202201 Math 122 Assignment 5

Due: Friday, April 1, 2022 at 23:59. Please submit on your section's Crowdmark page.

There are five questions of equal value (worth a total of 45 marks). There are 4 bonus marks available if the solutions are typeset with LaTeX. Information on obtaining and using LaTeX is available on the cross-listed Brightspace page.

Please feel free to discuss these problems with each other. You may not access any "tutoring" or "help" website in any way. In the end, each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

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1. (a) For which bases b is $(1331)_b$ the third power of an integer?
   (b) Find the base b if $(31)_b \times (14)_b = (464)_b$.
   (c) Find a base b in which $(56)_b$ and $(66)_b$ are squares of consecutive integers.
   (d) Find x if $(24x3)_5 = (x02x)_7$.

2. (a) Prove that $6 \mid (n^3 + 5n)$ for every integer $n \geq 1$.
   (b) Prove that for nonzero integers a and b and integer c if $\gcd(a, b) = 1$ and $a \mid c$ and $b \mid c$, then $ab \mid c$.

3. Find the prime factorization of the smallest integer that is divisible by 24 and 42, and is simultaneously a fourth power and a sixth power.

4. (a) Let p be a prime number such that $p \mid n^3$. Prove that $p^3 \mid n^3$.
   (b) Let a be a positive integer. Find, with proof, all possibilities for $\gcd(a, 63)$.
   (c) Let d and n be positive integers such that $d \mid n$ and $d \mid n + 9$. Prove that $d = 1$ or $d = 3$ or $d = 9$.

5. (a) Prove that if $a \equiv b \pmod{12}$ and $b \equiv c \pmod{18}$ then $a \equiv c \pmod{3}$.
   (b) Find an integer $m > 1$ and an integer x such that $x^2 \equiv 1 \pmod{m}$, but $x \not\equiv 1 \pmod{m}$ and $x \not\equiv -1 \pmod{m}$.
   (c) Find the last digit of 37^{37} in base 10. Then find the last digit of 37^{37} in base 7. Perform both calculations without explicitly computing 37^{37}.