MATH 100, Fall, 2021
Tutorial #4
Derivatives and Instantaneous Rates

Q1. a) Complete the following statement (so as to be true): A function $f = f(x)$ is differentiable at $x = c$ if and only if the **the right-hand derivative**[1] of f exists at $x = c$ and the **the left-hand derivative** of f exists at $x = c$ and ....

b) Find an example of an f and c as in part a) where both left- and right-hand derivatives exist, but f is NOT differentiable at $x = c$. Show that your example works by computing both one-sided derivatives and explaining why f is not differentiable.

Q2 Find the derivative of $y = e^x \left[\frac{1}{x^2} - x^{e-1} \right]$ as a function of $x > 0$, then $y'(1)$ as an exact expression. Finish up by computing an approximation to $y'(1)$ rounded to three decimal places.

Q3 Transport Canada developed a model for a car's stopping distance on dry, paved roads as follows

$$s(v) = 0.245v + 0.008v^2$$

where s = stopping distance in metres and v = speed in kilometers per hour.

a) Compute $s'(50)$. What are the units? Interpret the number $s'(50)$ in terms of increased stopping distance on a city street. Do the same for $s'(100)$ on a highway.

b) Use the computation in part a) and the tangent line to the curve s to **estimate** how much extra distance you will need to stop if you are speeding in a 50km per hour zone at 55km/hour (extra, compared to not speeding).

Q4 Let $y = \frac{1}{\cos x} + \frac{1}{\cot x}$ for $-\pi/4 < x < \pi/4$. Find (exact answer) $y'(\pi/6)$. Simplify as much as possible.

Q5 a) Find all points x on the interval $(-\pi, \pi)$ where the slope of the tangent line to the curve $y = \tan x$ is parallel to the line $y = 4x$.

b) Make a sketch of the tangent function and the line from part a) then add in all the tangent lines you found in part a). Use colours!

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[1] See textbook page 128 for definition of RH derivative.

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Tutorial Worksheet
Tutorial Section (A01, A02 etc) T00
Tutorial Instructor Name:
Question Number Attempted (Q1, Q2, etc) Q1

Your Name: KEY
Your Student Number: V00
Today's Date: Sept 29

a) A function $f = f(x)$ is differentiable at $x = c$ if and only if the RH derivative of f exists at $x = c$ and the LH derivative of f exists at $x = c$ and the LH derivative = RH derivative.

b) Standard example

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$c = 0$

RH derivative: $\displaystyle \lim_{h \to 0^+} \frac{|0+h| - |0|}{h}$

$$= \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

LH derivative: $\displaystyle \lim_{h \to 0^-} \frac{|0+h| - |0|}{h}$

$$= \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

These are not equal.

Therefore $\displaystyle \lim_{h \to 0} \frac{|0+h| - |0|}{h}$ DNE

$\Rightarrow f'(0)$ DNE.

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Tutorial Worksheet
Tutorial Section (T01, T02 etc) T00
Tutorial Instructor Name:
Question Number Attempted (Q1, Q2, etc) Q2

Your Name: KEY
Your Student Number: V00
Today's Date: Sept 29

$$y = e^x\left[\frac{1}{x^2} - x^{e-1}\right];\quad x > 0$$

$$y' = e^x[\text{same}] + e^x\left[\frac{-2}{x^3} - (e-1)x^{e-2}\right]$$

product rule

$$y'(1) = e[\cancel{1 - 1^{e-1}}]^{\,0} + e\left[-2 - (e-1)1^{e-2}\right]$$

$$= e(-1-e) = \underline{-e(1+e)} \quad \text{exact}$$

$$y'(1) \simeq \underline{-10.107}$$

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Tutorial Worksheet
Tutorial Section (A01, A02 etc)________
Tutorial Instructor Name:________________
Question Number Attempted (Q1, Q2, etc) Q3

Your Name: KEY
Your Student Number: V00________
Today's Date:________

a) $S'(v) = 0.245 + 0.016v$

$S'(50) = 0.245 + 0.016\cdot(50) \cong 1.045$

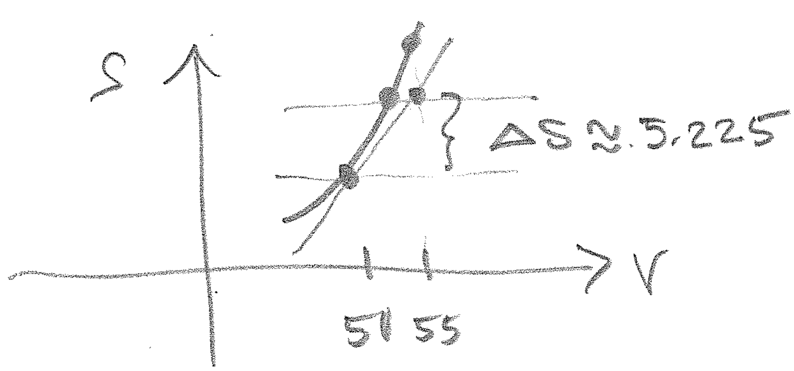
units are meters per km/hr (*) ie: for each additional km/hr of speed, distance increases by 1.045 metres. (think marginal rate).

$S'(100) = 0.245 + 1.600 = 1.845$

For each km/hr above 100 km/hr, braking distance increases 1.845 metres (almost 2 metres!)

b) $\Delta v = 5$ while $S'(50) \cong 1.045$

Expect: $\Delta S \cong S'(50)\cdot\Delta v = 5.225$ metres. extra stopping distance



(*) m/km/hr hours which seems strange but is correct.

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Tutorial Worksheet
Tutorial Section (A01, A02 etc) T00
Tutorial Instructor Name: ____
Question Number Attempted (Q1, Q2, etc) Q4

Your Name: KEY
Your Student Number: V00____
Today's Date: Sept 29

$$y = \frac{1}{\cos x} + \frac{1}{\cot x} \; ; \; x \in \left(-\pi/4, \pi/4\right)$$

$$y' = \frac{\cos x \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x} + \frac{\cot x (0) - 1 \cdot (-\csc^2 x)}{\cot^2 x}$$

quotient, trig

$$= \frac{\sin x}{\cos^2 x} + \frac{\csc^2 x}{\cot^2 x}$$

At $x = \pi/6$: $\sin = 1/2$, $\cos = \frac{\sqrt{3}}{2}$, $\csc x = \frac{1}{\sin x} = 2$

$\cot x = \sqrt{3}/2 \big/ \frac{1}{2} = \sqrt{3}$.

$$y'\left(\frac{\pi}{6}\right) = \frac{1/2}{3/4} + \frac{4}{3} = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = \boxed{2}$$

<u>Alternate</u>: $y(x) = \sec x + \tan x$

$y'(x) = \sec x \tan x + \sec^2 x$

$\sec(\pi/6) = 2/\sqrt{3}$; $\tan x = 1/2 \big/ \sqrt{3}/2 = \frac{1}{\sqrt{3}}$

$$\Rightarrow y'(\pi/6) = \frac{2}{\sqrt{3}} \frac{1}{\sqrt{3}} + \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{6}{3} = \boxed{2} \; ✓$$

MATH 100, Fall 2021
Tutorial Worksheet
Tutorial Section (A01, A02 etc) T00
Tutorial Instructor Name:
Question Number Attempted (Q1, Q2, etc) Q5

Your Name: KEY
Your Student Number: V00
Today's Date: Sept 29

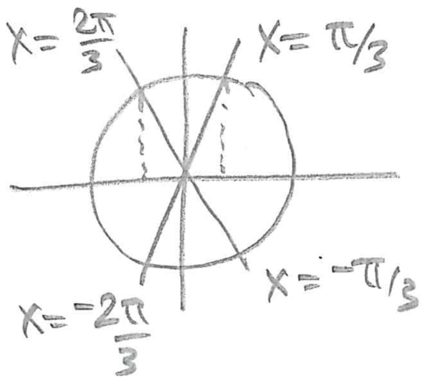
a) $y = \tan x$; $y' = \sec^2 x$ $\quad (-\pi, \pi) \ni x$

Want $y'(x) = 4$: $\sec^2 x = 4$

$\iff \frac{1}{\cos^2 x} = 4$

$\cos^2 x = \frac{1}{4} \quad x \in (-\pi, \pi)$.

$\cos x = \pm \frac{1}{2} \quad x \in (-\pi, \pi)$



$x = \pm \pi/3, \pm \frac{2\pi}{3}$ (4 pts)

b)