Math 110, Fall 2021, Test 1B Sample Answers

Note: Students were required to show their steps in solving the problems. For brevity these solutions omit intermediate steps of row reductions.

Instructions:

- You may use a calculator on this test, but the only permitted calculators are SHARP brand calculators with model numbers beginning EL-510R. No other electronic devices are permitted.
- No notes, textbooks, or other outside materials or aids are permitted.
- For questions with numerical answers, either give your answer in exact form or give it as a decimal to two decimal places.
- For **all** questions you must show your work to be given credit, even if your answer is correct.
- For questions 1–3, show your work and then enter your final answer in the box provided.
- This test is printed double-sided - be sure not to miss the questions on the back of the first page! For the long-answer questions the backs of the pages are additional space for your solution.

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(1 point) 1. Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Calculate $2\vec{v} + \vec{w} - 2(\vec{w} + \vec{v})$.

Solution:

$$2\vec{v} + \vec{w} - 2(\vec{w} + \vec{v}) = 2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 2\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$$

Answer: $\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$

(1 point) 2. Let $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Find $\cos(\theta)$.

Solution:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{3}{\sqrt{11}\sqrt{9}} = \frac{1}{\sqrt{11}}$$

(The last step of simplification was not required).

Answer: $\dfrac{1}{\sqrt{11}}$

(1 point) 3. Find all values of x such that $\begin{bmatrix} 0 \\ x \\ x+1 \end{bmatrix}$ has length 1.

Solution: We have

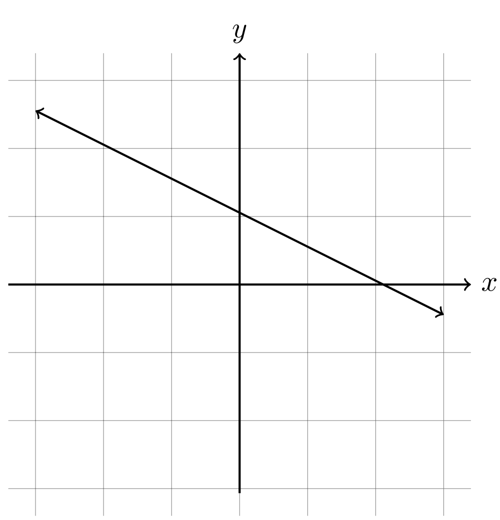
$$\left\| \begin{bmatrix} 0 \\ x \\ x+1 \end{bmatrix} \right\| = \sqrt{0^2 + x^2 + (x+1)^2} = \sqrt{2x^2 + 2x + 1}.$$

To give this vector length 1 we therefore need $2x^2 + 2x + 1 = 1$, so $2x(x+1) = 0$. That is, the possible values of x are $x = 0$ and $x = -1$.

Answer: $x = 0$ and $x = -1$

(1 point) 4. Sketch the line in $\mathbb{R}^2$ that has vector equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Very briefly (one sentence is enough) tell us how you decided where to draw the line.

Solution: By choosing $t = 0$ and $t = 1$ we find that the line passes through $(0, 1)$ and $(-2, 2)$, so we draw the only line through those two points.



(4 points) 5. Determine whether the following system of linear equations in variables x_1, x_2, x_3, x_4 has no solution, exactly one solution, or infinitely many solutions.

$$x_1 - 3x_2 + 2x_3 = 1$$

$$-2x_1 + 5x_2 + x_4 = 2$$

$$x_2 + x_3 + x_4 = 0$$

Solution: We set up an augmented matrix and row reduce.

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 1 \\ -2 & 5 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 3/5 & -4/5 \\ 0 & 0 & 1 & 2/5 & 4/5 \end{array}\right].$$

We see from the reduced row echelon form that x_4 is a free variable, and therefore there are infinitely many solutions.

(4 points) 6. Find all values of k for which $\begin{bmatrix}1\\2\\k\end{bmatrix}$ is a linear combination of $\begin{bmatrix}-1\\3\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

Solution: We want to know for which k there are a and b such that

$$a\begin{bmatrix}-1\\3\\2\end{bmatrix}+b\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\k\end{bmatrix}.$$

We treat this vector equation as a system of linear equations in variables a and b, and solve:

$$\left[\begin{array}{cc|c}-1 & 1 & 1\\3 & -1 & 2\\2 & 1 & k\end{array}\right]\to\left[\begin{array}{cc|c}1 & 0 & 3/2\\0 & 1 & 5/2\\0 & 0 & 2k-11\end{array}\right].$$

This system has a solution if and only if $2k-11=0$, that is, if and only if $k=11/2$. Therefore the only k for which $\begin{bmatrix}1\\2\\k\end{bmatrix}$ is a linear combination of $\begin{bmatrix}-1\\3\\2\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is $k=11/2$.

(4 points) 7. Let L be the line in $\mathbb{R}^3$ that passes through the points $(2,1,0)$ and $(0,1,1)$. Let P be the plane in $\mathbb{R}^3$ that is orthogonal to L and passes through the point $(1,2,1)$. Find, with justification, a vector equation for P.

Solution: Since P is orthogonal to L, a direction vector for L will be a normal vector to P. Such a vector is $\vec{n} = \begin{bmatrix} 2-0 \\ 1-1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. We know that P passes through $(1,2,1)$, so in normal form the equation for P is

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Expanding the dot products we obtain the general form

$$2x - z = 1.$$

We rearrange this equation to say

$$z = -1 + 2x,$$

and then by substituting we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -1+2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The equation above is a vector equation for P (it is not the only possible correct answer).

(4 points) 8. Suppose that $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$, and that c is a scalar. Show that

$$\text{proj}_{\vec{w}}(c\vec{v}) = c\,\text{proj}_{\vec{w}}(\vec{v}).$$

Note: In this question we want you to write a general argument, so you should not choose specific numbers for any of the objects in the question.

Solution: We use properties of the dot product to calculate:

$$\begin{aligned}
\text{proj}_{\vec{w}}(c\vec{v}) &= \left(\frac{\vec{w}\cdot(c\vec{v})}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= \left(\frac{c(\vec{w}\cdot\vec{v})}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= c\left(\frac{\vec{w}\cdot\vec{v}}{\vec{w}\cdot\vec{w}}\right)\vec{w} \\
&= c\,\text{proj}_{\vec{w}}(\vec{v})
\end{aligned}$$