MATHEMATICS 101 (Sections A01-A04),
Midterm # 3, March 19, 2015.
Time: 2 hours

Last name: ______________________ StudentID: $V00$______________

First name: ______________________ Lecture / Tutorial section numbers: A____/T____

Problems 2 - 10	.... 12 marks $(1 \cdot 6 + 2 \cdot 3 = 12)$
Problem 11	.... 4 marks
Problem 12	.... 4 marks
Problem 13	.... 5 marks
Total:	.... 25 marks

- The only calculators allowed on any examination are Sharp EL-510R, Sharp EL-510 RN and Sharp EL-510RNB.
- This test consists of 12 questions (numbered 2 through 13) and has 11 pages (including this cover) and a Formula sheet on the last page.
  - Questions 2 through 10 are multiple-choice. Enter your **final answer** in the bubble sheet and mark them in this paper as well. You need to **show your work for all answers**, as we may disallow any answer which is not properly justified.
  - Questions 11 through 13 are long-answer. Write your full answer in this booklet as indicated.
- For the multiple-choice questions, select the numerical answer closest to yours. If the answer is equidistant from two nearest choices, select the largest of the two choices.
- Before starting your test enter your Name (Last, First), student ID (starting from V00******), and tutorial section number (T01 - T43) on this page and on the bubble sheet.
- At the end of your test, turn in both this booklet and the bubble sheet.
- Enter "B" in the bubble sheet as your answer to Question 1 now.

1. Enter "B" in the bubble sheet as your answer to Question 1 now.

2. [1 point] Determine whether or not the series $\sum_{n=0}^{\infty} \frac{2^n - 1}{5^n}$ converges, and find the sum of the series if it converges.

$\sum_{n=0}^{\infty}$

geometric series

$\frac{2^1-1}{5^1} = \frac{2}{5}.$

$\frac{2^0-1}{5^0} = \frac{2^0}{5^0} = 1 = \frac{2}{5}$ $r<1$

$\frac{1}{1-\frac{2}{5}}.$

(A) 0.2 (B) 0.4 (C) 0.6 (D) 0.8 (E) 1.0

(F) 1.5 (G) 2.0 (H) 2.5 (I) 3.0 (J) Diverges

3. [1 point] Find the 31st power of $z = i$

z^{31}

(A) $z^{31} = 1$ (B) $z^{31} = -1$ (C) $z^{31} = i$ (D) $z^{31} = -i$ (E) None of those

4. [1 point] Determine whether or not the series $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$ converges, and find the sum of the series if it converges.

$$\left[\frac{e^n}{e^n+n}\right]^{\frac{1}{n}}$$

$$\frac{e}{(e^n+n)^{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{e}{1}$$

(A) 0.1	(B) 0.2	(C) 0.3	(D) 0.4	(E) 0.5
(F) 1.0	(G) 1.5	(H) 2.0	(I) 3.0	(J) Diverges

5. [1 point] Determine whether or not the series $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ converges or diverges, using direct comparison test.

0,12

$\frac{1+\cos n}{n^2}$ [illegible] 0.06.

$\frac{1}{n^2}$ → converge

$\frac{1}{n}$ diverg 0,25

(A) Diverges, since $0 \le \frac{2}{n} \le \frac{1+\cos n}{n^2}$;

(B) Converges, since $0 \le \frac{2}{n} \le \frac{1+\cos n}{n^2}$;

(C) Diverges, since $0 \le \frac{1+\cos n}{n^2} \le \frac{2}{n}$;

(D) Converges, since $0 \le \frac{1+\cos n}{n^2} \le \frac{2}{n}$;

(E) Diverges, since $0 \le \frac{2}{n^2} \le \frac{1+\cos n}{n^2}$;

(F) Converges, since $0 \le \frac{2}{n^2} \le \frac{1+\cos n}{n^2}$;

(G) Diverges, since $0 \le \frac{1+\cos n}{n^2} \le \frac{2}{n^2}$;

(H) Converges, since $0 \le \frac{1+\cos n}{n^2} \le \frac{2}{n^2}$;

6. [1 point] Determine whether or not the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

integral test $\sum_{n=1}^{\infty} ne^{-n^2}$

$u=-n^2$

$\frac{du}{dx}=-2n$

$\frac{du}{-2}=n$

$-\frac{1}{2}\int_1^{\infty} e^u \, du$

$\frac{1}{2}(e^u)\Big|_1^{\infty}$

e^{-n^2}

$e^{-(\infty)^2}$

$e^{-\infty}$

$\frac{1}{e^{\infty}}$ -

$\frac{x}{e^{x^2}}$

$u=x^2$

$\frac{du}{dx}=2x$

$\frac{du}{2}=x$

$\frac{1}{2}\int \frac{1}{e^u}$

(A) Diverges by Integral Test when compared to $\int_1^{+\infty} \frac{x}{e^{x^2}}\,dx$;

(B) Converges by Integral Test when compared to $\int_1^{+\infty} \frac{x}{e^{x^2}}\,dx$;

(C) Diverges by n-th term Test for Divergence;

(D) Converges by n-th term Test for Divergence;

(E) Diverges, because it is a geometric series with $|r| \geq 1$;

(F) Converges, because it is a geometric series with $|r| < 1$;

(G) Diverges, because it is a p-series with $p \geq 1$;

(H) Converges, because it is a p-series with $p < 1$;

7. [1 points] Identify all True statements among those listed below:

(i) Series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.

(ii) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent. ✗

(iii) The Ratio Test can be used to determine whether series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. ✓

(A) only (i) is true (B) only (ii) is true (C) only (iii) is true

(D) only (i) and (ii) (E) only (i) and (iii) (F) only (ii) and (iii)

(G) All three are true (H) None of the three are true

8. [2 point] Find the complex number $z = \frac{3+2i}{5-i}$ in the form: $z = x + iy$.

(A) $z = 1 - i$; (B) $z = \frac{1}{2} - \frac{1}{2}i$; (C) $z = 13 - 13i$; (D) $z = 26 - 26i$; (E) $z = 5$;

(F) $z = 1 + i$; (G) $z = \frac{1}{2} + \frac{1}{2}i$; (H) $z = 13 + 13i$; (I) $z = 26 + 26i$; (J) None of those.

9. [2 points] For the series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$ find the smallest integer value of N for which the Alternating Series test condition is satisfied:

$$a_n \geq a_{n+1} > 0, \text{ for all } n \geq N \text{ for some integer } N.$$

$$\frac{(\ln n+1)}{(n+1)^{\frac{1}{2}}} \cdot \frac{\sqrt{n}}{\ln(n)}$$

(A) 1	(B) 2	(C) 4	(D) 6	(E) 8
(F) 10	(G) 12	(H) 14	(I) 16	(J) Such number does not exist

10. [2 points] Find the polar coordinates (r, θ) of the complex number $z = -1 + i$.

(A) $\left(2, \frac{\pi}{4}\right)$ (B) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (C) $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ (D) $\left(2\sqrt{2}, \frac{\pi}{4}\right)$ (E) $\left(1, \frac{\pi}{2}\right)$

(F) $\left(2, \frac{3\pi}{4}\right)$ (G) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$ (H) $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$ (I) $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$ (J) None of those

11. [1 + 2 + 1 = 4 points]

telescoping: - partial sum

(a) Show that $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ diverges.

lim comparison $\sqrt{n}$

$\frac{\sqrt{n+1}-\sqrt{n}}{\frac{1}{\sqrt{n}}}$ $\frac{(\sqrt{n+1}-\sqrt{n})}{1}\cdot\frac{(\sqrt{n})}{1}$

$\sqrt{\infty+1} \Rightarrow \infty$

telescoping.

$\sqrt{1+1}-\sqrt{1}$

S_1 $\cancel{\sqrt{2}}-\sqrt{1}$

S_2 $\sqrt{2+1}-\sqrt{2} = \cancel{\sqrt{3}}-\cancel{\sqrt{2}}$

S_3 $\sqrt{3+1}-\sqrt{3} = \sqrt{4}-\cancel{\sqrt{3}}$

S_4 $\sqrt{4+1}-\sqrt{4} = \sqrt{\ }$

$\int_0^{\infty}$ $\int_{-\infty}^{\infty}$

always positive

(b) Show that $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$ converges. altenating series

bigger

$\frac{\sqrt{n+1+1}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}$ $\cdot$ $\frac{(\sqrt{n+2}-\sqrt{n+1})}{\sqrt{n+1}-\sqrt{n}}\frac{(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$

① decreasing

$\lim (-1)^n\sqrt{n+1}$

$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ decreasing

0

N

(c) Determine if $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$ converges absolutely or conditionally. Explain why.

12. [2 + 2 = 4 points]

(i) Find the Taylor series for $y = e^{x/2}$ generated at the point $x = 0$.

$y = e^{\frac{x}{2}}$ $\quad e^x = \frac{x^n}{n!}$ $\quad \frac{x^{\frac{1}{2}n}}{n!}$

$f(a) + f'(a)(x-0) + \frac{f''(a)(x-0)^2}{2!} \dots \frac{f^{(n)}(a)(x-a)^n}{n!}$

$e^{\frac{1}{2}x}$

$e^{\frac{\frac{1}{2}x(0)}{2!}}$

$1 + \frac{1}{2} + 0$

$e^{\frac{1}{2}x}\left(\frac{1}{2}\right)$

$e^{\frac{x}{2}}$

$\left(\frac{1}{2}\right)e^{\frac{1}{2}x^2}(2x)\left(\frac{1}{2}\right)$

$\frac{1}{2}e^{\frac{1}{2}x^2}x$

$\frac{x^n}{n!}$ $\quad e^{\frac{x}{2}}$ $\quad \frac{x^{\frac{1}{2}n}}{n!}$ $\quad \frac{(\frac{1}{2}x)^n}{n!}$

$\frac{1}{2}e^{\frac{1}{2}(0)}(0)$

(ii) Use series for $y = e^{x/2}$ and $y = \frac{1}{1+x}$, and power series operations to find the Taylor series at $x = 0$ for the function $y = e^{x/2} - \frac{1}{1+x}$.

$\frac{a}{1-r}$

Write your answer as $\sum_{n=1}^{\infty} c_n (x-a)^n$ or $c_0 + c_1(x-a) + c_2(x-a)^2 + +c_3(x-a)^3 + \dots$

$\frac{1}{1-x}$ $\quad x^n$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

x^{n-1}

$\frac{(\frac{1}{2}x)^n}{n!}$

$\frac{1}{1-(-1)(x)}$

$(-1)^n x^n$

[1+4 = 5 points]

For the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$ find:

(a) the series' radius of convergence;

(b) the interval of convergence (clearly identifying for each of the end points as included or not included in the interval).

$$\left| \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^2+3}} \right|$$

$$\frac{\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}}{\frac{x^n}{\sqrt{n^2+3}}} = \frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}$$

$n+1-n$

$$\lim_{n\to\infty} x^{n+1-n} \frac{x\sqrt{n^2+3}}{\sqrt{n^2+2n+4}} \quad |x| < 1$$

$$\frac{\sqrt{n^2}}{\sqrt{n}} \qquad |x| < 1$$

$(n+1)(n+1)$

$n^2+n+n+1$

$n^2+2n+1+3$

n^2+2n+4

$$x \lim_{n\to\infty} \sqrt{\frac{n^2+3}{n^2+2n+4}} \qquad \frac{n^2}{n^2} = 1$$

$-1 < x < 1$

-1 0 1

Radius $= 1$

plug it back in

direct comp.

$$\lim_{n\to\infty} \frac{1^n}{\sqrt{n^2+3}} \quad \text{compare to } \frac{1}{n}$$

diverges

diverge.

$$\frac{\frac{1}{\sqrt{n^2+3}}}{\frac{1}{n}} \qquad \lim_{n\to\infty} \frac{n}{\sqrt{n^2+3}}$$

$$\sum \frac{(-1)^n}{\sqrt{n^2+3}} \qquad (-1)^n \qquad \frac{\sqrt{n^2}}{\sqrt{n^2+3}} \qquad \sqrt{\frac{n^2}{n^2+3}}$$

1

alternating series test.

converge

$[-1, 1)$