Direct Comparison Test

Let $f(x)$ and $g(x)$ be functions that are both continuous and satisfying $0 \le f(x) \le g(x)$ on $x \ge a$. Then if...

- $\displaystyle\int_a^\infty g(x)dx$ converges THEN $\displaystyle\int_a^\infty f(x)dx$ converges.
- $\displaystyle\int_a^\infty f(x)dx$ diverges THEN $\displaystyle\int_a^\infty g(x)dx$

and the integrals satisfy the inequality

$$0 \le \int_a^\infty f(x)dx \le \int_a^\infty g(x)dx$$

Note: **The nature and position of the bounding functions is vital**. *If you want to show an integral is divergent, you need to bound it below by a divergent integral. If you want to show an integral is convergent, you need to bound it above by a convergent integral. Bounding an integral below by a convergent one does nothing. Bounding an integral above by a divergent one also does nothing.*

Steps for Direct Comparison

① Formulate another integrand to compare to.

② Justify and form the inequality over the region of integration.

③ Justify why the integral of the comparing function, over the same region converges or diverges.

④ State the theorem you are using.

⑤ Conclude the proper result based on your stated theorem.

Example: Consider the integral $\displaystyle\int_1^\infty \frac{|\sin(x)|+1}{x}dx$.

1. Sine is a bounded function, so the above integral looks like one of the form. $\displaystyle\int_1^\infty \frac{C}{x}dx$, which is a divergent p-integral where $p = 1$. We will thus attempt to bound it below.

2. Since $-1 \le \sin(x) \le 1$ then $0 \le |\sin(x)| \le 1 \Rightarrow 1 \le |\sin(x)| + 1 \le 2$ and we obtain

$$\frac{1}{x} \le \frac{|\sin(x)|+1}{x}$$

on the interval $x \geq 1$.

3. Since $\displaystyle\int_1^\infty \frac{dx}{x}$ is a p-integral where $p = 1 \leq 1$ then it diverges.

4. We have bounded the original integral below by $\displaystyle\int_1^\infty \frac{dx}{x}$ and invoke the Direct Comparison Test.

5. By the direct comparison test, as the original integral is bounded below by a divergent integral, it must diverge as well ■

Assuming this is out of five marks, since there are five main steps, to receive full marks on a test, this is the **LEAST** amount of writing you would need to show on an exam:

Since $-1 \leq \sin(x) \leq 1 \Rightarrow 0 \leq |\sin(x)| \leq 1$

$\Rightarrow 1 \leq |\sin(x)| + 1$ on $x \geq 1$.

So $\dfrac{1}{x} \leq \dfrac{|\sin(x)|+1}{x}$ on $x \geq 1$

Note that $\displaystyle\int_1^\infty \frac{dx}{x}$ is a p-integral with $p = 1 \leq 1$ thus diverges.

5/5

$\therefore$ By the DCT, since $\displaystyle\int_1^\infty \frac{|\sin(x)|+1}{x}\,dx$ is bounded below by a divergent integral, it diverges as well ■

If you write any less than this, you are willingly accepting a mark or a few off.

For comparison, below is an example from a previously marked math 101 exam:

$$\int_1^\infty \frac{dx}{x} \leq \int_1^\infty \frac{|\sin(x)|+1}{x}\, dx$$

why?

0.5/5

$$\therefore \int_1^\infty \frac{dx}{x} \text{ diverges}$$

why?

Theorem? Conditions?

$$\therefore \int_1^\infty \frac{|\sin(x)|+1}{x}\, dx \text{ diverges}$$

This response is riddled with problems. A reminder that you aren't being graded on your answer, but the justification and consistency of your argument. Here's a list of all things wrong:

1. The didn't justify the inequality, so it doesn't count. Otherwise there is no way to separate this from every other argument where a student makes up a random bound that doesn't work. It is not on the marker to make sure what you're saying is correct, it's on you to give a straightforward, non-debatable argument.
2. They stated the comparing integral diverges, but didn't state why. Therefore it doesn't count. I could make up any integral converges or diverges, and we should treat this the same regardless of whether it is true or not.
3. They concluded the original integral diverges, but they didn't state the theorem that supports it, they didn't say how the conditions of said theorem are satisfied, and they didn't show how this follows from their setup. It gets no marks.

Student: **"Wait! So if they did everything wrong in their argument, why even give them a half mark?"**

Instructor: *True that it is debatable whether to give them 0/5 or 0.5/5, but it's because they intuitively knew to bound it below by something. So, it shows some level of knowing where to start.*

Here's another comparison to a real answer from a previous term that a student provided that had an incorrect conclusion, but gets a higher mark than the previous:

Since $-1 \le \sin(x) \le 1 \Rightarrow 0 \le |\sin(x)| \le 1$

$\Rightarrow 1 \le |\sin(x)| + 1 \le 2$ 1/2

So $\dfrac{|\sin(x)|+1}{x} \le \dfrac{2}{x}$ 1/2 × on what interval? 3/5

as $\int_1^\infty \frac{2}{x}\,dx$ converges since it is a p-integral with $p = 1 \geq 1$ × should be $p>1$ does not converge 1/2

then by DCT ✓ $\int_1^\infty \frac{|\sin(x)|+1}{x}\,dx$ converges why? state it is bounded above 1/2

You can see that the argument is not as simple as "The answer is 5". You can write down everything true, but receive a 0/5. Compared to the above where the answer is wrong, but the argument is consistent, invokes the right theorem, and understands how to compare. Point being...

> You should not compare the marking scheme and answering of these questions to any other type of calculation argument you have done in the past. This is brand new and catches all new students off guard.

Unless a result is properly justified, it doesn't count. Otherwise I could make up anything I want, like the incorrect inequality:

$$\frac{|\cos(x)| + x^2}{x^7} \le \frac{1}{x^6}$$

on $x \ge 1$ to state that since $\displaystyle\int_1^\infty \frac{1}{x^6}dx$ is a p-integral with $p = 6 > 1$ then the original integral converges by DCT. While it is true that the original integral does converge, the inequality is incorrect and the justification is absent. There should be no reason to accept an argument based on this. Just because an inequality is correct or statement is correct, doesn't mean it's counted. This is true for any statement that isn't justified properly. The response "But it's true!" doesn't mean anything.

In other words, the truth is only the truth in context. Not by itself.