# The Cost-of-Capital in Economic Studies

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## Part I—Fundamental Relations

THE COST-OF-CAPITAL is the rental (return), plus the tax paid to retain the rental (income tax), plus the cost of recovery of the capital (depreciation). Some or all of these factors are often approximated roughly, with the rationalization that if there is an error, all alternates are affected alike. Unfortunately, when two or more alternates exist, the effect of the cost-of-capital is often the factor to be evaluated. In the example of Table I for instance, if a 12% fixed charge rate is used, there is no apparent economic difference; if a 13% fixed charge rate is used, there is a difference of $70,000 per year. The $70,000 per year difference is less than 3% difference in annual costs; it may be argued that this it too small to be significant, but a 3% difference between well considered engineering proposals is not unusually small.

The results of an economic study are not the only factors necessary for a management choice, nor is the cost-of-capital the only factor in an economic study. But the economic results are a vital part of a proper management choice, and the cost-of-capital is an essential part of a proper economic study. Since the cost-of-capital may have a controlling effect on the results of an economic study and hence on a management decision, a high degree of precision is justified in the computation of the cost-of-capital.

The purpose of this paper is to provide, from a few fundamental concepts, a unified mathematical development of the cost-of-capital. The correct treatment of depreciation is emphasized, since many common errors and controversies arise out of misconceptions about this essential item. Several formulas not generally available are derived, and certain commonly used formulas are shown to be only special cases.

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### Fixed Charges

The cost-of-capital (return, depreciation, and income taxes) is commonly lumped with the cost of insurance and ad valorem taxes, and the entire group referred to as "fixed charges." It is convenient to reduce all the fixed charges to an equivalent annual percentage of original cost. Thus, by multiplying this fixed percentage by the original cost, a fixed quantity of dollars per year is determined which is financially equivalent to the actual costs.

Insurance and ad valorem taxes must be determined for each type of project and each geographical location. These two items are commonly available as percentages of some base valuation. These percentages must then be converted to the original cost base normally used for fixed charges. This conversion is purely a matter of convenience for use. For example, suppose real property tax is $1.00 per $1,000 of valuation (0.10%), and valuation is at 60% of original cost; then the contribution to the fixed rate is $0.60\% \times 0.10\% = 0.06\%$ based on original cost. The remaining three items of the fixed charges are the cost-of-capital items, which are amenable to mathematical generalization.

### Methods of Economic Comparison

A wide variety of methods for making economic studies is available, the techniques of which are detailed in textbooks. The author has found the method of revenue requirements the most generally useful; it is exact and permits well-defined mathematical generalization.

The term "revenue requirements" is used to indicate that the quantities to be determined are the amounts of revenue which must be earned in order that a minimum acceptable return be earned after taxes.

For many comparisons, where anticipated revenues are identical or meaningless, the revenue requirements provide a direct comparison between plans. The annual difference in these requirements is the annual economic difference. The plan having the lowest total of revenue requirements is clearly the most economical.

For those studies in which anticipated revenues are meaningful, the total revenue requirement provides a standard against which the actual anticipated revenue can be compared. The ratio of anticipated revenue to revenue requirement may be taken as a measure of the probability of achieving the minimum acceptable return.

The revenue requirement method provides an exact and complete method of engineering comparison. The term "revenue requirements" is preferable to alternate terms, such as "annual costs"; it avoids the use of the word "cost," which is objectionable because of the special meaning attached to it by accountants and financial analysts.

### Rate of Return

The minimum acceptable rate of return on invested capital must be established before the complete cost of capital can be computed. This rate of return is the minimum return acceptable for the company after taxes. It includes earnings available for dividends and interest on bonds and other obligations.

The determination of a minimum acceptable rate of return may not be easy or simple. A management policy decision may be required. Note, however, that some such determination must be made for any economic study by any method. There is a particular virtue in determin-

**Table I. Effect of Cost-of-Capital Factor**

100-Mw Generating Plant for 1,000 Hours Per Year Operations

| Items on Annual Basis | Fixed Charge Rate 12% Plant A* | 12% Plant B† | 13% Plant A | 13% Plant B |
|---|---|---|---|---|
| Fixed cost | $2,100,000 | $1,260,000 | $2,275,000 | $1,365,000 |
| Operation | 300,000 | 1,140,000 | 300,000 | 1,140,000 |
| Total | $2,400,000 | $2,400,000 | $2,575,000 | $2,505,000 |
| Difference | | $0.0 | | $70,000 |

* Plant A at $175/kw; 3 mills/kw-hr.

† Plant B at $105/kw; 11.4 mills/kw-hr.

ing a minimum acceptable rate of return after taxes, as such. This rate can then properly be used for the value of money (interest rate) in any calculation involving the present value of future money. That is, if a 7% return on investment is acceptable, then 7% is an acceptable return on any investment within the company structure.

A choice of rate of return which is either too high or too low will bias the results of an engineering study. A high rate favors low capital cost items and, conversely, a low rate favors high capital cost items.

Ultimately, the proper value for the acceptable rate of return must be determined by someone making a decision; this much will be acceptable and no less. The decision may be guided by many factors.[1] One useful guidepost is the past history of the ratio of net return after taxes (dividends, interest, and retained earnings) per year to the total equity for the same year. Additional guidance may be obtained by breaking down the return into its components: dividends, interest, etc., and investigating the past history and future expectations of each component.

Caution must be used in determining the satisfactory rate of return to insure that the rate is based on actual dollars available for investment. For example, 4 5/8% 10-year bonds that are sold for $971 ($1,000 face value) actually provide bond money at 5%.

Similarly, if the legal maximum allowable return is to be used, which, let us say, is 6%, the actual rate of return on capital structure may be 6$^1/_2$% to 7$^1/_2$% in fair-value states because of the difference between capital structure and rate base. The dollars available for investment are those in the capital structure of the company.

It is convenient, and usually sufficiently accurate, to assume that the rate of return will be constant throughout a given study. The relationships to be developed here assume a constant rate of return. The rate of return can be varied with time, but the calculations will be extremely tedious and very specific for each kind of variation in the rate.

## Annual Return

The annual revenue requirement, necessary to provide an acceptable rate of return on the capital invested in a particular project, is in general not a constant. As depreciation accrues, part of the capital invested is recovered; hence, the satisfactory rate need only be earned on a progressively smaller remaining investment each year. For example, see Table II. Notice that in each year the money which has been previously recovered does not enter into the computation of return for that year.

Once capital has been recovered from a particular project, that project need no longer earn a return on the recovered portion, since the money is then available to retire the securities involved. The money will not normally be used to retire securities but to avoid the issuing of some securities for other projects. The effect, however, is the same: the original project need no longer be charged with return on the recovered investment. The exceptional case, for which this is not true, is discussed in Part II.

## Equivalent Equal Annual Values

The return and depreciation for each year will generally be different. The desired value of fixed charge rate is a constant for all years. Therefore, the variable annual costs must be converted into a set of annual costs which are the same for every year, and financially equivalent to the variable costs. Such constant annual costs or revenue requirements are termed "equivalent equal annual values," and are computed by straightforward applications of present-worth relations.

## Depreciation

Depreciation, reserve for depreciation, and reinvested funds from accumulated depreciation cause much unnecessary confusion. According to purpose, intent, and accounting definition, depreciation is simply the provision for recovery of invested capital.

The various "methods" of depreciation are simply techniques for spreading systematically the recovery of capital over the expected life of the equipment. It is important to observe that, logically and by accounting practice, the entire recovery of the invested capital (less salvage) must come from the revenues of the project in which the money is invested. Failure to observe this principle has led to the assumption that the "interest on the accumulated reserve" in the sinking fund method of depreciation is derived from reinvesting the accumulated reserve. This is not correct, since it would provide for recovery of capital in the original investment being charged to some other investment. Note also that by the sinking fund method of depreciation (see Table IIB) the total annual depreciation increases with time because of the "interest" which must be provided. The sum of the total annual depreciation column in any case must be the amount of the original investment.

"Sinking fund" is an unfortunate term since there is no actual fund. It represents an accounting item usually titled "Accumulated Reserve for Depreciation." The account is actually only an accounting device used for tabulating the amount of the original capital which has been recovered.

Capital recovered from the project in which it was orginally invested is (theoretically) available to retire securities. Hence, when dollars are made available through the mechanism of depreciation, they are simply dollars of capital made available for investment or retirement of securities.

**Table II. Depreciation**

Original Investment of $1,000; Return at 10% Per Year

| Year End | Remaining Investment | Return on Remaining Investment | Total Annual Depreciation | Present Value: Return | Present Value: Depreciation |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| **A—Straight-Line Depreciation Over 5 Years** | | | | | |
| 1 | $1,000.00 | $100.00 | $200.00 | $90.91 | $181.82 |
| 2 | 800.00 | 80.00 | 200.00 | 66.12 | 165.29 |
| 3 | 600.00 | 60.00 | 200.00 | 45.08 | 150.26 |
| 4 | 400.00 | 40.00 | 200.00 | 27.32 | 136.60 |
| 5 | 200.00 | 20.00 | 200.00 | 12.42 | 124.18 |
| 6 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Totals | | | $1,000.00 | $242.00 | $758.00 |
| **B—Sinking Fund Depreciation Over 5 Years** | | | | | |
| 1 | $1,000.00 | $100.00 | $163.80 | $90.91 | $148.91 |
| 2 | 836.20 | 83.62 | 180.18 | 69.10 | 148.91 |
| 3 | 656.02 | 65.60 | 198.20 | 49.28 | 148.91 |
| 4 | 457.82 | 45.78 | 218.02 | 31.27 | 148.91 |
| 5 | 239.80 | 23.98 | 239.80 | 14.89 | 148.91 |
| 6 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Totals | | | $1,000.00 | $255.00 | $745.00 |

When a new investment is made, all or some part of the dollars used may be those recovered from other older investments through the mechanism of depreciation. This has no effect on the revenue requirements for the new investment. All money invested is money provided by security holders for investment in the projects of the company. That the money might already have been used and recovered several times does not alter its status.

## Return Plus Depreciation

If the sums of the two present-value columns of the Table II are combined, the total present value of interest (return) plus recovery (depreciation) will be exactly the original investment. This is a logical result, since it amounts to saying that the present value of any investment is the amount of the investment. The effect of salvage is discussed in a later section.

Appendix II is a general mathematical proof that the present value of return plus depreciation is always the original investment. Note that the proof is completely general; any method of depreciation may be used, in particular, retirement accounting. That is, depreciation can be charged exactly as individual portions fail (for example, poles in a transmission line). Thus, it makes no difference what method of depreciation is used or in what manner retirements may be made; the present value of return plus depreciation is exactly equal to original investment.

It therefore follows that the equivalent equal annual cost of any investment (return plus depreciation) is the original cost multiplied by the factor, called the capital recovery factor, which will spread this over the recovery period:

$$\frac{1}{a_{\overline{n}|}}=\frac{i(1+i)^n}{(1+i)^n-1}$$

where $n$ is the number of years over which the recovery of the investment is to be made. The average economic life is commonly used for the value of $n$.

The capital recovery factor may be written in a different form:

$$\frac{1}{a_{\overline{n}|}}=i+\frac{i}{(1+i)^n-1}$$

The first term is just the rate of return, and the second term is the sinking fund factor, $1/s_{\overline{n}|}$. If the equivalent equal annual revenue requirement for return is written as the satisfactory rate of return and the equivalent equal annual revenue requirement for depreciation is written as the sinking fund factor, the sum of these two will always be exactly equal to the equivalent equal annual revenue requirement for return plus depreciation, regardless of the actual method of depreciation. These two terms are separately incorrect as labeled, but the sum is correct regardless of the actual method of depreciation used.

The argument which ensues shows the use of these two values to be reasonable. Assume that the sinking fund method of depreciation is used, with the satisfactory rate of return as the interest rate; then, return at this same rate will be charged on the declining investment, and interest at the same rate will be charged on the increasing depreciation account. But the sum of the remaining investment and the depreciation account is just the original investment; hence, the return plus interest will always be the same. Therefore $i+1/s_{\overline{n}|}$ is exactly the equal annual revenue requirement for return plus depreciatio ($\bar{r}+\bar{d}$).

This illustrative "proof" is sometimes useful, but it must be used with care. It is important not to lose sight of the fact that $i+1/s_{\overline{n}|}$ is the correct equivalent equal annual total for return plus depreciation, regardless of the method of depreciation or any arbitrary interest rate which is used to calculate an actual sinking fund.

## Income Tax

The revenue requirement for income tax, attributable to a given project, is the taxable income for that project multiplied by the income tax rate. The taxable income required to provide the acceptable rate of return after taxes is easily computed by working back from the return after taxes. The revenue requirement for income tax is computed in Appendix III-A for the case in which the actual method for recovery of investment is the same as the method of depreciation used for income tax.

The resulting relation is

$$\bar{l}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\times\left(i+\frac{1}{s_{\overline{n}|}}-\bar{d}\right)$$

The use of this relationship is correct only when the actual recovery method to be used is identical to the method used for income tax purposes.

The most common value used for $\bar{d}$ in this relationship is $1/n$, indicating that straight-line depreciation is used. The equivalent equal annual value for income tax then becomes

$$\bar{l}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\overline{n}|}}-\frac{1}{n}\right)$$

This is the most commonly used function for $\bar{l}$, but it is correct only when the same straight-line depreciation is used for both recovery and income tax purposes and both are equal to $1/n$.

Note that it is only for the calculation of the income tax component that a particular type of depreciation need be specified. The functions expressing the values of $\bar{d}$ for various methods of depreciation are given in Part II.

A popular misconception about the factor $(i+1/s_{\overline{n}|}-1/n)$ should be carefully avoided: the difference between the last two terms is often considered to be the difference between two kinds of depreciation, sinking fund and straight-line. One of the terms is an equivalent equal annual value and the other is not. Therefore, the difference between the two would have no more meaning than subtracting five orange from six grapes. The whole factor is actually

$$\bar{r}=\left(\frac{1}{a_{\overline{n}|}}-\bar{d}\right)=\left(\frac{1}{a_{\overline{n}|}}-\frac{1}{n}\right)$$

when $\bar{d}=1/n$.

Expressed in this manner, both $1/a_{\overline{n}|}$ and $1/n$ are equivalent equal annual values and the difference means that it is the equivalent equal annual value of return. The term $1/a_{\overline{n}|}$ can be broken up into the sum of two terms, $i$ and $1/s_{\overline{n}|}$, but these two separately have no more meaning than do $1/n^2$ and $(n-1)/n^2$, which add up to $1/n$.

## Summary of Basic Relations

With the minimum satisfactory rate of return determined, the cost-of-capital is readily determined with only four other factors:

1. Income tax rate.
2. Average bond interest rate.
3. Bond ratio.
4. Life of the equipment and the required depreciation schedule.

With these factors, the cost-of-capital is computed:

Return (pseudo value) $=i$

$$\text{Depreciation (pseudo value)} = \frac{1}{s_{\overline{n}|}}$$

$$\text{Income tax}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\overline{n}|}}-\bar{d}\right)$$

These relations are adequate for various studies, and for quickly estimating the order of magnitude. In more precise work, several other factors which should be included are discussed here and in Part II.

Note that the use of the pseudo equivalent equal annual values gives the correct total, yet preserves the convention that the return is $i$. Use of the actual values would confuse those not familiar with the details of the cost-of-capital factors.

## Refinements

Among the several refinements (of the basic relationships just discussed) which are important in particular applications, the most obvious is a consideration of the effect of salvage value or nondepreciable portion of property. This is a relatively simple process and should be used in any case of appreciable salvage or nondepreciable portion.

In some cases the method of depreciation used for income tax purposes will be different from that actually used for recovery of the investment, and the period of recovery may be different. This may be of less importance than consideration of salvage, but the correction is simple and straightforward, and there is little reason for neglecting this factor.

The income tax factors are calculated with the use of a single rate $t$, which is often simply the Federal income tax rate. In some cases there is an applicable state income tax rate. The two tax rates can be combined into a single composite rate for use in the income tax formulas. The two income tax factors cannot be calculated independently, since state taxes are deductible for Federal income tax purposes, and Federal taxes may be deductible for state income tax purposes.

In all the computations presented here, money is treated as though all transactions occurred at the end (or beginning) of a fiscal year. A very small change would result by attempting to account for smaller time periods. It is doubtful that this excessively complicated procedure would actually result in a more realistic picture. If such a correction is made, care must be exercised to adjust interest rates properly. The semiannual interest rate corresponding to 6% annually is not 3%, but approximately 2.956%.

The various refinements, except the last-mentioned, are discussed in detail in the ensuing sections; others are considered in Part II.

## Salvage

For many projects the salvage and nondepreciable portions of the investment are such a small percentage of the total that practically no error is introduced by neglecting the fact that such portions are not depreciable. When nondepreciable items are significant, the only exact method is to treat the depreciable portions as outlined above and the remainder separately. This is necessary because the capital recovery of positive salvage or nondepreciable portions is not from revenue.

The nondepreciable portion of an investment must earn a return at the rate $i$ throughout the life of the project; depreciation is zero, and income tax is based on the full return:

$$\bar{r} = i$$

$$\bar{d} = 0$$

$$\bar{l} = \left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)i$$

The only exception to this occurs when the actual life $L$ is different from the period of amortization $n$. Then the factors become

$$\bar{r}' = i\left(\frac{a_{\overline{L}|}}{a_{\overline{n}|}}\right)$$

$$\bar{d}' = 0$$

$$\bar{l}' = \left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)i\left(\frac{a_{\overline{L}|}}{a_{\overline{n}|}}\right)$$

The total annual cost-of-capital can thus be found by multiplying the cost-of-capital rates for depreciable portions by the fraction depreciable and adding the cost-of-capital rates for nondepreciable portions multiplied by the fraction not depreciable. If the depreciable fraction of the total is $k$, then the total cost-of-capital is

$$\text{Total return} = i$$

$$\text{Total depreciation} = \frac{k}{s_{\overline{n}|}}$$

$$\text{Total income tax} = \left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\times \left(i+k\left[\frac{1}{s_{\overline{n}|}}-\bar{d}\right]\right)$$

That is, the fact that some portion of the investment is not depreciable alters only the revenue requirement for income tax and depreciation. Since $\bar{d}$ is usually greater than $1/s_{\overline{n}|}$, the net effect is to increase the requirement for taxes and to reduce the requirement for depreciation, with a net total reduction in cost-of-capital.

It is often convenient to express the effect of salvage as an additive corrective factor thus:

Depreciation correction factor

$$= -(1-k)\left(\frac{1}{s_{\overline{n}|}}\right)$$

Income tax correction factor

$$= \left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(\bar{d}-\frac{1}{s_{\overline{n}|}}\right)(1-k)$$

## Multiple Income Tax

In states where corporations are subject to a state income tax, taxable income for state and Federal purposes will be essentially the same except for the effect of the income taxes themselves. The state taxes will be deductible as expenses for Federal tax purposes, and the converse may be true in some cases and not in others. Therefore, two cases are considered.

*Case 1.* Federal income tax is not deductible for state income tax purposes. In this case the composite rate $t$ is calculated:

$T_s = sI =$ state tax

$T_f = f(I-T_s) = (f-fs)I =$ Federal tax

$$t = \frac{T_f+T_s}{I} = (f-fs)+s$$

$$t = (f+s-fs)$$

where $f$ and $s$ are the Federal and state tax rates respectively, and $I$ is taxable income.

*Case 2.* Federal income tax is deductible for state income tax purposes. In this case, the composite rate $t$ is calculated:

$T_s = s(I-T_f) =$ state tax

$T_f = f(I-T)_s =$ Federal tax

$$t = \frac{T_f+T_s}{I} = \frac{(f-fs)+(s-fs)}{(1-fs)}$$

$$t = \left(\frac{f+s-2fs}{1-fs}\right)$$

## Multiple Depreciation

If the actual recovery of investment is to be different from that used for income tax purposes, the problem is not quite as simple as the basic case. When the actual recovery during any year is greater than that claimed for income tax purposes, tax is incurred on the difference. When the actual recovery is less than that claimed for tax purposes, a tax loss can be claimed, resulting in a reduction in the tax obligation. The return after taxes is unaltered in any case.

The relation given above for $\bar{l}$ is the tax obligation incurred as a result of the return. The increase or decrease in tax obligation arising from the difference between actual and tax depreciation is a separate, independent amount. The difference after taxes in any year is

$$D(y) - D_t(y)$$

Hence, the resulting income tax change is

$$\left(\frac{t}{1-t}\right)[D(y)-D_t(y)]$$

and the equivalent equal annual value change is

$$\left(\frac{t}{(1-t)}\right)(\bar{d}-\bar{d}_t)$$

This quantity is not affected by the bond ratio. Full credit for the bond interest is taken in the calculation of the tax obligation arising from the return.

The complete expression for the equivalent equal annual revenue requirement for income tax is then

$$\bar{l}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\overline{n|}}}-\bar{d}\right)+\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)$$

This relation is derived in toto in Appendix III-B.

Use of the relation

$$\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)$$

is straightforward if the only difference between actual desired recovery and income tax depreciation is the method. If the period of recovery is different for the two purposes, care must be exercised to insure that $d_t$ is on the same amortization period as $\bar{d}$. The simplest method for accomplishing this is to multiply the value for $d_t$ based on the recovery period to be used for income tax purposes by the ratio$(a_{\overline{t|}}/a_{\overline{n|}})$, where $t$ refers to the recovery period for tax purposes, and $n$ refers to the desired amortization period.

A hypothetical numerical example may serve to clarify this. Suppose straight-line depreciation is to be used for all purposes, and the amortization period for actual recovery is to be 10 years. Suppose further that income tax depreciation can cover 5, 10, or 15 years. If it is taken over 10 years, the problem follows the simple case. But the fast (5-year) write-off change in $\bar{l}$, for tax purposes, will be

$$\left(\frac{52}{100-52}\right)\left(0.100-\frac{4.212}{7.360}\,0.200\right)=-0.0156 \quad (i=0.100)$$

i.e., there will be a reduction in the cost-of-capital of 1.56 percentage points.

The slow write-off change in $\bar{l}$ is

$$\left(\frac{52}{100-52}\right)\left(0.100-\frac{9.712}{7.360}\,0.067\right)=+0.0126$$

i.e., there will be an increase in the cost-of-capital of 1.26 percentage points.

Including the effect of salvage in this relation results in multiplying the additive correction factor

$$\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)$$

by the depreciable fraction $k$.

## Summary

The basic relations combined with the effect of salvage and multiple depreciation can then be expressed as

*Return*

Basic (pseudo value) $=(i)$

*Depreciation*

Basic (pseudo value) $=\left(\frac{1}{s_{\overline{n|}}}\right)$

Salvage $=-(1-k)\left(\frac{1}{s_{\overline{n|}}}\right)$

*Income tax*

$$\text{Basic}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\overline{n|}}}-\bar{d}\right)$$

$$\text{Salvage}=+\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(\bar{d}-\frac{1}{s_{\overline{n|}}}\right)\times(1-k)$$

$$\text{Multiple depreciation}=+\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)k$$

The quantities below the basic value in each case are the additive corrections.

The use of the correct $\bar{d}$ rather than $1/n$ in the computation of income tax requirements is not a trivial distinction. The revenue requirements for income tax may differ by several percentage points for different methods of depreciation. The calculation of the value of $\bar{d}$ for several methods is covered in Part II.

An alternate treatment of invested capital recovered through depreciation is to say that it somehow remains associated with the original plant; but it is reinvested in other plants, and part of the return from other plants is credited to the original plant. This is the treatment used in most previous papers, and is the source of much confusion. Mathematically this approach is precisely equivalent to that used by the author and, correctly applied, will produce the same results. Correctly accounting for all the money is more difficult and fruitless discussions arise about the proper return rate to use on other plants, and the total cost of money.

# Part II—Depreciation, Retirement Dispersion, and Reinvestment

In Part I the fundamentals of the coat-of-capital were developed. It is the purpose here to extend these calculations in three vital areas: (1) The method of calculating the equivalent equal annual value of depreciation for any arbitrary method of depreciation is illustrated, and the formulas for several common methods of depreciation are developed. (2) Retirement dispersion is discussed and the correct formulas for calculating the effect of retirement dispersion on the cost-of-capital computed. (3) Possible failure of the assumption of reinvestment used in the development is discussed, and the correction factors necessary for taking this into account developed.

## General Depreciation

The calculation for depreciation follows a straightforward technique. By any method used, one knows, for each year, exactly how much of the original plant will be depreciated in that year. Let $D(y)$ be the depreciated portion of the original plant in the year $y$. The present worth of the depreciation for any year is just the product of the present-worth factor $v^y$, and the depreciation factor $D(y)$. The total present worth of all the depreciation equals the sum of all such products taken over the period during which depreciation occurs. Then, the equivalent equal annual value of depreciation for any amortization period, $n$ years, is the product of the capital recovery factor, $1/a_{\overline{n|}}$, and the total present worth of depreciation:

$$\bar{d}=\frac{1}{a_{\overline{n|}}}\sum_{y=0}^{m} v^y D(y)$$

Note that the upper limit of the summation may be equal to, greater than, or less than $n$.

The sum of the equivalent equal annual values for return and depreciation is exactly the capital recovery factor as shown in Part I.

$$\bar{r}+\bar{d}=\frac{1}{a_{\overline{n|}}}=i+\frac{1}{s_{\overline{n|}}}$$

or

$$\bar{d}=\frac{1}{s_{\overline{n|}}}+(i-\bar{r})$$

Now, if the entire investment were depreciated at the time of construction, then return would be zero for every year, and $\bar{r}$ would be zero.

$$\bar{d}=\frac{1}{s_{\overline{n|}}}+i=\frac{1}{a_{\overline{n|}}}$$

represents the maximum possible value of $\bar{d}$. If 100% depreciation were taken in year $n$ (retirement accounting) then $f=i$, and

$$\bar{d}=\frac{1}{s_{\overline{n}|}}$$

Therefore it will generally be correct to say that $\bar{d}$ lies between $1/a_{\overline{n}|}$ and $1/s_{\overline{n}|}$. Note, however, that $\bar{d}$ can be less than $1/s_{\overline{n}|}$.

## Common Methods

Four methods of depreciation are in common use. The value of $\bar{d}$ for each of these is derived in Appendix II following the procedure outlined above. The most useful results are outlined:

1. Straight Line

The straight-line method is the simplest. When the depreciation period and the amortization period are the same, the equivalent equal annual value factor equals the actual annual factor: $\bar{d}=1/n$.

One occasionally encounters the attitude that this method is improper for engineering economic studies because it overstates the total depreciation since, if the "reserve for depreciation" is invested, it will earn interest and the total on retirement will exceed the original capital. The fallacy contained here arises from the pernicious reasoning that treats the "reserve for depreciation" as though it represented a fund of money. It does not. On a utility balance sheet the reserve for depreciation (under one title or another) appears as a liability, which is balanced purely by overstating assets. The "reserve for depreciation" is only an accounting device for keeping track of the amount of capital which has been recovered from a given item.

2. Sum-of-Years Digits

The sum-of-years digits method yields slightly higher values for $\bar{d}$ than the straight-line method, since the depreciation is higher in the early years. When the depreciation period and the amortization period are equal,

$$\bar{d}=\frac{2}{(n+1)i}\left(\frac{1}{a_{\overline{n}|}}-\frac{1}{n}\right)$$

3. Sinking Fund

Because of the use of the sinking fund factor as the pseudo value (see Part I) of depreciation, the sinking fund method of depreciation is sometimes wrongly associated with the method of economic comparison called revenue requirements. The method of comparison is applicable no matter what method of depreciation is used.

The sinking fund method does *not* provide for a constant amount each year. The amount increases each year because of the interest which must be provided. Even the present worth of each year's depreciation is a constant only if the interest rate on the sinking fund is identical with the minimum satisfactory rate of return.

When the depreciation period and the amortization period are equal and the interest rate on the sinking fund is the minimum satisfactory rate

$$\bar{d}=\frac{1}{a_{\overline{n}|}}\,\frac{1}{s_{\overline{n}|}}\,\frac{n}{(1+i)}$$

4. Declining Balance

Of the two ways in which the declining balance method is used, the true method is the one which spreads the capital recovery over all future time, i.e., there is no cutoff date. It is sometimes used for group depreciation for income tax purposes. The equivalent equal annual value is

$$\bar{d}=\frac{1}{a_{\overline{n}|}}\left(\frac{e}{i+e}\right)\left(\frac{1+fi}{1+i}\right)$$

where $fe$ is the first-year actual depreciation rate, and $e$ is the actual rate for all other time.

The limited declining balance method is occasionally used in order to provide recovery in a finite period. For this method the declining balance method is used through a particular year $b$; then straight-line depreciation is used for the remainder. The value of $\bar{d}$ for this case is given in Appendix IV-D, as are more general forms of the $d$ values for all of the foregoing.

## Mortality Dispersion

Some assets may have a definite economic life dictated by obsolescence. In general, however, the actual life of a single object cannot be predicted. Given a large number of similar objects, and sufficient historical data, reasonable predictions can be made about the number which will still be useful after any given number of years. Such data can be represented as a curve called the "survivor curve." A number of useful theoretical curves, the Iowa series,[3] are available for use when the available historical data are inadequate.

Regardless of the dispersion or the shape of the survivor curve, the present value of return plus depreciation on the original investment is still exactly unity as demonstrated in Part I. Hence the equivalent equal annual revenue requirement for return plus depreciation depends only upon the value of the acceptable rate of return and the recovery or amortization period.

When the useful economic life of a project can be predicted, it is reasonable that the recovery period should be exactly this useful economic life. When the life can be predicted only from a survivor curve, the recovery period to be used is not so obvious.

The concept of a survivor curve is based on the assumption of an extremely large number of identical items. From such a curve, and its associated mortality curve, three different possible values for life can be directly determined: the mean, or average life; the mode, or year of greatest loss; and the median, or year in which just one half of the original number are still in service. These different values are sometimes coincident, although the average life is the most commonly used.

When only one or a limited number of objects are being considered, the survivor curve becomes a probability curve. The 50% survival point, for example, is the year for which the odds are even that a single object will still survive. Thus, when the mortality curve is symmetric, and all three of the factors mentioned above coincide, there is only a 50-50 chance that a given object will survive to the average age. For nonsymmetric mortality curves, the chances of a single object reaching the average age may increase or decrease slightly, but will be on the order of 50%.

It would be unfair to assign the same value to an object which has only a 50% chance of reaching age 10, for example, as to an object which is certain to reach age 10. True, there is also a 50% chance that the first item may exceed age 10, but the possible gain from service during the 11th year will not quite balance the loss if there is no service during the 10th year. Similarly, items having different mortality curves, but the same average life, do not have quite the same economic value.

The several approaches to this problem all lead to the same results. Two of these will be discussed.

The first approach assigns a recovery period $s$, which is less than the average life $n$, and represents that number of years for which the present worth of 100% service is just equal to the present worth of the service calculated from the survivor curve. The recovery period is a measure of the value of the expected service; it is derived in Appendix III as the equivalent economic life $s$.

The capital recovery factor $1/a_{\overline{g}|}$ for the equivalent economic life is just the factor

$$1/\sum_{1}^{m} v^y s(y)$$

Hence. the values of $\bar{r}$ and $\bar{d}$ calculated on average life are corrected by the factor $a_{\overline{n}|}/a_{\overline{g}|}$. Therefore, the cost-of-capital is

$$\text{Return} = i$$

$$\text{Depreciation} = \frac{1}{a_{\overline{g}|}} - i = \frac{1}{\sum_{1}^{m} v^y s(y)} - i = \frac{1}{s_{\overline{g}|}}$$

$$\text{Income tax} = \left(\frac{t}{1-t}\right)\left(1 - \frac{bB}{i}\right) \times \left(\frac{1}{a_{\overline{g}|}} - \frac{a_{\overline{n}|}}{a_{\overline{g}|}}\bar{d}\right)$$

or

$$\text{Income tax} = \left(\frac{t}{1-t}\right)\left(1 - \frac{bB}{i}\right) \times \left(i + \frac{1}{s_{\overline{n}|}} - \bar{d}\right)\left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)$$

where $\bar{d}$ is the equivalent equal annual value of depreciation based on average life.

The second approach is somewhat different. The factor

$$\left(\frac{1}{\sum_{1}^{m} v^y s(y)} - i\right)$$

which is equivalent to $1/s_{\overline{g}|}$, is a special kind of sinking fund factor. This factor can be applied to the surviving fraction of the original project, and if interest is charged at $i$ on the difference between accumulated total depreciation and accumulated retirements, the original value will be recovered over the entire life of the project. The interest on the depreciated, but not retired, portion, plus the return on the portion not yet depreciated will be exactly $is(y)$ for any year; thus the sum of depreciation (total) plus return for any year will be

$$R(y) + D(y) = \left(\frac{1}{\sum_{1}^{m} v^y s(y)} - i\right)s(y) + is(y) = \frac{s(y)}{\sum_{1}^{m} v^y s(y)}$$

and the total present worth of return plus depreciation will be

$$\sum_{1}^{m} R(y)v^y + \sum_{1}^{m} D(y)v^y = \sum_{1}^{m}\left(\frac{s(y)}{\sum_{1}^{m} v^y s(y)}\right)v^y = 1$$

as it should. We can now observe that

$$1/\sum_{1}^{m} v^y s(y)$$

equivalent to $1/a_{\overline{g}|}$, is a "capital recovery factor" applicable to the survivors in any year. If we assume that retirements will be continuously replaced at the original cost, then the "total survivors" in any year will always be exactly unity, and $1/a_{\overline{g}|}$ is a capital recovery factor applicable to original investment in any year. This does not require that the company actually replace the retirements, but it is the value adjustment necessary to place items with different survivor curves on the same basis.

Actual recovery of investment will generally be made on the average-life basis; but since recovery and return are effectively to be made on the original investment plus the replacements, the equivalent equal annual values based on original investment will be greater than for the certain-life case. Since

$$(\bar{r} + \bar{d}) = 1/a_{\overline{L}|}$$

for all cases, and $1/a_{\overline{g}|}$ is greater than $1/a_{\overline{n}|}$, it follows that the corresponding $\bar{r}$ and $\bar{d}$ must be greater. We note, however, that

$$\left(\frac{\bar{r}'}{\bar{r}' + \bar{d}'}\right) = \left(\frac{\bar{r}}{\bar{r} + \bar{d}}\right) \text{ and } \left(\frac{\bar{d}'}{\bar{r}' + \bar{d}'}\right) = \left(\frac{\bar{d}}{\bar{r} + \bar{d}}\right)$$

where the primed values correspond to the capital recovery factor

$$1/a_{\overline{g}|} = (\bar{r}' + \bar{d}')$$

and the unprimed values to

$$1/a_{\overline{n}|} = (\bar{r} + \bar{d})$$

But then

$$\bar{r}' = \left(\frac{\bar{r}' + \bar{d}'}{\bar{r} + \bar{d}}\right)\bar{r} = \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{r}$$

$$\bar{d}' = \left(\frac{\bar{r}' + \bar{d}'}{\bar{r} + \bar{d}}\right)\bar{d} = \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{d}$$

and the cost-of-capital is

$$\text{Return} = \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{r}$$

$$\text{Depreciation} = \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{d}$$

$$\text{Income tax} = \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{t}$$

Using the convention that the equivalent equal annual value of return is equal to $i$, the cost-of-capital is. as in the first approach,

$$\text{Return} = i$$

$$\text{Depreciation} = \frac{1}{s_{\overline{g}|}} = \left(\frac{1}{a_{\overline{g}|}} - i\right)$$

$$\text{Income tax} = \left(\frac{t}{1-t}\right)\left(1 - \frac{bB}{i}\right) \times \left(i + \frac{1}{s_{\overline{g}|}} - \left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)\bar{d}\right)$$

or

$$\text{Income tax} = \left(\frac{t}{1-t}\right)\left(1 - \frac{bB}{i}\right) \times \left(i + \frac{1}{s_{\overline{n}|}} - \bar{d}\right)\left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right)$$

In any case, then, the value adjustment necessary to put items with different survivor curves on the same basis is accomplished by using

$$\frac{1}{s_{\overline{g}|}} = \left(\frac{1}{\sum_{1}^{m} v^y s(y)} - i\right)$$

for the pseudo equivalent equal annual value of depreciation, and multiplying the income tax computed on an average life basis by the factor,

$$\left(\frac{a_{\overline{n}|}}{a_{\overline{g}|}}\right) = \left(\frac{a_{\overline{n}|}}{\sum_{1}^{m} v^y s(y)}\right)$$

This adjustment does not commit the company to a course of perpetual replacement. It is a theoretical computation designed to correct the inequity introduced in using average age as the amortization period.

A set of tables for $1/s_{\overline{g}|}$ derived from the Iowa curves is given in reference 4, along with a different derivation.

The income tax with mortality dispersion is not equal to

$$\left(\frac{t}{1-t}\right)\left(1 - \frac{bB}{i}\right)\left(i + \frac{1}{s_{\overline{g}|}} - \bar{d}\right)$$

This means that it is not correct to replace $1/s_{\overline{n}|}$ in the basic formula by $1/s_{\overline{g}|}$ and leave $\bar{d}$ unaltered. This common error amounts to saying that the value of $\bar{r}$ is increased but that $\bar{d}$ remains unaffected; i.e., depreciation is unchanged by mortality dispersion.

Additive correction factors for the effects of mortality dispersion are:

For depreciation:

$$\left[\frac{s_{\overline{n}|}}{s_{\overline{g}|}} - 1\right]$$

depreciation computed without regard to mortality dispersion

For income tax:

$$\left[\frac{a_{\overline{n}|}}{a_{\overline{g}|}} - 1\right]$$

income tax computed without regard to mortality dispersion.

The factors in braces should include all the other corrections.

### Summary of Results

Basic *Return* (pseudo values)

$(i)$

*Depreciation* (pseudo values)
Basic

$$\left(\frac{1}{s_{\overline{n}|}}\right)$$

Salvage

$$-\left(\frac{1-k}{s_{\overline{n}|}}\right)$$

Mortality dispersion

$$+\left(\frac{s_{\overline{n}|}}{s_{\overline{n}|}}-1\right)\left(\frac{k}{s_{\overline{n}|}}\right)$$

*Income Tax*
Basic

$$\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\overline{n}|}}-\bar{d}\right)$$

Salvage

$$+\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(\bar{d}-\frac{1}{s_{\overline{n}|}}\right)(1-k)$$

Multiple depreciation

$$+\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)k$$

Mortality dispersion

$$+\left(\frac{a_{\overline{n}|}}{a_{\overline{n}|}}-1\right)\times(\text{sum of other times})$$

### Recovered Capital Not Reinvested at i

Under some conditions it may not be possible to reinvest recovered capital at $i$ after taxes, or actually to retire securities. This may result from contractual or legal requirements, or from lack of growth. In such a case the original project is subject to additional revenue requirements in order to maintain the minimum satisfactory rate of return $i$.

The capital recovered through depreciation has in this case not really been recovered to the company, since the money is not available for use as capital. Hence it can be treated as though it is still a part of the original plant. It is convenient to treat these cases following the general method, and to correct for the fact that the recovery of depreciation has not been fully completed.

Let the actual return on accrued depreciation before taxes be $u$, and let the taxable fraction of $u$ be $x$. For instance, if the entire amount is invested in 2% tax-free bonds, then $u=0.02$ and $x=0$. Now, it is required that the net return after taxes on the accrued depreciation be at the rate $i$. From this, and the values of $u$ and $x$, we can determine the required value of the net return before income tax. Then the excess to be charged to the original plant is just the difference between this required value and the actual return.

The details are in Appendix V where it is shown that the excess revenue requirement to account for reinvesting accrued depreciation at less than $i$ is just equal to

$$\left(\frac{i-tbB}{1-t}-\frac{(1-xt)u}{1-t}\right)\left(\frac{i-\bar{r}}{i}\right)$$

When retirement depreciation is used, $\bar{r}\equiv i$, and the above factor is zero, as it should be. When $u$ and $x$ have values which yield $i$ after taxes, the factor is again zero.

### Final Comments

For estimating or for problems involving small sums of money, the fundamental relations derived in Part I are often adequate in the sense that the time necessary for more precise results is not justified. But modern companies often commit large sums of money as the result of engineering economic studies. Certainly the saving of an extra hour or two of engineering time is poor economy when the decision involves hundreds of thousands or millions of dollars.

The work involved in a precise determination of the cost-of-capital is actually not as great as the detailed work that this paper might suggest. The reason for this is simply that within the framework of one company only a few of the possible variations actually occur. Furthermore, a detailed analysis of some representative cases will usually show that the range of variables which must be considered is fairly restricted.

## Appendix I. Nomenclature

$\left(\frac{1}{a_{\overline{n}|}}\right)$ = capital recovery factor for recovery period $n$

$$=\frac{i(1+i)^n}{(1+i)^n-1}$$

$$\frac{1}{a_{\overline{n}|j}}=\frac{j(1+j)^n}{(1+j)^n-1}$$

If $j=i$, then

$$\frac{1}{a_{\overline{n}|i}}=\frac{1}{a_{\overline{n}|}}$$

$B$ = bond ratio
$b$ = average bond interest rate
$C$ = original investment in a project
$D(y)C$ = dollars of depreciation for year $y$ for a project
$D_t(y)C$ = dollars of depreciation claimed for income tax purposes for year $y$ for a project
$\bar{d}$, $\bar{d}_t$, $\bar{r}$, $\bar{t}$ = equivalent equal annual values of the quantities $D(y)$, $D_t(y)$, $R(y)$, $T(y)$ calculated as

$$\bar{d}=\left(\frac{1}{a_{\overline{n}|}}\right)\sum_{y=0}^{\infty}D(y)v^y$$

$e$ = depreciation factor for declining balance method
$fe$ = first year depreciation factor for declining balance method
$i$ = acceptable rate of return, interest rate
$k$ = fraction of original investment which will be depreciated
$m$ = smallest value of $y$, such that $D(y)=0$ for $y>m$
$n$ = amortization period, years
$P$ = total present worth
$R(y)C$ = dollars of return for year $y$ for a project
$\left(\frac{1}{s_{\overline{n}|}}\right)$ = sinking fund factor for recovery period $n$

$$=\frac{i}{(1+i)^n-1}$$

$$\frac{1}{s_{\overline{n}|j}}=\frac{j}{(1+j)^n-1}$$

If $j=i$, then

$$\frac{1}{s_{\overline{n}|i}}=\frac{1}{s_{\overline{n}|}}$$

$S(y)$ = fraction of original surviving at time $y$
$s(y)$ = average fraction of original surviving during year $y$; $S(y-1)\geq s(y)\geq S(y)$
$s$ = equivalent economic life of plant with survivor curve $S(y)$
$t$ = statutory income tax rate (composite Federal and state)
$T(y)C$ = dollars of income tax for year $y$ for a project
$u$ = rate of return on invested depreciation before income tax
$v=1/(1+i)$
$x$ = taxable fraction of $u$
$y$ = time in years, with beginning of use of a project taken as zero time

## Appendix II. Present Worth of Depreciation Is Original Value

Proof that the sum of the present worth of return plus depreciation is always original value:

$CD(y)$ = total depreciation for year $y$
$CR(y)$ = return for year $y$

Present worth of return plus depreciation for year $y$ is

$$\frac{CR(y)+CD(y)}{(1+i)^y}=Cv^yR(y)+Cv^yD(y)$$

Total present worth of all return plus depreciation is

$$P=\sum_{0}^{\infty}Cv^yR(y)+\sum_{0}^{\infty}Cv^yD(y)$$

$$P=\sum_{1}^{m}Cv^yR(y)+\sum_{0}^{m}Cv^yD(y)$$

where $m$ is the smallest value of $y$ giving $D(y)=0$ for all values of $y>m$.

$$P/C=\sum_1^m v^y R(y)+\sum_0^m v^y D(y)$$

But

$$R(y)=i\left[1-\sum_0^{y-1} D(y')\right]$$

i.e., annual return is rate of return times remaining investment.

$$P/C=\sum_1^m v^y i\left[1-\sum_0^{y-1} D(y')\right]+\sum_0^m v^y D(y)$$

$$=i\sum_1^m v^y-i\sum_1^m\left[v^y\sum_0^{y-1} D(y')\right]+\sum_0^m v^y D(y)$$

But note that

$$\sum_1^m v^y\sum_0^{y-1} D(y')=\sum_0^{m-1} D(y)\sum_{y-1}^m v^{y'}$$

$$=\sum_0^m D(y)\sum_1^m v^{y'}-\sum_0^m D(y)\sum_1^y v^{y'}$$

$$=\frac{1-v^m}{i}\sum_0^m D(y)-\sum_0^m \frac{1-v^y}{i} D(y)$$

since

$$\sum_1^k v^y=\frac{1-v^k}{i}$$

Hence

$$P/C=(1-v^m)-(1-v^m)\sum_0^m D(y)+\sum_0^m (1-v^y)D(y)+\sum_0^m v^y D(y)$$

Note that

$$\sum_0^m D(y)=1$$

$$P/C=1-v^m-1+v^m+1-\sum_0^m v^y D(y)+\sum_0^m v^y D(y)$$

$$P/C=1$$

$$P=C$$

The entire discussion applies only to the depreciable portion of an investment. For treatment of positive salvage and other nondepreciable portions; see the section "Salvage."

## Appendix III. Equivalent Equal Annual Charge for Income Tax

A. Assume that the method of depreciation used for income tax purposes is the same as the actual method used for recovery of investment.

Return before taxes is

$$[R(y)+T(y)]C$$

Taxable return before taxes is

$$\left[R(y)+T(y)-\frac{bB}{i}R(y)\right]C$$

Income tax is then

$$CT(y)=t\left[R(y)+T(y)-\frac{bB}{i}R(y)\right]C$$

$$CT(y)=\frac{t}{1-t}\left[R(y)-\frac{bB}{i}R(y)\right]C=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)R(y)C$$

Rate of revenue requirement is then

$$T(y)=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)R(y)$$

for the year $y$.

Equivalent equal annual charge is

$$\bar{t}=\frac{1}{a_{\bar{n}}}\sum_1^m v^y T(y)=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}$$

$$\bar{t}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}$$

or

$$\bar{t}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(\frac{1}{a_{\bar{n}}}-\bar{d}\right)$$

or

$$\bar{t}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\left(i+\frac{1}{s_{\bar{n}}}-\bar{d}\right)$$

B. Assume that the method of depreciation used for income tax purposes is not the same as the method used for recovery of investment.

Taxable return is

$$\left[R(y)+T(y)+D(y)-D_t(y)-\frac{bB}{i}R(y)\right]C$$

Income tax is then

$$CT(y)=t\left[\left(1-\frac{bB}{i}\right)R(y)+D(y)-D_t(y)+T(y)\right]C$$

$$CT(y)=\left(\frac{t}{1-t}\right)\left\{\left(1-\frac{bB}{i}\right)R(y)+[D(y)-D_t(y)]\right\}C$$

Rate of revenue requirement is then as follows:

$$T(y)=\left(\frac{t}{1-t}\right)\left\{\left(1-\frac{bB}{i}\right)R(y)+[D(y)-D_t(y)]\right\}$$

for the year $y$.

Equivalent equal annual charge is then

$$\bar{t}=\frac{1}{a_{\bar{n}}}\sum_1^m v^y T(y)=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}+\frac{t}{1-t}(\bar{d}-\bar{d}_t)$$

$$\bar{t}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}+\left(\frac{t}{1-t}\right)(\bar{d}-\bar{d}_t)$$

Note that when $\bar{d}=\bar{d}_t$, i.e., when only one method of depreciation is used, this reduces to the simple case discussed in section A, where

$$\bar{t}=\left(\frac{t}{1-t}\right)\left(1-\frac{bB}{i}\right)\bar{r}$$

The saving or loss on income tax, $t(1-t)\times(\bar{d}-\bar{d}_t)C$, is due to the difference in timing of the two methods of depreciation. The amount by which the revenue requirement is increased or reduced for income tax depends only on the methods of depreciation and the income tax rate. The amount is independent of the bond ratio, as it should be, since the difference in depreciation is independent of the bond ratio.

## Appendix IV. Equivalent Equal Annual Values for Depreciation

A. Straight line

$$D(y)=1/m,\ 1\le y\le m$$

$$\sum_{y=0}^m v^y D(y)=\frac{1}{m}\sum_{y=1}^m v^y=\frac{a_{\bar{m}}}{m}$$

$$\bar{d}=\frac{1}{a_{\bar{n}}}\left(\frac{a_{\bar{m}}}{m}\right)=\frac{a_{\bar{m}}}{a_{\bar{n}}}\frac{1}{m}$$

When $m=n$:

$$\bar{d}=\frac{1}{n}$$

B. Sum of year's digits

$$D(y)=\frac{m-(y-1)}{\sum_{y=1}^m y}=\frac{2(m-y+1)}{m(m+1)},\ 1\le y\le m$$

$$D(y)=\frac{2}{m}-\frac{2y}{m(m+1)}$$

$$\sum_{y=0}^m v^y D(y)=\frac{2}{m}\sum_{y=1}^m v^y-\frac{2}{m(m+1)}\sum_{y=1}^m y v^y=\frac{2a_{\bar{m}}}{(m+1)i}\left(\frac{1}{a_{\bar{m}}}-\frac{1}{m}\right)$$

$$\bar{d}=\frac{a_{\bar{m}}}{a_{\bar{n}}}\frac{2}{(m+1)i}\left(\frac{1}{a_{\bar{m}}}-\frac{1}{m}\right)$$

When $m=n$:

$$\bar{d}=\frac{2}{(n+1)i}\left(\frac{1}{a_{\bar{n}}}-\frac{1}{n}\right)$$

C. Sinking fund

$$D(y)=\frac{1}{s_{\bar{m}j}}+j\times(\text{accumulated depreciation account})$$

$$D(y)=\frac{1}{s_{\overline{m}|j}}+j\left(\frac{s_{\overline{y-1}|j}}{s_{\overline{m}|j}}\right)=\frac{1}{s_{\overline{m}|j}}\,(1+js_{\overline{y-1}|j})$$

$$D(y)=\frac{1}{s_{\overline{m}|j}}\left(\frac{j+j((1+j)^{y-1}-1)}{j}\right)=\frac{(1+i)^{y-1}}{s_{\overline{m}|j}}$$

$$\sum_{y=0}^{m} v^y D(y)=\frac{1}{s_{\overline{m}|j}}\sum_{y=1}^{m}\frac{(1+j)^{y-1}}{(1+i)^y}$$

$$=\frac{1}{(1+j)s_{\overline{m}|j}}\sum_{y=1}^{m}\left(\frac{1+j}{1+i}\right)^y$$

$$=\frac{1}{s_{\overline{m}|j}}\ \frac{(1+i)^m-(i+j)^m}{(1+i)^m(1-j)}$$

$$\bar{d}=\frac{1}{a_{\overline{n}|i}\,s_{\overline{m}|j}}\ \frac{(1+i)^m-(1+j)^m}{(1+i)^m(i-j)}$$

When $m=n$:

$$\bar{d}=\frac{1}{s_{\overline{n}|i}\,s_{\overline{n}|j}}\ \frac{(1+i)^n-(1+j)^n}{(i-j)}$$

If $j=i$:

$$\bar{d}=\frac{1}{a_{\overline{n}|i}\,s_{\overline{m}|j}}\ \frac{m}{(1+i)}$$

If $m=n$ and $j=i$:

$$\bar{d}=\frac{1}{a_{\overline{n}|i}}\ \frac{1}{s_{\overline{n}|i}}\ \frac{n}{(1+i)}$$

D. Declining balance

$D(0)=0,\ D(1)=fe,\ D(2)=(1-fe)e$

$D(3)=(1-fe)(1-e)e,\ D(4)=(1-fe)(1-e)^2e$

$$D(y)=\begin{cases}fe, & y=1\\(1-fe)(1-e)^{y-2}e, & y>1\end{cases}$$

$$\sum_{y=0}^{m} v^y D(y)=vfe+\sum_{2}^{\infty} v^y(1-fe)(1-e)^{y-2}e$$

$$=vfe+\frac{(1-fe)ve}{i+e}$$

$$=\frac{ve(1+fi)}{i+e}=\left(\frac{e}{i+e}\right)\left(\frac{1+fi}{1+i}\right)$$

$$\bar{d}=\frac{1}{a_{\overline{n}|}}\left(\frac{e}{i+e}\right)\left(\frac{1+fi}{1+i}\right)$$

If one uses the maximum income tax value for $e$; $e=2/n$:

$$\bar{d}=\frac{1}{(1+ni/2)a_{\overline{n}|}}\left(\frac{1+fi}{1+i}\right)$$

For limited declining balance method

$$D(y)=\begin{cases}fe, & y=1\\(1-fe)(1-e)^{y-2}e, & 1<y\le b\\K/(m-b), & b<y\le m\\0, & \text{otherwise}\end{cases}$$

where $K$ = remaining investment after $b$ years.

$$K=\left(\frac{1-fe}{1-e}\right)(1-e)^b$$

Now

$$\sum_{0}^{m} v^y D(y)=fev+\frac{(1-fe)e}{(1-e)^2}\times$$

$$\sum_{y=2}^{b}(1-e)^y v^y+\left(\frac{K}{m-b}\right)\sum_{y=b+1}^{m} v^y$$

$$=\left(\frac{e}{i+e}\right)\left(\frac{1+fi}{1+i}\right)-$$

$$\left(\frac{e}{i+e}\right)\left(\frac{1-e}{1+i}\right)^b\left(\frac{1-fe}{1-e}\right)+$$

$$\left(\frac{1-e}{1+i}\right)^b\frac{a_{\overline{m-b}|}}{m-b}\left(\frac{1-fe}{1-e}\right)$$

$$\bar{d}=\frac{1}{a_{\overline{n}|}}\left(\frac{e}{i+e}\right)\left(\frac{1+fi}{1+i}-\left[\frac{1-e}{1+i}\right]^b\frac{1-fe}{1-e}\right)+$$

$$\frac{a_{\overline{m-b}|}}{a_{\overline{n}|}}\left(\frac{1-e}{1+i}\right)^b\frac{1}{m-b}\left(\frac{1-fe}{1-e}\right)$$

## Appendix V. Depreciation Not Reinvested at i

Normal return on accrued depreciation after taxes is

$$i\sum_{0}^{y-1} D(y')=i-R(y),$$

$$\text{since } R(y)=i\left[1-\sum_{0}^{y-1} D(y')\right]$$

Let $Z(y)$ be the return before income tax corresponding to the normal return after taxes.

Actual return before income tax is

$$=u\sum_{0}^{y-1} D(y')=\frac{u}{i}\,[i-R(y)]$$

and the taxable portion of this is

$$=\frac{xu}{i}\,[i-R(y)]$$

Total taxable return is

$$=Z(y)-\frac{(1-x)u}{i}[i-R(y)]-bB\sum_{0}^{y-1} D(y')$$

Note that the bond interest credit from the capital structure of the company is not altered by the taxability of the investment return.

Income tax is

$$=tZ(y)-t((1-x)u-bB)\left(\frac{i-R(y)}{i}\right)$$

Hence the return after income tax is

$$=i-R(y)=Z(y)-tZ(y)+t((1-x)u-bB)\times\left(\frac{i-R(y)}{i}\right)$$

or

$$Z(y)=\left(\frac{i-tbB}{1-t}-\frac{(1-x)u}{1-t}\right)\left(\frac{i-R(y)}{i}\right)$$

Thus the excess revenue requirement is

$$=Z(y)-\frac{u}{i}\,[i-R(y)]$$

$$=\left(\frac{i-tbB}{1-t}-\frac{(1-xt)u}{1-t}\right)\left(\frac{i-R(y)}{i}\right)$$

and the equivalent equal annual value of this is

$$\left(\frac{i-tbB}{1-t}-\frac{(1-xt)u}{1-t}\right)\left(\frac{i-\bar{r}}{i}\right)$$

## Appendix VI. Retirement Dispersion and Equivalent Economic Life

On the survivor curve, the fraction of the original which has survived through but not beyond year $y$ is $S(y)$. For the determination of the useful service we can define a new function, $s(y)$, in such a way that

$$S(y-1)\geqslant s(y)\geqslant S(y)$$

The function $s(y)$ may be defined as the mean value

$$s(y)=\int_{y-1}^{y} S(x)dx$$

or, as the average value

$$s(y)=\frac{S(y)+S(y-1)}{2}$$

or some other reasonable value subject to

$$S(y-1)\geqslant s(y)\geqslant S(y)$$

The present worth of the service provided by items with survivor curve $S(y)$ is

$$P=[s(1)-s(2)](v)+[s(2)-s(3)](v-v^2)+\ldots+[s(m-1)-s(m)](v-v^2-\ldots-v^{m-1})+[s(m)-s(m+1)](v-v^2-\ldots-v^m)$$

Note that

$S(y)=0$ for $y\geqslant m$, but $s(y)=0$ for $y>m$.

Simplifying gives

$$P=\sum_{1}^{m} v^y s(y)$$

Present worth of the service provided by items with certain life $s$ is

$$P'=\sum_{1}^{s} v^y \text{ since } s(y)=\begin{cases}1, & y\le s\\0, & y>s\end{cases}$$

We can define the equivalent life $s$ from the relation $P=P'$, or

$$\sum_{1}^{m} v^y s(y)=\sum_{1}^{s} v^y=a_{\overline{s}|}$$

i.e., the equivalent economic life $s$ is that value of $n$ such that

$$a_{\overline{n}|}=\sum_{1}^{m} v^y s(y)$$

The pseudo equivalent equal annual depreciation for a life $s$ is

$$\frac{1}{s_{\overline{s}|}}=\frac{1}{a_{\overline{s}|}}-i$$

Hence is equal to

$$\frac{1}{\sum_{1}^{m} v^y s(y)}$$

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## Discussion

**Elwood A. Church** (Boston Edison Company, Boston, Mass.): This discusser believes that the basic premises upon which this commendable paper is based are valid; while the nomenclature is sometimes hard to follow, the results appear to be correct.

The author, in discussing the reserve for depreciation, implies that the concept of depreciation reserve considered as a fund earning interest is pernicious reasoning since it is not actually a fund. This discusser does not believe that assuming the depreciation reserve to be a fund earning interest is harmful or misleading. If the arithmetic is correct, the results of calculating return by this method are the same as when depreciation is deducted from original plant; thereby the net plant is obtained on which return is computed.

Common practice in utility accounting is to carry the plant on the books at its original cost until it is retired, then to write it off all at once instead of charging depreciation to it during its life by a suitable formula, such as a fixed percentage per year (straight-line depreciation), or other formula. When such an accounting practice is followed, sufficient earnings each year must be retained before paying dividends, so that the utility's assets will not become impaired over the years from normal wear and tear and obsolescence. These retained earnings could be deposited in a savings bank or otherwise invested outside the utility to increase their value year by year, and used to replace equipment when it is worn out or retired for other reasons.

It is, however, recognized that the most economical way to invest this money is to use it to buy new equipment immediately and not wait till equipment is replaced to make use of it. On the premise that money saved is equivalent to the same amount of money earned, the rate of return on retained earnings for depreciation can be assumed to be the same as the composite rate of return on the money invested by the stockholders and bondholders.

The retained earnings will appear on the utility's books on the asset side either as cash if not reinvested or in the plant account if used to purchase new equipment. They will also appear under the heading of Reserve for Depreciation or other descriptive title either as a deduction from the gross plant (first cost) to obtain the net plant or on the liability side as part of the total liabilities of the utility. When plant is retired, its original cost is deducted from both sides of the books (plant account and depreciation reserve).

Table III shows that either concept of the depreciation reserve may be used: i.e., (1) an actual fund which earns interest or (2) a deduction from the plant account which decreases required earnings on plant. The various quantities are computed by concept 1 in Table III for three types of depreciation accrual from the example which is shown in Tables I and II of the paper.

Table III's column headed Earnings Retained for Depreciation is the same as that headed Total Annual Depreciation in Table II. The column headed Cumulative Retained Earnings is what is usually called Depreciation Reserve or Reserve for Depreciation. It is suggested that these terms might clarify some of the misunderstandings about the true nature of the depreciation reserve.

Examination of columns 5, 6, and 7 will reveal that Return on Net Plant is the difference betweeen Return on Original Investment and Return on Retained Earnings at the same rate, namely, 10%. This is true regardless of the method of depreciation accrual which is used. Column 8 is the quantity commonly known as Capital Recovery which may be defined as the sum of the Return on Original Investment plus Retained Earnings less the Return thereon. It may also be defined as Retained Earnings plus Return on Net Plant, as was done by the author. This quantity is the same for both concepts of the reserve provided the rate of return on Retained Earnings is the same as the rate of return on Original Investment and on Net Plant

The Present Worth of Capital Recovery over the life of the investment will always equal the Original Investment, or $1,000 in the example shown, which is the same result as that obtained by the author for the sum of the last two columns of Table II. This is always true regardless of what method of depreciation accrual is used.

It will be observed that the annual capital recovery as shown in column 8 is constant in the case of Sinking Fund Depreciation accrual, namely, the sum of Return on Original Investment and the constant amount retained each year at the sinking fund rate.

**C. W. Watchorn** (Pennsylvania Power and Light Company, Allentown, Pa.): The basic arithmetical processes described are substantially the same as those that have been in fairly broad general use for many years. Would the author mention the specific contributions of the paper as related to or compared with references 2, 4, and 5?

Actually, economic comparisons of alternative facilities, except concerning the correctness of the arithmetic, are not precise determinations, since they depend

**Table III. Return on Original Investment, Retained Earnings, and Net Plant Per $1,000 Original Investment**

| Year End | Earnings Retained for Depreciation | Net Plant | Cumulative Retained Earnings | Return: On Original Investment | Return: On Retained Earnings | Return: On Net Plant | Capital Recovery, Columns (2)+(5)−(6) | Present Worth of Capital Recovery |
|---|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| | | | Straight-Line Depreciation, 5-Year Life | | | | | |
| 1 | $ 200 | $1,000 | $ 0 | $100 | $ 0 | $100 | $300 | $ 272.73 |
| 2 | 200 | 800 | 200 | 100 | 20 | 80 | 280 | 231.40 |
| 3 | 200 | 600 | 400 | 100 | 40 | 60 | 260 | 195.35 |
| 4 | 200 | 400 | 600 | 100 | 60 | 40 | 240 | 163.92 |
| 5 | 200 | 200 | 800 | 100 | 80 | 20 | 220 | 136.60 |
| Totals | $1,000 | | | $500 | $200 | $300 | | $1,000.00 |
| | | | Sum of Digits Depreciation, 5-Year Life | | | | | |
| 1 | $ 333.33 | $1,000.00 | $ 0 | $100 | $ 0 | $100.00 | $433.33 | $ 393.94 |
| 2 | 266.67 | 666.67 | 333.33 | 100 | 33.33 | 66.67 | 333.34 | 275.48 |
| 3 | 200.00 | 400.00 | 600.00 | 100 | 60.00 | 40.00 | 240.00 | 180.32 |
| 4 | 133.33 | 200.00 | 800.00 | 100 | 80.00 | 20.00 | 153.33 | 104.73 |
| 5 | 66.67 | 66.67 | 933.33 | 100 | 93.33 | 6.67 | 73.34 | 45.53 |
| Totals | $1,000.00 | | | $500 | $266.66 | $333.34 | | $1,000.00 |
| | | | Sinking Fund Depreciation, 5-Year Life | | | | | |
| 1 | $ 163.80 | $1,000.00 | $ 0 | $100 | $ 0 | $100.00 | $263.80 | $ 239.81 |
| 2 | 180.18* | 836.20 | 163.80 | 100 | 16.38 | 83.62 | 263.80 | 218.01 |
| 3 | 198.20* | 656.02 | 343.98 | 100 | 34.40 | 65.60 | 263.80 | 198.20 |
| 4 | 218.01* | 457.82 | 542.18 | 100 | 54.22 | 45.78 | 263.80 | 180.18 |
| 5 | 239.81* | 239.81 | 760.19 | 100 | 76.02 | 23.98 | 263.80 | 163.80 |
| Totals | $1,000.00 | | | $500 | $181.02 | $318.98 | | $1,000.00 |

* Includes return compounded annually at 10%.