within $\pm 5\%$ of the quoted value. A resistor with a target resistance of $1\,\mathrm{M\Omega}$ and a 5% precision is most likely going to have an actual resistance value within the range $R = 0.95 - 1.05\,\mathrm{M\Omega}$ ($\sim$ 67% likelihood), and may even have a resistance in the range $R = 0.9 - 1.1\,\mathrm{M\Omega}$ ($\sim$ 95% likelihood).

When designing a circuit, especially when mass producing circuitry, engineers and technicians have to account for the fact that the resistance they get may not be exactly the resistance they wanted. The designed product must have a tolerance so that it is still able to perform its function well enough despite the possibility of variations in the components. Where this is most noticeable is in cheaper products – cheaper versions of a product use cheaper components that have less precision, and as a result are more likely to be prone to failure when the variations in their components result in them operating outside of acceptable ranges.

Consider a case where a manufacturing plant produces 100 000 circuits using cheap, low precision resistors. They have saved significant money by doing so. Now imagine a few of those circuits are produced where all of the resistors happen to, by random chance, be ones that are significantly less resistive than the design specifications. Since power is given by $P = V^2/R_{eq}$, if the R_{eq} (the total resistance of the entire circuit) is too small, then the power usage will be higher. It could even be high enough that the power supply cannot provide enough current and the product fails – maybe it even catches fire!

Precision in manufacturing quickly becomes cost prohibitive, and the goal of many engineers/designers is not to improve precision, but to make a circuit that is robust enough to withstand the total possible variations. This argument holds true for car manufacturing, infrastructure design (bridges, roads), housing/building design, and all components of our economy.

In this lab, you aren't going to be designing a circuit that is robust, as that is a very advanced activity. However, you will be exploring the statistical nature of the manufacturing process by statistically examining a series of resistors.

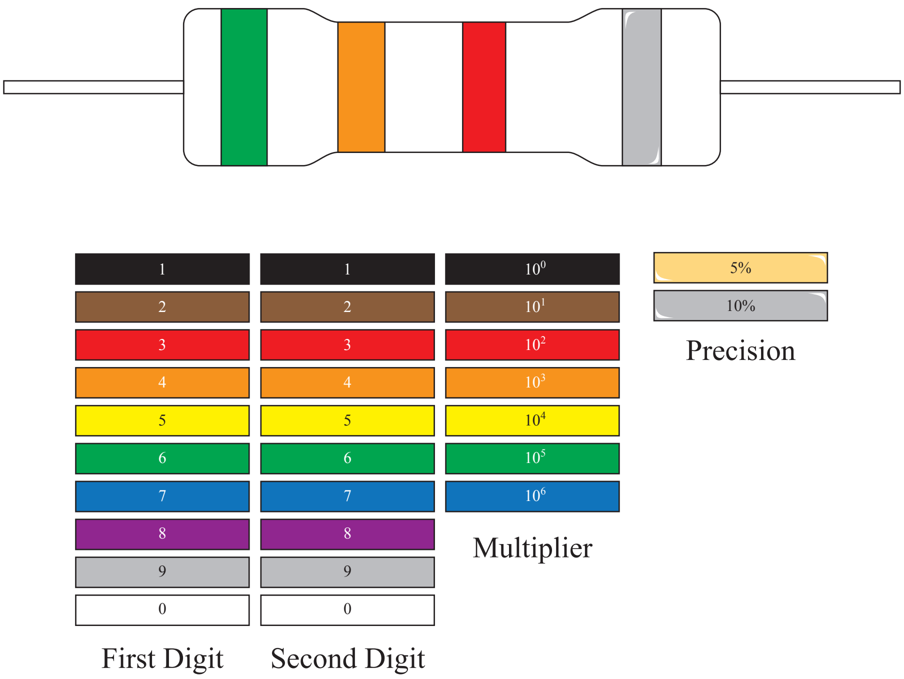


FIGURE 6.1: Resistor 4-bar colour coding system. The first bar indicates the first significant digit, the second bar indicates the second significant digit, the third bar indicates a factor of 10 multiplier, and the fourth bar indicates the precision of the production. The fourth bar typically has a metallic sheen to it, rather than being one of the possible colours from the other bars, to reduce confusion over which end is which.

6.3.1 Resistor Colour Coding

Some resistors use a colour coding system to communicate their ideal values. The most common colour code system uses 4 bars of colour, as illustrated in Fig. 6.1. The first two bars represent the number, without any decimals. So red-green would be 36, purple-blue would be 87, etc. The third digit is a multiplier, so red-green-blue would be $36 \times 10^7\,\Omega$ and purple-blue-yellow would be $87 \times 10^3\,\Omega$.

The last bar represents the precision. Typically, it is either 5% or 10% for most mass produced resistors. These are either silver (10%) or gold (5%) in colour. However, more precise manufacturing techniques have lead to brown (1%) and red (2%) colour bars.

Modern resistors do not always use this colour coding system, and can also be manufactured to be much smaller - so small, in fact, that colour codes couldn't be seen! Never-the-less, it is useful to be aware of the colour coding system. Colour coded resistors are still used frequently for prototyping and on-demand applications.

6.4 Activity & Analysis

6.4.1 Required Equipment

You will need the following equipment to complete this lab project:

- Resistors
- Digital multimeter

6.4.2 Activity Instructions

Obtain your resistor board. All of the resistors are already assembled onto a convenient testing board for you. The testing board has two sets of resistors on them - one set with one nominal resistance value and another set with a different nominal resistance value.

Measure the resistance of one set of resistors using a digital multimeter. Have your TA sign your list of resistance values. You may complete the rest of the analysis at home.

6.4.3 Analysis

Determine the mean and standard deviation of your resistors. Your goal will be to create a pair of histograms of your results. All of your histograms will be centred at your mean value and have a range of $\pm 2\sigma$ from your mean. For more information on histograms, review Appendix A.

The first histogram will have **eight segments** to it. Create a table that has the following columns:

i	R_i (Ω)	R_{i+1} (Ω)	R_{mid} (Ω)	n_i	p_i	$p(R_{mid})\Delta R$

where R_i and R_{i+1} are the edges of your segments, R_{mid} is the midpoint between the two end points, n_i is the number of resistors that have resistance values within the range $R_i < R < R_{i+1}$, and p_i is the fraction of all the resistors that have resistance values in that range (i.e. $p_i = n_i/N$). The last column is the predicted probability value for that segment using the rectangular method discussed in Appendix A. There is no reason that p_i and $p(R_{mid})\Delta R$ should be significantly similar to each other, so if they are different, do not be concerned.

Because your histogram is centred at the mean value, $\bar{R}$, and has eight segments in the range $\pm 2\sigma$, this means that it is easy to know where your bar centres should be. The following should help you set up your calculation for the first segment, and you should be able to extrapolate to fill in the rest.

- The lowest edge of your histogram will be at $\bar{R} - 2\sigma$ and the highest edge will be at $\bar{R} + 2\sigma$. Each segment of your histogram will have a range of $\Delta R = ((\bar{R} + 2\sigma) - (\bar{R} - 2\sigma))/8 = \frac{1}{2}\sigma$. Thus, your first semgnet, for example, should have edges at $\bar{R} - 2\sigma$ and $\bar{R} - 1.5\sigma$, and thus its centre should be located at $R_{mid} = \bar{R} - 1.75\sigma$ - the midway point between each edge. Because you are assuming a Gaussian probability distribution, $p(R_{mid})$ would be calculated as:
  $p(R_{mid}) = \left(\frac{1}{\sqrt{2\pi}}\right)\frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{R_{mid}-\bar{R}}{\sigma}\right)^2}$
  All you need to do is fill in the values of R_{mid} and σ and use a calculator (or a spreadsheet program)!

With your table, use it to create a histogram of your results. Your histogram should include both the p_i values and the $p(R_{mid})\Delta R$ values as separate traces.

The second histogram will have **16 segments** to it but will be based on the exact same data. The goal of this is to illustrate the differences in visualizing 50 data points into 8 segments versus 16 segments. Follow the same procedure as above, only now separate the $\bar{R} - 2\sigma$ to $\bar{R} + 2\sigma$ into 16 segments. This means your table will have 16 rows of data to it, instead of 8. If you use the mathematical tools in your spreadsheet program, this should be a very easy process. However, if you choose not to learn how to use your spreadsheet program very well, then you will find this tedious and laborious.

You should produce the same type of plot for the data in both tables.

The analysis worksheet includes a number of other questions you will need to answer based on your results.

Your worksheet should be completed in a word processing program (such as Microsoft Word, Google Docs, Apple Pages). If you are asked to perform calculations by hand, you may write them on a piece of paper, take a picture and place the image in the document. If you are asked to create a table, you should do so in Microsoft Excel, Google Sheets or Apple Numbers, and include sample calculations for any column that involves a calculation. You may NOT copy and paste any part of this lab manual for any part of your submission - doing so will result in no marks for that component of your submission.

Analysis Worksheet

1. Include a photo of your data that shows your TAs signature.

2. Calculate the mean and standard deviation of your set of resistors.

3. Include your table for your 8-segment histogram.

4. Show a sample calculation of how you calculated $p(R_{mid})\Delta R$, including all of the values that you used, for one of the segments of your histogram.

5. Include your plot of your 8-segment histogram data.

6. Include a table for your 16-segment histogram.

7. Include your plot of your 16-segment histogram.

8. Respond to the following questions/instructions:

    (a) Using either histogram, what fraction of all of your resistors are within $\bar{R} \pm 1\sigma$? within $\bar{R} \pm 2\sigma$? outside of $\bar{R} \pm 2\sigma$?

    (b) Based on your results, if your resistors had a colour coding what should that colour coding be? Justify your answer.

    (c) Which of the two histograms that you made looks most similar to the idealized histogram you were expecting? Justify your answer/explain why this is this way.

Appendix A: Statistics

A.1 Mean and Standard Deviation

Statistics is not incorporated commonly into the theory/lecture portion of physics courses, yet is one of the most important tools in scientific research. At the very least, when measurements are performed, there will always be some amount of statistical variation in the results such that the results will be slightly different if the experiment were performed again. Reducing the amount of this variation is one of the primary challenges of experimental scientists.

As a simple example, consider measuring the time it takes an object to fall down a tube. Even if we assume the timing device is perfectly accurate, there are subtle variations in the atmosphere and other environmental factors that can result in slight differences in the time it takes the object to fall. Subtle differences in the start and stop triggers for the timing device usually make up a larger component of the variation, resulting in a seemingly identically performed measurement but producing a slightly different value. When human judgment is involved, such as a human lining up a ruler, or starting/stopping a timer, or in some way reacting to a situation, the amount of statistical variation always increases.

In this course, we are going to assume that all of the random variation that occurs in an experiment follows a Normal or Gaussian distribution. A Gaussian distribution is readily characterized by a mean value (μ) and a standard deviation (σ), as shown in Figure A.1. The mean value represents the **the expected value of the measurement**, while the standard deviation is a **measure of how precise is the measurement** or how wide is the distribution. A smaller standard deviation is thus a more precise measurement, and a narrower curve. This is illustrated in Figure A.2.

In practice, it is not feasible to perform a measurement enough times to verify that it does follow a Gaussian distribution. Instead, scientists typically perform enough measurements to make an estimation of the mean value and the standard deviation. In this course, $N = 20$ measurements will be considered enough to estimate these values.

The mean value of a measurement is determined as the sum of all of the measurements, divided by the total number of measurements. If each measurement is identified as an x_i with the subscript indicating which measurement is being referred

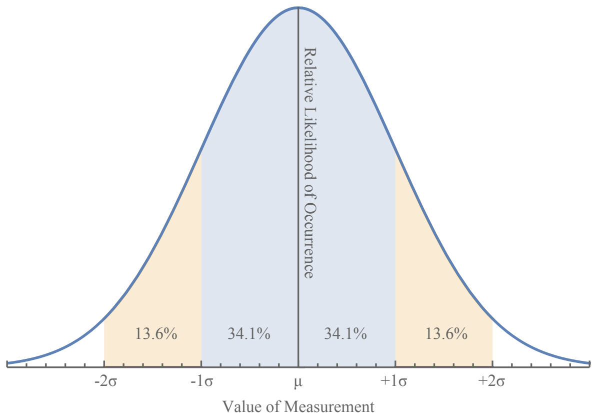


FIGURE A.1: Gaussian or Normal probability distribution. When applied to a measurement in an experiment, the x-axis represents the value of the measurement, while the y-axis represents the relative probability of measuring that value. Approximately 64% of all measurements will fall within one standard deviation ($\pm 1\sigma$) of the mean, while approximately 95% of all measurements will fall within two standard deviations.

to, then the formula would be:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ...}{N} \tag{A.1}$$

$$= \frac{1}{N}\sum_{i=1}^{N} x_i \tag{A.2}$$

The standard deviation is calculated by finding the mean square deviation and then taking the square root:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}} \tag{A.3}$$

$$= \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + ...}{N-1}} \tag{A.4}$$

You may have seen an equation for the standard deviation before that only has an N in the denominator under the square root. This is an approximation that is used for cases where N is really large, where $N - 1 \approx N$. However, the $N - 1$ is important in situations where the number of values used to calculate the mean is small.

Important note: Calculating the standard deviation from a small set of numbers is unreliable, and often under-estimates the value. As a result, if there are fewer than 10 values usable for calculating a standard deviation, then the the standard deviation should be treated as a quarter of the distance between the maximum and minimum values. In the form of an equation: $\sigma_x \approx \frac{x_{max} - x_{min}}{4}$.

Here is an example that you can reproduce if you want to check your understanding of these equations:

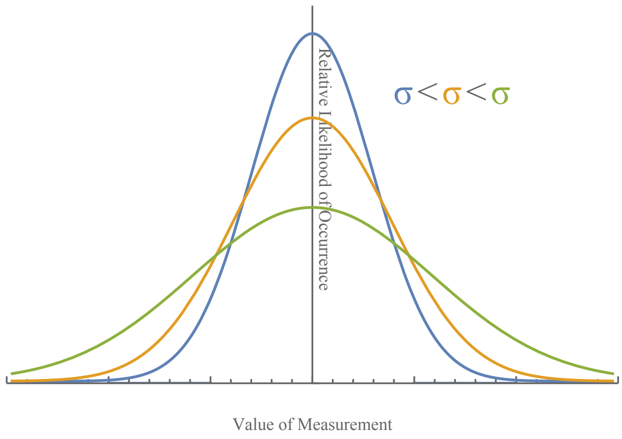


FIGURE A.2: Illustration of Gaussian distributions with different standard deviation (σ) values. The largest standard deviation corresponds to the green curve, which is the broadest of the three curves. In order for 64% of all measurements to fall within one standard deviation, no matter what is the standard deviation value, the curves are naturally shorter for a larger standard deviation.

EXAMPLE:
Calculate the mean and standard deviation of the following numbers:
18.8, 25.4, 19.5, 20.9, 16.3, 16.6, 20.8, 22.4, 16.5, 21.7, 19.1, 23.4, 21.1, 18.4, 22.4, 19.1, 18.8, 15.9, 25.0, 20.4

$$\begin{aligned}
\bar{x} &= (18.8 + 25.4 + 19.5 + 20.9 + 16.3 + 16.6 + 20.8 + 22.4 + 16.5 + 21.7 + 19.1 \\
&\quad +23.4 + 21.1 + 18.4 + 22.4 + 19.1 + 18.8 + 15.9 + 25.0 + 20.4)/20 \\
&= 20.125
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= ((18.8-20.125)^2 + (25.4-20.125)^2 + (19.5-20.125)^2 + (20.9-20.125)^2 \\
&\quad +(16.3-20.125)^2 + (16.6-20.125)^2 + (20.8-20.125)^2 + (22.4-20.125)^2 \\
&\quad +(16.5-20.125)^2 + (21.7-20.125)^2 + (19.1-20.125)^2 + (23.4-20.125)^2 \\
&\quad +(21.1-20.125)^2 + (18.4-20.125)^2 + (22.4-20.125)^2 + (19.1-20.125)^2 \\
&\quad +(18.8-20.125)^2 + (15.9-20.125)^2 + (25.0-20.125)^2 \\
&\quad +(20.4-20.125)^2)/19 \\
&= 7.5925 \\
\sigma_x &= 2.755
\end{aligned}$$

The mean value of this set of numbers is $\bar{x} = 20.125$ and the standard deviation is $\sigma_x = 2.755$.

A.2 Significant Digits, Uncertainty, Standard Deviation of the Mean

In math and science classes, it is common for students to be held to a standard when communicating the results of their calculations. This standard is commonly called **significant digits** or **significant figures**. Students are taught to round their answers to the same precision of the least-precise number in the calculation. For example, when adding $2.467 + 1.23 + 3.1$, the full answer is 6.797, but students are expected to write down 6.8 because the least precise number in the set has two significant digits, so the answer should have two significant digits.

On the surface, this looks like an arbitrary set of rules, or at least rules that are designed around simplifying the answer for easier marking. For example, $\sqrt{2.000} =$ 1.41421356237309505.... For a marker's purposes, 1.414 would be sufficient to indicate that the student calculated the number correctly, so all the extra digits just take up space on the paper.

However, the origin of significant figures is actually in the sciences and has to do with a part of statistics and experimentation known as **Uncertainty** or **Error**. More information about uncertainty and error calculations is in Appendix E. However, to understand that material, it is critical to first understand the **standard deviation of the mean** ($\sigma_{\bar{x}}$).

When performing an experiment, it is impossible to take every possible measurement. For example, there are too many dogs to measure the length or mass of all dogs in order to accurately describe the average length or mass of all dogs. While less obvious, measuring the length of an object can be done multiple times in multiple different ways with multiple different measuring devices – performing all of these measurements is not feasible. Instead, we accept that the best that can be done is to take a *sample* of measurements out of the entire *population* of possible measurements.

The standard deviation value is a measure representing the likelihood of new individual measurements being in proximity to the mean value of our sample. The standard deviation *of the mean* is a measure representing how close we might get to our original value if we were to repeat all of the measurements over again – how likely two subsequent experiments are to produce the same result. The standard deviation of the mean is also referred to as the **uncertainty** or **error** value. The use of the term *error* does not refer to a mistake, but rather the likelihood that the value is not representative of the population.

The standard deviation of the mean can be determined in two ways. The obvious way would be to repeat the experiment many times and then calculate the standard deviation of the mean values from all of the different experiments. This is long and laborious. A simpler way is to *estimate* the standard deviation of the mean by using the standard deviation value from a single experiment using the formula:

$$\delta_x = \sigma_x / \sqrt{N} \approx \sigma_{\bar{x}} \tag{A.5}$$

This estimation equation is not meant to replace performing experiments multiple times – replication of scientific studies is important to the scientific method. Instead, it is used to assess the precision of the mean value that was determined in the experiment.

This might make more sense with an example:

EXAMPLE:
Continuing our example from the previous section, here are the results from collecting multiple sets of random numbers that follow the same distribution.

Set 1: 23.3, 14.9, 24.9, 20.8, 16.3, 14.5, 19., 22.6, 18.2, 20.9, 23.9, 16.9, 18.1, 11.4, 21.9, 23.1, 20.7, 24.2, 16.4, 17.1

Set 2: 18.6, 27.5, 18.5, 15.9, 23.7, 18.4, 19.2, 18.3, 22.8, 19.1, 20.1, 17.3, 25.6, 19.3, 22.1, 20.8, 13.8, 22.0, 17.6, 25.2

Set 3: 17.5, 17.2, 15.7, 24.3, 19.1, 18.8, 19.0, 25.7, 17.0, 18.6, 24.2, 18.0, 18.1, 20.2, 19.5, 22.4, 25.6, 18.6, 17.3, 24.4

Set 4: 16.6, 18.8, 18.0, 19.9, 22.4, 18.2, 21.0, 24.6, 20.1, 22.9, 19.2, 23.2, 21.6, 19.9, 17.1, 19.2, 16.3, 15.0, 20.4, 18.0

Set 5: 19.2, 20.7, 20.9, 16.4, 15.9, 22.7, 20.2, 20.3, 17.6, 20.0, 22.5, 20.7, 21.8, 19.0, 17.8, 19.0, 21.5, 17.2, 16.6, 18.2

Set 6: 20.7, 17.8, 21.5, 22.5, 18.7, 21.3, 12.4, 21.9, 21.6, 23.8, 15.0, 21.1, 20.2, 21.1, 22.1, 19.7, 14.3, 20.2, 22.0, 16.9

Set 7: 21.2, 20.0, 19.5, 23.4, 17.3, 20.7, 19.7, 21.9, 24.4, 23.6, 20.8, 21.6, 14.8, 21.8, 20.3, 22.9, 17.6, 21.6, 25.6, 18.1

Set 8: 18.8, 21.1, 21.3, 26.1, 23.7, 14.6, 20.7, 14.0, 21.4, 26.4, 17.6, 19.9, 18.1, 18.8, 15.8, 12.6, 19.0, 24.5, 20.9, 20.4

Set 9: 18.6, 23.6, 20.9, 22.8, 19.5, 21.8, 23.7, 23.3, 17.7, 13.7, 15.5, 22.5, 22.9, 23.5, 20.4, 14.1, 16.1, 18.7, 18.8, 22.6

Set 10: 16.0, 21.5, 13.2, 18.1, 16.3, 25.9, 22.4, 19.1, 26.0, 23.0, 17.0, 24.5, 22.0, 18.5, 18.7, 15.0, 18.1, 19.0, 19.3, 17.8

The mean values for each of these sets is: 19.945, 20.485, 18.960, 18.500, 21.450, 20.475, 19.145, 20.030, 20.140, 19.045, respectively. The average value of this set of mean values is 19.8775.

The standard deviation of these mean values is:

$$\begin{aligned}\sigma_{\bar{x}} &= \big((19.945-19.8775)^2+(20.485-19.8775)^2+(18.960-19.8775)^2 \\ &\quad+(18.500-19.8775)^2+(21.450-19.8775)^2+(20.475-19.8775)^2+(19.145-19.8775)^2 \\ &\quad+(20.030-19.8775)^2+(20.140-19.8775)^2+(19.045-19.8775)^2\big)\,/9 \\ &= 0.68767 \approx 0.69\end{aligned}$$

If we use our estimation formula on the dataset from the previous example, we get:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \frac{2.75545}{\sqrt{20}} = 0.61614 \approx 0.62$$

Note: This technique only works because all of the sets of numbers were drawn from the same population.

As you can see in the example, the two methods to determine the standard deviation of the mean produce very similar values. Remember that both of these are *estimations* of the theorized *true* value, so neither one of them should be treated as the correct value to compare against. Since both calculations return similar results, it is reasonable that the simpler one can be used for practical purposes.

Significant figures arise when communicating a scientifically determined quantity. In the above example, the standard deviation of the mean was 0.61614 – but remember that this was simply an estimate. The first significant digit was the same between our long method and our short method, so the first digit is the only digit that is important. Thus, we truncate our standard deviation of the mean to the **first significant digit**. In some cases, for advanced and highly precise science, two significant digits of the standard deviation of the mean can be used. We will only use one in this class.

With this value truncated, we then report the mean value rounded to the same precision as the standard deviation of the mean/uncertainty/error (all three terms refer to the same quantity). If $\sigma_{\bar{x}} = 0.6$, then we need to round the mean value (20.125) to one decimal place: $\bar{x} = 20.1$. The last thing we do is report the value using the following format:

$$\bar{x} = 20.1 \pm 0.6$$

This makes the statement that the mean value was determined to be 20.1, and it is believed that there is a $\sim 64\%$ chance that the true mean value of the population is within the range $20.1 - 0.6$ and $20.1 + 0.6$.

Thus, significant figures tell us about how precisely we know the value of an object. Including more significant figures in the value than the leading digit(s) of the uncertainty would make a claim about knowledge that is not supportable by evidence. In math and physics classes, the use of significant figures is meant to prepare students for their application in engineering and the sciences. While it is true that we can know the value of radicals and fractions to infinite precision, the use of that precision is irrelevant in practice – typically only the precision of the uncertainty matters.

EXAMPLE:
Given the following information on mean values, standard deviations and sample sizes, write the mean values and the errors in the correct format.

a) $\bar{x} = 0.4512858123\text{m}$, $\sigma_x = 0.0155123\text{m}$, $N = 10000$

b) $\bar{t} = 5.6235123 \times 10^5\text{s}$, $\sigma_t = 45.9725\text{s}$, $N = 50$

c) $\bar{m} = 1.2459001 \times 10^{-2}\text{g}$, $\sigma_m = 4.578 \times 10^{-4}\text{g}$, $N = 1000$

Solutions:

a) $\sigma_{\bar{x}} = 0.0155123/\sqrt{10000}$ m $= 0.000155123$ m ≈ 0.0002 m

$$\bar{x} = (0.4513 \pm 0.0002) \text{ m}$$

b) $\sigma_{\bar{t}} = 45.9725/\sqrt{50}$ s $= 6.50149$ s ≈ 7 s

$$\bar{t} = (5.62351 \pm 0.00007) \times 10^5 \text{ s}$$

c) $\sigma_{\bar{m}} = 4.578 \times 10^{-4}/\sqrt{1000}$ g $= 0.0000144769$ g $\approx 1 \times 10^{-5}$g

$$\bar{m} = (1.246 \pm 0.001) \times 10^{-2} \text{ g}$$

A.3 Histograms and Distributions

A.3.1 Probability Distribution Functions

The normal curve, bell curve, or Gaussian curve – all terms for the same thing – is an example of a probability distribution function. A probability distribution function is a function of a variable that has a total area of 1, and relates the likelihood of a value occurring in a measurement. The Gaussian curve is commonly discussed because it is very common in physics for a very good reason – the mean value of a sum of random numbers that follows any probability distribution is going to be Gaussian. This is known as the Central Limit Theorem, but the derivation and explanation of this will be left to future courses. What it means, however, is that the Gaussian distribution dominates wherever repeated measurements are taken, and repeated measurements is a central feature of the scientific method.

To understand probability distribution functions, it will be useful to use the example of the Gaussian distribution. Consider the formula for the Gaussian distribution:

$$p(x) = \left(\frac{1}{\sqrt{2\pi}}\right)\frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} \tag{A.6}$$

where x is the value of the variable, $\bar{x}$ is the mean value of the Gaussian curve, and σ is the standard deviation.

A common misconception of probability distribution functions like this is that $p(x)$ is the probability of finding x at that value. Because x is a continuous quantity, that would be silly – $p(0.5)$ and $p(0.5000001)$ would have very similar values, but it seems silly to think that the probability of finding $x = 0.5$ and the probability of finding $x = 0.5000001$ could be similarly large for $\bar{x} = 0.5$ and $\sigma = 0.2$ (or any other such mean and standard deviations).

Above, it was stated that a probability distribution function "relates the likelihood of a value occurring", but does not say the function gives the probability. A probability function, properly used, requires integration to communicate probabilities. For example, the probability of finding a value of x in the range of $a < x < b$ would be:

$$P(a < x < b) = \int_a^b p(x)dx \tag{A.7}$$

where $P(a < x < b)$ uses a capital P to refer to probability, rather than a probability distribution.

One of the most common ways of integrating something numerically (that is, using an algorithm rather than calculus) is by approximating a curve with rectangles, as shown in Fig. A.3. To integrate a curve using the rectangle method, the function value in the centre of each rectangle must be known, as well as the width of the rectangle. The integral then is approximated as a sum:

$$\int_a^b p(x)dx \approx \sum_{i=1}^{N} p(x_i)\Delta x \tag{A.8}$$

where Δx is the width of the rectangle and is equal to $\Delta x = (b-a)/N$.

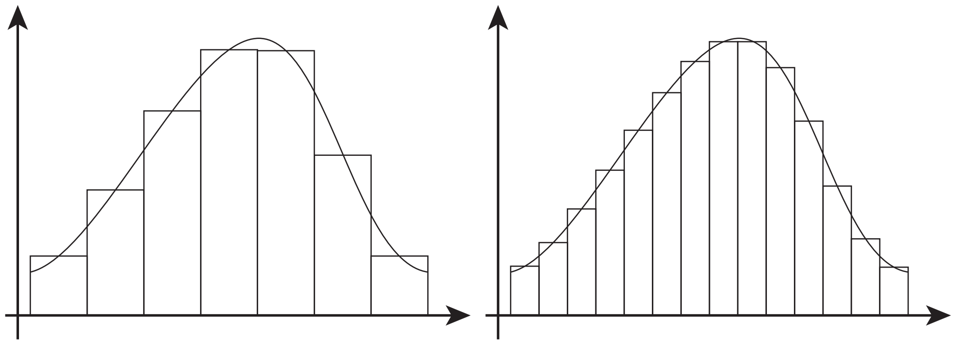


FIGURE A.3: Rectangular integration shown for the same curve but two different widths of rectangles.

There are easier ways of doing this, however, but to understand them, we must first discuss **histograms**.

A.3.2 Histograms

A histogram is like a bar chart/graph in that it discretizes the x-axis, and creates a series of step functions along the y-axis. An example of a histogram is shown in Fig. A.4. For each step in the graph, the horizontal line represents the value for the entire span of that step along the x-axis. In Fig. A.4, the first step occurs between x_1 and x_2. The value is assumed to be the same for that entire region, so that $f(x_1 < x < x_2)$ is a constant.

Histograms are commonly used to represent probabilities for discrete or discretized data. For example, consider rolling a 6-sided die 100 times. You might use a histogram to graph the number of times each number occurred, but this is using a histogram exactly like a bar chart. A more appropriate use of the histogram would be to represent the fraction of times each number occurred. If the number 1 occurred 15 times, the number 2 occurred 22 times, the number 3 occurred 11 times, the number 4 occurred 18 times, the number 5 occurred 14 times, and the number 6 occurred 20 times, then the fraction would be 1: 0.15, 2: 0.22, 3: 0.11, 4: 0.18, 5: 0.14, 6: 0.20. The sum of each the values equals to 1, which represents the probability of rolling a number in the range of 1-6 using a 6-sided die. That seems like a trivial statement for this example, but these kinds of statements get more complicated when studying more advanced statistics.

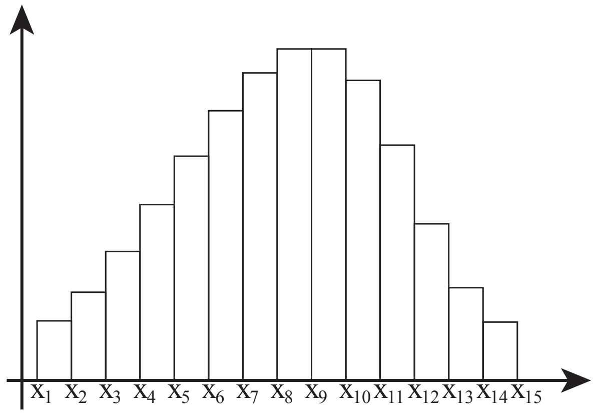


FIGURE A.4: A histogram shows a series of values that are joined by step-function-like jumps, rather than a smooth, continuous curve. The vertical lines are not necessary, and are often not included.

The 6-sided die example was obvious. But a histogram can also be used for data that is not so obviously discrete. Consider as an example a case of measuring the distances that airplanes take to stop when landing. Obviously the distance is a continuous quantity, and with precise enough measurements you might be able to determine the distance down to, say, a micrometre (μm). But equipment is rarely that precise, and maybe the exact landing point that the measurement starts from is difficult to determine, or perhaps you really only care about approximate distances. In such cases, you might round to the nearest 10 m. You could create a series of categories: 820 m, 830 m, 840 m, 850 m, 860 m, etc... And then instead of recording your values to high precision, you simply take your measurement or 833.5 m and say "This is greater than 825 m and smaller than 835 m, so I will assign it to the 830 m category." When done, the histogram would be created by graphing steps from 815 m to 825 m, 825 m to 835 m, 835 m to 845 m, etc... and then graphing the value in the corresponding category divided by the total number of measurements.

A.3.3 Comparing Histograms and Probability Distributions

A probability distribution can be turned into a histogram with relative ease. Note that a histogram of a probability distribution function is only an approximation. Recall that with a histogram, the sum of all the values should add to 1. For a probability distribution function, the integral equals to 1. If we use the rectangle method for numerical integration, then we are converting the integral into a sum. Recall:

$$\int_a^b p(x)dx \approx p(x_1)\Delta x + p(x_2)\Delta x + p(x_3)\Delta x + ... \tag{A.9}$$

If instead of graphing $p(x)$, you graph $p(x)\Delta x$, where Δx is the spacing between points on the graph, then you are effectively creating a histogram-like plot where the sum of all of the values (rather than the integral) is approximately equal to 1.

EXAMPLE:

A Gaussian probability distribution with a mean of $\bar{x} = 22$ and standard deviation of $\sigma = 2$ is represented by the equation:

$$p(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} \frac{1}{2} e^{-\frac{1}{2}\left(\frac{x-22}{2}\right)^2} \tag{A.10}$$

Plot a histogram of this and use that to estimate the integral of the Gaussian distribution in the range of $\pm 3\sigma$ about the mean.

The actual integral on this range is independent of the values of $\bar{x}$ and σ:

$$\int_{\bar{x}-3\sigma}^{\bar{x}+3\sigma} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx \approx 0.9973$$

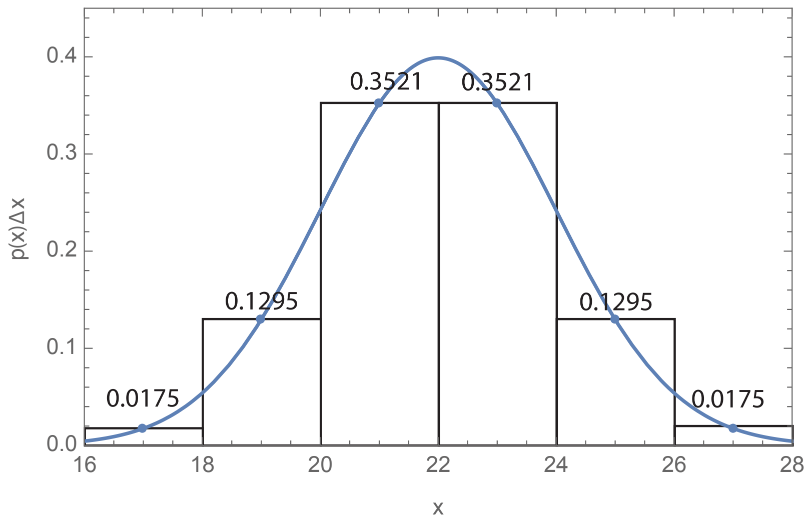
Integrating a Gaussian is an advanced technique and this solution is provided just to illustrate the idealized value. The reason that this integral is not equal to 1 is because the Gaussian extends from $x = -\infty$ to $x = +\infty$. Thus, the portion of the integral outside of the $\bar{x} \pm 3\sigma$ range only amounts to approximately 0.0027. Put statistically, this means that a random number that follows a Gaussian distribution has a 0.27% chance of occurring outside of the 3σ range about the mean.

First, consider performing this integration with a spacing of $\Delta x = 1\sigma$. This is a large spacing, and isn't going to give the best results, but it is illustrative to show. The lowest end on the histogram will be at $x_1 = \bar{x} - 3\sigma$. The start of the next section will be $x_2 = x_1 + \Delta x$ because we are using a 1σ spacing for this solution. Continuing like this, the upper end of the histogram is at $x_7 = \bar{x} + 3\sigma$.

Creating a table for each segment of the histogram would look like this:

i	x_i	x_{i+1}	x_{mid}	$f(x_{mid})$	$f(x_{mid})\Delta x$
1	16	18	17	0.00876	0.0175
2	18	20	19	0.06476	0.1295
3	20	22	21	0.17603	0.3521
4	22	24	23	0.17603	0.3521
5	24	26	25	0.06476	0.1295
6	26	28	27	0.00876	0.0175

The histogram looks like this:

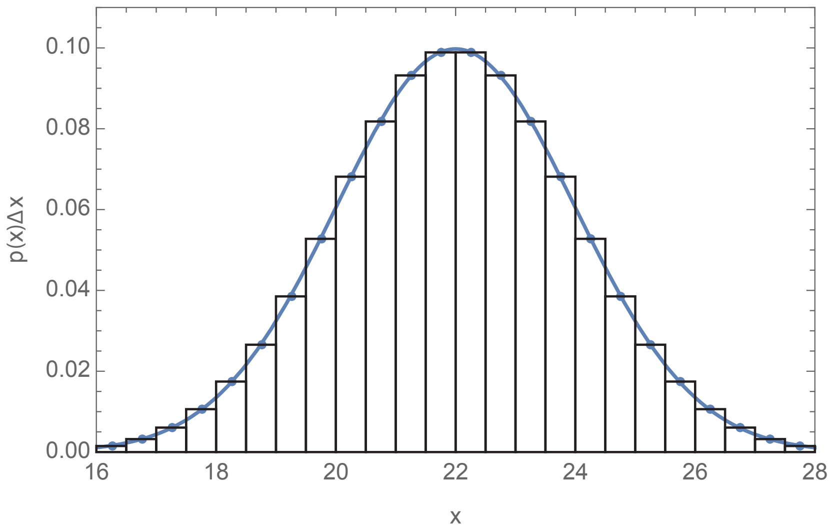


And the sum of the six segments of the histogram is 0.9982. This is surprisingly close to the actual value of the integral, despite having such large spacing.

Now, consider doing the integration with a spacing of 0.25σ. There will be four times as many segments to the histogram, but the start and end point remain the same. The table for the histogram would look like:

i	x_i	x_{i+1}	x_{mid}	$f(x_{mid})$	$f(x_{mid})\Delta x$
1	16.0	16.5	16.25	0.00320	0.00320
2	16.5	17.0	16.75	0.00636	0.00318
3	17.0	17.5	17.25	0.01189	0.00594
4	17.5	18.0	17.75	0.02086	0.01043
5	18.0	18.5	18.25	0.03439	0.01720
6	18.5	19.0	18.75	0.05327	0.02663
7	19.0	19.5	19.25	0.07751	0.03875
8	19.5	20.0	19.75	0.10594	0.05297
9	20.0	20.5	20.25	0.13603	0.06801
10	20.5	21.0	20.75	0.16408	0.08204
11	21.0	21.5	21.25	0.18593	0.09296
12	21.5	22.0	21.75	0.19792	0.09896
13	22.0	22.5	22.25	0.19792	0.09896
14	22.5	23.0	22.75	0.18593	0.09296
15	23.0	23.5	23.25	0.16408	0.08204
16	23.5	24.0	23.75	0.13603	0.06801
17	24.0	24.5	24.25	0.10594	0.05297
18	24.5	25.0	24.75	0.07751	0.03875
19	25.0	25.5	25.25	0.05327	0.02663
20	25.5	26.0	25.75	0.03439	0.01720
21	26.0	26.5	26.25	0.02086	0.01043
22	26.5	27.0	26.75	0.01189	0.00594
23	27.0	27.5	27.25	0.00636	0.00318
24	27.5	28.0	27.75	0.00320	0.00160

The histogram looks like this:



And the sum of the 24 segments of the histogram is 0.99737. This is clearly much closer to the actual value of the integral.

Appendix B: Using Spreadsheets

Spreadsheet programs are fairly ubiquitous software – the major ones are Microsoft Excel, Apple Numbers and Google Sheets. All of them have the same basic functionality, with some slight differences between them that are either cosmetic or require significantly advanced knowledge to use. For this reason, only Google Sheets will be described in this appendix – Google Sheets is free and available on all platforms, and so is the most widely usable one by students. However, there will be no consequences for choosing to use a different one.

The view of a spreadsheet looks like Figure B.1. It is effectively a table with columns labeled with letters and the rows labeled with numbers. A particular cell within the spreadsheet is thus identified as a combination of a letter with a number. For example, the upper left most cell is referenced as $A1$. Cells can have text within them, numbers, or they can perform mathematical operations based on the values in other cells.

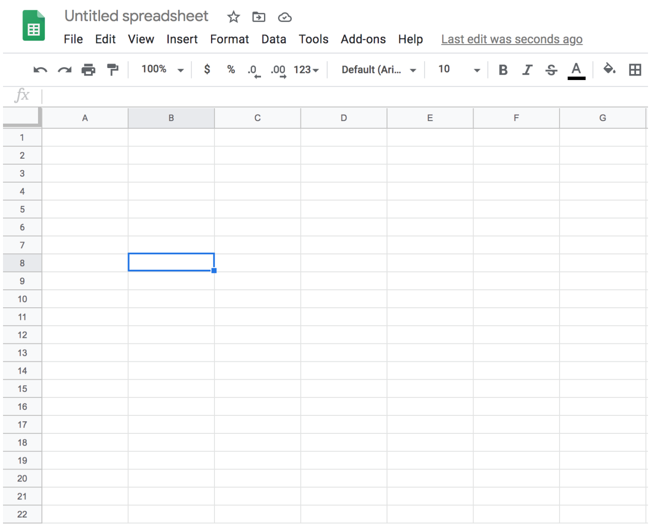
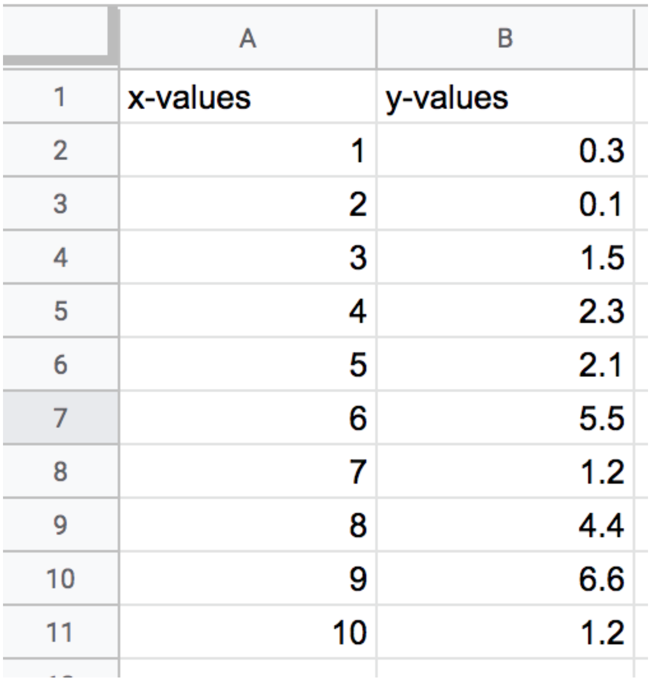


FIGURE B.1: Image of a spreadsheet showing the first few rows and columns. The cell $B8$ is selected in this case.

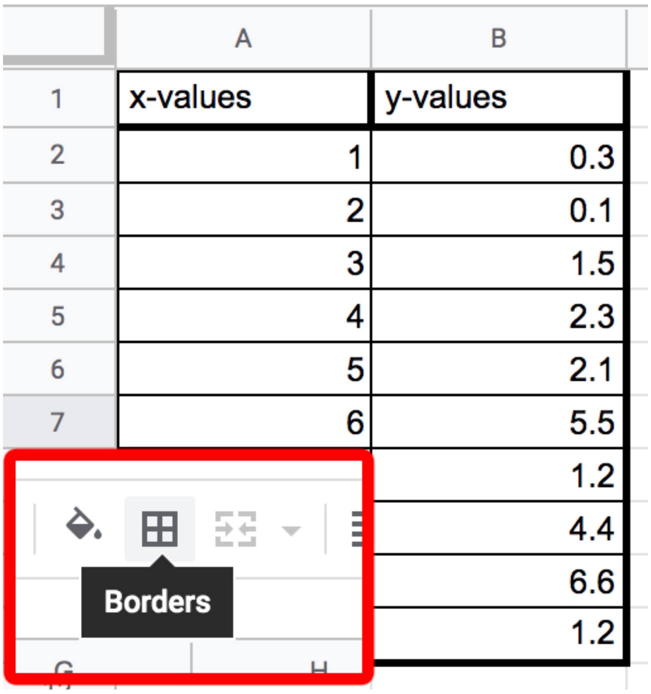
B.1 Creating Tables

Spreadsheets can be used to easily create tables of data that can be copied and pasted into a document. Figure B.2 shows a simple array of values with column headings. By pressing the Borders button, shown in the bottom left of Figure B.3, it is possible to add structural lines around cells to help with organization. Typically, these properties will copy over to any document for displaying data.



	A	B
1	x-values	y-values
2	1	0.3
3	2	0.1
4	3	1.5
5	4	2.3
6	5	2.1
7	6	5.5
8	7	1.2
9	8	4.4
10	9	6.6
11	10	1.2

FIGURE B.2: A simple spreadsheet showing multiple columns of data with multiple rows. Column headings are included at the top of each column.



	A	B
1	x-values	y-values
2	1	0.3
3	2	0.1
4	3	1.5
5	4	2.3
6	5	2.1
7	6	5.5
		1.2
		4.4
		6.6
		1.2

FIGURE B.3: The same simple spreadsheet, but now with thick lines around it and around the column headings, with thin lines separating the data cells. This is just one way to improve visibility of the data. There is no fixed style that must be used. However, some kind of organizational style is helpful to improve readability.

B.2 Formula Basics

A cell can contain text/numbers, or it can contain the results of formulas. To use a formula, the first character in the cell must be an equal sign, =. Common mathematical functions include:

Name	Symbol	Example	Approx. Result
Addition	$+$	$= 3.2 + 2.3$	5.5
Subtraction	$-$	$= 3.2 - 2.3$	0.9
Multiplication	$*$	$= 3.2 * 2.3$	7.36
Division	$/$	$= 3.2/2.3$	1.391
Exponents	ˆ	$= 3.2\hat{}2.3$	14.52
Square Root	sqrt()	$= \text{sqrt}(4)$	2
Powers of Euler's Number	exp()	$= \exp(3.2)$	24.53
Log Base 10	log()	$= \log(3.2)$	0.505
Natural Log	ln()	$= \ln(3.2)$	1.16
Cosine (radians)	cos()	$= \cos(3.2)$	-0.998
Sine (radians)	sin()	$= \sin(3.2)$	-0.058
Tangent (radians)	tan()	$= \tan(3.2)$	0.058
Arccos (radians)	acos()	$= \text{acos}(0.3)$	1.266
Arcsin (radians)	asin()	$= \text{asin}(0.3)$	0.305
Arctan (radians)	atan()	$= \text{atan}(0.3)$	0.291
Round	round()	$= \text{round}(0.39, 1)$	0.4

The strength of using a spreadsheet is not in performing manually entered formulas, but in performing mathematical calculations based on the values entered into specific cells. All of the above formulas can be combined with data entered in cells. For example, if cell $A1$ has contents 3.2 and cell $B1$ has contents 2.3, then the addition example would be just as easily accomplished with $= A1 + B1$.

Some functions require ranges of cells as input. A range of cells is accomplished with a semi-colon between the upper left cell and the lower right cell. For example $A1 : B4$ would be a selection of all cells between $A1$ and $B4$ – a total of 8 cells. Two common functions that you will find useful for your lab reports, summation and averaging, are shown in Figure B.4.

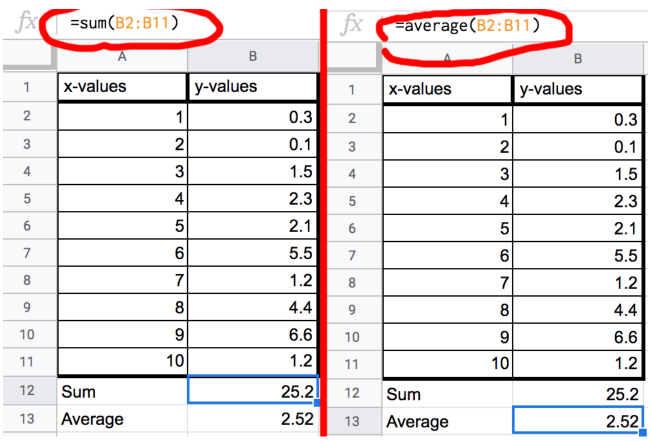
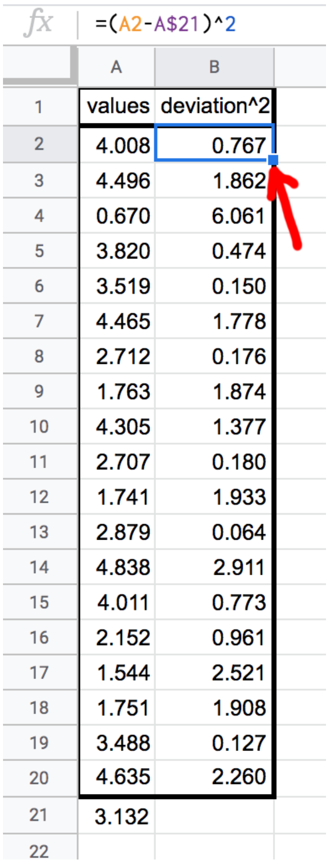


FIGURE B.4: Summation and addition are very commonly used in spreadsheets. These will be helpful when calculating averages and standard deviations.

EXAMPLE:
Use a spreadsheet to calculate the Standard Deviation of the following series of values:
4.008, 4.496, 0.670, 3.820, 3.519, 4.465, 2.712, 1.763, 4.305, 2.707, 1.741, 2.879, 4.838, 4.011, 2.152, 1.544, 1.751, 3.488, 4.635

To do this, we need to create a column of these values. The top-most cell in the column, $A1$, will be the label for the data. Then all of the values will be entered beneath that label heading in cells $A2 : A20$. Below the list of values, calculate the average for that column by entering the formula $= Average(A2 : A20)$.

The next thing to do is create a second column beside the one with our values, to contain the square of the deviation ($(x - \bar{x})^2$). This is accomplished by entering the formula $= (A2 - A21)\hat{}2$ into cell $B2$. This needs to be repeated for cell $B3$ through $B20$. There is an easier way to do this, however, by autocompleting the entering of values in the cells below $B2$. Change the formula to be $= (A2 - A\$21)\hat{}2$, where the \$ will lock the location of cell $A21$ into the formula, but allow other cells to change. Then click and drag the blue square in the bottom right corner of cell $B2$ down to the bottom. This is shown in Figure B.5.


fx =(A2-A$21)^2

	A	B
1	values	deviation^2
2	4.008	0.767
3	4.496	1.862
4	0.670	6.061
5	3.820	0.474
6	3.519	0.150
7	4.465	1.778
8	2.712	0.176
9	1.763	1.874
10	4.305	1.377
11	2.707	0.180
12	1.741	1.933
13	2.879	0.064
14	4.838	2.911
15	4.011	0.773
16	2.152	0.961
17	1.544	2.521
18	1.751	1.908
19	3.488	0.127
20	4.635	2.260
21	3.132	
22		

FIGURE B.5: The values are entered into column A, and then the deviation squared is calculated in column B. The red arrow shows where the blue box is that can be used to auto-fill the cells beneath.

The formula for the standard deviation is:

$$\sigma_x = \sqrt{\frac{(\sum(x-\bar{x})^2}{N-1}}$$

Column B contains the individual square deviations. Thus, we want to add up all of the values in column B, divide by $N-1$, and then take the square root. This can be accomplished by entering the formula $= sqrt(sum(B2 : B20)/18)$ into cell $B21$. This is shown in Figure B.6.

fx =sqrt(sum(B2:B20)/18)

	A	B	C
3	4.496	1.862	
4	0.670	6.061	
5	3.820	0.474	
6	3.519	0.150	
7	4.465	1.778	
8	2.712	0.176	
9	1.763	1.874	
10	4.305	1.377	
11	2.707	0.180	
12	1.741	1.933	
13	2.879	0.064	
14	4.838	2.911	
15	4.011	0.773	
16	2.152	0.961	
17	1.544	2.521	
18	1.751	1.908	
19	3.488	0.127	
20	4.635	2.260	
21	3.132	1.251	
22			

FIGURE B.6: The values are entered into column A, and then the deviation squared is calculated in column B. The red arrow shows where the blue box is that can be used to auto-fill the cells beneath.

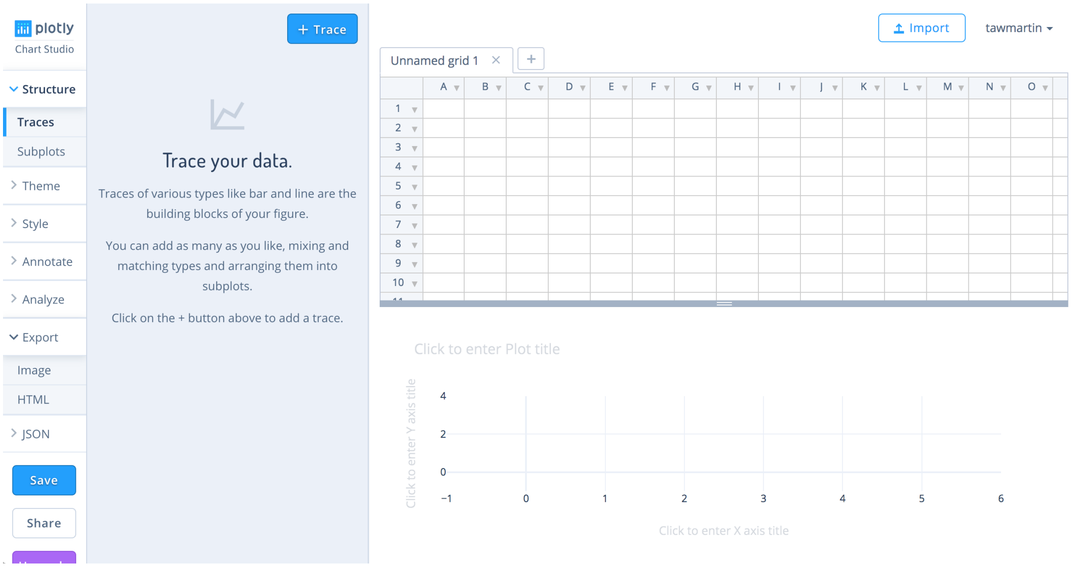
B.3 Plotting

While Microsoft Excel will do plotting well enough for the purposes of these labs, Google Sheets will not. As a result, it is recommended that you use Plotly. It is free to use by visiting https://chart-studio.plotly.com/create/#/. It also has some spreadsheet capabilities, however it does not perform calculations. Thus, the calculations will need to be done in Excel, Numbers or Sheets, and then the plotting can be done in Plotly.

Plotting is much more easily explained by example. Thus, all explanations below will be based on the following data, that is assumed to be entered into a spreadsheet like Google Sheets:

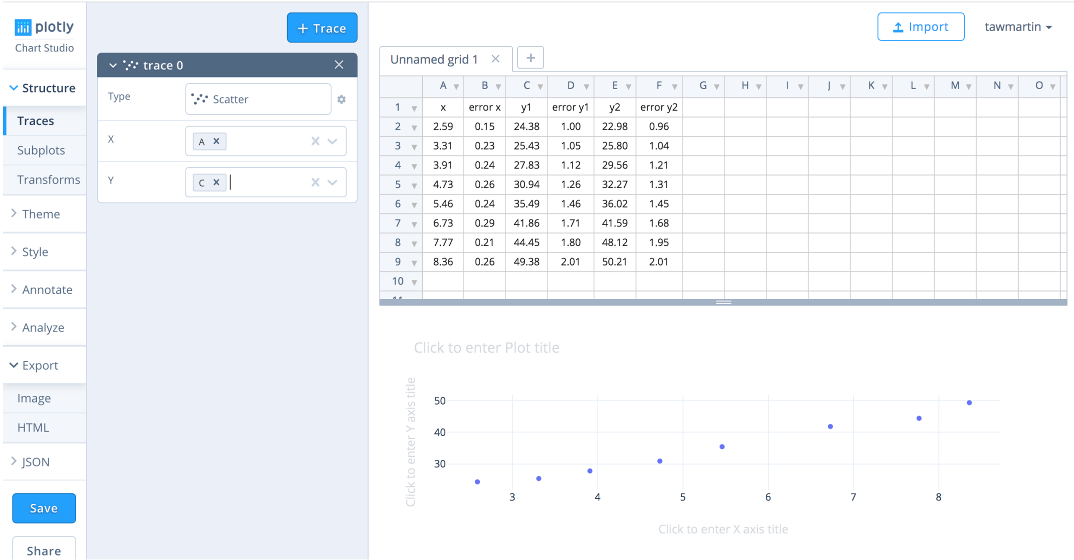
	A	B	C	D	E	F
1	x	error x	y1	error y1	y2	error y2
2	2.59	0.15	24.38	1.00	22.98	0.96
3	3.31	0.23	25.43	1.05	25.80	1.04
4	3.91	0.24	27.83	1.12	29.56	1.21
5	4.73	0.26	30.94	1.26	32.27	1.31
6	5.46	0.24	35.49	1.46	36.02	1.45
7	6.73	0.29	41.86	1.71	41.59	1.68
8	7.77	0.21	44.45	1.80	48.12	1.95
9	8.36	0.26	49.38	2.01	50.21	2.01

When you first open Plotly, you will see a screen that looks similar to:



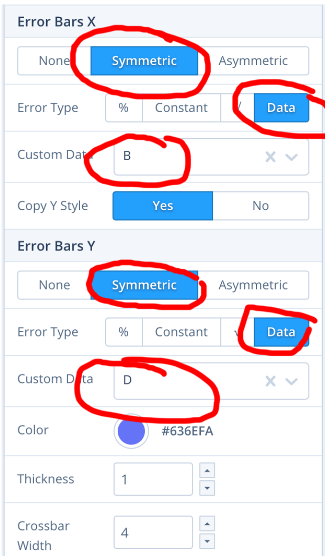
Copy your data from your spreadsheet and paste it into the spreadsheet area of Plotly.

To create a graph, click on the *Structure* button on the left, and then *Traces*, then on the blue *+Trace* button. For the x values, select column A, and for the y values select column C. The result should look something like this:



Repeat this same procedure but for column *A* and column *E* to create a second graph.

To add error bars, click on the *Style* button on the left, and then *Traces*, which appears underneath it. Scroll down in the middle menu until you find the *Error Bars X* and *Error Bars Y* entries. Choose *Symmetric*, then *Data*, and then enter the column of your x error bars for the *Error Bars X* entry. Repeat for *Error Bars Y*. Continue scrolling to find the same entries for your other trace(s). This should look like:



The last few things to finish up the plot will be to add labels to the x and y axes, as well as labels to the datasets in the legends.

Also, it is possible to use Plotly to automatically determine the line-of-best-fit and include it on the plot. To do so, choose the *Analyze* and then *Curve Fitting* option, and click on the *Run* button for the dataset that you want to produce a line for. Note that Plotly will give a value for the slope and the intercept, however you will be expected to calculate these yourself and show your calculation in your report. You can use Plotly's values to check if you did it correctly, though. Plotly does not give estimates of the uncertainty in the line parameters (slope and intercept), which you will be expected to include in your reports.

A completed plot might look like Figure B.7.

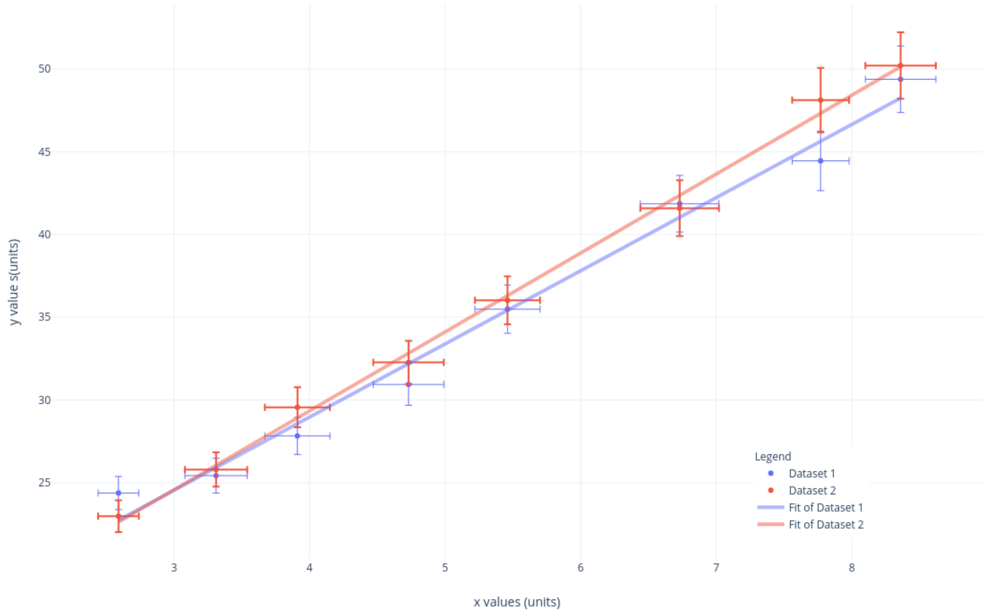


FIGURE B.7: Example of a completed plot produced using Plotly. Plot includes two different datasets, best fit lines, a legend, and proper labeling of the axes. The equation for the line of best fit can be included on the graph or in the caption.

Appendix C: Hypothesis Testing

A central feature of the scientific method is that hypotheses and theories are never proven. This is contrary to much of what you might read in the media, where words like *prove* and *cause* are used frequently to describe the results of a scientific experiment. The use of these terms misrepresent the goal of science altogether – the scientific method is used to find predictive patterns among natural phenomena, and that is all.

Consider the advancement of astronomy over the last several hundred years. For thousands of years, our understanding of the "heavens" was dominated by geocentric theory – where Earth is the centre of the universe – and the motions of celestial objects was predicted by the Ptolomaic model. The predictions of the Ptolomaic model were accurate over short periods of time, to the precision of the instruments that were available during that era, but advancements in technology changed human interpretation of the observed data of the stars. After geocentric theory came heliocentric theory, dominated by the contributions of Kepler and Galileo. Heliocentric theory posited the Sun at the centre of the universe, and allowed for orbits to be elliptical rather than just circular (as in geocentric theory). When Newton's Laws were developed, it displaced heliocentric theory, providing an explanation that the centre of orbits was the centre-of-mass of the system of orbiting objects, and no longer required that the Sun was the centre of the universe. Later observations of the movement of the stars and Mercury showed that Newton's Laws were also insufficient/inaccurate at predicting the future positions of objects, and the Theory of Relativity became dominant as our understanding of the motion of objects.

In each case, an existing theory was disproven, and a new theory was adopted. The reason the new theory was adopted was because the predictions of the new theory more accurately matched the observations. In turn, each new theory was disproven by later, more precise observations.

Scientific knowledge, when developed correctly, is the body of knowledge that makes useful predictions about future phenomena, and has not yet been disproven. For many, this understanding is disconcerting and they misuse the term *theory* to mean something other than how it is used in science. To be clear: a hypothesis is an idea about nature that has not yet been tested, or has been insufficiently tested – this is the word that should be used by people who deny scientific knowledge. A theory is an idea about nature that is worded in such a way as to produce predictions (hypotheses) about nature that, when tested, would conclusively disprove the idea if the predictions did not match reality, and that has been tested multiple times and in multiple different ways in attempts to disprove it.

Developing a theory is very difficult, and well beyond the activities of undergraduate students. However, testing a hypothesis – the first step in developing a theory, and the most important step – is a useful activity in which to participate. Each person should be empowered to test hypotheses, rather than simply trust the words of others. This appendix will provide an introduction into performing such tests.

Important Note: Not every experiment focuses on hypothesis testing. Scientific endeavours are often broken down into smaller subsections. Sometimes experiments are performed to measure an experimental or phenomenological constant, for example, where there is no known value. In these studies, typically the first study testing a hypothesis, there is nothing to compare against, and so hypothesis testing is not necessarily an intrinsic activity done. However, measures of precision are still critical in these studies, even if accuracy is not.

C.1 Accuracy and Precision

Intrinsically related to the concept of hypothesis testing is the idea of accuracy and precision. These terms are often used interchangeably in society, but they should not be – they have very different scientific meanings:
Precision – a measure of the reproducibility of a result of a measurement or experiment. **Accuracy** – a measure of how close a measured value is to another version of that value (measured or theoretical).

C.1.1 Precision

In practice, the term precision describes the uncertainty of a value. For example, $x = 42.3 \pm 0.3$cm has an uncertainty of $\delta_x = 0.3$cm. This is a description of the precision of the value of x. Sometimes, the precision is quoted as a percentage. In the previous example, the percent precision would be $0.3/42.3 = 0.7\%$, and the number could just as correctly be quoted as $x = 42.3\,\mathrm{cm} \pm 0.7\%$.

However, using percentage quotes of precision can give the wrong impression, and so the two methods are not completely interchangeable. This is best explained by example: if a metre-stick has a measurement uncertainty of ± 0.1cm, then measuring a distance of 10cm versus 80cm would have the exact same measurement uncertainty, and yet a different percentage uncertainty. This gives the false impression that the distance of 80cm is more accurately measured, when it is, in fact, measured to the same precision.

To that end, measurement uncertainties should never be quoted as a percentage. Percentage uncertainties should only be used for values that are calculated from multiple different sources, and error propagation has been used to determine the overall uncertainty on the value. In such cases, either method of communicating precision is valid. Alternatively, percentage values can be used for statistical uncertainties. In all cases, using the direct value (the non-percentage one) is always valid.

C.1.2 Accuracy

In practice, the term accuracy describes how similar two distinct, independently measured/calculated values are to each other. For example, if the gravitational constant, G, is measured in two completely distinct experiments, accuracy would be a descriptor of how similar the two values are. Thus, accuracy takes into account both the difference in the two values, and the uncertainty.

For two separate values, $x_1 \pm \delta_{x_1}$ and $x_2 \pm \delta_{x_2}$, that follow a Gaussian probability distribution, a numerical measure of the accuracy would be:

$$t = \frac{x_1 - x_2}{\sqrt{\delta_{x_1}^2 + \delta_{x_2}^2}} \tag{C.1}$$

A larger value of t indicates a worse accuracy – improving accuracy means decreasing the value of t. This can be done in two ways: a smaller difference between x_1 and x_2 will produce a smaller t, but larger uncertainty values can also result in a smaller value of t. Obviously, the first one is more ideal. However, this tells an important truth about science – the larger the uncertainty (the lower the precision), the more likely two things are to agree but the less useful is the result.

In the case that one of the two values has a very high precision, the denominator of the t value will be dominated by the larger of the two uncertainties. In such cases, it is reasonable to approximate the t value as:

$$t \approx \frac{x_1 - x_2}{\delta_{x_1}} \tag{C.2}$$

assuming $\delta_{x_1} \gg \delta_{x_2}$. This is particularly used when calibrating, or when verifying an experimental apparatus/technique, as these are cases when the experimental value is viewed as less reliable than an accepted or target value.

A graphical illustration of the differences between precision and accuracy are shown in Figure C.1.

C.2 Disproving Hypotheses - Statistical Testing

In science, when testing a new hypothesis, there must always be a *null hypothesis* as an alternative. The *null hypothesis* is typically simply the statement that the proposed idea does not match nature. Null hypotheses is more complicated than what is necessary for this course. Instead, we will simply focus on basic hypothesis testing.

Note: There are a wide variety of statistical tests that are used for the multitude of difference types of scenarios that can arise. The one you will be doing in this class is related to the Student's T test, which is a comparison of mean values for samples. If you take a more advanced statistics course, you may be doing Z tests, p-value tests, and a variety of others.

The basic hypothesis we will be using is that the measure of accuracy discussed above, t, should follow a Gaussian probability distribution with a mean value of 0 and a standard deviation of 1. What this means in practice is that in a comparison of two values, $x_1 \pm \delta_{x_1}$ and $x_2 \pm \delta_{x_2}$, there is a $\sim$ 5% chance that x_1 and x_2 are distinct values (do not *agree* with each other) if $|t| > 2$. Thus, if $|t| < 2$, then we can say that x_1 and x_2 are both measuring the same quantity to the precision of the experiment(s).

In short, if $|t| > 2$, there is only a $\sim$ 5% chance that the two mean values "agree with each other". Thus, we say that the two measurements "do not agree" with each other (with 95% probabilistic confidence).

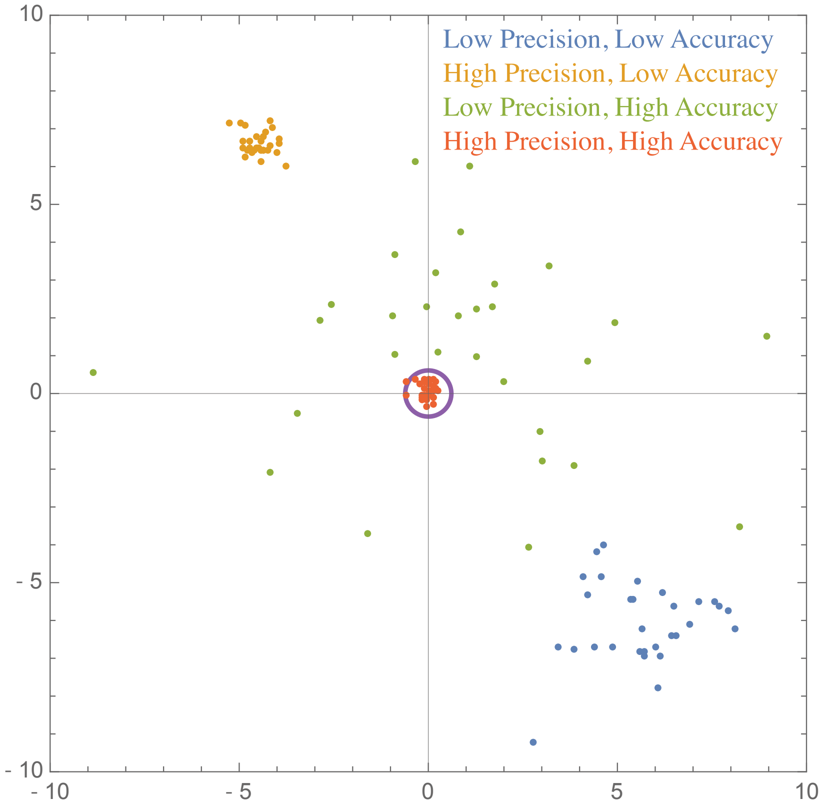


FIGURE C.1: This graph illustrates precision and accuracy. Points clustered close together (orange and yellow) represent higher precision than points that have a wide distribution (green and blue). Points whose mean value is closer to the centre of the purple circle (green and orange) represent higher accuracy than points that have a central value further from the purple circle (blue and yellow).

Remember, if t is small because the precision is poor, that is not a useful or helpful outcome when trying to push the boundaries of science. However, for home experiments, such as in this laboratory, it is an indicator that we aren't neglecting anything in the analysis.

Typically, if the theory/model is sound and applied correctly, then $|t| > 2$ should only occur when there is something that isn't properly accounted for. For example, if we try to model an object sliding down a slope without including the coefficient of friction, the acceleration will be wrong and our prediction of time taken to travel some distance will be wrong. Or if we assume that friction is a fixed value and independent of the speed of the object, then our model might similarly fail to predict the results.

Science is more interesting when hypotheses are disproven (e.g. $|t| > 2$), because that is when the investigation into why the model/theory was not accurate at predicting results occurs. This is the fecund nature of science – it must always push precision and accuracy boundaries to improve predictive power. In your lab reports, if you find that you get an unexpected result, you will be tasked with trying to explain why your result was unexpected and proposing ways to fix the problem for future experimentation.

Appendix D: Graphical Analysis

When taking measurements, there are many things that can impact the results – some are random in nature and some are biased in some way. For example, imagine measuring the speed of an object by measuring the time it takes to travel some distance. If your timing method is consistently late in starting the timer but not stopping the timer, then the speed will consistently be measured too high. It doesn't matter how many times you repeat the experiment, you will get the same results. However, through experimental design it is possible to eliminate these factors. This is illustrated in Figure D.1.

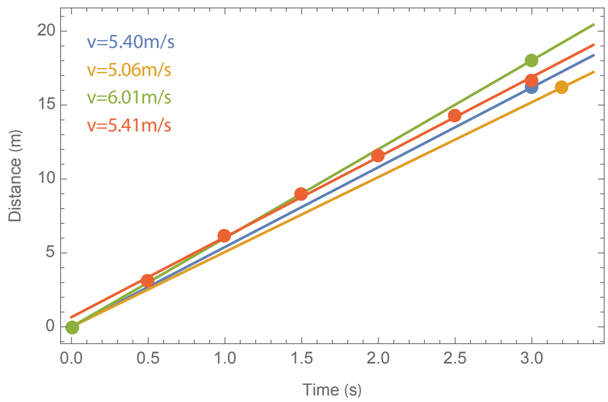


FIGURE D.1: This illustrates four different measurements of distance and time – the blue represents a perfect measurement scenario, with no random fluctuations or errors in the experimental process. The green and yellow lines represent the effect of some type of influence on the single measurement of distance and time. The red represents the effect of measuring multiple time/distance pairs. Speeds determined from this data are shown in the upper left of the figure.

In Figure D.1, there is one critical mistake that is incorporated into the experimental design for all but the red line: the assumption that there is a fixed origin point for $t = 0$s and $d = 0$m. The origin in an experiment is not actually a measured value, it is an assumed value. It is impossible to make a measurement of a length of time of 0s, nor a distance measurement of 0m. Thus, it is critically important to never assume that the origin is a value in your graph. This is the first lesson of graphical analysis. In the figure, it is also possible to see that the red data points have some statistical fluctuations to them, and yet these effects are averaged out over the multiple data points to give a good measure of the speed. This is the same effect as taking multiple measurements of the same quantity. This is the second lesson of graphical analysis.

This approach to analyzing data is part of a larger technique called *model fitting*, and there are a wide variety of mathematical approaches that can be used for very complicated mathematical models. In this course, we will be using a *least squares fit* on linearized data. That means we will simply be looking at determining the slope of a straight line using a technique that chooses the slope by minimizing the square of the deviation of all the data points from the line.

D.1 Plotting Data

When plotting data, it isn't always obvious which is the *controlled variable* and which is the *responding variable*. In this type of analysis, it isn't critically important which is put on the x-axis and which is put on the y-axis. The thing to keep in mind is the results of the analysis – when performing a linear line fit to data, you are determining the slope: $\frac{dy}{dx}$. This will tie in to the section on Interpreting the Results.

Each data point that is measured has a corresponding uncertainty to it. These uncertainties must be incorporated into the graph in a particular way. If a measurement gives a value of $y \pm \delta_y$, then there are three points of interest that need to be indicated on the graph: y, $y + \delta_y$ and $y - \delta_y$. The same is true for the x value, where there is a known uncertainty. The $\pm\delta$ is shown with *error bars* that illustrate the confidence range of the location of the value. The way this is drawn for a single data point is shown in Figure D.2. This is effectively making the statement "The best estimate of the location of this data point is at the point shown, but I am confident that the true value lies somewhere within the range shown."

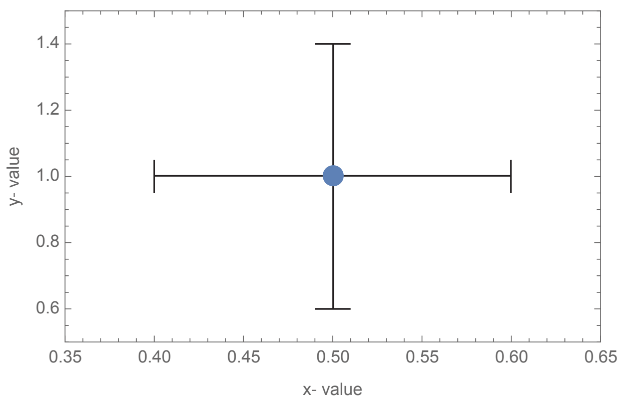


FIGURE D.2: A single data point with the corresponding error bars. This data point illustrates $\delta_x = 0.1$ and $\delta_y = 0.2$. Units are not included because this is an arbitrary example illustrating the graphical approach to error bars.

When put together with all data points, the graph should look like Figure D.3. For some experiments, the precision of some measurements may be very high. For example, when using highly accurate timing equipment, the uncertainty might be $\delta_t = 0.01$s while the data points are plotted on the range 0s to 10s. In such cases, the uncertainty bars parallel to the time axis would be too small to show, and thus it is reasonable to be neglect them. This should only be done if the error bars themselves are smaller than the width of the marker that is used to illustrate the point on the figure, or are otherwise visually unreasonable. Some judgment in the scientist is allowed when making this determination.

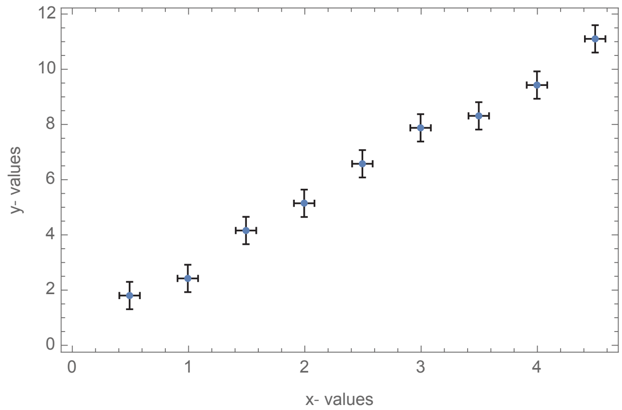


FIGURE D.3: This graph illustrates plotting an entire dataset along with uncertainties in the x and y axes. The uncertainties displayed here are $\delta_x = 0.1$ and $\delta_y = 0.5$. Uncertainties of $\delta_x < 0.05$ or $\delta_y < 0.2$ might be reasonable to leave off on this scale.

D.2 Least Squares Analysis

Least squares analysis is performed by minimizing the square of the deviation. That is a statement which is best explained mathematically and visually. A linear fit makes the prediction that the data follows a linear line of the formula: $y = Ax + B$. The parameter A is the slope and the parameter B is the intercept. The *deviation* is the vertical distance between the data point (x_i, y_i) and the value predicted by the linear line, $y(x_i) = Ax_i + B$:

$$\Delta y_i = y_i - y(x_i) = y_i - (Ax_i + B) \tag{D.1}$$

This is illustrated in Figure D.4. Keep in mind that before any calculations are done, the values of A and B are unknown – so at this point, we assume that there must exist some *best-fit* values of A and B, and our goal is to find them.

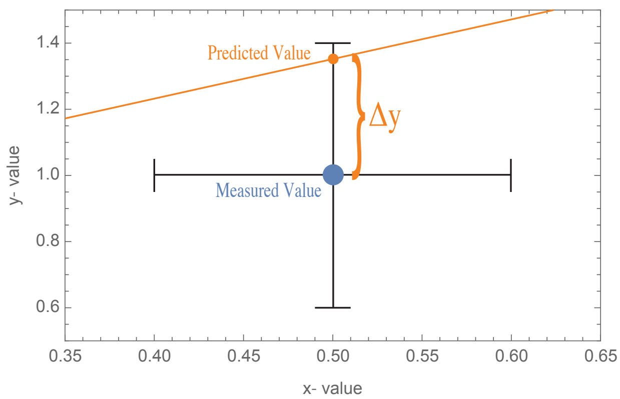


FIGURE D.4: Data point and hypothetical best-fit line illustrating what is meant by the term *deviation*. The deviation, Δy is shown on the graph.

If A and B were known values, the deviation could be calculated for every single data point (x_i, y_i). The total deviation square is the quantity that we want to minimize:

$$\Delta_{total} = \sum_{i=1}^{N} (\Delta y_i)^2 = \sum_{i=1}^{N} (y_i - (Ax_i + B))^2 = (y_1 - (Ax_1 + B))^2 + (y_2 - (Ax_2 + B))^2 + ... \tag{D.2}$$

For poorly fitting values of A and B, the square of the deviation will always increase. For example, imagine the correct slope but an incorrect intercept – the deviation will be larger for all values. Alternatively, if the slope is incorrect, the deviation will at best be small for one or two points where the line passes near the data points, but will be larger for all other values. This is illustrated in Figure D.5. Thus, given the equation for the total deviation square, it is clear that it is a concave up parabola in both A and B. Thus, to find the minimum, we simply need to find the value of A and B that satisfies:

$$\frac{d\Delta_{total}}{dA} = 0 \qquad \frac{d\Delta_{total}}{dB} = 0 \tag{D.3}$$

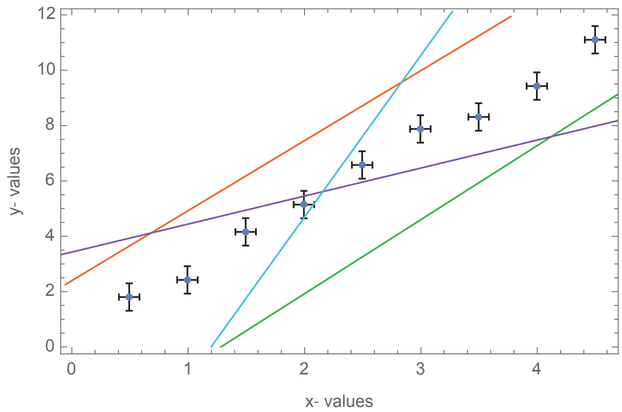


FIGURE D.5: This graph illustrates poor choices of lines to fit the data. When the slope and/or intercept are off, the sum of the square of the deviations increases.

Expanding out the terms in the sum, we get a different expression for the total deviation square:

$$\begin{aligned}
\Delta_{total} &= \sum_{i=1}^{N} (y_i^2 + A^2 x_i^2 + B^2 - 2Ay_i x_i - 2By_i + 2ABx_i) && \text{(D.4)}\\
&= \sum_{i=1}^{N} y_i^2 + A^2 \sum_{i=1}^{N} x_i^2 + NB^2 - 2A \sum_{i=1}^{N} x_i y_i - 2B \sum_{i=1}^{N} y_i + 2AB \sum_{i=1}^{N} x_i && \text{(D.5)}\\
&= N(\overline{y^2} + A^2 \overline{x^2} + B^2 - 2A\overline{xy} - 2B\overline{y} + 2AB\overline{x}) && \text{(D.6)}
\end{aligned}$$

Thus, the derivatives are:

$$\begin{aligned}
\frac{d\Delta_{total}}{dA} &= 2A \sum_{i=1}^{N} x_i^2 - 2 \sum_{i=1}^{N} x_i y_i + 2B \sum_{i=1}^{N} x_i = 0 && \text{(D.7)}\\
\frac{d\Delta_{total}}{dB} &= 2NB - 2 \sum_{i=1}^{N} y_i + 2A \sum_{i=1}^{N} x_i = 0 && \text{(D.8)}
\end{aligned}$$

This is two equations with two unknowns – we want to solve these equations simultaneously for A and B. The simplest way to do this would be to solve one of them for A in terms of B, then substitute that into the other equation to eliminate any factors of A and produce an equation that depends only on B. Solving for B then allows solving for A in the original equation. This results in the following:

$$A = \frac{N\sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} y_i\right)}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \tag{D.9}$$

$$B = \frac{\left(\sum_{i=1}^{N} x_i^2\right)\left(\sum_{i=1}^{N} y_i\right) - \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{i=1}^{N} x_i y_i\right)}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2} \tag{D.10}$$

Dividing the numerator and denominator simultaneously by N^2 gives a way of writing these equations in terms of the averages $\overline{x}$, $\overline{y}$, $\overline{xy}$ and $\overline{x^2}$:

$$A = \frac{(\overline{xy}) - (\overline{x})(\overline{y})}{(\overline{x^2}) - (\overline{x})^2} \tag{D.11}$$

$$B = \frac{(\overline{x^2})(\overline{y}) - (\overline{x})(\overline{xy})}{(\overline{x^2}) - (\overline{x})^2} \tag{D.12}$$

These calculations would be tedious if done by hand, but are simple when done with the aid of computing. For these labs, you will be expected to use a spreadsheet program such as Microsoft Excel, Apple Numbers, or Google Sheets. Information on how to do this is included in Appendix B. As a side note, it turns out that the point $(\bar{x}, \bar{y})$ is a point that satisfies the equation $y = Ax + B$ for the best fit values of A and B. Thus, the *best-fit line* has slope A, intercept B, and passes through the point $(\bar{x}, \bar{y})$.

A careful observer might be wondering why there is little attention paid to the error bars when performing this analysis. Incorporating uncertainties in calculating A and B is more complicated than necessary for this course. However, the error bars are still important. For the purposes of these labs, a visual inspection of the graph and best-fit line should be done to ensure that none of the data points are more than two error bars distances away from the best fit line. The likelihood of this happening purely randomly is less than 3%, and such a situation is more likely caused by a bias or mistake in measurement or calculation. If you find that any of your points lie further than two error bar distances away from the best fit line, the data point should either be re-measured or re-calculated, depending on where the mistake lies, and the analysis should be repeated. This should emphasize the importance of proper data collection and calculations to minimize the effort involved in performing the analysis.

This does not mean that there is no uncertainty at all in the values of A and B. Instead, the uncertainty is calculated in a different manner. The derivation of the uncertainties in A and B can be found in uncertainty texts and will not be reproduced here. These uncertainties depend on the uncertainty in the value of Δ_{total}:

$$\delta_\Delta = \sqrt{\frac{\Delta_{total}}{N-2}} \tag{D.13}$$

With this value, the uncertainties in A and B are given by:

$$\delta_A = \delta_\Delta \sqrt{\frac{N}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}} = \delta_\Delta \sqrt{\frac{1}{N(\overline{x^2} - \overline{x}^2)}} \tag{D.14}$$

$$\delta_B = \delta_\Delta \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2}} = \delta_\Delta \sqrt{\frac{\overline{x^2}}{N(\overline{x^2} - \overline{x}^2)}} \tag{D.15}$$

A completed graph illustrating the results is shown in Figure D.6.

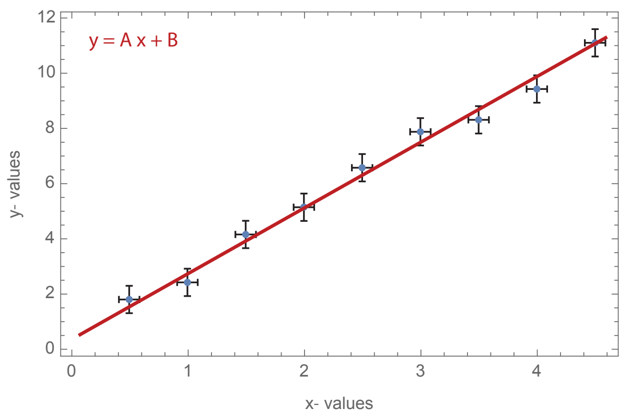


FIGURE D.6: This graph illustrates all of the necessary components of graphically displaying your data, including error bars, the best-fit line, and the equation for the best fit line. If there are multiple lines shown on the graph, the equation for the best-fit line should be kept separate from the graph.

D.3 Interpreting the Results

In graphing an objects distance from an origin versus time, it should be obvious that the slope of the graph is the speed of the object and the intercept is the distance from the origin at $t = 0$s. The interpretation of the slope and intercept is more difficult for more complicated experiments.

Consider Newton's Law of Gravitation, $\vec{F} = \frac{Gm_1m_2}{r^2}\hat{r}$. The letter G represents the universal gravitational constant, but all other components of the equation could potentially be measurable. If we measure the magnitude of the force F and vary only one of the masses while keeping the other mass and the distance, r, constant, then we have the following relationship:

$$F = \left(\frac{Gm_2}{r^2}\right) m_1$$

Keeping m_2 and r constant, and varying m_1 allows us to identify the y-axis with the force, F, and the x-axis with the mass, m_1. In other words:

$$\begin{array}{ccccccc} F & = & \left(\frac{Gm_2}{r^2}\right) & m_1 & + & ? \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ y & = & A & x & + & B \end{array}$$

Thus, a graph of F versus m_1 would produce a slope that can be equated with Gm_2/r^2. If the whole point of the experiment is to determine the gravitational constant, G, then our slope could be used as:

$$\begin{aligned} A &= \frac{Gm_2}{r^2} \\ G &= \frac{Ar^2}{m_2} \end{aligned}$$

The uncertainty of A is determined from the graph data, as shown in the previous section. Given measurement uncertainties on r and m_2, the uncertainty of the constant G can be determined using uncertainty propagation.

The likelihood of the data producing an intercept that is zero for a model that predicts a zero intercept is effectively negligibly small. Thus, it is important to consider the meaning of an intercept where the model (the physics equation) being used does not include one. In the situation proposed above, determining G, a small intercept is likely due to random fluctuations of the data, but a larger intercept is possible without ruining the experiment altogether. A negative intercept could be caused by a minimum force needed to produce a measurement in the force-meter. A positive intercept could be caused by a force-meter that is not properly calibrated (the zero point is not set correctly). So long as the force varies correctly with the mass, the presence of such an intercept has no impact on the results. This is one of the reasons why graphical analysis is so useful – a single measurement of the force and mass would not account for these kinds of biased influences, and would give a drastically incorrect value for G.

There are times when the model has non-linear dependencies. Again returning to the situation above, F is linearly proportional to the masses m_1 and m_2, but it is proportional to square inverse of distance. Instead of using F and m_1 to determine G, it would be possible to fix m_1 and m_2 and vary r and still use the least square technique. In order to do this, we do not plot F versus r, but rather F versus r^{-2}. Thus, we see:

$$\begin{array}{ccccccc} F & = & (Gm_2m_1) & \frac{1}{r^2} & + & ? \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ y & = & A & x & + & B \end{array}$$

and

$$\begin{aligned} A &= Gm_1 m_2 \\ G &= \frac{A}{m_1 m_2} \end{aligned}$$

This is known as *linearizing* the graph. When doing this kind of analysis, it is important to remember that the uncertainties on the linearized data need to be properly propagated. The uncertainty for r^{-2} is not the same as δ_r (the uncertainty on r). Instead:

$$\delta_{\frac{1}{r^2}} = \left| \frac{2\delta_r}{r^3} \right|$$

These values would need to be calculated for each value of r that is measured, and used in plotting the data.

Appendix E: Uncertainty Analysis

In many texts, the topic of uncertainty is referred to as *error*. For many students, this conveys an incorrect understanding. The term *error* is historically applied to this analysis as a reference to the likelihood that the experimentally determined value is inaccurate or incorrect, or that the theory that the experiment is testing is incorrect. The use of the term does not imply any error on the part of the experiment or the experimentor: any errors or mistakes should be corrected, and are in no way admissible by the use of uncertainty calculations.

A very brief overview of the origin of uncertainty in the context of Gaussian statistics was given in Appendix A. This appendix will focus on the application of uncertainty in more complicated systems. It is important to note that the study of uncertainty has great depth that is not fully explored or used in this class. These labs and this lab manual are meant as an introduction to the important concepts only.

Instead of the more laborious notation of standard deviation of the mean, $\sigma_{\bar{x}}$, and to acknowledge that we are approximating standard deviations of the mean using the formula $\sigma_{\bar{x}} \approx \sigma_x/\sqrt{N}$, we will be using a slightly different notation to refer to the *uncertainty* of a quantity:

$$\sigma_{\bar{x}} = \delta_x \tag{E.1}$$

Thus a δ_x will refer to the uncertainty in quantity x and σ_x will refer to the standard deviation of a series of measurements of x. The overbar on the x in a subscript can be both hard to write and hard to read.

In cases where there is both a statistical uncertainty and a measurement uncertainty, these should be combined as:

$$\delta_{combined} = \sqrt{\delta^2_{statistical} + \delta^2_{measurement}} \tag{E.2}$$

For example, if the distance an object travels in projectile motion is measured multiple times, under apparent identical conditions, then the location of impact has multiple measurements of it, from which to calculate the statistical uncertainty. However, the measuring device probably has an uncertainty on the order of 0.5-1mm. Thus, these two components of uncertainty are independent of each other and should be combined.

If $\delta_{statistical} > 4\delta_{measurement}$, the measurement uncertainty can be neglected.

E.1 Origin of Uncertainty Propagation

There are few things which can be measured directly – distance, time, mass are three of them. Other types of quantities, such as forces and energies, must be calculated from measurements of more fundamental quantities. Even devices that purport to measure these quantities directly are, in fact, using more fundamental measurements and translating the fundamental measurement into the desired quantity using a calibrated conversion. In the Appendix A, the origin of uncertainty in measured quantities was discussed, and now we want to explore how these uncertainties propagate through as we use the measurements in contexts like calibrated conversions, and determining calculated values.

The formula for propagation of uncertainty has as its origin the Taylor Series for approximating a function using a polynomial. For those that are unfamiliar with Taylor Series, a brief description is included in this lab manual. Given a function of one variable, $f(x)$, for which we know the value of the function and all of its derivatives at location $x = a$, we can approximate the value for $f(x)$ at values of x that are very close to a with the following formula:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k \end{aligned} \tag{E.3}$$

The accuracy of this approximation is shown in Figure E.1. Clearly the more terms included in the expression, the more similar the polynomial is to the original function.

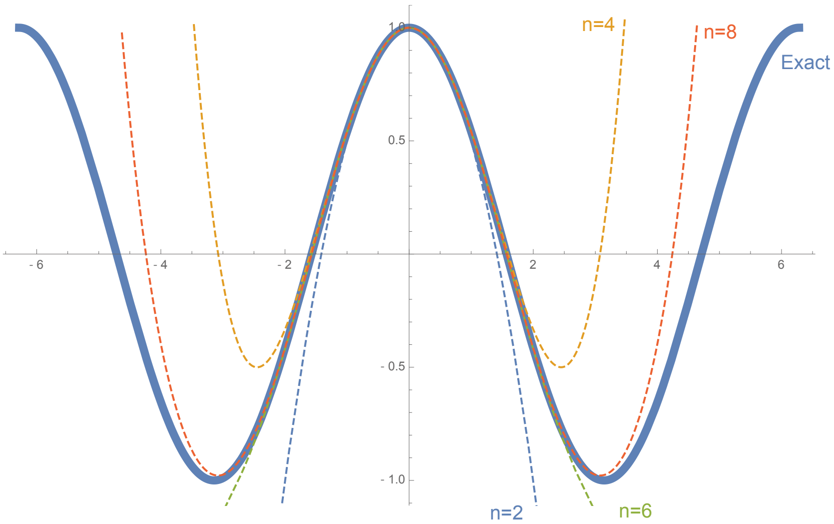


FIGURE E.1: The function $\cos x$ has a Taylor series about the point $x = 0$ of $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \ldots$. In this figure, the thick blue line is the exact value for $\cos x$ over the range $[-2\pi, 2\pi]$. The blue dashed line is the Taylor series truncated at the x^2 term, the yellow dashed line is the series truncated at the x^4 term, the green dashed line is the series truncated at the x^6 term, and the red dashed line is the series truncated at the x^8 term.

If we change notation slightly, its application to uncertainties becomes more obvious: let $\delta_f = f(x) - f(a)$ and $\delta_x = x - a$, the differences between the approximated values and the hypothetically true values. Rearranging the equation, we then get:

$$\delta_f \approx f'(a)\delta_x + \frac{1}{2}f''(a)(\delta_x)^2 + \frac{1}{6}f'''(a)(\delta_x)^3 + ... \tag{E.4}$$

If δ_x is a small value (say $\delta_x = 0.01$) then $(\delta_x)^2$ is a very small number ($(\delta_x)^2 = 0.0001$). Thus, the terms in the sum have a smaller and smaller effect as the power of the δ_x factor increases.

We can perform a *linear approximation* by truncating the terms with $(\delta_x)^k$ for $k \geq 2$ to get much more simple formula:

$$\delta_f \approx \frac{df}{dx}\delta_x \tag{E.5}$$

where the derivative of f with respect to x is evaluated at the measured value. This is the most basic formula that can be applied to uncertainties – the uncertainty in a *function* of x depends on the value of the derivative of that function and the value of the uncertainty of x. Since uncertainty applies equally on either side of the central value, a negative value for the slope is irrelevant: $-1 \times (\pm\delta_x) = \mp\delta_x$, which has the same effect. Thus, we typically write the derivative factor in absolute values:

$$\delta_f \approx \left|\frac{df}{dx}\right|\delta_x \tag{E.6}$$

EXAMPLE:
The surface of sand on the beach looks like a cosine function with a wavelength of 12.3cm and an amplitude of 3.1cm, and you want to test this hypothesis. At a horizontal distance of 4.0 ± 0.1cm from one of the peaks, the height of the sand has dropped 4.4 ± 0.1cm relative to the nearest peak. If the surface of the sand really does follow a cosine function, how much should the height of the sand drop at a point 4.0 ± 0.1cm from the nearest peak? (i.e. Does $y_{measured}(4.0\text{cm}) = -4.4\text{cm}$ match our prediction that $y_{theory}(x) = 3.1\,\text{cm}(\cos(2\pi x/12.3\text{cm}) - 1)$?)

The modelled height of the sand relative to the nearest peak is $y_{theory}(x) = 3.1\,\text{cm}(\cos(2\pi x/12.3\,\text{cm}) - 1)$cm, while the measured value at $x = 4.0 \pm 0.1$cm is $y = -4.4 \pm 0.1$cm. The goal of the calculation is to determine what the height *should* be if it followed the cosine model. We know what is height the sand at that point, so we want to test against what it should be.

$$y_{theory}(4.0) \quad = \quad 3.1(\cos(2\pi 4.0\,\text{cm}/12.3\,\text{cm}) - 1)\,\text{cm} = -4.5109\,\text{cm}$$

We know the value of the height of the sand, but we also need to know the uncertainty.

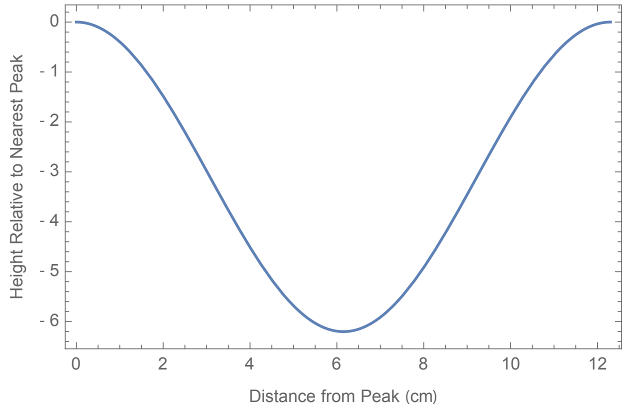


FIGURE E.2: The height of the sand relative to the nearest peak follows a $\cos(x) - 1$ type function. Given the wavelength and amplitude, the equation for the height of the sand should be $y_{theory}(x) = 3.1(\cos(2\pi x/12.3\,\text{cm}) - 1)$cm. At $x = 4.0$cm, the height of the sand was measured to be $y = -4.4$cm.

$$\begin{aligned}
\frac{df}{dx} &= -\frac{2\pi}{12.3\,\text{cm}} 3.1\text{cm}\sin(2\pi x/12.3\,\text{cm}) \\
\left.\frac{df}{dx}\right|_{x=4.0\,\text{cm}} &= -\frac{2\pi}{12.3\,\text{cm}} 3.1\,\text{cm}\sin(2\pi 4.0\,\text{cm}/12.3\,\text{cm}) = -1.41005 \\
\delta_f &= \left|\left.\frac{df}{dx}\right|_{x=a}\right| \delta_x = (1.41005)(0.1\,\text{cm}) = 0.141005\,\text{cm}
\end{aligned}$$

Thus, the height of the sand should be -4.5 ± 0.1cm according to the cosine model. The height that was measured was -4.4 ± 0.1cm. As shown in Figure E.3, there is significant similarity between these two values. Without performing a calculation to see if these two values agree with each other, it is reasonable to have intuition that the measured value supports the modelled equation. Another way to say this is, "We cannot exclude the possibility that the theoretical model is accurate based on the measurements taken and their precision." Mathematical comparisons of these results will be discussed in more detail in Appendix C.

E.2 Multivariate Uncertainty Propagation

The Taylor expansion explanation for uncertainty analysis is simple for a single variable. Taylor expanding for multiple variables is more complicated. However, it is easy to write up to the linear term. In one dimension, the expansion was:

$$f(x) \approx f(a) + f'(a)(x - a) + ... \tag{E.7}$$

For multiple variables, we just add on extra terms for each of the other variables:

$$f(x, y, z, ...) = f(a, b, c, ...) + \frac{df}{dx}(x - a) + \frac{df}{dy}(y - b) + \frac{df}{dz}(z - c) + ... \tag{E.8}$$

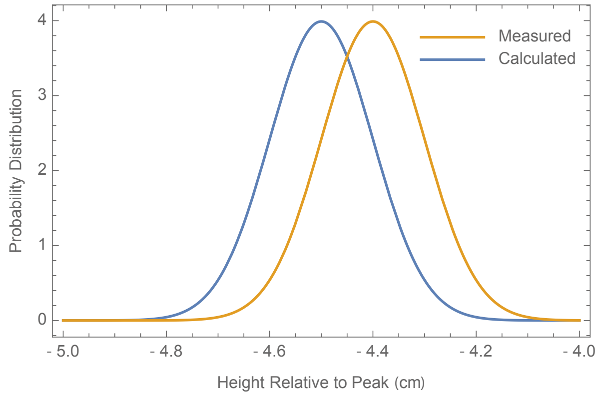


FIGURE E.3: The probability distributions for the measured and modelled values.

Shifting the notation to include the δ's, it becomes:

$$\delta_f = \frac{df}{dx}\delta_x + \frac{df}{dy}\delta_y + \frac{df}{dz}\delta_z + ... \tag{E.9}$$

Note: In the derivatives, the variables x, y, z, etc are evaluated at the measured values.

However, this isn't the end of the explanation. In the previous section, we simply took the absolute value of the derivative, because the uncertainty is symmetric – the Gaussian curve behaves identically on either side of the mean. Now we have to account for the possibility that some uncertainties could overlap. For example, if we measure distance and time to determine velocity, then longer distances will also correspond to longer times. Using the formula above would miscount these overlapping effects. Thus, the uncertainties between these two have a component that is related to each other in some way.

To do this, we first square each side of the equation and expand:

$$\delta_f^2 = (\frac{df}{dx}\delta_x)^2 + (\frac{df}{dy}\delta_y)^2 + (\frac{df}{dz}\delta_z)^2 + 2\frac{df}{dx}\frac{df}{dy}\delta_x\delta_y + 2\frac{df}{dx}\frac{df}{dz}\delta_x\delta_z + 2\frac{df}{dy}\frac{df}{dz}\delta_y\delta_z + ... \tag{E.10}$$

This equation now represents the uncertainty in f, δ_f, if the measurements are completely positively correlated (e.g. any increase in x results in a proportionally identical increase in y). A negative sign in front of the factors of 2 in the cross terms would be if the measurements were completely anti-correlated (e.g. any increase in x results in a proportionally identical decrease in y). And a factor of 0 instead of 2 would indicate that the uncertainties are completely uncorrelated. In general, we can replace the factors of 2 with a number ρ_{ij} to represent the correlation between factor i and factor j. However, in this course, we are going to assume that the uncertainties are uncorrelated and leave correlation factors for more advanced courses.

Thus, the equation to use for calculating the uncertainty of a multivariate theoretical equation is:

$$\delta_f = \sqrt{\left(\frac{df}{dx}\delta_x\right)^2 + \left(\frac{df}{dy}\delta_y\right)^2 + \left(\frac{df}{dz}\delta_z\right)^2 + ...} \tag{E.11}$$

This is the most general expression that can always be used for uncorrelated uncertainties. However, physics has many formulas that are very similar in mathematical structure. Thus, it is sometimes easier to use the template method to calculate the uncertainties. This is covered in the next section.

EXAMPLE:
You want to determine the density of a uniform rectangular prism. The density formula is $\rho = m/V$. The mass is measured to be 22.3 ± 0.4kg, the length is measured to be 0.42 ± 0.02m, the width is measured to be 0.12 ± 0.02m, and the height is measured to be 0.23 ± 0.02m. What is the density and uncertainty in the density?

The volume of the prism is given by the formula $V = \ell wh$, where ℓ is the length, w is the width and h is the height. Thus, the combined expression for density is:

$$\rho = \frac{m}{\ell wh} = 22.3\,\mathrm{kg}/0.42\,\mathrm{m}\cdot 0.12\,\mathrm{m}\cdot 0.23\,\mathrm{m} = 1923.74\,\mathrm{kg\,m^{-3}}$$

We first need to take a series of derivatives:

$$\begin{aligned}
\frac{d\rho}{dm} &= \frac{1}{\ell wh} = \frac{1}{0.42\,\mathrm{m}\cdot 0.12\,\mathrm{m}\cdot 0.23\,\mathrm{m}} = 86.27\,\mathrm{m^{-3}}\\
\frac{d\rho}{d\ell} &= -\frac{m}{\ell^2 wh} = \frac{22.3\,\mathrm{kg}}{0.12\,\mathrm{m}\cdot 0.23\,\mathrm{m}} = 807.97\,\mathrm{kg\,m^{-2}}\\
\frac{d\rho}{dw} &= -\frac{m}{\ell w^2 h} = \frac{22.3\,\mathrm{kg}}{0.42\,\mathrm{m}\cdot 0.23\,\mathrm{m}} = 230.85\,\mathrm{kg\,m^{-2}}\\
\frac{d\rho}{dh} &= -\frac{m}{\ell wh^2} = \frac{22.3\,\mathrm{kg}}{0.42\,\mathrm{m}\cdot 0.12\,\mathrm{m}} = 442.46\,\mathrm{kg\,m^{-2}}
\end{aligned}$$

Putting this together in the uncertainty formula, we find:

$$\begin{aligned}
\delta_f &= \sqrt{\left(\frac{d\rho}{dm}\delta_m\right)^2 + \left(\frac{d\rho}{d\ell}\delta_\ell\right)^2 + \left(\frac{d\rho}{dw}\delta_w\right)^2 + \left(\frac{d\rho}{dh}\delta_h\right)^2}\\
&= \big((86.27\,\mathrm{m^{-3}}\cdot 0.4\,\mathrm{kg})^2 + (807.97\,\mathrm{kg\,m^{-2}}\cdot 0.02\,\mathrm{m})^2 + (230.85\,\mathrm{kg\,m^{-2}}\cdot 0.02\,\mathrm{m})^2\\
&\quad +(442.46\,\mathrm{kg\,m^{-2}}\cdot 0.02\,\mathrm{m})^2\big)^{1/2}\\
&= 39.39\,\mathrm{kg\,m^{-3}}
\end{aligned}$$

Thus, rounding our value to the same precision as the leading digit of the uncertainty gives us:

$$\rho = 1920 \pm 40\,\mathrm{kg\,m^{-3}}$$

E.3 Uncertainty Propagation Templates

In physics, there are a number of very common types of equations that are used. In particular, summations occur frequently (e.g. in superposition), as well as polynomials, and trig functions and exponentials are also common. To simplify your analysis, the uncertainties for these types of template functions are shown below. If you don't want to take derivatives and use the full formula – which always works – you can use these templates. The risk is in whether you have used the template correctly.

Type	Function	Uncertainty
Scalar Multiplication	$f(x) = Ax$	$\delta_f = \|A\|\delta_x$
Addition/ Subtraction	$f(x, y, ...) = \pm Ax \pm By \pm ...$	$\delta_f = \sqrt{(A\delta_x)^2 + (B\delta_y)^2 + ...}$
Multiplication	$f(x, y, ...) = Axy...$	$\delta_f = \|f\|\sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2 + ...}$
Division	$f(x, y, ...) = Axy^{-1}...$	$\delta_f = \|f\|\sqrt{\left(\frac{\delta_x}{x}\right)^2 + \left(\frac{\delta_y}{y}\right)^2 + ...}$
Exponents	$f(x) = Ax^n$	$\delta_f = \|n \cdot f\| \left\|\frac{\delta_x}{x}\right\|$
Exponents w/ Multiplication	$f(x, y, ...) = Ax^n y^m ...$	$\delta_f = \|f\|\sqrt{\left(n\frac{\delta_x}{x}\right)^2 + \left(m\frac{\delta_y}{y}\right)^2 + ...}$
Exponentials	$f(x) = Ae^{Bx}$	$\delta_f = \|fB\|\delta_x$
Logarithms	$f(x) = A\log(Bx)$	$\delta_f = \left\|A\frac{\delta_x}{x}\right\|$
Sine	$f(x) = A\sin(Bx)$	$\delta_f = \|AB\cos(Bx)\|\delta_x$
Cosine	$f(x) = A\cos(Bx)$	$\delta_f = \|AB\sin(Bx)\|\delta_x$

Note: This table assumes that only x, y and z are values that have uncertainties. The constants A, B and n should be taken as values without uncertainty, such as π or integers.