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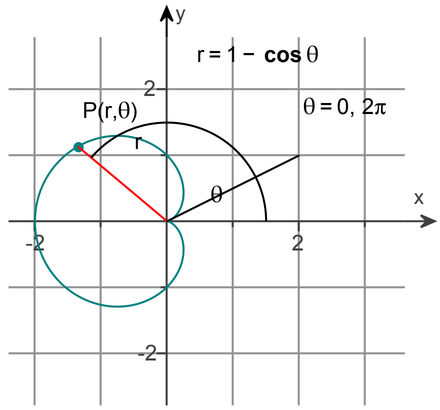
Date: 04/20/22	**Course:** Math 101 A04 Spring 2022	Sections 11.4 & 11.5 [Not f

Find the length of the cardioid $r = 1 - \cos\theta$.

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r,\theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Sketch the loop to determine the limits of integration. The point $P(r,\theta)$ traces the curve once, counterclockwise as θ runs from α to β. Note that the derivative is not continuous at the origin. Therefore, the range for θ must start and end at the values for the origin.



cardioid

Along the positive x-axis, $\theta = 0$. Then, $\alpha = 0$.

Along the positive x-axis, $\theta = 0$ and 2π. In one rotation, θ increases from 0 to 2π. Then, $\beta = 2\pi$.

In the general case, $\frac{d}{d\theta}(\cos n\theta) = -n \sin n\theta$. So, if $r = 1 - \cos\theta$, then, $\frac{dr}{d\theta} = \sin\theta$.

Substituting for α and β, the length, L, of the cardioid is given by the following.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + (\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \end{aligned}$$

Simplify using $\sin^2\theta + \cos^2\theta = 1$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \end{aligned}$$

Simplify using $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$. Notice that for all values of θ on the interval $[0,2\pi]$, $\sin\frac{\theta}{2} \ge 0$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2} d\theta \end{aligned}$$

Integrate using $\frac{d}{d\theta}\left(\cos\frac{\theta}{2}\right) = -\frac{1}{2}\sin\frac{\theta}{2}$.

$$\begin{aligned} L &= \int_0^{2\pi} 2\sin\frac{\theta}{2} d\theta \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} \\ &= (-4\cos(\pi) + 4\cos(0)) \end{aligned}$$

Simplify, using $\cos(\pi) = -1$ and $\cos(0) = 1$, to find the length of the cardioid.

$$\begin{aligned} L &= (-4\cos(\pi) + 4\cos(0)) \\ &= (4 + 4) \\ &= 8 \end{aligned}$$