Math 110 - Final Exam

December 2021

To receive credit for any question you must show all of your work, and must also write enough explanation so that the person reading your work can follow the logic you applied in deciding what steps to perform. Correct final answers given without justification will receive no credit.

You may use a computer to calculate RREFs of matrices, multiply matrices, and factor polynomials. **All** other calculations are to be done by hand, with steps shown. Clearly indicate any places where you used a computer to assist you.

BEFORE YOU BEGIN: Write the following with your information filled in, sign beneath it, and upload to the "Pledge" question on Crowdmark:

"I (*name, V-number*) affirm that I will not give or receive any aid on this exam, and that all work I submit will be my own. I have read and understood the exam instructions posted on Brightspace."

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1. (2 points) Find $\cos(\theta)$, where θ is the angle between the vector $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and the normal vector of the plane $2x + 3y - z = 5$.

2. (2 points) Determine whether or not $\begin{bmatrix}1\\1\\4\end{bmatrix}$ is in span $\left(\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

3. (2 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\2\\-3\end{bmatrix}$. Find $T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$ or explain why it is impossible to do so with the given information.

4. (2 points) Let $A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$. Find a matrix B such that $\det(AB) = 50$ and explain how you know that your matrix works.

5. (2 points) Suppose that A and B are invertible $n \times n$ matrices, and that X is an $n \times n$ matrix such that $AXA = BA + I_n$. Write an equation that expresses X in terms of A, B, A^{-1}, B^{-1}, and I_n (you might not need to use all of these matrices in your equation).

6. (2 points) Let $A = \begin{bmatrix}2 & 1 & 1\\2 & 2 & -1\\2 & 1 & 1\end{bmatrix}$. Find the dimension of $\text{null}(A)^{\perp}$.

7. (4 points) Let P be the plane in $\mathbb{R}^3$ that passes through $(1, 2, 1)$, $(0, -2, 1)$, and $(1, 2, 3)$. Does P also pass through $(0, 0, 0)$? Justify your answer. Do not use the cross product in your solution.

8. (4 points) Let W be the collection of vectors in $\mathbb{R}^4$ defined by saying that $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is in W if and only if $z = 2x - 3y$ and $w = 3x + z$. Is W a subspace of $\mathbb{R}^4$? If it is, prove it. If not, explain why not.

9. (4 points) Find a formula for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix},$$

and then prove that your formula is actually a linear transformation.

10. (4 points) Let $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -2 & -1 \end{bmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $B = PDP^{-1}$, or explain why it is impossible to do so. If you find matrices P and D you must also calculate P^{-1}.

11. (4 points) Let $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}\right)$.
    (a) Find a basis for $W^{\perp}$.
    (b) Is the basis you found in part (a) an orthogonal basis? Explain.

12. (5 points) Determine whether each of the following statements are true or false. For each question provide a **brief** explanation of your answer.
    (a) If $\text{RREF}(A) = \text{RREF}(B)$ then $\det(A) = \det(B)$.
    (b) If A is a matrix such that $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$ are in $E_A(-6)$ then $\begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$ is an eigenvector of A.
    (c) Every 4×4 matrix with characteristic polynomial $\lambda(\lambda - 3)(\lambda + 2)(\lambda - 1)$ is diagonalizable.
    (d) There is an orthogonal matrix Q such that $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} Q^t$.
    (e) There is a non-zero square matrix A such that the columns of A are linearly dependent and $\det(A) = \operatorname{rank}(A)$.

13. (4 points) Suppose that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent vectors in $\mathbb{R}^n$. Prove that

$$\dim(\operatorname{span}(\vec{v_1} - \vec{v_2}, \vec{v_1} - \vec{v_3}, \vec{v_2} - \vec{v_3})) = 2.$$

Note: In this question we want a general proof, so you should not pick a specific value for n and you should not pick specific vectors for $\vec{v_1}, \vec{v_2}, \vec{v_3}$.