MATH 100, Fall, 2021
Tutorial #8
Story Problems and Newton's method

Q1 A printed page contains 24 in^2 of text with 1 in blank side margins and 1.5 inch top and bottom margins each. Find the dimensions of the page with smallest area that satisfies these constraints. In your group, make a sketch of the page and apply labels x and y to width and height of the printed area respectively. Convince yourself that the problem becomes: minimize $A = (x + 2)(y + 3)$ where $xy = 24$.

Assignment:

1. Formulate the problem as a single variable optimization problem (in terms of x or y, but not both) and then solve the problem via calculus to find the optimal page dimension. Explain how you know your solution is a minimum. State your conclusion in a grammatically correct sentence that tells the printer what size paper to order.

Q2 A business manufactures and sells $x \geq 0$ tons of a product each week with revenue $r(x)$ and operating costs $c(x)$ measured in thousands of \$. Its weekly profit at production level x is therefore $P(x) = r(x) - c(x)$. In your group, discuss the economist's rule-of-thumb that says the company's profit is maximized at output x_* when $r'(x_*) = c'(x_*)$ and interpret this in terms of a critical point for the profit function P.

Assignment:

1. Suppose that $r(x) = 50\sqrt{x}$ and that $c(x) = 10 + 6x^{3/2}$. Find the output x_* that maximizes profit for the firm. Summarize your findings in a grammatically correct sentence that can be understood by the president of the company.

Final thought: is there a way to be sure you have found a maximizer, and not a minimizer? The president might ask.

Q3 In your group, discuss how the function $g(x) = x^{12} - 2$ can be used to approximate the number $2^{\frac{1}{12}}$ by Newton's method.

Assignment:

1. If your current estimate of the root is x_n find Newton's formula for computing the next approximation x_{n+1}.
2. Set $x_0 = 1$, and derived the approximations $x_1, x_2 \ldots$ until your estimate agrees with the first 3 decimals of your calculator's estimate for $\sqrt[12]{2}$. Report back: calculator $= \ldots$; Newton after n steps$= \ldots$. Tell us what n you stopped at.

$\sqrt[12]{2}$ turns out to be an important number in music theory, so having a decimal approximation is actually a helpful thing to know: (Eg, see `https://en.wikipedia.org/wiki/Twelfth_root_of_two`)

Q4 As many of you have realized, Newton's method does not always work first time. Consider the function $f(x) = x^3 - 2x + 2$. Discuss in your group how you **know** that f has a root and set up a Newton iteration formula to compute its approximation x_{n+1} from the estimate x_n.

Assignment: Let $x_0 = 1$. Compute the Newton iteration x_1. Now compute x_2. What do you notice happening?

Q5 Consider the same function $f(x) = x^3 - 2x + 2$ from Q4. Despite your experience in Q4, discuss again why you are sure there must be a root for f. Can you find it?

Assignment: Set $x_0 = -2$. Compute x_1 and x_2 along with $f(x_1)$ and $f(x_2)$. Does the method now seem to be working?