Q:1 (a). $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0$

Q:2 (a). Since $f(x) = \frac{1}{x}-\frac{1}{x^2}$ is decreasing for $x \geq 2$. As,

$\int_2^{\infty}\left(\frac{1}{x}-\frac{1}{x^2}\right) dx$ diverges, by the integral test, so

does the series $\sum_{n=1}^{\infty} a_n$.

Q: 1 (b).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{(n)^{5/2}} + \lim_{n\to\infty} \frac{\sin[e^n]}{(n)^{5/2}}$$

$$\lim_{n\to\infty} \frac{n}{n^{5/2}} = \lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$$

$$\lim_{n\to\infty} \frac{\sin[e^n]}{n^{5/2}} = 0 \quad \text{as} \quad -1 \leq \sin[e^n] \leq 1$$

$$\Rightarrow \frac{-1}{n^{5/2}} \leq \frac{\sin[e^n]}{n^{5/2}} \leq \frac{1}{n^{5/2}}$$

Q: 2 (b).

$$\lim_{n\to\infty} \frac{\frac{n+\sin[e^n]}{n^{5/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} 1 + \frac{\sin[e^n]}{n}$$

$= 1$, Converges by the Limit Comparison test with the series $\sum_n 1/n^{3/2}$

Q.1 (c) Using Logarithm, we write

$$\ln[a_n] = n\ln[\cos(1/n)] = \frac{\ln[\cos(1/n)]}{1/n}$$

Therefore, $$\lim_{n\to\infty} \ln[a_n] = \lim_{n\to\infty} \frac{\ln[\cos(1/n)]}{1/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{-\tan(1/n)\,\cancel{(-1/n^2)}}{\cancel{(-1/n^2)}}$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = e^0 = 1.$$

Q.2 (c) Since $\lim_{n\to\infty} a_n = 1$, the series $\sum_n a_n$ diverges by the $n^{th}$-term divergence test.

Q:1 (d)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{-n} + \left(\frac{1}{2}\right)^n$$

$$= \lim_{n\to\infty} e^{-n} + \lim_{n\to\infty} \frac{1}{2^n}$$

$$= 0$$

Q:2 (d) The series $\sum_n a_n = \sum_n \frac{1}{e^n} + \sum_n \frac{1}{2^n}$

As $\sum_n \frac{1}{e^n}$ and $\sum_n \frac{1}{2^n}$ are both geometric with common ratio less than 1, both are convergent.

Therefore $\sum_n a_n$ also converges.

Q. 1. (e).

$$\frac{2^n}{(n+1)!} = \frac{2\cdot 2\cdot 2 \cdots 2}{1\cdot 2\cdot 3\cdot 4 \cdots n\cdot(n+1)}$$

$$= \frac{1}{\cancel{1}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{2}\cdots\frac{n+1}{2}}$$

$$= \frac{1}{\frac{3}{2}\cdot\frac{4}{2}\cdot\frac{5}{2}\cdots\frac{n+1}{2}}$$

$$\leq \frac{1}{\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdots\frac{3}{2}} = \frac{1}{\left(\frac{3}{2}\right)^{n-1}} = \left(\frac{2}{3}\right)^{n-1}; \quad \lim_{n\to\infty}\left(\frac{2}{3}\right)^{n-1} = 0$$

As $0 \leq \frac{2^n}{(n+1)!} \leq \left(\frac{2}{3}\right)^{n-1}$

Therefore, $\lim_{n\to\infty} \frac{2^n}{(n+1)!} = 0$ by the Sandwich theorem.

Q. 2 (e). As $\frac{2^n}{(n+1)!} \leq \frac{3}{2}\left(\frac{2}{3}\right)^n$ and $\sum_n \left(\frac{2}{3}\right)^n$; Converges [Geometric with $r = \frac{2}{3}$]

By direct comparison test, $\sum_n \frac{2^n}{(n+1)!}$ also converges.

Q:1 (f) $a_n = \dfrac{(n+1)!}{(n+3)!} = \dfrac{1\cdot 2\cdot 3 \cdots n\cdot(n+1)}{1\cdot 2\cdot 3\cdots n\cdot(n+1)\cdot(n+2)(n+3)}$

$$= \frac{1}{(n+2)(n+3)}$$

therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{(n+2)(n+3)} = 0.$

Q:2 (f) $$\frac{1}{(n+2)(n+3)} \leq \frac{1}{n^2}$$

Since the series $\sum_n \left(\frac{1}{n^2}\right)$ converges [by p-test], the series $\sum_n \dfrac{(n+1)!}{(n+3)!}$ also converges.