Math 122 Assignment 3 Solution Ideas

* (Assignment 2, Question 5)
  (a) False. The elements of A are $1, \{1\}, 2, \{\emptyset\}, \{\{1\},\{2\}\}, \{\{1\}, 2\}$, and $\{2\}$ isn't one of these.
  (b) True. Both 1 and 2 are elements of A, and $A \neq \{1, 2\}$.
  (c) False. As in (a), $\{1, \{2\}\}$ is not an element of A,
  (d) False. Similar reason as in (a).
  (e) False. Both $\{1\} \subseteq A$ and $\{1\} \in A$, so $\{1\} \in A \cap \mathcal{P}(A)$.
  (f) True. $\{2\} \subsetneqq A$.
  (g) False. $\{1\}$ has no non-empty proper subset.

1. (a)

$$\begin{aligned}
A \setminus (B \cup C) &= \{x : (x \in A) \wedge (x \notin B \cup C)\} \\
&= \{x : (x \in A) \wedge \neg(x \in B \cup C)\} \\
&\qquad \text{Definition} \\
&= \{x : ((x \in A) \wedge (x \in A)) \wedge ((x \in B^c) \wedge (x \in C^c)\} \\
&\qquad \text{DeMorgan, Conj. Idemp.} \\
&= \{x : ((x \in A) \wedge (x \in B^c)) \wedge ((x \in A) \wedge (x \in C^c))\} \\
&\qquad \text{Associative and Commutative (several times)} \\
&= \{x : (x \in A \cap B^c) \wedge (x \in A \cap C^c)\} \\
&\qquad \text{Definition} \\
&= \{x : x \in (A \setminus B) \cap (A \setminus C)\} \\
&\qquad \text{Definition} \\
&= (A \setminus B) \cap (A \setminus C)
\end{aligned}$$

   (b)

$$\begin{aligned}
A \setminus (B \cup C) &= A \cap (B \cup C)^c && \text{known} \\
&= (A \cap A) \cap (B^c \cap C^c) && \text{Idempotent} \\
&= (A \cap B^c) \cap (A \cap C^c) && \text{Associative and Commutative (several times)} \\
&= (A \setminus B) \cap (A \setminus C) && \text{known}
\end{aligned}$$

2. (a) We prove the contrapositive: if $A \setminus B \neq \emptyset$, then $A \not\subseteq B$. Suppose $A \setminus B \neq \emptyset$. Then, by definition, there exists an element of A which is not in B. Thus $A \not\subseteq B$. Hence if $A \subseteq B$, then $A \setminus B = \emptyset$.
   (b) Suppose $A \setminus B = \emptyset$. Since $B \subseteq A \cup B$ by definition, it suffices to show that $A \cup B \subseteq B$. Take any $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in B$ there is nothing to show. If $x \in A$ then, since $A \setminus B = \emptyset$, we have $x \in B$. In either case $x \in B$. Thus $A \cup B \subseteq B$ and hence $A \cup B = B$.
   (c) Suppose $A \cup B = B$. Take any $x \in A$. Then $x \in A \cup B$ by definition of union. Since $A \cup B = B$, we have $x \in B$. Therefore $A \subseteq B$.

(d) Yes. In parts (a), (b), (c) above we have shown (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i). The three statements are then logically equivalent by Assignment 1, Question 3.

3. (a) We need to show (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a).

   (a) $\Rightarrow$ (b). Suppose $A = B$. Then $A \cup B = A \cup A = A = A \cap A = A \cap B$.

   (b) $\Rightarrow$ (c). We prove the contrapositive: If $A \oplus B \neq \emptyset$ then $A \cup B \neq A \cap B$. Suppose $A \oplus B \neq \emptyset$. Then there exists $x \in A$ such that $x \notin B$, or there exists $x \in B$ such that $x \notin A$. In either case, the element x belongs to $A \cup B$ but not to $A \cap B$. Thus $A \cup B \neq A \cap B$. Hence if $A \cup B = A \cap B$, then $A \oplus B = \emptyset$.

   (c) $\Rightarrow$ (a). We prove the contrapositive: if $A \neq B$ then $A \oplus B \neq \emptyset$. Suppose $A \neq B$. Then either there exists $x \in A$ such that $x \notin B$, or there exists $x \in B$ such that $x \notin A$. In either case, $x \in A \oplus B$, so $A \oplus B \neq \emptyset$. Hence if $A \oplus B = \emptyset$, then $A = B$.

   The proof is now complete.

4. (a) It is possible to generate a counterexample from a Venn diagram, as shown in class. Here is a different one. Let $A = B = C = \{1\}$. Then $A \setminus (B \setminus C) = A \setminus \emptyset = A = \{1\}$, while $(A \setminus B) \setminus C = \emptyset \setminus C = \emptyset \neq \{1\}$. Thus $A \setminus (B \setminus C)$ is not equal to $(A \setminus B) \setminus C$ for all sets A, B, C. But the Venn diagram suggests $(A \setminus B) \setminus C$ is a subset of $A \setminus (B \setminus C)$.

   (b) The two sets are equal. We show LHS $\subseteq$ RHS and RHS $\subseteq$ LHS.

   LHS $\subseteq$ RHS. Take any $x \in (A \oplus B^c) \oplus C$. Then $x \in A \oplus B^c$ and $x \notin C$ or $x \notin A \oplus B^c$ and $x \in C$. In the first case $x \notin C$ and either $x \in A$ and $x \notin B^c$, or $x \in B^c$ and $x \notin A$. In all cases, $x \in A \oplus (B^c \oplus C)$.

   RHS $\subseteq$ LHS. Take any $x \in A \oplus (B^c \oplus C)$. Then $x \in A$ and $x \notin (B^c \oplus C)$, or $x \notin A$ and $x \in (B^c \oplus C)$. In the first case, $x \in A, x \in B^c$ and $x \in C$, or $x \in A, x \notin B^c$ and $x \notin C$. In the second case $x \notin A, x \in B^c$ and $x \notin C$, or $x \in A, x \notin B^c$ and $x \notin C$. In all cases, $x \in (A \oplus B^c) \oplus C$.
   Thus $(A \oplus B^c) \oplus C = A \oplus (B^c \oplus C)$.

5. (a) Let A_1, A_2, A_3 and A_4 be sets. We define

   $$A_1 \cup A_2 \cup A_3 \cup A_4 = \{x : (x \in A_1) \vee (x \in A_2) \vee (x \in A_3) \vee (x \in A_4\}$$

   and

   $$A_1 \cap A_2 \cap A_3 \cap A_4 = \{x : (x \in A_1) \wedge (x \in A_2) \wedge (x \in A_3) \vee (x \in A_4\}.$$

   (b) We show LHS $\subseteq$ RHS and RHS $\subseteq$ LHS.

   LHS $\subseteq$ RHS. Take any $x \in (A_1 \cup A_2 \cup A_3 \cup A_4)^c$. Then, by definition, $x \notin A_1$, $x \notin A_2$, $x \notin A_3$ and $x \notin A_4$. That is, $x \in A_1^c$, $x \in A_2^c$, $x \in A_3^c$ and $x \in A_4^c$. Therefore, $x \in (A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)$.

RHS $\subseteq$ LHS. Take any $x \in (A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)$. Then, by definition, $x \notin A_1$, $x \notin A_2$, $x \notin A_3$ and $x \notin A_4$. Therefore, $x \notin (A_1 \cup A_2 \cup A_3 \cup A_4)$. Hence $x \in (A_1 \cup A_2 \cup A_3 \cup A_4)^c$.

Therefore $(A_1 \cup A_2 \cup A_3 \cup A_4)^c = (A_1^c \cap A_2^c \cap A_3^c \cap A_4^c)$.

(c) $(A_1 \cap A_2 \cap A_3 \cap A_4)^c = (A_1^c \cup A_2^c \cup A_3^c \cup A_4^c)$.