Math 122 Lecture Notes
Sections 1.1 and 1.2 - Statements, Compound Statements and Logical Connectives

A **statement** or **proposition** is an assertion that is either true or false - it has a truth value.

Example 1:

- $-3 < 42$
- π is exactly 3
- Michelle likes pizza
- For all integers $n \geq 0$, the integer $2n + 1$ is prime.
- Every even positive integer except 2 is the sum of two prime numbers.

We use letters to represent statements (p, q, r).
Thing of statements like variables that can that can take on the values "true" (1) or "false" (0).

A **compound statement** is a statement formed from two other statements with a **logical connective**.

Example 2: $25 = 5^2$ and $3 < 4$.

Today we will learn math or today we will play with puppies.

The logical connectives above are "and" and "or".

Say p and q are statements. Here are the logical connectives we will use in Math 122:

- **conjunction**, $p \wedge q$
  read: "p and q"
  true only when both p and q are true

- **disjunction**, $p \vee q$
  read: "p or q"
  true when either p or q or both are true

- **implication**, $p \rightarrow q$
  read: "if p then q", or "p implies q"
  is false only when p is true and q is false
  all other situations are true
  p is called the hypothesis, q is called the conclusion

- **biconditional**, $p \leftrightarrow q$
  read: "p if and only if q"
  true whenever the truth values of p and q match

To remember the implication $p \rightarrow q$, think of the following: If you get an A on the midterm, I will pay you \$5.
Possible Outcomes:

Below we will look at some compound statements and decide on their truth value.

Example 3:

(a) $(3 < 10) \quad \wedge \quad (\sqrt{2}$ is rational$)$

(b) $(\sqrt{10} > 3) \quad \vee \quad$ (for every real number x, $x^2 \geq 0$)

(c) (There is an x such that $3x + 1 = 7$) $\quad \vee \quad$ (1 is an even number)

(d) $(25 = (-5)^2) \;\rightarrow\; (-1 > 2)$

(e) $(-1 > 2) \;\rightarrow\; (25 = (-5)^2)$

(f) $(4 \text{ is even}) \;\leftrightarrow\; (7 < 10)$

(g) $(2 + 2 = 4) \;\leftrightarrow\; (\sqrt{50} < 7)$

(h) $(2 + 2 = 5) \;\leftrightarrow\; (\sqrt{50} < 7)$

Note: Examples (d) and (e) show that $p \rightarrow q$ and $q \rightarrow p$ may not have the same truth value.

Second Class, First Week
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Math 122 Lecture Notes
Section 1.3 - Negation of Statements

The **negation** of statement p is $\neg p$. (read: "not p")

"$\neg$" is not a logical connective since it does not join two statements

$\neg$ has precedence over connectives, so

$$\neg p \vee q \qquad \text{means} \qquad (\neg p) \vee q$$

Note: $\neg(\neg p)$ is the same as p

Rules:

- negation of $p \wedge q$ is
- $\neg(p \vee q)$ is the same as
- $\neg(p \to q)$ is the same as
- $\neg(p \leftrightarrow q)$ is the same as

We can also see this by using truth tables:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
0	0					
0	1					
1	0					
1	1					

Math 122 Lecture Notes
Section 1.4 - Truth Tables

Example 1: Show that $p \to q$ is the same as $\neg p \vee q$.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
0	0			
0	1			
1	0			
1	1			

Class Note

Example 2: Show that $p \leftrightarrow q$ is the same as $(p \to q) \wedge (q \to p)$.

Class Note

From Example 1 we can now also say that $p \leftrightarrow q$ is the same as $(\neg p \vee q) \wedge (\neg q \vee p)$.

Example 3: Write the truth table for $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$(p \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

For a compound statement comprised of n statements, how many rows would the truth table have?

Try on your own: write the truth table for

$$(\neg p \to r) \to (q \vee \neg r)$$

(The answer is in the typed Chapter 1 notes.)

Sometimes we don't need the entire truth table to find what we want.

Example 4: Find the possible truth values of $(a \vee b) \wedge (\neg b \vee \neg c)$ if we are given the extra information that $\neg a \to (b \leftrightarrow \neg c)$ is false.

Math 122 Lecture Notes

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Sections 1.5 and 1.6 - Tautologies and Contradictions, Logical Equivalence

A statement which is always true is called a **tautology**.

A statement which is always false is called a **contradiction**.

Example 1:

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1		
1	0		

Example 2: Look at the truth table for $(p \to q) \leftrightarrow (\neg p \vee q)$.

$(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology since the truth values of $p \to q$ and $\neg p \vee q$ always agree. (i.e. $p \to q$ is "the same" as $\neg p \vee q$.)

Two statements a and b are **logically equivalent** exactly when $a \leftrightarrow b$ is a tautology.
(Another way to say this: a and b are logically equivalent if and only if $a \leftrightarrow b$ is a tautology.)

Notation: $a \Leftrightarrow b$ (read: a is logically equivalent to b)

In other words: $a \Leftrightarrow b$ exactly when a and b have matching columns in the truth table.

So in Example 2 we saw that $(p \to q) \Leftrightarrow (\neg p \vee q)$ since $(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology. Another way to see that $(p \to q) \Leftrightarrow (\neg p \vee q)$ is to say that $(p \to q)$ and $(\neg p \vee q)$ produced matching columns in the truth table.

The typed notes for Chapter 1 (page 13, 3rd paragraph) gives a good explanation of the difference between $a \leftrightarrow b$ and $a \Leftrightarrow b$. The first one is a statement that has a truth value of true or false. The second one is a higher level fact that says that a and b always have the same truth value.

Since we can think of logical equivalence as two statements being "the same", we can use one statement to substitute for another statement it is logically equivalent to. In other words, we can simplify logical expressions through this substitution.

Math 122 Lecture Notes
Section 1.7 - Converse and Contrapositive of an Implication

The **converse** of $p \to q$ is $q \to p$.

The **contrapositive** of $p \to q$ is $\neg q \to \neg p$.

Example 1: Consider the statement "The university is closed when it is snowing." Find the converse and the contrapositive.

Example 2: Look at the truth table for the three statements $(p \to q)$, $q \to p$, and $\neg q \to \neg p$.

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg q \to \neg p$
0	0					
0	1					
1	0					
1	1					

Here we can see that $p \to q$ and $\neg q \to \neg p$ are the same, but $q \to p$ is not the same. So the implication and the contrapositive are the same, but the converse is not the same.

The **inverse** of $p \to q$ is $\neg p \to \neg q$. The inverse is the contrapositive of the converse (so $q \to p$ is the same as $\neg p \to \neg q$).

Math 122 Lecture Notes
Section 1.8 - Necessity and Sufficiency

~~Please go watch the screencast on the shared Math 122 Brightspace page.~~

Summary of this section:

- "q is necessary for p" is the same as $p \to q$
- "p is sufficient for q" is the same as $p \to q$
- "p is necessary and sufficient for q" is the same as $(p \to q) \wedge (q \to p)$, which is the same as $p \leftrightarrow q$.

Cut off for QUIZ 1 on Week 2

Math 122 Lecture Notes
Section 1.9 - The Laws of Logic

Idea of this section: We can simplify logical expressions by replacing a statement with another logically equivalent statement.

See the handout: The Laws of Logic

Four of the most useful logical equivalences that we have seen so far:

- $(p \rightarrow q) \Leftrightarrow \neg p \vee q$
- $(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
- $(p \vee \neg p) \Leftrightarrow \mathbf{1}$
- $(p \wedge \neg p) \Leftrightarrow \mathbf{0}$

When using any of these four rules, you can give "known logical equivalence" as the justification. The use of any of the rules from the Laws of Logic handout should state the name of the rule as the justification.

Example 1: Simplify $(p \wedge q) \vee (p \wedge \neg q)$.

Example 2: Show that $\neg(p \leftrightarrow q) \Leftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg p)]$

Example 3: Show that $(\neg p) \rightarrow (p \rightarrow q)$ is a tautology.

If we look at the Laws of Logic handout, we can see that many of the rules come in pairs. The **dual** of a statement is found by swapping $\vee$ for $\wedge$ (and vice versa), and **0** for **1** (and vice versa).

Example 4: Find the dual of $\neg(p \wedge q) \vee (\neg p \wedge q)$

Rule: If two statements are logically equivalent, then their duals are logically equivalent too.

So since $\neg(p\wedge q)\vee(\neg p\wedge q) \Leftrightarrow (\neg p\vee\neg q)$ (try show this logical equivalence on your own, either with the Laws of Logic - which would be good practice - or with a truth table), we can also say that $\neg(p\vee q)\wedge(\neg p\vee q) \Leftrightarrow (\neg p\wedge\neg q)$ (and even though we know this equivalence holds from our rule about duals, it would also be good practice to try show this equivalence with the Laws of Logic, and also with a truth table).

Math 122 Lecture Notes
Section 1.10 - Using Only $\wedge$, $\vee$, $\neg$

Any statement can be written using only $\wedge$, $\vee$, and $\neg$.
The way we can find this version of the statement comes from the truth table.

Look at the truth table below for the statment s.

p	q	s
0	0	1
0	1	1
1	0	1
1	1	0

Here's our method:

- For each row where the statement s is true, write an expression with $p, q, \wedge, \neg$ that would also be true for this combination of inputs of p and q.
- To find the statement that is overall logically equivalent to s, take the $\vee$ of all the statements created in the first step.

So for the above statement s we would have the equivalent statement:

Note: The d.n.f. is not necessarily the most simple way to write our statement. If it is not, we could further simplify by using our Laws of Logic.

Example 2: Write the statement s using only $\wedge$, $\vee$, $\neg$.

p	q	r	s
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Math 122 Lecture Notes
Section 1.11 - Logical Implication

We saw before that a and b are logically equivalent when $a \leftrightarrow b$ is a tautology. For this we wrote $a \Leftrightarrow b$.

In the same way, we say that a **logically implies** b if $a \to b$ is a tautology. Here we write $a \Rightarrow b$.

Remember that $a \leftrightarrow b \Leftrightarrow (a \to b) \wedge (b \to a)$.
If we had that $a \Rightarrow b$ and $b \Rightarrow a$, this would mean that $a \to b$ is a tautology and $b \to a$ is a tautology, and therefore $(a \to b) \wedge (b \to a)$ is a tautology too.
This then tells us that $a \Leftrightarrow b$.

This idea will be important for writing proofs later in the course.

Math 122 Lecture Notes
Section 1.12 - Valid Arguments and Inference Rules

An **argument** is a collection of statements $A_1, A_2, A_3, \ldots, A_n$ called **premises** or **hypotheses**, followed by a statement B called the **conclusion**.

An argument is **valid** if whenever all the premises $A_1, A_2, \ldots, A_n$ are true, then the conclusion B is also true.

An argument is **invalid** if we can find truth values so that all the premises $A_1, A_2, \ldots, A_n$ are true, but the conclusion B is false.

We write arguments in the format:

Another way of saying that an argument is valid is that
$(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to B$ is a tautology.
This is the same as saying $(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \Rightarrow B$.

Example 1: Show that the argument below is not valid.

$$\begin{array}{rc} & p \to q \\ & q \vee r \\ \hline \therefore & r \to \neg q \end{array}$$

See the Rules of Inference handout. Note that Modus Ponens and the Chain Rule are the most commonly used - in fact, the other rules can be proved starting from just these two rules.

Example 2: Show that the Modus Tollens rule is valid, only assuming we know Modus Ponens and the Chain Rule.
That is, show that $[(p \to q) \wedge \neg q] \Rightarrow \neg p$.

Example 3: Show that the Disjunctive Syllogism rule is valid. That is, show that

$$\begin{array}{rc} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

is valid.

Example 4: Show that the following argument is valid.

$$\begin{array}{rl} & p \to r \\ & \neg p \to q \\ & q \to s \\ \hline \therefore & \neg r \to s \end{array}$$

Note: There may be more than one way to show that an argument is valid!

Let's try a proof by contradiction. The idea here is that we assume our premises, and the negation of the conclusion. We show that we have a contradiction, so the negation of the conclusion is not correct. Therefore, our conclusion should be true.

Example 5: Use proof by contradiction to show that the following argument is valid.

$$
\begin{array}{rc}
 & \neg p \leftrightarrow q \\
 & q \rightarrow r \\
 & \neg r \\
\hline
\therefore & p
\end{array}
$$