Math 110 - Homework 2
Topic: Vectors in $\mathbb{R}^n$

Due at 6:00pm (Pacific) on Friday, September 24, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 1.2 and 2.1 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- Declaring a vector in MATLAB is the same process as declaring a matrix, but with only one column. For instance, the vector $\overrightarrow{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can be entered in MATLAB as `v = [1; 2; 3]`.
- Addition of vectors of the same size is accomplished in MATLAB by putting a + between the vectors; for instance, if `v` and `w` are vectors of the same size then `v + w` produces their sum.
- To multiply a vector by a scalar, use the operator $*$. For instance, to compute $2\overrightarrow{v}$, enter `2 * v`.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

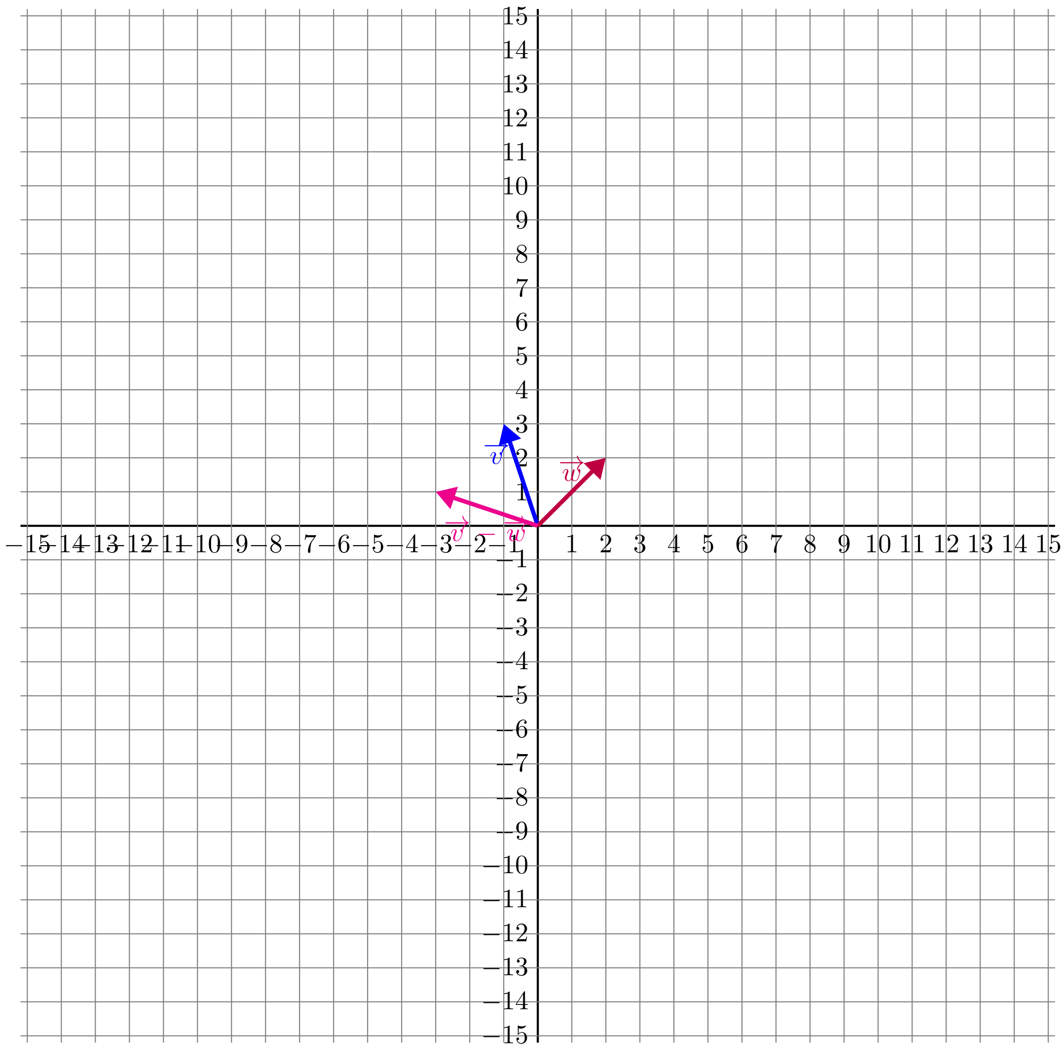
For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $\overrightarrow{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\overrightarrow{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. For each of the following, calculate the requested vector and draw a diagram that includes $\overrightarrow{v}$, $\overrightarrow{w}$, and the requested vector (all in standard position).

   (a) $\overrightarrow{v} - \overrightarrow{w}$

Solution:

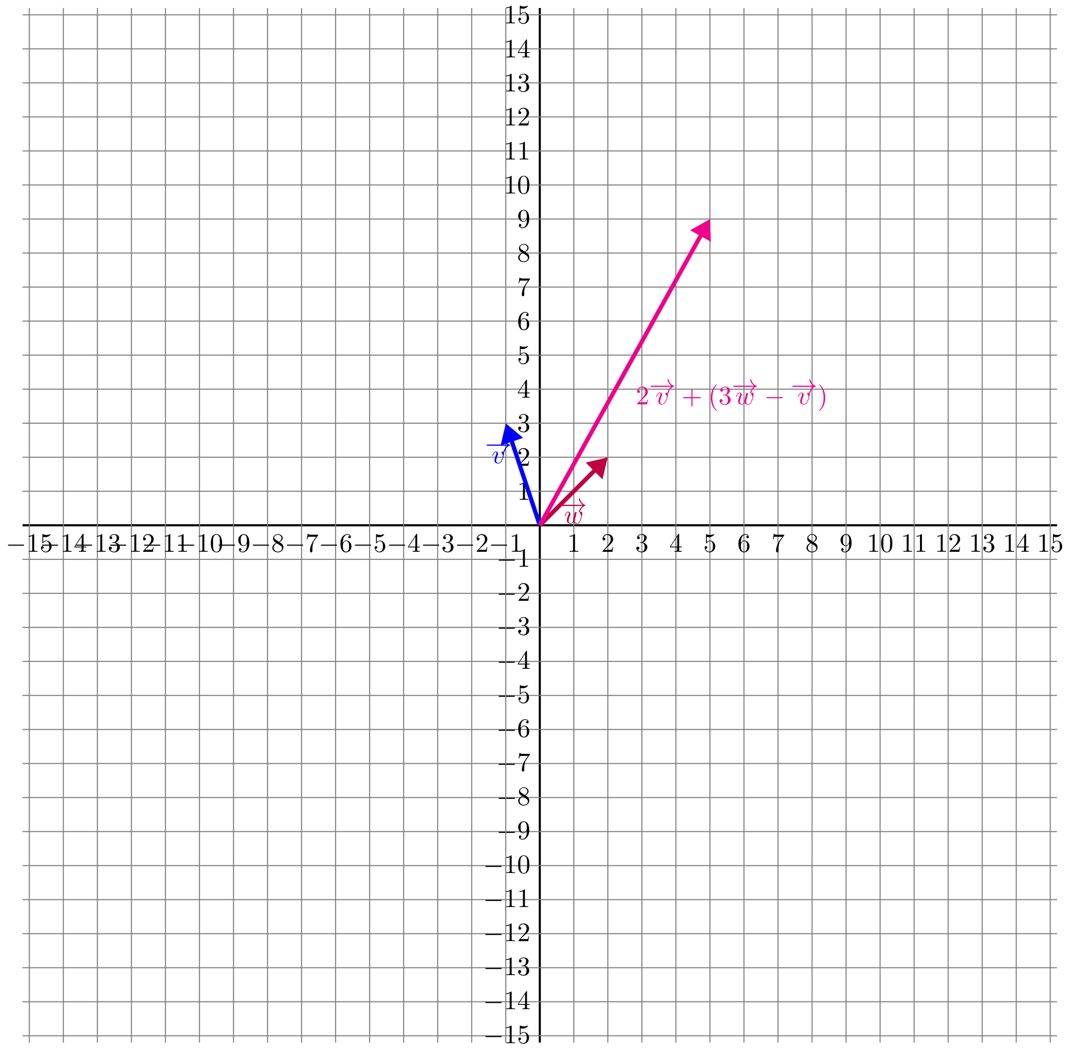
$$\overrightarrow{v} - \overrightarrow{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$



(b) $2\overrightarrow{v} + (3\overrightarrow{w} - \overrightarrow{v})$

Solution:

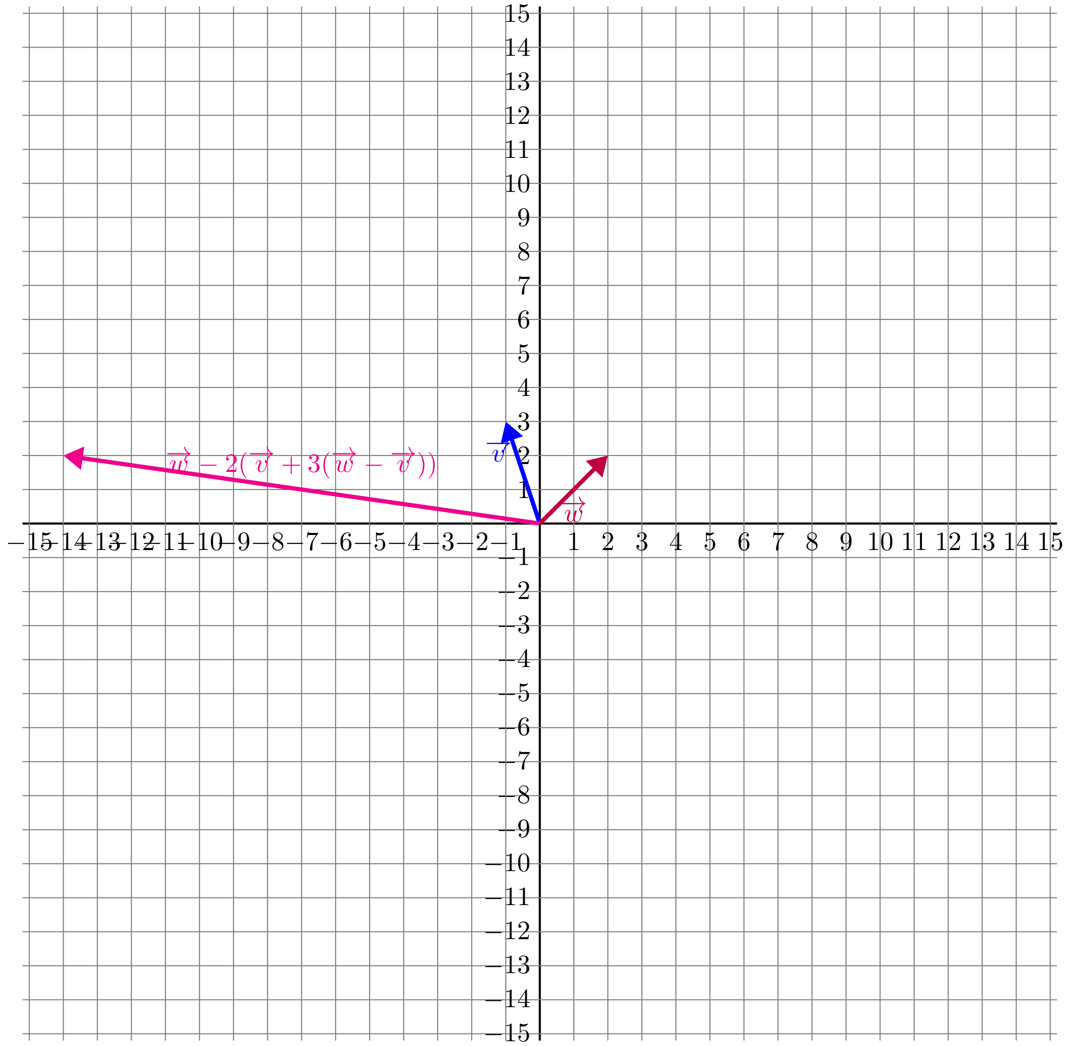
$$2\overrightarrow{v} + (3\overrightarrow{w} - \overrightarrow{v}) = \overrightarrow{v} + 3\overrightarrow{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$



(c) $\overrightarrow{w} - 2(\overrightarrow{v} + 3(\overrightarrow{w} - \overrightarrow{v}))$

Solution:

$$\overrightarrow{w} - 2(\overrightarrow{v} + 3(\overrightarrow{w} - \overrightarrow{v})) = \overrightarrow{w} - 2\overrightarrow{v} - 6\overrightarrow{w} + 6\overrightarrow{v} = 4\begin{bmatrix}-1\\3\end{bmatrix} - 5\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-14\\2\end{bmatrix}.$$



2. Find a vector in $\mathbb{R}^3$ that cannot be written as a linear combination of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$. Explain both how you found your vector and why it cannot be written as a linear combination of these vectors.

Solution: In order to solve this problem we will first find out which vectors *can* be written as a linear combination of the given vectors, and then simply choose any vector that does not fit that description. To do that, suppose that we have a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}.$$

This gives us a system of equations, where the variables are a, b, c. We want to know what the conditions are on x, y, z to make this system consistent, so we start row-reducing:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & x \\ -3 & 1 & -7 & y \\ 2 & 1 & 3 & z \end{array}\right] \to_{\substack{R_3-2R_1 \\ R_2+3R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 4 & x \\ 0 & -5 & 5 & y+3x \\ 0 & 5 & -5 & z-2x \end{array}\right]$$
$$\to_{R_3+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & x \\ 0 & -5 & 5 & 3x+y \\ 0 & 0 & 0 & x+y+z \end{array}\right]$$

We've reached row echelon form, which is far enough for us to be able to see that this system has a solution if and only if $x + y + z = 0$.

Since our goal was to find a vector for which this system does *not* have a solution, we can choose *any* x, y, z such that $x + y + z \neq 0$. Absolutely any such choice would be fine, so we'll choose the simplest one: $x = 1, y = 0, z = 0$. Thus $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. Note that there may be questions that can be solved without doing any calculations, or where MATLAB is not helpful for the calculations you need to do; in such cases, do the calculations by hand. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Determine whether or not the vector $\begin{bmatrix} -6 \\ 4.5 \\ 15 \\ -24 \\ 15 \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -1 \\ -2 \\ 6 \\ -2 \end{bmatrix}$. If it can be, then find such a linear combination. If it cannot be, then explain why not.

   Hint: You can turn this question into a question about a system of equations, and then solving that system would be a great place to use MATLAB.

   Solution: The question asks whether or not there are x, y, z such that

   $$\begin{bmatrix} -6 \\ 4.5 \\ 15 \\ -24 \\ 15 \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 5 \end{bmatrix} + z \begin{bmatrix} 2 \\ -1 \\ -2 \\ 6 \\ -2 \end{bmatrix}.$$

   Equivalently, we're looking for a solution to the system represented by the following matrix, which we row reduce using the MATLAB `rref` command:

   $$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -6 \\ -1 & 1 & -1 & 4.5 \\ 2 & 3 & -2 & 15 \\ -1 & -1 & 6 & -24 \\ -2 & 5 & -2 & 15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   From here we see that the unique solution to this system is $x = 1, y = 2, z = -3.5$. Therefore $\begin{bmatrix} -6 \\ 4.5 \\ 15 \\ -24 \\ 15 \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -1 \\ -2 \\ 6 \\ -2 \end{bmatrix}$, and that linear combination is:

   $$\begin{bmatrix} -6 \\ 4.5 \\ 15 \\ -24 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \\ 5 \end{bmatrix} - 3.5 \begin{bmatrix} 2 \\ -1 \\ -2 \\ 6 \\ -2 \end{bmatrix}.$$

2. Suppose that $\overrightarrow{w_1}$, $\overrightarrow{w_2}$, and $\overrightarrow{w_3}$ are vectors in $\mathbb{R}^n$, and that $\overrightarrow{w_3}$ can be written as a linear combination of $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$. Show that $3\overrightarrow{w_1} + 5\overrightarrow{w_2} - 4\overrightarrow{w_3}$ can also be written as a linear combination of $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$.

   Note: In this question we are asking you to give a general explanation, not a specific example. Do not choose a specific value of n, and also do not choose specific vectors $\overrightarrow{w_1}$, $\overrightarrow{w_2}$, and $\overrightarrow{w_3}$. Since we're not working with specific numbers here, MATLAB will not be helpful.

Solution: The assumption that $\overrightarrow{w_3}$ can be written as a linear combination of $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ means that there are scalars a and b such that $\overrightarrow{w_3} = a\overrightarrow{w_1} + b\overrightarrow{w_2}$. Therefore:

$$\begin{aligned} 3\overrightarrow{w_1} + 5\overrightarrow{w_2} - 4\overrightarrow{w_3} &= 3\overrightarrow{w_1} + 5\overrightarrow{w_2} - 4(a\overrightarrow{w_1} + b\overrightarrow{w_2}) \\ &= (3-4a)\overrightarrow{w_1} + (5-4b)\overrightarrow{w_2}, \end{aligned}$$

and this last expression is a linear combination of $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$.