CSC 225: Lab #9

Walks, Trails and Paths

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let x, y be two (not necessarily distinct) verticies of G.

- **Walk:** A walk from a to b in graph G is an alternating sequence of vertices and edges starting from a and ending at b. The sequence may look like this:

$$a = v_0, e_0, v_1, e_1, \ldots, e_n, v_{n+1} = b$$

  The length of a walk is the number of edges in the walk. In the example above the length of the $a - b$ walk is n. There might be repeated verticies and/or repeated edges in a walk.

- **Closed and Open Walks:** An $a - b$ walk is closed if $a = b$, otherwise it is open.
- **Trail and Circuit:** A *trail* is an $a - b$ walk where no edge is repeated. A closed trail is called a *circuit*.
- **Path and Cycle:** A *path* is an $a - b$ open trail where no vertex is repeated. A closed trail (where only a is visited twice) is called a *cycle*.

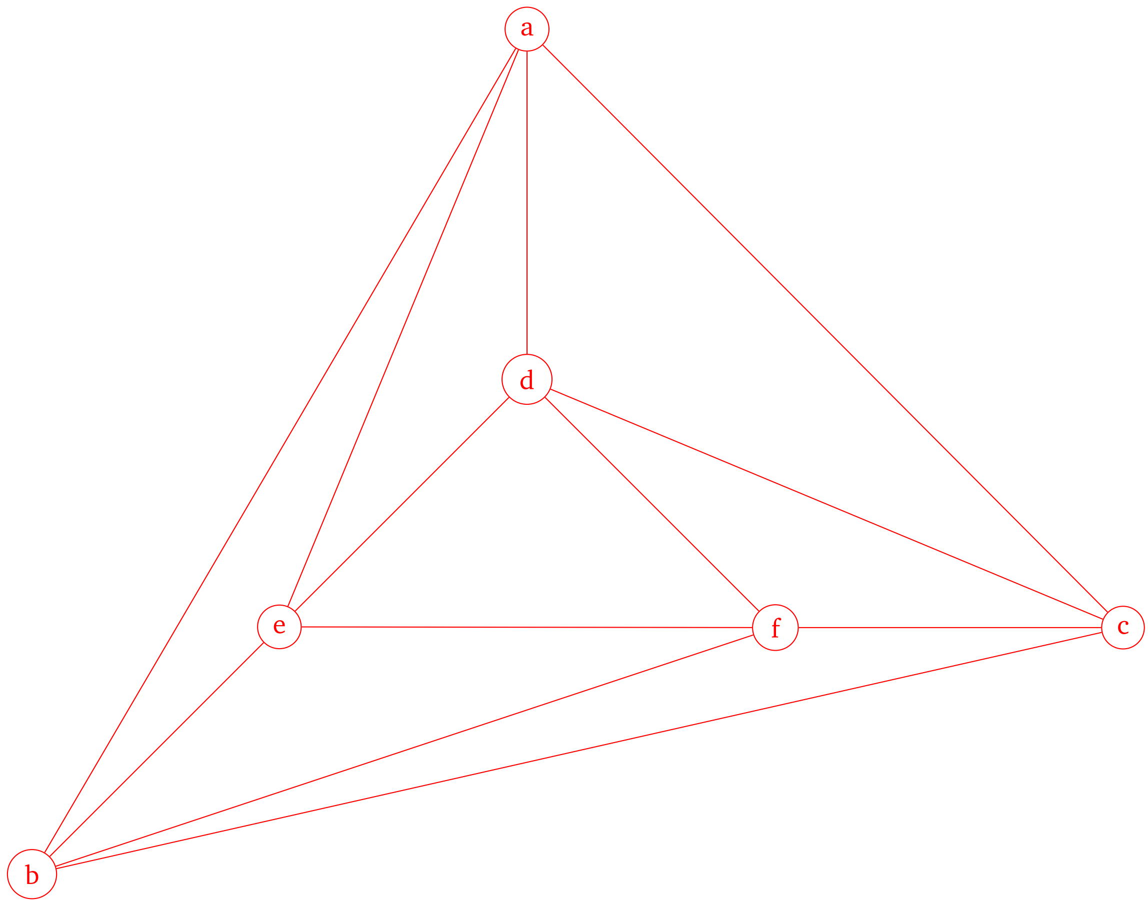
Based on the definitions above, answer the following questions:

1. In each of the following pairs, which one is a subset of the other? For example, in the pair "path, circuit", is a path always a circuit? Or is a circuit always a path? Or is neither true?
   (a) path, circuit Neither - A path is defined as an open trail; it does not revisit any verticies. A circuit revisits its starting vertex.
   (b) cycle, trail Many trails are not cycles - if a trail does not end at its beginning vertex, or if it otherwise repeats a vertex, it is not a cycle. All cycles are trails, since a cycle is defined as a closed trail.
   (c) trail, open walk Some trails are closed - they end at their starting vertex. Some open walks cross an edge multiple times, and so are not trails.

2. Draw the graph with the following edges and call it T_t. Try to draw it without crossing edges.

$$E_t = \{(a,b),(b,c),(c,a),(d,e),(e,f),(f,d),(a,d),(b,e),(c,f),(a,e),(b,f),(c,d)\}$$

There are a large number of drawings possible. Here is one:



3. How many $a - c$ paths are there in graph T_t (from question 2 above)? How many of those paths have length 4?

Path number	Path	Length
1	a - c	1
2	a - b- c	2
3	a-b-f-c	3
4	a-b-e-f-c	4
5	a-b-e-d-c	4
6	a-b-f-d-c	4
7	a-b-f-e-d-c	5
8	a-b-e-f-d-c	5
9	a-b-e-d-f-c	5
10	a-d-c	2
11	a-d-f-c	3
12	a-d-e-b-c	4
13	a-d-e-f-c	4
14	a-d-f-b-c	4
15	a-d-e-b-f-c	5
16	a-d-e-f-b-c	5
17	a-d-f-e-b-c	5
18	a-e-b-c	3
19	a-e-d-c	3
20	a-e-f-c	3
21	a-e-b-f-c	4
22	a-e-d-f-c	4
23	a-e-f-b-c	4
24	a-e-f-d-c	4
25	a-e-b-f-d-c	5
26	a-e-d-f-b-c	5

There are 26 paths, of which 10 are length 4.

4. Let G be an undirected graph and let x, y be two distinct verticies of G. If there is an $x - y$ trail in G, prove that there is an $x - y$ path in G.

Proof by contradiction.

Select the shortest length trail t_1 from x to y, or if multiple trails are the shortest, select t_1 as any such shortest trail.

This shortest trail is of the form $(x, x_1), (x_1, x_2), \ldots, (x_n, y)$. Assume this trail is not a path.

Then we have $(a, x_1), (x_1, x_2), \ldots, (x_k, x_{k+1}), \ldots, (x_m, x_{m+1}), \ldots, x_n, b$, where $k < m$ and $x_k = x_m$. Since $x_k = x_m$, $(x_k, x_{k+1}), \ldots, (x_{m-1}, x_m)$ is a cycle. Remove this cycle from the trail giving t_2.

Since edges were removed from t_1 to create t_2, the length of t_2 is less than the length of t_1.

But t_1 was the shortest trail $a - b$ in G. $\Rightarrow\Leftarrow$