Math 110 (A01, A02, A03)
Test 1
Version: A
September 27, 2019
Time: 45 minutes

Student ID: V00________________ Family (Last) Name: ________________

Given (First) Name: ________________

Tutorial sections (check one):

☐ T01 (Jaimes Joschko, 2:30, CLE A127)
☐ T02 (MacKenzie Carr, 2:30, CLE A308)
☐ T03 (Jacob Nagrocki, 2:30, HHB 110)
☐ T04 (Jacob Nagrocki, 3:30, HHB 110)
☐ T05 (Jaimes Joschko, 3:30, CLE C112)
☐ T06 (MacKenzie Carr, 3:30, CLE A203)
☐ T07 (Jaimes Joschko, 4:30, CLE C112)
☐ T08 (Jacob Nagrocki, 4:30, HHB 110)
☐ T12 (MacKenzie Carr, 4:30, CLE A203)

Question(s)	Value	Score
Question 1	1	
Question 2	1	
Question 3	1	
Question 4	1	
Question 5	4	
Question 6	4	
Question 7	4	
Question 8	4	
Total	**20**	

Instructions:

1. Identifying information:
   (a) Enter your Student ID and name at the top of this page now.
   (b) Select your tutorial section above now.
2. Only the following materials are permitted:
   (a) Pens, pencils, erasers, and a ruler are permitted at your desk. If you have a pencil case it must be stored with your belongings in the front of the room.
   (b) You may use a Sharp calculator with a model number beginning with EL510-R. No other calculators are acceptable on this examination.
3. No notes, outside paper, or aid other than the ones listed above is permitted. You are responsible for ensuring that any unauthorized material is stored with your belongings at the front of the room.
4. Show all calculations on this paper for all problems. We may disallow any answer given without appropriate justification.
5. If you require extra space, use the backs of test pages or the blank page at the end of the test. Be sure to clearly indicate where you have done so. No outside paper is permitted.
6. If you need to leave the room during the test, raise your hand until an invigilator comes to you.
7. This test has 8 pages, including this cover and the blank page at the end.

For questions 1–4, enter your final answer in the box provided. You must show your work to be given credit, even if your answer is correct.
Leave all answers in exact form - do not give decimal approximations.

(1 point) 1. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Calculate $3(\mathbf{v} - 2\mathbf{w})$.

Solution:

$$3(\mathbf{v} - 2\mathbf{w}) = 3\left(\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right) = 3\begin{bmatrix} -1 \\ 1 \\ -9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -27 \end{bmatrix}.$$

Answer: $\begin{bmatrix} -3 \\ 3 \\ -27 \end{bmatrix}$

(1 point) 2. Find all real numbers x such that $\begin{bmatrix} x \\ x \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ 2 \\ -1 \end{bmatrix}$ are orthogonal.

Solution: For these vectors to be orthogonal we must have

$$0 = \begin{bmatrix} x \\ x \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ 2 \\ -1 \end{bmatrix} = x^2 + 2x - 3.$$

This equation has solutions $x = -3$ and $x = 1$.

Answer: $x = -3$ and $x = 1$

(1 point) 3. Write a vector equation for the line in $\mathbb{R}^3$ passing through the point $(1, 4, 2)$ and meeting the plane $2x - 5y + z = 3$ at a right angle.

Solution: To meet that plane in a right angle the direction vector of our line must be the normal vector of the plane, which is $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$. The line with this direction vector that passes through $(1, 4, 2)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}.$$

Answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

(1 point) 4. Find a unit vector in the same direction as $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$.

Solution: The vector we are given has length

$$\left\| \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \right\| = \sqrt{3^2 + 2^2 + (-5)^2} = \sqrt{38}.$$

Therefore the vector we want is

$$\frac{1}{\sqrt{38}} \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{38} \\ 2/\sqrt{38} \\ -5/\sqrt{38} \end{bmatrix}.$$

Answer: $\begin{bmatrix} 3/\sqrt{38} \\ 2/\sqrt{38} \\ -5/\sqrt{38} \end{bmatrix}$

(4 points) 5. Find all values of k for which $\begin{bmatrix} 1 \\ 4 \\ k \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$.

Solution: We are interested in whether or not the following equation has solutions:

$$a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} + c \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ k \end{bmatrix}.$$

Equivalently, we are interested in the linear system with the following augmented matrix, which we then row-reduce:

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 2 & 5 & 4 \\ -1 & 4 & 8 & k \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 2 & 5 & 4 \\ 0 & 0 & 0 & k-3 \end{array}\right].$$

From here we can see that this system will have a solution when $k = 3$, and otherwise will not. Thus $\begin{bmatrix} 1 \\ 4 \\ k \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$ whenever $k \neq 3$.

(4 points) 6. Find, with justification, the general equation for the plane in $\mathbb{R}^3$ that passes through the points $(1, -1, 2)$, $(0, 0, 2)$, and $(2, 0, 3)$.

Solution: We begin by finding two direction vectors for this plane, namely

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{d}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

(Other directions vectors are also possible). To find the general form we need a normal vector that is orthogonal to both of these direction vectors. Either by using the cross product, or by solving an appropriate system of equations, or by inspection (because the numbers happened to work out nicely!) we find that $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$ (in fact, any multiple of this $\mathbf{n}$ can also be used as the normal vector).

The normal equation for our plane is thus

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix},$$

which expands to the general equation

$$x + y - 2z = -4.$$

(4 points) 7. Find the distance between the point $(0, 2)$ and the line $2x + 3y = 1$ in $\mathbb{R}^2$. Your solution should use the techniques developed in this course; in particular, solutions that use calculus will not receive credit.

Solution: There are several ways to solve this question; we present only one of the possibilities.

The normal vector for our line is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $(-1, 1)$ is a point on the line. The length of the vector between the line and $(0, 2)$ that meets the line at a right angle is therefore given by

$$\begin{aligned}
&\left\| \operatorname{proj}_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \right\| \\
&\qquad = \left\| \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\| \\
&\qquad = \left\| \begin{bmatrix} 10/13 \\ 15/13 \end{bmatrix} \right\| \\
&\qquad = \sqrt{\left(\frac{10}{13}\right)^2 + \left(\frac{15}{13}\right)^2}.
\end{aligned}$$

Therefore the distance is $\sqrt{\left(\frac{10}{13}\right)^2 + \left(\frac{15}{13}\right)^2}$ (this simplifies to $\frac{5}{\sqrt{13}}$, but you did not need to simplify to get full credit).

(4 points) 8. In this question we are looking for an answer that applies in general, so no credit will be given for answers that choose specific vectors $\mathbf{w}$ or $\mathbf{v}$.

Let $\mathbf{w}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that $\mathbf{w} \perp (\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v}))$.
Hint: Start by writing down the formula for $\mathrm{proj}_{\mathbf{w}}(\mathbf{v})$, and what the symbol $\perp$ means.

Solution: We need to show that $\mathbf{w} \cdot (\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v})) = 0$. We calculate:

$$\begin{aligned}
\mathbf{w} \cdot (\mathbf{v} - \mathrm{proj}_{\mathbf{w}}(\mathbf{v})) &= \mathbf{w} \cdot \left(\mathbf{v} - \left(\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}\right) \\
&= \mathbf{w} \cdot \mathbf{v} - \left(\frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{w}}\right) (\mathbf{w} \cdot \mathbf{w}) \\
&= \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} \\
&= 0
\end{aligned}$$

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[END]