Part Question 1

Question 1

$$A = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

(a) Basis for row(A)

$$\text{row}(A) = \text{span}(r_1, r_2, r_3) \quad \text{where } r_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad r_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad r_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

For finding the basis of row(A), we want to find the non-zero rows of the rref (reduced row echelon form) of Matrix A.
or ref

Hence,

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & \\ 0 & 2 & 0 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{\left(-\frac{1}{3}\right)R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 2 & 0 \\ 0 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 - 2R_2,\ R_4 + 3R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\left(\frac{3}{2}\right)R_3} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 + \left(\frac{1}{3}\right)R_3]{R_1 + \left(\frac{1}{3}\right)R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the 4th Row is not a non-zero row,

Basis for row(A) = span(

(b) basis for $col(A)$

$$A = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix} \quad rref(A) = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 9/2 \end{bmatrix}$$

since there are more columns than rows, the column space can be expressed as the span of the three linearly independent vectors, which we can also refer to, as pivot columns.

We have $c_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $c_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $c_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $c_4 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Here, from $rref(A)$, we already get that

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (-6)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (9/2)\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Basis for
so, $col(A) = span\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right)$

= pivot columns of Matrix A.

(c) Basis for Null(A)

say $\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. $A\vec{V} = \vec{0}$

or, $\begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Rearranging into augmented matrix

or, $\left[\begin{array}{cccc|c} 2 & -1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array}\right]$

rref (reduced row echelon form from "A")

or, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & 9/2 & 0 \end{array}\right]$

We can write it as parametric equation

(We notice x_4 is a free variable)

or, $x_1 - 2x_4 = 0$;

$x_2 - 6x_4 = 0$;

$x_3 + \left(\frac{9}{4}\right)x_4 = 0$.

or, $x_1 = 2x_4$

$x_2 = 6x_4$

$x_3 = -\frac{9}{4}x_4$

or, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ 6 \\ -9/4 \\ 1 \end{bmatrix}$

Basis for Null(A) = span $\left(\begin{bmatrix} 2 \\ 6 \\ -9/4 \\ 1 \end{bmatrix} \right)$

Question 2

2a) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

$$= 1\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} - 1\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} + 0\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix} - 0\begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= 1\left(2\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} - 0 + 0\right) - 1\left(2\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} - 0 + 0\right) + 0 - 0$$

Multiplying by 0 is going to give us 0 in the end. So I didn't write the whole determinant matrices

$$= 1\left(2\cdot(-2-2)\right) - 1\left(2\cdot(-2-2)\right)$$

$$= 1\left(2(-4)\right) - 1\left(2\cdot(-4)\right)$$

$$= 1(-8) - 1(-8)$$

$$= -8 + 8$$

$$= 0$$

Here, we can see that A_3 can be expressed as the linear combination of the other vectors in the system. So, by

$\text{span}(A_1, A_2, A_3, A_4, A_5) = \text{span}(A_1, A_2, A_4, A_5)$

Number of Independent vectors in the span
= Number of non-zero rows in the system/space
= Number of pivot columns in the system/space
= Number of dimention of the column space
= Number of basis vectors in the column span
= Number of dimentions in the span of vectors

highest possible

For any matrix A,
dim(col A) = Basis Vectors of colspan
~~dim(row A)~~
= rank(A)

∴ The dim(S) = 4