Math 101 (A01-A04)
Test 2
Version: B
February 28, 2017
Time: 120 minutes
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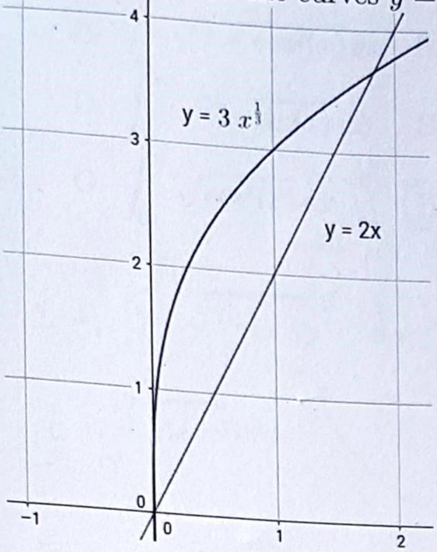
Test Score		
Question	**Points**	**Score**
Multiple Choice	16	16
Question 9	10	10
Question 10	5	5
Question 11	3	3
Question 12	2	2
Question 13	4	4
Total	**40**	40

Instructions:

- Before beginning the test, enter your name and ID number on this cover and on the bubble sheet. Be sure to fill in the bubbles for your ID number.
- The only items you should have with you are writing implements, your OneCard, and your calculator. The only calculators permitted are Sharp EL-510R, Sharp EL-510RN, and Sharp EL-510RNB. No notes or any other aids are permitted. You are responsible for ensuring that you do not have any prohibited items with you during the test.
- Write out your solutions carefully and completely on the question paper provided. Marks will not be awarded for final answers that are not supported by appropriate work. This includes multiple choice problems.
- For multiple choice questions, the exact answer may not appear as one of the options. Solve the question, then select the answer *closest* to your answer. If your answer is exactly equidistant from two options, choose the larger answer.
- If you require extra space, use the backs of test pages or the blank page at the end of the test. Be sure to clearly indicate where you have done so. No outside paper is permitted.
- This test has 14 pages, including this cover and the blank page at the end.
- **Fill in "B" in the "Form" field of the bubble sheet now.**

For **both** questions on this page, let R be the region in the first quadrant of the plane bounded between the curves $y = 2x$ and $y = 3x^{1/3}$.



(2) 1. Suppose that R is to be rotated around the x-axis, and we wish to use the washer method to find the volume of the resulting solid. What is the **inner radius** of the washer at x?

A. $2x$ B. $3x^{1/3}$ C. $3x^{1/3} - 2x$ D. $2x - 3x^{1/3}$

E. $\frac{1}{2}x$ F. $\left(\frac{x}{3}\right)^3$ G. $\frac{1}{2}x - \left(\frac{x}{3}\right)^3$ H. $\left(\frac{x}{3}\right)^3 - \frac{1}{2}x$

I. Washers cannot be used in this situation
J. None of the other answers

(2) 2. Suppose that R is to be rotated around the y-axis, and we wish to use the cylindrical shells method to find the volume of the resulting solid. What is the **height** of the shell at x?

A. $2x$ B. $3x^{1/3}$ C. $3x^{1/3} - 2x$ D. $2x - 3x^{1/3}$

$V = \int 2\pi (x)(3x^{1/3} - 2x)$

E. $\frac{1}{2}x$ F. $\left(\frac{x}{3}\right)^3$ G. $\frac{1}{2}x - \left(\frac{x}{3}\right)^3$ H. $\left(\frac{x}{3}\right)^3 - \frac{1}{2}x$

I. Cylindrical shells cannot be used in this situation
J. None of the other answers

(2) 3. Which of the following integrals represents the arc length of $y = \sin(x)$ from $x = 0$ to $x = \pi$?

A. $\int_0^\pi \sqrt{1+\cos^2(x)}\,dx$ B. $\int_0^\pi \sqrt{1+\cos(x)}\,dx$ C. $\int_0^\pi \sqrt{1+\sin^2(x)}\,dx$

D. $\int_0^\pi \sqrt{1-\sin(x)}\,dx$ E. $\int_0^\pi \sqrt{1-\sin^2(x)}\,dx$ F. $\int_0^\pi \sqrt{\sin^2(x)}\,dx$

G. $\int_0^\pi \sqrt{\cos^2(x)}\,dx$ H. $\int_0^\pi \sqrt{1-\cos^2(x)}\,dx$ I. None of the other answers

$L = \int_0^\pi \sqrt{1+(f'(x))^2}\,dx$

$f(x) = \sin(x)$

$f'(x) = \cos x$

$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$

$1+[f'(x)]^2 = 1+\cos^2 x$

$1+[f'(x)]^2$

(2) 4. Find the real part of the complex number $\frac{1+2i}{\sqrt{11}+i}$.

$(a+bi)(c+di) = (ac-bd)+i(ad+bc)$

A. -1 B. -0.8 C. -0.6 D. -0.4 E. -0.2

F. 0 G. 0.2 H. 0.4 I. 0.6 J. 0.8

$\frac{1+2i}{\sqrt{11}+i}\left(\frac{\sqrt{11}-i}{\sqrt{11}-i}\right) = \frac{[\sqrt{11}-(-2)]+i(-1+2\sqrt{11})}{11-(i^2)} = \frac{(\sqrt{11}+2)+i(-1+2\sqrt{11})}{11-(-1)} = \left(\frac{\sqrt{11}+2}{12}\right) + i\left(\frac{-1+2\sqrt{11}}{12}\right)$

real part $\frac{\sqrt{11}+2}{12} = 0.443$

(2) 5. Given $z = 3 - 4i$ and $w = 1 + i$, find $|z - w|$. $|z| = |x + iy| = \sqrt{x^2 + y^2}$

A. $2 - 5i$ B. $2 + 5i$ C. 2 D. 4.58 E. 5.39

F. 6.64 G. 10 H. 17 I. 24.13 J. 29

$|z - w| = (3 - 4i) - (1 + i) = 3 - 4i - 1 - i = 2 - 5i$

$|z - w| = |2 - 5i| = \sqrt{2^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29} = 5.385$

(2) 6. If the complex number $z = (1 + i\sqrt{3})^2$ is represented in polar form as $z = re^{i\theta}$, which of the following is θ?

A. 0 B. $\pi/4$ C. $\pi/3$ D. $\pi/2$ E. $2\pi/3$

F. $3\pi/4$ G. π H. $4\pi/3$ I. $3\pi/2$ J. $7\pi/4$

$z = (1 + i\sqrt{3})^2 = 1 + 2(1)i\sqrt{3} + i^2(\sqrt{3})^2 = 1 + 2\sqrt{3}i + -1(3) = 1 - 3 + 2\sqrt{3}i = -2 + 2\sqrt{3}i$

$z = -2 + 2\sqrt{3}i = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{2}$

$= 4e^{i\frac{2\pi}{3}}$

(2) 7. Find $\displaystyle\int_0^3 \frac{\log_5(x+2)}{x+2}\,dx$.

A. 0.60 B. 0.66 C. 0.69 D. 0.80 E. 0.87

F. 0.96 G. 1.11 H. 1.30 I. 1.93 J. Does not exist

$u(3) = \ln 5$

$u(0) = \ln 2$

$u = \ln(x+2)$

$du = \frac{1}{x+2}\,dx$

$$\int_0^3 \frac{\log_5 x+2}{x+2}\,dx = \int_0^3 \frac{\left(\frac{\ln(x+2)}{\ln 5}\right)}{x+2}\,dx = \frac{1}{\ln 5}\int_0^3 \frac{\ln(x+2)}{x+2}\,dx$$

$$= \frac{1}{\ln 5}\int_{\ln 2}^{\ln 5} u\,du = \frac{1}{\ln 5}\left[\frac{u^2}{2}\right]_{\ln 2}^{\ln 5} = \frac{1}{2\ln 5}\left[(\ln 5)^2 - (\ln 2)^2\right] = 0.66$$

$$\frac{1}{3.22}(2.59 - 0.48) = 0.655$$

(2) 8. Suppose that $y(x)$ is a positive function satisfying $\dfrac{dy}{dx} = \dfrac{\cos(x)}{y}$, and $y(0) = \sqrt{34}$. Find $y(\pi/2)$.

A. 0 B. 1 C. 2 D. 3 E. 4

F. 5 G. 6 H. 7 I. 8 J. 9

$$\frac{dy}{dx} = \frac{\cos(x)}{y}$$

$$\int y\,dy = \int \cos(x)\,dx$$

$$\frac{y^2}{2} = \sin(x) + C$$

$$y^2 = 2\sin(x) + C$$

$$y = \sqrt{2\sin(x) + C}$$

$$y(0) = (\sqrt{34})^2 = \left(\sqrt{2\sin(0) + C}\right)^2$$

$$34 = 2(0) + C$$

$$34 = C$$

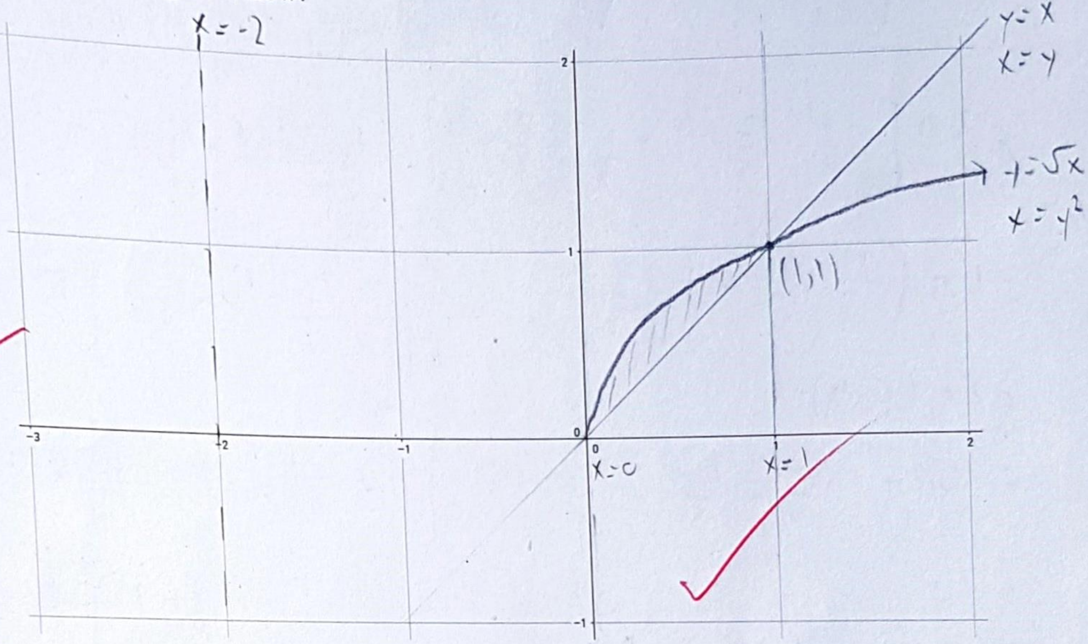
$$y(x) = \sqrt{2\sin(x) + 34}$$

$$y\left(\tfrac{\pi}{2}\right) = \sqrt{2\sin\left(\tfrac{\pi}{2}\right) + 34} = \sqrt{2(1) + 34} = \sqrt{36} = 6$$

$y = \sqrt{x}$
$y^2 = x$

9. Let R be the region in the first quadrant of the plane bounded between $y = x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 1$. Let S be the solid obtained by rotating R around the line $x = -2$.

(2) (a) Sketch R on the grid below.



2

(3) (b) Set up, but **do not evaluate**, the integral used to calculate the volume of S using the washer method.

$R(y) = 2 + y$ $\quad$ $r(y) = 2 + y^2$

$R(y)^2 = 4 + 4y + y^2$ $\quad$ $r(y)^2 = 4 + 4y^2 + y^4$

$V = \int_a^b \pi \, (R(y))^2 - (r(y))^2 \, dy$

$V = \int_0^1 \pi \, (4y - 3y^2 - y^4) \, dy$

$R(x)^2 - r(x)^2 = \cancel{4} + 4y + y^2 - \cancel{4} - 4y^2 - y^4$

$4y - 3y^2 - y^4$

3

(3) (c) Set up, but **do not evaluate**, the integral used to calculate the volume of S using the shell method.

radius $(2 + x)$ $\quad$ height $(\sqrt{x} - x)$

$V = \int_a^b 2\pi (\text{radius})(\text{height})$

$= \int_0^1 2\pi (x+2)(\sqrt{x} - x) \, dx$

3

Question 9 continued

(2) (d) Find the volume of the solid S from the previous page using whichever method you prefer. Leave your answer in exact form (do not give a decimal approximation).

$$V = \int_0^1 \pi \left(4y - 3y^2 - y^4\right) \, dx$$

$$V = \pi \left[4\frac{y^2}{2} - \frac{3y^3}{3} - \frac{y^5}{5} \right]_0^1$$

$$V = \pi \left[2y^2 - y^3 - \frac{y^5}{5} \right]_0^1$$

$$V = \pi \left[\left(2 - 1 - \frac{1}{5}\right) - 0 \right]$$

2

$$V = \pi \left(1 - \frac{1}{5} \right)$$

$$V = \frac{4}{5}\pi$$

$\sec'(x) = \left(\frac{1}{\cos}(x)\right)' = \frac{0\cos(x) + 1\cdot\sin(x)}{\cos^2(x)} = \frac{+\sin x}{\cos^2 x}$

$\sec'(x) = \sec x \tan x$

$= \frac{1}{\cos x}\,\frac{\sin x}{\cos x} = \sec x \tan x$

(5) 10. Find the length of the curve $y = \ln(\sec(x))$ between $x = 0$ and $x = \pi/4$. Leave your answer in exact form (do not give a decimal approximation).
You may use, without proof, the fact that $\sec(x) > 0$ for all x in the interval $[0, \pi/4]$.

$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + [f'(x)]^2}\, dx$

$f(x) = \ln(\sec(x))$

$f'(x) = \frac{1}{\sec(x)} \sec x(\tan x) = \tan x$

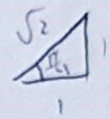
$L = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)}\, dx$

$1 + (f'(x))^2 = 1 + \tan^2 x = \sec^2 x$ ✓

$L = \int_0^{\frac{\pi}{4}} \sec(x)\, dx$ ✓

$\int \sec(x) = \ln|\sec(x) + \tan(x)| + C$

$L = \left[\ln|\sec(x) + \tan(x)|\right]_0^{\frac{\pi}{4}}$



$L = \left[\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|\right]$

$L = \ln|\sqrt{2} + 1| - \ln|1|$

5

$L = \ln|\sqrt{2} + 1|$ ✓

(3) 11. When the internet was initially becoming widely available, the number of internet users P (measured in millions of people) at time t (measured in years after 1995) could be modelled by the differential equation

$k = 0.745$

$$\frac{dP}{dt} = 0.745P.$$

$P = Pe^{kt}$

$\frac{dP}{dt} = Pke^{kt}$ → t small

In 1995 (time $t = 0$) there were approximately 15 million internet users. According to this model, how many internet users were there in 1997? Round your answer to the nearest million.

$e^{kt} \to 1$

$\frac{dP}{dt} = Pk$

$\frac{dP}{dt} = 0.745P$ $\quad \frac{dP}{dt} = kP$ $\quad 0.745P = kP$

$0.745 = k$

$P = Ie^{kt}$

$15 = Ie^{k(0)}$

$I = 15$ million users

$P = 15e^{0.745(2)}$

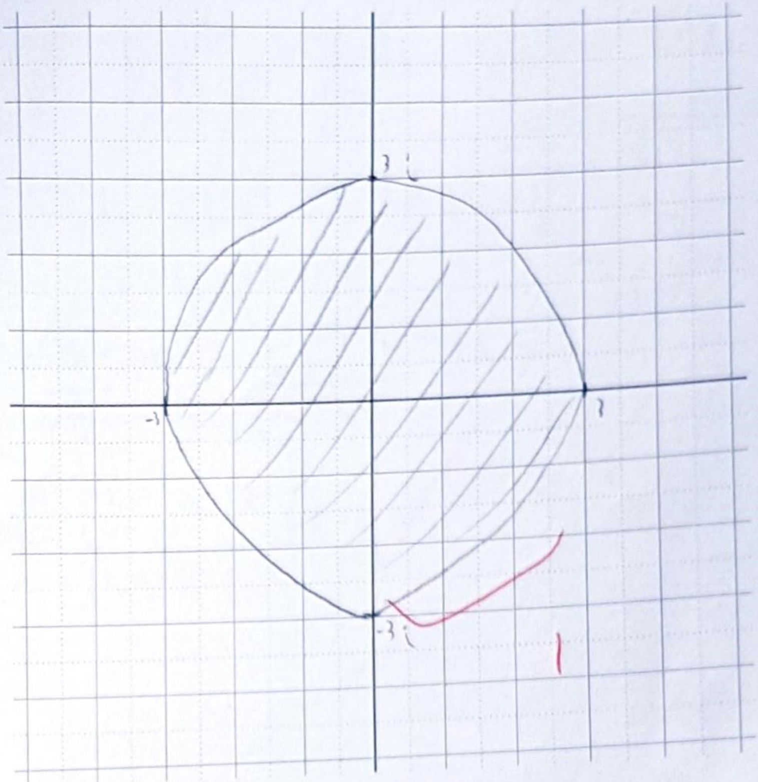
$= 15e^{1.49}$

$= 66.56$ million

$= 67$ million

There are 67 million internet users in 1997

12. Sketch the region in the complex plane defined by $z\overline{z} \leq 9$. Below the grid give a **brief** explanation of why your sketch corresponds to $z\overline{z} \leq 9$.



$z = x + iy \quad \bar{z} = x - iy$

$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$

$z\bar{z} = |z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$

$z\bar{z} \leq 9$

$x^2 + y^2 \leq 9 \longrightarrow$ for any point z on or within the circle radius 3 centered at (0,0)

$z\bar{z} \leq 9$ because

1

(4) 13. Find $\int 3^x 5^x\,dx$. *Hint*: Either integrate by parts, or use your knowledge of exponentials to write the integrand in a form that is easier to antidifferentiate.

$$\int 3^x 5^x\,dx = \int e^{\ln(3)^x} e^{\ln(5)^x}\,dx = \int e^{x\ln 3} e^{x \ln 5}\,dx$$

$$= \int e^{x\ln 3 + x\ln 5}\,dx = \int e^{x(\ln 3 + \ln 5)}\,dx$$

$u = \overbrace{(\ln 3 + \ln 5)}^{\text{constant}}\,x$

$du = (\ln 3 + \ln 5)\,dx$

$\frac{du}{(\ln 3 + \ln 5)} = dx$

$$= \int e^u \frac{du}{\ln 3 + \ln 5} = \frac{e^u}{\ln 3 + \ln 5} + C = \frac{e^{(\ln 3 + \ln 5)x}}{(\ln 3 + \ln 5)} + C$$

$$\int 3^x 5^x\,dx = \frac{e^{(\ln 3 + \ln 5)x}}{(\ln 3 + \ln 5)} + C$$

4

Math 101
Formula Sheet
[Do not remove]

1 Trigonometric Identities

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sec^2(\theta) = 1 + \tan^2(\theta) \qquad \csc^2(\theta) = 1 + \cot^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \qquad \cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\sin\left(A - \frac{\pi}{2}\right) = -\cos(A) \qquad \cos\left(A - \frac{\pi}{2}\right) = \sin(A)$$

$$\sin\left(A + \frac{\pi}{2}\right) = \cos(A) \qquad \cos\left(A + \frac{\pi}{2}\right) = -\sin(A)$$

$$\sin(A)\sin(B) = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B)$$

$$\cos(A)\cos(B) = \tfrac{1}{2}\cos(A-B) + \tfrac{1}{2}\cos(A+B)$$

$$\sin(A)\cos(B) = \tfrac{1}{2}\sin(A-B) + \tfrac{1}{2}\sin(A+B)$$

2 Hyperbolic identities

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \tanh^2(x) = 1 - \operatorname{sech}^2(x) \qquad \coth^2(x) = 1 + \operatorname{csch}^2(x)$$

$$\sinh(2x) = 2\sinh(x)\cosh(x) \qquad \cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\cosh^2(x) = \frac{\cosh(2x)+1}{2} \qquad \sinh^2(x) = \frac{\cosh(2x)-1}{2}$$

3 Integrals

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C \quad (0 < x < a)$$

$$\int \frac{1}{x\sqrt{a^2 + x^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C \quad (x \neq 0, a > 0)$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$