CSC 225 FALL 2022
ALGORITHMS AND DATA STRUCTURES I
ASSIGNMENT 2 - SOLUTIONS
UNIVERSITY OF VICTORIA

1. (a) $T(n) = 1A + 1A + (n^2+2)C + (n^2+1)(2A) = 3n^2 + 6$

   Or

   $$T(n) = 3 + \sum_{i=1}^{n^2+1} 3 = 3 + 3(n^2+1) = 3n^2 + 6$$

   (b) $T(n) = 1A + 1A + (n^2+2)C + (n^2+1)A + (n^2+1)A + \big(2 + 3 + \cdots + (n^2+2)\big)C + \big(1 + \cdots + (n^2+1)\big)A + \big(1 + \cdots + (n^2+1)\big)A$

   $$= 1 + 1 + n^2 + 2 + n^2 + 1 + n^2 + 1 + \frac{(n^2+2)(n^2+3)}{2} - 1 + \frac{(n^2+1)(n^2+2)}{2} + \frac{(n^2+1)(n^2+2)}{2}$$

   $$= 5 + 3n^2 + \frac{n^4 + 5n^2 + 6}{2} + n^4 + 3n^2 + 2 = \frac{3}{2}n^4 + \frac{17}{2}n^2 + 10$$

   Or

   $$T(n) = 3 + \sum_{i=1}^{n^2+1}\left(4 + \sum_{j=1}^{i} 3\right) = 3 + 4\sum_{i=1}^{n^2+1} 1 + 3\sum_{i=1}^{n^2+1} i$$

   $$= 3 + 4(n^2+1) + 3\frac{(n^2+1)(n^2+2)}{2} = 7 + 4n^2 + 3\left(\frac{n^4 + 3n^2 + 2}{2}\right)$$

   $$= 7 + 4n^2 + \frac{3}{2}n^4 + \frac{9}{2}n^2 + \frac{6}{2} = \frac{3}{2}n^4 + \frac{17}{2}n^2 + 10$$

2. Note that there are many ways to write this and that each of those will have a different runtime.

   Algorithm recMinMax(A,n):

   Input: An array A storing $n \geq 1$ elements.

   Output: The pair (a, b) where $a = \min A$ and $b = \min A$.

   if $n = 1$ **then**

   return $(A[0], A[0])$

   else

   $(a, b) \leftarrow$ recMinMax(A,$n-1$)

   if $A[n-1] < a$ **then**

   $a \leftarrow A[n-1]$

if $A[n-1] > b$ **then**

$b \leftarrow A[n-1]$

return (a, b)

Here, for my version

$$T(n) = \begin{cases} 2, & \text{if } n = 1 \\ T(n-1) + 6, & \text{if } n \geq 2 \end{cases}$$

3. a) Let $T(n) = \begin{cases} 1, & \text{if } n = 1 \\ T(n-1) + n, & \text{if } n \geq 2 \end{cases}$. We want to show that $T(n) = n(n+1)/2$.

For the base case, let $n = 1$, then according to the recurrence equation $T(1) = 1$. Also, the closed formula gives us $T(1) = \frac{1(2)}{2} = 1$. So, the base case holds.
Assume now that it is true for some $n = k \geq 1$. That is, $T(k) = k(k+1)/2$.
Now, we consider when $n = k + 1 \geq 2$. By the recurrence,

$$T(k+1) = T(k) + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k^2 + k}{2} + \frac{2(k+1)}{2}$$
$$= \frac{k^2 + 3k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

Therefore, by induction, $T(n) = n(n+1)/2$ for all $n \geq 1$.

b) Let $T(n) = \begin{cases} 1, & \text{if } n = 0 \\ T(n-1) + 2^n, & \text{if } n \geq 1 \end{cases}$. We want to show that $T(n) = 2^{n+1} - 1$.

For the base case, let $n = 0$, then according to the recurrence equation $T(0) = 1$. Also, the closed formula gives us $T(0) = 2^{0+1} - 1 = 2 - 1 = 1$. So, the base case holds.
Assume now that it is true for some $n = k \geq 0$. That is, $T(k) = 2^{k+1} - 1$.
Now, we consider when $n = k + 1 \geq 1$. By the recurrence,

$$T(k+1) = T(k) + 2^{k+1}$$
$$= 2^{k+1} - 1 + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

Therefore, by induction, $T(n) = 2^{n+1} - 1$ for all $n \geq 0$.

4. Using the definition of Big-Oh, I will prove each:

a) We want to show that there exists a $c, n_0 > 0$ such that $3n^2 - 100n + 6 \leq cn^2$ for all $n \geq n_0$.

$3n^2 - 100n + 6 \leq 3n^2 + 6 \leq 3n^2 + 6n^2 = 9n^2$

For all $n \geq 1$. Therefore, for $c = 9, n_0 = 1$, $3n^2 - 100n + 6 \leq cn^2$, for all $n \geq n_0$

b) We want to show that there exists a $c, n_0 > 0$ such that $2n^3 + n\sqrt{n} \leq cn^3$ for all $n \geq n_0$.

$2n^3 + n\sqrt{n} = 2n^3 + n^{3/2} \leq 2n^3 + n^3 = 3n^3$

For all $n \geq 1$. Therefore, for $c = 3, n_0 = 1$, $2n^3 + n\sqrt{n} \leq cn^3$, for all $n \geq n_0$.

c) We want to show that there exists a $c, n_0 > 0$ such that $3n \log n + 2n\sqrt{n} \leq cn\sqrt{n}$ for all $n \geq n_0$.

Here, we note that $\log n = \log\left(\left(\sqrt{n}\right)^2\right) = 2\log\sqrt{n} \leq 2\sqrt{n}$ for all $n \geq 1$. So,

$3n \log n + 2n\sqrt{n} \leq 3n\left(2\sqrt{n}\right) + 2n\sqrt{n} = 8n\sqrt{n}$

For all $n \geq 1$. Therefore, for $c = 8, n_0 = 1$, $3n \log n + 2n\sqrt{n} \leq cn\sqrt{n}$ for all $n \geq n_0$.

d) We want to show that there exists a $c, n_0 > 0$ such that $(x + y)^2 \leq c(x^2 + y^2)$ for all $x, y \geq n_0$.

$(x + y)^2 = x^2 + 2xy + y^2$
$\leq x^2 + 2\max(x^2, y^2) + y^2$
$\leq x^2 + 2x^2 + 2y^2 + y^2$
$= 3x^2 + 3y^2$
$= 3(x^2 + y^2)$

For all $x, y \geq 1$. Therefore, for $c = 3, n_0 = 1$, $(x + y)^2 \leq c(x^2 + y^2)$ for all $x, y \geq n_0$

5. The order is as follows, from fastest to slowest, where if they are on the same line, they are Big-Theta of each other (I have also highlighted them):

$\log\log n$
$\log n$, $\ln n$
$(\log n)^2$
$\sqrt{n}$
n
$n \log n$
$n^{1.375}$

$n^2, n^2 + \log n$
n^3
$n - n^3 + 7n^5$
$2^n, 2^{n-1}$
e^n
$n!$