Q:1 (a)

$$a_n = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \frac{\ln\left[\frac{1+n}{n}\right]}{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln\left[\frac{1+n}{n}\right]}{\frac{1}{n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{\frac{1+n}{n}}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{n}{1+n}$$

$$= 1$$

(b).

$$a_n = \sqrt{n}\left(\sqrt{n+3} - \sqrt{n}\right)$$

$$= \frac{\sqrt{n}\left(\sqrt{n+3} - \sqrt{n}\right)}{\left(\sqrt{n+3} + \sqrt{n}\right)} \times \left(\sqrt{n+3} + \sqrt{n}\right)$$

$$= \frac{3\sqrt{n}}{\sqrt{n+3} + \sqrt{n}}$$

$$= \frac{3}{\sqrt{1+\frac{3}{n}} + 1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{\sqrt{1+\frac{3}{n}} + 1}$$

$$= \frac{3}{2}$$

Q: 2. Suppose the sequence converges to L, then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n}$$

gives,

$$L = \sqrt{1+L}$$

or $$L^2 - L - 1 = 0$$

$$\Rightarrow \quad L = \frac{1 \pm \sqrt{1+4}}{2}$$

or $$L = \frac{1 \pm \sqrt{5}}{2}$$

Since this is a sequence of positive terms,

$$L = \frac{1+\sqrt{5}}{2}$$

Q.3. Here $\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2} (x-1)^{\frac{3}{2}-1} = (x-1)^{1/2}$

$$\Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (x-1)}$$

$$= \sqrt{x}$$

$$\Rightarrow \text{Arc length} = s = \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{x}\, dx$$

$$= \frac{2}{3} x^{3/2} \Big|_{x=1}^{x=4}$$

$$= \frac{14}{3}$$

Q: 4.

We have $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_2^6 \sqrt{1 + \frac{16}{x^6}}\, dx$

This gives, $\left(\frac{dy}{dx}\right)^2 = \frac{16}{x^6}$

or $\frac{dy}{dx} = \frac{4}{x^3} \Rightarrow y(x) = \frac{-2}{x^2} + C$

We are given that $y(1) = 5$

$\Rightarrow 5 = \frac{-2}{1^2} + c \Rightarrow c = 7.$

Therefore, $y(x) = -\frac{2}{x^2} + 7$

Q.5: We have

$$y(x) = -315\left[e^{x/240} + e^{-x/240}\right] + 1260$$

$$\Rightarrow \frac{dy}{dx} = \frac{-315}{240}\left[e^{x/240} - e^{-x/240}\right] \rightarrow (1)$$

Therefore,

$$s = \int_{-315}^{315} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

with $\frac{dy}{dx}$ given in (1)

Q: 6. (a) We have $\frac{dN}{dt} = kN$, which is separable.

$$\Rightarrow \frac{dN}{N} = k\,dt$$

$$\Rightarrow \int \frac{1}{N}\,dN = k\int dt$$

$$\Rightarrow \ln(N) = kt + c$$

$$\Rightarrow N(t) = C_1 e^{kt}, \quad C_1 = e^{c}$$

Let N_0 be the initial number of Bacteria.

i.e, at $t=0$, $N = N_0$

$$\Rightarrow N_0 = C_1 e^{k(0)} = C_1$$

therefore, $N(t) = N_0 e^{kt}$

We are given that

$N(t) = 4.5\,N_0$

when $t = 60$

$$\Rightarrow N(60) = 4.5\,N_0 = N_0 e^{k(60)}$$

$$\Rightarrow \frac{\ln(4.5)}{60} = k$$

or $k \approx 0.025067$

(b) we find t for which $N(t) = 8N_0$

$$\Rightarrow 8N_0 = N_0 e^{(0.025067)t}$$

$$\Rightarrow t = \frac{\ln(8)}{0.025067} \approx 82.95 \text{ mints.}$$