CSC 225

Algorithms and Data Structures I

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Big-Omega Notation

Let $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$. $f(n)$ is $\Omega(g(n))$ if and only if

there exists a real constant $c > 0$

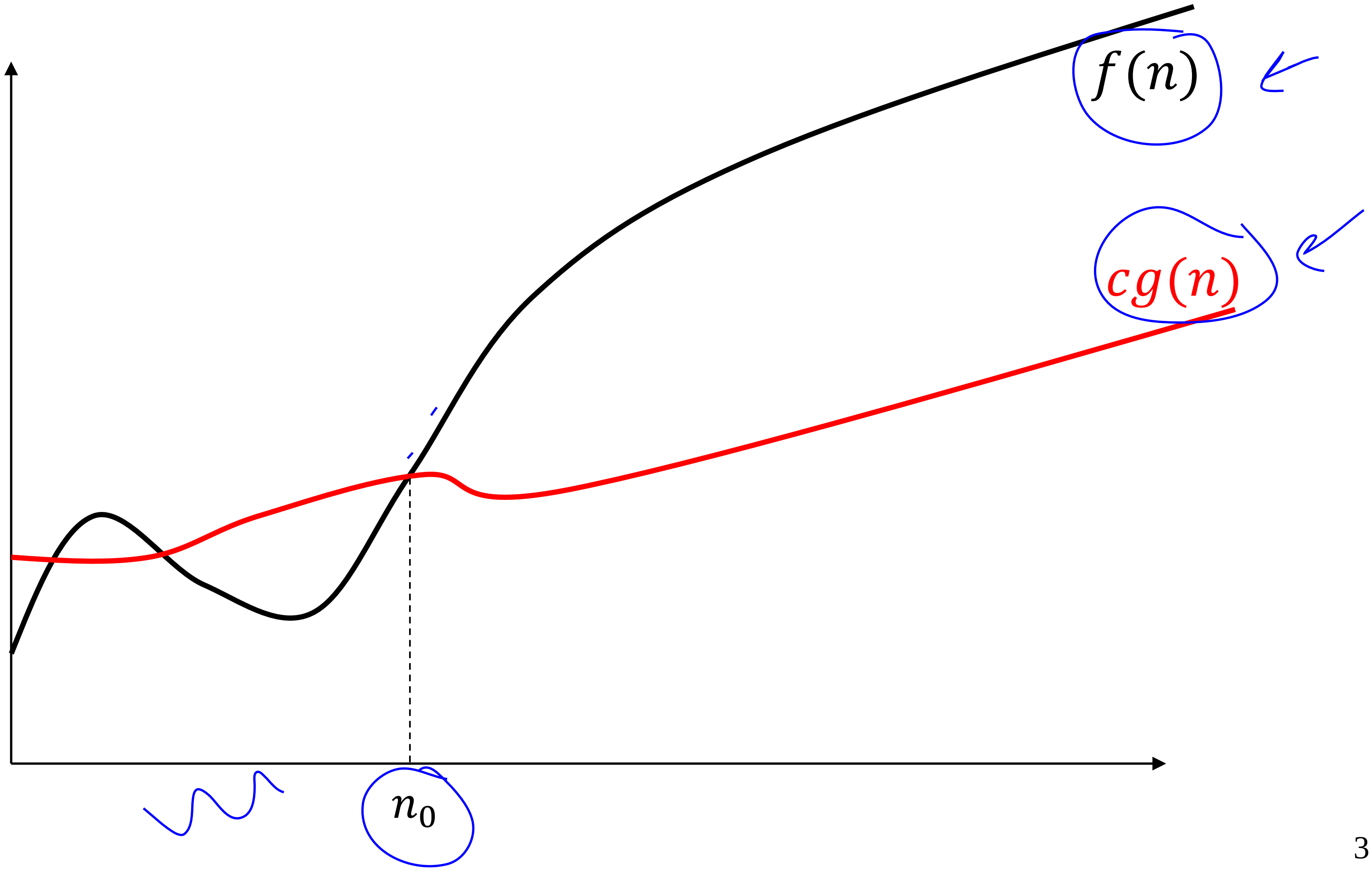
and an integer constant $n_0 > 0$

such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

$\mathbb{N}$: non-negative integers

$\mathbb{R}$: real numbers

$f(n)$ is $\Omega(g(n))$



Big-Omega Notation

Let $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$. Then, $f(n)$ is $\Omega(g(n))$ if and only if $g(n)$ is $O(f(n))$.

$\Leftarrow$ Let $g(n) \in O(f(n))$ + show that $f(n) \in \Omega(g(n))$

There exists $c, n_0 > 0$, s.t. $g(n) \le cf(n)$; $\forall n \ge n_0$

want to show $\exists k, n_1 > 0$ s.t. $f(n) \ge kg(n)$ $\forall n \ge n_1$

$\Rightarrow g(n) \le cf(n) \quad \forall n \ge n_0$

$\Rightarrow \frac{1}{c} g(n) \le f(n)$? $\frac{1}{c} > 0$ Let $k = 1/c$, $n_1 = n_0$

$\therefore f(n) \ge k g(n) \quad \forall n \ge n_1$

Quiz: What is true, what is false?

1. 2^n is $\Omega(n!)$

$\nexists$ $n! \notin O(2^n)$

$\therefore$ $2^n \notin \Omega(n!)$

2. $n!$ is $\Omega(2^n)$

since $2^n \in O(n!)$

therefore $n! \in \Omega(2^n)$

Big-Theta Notation

Let $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$.

$f(n)$ is $\Theta(g(n))$ if and only if

there exists $c_1, c_2 > 0$ and $n_0 > 0$ such that

$n_1, n_2 > 0$

$$c_1 g(n) \le f(n) \le c_2 g(n)$$

$f(n) \in \Omega(g(n))$ $\quad$ $f(n) \in O(g(n))$

for all $n \ge n_0$.

Big-Theta Notation

Let $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$. Then, $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

$\Rightarrow$ $f(n) \in \Theta(g(n))$, then $\exists\, c_1, c_2 > 0, n_0 > 0$

s.t. $c_1 g(n) \le f(n) \le c_2 g(n)$, $\forall n \ge n_0$

w.t.s. $f(n) \in O(g(n))$, $\exists\, c_2 > 0, n_0 > 0$, s.t.

$f(n) \le c_2 g(n)$ $\forall n \ge n_0$

& $\exists\, c_1 > 0, n_0\, 0$, s.t. $f(n) \ge c_1 g(n)$ $\forall n \ge n_0$.

$\therefore f(n)$ is $O(g(n))$ and $\Omega(g(n))$

Big-Theta: Examples

1. $3n + 1$ is $\Theta(n)$

W.T.S. $\exists\, c_1, c_2, n_0 > 0$ s.t.

$c_1 n \le 3n+1 \le c_2 n \quad \forall\, n \ge n_0$

Let $c_1 = 3$, then $3n \le 3n+1 \quad \forall n \ge 1$

$c_2 = 4$, $3n+1 \le 3n+n = 4n \quad \forall n \ge 1$

Let $n_0 = 1$, then

$3n \le 3n+1 \le 4n \quad \forall n \ge 1$

Big-Theta: Examples

2. $\sum_{i=1}^{n} \log_2 i$ is $\Theta(n \log n)$, w.t.s. (a) $\sum_{i=1}^{n} \log i \in O(n \log n)$

(b) $\sum_{i=1}^{n} \log i \in \Omega(n \log n)$

$\sum_{i=1}^{n} \log i = \log(n!)$

(a) $\exists\, c, n_0 > 0$, s.t. $\sum_{i=1}^{n} \log i \le c n \log n, \forall n \ge n_0$

$\sum_{i=1}^{n} \log i = \log 1 + \log 2 + \log 3 + \dots + \log n$

$\le \underbrace{\log n + \log n + \log n + \dots + \log n}$

$\forall n \ge 1$

$= n \log n$, $c = 1, n_0 = 1$

Big-Theta: Examples

2. $\sum_{i=1}^{n} \log_2 i$ is $\Theta(n \log n)$ (b) $\sum_{i=1}^{n} \log i \in \Omega(n \log n)$

Assume n is even.

So, w.t.s. $\exists\, c, n_0 > 0$, such that $\sum_{i=1}^{n} \log i \geq cn \log n$ $\forall n \geq n_0$.

$$\sum_{i=1}^{n} \log i = \underbrace{\log 1 + \log 2 + \dots + \log(\tfrac{n}{2})}_{n/2} + \underbrace{\log(\tfrac{n}{2}+1) + \dots + \log n}_{n/2}$$

$\log(\frac{n}{2}+1) \geq \log \frac{n}{2}$, $\log(\frac{n}{2}+2) \geq \log \frac{n}{2}$, ..., $\log n \geq \log(\frac{n}{2})$

$$\geq \sum_{i=1}^{n/2} \log i + \frac{n}{2}\log(\frac{n}{2}) \geq \frac{n}{2}\log(\frac{n}{2})$$

$$= \frac{n}{2}(\log n - \log 2) = \frac{n}{2}(\log n - 1)$$

Stirling's Formula

- Another useful formula for ordering functions by growth rate is Stirling's Formula (1730)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$n! \in O(n^n)$

$n! \notin \Omega(n^n)$

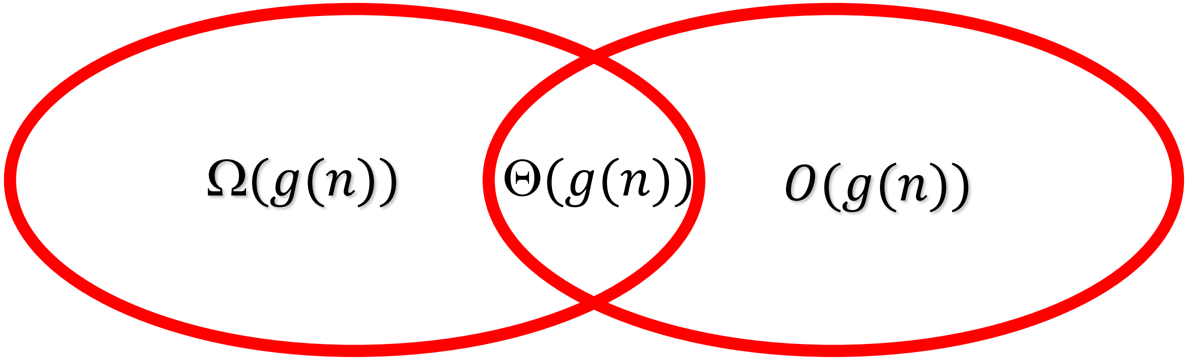
- Alternatively,

$$\log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}\right) < \log(n!) < \log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}\right)$$

$= \log(\sqrt{2\pi n}) + \log(n^n) - \log(e^n) + \log(e^{\frac{1}{12n}})$

Intuition of Asymptotic Terminology

- Big-Oh: $O(g(n))$ upper bound; functions that grow no faster than $g(n)$
- Big-Omega: $\Omega(g(n))$ lower bound; functions that grow at least as fast as $g(n)$
- Big-Theta: $\Theta(g(n))$ asymptotic equivalence; functions that grow at the same rate as $g(n)$



Little-Oh Notation

Let $f: \mathbb{N} \rightarrow \mathbb{R}$ and $g: \mathbb{N} \rightarrow \mathbb{R}$.

$f(n)$ is $o(g(n))$

if and only if

for any constant $c > 0$ there is a constant $n_0 > 0$ such that $f(n) \leq cg(n)$ for $n \geq n_{0.}$

Recall: Analogous to "f(n) < g(n)".

Examples: Little-Oh

1. $2n$ is $o(n^2)$ w.t.s. $\forall c>0, \exists n_0>0$, s.t. $2n \le cn^2$ $\forall n \ge n_0$.

$2n \le cn^2$?

$2 \le cn$

$\boxed{n \ge 2/c}$

$\forall c$, if $n_0 \ge 2/c$

then $2n \le cn^2$

$\therefore 2n \in o(n^2)$

2. $2n^2$ is ***not*** $o(n^2)$! [but $2n^2$ is $O(n^2)$]

If $2n^2 \in o(n^2)$, $\forall c>0, \exists n_0>0$, s.t. $2n^2 \le cn^2$

$\underline{c \ge 2}$

$\exists c>0$, where $2n^2 \not\le cn^2$, for any $n_0 \ge 1$

Little-Omega Notation

Let $f\colon \mathbb{N} \to \mathbb{R}$ and $g\colon \mathbb{N} \to \mathbb{R}$.

$f(n)$ is $\omega(g(n))$

if and only if

for any constant $c > 0$ there is a constant $n_0 > 0$ such that $f(n) \geq cg(n)$ for $n \geq n_0$.

Recall: Analogous to "f(n) > g(n)".

Little-Omega Notation

Let $f: \mathbb{N} \to \mathbb{R}$ and $g: \mathbb{N} \to \mathbb{R}$. Then, $f(n)$ is $\omega(g(n))$ if and only if $g(n)$ is $o(f(n))$.

$\Rightarrow$ Let $f(n) \in \omega(g(n))$, for all $c>0$, $\exists n_0 > 0$

s.t. $f(n) \geq c g(n)$, $\forall n \geq n_0$.

wts. $g(n) \in o(f(n))$, that is, $\forall k>0$, $\exists n_1 > 0$

s.t. $g(n) \leq k f(n)$, $\frac{1}{k} > 0$ does this exist?

is true for $\frac{1}{k}$ that $f(n) \geq \frac{1}{k} g(n)$, $\forall n \geq n_0$

$k f(n) \geq g(n)$

Alternate Limit Definitions

- The limit as can reveal the asymptotic relationship provided it exists.

$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$	$\Rightarrow$	$f(n) \in O(g(n))$
$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$	$\Rightarrow$	$f(n) \in \Omega(g(n))$
$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$	$\Rightarrow$	$f(n) \in \Theta(g(n))$
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$	$\Rightarrow$	$f(n) \in o(g(n))$
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$	$\Rightarrow$	$f(n) \in \omega(g(n))$

Examples using Limits

1. $2n^2$ is $\omega(n)$

$$\lim_{n\to\infty} \frac{2n^{\cancel{2}}}{\cancel{n}} = \lim_{n\to\infty} 2n = \infty$$

$$\therefore\ 2n^2 \in \omega(n)$$

2. $n \log n$ is $o(n^2)$

$$n \log n \le c\, n \cdot n$$

$$\lim_{n\to\infty} \frac{\cancel{n}\log n}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{n\ln 2} = 0 \qquad \therefore\ n\log n \in o(n^2)$$

Examples using Limits

3. Show that $\log^x n \in O(n^y)$ for any fixed constants $x, y > 0$.

$$\lim_{n\to\infty} \frac{(\log n)^x}{n^y} = \lim_{n\to\infty} \left(\frac{\log n}{n^{y/x}} \right)^x$$

$$= \left(\lim_{n\to\infty} \frac{\log n}{n^{y/x}} \right)^x = \left(\lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{y/x \, n^{y/x - 1}} \right)^x$$

$$= \left(\lim_{n\to\infty} \frac{1}{(y/x) n \ln 2 \; n^{y/x-1}} \right)^x = 0^x = 0$$

$$n^{(1 + y/x - 1)} = n^{y/x}$$