**Math 110 (A01, A02, A03)**
**Test 2**
**Version: A**
**October 25, 2019**
Time: 45 minutes

Student ID: V00 ______________ Family (Last) Name: ______________

Given (First) Name: ______________

**Tutorial sections (check one):**

- ☐ T01 (Jaimes Joschko, 2:30, CLE A127)
- ☐ T02 (MacKenzie Carr, 2:30, CLE A308)
- ☐ T03 (Jacob Nagrocki, 2:30, HHB 110)
- ☐ T04 (Jacob Nagrocki, 3:30, HHB 110)
- ☐ T05 (Jaimes Joschko, 3:30, CLE C112)
- ☐ T06 (MacKenzie Carr, 3:30, CLE A203)
- ☐ T07 (Jaimes Joschko, 4:30, CLE C112)
- ☐ T08 (Jacob Nagrocki, 4:30, HHB 110)
- ☐ T12 (MacKenzie Carr, 4:30, CLE A203)

| Question(s) | Value | Score |
|---|---|---|
| Question 1 | 1 | |
| Question 2 | 1 | |
| Question 3 | 1 | |
| Question 4 | 1 | |
| Question 5 | 4 | |
| Question 6 | 4 | |
| Question 7 | 4 | |
| Question 8 | 4 | |
| **Total** | **20** | |

<u>Instructions:</u>

1. Identifying information:
   (a) Enter your Student ID and name at the top of this page now.
   (b) Select your tutorial section above now.
2. Only the following materials are permitted:
   (a) Pens, pencils, erasers, and a ruler are permitted at your desk. If you have a pencil case it must be stored with your belongings in the front of the room.
   (b) You may use a Sharp calculator with a model number beginning with EL510-R. No other calculators are acceptable on this examination.
3. No notes, outside paper, or aid other than the ones listed above is permitted. You are responsible for ensuring that any unauthorized material is stored with your belongings at the front of the room.
4. Show all calculations on this paper for all problems. We may disallow any answer given without appropriate justification.
5. If you require extra space, use the backs of test pages or the blank page at the end of the test. Be sure to clearly indicate where you have done so. No outside paper is permitted.
6. If you need to leave the room during the test, raise your hand until an invigilator comes to you.
7. This test has 8 pages, including this cover and the blank page at the end.

For questions 1–4, enter your final answer in the box provided. You must show your work to be given credit, even if your answer is correct.
**Leave all answers in exact form** - do not give decimal approximations. If it is impossible to answer a question using the given information, write "NA" in the box.

(1 point) 1. Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 7 & -3 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & -1 & -1 \\ 3 & 3 & 1 \end{bmatrix}$. Find the $(2,1)$ entry of the matrix $A + 2B$.

**Solution:** The $(2,1)$ entry of $A$ is $-1$, and the $(2,1)$ entry of $B$ is 3, so the $(2,1)$ entry of $A + 2B$ is $-1 + 2 \cdot 3 = 5$.

Answer: 5

(1 point) 2. Suppose that $A$ is a $5 \times 3$ matrix, and $B$ is a matrix such that $AB$ is defined. How many rows does $B$ have?

**Solution:** For $AB$ to be defined when $A$ is $5 \times 3$, $B$ must be $3 \times k$ for some $k$, so $B$ has 3 rows.

Answer: 3

(1 point) 3. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and that
$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\5\end{bmatrix}$, and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$.

Find $T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)$.

**Solution:**

$$\begin{aligned}
T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\
&= T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\
&= \begin{bmatrix}1\\-3\end{bmatrix} + 2\begin{bmatrix}-2\\5\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} \\
&= \begin{bmatrix}-2\\8\end{bmatrix}
\end{aligned}$$

Answer: $\begin{bmatrix}-2\\8\end{bmatrix}$

(1 point) 4. Let $A = \begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}$ and $B = \begin{bmatrix}2 & -1\\-1 & 5\end{bmatrix}$. Calculate $AB$.

**Solution:** Using the algorithm learned from class,

$$AB = \begin{bmatrix}1\cdot 2 + 3\cdot(-1) & 1\cdot(-1) + 3\cdot 5\\(-2)\cdot 2 + 0\cdot(-1) & (-2)\cdot(-1) + 0\cdot 5\end{bmatrix} = \begin{bmatrix}-1 & 14\\-4 & 2\end{bmatrix}.$$

Answer: $\begin{bmatrix}-1 & 14\\-4 & 2\end{bmatrix}$

(4 points) 5. Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$. Determine whether or not $C$ is invertible. If $C$ is invertible, calculate $C^{-1}$. If $C$ is not invertible, explain why not.

**Solution:** We set up the matrix $[C|I_3]$ and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

We see that the reduced row echelon form of $C$ is not $I_3$, and therefore $C$ is not invertible.

It is also possible to show that $C$ is not invertible in other ways (for instance, you could show that $\det(C) = 0$, or you could show that $C\mathbf{v} = \mathbf{0}$ has non-trivial solutions, or...).

(4 points) 6. Find all values of $a$ for which the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ 1 \\ -a \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2a \\ 3a+1 \end{bmatrix}$ are linearly independent.

**Solution:** As we know from class, these vectors will be linearly independent exactly when the homogeneous linear system with these vectors as columns has only one solution. So we row reduce:

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 2a & 0 \\ 0 & -a & 3a+1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1-2a^2 & 0 \\ 0 & 1 & 2a & 0 \\ 0 & 0 & 2a^2+3a+1 & 0 \end{array}\right].$$

From here we see that the system will have a unique solution as long as $2a^2+3a+1 \neq 0$, that is, as long as $a \neq -1$ and $a \neq -1/2$. Therefore the given vectors are linearly independent for all values of $a$ except for $a = -1$ and $a = -1/2$.

(4 points) 7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ -3x+y \end{bmatrix}$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $[S] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Calculate $(T \circ S)\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right)$.

**Solution:** There are several ways to complete this problem. We could, for instance, use the matrix form of $S$ to find a formula for $S$, and then compose the two formulas. Here is another solution.

We first observe that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Therefore $[T] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$. We therefore have

$$[T \circ S] = [T][S] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 0 & -2 \end{bmatrix}.$$

Finally,

$$(T \circ S)\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = [T \circ S]\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 71 \\ -10 \end{bmatrix}.$$

(4 points) 8. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$. Show that $\mathbf{v}$ is in $\mathrm{span}(2\mathbf{v}+\mathbf{w}, \mathbf{v}-\mathbf{w})$.

**Solution:** We want to show that there are scalars $a, b$ such that

$$\mathbf{v} = a(2\mathbf{v}+\mathbf{w}) + b(\mathbf{v}-\mathbf{w}).$$

Rearranging, we obtain

$$\mathbf{v} = (2a+b)\mathbf{v} + (a-b)\mathbf{w}.$$

We therefore would like to arrange for $2a+b=1$ and $a-b=0$. You might be able to see by inspection that $a=b=\frac{1}{3}$ works. If you don't see that immediately, we can find it by row reducing:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right].$$

What we have shown is that

$$\mathbf{v} = \frac{1}{3}(2\mathbf{v}+\mathbf{w}) + \frac{1}{3}(\mathbf{v}-\mathbf{w}).$$

Thus $\mathbf{v}$ is a linear combination of $2\mathbf{v}+\mathbf{w}, \mathbf{v}-\mathbf{w}$, and so $\mathbf{v}$ is in $\mathrm{span}(2\mathbf{v}+\mathbf{w}, \mathbf{v}-\mathbf{w})$.

[Blank - Do not remove]

[END]