SIGMAS
Students in Graduate Mathematics and Statistics

EXAM SALES

Course: MATH 101
Semester: August 2013
Instructor: E. Moore

More Information: www.math.uvic.ca/~sigmas
Email: SIGMAS.exams@gmail.com

UNIVERSITY OF VICTORIA
EXAMINATIONS AUGUST 2013
MATHEMATICS 101 – SECTION [A01-A03]
Calculus II

Name: ____________________ ID No.: ____________________

Section: ____________________

Problem	max points	marks
Multiple Choice	2 x 18 = 36	
#19	4	
#20	4	
#21	4	
#22	4	
#23	4	
Total:	56	

Instructor:
A01 CRN 30460
A02 CRN 30461 } Edward Moore
A03 CRN 30462

DURATION: 3 hours

TO BE ANSWERED ON THE PAPER AND ON N.C.S SHEETS

STUDENTS MUST COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATOR.

THIS QUESTION PAPER HAS 14 PAGES PLUS COVER AND BLUE SHEETS.

1. Find the coefficient of x^4 in the power series centred at 0 for $e^{-x^2/2}$.

(A) -0.4 (B) -0.3 (C) -0.2 (D) -0.1 (E) 0
(F) 0.1 (G) 0.2 (H) 0.3 (I) 0.4 (J) 0.5

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\text{So } e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{(-x^2/2)^2}{2!} + \dots$$

$$= 1 - \frac{x^2}{2} + \boxed{\frac{x^4}{8}} + \dots$$

$$\frac{1}{8} = 0.125$$

2. Compute the arc length of $f(x) = x^{3/2}$ on [0,3].

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5
(F) 6 (G) 7 (H) 8 (I) 9 (J) 10

$$AL = \int_0^3 \sqrt{1 + f'(x)^2}\, dx$$

$$= \int_0^3 \sqrt{1 + 9/4\, x}\, dx$$

$$= 4/9 \int_{u(0)}^{u(3)} \sqrt{u}\, du$$

$$= 4/9 \left[2/3\, u^{3/2} \right]_{u(0)}^{u(3)}$$

$$= 4/9 \left[2/3 (1 + 9/4\, x)^{3/2} \right]_0^3 = 6.0963$$

$f'(x) = \frac{3}{2} x^{1/2}$

$1 + f'(x)^2 = \frac{9}{4} x + 1$

let $u = 1 + 9/4\, x$

$du = 9/4\, dx$

3. Evaluate $\int_1^2 \frac{sin\sqrt{x}}{\sqrt{x}}\,dx$

(A) -0.6 (B) -0.4 (C) -0.2 (D) -0.1 (E) 0
(F) 0.2 (G) 0.4 (H) 0.6 (I) 0.8 (J) 1.0

let $u=\sqrt{x}$
$du=\frac{1}{2\sqrt{x}}dx$

$$\int_{u(1)}^{u(2)} \frac{\sin u \cdot 2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}\,du$$
$$= 2\int_{u(1)}^{u(2)} \sin u\, du$$
$$= -2\cos u \Big|_{u(1)}^{u(2)}$$
$$= -2\cos\sqrt{x}\Big|_1^2$$
$$= -2\cos\sqrt{2} + 2\cos\sqrt{1}$$
$$= 0.769$$

4. Compute $\int_1^2 x^2 \ln x\, dx$

(A) 0 (B) 0.5 (C) 1 (D) 1.5 (E) 2
(F) 2.5 (G) 3 (H) 3.5 (I) 4 (J) 4.5

let $u=\ln x$ and $dv = x^2 dx$
$du = \frac{1}{x}dx$ $v=\frac{x^3}{3}$

$$\int_1^2 x^2 \ln x\, dx = uv - \int v\,du$$
$$= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}dx$$
$$= \frac{8}{3}\ln 2 - \left[\frac{x^3}{9}\right]_1^2$$
$$= 1.070622$$

5. Evaluate the derivative of $\tan^{-1}(\tanh x)$ for $x = 1$.

(A) 0 (B) 0.1 (C) 0.2 (D) 0.3 (E) 0.4
(F) 0.5 (G) 0.6 (H) 0.7 (I) 0.8 (J) 0.9

$$y = \tan^{-1}(\tanh x)$$

$$\frac{dy}{dx} = \frac{1}{1+(\tanh x)^2} \cdot (\operatorname{sech}^2 x)$$

$$= \frac{1}{\cosh^2 x\left(1 + \frac{\sinh^2 x}{\cosh^2 x}\right)}$$

$$= \frac{1}{\cosh^2 x + \sinh^2 x}$$

$$= \frac{1}{\left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{2.381098 + 1.381098} \doteq 0.266$$

6. Suppose a particle moves in a straight line so that its velocity is $v = t^2 - 9t + 14$; find the total distance travelled in the time interval $2 \le t \le 10$.

(A) 10 (B) 20 (C) 30 (D) 40 (E) 50
(F) 60 (G) 70 (H) 80 (I) 90 (J) 100

$$TD = \int_2^{10} |t^2 - 9t + 14|\, dt$$

$$= \int_2^{10} |(t-7)(t-2)|\, dt$$

	2		7	
	+	−		+
$t-7$	−	−		+
$t-2$	−	+		+

$$= \int_2^7 -(t^2 - 9t + 14)\, dt + \int_7^{10} t^2 - 9t + 14\, dt$$

$$= -\frac{t^3}{3} + \frac{9t^2}{2} - 14t \Big|_2^7 + \frac{t^3}{3} - \frac{9t^2}{2} + 14t \Big|_7^{10}$$

$$= \frac{157}{3} \doteq 52.333$$

7. Evaluate $\int_1^2 \frac{x-5}{x^2(x+1)}\,dx$

(A) -1 (B) -0.8 (C) -0.6 (D) -0.4 (E) -0.2
(F) 0 (G) 0.2 (H) 0.4 (I) 0.6 (J) 0.8

want $\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \Leftrightarrow Ax(x+1) + B(x+1) + Cx^2 = x-5$

$x=0$: $B=-5$
$x=-1$: $C=-6$
$x=1$: $A=6$

So $\int_1^2 \frac{x-5}{x^2(x+1)}\,dx$

$= \int_1^2 \frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1}\,dx$

$= 6\ln|x| + \frac{5}{x} - 6\ln|x+1| \Big|_1^2$

$\doteq -0.774$

8. Compute $\int_0^{\pi/4} 2\sin 2\theta \cos 2\theta \, d\theta$

(A) 0 (B) 0.1 (C) 0.2 (D) 0.3 (E) 0.4
(F) 0.5 (G) 0.6 (H) 0.7 (I) 0.8 (J) 0.9

let $u = \sin(2\theta)$
$du = 2\cos 2\theta \, d\theta$

So $\int_{u(0)}^{u(\pi/4)} u\,du$

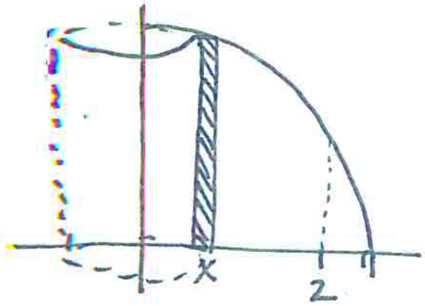
$= \frac{u^2}{2} \Big|_{u(0)}^{u(\pi/4)}$

$= \frac{(\sin 2\theta)^2}{2} \Big|_0^{\pi/4}$

$= \frac{1}{2}$

9. Use the cylindrical shell method to find the volume of the solid obtained by rotating around the y-axis the region between the graph of $f(x) = -x^2 - x + 8$ and the x-axis on [0,2].

(A) 10	(B) 20	(C) 30	(D) 40	(E) 50
$\boxed{\text{(F) 60}}$	(G) 70	(H) 80	(I) 90	(J) 100



$r(x) = x$

$h(x) = -x^2 - x + 8$

$$\therefore V = \int_0^2 2\pi r(x) h(x)\, dx$$

$$= \int_0^2 2\pi (x)(-x^2 - x + 8)\, dx$$

$$= 2\pi \int_0^2 -x^3 - x^2 + 8x\, dx$$

$$= 2\pi \left[-\frac{x^4}{4} - \frac{x^3}{3} + \frac{8x^2}{2} \right]_0^2$$

$$= 2\pi (9.333) = 58.64$$

10. Let $a_n = 3\left(\frac{n+1}{n}\right)^n$; compute $\lim a_n$ as $n \to \infty$.

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4
(F) 5	(G) 6	(H) 7	$\boxed{\text{(I) 8}}$	(J) ∞

Let $\lim_{n\to\infty} a_n = L$, then,

$$\ln L = \lim_{n\to\infty} \ln\left(3\left(\frac{n+1}{n}\right)^n\right)$$

$$= \lim_{n\to\infty} \ln(3) + \ln\left(\frac{n+1}{n}\right)^n$$

$$= \ln(3) + \lim_{n\to\infty} n \ln\left(\frac{n+1}{n}\right)$$

$$= \ln(3) + \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}$$

$$\overset{l'Hop}{=} \ln(3) + \lim_{n\to\infty} \frac{-1/n^2 \cdot \frac{1}{1+1/n}}{-1/n^2}$$

$$\ln L = \ln(3) + \lim_{n\to\infty} \frac{1}{1+1/n}$$

$$= \ln(3) + \frac{1}{1+0}$$

$$= \ln(3) + 1$$

$$\therefore L = e^{\ln(3)+1}$$

$$L = e^{\ln(3)} \cdot e^1$$

$$L = 3e \approx 8.155$$

11. Evaluate $\int_0^{\sqrt{5}} \frac{1}{(x^2+5)^{3/2}}\,dx$

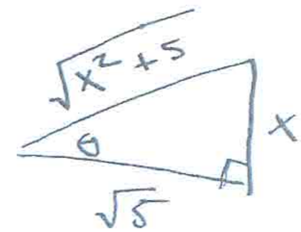
(A) 0 (B) 0.1 (C) 0.2 (D) 0.3 (E) 0.4
(F) 0.5 (G) 0.6 (H) 0.7 (I) 0.8 (J) 0.9

Let $x = \sqrt{5}\tan\theta$ $\quad \tan\theta = \frac{x}{\sqrt{5}}$

$dx = \sqrt{5}\sec^2\theta\, d\theta$

$$\int_{0*}^{\sqrt{5}*} \frac{\sqrt{5}\sec^2\theta\, d\theta}{5^{3/2}(\tan^2\theta+1)^{3/2}}$$

$$= \frac{1}{5}\int_{0*}^{\sqrt{5}*} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta$$



$$= \frac{1}{5}\int_{0*}^{\sqrt{5}*} \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{5}\int_{0*}^{\sqrt{5}*} \cos\theta\, d\theta$$

$$= \frac{1}{5}\sin\theta \Big|_{0*}^{\sqrt{5}*} = \left(\frac{1}{5}\right)\left(\frac{x}{\sqrt{x^2+5}}\right)\Big|_0^{\sqrt{5}} = \frac{\sqrt{5}}{5\sqrt{5+5}} = 0.14142$$

12. The number of bacteria y at time t in a certain culture is growing according to the differential equation $\frac{dy}{dt} = ky$. If 100 bacteria are present initially and 400 are present after 1 hour, how many are present after 3½ hours?

(A) 2,000 (B) 4,000 (C) 6,000 (D) 8,000 (E) 10,000
(F) 12,000 (G) 14,000 (H) 16,000 (I) 18,000 (J) 20,000

$dy/dt = ky$ $\quad$ let $e^c = m$

$\int \frac{1}{y}\,dy = \int k\,dt$ $\quad$ Initial Values $\quad 100 = me^0 = m$

$\ln y = kt + c$ $\quad \& \; 400 = me^k$

$y = e^{kt+c}$ $\quad 400 = 100e^k$

$y = e^c e^{kt}$ $\quad 4 = e^k$

$y = me^{kt}$ $\quad \ln 4 = k$

So $y = 100e^{(\ln 4)t}$

When $t = \frac{7}{2}$: $y = 100e^{\ln 4(7/2)} = 12800$

13. Find $y(1)$ given $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$; $y(0) = 1$ * Note there is an error in this question

(A) 0 (B) 0.5 (C) 1 (D) 1.5 (E) 2
(F) 2.5 (G) 3 (H) 3.5 (I) 4 (J) 4.5

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$$

$$\tan^{-1} y = \tan^{-1} x + C$$

$$y = \tan(\tan^{-1} x + C)$$

Initial Value:

$$1 = \tan(\tan^{-1}(0) + C)$$

$$1 = \tan(C)$$

$$\frac{\pi}{4} = C$$

So $y = \tan(\tan^{-1} x + \pi/4)$

$$y(1) = \tan(\tan^{-1} 1 + \pi/4)$$

$$= \tan(\pi/4 + \pi/4)$$

$$= \tan(\pi/2)$$

$$= \infty$$

which is not one of the options.

(likely question should have asked to find different y-value)

14. Evaluate the improper intergral $\int_0^1 \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) dx$.

(A) 0 (B) 0.1 (C) 0.2 (D) 0.3 (E) 0.4
(F) 0.5 (G) 0.6 (H) 1 (I) 2 (J) DVGT.

$$\int_0^1 \frac{1}{x} - \frac{1}{\sqrt{x}}\,dx$$

$$= \ln|x| - 2x^{1/2} \Big|_0^1$$

$$= \ln 1 - 2(\sqrt{1}) - \left(\lim_{t \to 0} \ln|t| - 2t^{1/2}\right)$$

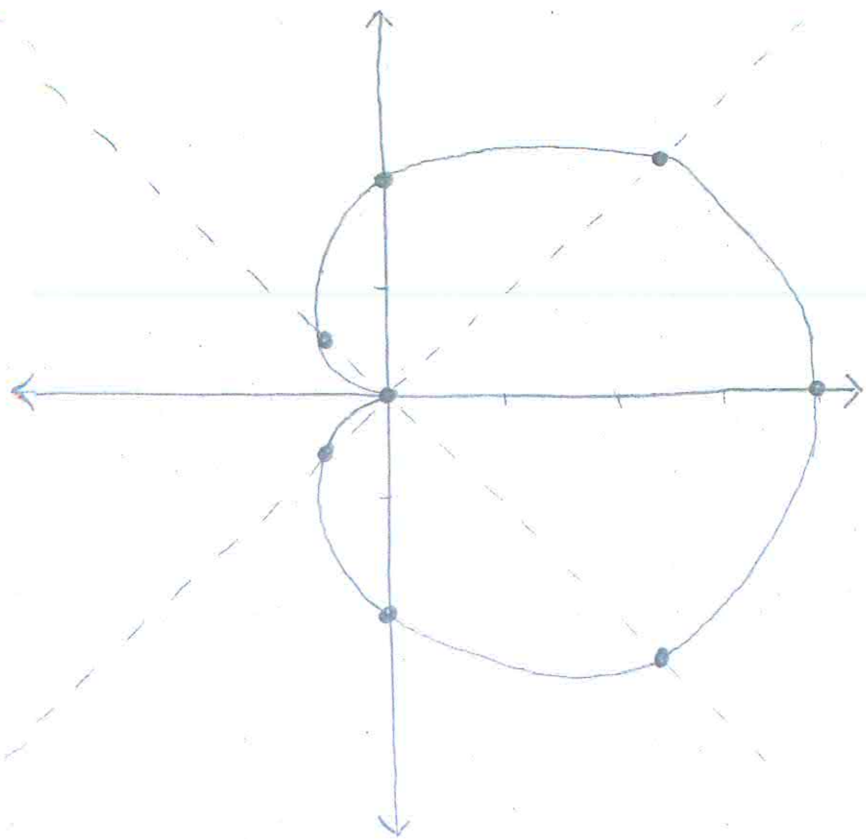
$$= -2 - \lim_{t \to 0} \ln|t|$$

$$= \infty$$

FULL-ANSWER QUESTIONS

19. Consider the polar curve $r = 2 + 2\cos\theta$

(a) Sketch the curve. Indicate the coordinates of at least 3 key points.



θ	r
0	4
$\pi/4$	3.414
$\pi/2$	2
$3\pi/4$	0.586
π	0
$5\pi/4$	0.586
$3\pi/2$	2
$7\pi/4$	3.414
2π	4

(b) Find the area inside the curve

$$A = 2\left[\frac{1}{2}\int_0^{\pi}(2+2\cos\theta)^2\,d\theta\right]$$

$$= \int_0^{\pi} 4 + 4\cos\theta + 4\cos^2\theta\,d\theta$$

$$= 4\theta + 4\sin\theta\Big|_0^{\pi} + 4\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 4\pi + \Big[2\theta + \sin(2\theta)\Big]_0^{\pi}$$

$$= 4\pi + 2\pi$$

$$= 6\pi$$

20. Find the interval of convergence for the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+2)^n}{n}$. Check the endpoints of the interval for convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{\cancel{n+1}}}{n+1}\cdot\frac{n}{\cancel{(x+2)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+2)\cancel{n}}{\cancel{n}(1+1/n)}\right|$$

$$= |x+2| < 1$$

$$-1 < x+2 < 1$$

$$-3 < x < -1$$

Check endpoints:

$\underline{x=-3}$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1+n}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$

which is just $(-1)\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{the harmonic series}}$

so diverges when $x=-3$

$\underline{x=-1}$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which is the alternating harmonic series so converges.

Thus the interval of convergence is $(-3,-1]$

21. Evaluate the sum of the all three roots of the equation $z^3 + 4 = 0$.

$z^3 + 4 = 0$

$\boxed{z^3 = -4}$

$z = (-4)^{1/3}$

$w_k = 4^{1/3}\left(e^{i\pi\left(\frac{2k+1}{3}\right)}\right)$, $k = 0, 1, 2$

so $\underbrace{w_0 = 4^{1/3} e^{i\pi/3}, \quad w_1 = 4^{1/3} e^{i\pi}, \quad w_2 = 4^{1/3} e^{i5\pi/3}}_{\text{these are the 3 roots in exponential form}}$

Convert to rectangular form:

$$4^{1/3} e^{i\pi/3} = 4^{1/3}\left(\cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right)\right)$$
$$= 4^{1/3}\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$$

$$4^{1/3} e^{i\pi} = 4^{1/3}(\cos\pi + i\sin\pi)$$
$$= 4^{1/3}(-1 + i(0))$$
$$= -4^{1/3}$$

$$4^{1/3} e^{i\frac{5\pi}{3}} = 4^{1/3}\left(\cos\left(\tfrac{5\pi}{3}\right) + i\sin\left(\tfrac{5\pi}{3}\right)\right)$$
$$= 4^{1/3}\left(\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)$$

So the sum of the three roots is

$$4^{1/3}\left[\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} - 1 + \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right] = 4^{1/3}(0) = 0$$

22. (a) State the definition of conditional convergence for a series $\sum_{n=1}^{\infty} a_n$.

A series $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

(b) Determine if the series $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n)!}$ is absolutely convergent, conditionally convergent, or divergent (or any combination of these).

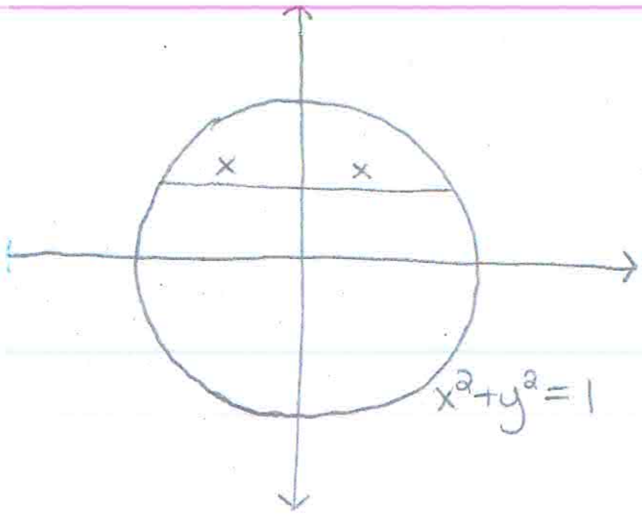
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{3^{\cancel{n+1}}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{3^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3}{(2n+2)(2n+1)} \right| = 0 < 1$$

Thus by the ratio test, the series converges absolutely.

23. The base of a solid is the region enclosed by the circle $x^2 + y^2 = 1$. Find the volume of the solid given that each cross-section perpendicular to the x-axis is an isosceles triangle with its base in the region and altitude equal to one-half of the base.



Area of a triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2x)(x) = x^2$$

$$b = 2x$$

$$h = \frac{1}{2}b = \frac{1}{2}(2x) = x$$

Since $x^2+y^2=1$

$$x = \sqrt{1-y^2}$$

So $A = (\sqrt{1-y^2})^2 = 1-y^2$

$$V = 2\cdot\int_0^1 1-y^2\,dy = 2\left(y - y^3/3\right)\Big|_0^1 = 2\left[1-\tfrac{1}{3}\right] = \frac{4}{3}$$