CSC 225 : Spring 2021 : Lab 3 Solutions

1a. $T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(n-1)+n & \text{if } n \geq 2 \end{cases}$

"Bottom up" method:

$T(2) = T(1) + 2 = 1 + 2$

$T(3) = T(2) + 3 = 1 + 2 + 3$

$T(4) = T(3) + 4 = 1 + 2 + 3 + 4$

$\vdots$

$T(n) = 1 + 2 + 3 \dots + n = \boxed{\sum_{i=1}^{n} i}$

"Top down" method:

using the given equation:

$$T(n) = T(n-1) + n \longrightarrow T(n-1) = T((n-1)-1) + (n-1)$$

$$= T(n-2) + n + (n-1) \rightarrow T(n-2) = T(n-3) + (n-2)$$

$$= T(n-3) + n + (n-1) + (n-2)$$

$$= T(n-4) + n + (n-1) + (n-2) + (n-3)$$

$$\vdots$$

$$= T(n-i) + n + (n-1) + \dots (n-(i-1))$$

$$= T(n-i) + \sum_{i=n-i+1}^{n} i \rightarrow \text{plug in } n-1 \text{ for } i$$

$$= T(n-(n-1)) + \sum_{i=n-(n-1)+1}^{n} i$$

$$= T(1) + \sum_{i=2}^{n} i = 1 + \sum_{i=2}^{n} i = \boxed{\sum_{i=1}^{n} i}$$

1b. $T(n) = \begin{cases} 1 & \text{if } n=0 \\ 2T(n-1) & \text{if } n \geq 1 \end{cases}$

$T(1) = 2 \cdot 1$

"Bottom up":

$T(2) = 2T(1) = 2 \cdot 2 = 2^2$

$T(3) = 2T(2) = 2 \cdot 2 \cdot 2 = 2^3$

$T(4) = 2T(3) = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$

$\vdots$

$T(n) = \boxed{2^n}$

"Top down":

$$\begin{aligned} T(n) &= 2T(n-1) && \to T(n-1) = 2T(n-2) \\ &= 2(2T(n-2)) && \to T(n-2) = 2T(n-3) \\ &= 2(2(2T(n-3))) \\ &\vdots \\ &= 2^i T(n-i) \to \text{plug in } i=n \\ &= 2^n T(n-n) \\ &= 2^n T(0) = \boxed{2^n} \end{aligned}$$

2a. $\sum_{i=1}^{n} (2i-1) = n^2$ for all $n \geq 1$

① Base case: $n=1 \Rightarrow \sum_{i=1}^{1} (2i-1) = 1^2 \to 1 = 1$ ✓

② IH: Assume $\sum_{i=1}^{k} (2i-1) = k^2$ for $n=k$

③ show that $\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$

LHS: $\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + (2(k+1)-1)$

↳ substitute k^2 for the sum using the I.H.

LHS $= k^2 + (2k+1) = (k+1)^2$

LHS = RHS ✓ ∴ proven by induction □

2b. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$

① Base case: $n=0 \Rightarrow \sum_{0}^{0} i^2 = \frac{0(1)(1)}{6} \Rightarrow 0^2 = 0$ ✓

② IH: Assume $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ for $n=k$

③ show that $\sum_{i=0}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\begin{aligned} \text{LHS: } \sum_{i=0}^{k+1} i^2 &= \sum_{i=0}^{k} i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{2k^3 + 9k^2 + 13k + 6}{6} = \text{RHS} \checkmark \end{aligned}$$

∴ proven by induction

3. Loop Invariant: S_i: $x \neq$ any of the first i elements of A
(at the start of the iteration with index i, element x
has not yet been found)

① initialization / base case:
at the start of the first loop $i=0 \Rightarrow S_0$ is true since
x cannot have been found if no elements have been checked. ✓

② Matinence / I.H:
at iteration i, we compare x to $A[i]$
If $x = A[i]$ then we return i and the loop terminates ✓

If $x \neq A[i]$ then we increment i and go to the next loop.
to show that the invariant is still true at the beginning
of iteration $i+1$:
Assume that S_i is true for $i=k$ → I.H.
show that S_i is true for $i=k+1$:
↳ when $i=k+1$
from our I.H. we know elements $A[0] \dots A[k]$
are not equal to x and therefore that x has not been
found before entering iteration $i=k+1$ of the loop.
If $x = A[i] = A[k+1]$, return i and terminate
If not, we increment i and go to the next loop ✓
$\therefore$ S_i is true for $i=k+1$

③ termination:
If the while loop terminates withought returning
an index, then $i=n$ and x was not found in
the n-element array. Then we return. -1
$\therefore$ S_n is true

$\therefore$ ArrayFind is correct by proof by loop invariant