Math110

Part 1

① $A = \begin{bmatrix} 9 & -4 & -8 \\ 12 & -7 & -8 \\ 0 & 0 & -3 \end{bmatrix}$ for finding a matrics consisting of all the eigenvectors and a matrics having eigenvalues in it's diagonals,

P and D, we have to find the eigenvalues and eigenvectors first

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} - \lambda\vec{v} = 0$

$A\vec{v} - \lambda I\vec{v} = 0$

$\vec{v}(A - \lambda I) = 0$

we need to find the values for which $\det(A - \lambda I) = 0$

$\Rightarrow \det\left(\begin{bmatrix} 9-\lambda & -4 & -8 \\ 12 & -7-\lambda & -8 \\ 0 & 0 & -3-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow 0 - 0 + (-3-\lambda)\begin{bmatrix} 9-\lambda & -4 \\ 12 & -7-\lambda \end{bmatrix} = 0$

$\Rightarrow (-3-\lambda)\left\{(9-\lambda)(-7-\lambda) - 12(-4)\right\} = 0$

$\Rightarrow (-3-\lambda)\left\{9(-7-\lambda) - \lambda(-7-\lambda) + 48\right\} = 0$

$\Rightarrow (-3-\lambda)\left\{-63 - 9\lambda + 7\lambda + \lambda^2 + 48\right\} = 0$

$\Rightarrow (-3-\lambda)(-2\lambda + \lambda^2 - 15) = 0$

$\Rightarrow (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$\Rightarrow (\lambda+3)(\lambda^2 + 3\lambda - 5\lambda - 15) = 0$

$\Rightarrow (\lambda+3)\left\{\lambda(\lambda+3) - 5(\lambda+3)\right\} = 0$

$\Rightarrow (\lambda+3)(\lambda-5)(\lambda+3) = 0$

$\Rightarrow (\lambda+3)^2(\lambda-5) = 0$

$\therefore \lambda = -3, -3, 5$

$\boxed{Alg(-3) = 2}$ $\boxed{Alg(5) = 1}$

Hibroy

Eigenvectors for $\lambda = -3$

Putting the eigenvalue $\lambda = -3$ in matrics $(A - \lambda I)$ we get

$$\begin{bmatrix} 12 & -4 & -8 \\ 12 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[(1/12)R_2]{(1/12)R_1} \begin{bmatrix} 1 & -1/3 & 2/3 \\ 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + (-\tfrac{1}{3})x_2 + (-\tfrac{2}{3})x_3 = 0$$

or, $$x_1 = \frac{x_2}{3} + \frac{2x_3}{3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_2 + \frac{2}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors : $\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 0 \\ 1 \end{bmatrix}$

$geo(-3) = 2$

Eigenvalues for $\lambda = 5$

$$\begin{bmatrix} 4 & -4 & -8 \\ 12 & -12 & -8 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\xrightarrow[(1/12)R_2,\ (1/8)R_3]{(1/4)R_1} \begin{bmatrix} 1 & -1 & -2 \\ 1 & -1 & -2/3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[(-1)R_3]{R_2 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - \frac{4}{3}R_3]{R_1 + 2R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 - x_2 = 0$ $\quad \boxed{x_1 = x_2}$

$\boxed{x_3 = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$geo(5) = 1$

$\therefore$ $alg(-3) = geo(-3)$
$alg(5) = geo(5)$

This means that the matrix A is diagonalizable for all it's eigenvalues.

$$P = \begin{bmatrix} 1/3 & 2/3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Each columns consisting of corresponding eigenvectors for eigenvalues $\lambda_i = -3, -3, 5$

$$\therefore D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Matrix having diagonal entries of eigenvalues

$$\therefore A = PDP^{-1}$$

$$\therefore A = \begin{bmatrix} 1/3 & 2/3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Part 1

② (a) Given L is a line that passed through $(0,0,0)$ and $(3,3,-1)$

$\vec{d}$, directional vector is $\begin{bmatrix}3\\3\\-1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}3\\3\\-1\end{bmatrix}$

Given $L = \left\{ l\begin{bmatrix}3\\3\\-1\end{bmatrix}, \ l \in R \right\}$

$$L^{\perp} = \left\{ \vec{v} : \vec{v}\cdot\vec{w} = 0 \ ; \ \vec{w} = l\vec{d} = l\begin{bmatrix}3\\3\\-1\end{bmatrix} \right\}$$

$$= \left\{ \vec{v} : \vec{v}\cdot\left(l\begin{bmatrix}3\\3\\-1\end{bmatrix}\right) = 0 \right\}$$

$$= \left\{ \vec{v} : \vec{v}\cdot\begin{bmatrix}3\\3\\-1\end{bmatrix} = 0 \right\}$$

$$\begin{bmatrix}3\\3\\-1\end{bmatrix}\vec{v} = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & 3 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{(1/3)R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1/3 & 0 \end{array}\right]$$

$$a + b - \frac{c}{3} = 0$$

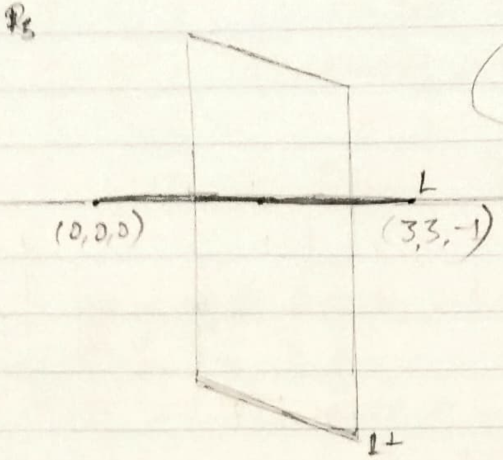
or, $\boxed{a = c/3 - b}$

$$\therefore \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}\frac{1}{3}c - b\\b\\c\end{bmatrix} = c\begin{bmatrix}1/3\\0\\1\end{bmatrix} + b\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$\therefore$ The linearly independent basis vectors for $\vec{v}$ are $\begin{bmatrix}1/3\\0\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\0\end{bmatrix}$

Part 1

2(b)



• L = All vectors in $l\vec{d}$, l being any scalar real number

• $L^\perp$ = All vectors orthogonal to L in our $\mathbb{R}_3$ space.

$$= \text{span}\left(\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

in the given three dimentional co-ordinate system.

Part 2

1(a) We know for a matrix to be invertible the determinant cannot be 0.

$\boxed{\det(A) \neq 0}$

And for it to be diagonalizable, $\boxed{\det(A-\lambda I)=0}$ and $\boxed{alg(\lambda)=geo(\lambda)}$

It is not true that every invertible matrix is diagonalizable

A 3X3 matrix with $\det(A) \neq 0$ would be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

For it to be diagonal, algebraic multiplicity must be equal to the geometric multiplicity.

$\det(A-\lambda I)=0$

or, $-\lambda^3+\lambda^2+\lambda-1=0$

or, $(\lambda+1)(\lambda-1)^2=0$

$\lambda = -1, 1, 1$

$\therefore alg(-1)=1;\ alg(1)=2$

Calculating eigenvectors for $\lambda=1$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{a=0}$

$b-c=0$

$\boxed{b=c}$

$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ c \end{bmatrix} = c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\therefore geo(1)=1$

Hikhor

Since $alg(1) \neq geo(1)$, the invertible matrix is not diagonal.

Part 2

1(b).

For a matrix to be invertible, $\det(A) \neq 0$.

Let's pick a matrix which is diagonal, say $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The eigenvalues of the matrix is 0 and the

eigenvectors is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Here, $alg(0) = 3 = geo(0)$.

So, the matrix is diagonalizable.

However, $\det(A) = 0 \Rightarrow$ Matrix A is not invertible.

This proves that it's not always true that every diagonalizable matrix is invertible.

(Ans)

Part 2

Q2

Given, $V_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$

Let's take four more vectors orthogonal to the vectors, in $\mathbb{R}^4$ containing vecture $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

$$V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad V_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad V_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here taking rref(A), we can see at the 5th column is a free variable, thus

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\therefore$ These 4 vectors are linearly independent vectors in $\mathbb{R}^4$, and thus they form a basis.

$$\text{thus } W_1 = V_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \qquad \vec{w_2} = \vec{v_2} - \left(\frac{\vec{w_1} \cdot \vec{v_2}}{\vec{w_1} \cdot \vec{w_1}} \right) \vec{w_1}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left\{ \frac{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}} \right\} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left\{ \frac{1}{15} \right\} \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/15 \\ 2/15 \\ -1/5 \\ 1/15 \end{bmatrix} = \begin{bmatrix} 14/15 \\ -2/15 \\ 1/5 \\ -1/15 \end{bmatrix}$$

$$\therefore \vec{w_2} \begin{bmatrix} 14/15 \\ -2/15 \\ 1/5 \\ -1/15 \end{bmatrix} = \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{w_3} = \vec{v_3} - \text{proj}_{\vec{w_2}}\vec{v_3} = \vec{v_3} - \left(\frac{\vec{v_3}\cdot\vec{w_2}}{\|\vec{w_2}\|^2}\right)\vec{w_2}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix} \right\|^2} \right) \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{(-2)}{(210)} \right) \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{1}{105} \right) \begin{bmatrix} 14 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 14\times\frac{1}{105} \\ -2\times\frac{1}{105} \\ 3\times\frac{1}{105} \\ -1\times\frac{1}{105} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/15 \\ -2/105 \\ 1/35 \\ -1/105 \end{bmatrix}$$

$$= \begin{bmatrix} -2/15 \\ 103/105 \\ -1/35 \\ 1/105 \end{bmatrix} = \begin{bmatrix} -14 \\ 103 \\ -3 \\ 1 \end{bmatrix}$$

$$w_4 = \vec{v_4}' - \text{proj}_{\vec{w_3}} \vec{v_4}' = \vec{v_4}' - \left(\frac{\vec{v_4}' \cdot \vec{w_3}}{\|w_3\|^2} \right) \vec{w_3}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -14 \\ 103 \\ -3 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} -14 \\ 103 \\ -3 \\ 1 \end{bmatrix} \right\|^2} \right) \begin{bmatrix} -14 \\ 103 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{-3}{10815} \right) \begin{bmatrix} -14 \\ 103 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -14 \times -\frac{3}{10815} \\ 103 \times -3/10815 \\ -3 \times -3/10815 \\ 1 \times -3/10815 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2/515 \\ -1/35 \\ 3/3605 \\ -1/3605 \end{bmatrix}$$

$$= \begin{bmatrix} -2/512 \\ -1/35 \\ 3602/3605 \\ 1/3605 \end{bmatrix}$$