202201 Math 122 Assignment 2

Due: Friday, Feb. 11, 2022 at 23:59. Please submit on your section's Crowdmark page.

There are five questions of equal value (worth a total of 45 marks), and one bonus question (worth 4 marks). Please feel free to discuss these problems with each other. You may not access any "tutoring" or "help" website in any way. In the end, each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

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1. First write the argument below in symbolic form, and then determine if it is valid or invalid. If the argument is valid, prove it using known logical equivalences and inference rules. If it is invalid, demonstrate that by giving a counterexample.

   If I go motorcycling, then it rains. If it rains and is warm, then I wear rain gear or decide to get wet. If my friends phone me to go riding, then I go motorcycling and it is warm. I do not decide to get wet. Therefore, if my friends phone me to go riding, then I wear rain gear.

2. (a) Suppose the universe is the real numbers. Write the following statement in English, and also determine its truth value. Do not literally translate the symbols, as in "*For all x, there exists y, such that ...*" Instead, write an understandable sentence in plain English that accurately describes the mathematical property being precisely specified, and that starts with "*Every positive real number has ...*".

   $$\forall x, (x > 0) \to [(\exists y_1, \exists y_2, (y_1^2 = x) \wedge (y_2^2 = x) \wedge (y_2 \neq y_1)) \\ \wedge(\forall z, (z \neq y_1) \wedge (z \neq y_2) \to (z^2 \neq x))].$$

   (b) Write the sentence "*there is no smallest integer*" in symbols, making all quantifiers explicit. Remember to state the universe.

   (c) Suppose the universe is the integers. Write the negation of $\forall n, \exists m, m \cdot n = n$ in symbols, with quantifiers, and without using negation ($\neg$) or any negated mathematical symbols like $\neq$ or $\not>$.

3. Consider the two statements $s_1 : \forall n, p(n) \vee q(n)$ and $s_2 : [\forall n, p(n)] \vee [\forall n, q(n)]$.

   (a) Explain why s_2 logically implies s_1.

   (b) Are s_1 and s_2 logically equivalent (no matter what universe and the open statements $p(n)$ and $q(n)$)? Why of why not?

   (c) Are the statements $s_1 : \exists n, p(n) \vee q(n)$ and $s_2 : [\exists n, p(n)] \vee [\exists n, q(n)]$ logically equivalent? Why pr why not?

4. (a) Let n be an integer. Prove that if n^4 is a multiple of 3 then n is a multiple of 3 by stating the contrapositive, and then proving it by cases. (The possible remainders on division by 3 are 0, 1 and 2.)

   (b) Prove that $\sqrt{8}$ is irrational. (Hint: look at the corresponding proof for $\sqrt{2}$.)

   (c) Use the fact that $\sqrt{2}$ is irrational to show that $\sqrt{2^{2m+1}}$ is irrational for any positive integer m.

5. Answer each question True or False, and write a sentence or two to briefly explain your reasoning. Let $A = \{1, \{1\}, 2, \{\emptyset\}, \{\{1\}, \{2\}\}, \{\{1\}, 2\}\}$.

   (a) $\{2\} \in A$

   (b) $\{1, 2\} \subsetneq A$

   (c) $\{\{1, \{2\}\}\} \subseteq A$

   (d) $\emptyset \in A$

   (e) $A \cap \mathcal{P}(A) = \emptyset$

   (f) $\{2\} \in \mathcal{P}(A)$.

   (g) $\emptyset$ is the only set with no non-empty proper subset.

6. (Bonus question, 4 marks) Suppose you are given a 2×2 array of lights such that for each row of the array there is a switch that changes the state of every light in the row (from off to on, and on to off), and similarly for each column. Explain why the following statement is true: *The switches can be flipped so that all lights are eventually on if and only if there are an even number of lights on.* Suppose you are given a similar 3×3 array of lights with two lights on in each row. Is it always possible to flip the switches so that all lights are eventually off? Why or why not?