Math 110 - Homework 8
Topic: Matrix inverses

Distributed on October 29

Practice

Before beginning the rest of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 4.4 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` is a matrix then the transpose of `A` can be calculated by using `transpose(A)`, or by using `A.'`
- If `A` is a square matrix, you can find the inverse by using inv(`A`). Be careful, though! Sometimes MATLAB will still give you an answer even if A is not invertible.

Part I: Calculation by hand

1. Let $A = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Calculate A^{-1}.

   Solution: We set up an augmented matrix and row reduce. Students should show the row reduction steps.

   $$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/4 & 0 & 1/4 \end{array}\right].$$

   So

   $$A^{-1} = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 0 & 1/2 & -1/2 \\ -1/4 & 0 & 1/4 \end{bmatrix}.$$

   (b) Calculate B^{-1}.

   Solution: By the same method as part (a), we obtain the following (again, students were expected to show the row reduction steps).

   $$B^{-1} = \begin{bmatrix} -1/3 & -1 & 2 \\ 0 & 1 & -1 \\ 1/3 & 0 & 0 \end{bmatrix}.$$

   (c) Use your results to (a) and (b) to calculate $(AB)^{-1}$ without doing any more row operations.

   Solution:

   $$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -1/3 & -1 & 2 \\ 0 & 1 & -1 \\ 1/3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 3/4 \\ 0 & 1/2 & -1/2 \\ -1/4 & 0 & 1/4 \end{bmatrix} = \begin{bmatrix} -7/12 & -1/2 & 3/4 \\ 1/4 & 1/2 & -3/4 \\ 1/12 & 0 & 1/4 \end{bmatrix}.$$

Part II: Concepts and connections

1. Find an example of 2×2 matrices A and B such that A and B are both invertible, but $A + B$ is not invertible. Remember to justify that your example has the required properties.

   Solution: There are many solutions available. Perhaps the easiest to find is $A = I_2$ and $B = -I_2$. Then A and B are both invertible (with $A^{-1} = A$ and $B^{-1} = B$), but $A + B = 0_{2\times 2}$, which is certainly not invertible.

2. It is a fact, which you do not need to prove, that

$$\begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}^{-1} = \frac{1}{15}\begin{bmatrix} 6 & -3 & -3 \\ 13 & 1 & -9 \\ -2 & 1 & 6 \end{bmatrix}.$$

Use this fact to find all solutions to the following system of equations.

$$x + y + 2z = 15$$

$$-4x + 2y + z = 0$$

$$x + 3z = 1$$

Solution: The given system of equations is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 1 \end{bmatrix},$$

and since the matrix there is invertible, we get

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 15 \\ 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{15}\begin{bmatrix} 6 & -3 & -3 \\ 13 & 1 & -9 \\ -2 & 1 & 6 \end{bmatrix}\begin{bmatrix} 15 \\ 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{15}\begin{bmatrix} 87 \\ 186 \\ -24 \end{bmatrix} \\ &= \begin{bmatrix} 29/5 \\ 62/5 \\ -8/5 \end{bmatrix} \end{aligned}$$

Therefore the (unique) solution to the system of equations is $x = \frac{29}{5}$, $y = \frac{62}{5}$, $z = -\frac{8}{5}$.

3. Suppose that A is a 3×2 matrix and B is a 2×3 matrix.

(a) Explain why, no matter how A and B are chosen, the 3×3 matrix AB can never be invertible.

Solution: The system of equations $[B|\vec{0}]$ is a homogeneous system with two rows and three columns. It therefore has infinitely many solutions. Let $\vec{v}$ be a non-zero vector that is a solution to the system $[B|\vec{0}]$. Then $B\vec{v} = \vec{0}$, so

$$(AB)\vec{v} = A(B\vec{v}) = A\vec{0} = \vec{0}.$$

Therefore the system of equations $[AB|\vec{0}]$ has a non-zero solution (namely $\vec{v}$), and by the fundamental theorem of linear algebra AB is not invertible.

(b) Find an example of such an A and B where the 2×2 matrix BA is invertible.

Solution: Many examples are possible. Here's one that you might find by guessing and checking:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Certainly, BA is invertible.