MATHEMATICS 101 (all sections), Spring 2012,
Midterm # 3.
March 23rd, 2012 — Happy birthday, Pierre-Simon Laplace!
Time: 2 hours

Last name: [redacted]

First name: [redacted]

Student number: [redacted]

Section number: A05

Questions	Score	Out of
2 to 11	18	20
12	4	4
13	3	4
14	0.5	4
Total	25 1/2	**32**

- As stated in the course outline, the only calculator we allow on any examination is the Sharp EL-510R.
- This test consists of 13 questions (numbered 2 through 14) and has 11 pages (including this cover and an integral table at the end).
  - Questions 2 through 11 are multiple choice. **Enter your final answer in the bubble sheet and mark them in this paper as well. You need to show your work for all answers, as we may disallow any answer which is not properly justified.**
  - Questions 12 through 14 are long answer. You need to show your work and justify all of your answers.
- For the multiple-choice questions with numerical answers, select the answer closest to yours. If the answer is equidistant from two nearest choices, select the largest of the two choices.
- Before starting your test enter your name, student number, and section number on this page and on the bubble sheet.
- At the end of your test, turn in both this paper and the green bubble sheet.
- Enter "J" as your answer to Question 1 now.

Multiple Choice Questions

1. Enter "J" as your answer to Question 1 now.

2. Find the coefficient of the cubic term in the Maclaurin Series for xe^x.

$A)$ 1.0	$B)$ 0.5	$C)$ 0.25	$D)$ 0.15	$E)$ 0.0
$F)$ -0.15	$G)$ -0.25	$H)$ -0.5	$I)$ -1.0	$J)$ diverges

$f(x) = xe^x$ $\qquad f(0) = 0$

$f'(x) = xe^x + e^x$ $\qquad f'(0) = 1$

$= e^x(x+1)$ $\qquad f''(0) = 2$

$f''(x) = e^x + (x+1)e^x$ $\qquad f'''(0) = 3$

$= e^x(1 + x + 1)$

$= e^x(x+2)$

$f'''(x) = e^x + (x+2)e^x$

$= e^x(1 + x + 2)$

$= e^x(x+3)$

$= 0 + x + \frac{2x^2}{2!} + \frac{3x^3}{3!}$

$= \frac{1}{2}x^3$

$\frac{1}{x} = x^{-1} \qquad -x^{-2} = -\frac{1}{x^2}$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

3. Evaluate $\displaystyle\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx$.

$A)$ 2.0	$B)$ 1.75	$C)$ 1.5	$D)$ 1.25	$E)$ 1.0
$F)$ 0.75	$G)$ 0.5	$H)$ 0.25	$I)$ 0.0	$J)$ diverges

$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx$ $\qquad$ let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$\frac{\sin x}{\cos x}$ $\qquad \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$ $\qquad \frac{1}{\cos x}$

$= \int_0^1 \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$ $\qquad = \int \frac{1}{\sec\theta}\, d\theta$

$x = \tan\theta$

$1 = \tan\theta$

$= \int_0^{\pi/4} \cos\theta\, d\theta$

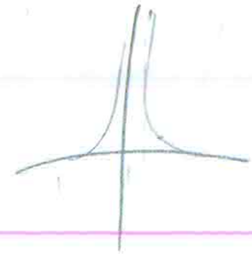
$= \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$

$= \sin\theta \Big|_0^{\pi/4}$

$= \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$

$= \sin(\pi/4) - \sin(0) = \frac{1}{\sqrt{2}}$

4. Evaluate $\int_1^4 \frac{1}{(x-2)^2}\,dx$.



$(x-2)^{-2}$

$-(x-2)^{-1}$

$A)$ 2.0	$B)$ 1.5	$C)$ 1.0	$D)$ 0.5	$E)$ 0.0
$F)$ -0.5	$G)$ -1.0	$H)$ -1.5	$I)$ -2.0	$J)$ diverges

$$\int_1^4 \frac{1}{(x-2)^2} = \int_1^4 \frac{A}{x-2} + \frac{B}{(x-2)^2} \Rightarrow A(x-2) + B = 1$$

$Ax - 2A + B = 1$

$A = 0$

$-2A + B = 1$

$B = 1$

$$\lim_{t\to 2} \int_1^t \frac{dx}{(x-2)^2} + \lim_{t\to 2} \int_t^4 \frac{1}{(x-2)^2}\,dx$$

$$\lim_{t\to 2}\left[\left(-\frac{1}{x-2}\right)\Big|_1^t + \left(\frac{-1}{x-2}\right)\Big|_t^4\right]$$

$$\lim_{t\to 2}\left[\left(\frac{-1}{t-2}\right) - \left(\frac{-1}{1-2}\right) + \left(\frac{-1}{4-2}\right) - \left(\frac{-1}{t-2}\right)\right]$$

$$= \lim_{t\to 2}\left[\left(\cancel{\frac{-1}{t-2}} - 1\right) + \left(-\frac{1}{2} + \cancel{\frac{1}{t-2}}\right)\right]$$

$$-1 + -\frac{1}{2} = -\frac{3}{2}$$

5. Evaluate $\int_0^\infty \frac{1}{(x+1)^5}\,dx$.

$A)$ 5.0	$B)$ 2.5	$C)$ 1.0	$D)$ 0.5	$E)$ 0.3
$F)$ 0.25	$G)$ 0.2	$H)$ 0.1	$I)$ 0.0	$J)$ diverges

$(x+1)^{-5}$

$$\lim_{t\to\infty} \int_0^t \frac{1}{(x+1)^5}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{4}(x+1)^{-4}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left[\left(-\frac{1}{4}(t+1)^{-4}\right) - \left(\frac{-1}{4}(0+1)^{-4}\right)\right]$$

$$= \lim_{t\to\infty} \frac{-1^{\to 0}}{4(t+1)^4} + \frac{1}{4}$$

6. Evaluate $\sum_{n=1}^{\infty} \frac{5^n + (-4)^n}{6^n}$

$A)$ 23	$B)$ 5	$C)$ 4	$D)$ 3	$E)$ 2
$F)$ 1.2	$G)$ 0.8	$H)$ 0.5	$I)$ 0.2	$J)$ 0

$$\sum_{n=1}^{\infty} \left[\frac{5^n}{6^n} + \frac{(-4)^n}{6^n}\right]$$

$$\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$$

$$\frac{\frac{5}{6}}{1-\frac{5}{6}} + \frac{-\frac{2}{3}}{1-\left(\frac{-2}{3}\right)}$$

$$\frac{\frac{5}{6}}{\frac{1}{6}} + \frac{-\frac{2}{3}}{\frac{5}{3}}$$

$$5 + \frac{-2}{5}$$

$$\frac{25}{5} - \frac{2}{5}$$

$$\frac{23}{5}$$

7. Evaluate $\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{n} - \frac{\ln(n+1)}{n+1}\right)$

$3.11 \times 10^{-85} = 0.\ldots$

$A)$ 2.8	$B)$ 1.0	$C)$ 0.5	$D)$ 0.25	$E)$ 0
$F)$ -0.25	$G)$ -0.5	$H)$ -1.0	$I)$ -2.8	$J)$ diverges

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} - \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$$

$$\left(\frac{\ln(1)}{1} - \frac{\ln(2)}{2}\right) + \left(\frac{\ln 2}{2} - \frac{\ln 3}{3}\right) + \left(\frac{\ln 3}{3} - \frac{\ln 4}{4}\right) + \left(\frac{\ln 4}{4} - \frac{\ln 5}{5}\right)$$

$$+ \left(\frac{\ln(n-2)}{n-2} - \frac{\ln(n-1)}{(n-1)}\right) + \left(\frac{\ln(n-1)}{n-1} - \frac{\ln(n)}{n}\right) + \left(\frac{\ln(n)}{n} - \frac{\ln(n+1)}{n+1}\right)$$

$$\overset{\to 0}{\ln 1} - \frac{\ln n+1}{n+1} = -\frac{\ln(n+1)}{n+1}$$

$$\lim_{n\to\infty} -\frac{\ln(n+1)}{n+1} = 0$$

8. Ind the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{8^n x^n}{n}$.

$A)$ ∞	$B)$ 16.0	$C)$ 8.0	$D)$ 4.0	$E)$ 2.0
$F)$ 1.0	$G)$ 0.5	$H)$ 0.25	$I)$ 0.125	$J)$ 0

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| < 1$

$= \lim_{n\to\infty} \frac{\frac{8^n x^n}{n}}{\frac{8^{n+1} x^{n+1}}{n+1}}$

$= \lim_{n\to\infty} \frac{8^n x^n}{n} \cdot \frac{n+1}{8^{n+1} x^{n+1}}$

$= \lim_{n\to\infty} \frac{n+1}{8xn}$

$= \frac{1}{8x} \lim_{n\to\infty} \frac{n+1}{n} \frac{\frac{1}{n}}{\frac{1}{n}}$

$= \frac{1}{8x} \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1}$

$= \frac{1}{8x}(1)$

$\left|\frac{1}{8x}\right| < 1$

$-1 < \frac{1}{8x} < 1$

$-1 < 8x < 1$

$-1/8 < x < 1/8$

9. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{8^n x^{3n}}{n}$.

$A)$ ∞	$B)$ 16.0	$C)$ 8.0	$D)$ 4.0	$E)$ 2.0
$F)$ 1.0	$G)$ 0.5	$H)$ 0.25	$I)$ 0.125	$J)$ 0

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| < 1$

$\lim_{n\to\infty} \frac{\frac{8^n x^{3n}}{n}}{\frac{8^{n+1} x^{3(n+1)}}{n+1}}$

$= \lim_{n\to\infty} \frac{8^n x^{3n}}{n} \cdot \frac{n+1}{8^{n+1} x^{3n+3}}$

$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{8x^3} \right|$

$\frac{1}{8x^3} \lim_{n\to\infty} \frac{n+1}{n}$

$= \left|\frac{1}{8x^3}\right| < 1$

$x = 0.5$

$-1 < \frac{1}{8x^3} < 1$

$-1 < 8x^3 < 1$

$-\frac{1}{8} < x^3 < \frac{1}{8}$

$\sqrt[3]{-\frac{1}{8}} < x < \sqrt[3]{1/8}$

10. Find a number a such that $\sum_{n=0}^{\infty} \frac{1}{a^n} = 5$

A) 5 B) 2 C) 1.5 D) 1.25 E) 1.2

F) 1.15 G) 1.1 H) 1.05 I) 0.5 J) 0.2

$\sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^n = 5$

$\frac{1}{1-\frac{1}{a}} = 5$

$1 = 5\left(1-\frac{1}{a}\right)$

$= 5 - \frac{5}{a}$

$4 = \frac{5}{a}$

$4a = 5$

$a = \frac{5}{4}$

$\frac{1}{(5/4)^n} = \left(\frac{4}{5}\right)^n$

$\frac{1}{1-\frac{4}{5}}$

$\frac{1}{1/5} = 5$

Evaluate the sum of the 3 complex cube roots of (-1).

A) 1 B) 0 C) -1 D) $-\sqrt{3}+i$ E) $-1+\sqrt{3}i$

F) $\sqrt{3}+i$ G) $1+\sqrt{3}i$ H) $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ I) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ J) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$= (-1)$

$e^{i2\pi/3}$ $\cos 2\pi/3 + i\sin\frac{2\pi}{3}$

$e^{i4\pi/3}$ $+ \cos 4\pi/3 + i\sin 4\pi/3$

1 1

$0.5 + -0.5 + 1) + \left(\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right)i$

$0 + 0i = 0$

14. A cylindrical tank contains water to a depth of 4 metres before a plug is removed from the base to allow the water to drain. If it takes 30 minutes for all the water to drain out, use Toricelli's Law to find how long it took for the water depth to reach 3 metres.

Toricelli's Law says that $\frac{dy}{dt} = -k\sqrt{y}$, where $y(t)$ is the depth of water after t minutes and k is a constant (which depends on the size of the tank and the size of the hole) that you will have to determine in the process of answering the question.

$\frac{dy}{dt} = -k\sqrt{y}$

$\frac{dy}{\sqrt{y}} = -k\,dt$

$\int \frac{dy}{\sqrt{y}} = -k\int dt$

$\int y^{-1/2}\,dy = -kt$

$\boxed{2y^{1/2} = -kt + c}$

$\sqrt{y} = \frac{-kt}{2}$

$y = \frac{(-kt)^2}{(2)^2}$

$y = \frac{k^2t^2}{4} + c$

$0 = k^2($

$k = \frac{-2\sqrt{y}}{t}$

~~$\frac{dy}{dt} = -k\left(\frac{-kt}{2}\right)$~~

~~$\frac{dy}{dt} = \frac{k^2 t}{2}$~~

~~$\int dy = \frac{k^2}{2}\int t\,dt$~~

~~$y = \frac{k^2}{2}\cdot\frac{t^2}{2}$~~

~~$\frac{dy}{dt}$ at top $= -k(2)$~~

~~$\frac{dy}{dt}$ at bottom $= 0$~~

~~$3 = \frac{k^2t^2}{4}$~~

~~$t^2k^2 = 12$~~

as far as I got

$2\sqrt{y} = -kt$

$\frac{dy}{dt} = -k\sqrt{y}$

Takes about

$y(t) = \int -k\sqrt{y}$

$k = \frac{-2\sqrt{y}}{t}$

$k = \frac{\frac{dy}{dt}}{\sqrt{y}} = \frac{+2\sqrt{y}}{t}$

$k = -\frac{dy}{\sqrt{y}\,dt} = -\frac{2\sqrt{y}}{t}$

$\frac{dy}{dt} = \frac{2y}{t}$

0.5 / 4.

$\frac{dy}{2y} = \frac{dt}{t}$

$\int \frac{dy}{2y} = \int \frac{dt}{t}$

$\frac{1}{2}\ln y = \ln t$

$e^{\ln\sqrt{y}} = t = \frac{-2\sqrt{y}}{k}$

$k = \frac{-2\sqrt{y}}{e^{\ln\sqrt{y}}}$

$k =$

$$\frac{dy}{dx} = \frac{-\sqrt{y^2+4}}{x}$$

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$$\frac{1}{dx} = \frac{-\sqrt{y^2+4}}{x\,dy}$$

$$\frac{}{x} = \frac{-\sqrt{y^2+4}}{dy}$$

$$x = \int \frac{-dy}{\sqrt{y^2+4}}$$

$$= -\int \frac{dy}{\sqrt{y^2+4}}$$

$$-\int \frac{1}{2\sqrt{y^2/4+1}}\,dy$$

let $y^2/4 = u^2$

$u = y/2$

$du = \frac{1}{2}dy$

$$-\frac{1}{2}\int \frac{2\cdot 1/2\,dy}{\sqrt{u^2+1}}$$

$$-\int \frac{du}{\sqrt{u^2+1}}$$

$$k = \frac{-2\sqrt{y}}{t}$$

$k =$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \sec^{-1} |u| + C$$

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C & \text{if } |u| < 1 \\ \coth^{-1} u + C & \text{if } |u| > 1 \end{cases}$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C$$

$$\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$$