SIGMAS

Students in Graduate Mathematics and Statistics

EXAM SALES

Course: MATH 101
Semester: August 2014
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UNIVERSITY OF VICTORIA
EXAMINATIONS AUGUST 2014
MATH 101, SECTIONS [A01] and PATHWAYS [A02]

Last Name:________________________ Student ID:______________

First Name:________________________ Section :______________

TO BE ANSWERED ON THE PAPER

Duration: 3 hours

Instructors:
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Questions	Marks	Score
1 to 16		32
17		4
18		4
19		4
20		4
21		4
Total		**52**

COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATORS.

THIS EXAMINATION PAPER HAS 13 PAGES PLUS THIS COVER SHEET.

INSTRUCTIONS:

1. Make sure your name is in the following places
   (a) on the top of this page
   (b) back of the last page of this exam
2. You may only use a Sharp EL-510R calculator - no other calculator is acceptable on this examination.
3. The examination consists of **16 multiple choice questions** to be answered on the bubble sheet and **5 long-answer questions** to be answer directly on the booklet. Each multiple choice question is worth **2 marks** and each long answer question is worth **4 marks**. The maximum total score is 52 points.
4. For verification purposes, show all calculations on this paper, including those for the multiple choice questions. Do all work on the test pages using the backs if necessary. Use no extra paper. We may disallow any answer not properly justified.
5. Cell phones should be turned off during the exam and not be kept with you. We do not allow use of cell phones or head phones in any manner.

21. *[4 points]* Find the first three **non-zero** terms of the Taylor Series for the function $f(x) = \sin^2(x)$ centred at the point $x = \frac{\pi}{4}$.

$f'(x) = 2\sin x \cos x$

$f''(x) = 2\cos^2 x - 2\sin^2 x$

$f'''(x) = -4\cos x \sin x - 4\sin x \cos x$

$= -8 \sin x \cos x$

$f^{(4)}(x) = -8\cos^2 x + 8\sin^2 x$

$= 8(\sin^2 x - \cos^2 x)$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/4)}{k!} = \sin^2(\pi/4) + 2\sin(\pi/4)\cos(\pi/4)(x-\pi/4)$$

$$+ \frac{[2\cos^2(\pi/4) - 2\sin^2(\pi/4)]}{2!}(x-\pi/4)^2$$

$$+ \frac{(-8\sin x \cos x)}{3!}(x-\pi/4)^3$$

$$+ \frac{8(\sin^2(\pi/4) - \cos^2(\pi/4))}{4!}(x-\pi/4)^4 + \ldots$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(x-\frac{\pi}{4}) + (0)(x-\pi/4)^2$$

$$+ \frac{(-8)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{3!}(x-\frac{\pi}{4})^3 + \ldots$$

$$= \frac{1}{2} + (x-\pi/4) + (-2/3)(x-\pi/4)^3 + \ldots$$

THE END

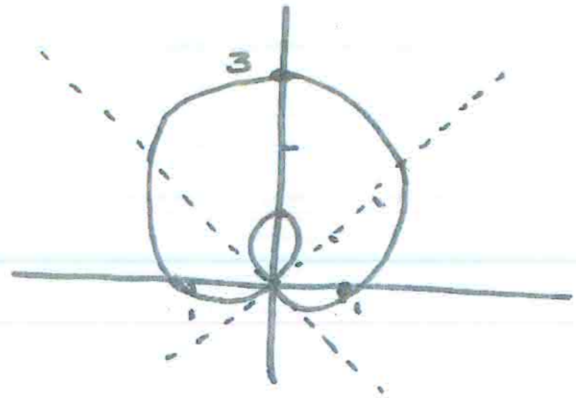
19. Consider the limaçon, $r = 1 + 2\sin(\theta)$.

(a) *[2 points]* Sketch the given curve, indicating the coordinates of a few points.

(b) *[2 points]* Calculate the area bounded by the inner loop of this limaçon.

θ	r
0	1
$\pi/4$	2.41
$\pi/2$	3
$3\pi/4$	2.41

θ	r
π	1
$5\pi/4$	−0.414
$3\pi/2$	−1
$7\pi/4$	−0.414



b) Bounds: need to find θ where $r<0$.

Solve $0 = 1 + 2\sin\theta$

$-\frac{1}{2} = \sin\theta$

$\therefore \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(1+2\sin\theta)^2\, d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(3 + 4\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left[\ + 4\sin\theta + 2(1-\cos 2\theta)\right] d\theta$$

$$= \left(\frac{1}{2}\right) \left[\theta - 4\cos\theta + 2\theta + \sin 2\theta\right]_{7\pi/6}^{11\pi/6}$$

$$= \pi - \frac{3}{2}\sqrt{3} \approx 0.544$$

20. *[4 points]* Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{(-4)^n \cdot \sqrt{n}}$. Remember to check the **endpoints** of the interval for convergence.

Use Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1} (-4)^n \sqrt{n}}{(-4)^{n+1} \sqrt{n+1}\ x^n} \right| = \lim_{n\to\infty} \left| \frac{x\sqrt{n}}{(4)\sqrt{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}\ (4)} |x| = \lim_{n\to\infty} \left(\frac{1}{4}\right) \sqrt{\frac{n}{n+1}}\ |x|$$

$$= \lim_{n\to\infty} \left(\frac{1}{4}\right) \sqrt{\frac{n}{n(1+\frac{1}{n})}}\ |x| = \lim_{n\to\infty} \left(\frac{1}{4}\right) \sqrt{\frac{1}{1+\frac{1}{n}}}\ |x| =$$

$$= \left(\frac{1}{4}\right) \sqrt{\frac{1}{1+0}}\ |x| = \frac{|x|}{4}$$

Converges when $\frac{|x|}{4} < 1 \Rightarrow |x| < 4$

Endpoint Tests:

- $x = -4$: $\sum_{n=1}^{\infty} \frac{(-4)^n}{(-4)^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ ← diverges (p-series with $p = 1/2$)

- $x = 4$: $\sum_{n=1}^{\infty} \frac{4^n}{(-4)^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Alternating Series Test:

(i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

(ii) $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$

∴ Converges by Alt. Series Test.

∴ Interval of Convergence: $-4 < x \leq 4$

17. *[4 points]*

Natural Growth Model: A growing population is modeled by the differential equation

$$\frac{dP}{dt} = kP,$$

where $P(t)$ is the population at the time t (measured in years), and k is the growth rate constant.

In 2002, there were 2 happy rabbits in a happy meadow. In 2003, there were 9 happy rabbits in the same happy meadow. Assuming this growth rate continues and is following the natural growth model above, how many happy rabbits are there in 2014? (assume all new rabbits are happy)

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\Rightarrow \ln P = kt + C$$

$$\therefore P(t) = e^{kt+C} = Me^{kt} \text{ for some constant } M$$

Initial Conditions

- In 2002 $(t=0)$:

$$P(0) = Me^{k(0)} = M = 2$$

$$\therefore P(t) = 2e^{kt}$$

- In 2003 $(t=1)$

$$P(1) = 2e^{k(1)} = 9 \Rightarrow 9/2 = e^{k} \Rightarrow k = \ln(9/2)$$

$$\therefore P(t) = 2e^{\ln(9/2)t}$$

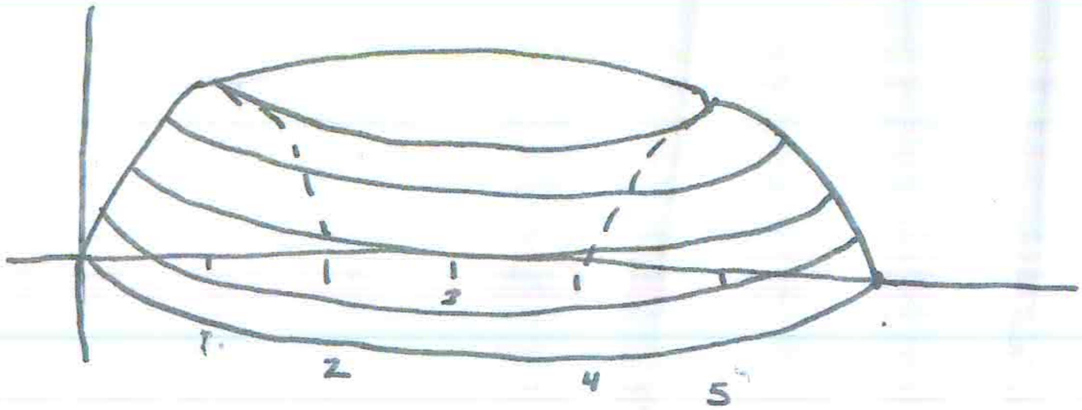
Therefore, in 2014 $(t=12)$:

$$P(12) = 2e^{\ln(9/2)(12)} = 2\,(9/2)^{12}$$

$$= 137905047.1$$

$\therefore$ 137905047 happy rabbits in the happy meadow in 2014.

18. *[4 points]* Consider the region R bounded by the graph of $y = 4x - 2x^2$ and the x-axis. Find the volume of the solid that is generated when R is revolved around the line $x = 3$



$$V = \int_a^b 2\pi r(x) h(x)\, dx$$

Radius: $r(x) = 3 - x$

Height: $h(x) = 4x - 2x^2$

Bounds: from $x = 0, 2$

$$V = 2\pi \int_0^2 (3-x)(4x-2x^2)\, dx$$

$$= 2\pi \int_0^2 12x - 10x^2 + 2x^3\, dx$$

$$= 2\pi \left[\frac{12x^2}{2} - \frac{10x^3}{3} + \frac{2x^4}{4} \right]_0^2$$

$$= 2\pi \left(\frac{16}{3}\right)$$

$$= \frac{32\pi}{3} = 33.51$$

13. Evaluate $\int_{-1}^{0} \frac{1}{x^2+5x+6} dx$ $= \int_{-1}^{0} \frac{dx}{(x+2)(x+3)}$

$$\frac{1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)} \leftarrow \text{Partial Fractions}$$

$$\Rightarrow 1 = A(x+3) + B(x+2)$$

$$1 = x(A+B) + (3A+2B)$$

$$\Rightarrow A = -B, \quad 1 = 3A+2B = 3A-2A = A \Rightarrow A=1, B=-1$$

$$\therefore \int_{-1}^{0} \frac{dx}{(x+2)(x+3)} = \int_{-1}^{0} \frac{1}{x+2} + \frac{-1}{x+3}\, dx = \Big[\ln(x+2) - \ln(x+3)\Big]_{-1}^{0}$$

$$= (\ln 2 - \ln 3) - (\ln 1 - \ln 2) = 0.288$$

(A) −0.5; (B) −0.4; (C) −0.3; (D) −0.2; (E) −0.1;
(F) 0.0; (G) 0.1; (H) 0.2; $\boxed{\text{(I) 0.3;}}$ (J) Diverges

14. Determine whether the sequence $\{ (1 + \frac{2}{n})^n \}_{n=0}^{\infty}$ converges. If the sequence converges, find what it converges to.

$$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{2}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{2}{n}\right)}$$

$$= e^{\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right)} = e^{\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{1/n}} \quad \text{use L'Hôpital's}$$

$$= e^{\lim_{n\to\infty} \left(\frac{1}{1+2/n}\right)\left(\frac{-2}{n^2}\right)(-n^2)} = e^{\lim_{n\to\infty}\left(\frac{2}{1+2/n}\right)}$$

$$= e^2 = 7.389$$

(A) Diverges; (B) 1.0; (C) 2.0; (D) 3.0; (E) 4.0;
(F) 5.0; (G) 6.0; $\boxed{\text{(H) 7.0;}}$ (I) 8.0; (J) 9.0

11. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2^n}$.

$$= 3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 3 \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \leftarrow \text{Geometric series}$$

$$= 3\left(\frac{1}{1-\left(\frac{-1}{2}\right)}\right) = 3\left(\frac{2}{3}\right) = 2$$

$\boxed{\text{(A) 2.0;}}$ (B) 3.0; (C) 4.0; (D) 5.0; (E) 6.0;
(F) 7.0; (G) 8.0; (H) 9.0; (I) 10.0; (J) 11.0

12. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n(x-3)^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}\, n!}{(n+1)!\, n(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-3)}{(n+1)\, n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n} |x-3| = 0$$

$\therefore$ Converges for any value of x

(A) 1.0; (B) 2.0; (C) 3.0; (D) 4.0; (E) 5.0;
(F) 6.0; (G) 7.0; (H) 8.0; (I) 9.0; $\boxed{\text{(J) } \infty}$

7. Evaluate $\int_1^e \ln(x)dx$

$$\left[\begin{array}{l}\text{Let } u = \ln x,\ dv = dx \\ \text{Then } du = \frac{1}{x}dx,\ v = x\end{array}\right]$$

$$= \left[x \ln x\right]_1^e - \int_1^e x\left(\frac{1}{x}\right)dx = \left[x\ln x\right]_1^e - \int_1^e dx$$

$$= \left[x\ln x\right]_1^e - \left[x\right]_1^e = (e\ln e - \ln 1) - (e-1)$$

$$= (e-0) - (e-1) = 1$$

(A) Diverges; $\boxed{\text{(B) 0.0;}}$ (C) 5.0; (D) 7.5; (E) 10.0;
(F) 11.0; (G) 12.0; (H) 13.0; (I) 14.0; (J) 15.0

8. Evaluate $\int_2^4 \frac{1}{(x-2)^3}dx$.

$$= \lim_{a\to 2^+} \int_a^4 \frac{dx}{(x-2)^3} = \lim_{a\to 2^+}\left[\frac{-1}{2(x-2)^2}\right]_a^4$$

$$= \lim_{a\to 2^+} \frac{-1}{2(4-2)^2} + \frac{1}{2(a-2)^2}$$

$$= \lim_{a\to 2^+} \frac{-1}{8} + \frac{1}{2(a-2)^2} = \frac{-1}{8} + \infty$$

$$= \infty$$

$\boxed{\text{(A) Diverges;}}$ (B) 0.0; (C) 0.1; (D) 0.2; (E) 0.3;
(F) 0.4; (G) 0.5; (H) 1.0; (I) 1.5; (J) 2.0

5. Consider the implicitly defined function $(x-1)^2+y^2=1$. Convert it to a polar function $r=f(\theta)$ and compute $f\left(\frac{\pi}{3}\right)$.

$x = r\cos\theta \quad y = r\sin\theta$

$\therefore 1 = (r\cos\theta - 1)^2 + (r\sin\theta)^2$

$\Rightarrow 1 = r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta$

$\Rightarrow 0 = r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta$

$\Rightarrow 0 = r^2(1) - 2r\cos\theta$

$\therefore r = 2\cos\theta = f(\theta)$

$f\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$

(A) 0.1; (B) 0.2; (C) 0.3; (D) 0.4; (E) 0.5;
(F) 0.6; (G) 0.7; (H) 0.8; (I) 0.9; $\boxed{\text{(J) 1.0}}$

6. Find the arc length of $y = \frac{1}{3}x^{3/2}$ from $x=0$ to $x=5$

$f'(x) = \frac{1}{3}\left(\frac{3}{2}\right)x^{1/2} = \frac{1}{2}x^{1/2}$

Arc Length $= \int_a^b \sqrt{1+f'(x)^2}\,dx$

$= \int_0^5 \sqrt{1+\left(\frac{1}{2}x^{1/2}\right)^2}\,dx$

$= \int_0^5 \sqrt{1+\frac{1}{4}x}\,dx \longrightarrow \int_*^{**} \sqrt{u}\,4\,du$

$= \int_*^{**} 4u^{1/2}\,du$

$= \left[4\left(\frac{2}{3}\right)u^{3/2}\right]_*^{**}$

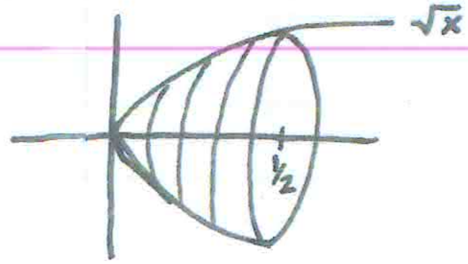
$= \left[\frac{8}{3}\left(1+\frac{1}{4}x\right)^{3/2}\right]_0^5$

$= \frac{19}{3} \approx 6.333$

$\left[\begin{array}{l}\text{Let } u = 1+\frac{1}{4}x \\ \text{Then } du = \frac{1}{4}dx\end{array}\right]$

(A) 1.0; (B) 2.0; (C) 3.0; (D) 4.0; (E) 5.0;
$\boxed{\text{(F) 6.0;}}$ (G) 7.0; (H) 8.0; (I) 9.0; (J) 10.0

3. Consider the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = \frac{1}{2}$. Calculate the volume of the solid obtained by revolving the region around the x-axis.



$$V = \int_a^b \pi r(x)^2\,dx$$
$$= \int_0^{1/2} \pi (\sqrt{x})^2\,dx$$
$$= \pi \int_0^{1/2} x\,dx$$
$$= \pi \left[\frac{x^2}{2} \right]_0^{1/2}$$
$$= \pi \left(\frac{1}{8}\right) \approx 0.3927$$

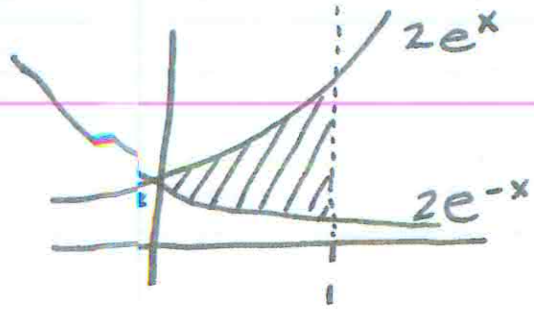
(A) 0.1; (B) 0.2; (C) 0.3; $\boxed{\text{(D) 0.4;}}$ (E) 0.5;
(F) 0.6; (G) 0.7; (H) 0.8; (I) 0.9; (J) 1.0

4. Consider the function $f(x) = x^{2x}$. Compute $f'(1)$.

$$f(x) = e^{\ln x^{2x}} = \exp[2x \ln x] = e^{2x \ln x}$$
$$f'(x) = e^{2x \ln x} \cdot (2 \ln x + 2x (\tfrac{1}{x}))$$
$$= x^{2x} [2 \ln x + 2] = 2x^{2x} [\ln x + 1]$$
$$f'(1) = 2(1)^{2(1)} [\ln(1) + 1]$$
$$= 2[1] = 2$$

(A) 1.0; $\boxed{\text{(B) 2.0;}}$ (C) 3.0; (D) 4.0; (E) 5.0;
(F) 6.0; (G) 7.0; (H) 8.0; (I) 9.0; (J) 10.0

1. Evaluate the area bounded by the curves $y = 2e^{-x}$, $y = 2e^{x}$, $x = 0$ and $x = 1$.



$$\text{Area} = \int_0^1 2e^x - 2e^{-x}\,dx$$

$$= 2\left[\int_0^1 e^x\,dx - \int_0^1 e^{-x}\,dx\right]$$

$$= 2\left[e^x\right]_0^1 + 2\left[e^{-x}\right]_0^1$$

$$= 2\left[e^1 - 1\right] + 2\left[e^{-1} - 1\right]$$

$$= 2.172$$

(A) -2.4; (B) -1.8; (C) -1.2; (D) -0.6; (E) 0.0;
(F) 0.6; (G) 1.2; (H) 1.8; $\boxed{\text{(I) 2.4;}}$ (J) Diverges

2. Evaluate $\int_0^1 e^x\sqrt{e^x - 1}\,dx$.

$$\int_0^1 e^x\sqrt{e^x - 1}\,dx$$

$$\left[\begin{array}{l}\text{Let } u = e^x - 1 \\ \text{Then } du = e^x\,dx\end{array}\right]$$

$$= \int_{*}^{**} \sqrt{u}\,du$$

$$= \left[\frac{2}{3}u^{3/2}\right]_{*}^{**}$$

$$= \left[\frac{2}{3}(e^x - 1)^{3/2}\right]_0^1 = 1.5016$$

(A) 0.0; (B) 0.5; (C) 1.0; $\boxed{\text{(D) 1.5;}}$ (E) 2.0;
(F) 2.5; (G) 3.0; (H) 3.5; (I) 4.0; (J) Diverges

15. Find the **real part** of the complex number $z_1 \cdot z_2$, where $z_1 = 1 + i$ and $z_2 = \sqrt{5} - i$

$$z_1 \cdot z_2 = (1+i)(\sqrt{5}-i)$$
$$= \sqrt{5} - i + i\sqrt{5} + 1$$
$$= (1+\sqrt{5}) + i(\sqrt{5}-1)$$

$$\mathrm{Re}(z_1 \cdot z_2) = (1+\sqrt{5}) \approx 3.24$$

(A) −5.0; (B) −4.0; (C) −3.0; (D) −2.0; (E) −1.0;
(F) 0.0; (G) 1.0; (H) 2.0; $\boxed{\text{(I) 3.0;}}$ (J) 4.0

16. Find the slope $\frac{dy}{dx}$ of the parametric curve $x = 1 + \cos(t)$, $y = \sin(t)$ at the point $(x, y) = (1, 1)$, which occurs when $t = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dx} = \frac{\cos(t)}{-\sin(t)}$$

slope at $(1,1)$:

$$\frac{\cos(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = \frac{-0}{-1} = 0$$

(A) −1.5; (B) −1.1; (C) −0.7; (D) −0.3; $\boxed{\text{(E) 0.0;}}$
(F) 0.3; (G) 0.7; (H) 1.1; (I) 1.5; (J) ∞

9. Evaluate $\int_4^{+\infty} \frac{96}{(2x-4)^4}dx$ $= \lim_{a\to\infty} \int_4^a \frac{96}{(2x-4)^4} dx$

$\left[\begin{array}{l}\text{Let } u = 2x-4 \\ \text{Then } du = 2\,dx\end{array}\right]$ $= \lim_{a\to\infty} 96 \int_*^{**} \frac{1}{u^4}\left(\frac{1}{2}\right) du$

$= \lim_{a\to\infty} \left[48 \frac{u^{-3}}{-3}\right]_*^{**} = \lim_{a\to\infty} \left[\frac{-16}{(2x-4)^3}\right]_4^a$

$= \lim_{a\to\infty} -16 \left[\frac{1}{(2a-4)^3} - \frac{1}{64}\right]$

$= -16\left(0 - \frac{1}{64}\right) = \frac{1}{4}$

(A) −1.00; (B) −0.75; (C) −0.50; (D) −0.25; (E) 0.0;
$\boxed{\text{(F) 0.25;}}$ (G) 0.50; (H) 0.75; (I) 1.00; (J) Diverges

10. Evaluate $\int_0^{1/2} \frac{3x^2}{\sqrt{1-x^2}}dx$

$\left[\begin{array}{l}\text{Let } x = \sin\theta \\ \text{Then } dx = \cos\theta\, d\theta \\ \text{When } x = 0,\ \theta = 0 \\ \qquad x = 1/2\ \ \theta = \frac{\pi}{6}\end{array}\right]$

$= 3\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx$

$= 3\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$

$= 3\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$

$= 3\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$

$3\int_0^{\frac{\pi}{6}} \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta$

$= 3\left[\frac{1}{2}\theta - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\sin 2\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{4} - \frac{3\sqrt{3}}{8} \approx 0.136$

$\boxed{\text{(A) 0.0;}}$ (B) 0.3; (C) 0.6; (D) 0.9; (E) 1.2;
(F) 1.5; (G) 1.8; (H) 2.1; (I) 2.4; (J) 2.7