Math 101 Series Tests

Definition of Convergence

Definition: We say a series $\sum_{n=k}^{\infty} a_n$ converges provided (as a sequence) the partial sums $S_N = \sum_{n=k}^{N} a_n$ converge. When $\lim_{N\to\infty} S_N = L$ we define $\sum_{n=k}^{\infty} a_n = L$. If the partial sums diverge we say the series diverges.

Special Series

Geometric Series: A Geometric series is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$ where a and r are constants. A Geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Telescoping Series: A Telescoping series is a series of the form $\sum_{n=k}^{\infty} (a_{f(n)} - a_{f(n-1)})$. It converges if the partial sums converge ortherwise it diverges.

p**-Series**: A p-series is a series of the form $\sum_{n=k}^{\infty} \frac{1}{n^p}$. It converges if $p > 1$ and diverges if $p \leq 1$.

Tests Restricted to Positive Series

Integral Test: Let $\{a_n\}_{n=k}^{\infty}$ be a sequence and suppose that $a_n = f(n)$ where

1. $f(x)$ is positive for $x \geq k$
2. $f(x)$ is continuous for $x \geq k$, and
3. $f(x)$ is decreasing for all $x \geq k_0$ where $k_0 \geq k$ (eventually decreasing).

then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)dx$ either both converge or diverge together.

Direct Comparison Test: Let $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty}$ with $0 \leq a_n \leq b_n$ for all $n \geq k$. Then:

- If $\sum_{n=k}^{\infty} b_n$ converges then $\sum_{n=k}^{\infty} a_n$ converges.
- If $\sum_{n=k}^{\infty} a_n$ diverges then $\sum_{n=k}^{\infty} b_n$ diverges.

Limit Comparison Test: Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq k$. Then let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

then the following hold:

- If $0 < L < \infty$ then $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$ either both converge or diverge together.
- If $L = 0$ and $\sum_{n=k}^{\infty} b_n$ converges then $\sum_{n=k}^{\infty} a_n$ converges
- If $L = \infty$ and $\sum_{n=k}^{\infty} b_n$ diverges then $\sum_{n=k}^{\infty} a_n$ diverges

Tests for Negative or Positive Series

n^{th} **- Term Divergence Test**: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=k}^{\infty} a_n$ diverges.

Absolute Convergence Test: If $\sum_{n=k}^{\infty} |a_n|$ converges then $\sum_{n=k}^{\infty} a_n$ converges.

Ratio Test: Let $\sum_{n=k}^{\infty} a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R$$

Then:

- If $R < 1$ then the series converges (absolutely)
- If $R > 1$ then the series diverges
- If $R = 1$ the result is inconclusive and a different test must be used

Root Test: Let $\sum_{n=k}^{\infty} a_n$ be any series and suppose that

$$\lim_{n\to\infty} |a_n|^{1/n} = R$$

Then:

- If $R < 1$ then the series converges (absolutely)
- If $R > 1$ then the series diverges
- If $R = 1$ the result is inconclusive and a different test must be used

Alternating Series Test: Let $\sum_{n=k}^{\infty} (-1)^{n+1} a_n$ be any alternating series. Provided that

1. $a_n \geq 0$ for all $n \geq k$
2. $\{a_n\}_{n=k}^{\infty}$ is an (eventually) non-increasing sequence, and
3. $\lim_{n\to\infty} a_n = 0$

   then the series converges.

Some Notes

Geometric Series:

- The convergence results for this apply regardless of whether the series is positive or alternating.
- This and Telescoping series are the only series you can find the sum of in this course.

  (i.e. what the partial sums converge to)
- What matters is the sum $a + ar + ar^2 + ar^3 + \cdots$. This may also be written as $\sum_{n=0}^{\infty} ar^n$.

Telescoping Series:

- This and Geometric series are the only series you can find the sum of in this course.

  (i.e. what the partial sums converge to)
- The trick to determining convergence or divergence of this series is to find a formula for the partial sum and take a limit.

Integral Test:

- You must check the three conditions of this theorem before you investigate the integral.
- To form the function $f(x)$ you may identify a_n and replace $n \to x$. If we have that the sum is over $n \geq k$ then the domain of the function is $x \geq k$.
- An easy way to determine if the function $f(x)$ is decreasing over $x \geq k$ is to show $f'(x) < 0$ for $x \geq k$.

Direct Comparison Test:

- Forming the inequality for the comparison is often difficult.
- The inequality formed must be justified in some manner.
- To obtain convergence you must aim to bound the expression above by a convergent series. To obtain divergence you must aim to bound the expression below by a divergent series. (i.e. Bounding a series above by a divergent one or below by a convergent one is meaningless)
- Generally most tests can be used in place of a direct comparison. It might be more work but will conceptually often be easier. Using this test often means you have a sense of clairvoyance about the nature of the series. There are three hurdles to overcome:
  - You need to know how to bound the inequality. Claims of an inequality without justification render the argument meaningless so you must have a way to justify whatever inequality you form. A graph is not an acceptable justification.
  - When forming the inequality you need to know which direction you are forming the inequality. In a sense, this means when using this you know the nature of convergence (already) for the series you are trying to justify the nature of. If convergent bound above by convergent, if divergent bound below by divergent.
  - Like any comparison type test you need to be able to determine the nature of the test you are comparing the original one to. This might require a further test such as the integral test.

Limit Comparison Test:

- Like any comparison type test you need to be able to determine the nature of the test you are comparing the original one to. This might require a further test such as the integral test.

- The limit value L can intuitively be understood as a measure of growth between the two sequences. If $0 < L < \infty$ then it tells you both sequences grow at essentially the same rate (proportional in a sense) in the long run (thus this result will tell you they both, as series, converge or diverge together). If $L = 0$ then it tells you that in the long run, b_n is bigger than a_n, so just like direct comparison, if $\sum b_n$ converges then $\sum a_n$ converges. Lastly, if $L = \infty$ then it tells you that in the long run, b_n is smaller than a_n, so just like direct comparison, if $\sum b_n$ diverges then $\sum a_n$ diverges.

Divergence Test:

- A common mistake is that students conclude a result when $\lim_{n\to\infty} a_n = 0$. This is false. You can only make a claim about the series of a sequence when that sequence approaches something non-zero. There are series that diverge when the limit of the sequence is zero, there are also series that converge when the limit is zero. This result does not allow you to differentiate which is which and a different test must be used.

- The limit does not have to approach a number that is non-zero. The result also applies if the limit $\lim_{n\to\infty} a_n$ does not exist.

Absolute Convergence Test:

- If the sum of the absolute values $\sum |a_n|$ diverges then there is nothing you can conclude about the original series $\sum a_n$ and a different test must be used. If the series is an alternating series, it is often recommended to use the Alternating Series Test in this circumstance.

- As with most comparison type tests, you need to be able to determine the nature of $\sum |a_n|$. Further tests like the Integral Test on $\sum |a_n|$ may be required.

Ratio Test:

- If the result is inconclusive it is recommended to use the Integral Test or Comparison Test if the series is positive. Use an Alternating Series Test if the series is alternating.

- The absolute values are important, do not discard them.

Root Test:

- If the result is inconclusive it is recommended to use the Integral Test or Comparison Test if the series is positive. Use an Alternating Series Test if the series is alternating.

- The absolute values are important, do not discard them.

- If the result is inconclusive it will also be inconclusive for the Ratio Test.

Alternating Series Test:

- All three conditions must be checked. Like the integral test, an easy way to show $\{a_n\}$ is decreasing is to form a function $f(x)$ obtained by replacing $n \to x$ in a_n and show that $f'(x) < 0$ over the appropriate domain.

- If any of the conditions fail then this does not allow you to conclude anything immediately, but it does allow you to move onto a test that you know will work. If either the second or third condition fail then use the divergence test.

Identifying the Appropriate Series for a Test

The following is not a guarantee on what to use but does hit a majority of appropriate tests to use on a series. The following also does not always give you the most efficient way to determine the convergence or divergence of a series, but does allow you to have reserved techniques that work well to predict the outcome of series. By getting good with series you may blend techniques.

Geometric Series: You may identify a Geometric Series by the fact that every term in it is a product and ratio of exponential terms. You may use exponential rules to compactly write them as a single exponential term that is being summed. For example:

$$\sum_{n=5}^{\infty}(-1)^{n-3}\frac{4^{n+2}}{e^n+4}=\sum_{n=5}^{\infty}(-1)^n(-1)^3\frac{4^n\cdot 4^2}{e^n\cdot e^4}=\sum_{n=3}^{\infty}\left(-\frac{4^2}{e^4}\right)\left(\frac{4}{e}\right)^n$$

where we see that $a=-\frac{4^2}{e^4}$ and $r=\frac{4}{e}$. Of course, from this you will only be able to determine the nature of convergence or divergence by investigating $|r|$. Since the index does not start at $n=0$ you can not use the convergent value $\frac{a}{1-r}$ and must include the missing terms or re-index and put into the appropriate form.

Telescoping Series: These are relatively easy to spot in nature as they are a difference of terms in the same sequence that are being summed up. For instance

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n-2}\right)$$

where we see that if $c_n=\frac{1}{2n}$ then the above term in the sum is just

$$c_{n+1}-c_{n-1}=\frac{1}{2(n+1)}-\frac{1}{2(n-1)}=\frac{1}{2n+2}-\frac{1}{2n-2}$$

So you spot them as a "difference of similar looking terms". There are two ways people like to *hide* Telescoping Series:

- By requiring partial fractions to write in the appropriate form

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\text{Partial Fraction Math}\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

- By requiring logarithmic properties to write in the appropriate form

$$\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)=\sum_{n=1}^{\infty}(\ln(n)-\ln(n+1))$$

Once you have identified a Telescoping Series, the trick then is to write out partial sums, find a closed formula by canceling out terms, then take a limit of the closed form.

Integral Test: This test can only be used to determine the convergence or divergence for a sum of positive terms. This technique is often used for series involving the ratio of logarithms with monomials, and the product of a polynomial with exponentials. Essentially I reserve this technique for whenever I notice that when "swapping out a Sigma for an Integral" the result is something that needs to be computed using an integration technique (such as, and mainly, u-substitution or Integration by Parts). Ratios of logarithms are very big giveaways. For example,

$$\sum_{n=5}^{\infty}\frac{\ln(\ln(n))}{n\ln(n^3)}\qquad\sum_{n=2}^{\infty}\frac{1}{n\ln(n)^2}\qquad\sum_{n=7}^{\infty}(n^2+3n+1)e^{-4n}$$

Direct Comparison Test: Because these are typically so tricky to use in practice and be successful with when first experiencing them I mainly reserve them as a shortcut to the Limit Comparison Test for ratios of sums of functions

$$\sum_{n=k}^{\infty} \frac{f_1(n) + \cdots + f_m(n)}{g_1(n) + \cdots + g_l(n)}$$

where an easy inequality to obtain is to:

- Decrease the term by increasing the denominator (to bound below by something divergent)
- Increase the term by decreasing the denominator (to bound above by something convergent)

For instance it is easy enough to see this technique in practice by

$$\frac{1}{n^2+5} \leq \frac{1}{n^2} \leq \frac{1}{n^2-5}$$

You may also use these in the rarer circumstance of dealing with a bounded function (keep in mind everything needs to be positive for a comparison test to be used)

$$\frac{|\arctan(n)|}{1+n^2} \leq \frac{\pi/2}{n^2+1}$$

I mainly reserve them as shortcuts to rational expressions in this course. Keep in mind you need to know the outcome (sort of) in advance when using Direct Comparison. So for the following

$$\sum_{n=1}^{\infty} \frac{1}{n^3+7}$$

I see that as $n \to \infty$ the expression in the sum behaves like $\frac{1}{n^3}$ which when summed over represents a convergent p-series. So my goal is to bound it above. To get a bigger expression I can decrease the denominator to obtain

$$\frac{1}{n^3+7} \leq \frac{1}{n^3+7-7} = \frac{1}{n^3}$$

and presto, I get to proceed. Keep in mind there are several limitations to this for some rational sums like

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$$

where we see that as $n \to \infty$ the expression in the sum behaves like $\frac{1}{\sqrt{n}}$ which when summed over represents a divergent p-series. My goal, if using Direct Comparison, will be to bound it below by a divergent series. However there is absolutely no convenient or immediately obvious way for me to add something positive to the denominator to get $\frac{1}{\sqrt{n}}$. So this technique of direct comparison is very circumstantial even with a game plan in mind.

Limit Comparison Tests: A huge giveaway to use this technique is when the numerator or denominator involves sums of different functions (similar to the above direct comparison) except this technique is guaranteed to work most of the time.

$$\sum_{n=k}^{\infty} \frac{f_1(n) + \cdots + f_m(n)}{g_1(n) + \cdots + g_l(n)}$$

There is a "Hierarchy of Function Growth" that determines what a function looks like in the long run (for large values). For instance, the function $f(x) = x^2 + x + 1$ looks like $f(x) \approx x^2$ for **very** large values of x and the other terms become *negligible*. The growth works like this:

For very large values of n,

$$n^n >> n! >> a^n >> b^n(\text{ where } a > b) >> n^c >> n^{c-1} >> \cdots >> n >> \log_d(n) >> 1$$

where the inequalities flip if the reciprocals of everything are taken.

For instance for large values of x,

$$x^7 + x^{3/2} + e^{-1} \approx x^7$$

$$\frac{1+\ln(x)}{2^x+3^x} \approx \frac{\ln(x)}{3^x}$$

and so forth. Ultimately this tells you that

$$\frac{f_1(n)+\cdots+f_m(n)}{g_1(n)+\cdots+g_l(n)} \approx \frac{\max(f_i(n))}{\max(g_j(n))}$$

in the long run and gives you something to compare to. For instance,

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}+e^{-n}+1}{n^2+\ln(n)}$$

would be a prime candidate for Limit Comparison, in which I would compare $a_n = \frac{\sqrt{n}+e^{-n}}{n^2+\ln(n)}$ to $\approx b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ in which case I would move on forward to taking a ratio a_n/b_n and form a limit.

When trigonometric functions are involved in these I resort to using the Direct Comparison test.

Divergence Test: I honestly use this when most other series tests fail. The only time I use this right-off the bat in front of students is when one of two things is satisfied:

- The terms a_n are (VERY CLEARLY) not decreasing, or
- The limit is recognized as being something very similar to a well known non-zero limit like

$$\lim_{n\to\infty}\left(\frac{n+a}{n}\right)^n = e^a$$

This test is mainly used in my mind as a last resort if it's not **immediately** recognizable. The most common times that I use this test are when something fails under the Alternating Series Test.

Ratio Test: Mainly, the Limit Comparison test works well for fractions involving sums of functions and the ratio test works well for fractions involving products of functions. So you could consider it the cousin to the Limit Comparison Test. Good functions in the product of the fraction are factorials, polynomials and exponentials. The following are great candidates for the ratio test:

$$\sum_{n=1}^{\infty} \frac{e^n n^2}{2^{n+1}} \qquad \sum_{n=3}^{\infty} \frac{n^3}{(2n)!}$$

Since these work really well with factorials, exponentials, polynomials, and I guess technically constants...the only thing left out of this picture are logarithms and trig functions. This is why I reserve logarithms for the Integral Test, it essentially picks up what the Ratio test can't handle. The trigonometric functions are a little trickier and usually require an absolute convergence test (if negatives are allowed) or a comparison test (if say an absolute value of cosine or sine is already present).

When the exponents are pretty bad for the exponentials, I will sometimes move onto using the root test instead. The one exception being that, no matter the series, if it's a product of functions on the numerator and denominator of a ratio, I will always use a ratio test first if I see a factorial.

Note: If this test fails and the series is alternating move onto the Alternate Series Test. If it fails and everything is positive move onto either the Integral Test or Comparison Test.

Root Test: I use this tests for series with bad exponents like

$$\sum_{n=1}^{\infty}\left(\frac{n+3}{n}\right)^{n^2} \qquad \sum_{n=2}^{\infty}\frac{2^{n^2+2n}}{n^n}$$

Essentially, where your standard functions that look like 2^n now look like 2^{n^2+2n} or n now looks like n^n. There are exceptions though...if I see a factorial, I will still always resort to a ratio test.

Alternating Series Test: To play it safe, since the requirements of this test are easy enough to check, these are generally a safe bet to immediately use whenever you see an alternating series. The only thing you will gain by using another test method is efficiency. If you can tell this thing converges by taking the absolute value (if the absolute value sum diverges a reminder that it means nothing) then you can cut down on the time it will take to prove its convergence by taking an Absolute Convergence Test. Only do this though if you're at a point where you can see seven steps ahead and it is immediately obvious to you.

If the Alternating Series Test fails, use the divergence test.

Quick Guide Summary

- Be good at recognizing p-Series, Geometric Series and Telescoping Series. Those stand on their own.
- If your series is an alternating series try the Alternating Series Test unless there is a factorial present. If one of the conditions fails move onto using the divergence test.
- If your series involves a fraction $\sum \frac{f_1 f_2 \cdots f_m}{g_1 g_2 \cdots g_l}$ where there is a product of everything on the numerator and/or denominator use the Ratio Test **unless**
  - The product involves logarithms, in which case check to see if the Integral Test applies.
  - The product involves exponents with bad powers, in which case check to see if the Root Test is better unless there are factorials present, in which case stick with the ratio test.
  - Trigonometric Functions are involved in which case resort to a direct comparison if applicable.

  If the result is inconclusive and the series is alternating move onto the Alternating Series Test. If the result is inconclusive and everything is positive move onto the integral test.
- If your series involves a fraction $\sum \frac{f_1 + \cdots f_m}{g_1 + \cdots g_l}$ where there is a sum of everything on the numerator and/or denominator then use a Limit Comparison and follow the "hierarchy". This works as well if terms are displaced like $n\sqrt{n+1} \approx n\sqrt{n} = n^{3/2}$.
- The Direct Comparison can sometimes be used as a quick shortcut to the Limit Comparison in appropriate scenarios. It is mainly reserved for dealing with bounded trigonometric functions if they contained in a way to be positive.
- If your expression involves bad powers go for the root test.

There are of course exceptions but this gives you some solid ground and a game plan to work with. Some exceptions include

$$\sum_{n=2}^{\infty}\frac{\ln(n)}{\sqrt{n}e^n}$$

in which case, this is a product of terms with no summation so you would think Ratio test. Then we see a logarithm so you might think integral test, but integrating this is a nightmare. It will be better to stick with the ratio test and bust out limit techniques like l'Hôpital's rule.

Examples

For all the following I am omitting the starting and ending index. Assume all of the following are infinite series with an index big enough so the sum is properly defined. All the following are just, by following the guidelines and intuition of above, the first guess of a method to use for each series.

1. $\sum e^{-n}$: It's a Geometric Series
2. $\sum \frac{1}{\sqrt{n}}$: It's a p-series
3. $\sum \frac{(-1)^n}{\sqrt{n}}$: Use an Alternating Series Test
4. $\sum \frac{1}{n(\ln(n))^2}$: Use an Integral Test
5. $\sum \frac{n+1}{n!}$: Use a Ratio Test
6. $\sum \frac{(-1)^n(n^2+1)}{2n^2+n-1}$: Use an Alternating Series Test. It will fail, move on to the divergence test.
7. $\sum \frac{(-3)^n}{n!}$: Use a Ratio Test
8. $\sum \frac{2^n 3^n}{n^n}$: Use a Root Test
9. $\sum \frac{1}{\sqrt{n(n+1)(n+2)}}$: Use the Limit Comparison Test
10. $\sum \frac{1}{n\sqrt{n^2-1}}$: Use the Limit Comparison Test
11. $\sum \frac{\ln(n)}{n^3}$: Use the Integral Test
12. $\sum \frac{(-1)^n}{\ln(n+1)}$: Use the Alternating Series Test
13. $\sum \frac{1}{n2^n}$: Use the Ratio Test
14. $\sum \frac{n-4}{n^3+2n}$: Use the Limit Comparison Test
15. $\sum \frac{1+\cos(n)}{e^n}$: Use the Direct Comparison Test
16. $\sum \frac{n^2}{\sqrt{n^3-1}}$: Use the Limit Comparison Test
17. $\sum (-1)^n 3^{-n/3}$: It's a Geometric Series
18. $\sum \left(\frac{n}{n+1}\right)^{n^2}$: Use the Root Test
19. $\sum \frac{|\sin(n)|}{n}$: Use the Direct Comparison Test
20. $\sum \frac{2^n(2n)!}{n!(n^2+n)}$: Use the Ratio Test
21. $\sum \frac{1}{n^2-1}$: Use the Limit Comparison Test