UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2021 (Adapted)

MATH 122: Logic and Foundations

CRN: 12133 (A01), 12135 (A03), 12136 (A04), 12137 (A05)

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Duration: 3 Hours.

Answers are to be written on the exam paper. Please write within the page borders.
No calculator is necessary, but a Sharp EL-510R (plus some letters) calculator is allowed.

This exam consists of 20 questions for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed. Question 1 consists of 28 true/false questions labelled TF 1 to TF 28 to be answered on the bubble sheet at the end of the booklet.

There are 12 numbered pages, not including covers, and one bubble sheet.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

Questions marked with an asterisk (*) have been adapted to suit the course coverage of the Spring 2022 semester.

Do your rough work here. Nothing written here will be marked.

Your T/F questions will be formatted like your quizzes – they will be scattered throughout the test in groups of four questions, and you will answer directly on the paper (not on a bubble sheet).

1. [worth 14 points] Use the **bubble sheet** provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**. No justification is necessary.

p	q	$\neg p \wedge \neg q$ (A)	$p \to q$ (B)	$A \to B$
0	0	1	1	1
0	1	0	1	1
1	0	0	0	1
1	1	0	1	1

T **[TF 1]** $\neg p \wedge \neg q$ logically implies $p \to q$.

$\neg p \wedge \neg q \Rightarrow p \to q$ means that $(\neg p \wedge \neg q) \to (p \to q)$ is a tautology

F **[TF 2]** The negation of "*If I buy groceries and watch TV today, then I don't have enough time to study*" is "*If I don't buy groceries or I don't watch TV today, then I have enough time to study*".

$(g \wedge t) \to \neg s$ $\quad \neg(\neg(g \wedge t) \vee \neg s)$ $\quad$ if $\quad (g \wedge t) \wedge s$

T **[TF 3]** The converse of "*The university will close when it it snowing or there is a heatwave*" is "*If the university is closed, then it is snowing or there is a heatwave*".

$(s \vee h) \to c \qquad c \to (s \vee h)$

F **[TF 4]** The contrapositive of "*All math majors must take a calculus course*" is "*Some people who take calculus courses are not math majors*".

$\forall \qquad \exists$

$\forall m \;\; m \to c \qquad \exists m \;\; \neg c \to \neg m$

quantifiers don't change when finding contrapositive, or converse, or inverse, or something logically equivalent to the statement.

In questions TF 5 to TF 8, let $S = \{1, 2, 3, \{\emptyset, 1, 2, 3\}\}$.

F **[TF 5]** $\{1, 2, 3\} \in S$

T **[TF 6]** $\{1, 2, 3\} \subseteq S$

F **[TF 7]** $\emptyset \in S \cap \mathcal{P}(S)$

F $\qquad$ T

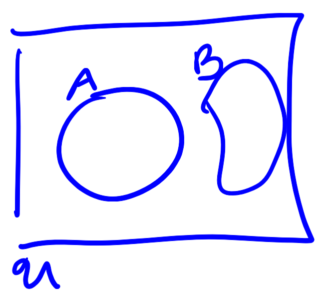
$\emptyset \in S$ and $\emptyset \in \mathcal{P}(S)$

overlap.

F **[TF 8]** $|S| = 3$

$|S| = 4$

T **[TF 9]** For all sets A and B, if $B \subseteq A^c$, then $A \cap B = \emptyset$.



F **[TF 10]** For all sets A, if $a \in A$, then $\{a\} \subseteq \mathcal{P}(A)$.

$\{a\} \subseteq A \qquad \{a\} \in \mathcal{P}(A)$

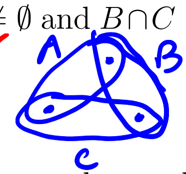
$A = \{1,2\} \qquad A \cap B = \{2\}$

F **[TF 11]** For all sets A, B and C, if $B \neq C$, then $A \cap B \neq A \cap C$.

$B = \{2,3\} \qquad A \cap C = \{2\}$

$C = \{2,4\}$

T **[TF 12]** There exist sets A, B and C such that $A \cap B \neq \emptyset$, $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$, but $A \cap B \cap C = \emptyset$.



$A = \{1,2\}$

$B = \{2,3\}$

$C = \{3,1\}$

F **[TF 13]** Let $a, b \in \mathbb{Z}$. If p is prime and $p^2 \mid ab$, then $p \mid a$ and $p \mid b$.

$p = 3 \quad a = 9 \quad b = 7$

T **[TF 14]** If $n \in \mathbb{N}$, then there are $\lfloor \frac{n}{5} \rfloor$ positive integers less than or equal to n that are multiples of 5.

T **[TF 15]** $\gcd(a, b) \mid \operatorname{lcm}(a, b)$.

$\gcd(a,b) \mid a$, $\gcd(a,b) \mid b$

lcm is a multiple of a and b, so $a \mid \operatorname{lcm}(a,b)$.

$\therefore \gcd(a,b) \mid \operatorname{lcm}(a,b)$

T **[TF 16]** If $\gcd(a, b) = 2$ and n is even, then there exist integers x and y such that $ax + by = n$.

$ax_1 + by_1 = 2$ multiply by $\frac{n}{2}$ to get $ax + by = n$.

F **[TF 17]** $(222)_3 = (111)_6$

$\underbrace{2 \times 3^2 + 2 \times 3 + 2}_{= 26}$ $1 \times 6^2 + 1 \times 6 + 1$

T **[TF 18]** If $a^2 = b^3$, then every prime divisor of a^2 is a divisor of b.

a^2 and b^3 have same prime divisors.

T **[TF 19]** The last digit of 33^{33} is 3.

$33^{33} \equiv 3^{33} \equiv 3 \cdot 3^{32} \equiv 3 \cdot (9)^{16} \equiv 3 \cdot (-1)^{16} \pmod{10}$
$\equiv 3 \pmod{10}$

T **[TF 20]** $(d_2d_1d_0)_{10} \equiv d_2 + d_1 + d_0 \pmod 9$.

$d_2 \times 10^2 + d_1 \times 10 + d_0 \equiv d_2(1)^2 + d_1(1) + d_0 \pmod 9$

F **[TF 21]** There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $(1, 1) \in f$ and $(1, 2) \in f$.

1 used twice as first entry of ordered pair.

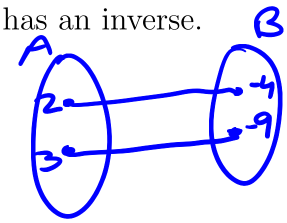
T **[TF 22]** If $f : A \to B$ is a function, then every element of the range of f has a pre-image.

elements of B used by f.

F **[TF 23]*** The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is onto.

$2x$ is even, don't map to odd integers

T **[TF 24]** The function $f : \overset{A}{\{2, 3\}} \to \overset{B}{\{-4, -9\}}$ defined by $f(x) = -x^2$ has an inverse.



is f 1-1 and onto? Yes! So f has an inverse.

T **[TF 25]** Any subset of $\mathbb{N} \times \mathbb{N}$ is countable.

$\mathbb{N} \times \mathbb{N}$ is countable (can list all the elements)

T **[TF 26]*** The set $A = \{2n + 1 : n \in \mathbb{Z}\}$ is countable.

$A \subseteq \mathbb{Z}$ and $\mathbb{Z}$ is countable so A is countable, or show that we can list all elements of A.

T **[TF 27]** Every uncountable set has a countable subset.

$\emptyset$ is always a subset. $\emptyset$ is countable.

T **[TF 28]*** There exists a one-to-one and onto function $f : \mathbb{Z} \to \mathbb{N}$.

$\mathbb{Z}$ is countably infinite, which means there is a 1-1 and onto function $f : \mathbb{Z} \to \mathbb{N}$.

2. [3] Use any method to determine whether $(p \leftrightarrow q) \rightarrow (p \wedge \neg q)$ is a tautology. Write a sentence that explains how your work shows your conclusion.

p	q	$\neg q$	$p \leftrightarrow q$	$p \wedge \neg q$	$(p \leftrightarrow q) \rightarrow (p \wedge \neg q)$
0	0	1	1	0	0
0	1	0	0	0	1
1	0	1	0	1	1
1	1	0	1	0	0

$(p \leftrightarrow q) \rightarrow (p \wedge \neg q)$ is not a tautology since there is an instance where it is false.

(Note: pointing out that $(p \leftrightarrow q) \rightarrow (p \wedge \neg q)$ is false when p is 0 and q is 0 is enough, don't need the full truth table.)

3. [4] Use known logical equivalences to show that $p \rightarrow \neg(q \wedge \neg r)$ is logically equivalent to $(p \wedge q) \rightarrow r$.

$$
\begin{aligned}
p \rightarrow \neg(q \wedge \neg r) &\Leftrightarrow \neg p \vee \neg(q \wedge \neg r) && \text{known LE.}\\
&\Leftrightarrow \neg p \vee (\neg q \vee \neg\neg r) && \text{De Morgan's}\\
&\Leftrightarrow \neg p \vee (\neg q \vee r) && \text{Double negation.}\\
&\Leftrightarrow (\neg p \vee \neg q) \vee r && \text{Associative}\\
&\Leftrightarrow \neg(p \wedge q) \vee r && \text{De Morgan's}\\
&\Leftrightarrow (p \wedge q) \rightarrow r && \text{known LE.}
\end{aligned}
$$

4. [2] Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\forall x, \exists y, xy \leq x+y$. Explain your reasoning.

For $x=1$, $y=1$ has $1 \cdot 1 \leq 1+1$.
For $x=2$, $y=1$ has $2 \cdot 1 \leq 2+1$.
For $x=3$, $y=1$ has $3 \cdot 1 \leq 3+1$.
For $x=4$, $y=1$ has $4 \cdot 1 \leq 4+1$.

So for every x value we can find a y such that $xy \leq x+y$.

The statement is true.

5. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{rl} & \neg p \vee q \\ & q \to \neg r \\ & r \\ \hline \therefore & \neg p \end{array}$$

Steps	Reasons
1) $\neg p \vee q$	premise
2) $q \to \neg r$	premise
3) r	premise
4) $p \to q$	1), known LE
5) $p \to \neg r$	2),4) Chain Rule
6) $r \to \neg p$	5), contrapositive
7) $\neg p$	3),6), Modus Ponens

6. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{lrl} 1 & & r \vee \neg s \\ 1 & & p \leftrightarrow \neg q \\ 1 & & s \to p \\ \hline 0 & \therefore & q \end{array}$$

$p:1$, $q:0$, $r:1$, $s:1$ gives that $r \vee \neg s$ is true, $p \leftrightarrow \neg q$ is true, $s \to p$ is true, but q is false. This shows the argument is invalid since we have all premises true but the conclusion is false.

7. [4] Let A and B be sets such that $A \subseteq B$. Prove that $A \oplus B \subseteq B$, using an argument that starts with "Take any $x \in A \oplus B$ ...". Then use the universe $\mathcal{U} = \{1, 2\}$ to demonstrate that $A \oplus B \subsetneq B$ is possible.

Suppose $A \subseteq B$.

Take any $x \in A \oplus B = (A \setminus B) \cup (B \setminus A)$. So $x \in A \setminus B$ or $x \in B \setminus A$.

If $x \in A \setminus B$, we have $x \in A$ and $x \notin B$. Since $A \subseteq B$ and $x \in A$ this gives $x \in B$. This case is not possible since we cannot have $x \in B$ and $x \notin B$.

If $x \in B \setminus A$, we have $x \in B$ and $x \notin A$. So $x \in B$.

Therefore $A \oplus B \subseteq B$.

$\mathcal{U} = \{1, 2\}$ $A = \{1\}$ $B = \{1, 2\}$ gives $A \oplus B = A \setminus B \cup B \setminus A$
$= \emptyset \cup \{2\} = \{2\}$.

So here $A \oplus B \subsetneq B$.

8. [4] Let A, B and C be sets. Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. Hint: there is a short argument that uses set-theoretic identities.

$$
\begin{aligned}
A \setminus (B \cup C) &= A \cap (B \cup C)^c && \text{definition.} \\
&= A \cap (B^c \cap C^c) && \text{DeMorgan's} \\
&= (A \cap B^c) \cap (A \cap C^c) && \text{Distributive} \\
&= (A \setminus B) \cap (A \setminus C) && \text{definition}
\end{aligned}
$$

9. [3] A set A consisting of 40 integers contains 20 even numbers, four multiples of 10 and eight numbers that are relatively prime to both 2 and 5. How many odd numbers in A are multiples of 5?

20 of the 40 integers are even, so the other 20 are odd.

The multiples of 10 are even integers.

Numbers that are relatively prime to 2 and 5 are odd (since they cannot have 2 as a factor), and they are not multiples of 5 (since they cannot have 5 as a factor).

So of the 20 odd integers, 8 are not multiples of 5, which leaves $20 - 8 = 14$ which <u>are</u> multiples of 5.

10. [2] Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

Suppose $a \mid b$ and $b \mid c$.

Then $\exists\, k \in \mathbb{Z}$ such that $b = ak$.

Also $\exists\, \ell \in \mathbb{Z}$ such that $c = b\ell$.

Now $c = b\ell = (ak)\ell = a(k\ell)$.

Since $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we have $k\ell \in \mathbb{Z}$.

Therefore $a \mid c$. $\blacksquare$

11. [4] Use the Euclidean Algorithm to find $d = gcd(1350, 456)$ and then use your work to find integers x and y such that $1350x + 456y = d$.

$1350 = 456(2) + 438$

$456 = 438(1) + 18$

$438 = 18(24) + 6$

$18 = 6(3) + 0$

So $\gcd(1350, 456) = 6$.

$438 = 1350 - 456(2)$

$18 = 456 - 438(1)$

$6 = 438 - 18(24)$

$6 = 438 - 18(24)$

$6 = 438 - [456 - 438(1)](24)$

$6 = 438(25) + 456(-24)$

$6 = [1350 - 456(2)](25) + 456(-24)$

$6 = 1350(25) + 456(-74)$

So $d = 6$, $x = 25$, $y = -24$.

12. [3] Find the base 7 representation of 2021.

$2021 = 7(288) + 5$

$288 = 7(41) + 1$

$41 = 7(5) + 6$

$5 = 7(0) + 5$

So $2021 = (5615)_7$.

13. Let k be an integer such that $k \equiv -1 \pmod 5$.

(a) [2] What is the remainder when $50k^{10} + 26k^3 - 17$ is divided by 5?

work (mod 5)

$$50k^{10} + 26k^3 - 17 \equiv 50(-1)^{10} + 26(-1)^3 - 17 \pmod 5$$
$$\equiv 0(1) + 1(-1) - 2 \pmod 5$$
$$\equiv -3 \pmod 5$$
$$\equiv 2 \pmod 5$$

The remainder is 2.

(b) [2] What is the last digit of the base 5 representation of k? Why?

To find the last digit in base 5, we want the remainder of the integer divided by 5, which is the same as the least residue (mod 5). $k \equiv -1 \equiv 4 \pmod 5$, so the last digit of k in the base 5 representation is 4.

14. [4] Use induction to prove that $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Basis: $n=1$ LHS $= \frac{1}{1(1+1)} = \frac{1}{2}$ RHS $= \frac{1}{1+1} = \frac{1}{2}$

Induction Hypothesis:

Suppose $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ for some $k \geq 1$.

Induction Step: Look at $n = k+1$.

$$\text{LHS} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}.$$

Conclusion: By PMI, $\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \geq 1$. ∎

15. [3] Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 3$, and $a_n = 4a_{n-1} + 3$ for $n \geq 2$. Express each of a_2, a_3, a_4, and a_5 as a sum of terms that involve the numbers 3, 4 and exponents. Then, use your work to guess a (correct) formula for a_n. You do not need to prove that your formula is correct. (Suggestion: first use your work to express a_n as a sum of n terms, as above, and then use that sum to obtain a formula.)

$$a_2 = 4a_1 + 3 = 4(3) + 3$$
$$a_3 = 4a_2 + 3 = 4[4(3)+3] + 3 = 4^2(3) + 4(3) + 3$$
$$a_4 = 4a_3 + 3 = 4[4^2(3) + 4(3) + 3] + 3 = 4^3(3) + 4^2(3) + 4(3) + 3$$
$$a_5 = 4a_4 + 3 = 4[4^3(3) + 4^2(3) + 4(3) + 3] + 3 = 4^4(3) + 4^3(3) + 4^2(3) + 4(3) + 3$$

$$a_n = 4^{n-1}(3) + 4^{n-2}(3) + \cdots + 4^2(3) + 4(3) + 3$$
$$= 3[4^{n-1} + 4^{n-2} + \cdots + 4^2 + 4 + 1]$$
$$= 3\left(\frac{4^n - 1}{4-1}\right) = 4^n - 1.$$

16. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 0$, $b_1 = 1$ and $b_n = 2b_{n-1} - b_{n-2} + 2$ for all $n \geq 2$. Use induction to prove that $b_n = n^2$ for all $n \geq 0$.

Basis: $n = 0 \quad b_0 = 0 \quad n^2 = 0^2 = 0$
$n = 1 \quad b_1 = 1 \quad n^2 = 1^2 = 1$

Induction Hypothesis: Suppose $b_n = n^2$ for all n in $0, 1, \ldots, k$ for some $k \geq 1$.
That is, suppose $b_0 = 0^2$, $b_1 = 1^2$, $\ldots$, $b_k = k^2$ for some $k \geq 1$.

Induction Step: Look at $n = k+1$.

$$b_{k+1} = 2b_k - b_{k-1} + 2$$
$$= 2k^2 - (k-1)^2 + 2$$
$$= 2k^2 - (k^2 - 2k + 1) + 2$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

Conclusion: By PMI, $b_n = n^2$ for all $n \geq 0$.

17. [4] Let $f : [0, \infty) \to \mathbb{N}$ be defined by $f(x) = \lfloor x + 1 \rfloor$. Show that f is onto, but not 1-1. (Note: onto means surjective and 1-1 means injective; also, $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$.)

not 1-1: $f(1) = \lfloor 1+1 \rfloor = \lfloor 2 \rfloor = 2$

$f(1.1) = \lfloor 1.1 + 1 \rfloor = \lfloor 2.1 \rfloor = 2$

$f(1) = f(1.1)$ but $1 \neq 1.1$ so f is not one-to-one.

onto: Take $b \in \mathbb{N}$. If $f(x) = b$

we have $\lfloor x+1 \rfloor = b$.

Let's find an x that maps to b.

Say $x = b - 1$

$\lfloor x + 1 \rfloor = \lfloor b - 1 + 1 \rfloor = \lfloor b \rfloor = b$, since $b \in \mathbb{N}$.

Check that $x \in [0, \infty)$: if $b = 1$, $x = 0$

if $b > 1$, $x > 0$

so $x \in [0, \infty)$.

Since b was an arbitrary element of $\mathbb{N}$, we have that for every $b \in \mathbb{N}$ there exists an $x \in [0, \infty)$ such that $f(x) = b$, so f is onto. ∎

Use a proof to show statement is true.
Use a counterexample to show statement is false.

$a \mathcal{R} b$

18. [4]* Consider the relation $\mathcal{R}$ on the set $\mathbb{Z}$ defined by $(a, b) \in \mathcal{R}$ if and only if $a + b$ is an even integer. Answer each of the following and provide a proof or counterexample as an explanation.

(a) Is $\mathcal{R}$ reflexive? Yes. (Show $(x,x) \in \mathcal{R}$ for all $x \in \mathbb{Z}$)

$x + x = 2x$ is an even integer, so $(x,x) \in \mathcal{R}$ $\forall x \in \mathbb{Z}$.

(b) Is $\mathcal{R}$ symmetric? Yes (Show if $(x,y) \in \mathcal{R}$, then $(y,x) \in \mathcal{R}$)

Suppose $(x,y) \in \mathcal{R}$, so $x+y$ is even.
This means $y+x$ is also even, so $(y,x) \in \mathcal{R}$

(c) Is $\mathcal{R}$ antisymmetric? No. (If $(x,y) \in \mathcal{R}$, then $(y,x) \notin \mathcal{R}$ when $x \neq y$)

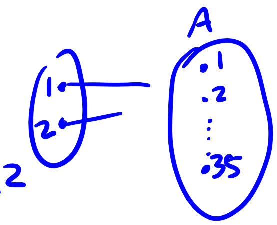
Counterexample. $(1,3) \in \mathcal{R}$ since $1+3$ is even
$(3,1) \in \mathcal{R}$ since $3+1$ is even
since $3 \neq 1$, $\mathcal{R}$ is not antisymmetric.

(d) Is $\mathcal{R}$ transitive? Yes. (If $(x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$, then $(x,z) \in \mathcal{R}$)

Suppose $(x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$, so $x+y$ is even and $y+z$ is even. x & y are either both even or both odd, y & z are either both even or both odd. So x & z are both even or both odd. So $x+z$ is even, $(x,z) \in \mathcal{R}$.

(This is not the only possible proof!)

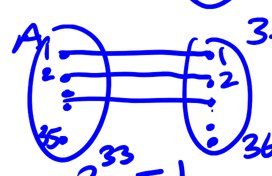
19. Let $A = \{1, 2, \ldots, 35\}$. Fill in the blanks. No justification is necessary.



(a) [1] The number of functions $f : \{1, 2\} \to A$ is $35 \cdot 35$.

(b) [1] The number of subsets of $A \times A$ is 2^{35^2}.

$|A \times A| = 35^2$



$36 \cdot 35 \cdot 34 \cdot \cdots \cdot 2 = 36!$

(c) [1] The number of 1-1 functions $f : A \to \{1, 2, \ldots, 36\}$ is $36!$.

(d) [1] The number of non-empty subsets of A that contain neither 2 nor 3 is $2^{33} - 1$.

Methods to show a set A is countably infinite:

- list all the elements of A
- find a 1-1 and onto function $f: A \to \mathbb{N}$
- A is a subset of a countable set
- use diagonal sweeping. (list all the elements).

20. [3] Use a diagonal sweeping argument to prove that $\mathbb{N} \times \mathbb{Z}$ is countable.

Proof idea:

To show $A \times B$ is countable:

- point out that A is countable, list all elements
  point out that B is countable, list all elements
- make a grid showing all the (a, b) pairs
- diagonal sweep to list all elements in $A \times B$.

Proof:

$\mathbb{N}$ is countable since we can list all the elements

$A = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ is countable since we can list all the elements

$B = \mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$

B $\mathbb{Z}$

A $\mathbb{N}$	0	-1	1	-2	2	…
1	(1,0)	(1,-1)	(1,1)	(1,-2)	(1,2)	…
2	(2,0)	(2,-1)	(2,1)	…		
3	(3,0)	(3,-1)	(3,1)	…		
4	(4,0)	(4,-1)	(4,1)	…		
⋮	⋮	⋱				

$\mathbb{N} \times \mathbb{Z} = \{(1,0), (2,0), (1,-1), (1,1), (2,-1), (3,0), (4,0) \ldots\}$

Because we can list all elements of $\mathbb{N} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{Z}$ is countable. ∎

/END