$\sum$ Math 110 Lecture Notes
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2.1 Vectors and basic operations

In this section we introduce *vectors*, which are the fundamental objects of study in linear algebra. We begin in the plane, $\mathbb{R}^2$, where we can easily interpret vectors both algebraically and geometrically. We then move on to the general setting of $\mathbb{R}^n$. This section also introduces the two most important operations on vectors: Addition and scalar multiplication.

2.1.1 Vectors in $\mathbb{R}^2$.

Definition 2.1.1. A *vector* in $\mathbb{R}^2$ is a column with two entries, each of which is a real number. $\mathbb{R}^2$ is the collection of all such vectors.
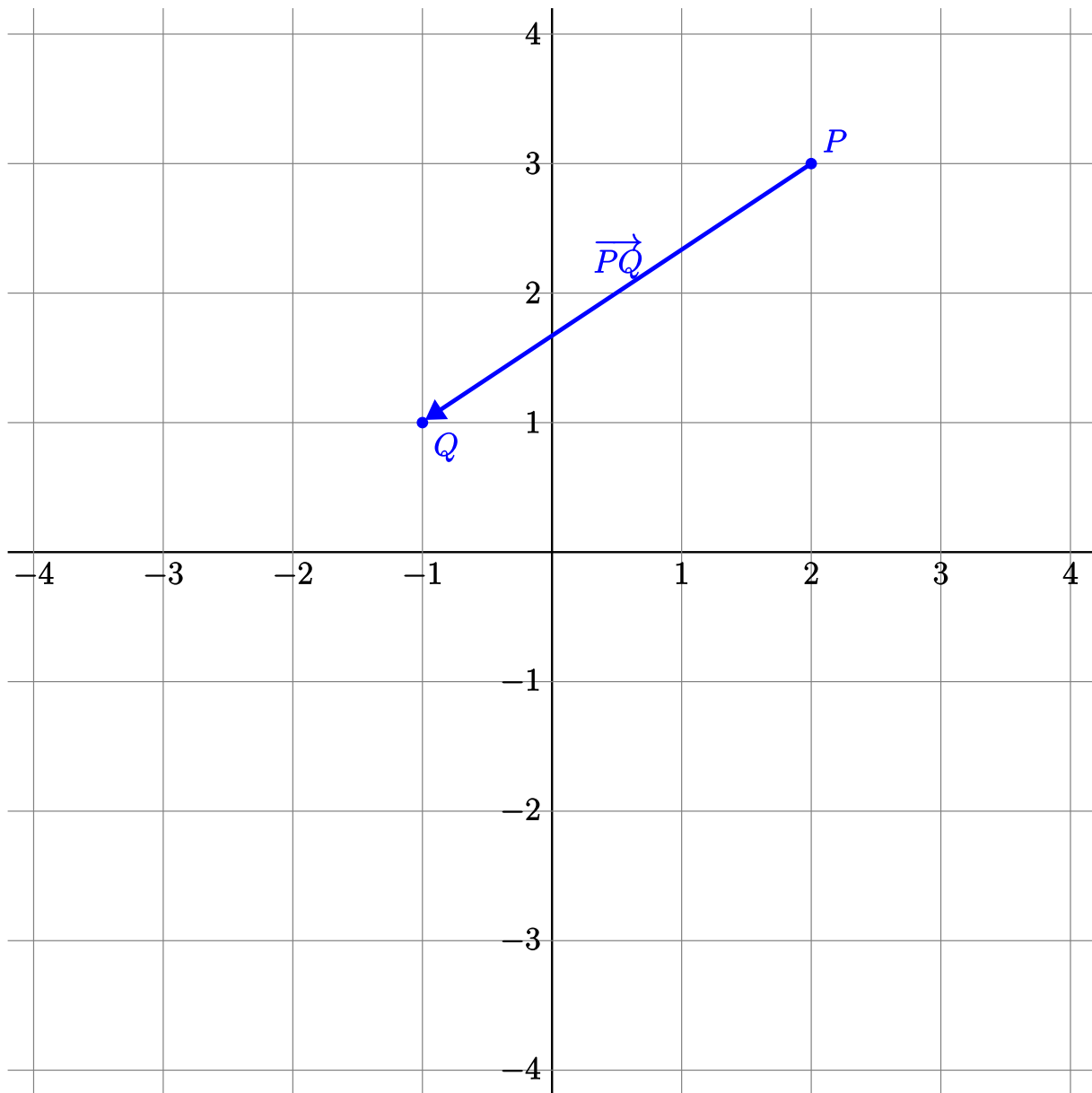
Example 2.1.2.

Here are some examples of vectors in $\mathbb{R}^2$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \pi \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Just like in the ordered pair notation for points, the order of the numbers matters. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is *not* the same vector as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Geometrically, vectors arise as *directed line segments* describing how to get from one point in the plane to another. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points in the plane then the vector $\overrightarrow{PQ}$ is the vector $\overrightarrow{PQ} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$.

Example 2.1.3.

Let $P = (2, 3)$ and $Q = (-1, 1)$. Then $\overrightarrow{PQ} = \begin{bmatrix} -1 - 2 \\ 1 - 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$. Geometrically, we have the following picture:
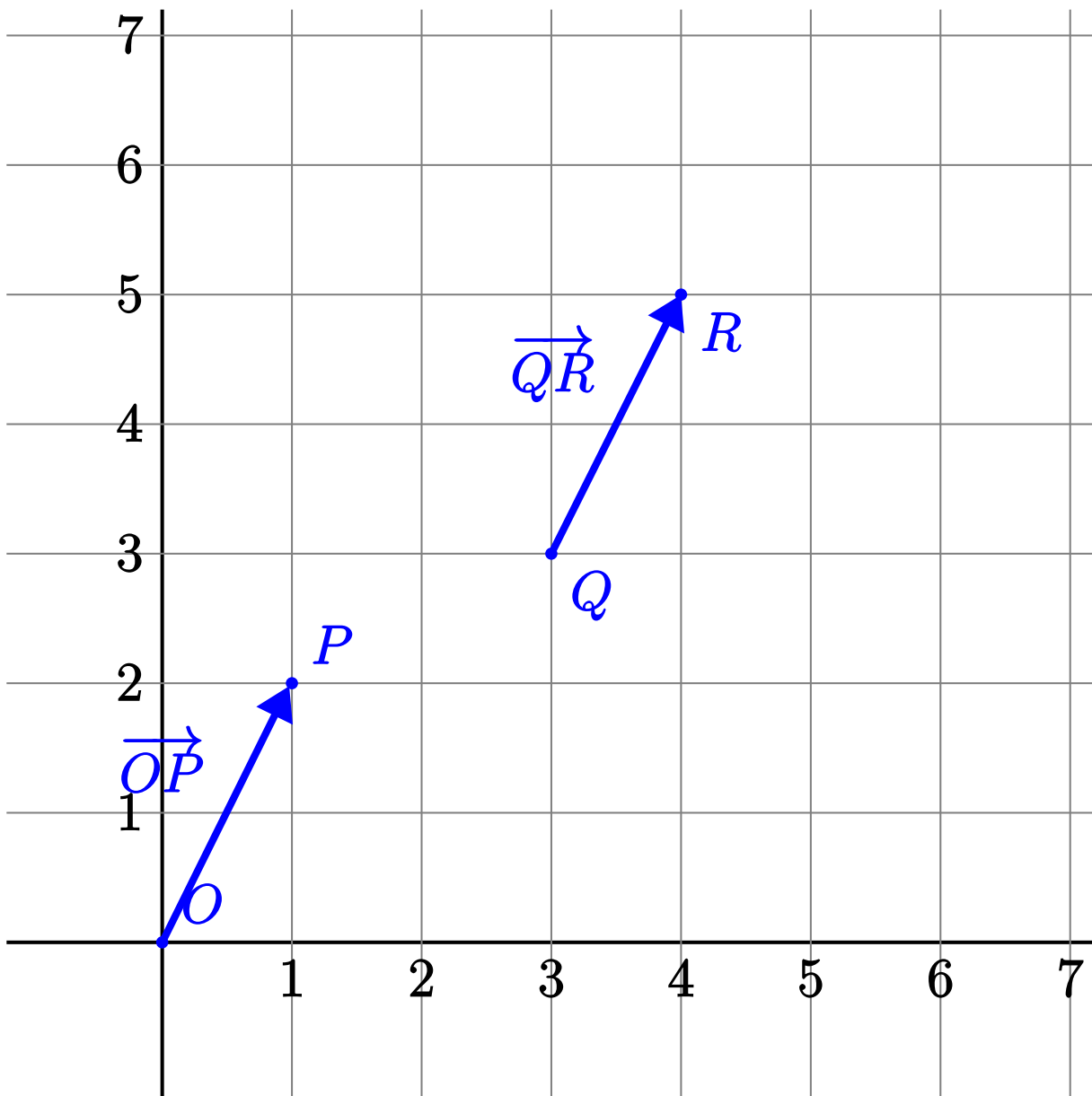
Figure 2.1.4. The vector from P to Q.

The next example shows that our algebraic representation of vectors does not determine the starting point or ending point of the geometric representation.

Example 2.1.5.

Let $O = (0,0), P = (1,2), Q = (3,3), R = (4,5)$. Then $\overrightarrow{OP} = \begin{bmatrix} 1-0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and

$\overrightarrow{QR} = \begin{bmatrix} 4-3 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Figure 2.1.6. The vectors $\overrightarrow{OP}$ and $\overrightarrow{QR}$.

The geometric interpretation of two vectors being equal is that they have the same length and point in the same direction (we will define "length" and "direction" carefully in the next section). This means that we can move a vector around without changing which vector it is, as long as we do not change the length or direction. It is helpful to think of vectors as "instructions": Knowing a list of instructions does not tell you where you will end up, only how that place relates to where you started.

Definition 2.1.7. Suppose that $\vec{v}$ is a vector in $\mathbb{R}^2$. We say that $\vec{v}$ is drawn in *standard position* if it is drawn in the plane so that the starting point of the vector is the origin, $(0, 0)$.

2.1.2 Addition and scalar multiplication

Definition 2.1.8. Suppose that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. We define *vector addition* "$\vec{v}$ plus $\vec{w}$" by defining

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

Geometrically, following the vector $\vec{v} + \vec{w}$ amounts to first following $\vec{v}$ and then following $\vec{w}$.

Example 2.1.9.

Let $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. If we begin at the origin $O = (0,0)$ and follow $\vec{v}$ we will end up at $P = (1, -1)$. Then starting from P and following $\vec{w}$ will take us to $Q = (4, 1)$. On the other hand, $\vec{v} + \vec{w} = \begin{bmatrix} 1 + 3 \\ -1 + 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus $\vec{v} + \vec{w} = \overrightarrow{OQ}$.
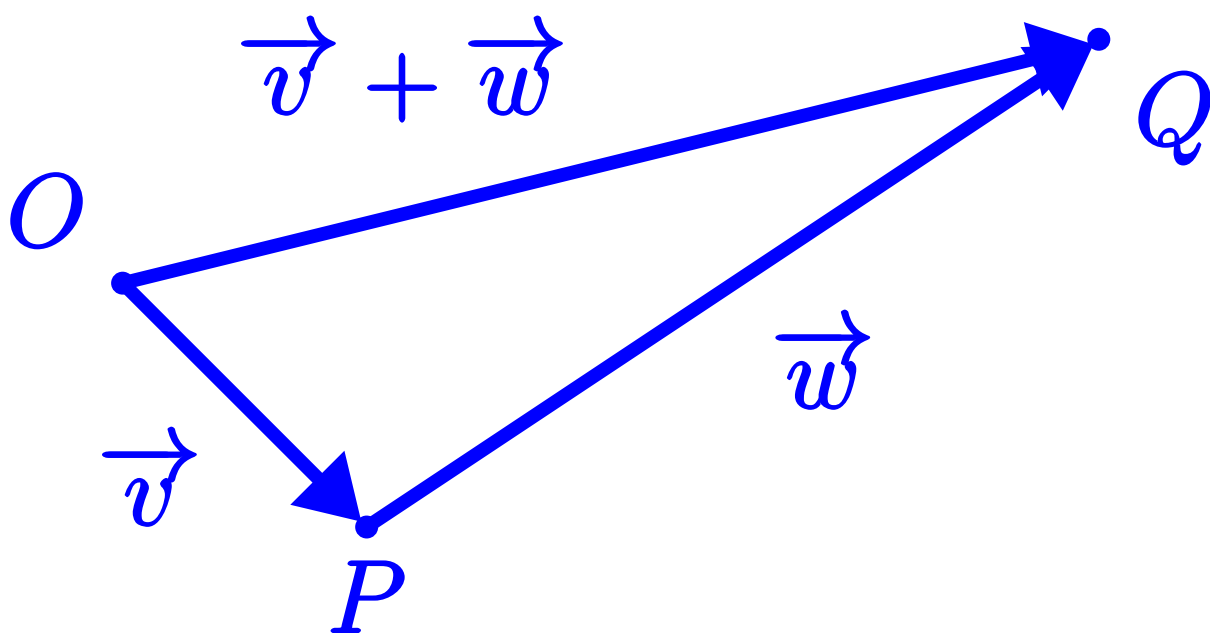
Figure 2.1.10. The vectors $\vec{v}, \vec{w}, \vec{v} + \vec{w}$.

Addition of vectors in $\mathbb{R}^2$ shares many properties with addition of numbers. We list some of the most important properties in the following theorem, but before that we need one more definition.

Definition 2.1.11. The *zero vector* in $\mathbb{R}^2$ is the vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Theorem 2.1.12. *Suppose that $\vec{v}, \vec{w},$ and $\vec{z}$ are vectors in $\mathbb{R}^2$. Then:*

- $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- $(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$
- $\vec{v} + \vec{0} = \vec{v}$

Proof.

We will only prove the second property, as an illustration of what it takes to prove a statement like this. We emphasize that the theorem statement is supposed to apply to any vectors we might have, so a proof needs to handle *all possible choices* for $\vec{v}, \vec{w}, \vec{z},$ so working with a numerical example is not enough. Instead, we assign variables to the entries of our vectors, and thus do a calculation that will work for all possible vectors. Suppose that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. Then we calculate:

$$
\begin{aligned}
(\vec{v} + \vec{w}) + \vec{z} &= \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
&= \begin{bmatrix} (v_1 + w_1) + z_1 \\ (v_2 + w_2) + z_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 + (w_1 + z_1) \\ v_2 + (w_2 + z_2) \end{bmatrix} \\
&= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 + z_1 \\ w_2 + z_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) \\
&= \vec{v} + (\vec{w} + \vec{z})
\end{aligned}
$$

The proofs of the other two statements of the theorem are similar, and are good exercises to make sure you're comfortable with what a proof needs to do.

The second of the two main things we do with vectors is *scalar multiplication*, which allows us to change the length of a vector without changing its direction.

Definition 2.1.13. Suppose that $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a vector in $\mathbb{R}^2$, and r is a scalar

(that is, a real number). We define the *scalar multiplication* "r times $\vec{v}$" by
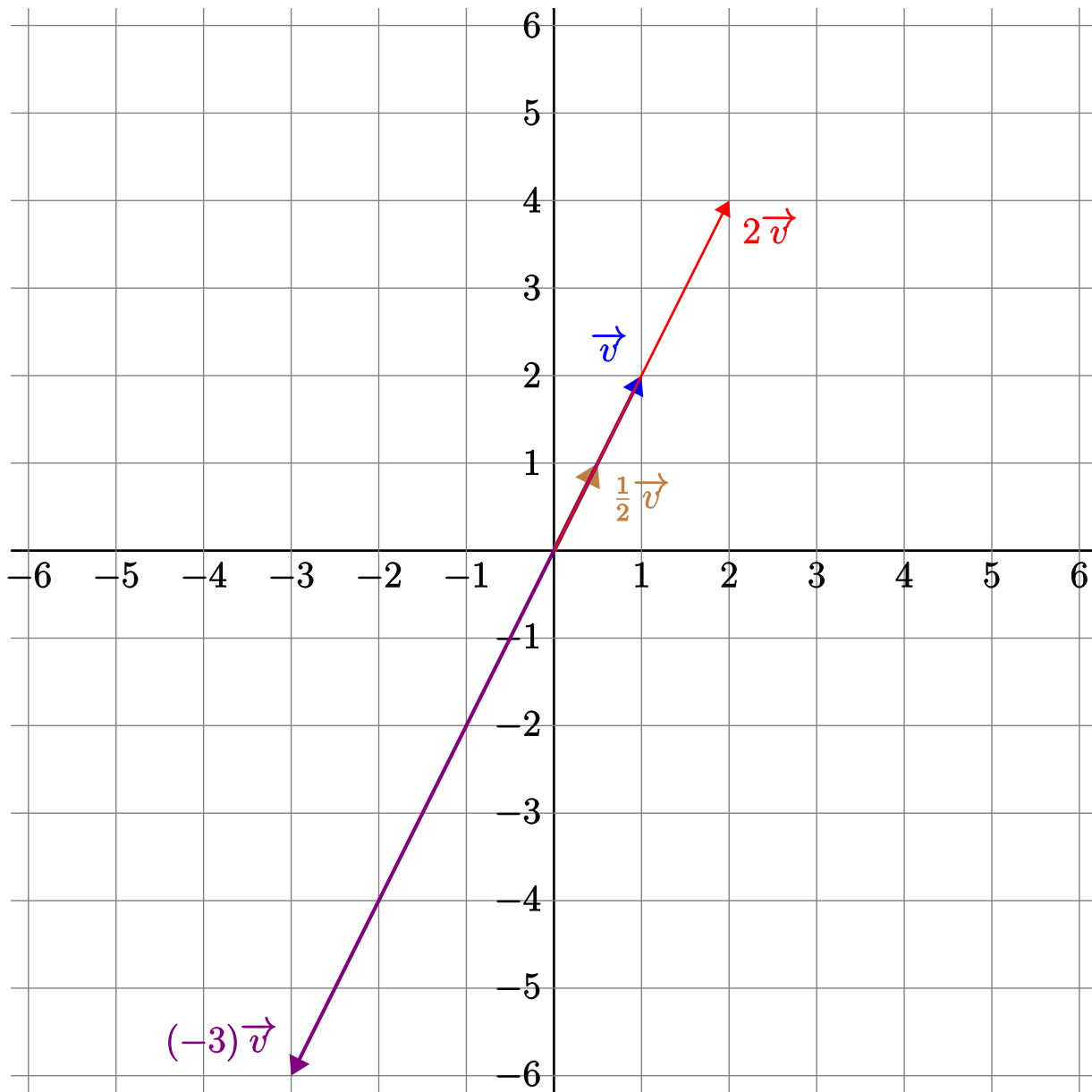defining

$$r\vec{v} = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix}.$$

Definition 2.1.14. If $\vec{v}$ and $\vec{w}$ are vectors, and there is a scalar c such that
$\vec{v} = c\vec{w}$, then we say that $\vec{v}$ is *parallel* to $\vec{w}$.

Example 2.1.15.

Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then:

- $2\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

- $\dfrac{1}{2}\vec{v} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

- $-3\vec{v} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$

- $0\vec{v} = \vec{0}$

Figure 2.1.16. The vectors $\vec{v}$, $2\vec{v}$, $\frac{1}{2}\vec{v}$, and $-3\vec{v}$ drawn in standard position.

Addition and scalar multiplication of vectors interact in the ways that you probably expect based on how addition and multiplication of numbers work. However, we must be a little bit careful! We are *not* defining what it means to multiply a vector by another vector; an expression like $r(\overrightarrow{vw}) = (\overrightarrow{rv})\vec{w}$ isn't even false, it is meaningless.

Theorem 2.1.17. *Suppose that $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^2$, and r and s are scalars. Then:*

- $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$
- $(r + s)(\vec{v}) = r\vec{v} + s\vec{v}$

- $(rs)(\vec{v}) = r(s\vec{v})$
- $0\vec{v} = \vec{0}$
- $r\vec{0} = \vec{0}$

Definition 2.1.18. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, then we define subtraction by

$$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix}.$$

2.1.3 Vectors in $\mathbb{R}^n$

So far we have considered vectors in the plane, $\mathbb{R}^2$. In applications of linear algebra one often needs to work in higher dimensions for a variety of reasons. First, the space around us is three-dimensional, so for physical applications we often work in $\mathbb{R}^3$. Other applications of linear algebra think of a vector as a list of data (rather than as a geometric object), and so may need even higher dimensions. For instance, if you are taking five courses this term, then when the term is over your grades for the term could be listed as a vector with 5 entries (that is, a vector in $\mathbb{R}^5$).

If you look back at the previous section, you will see that the algebraic definitions we gave make perfect sense no matter how many dimensions we are in. Thus, for example, if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ then it still makes sense to

define $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ v_4 + w_4 \end{bmatrix}$. You can also verify that the properties of addition

and scalar multiplication continue to work in any number of dimensions. This will become a recurring theme in the course: We will take geometric ideas from $\mathbb{R}^2$ or $\mathbb{R}^3$, translate them into algebraic language, and then use that to generalize to $\mathbb{R}^n$ for higher values of n.

We will usually state general results for vectors in $\mathbb{R}^n$, leaving n unspecified so that our results can be applied in any number of dimensions.

Note 2.1.19. There are a few subtle details that we should be careful about.

- It does not make sense to combine vectors with different numbers of entries. If $\vec{v}$ is in $\mathbb{R}^4$ and $\vec{w}$ is in $\mathbb{R}^6$ then the expression $\vec{v} + \vec{w}$ does not make sense.

- The zero vector in $\mathbb{R}^2$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, while the zero vector in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and these are different! Much of the time it will be clear from context what the symbol $\vec{0}$ means, but sometimes we will write $\vec{0_n}$ if we need to emphasize that we are talking about the zero vector in $\mathbb{R}^n$.

2.1.4 A first look at linear combinations

Definition 2.1.20. Suppose that $\vec{v_1}, \ldots, \vec{v_k}$ are vectors in $\mathbb{R}^n$. A *linear combination* of $\vec{v_1}, \ldots, \vec{v_k}$ is a vector that can be written in the form

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k},$$

where $c_1, \ldots, c_k$ are scalars.

Example 2.1.21.

The vector $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and

$\vec{v_2} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, because $\vec{w} = 2\vec{v_1} + 3\vec{v_2}$.

Example 2.1.22.

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

because for any $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example 2.1.23.

Which vectors in $\mathbb{R}^3$ can be expressed as linear combinations of

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}?$$

Solution.

2.1.5 Exercises

1. Given the points $P = (2, 0, 4)$ and $Q = (5, -2, 1)$, find $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

▼ Hint. 1   ▼ Hint. 2   ▼ Answer.   ▼ Solution.

The components of $\overrightarrow{PQ}$ are the difference of the respective components of the point Q and of the point P. Therefore,

$$\overrightarrow{PQ} = \begin{bmatrix} 5 - 2 \\ -2 - 0 \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}.$$

Similarly,

$$\overrightarrow{QP} = \begin{bmatrix} 2 - 5 \\ 0 - (-2) \\ 4 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}.$$

Notice that $\overrightarrow{PQ} = (-1)\overrightarrow{QP}$.

$$\overrightarrow{PQ} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \text{ and } \overrightarrow{QP} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}.$$

Be careful with the order: for example, $\overrightarrow{PQ}$ has first component $q_1 - p_1 = 5 - 2 = 3$.

By definition, the components of $\overrightarrow{PQ}$ are the difference of the respective components of the point Q and of the point P.

2. In each case, find a point Q such that $\overrightarrow{PQ}$ has

   i.   the same direction as $\mathbf{v}$;   ii.   the opposite direction of $\mathbf{v}$.

a.
$$P = (-1, 2, 2), \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

b.
$$P = (3, 0, -1), \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

▼ Hint. 1   ▼ Hint. 2   ▼ Hint. 3   ▼ Answer.   ▼ Solution.

First, recall that the vector $\overrightarrow{PQ}$ can be computed as

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}, \text{ where } O \text{ denotes the origin.}$$

Since we are looking for a point Q, we should rephrase this equation:

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}. \tag{2.1.1}$$

Secondly, recall that $\overrightarrow{PQ}$ has the same (resp. the opposite) direction of $\mathbf{v}$ if $\overrightarrow{PQ} = c\mathbf{v}$ for some positive (resp. negative) real number c. Plugging this back into (2.1.1), we see that we would like Q to satisfy

$$\overrightarrow{OQ} = \overrightarrow{OP} + c\mathbf{v}$$

for some positive (resp. negative) number c. Therefore, we should compute $\overrightarrow{OP} + \mathbf{v}$ (resp. $\overrightarrow{OP} - \mathbf{v}$) for the given point P and the given vector $\mathbf{v}$. (Notice that we could also compute $\overrightarrow{OP} + 5\mathbf{v}$, $\overrightarrow{OP} + \pi\mathbf{v}$, or many other things for same direction, but the positive number $+1$ is an easy choice. Similarly, for opposite direction, we could also compute $\overrightarrow{OP} - 5\mathbf{v}$, $\overrightarrow{OP} - \frac{1}{8}\mathbf{v}$, etc. - there are many correct answers.)

First, our solution for $P = (-1, 2, 2)$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$:

same direction: We compute

$$\overrightarrow{OP} + \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+1 \\ 2+3 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}.$$

We have argued before that this should equal $\overrightarrow{OQ}$, so we conclude:

$$Q = (0, 5, 3).$$

Let us check our solution: we have

$$\overrightarrow{PQ} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 - (-1) \\ 5 - 2 \\ 3 - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \mathbf{v},$$

so that $\overrightarrow{PQ}$ indeed has the same direction as $\mathbf{v}$ (since they are equal).

opposite direction : We compute

> **Item 2.1.5.2.ii.** the opposite direction of $\mathbf{v}$.
>
> in-context

$$\overrightarrow{OP} - \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - 1 \\ 2 - 3 \\ 2 - 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

We have argued before that this should equal $\overrightarrow{OQ}$, so we conclude:

$$Q = (-2, -1, 1).$$

Let us check our solution: we have

$$\overrightarrow{PQ} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 - (-1) \\ -1 - 2 \\ 1 - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = (-1)\mathbf{v},$$

so that $\overrightarrow{PQ}$ indeed has the opposite direction as $\mathbf{v}$.

Next, our solution for $P = (3, 0, -1)$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$:

> **Item 2.1.5.2.b.** $P = (3, 0, -1)$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$
>
> in-context

same direction : We compute

Item 2.1.5.2.i. the same direction as $\mathbf{v}$;

in-context

$$\overrightarrow{OP} + \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 0+(-1) \\ -1+3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}.$$

We have argued before that this should equal $\overrightarrow{OQ}$, so we conclude:

$$Q = (5, -1, 2).$$

Let us check our solution: we have

$$\overrightarrow{PQ} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5-3 \\ -1-0 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \mathbf{v},$$

so that $\overrightarrow{PQ}$ indeed has the same direction as $\mathbf{v}$ (since they are equal).

opposite direction : We compute

Item 2.1.5.2.ii. the opposite direction of $\mathbf{v}$.

in-context

$$\overrightarrow{OP} - \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0-(-1) \\ -1-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}.$$

We have argued before that this should equal $\overrightarrow{OQ}$, so we conclude:

$$Q = (1, 1, -4).$$

Let us check our solution: we have

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 1-0 \\ -4-(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = (-1)\mathbf{v},$$

so that $\overrightarrow{PQ}$ indeed has the opposite direction as $\mathbf{v}$.

There are multiple correct answers, for example:

a. *same direction*: $Q = (0, 5, 3)$.

  opposite direction: $Q = (-2, -1, 1)$.

b. _same direction_: $Q = (5, -1, 2)$.

  opposite direction: $Q = (1, 1, -4)$.

For $c = 1$, what does Hint 2 tell you about $\overrightarrow{PQ}$? Can you then use Hint 1 to find Q for Part i?

A vector $\mathbf{w}$ has _the same direction as_ $\mathbf{v}$ if $\mathbf{w} = c\mathbf{v}$ for some positive real number c. It has _the opposite direction_ if $\mathbf{w} = c\mathbf{v}$ for some negative real number c.

The vector $\overrightarrow{PQ}$ can be computed as

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP},$$

where O denotes the origin.

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3. Decide if the two vectors are equal.

a. The vector from $(5, 3)$ to $(6, 2)$ and the vector from $(1, -2)$ to $(1, 1)$.

b. The vector from $(2, 1, 1)$ to $(3, 0, 4)$ and the vector from $(5, 1, 4)$ and $(6, 0, 7)$.

▼ Hint.   ▼ Answer.   ▼ Solution.

a. By definition, the vector $\overrightarrow{v}$ from $(5, 3)$ to $(6, 2)$ is given by

$$\overrightarrow{v} = \begin{bmatrix} 6 - 5 \\ 2 - 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

  Similarly, the vector $\overrightarrow{w}$ from $(1, -2)$ to $(1, 1)$ is given by

$$\overrightarrow{w} = \begin{bmatrix} 1 - 1 \\ 1 - (-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

  Since $\overrightarrow{v}$ and $\overrightarrow{w}$ do not have the same components, we conclude that they are different.

b. By definition, the vector $\overrightarrow{v}$ from $(2, 1, 1)$ to $(3, 0, 4)$ is given by

$$\vec{v} = \begin{bmatrix} 3 - 2 \\ 0 - 1 \\ 4 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Similarly, the vector $\vec{w}$ from $(5, 1, 4)$ and $(6, 0, 7)$ is given by

$$\vec{w} = \begin{bmatrix} 6 - 5 \\ 0 - 1 \\ 7 - 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Since $\vec{v}$ and $\vec{w}$ have the same components, we conclude that they are the same.

a. The two vectors are different.

b. The two vectors are the same.

It might be helpful to look back at Hint 2.1.5.1.1 and Hint 2.1.5.1.2.

Hint 2.1.5.1.2.

Be careful with the order: for example, $\overrightarrow{PQ}$ has first component $q_1 - p_1 = 5 - 2 = 3$.

in-context

Hint 2.1.5.1.1.

By definition, the components of $\overrightarrow{PQ}$ are the difference of the respective components of the point Q and of the point P.

in-context

4. Let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of points P and Q, respectively, and let R be the point whose vector is $\mathbf{p} + \mathbf{q}$. Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$:

a. $\overrightarrow{QP},$

b. $\overrightarrow{QR},$

c. $\overrightarrow{RP},$

d. $\overrightarrow{RO}$ where O is the origin.

▼ Hint.   ▼ Answer.   ▼ Solution.

In the following, let $\mathbf{r}$ be the vector of the point R. In other words,

$$\mathbf{r} = \mathbf{p} + \mathbf{q}$$

by the choice of R.

a. By definition, the components of $\overrightarrow{QP}$ are the difference of the respective components of the point P and of the point Q, or in other words:

$$\overrightarrow{QP} = \mathbf{p} - \mathbf{q}.$$

b. Similarly, we have

$$\overrightarrow{QR} = \mathbf{r} - \mathbf{q} = (\mathbf{p} + \mathbf{q}) - \mathbf{q} = \mathbf{p}.$$

c. Just as above, we have

$$\overrightarrow{RP} = \mathbf{p} - \mathbf{r} = \mathbf{p} - (\mathbf{p} + \mathbf{q}) = -\mathbf{q}.$$

d. If we write $\mathbf{0}$ for the vector of the origin O, all of whose components are zero, then

$$\overrightarrow{RO} = \mathbf{0} - \mathbf{r} = -(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q}.$$

a.

$$\overrightarrow{QP} = \mathbf{p} - \mathbf{q}.$$

b.

$$\overrightarrow{QR} = \mathbf{p}.$$

c.

$$\overrightarrow{RP} = -\mathbf{q}.$$

d.

$$\overrightarrow{RO} = -\mathbf{p} - \mathbf{q}.$$

If $\mathbf{a}$ and $\mathbf{b}$ are the vectors of points A and B, respectively, and if O denotes the origin, then by definition $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

5. Find

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}.$$

▼ Hint.   ▼ Answer.   ▼ Solution.

By definition of vector addition, we have to add the numbers componentwise, so:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 + 1 + (-1) \\ 2 + 5 + 2 \\ 3 + 1 + (-4) \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 0 \end{bmatrix}.$$

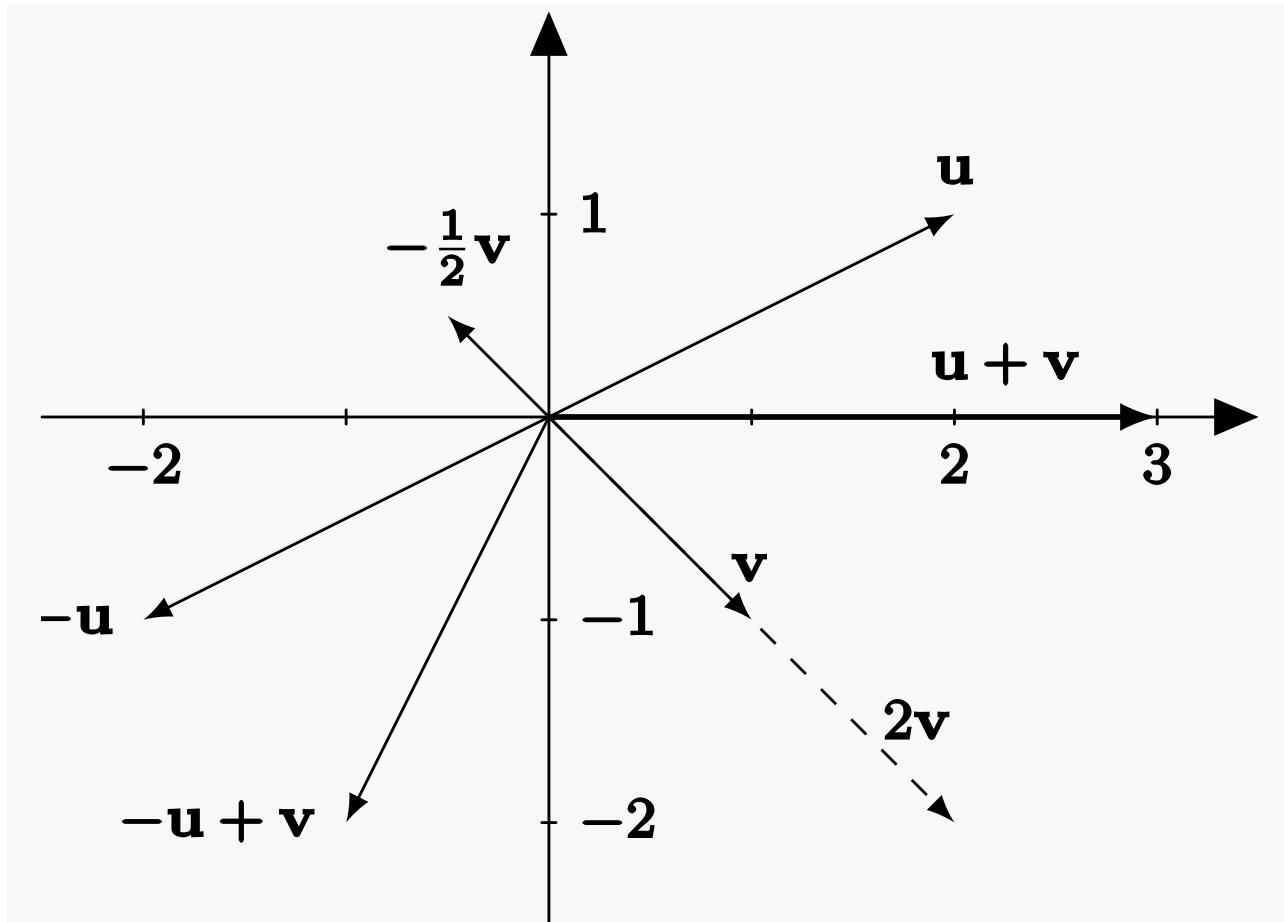Vector addition is defined componentwise. For example,

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 + (-2) \\ 6 + 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

6. Draw the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then draw $-\mathbf{u}, 2\mathbf{v}, -1/2\mathbf{v}, \mathbf{u} + \mathbf{v}$, and $-\mathbf{u} + \mathbf{v}$ on the same set of coordinate axes.
▼ Solution.

We choose to have all seven vectors anchored at the origin.

Figure 2.1.24. A picture of $\mathbf{u}, \mathbf{v}, -\mathbf{u}, 2\mathbf{v}, -1/2\mathbf{v}, \mathbf{u} + \mathbf{v}$, and $-\mathbf{u} + \mathbf{v}$

Notice that $\mathbf{u} + \mathbf{v}$ represents the diagonal of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$, and similarly, $-\mathbf{u} + \mathbf{v}$ represents the diagonal of the parallelogram with sides $-\mathbf{u}$ and $\mathbf{v}$. Furthermore, $-1/2\mathbf{v}$ has the opposite direction of $\mathbf{v}$, and $-\mathbf{u}$ the opposite direction of $\mathbf{v}$.

7. Let $A, B, C, D, E,$ and F be the vertices of a regular hexagon, taken in order. Show that

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}.$$

▼ Hint. 1   ▼ Hint. 2   ▼ Solution.

Denote by X the barycentre of the hexagon. By construction,

$$\overrightarrow{XA} = -\overrightarrow{XD}, \; \overrightarrow{XB} = -\mathbf{e}, \; \overrightarrow{XC} = -\overrightarrow{XF}.$$

Now, by definition $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, where O denotes the origin. If we add and substract $\overrightarrow{OX}$ on the right hand-side, we do not change the equality, so

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} + \overrightarrow{OX} - \overrightarrow{OX} = (\overrightarrow{OB} - \overrightarrow{OX}) - (\overrightarrow{OA} - \overrightarrow{OX}) = \overrightarrow{XB} - \overrightarrow{XA}.$$

For convenience, let us denote $\overrightarrow{XA}, \ldots, \overrightarrow{XF}$ by $\mathbf{a}, \ldots, \mathbf{f}$. We can now rewrite the left-hand side of the equation in question as follows:

$$\begin{aligned}
\overrightarrow{AB} &+ \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} \\
&= (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) + (\mathbf{d} - \mathbf{a}) + (\mathbf{e} - \mathbf{a}) + (\mathbf{f} - \mathbf{a}) \\
&= (\mathbf{b} + \mathbf{d}) + (\mathbf{c} + \mathbf{d}) + (\mathbf{d} + \mathbf{d}) + (\mathbf{e} + \mathbf{d}) + (\mathbf{f} + \mathbf{d}) \qquad \text{since } \mathbf{a} = -\mathbf{d} \\
&= \mathbf{b} + \mathbf{c} + \mathbf{e} + \mathbf{f} + 6\mathbf{d}
\end{aligned}$$

Since $\mathbf{b} = -\mathbf{e}$ and $\mathbf{c} = -\mathbf{f}$, we conclude:

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 6\mathbf{d}.$$

Now, since $\mathbf{a} = -\mathbf{d}$, we can write $2\mathbf{d} = \mathbf{d} - \mathbf{a}$, so that

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3(\mathbf{d} - \mathbf{a}) = 3\overrightarrow{AD},$$

as claimed.

If X denotes the barycentre of the hexagon, what is the relationship between $\overrightarrow{XA}$ and $\overrightarrow{XD}$? What about $\overrightarrow{XB}$ and $\mathbf{e}$?

If X denotes the barycentre of the hexagon, what is the relationship between $\overrightarrow{AB}$, and $\overrightarrow{XA}$ and $\overrightarrow{XB}$?

8. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$. In each case, find $\mathbf{x}$ such that:

    a. $3(2\mathbf{u} + \mathbf{x}) + \mathbf{w} = 2\mathbf{x} - \mathbf{v}$ b. $2(3\mathbf{v} - \mathbf{x}) = 5\mathbf{w} + \mathbf{u} - 3\mathbf{x}$.

▼ Hint.   ▼ Answer.   ▼ Solution.

We will make extensive use of basic properties of vector addition.

a.

$$3(2\mathbf{u} + \mathbf{x}) + \mathbf{w} = 2\mathbf{x} - \mathbf{v}$$

$$\Longleftrightarrow \qquad 6\mathbf{u} + 3\mathbf{x} + \mathbf{w} = 2\mathbf{x} - \mathbf{v}$$

$$\Longleftrightarrow \qquad 6\mathbf{u} + 3\mathbf{x} + \mathbf{w} - 2\mathbf{x} = 2\mathbf{x} - \mathbf{v} - 2\mathbf{x} \qquad (-2\mathbf{x} \text{ on both sides})$$

$$\Longleftrightarrow \qquad 6\mathbf{u} + 1\mathbf{x} + \mathbf{w} = -\mathbf{v}$$

$$\Longleftrightarrow \qquad \mathbf{x} = -\mathbf{v} - 6\mathbf{u} - \mathbf{w}$$

Using that

$$-6\mathbf{u} = (-6)\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-6)3 \\ (-6)(-1) \\ (-6)0 \end{bmatrix} = \begin{bmatrix} -18 \\ 6 \\ 0 \end{bmatrix}$$

and so forth, we can now just plug in the values in the equation for $\mathbf{x}$:

$$\mathbf{x} = -\mathbf{v} - 6\mathbf{u} - \mathbf{w} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} (-6)3 \\ (-6)(-1) \\ 0 \end{bmatrix} + \begin{bmatrix} (-1)(-1) \\ (-1)1 \\ (-1)5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 + (-18) + 1 \\ 0 + 6 + (-1) \\ -1 + 0 + (-5) \end{bmatrix}$$

$$= \begin{bmatrix} -21 \\ 5 \\ -6 \end{bmatrix}$$

b. First, we compute:

$$2(3\mathbf{v} - \mathbf{x}) = 5\mathbf{w} + \mathbf{u} - 3\mathbf{x}$$

$$\Longleftrightarrow \qquad 6\mathbf{v} - 2\mathbf{x} = 5\mathbf{w} + \mathbf{u} - 3\mathbf{x}$$

$$\Longleftrightarrow \qquad -2\mathbf{x} + 3\mathbf{x} = 5\mathbf{w} + \mathbf{u} - 6\mathbf{v}$$

$$\Longleftrightarrow \qquad \mathbf{x} = 5\mathbf{w} + \mathbf{u} - 6\mathbf{v}$$

Plugging in the vectors, we get:

$$\mathbf{x} = 5\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 5 \\ 25 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -24 \\ 0 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 + 3 - 24 \\ 5 - 1 + 0 \\ 25 + 0 - 6 \end{bmatrix} = \begin{bmatrix} -26 \\ 4 \\ 19 \end{bmatrix}$$

a.

$$\mathbf{x} = \begin{bmatrix} -21 \\ 5 \\ -6 \end{bmatrix}$$

b.

$$\mathbf{x} = \begin{bmatrix} -26 \\ 4 \\ 19 \end{bmatrix}$$

Substracting some vector $\mathbf{y}$ from a vector $\mathbf{z}$ is the same as adding the vector $-\mathbf{y}$ to $\mathbf{z}$.

9. Decide whether

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

▾ Hint. 1   ▾ Hint. 2   ▾ Answer.   ▾ Solution.

We want to find real numbers c_1, c_2 such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

We compute the right-hand side:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = c_1 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 3 \\ c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} c_2 2 \\ c_2(-2) \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 3 + c_2 2 \\ c_1 - c_2 2 \\ -c_1 + c_2 \end{bmatrix}.$$

We want this to equal $\mathbf{v}$, i.e. we want

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} c_1 3 + c_2 2 \\ c_1 - c_2 2 \\ -c_1 + c_2 \end{bmatrix}.$$

In other words, we need all of the following to be true:

$$\begin{aligned} 4 &= c_1 3 + c_2 2 \\ 4 &= c_1 - c_2 2 \\ -3 &= -c_1 + c_2 \end{aligned} \tag{2.1.2}$$

The second equation (2.1.2) yields $c_1 = 4 + c_2 2$, which we will plug into the other two equations to get:

$$\begin{aligned} 4 &= (4 + c_2 2)3 + c_2 2 \tag{2.1.3} \\ -3 &= -(4 + c_2 2) + c_2 \tag{2.1.4} \end{aligned}$$

The first equation (2.1.3) yields:

$$4 = (4 + c_2 2)3 + c_2 2 = 12 + c_2 8, \text{ i.e. } c_2 = -1,$$

and the second equation (2.1.4) yields

$$-3 = -(4 + c_2 2) + c_2 = -4 - c_2, \text{ i.e. } c_2 = -1.$$

We got the same value for c_2, so the answer to our question is: Yes, the vector $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

For good measure, let us check that we did not make a mistake. With $c_2 = -1$, we get $c_1 = 4 + c_2 2 = 4 + (-1)2 = 2$. For these values, we compute:

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 = \begin{bmatrix} 2 \cdot 3 + (-1)2 \\ 2 - (-1)2 \\ -2 + (-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} = \mathbf{v},$$

as claimed.

$$\begin{aligned} 4 &= c_1 3 + c_2 2 \\ 4 &= c_1 - c_2 2 \\ -3 &= -c_1 + c_2 \end{aligned} \tag{2.1.2}$$

in-context

$$\begin{aligned} 4 &= (4 + c_2 2)3 + c_2 2 \tag{2.1.3} \\ -3 &= -(4 + c_2 2) + c_2 \tag{2.1.4} \end{aligned}$$

in-context

$$4 = (4 + c_2 2)3 + c_2 2 \tag{2.1.3}$$
$$-3 = -(4 + c_2 2) + c_2 \tag{2.1.4}$$

in-context

Yes, the vector $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. To be precise,

$$\mathbf{v} = 2\mathbf{u}_1 + (-1)\mathbf{u}_2.$$

Two vectors are equal if and only if they have the same components.

$\mathbf{v}$ is a *linear combination* of $\mathbf{u}_1$ and $\mathbf{u}_2$ if and only if there exists real numbers c_1, c_2 such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

10. Decide whether

$$\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\mathbf{u_1} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{u_3} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

▼ Hint. 1   ▼ Hint. 2   ▼ Answer.   ▼ Solution.

We want to find real numbers c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

We compute the right-hand side:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = c_1\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 3 \\ c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} c_2 8 \\ 0 \\ -c_2 \end{bmatrix} + \begin{bmatrix} c_3 2 \\ c_3(-2) \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 3 + c_2 8 + c_3 2 \\ c_1 + 0 - c_3 2 \\ -c_1 - c_2 + c_3 \end{bmatrix}$$

We want this to equal $\mathbf{v}$, i.e. we want

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 3 + c_2 8 + c_3 2 \\ c_1 + 0 - c_3 2 \\ -c_1 - c_2 + c_3 \end{bmatrix}.$$

In other words, we need

$$\begin{aligned} 4 &= c_1 3 + c_2 8 + c_3 2 \\ 4 &= c_1 - c_3 2 \\ 4 &= -c_1 - c_2 + c_3 \end{aligned} \tag{2.1.5}$$

The second equation (2.1.5) yields $c_1 = 4 + c_3 2$. Plugging this into the other two equations, we get:

$$\begin{aligned} 4 &= (4 + c_3 2)3 + c_2 8 + c_3 2 \\ 4 &= -(4 + c_3 2) - c_2 + c_3 \end{aligned} \tag{2.1.6} \tag{2.1.7}$$

The first equation (2.1.6) yields:

$$4 = (4 + c_3 2)3 + c_2 8 + c_3 2 = 12 + c_3 6 + c_2 8 + c_3 2 = 12 + c_2 8 + c_3 8,$$

i.e. $c_2 = -1 - c_3$. Plugging this, in turn, into the last remaining equation (2.1.7) yields:

$$4 = -(4 + c_3 2) - c_2 + c_3 = -4 - c_3 2 - (-1 - c_3) + c_3 = -3.$$

Since this is false, we conclude: No, the vector $\mathbf{v}$ is not a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

No, the vector $\mathbf{v}$ is not a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

Two vectors are equal if and only if they have the same components.

$\mathbf{v}$ is a *linear combination* of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ if and only if there exists real

numbers c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_2.$$

numbers c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_2.$$