Q:1 (a)

$$\sum_n \frac{2^n + 3^n}{3^n + 4^n}$$

$$\lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{3^{n+1} + 4^{n+1}} \times \frac{3^n + 4^n}{2^n + 3^n} = \lim_{n\to\infty} \frac{2(2/3)^n + 3}{3(3/4)^n + 4} \cdot \frac{(3/4)^n + 1}{(2/3)^n + 1}$$

$$= \lim_{n\to\infty} \frac{2(2/3)^n + 3}{2(3/4)^n + 4} \cdot \lim_{n\to\infty} \frac{(3/4)^n + 1}{(2/3)^n + 1}$$

$$= \frac{3}{4} < 1$$

Convergent.

(b)

$$\sum_n (-1)^n\left[\sqrt{n^2+n}-n\right] \times \frac{\left(\sqrt{n^2+n}+n\right)}{\left(\sqrt{n^2+n}+n\right)}$$

$$= \sum_n \frac{(-1)^n\left[(\cancel{n^2}+n)-\cancel{n^2}\right]}{\sqrt{n^2+n}+n}$$

$$= \sum_n \frac{(-1)^n\, n}{\sqrt{n^2+n}+n}$$

Now, $\displaystyle \lim_{n\to\infty} \frac{(-1)^n\, n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{1+\frac{1}{n}}+1}$

$$= \begin{cases} -1/2, & n \text{ odd} \\ 1/2, & n \text{ even} \end{cases}$$

divergent.

(c) $\displaystyle\sum_n \frac{\sqrt{n}}{n^2+1}$

Notice $\quad n^2+1 > n^2$

$$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$$

$$\Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

As $\displaystyle\sum_n \frac{1}{n^{3/2}}$ converges by the p-series test, so does $\displaystyle\sum_n \frac{\sqrt{n}}{n^2+1}$ by the Comparison test.

(d) $\sum_n \frac{(-1)^n n^n}{(2^n)^3}$

Notice $\sum_n \frac{(-1)^n n^n}{(2^n)^3} = \sum_n (-1)^n \left[\frac{n \cdot n \cdot n \cdots n}{8 \cdot 8 \cdot 8 \cdots 8}\right]$

$= \sum_n (-1)^n \left[\left(\frac{n}{8}\right) \cdot \left(\frac{n}{8}\right) \cdots \left(\frac{n}{8}\right)\right]$

$= \sum_n (-1)^n \left(\frac{n}{8}\right)^n$

As $n \to \infty$, $\left(\frac{n}{8}\right)^n \to \infty$

divergent.

(e).

$$\sum_{n} \frac{(-100)^n}{n!} = \sum_{n} \frac{(-1)^n\, 100^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, 100^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n\, 100^n} \right|$$

$$= \lim_{n\to\infty} \frac{100}{(n+1)}$$

$$= 0$$

Convergent

(f).

$$\sum_{n} \left(1 - \frac{1}{3n}\right)^{n}$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{(-3n)}\right)^{n} = \lim_{n\to\infty} \left[\left(1 + \frac{1}{(-3n)}\right)^{-3n}\right]^{-\frac{1}{3}}$$

$$= e^{-\frac{1}{3}}$$

$$= \frac{1}{\sqrt[3]{e}}$$

$$\neq 0$$

divergent.

(g). $\sum_{n} \frac{n \cdot 2^n (n+1)!}{3^n \cdot n!}$

$$\lim_{n\to\infty} \left[\frac{(n+1)\, 2^{n+1} (n+2)!}{3^{n+1} \cdot (n+1)!} \times \frac{3^n n!}{n \cdot 2^n \cdot (n+1)!} \right]$$

$$= \lim_{n\to\infty} \frac{2(n+2)}{3n}$$

$$= \frac{2}{3}$$

$$< 1$$

Convergent.

(h). $\displaystyle\sum_n \frac{1}{1^2+2^2+\cdots+n^2}$

Notice $\displaystyle\frac{1}{1^2+2^2+\cdots+n^2} < \frac{1}{n^2}$

As $\displaystyle\sum_n \frac{1}{n^2}$ converges, so does $\displaystyle\sum_n \frac{1}{1^2+2^2+\cdots+n^2}$

Q. 2. (a). $a_1 = 3$, $a_{n+1} = \left(\frac{n}{n+1}\right) a_n$

$$\Rightarrow \quad a_n = \left(\frac{n-1}{n}\right) a_{n-1}$$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right) a_{n-2}$$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{n-3}{n-2}\right) a_{n-3}$$

$$\vdots$$

$$\Rightarrow \quad a_n = \frac{(n-1)(n-2)(n-3)\cdots(1)}{(n)(n-1)\cdots(2)} \underbrace{a_1}_{=3} = \frac{3}{n}$$

diverges by p-series test.

Q.2 (b). $a_1 = 1$, $a_{n+1} = \left[\dfrac{1+\tan^{-1}(n)}{n}\right] a_n$

Since $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \dfrac{1+\tan^{-1}(n)}{n}$

$= 0$

< 1

Convergent.