The integral $\int \tan^2(x)\sec^3(x)dx$ is the case of even tangent and odd secant. Start by transforming this into a sum of secant powers. Using the identity $\sec^2(x) - 1 = \tan^2(x)$ we obtain

$$\begin{aligned}\int \tan^2(x)\sec^3(x)dx &= \int (\sec^2(x)-1)\sec^3(x)dx \\ &= \int \sec^5(x)dx - \int \sec^3(x)dx\end{aligned}$$

We will label these integrals $I_1 = \int \sec^5(x)dx$ and $I_2 = \int \sec^3(x)dx$.

Computing I_2

As this is a case of lonely secant powers, pull off a $\sec^2(x)$ term and perform integration by parts with $dv = \sec^2(x)dx$.

$$I_2 = \int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx$$

Then $u = \sec(x)$ and so we compute

$$\begin{array}{ccc} u = \sec(x) & & du = \sec(x)\tan(x)dx \\ & \searrow & \uparrow \\ dv = \sec^2(x)dx & & v = \tan(x) \end{array}$$

and thus

$$\begin{aligned}I_2 &= \int \sec^3(x)dx \\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx \\ &= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \\ &= \sec(x)\tan(x) - I_2 + \ln|\sec(x)+\tan(x)|\end{aligned}$$

To which we collect and solve for I_2 to obtain

$$I_2 = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C$$

Computing I_1

We perform integration by parts as above to express this integral in terms of an integral of $\sec^{5-2}(x) = \sec^3(x)$. By pulling off a $\sec^2(x)$ and setting $dv = \sec^2(x)dx$ we obtain

$$_1 = \int \sec^5(x)dx = \int \sec^3(x)\sec^2(x)dx$$

Then $u = \sec^3(x)$ and so we compute

$$\begin{array}{ccc} u = \sec^3(x) & & du = 3\sec^3(x)\tan(x)dx \\ & \searrow & \uparrow \\ dv = \sec^2(x)dx & & v = \tan(x) \end{array}$$

and thus

$$\begin{aligned} I_1 &= \int \sec^5(x)dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^3(x)\tan^2(x)dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^3(x)(\sec^2(x)-1)dx \\ &= \sec^3(x)\tan(x) - 3\int \sec^5(x)dx + 3\int \sec^3(x)dx \\ &= \sec^3(x)\tan(x) - 3I_1 + 3I_2 \\ &= \sec^3(x)\tan(x) - 3I_1 + 3\left\{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\right\} \\ &= \sec^3(x)\tan(x) + \frac{3}{2}\sec(x)\tan(x) + \frac{3}{2}\ln|\sec(x)+\tan(x)| - 3I_1 \end{aligned}$$

To which we collect and solve for I_1 to obtain

$$I_1 = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln|\sec(x)+\tan(x)| + C$$

Finally we combine both integrals to complete the integral as

$$\begin{aligned} \int \tan^2(x)\sec^3(x)dx = I_1 - I_2 \\ &= \left(\frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln|\sec(x)+\tan(x)|\right) \\ &\quad - \left(\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|\right) + C \\ &= \frac{1}{4}\sec^3(x)\tan(x) - \frac{1}{8}\sec(x)\tan(x) - \frac{1}{8}\ln|\sec(x)+\tan(x)| + C \end{aligned}$$