Math 122 Lecture Notes
Section 2.1 - Open Statements

When a statement includes a variable, we need to know something about the variable before we can tell the truth value of the statement.

Example 1:

- $x^2 + 3x + 2 = 0$

- for all x in the real numbers, $x^2 + 3x + 2 = 0$

- for some x in the real numbers, $x^2 + 3x + 2 = 0$

- for $x = -1$, $x^2 + 3x + 2 = 0$

An **open statement** is an assertion containing one or more variables.

Notation:
$p(x)$ is a statement that contains the variable x
$p(x, y)$ is a statement that contains the variables x and y

The Laws of Logic apply to open statements (because they will apply once the variables are assigned values).

For example, we can say that the contrapositive of $p(x) \to q(x)$ is:

Also, $\neg(p(x) \vee q(x))$ is equivalent to:

Math 122 Lecture Notes
Section 2.2 - Quantifiers

We need to give values to variables in statements in order to decide the truth value. The options are:

- Give a specific value. (e.g. $x = 3$)
- Specify the quantity of allowed replacements to make the statment true.

That second option leads us to the use of quantifiers.

The **universe** of a variable is the collection of values it is allowed to take. (e.g. the real numbers, the integers, positive rational numbers, etc.)

The **universal quantifier** $\forall$ says that the statement is true **for all** allowed replacements of the variable.
Watch for: “for all”, “all”, “every”, “for each”.

The **existential quantifier** $\exists$ says that **there exists** at least one allowed replacement to make the statement true.
Watch for: “there exists”, “there is”, “there are”, “some”, “at least one”.

Example 1: For each statement, write in symbolic form using quantifiers and decide on the truth value of the statement.

(a) statement: "for all integers n, n^2 is not negative".

(b) statement: "for the universe of integers 1, 2, and 3, for all x in this universe, the value x^2 is odd".

(c) statement: "there is an integer n such that $n^2 + 3n + 2 = 0$".

We can nest quantifiers. Here the ordering is important, and we read the statement from left to right.

Example 2: Determine the truth value of the statement $\forall x, \exists y, x + y = 0$.

Example 3: Determine the truth value of the statement $\exists y, \forall x, x + y = 0$.

Example 4: Suppose the universe consists only of the integers 1 and 3. Determine the truth value of the statement $\forall x, \exists y, xy = 3$.

Example 5: Suppose the universe consists only of the integers 1 and 3. Determine the truth value of the statement $\exists y, \forall x, xy = 3$.

Be careful! Sometimes quantifiers can be hidden!

Look at the quadratic formula:

$$\text{if } a \neq 0 \text{ and } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This rule is mean to apply to **all** real numbers x, so the quantifier $\forall$ is hidden.

Explicitly stated:

$$\forall x \in \mathbb{R}, \text{ if } a \neq 0 \text{ and } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Final note: if the universe contains at least one element and we know the statement "$\forall x, p(x)$" is true, then we know that the statement "$\exists x, p(x)$" is also true.

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Section 2.3 - Negating Statements Involving Quantifiers

The negation of "for all *something*, p" is: "there exists *something* such that $\neg p$".

For all - Quantifier

The negation of "there exists *something* such that p" is: "for all *something*, $\neg p$".

Example 1: Write the negation of each statement

(a) statement: "$\forall$ integers $x \geq 2$, x is divisible by a prime".

(b) statement: "$\exists$ a dog that is purple".

(c) statement: "$\forall x, \exists y, \big((x^2 > y) \wedge (x < y)\big)$"

Math 122 Lecture Notes
Section 2.4 - Some Examples of Written Proofs

In this section we'll talk about some common methods of demonstrating a mathematical proof:

- direct proof
- contrapositive
- proof by contradiction
- proof by cases

The Direct Proof Method: Here we build our conclusion directly from the premises.

Proposition 2.4.1: If the integer n is even, then n^2 is even.

The Contrapositive Method: If our statement was $p \Rightarrow q$, then instead we try to show that $\neg q \Rightarrow \neg p$. Our premise then is $\neg q$ (the negation of the conclusion in our original statement) and we try build $\neg p$ (the negation of the original premise).

Proposition 2.4.2: If the integer n^2 is even, then n is even.

The Proof by Contradiction Method: Here we assume the negation of the conclusion along with the original premises. We want to show that we have a contradiction. Therefore, our extra assumption (the negation of the conclusion) was wrong, so we actually have the intended conclusion.

Proposition 2.4.3: $\sqrt{2}$ is not rational.

The Proof by Cases Method: Here we do a proof for each case to demonstrate that the statement hold in all cases.

Proposition 2.4.4: If the integer n^2 is a multiple of 3, then n is a multiple of 3.