CSC 225: Fall 2022: Lab 2

1 Solving Recurrence Equations

Determine the closed form of the following recurrence equations.

a) $T(n) = \begin{cases} 1, & \text{if } n = 1 \\ T(n-1) + n, & \text{if } n \geq 2 \end{cases}$

b) $T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 2T(n-1), & \text{if } n \geq 1 \end{cases}$

2 Proof Techniques

Prove each of the following identities using induction.

a) $\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \geq 1$.

b) $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 0$.

3 Loop Invariants

Consider the Algorithm `arrayFind`, given below, which searches an array A for an element x. Prove that `arrayFind` is correct using induction (loop invariants).

Algorithm `arrayFind`(x, A, n):
  Input: An element x and an n-element array, A.
  Output: The index i such that $x = A[i]$ or -1 if no element of A is equal to x.

$i \leftarrow 0$
while $i < n$ **do**
  if $x = A[i]$ **then**
    return i
  else
    $i \leftarrow i + 1$
return -1

ie. S_1 must hold for the first iteration of the loop, S_2 for the second, and so on.
If there is a k^{th} iteration, the S_k is called into existence and must hold.

these statements mimic the loop itself.

Define statements S_i = x is not any of the first i elements of A.

Check: S_0 means the first 0 elements of A cannot contain x. Trivially true.
↳ can also think of it as the 0^{th} iteration of the loop. x clearly has not been found yet.

Consider S_i. At the beginning of the iteration we check if x is the i^{th} element of A. If so, we return the index and terminate.
Only if x is not the i^{th} element do we continue to the next iteration, in which case S_{i+1} must hold. S_{i+1} clearly does hold since we did not find x yet.

1. a) $T(n) = \begin{cases} 1, & \text{if } n = 1 \\ T(n-1) + n, & \text{if } n \geq 2 \end{cases}$

Consider $n \gg 2$.

$T(n) = \underbrace{T(n-1)}_{} + n.$

we can use the definition of $T(n)$ itself to solve what $T(n-1)$ is.

ie. $T(n-1) = T(n-2) + (n-1)$

Thus, $T(n) = \underbrace{T(n-2)}_{} + (n-1) + n$. We can continue to substitute...

$T(n) = \overbrace{T(n-3) + (n-2)} + (n-1) + n$

And we can extrapolate down to the base case of $n=1$.

$T(n) = T(1) + 2 + \ldots + (n-2) + (n-1) + n.$ Since we know that $T(1) = 1$ by the question definition,

We can write

$$T(n) = 1 + 2 + \ldots + (n-1) + n = \sum_{i=1}^{n} i$$

b) $T(n) = \begin{cases} 1, & n = 0 \\ 2T(n-1), & n \geq 1 \end{cases}$

Again, consider when $n \gg 0$.

$$\begin{aligned} T(n) &= 2T(n-1) \\ &= 2(2T(n-2)) = 2^2 T(n-2) \\ &= 2(2(2T(n-3))) = 2^3 T(n-3) \\ &= 2^4 T(n-4) \\ &= 2^n T(n-n) = 2^n T(0) = 2^n \end{aligned}$$

we can use this pattern to see the answer

$\boxed{Q2}$ a) $\sum_{i=1}^{n} 2i-1 = n^2 \quad \forall\, n \geq 1$

Induction ALWAYS has 4 steps! Very formulaic!

Step ①: Base case(s). [Check that the formula works for at least some number(s)].

Consider $n = 1$.

LHS	RHS
$\sum_{i=1}^{n} 2i-1 = \sum_{i=1}^{1} 2i-1$	$n^2 = (1)^2$
$= 2(1)-1$	$= 1$
$= 1$	

both sides match so the formula works for $n = 1$.

Step ②: Inductive hypothesis. [Claim that the formula will hold for some random number.]

Assume $\sum_{i=1}^{l} 2i-1 = l^2$ holds for $\underbrace{1 \leq l < n}$

the random number can be any number from your base case(s) up to n

Step ③: Inductive step. [Prove that if it holds for l, it must also hold for $l+1$].

We want to show $\sum_{i=1}^{l+1} 2i-1 = (l+1)^2$.

Start with LHS.

$$\sum_{i=1}^{l+1} = 2i-1 = \sum_{i=1}^{l}(2i-1) + 2(l+1)-1$$

$$= l^2 + 2l + 2 - 1$$

$$= l^2 + 2l + 1$$

$$= (l+1)(l+1)$$

$$= (l+1)^2 \checkmark$$

Step ④: Conclusion. [Write a nice conclusion! Free marks!]

Since the inductive step proves that if the formula holds for l, it also holds for $l+1$, by induction the claim must hold for any $n \geq 1$.

b) $$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall \; n \geq 0$$

1. <u>Base cases.</u>

$n=0 \rightarrow \sum_{i=0}^{0} i^2 = 0 \qquad \frac{0(0+1)(2(0)+1)}{6} = 0 \quad \checkmark$

$n=1 \rightarrow \sum_{i=0}^{1} i^2 = 1 \qquad \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1 \quad \checkmark$

2. <u>Inductive hypothesis</u>

Let $$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{for} \quad 0 \leq k < n.$$

3. <u>Inductive step</u>

Then $$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)\left(k(2k+1) + 6(k+1)\right)}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

LHS

this is difficult to factor so we'll just check to see if it matches the answer.

RHS

We want $\dfrac{(k+1)(k+2)(2(k+1)+1)}{6}$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(2k^2 + 4k + 3k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6} \checkmark$$

It matches the LHS so we good!

Step ④: Conclusion

$\therefore$ By induction, $\displaystyle\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall \; n \geq 0$