Math101 - Midterm II

Version A - March 26

Sections: [A01 - A05]

Last Name, First Name : ___________________________________

Lecture Section : ____

Student ID : ____

- This examination has 13 problems [10 multiple choice and 3 long answer questions], worth a total of 25.
- It consists of 9 pages, excluding this one. Make sure your exam copy has the correct number of pages.
- You have 90 minutes to complete the exam.
- Answer each question in the appropriate space immediately below that question. Use the backsides as an extra space for rough work. Show all your work to get full marks. Unsupported correct answers may get zero marks. Also, you must fill the bubble sheet for multiple choice questions.
- No Textbooks, No Class Notes, and No Formula Sheets allowed.
- Only **Sharp EL-510R** or its variant calculator is allowed. No other calculator allowed.
- **No Electronic Communication Device of any sort (e.g. cell phones, laptops, iPods, translators, pagers) are allowed during the exam.**

Question	1 – 10[10]	11[5]	12[5]	13[5]	**Total**
Marks					

Multiple Choice Part [Q1 - Q10]

We will only grade these questions by the final answers but you must show your complete mathematical work as well. Final answers with work missing may get zero on the question.

Q1. Consider the convergent sequence, recursively defined as $a_{n+1} = \dfrac{5 + a_n}{7}$ for $n \geq 1$ and $a_1 = 1$. This sequence converges to:

(a) $\frac{5}{8}$; (b) $\frac{5}{6}$; (c) 0; (d) $\frac{5}{7}$; (e) None of these.

Let $\lim_{n\to\infty} a_n = L$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5+a_n}{7}$$

$$\Rightarrow L = \frac{5+L}{7}$$

$$\Rightarrow 7L = 5 + L \Rightarrow L = \frac{5}{6}$$

Q2. Consider the series $\sum_{n=1}^{\infty} \dfrac{(n+9)!}{9!\ n!\ 9^n}$. Select the appropriate statement that describes whether this series converges or diverges.

(a) The series converges because the limit in the ratio test is less than 1;
(b) The series diverges because the limit in the ratio test is $L = 1$;
(c) The series converges by the root test because the limit is greater than 1;
(d) The series diverges because the limit in the root test is $L = 1$;
(e) The series diverges by the n^{th} term divergence test.

Using the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+10)!}{\cancel{9!}(n+1)!\ 9^{n+1}} \cdot \frac{\cancel{9!}\ n!\ 9^n}{(n+9)!}\right| = \lim_{n\to\infty} \left|\frac{(n+10)}{9\cdot(n+1)}\right|$$

$$= \frac{1}{9} < 1$$

Converges by the Ratio test.

Q3. Consider the series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^3}$. Select the appropriate statement that describes whether this series converges or diverges.

(a) The series diverges by the divergence test;

(b) The series converges by the integral test because the integral $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$ converges;

(c) The series diverges by the integral test because the integral $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$ diverges;

(d) The series converges by the integral test because the integral $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$ converges and they have the same value;

(e) The series diverges because the sequence $a_n = \dfrac{1}{n(\ln n)^3}$ is non-decreasing.

Let $f(x) = \dfrac{1}{x(\ln x)^3}$. For $x > 4$, $f(x) > 0$ and $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x}\right)(\ln x)^{-3} + \dfrac{1}{x}\dfrac{d}{dx}(\ln x)^{-3}$

$$= -\frac{1}{x^2}\cdot\frac{1}{(\ln x)^3} - \frac{3}{x}(\ln x)^{-4}\cdot\frac{1}{x}$$

$$= -\frac{1}{x^2(\ln x)^3}\left[1 + \frac{3}{\ln x}\right]$$

< 0 for $\ln x < -3$

$\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$

Let $u = \ln x$

$$\Rightarrow \int_4^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{b\to\infty} \int_{\ln 4}^{b} \frac{1}{u^3} du = -\lim_{b\to\infty} \frac{1}{2u^2}\Big|_{\ln 4}^{b}$$

$= \dfrac{1}{2(\ln 4)^2} < \infty$, Converges by the integral test

Q4. Consider the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{15}\right)^n$. Choose the correct statement on the nature of convergence of the series by the appropriate test.

(a) Converges as it is a geometric series with $r = \frac{n}{15}$;
(b) Converges because it is a p-series with $p = -n$;
(c) Diverges by the root test because the limit from the root test is infinity;
(d) Converges by the alternating series test;
(e) Diverges because it is a p-series with $p = -n$.

Using the root test;

$$\lim_{n\to\infty}\left|(-1)^{n+1}\left(\frac{n}{15}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n}{15}\right) = \infty,$$ Diverges by the root test.

Q5. Consider the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$. Select the appropriate statement that describes whether this series converges absolutely, conditionally, or diverges by the appropriate test.

(a) The series converges conditionally because the limit in the root test is $L = 1$;
(b) The series converges conditionally as it converges by the alternating series test but the series of absolute values diverges by the limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$;
(c) The series converges absolutely by the limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$;
(d) The series diverges because the limit in the ratio test is $L = 1$;
(e) The series diverges by the divergence test.

Alternating series test.

① It is alternating.

② $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n^2+5} = 0$

③ $\displaystyle\frac{d}{dx}\left[\frac{x}{x^2+5}\right] = \frac{5-2x^2}{(x^2+5)^2}$

$5-2x^2 < 0 \Rightarrow x^2 > \frac{5}{2}$

$\Rightarrow x > \sqrt{\frac{5}{2}}$

Converges by AST.

Series of Absolute Values:

$$\sum_{n=1}^{\infty}\frac{n}{n^2+5}$$

Taking $\sum_n b_n = \sum_n \frac{1}{n}$ (which is divergent)

$\displaystyle\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n^2+5} = 1 > 0$

$\sum_n \frac{n}{n^2+5}$ diverges by the L.C.T.

Given series converges conditionally.

Q6. What is the largest interval for which the series $\displaystyle\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$ converges absolutely?

(a) $(-1, 1)$; (b) $[-1, 1]$; (c) $(-2, 2]$; (c) $[-2, 2)$; (e) $(-\infty, \infty)$.

Using the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^{n+1}}\,\cancel{x^{n+1}}}{(n+1)!}\cdot\frac{n!}{\cancel{2^n}\cdot\cancel{x^n}}\right| = \lim_{n\to\infty}\frac{|x|}{(n+1)} = 0 < 1$$

Series converges for all x.

Q7. Using the fact that the Maclaurin series for the tangent function over the interval $-\pi/2 < x < \pi/2$ is:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots,$$

determine the first non-zero term of the Maclaurin series for $\ln|\sec(x)|$.

(a) 1; (b) x; (c) $\frac{x^3}{3}$; (d) $\frac{x^2}{2}$; (e) $\frac{x^4}{12}$.

We use:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Integrating:

$$\Rightarrow -\ln|\cos x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots + C$$

or $$\ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots + C$$

At $x=0$

$$\Rightarrow \ln(1) = 0 = \frac{0^2}{2} + \frac{0^4}{12} + \cdots + C \Rightarrow C=0$$

$$\Rightarrow \ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

Q8. Using series expansion, find $\lim_{t\to 0} \frac{1-\cos(t)-\frac{1}{2}t^2}{3t^4}$.

(a) $-\frac{1}{48}$; (b) $\frac{1}{72}$; (c) $-\frac{1}{8}$; (d) $-\frac{1}{72}$; (e) $\frac{1}{2}$.

$$\lim_{t\to 0} \frac{1-\cos t - \frac{t^2}{2}}{3t^4} = \lim_{t\to 0} \frac{\cancel{1} - \left(\cancel{1} - \cancel{\frac{t^2}{2}} + \frac{t^4}{4!} - \cdots\right) - \cancel{\frac{t^2}{2}}}{3t^4}$$

$$= \lim_{t\to 0} \frac{-\frac{t^4}{4!} + \frac{t^6}{6!} - \cdots}{3t^4}$$

$$= \lim_{t\to 0} \frac{-\frac{1}{4!} + \frac{t^2}{6!} - \cdots}{3}$$

$$= \frac{-1}{4!}/3 = -\frac{1}{72}$$

Q9. The binomial series for $\left(1+\frac{x}{2}\right)^{-2}$ starts with:

(a) $1-\frac{1}{2}x+\frac{1}{6}x^2$; (b) $1-x+\frac{2}{3}x^2$; (c) $1+x-\frac{3}{4}x^2$; (d) $1-x+\frac{3}{4}x^2$ (e) None of these.

$$\left(1+\frac{x}{2}\right)^{-2} = 1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{x}{2}\right)^2 + \cdots$$

$$= 1 - x + -(-3)\frac{x^2}{4} + \cdots$$

$$= \underbrace{1 - x + \frac{3x^2}{4}} + \cdots$$

Q10. The solution $x(t)$ of the separable differential equation $\frac{dx}{dt} = 2t\sqrt{1-x^2}$ with $-1 < x < 1$ is given by:

(a) $\sin(2t^2+C)$; (b) $\sin(t^2+C)$; (c) $C\tan t^2$; (d) $C\tan t$; (e) None of these.

Separating variables:

$$\frac{dx}{\sqrt{1-x^2}} = (2t)\, dt$$

Integrating:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int (2t)\, dt$$

$$\Rightarrow \quad \sin^{-1}(x) = t^2 + C$$

$$\Rightarrow \quad x(t) = \sin[t^2 + C].$$

Long Answer Part [Q11 - Q13]

You must show complete mathematical work to justify your answers to the following problems as part marks may be awarded for your correct work on these questions. Direct answers without supporting work may get zero marks.

Q11. (a) Determine the convergence of the **sequence** whose n^{th} term is defined by:

$$a_n = 3n \cdot \ln\left(\frac{n}{n+3}\right)$$

.

Taking the limit of the n^{th} term (and using **L'Hopital's rule**), we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(3n)\cdot\ln\left(\frac{n}{n+3}\right) = 3\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+3}\right)}{1/n} = \lim_{n\to\infty}\frac{-9n^2}{n^2+n} = -9$$

The sequence therefore converges.

(b) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{(n+3)^{2n}}{(n^2-3)^{3n}}$ converges or diverges.

Applying the Root test, we get

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(n+3)^{2n}}{(n^2-3)^{3n}}\right|} = \lim_{n\to\infty}\frac{(n+3)^2}{(n^2-3)^3} = 0 < 1$$

Thus the series coverges absolutely by the Root test.

Q12. Find the Taylor Polynomial $P_3(x)$ of order 3 generated by $f(x) = \dfrac{4}{x}$ at $a = 2$.

Solution: We create the following table of derivatives and evaluating at $a = 2$ to obtain

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\frac{4}{x}$	$\frac{4}{2} = 2$
1	$-\frac{4}{x^2}$	$-\frac{4}{4} = -1$
2	$\frac{8}{x^3}$	$\frac{8}{8} = 1$
3	$-\frac{24}{x^4}$	$-\frac{24}{16} = -\frac{3}{2}$

and therefore

$$\begin{aligned} P_3(x) &= \frac{2}{0!} + \frac{-1}{1!}(x-2) + \frac{1}{2!}(x-2)^2 + \frac{-3/2}{3!}(x-2)^3 \\ &= 2 - (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{4}(x-2)^3 \end{aligned}$$

Q13. Consider the infinite power series:

$$\sum_{n=1}^{\infty} \frac{8^n x^n}{n} \tag{1}$$

(a) Find the interval of convergence of the power series given in (1). ***Note:*** *Make sure to check your series at the end points.*

Let x be fixed and we construct

$$L = \lim_{n\to\infty} \left| \frac{8^{n+1}x^{n+1}}{n+1} \div \frac{8^n x^n}{n} \right| = \lim_{n\to\infty} \left| \frac{8^{n+1}}{8^n} \frac{x^{n+1}}{x^n} \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| 8x \frac{n}{n+1} \right| = 8|x|$$

and thus the series converges absolutely for $L = 8|x| < 1$ which gives the interval $-\frac{1}{8} < x < \frac{1}{8}$. Evaluating at the endpoint $x = -8$ gives

$$\sum_{n=1}^{\infty} \frac{8^n(-1)^n/8^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges as it is the alternating Harmonic series. On the other hand, at $x = 8$ we get

$$\sum_{n=1}^{\infty} \frac{8^n \cdot 1/8^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges as it is the Harmonic series.
Thus in conclusion the interval of convergence is $-\dfrac{1}{8} \leq x < \dfrac{1}{8}$.

(b) What is the radius of convergence of the series in (1)?

The radius is $R = \dfrac{1/8 - (-1/8)}{2} = \dfrac{1}{8}$.

– Rough Work –