MATHEMATICS 101 (Sections A01-A04),
Midterm # 3, March 23, 2016.
Time: 2 hours

Last name: ______________________ StudentID: $V00$______________

First name: ______________________ Lecture / Tutorial section numbers: A____/T____

Problems 1 - 8	.... 1 mark for each (maximum of 8)
Problems 9 - 12	.... 2 marks for each (maximum of 8)
Problem 13	.... 3 marks
Problem 14	.... 2 marks
Problem 15	.... 4 marks
Total:	.... 25 marks

- The only calculators allowed on any examination are Sharp EL-510R, RN and RNB.
- This test consists of 15 questions and has 14 pages (including this cover) and a Formula sheet on the last page.
  - Questions 1 through 12 are multiple-choice. **Enter your final answer in the bubble sheet and mark them in this paper as well. You need to show your work for all answers, as we may disallow any answer which is not properly justified.**
  - Questions 13 through 15 are long-answer. Write your full answer in this booklet as indicated.
- For the multiple-choice questions, select the numerical answer closest to yours. If the answer is equidistant from two nearest choices, select the largest of the two choices.
- Before starting your test enter your Name (Last, First), student ID (starting from V00******), and lecture (A01-A04) and tutorial section (T01 - T40) on this page, and on the bubble sheet.
- At the end of your test, turn in both this booklet and the bubble sheet.
- **Enter** "C" in the bubble sheet as your version of the exam in the "form" section now.

1. [1 point] Find the limit of the sequence $\{a_n\}$:

$$\lim_{n\to\infty} \frac{\ln n}{n}$$

sequence $\frac{\ln n}{n}$

$\frac{\ln(\infty)}{\infty}$ $\frac{\infty}{\infty}$ L'Hop

$\frac{\frac{1}{n}}{1} = \frac{1}{n}$ ∞

(A) $-\infty$ (B) -3.0 (C) -2.0 (D) -1.0 (E) 0.0

(F) 1.0 (G) 2.0 (H) 3.0 (I) $+\infty$ (J) Diverges by oscillation

2. [1 point] Find the limit of the sequence $\{a_n\}$:

$$\lim_{n\to\infty} \frac{n+(-1)^n}{n}$$

$\frac{(-1)^n + n}{n}$ $\frac{n}{n} = 1$

$\frac{n}{n}$

$\frac{(-1^n)+n}{n}$ $\frac{(-1)^\infty + n}{n}$ $\frac{1+n}{n}$

(A) $-\infty$ (B) -3.0 (C) -2.0 (D) -1.0 (E) 0.0

(F) 1.0 (G) 2.0 (H) 3.0 (I) $+\infty$ (J) Diverges by oscillation

3. [1 point] Which of the given series could be used for Direct Comparison Test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{5/2}}$ converges or diverges.

2.5

compare $\frac{\sin^2 n}{n} = 1$

compare to $n^{\frac{5}{2}} = n^{2.5}$

converges

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (B) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$ (C) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$

conv.

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ 0.176 (E) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^{2/5}}$ (F) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{2/5}}$

(G) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$ (H) None of the above (J) All of the above

4. [1 point] Assume that this sequence converges and find its limit:

$a_1 = -4, \quad a_{n+1} = \sqrt{9 + 2a_n}$

monotonic & upper/lower bound

$\sqrt{9+2(-4)}$

$\sqrt{9-8}$

$\sqrt{1} = 1$

$\sqrt{9+2(1)}$

$\sqrt{9+2}$ $\sqrt{11}$ 3.3

$\frac{1}{n+\infty}$

$a = \sqrt{9+2a}$

$a^2 = 9 + 2a$

$a^2 - 2a - 9 = 0$

(A) 4.160 (B) 4.161 (C) 4.162 (D) 4.163 (E) 4.164

(F) 4.165 (G) 4.166 (H) 4.167 (I) 4.168 (J) 4.169

4.8

$\frac{-(-b) \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{9 \pm \sqrt{4 - 2(1)(-9)}}{2(1)}$

$\frac{9 \pm \sqrt{4+81}}{4}$

$\frac{9 \pm \sqrt{85}}{4}$

5. [1 point] Determine whether the series is convergent or divergent. If it is convergent, find its sum:

$$\sum_{n=0}^{\infty} \frac{2}{10^n}$$

$r = \frac{0.2}{2}$

$\left(\frac{2^1}{10^n}\right)^{\frac{1}{n}}$

$a_1 = 2$

$\frac{2^{\frac{1}{n}}}{10}$ $\frac{1}{10}$ convergen

$a_2 \; 0.2$

$\frac{2}{10^0} = \frac{2}{1}$

geometric series

$\frac{a}{1-r}$ $\frac{2}{1-0.1}$

$a_1 = \frac{2}{10^1} = \frac{1}{5}$

(A) 0.0 (B) 0.3 (C) 0.7 (D) 1.3 (E) 1.7

(F) 2.3 (G) 2.7 (H) 3.0 (I) 3.3 (J) Divergent

6. [1 point] Determine whether the series is convergent or divergent. If it is convergent, find its sum:

$$\sum_{n=1}^{\infty} \sqrt[n]{2}$$

$2^{\frac{1}{n}}$

$\frac{1}{n \to \infty}$ 2^0

1

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5

(F) 1.0 (G) 1.2 (H) 1.4 (I) 2.0 (J) Divergent

7. [1 point] Determine whether the series is convergent or divergent by expressing partial sum S_n as a telescoping sum. If it is convergent, find its sum:

$$\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$$

$\frac{2}{(n-1)(n+1)} = \frac{A}{(n-1)} + \frac{B}{n+1} = 2.$

telescoping

$A(n+1) + B(n-1) = 2$

$(A+B)n + A - B = 2$

$A+B=0$ $\qquad -B-B=2$

$A=-B$ $\qquad \frac{-2B=2}{-2}$

$A=1$ $\qquad B=-1$

$\frac{1}{n-1} - \frac{1}{n+1}$

$\frac{1}{2-1} = \frac{1}{1} - \frac{1}{2+1} = \frac{1}{3}$

$\frac{1}{4-1} = \frac{1}{3} - \frac{1}{4+1} = \frac{1}{5}$

$\frac{1}{5-1} = \frac{1}{4} - \frac{1}{5+1} = \frac{1}{6}$

$\frac{\;}{n+2}$ $\qquad 1 - \frac{1}{n+1}$

$1 - \frac{1}{\infty + 1} = \frac{1}{\infty}$ $\quad 1$

(A) Divergent (B) 0.0 (C) 0.5 (D) 1.0 (E) 1.5

(F) 2.0 (G) 2.5 (H) 3.0 (I) 3.5 (J) 4.0

8. [1 point] List all values of p for which the series $\sum_{n=7}^{\infty} \frac{1}{n^p}$ is divergent. Why?

divergent

(A) $p < -1$ (B) $p = -1$ (C) $p = 1$ (D) $p < 1$ (E) $p \leq 1$

(F) $|p| < 1$ (G) $|p| = 1$ (H) $p \geq 1$ (I) $p > 1$ (J) Divergent for all values of p

9. [2 points] Find **interval** of convergence of the series $\sum_{n=1}^{\infty} \frac{4^n x^n}{n}$.

$$\frac{\frac{4^{n+1} x^{n+1}}{n+1}}{\frac{4^n x^n}{n}}$$

$$\frac{4^{n+1} x^{n+1}}{n+1} \quad \frac{n}{4^n x^n}$$

$4^{\cancel{n}+1-\cancel{n}} \quad \cancel{n}+1-n$

$\lim_{n\to\infty} \frac{4xn}{(n+1)} \quad \frac{n}{n+1} \quad 1$

$|4x| < 1$

$-\frac{1}{4} < x < \frac{1}{4}$

(A) $-4 < x < 4$ (B) $-4 < x \le 4$ (C) $-\frac{1}{4} < x < \frac{1}{4}$ (D) $-\frac{1}{4} < x \le \frac{1}{4}$

(E) $-4 \le x \le 4$ (F) $-4 \le x < 4$ (G) $-\frac{1}{4} \le x \le \frac{1}{4}$ (H) $-\frac{1}{4} \le x < \frac{1}{4}$

(I) Divergent for all x (J) None of those

10. [2 points] Find **radius** of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n6^n}$.

$$\left(\frac{(x+3)^n}{n\,6^n}\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{(x+6)}{n^{\frac{1}{n}}\,6} \qquad \frac{x+6}{1(6)} < 1$$

$$-1 < \frac{x+6}{6} < 1$$

$$-6 < x+6 < 6$$

$$-6-6 < x < 6-6$$

$$12 < x < 0$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) Converges for a single value of x
(F) 5 (G) 6 (H) 7 (I) 8 (J) Diverges for all values of x

11. [2 points] Determine whether or not the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. Justify.

integral test $\int_1^{\infty} n e^{-n^2}$

$u = -n^2$

$\frac{du}{dx} = -2n$

2

$\frac{du}{2} = n\,dx$

$\sum_{n=1}^{\infty} \frac{1}{2} \int_1^{\infty} e^u$

$\frac{e^u}{2} \Big|_1^{\infty} \quad \frac{e^{-\infty^2}}{2} - \frac{e^{(u)^{-1}}}{2} \quad (\infty)^2 \quad -\infty \quad \frac{1}{2e^{\infty}} \quad \frac{1}{0} \quad 0 - e^{(-n2)}$

$0 \quad -1$

(A) Diverges by n-th term Test for Divergence;

(B) Converges by n-th term Test for Divergence;

(C) Diverges by Integral Test when compared to $\int_1^{+\infty} \frac{x}{e^{x^2}}\,dx$;

(D) Converges by Integral Test when compared to $\int_1^{+\infty} \frac{x}{e^{x^2}}\,dx$;

(E) Diverges, because it is a p-series with $p \geq 1$;

(F) Converges, because it is a p-series with $p < 1$;

(G) Diverges, because it is a geometric series with $|r| \geq 1$;

(H) Converges, because it is a geometric series with $|r| < 1$.

12. [2 points] Find the constant c_2 of the binomial series

$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \ldots, \; -1 < x < 1$ for the function $y = (1 + 2x^2)^{-1}$.

(A) -8 (B) -4 (C) -2 (D) -1 (E) 0

(F) 1 (G) 2 (H) 4 (I) 8 (J) None of those

13. [3 points] Determine if given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2+n)}{n^2}$ converges absolutely, converges conditionally, or diverges.

$\lim_{n\to\infty} \frac{2+n}{n^2} \quad \frac{1}{\infty} \; 0$

Converges

$$\frac{\frac{2+n+1}{(n+1)^2}}{\frac{2+n}{n^2}}$$

$\frac{(n+3)}{(n+1)^2} \cdot \frac{n^2}{n+2}$ $\qquad n^2(n+3)$

n^3+3n^2

$(n+1)(n+1)$ $\qquad \frac{n^3+3n^2}{n^3+2n^2+n^2+4n+n+2} \Rightarrow \frac{n^3+n^2}{n^3+3n^2+5n+2}$

$(n^2+2n+1)(n+2)$

$n^3+2n^2+n^2+4n+n+2$

$\lim_{n\to\infty} 1 \neq$

$\frac{2+n}{n^2}$ lim compar $\frac{1}{n^2}$

$$\frac{\frac{2+n}{n^2}}{\frac{1}{n^2}} \qquad \frac{2+n}{\cancel{n^2}} \cdot \frac{\cancel{n^2}}{1} = 2+n \qquad \text{diverges}$$

$\lim_{n\to\infty} 2+n$

$y = \sin x$ is estimated by $\int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ to estimate:

power series

$$\sum_{n=0}^{\infty} (-1)^n x^n$$

$$\sin\left(x - \frac{x^3}{3!} + x^5\right.$$

$\sqrt{t}$

$$\int \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} \frac{(\sqrt{t})^7}{7!} \Big|_0^1$$

$1 -$

$\frac{1}{1+x}$

$(-1)^n x^n$

$(-1)^{n-1} x^{n-1}$

14. [2 points] Use the fact that $y = \sin x$ is estimated by $\int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ to estimate:

$\int_0^1 \sin\sqrt{t}\, dt$

sin √t power series

$\sum_{n=0}^{\infty} (-1)^n x^n$

$\int_0^1 \sin x$

$-\cos x \Big|_0^1$

$\sin\left(x - \frac{x^3}{3!} + x^5\right.$

$\sin\sqrt{t}$ $\int \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} \frac{(\sqrt{t})^7}{7!} \Big|_0^1$

$1 -$

$\frac{1}{1+x}$ $(-1)^n x^n$ $\frac{1}{n-1} x^{n-1}$

15. [4 points]

(i) Write general formula the Taylor series for function $y = f(x)$ generated at the point $x = a$.

$$f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\frac{f'''(a)(x-a)^3}{3!}+\frac{f'(a)(x-a)^n}{n!}$$

$$f(3)+\frac{f'(3)(x-3)}{1!}+\frac{f''(3)(x-3)^2}{2!}+$$

(ii) Find the Taylor series for $y = \frac{1}{x}$ generated at the point $a = 3$.

Find the first three non-zero terms of this series, and then write n-th ~~term of the series~~.

$y=\frac{1}{x}$ x^{-1} $-x^{-2}$ $-\frac{1}{x^2}$ $-(-2)x^{-3}$

$\frac{1}{3}+$ $f'(a)$ $-\frac{1}{9}$ $\frac{f'(3)(x-3)^n}{n!}$

$\frac{1}{x}$ x^{-1} $f''(a)$ $2\frac{1}{x^3}$

$\frac{1}{3}-\frac{1}{9}(x-3)$ $-\frac{1}{x^2}$ $-\frac{1}{9}$ $\frac{2}{27}$

(iii) Using the Maclauring series for e^x and $\sin x$ given at the bottom of this page, find first three terms of Maclaurin series for the function: $a=0$

$y = e^x \sin x$ = ~~(1+~~

$1+x+$

Maclaurin Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\left(1+x+\frac{x^2}{2!}\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)$$

$f(a)+f'(a)$

$\left(\frac{1}{2}-\frac{1}{3}\right)x^3$

[BLANK]

$\frac{1}{1+x}$

$\frac{x}{1+x^3}$ $\sum_{n=0}^{\infty} (-1)^n x^{3n}$