202201 Math 122 Assignment 4

Due: Friday, March 18, 2022 at 23:59. Please submit on your section's Crowdmark page.

There are five questions of equal value (worth a total of 45 marks). There are 4 bonus marks available if the solutions are typeset with LaTeX. Information on obtaining and using LaTeX is available on the cross-listed Brightspace page.

Please feel free to discuss these problems with each other. You may not access any "tutoring" or "help" website in any way. In the end, each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

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1. Let $f_1, f_2, \ldots$ be the sequence of Fibonacci numbers. Suppose f_n is computed by repeatedly applying the defining recurrence.

   (a) The number f_n is eventually expressed as $a_n f_2 + b_n f_1$ for some integers a_n and b_n. For example, $f_3 = 1f_2 + 1f_1$, so $a_3 = b_3 = 1$ and $f_4 = f_3 + f_2 = 1f_2 + 1f_1 + f_2 = 2f_2 + f_1$, so $a_4 = 2$ and $b_4 = 1$.

   Give a recursive definition of the sequences $a_3, a_4, \ldots$ and $b_3, b_4, \ldots$. What are the numbers a_n and b_n really?

   (b) Let $c_1, c_2, \ldots$ be the total number of times the defining recurrence is applied in the computation of f_n. Then $c_1 = c_2 = 0$, and further computation shows $c_3 = 1, c_4 = 2$ and $c_5 = 4$. Using the values of c_1 and c_2 as base cases, give a recursive formula for the number c_n that is valid for all $n \geq 1$.

   (c) Compute c_7 using your formula in (b).

2. Find, with proof, the smallest positive integer n_0 such that $5^n > n^5$ for all $n \geq n_0$. (Note: $(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$.)

3. Let $a_0, a_1, \ldots$ be the sequence recursively defined by

$$a_0 = 6,\ a_1 = -13,\ \text{and } a_n = -5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2.$$

   Prove that $a_n = 5(-2)^n + (-3)^n$ for all $n \geq 0$.

4. Let $f_1, f_2, \ldots$ be the sequence of Fibonacci numbers. Prove that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

   for all integers $n \geq 1$.

5. Let $a, b \in \mathbb{R}$, and let $t_0, t_1, \ldots$ be the sequence recursively defined by $t_0 = b$, and $t_n = a \cdot t_{n-1} + b$ for $n \geq 1$. Conjecture a formula (not involving a sum of about n terms) for t_n that holds for all $n \geq 0$, and then prove that your conjecture is correct.