PHYS 110 Lecture Workbook

UVic 2017-2018

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Preface

0.1 Introduction

Physics is the art of predicting the future based on the application of a few simple principles to knowledge about the present. This description also applies to many other subjects, including things like astrology, card reading, palmistry, and divination. The thing that sets Physics apart from other pursuits with similar goals is that Physics works.

When we say that Physics works we mean something specific: You can use Physics to make both qualitatively and quantitatively accurate predictions of observations. These observations are typically, but not always, the results of experiments: the process of trying something under controlled and documented conditions to see what will happen.

The methodology of these predictions takes a simple form: We make a simplified, abstract model of whatever we want to explain; we measure the quantities we know; we express relations between those quantities and physical principles in mathematical form; we then use those relationships together with mathematical rules to derive other relationships and predict future behavior of the system we are considering.

The general way that we will approach problems in this text is to consider three main steps: We first do Physics by expressing known relationships in a mathematical form, we then use mathematics to implement a solution to the question posed, and we finally do more Physics by interpreting the derived relationship.

0.2 How the course is organized

There are several different resources that we will be using to teach you Physics in this course:

- The course textbook. We have written a textbook to specifically cover the material of this course at the level we expect you to learn,

- the lectures,
- the schedule posted on our course website,
- the laboratory exercises,
- the regular online homework,
- the midterm exams, and
- this workbook (of course).

Each of the resources is a different part of our plan to help you to learn Physics. The resources are good for different things – we expect that when they are taken together they will reinforce each other and give you different ways to engage with the subject matter.

The textbook is the starting place for our course. The way you should use it is to read it (surprise!) We strongly recommend that you read it *before* the corresponding lectures. The text we wrote goes sequentially through the material in the order we present it. The text is presented at a formal level, with limited examples, which is why we have prepared this workbook. Something you may notice if you have access to another physics textbook is that we spend a lot of time doing things with vectors before we turn to kinematics. This is a slightly different order from most other textbooks. The reason for this difference is that in our experience dealing with vectors is a challenging new concept for many students, and also so that your MATH courses (normally MATH 100) have a chance to 'get ahead' so you have been exposed to more material on differentiation before we present it in this class in the context of velocity and acceleration.

The purpose of the lectures is for us (the instructors) to highlight the important concepts, explain the points that often confuse people, and generally to give you an insight into how physicists think. Learning to think like a physicist is, at the most basic level, the reason your degree program wants you to take this course. Another way to think of the purpose of the lectures is to communicate what you are expected to learn and master to successfully complete the course. A common question is 'what will we be expected to know?' The answer is 'the material we have covered in lecture, and what we think you should be able to deduce based on that.'

The schedule that we post has two purposes. The first is to give you some information about what is coming up, so that you can prepare prior to lectures. Related to this, we also provide the schedule so that you have a basis from which to review, and so that if there are some omissions from your notes, or if you missed a lecture because of illness, you will know where to look for more information.

The laboratory exercises are chosen to reinforce some key concepts, and to illustrate for you the difference between the abstractions we discuss in class and the real-world things they represent. You will see how to deal with uncertainty and inaccuracy in data, and how those imperfections affect the conclusions you can draw about physical processes.

The homework we assign is done online. We do this for several reasons: The online format allows us to give you quick feedback about whether you are successful at doing calculations on the topics we cover. Use this information to prioritize your studying on subjects you struggle with. Another reason for the regular homework is that it forces you to do some studying (to help answer the homework problems) each week. Physics takes time! You need to read and hear something, think about it, try to use that information, make mistakes, go back and read or study more, and try again and again. You can't rush it. We have you do homework each week in an attempt to force you to use a strategy of regular studying. We believe (and there are lots of studies about this) that studying regularly will make you learn the material better. Assignments are our paternalistic attempt to make you do just that.

The midterms are another learning tool. Their purpose isn't to generate marks, but rather to give you some important information and feedback: You will learn what our exams look like and what style of questions we ask; you will learn about the exam environment, and how you respond to it; you will learn whether you have a strong grasp of the material; and you will learn whether the study strategy you have been using is working for you. We have the midterm exams early enough that you can make changes prior to the end of term. The key idea is that the midterms are intended to give you information about whether you are 'on track.' It is only in the final exam that we assess whether you have learned the course material.

This workbook deserves a section of its own:

0.3 How to use this book

Your high school experience in Physics likely had the general pattern that you were told some physics facts, and then your instructor did some examples. The things that were noteworthy about the examples was that they showed you how to put ‘numbers’ into ‘formulæ’. You were likely shown at least one example in class of any problem you could be expected to solve. You were also probably expected to get *numerical* rather than *symbolic* answers to questions.

This Physics class will not be like that.

Really.

The breadth of the subject is such that we could not show you an example of ‘every’ problem even if we had three times as many lecture hours as we do. We *assume* that you are fairly competent at substituting numbers into formulæ, so we are trying to teach you *how* to come up with formulæ that are appropriate for different situations. Since we are trying to illustrate general principles, we are going to emphasize getting symbolic relations as much as possible, and examining those will give us information about how the world works.

You should use this book as an adjunct to your studying after we have covered the relevant material in class.

The way we’ve structured this book is as a series of examples. We have chosen the examples because they typically have several concepts that you have to bring together, or ‘synthesize’, to get whatever information the question asks for. These examples are intended to illustrate the kind of thought process you need to have as you are solving problems – the examples are also a sort of ‘guided tour’ of physics: we will sometimes point out an important concept as we ‘drive past’ it while doing a solution.

After each example are some questions. The questions are chosen to have some important similarites to the material that was just covered.

If you can do all the questions in this workbook, you should be well prepared for any exam in this class.

January 18, start here

Chapter 1

Vectors

1.1 Summary

The first chapter explains the concept of vectors and gives examples of how they are used in Physics. The key points from the chapter are:

- Vectors are a mathematical construct that encode information about *magnitude* and *direction*.
- You can express vectors in different coordinate systems. Different coordinate systems have different 'unit vectors' (also called 'basis vectors').
- When you express a vector in two different coordinate systems, the vector is the same but the components you get will be different. The end result is always the same. Sometimes a clever choice of basis will make the algebra easier.
- Vectors have properties that makes it possible to manipulate them very much like numbers in most situations. You can typically add them together in any order; they are commumative with respect to addition. They obey the associative rules you would normally expect for multiplication. The case in this course that you will encounter slightly different formal rules is in the case of the cross-product: for the cross-product the *order* of the vectors multiplied is important.
- It is often convenient to express vectors in terms of magnitudes and directions, but when you are calculating with them it is almost always easiest to express them in terms of components and unit vectors.

- It is critically important to distinguish between the magnitude of a vector and the vector itself.

- In the regular xyz coordinate system with corresponding unit vectors $\hat{\imath},\ \hat{\jmath},\ \hat{k}$ a vector can be expressed as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \tag{1.1}$$

  The quantity A_x is the 'x-component of the vector A', with similar names for A_y and A_z.

- The notation for the *magnitude* of $\vec{A}$ is $\left|\vec{A}\right|$.

- The rules for adding two vectors in terms of their components is that the x-component of $\vec{A}+\vec{B}$ is $A_x + B_x$. The rule for the y- and z-components are similar. You add vectors by putting them in the *same* coordinate system and adding up each component in turn.

- The rules for multiplying a vector by a number (also called a 'scalar') are that the x-component of $c\vec{A}$ is cA_x. You multiply a vector by a scalar by multiplying all components of the vector by that same amount.

- The scalar product (also known as the 'dot product' or the 'inner product') of two vectors is *defined* as

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta \tag{1.2}$$

  where θ is the angle between $\vec{A}$ and $\vec{B}$. When $\vec{A}$ and $\vec{B}$ are written in the xyz coordinate system an equivalent expression for the scalar product is

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.3}$$

- The magnitude of a vector can be related to its components as

$$\left|\vec{A}\right| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.4}$$

- The vector product (also known as the 'cross product' or the 'outer product') of two vectors *in three dimensions* is defined as

$$\begin{aligned}\vec{A}\times\vec{B} \quad = \quad & (A_yB_z - A_zB_y)\,\hat{\imath} \\ & + (A_zB_x - A_xB_z)\,\hat{\jmath} \\ & + (A_xB_y - A_yB_x)\,\hat{k}\end{aligned} \tag{1.5}$$

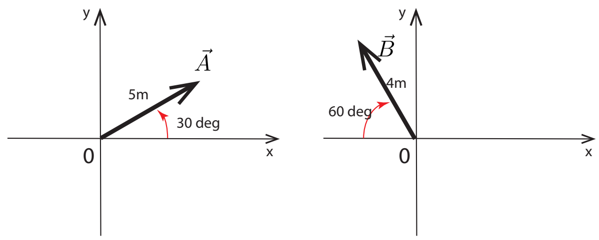


Figure 1.1: Two different vectors are shown. In (a) the vector $\vec{A}$ has a magnitude of $5m$ and makes an angle of $30°$ counterclockwise from the positive x-axis. In (b) the vector $\vec{B}$ has a magnitude of $4m$ and makes an angle of $60°$ clockwise from the negative x-axis.

It takes a pair of vectors and produces a third vector. The magnitude of the cross product satisfies

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta \tag{1.6}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is at right angles to both $\vec{A}$ and $\vec{B}$, in the direction described by the right-hand rule.

1.2 Components, Magnitude, and Direction

The most important skill that you have to master with vectors is the ability to manipulate them to either get magnitude and direction from the components, or to get the components from information about the magnitude and direction. Most of the material we will talk about here is particularly important for cases where we can restrict our attention to vectors in two dimensions; if you want to express a vector in three dimensions you (in principle) have to give three pieces of information which is just as much work as specifying three components.

1.2.1 Going from magnitude and direction to components

Example Express the vectors $\vec{A}$ and $\vec{B}$ given in Figure 1.1 in terms of their components in the usual x-y coordinate system and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

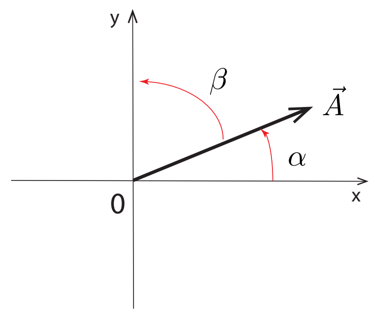


Figure 1.2: Vector $\vec{A}$ makes an angle of α with the x-axis and β with the y-axis

Worked Solution We will try and get the components of the vector $\vec{A}$ first. The general method of getting components for a vector is to take the *dot product* of the vector with the unit vectors of the basis we are using. In this case, it would mean that we can easily write

$$\vec{A} = \left(\vec{A}\cdot\hat{\imath}\right)\hat{\imath} + \left(\vec{A}\cdot\hat{\jmath}\right)\hat{\jmath} \tag{1.7}$$

Remember that $\hat{\imath}$ is the unit vector along the positive x-axis and that $\hat{\jmath}$ is the unit vector along the positive y-axis.

Now we just have to calculate those dot products. The general rule for the dot product is that (for arbitrary vectors $\vec{A}$ and $\vec{C}$) we have $\vec{A}\cdot\vec{C} = \left|\vec{A}\right|\left|\vec{C}\right|\cos\phi$ where ϕ is the angle between $\vec{A}$ and $\vec{C}$. In our case, the vector $\vec{C}$ is replaced by either $\hat{\imath}$ or $\hat{\jmath}$ as appropriate, and both $|\hat{\imath}| = |\hat{\jmath}| = 1$. Let's re-draw the first part of Figure 1.1: We can use this to refine our expression for $\vec{A}$:

$$\vec{A} = \left|\vec{A}\right|\cos\alpha\hat{\imath} + \left|\vec{A}\right|\cos\beta\hat{\jmath} \tag{1.8}$$

When we compare Figure 1.2 with Figure 1.1 we can see that $\alpha = 30^\circ$. Since the angle between the x and y axes is 90°, this means that $\beta = 90^\circ - \alpha = 60^\circ$, and with $\left|\vec{A}\right| = 5m$ we have

$$\begin{aligned}\vec{A} &= 5m\cos 30^\circ\hat{\imath} + 5m\cos 60^\circ\hat{\jmath} \\ &= 4.33m\hat{\imath} + 2.5m\hat{\jmath} \end{aligned} \tag{1.9}$$

We should check that this is consistent with what we know about the mag-

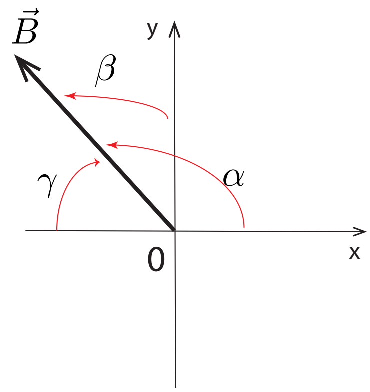


Figure 1.3: Vector $\vec{B}$ makes an angle of α with the positive x-axis and β with the y-axis

nitudes of a vector. The definition of a vector's magnitude is

$$\begin{aligned} \left|\vec{A}\right| &= \sqrt{\vec{A}\cdot\vec{A}} \\ &= \sqrt{A_x^2 + A_y^2 + A_z^2} \end{aligned} \tag{1.10}$$

We know that $\vec{A}$ is supposed to have a magnitude of $5m$, so we check that $\sqrt{A_x^2 + A_y^2} = \sqrt{(4.33m)^2 + (2.5m)^2}$ which works out to $5m$, as expected.

We can use the same approach for vector $\vec{B}$. In this case, we have something very similar. We can compare Figure 1.3 with Figure 1.1 and see that the angle labelled as γ is $60°$, and some trigonometry tells us that $\alpha = 180° - \gamma = 120°$ and $\beta = 90° - \gamma = 30°$. We can then write our desired vector as

$$\begin{aligned} \vec{B} &= \left|\vec{B}\right|\cos\alpha\hat{\imath} + \left|\vec{B}\right|\cos\beta\hat{\jmath} \\ &= 4m\cos 120°\hat{\imath} + 4m\cos 30°\hat{\jmath} \\ &= -2m\hat{\imath} + 3.46m\hat{\jmath} \end{aligned} \tag{1.11}$$

In this case, note that the x-component of $\vec{B}$ is negative while the y-component is positive; $B_x = -2m$ and $B_y = 3.46m$. This tells you that the vector is pointing to the left (since the x-component is negative) and up on the page (since the y-component is positive). Whenever you calculate components of vectors it is important to check that the signs of the components correspond to the direction you expect the vectors to be going.

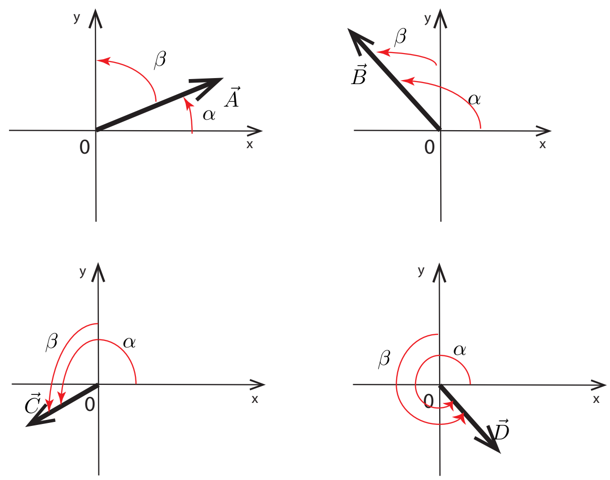


Figure 1.4: A vector is drawn in each of the four quadrants. The angles α is measured clockwise from the positive x-axis and β is drawn as in previous figures.

Something you should notice as we have been doing this is that we haven't used sin anywhere. However you may be used to seeing components given using both sin and cos as is typical high-school physics courses. The two methods are equivalent – here's why: When we look at Figure 1.4 we can see that the x-component of $\vec{C}$ is going to be $C_x = \left|\vec{C}\right| \cos\alpha$. (Note that if $\alpha > 180°$ the fact that $\cos(\theta) = \cos(360° - \theta)$ means that $\cos\alpha$ is the same as cos of the angle measured the other way around.) In parts (b), (c), and (d) of Figure 1.4 it is clear that $\beta = \alpha - 90°$, and in general $\sin(\theta) = \cos(\theta - 90°)$. In part (a) of Figure 1.4 we have $\beta = 90° - \alpha$, and there's a similar identity that $\sin(\theta) = \cos(90° - \theta)$. When we put this together, it means that $\cos(\beta) = \sin(\alpha)$ all the time, provide α is measured counterclockwise from the positive x-axis as shown in the diagram. This means that, as long as you measure the angle α counterclockwise from the x-axis, for a general vector $\vec{C}$:

$$\vec{C} = \left|\vec{C}\right| \cos\alpha\hat{\imath} + \left|\vec{C}\right| \sin\alpha\hat{\jmath} \tag{1.12}$$

Student Exercises

- Use the relationship in equation 1.12 to get the components of $\vec{B}$ from the worked problem.

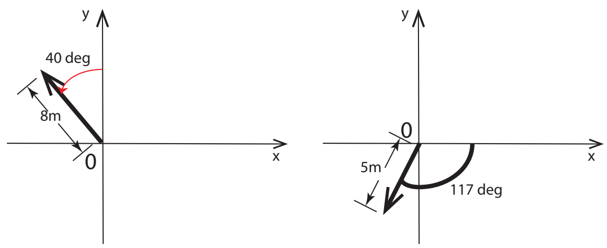


Figure 1.5: Two vectors are drawn with angle and magnitude specified.

- Verify that the components of vector $\vec{B}$ have the relationship with its magnitude suggested by equation 1.10.

- Consider the vectors in Figure 1.5. Find their x and y components. *They are* $5.14m\hat{\imath} + 6.13m\hat{\jmath}$ *and* $-2.27m\hat{\imath} - 4.46m\hat{\jmath}$ *respectively.*

1.2.2 Getting direction angle from components

Example For the vectors $\vec{A} = 3m\hat{\imath} - 4m\hat{\jmath}$, $\vec{B} = -3m\hat{\imath} + 4m\hat{\jmath}$, and $\vec{C} = -3m\hat{\imath} - 4m\hat{\jmath}$, find the angle each of these makes with the positive x-axis (measured counterclockwise). These vectors are illustrated in Figure 1.6.

Worked Solution We are going to approach this in general, and then look at the specific problem. When we worked out how you get the components of a vector from its magnitude and the angle measured counterclockwise with respect to the x-axis we found the relationships implied in 1.12:

$$\begin{aligned} C_x &= \left|\vec{C}\right| \cos\alpha \\ C_y &= \left|\vec{C}\right| \sin\alpha \end{aligned} \tag{1.13}$$

In our case, we have the x and y components of the vector; from these we can determine the magnitude of the vector, and then solve for the angle α.

For the two-dimensional vectors we have, the magnitude is $\left|\vec{C}\right| = \sqrt{C_x^2 + C_y^2}$ (from relationship 1.10), so we can calculate the values for sin and cos of

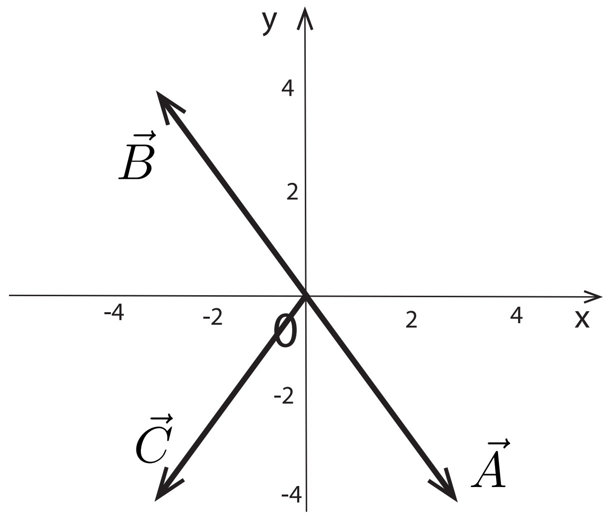


Figure 1.6: The three vectors $\vec{A} = 3m\hat{\imath} - 4m\hat{\jmath}$, $\vec{B} = -3m\hat{\imath} + 4m\hat{\jmath}$, and $\vec{C} = -3m\hat{\imath} - 4m\hat{\jmath}$ drawn on the same set of axes.

the angle as

$$\boxed{\begin{aligned} \cos\alpha &= \frac{C_x}{\left|\vec{C}\right|} \\ \sin\alpha &= \frac{C_y}{\left|\vec{C}\right|} \end{aligned}} \tag{1.14}$$

Alternatively, we can take the ratio of the two relationships in 1.13 and derive that

$$\boxed{\frac{C_y}{C_x} = \frac{\left|\vec{C}\right| \sin\alpha}{\left|\vec{C}\right| \cos\alpha} = \tan\alpha} \tag{1.15}$$

Let's apply this to the vector $\vec{A}$ given above. We find that $\left|\vec{A}\right| = \sqrt{(3m)^2 + (-4m)^2} = 5m$. This means that using 1.14 we have that $\cos\alpha = 0.6$ and $\sin\alpha = -0.8$. Similarly, from 1.15 we can see that $\tan\alpha = -1.33$.

When we invert using our hand calculator, these relationships seem to imply

$$\boxed{\begin{aligned} \cos\alpha = 0.6 &\quad\rightarrow\quad \alpha = 53.1^\circ \\ \sin\alpha = -0.8 &\quad\rightarrow\quad \alpha = -53.1^\circ \\ \tan\alpha = -1.33 &\quad\rightarrow\quad \alpha = -53.1^\circ \end{aligned}} \tag{1.16}$$

Of course, there is only one value for α; which one is it? The crucial thing to remember is that in the interval between 0° and 360° there are always *two* solutions to a trigonometric equality. If the calculator gives x, then

- for cos the other possible value is $-x$,
- for sin the other possible value is $180^\circ - x$, and
- for tan the other possible value is $180^\circ + x$.

This means we can re-write the previous chart as

$$\boxed{\begin{aligned} \cos\alpha = 0.6 &\quad\rightarrow\quad \alpha = 53.1^\circ \text{ or } -53.1^\circ \\ \sin\alpha = -0.8 &\quad\rightarrow\quad \alpha = -53.1^\circ \text{ or } 233.1^\circ \\ \tan\alpha = -1.33 &\quad\rightarrow\quad \alpha = -53.1^\circ \text{ or } 126.9^\circ \end{aligned}} \tag{1.17}$$

Note that there's only one possibility that appears in all three: -53.1°. The last thing is to notice that angles are periodically identified: an angle of θ and an angle of $360^\circ + \theta$ are the same, so this angle is the same as 306.9° measured counterclockwise from the positive x-axis. We don't strictly need to calculate all of cos, sin and tan to get the angle for certain – two are enough.

Performing the same analysis for the vector $\vec{B}$ given above, we find that

$$\begin{aligned} \cos\alpha = -0.6 &\quad\rightarrow \\ \sin\alpha = 0.8 &\quad\rightarrow \\ \tan\alpha = -1.33 &\quad\rightarrow \end{aligned} \tag{1.18}$$

(theta) from cos = 126.869 or -126.869
(theta) from sin = 53.13 or 126.869
(theta) from tan = -53.13 or 126.869

Again, there's only one angle which is in all three, so $\vec{B}$ makes an angle of 126.9° with respect to the positive x-axis. Note that $\vec{B} = -\vec{A}$. This means that they point in the opposite directions, so the angle between them is 180°.

We can also analyze the vector $\vec{C}$. In this case we find

$$\begin{aligned} \cos\alpha = -0.6 &\quad\rightarrow\quad \alpha = 126.9^\circ \text{ or } -126.9^\circ (\text{which is } 233.1^\circ) \\ \sin\alpha = -0.8 &\quad\rightarrow\quad \alpha = -53.1^\circ \text{ or } 233.1^\circ \\ \tan\alpha = 1.33 &\quad\rightarrow\quad \alpha = 53.1^\circ \text{ or } 233.1^\circ \end{aligned} \tag{1.19}$$

so this vector makes an angle of $233.1°$ measured counterclockwise from the positive x-axis.

If you look at the Figure 1.6 you'll see that the quoted results appear reasonable. It is important to check that the angles you get in an exercise like this match up with what you know about the vectors themselves.

Student Exercises

- Verify that the x and y components of the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ given above can be obtained from the magnitude and the angle we derived. *$\vec{A}$ had $\left|\vec{A}\right| = 5m$ and $\theta = 306.9°$, so $A_x = \left|\vec{A}\right| \cos 306.9° = 3m$. The relations for all the other vectors and components are similar.*

- Find the magnitude, and angle made counterclockwise with the positive x-axis, for the following vectors:

  1. $4m\hat{\imath} + 3m\hat{\jmath}$ *$5m$ and $37°$.*
  2. $5m\hat{\imath} - 5m\hat{\jmath}$ *$7.1m$ and $315°$.*
  3. $-8m\hat{\imath} + 6m\hat{\jmath}$ *$10m$ and $143°$.*
  4. $-7m\hat{\imath} - 5m\hat{\jmath}$ *$8.6m$ and $215°$.*

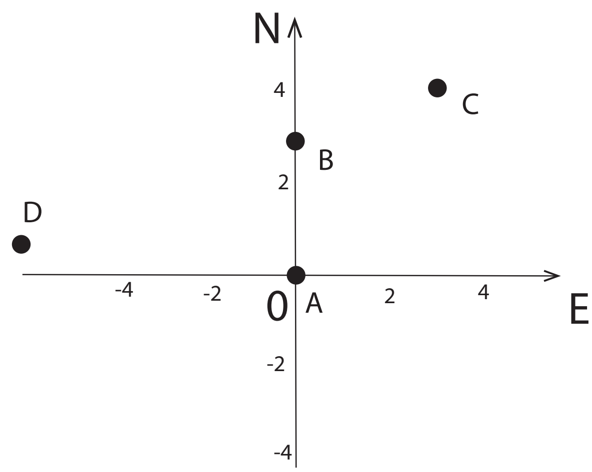
1.3 Adding, Subtracting, and Scaling Vectors

Throughout this course, we are going to be doing a lot of algebra with vectors. This includes things like adding, subtracting, and multiplying vectors by scalars (i.e. by numbers). Finding the vector from one location to another is a key skill. Below, we do an involved problem that uses a lot of these concepts.

Example Town B is $3.0km$ due north of town A, and town C is $5.0km$ away from town A at an angle of $53.1°$ North of East. Town D is twice as far from B as C is, but in the opposite direction. How far is it from A to D, and what is the direction of that vector? The location of these towns is sketched in figure 1.7.

Worked Solution There are two parts to this problem, the first is translating the statement of the problem into mathematical language, and the second is executing the solution. We are going to (obviously) attempt both parts.

Figure 1.7: The approximate locations of towns A, B, C, and D



First, let's draw a diagram (figure 1.8), with a number of vectors on it; we use the notation $\Delta\vec{x}_{AB}$ to mean the vector from A to B, with similar names for the other points. (Note that we are using this language because Δ generally means 'change', and $\vec{x}$ normally is talking about the location of something, so the meaning of $\Delta\vec{x}_{AB}$ is 'the change in $\vec{x}$ as you go from A to B'.)

What we have been asked for is a description (magnitude and direction) of the vector from A to D ($\Delta\vec{x}_{AD}$). We were told the locations of B and C with respect to A, so we know $\Delta\vec{x}_{AB}$ and $\Delta\vec{x}_{AC}$. We also know that D is twice as far from B in the opposite direction as C is, so we know a relationship between $\Delta\vec{x}_{BC}$ and $\Delta\vec{x}_{BD}$. We don't immediately know $\Delta\vec{x}_{BC}$, but we can get it.

To get from A to C, we can either go directly, or we can go to B first: From A to B, and then from B to C. This can be expressed mathematically as

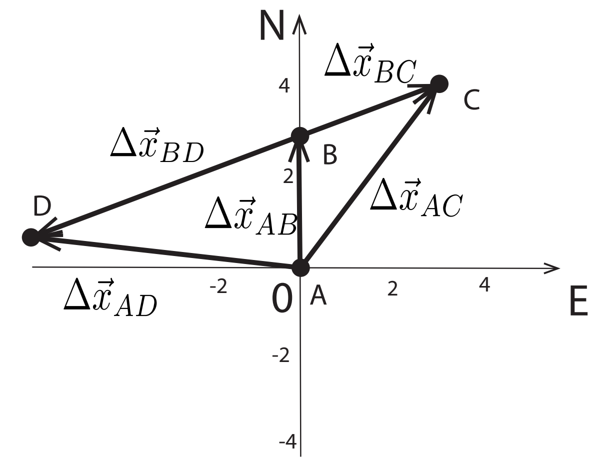
$$\Delta\vec{x}_{AC} = \Delta\vec{x}_{AB} + \Delta\vec{x}_{BC} \tag{1.20}$$

We can rearrange this to become

$$\Delta\vec{x}_{BC} = \Delta\vec{x}_{AC} - \Delta\vec{x}_{AB} \tag{1.21}$$

This relationship is a *KEY* idea: the vector from one place to another is the vector describing where you end up with the vector describing where you

Figure 1.8: The approximate locations of towns A, B, C, and D, and the vectors separating them.



start subtracted off. This can be paraphrased as 'Change in where you are' is given by 'Where you end up' minus 'Where you started'.

Similarly,

$$\Delta\vec{x}_{AD} = \Delta\vec{x}_{AB} + \Delta\vec{x}_{BD} \tag{1.22}$$

In this, we know $\Delta\vec{x}_{AB}$ from the statement of the problem, and the fact that D is twice as far in the opposite direction from B as C means that $\Delta\vec{x}_{BD} = -2\Delta\vec{x}_{BC}$. The negative sign $(-)$ means it is in the opposite direction, and the 2 encodes that it is twice as far.

Putting this together, we can make our plan:

- Express $\Delta\vec{x}_{AB}$ and $\Delta\vec{x}_{AC}$ in components.
- Use that component relationship to get $\Delta\vec{x}_{BC}$
- Use what we know about $\Delta\vec{x}_{BC}$ to get $\Delta\vec{x}_{BD}$
- Knowing $\Delta\vec{x}_{AB}$ and $\Delta\vec{x}_{BD}$ get $\Delta\vec{x}_{AD}$
- Use our magnitude and component magic to get the distance and angle.

We are going to assume that the x-axis has its positive direction to the East, and the y-axis has its positive direction to the North. With this, we find that

$$\begin{aligned}\Delta\vec{x}_{AB} &= 3.0km\hat{\jmath} \\ \Delta\vec{x}_{AC} &= 3.0km\hat{\imath} + 4.0km\hat{\jmath} \qquad (1.23)\end{aligned}$$

Then using relation 1.21 we obtain

$$\Delta\vec{x}_{BC} = (3.0km\hat{\imath} + 4.0km\hat{\jmath}) - (3.0km\hat{\jmath}) = 3.0km\hat{\imath} + 1.0km\hat{\jmath} \qquad (1.24)$$

We can now find that (because, as discussed, D is twice as far from B as C, but in the opposite direction)

$$\begin{aligned}\Delta\vec{x}_{BD} &= -2\Delta\vec{x}_{BC} \\ &= -2\,(3.0km\hat{\imath} + 1.0km\hat{\jmath}) \\ &= -6.0km\hat{\imath} - 2.0km\hat{\jmath} \qquad (1.25)\end{aligned}$$

so finally

$$\begin{aligned}\Delta\vec{x}_{AD} &= \Delta\vec{x}_{AB} + \Delta\vec{x}_{BD} \\ &= (3.0km\hat{\jmath}) + (-6.0km\hat{\imath} - 2.0km\hat{\jmath}) \\ &= -6.0km\hat{\imath} + 1.0km\hat{\jmath} \qquad (1.26)\end{aligned}$$

This tells us that the distance from A to D is $6.1km$ (i.e. $|\Delta\vec{x}_{AD}|$), at an angle of 170.4° measured clockwise from the x-axis. Since the x-axis ran positive to the East, this is the same as 9.6° North of West. A comparison with figure 1.7 suggests this is reasonable.

Student Exercises

- Repeat the question above taking the x-axis as going along the line from A to C. You'll have to do a bit of trigonometry to figure out the angle the line from A to B makes with your x-axis. You should get exactly the same distance from A to D; the angle you get *measured from the x-axis* should be different, but when you express the angle with respect to the direction West, it should be the same.

  This exercise is important because it shows that the coordinate system (i.e. the direction of the x and y axes) isn't critical. As long as you express things carefully in components, you'll get the same final answer no matter what coordinates you decide to use.

  In this, you should find that $\Delta\vec{x}_{AC} = 5km\hat{\imath}$, *so* $\Delta\vec{x}_{AB} = 2.4km\hat{\imath} + 1.8km\hat{\jmath}$, *and get that* $\Delta\vec{x}_{AD} = -2.8km\hat{\imath} + 5.4km\hat{\jmath}$. *This vector has* *a magnitude of* $6.1km$*, as expected. It makes an angle of* $117°$ *counterclockwise from the line from A to C, and since the line from A to C makes an angle of* $53°$ *with 'due East', comparison makes it clear that the angle with 'due West' is exactly what was calculated in the solution.*

- Towns A, B, and C are at $2.0km\hat{\imath} - 3.0km\hat{\jmath}$, $-4.0km\hat{\imath} + 2.0km\hat{\jmath}$, and $1.0km\hat{\imath} + 1.0km\hat{\jmath}$ respectively. Town D is half as far from C as B is from C, and the direction from D to C is the same direction as B is from A. How far is D from B, and what angle does the vector from B to D make with the x-axis?

  After a bit of work we find that $\vec{r}_D = -0.96km\hat{\imath}+2.63km\hat{\jmath}$*. This means that the distance from B to D is* $3.1km$*. The vector from B to D makes an angle of about* $11°$ *with the x-axis.*

1.4 Multiplying Vectors by other Vectors

We have talked about algebra with vectors, in particular adding and multiplying vectors by scalars. Those rules are designed to be as similar to the 'normal' rules for addition and multiplication as possible. However, when we multiply vectors together, we have to take into account their direction as well as their magnitude. There are two different ways to do this, depending on the circumstances. You can multiply a vector by another vector and get a *scalar* (i.e. a number) if you use the dot product (also called the 'inner product' in math). You can multiply a vector by another vector and get a *vector* if you use the cross product (sometimes called the 'vector product' or 'outer product') – interestingly, this only works in 3-dimensions, and that's because two vectors define a plane, and there is a unique direction which is normal to a plane.

1.4.1 Calculating the Dot Product

Example Vector $\vec{A}$ has a magnitude of $5.0m$ and makes an angle of $53.1°$ counterclockwise from the positive x-axis, vector $\vec{B}$ has components $B_x = -6.0m$ and $B_y = 8.0m$, and vector $\vec{C}$ has magnitude of $5.0m$ and lies along the positive x-axis.

Find $\vec{A} \cdot \vec{B}$, $\vec{A} \cdot \vec{C}$ and $\vec{B} \cdot \vec{C}$.

Worked Solution There are two ways to calculate the Dot product: to use the vectors' components, or to use the angle between the vectors and

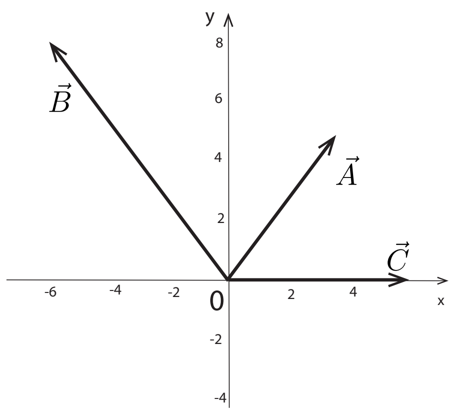


Figure 1.9: Vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ drawn on a common axis

their magnitude. The two approaches need to give the same result.

First, we can express the three vectors, $\vec{A}$, $\vec{B}$, and $\vec{C}$ in components. We find that

$$\begin{aligned}\vec{A} &= 3.0m\hat{\imath} + 4.0m\hat{\jmath} \\ \vec{B} &= -6.0m\hat{\imath} + 8.0m\hat{\jmath} \\ \vec{C} &= 5.0m\hat{\imath}. \end{aligned} \tag{1.27}$$

Using the fact that $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ (the z-components are zero in this case. We included them to be explicit about what happens in 3-dimensions) we can immediately calculate

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (3.0m)(-6.0m) + (4.0m)(8.0m) = 14.0m^2 \\ \vec{A}\cdot\vec{C} &= (3.0m)(5.0m) + (4.0m)(0m) = 15.0m^2 \\ \vec{B}\cdot\vec{C} &= (-6.0m)(5.0m) + (8.0m)(0m) = -30.0m^2 \end{aligned} \tag{1.28}$$

This highlights the important point that the inner product can be positive or negative (if the angle between the two vectors is 90° then the dot product is exactly 0)

The angle method works just as well. The problem states that the angle between $\vec{A}$ and $\vec{C}$ is 53.1°, and their magnitudes are given. We can use the components of $\vec{B}$ to get that $\left|\vec{B}\right| = 10.0m$, and that $\vec{B}$ makes an angle of 126.9° measured from the positive x-axis. This means the angle between $\vec{A}$

and $\vec{B}$ is 73.8° also. So:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (5.0m)(10.0m)\cos 73.8^\circ = 14m^2\\ \vec{A}\cdot\vec{C} &= (5.0m)(5.0m)\cos 53.1^\circ = 15m^2\\ \vec{B}\cdot\vec{C} &= (10.0m)(5.0m)\cos 126.9^\circ = -30m^2\end{aligned} \tag{1.29}$$

If we look at Figure 1.9 we see that $\vec{A}$ and $\vec{B}$, and $\vec{A}$ and $\vec{C}$ point in similar directions, so their dot product is positive. However, $\vec{B}$ and $\vec{C}$ point in directions that are more or less opposite to each other; their dot product is negative.

Student Exercises

- $\vec{A}$ has a magnitude of $3.0m$ and points at an angle of 30° counterclockwise from the x-axis, $\vec{B}$ has a magnitude of $5.0m$ and points at an angle of 225° counterclockwise from the x-axis, and $\vec{C} = -3.0m\hat{\imath} + 2.0m\hat{\jmath}$. Find $\vec{A}\cdot\vec{B}$, $\vec{A}\cdot\vec{C}$ and $\vec{B}\cdot\vec{C}$. Make sure you can do it with both components and magnitudes and angles. *We find that* $\vec{A}\cdot\vec{B} = -14.5m^2$, $\vec{A}\cdot\vec{C} = -4.8m^2$, *and* $\vec{B}\cdot\vec{C} = 3.5m^2$.

- Find the angle between $\vec{A} = 3m\hat{\imath} + 4m\hat{\jmath}$ and $\vec{B} = -4m\hat{\imath} + 3m\hat{\jmath}$. *It's* 90°.

Page 16 [Workbook] Question Solved

- Find a point which is $2.0m$ away from $\vec{A} = 3.0m\hat{\imath} + 2.0m\hat{\jmath}$ along a line which is at 90° to the line between $\vec{A}$ and the point $\vec{B} = 4.0m\hat{\imath}+3.0m\hat{\jmath}$. *We find that the unit vector which is at 90° to the line from* $\vec{A}$ *to* $\vec{B}$ *is* $-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$ – *assuming we are restricted to the xy plane. This means that there are two points which satisfy the criterion specified:* $1.6m\hat{\imath} + 3.4m\hat{\jmath}$ *and* $4.4m\hat{\imath} + 0.6m\hat{\jmath}$.

1.4.2 Calculating the Cross Product

Example For the vectors $\vec{A} = 5.0m\hat{\imath}$, $\vec{B} = 2.0m\hat{\imath} + 2.0m\hat{\jmath}$, $\vec{C} = 3.0m\hat{\imath} - 4.0m\hat{\jmath}$, and $\vec{D} = 5.0m\hat{\imath} + 12.0m\hat{k}$ calculate: $\vec{A}\times\vec{B}$, $\vec{B}\times\vec{A}$, $\vec{A}\times\vec{C}$, $\vec{C}\times\vec{A}$, $\vec{A}\times\vec{D}$, and $\vec{D}\times\vec{A}$.

Worked Solution In this, we have to remember that the vector product depends on order. For any vectors $\vec{A}$ and $\vec{B}$, it will always be true that $\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$.

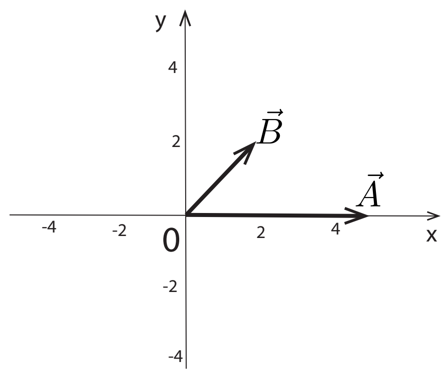


Figure 1.10: Vectors $\vec{A} = 5m\hat{\imath}$ and $\vec{B} = 2m\hat{\imath} + 2m\hat{\jmath}$ drawn on a common axis

The magnitude of the vector $\vec{B}$ given in the problem is $2.83m$ and it makes an angle of $45°$ from the x-axis in the x-y plane as shown in Figure 1.10. If we calculate the magnitude, and use the right-hand rule to find direction, we know that

$$\left|\vec{A} \times \vec{B}\right| = (5.0m)(2.83m)\sin 45° = 10.0m^2 \tag{1.30}$$

This is the same as the magnitude of $\vec{B} \times \vec{A}$. Using the right-hand rule, the direction of $\vec{A} \times \vec{B}$ will be (as drawn) out of the page, so in the positive z-direction, giving $\vec{A} \times \vec{B} = 10.0m^2\hat{k}$. This means that $\vec{B} \times \vec{A}$ will be in the negative z-direction.

We can use the component definition of the cross product too; if $\vec{C} = \vec{A} \times \vec{B}$ then

$$\begin{aligned}
C_x = \left(\vec{A} \times \vec{B}\right)_x &= A_yB_z - A_zB_y \\
C_y = \left(\vec{A} \times \vec{B}\right)_y &= A_zB_x - A_xB_z \\
C_z = \left(\vec{A} \times \vec{B}\right)_z &= A_xB_y - A_yB_x
\end{aligned} \tag{1.31}$$

So, with the $\vec{A}$ and $\vec{B}$ given:

$$\begin{aligned}
\left(\vec{A} \times \vec{B}\right)_x &= A_yB_z - A_zB_y = (0m)(0m) - (0m)(2m) = 0m^2 \\
\left(\vec{A} \times \vec{B}\right)_y &= A_zB_x - A_xB_z = (0m)(2m) - (5m)(0m) = 0m^2 \\
\left(\vec{A} \times \vec{B}\right)_z &= A_xB_y - A_yB_x = (5m)(2m) - (0m)(2m) = 10m^2
\end{aligned} \tag{1.32}$$

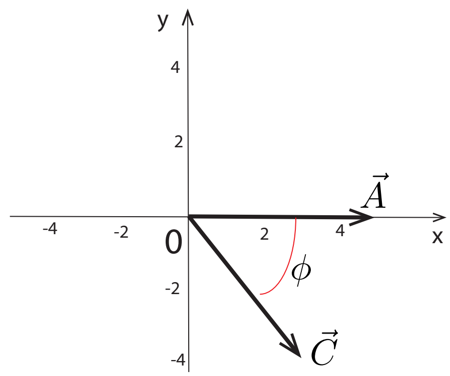


Figure 1.11: Vectors $\vec{A} = 5m\hat{\imath}$ and $\vec{C} = 3m\hat{\imath} - 4m\hat{\jmath}$ drawn on a common axis

this matches the result we got above.

Figure 1.11 shows the situation for the given vectors $\vec{A}$ and $\vec{C}$. Note $\vec{C}$ is the vector given in the initial statement of the problem and not the one we just worked out. In this case, the angle between the two vectors is 53.1° (note that the angle between $\vec{C}$ and the positive x-axis is 306.9° when measured counterclockwise). The important thing to note here is that the angle ϕ between the two vectors is always measured in such a way (for cross products) that it is between 0° and 180°. Applying the relationship for the magnitude of the cross-product, we find that

$$\left|\vec{A} \times \vec{C}\right| = (5m)(5m)\sin 53.1^\circ = 20m^2. \tag{1.33}$$

The right-hand rule gives the direction of the cross product as into the page. The figure is drawn with the x-axis horizontally and the y-axis up the page; in this configuration the positive z-axis will be out of the page, so a vector *into* the page points in the *negative* z-direction. This means that $\vec{A} \times \vec{C} = -20m^2\hat{k}$.

We can similarly find the components:

$$\begin{aligned}
\left(\vec{A} \times \vec{C}\right)_x &= A_y C_z - A_z C_y = (0m)(0m) - (0m)(-4m) = 0m^2 \\
\left(\vec{A} \times \vec{C}\right)_y &= A_z C_x - A_x C_z = (0m)(3m) - (5m)(0m) = 0m^2 \\
\left(\vec{A} \times \vec{C}\right)_z &= A_x C_y - A_y C_x = (5m)(-4m) - (0m)(3m) = -20m^2
\end{aligned} \tag{1.34}$$

this matches the result we got above.

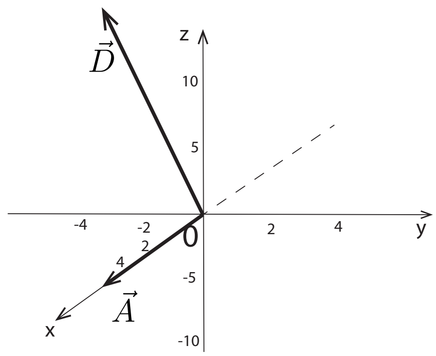


Figure 1.12: Vectors $\vec{A} = 5m\hat{\imath}$ and $\vec{D} = 5m\hat{\imath} + 12m\hat{k}$ drawn on a common axis

Figure 1.12 gives a three-dimensional representation of the vectors $\vec{A}$ and $\vec{D}$. With the x-axis coming out of the paper (as drawn, obliquely) the vector $\vec{A} \times \vec{D}$ is going to point in the negative y-direction. We see this using the component expression:

$$\begin{aligned}
\left(\vec{A} \times \vec{D}\right)_x &= A_y D_z - A_z D_y = (0m)(12m) - (0m)(0m) = 0m^2 \\
\left(\vec{A} \times \vec{D}\right)_y &= A_z D_x - A_x D_z = (0m)(5m) - (5m)(12m) = -60m^2 \\
\left(\vec{A} \times \vec{D}\right)_z &= A_x D_y - A_y D_x = (5m)(0m) - (0m)(5m) = 0m^2 \qquad (1.35)
\end{aligned}$$

Student Exercises

- Use the component method to check that if you change the order of the things multiplied in the cross product you will get the opposite vector. *The key thing you'll see is that since the order of the vectors being multiplied changed the pair that gets the negative sign changes, so all components swap signs.*

- Find the angle between $\vec{A}$ and $\vec{D}$ above, and verify that the magnitude of their cross-product is $60m^2$. *The angle is* $67.4°$*, and so the magnitude of the cross product (using* sin *of the angle between the vectors) is* $(5m)(13m)\sin 67.4° = 60m^2$.

- Consider the following 6 points: O at the origin, A at $1m\hat{\imath}$, B at $1m\hat{\jmath}$, C at $1m\hat{k}$, D at $1m\hat{\imath} + 1m\hat{\jmath}$, and E at $1m\hat{\imath} + 1m\hat{\jmath} + 1m\hat{k}$. Call the vector from O to A $\Delta\vec{x}_{OA}$, and name a bunch of the others similarly. Calculate the various cross-products you can. What is the difference between $\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OB}$ and $\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OD}$?

There are a lot of possible cross products, here are a few:

$$\begin{aligned}
\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OB} &= 1m^2\hat{k} \\
\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OC} &= -1m^2\hat{\jmath} \\
\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OD} &= 1m^2\hat{k} \\
\Delta\vec{x}_{OA} \times \Delta\vec{x}_{OE} &= -1m^2\hat{\jmath} + 1m^2\hat{k} \\
\Delta\vec{x}_{OB} \times \Delta\vec{x}_{OC} &= 1m^2\hat{\imath} \\
\Delta\vec{x}_{OB} \times \Delta\vec{x}_{OD} &= -1m^2\hat{k} \\
\Delta\vec{x}_{OB} \times \Delta\vec{x}_{OE} &= 1m^2 - 1m^2\hat{k}\hat{\imath} \\
&\ldots
\end{aligned} \tag{1.36}$$

Note that some of these cross products are the same, since parallel components do not contribute to a cross product.

1.5 Questions

1. Consider the vector $\vec{A} = 3m\hat{\imath} - 5m\hat{\jmath} + 4m\hat{k}$. What is
   - Its magnitude?
   - Its x-component? Y-component? Z-component?
   - the angle between it and the unit vector $\hat{\imath}$? $\hat{\jmath}$? $\hat{k}$?
2. What is the vector from $\vec{r}_1 = 3m\hat{\imath} - 4m\hat{\jmath}$ to $\vec{r}_2 = 6m\hat{\imath} + 2m\hat{\jmath}$? What is its magnitude? What is the *unit vector* from $\vec{r}_1$ to $\vec{r}_2$?
3. What is the magnitude of $3\vec{A} + 2\vec{B}$ where $\vec{A} = 5m\hat{\imath} + 2m\hat{\jmath}$ and $\vec{B} = -4m\hat{\imath} + 4m\hat{\jmath}$? What is the angle between this vector and the y-axis (ie $\hat{\jmath}$)? What is its x-component?
4. Consider $\vec{A} = 3m\hat{\imath} + 4m\hat{\jmath} - 5m\hat{k}$ and $\vec{B} = 2m\hat{\imath} - 5m\hat{\jmath} + 3m\hat{k}$. What is $\vec{A} \cdot \vec{B}$? What is the angle between $\vec{A}$ and $\vec{B}$?
5. Consider $\vec{A} = 3m\hat{\imath} + 4m\hat{\jmath} - 5m\hat{k}$ and $\vec{B} = 2m\hat{\imath} - 5m\hat{\jmath} + 3m\hat{k}$. What is $\vec{A} \times \vec{B}$? What is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$?

1.6 Answers

1. They are
   - Magnitude: $7.1m$
   - X-component: $3m$. Y-component: $-5m$. Z-component: $4m$.
   - Angle with $\hat{\imath}$: $64.9°$. With $\hat{\jmath}$: $135°$. With $\hat{k}$: $55.6°$.
2. The vector from $\vec{r}_1$ to $\vec{r}_2$ is $3m\hat{\imath} + 6m\hat{\jmath}$. Its magnitude is $6.7m$, and the unit vector in that direction is $0.447\hat{\imath} + 0.894\hat{\jmath}$.
3. The magnitude is $15.7m$, and the angle between that vector and the y-axis is $26.6°$. The x-component is $7m$.
4. The inner product is $-29m^2$, so the angle between the vectors is $131.7°$.
5. The cross product is $-13m^2\hat{\imath} - 19m^2\hat{\jmath} - 23m^2\hat{k}$. The corresponding unit vector is $-0.399\hat{\imath} - 0.584\hat{\jmath} - 0.707\hat{k}$.

Chapter 2

Translational Equilibrium

2.1 Summary

The second chapter of the text talks about the conditions for equilibrium. This material is presented after the material on vectors because the conditions for equilibrium are conditions on *Forces*. Since forces are vector quantities, the conditions on forces are themselves vector conditions.

The key points from the chapter are:

- An object which is in (translational) equilibrium is subject to no net force. This means that

$$\vec{F}_{net} = \sum_{all\ forces} \vec{F}_i = 0 \tag{2.1}$$

  This is Newton's first law.

- Translational equilibrium means that if the object is at rest it stays at rest, and if it is moving it moves with constant speed and direction. We will learn that motion with constant speed and direction is called constant 'velocity' motion, but that discussion will happen in a later chapter.

- If an object is subject to no force we can infer it is in equilibrium. If we see an object moving with constant velocity we can infer it is subject to no net force.

- When two objects interact, A and B, interact, they exert forces of equal magnitude in opposite directions on each other.

$$\vec{F}_{A\ on\ B} = -\vec{F}_{B\ on\ A} \tag{2.2}$$

This is Newton's third law.

- When an object of mass m is near the surface of the Earth, it is subject to a constant gravitional force

$$\vec{F}_g = -mg\hat{k} \tag{2.3}$$

The numerical value of g is about $9.8\frac{m}{s^2} = 9.8\frac{N}{kg}$. On a different planet, g would be different.

- When an object is touching another, they exert 'contact' forces on each other. A very common example of this contact force is the *Normal Force*. The normal force is exerted by a surface on an object resting on it. The normal force can be *whatever is needed* to make sure that the resting object does not fall through the surface. A smooth surface can only exert a normal force.

- We will often denote the normal force as $\vec{F}_n$ and the direction of the normal force as $\hat{n}$. The normal force is always at 90° to the surface. Note that if you know the vectors along the surface you can use a cross product to determine the vector normal to the surface.

- A rough surface can exert a force parallel to it. This force is called the force of friction.

- If an object is stationary relative to a rough surface, it can be subject to the force of *static* friction. The magnitude of this force satisfies

$$\left|\vec{F}_s\right| \leq \mu_s \left|\vec{F}_n\right| \tag{2.4}$$

The value of μ_s, the coefficient of static friction, depends on the surfaces which are in contact. It is very important to notice that the force of static friction can be whatever it needs to be up to a maximum given by equation 2.4, and it will be in whatever direction (along the surface) is needed to keep the object in equilibrium.

- If an object is moving relative to a rough surface, it will be subject to the force of *kinetic* friction. The magnitude of this force is

$$\left|\vec{F}_k\right| = \mu_k \left|\vec{F}_n\right| \tag{2.5}$$

The force is in the opposite direction of motion.

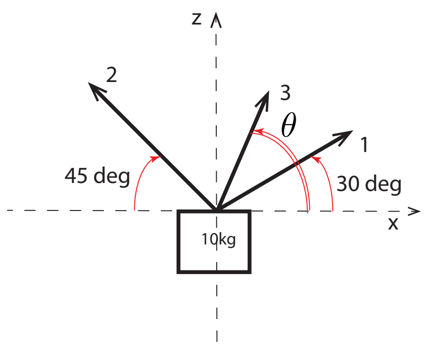


Figure 2.1: A box is suspended by three different ropes. Rope 1 has a tension of $70N$ and pulls at an angle $30°$ above the positive x-axis. Rope 2 has a tension of $70.7N$ and pulls at an angle of $45°$ above the negative x-axis. Rope 3 has an unknown tension T and pulls at an unknown angle θ.

2.2 Finding the force required to keep an object in equilibrium

When we say that an object is in equilibrium, this means that the net force on it vanishes. That the net force is zero is the same as saying that the total x, y, and z components of the force are all individually zero. One of the mathematical techniques we would like you to develop is the habit of always treating forces as vectors, and so carefully expressing them in terms of components.

2.2.1 A box suspended from some ropes

Example A box of mass $10kg$ is suspended by three ropes, as shown in figure 2.1. Find the tension in rope 3, and find the angle that rope 3 makes with respect to the positive x-axis.

Worked Solution First, we need to have a plan about how to solve this problem, and then we will set about executing it. The general principle is that the box is in equilibrium, and since it is in equilibrium we know that it must be subject to zero net force. We know what three of the applied forces are – those from the first two ropes and gravity, and we can use that to get the force that the third rope must exert. Once we know the force that the third rope exerts in component form, we can find the magnitude and direction of the force.

Note that this solution uses $\hat{k}$ to descripe the unit vector upwards - we are calling the vertical direction z. We could have called it y without breaking anything, but as a pedagogical choice, we choose to call it z with a view to future problems where things will be explicitly three-dimensional.

The first thing we need to do is apply the concept of equilibrium, so the net force is zero:

$$\begin{aligned}
0 &= \vec{F}_{net} \\
\vec{F}_{net} &= \vec{F}_{rope\ 1} + \vec{F}_{rope\ 2} + \vec{F}_{rope\ 3} + \vec{F}_g \\
0 &= \vec{F}_{rope\ 1} + \vec{F}_{rope\ 2} + \vec{F}_{rope\ 3} + \vec{F}_g \\
\vec{F}_{rope\ 3} &= -\vec{F}_{rope\ 1} - \vec{F}_{rope\ 2} - \vec{F}_g \qquad (2.6)
\end{aligned}$$

With this, we have, in principle, completely solved the problem, since we *know* the forces due to the two ropes and gravity. Let us express the forces due to the ropes and gravity in components. (If you have trouble remembering how to do this, re-read the previous workbook chapter and the corresponding , and look at section 1.2.) We find that

$$\begin{aligned}
\vec{F}_{rope\ 1} &= 60.6N\hat{\imath} + 35.0N\hat{k} \\
\vec{F}_{rope\ 2} &= -50.0N\hat{\imath} + 50.0N\hat{k} \\
\vec{F}_g &= -mg\hat{k} = -98N\hat{k} \qquad (2.7)
\end{aligned}$$

Applying this to what we derived for the force in the third rope 2.6, we find that

$$\begin{aligned}
\vec{F}_{rope\ 3} &= -\left(60.6N\hat{\imath} + 35.0N\hat{k}\right) - \left(-50.0N\hat{\imath} + 50.0N\hat{k}\right) - \left(-98N\hat{k}\right) \\
&= -10.6N\hat{\imath} + 13N\hat{k} \qquad (2.8)
\end{aligned}$$

We now know that the x-component of $\vec{F}_{rope\ 3}$ is $F_{rope\ 3,x} = -10.6N$, and the z-component is $F_{rope\ 3,z} = 13N$. Now that we have the components, we can find the magnitude: $\left|\vec{F}_{rope\ 3}\right| = 16.8N$, and find that the angle it makes with the positive x-axis is $129.2°$ in the xz plane.

A couple comments about the solution We could have done this solution, and gotten the same answer, by writing from the start that the total force in the x-direction was zero, and solved for the x-component of force due to the third rope, and then doing the same thing in the z-direction. The reason we wrote it using vectors was that this way all the components come along 'at once'.

You may have learned to solve problems like this by drawing triangles and using some trigonometry (like the cosine law) to get magnitudes of the unknown force. The reason we did not do that here is that triangle-based technique is hard to scale up to problems with more than three forces. Writing things using vectors will always work.

In this problem there were two things we did not know (the magnitude and angle of the vector). When you need to get a numerical answer for two things, you typically need two pieces of information; in our case, we knew three of the applied forces, and we knew that the total force in both the x-direction and the z-direction was zero. Knowing the total force in one direction is one piece of information.

Student Exercises

- Consider the situation in Figure 2.1. What would the unknown tension and angle be if:
  - The tension in rope 2 was $141.4N$? *The tension would be* $54.5N$ *at an angle of* $43°$ below *the horizontal. The rope would be pulling down and to the right.*
  - The mass of the box was $6.0kg$? *The tension would be* $28.3N$ *and the angle would be* $68°$ *below the* $-\hat{\imath}$ *axis. This means the rope pulls down and to the left.*
- A $20.0kg$ box is suspended by three ropes. One pulls up and to the right making an angle of $30°$ with the vertical with a tension of $200N$. Another pulls directly left with a tension of $130N$. What is the direction and magnitude of the tension in the third rope? *The tension is* $38N$ *at an angle of* $37°$ *above the positive x-axis* ($\hat{\imath}$).

2.2.2 Equilibrium on a slope with friction

Example A box of mass m rests on a slope which makes an angle θ with the horizontal as shown in figure 2.2. There is a coefficient of static friction of μ_s between the box and the slope.

1. If $m = 10kg$, $\theta = 30°$, and $\mu_s = 0.8$, what is the magnitude of the friction force on the box?
2. If $\mu_s = 0.8$, what is the biggest θ can be before the box starts to slide?

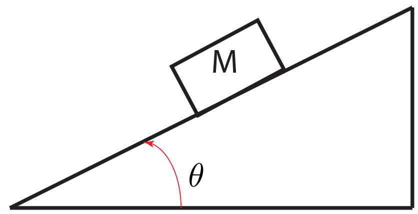


Figure 2.2: A box of mass m is on a rough slope which makes an angle θ with the horizontal.

Worked Solution We will follow a general pattern in this solution: we want to solve the problem *in general* and only add in numbers at the very last place we can. The way to think of this is that *we're not showing you how to put numbers into formulae; we're showing you how to figure out formulae that you can then put numbers into.*

This problem is also an equilibrium problem: there are a total of three forces on the box - the force of gravity, the normal force, and the force of friction. Their sum, as a vector, must be equal to zero. This means that we are going to have to figure out the magnitudes and directions of the normal force and the force of friction.

When we were talking about vectors, one of the lessons from the student exercises in section 1.3 was that it does not matter what coordinate system you pick, as long as you are careful to do the algebra correctly. We are going to solve this problem in two different coordinate systems - it will turn out that one is *calculationally* simpler, but you need to be a little 'inspired' to think of it. Physics is wonderful in that there are lots of ways to get to correct solutions; it is just that some require more work *implementing* and some require more work *setting up*.

We can set up the free-body diagram as shown in part (a) of Figure 2.3. In this, the condition of equilibrium is that $\vec{F}_{net} = 0$, so

$$\vec{F}_g + \vec{F}_n + \vec{F}_f = 0 \tag{2.9}$$

As in the previous question, we have to spend some time finding the components of the vectors. We do this (first) in the x-z coordinate system given in part (b) of Figure 2.3. We do not have a *number* for what θ is, but that is OK; we treat it as though it is given. We also do not know the magnitudes of

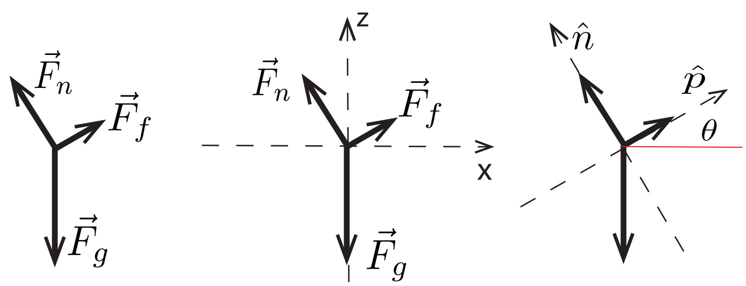


Figure 2.3: Free body diagrams for a box of mass m on a rough slope which makes an angle θ with the horizontal.

either the friction force or the normal force, but that is also not a problem; we decide to call them $\left|\vec{F}_f\right|$ and $\left|\vec{F}_n\right|$ respectively - they are variables we need to solve for. Note that the normal force $\vec{F}_n$ makes an angle of $90^\circ + \theta$ with the positive x-axis measured counterclockwise.

$$\begin{aligned}
\vec{F}_g &= -mg\hat{k} \\
\vec{F}_f &= \left|\vec{F}_f\right| \cos\theta\hat{\imath} + \left|\vec{F}_f\right| \sin\theta\hat{k} \\
\vec{F}_n &= \left|\vec{F}_n\right| \cos\left(90^\circ + \theta\right)\hat{\imath} + \left|\vec{F}_n\right| \sin\left(90^\circ + \theta\right)\hat{k} \\
&= -\left|\vec{F}_n\right| \sin\theta\hat{\imath} + \left|\vec{F}_n\right| \cos\theta\hat{k}
\end{aligned} \tag{2.10}$$

The last equality in 2.10 comes from a trigonometric identity, but you could have gotten it in other ways.

Now, we will put the component expressions from 2.10 into the expression for equilibrium. This says that

$$\begin{aligned}
0 &= \vec{F}_g + \vec{F}_f + \vec{F}_n \\
&= -mg\hat{k} + \left(\left|\vec{F}_f\right| \cos\theta\hat{\imath} + \left|\vec{F}_f\right| \sin\theta\hat{k}\right) + \left(-\left|\vec{F}_n\right| \sin\theta\hat{\imath} + \left|\vec{F}_n\right| \cos\theta\hat{k}\right) \\
&= \left(\left|\vec{F}_f\right| \cos\theta - \left|\vec{F}_n\right| \sin\theta\right)\hat{\imath} + \left(-mg + \left|\vec{F}_f\right| \sin\theta + \left|\vec{F}_n\right| \cos\theta\right)\hat{k}
\end{aligned} \tag{2.11}$$

Notice that the last line of 2.11 is in fact two relationships, since the x-component and the z-component both vanish; we need two relationships because there are two unknowns we have to solve for.

There are two things we do not know: $\left|\vec{F}_f\right|$ and $\left|\vec{F}_n\right|$. We have two relations that involve both of them. We will solve them together. The two relations implied by 2.11 are

$$\begin{aligned}
\text{x component}:\ 0 &= \left|\vec{F}_f\right|\cos\theta - \left|\vec{F}_n\right|\sin\theta \\
\text{which gives } \left|\vec{F}_f\right| &= \left|\vec{F}_n\right|\frac{\sin\theta}{\cos\theta} \qquad (2.12) \\
\text{z component}:\ 0 &= -mg + \left|\vec{F}_f\right|\sin\theta + \left|\vec{F}_n\right|\cos\theta \\
\text{which gives } mg &= \left|\vec{F}_f\right|\sin\theta + \left|\vec{F}_n\right|\cos\theta \qquad (2.13)
\end{aligned}$$

Our plan is to use the two equations 2.12 and 2.13 to get the magnitudes of the two vectors. We substitute $\left|\vec{F}_f\right|$ from 2.12 into 2.13 to get $\left|\vec{F}_n\right|$, and then substitute that back in. We have

$$\begin{aligned}
mg &= \left|\vec{F}_f\right|\sin\theta + \left|\vec{F}_n\right|\cos\theta \\
mg &= \left|\vec{F}_n\right|\frac{\sin\theta}{\cos\theta}\sin\theta + \left|\vec{F}_n\right|\cos\theta \\
mg\cos\theta &= \left|\vec{F}_n\right|\sin^2\theta + \left|\vec{F}_n\right|\cos^2\theta \\
\left|\vec{F}_n\right| &= mg\cos\theta \qquad (2.14)
\end{aligned}$$

Finally, putting 2.14 into 2.12 we get

$$\begin{aligned}
\left|\vec{F}_f\right| &= \left|\vec{F}_n\right|\frac{\sin\theta}{\cos\theta} \\
&= mg\cos\theta\frac{\sin\theta}{\cos\theta} = mg\sin\theta \qquad (2.15)
\end{aligned}$$

There! We are done. We have $\left|\vec{F}_f\right|$ and $\left|\vec{F}_n\right|$.

You might be wondering *"that seemed like a lot of work, is there an easier way?"* We could instead have used the coordinate system in part (c) of Figure 2.3. In this, we define the $\hat{n}$ direction to be along the line at 90° to the surface, and the $\hat{p}$ direction to be along the surface (up it, in fact). Then, the expression of equilibrium is the same, but the components of the vectors are different. The key thing to notice is that the force of gravity makes an angle of $90° + \theta$ measured clockwise from $\hat{p}$, so it makes an angle of $270° - \theta$ counterclockwise from $\hat{p}$. This means that, in terms of the unit vectors $\hat{n}$ and $\hat{p}$ we have:

$$\begin{aligned}
\vec{F}_f &= \left|\vec{F}_f\right| \hat{p} \\
\vec{F}_n &= \left|\vec{F}_n\right| \hat{n} \\
\vec{F}_g &= mg\cos\left(270^\circ - \theta\right)\hat{p} + mg\sin\left(270^\circ - \theta\right)\hat{n} \\
&= -mg\sin\theta\hat{p} - mg\cos\theta\hat{n} \qquad (2.16)
\end{aligned}$$

So, the condition of equilibrium is

$$\begin{aligned}
0 &= \vec{F}_g + \vec{F}_f + \vec{F}_n \\
&= \left(-mg\sin\theta\hat{p} - mg\cos\theta\hat{n}\right) + \left|\vec{F}_f\right|\hat{p} + \left|\vec{F}_n\right|\hat{n} \\
&= \left(\left|\vec{F}_f\right| - mg\sin\theta\right)\hat{p} + \left(\left|\vec{F}_n\right| - mg\cos\theta\right)\hat{n} \qquad (2.17)
\end{aligned}$$

So, since a vector only vanishes if its components all individually vanish, 2.17 tells us, just as 2.15 and 2.14 did, that:

$$\begin{aligned}
\left|\vec{F}_f\right| &= mg\sin\theta \\
\left|\vec{F}_n\right| &= mg\cos\theta \qquad (2.18)
\end{aligned}$$

This puts us into the position of being able to answer part one of the original question: we find that, for the given values of m and θ we have $\left|\vec{F}_f\right| = 49N$, and $\left|\vec{F}_n\right| = 84.9N$. You might wonder why we did not use the information about the coefficient of friction? Well, all that the coefficient of static friction tells us is a maximum value for the magnitude of the force of friction. We can check that, in this case $49N = \left|\vec{F}_f\right| \leq \mu_s \left|\vec{F}_n\right| = 0.8 \times 84.9N = 67.9N$. This means that the amount of friction *required* for equilibrium is less than the maximum force that static friction could exert.

Now, we turn our attention to the second part: what is the maximum angle θ (as illustrated in Figure 2.2) before the box starts to slide? The free-body diagram is exactly the same as before; there are three forces: gravity, static friction, and the normal force. If we know m and θ we can determine that

$$\begin{aligned}
\left|\vec{F}_f\right| &= mg\sin\theta \\
\left|\vec{F}_n\right| &= mg\cos\theta
\end{aligned}$$

just like we did before. We need to think about what happens as θ changes: As θ increases, $\sin\theta$ increases, and $\cos\theta$ decreases. This means that the *needed* magnitude of the force of friction is going to increase, while the magnitude of the normal force decreases. However, since the magnitude of the normal force determines the maximum possible force of friction that means that the maximum possible force of friction will *decrease*. At some point, in order for there to be equilibrium the force of friction would need to be *more than it can be*. Then the block won't have a net force of zero - we are going to learn explicitly what happens when the net force is non-zero later, but for now, we can just say it starts to slide.

We know that for the box not to slide we need

$$\begin{aligned}
\text{Force of friction} &\leq \text{Max possible force of friction} \\
\left|\vec{F}_f\right| &\leq \mu_s \left|\vec{F}_n\right| \\
mg\sin\theta &\leq \mu_s mg\cos\theta \\
\frac{\sin\theta}{\cos\theta} &\leq \mu_s \qquad (2.19)
\end{aligned}$$

so, the biggest θ where the box won't slide is one where $\tan\theta = \mu_s$. In the case we had, with $\mu_s = 0.8$, the maximum angle is $\theta = 38.7°$.

Some of the key things you should see in the solution We did this example for a couple of reasons:

- We wanted to point out that you can choose any coordinate system to do your solution, and it will work out as long as you're careful.
- Another key thing is notice that we refrained from putting numbers in right away: we kept everything as variables for as long as possible. This is a good habit that will make physics (and similar disciplines) easier in the long run.
- Also notice that we saw that the normal force does not always have the magnitude mg. As well, we saw that the force of static friction isn't always $\mu_s \left|\vec{F}_n\right|$; instead it can vary *up to* that amount.

Student Exercises

- Consider the box of mass $m = 10kg$ shown in Figure 2.4. It is being pulled by a force of magnitude T at an angle of θ above the horizontal, as shown, and it is on a rough surface with which it has a coefficient of static friction of $\mu_s = 0.6$.

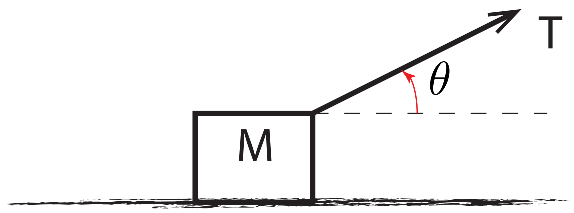


Figure 2.4: A box of mass m is on a rough surface with which it has a coefficient of static friction μ_s. It is stationary, but is being pulled by a force of magnitude T which makes an angle θ with the positive x-axis.

- If $T = 50N$, and $\theta = 45°$, what are the magnitudes of the normal and friction forces? *The magnitude of the normal force is* $62.6N$ *and of the friction force is* $35.4N$.
- If $\theta = 30°$, what is the maximum T that can be applied before the box starts to slide? *The box will start to slide when* $T > 50.4N$.

• Consider the box shown in figure 2.5. It is on a rough surface with which it has a coefficient of static friction μ.

- If $m = 20kg$, $\theta = 15°$, and $T = 65N$, what is the magnitude *and direction* of the friction force? *The friction force has a magnitude* $14.3N$ *down the slope.*
- If $m = 5kg$, $\mu_s = 1.0$, and the maximum tension T before the string breaks is $25N$, what is the largest angle at which the block can be held in equilibrium? What if $\mu_s = 0.4$? Hint: this is hard to solve analytically, create the inequality you need to solve and then graph it or use a linear approximation. *You end up finding that the angle is about* $66.1°$ *for* $\mu_s = 1.0$ *and about* $50.1°$ *for* $\mu_s = 0.4$.

2.2.3 Equilibrium of a moving object

Example A box of mass m is pulled at a constant speed in a straight line along a rough surface with which it has a coefficient of static friction of μ_k by a force of magnitude T at an angle of θ with the positive x-axis, as shown in figure 2.6.

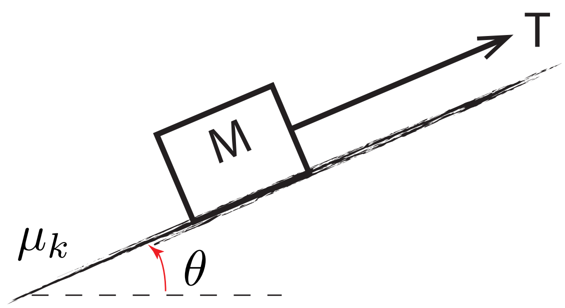


Figure 2.5: A box of mass m is stationary on a rough surface with which it has a coefficient of static friction μ_s. It is being pulled by a force of magnitude T which is parallel to the slope. The slope makes an angle θ with the positive x-axis.

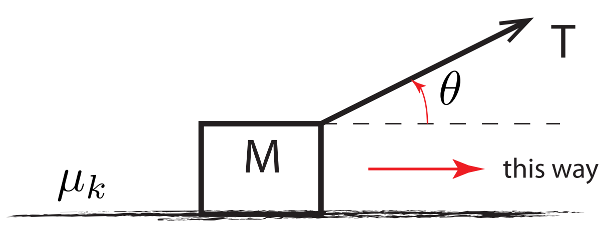


Figure 2.6: A box of mass m is on a rough surface with which it has a coefficient of static friction μ_s. It is being pulled by a force of magnitude T which makes an angle θ with the positive x-axis.

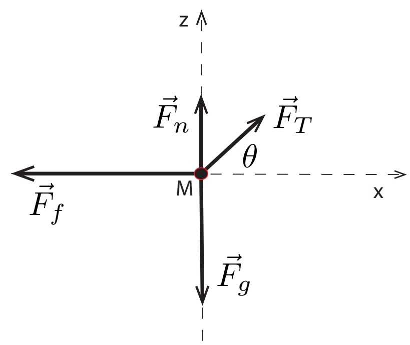


Figure 2.7: A free body diagram for the situation shown in Figure 2.6.

1. If $m = 10kg$, $\mu_k = 0.6$, and $\theta = 30°$, what is the required tension T pulling the box?
2. If $m = 20kg$, $\mu_k = 0.8$, and $T = 200N$, what is the angle it is pulled at?

Worked Solution This is another equilibrium problem, and as always, we express the idea that an object in equilibrium is subject to zero net force. There are four forces on box: the normal force, the force of friction, gravity, and the tension force, shown in a free-body diagram in Figure 2.7 The condition for equilibrium is that

$$0 = \sum \vec{F} = \vec{F}_n + \vec{F}_f + \vec{F}_g + \vec{F}_T \tag{2.20}$$

We write the forces in their components:

$$\begin{aligned}
\vec{F}_n &= \left|\vec{F}_n\right| \hat{k} \\
\vec{F}_f &= -\left|\vec{F}_f\right| \hat{\imath} \\
&= -\mu_k \left|\vec{F}_n\right| \hat{\imath} \\
\vec{F}_g &= -mg\hat{k} \\
\vec{F}_T &= T\cos\theta\hat{\imath} + T\sin\theta\hat{k}
\end{aligned} \tag{2.21}$$

There are two things to notice about the expression for the force of friction: it is in the *negative* x-direction because the force of friction opposes motion, and it is *equal* in magnitude to $\mu_k \left|\vec{F}_n\right|$ because it is kinetic friction, not static friction (whose magnitude can vary).

First, we solve the part where we want to find T. Knowing the components of the various forces gives that

$$\begin{aligned} 0 &= \vec{F}_n + \vec{F}_f + \vec{F}_g + \vec{F}_T \\ &= \left|\vec{F}_n\right| \hat{k} + \left(-\mu_k \left|\vec{F}_n\right| \hat{\imath}\right) + \left(-mg\hat{k}\right) + \left(T\cos\theta\hat{\imath} + T\sin\theta\hat{k}\right) \\ &= \left(T\cos\theta - \mu_k \left|\vec{F}_n\right|\right)\hat{\imath} + \left(\left|\vec{F}_n\right| + T\sin\theta - mg\right)\hat{k} \end{aligned} \tag{2.22}$$

By looking at the two components of 2.22 we see that

$$\begin{aligned} 0 &= T\cos\theta - \mu_k \left|\vec{F}_n\right| \\ 0 &= \left|\vec{F}_n\right| + T\sin\theta - mg \end{aligned} \tag{2.23}$$

This is general, but we know θ, μ_k, and m, so in principle, this is a pair of equations in two unknown quantities: T and $\left|\vec{F}_n\right|$. When there are two things we don't know, the procedure is to find one *in terms of the other* and then substitute that into the second relation. We have that

$$\left|\vec{F}_n\right| = T\frac{\cos\theta}{\mu_k} \tag{2.24}$$

and so from 2.23

$$\begin{aligned} 0 &= \left|\vec{F}_n\right| + T\sin\theta - mg \\ &= T\frac{\cos\theta}{\mu_k} + T\sin\theta - mg \\ T &= \frac{mg}{\frac{\cos\theta}{\mu_k} + \sin\theta} \\ &= \frac{\mu_k mg}{\cos\theta + \mu_k\sin\theta} \end{aligned} \tag{2.25}$$

I GOT THIS

Putting in the numbers given, we get that $T = 50.4N$. Note that this implies that the $\left|\vec{F}_n\right| = 72.7N$, which is less than mg – this is because the rope was pulling *up* too, which reduced the force needed from the floor to keep the mass in equilibrium vertically.

Now, we will attack the second problem, where we need to get the angle that the box is pulled. It is obviously an equilibrium problem, so the *general* set-up is the same, as described in 2.22 and 2.23. This time there are (again) two things we don't know: $\left|\vec{F}_n\right|$, and θ. Following the prescription from the previous work, we solve for $\left|\vec{F}_n\right|$ in terms of θ and eliminate giving us

$$0 = T\frac{\cos\theta}{\mu_k} + T\sin\theta - mg \tag{2.26}$$

There is only one unknown thing here (θ) but solving this relationship is not easy and involves knowing a lot of trigonometry. We can approach the problem the other way around: try to solve for $\left|\vec{F}_n\right|$, and then use that to get θ. Here's how: We know that

$$\begin{aligned} 0 &= \vec{F}_n + \vec{F}_f + \vec{F}_g + \vec{F}_T \\ -\vec{F}_T &= \vec{F}_n + \vec{F}_f + \vec{F}_g \\ \left|-\vec{F}_T\right| &= \left|\vec{F}_n + \vec{F}_f + \vec{F}_g\right| \\ T &= \sqrt{\left(\vec{F}_n + \vec{F}_f + \vec{F}_g\right)\cdot\left(\vec{F}_n + \vec{F}_f + \vec{F}_g\right)} \end{aligned} \tag{2.27}$$

The second equality in 2.27 is from the principle that if two vectors are equal, their magnitudes are too, and the third is from the definition of magnitude. (Recall that $\left|\vec{A}\right| = \sqrt{\vec{A}\cdot\vec{A}}$.) Since we know the components of the vector $\vec{F}_n + \vec{F}_f + \vec{F}_g$, we can evaluate the magnitude easily:

$$\begin{aligned} \vec{F}_n + \vec{F}_f + \vec{F}_g &= \left(\left|\vec{F}_n\right| - mg\right)\hat{k} - \mu_k\left|\vec{F}_n\right|\hat{\imath} \\ \left|\vec{F}_n + \vec{F}_f + \vec{F}_g\right| &= \sqrt{\left(\left|\vec{F}_n\right| - mg\right)^2 + \left(-\mu_k\left|\vec{F}_n\right|\right)^2} \end{aligned} \tag{2.28}$$

Combining 2.27 and 2.28 give us

$$T^2 = \left|\vec{F}_n\right|^2 - 2\left|\vec{F}_n\right| mg + (mg)^2 + \mu_k^2\left|\vec{F}_n\right|^2 \tag{2.29}$$

Since we know all the variables in this except $\left|\vec{F}_n\right|$, this is a quadratic relation for $\left|\vec{F}_n\right|$, which we can solve:

$$\begin{aligned} T^2 &= \left|\vec{F}_n\right|^2\left(1+\mu_k^2\right) - 2\left|\vec{F}_n\right| mg + (mg)^2 \\ (200N)^2 &= \left|\vec{F}_n\right|^2\left(1+0.8^2\right) - 2\left|\vec{F}_n\right|(196N) + (196N)^2 \\ 0 &= \left|\vec{F}_n\right|^2 1.64 - 392N\left|\vec{F}_n\right| - 1584N^2 \end{aligned} \tag{2.30}$$

Using the quadratic formula, we get that the possible values of $\left|\vec{F}_n\right|$ are $243N$ or $-4N$, and since the *magnitude* of a vector can't be negative, only the first makes sense. Finally then, $\cos\theta = \frac{\mu_k\left|\vec{F}_n\right|}{T} = \frac{0.8\ 243N}{200N}$ and so we have $\cos\theta = 0.972 \rightarrow \theta = 13.6°$.

Some things to note about this question We mentioned it in the text, but it is really important to notice that the magnitude of the normal force was not just mg. The normal force is a force 'of constraint' which keeps the object from falling through the surface. If the surface can't supply the required force the object will move off the surface - like if somebody is walking on ice and falls through; the ice couldn't supply the force needed to keep the person up. In this way the normal force is like the force of static friction – *it is whatever it needs to be to keep the object where it has to be.* For both forces there is a maximum possible magnitude. If the forces would need to be larger than this magnitude then something different will happen.

The second part of this question is quite involved. The reason chose to show it to you is to emphasize that sometimes you need to use very different *mathematics* techniques to solve problems that are essentially identical *physics*. Our goal for you in this class is to learn how to do the setting up of a problem, and the interpretation of the results - these are the parts that are 'physics'. The intermediate manipulations - how you get from 'set up' to 'result' - is something that you will learn by practice. We are going to highlight some techniques; you are expected to learn others in your concurrent mathematics courses. We are trying to teach you problem solving strategies, not recipes.

Student Exercises

- For the problem set up in Figure 2.6, if $m = 15kg$, $T = 75N$, and $\theta = 15°$, find μ_k and the magnitude of the normal force. *We find that the magnitude of the normal force is* $128N$*, and the coefficient of kinetic friction is* $\mu_k = 0.57$.

- A box of mass m is being pulled at a constant speed along a rough slope with which it has a coefficient of kinetic friction μ_k. The slope makes an angle of θ with the horizontal, and the box is pulled by a force T parallel to the slope. This is illustrated in Figure 2.8.

  – For this equilibrium situation, find a relationship (equation) which gives T in terms of m, μ_k, (functions of) θ, and any appropriate

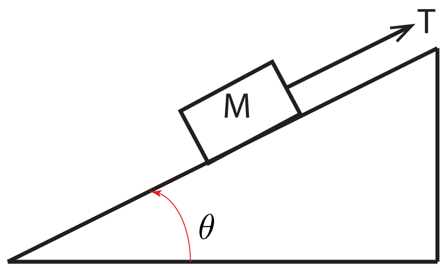


Figure 2.8: A box of mass m is pulled by a force T parallel to a rough surface. The surface makes an angle θ with the horizontal, and there is a coefficient of friction of μ_k between the slope and the box.

constants. *Check your answer by comparing if you get the right number for each of the next two questions.*

- If $\mu_k = 0.5$, $m = 12kg$, and $\theta = 30°$ find the T required to pull the box up at a constant speed. *The required tension is $T = 109.7N$.*
- If $T = 150N$, $m = 20kg$, and $\theta = 30°$, what value of μ_k is consistent with the box moving at constant speed? *The required value is $\mu_k = 0.306$.*

2.3 Applying Newton's Third Law to Equilibrium Problems

Newton's third law says that when two objects interact, they will exert forces of equal magnitudes in opposite directions on each other. The place this often shows up in problems is as an additional constraint, or piece of information.

2.3.1 Two Boxes on top of each other

Example A box of mass m_1 rests on a surface with coefficients of both static and kinetic friction (they are the same) given as μ. A box of mass m_2 rests on top of the first box, and the second box has the same coefficients of friction between it and the first box. The first box is pushed by a force of magnitude F over a level surface at a constant speed. This is shown in figure 2.9. What is the magnitude of the applied force F, given that $m_1 = 50kg$, $m_2 = 20kg$, and $\mu = 0.2$?

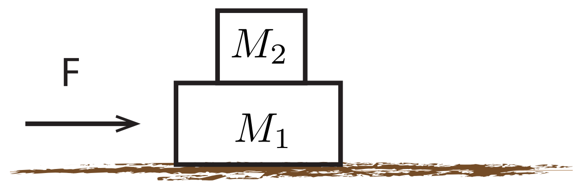


Figure 2.9: A box of mass m_2 rests on a box of mass m_1 which is being pushed horizontally by a force F.

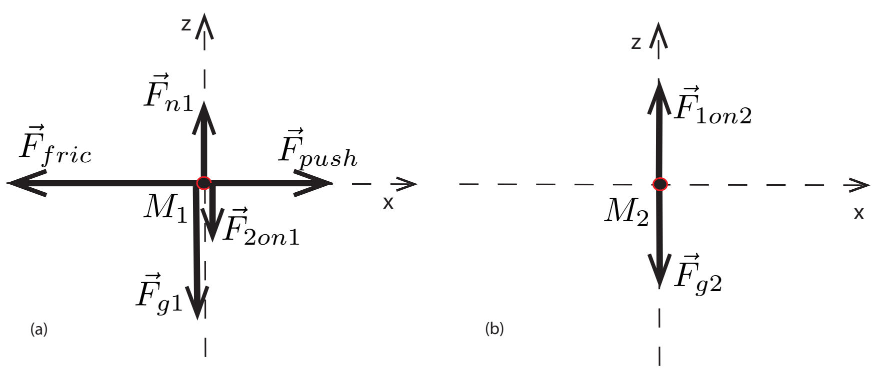


Figure 2.10: A free body diagram for the situation shown in Figure 2.9 for (a) the first box, and (b) the second box.

Worked Solution Since we know that the boxes are moving at constant speed in a constant direction, we recognize this as an equilibrium problem. Since each box *individually* is in equilibrium, we can draw a free body diagram for each. These are shown in Figure 2.10. We will denote the forces acting on box 1 and box 2 with subscripts 1 and 2 respectively. The condition for equilibrium is

$$\text{For box 1}: \quad 0 = \vec{F}_f + \vec{F}_{push} + \vec{F}_{n1} + \vec{F}_{g1} + \vec{F}_{2\ on\ 1} \tag{2.31}$$

$$\text{For box 2}: \quad 0 = \vec{F}_{g2} + \vec{F}_{1\ on\ 2} \tag{2.32}$$

The condition of Newton's third law tells us that $\vec{F}_{2\ on\ 1} = -\vec{F}_{1\ on\ 2}$, but from 2.32 we know that $\vec{F}_{1\ on\ 2} = -\vec{F}_{g2}$. Substituting this into 2.31 we have

$$0 = \vec{F}_f + \vec{F}_{push} + \vec{F}_{n1} + \vec{F}_{g1} + \left(-(-\vec{F}_{g2})\right) \tag{2.33}$$

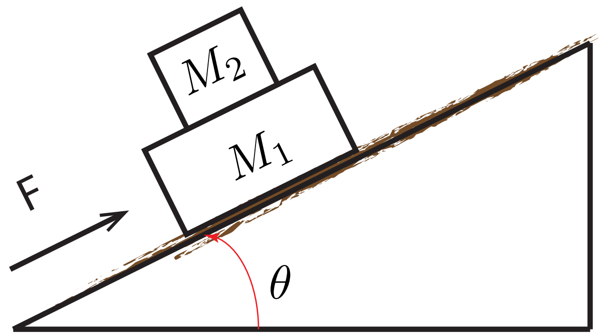


Figure 2.11: One box is being pushed up a slope angled at θ above the horizontal, with a second box on it.

Expressing this in terms of components as we have done in previous examples (note that they are all in the horizontal or vertical direction) we have

$$0 = -\mu_k \left|\vec{F}_{n1}\right| \hat{\imath} + F\hat{\imath} + \left|\vec{F}_{n1}\right| \hat{k} - m_1 g\hat{k} - m_2 g\hat{k} \tag{2.34}$$

so $\left|\vec{F}_{n1}\right| = (m_1 + m_2)g$, and so $F = 137N$.

A couple quick notes We were a lot less explicit about breaking things into components in this example. We have already done it a lot.

Notice that we were able to use the vector form of Newton's third law to substitute through and replace $\vec{F}_{2\ on\ 1}$ with $\vec{F}_{g2}$. This avoided any problems like us having to break the contact forces between the boxes into their components; we treated the contact force as a single force, not as a normal force and a friction force.

Student Exercises

- Consider the situation shown in figure 2.11. It shows a box of mass m_1 being pushed up a slope by a force of magnitude F with a box of mass m_2 on it. There is a coefficient of kinetic friction μ_k between box 1 and the slope, and a coefficient of static friction μ_s between box 2 and the slope. If $m_1 = 10kg$, $m_2 = 5kg$, $\mu_k = 0.4$, $\mu_s = 0.8$, and $\theta = 20°$, what value of F is required for constant speed? *The magnitude of the pushing force must be* $105.5N$.

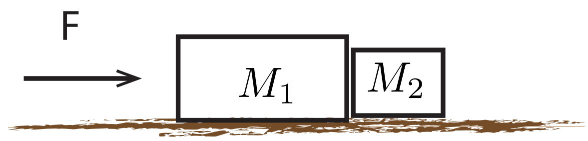


Figure 2.12: Two boxes of mass m_1 and m_2 are in contact with each other being pushed at constant speed over a rough surface.

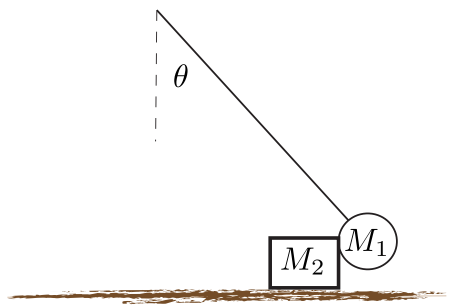


Figure 2.13: A ball of mass m_1 is supported away from the vertical by a block of mass m_2 on a rough surface.

- One box of mass m_1 is in contact with a second box of mass m_2. The first box is being pushed, as shown in figure 2.12 by a horizontal force of magnitude F. The coefficient of kinetic friction between the second box and the ground is μ_{k2}. If $m_1 = 5kg$, $m_2 = 10kg$, $\mu_{k2} = 0.4$, and $F = 50N$, what value of the coefficient of friction μ_{k1} between the first box and the surface is consistent with the boxes moving at a constant speed? *A coefficient of kinetic friction* $\mu_{k1} = 0.22$ *works.*

2.3.2 More contact problems

Example A ball of mass m_1 is supported by a rope which makes an angle of θ with the vertical and touching the smooth surface of a block of mass m_2 which rests on a rough surface with which it has coefficient of friction μ_s. This situation is shown in figure 2.13. If $m_1 = 4kg$, $m_2 = 8kg$, and $\mu_s = 0.3$, what is the largest that θ can be before the block starts to slide?

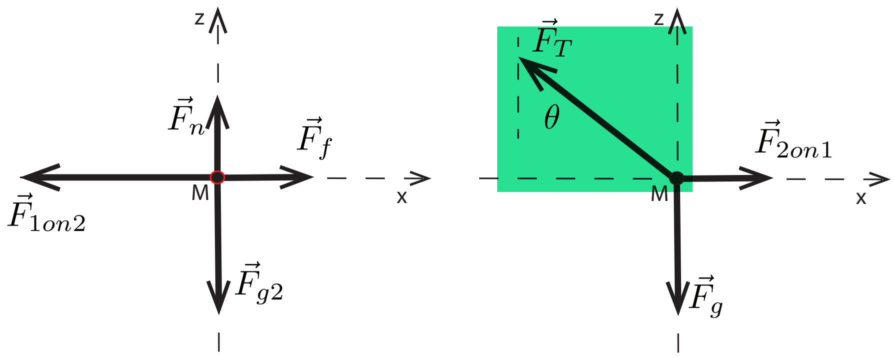


Figure 2.14: Free body diagrams for the problem outlined in figure 2.13. (a) is for the box and (b) is for the ball.

Note that the block would slide to the left as drawn.

Worked Solution The key insight for solving the problem is that if we know the maximum possible force of friction, then we will know the maximum magnitude that $\vec{F}_{1\ on\ 2}$ can have, and hence the maximum magnitude of $\vec{F}_{2\ on\ 1}$, which is related to the horizontal component of the force from the rope. We know that the vertical component of the force from the rope must support the ball and prevent it from going down.

The information (from the question) that the point of contact between the ball and the block is 'smooth' means there is no friction, and hence no vertically-directed forces between them.

We will be somewhat more terse in this solution. Both masses are in equilibrium. Since they are in equilibrium, we can apply that the forces will sum to zero. Looking at the vertical component of the forces in 2.14 part (a), we see that $\left|\vec{F}_n\right| = m_2 g$, so the maximum magnitude of the force of friction is $\left|\vec{F}_f\right| = \mu_s m_2 g$. This means that *just* before the block slips, $\vec{F}_{2\ on\ 1} = -\mu_s m_2 g\hat{\imath}$, so since $\vec{F}_{1\ on\ 2} = -\vec{F}_{2\ on\ 1}$ the expression for equilibrium for the ball is

$$\begin{aligned} 0 &= \vec{F}_T + \mu_s m_2 g\hat{\imath} - m_1 g\hat{k} \\ -\mu_s m_2 g\hat{\imath} + m_1 g\hat{k} &= \vec{F}_T \end{aligned} \tag{2.35}$$

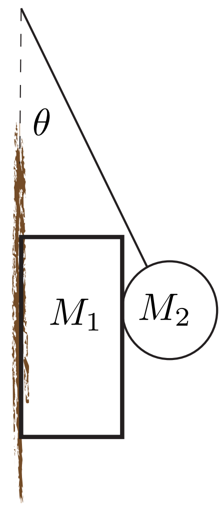


Figure 2.15: A ball is supported by a string and is pushing a box into a rough wall.

and since we can go from components to angles, we find that

$$\tan\theta = \mu_s \frac{m_2}{m_1} \tag{2.36}$$

and with the given values for the masses, we find $\theta = 31.0°$.

Student Exercises

- Go through and verify the result claimed in this example. Use the techniques from previous examples. *Really, what we're saying is "work through this example, and fill in the parts you didn't initially understand."*

- As shown in figure 2.15 a ball of mass m_2 is supported by a string which makes an angle θ with the vertical. The ball is smooth and is in contact with a box of mass m_1 which is pushed against a vertical wall with which the box has coefficient of friction μ_s. If $\theta = 10°$, $\mu_s = 0.5$, and $m_1 = 5kg$ what is the smallest mass m_2 of the ball that will keep the box in place? What about if $\theta = 30°$? *The minimal possible values for m_2 are $56.7kg$ and $17.3kg$ respectively.*

2.4 Questions

1. Two masses, m_1 and m_2, are each suspended by a single rope which make angles θ_1 and θ_2 with the vertical respectively. The two masses are connected by a horizontal rope. *Note that this means each mass is subject to three forces, two from ropes, and the downwards force of gravity.*

   - What is the equation that gives θ_2 in terms of θ_1, m_1, and m_2?
   - If $m_1 = 5kg$, $m_2 = 10kg$, and $\theta_1 = 30°$, what is the tension in the rope pulling up on mass m_2?

2. A block of mass $m = 8kg$ is on a horizontal surface with which it has a coefficient of static friction of 0.4. It is being pulled horizontally to the left by a rope which exerts a force of $15N$. It is being pulled upwards by a rope under tension T. For which values of T can the block be in equilibrium?

3. A block of mass $m = 7kg$ is placed on a slope which rises to the right and which makes an angle of 15° above the horizontal. The block is being pulled by a rope which extends horizontally to the left under tension T.

   - If the coefficient of static friction between the block and the slope is 0.8, what values of T are consistent with static equilibrium?
   - If the block is being pulled by tension $T = 15N$ and is sliding down the slope at a constant speed what is the coefficient of kinetic friction?

2.5 Answers

1. We find:

   - $\theta_2 = \tan^{-1}\left(\frac{m_1}{m_2}\tan\theta_1\right)$
   - $102N$

2. Equilibrium is possible if $T < 40.9N$.

3. For this case:

   - Equilibrium requires $T < 30.1N$.
   - $\mu_k = 0.517$

Chapter 3

Rotational Equilibrium

3.1 Summary

The third chapter of the text discusses the conditions for rotational equilibrium. This is an extension of the conditions we talked about in the previous chapter for translational equilibrium. The objects that are being considered are no longer conceptualized as simple points, but rather as rigid objects with spatial extent.

- For an object to be in rotational equilibrium it must be subject to zero *net* torque.
- Torque is a vector quantity. The units of torque are Nm. In our later chapter on energy, we will encounter another quantity that has the same dimensions (the Joule); the reason we express the units differently is to remind us that torque is a vector.
- The torque a force exerts 'about' a particular point is given by $\vec{\tau} = \vec{r} \times \vec{F}$. In this, $\vec{r}$ is the vector from the point around which you are calculating the torque *to* the place where the force is applied. The expression for torque depends on which point you calculate torque around; in other words which point you use as your origin.
- An object in static equilibrium (ie not moving, and not rotating, and not changing how it moves or rotates) is subject to $\vec{F}_{net} = 0$ *and* $\vec{\tau}_{net} = 0$.
- Any object which experiences a net force of 0 will have a net torque which does not depend on the choice of origin. This fact can often be used to simplify calculations involving torque.

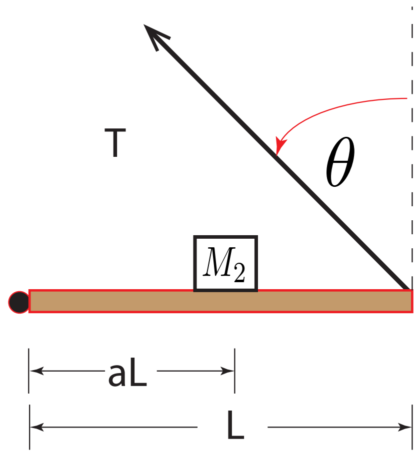


Figure 3.1: A uniform beam of mass m_1 and length L is supported horizontally at the left end with a pin and at the right end by a rope with tension T which makes an angle θ to the left of vertical. A small mass m_2 is placed a distance aL from the left end of the beam.

- A rigid object which is subject to the force of gravity can be treated as though gravity acts at its center of mass. The center of mass is the 'average' position of the stuff that makes the object up. For an *uniform* object – one with the mass evenly distributed – this is the same as the geometric center. (This only works for the constant gravitational force near the surface of the Earth)

3.2 A beam held by a rope at one end

Example This is a classic problem which illustrates a lot of the things which you'll need to know about how to approach torque problems.

A uniform beam of mass m_1 and length L is supported at one end by a pin, and at the other end by a rope which makes an angle of θ with the vertical as shown in figure 3.1. The rope is under tension T, and the beam supports a small mass m_2 a distance aL from the left end. (A word of explanation: a is a number. If a is 0 then the mass is at the left end of the beam, if a is 1 then it is at the right end, as drawn a is about 0.7)

1. Assuming that $m_1 = 10kg$, $m_2 = 4.0kg$, $\theta = 30°$ and $a = 0.8$, what is

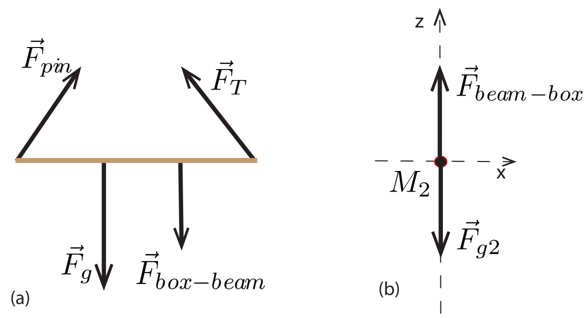


Figure 3.2: The free-body diagrams for (a) the beam and (b) the box described in Figure 3.1

the value of T, the tension in the rope?

2. What is the force exerted by the pin?

Worked Solution This problem is one involving the concepts of translational and rotational equilibrium. There are two conditions: that the net force on the beam is zero, and that the net torque is zero. These two conditions will form a set of linear equations which we can use to find the tension T and the force exerted by the pin.

We can start by making a diagram of the situation showing both the forces and the places the forces are exerted. This is shown in figure 3.2. In part (a) of this figure, the free-body diagram for the beam shows that there are four forces acting on it: the force from the pin, the force due to tension, the force due to gravity, and the force due to the box – a reaction force since the beam is supplying the force which holds the box in equilibrium. In part (b), we can see that $\vec{F}_{beam\ on\ box} = m_2 g\hat{k}$ (remember that this means the force is upward) since the box is in equilibrium and there are only two forces on it. This means that $\vec{F}_{box\ on\ beam} = -m_2 g\hat{k}$.

Since the forces that the beam experience are not all exerted at the same point, it will help us to tabulate the forces and the locations at which they are exerted. In this, we take the origin as being at the location of the pin at the left end of the beam.

$$
\begin{aligned}
\vec{F}_{pin} &= \text{unknown} & \vec{r}_{pin} &= 0 \\
\vec{F}_g &= -m_1 g\hat{k} & \vec{r}_g &= \frac{1}{2}L\hat{\imath} \\
\vec{F}_{box} &= -m_2 g\hat{k} & \vec{r}_{box} &= aL\hat{\imath} \\
\vec{F}_T &= -T\sin\theta\hat{\imath} + T\cos\theta\hat{k} & \vec{r}_T &= L\hat{\imath}
\end{aligned}
\tag{3.1}
$$

The conditions for translational and rotational equilibrium as

$$
\begin{aligned}
0 &= \vec{F}_{pin} + \vec{F}_g + \vec{F}_{box} + \vec{F}_T \\
0 &= \vec{\tau}_{pin} + \vec{\tau}_g + \vec{\tau}_{box} + \vec{\tau}_T
\end{aligned}
\tag{3.2}
$$

When we compare 3.1 with 3.2 we see that the horizontal components of the force supplied by the pin and by the tension of the rope must be equal and opposite, and that their vertical components must be enough to counteract the downwards force supplied by gravity on the two masses.

Since this is a situation where both the net force is zero and the net torque is zero, we can calculate the torque around any point. However, since we do not know either component of the force due to the pin, it will make our job easier if we decide to calculate the torque around the location of the pin - since the torque it exerts must be zero. Using this as our origin, we have the torques (using the relations in 3.1):

$$
\begin{aligned}
\vec{\tau} &= \vec{r} \times \vec{F} \\
\vec{\tau}_{pin} &= (0\hat{\imath}) \times (\text{unknown}) = 0 \\
\vec{\tau}_g &= \left(\frac{1}{2}L\hat{\imath}\right) \times \left(-m_1 g\hat{k}\right) = \frac{m_1 g L}{2}\hat{\jmath} \\
\vec{\tau}_{box} &= (aL\hat{\imath}) \times \left(-m_2 g\hat{k}\right) = a m_2 g L\hat{\jmath} \\
\vec{\tau}_T &= (L\hat{\imath}) \times \left(-T\sin\theta\hat{\imath} + T\cos\theta\hat{k}\right) = -LT\cos\theta\hat{\jmath}
\end{aligned}
\tag{3.3}
$$

If you have trouble remembering how to get those torques in their components, look at Section 1.4.2 again. Applying the relation that the net torque is zero we have

$$
\begin{aligned}
0 &= \vec{\tau}_{pin} + \vec{\tau}_g + \vec{\tau}_{box} + \vec{\tau}_T \\
&= 0 + \frac{m_1 g L}{2}\hat{\jmath} + a m_2 g L\hat{\jmath} + (-LT\cos\theta\hat{\jmath}) \\
T\cos\theta &= \frac{1}{2}m_1 g + a m_2 g.
\end{aligned}
\tag{3.4}
$$

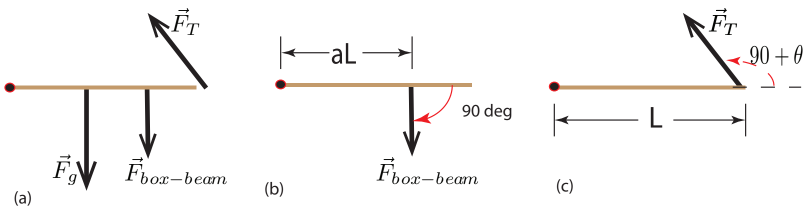


Figure 3.3: The beam is supported by three forces that act at places other than the center of rotation. (a) shows the location of the three forces, (b) shows the angle between $\vec{r}$ and $\vec{F}$ for the force supplied by the box, and (c) shows the angle between $\vec{r}$ and $\vec{F}$ for the rope.

In the second equality of 3.4 we got rid of the $\hat{\jmath}$ because all vectors were in the $\hat{\jmath}$ direction – saying that the total vector was 0 is the same as saying that the y-component vanishes. We can solve 3.4 for the unknown T (since θ, m_1, and m_2 are known), and we get that $T = 92.8N$.

Note that we could have gotten the same torque result using the 'high school' method of keeping track of whether the torque is clockwise or counterclockwise, and the magnitude. In that case, we would have examined figure 3.3. We identify that the force of gravity on the beam, and the force of the box on the beam both exert *clockwise* torques, while the force from the rope exerts a *counterclockwise* torque. The magnitude of the clockwise torque from the box on the beam is $|\vec{\tau}_{box}| = |\vec{r}|\left|\vec{F}\right|\sin\phi = (aL)(m_2 g)\sin 90° = aLm_2 g$. Similarly, the magnitude of the clockwise torque from gravity on the beam is $\frac{1}{2}Lm_1 g$. Finally, the magnitude of the counterclockwise torque from the rope is $|\vec{r}|\left|\vec{F}\right|\sin\phi = (L)(T)\sin(90° + \theta) = LT\cos\theta$. When we equate the total clockwise torque with the total counterclockwise torque we will get

$$\begin{aligned} \tau_{cw} &= \tau_{ccw} \\ aLm_2 g + \frac{1}{2}Lm_1 g &= LT\cos\theta \\ T\cos\theta &= am_2 g + \frac{1}{2}m_1 g \end{aligned} \qquad (3.5)$$

which is exactly the relationship we got in 3.4, so we used this other method of calculating to get $T = 92.8N$. The reason we always use the vector method is that it is applicable to any situation whereas discussing the direction something will rotate only makes sense if all forces and position vectors are confined to the same plane.

Knowing the force supplied by the rope, we can now use the *translational* equilibrium relationship to impose that

$$\begin{aligned} 0 = \vec{F}_{net} &= \vec{F}_{pin} + \vec{F}_g + \vec{F}_{box} + \vec{F}_T \\ \vec{F}_{pin} &= -\left(-m_1 g\hat{k}\right) - \left(-m_2 g\hat{k}\right) - \left(-T\sin\theta\hat{\imath} + T\cos\theta\hat{k}\right) \end{aligned} \tag{3.6}$$

Since every variable is know, we can substitute in for m_1, m_2, T, and θ, and find that $\vec{F}_{pin} = 46.4N\hat{\imath} + 56.8N\hat{k}$.

Some comments about the problem

- We showed that you can either use the 'component' method for torques or the 'clockwise/counterclockwise' method. In cases like this, when the forces and displacements are all in a plane, they work the same way - the sign of the component keeps track of whether the torque was clockwise or counterclockwise. If there's a problem with forces going in all three directions, you have to use components of torque.

- In this case, we made a 'clever' choice of the origin - it was at the point of an unknown force (the force of the pin) that we didn't need to solve part of the problem. The strategy of putting the origin about which you calculate torques at the location of an unknown force is one that usually helps.

- The vertical component of the force the pin exerted is *less* than the vertical component of the force the rope exerted.

- The length L of the beam did not end up in the solution - the only important part was the fraction along the beam that the various forces were exerted. This typically happens.

Student Exercises

- This is a rich problem, and is worth exploring a bit more.

    1. We solved for the various forces and their components. Use this to check that the net torque around the location the box touches the beam is zero. *It is; just make sure you can show that.*

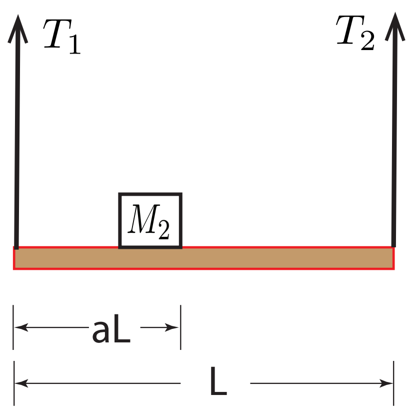


Figure 3.4: A beam of length L and mass m_1 is supported by two vertical ropes with tensions T_1 and T_2 respectively. A small box of mass m_2 is placed a distance aL from the left end of the horizontal beam.

2. Find the location of the center of mass of the box and beam. Repeat the analysis we did to find the tension in the rope when you treat the beam and box as a single non-uniform object with the force of gravity exerted at its center of mass. *The location of the center of mass (with the left end of the bar as the origin) is* $\left(\frac{1}{2}\frac{m_1}{m_1+m_2} + a\frac{m_2}{m_1+m_2}\right)L\hat{\imath}$. *For the same numbers you get the same tension.*
3. What is T if the problem is identical except that $a = 0.2$? *It is* $65.6N$.

- Consider the situation depicted in Figure 3.4. A box of mass m_2 sits a distance aL from the left end of a horizontal beam of mass m_1 and length L. The beam is supported by a vertical rope of tension T_1 at the left end and by a vertical rope of tension T_2 at the right end.

1. Find expressions for T_1 and T_2 in terms of m_1, m_2, a, and g. *Check* that $T_1 + T_2 = (m_1 + m_2)\,g$. *If you got them right your answers for the next part will be correct.*
2. If $m_1 = 5kg$, $m_2 = 1kg$, $a = 0.9$ find T_1 and T_2. *We find that* $T_1 = 25.5N$ *and* $T_2 = 33.3N$.

- Consider the situation shown in figure 3.5. A light beam is held at

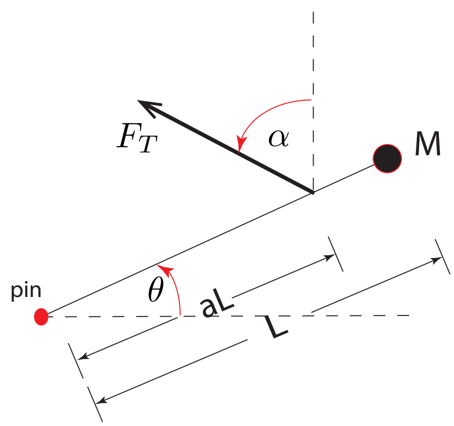


Figure 3.5: A light (massless) beam of length L is held in place by a pin at the left end, and supports a ball of mass m at the other. The beam makes an angle of θ above the horizontal, and is held in place by a force which makes an angle of α with the vertical at a point aL along the beam.

one end by a pin and supports a mass m at the other end. A rope is attached $4m$ from the end making an angle $\alpha = 30°$ with the vertical (as shown). The beam is $5m$ long, the mass is $8kg$, the beam makes an angle $\theta = 15°$ with the horizontal. What is the tension in the rope? *It turns out to be* $98N$.

3.3 Ladders slipping because of torque

There are a whole family of problems which can be paraphrased as 'something is leaning against a wall; under what conditions will it start to slip'? The critical idea here is that you have to combine information about both rotational and translational equilibrium.

Example A ruler of length L and mass m is balanced vertically on a table with which it has coefficent of static friction μ_s. It is held in place by a strong rope pulling to the left at the top and a second rope pulling to the right at an angle of θ above the horizontal with a tension T. This situation is shown in figure 3.6. If $a = 0.25$, $\theta = 30°$, $m = 5kg$, and $\mu_s = 0.25$ what is the maximum tension T before the stick starts to slip?

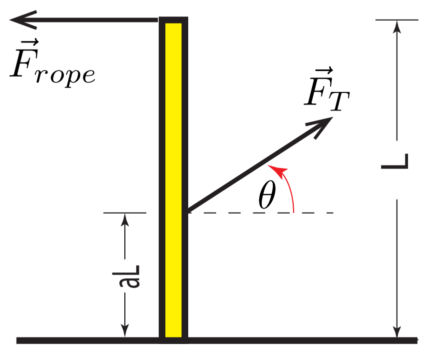


Figure 3.6: A ruler of length L and mass m is oriented vertically over a surface with which it has a coefficient of static friction μ_s. There is a rope attached at the upper end pulling to the left, and a rope attached aL from the lower end pulling up and to the right at an angle of θ with the horizontal.

Worked Solution This is another equilibrium problem, and it is similar to the problems in the equilibrium section where friction was involved. We typically had to find out the normal force to find out the force of friction, which would then be used in conjunction with Newton's first law to find the required force to keep the object in equilibrium. Here, our strategy will be to use the translational equilibrium requirements to figure out what the normal force is, hence what the maximum force of friction. We will then use the rotational equilibrium conditions to find a relationship between the applied tension T and the force of friction. Knowing the maximum force of friction will then tell us the maximum applied tension.

We will start by looking at the free-body diagram for this situation, annotated to show where the forces act on the rigid object. This is shown in figure 3.7. The two conditions we have to satisfy are that the net force and net torque vanish.

$$\begin{aligned} 0 &= \vec{F}_{rope} + \vec{F}_g + \vec{F}_T + \vec{F}_n + \vec{F}_f \\ 0 &= \vec{\tau}_{rope} + \vec{\tau}_g + \vec{\tau}_T + \vec{\tau}_n + \vec{\tau}_f \end{aligned} \tag{3.7}$$

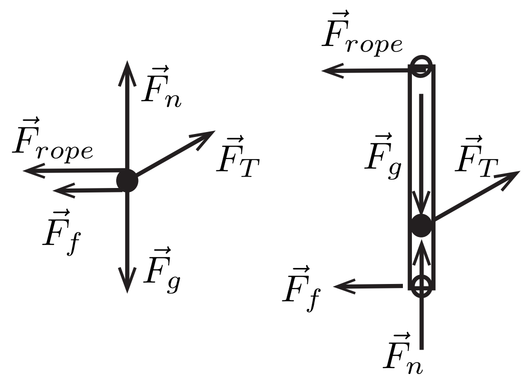


Figure 3.7: The ruler from figure 3.6 is shown in a free-body diagram. There is a horizontal (unknown) force at the top, the force of gravity acts at the midpoint, the tension force acts a distance aL from the bottom, and the normal force and friction act at the bottom.

As before, we express the forces in terms of their components.

$$
\begin{aligned}
\vec{F}_{rope} &= -F_r\hat{\imath} \quad F_r \text{ is unknown} \\
\vec{F}_g &= -mg\hat{k} \\
\vec{F}_T &= T\cos\theta\hat{\imath} + T\sin\theta\hat{k} \\
\vec{F}_n &= N\hat{k} \quad N \text{ is unknown} \\
\vec{F}_f &= -F_f\hat{\imath} \quad F_f \text{ is unknown}
\end{aligned} \tag{3.8}
$$

Note that we have given each force a variable describing the force's magnitude. We do not know the magnitudes, but we are going to use those to get relationships between them.

We need to calculate the torques as well; for this we need to choose a pivot point. Since we do not need to know the force supplied by the horizontal rope, we choose the top of the ruler as the pivot point (we could have chosen lots of different spots and we would get the same results). Recalling that torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$, we have

$$\begin{aligned}
\Delta\vec{r}_{rope} = 0 \quad &\rightarrow \quad \vec{\tau}_{rope} = 0 \\
\Delta\vec{r}_g = -\frac{1}{2}L\hat{k} \quad &\rightarrow \quad \vec{\tau}_g = \left(-\frac{1}{2}L\hat{k}\right) \times \left(-mg\hat{k}\right) = 0 \\
\Delta\vec{r}_T = -(1-a)L\hat{k} \quad &\rightarrow \quad \vec{\tau}_T = \left(-(1-a)L\hat{k}\right) \times \left(T\cos\theta\hat{\imath} + T\sin\theta\hat{k}\right) \\
&\qquad\quad = -(1-a)LT\cos\theta\hat{\jmath} \\
\Delta\vec{r}_n = -L\hat{k} \quad &\rightarrow \quad \vec{\tau}_n = \left(-L\hat{k}\right) \times \left(N\hat{k}\right) = 0 \\
\Delta\vec{r}_f = -L\hat{k} \quad &\rightarrow \quad \vec{\tau}_f = \left(-L\hat{k}\right) \times \left(-F_f\hat{\imath}\right) = LF_f\hat{\jmath}
\end{aligned} \tag{3.9}$$

Now, we combine the expressions in 3.8 with the condition that the total force is $\vec{F}_{net} = 0$. The vertical component of that gives us

$$0 = -mg + T\sin\theta + F_n \tag{3.10}$$

so $F_n = mg - T\sin\theta$. Note in passing, that this makes sense and we have seen similar things before: the normal force is smaller when there is another force pulling up. This means that $\left|\vec{F}_f\right| = F_f \leq \mu_s\left(mg - T\sin\theta\right)$.

Next, we turn our attention to the torque.

$$\begin{aligned}
0 &= \vec{\tau}_{rope} + \vec{\tau}_g + \vec{\tau}_T + \vec{\tau}_n + \vec{\tau}_f \\
&= 0 + 0 + (-(1-a)LT\cos\theta\hat{\jmath}) + 0 + (LF_f\hat{\jmath}) \\
F_f &= (1-a)T\cos\theta
\end{aligned} \tag{3.11}$$

So, for there to be rotational equilibrium, we have that $F_f = (1-a)T\cos\theta$, but from our knowledge about how friction works, $F_f \leq \mu_s\left(mg - T\sin\theta\right)$, so

$$\begin{aligned}
(1-a)T\cos\theta &\leq \mu_s\left(mg - T\sin\theta\right) \\
T &\leq \frac{\mu_s mg}{(1-a)\cos\theta + \mu_s\sin\theta}
\end{aligned} \tag{3.12}$$

and so we have that the maximum value for T is given by equating the two sides of this relationship.

For the given values of $a = 0.25$, $\theta = 30°$, $m = 5kg$, and $\mu_s = 0.25$, we can calculate $T = 15.8N$.

Some comments about the solution The key thing here, and it has been a theme in the other questions we have asked is the reasoning process: You start with a general idea (in this case, that we have equilibrium) and then get relationships between the various forces and torques. Once you know the relationships you can use those to find the values of the particular unknown you're looking for.

What you need to realize is that there isn't a fixed recipe for this - each problem is, computationally, different, but there is a general pattern. You use things like the fact that the system is in equilibrium to find relationships between the various things that are unknown.

Student Exercises

- Work through the problem again, using the bottom of the ruler as the pivot point. As you are working this through, pay some attention to how you determine what the friction force is: you can't get a condition on it *directly* from torque considerations, but you can from *translational* equilibrium considerations. *The friction force will not show up in the expression for torque, but it can be determined since the net horizontal force has to be zero.*

- Look again at the problem illustrated in figure 3.6. If $\left|\vec{F}_{rope}\right| = 30N$, $m = 5kg$, $\mu_s = 0.2$, and $\theta = 0°$, what is the smallest value of a such that the ruler can be in equilibrium? *We find that* $a = 0.734$.

3.4 Questions

1. Consider the situation illustrated in Figure 3.8 where a uniform ruler is held by friction at the left end and a rope at the right end.
   - If $m = 4kg$ and $\theta = 30°$ what is the tension in the rope? $60°$?
   - If $\mu_s = 0.5$ what is the range of angles θ for which the ruler can be in equilibrium?

2. A child climbs a ladder which is resting against a smooth wall and initially makes an angle of θ with the vertical. This is illustrated in figure 3.9.
   - Given that the ladder is massless and that the child is able to climb three-quarters of the way to the top when $\theta = 15°$ before

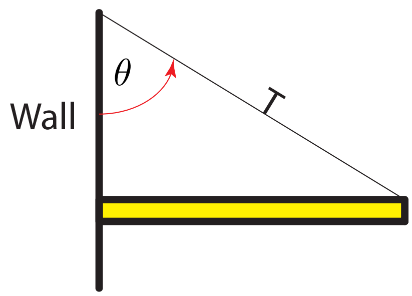


Figure 3.8: A uniform ruler of mass m and length L is held against a rough wall by a rope which makes an angle θ with the vertical. The coefficient of friction between the left end of the ruler and the wall is μ_s.

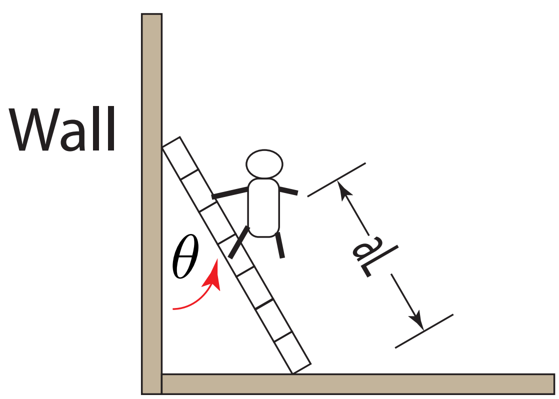


Figure 3.9: A child climbs a distance aL along a ladder of length L. The ladder rests on a smooth wall (so the coefficient of static friction between the top and the wall is 0) and a rough floor with which it has a coefficient of static friction μ_s.

the ladder slips, what is the coefficient of friction between the ladder and the floor?

- If the child has the same mass as the ladder, and the ladder is uniform, what inequality relates the fraction that the child has climbed the ladder (a), the angle at which the ladder is set, and the coefficient of static friction μ_s?

3.5 Answers

1. For the ruler we find
   - For $30°$ the tension is $22.6N$; for $60°$ the tension is $39.2N$.
   - If $\mu_s = 0.5$ we must have $\theta \geq 63.4°$ for equilibrium.
2. For the child climbing up the the ladder
   - If the child is able to climb $a = \frac{3}{4}$ of the way up the ladder before it slips and the angle is $15°$, then $\mu_s = 0.201$.
   - The inequality is $\left(\frac{2a+1}{2}\right) \tan\theta \leq 2\mu_s$.

Chapter 4

Differential Calculus

4.1 Summary

The fourth chapter of the text discusses differential calculus. The ability to use differential calculus will enable us to do things like calculate the velocity and acceleration of an object (as discussed in chapter 5), to approximate a function (to linear, quadratic, or higher order as necessary), and to calculate displacement vectors from parametric curves (relevant for calculations of work in chapter 11). In many cases the *details* of the calculations you'll be required to do are simpler than those in the concurrent mathematics course you are probably taking, however you will need to understand the meaning of the operations better in order to apply them to physics.

- The slope of tangent line the curve $y = f(x)$ (plotted in the xy plane) is given by the derivative of the function $f(x)$. The derivative is calculated by a limiting process which measures the ratio of the change in the function f as its argument changes by a little bit δx to that small change:

$$\frac{d}{dx}f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{4.1}$$

- The deriviative of a function is itself a function that can be differentiated.

- There are a number of basic derivatives that it is useful to know:

$$\begin{aligned} \frac{d}{dx}x^n &= nx^{n-1} \\ \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \\ \frac{d}{dx}\ln x &= \frac{1}{x} \end{aligned} \tag{4.2}$$

- The derivative operator is linear. The derivative of the sum of two functions is the sum of their derivatives.

$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x) \tag{4.3}$$

- When functions are combined there are two very useful facts which enable you to calculate derivatives of more complicated functions. These facts are called the product and chain rule:

$$\begin{aligned} \frac{d}{dx}\left[f(x)g(x)\right] &= f'(x)g(x)+f(x)g'(x) \quad \text{Product Rule} \\ \frac{d}{dx}f\left(g(x)\right) &= f'\left(g(x)\right)g'(x) \quad \text{Chain Rule} \end{aligned} \tag{4.4}$$

  In both of these the 'prime' symbol denotes differentiation.

- The linear approximation to a function $f(x)$ around the point x_0 is

$$f(x) \approx f(x_0) + f'(x_0)\left[x-x_0\right] \tag{4.5}$$

  The second-order (or quadratic) approximation to the function is

$$f(x) \approx f(x_0) + f'(x_0)\left[x-x_0\right] + \frac{1}{2}f''(x_0)\left[x-x_0\right]^2 \tag{4.6}$$

  and higher order approximations are given by appropriate generalizations of this.

- Since differentiation is linear, the derivative of a vector with respect to its parameter is going to be a vector whose components are the derivatives of the components of the original vector.

4.2 Applying the product rule

Example Consider the function $h(x) = \left(3x^2+1\right)\left(2x^3-4x+1\right)$. Evaluate the derivative of $h(x)$ at $x = 0.1$.

Worked Solution Here we apply the chain rule and check that *using* the chain rule will produce the same thing that we would expect from simply taking the derivative of a polynomial.

The function we are considering is $h(x) = \left(3x^2+1\right)\left(2x^3-4x+1\right) = 6x^5 - 10x^3 + 3x^2 - 4x + 1$. We can recast this as

$$h(x) = f(x)g(x) \tag{4.7}$$

with

$$\begin{aligned} f(x) &= 3x^2+1 \\ g(x) &= 2x^3-4x+1 \end{aligned} \tag{4.8}$$

Their derivatives are

$$\begin{aligned} \frac{d}{dx}f(x) &= 6x \\ \frac{d}{dx}g(x) &= 6x^2-4 \end{aligned} \tag{4.9}$$

so applying the product rule we have

$$\begin{aligned} \frac{d}{dx}h(x) &= \frac{d}{dx}\left(f(x)g(x)\right) \\ &= f'(x)g(x) + f(x)g'(x) \\ &= (6x)\left(2x^3-4x+1\right) + \left(3x^2+1\right)\left(6x^2-4\right) \\ &= 30x^4 - 30x^2 + 6x - 4 \end{aligned} \tag{4.10}$$

Note that this is the same calculating the derivative of $6x^5 - 10x^3 + 3x^2 - 4x + 1$.

The value of the derivative is $h'(0.1) = -3.697$.

Comments about the problem As you can tell, calculating the derivative of a polynomial is about the simplest possible problem from a calculus course. The reason we did this problem was so that you can *see* that the product rule works. We used the new way of calculating to solve a problem you could have solved with just the primitive rules for derivatives, but now that you've seen it work in a situation where you know what is happening you will believe that it works in situations where it is a little harder to apply only what you know about polynomials.

Student Exercises Calculate the derivatives of the following functions:

1. $f(x) = x^2 e^x$ *we find that* $f'(x) = \left(2x + x^2\right) e^x$
2. $f(x) = x^3 \ln x$ *we find that* $f'(x) = x^2 \left(1 + \ln x^3\right)$
3. $f(x) = e^x \sin x$ *we find that* $f'(x) = e^x (\sin x + \cos x)$
4. $f(x) = \cos x \sin x$ *we find that* $f'(x) = \cos^2 x - \sin^2 x$. *Note that trigonometric identities give that* $\cos x \sin x = \frac{1}{2} \sin(2x)$ *and* $\cos^2 x - \sin^2 x = cos\,(2x)$*; this will be important soon.*

4.3 Applying the chain rule

The chain rule is a calculus application that occurs all the time in Physics. The reason for this is that very often quantities – for example angles – are *parametrized* in terms of other quantities (such as positions). To learn how an angle changes in time, you would need to know how the angle changes in terms of the change in position and then how the position changes in time.

Example Consider the function $h(x) = \left(x^3 - 3\right)^2$. Evaluate the derivative $h'(x)$.

Worked Solution This is a direct application of the chain rule. Looking at $h(x)$ we see that we can write $h(x) = f(g(x))$ with

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= x^3 - 3 \end{aligned} \tag{4.11}$$

Since $f'(x) = 2x$ and $g'(x) = 3x^2$, we can immediately write that

$$\begin{aligned} \frac{d}{dx} h(x) &= \frac{d}{dx} f(g(x)) \\ &= f'(g(x))\, g'(x) \\ &= [2g(x)]\, g'(x) \\ &= \left(2\left(x^3 - 3\right)\right)\left(3x^2\right) \\ &= 6x^5 - 18x^2 \end{aligned} \tag{4.12}$$

Note that this result could have been obtained by expanding the polynomial first. Evaluating the 'square' gives that $h(x) = x^6 - 6x^3 + 9$, and the derivative of this polynomial is clearly the same as the result above. Here, as in the previous section, we demonstrate the application of a new rule (the chain rule) in a context where the answer is familiar.

Example Consider the function $h(x) = \cos\left(3x - 2x^2\right)$. If this function were graphed as $y = h(x)$, would the graph be going up or down at $x = 1$?

Worked Solution As outlined in the name of this section we are expecting to have to apply the chain rule. In this case $h(x)$ has the form $h(x) = f\left(g(x)\right)$ with

$$\begin{aligned} f(x) &= \cos x \\ g(x) &= 3x - 2x^2 \end{aligned} \tag{4.13}$$

Then applying the rule that

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) g'(x) \tag{4.14}$$

and noting that

$$\begin{aligned} f'(x) &= -\sin x \\ g'(x) &= 3 - 4x \end{aligned} \tag{4.15}$$

This gives that

$$\begin{aligned} \frac{d}{dx} h(x) &= \cos\left(3x - 2x^2\right) \\ &= -\sin\left(3x - 2x^2\right)[3 - 4x] \end{aligned} \tag{4.16}$$

This is the general expression for the derivative, and to determine whether the graph is rising or falling we examine the *sign* of the derivative: $h'(1) = -\sin(3-2)[3-4] = 0.841$, so the function is increasing because its derivative is positive. Note that the argument for sin is in *radians*.

Student Exercises Calculate the derivatives of the following functions:

1. $f(x) = 2\sin^3 x$. *We find that* $f'(x) = 6\sin^2 x \cos x$.
2. $f(x) = 5e^{-x^2}$. *We find that* $f'(x) = -10xe^{-x^2}$.
3. $f(x) = (\cos x)^{-1}$. *We find that* $f'(x) = \frac{\sin x}{\cos^2 x}$. *Note that this is the same fact as that the derivative of* $\sec x$ *is* $\sec x \tan x$.
4. $f(x) = \frac{1}{2}\sin(2x)$. *We find that* $f'(x) = \cos(2x)$. *Note that this is the same as the derivative we took in the previous section, just written differently using trigonometric identities.*

4.4 Linear Approximations

In physics derivatives are used to estimate values of functions in some region very often. As you will see in subsequent courses that we use approximate expansions to estimate complicated functions so that we can analyze them. Here, we give a simple example which illustrates how a polynomial expansion can be generated.

Example Estimate the value of x for which $\cos x = x$ if you know that x is in the range between 0 and $\frac{\pi}{2}$.

Worked Solution Different from the examples above, in this case we need to spell out the *plan* before we implement it. We need to get an approximate value for a number x such that $\cos x = x$. This is fundamentally the same as saying that we wish to find a value for x such that the function $f(x) = \cos x - x$ vanishes. The plan is to find a value of x_0 such that $f(x_0)$ is fairly close to zero, and then use a linear approximation, which is to say that

$$f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)\,. \tag{4.17}$$

We then look for the value of x such that $f(x_0) + f'(x_0)\,(x - x_0) = 0$. This will be an approximation of the x for which $f(x) = 0$; if $f(x_0)$ isn't close enough to zero, we can repeat the process. This is the root-finding technique that is typically taught in introductory calculus.

So, to start we guess. Noting that $\cos(0.70) = 0.7648$ and $\cos(0.75) = 0.7317$ the root we want is between the two values, so we will take as the initial point $x_0 = 0.70$. Now, with

$$\begin{aligned} f(x) = \cos x - x \quad &\rightarrow \quad f'(x) = -\sin x - 1 \\ f(0.7) = 0.0648 \quad &\text{and} \quad f'(0.7) = -1.6442 \end{aligned} \tag{4.18}$$

this gives us the approximation

$$f(x) \approx 0.0648 - 1.6442\,(x - 0.7)\,. \tag{4.19}$$

Solving for the value of x for which $f(x) = 0$ gives $x = 0.7394$.

We check: $f(0.7394) = \cos 0.7394 - 0.7394 = -0.0005$. This is obviously fairly close to 0; at least much closer than the value for $f(0.70)$; the context of the problem determines if this value is acceptably close. If it were not, we could *iterate* by using $x_0 = 0.7394$. In this case

$$f(0.7394) = -5.27 \times 10^{-4} \quad \text{and} \quad f'(0.7394) = -1.6738 \tag{4.20}$$

so we have the approximation

$$f(x) \approx -5.27 \times 10^{-4} - 1.6738\,(x - 0.7394) \tag{4.21}$$

This gives $x = 0.739085$ as the value for which $f(x) = 0$, and $f(0.739085) = 2.23 \times 10^{-7}$. Note that the subsequent iteration still left us close to the original approximation, and our f is much closer to zero.

Student Exercises

- Using the linear approximation described above, what is the approximate value of x where $f(x) = 0$ starting with $x_0 = 0.75$ and $f(x) = \cos x - x$. *We get* 0.73911.

- Find an approximate value for $\ln 2$ by using a linear approximation to solve the equation $e^x = 2$, noting that $e^{0.7} \approx 2.0138$. *Using* $f(x) = e^x - 2$, *and following the pattern above get* 0.69317 *after one step; note that* $\ln 2 = 0.693147$

- What is the smallest positive value of x for which $\sin x + \cos x = 1.3$? *Starting from* $x = 0.3$ *where* $\sin 0.3 + \cos 0.3 = 1.25086$, *we get* $x = 0.37448$; *starting from* $x = 0.4$ *where* $\sin 0.4 + \cos 0.4 = 1.31048$ *we get* $x = 0.38023$, *and we could iterate farther.*

4.5 Tangent line to parametric curves

The reason this is important is because we will be considering velocity next chapter. In that chapter, the position of objects will be parametrized in terms of *time*, and the velocity will be the derivative of their position with respect to time. However, velocity is in the *direction* of travel, so knowing the tangent line gives us an expression for where the object will be next.

Example Suppose a curve is given in parametric form as $\vec{r}(s) = s\cos s\hat{\imath} + s\sin s\hat{\jmath}$. Find the expression for the tangent line to the curve at $s = 1$.

Worked Solution The strategy for this is simple. We *evaluate* $\vec{r}(1)$ and then evaluate $\frac{d}{ds}\vec{r}(s)$. This derivative will give $\frac{d}{ds}x(s)$ as the x-component and $\frac{d}{ds}y(s)$ as the y-component. These will give the rate that the x- and y-components change as s changes. This *vector* is the tangent vector, so just like in the linear approximation section, the tangent line will be given as

$$\text{Tangent line} = \vec{r}(s_0) + \frac{d}{ds}\vec{r}(s_0)\,(s - s_0) \tag{4.22}$$

The x- and y-components are just the linear approximations of the components of $\vec{r}$ around s_0.

Now that we have a plan, the implementation is straightforward. We have $s_0 = 1$, so we need to calculate $\vec{r}(1) = \cos 1\hat{\imath} + \sin 1\hat{\jmath} = 0.5403\hat{\imath} + 0.8415\hat{\jmath}$ Similarly,

$$\begin{aligned} \frac{d}{ds}\vec{r}(s) &= (\cos s - s\sin s)\,\hat{\imath} + (\sin s + s\cos s)\,\hat{\jmath} \\ \frac{d}{ds}\vec{r}(1) &= -0.3012\hat{\imath} + 1.3818\hat{\jmath} \end{aligned} \tag{4.23}$$

This means that the tangent line to the curve given above at $s = 1$ is

$$\vec{r}_{tangent\ line} = (0.5403\hat{\imath} + 0.8415\hat{\jmath}) + (-0.3012\hat{\imath} + 1.3818\hat{\jmath})\,(s - 1) \tag{4.24}$$

This is true for *any* value of s, so the line is the set of all possible points which come from the different possible values of s.

We did this example to motivate the fact that the velocity vector will be tangent to the position as a function of time curve.

Student Exercises

- Find the unit vector tangent to the curve parametrized as $\vec{r}(s) = r_0 \cos s\hat{\imath} + r_0 \sin s\hat{\jmath}$ at the point given by s_0. *We find that the unit vector is* $-\sin s_0\hat{\imath} + \cos s_0\hat{\jmath}$.

- Approximately how far does an object whose position is given as $\vec{r}(s) = \frac{e^s+e^{-s}}{2}\hat{\imath} + \frac{e^s-e^{-s}}{2}\hat{\jmath}$ move as s goes from $s = 0.5$ to $s = 0.51$? *We find that the derivative with respect to s at* 0.5 *is* $1.12763\hat{\imath} - 0.52109\hat{\jmath}$ *so the displacement is about* $0.0113\hat{\imath} - 0.0052\hat{\jmath}$, *which has a magnitude of* 0.0124.

4.6 Implicit Differentiation and Related Rates

This topic shows up in a large number of mathematics courses, however in a physics course this kind of problem is often presented as simply an obvious application of the idea of differentiation.

Example Suppose that a ladder of length L is standing against a vertical wall. The upper end of the ladder is at $z\hat{k}$ and the lower end of the ladder is at $x\hat{\imath}$. Suppose that x is changing at a rate $\frac{dx}{dt} = k$; what is the rate at which z changes?

Worked Solution The essential plan for any problem like this is to construct a relationship between the relevant variables, and then use calculus.

In this case, the length of the ladder is L, and the *vector* from the bottom to the top is $-x\hat{\imath} + z\hat{k}$, so using the length of the vector we have

$$x^2 + z^2 = L^2 \tag{4.25}$$

Differentiating both sides with respect to x, treating z as a function of x, gives

$$\begin{aligned} \frac{d}{dx}\left(x^2 + z^2\right) &= \frac{d}{dx}L^2 \\ 2x + 2z(x)z'(x) &= 0 \end{aligned} \tag{4.26}$$

When we reorganize we get that

$$z'(x) = -\frac{x}{z(x)} \tag{4.27}$$

or, since we can solve for $z = \sqrt{L^2 - x^2}$ this is

$$z'(x) = -\frac{x}{\sqrt{L^2 - x^2}}. \tag{4.28}$$

Now, knowing the rate at which x is changing tells us that in a small amount of time δt x will change by $\delta x = k\delta t$. This small change in x will induce a correspondingly small change in z which is $\delta z = z'(x)\delta x$ or $\delta z = -\frac{x}{\sqrt{L^2-x^2}}k\delta t$, so the rate of change of z with respect to time is $-\frac{x}{\sqrt{L^2-x^2}}k$.

Note that we could have gotten the expression for $\frac{d}{dx}z(x)$ by solving for z and then simply differentiating. This technique is most useful when it is hard to explicitly solve the relation for one variable in terms of another.

Student Exercises

- If the angle the ladder makes with the horizontal is θ what is the rate at which θ changes if x increases at rate k? *We find that the rate of change of θ is $-\frac{k}{\sqrt{L^2-x^2}}$.*

- A hollow sphere has radius R. It is filled with water which drains out the bottom at a rate of L liters per minute. If z is the height of water measured above the bottom – so the maximum z is $2R$ – what is the rate at which z changes? *This might make more sense to solve by simply considering the change in volume, but we get that* $\frac{dz}{dt} = \frac{L}{\pi(2Rz-z^2)}$.

Chapter 5

Kinematics

5.1 Overview

In the fifth chapter of the text, and here we take the idea of using a vector to describe the position of an object and try and develop a mathematical description of what happens when the object's position changes in time. We will define *velocity* and *acceleration* in terms of derivatives of the position vector. While there are *general* relationships between these quantities, there are also some special cases where there are results that are useful to remember (and to be able to work out on your own!)

- The position of an object can be described mathematically as $\vec{r}(t)$, and when we use the x, y, and z coordinate system we can write the components as $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. In this $x(t)$ is the x-component of the vector $\vec{r}(t)$. The notation '(t)' is intended to explicitly remind us that the position (and hence some of the individual components of the position vector) changes in time.

- The *velocity* of an object is the rate at which the position vector changes. This is defined as $\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$. The individual components of $\vec{v}(t)$ are called $v_x(t)$, $v_y(t)$, and $v_z(t)$. These components are related to the components of $\vec{r}(t)$ since the vector nature of the position implies $v_x(t) = \frac{d}{dt}x(t)$, and similar definitions for the y and z components.

- The *acceleration* of an object is the rate at which the velocity vector changes. This is defined as $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$, or equivalently as $\vec{a}(t) = \frac{d^2}{dt^2}\vec{r}(t)$. This means that the components have the relationship $a_y(t) =$

$\frac{d}{dt}v_y(t) = \frac{d^2}{dt^2}y(t)$, with similar definitions for the x and z components.

- When we write the position, velocity, and acceleration vectors, we sometimes (honestly: often) omit the explicit time dependence in our notation. It is extremely important to remember that *this time dependence is always there*! When physicists write things, we often 'abbreviate' the mathematical descriptions because we remember they are there - this could be things like omitting the time dependence, or not being as pedantic as we could be about vectors. We will try and be very explicit in this workbook, but watch out for this as you read the text and your lecture notes.

- For the special case of constant acceleration, there are relationships between the position, velocity, and acceleration vectors at the beginning and end of some period of length Δt:

$$\begin{aligned} \vec{r}_f &= \vec{r}_i + \vec{v}_i \Delta t + \frac{1}{2}\vec{a}\,(\Delta t)^2 \\ \vec{v}_f &= \vec{v}_i + \vec{a}\Delta t \\ |\vec{v}_f|^2 &= |\vec{v}_i|^2 + 2\vec{a}\cdot(\vec{r}_f - \vec{r}_i) \end{aligned} \tag{5.1}$$

  In this set of expressions, the subscript i refers to the quantity at the *start* of the time interval, and the subscript f refers to the quantity at the end of the interval.

- If an object is known to be undergoing constant acceleration motion, the relationships 5.1 are often re-written as:

$$\begin{aligned} \vec{r}(t) &= \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2 \\ \vec{v}(t) &= \vec{v}_i + \vec{a}t \\ |\vec{v}(t)|^2 &= |\vec{v}_i|^2 + 2\vec{a}\cdot(\vec{r}(t) - \vec{r}_i) \end{aligned} \tag{5.2}$$

  where $\vec{r}_i$ and $\vec{v}_i$ are the position and velocity at time $t = 0s$.

- An object which is undergoing *uniform* circular motion in a circle of radius R has an acceleration with magnitude $|\vec{a}| = \frac{|\vec{v}|^2}{r}$. This acceleration is towards the center of the circle, and is constant in *magnitude* but not direction. Uniform circular motion refers to a case where the object is moving with a constant speed.

- If $\vec{a}\cdot\vec{v} > 0$ the object is speeding up; if $\vec{a}\cdot\vec{v} < 0$ the object is slowing down. If $|\vec{a}\cdot\vec{v}| \neq |\vec{a}|\,|\vec{v}|$ then the object is turning – changing direction.

- Objects moving near the surface of the earth experience a constant acceleration due to gravity of magnitude $g = 9.8\frac{m}{s^2}$ and directed down.

- If you know the *velocity* as a function of time, you can calculate the object's displacement between two known times t_1 and t_2. The displacement is given by

$$\vec{r}(t_2) - \vec{r}(t_1) = \int_{t_1}^{t_2} \vec{v}(t)dt. \tag{5.3}$$

- If you know the *acceleration* as a function of time, you can calculate the change in the object's velocity between times t_1 and t_2 as

$$\vec{v}(t_2) - \vec{v}(t_1) = \int_{t_1}^{t_2} \vec{a}(t)dt. \tag{5.4}$$

5.2 Position, Velocity, and Acceleration vectors

Example Consider an object which has a position which is a function of time and is given by

$$\begin{aligned} \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ x(t) &= 2m - 3\frac{m}{s}t + 0.45\frac{m}{s^2}t^2 \\ y(t) &= -4\frac{m}{s}t + 0.5\frac{m}{s^2}t^2 \end{aligned} \tag{5.5}$$

For this object:

1. Find the velocity as a function of time.

2. Find the acceleration as a function of time.

3. Sketch the x and y components of the object's position as a function of time.

4. Sketch the *trajectory* of the object.

Worked Solution The first thing we have to do is find the *velocity* and *acceleration*. We know that since we have the position as a function of time, all that we need to do to find the velocity and acceleration is to take the (appropriate) derivatives of the position.

At this point, it is worthwhile to remind ourselves of a couple properties of derivatives: The first is that derivative is a *linear* operator – what this means is that the derivative of the sum of two things is the sum of their derivatives. The second is that the derivative of a constant multiplied by a function is just that same constant multiplied by the derivative of the function. This is going to be the justification for what we have claimed is the relationship between the components of velocity (and acceleration) and the derivatives of the components of the position vector.

We will start with the definition of velocity, and work from there:

$$\begin{aligned}
\vec{v}(t) &= \frac{d}{dt}\left(x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}\right) \\
&= \frac{d}{dt}\left(x(t)\hat{\imath}\right) + \frac{d}{dt}\left(y(t)\hat{\jmath}\right) + \frac{d}{dt}\left(z(t)\hat{k}\right) \\
&= \left(\frac{d}{dt}x(t)\right)\hat{\imath} + \left(\frac{d}{dt}y(t)\right)\hat{\jmath} + \left(\frac{d}{dt}z(t)\right)\hat{k} \qquad (5.6)
\end{aligned}$$

and in the last equality, we have that the (for example) z-component of the velocity vector (which we would denote $v_z(t)$) has to be identified with the z-component of the right-hand side. This means that

$$\begin{aligned}
v_x(t) &= \frac{d}{dt}x(t) \\
v_y(t) &= \frac{d}{dt}y(t) \\
v_z(t) &= \frac{d}{dt}z(t) \qquad (5.7)
\end{aligned}$$

For the case we have, we know that

$$\begin{aligned}
x(t) &= 2m - 3\frac{m}{s}t + 0.45\frac{m}{s^2}t^2, \\
y(t) &= -4\frac{m}{s}t + 0.5\frac{m}{s^2}t^2, \text{ and} \\
z(t) &= 0. \qquad (5.8)
\end{aligned}$$

Note, in passing, that the constants that appear in these all have different units. Since t has units of seconds (s) the overall units of each part are just meters. You should convince yourself of this if it isn't obvious to you. Sometimes, the units are combined and written after the expression, or omitted entirely, and the reader is supposed to understand the individual units. In this problem, we are being very explicit about the inclusion of units, because they provide a useful cross-check or mnemonic to make sure that the quantities you substitute in to relations are the correct ones.

Since we know that we have to take derivatives, we should remind you of the basic rules associated with derivatives: remember that derivatives measure the *change* in something with respect to something else. In your math classes, you have probably learned how to calculate the change in a function $y(x)$ with a small change in x. What we do in Physics is exactly the same, except at this point our independant variable is t. Some derivatives that you should know or recall are:

$$\begin{aligned}
\frac{d}{dx}\left(af(x)\right) &= a\frac{d}{dx}f(x) \qquad a \text{ a constant} \\
\frac{d}{dx}x^n &= nx^{n-1} \\
\frac{d}{dx}e^{ax} &= ae^{ax} \\
\frac{d}{dx}\sin\left(ax+b\right) &= a\cos\left(ax+b\right) \\
\frac{d}{dx}\cos\left(ax+b\right) &= -a\sin\left(ax+b\right).
\end{aligned} \tag{5.9}$$

When we go ahead and take the derivative of the x-component of position given in our problem to get the x-component of velocity, we have

$$\begin{aligned}
v_x(t) &= \frac{d}{dt}x(t) \\
&= \frac{d}{dt}\left(2m - 3\frac{m}{s}t + 0.45\frac{m}{s^2}t^2\right) \\
&= \frac{d}{dt}\left(2m\right) - \frac{d}{dt}\left(3\frac{m}{s}t\right) + \frac{d}{dt}\left(0.45\frac{m}{s^2}t^2\right) \\
&= 2m\frac{d}{dt}1 - 3\frac{m}{s}\frac{d}{dt}t + 0.45\frac{m}{s^2}\frac{d}{dt}t^2 \\
&= 2m\frac{d}{dt}t^0 - 3\frac{m}{s}\frac{d}{dt}t^1 + 0.45\frac{m}{s^2}\frac{d}{dt}t^2 \\
&= 0 - 3\frac{m}{s}\left(1t^0\right) + 0.45\frac{m}{s^2}\left(2t^1\right) \\
&= -3\frac{m}{s} + 0.9\frac{m}{s^2}t
\end{aligned} \tag{5.10}$$

and we can similarly get

$$v_y(t) = -4\frac{m}{s} + 1\frac{m}{s^2}t \tag{5.11}$$

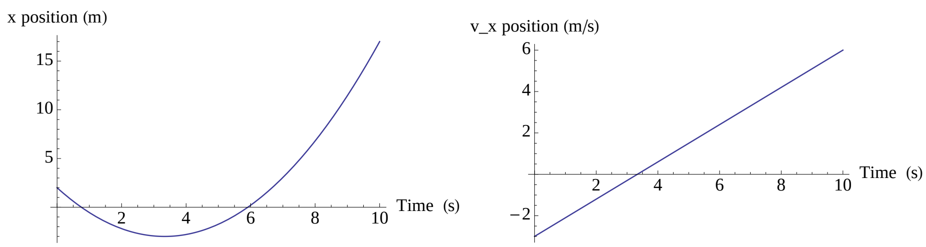


Figure 5.1: On the left, the x-component of $\vec{r}$, $x(t) = 2m - 3\frac{m}{s}t + 0.45\frac{m}{s^2}t^2$ is plotted. The slope of this function is the x component of the velocity, v_x. This slope is plotted on the right. Note that the slope - which represents the x-component of velocity - is not constant but rather changes with time.

and since $z(t) = 0$ (a constant) $v_z(t) = 0$ as well. This means we can write

$$\vec{v}(t) = \left(-3\frac{m}{s} + 0.9\frac{m}{s^2}t\right)\hat{\imath} + \left(-4\frac{m}{s} + 1\frac{m}{s^2}t\right)\hat{\jmath} \tag{5.12}$$

We can find the components of acceleration by taking the time derivative of the velocity; the component-by-component reasoning is the same here:

$$\begin{aligned}\vec{a}(t) &= \frac{d}{dt}\vec{v}(t)\\ &= \left(\frac{d}{dt}v_x(t)\right)\hat{\imath} + \left(\frac{d}{dt}v_y(t)\right)\hat{\jmath}\\ &= 0.9\frac{m}{s^2}\hat{\imath} + 1\frac{m}{s^2}\hat{\jmath}\end{aligned} \tag{5.13}$$

Since we know $x(t)$, and $y(t)$, we can plot them as functions of time. They are shown in figures 5.1 and 5.2. Given the expression for $\vec{r}(t)$, it is possible to calculate $\vec{r}$ at various instants in time, and plot them. This is done in figure 5.3. Notice that this is a plot of a *parametric* curve, with position being specified as a function of the *parameter* t.

Things you should notice There are a number of points that we really have to emphasize:

- The functions $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ are all vectors that can change in time. We can substitute in a particular value for t and get the position, velocity, and acceleration of the particle at that instant.

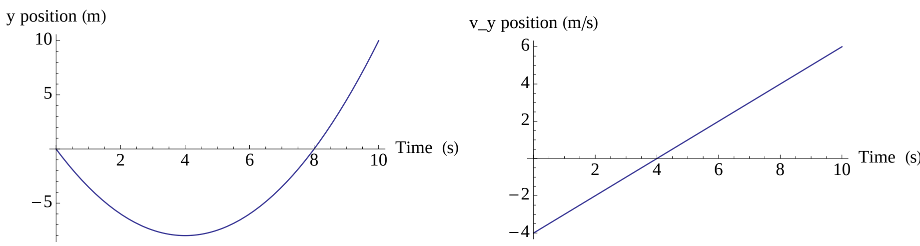


Figure 5.2: On the left, the y-component of $\vec{r}$, $y(t) = -4\frac{m}{s}t + 0.5\frac{m}{s^2}t^2$ is plotted. The slope of this function is the y component of the velocity, v_y, which is plotted on the right.

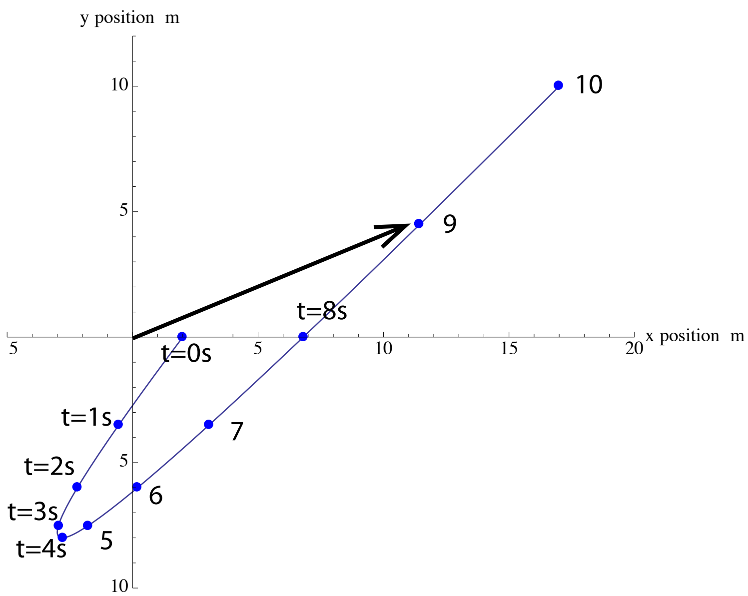


Figure 5.3: The trajectory $\vec{r}$ is plotted. The times at which the object reaches specific locations are indicated for 11 points starting at $t = 0$. The position vector $\vec{r}(9s)$ has also been drawn.

t	x	y	v_x	v_y	a_x	a_y
0s	2.00m	0.0m				
1s	-0.55m	-3.5m				
2s	-2.20m	-6.0m				
3s						
4s						
5s						
6s						
7s						
8s						
9s	11.45m	4.5m				
10s	17.00m	10m				

Table 5.1: The x and y positions for several times are given in the table below. Space has been made for the velocity and acceleration components as well.

- We calculate $\vec{v}(t)$ and $\vec{a}(t)$ by taking the derivative with respect to time of $\vec{r}(t)$ and $\vec{v}(t)$ respectively. We can get a *component* of the velocity or acceleration by taking the time derivative of the same component of the position or velocity vectors.
- We often represent, or visualize, motion by considering individual components as a function of time (as in figures 5.1 and 5.2) and we can also represent the motion as a plot where the individual points represent different times (as in figure 5.3).

Student Exercises

- For the vector $\vec{r}(t)$ given in the problem above, find the x and y components of position, velocity, and acceleration for times between $t = 0s$ and $t = 10s$. You can fill them in on table 5.1. Look at the direction of the velocity at each of those time intervals - convince yourself by looking at figure 5.3 that the direction of the velocity at a particular time goes from the current location to the location at the next instant. *This is straightforward. The expressions for $\vec{r}(t)$ and $\vec{v}(t)$ are earlier, and the acceleration should be constant.*
- Consider a particle which moves with position

$$\vec{r}(t) = \left(3\frac{m}{s}t - 1m\cos\left(2s^{-1}t\right)\right)\hat{\imath} + 1m\sin\left(2s^{-1}t\right)\hat{k} \tag{5.14}$$

– Sketch the trajectory. *It should look 'loopy', like circles whose center is moving as they are being drawn.*
– Find the velocity at $t = 3.14s$. *It is* $3\frac{m}{s}\hat{\imath} + 2\frac{m}{s}\hat{k}$.
– Find the acceleration at $t = 1.57s$. *It is* $-4\frac{m}{s^2}\hat{\imath}$.
– Find the *angle* between the velocity and the acceleration at $t = 1s$. Hint: You will need to find the velocity and acceleration, obviously, and you will also need their magnitudes and their component forms – remember the relationships between angles and dot products. *It is* 1.829 *radians, or* $105°$. *When you calculate this make sure your calculator was in* radians *for the trigonometric parts, otherwise you will not have taked the derivative correctly.*
– At what time(s) are the velocity and acceleration perpendicular? *There are an infinite number, but starting from* $t = 0s$ *the first few are* $0.785s,\ 2.356s,\ 3.927s, \ldots$.

5.3 Interpreting physical quantities to determine a trajectory

Example A ball is thrown from position $\vec{r} = 2m\hat{k}$ at an angle of $30°$ above the horizontal. It reaches the top of the arc along which it travels $2s$ later. What is the ball's location $3s$ after it was thrown?

Worked Solution This kind of problem is very typical for introductory physics classes. We are given a number of pieces of information, and need to put them together using 'detective work' to get an expression for the position as a function of time. We can then find the position at the desired time.

The information we got was:

- The initial position of the ball.
- The angle of the ball's flight when launched.
- The angle of the ball's flight at some fixed time ($2s$ after launch).
- The acceleration, and that this is a problem with constant acceleration.

The fact that this is a problem that has constant acceleration due to gravity is something you have to recognize from some of the key words in the statement of the problem – in general unless the problem states otherwise, physicists idealize 'thrown' objects as ones that undergo constant acceleration due to gravity and we neglect air resistance. The problems that can be done this way are moderately interesting in themselves, but are chiefly interesting as they provide a place to train the use of vectors in solving physical problems.

First we find and express the acceleration. The acceleration is due to gravity, and it is downwards. Here, we are using the $\hat{k}$ direction as the vertical, so $\vec{a} = -g\hat{k} = -9.8\frac{m}{s^2}\hat{k}$. Remember that the acceleration is downwards and that $\hat{k}$ is upwards. This explains the negative sign in the expression for acceleration.

Here's what we know: Since the particle is travelling with constant acceleration, its position and velocity are given as a function of time as:

$$\begin{aligned} \vec{r}(t) &= \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2 \\ \vec{v}(t) &= \vec{v}_i + \vec{a}t \end{aligned} \tag{5.15}$$

We take the time when the ball is thrown to be $t = 0s$.

The location when the ball is thrown is (from the statement of the problem)

$$\vec{r}(0s) = 2m\hat{k} \tag{5.16}$$

but if we put $t = 0s$ into the position part of equations 5.15 we get

$$\vec{r}(0s) = \vec{r}_i + \vec{v}_i\,[0s] + \frac{1}{2}\vec{a}\,[0s]^2 = \vec{r}_i \tag{5.17}$$

so $\vec{r}_i = 2m\hat{k}$. That was easy! We now know $\vec{r}_i$ and $\vec{a}$. What else can we do?

We claimed earlier that we know the angle the ball's velocity makes with a particular axis at $t = 2s$. To see this takes a bit of thinking: We know that the ball is at the top of the flight at this instant. We know that the velocity vector is *tangent* to the trajectory at every point. This means that at the top of the ball's flight its velocity must be horizontal: If the velocity had an upwards component the ball would be travelling up, and it would not be at the top. If the velocity had a downwards component, the ball would be travelling down, and so would have come from higher up. In either case, it would not be at the top unless the velocity has *no* vertical component. At $t = 2s$ the ball's velocity is along the x-axis (along $\hat{\imath}$).

We have left the job of finding the initial velocity to last. We know from equations 5.15 that the form of the velocity is $\vec{v}(t) = \vec{v}_i + \vec{a}t$. We have two pieces of information: at $t = 0s$ (the launch time) the velocity makes an angle of $30°$ above the horizontal, and at $t = 2s$ the velocity is only horizontal. We can write this as:

$$\begin{aligned} \vec{v}(0s) &= v\cos\theta\hat{\imath} + v\sin\theta\hat{k} \\ \vec{v}(2s) &= v_{top}\hat{\imath} \end{aligned} \tag{5.18}$$

Note that we'll put in that $\theta = 30°$ soon enough, but for now, the thing to notice is that there are two things we do not know: the speed v when launched and the speed v_{top} when the ball is at the top of its flight. We can compare these two pieces of information to our expression for the velocity as a function of time:

$$\begin{aligned} \vec{v}(0s) &= \vec{v}_i + \vec{a}\,[0s] = v\cos\theta\hat{\imath} + v\sin\theta\hat{k} \\ \vec{v}(2s) &= \vec{v}_i + \vec{a}\,[2s] = v_{top}\hat{\imath} \end{aligned} \tag{5.19}$$

The first equality tells us that $\vec{v}_i = v\cos\theta\hat{\imath} + v\sin\theta\hat{k}$, and we can substitute this (and our expression for $\vec{a}$) into the second relation and get

$$v_{top}\hat{\imath} = v\cos\theta\hat{\imath} + v\sin\theta\hat{k} + \left(-g\hat{k}\right)[2s] \tag{5.20}$$

so, by equating components: $v_{top} = v\cos\theta$, and $0 = v\sin\theta - g\,[2s]$. Since we know θ and g, we can solve the second expression for $v = 39.2\frac{m}{s}$, and so that tells us that $\vec{v}_i = v\cos\theta\hat{\imath} + v\sin\theta\hat{k} = 33.9\frac{m}{s}\hat{\imath} + 19.6\frac{m}{s}\hat{k}$.

Now we have $\vec{r}_i$, $\vec{v}_i$, and $\vec{a}$. We can write the position as a function of time:

$$\begin{aligned} \vec{r}(t) &= \left(2m\hat{k}\right) + \left(33.9\frac{m}{s}\hat{\imath} + 19.6\frac{m}{s}\hat{k}\right)t + \frac{1}{2}\left(-9.8\frac{m}{s^2}\right)t^2 \\ &= \left(33.9\frac{m}{s}t\right)\hat{\imath} + \left(2m + 19.6\frac{m}{s}t - 4.9\frac{m}{s^2}t^2\right)\hat{k} \end{aligned} \tag{5.21}$$

and so $\vec{r}(3s) = 102m\hat{\imath} + 16.7m\hat{k}$.

Some comments about this The process of going from information given to an expression for position as a function of time is one of the skills you have to develop. The general pattern for this kind of exercise is:

- You have – or guess – a general form for the position as a function of time. This is sometimes referred to as an *ansatz* (especially if your instructor is trying to seem smart). There will be some unknowns in it; in the example we had, the initial unknowns were the initial position, initial velocity, and acceleration.

- You can take derivatives of the position as a function of time to get the velocity and acceleration in terms of the unknowns in your *ansatz*. (There, we used the fancy word right away. It made you feel smart, right? That's why we do it.)

- You take the information you are given about the position, velocity, acceleration, at different times and substitute in. This will give you some equations for the unknowns you have.

- Solve the various equalities, and you should end up with an expression for the position where the only unknown is t. Use this for fun and profit.

Student Exercises

- Repeat the exercise we just did, but assume that $t = 0s$ happens when the ball is at the top of its flight. Make sure that you get the same result we did. Hint: The thing that will be different is that t will change. The ball was thrown $2s$ before it made it to the top, and we want the location $1s$ after the ball is at the top. This should change what the $\vec{r}_i$ and $\vec{v}_i$ are, but they should not change the *final answer. You should have found that* $\vec{r}_i = 67.8m\hat{\imath} + 21.6m\hat{k}$ *and* $\vec{v}_i = 33.9\frac{m}{s}\hat{\imath}$.

- A ball moves in a circle at a constant speed. Its position as a function of time is given by

$$\vec{r}(t) = 2.5m\cos(\omega t + \phi)\,\hat{\imath} + 2.5m\sin(\omega t + \phi)\,\hat{\jmath} \tag{5.22}$$

  At time $t = 0s$ the ball is at $1.77m\hat{\imath} - 1.77m\hat{\jmath}$ and it is travelling at a speed of $5.0\frac{m}{s}$. Find the next time after $t = 0s$ that the ball is at $\vec{r} = 2.5m\hat{\imath}$. Note that you have two unknowns: ω and ϕ, and you have two pieces of information: the position at $t = 0s$ and the speed. *The next time the ball is on the x-axis is* $t = 0.39s$.

5.4 Projectile Motion

Example A ball is thrown from level ground at a speed of $30\frac{m}{s}$ at an angle of $60°$ above the horizontal. How fast is it going when it is $10m$ higher than the point from which it was thrown?

Worked Solution There are two ways we can approach this question; one makes use of the techniques we've developed for projectile motion, and the other uses a different result from the study of constant acceleration motion.

We will first approach the problem using the techniques we discussed in the previous question. Our starting point is that the ball will move with constant acceleration, and we can infer from the statement of the question information about the initial position, velocity and acceleration. We know that

$$\begin{aligned}\vec{r}(t) &= \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2 \\ &= (0) + \left(v\cos(\theta)\hat{\imath} + v\sin(\theta)\hat{k}\right)t + \frac{1}{2}\left(-g\hat{k}\right)t^2 \\ &= v\cos(\theta)t\hat{\imath} + \left(v\sin(\theta)t - \frac{1}{2}gt^2\right)\hat{k} \qquad (5.23)\end{aligned}$$

We have put in the initial condition information already, note that v and θ were given in the statement of the question, but we don't need to use them yet, so we will leave those variables written in a symbolic way. This should help us see the structure of the solution before we get the numerical value that we are asked for.

The quantity we want is the *speed* at the time when the ball is a distance d above the launch point. What this means is we need to find a time t_1 such that

$$\vec{r}(t_1) = (\text{we do not care})\hat{\imath} + d\hat{k} \qquad (5.24)$$

but we already have an expression for $\vec{r}(t)$, so we compare and get

$$\begin{aligned}\vec{r}(t_1) = v\cos\theta t_1\hat{\imath} + \left(v\sin\theta t_1 - \frac{1}{2}gt_1^2\right)\hat{k} &= (\text{we do not care})\hat{\imath} + d\hat{k} \\ v\sin\theta t_1 - \frac{1}{2}gt_1^2 &= d \qquad (5.25)\end{aligned}$$

The second equality is from the z-component of the vectors given, and is a *quadratic* relation for t_1 (note that v, θ, g, and d are all known quantities).

We can use the quadratic formula to get t_1:

$$t_1 = \frac{v\sin\theta \pm \sqrt{(v\sin\theta)^2 - 4\left(\frac{1}{2}g\right)d}}{2\left(\frac{1}{2}g\right)} \rightarrow t_1 = 0.418s \text{ or } 4.884s \qquad (5.26)$$

We find there are two possible values for t_1, the time at which the ball is $10m$ up. The question does not provide much guidance about which value of t_1 is the one we are interested in, so we will try to get the speed in both cases.

We know that $\vec{v}(t) = \vec{v}_i + \vec{a}t$, and in this case $\vec{v}_i = 15.0\frac{m}{s}\hat{\imath} + 26.0\frac{m}{s}\hat{k}$ for the given values of $v = 30\frac{m}{s}$ and $\theta = 60°$. In this case:

$$\begin{aligned}
\vec{v}(0.42s) &= \vec{v}_i + \vec{a}(0.42s) \\
&= 15.0\frac{m}{s}\hat{\imath} + 26.0\frac{m}{s}\hat{k} + \left(-9.8\frac{m}{s^2}\hat{k}\right)(0.42s) \\
&= 15.0\frac{m}{s}\hat{\imath} + 21.9\frac{m}{s}\hat{k} \qquad (5.27) \\
\vec{v}(4.88s) &= \vec{v}_i + \vec{a}(4.88s) \\
&= 15.0\frac{m}{s}\hat{\imath} + 26.0\frac{m}{s}\hat{k} + \left(-9.8\frac{m}{s^2}\hat{k}\right)(4.88s) \\
&= 15.0\frac{m}{s}\hat{\imath} - 21.9\frac{m}{s}\hat{k} \qquad (5.28)
\end{aligned}$$

The velocities at these two times are *different*, since in one case there is a component of the velocity pointing downward, and in the other there is an upward component, but since they have components of the same magnitude, the speed is the same in either case. We find that

$$|\vec{v}(0.42s)| = |\vec{v}(4.88s)| = 26.5\frac{m}{s} \qquad (5.29)$$

The other way we could approach the question is to use a different relationship to help find the speed. We know that for constant acceleration

$$|\vec{v}(t)|^2 = |\vec{v}_i|^2 + 2\vec{a}\cdot(\vec{r}(t) - \vec{r}_i) \qquad (5.30)$$

In this, $|\vec{v}_i| = 30\frac{m}{s}$, $\vec{a} = -g\hat{k}$, and the displacement

$$\vec{r}(t) - \vec{r}_i = (\text{we don't care})\hat{\imath} + d\hat{k}. \qquad (5.31)$$

This means that

$$\begin{aligned}
|\vec{v}(t)|^2 &= |\vec{v}_i|^2 + 2\left(-g\hat{k}\right)\cdot\left((\text{we don't care})\hat{\imath} + d\hat{k}\right) \\
&= |\vec{v}_i|^2 - 2gd \\
&= 900\frac{m^2}{s^2} - 2\left(9.8\frac{m}{s^2}\right)10m \\
|\vec{v}(t)| &= 26.5\frac{m}{s} \qquad (5.32)
\end{aligned}$$

In this, the t is (by implication) the time at which the vertical displacement is $10m$.

Why we did this example: This illustrates the general method we can use to do constant acceleration problems (such as projectile motion problems). If you want to know a quantity such as velocity, speed, or location, at some later time, you often need to first solve for the time taken. This will usually require the solution of a quadratic equation relating to the vertical component of position, and you normally need to interpret which of the two solutions correspond to the situation you care about. You can then use that time information to get the horizontal distance travelled, or, as in this example, the velocity. We also showed how you can use the relation between acceleration and displacement to determine the change in the (square of) the speed.

When you are working on problems using constant acceleration, there are some general strategies you can use:

- Do not substitute numbers in until the latest possible moment. Keeping the problem with algebraic quatities will help you spot mistakes. This is a very important habit to develop.
- For projectile motion questions, draw a diagram showing a parabolic trajectory and indicate the known quantites on the diagram.
- Identify the quantitities in the expressions for constant acceleration motion that you are given or that you can derive from what you are given.
- Rearrange the equations to get *what you want* in terms of *what you know*.
- Note that sometimes you are only interested in what happens in one of the x, y or z directions. In this case you only need to consider the equations that use those variables.

Student Exercises

- A ball is thrown from an initial position of $3m\hat{\imath}+2m\hat{k}$ over level ground as shown in figure 5.4. It is thrown with a speed of $20\frac{m}{s}$ at an angle of $25°$ above the horizontal.
  - How long is the particle in the air? *It is in the air for* $1.94s$*; the other value the quadratic expression gives is* $-0.21s$*.*
  - What is the x-component of position when the ball lands? *It lands at* $38.2m\hat{\imath}$*.*

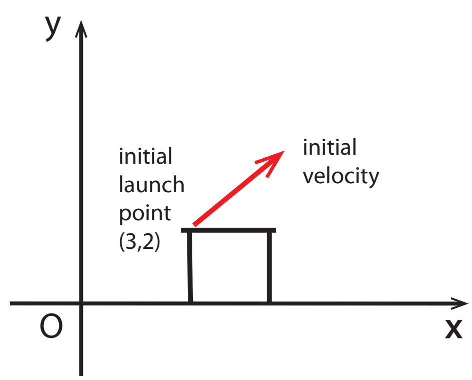


Figure 5.4: The particle is launched with an initial x and y value. There are also an initial x and y components of the velocity.

– What is the maximum height the ball reaches? *The ball reaches a height of* $5.65m$ *at a time of* $0.862s$.

– What is the x-component of position on landing and maximum height if the launch angle is 65° instead?

5.5 Circular Motion

Problem A particle travelling in uniform circular motion travels with a position given by

$$\vec{r}(t) = (x_0 + r\cos(\omega t + \phi))\,\hat{\imath} + (y_0 + r\sin(\omega t + \phi))\,\hat{\jmath} \tag{5.33}$$

Show that the particle's acceleration points towards the center of the circle in which it is travelling, and also show that the particle's speed is a constant for this expression for $\vec{r}(t)$.

Worked Solution This problem is fairly easy *mechanically* but there is a lot of content in understanding how the solution works. The first thing to do is to identify the center of the circle: The value for the x-component of $\vec{r}(t)$ is between $x_0 - r$ and $x_0 + r$ (since cos can vary between -1 and 1), and similarly the y-component can vary between $y_0 - r$ and $y_0 + r$. The middle of the circle is going to be the mid-point of each of those ranges, so the middle of the circle is at $x_0\hat{\imath} + y_0\hat{\jmath}$. This means that the vector to the particle *from* the center of the circle is

$$\vec{r}(t) - (x_0\hat{\imath} + y_0\hat{\jmath}) = r\cos(\omega t + \phi)\hat{\imath} + r\sin(\omega t + \phi)\hat{\jmath} \tag{5.34}$$

This relation will be important; we can visualize it as in figure 5.5.

We can differentiate $\vec{r}(t)$ to get the velocity. We get

$$\begin{aligned}
\vec{v}(t) &= \frac{d}{dt}\vec{r}(t) \\
&= \frac{d}{dt}\left[(x_0 + r\cos(\omega t + \phi))\,\hat{\imath} + (y_0 + r\sin(\omega t + \phi))\,\hat{\jmath}\right] \\
&= -r\omega\sin(\omega t + \phi)\hat{\imath} + r\omega\cos(\omega t + \phi)\hat{\jmath}
\end{aligned} \tag{5.35}$$

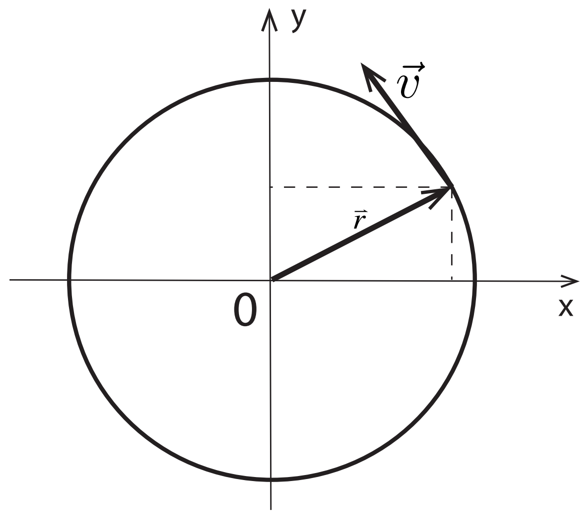


Figure 5.5: The particle moves in a circle of radius r in the x-y plane. The origin is the center of the circle illustrated

The speed (the magnitude of $\vec{v}(t)$) is

$$\begin{aligned} v &= |\vec{v}(t)| \\ &= \sqrt{\vec{v}(t)\cdot\vec{v}(t)} \\ &= \sqrt{(-r\omega\sin(\omega t+\phi))^2 + (r\omega\cos(\omega t+\phi))^2} \\ &= \sqrt{r^2\omega^2\left[\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)\right]} \\ &= r\omega \end{aligned} \tag{5.36}$$

so the speed is a constant since r and ω themselves are constants.

We can differentiate $\vec{v}(t)$ to get $\vec{a}(t)$:

$$\begin{aligned} \vec{a}(t) &= \frac{d}{dt}\vec{v}(t) \\ &= \frac{d}{dt}\left(-r\omega\sin(\omega t+\phi)\hat{\imath} + r\omega\cos(\omega t+\phi)\hat{\jmath}\right) \\ &= -r\omega^2\cos(\omega t+\phi)\hat{\imath} - r\omega^2\sin(\omega t+\phi)\hat{\jmath} \end{aligned} \tag{5.37}$$

Note that the vector $\vec{a}(t)$ is the same as the right hand side of equation 5.34 multiplied by a constant (specifically $-\omega^2$). The negative sign means that

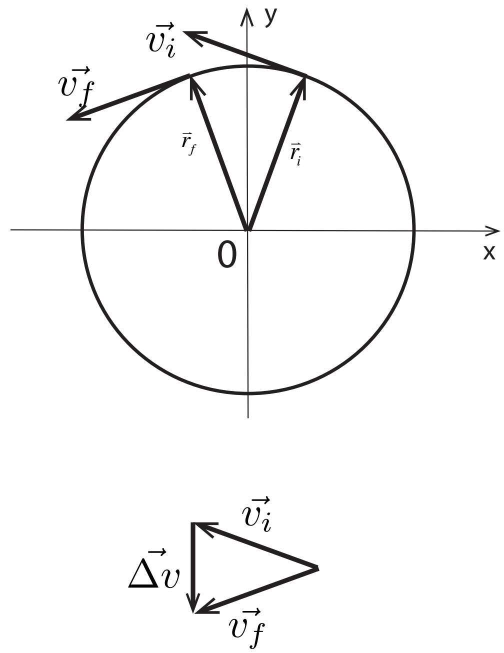


Figure 5.6: A particle moves in a circle with constant speed. The velocity at two instants is illustrated. The change in velocity is proportional to the acceleration, and points in the direction of the center of the circle at a time in between the two instants the velocities were illustrated.

the vector $\vec{a}(t)$ points in the opposite direction as the vector from the center of the circle to the object.

It is also possible to see *graphically* that the acceleration points towards the center of the circle. In figure 5.6 the velocity is illustrated at two instants in time. The difference between these velocities must be proportional to the acceleration, and you can see from the figure that this difference points to the center of the circle. Since as illustrated there is a finite time interval between when the two velocities are shown, we imagine that the points where they are taken get closer together, and then the change in velocity points more exactly towards the center of the circle.

Some things to remember We did this symbolically. Remember that:

- r is the radius of the circle.

- ω is a quantity with units s^{-1} which carries information about the speed at which the particle in uniform circular motion goes around its circle. ω is related to the speed as $v = r\omega$.
- The magnitude of the acceleration is $|\vec{a}| = r\omega^2 = \frac{v^2}{r}$.
- The quantity ϕ indicates where the particle is at time $t = 0$. If $t = 0$ then $\vec{r}(0s) = r\cos\phi\hat{\imath} + r\sin\phi\hat{\jmath}$.

When we calculate sin and cos in this context, we use radians, a dimensionless quantity. In general, the quantity in the argument of sin, cos, the exponential function, or ln has to be dimensionless.

Student Exercises

- Verify that the speed is constant by showing that $\vec{a}(t)$ and $\vec{v}(t)$ are always at $90°$ to each other. *Hint, what is their dot product, and how does knowing the dot product show that they are at $90°$?*
- A particle travels around a circle of radius r in time T. Verify that the relationship between T and ω is that $T = \frac{2\pi}{\omega}$. *Hint: Use the parametrization given, and the fact that goes around once means that it is at the same place at $t = T$ as it was at $t = 0s$.*
- A ball travels around a circle of radius $2m$ in time $0.5s$. What it the magnitude of the acceleration it experiences? *The magnitude of the acceleration is $316\frac{m}{s^2}$.*
- A ball travels around a circle of radius $3m$ at a speed of $4\frac{m}{s}$. What is the magnitude of the ball's acceleration? *The magnitude of the acceleration is $5.3\frac{m}{s^2}$.*

5.6 Questions

1. Consider a particle which moves with position given by

$$\vec{r}(t) = \left(7m - 3\frac{m}{s}t\right)\hat{\imath} + \left(-3m + 4\frac{m}{s}t\right)\hat{\jmath} \tag{5.38}$$

    - Find the velocity as a function of time.
    - Find the acceleration as a function of time.
    - Draw the trajectory for this particle.

2. A ball moves with constant acceleration in the x-y plane. The ball is initially at the origin, and $30s$ later the ball is at $10m\hat{\imath} + 10m\hat{\jmath}$, and travelling at $2\frac{m}{s}\hat{\imath} - 1\frac{m}{s}\hat{\jmath}$.
   - What is the ball's acceleration?
   - What is the ball's velocity when it is at the origin?
   - What is the ball's location $20s$ after it is at the origin?
3. A ball initially travels at $10\frac{m}{s}\hat{\imath}$, and experiences a constant acceleration of $-2\frac{m}{s^2}\hat{\imath}$.
   - How fast is it going when it has undergone a displacement of $16m\hat{\imath}$?
   - How long does it take for the ball to undergo this displacement?
   - How fast is it going and how long does it take if the acceleration is instead $2\frac{m}{s^2}\hat{\imath}$?
4. A ball travels in uniform circular motion around the origin. It is at $3m\hat{\imath} + 4m\hat{\jmath}$ and travelling $5\frac{m}{s}$ counterclockwise at time $t = 0$. What are r, ω, ϕ, and what is the next time it will be on the positive x-axis?

5.7 Answers

1. The particle ...
   - The velocity is $-3\frac{m}{s}\hat{\imath} + 4\frac{m}{s}\hat{\jmath}$.
   - The acceleration is 0.
   - The trajectory looks like the figure 5.7.
2. The ball ...
   - has a constant acceleration of $\vec{a} = 0.111\frac{m}{s^2}\hat{\imath} - 0.089\frac{m}{s^2}\hat{\jmath}$.
   - has a velocity of $\vec{v} = -1.33\frac{m}{s}\hat{\imath} + 1.67\frac{m}{s}\hat{\jmath}$ when it is at the origin.
   - is at $\vec{r} = -4.4m\hat{\imath} + 15.6m\hat{\jmath}$ at $t = 20s$.
3. The next ball ...
   - has a velocity of $\pm 6\frac{m}{s}\hat{\imath}$ when it has a displacement of $16m\hat{\imath}$. Note that the speeds associated with the two velocities are the same.

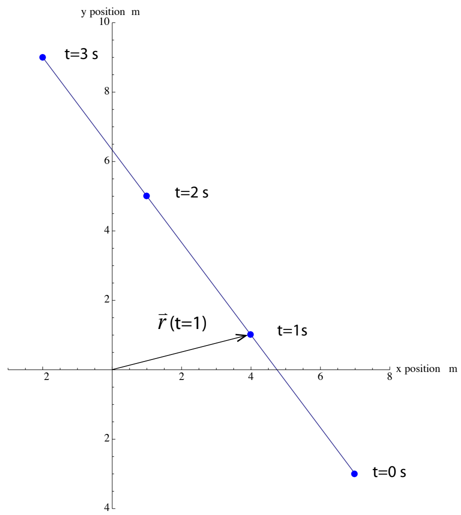


Figure 5.7: The trajectory for $\vec{r}(t) = \left(7m - 3\frac{m}{s}t\right)\hat{\imath} + \left(-3m + 4\frac{m}{s}t\right)\hat{\jmath}$ is plotted and the vector $\vec{r}(1s)$ is drawn.

- takes either $2s$ or $8s$ to undergo that displacement (for $6\frac{m}{s}\hat{\imath}$ and $-6\frac{m}{s}\hat{\imath}$ respectively.)
- would take $1.41s$ and its velocity would be $12.8\frac{m}{s}\hat{\imath}$ if the acceleration were $2\frac{m}{s^2}\hat{\imath}$.

4. It will be at $5m\hat{\imath}$ next at $t = 5.36s$. $r = 5m$, $\omega = 1s^{-1}$, and $\phi = 0.927$. Note that the argument of trigonometric functions is in *radians*.

Chapter 6

Newton's Second Law

6.1 Overview

Read the sixth chapter of the text which introduces Newton's second law: the relationship between the net force on an object and the acceleration.

- An acceleration can cause the speed of an object to either increase or decrease, it can cause the direction of movement to change, or it can cause the speed and direction to change. An acceleration changes the velocity of an object.
- Knowledge of the net force tells us about the acceleration, and conversely knowledge of the acceleration tells us about the net force.
- The colloquial expression $\vec{F} = m\vec{a}$ refers to the *net* force on an object.
- It is extremely important to treat all the forces acting on an object as vectors.

6.2 Sliding along a slope

Example A box of mass m is sliding *up* a slope which makes an angle of θ with the horizontal. The surface of the slope is rough. The box has a coefficient of kinetic friction with the slope of μ_k. This situation is illustrated in figure 6.1. If $\theta = 15°$ and $\mu_k = 0.2$, and the box was initially moving at a speed $|\vec{v}_i| = 10\frac{m}{s}$, how far does the box travel up the slope before it comes to a stop?

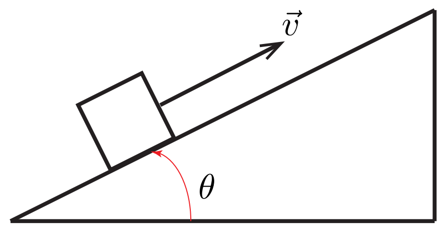


Figure 6.1: A box of mass m is sliding up a slope which makes an angle of θ with the horizontal at speed v.

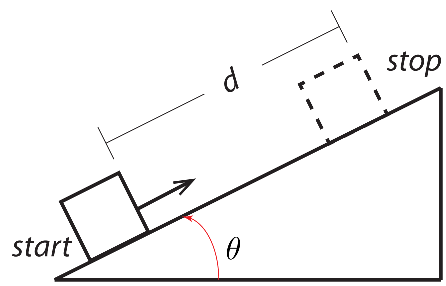


Figure 6.2: This figure shows the starting and ending position of the box described in figure 6.1. The box has travelled a distance d along the slope.

Worked Solution This is a problem where we have to put together information we know now (about the relationship between net force and acceleration) with information we learned before (the relationship between acceleration, velocity, and distance travelled). We have done problems involving things on slopes before so we can figure out the net force. Knowing the net force allows us to find the acceleration, and since the acceleration is constant, we can find how far the box has travelled.

The problem that we are trying to solve is illustrated in figure 6.2. We are trying to calculate the value of d. As this is one-dimensional motion with constant acceleration, we will need to determine the acceleration, and that can be worked out by finding the net force.

When we try to find the net force on anything, what we usually do is make a free-body diagram. In figure 6.3 we show the orientation of the three forces on the box: gravity, the normal force, and the force of friction. In

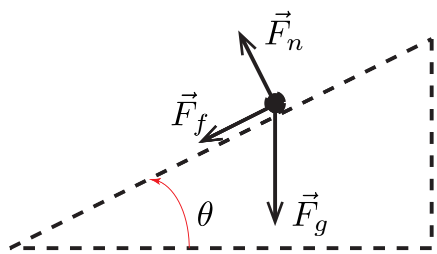


Figure 6.3: This figure shows forces acting on the box described in figure 6.1. The force of friction acts *down* the slope because the box is moving *up* the slope, and the force of friction opposes motion.

analyzing this set of forces, it will be useful to use the coordinate system shown in figure 6.4. We will use the $\hat{n}$, $\hat{p}$ coordinate system to decompose $\vec{F}_n$, $\vec{F}_g$, and $\vec{F}_f$. We find that

$$\begin{aligned} \vec{F}_n &= \left|\vec{F}_n\right| \hat{n} \\ \vec{F}_f &= -\mu_k \left|\vec{F}_n\right| \hat{p} \\ \vec{F}_g &= -mg\sin\theta\hat{p} - mg\cos\theta\hat{n} \end{aligned} \tag{6.1}$$

If this decomposition of the vectors is not immediately obvious, you should go back and review how to break vectors up into their components. In particular look at sections 1.2 and 2.2.2.

Using the decomposition in equations 6.1, we can find the net force on the box:

$$\begin{aligned} \vec{F}_{net} &= \vec{F}_n + \vec{F}_f + \vec{F}_g \\ &= \left|\vec{F}_n\right| \hat{n} + \left(-\mu_k \left|\vec{F}_n\right| \hat{p}\right) + \left(-mg\sin\theta\hat{p} - mg\cos\theta\hat{n}\right) \\ &= \left(\left|\vec{F}_n\right| - mg\cos\theta\right)\hat{n} + \left(-\mu_k\left|\vec{F}_n\right| - mg\sin\theta\right)\hat{p} \end{aligned} \tag{6.2}$$

Since we know the net force from equation 6.2 we can now find the acceleration, but we are going to have to use a bit more physics knowledge and intuition about the situation. The net force has two components. One of them describes the net force *along* the slope, and the other describes the net force *perpendicular to* the slope. Since the box slides along the slope, it doesn't change the position in the direction perpendicular to the slope, so

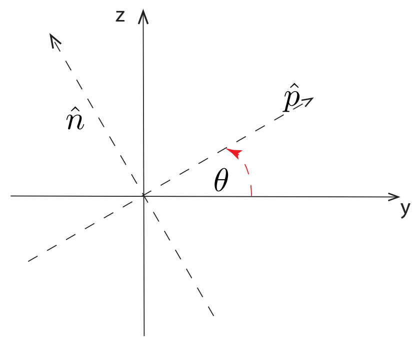


Figure 6.4: This figure shows a useful set of coordinates to analyze the motion of the box described in figure 6.1. The vector $\hat{n}$ is perpendicular to the slope, and the vector $\hat{p}$ is along the slope, pointing up.

it is in equilibrium in that direction. This means that the $\hat{n}$ component of the net force is 0, or that

$$0 = \left|\vec{F}_n\right| - mg\cos\theta \tag{6.3}$$

and substituting this into equation 6.2 we find that

$$\begin{aligned}
\vec{F}_{net} &= \left(\left|\vec{F}_n\right| - mg\cos\theta\right)\hat{n} + \left(-\mu_k\left|\vec{F}_n\right| - mg\sin\theta\right)\hat{p} \\
&= (mg\cos\theta - mg\cos\theta)\,\hat{n} + (-\mu_k mg\cos\theta - mg\sin\theta)\,\hat{p} \\
&= -mg\,(\mu_k\cos\theta + \sin\theta)\,\hat{p} \qquad (6.4)
\end{aligned}$$

We combine the result of with Newton's second law and get

$$\begin{aligned}
m\vec{a} &= \vec{F}_{net} \\
&= -mg\,(\mu_k\cos\theta + \sin\theta)\,\hat{p} \\
\vec{a} &= -g\,(\mu_k\cos\theta + \sin\theta)\,\hat{p} \qquad (6.5)
\end{aligned}$$

Now that we have found the acceleration, we are in a position to use kinematics. We know that for constant acceleration

$$\begin{aligned}
\Delta\vec{r} &= \vec{v}_{init}t + \frac{1}{2}\vec{a}t^2 \\
\vec{v}_{final} &= \vec{v}_{init} + \vec{a}t \qquad (6.6)
\end{aligned}$$

By looking at figure 6.2 we can see that the box's displacement is going to be $\Delta\vec{r} = d\hat{p}$. The final velocity will vanish because since the box comes to a stop. Note that since the original velocity was v_i up the slope, this means that we can write the original velocity as $v_i\hat{p}$. This means that 6.6 can be expressed for our box as

$$\begin{aligned} d\hat{p} &= (v_i\hat{p})\,t + \frac{1}{2}\left(-g(\mu_k\cos\theta + \sin\theta)\hat{p}\right)t^2 \\ 0 &= \vec{v}_i\hat{p} + \left(-g(\mu_k\cos\theta + \sin\theta)\hat{p}\right)t \end{aligned} \tag{6.7}$$

Since both of the relationships in 6.7 are in only the $\hat{p}$ direction, these are easy to solve. We know the numerical values of everything except t and d. We can use the second equation to solve for t, and the first to solve for d (alternatively, we could solve algebraically for t in terms of variables, and substitute that into the expression giving d, and then put in the numbers.)

With the given values $v_i = 10\frac{m}{s}$, $\theta = 15°$, and $\mu_k = 0.2$ we get $t = 2.26s$. This is the length of time for it to come to rest. Using this value for t gives us $d = 11.3m$.

Some things to notice There are several important things which we had to do in this problem:

- We expressed the forces in terms of their components in the $\hat{n}$ and $\hat{p}$ direction. We could have done this in $\hat{\imath}$, $\hat{k}$, but the algebra would have been more complicated.
- We used the fact that the box was not accelerating in the direction perpendicular to the slope to figure out the normal force. You cannot get the normal force from a formula that you simply memorize – you have to figure out what it needs to be to keep the object not going through the surface.
- We combined our result for the acceleration with information about kinematics to find out how far the object moved. This is typical in physics - you need to be able to combine concepts from different places together to attack problems.
- We wrote an expression for the distance travelled and the final position even though we didn't initially know what it is. This is a very valuable strategy: you give a name to quantities that you do not know, and often you can find out things about them.
- This is a problem that we could also solve using the concepts of work and energy. We will learn how to do this in chapter 11.

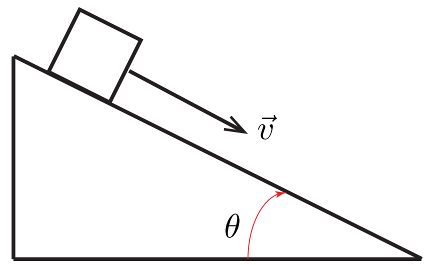


Figure 6.5: A box of mass m slides down a rough slope with which it has a coefficient of kinetic friction μ_k. The slope makes an angle of θ with the horizontal, as shown.

Student Exercises Consider a very similar problem: a box sliding down a rough surface with which it has a coefficient of kinetic friction μ_k has an initial speed of v_i. This is shown in figure 6.5.

- If $\mu_k = 0.2$ and $\theta = 20°$, what is the acceleration? *We find it to be* $1.51\frac{m}{s^2}$ *down the slope.*
- If $\mu_k = 0.3$, $\theta = 10°$, and the box initially goes at $5\frac{m}{s}$, how far down the slope does it travel? *We find that it takes the box* $4.19s$ *to travel* $10.5m$ *down the slope, at which point it stops.*
- Given μ_k, find the value of θ such that there is no acceleration of the box. If θ is bigger than this value, what is the direction of acceleration? *We find that there is no acceleration if* $\theta = \tan^{-1}\mu_k$*, and that if* $\theta > \tan^{-1}\mu_k$ *then* $\vec{a}$ *is down the slope.*

6.3 Moving in an accelerating vehicle

Example A block of mass m is suspended from a string in an elevator. What is the tension in the string, and what angle does the string make with the vertical if:

1. The elevator accelerates upwards with magnitude a_0?
2. The elevator accelerates sideways with magnitude a_0?

When calculating the numerical result, use $m = 5kg$ and $a_0 = 2.5\frac{m}{s^2}$. These situations are shown in figure 6.6.

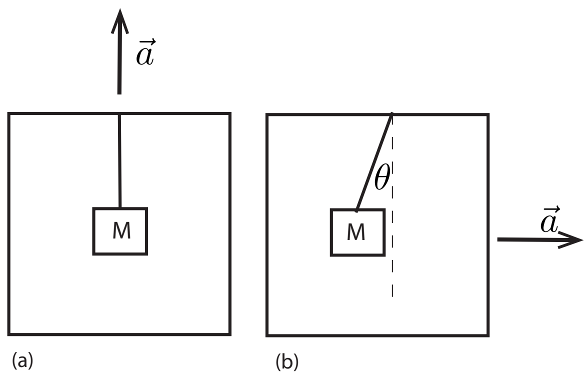


Figure 6.6: A box of mass m is suspended by a rope in an elevator accelerating (a) upwards or (b) towards the right.

Worked Solution We approach this as a Newton's law question from the point of view of the suspended mass. We know what its acceleration is, and we can use that to find the unknown force due to the tension.

The free-body diagram for this system is very simple, as shown in 6.7. From this, we have that the net force is

$$m\vec{a} = \vec{F}_{net} = \vec{F}_T + \vec{F}_g \tag{6.8}$$

Since we know what $\vec{a}$ is for the mass from the statement of the problem, and we know the force of gravity, the only thing that is unknown is the force supplied by the tension. We solve for it as:

$$\vec{F}_T = m\vec{a} - \vec{F}_g \tag{6.9}$$

For the first case (vertical acceleration) from the result in equation 6.9 and the vector decomposition of $\vec{a}$ and $\vec{F}_g$ we have

$$\begin{aligned} \vec{F}_T &= m\left(a_0\hat{k}\right) - \left(-mg\hat{k}\right) \\ &= m\left(a_0 + g\right)\hat{k} \end{aligned} \tag{6.10}$$

Using the numbers we have, $T = 61.5N$ and the rope is vertical.

Similarly, for the case of horizontal acceleration, the result in equation 6.9 gives us

$$\begin{aligned} \vec{F}_T &= m\left(a_0\hat{\imath}\right) - \left(-mg\hat{k}\right) \\ &= m\left(a_0\hat{\imath} + g\hat{k}\right) \end{aligned} \tag{6.11}$$

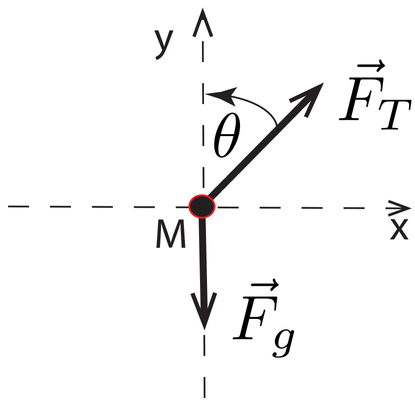


Figure 6.7: The free-body diagram for the mass shown in figure 6.6 showing the force from the tension in the rope and from gravity.

Since we know how to find magnitudes and angles, this vector has a magnitude of $T = 50.6N$, and makes an angle of $75.7°$ with the positive x-axis, or an angle of $14.3°$ as illustrated in figure 6.7.

A comment This is not as involved a problem as some: the key thing to notice here is that we knew about the *acceleration* and that, together with the relation between net forces and acceleration let us figure out what an unknown force was.

Student Exercises Repeat the exercise above for an acceleration which makes an angle of ϕ with the positive x-axis. The situation is shown in figure 6.8.

- Given that $m = 5kg$, $a_0 = 2\frac{m}{s^2}$, and $\phi = 30$, find the tension in the rope. *It turns out to be* $T = 54.7N$.
- Given that $m = 5kg$, $a_0 = 2\frac{m}{s^2}$, and $\phi = 30$, find the tangent of the angle θ that the rope makes with the vertical. *We find that* $\tan\theta = 0.160$.
- Repeat the previous two questions symbolically. Find the tension in the rope and $\tan\theta$ in terms of m, g, a_0, and ϕ. Check that your expression is correct by comparing the results for $\phi = 0$, $30°$, $90°$, and $-90°$ with the values obtained above.

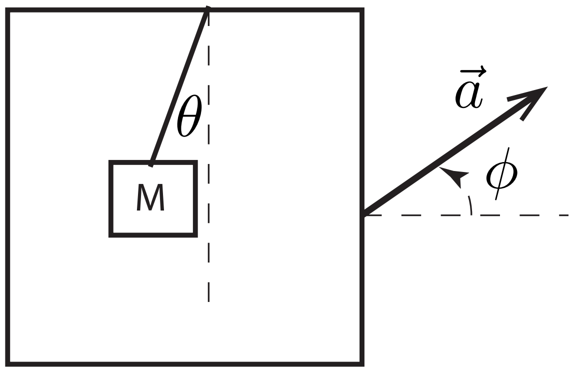


Figure 6.8: A box of mass m is in an elevator which accelerates at a rate a_0 at an angle of ϕ above the positive x-axis.

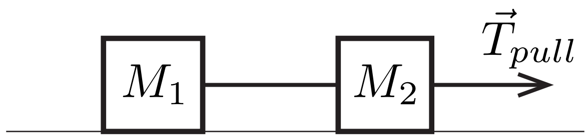


Figure 6.9: A box of mass m_1 is pulled over a smooth surface by rope attached to a second box of mass m_2. The box of mass m_2 is being pulled by a horizontal rope under tension T_{pull} .

The first two questions here are to encourage you to build up your skills in solving problems symbolically. Checking that your result matches known answers (which is what you did in the third question) is important because it helps you check that you got the right answer.

6.4 Atwood Machines

6.4.1 Two masses being pulled

Problem A mass m_1 is attached by a rope to a second mass m_2 on a frictionless plane. The second mass is pulled to the right by a rope acting horizontally under tension T_{pull}. This situation is shown in figure 6.9. If $m_1 = 2kg$, $m_2 = 3kg$, and $T_{pull} = 10N$, what is the tension in the rope

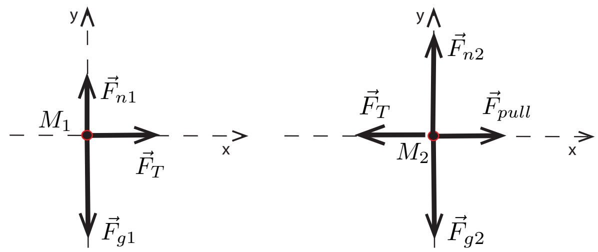


Figure 6.10: Free body diagrams for mass m_1 and m_2 described in figure 6.9.

between the two boxes?

Worked Solution For this, we choose to call the tension in the rope between the masses T. Based on this we can draw free-body diagrams for each mass as shown in figure 6.10. We will explicitly write out the contents of Newton's second law for each of the masses.

For box 1,

$$\begin{aligned} m_1\vec{a}_1 &= \vec{F}_{net,1} \\ &= \vec{F}_{n1} + \vec{F}_{g1} + \vec{F}_T \\ &= \left|\vec{F}_{n1}\right|\hat{k} + \left(-m_1 g\hat{k}\right) + T\hat{\imath} \end{aligned} \tag{6.12}$$

Since we assume that the box does not bounce off the surface, the equilibrium condition will make $\vec{F}_{g1}$ be cancelled by $\vec{F}_{n1}$, so

$$m_1\vec{a}_1 = T\hat{\imath} \tag{6.13}$$

Similarly, for box 2 we have:

$$\begin{aligned} m_2\vec{a}_2 &= \vec{F}_{net,2} \\ &= \vec{F}_{n2} + \vec{F}_{g2} + \vec{F}_T + \vec{F}_{pull} \\ &= \left|\vec{F}_{n2}\right|\hat{k} + \left(-m_2 g\hat{k}\right) + (-T\hat{\imath}) + T_{pull}\hat{\imath} \end{aligned} \tag{6.14}$$

Note that the force from the rope between the two boxes is now pulling to the left on this second box. The condition that there is equilibrium in the vertical direction (the same one we imposed for the first box) gives

$$m_2\vec{a}_2 = (T_{pull} - T)\,\hat{\imath} \tag{6.15}$$

When we look at equations 6.13 and 6.15, our chances of solving them do not initially look very good. There are three unknown quantities: $\vec{a}_1$, $\vec{a}_2$, and T, and only two relations between them. We can make one more observation: since they are joined by a rope if one moves, the other will; if one speeds up, the other will too. As long as the rope does not stretch the magnitudes of their accelerations are the same: $|\vec{a}_1| = |\vec{a}_2|$. We will call this acceleration magnitude a, so

$$a = |\vec{a}_1| = |\vec{a}_2| \tag{6.16}$$

and we will assume that both accelerate in the same direction (the x-direction) so we can write 6.13 and 6.15 as

$$\begin{aligned} m_1\,(a\hat{\imath}) &= T\hat{\imath} \\ m_2\,(a\hat{\imath}) &= (T_{pull} - T)\,\hat{\imath} \end{aligned} \tag{6.17}$$

This gives a pair of linear equations:

$$m_1 a = T \quad m_2 a = T_{pull} - T \rightarrow a = \frac{T_{pull}}{m_1 + m_2}. \tag{6.18}$$

Knowing what a is gives us that $T = T_{pull}\frac{m_1}{m_1+m_2}$. In the case we have here, $a = 2\frac{m}{s^2}$ and $T = 4N$.

Comment This *looks* like a simple application of Newton's second law, but there was a highly non-trivial thing that we did part way through: We said that the two masses had the same accelerations. This allowed us to make a relationship between the two accelerations and then we could solve for the unknown tension and acceleration.

This is an example of using *physics* to solve a problem. The physics part of this was applying Newton's law to each, and setting the two accelerations equal. The rest was algebra.

Student Exercises

- Work through the probem shown in figure 6.9. Find the acceleration of each mass and tension in the rope if

  1. $m_1 = m_2 = 4kg$, $T_{pull} = 16N$, and there is friction with $\mu_k = 0.1$ between m_1 and the ground. *We find the acceleration's magnitude is* $1.51\frac{m}{s^2}$ *and the tension is* $T = 9.96N$.

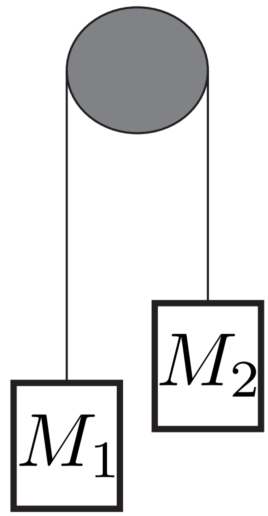


Figure 6.11: An Atwood machine consisting of two masses m_1 and m_2 attached by a rope which goes over a massless, frictionless pulley.

2. $m_1 = m_2 = 4kg$, $T_{pull} = 16N$, and there is friction with $\mu_k = 0.1$ between m_2 and the ground. *We find the acceleration's magnitude is* $1.51\frac{m}{s^2}$ *and the tension is* $T = 6.04N$.
3. $m_1 = m_2 = 4kg$, $T_{pull} = 16N$, and there is friction with $\mu_k = 0.1$ between both m_1 and m_2 and the ground. *We find the acceleration's magnitude is* $1.02\frac{m}{s^2}$ *and the tension is* $T = 8.0N$.

6.4.2 Classic Atwood Machine

Problem A mass m_1 is suspended against gravity by a rope which goes over a massless, frictionless pulley to another mass m_2. This is shown in figure 6.11. If $m_1 = 3kg$ and $m_2 = 2kg$, what is the tension in the rope, and what is the acceleration of the $3kg$ mass?

Worked Solution This problem is very similar to the problem we just encountered with the two boxes being pulled. We will set it up similarly, first doing a free-body diagram 6.12. The piece of *physics* that is hidden in the definition of the problem is when we say that the pulley is massless and frictionless. This means that there is the same tension in the rope on either

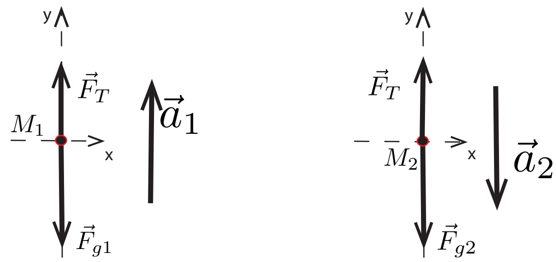


Figure 6.12: Free body diagrams for mass m_1 and m_2 described in figure 6.11. We also indicate a direction of acceleration; if mass m_1 goes up, the m_2 will go down.

side. If the pulley had mass or friction this would not be true. We will see this more explicitly when we talk about the kinetic energy of rotating objects, and when we talk about how torque relates to angular momentum, but for now, we will take it as given.

We write Newton's second law for each mass. For mass 1 we have:

$$\begin{aligned} m_1\vec{a}_1 &= \vec{F}_{net,1} \\ &= \vec{F}_{g1} + \vec{F}_T \\ &= -m_1 g\hat{k} + T\hat{k} \end{aligned} \tag{6.19}$$

Similarly for mass 2:

$$\begin{aligned} m_2\vec{a}_2 &= \vec{F}_{net,2} \\ &= \vec{F}_{g2} + \vec{F}_T \\ &= -m_2 g\hat{k} + T\hat{k} \end{aligned} \tag{6.20}$$

With this pair of relationships, we are in a similar position with equations 6.19 and 6.20 as we were with 6.13 and 6.15: There are three unknowns $\vec{a}_1$, $\vec{a}_2$, and T, and there are only two equations relating them. We have to relate the two accelerations, as we did before.

Considering mass 1, we see that the forces are in the vertical direction, so the acceleration will be in that direction as well so we decide to write $\vec{a}_1 = a\hat{k}$. We do not know what a is, but the direction of $\vec{a}_1$ is sure to be vertical. Now, if m_1 moves up, m_2 moves down, and if m_1 moves up *faster*, then m_2 will move *down* faster. Since they both start from rest this tells us that $\vec{a}_2 = -a\hat{k}$ (in other words that $\vec{a}_2 = -\vec{a}_1$).

Using this, we have

$$\begin{aligned} m_1\left(a\hat{k}\right) &= (T - m_1 g)\,\hat{k} \\ m_2\left(-a\hat{k}\right) &= (T - m_2 g)\,\hat{k} \end{aligned} \tag{6.21}$$

These give rise to the relations

$$\begin{aligned} m_1 a &= T - m_1 g \\ -m_2 a &= T - m_2 g \end{aligned} \tag{6.22}$$

and solving for a gives $a = \frac{m_2 - m_1}{m_1 + m_2} g$, and

$$T = m_1 g + m_1 \frac{m_2 - m_1}{m_1 + m_2} g = \frac{2 m_1 m_2}{m_1 + m_2} g \tag{6.23}$$

For the values of m_1 and m_2 we had, we find that $a = -1.96\frac{m}{s^2}$, and $T = 23.5N$.

This means that the mass m_1 accelerates at $\vec{a}_1 = -1.96\frac{m}{s^2}\hat{k}$. Note that this acceleration is *downward* even though in our picture 6.12 we guessed that the acceleration was upward. Even though we made a 'bad' guess about the direction of the acceleration, the algebra saved us.

Some comments: There are really two novel things to say here:

- This was a variation on the question where the two boxes were being pulled horizontally. In both, we had to use Newton's second law and a relationship between the two accelerations to get a system of equations we could solve.

- We (deliberately) made a mistake about which mass was going to go up and which was going to go down. Notice that we could have done the same question with a guess for m_1's acceleration in the other direction, and aside from $-$ signs, everything else would have *worked* the same way. We noticed at the end that the negative meant it was accelerating downward: It is a critical step in checking an answer to consider what the number or expression you obtained means.

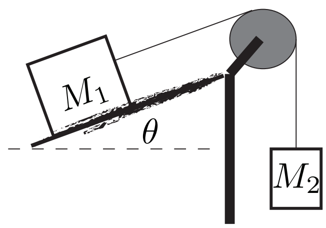


Figure 6.13: A mass m_1 is on a surface which makes an angle θ with the horizontal with which it has a coefficient of kinetic friction μ_k. It is attached via a horizontal rope which goes over a masslessfrictionless pulley to a second mass m_2 suspended against gravity.

Student Exercises

- Repeat the analysis of figure 6.11 with $m_1 = 5kg$ and $m_2 = 8kg$. Find T and the acceleration of m_1. Check that both m_1 and m_2 satisfy Newton's law with the T you calculated. *We find that* $\vec{a}_1 = 2.26\frac{m}{s^2}\hat{k}$ *and that the tension is* $T = 60.3N$.

- Consider the modified Atwood Machine depicted in Figure 6.13. It has m_1 moving over a rough sloped surface, pulled by a rope over a massless, frictionless pulley, which is connected to mass m_2 which is suspended against gravity.

1. Find T assuming that $m_1 = 2kg$, $m_2 = 3kg$, $\mu_k = 0.2$, and $\theta = 30°$. You can assume that the m_1 is initially moving up the slope. *You should find that the magnitude of the acceleration is* $3.24\frac{m}{s^2}$ *and that the tension is* $T = 19.7N$.
2. Calculate the tension symbolically in terms of m_1, m_2, μ_k, and θ. Check that the formula you got matches the value above for those parameters. You can also check this against one of the end-of-chapter questions. .

6.5 Circular Motion

6.5.1 A ball moving in a circle supported by two ropes

Problem A ball of mass m is rotated in a horizontal circle of radius L. It is held in this circle by two ropes, one horizontal, under tension T_1, and one which makes an angle of θ with the vertical under tension T_2. This is shown in figure 6.14. If $\theta = 45°$, $m = 4kg$, $L = 2m$, and $v = 6\frac{m}{s}$, what are the tensions T_1 and T_2?

Worked Solution Here we have a case where the object is going in circular motion. As we can see in figure 6.15, the ball is going in a circle of radius L, at speed v. We know from chapter 5 that this means that the magnitude of the acceleration is $|\vec{a}| = \frac{v^2}{L}$, and since we know the magnitude of the acceleration, we can then use Newton's second law to determine the unknown applied forces. It's really important (critical, in fact) to notice that the direction of the acceleration is in towards the center of the circle. In this case that direction is along the rope providing tension T_1.

We can write out what Newton's second law says:

$$
\begin{aligned}
m\vec{a} &= \vec{F}_{net} \\
m\left(-\frac{v^2}{L}\hat{\imath}\right) &= \vec{F}_{T1} + \vec{F}_{T2} + \vec{F}_g \\
-\frac{mv^2}{L}\hat{\imath} &= (-T_1\hat{\imath}) + \left(-T_2\sin\theta\hat{\imath} + T_2\cos\theta\hat{k}\right) + \left(-mg\hat{k}\right) \\
&= -\left(T_1 + T_2\sin\theta\right)\hat{\imath} + \left(T_2\cos\theta - mg\right)\hat{k} \qquad (6.24)
\end{aligned}
$$

As usual, the vector equation 6.24 contains information that can be used to solve for the unknown quantities. Since the left-hand side has no vertical components, we have that

$$T_2\cos\theta - mg = 0, \qquad (6.25)$$

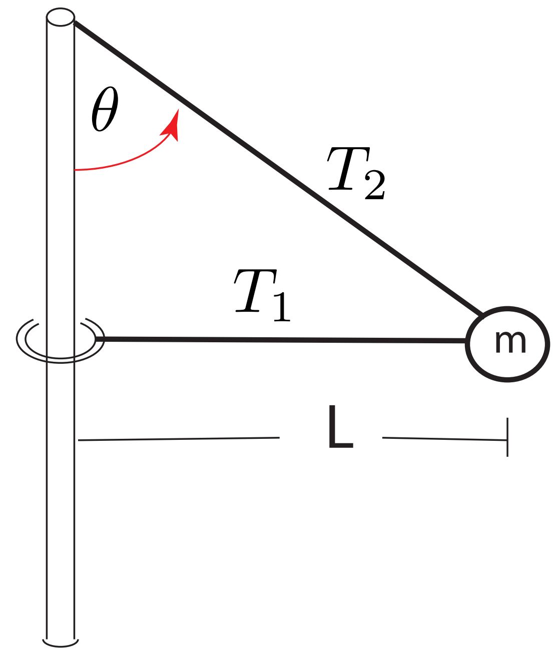


Figure 6.14: A ball of mass m is held by two ropes, one horizontal under tension T_1, and one under tension T_2 which makes an angle of θ with the vertical. The ball is going in a circle of radius L at speed v.

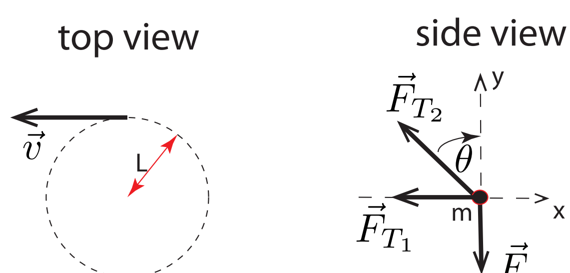


Figure 6.15: The direction of the velocity vector, tangent to a circle of radius L, ic shown in the top view, while the side view shows a snapshot of the forces applied, two tensions and the force of gravity.

and then the horizontal component gives

$$m\frac{v^2}{L} = T_1 + T_2 \sin\theta. \tag{6.26}$$

To solve this, we put in the numerical values for m, v, L, and θ. We get that

$$T_2 = \frac{mg}{\cos\theta} \rightarrow T_2 = 55.4N \tag{6.27}$$

and also

$$T_1 = m\frac{v^2}{L} - T_2 \sin\theta \rightarrow T_1 = 32.8N. \tag{6.28}$$

Things to notice: The hardest part of this was deciding the direction of the net force. We know its magnitude since the ball was moving in a circle, and we know that it always points in towards the center of the circle in which the ball is moving. The concept we are really applying is that the free-body diagram is done at a particular instant. We choose the x-axis to correspond to the direction from the center of the circle to the ball, and decompose from there.

Student Exercises

- For the ball illustrated in Figure 6.14 the tension is $T_1 = 60N$. $m = 5kg$ and $L = 3m$. What is the speed of the ball? *The ball's speed is* $|\vec{v}| = 7.28\frac{m}{s}$.
- For the ball illustrated in Figure 6.14 what is the slowest the ball can go and still have $T_1 > 0$ (for $L = 3m$ and $\theta = 30°$)? Does this depend on m? *The slowest the ball can go and have a positive* T_1 *is* $|\vec{v}| = 2.94\frac{m}{s}$.
- A ball is suspended from a single rope of length L and goes around in a horizontal circle. This is illustrated in figure 6.16. Find the tension T in the rope and the angle θ at which it swings in terms of v, the ball's speed, and m. Does your result agree with the previous results for the special case where the lower rope has a tension of $0N$?

6.5.2 A car going around a curve

Problem A car is on a banked curve, following a path which is part of a circle with radius R. The curve is banked at an angle of θ with the horizontal, and is very slippery, so there is no force of friction between the wheels and

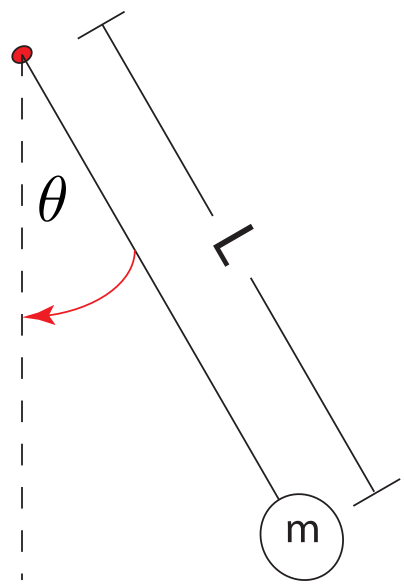


Figure 6.16: A ball of mass m swings in a pattern like a conical pendulum from a rope of length L.

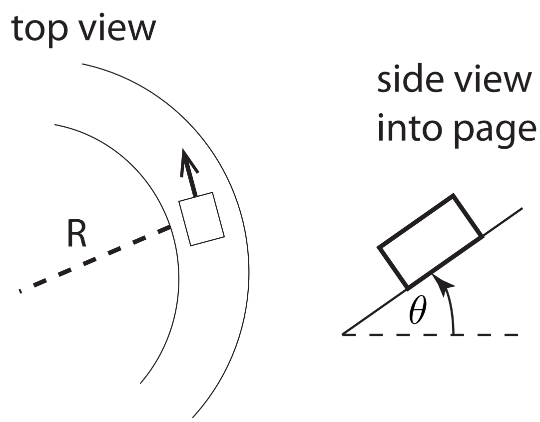


Figure 6.17: A car travels around a curved section of road with radius of curvature R. The road is banked at an angle of θ.

the road. What is the speed at which the car must go to accomplish this? This is illustrated in figure 6.17.

Worked Solution There are only two forces acting on the car. The normal force, and the force of gravity. This is illustrated in figure 6.18. The acceleration must be in towards the center of the circle in which it's travelling as illustrated in 6.18.

We apply Newton's second law and find

$$\begin{aligned} m\vec{a} &= \vec{F}_{net} \\ m\left(-\frac{v^2}{R}\hat{\imath}\right) &= \vec{F}_n + \vec{F}_g \\ &= \left|\vec{F}_n\right|\left(-\sin\theta\hat{\imath} + \cos\theta\hat{k}\right) + \left(-mg\hat{k}\right) \\ &= -\left|\vec{F}_n\right|\sin\theta\hat{\imath} + \left(\left|\vec{F}_n\right|\cos\theta - mg\right)\hat{k} \end{aligned} \tag{6.29}$$

Looking at the components of this, we find that (vertical component) $\left|\vec{F}_n\right| =$

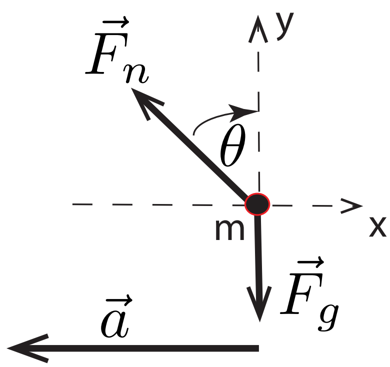


Figure 6.18: The normal force and force of gravity act on the car. The direction of the acceleration is indicated.

$\frac{mg}{\cos\theta}$, and so

$$\begin{aligned} m\frac{v^2}{R} &= \left|\vec{F}_n\right| \sin\theta \\ &= \frac{mg}{\cos\theta}\sin\theta \\ v^2 &= gR\tan\theta \end{aligned} \tag{6.30}$$

and so the required speed is $v = \sqrt{gR\tan\theta}$.

Things to note Well, that answer is a bit funny: we did not obtain a number, but rather just an expression. You should not worry, it matches the way we have done other things: we calculate in terms of variable, and then put in the numbers. This time there were no numbers. If we had them, we would use them.

We knew what the magnitude of the centripetal acceleration had to be in terms of a quantity we wanted (v) and one we knew (R). We then compared that to the known forces applied to the car.

Notice that the normal force was bigger than it is in the case where a block is resting on the slope. The reason for this is that the horizontal component of the normal force had to provide the centripetal acceleration.

Student Exercise

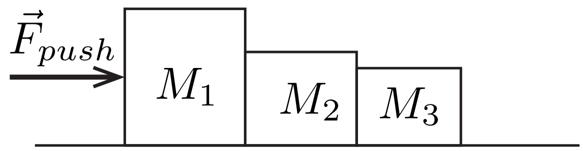


Figure 6.19: Three boxes of mass m_1, m_2, and m_3 are in contact on a horizontal frictionless surface. The leftmost box of mass m_1 is being pushed by a horizontal force $\vec{F}_{push}$.

- A car goes around a flat curve with which the wheels have a coefficient of static friction $\mu_s = 0.4$. If the radius of the curve is $R = 30m$, what is the fastest the car can go around the curve without slipping? *We find that the fastest possible speed without slipping on a flat curve is* $10.8\frac{m}{s}$.

6.6 Questions

1. Consider the set of three boxes shown in 6.19. If $m_1 = 3kg$, $m_2 = 2kg$, $m_3 = 1kg$, and $\vec{F}_{push} = 18N\hat{\imath}$, what is the force *(Magnitude and direction)* that the $1kg$ mass exerts on the $2kg$ mass? What is the force that the $3kg$ mass exerts on the $2kg$ mass?

2. Consider a modified Atwood machine depicted in 6.20. In it, the mass m_2 is suspended by a rope which pulls on mass m_1, which is on a rough horizontal surface.

   (a) If $m_1 = 5kg$, $m_2 = 1kg$, and $\mu_k = 0$, what are the tension in the rope and the acceleration of mass m_1?

   (b) If $m_1 = 4kg$, $m_2 = 2kg$, and $\mu_k = 0.25$, what are the tension in the rope and the acceleration of mass m_1?

3. Consider the ball which is attached to a pole by two ropes shown in figure 6.21. The length of each rope is L, and both ropes make an angle θ with the vertical. If $L = 4m$, $\theta = 30°$, $v = 7\frac{m}{s}$, and $m = 3kg$, what are the tensions in the top and bottom rope?

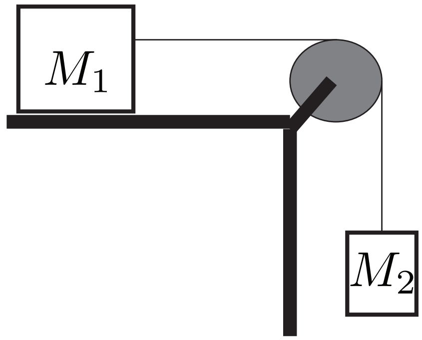


Figure 6.20: A mass m_1 is on a horizontal surface with which it has a coefficient of kinetic friction μ_k. It is attached via a horizontal rope which goes over a masslessfrictionless pulley to a second mass m_2 suspended against gravity.

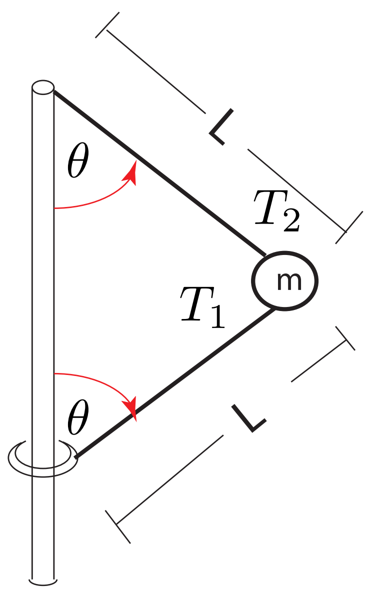


Figure 6.21: A ball of mass m rotates around a pole at speed v. It is held in place by two ropes of length L, which each make an angle of θ with the vertical.

6.7 Answers

1. We find that $\vec{F}_{1on2} = -3N\hat{\imath}$, and that $\vec{F}_{3on2} = 9N\hat{\imath}$.
2. For the modified Atwood machine we find
   (a) In the first case $\vec{a}_1 = 1.63\frac{m}{s^2}\hat{\imath}$ and $T = 8.17N$.
   (b) In the second case we find $\vec{a}_1 = 1.63\frac{m}{s^2}\hat{\imath}$ and $T = 16.3N$.
3. We find the tension in the bottom rope $T_1 = 56.5N$, and the tension in the top rope is $T_2 = 90.5N$.

Chapter 7

Forces

7.1 Overview

Read the seventh chapter of the text. In previous chapters, you have learned about the forces of static and kinetic friction, the force of gravity (near Earth's surface), as well as how to apply tension and contact forces. In this chapter we learn about Hooke's law, Newtonian Gravity, the Coulomb force, and the Lorentz force.

- Many objects exhibit a *restoring* force that makes them pull or push back towards their equilibrium length. The restoring force is described as a *linear* restoring force if the magnitude of the force is proportional to the displacement. The example of this kind of object that we will consider is the spring.

- Springs can be stretched or compressed. When they are, they obey Hooke's Law, which says that the magnitude of the force required to keep the deformed spring in equilibrium is $\left|\vec{F}\right| = k\left|\Delta\vec{x}\right|$. In this, k is called the 'spring constant' and is a physical quantity that has to be measured for each spring, and $\Delta\vec{x}$ is the amount that the spring is stretched or compressed from equilibrium. Obviously, if you push something far enough, it will break, but for small displacements this *linear approximation* is close enough to correct. In this course, we only consider what happens to springs when they are compressed or stretched along their length - what happens when you also consider what happens when something is pushed to one side is more complicated and can be expressed using a kind of 'super-vector' known as a 'tensor'.

- The direction of the force that the spring exerts is in a direction that gets it back to its normal length. If the spring is stretched past equilibrium then it pulls back; if the spring is compressed from equilibrium length then it pushes outward.

- The magnitude of the force of gravitation between two point objects of mass m_1 and m_2 is $\left|\vec{F}_g\right| = G\frac{m_1 m_2}{r^2}$, where $G = 6.67 \times 10^{-11}\frac{Nm^2}{kg^2}$ is a universal constant, and r is the distance between the centers of mass. The direction of the force of gravity pulls the two objects together.

- Written as a vector, the force exerted by mass m_1 on a second mass m_2 is

$$\begin{aligned} \vec{F}_{g,1on2} &= -G\frac{m_1 m_2}{r^2}\hat{r}_{1to2} \\ &= G\frac{m_1 m_2}{r^2}\hat{r}_{2to1} \end{aligned} \tag{7.1}$$

  In this, $\hat{r}_{1\ to\ 2}$ is the *unit* vector from mass m_1 to mass m_2. This means that mass m_2 feels a force pulling it *towards* mass m_1.

- In addition to being true for point masses, the law of gravity works for spherically symmetric extended objects. At the level of accuracy we care about for this course, humans are spherical.

- For an object of mass m that is in the presence of a whole bunch of other masses, we can express the force of gravity in terms of the *gravitational field*. For this mass $\vec{F}_g = m\vec{g}$ where $\vec{g}$ is the gravitational field. Note that this means that near the surface of the earth, $\vec{g} \approx -g\hat{k}$.

- The force of one point charge q_1 on a second point charge q_2 has magnitude given by Coulomb's law: $\left|\vec{F}_e\right| = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}$. This force is attractive if the two charges have opposite signs, and repulsive if the charges have the same signs.

- The Coulomb force of charge q_1 on charge q_2 can be expressed in vector form as

$$\begin{aligned} \vec{F}_{e,1on2} &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{1to2} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{2to1} \end{aligned} \tag{7.2}$$

  where $\hat{r}_{1\ to\ 2}$ is the unit vector from charge 1 to charge 2. This means that if both q_1 and q_2 are positive then q_2 will feel a force pushing it away from q_1, as it will if both are negative. If one is positive and the other negative q_2 is pulled towards q_1.

- In addition to being true for point charges, the Coulomb law given is also true for spherically symmetric charge distributions. Since the fundamental strength of the Coulomb force is much larger than that of the gravitational force, we normally have to care a lot more about the distribution of the charge. For a 'test charge' q in the presence of many other charges (a 'charge distribution'), the net force can be expressed in terms of the *electric field* as $\vec{F} = q\vec{E}$. The electric field depends on location; it could be written as $\vec{E}(\vec{r})$.

- A moving charge in the presence of a magnetic field $\vec{B}$ experiences the Lorentz force. This force is given by $\vec{F_b} = q\vec{v} \times \vec{B}$. The magnetic field depends on location; it could be written as $\vec{B}(\vec{r})$.

- It turns out that moving charges also create their own magnetic fields. The phenomenon of permanent magnets comes about because the electrons orbiting their atomic nucleii line up a particular way in certain materials (ferromagnets). The orbiting electrons all being oriented the same way creates a large magnetic field, which exerts a force on the orbiting in other magnets.

7.2 Springs

7.2.1 Springs in Parallel and Series

Example A box of mass m is free to move on a horizontal frictionless surface. It is connected to a spring with spring constant k_1 to the left, and a spring of spring constant k_2 to the right. This situation is shown in figure 7.1.

If $k_1 = 80\frac{N}{m}$, $k_2 = 120\frac{N}{m}$, $m = 5kg$ and the box is $l = 0.1m$ to the right of its equilibrium position, what is the magnitude of the acceleration the box experiences?

Worked Solution This problem is chosen to illustrate the vector nature of the spring force – that's all. The strategy for this problem is going to be to apply Newton's second law to the net sum of the forces, and hence find the acceleration of the box.

We can identify four possible forces on the box: The force of gravity (which will be down), the normal force (which will be perpendicular to the

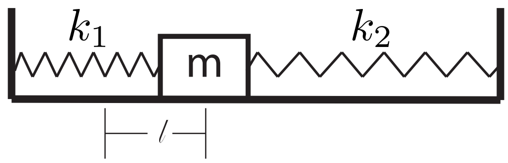


Figure 7.1: A box of mass m is between two springs of spring constants k_1 and k_2. It is a distance l to the right of its equilibrium position.

surface – in this case up), and the forces from the two springs. The spring forces are the ones we are interested in. The spring to the left of the box (labelled k_1 in figure 7.1) is stretched so it will try to pull back towards its natural length; this means it will pull the box towards the left. Similarly, the spring to the right of the box is compressed from equilibrium length, which means that it will try to push back towards its normal length, and hence the force it exerts will be to the left also.

Now that we have identified the direction of the forces, we can draw the free body diagram shown in figure 7.2. Since the forces from the springs are given by Hooke's law, $\left|\vec{F}_1\right| = k_1 l$ and $\left|\vec{F}_2\right| = k_2 l$, and so we have that the total force is

$$\begin{aligned}
\vec{F}_{net} &= \vec{F}_g + \vec{F}_n + \vec{F}_1 + \vec{F}_2 \\
&= \left(-mg\hat{k}\right) + \left|\vec{F}_n\right| \hat{k} + (-k_1 l\hat{\imath}) + (-k_2 l\hat{\imath}) \\
&= \left(\left|\vec{F}_n\right| - mg\right) \hat{k} + (-k_1 l - k_2 l)\,\hat{\imath} \qquad (7.3)
\end{aligned}$$

The usual consideration that the system is in equilibrium vertically tells us that $\left|\vec{F}_n\right| = mg$, and so using Newton's second law we have

$$m\vec{a} = \vec{F}_{net} = -(k_1 + k_2)l\hat{\imath} \rightarrow \vec{a} = -\frac{k_1 + k_2}{m} l\hat{\imath} \qquad (7.4)$$

Using the values given in the statement of the question, that means the acceleration is $-4.0\frac{m}{s^2}\hat{\imath}$, or $4.0\frac{m}{s^2}$ to the left.

Something to notice: Since one spring was compressed (and trying to get back to its original length) and the other was stretched the force they

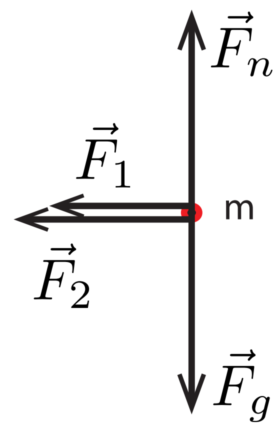


Figure 7.2: The free-body diagram for the box between the two springs shown in figure 7.1.

both exerted was in the same direction. That meant that the magnitude of the forces added. Another key thing is that it does not matter what the original lengths of the two springs were. The reason for this is that we have defined the position of the mass relative to the equilibrium location. This means that if the springs are stretched, they arrange themselves in equilibrium so that the forces pulling each way are equal, and if they are compressed, they arrange so that the the forces pushing each way are equal in magnitude. The motion of the box to the left or right will increase or decrease (as appropriate) these forces, however, the *net* force will be as we described.

A way to think of this is that Hooke's law tells us the force the spring exerts as it is moved relative to the equilibrium position for whatever situation we are examining.

Student Exercises Finding the force from a spring that is stretched or compressed is relatively easy. Consider the two situations shown in figure 7.3.

- For the situation depicted in part (a) of figure 7.3, find the magnitude of the force exerted by the springs when the box is displaced $\Delta x = 0.1m$ from its equilibrium position if $k_1 = 100\frac{N}{m}$ and $k_2 = 200\frac{N}{m}$. *In this case the force magnitude is* $30N$.

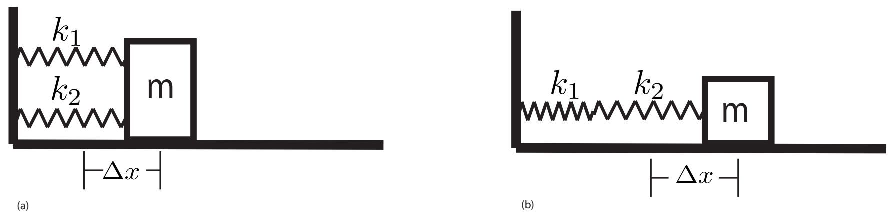


Figure 7.3: A box on a horizontal, frictionless surface is attached to two springs of spring constant k_1 and k_2. In part (a), the two springs are both connected to a rigid holder, and to the box. In part (b), one spring is connected to the box and the second spring, and the second spring is rigidly attached to a holder.

- For the situation depicted in part (b) of figure 7.3, find the magnitude of the force exerted by the springs when the box is displaced by $\Delta x = 0.20m$ from the equilibrium position if $k_1 = 100\frac{N}{m}$ and $k_2 = 200\frac{N}{m}$. *In this case the magnitude of the force is* $13.3N$.

- Find the *effective* spring constants algebraically in terms of k_1 and k_2 for each of these two cases. For part (b), call the amount that each individual spring is stretched Δx_1 and Δx_2, and note that $\Delta x = \Delta x_1 + \Delta x_2$. Assuming that the springs are massless if you find the relation between the force spring 2 exerts on the box and the force spring 1 exerts on spring 2 that will tell you something about the total force. *In this case we won't write down the answers, but the algebra is the same as that for resistors in series and in parallel respectively.*

7.2.2 Springs and Torque

Example A diving board of length L is held horizontally by a pin at one end, and by a spring pushing upwards placed a distance aL from the pin. When a diver of mass m walks out to the end, the tip of the board is pushed down by a distance Δz. This is illustrated in figure 7.4.

If the mass of the person is $80kg$, $L = 3.0m$, the spring is $1.0m$ from the left end, and the end of the board is depressed by $6.0cm$ when the person stands at the end, what is the spring constant of the spring that holds the board up?

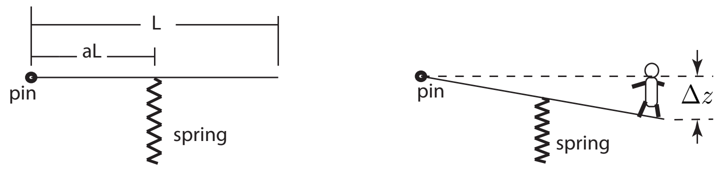


Figure 7.4: A diving board is held in place by a spring aL from the end. When a person of mass m walks out onto the board, the end is depressed by Δz.

Worked Solution The answer which we will obtain is quite simple to state, but there are a few points that should jump out at you as potential sources of confusion:

- We do not talk about the mass of the board, or the uncompressed length of the spring.
- Since the board is rotating a bit, the end moves in part of a circle; we are only given the vertical change part.
- Since the angle of the board has changed to something unknown, we expect the angle to be related to the torques we need to calculate.
- Both the compression of the spring and the torque due to the spring depend on the angle the board makes with the horizontal.
- Because of the previous observations, we expect some complicated trigonometric nightmare to solve.

The approach that we are going to take essentially ignores all of these problems. We will check along the way (and after) to make sure that what we have done is reasonable. A critical skill that you develop in physics to make simplifying assumptions to make the problem mathematically tractable – the real 'art' of physics is getting mathematical model to be complicated enough to tell you something, but simple enough to be solvable.

We will start by pretending that we can ignore the effect of the change in angle on the torque – that is to say that the force the spring exerts and the force the person exert both act at 90° to the diving board. We will check back later to see if this is reasonable.

Our next step is to draw a free-body diagram as in figure 7.5. In part

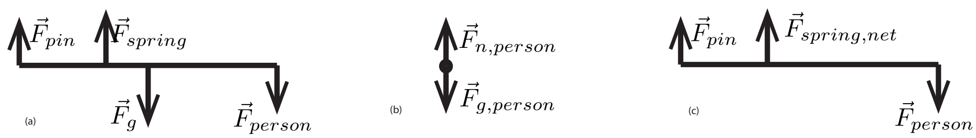


Figure 7.5: Part (a) shows the free-body diagram of the diving board shown in figure 7.4. Part (b) is the free body diagram for the person at the end of the diving board, and part (c) is the 'reduced' free body diagram which only considers the excess force of the spring in equilibrium.

(b) of this diagram we see that there are two forces on the person; the force of gravity downwards and the normal force from the diving board upwards. This means that the magnitude of the force the person exerts on the diving board is mg, whatever m is. The condition on the diving board for translational equilibrium is

$$0 = \vec{F}_{net} = \vec{F}_{pin} + \vec{F}_{spring} + \vec{F}_g + \vec{F}_{person} \tag{7.5}$$

and because of rotational equilibrium the torque satisfies

$$0 = \vec{\tau}_{net} = \vec{\tau}_{pin} + \vec{\tau}_{spring} + \vec{\tau}_g + \vec{\tau}_{person} \tag{7.6}$$

When the person isn't standing on the board (so $\vec{F}_{person} = 0$) the forces of the pin, spring, and gravity, and their torques are in equilibrium. Since (like in the last problem) we are interested in the change in spring length from the equilibrium position, we can subtract these off from the total problem and be left only with the following 'net' forces: the force due to the person, the 'extra' from the spring, and the 'extra' from the pin.

This brings us to an effective problem which we have to solve, illustrated in part (c) of figure 7.5:

$$0 = \vec{F}_{net} = \vec{F}_{pin,net} + \vec{F}_{spring,net} + \vec{F}_{person} \tag{7.7}$$

We do not really care what the force the pin exerts is, so we will use that as our pivot point, our 'origin'. *Look back at the material on torque in chapter 3 if you need to be reminded how to choose a pivot point or calculate torque.*

The spring exerts a force force $\vec{F}_{spring,net} = k\Delta l\hat{k}$. In this Δl is the amount the spring is compressed. The vector from the pin to where the spring exerts its force is $aL\hat{\imath}$. The diver exerts its force $\vec{F}_{person} = -mg\hat{k}$

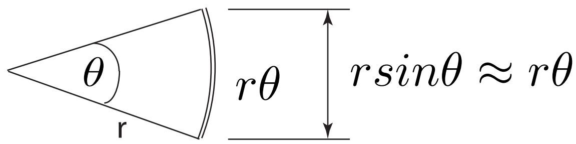


Figure 7.6: The segment of a circle of radius r subtended by an angle θ has length $r\theta$. This gives an approximate straight-line length of $r\theta$.

at a displacement of $L\hat{\imath}$ from the pin. Those vectors are written ignoring the small change in angle, and assuming that the spring pushes straight up. The torque around the pin is then

$$\begin{aligned}
0 &= \vec{\tau}_{net} \\
&= \vec{\tau}_{spring,net} + \vec{\tau}_{person} \\
&= (aL\hat{\imath}) \times \left(k\Delta l\hat{k}\right) + (L\hat{\imath}) \times \left(-mg\hat{k}\right) \\
&= (-k\Delta laL\hat{\jmath}) + (mgL\hat{\jmath}) \qquad (7.8)
\end{aligned}$$

This condition says that $k\Delta l = \frac{mg}{a}$. We know everything except k (the desired quantity) and Δl.

We have to figure out what Δl is in terms of the known quantities. For this, we have to make use of a little bit of analytic geometry, shown in figure 7.6. We know from figure 7.4 (and the statement of the problem) the quantity Δz. The radius of the circle we care about is L. This means that the angle (measured in radians) that the board dips down satisfies

$$\Delta z = L\theta \text{ so } \theta = \frac{\Delta z}{L} \qquad (7.9)$$

This means that since the spring is connected at aL from the pin, it will have a similar relationship with Δl, the amount that the spring is compressed:

$$\Delta l = (aL)\theta = aL\frac{\Delta z}{L} = a\Delta z \qquad (7.10)$$

This should make sense, if the very end is pushed down by a fixed amount, then half-way along the board will be pushed down by half as much. Putting this together, we have

$$k\Delta l = \frac{mg}{a} \text{ so } k = \frac{mg}{a\Delta l} = \frac{mg}{a^2\Delta z} \qquad (7.11)$$

With the values that we have for m and Δz, together with the fact that the spring is $1m$ along a $3m$ board which says that $a = \frac{1}{3}$, we have $k = 1.2\times10^5 \frac{N}{m}$

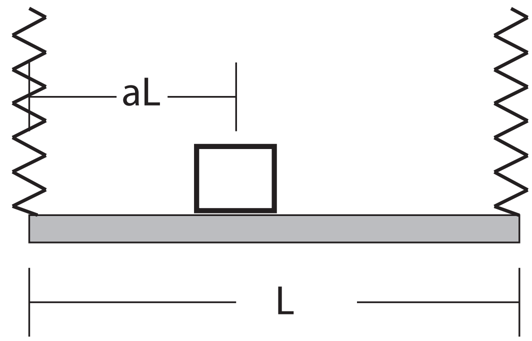


Figure 7.7: A beam of mass M and length L is suspended from two identical vertical springs of spring constant k at each end of the beam. A small block of mass m is placed a distance aL from the right end of the beam.

Some sanity checking: We made an idealizing assumption that the forces act straight up and down, and at 90 degrees to the board. An alternative way of saying the same thing is that we assumed the board was horizontal (almost). What if we hadn't done that?

If we had honestly taken into account the change in the angle, the vector from the pin to where the diver was standing would have been $L\cos\theta\hat{\imath} - L\sin\theta\hat{k}$, and similarly the vector from the pin to the point where the spring was would have been $aL\cos\theta\hat{\imath} - aL\sin\theta\hat{k}$. Calculating the torques if the spring is pushing straight up would have given us $0 = (-k\Delta l aL\cos\theta\hat{\jmath}) + (mgL\cos\theta\hat{\jmath})$ – there's a common factor of $\cos\theta$, so we didn't cheat there.

The place where we could have a problem is the spring itself: you imagine that one end is fixed, and the other end is attached to the diving board. For small angles like the ones we considered, as long as the spring is reasonably long (much bigger than the amount it was compressed) the angle the spring makes with the vertical would not change too much.

Student Exercise Consider the situation depicted in figure 7.7. A pair of springs each with the same spring constant hold a uniform beam against gravity with a small mass sitting a distance aL from the left end.

- If $M = 10kg$, $k = 100\frac{N}{m}$, $m = 0.5kg$, and $a = 0.7$, how much is each spring stretched relative to its equilibrium length? *We find that* $\Delta x_1 = 0.5047m$, *and* $\Delta x_2 = 0.5243m$.

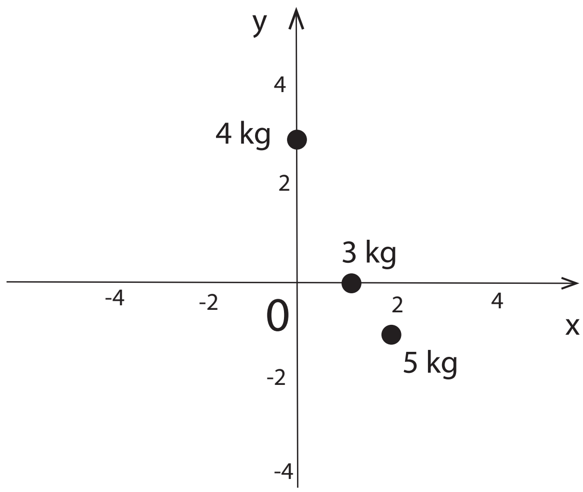


Figure 7.8: A $3kg$ mass is at $1m\hat{\imath}$, a $4kg$ mass is at $3m\hat{\jmath}$, and a $5kg$ mass is at $2m\hat{\imath} - 1m\hat{\jmath}$.

- If the beam is $0.80m$ long, for the variables in the previous question what angle does it make with the horizontal? *We find the angle is $1.4°$.*

7.3 Forces that depend on $\frac{1}{r^2}$

The gravitational and electrostatic force both are proportional to $\frac{1}{r^2}$ and have a radial direction. Practice with one gives practice with the other.

Example Three masses are located as shown in figure 7.8. Find the net force on the $3kg$ mass illustrated.

Worked Solution The key thing to remember here is that the force of gravity can be expressed as

$$\vec{F}_{A\ on\ B} = -G\frac{m_A m_B}{r^2}\hat{r}_{A\ to\ B} \tag{7.12}$$

where r is the magnitude of the vector from mass A to mass B.

Our strategy for this is going to be to find the force of the $4kg$ mass and the $5kg$ mass on the $3kg$ mass and add them as vectors. To get each force, we have to get the unit vectors $\hat{r}_{A\ to\ B}$; and to do this, we will have to find the vector from the mass exerting the force, to the mass which feels the force. Then we find the length of this vector, which gives us r, and we can divide the vector by its length to get the unit vector we need.

First, the force from the $4kg$ mass. The vector from the $4kg$ mass to the $3kg$ mass is

$$\begin{aligned}\vec{r}_{4\ to\ 3} &= \vec{x}_3 - \vec{x}_4 \\ &= (1m\hat{\imath}) - (3m\hat{\jmath}) \\ &= 1m\hat{\imath} - 3m\hat{\jmath} \end{aligned} \tag{7.13}$$

Notice that this follow the general rule that a vector from one place to another is the location where you end, and you subtract where you started. This is because $\vec{r}$ measures the change in position from where you start to where you end.

Now, we can use $\vec{r}_{4\ to\ 3}$ to get $\hat{r}_{4\ to\ 3}$, and the r used in the expression for gravitational force.

$$r = |\vec{r}_{4\ to\ 3}| = |1m\hat{\imath} - 3m\hat{\jmath}| = \sqrt{(1m)^2 + (-3m)^2} = 3.16m \tag{7.14}$$

We can use this to get the unit vector as well:

$$\hat{r}_{4\ to\ 3} = \frac{\vec{r}_{4\ to\ 3}}{|\vec{r}_{4\ to\ 3}|} = \frac{1m\hat{\imath} - 3m\hat{\jmath}}{3.16m} = 0.32\hat{\imath} - 0.95\hat{\jmath} \tag{7.15}$$

As a side note, this makes sense, since the *unit* vector doesn't have any dimensional content, and it points down and to the left, as shown in figure 7.9.

Now, we calculate the gravitational force.

$$\begin{aligned}\vec{F}_{4\ on\ 3} &= -G\frac{m_3 m_4}{r^2}\hat{r}_{4\ to\ 3} \\ &= -\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)\frac{3kg\ 4kg}{(3.16m)^2}\hat{r}_{4\ to\ 3} \\ &= -8.0\times10^{-11}N\,(0.32\hat{\imath} - 0.95\hat{\jmath}) \\ &= -2.5\times10^{-11}N\hat{\imath} + 7.6\times10^{-11}N\hat{\jmath} \end{aligned} \tag{7.16}$$

This shows us that the magnitude of the gravitational force from the 4kg mass is $8.0\times10^{-11}N$, and it is, as illustrated, up and to the left.

Similarly, we can calculate the force on the $3kg$ mass from the $5kg$ mass. We follow the same rules and procedures:

$$\vec{r}_{5\ to\ 3} = \vec{x}_3 - \vec{x}_5 = (1m\hat{\imath}) - (2m\hat{\imath} - 1m\hat{\jmath}) = -1m\hat{\imath} + 1m\hat{\jmath} \tag{7.17}$$

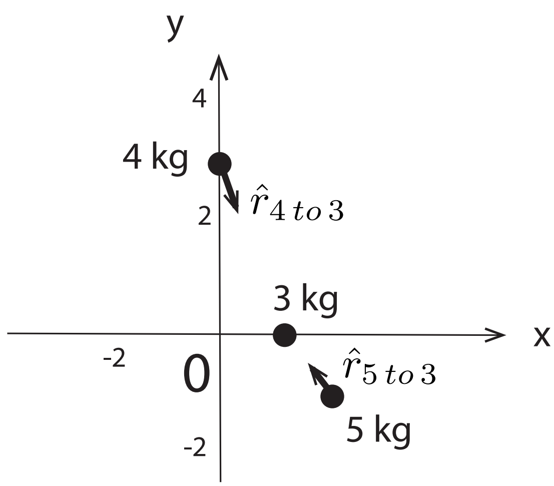


Figure 7.9: The vectors $\hat{r}_{4\ to\ 3}$ and $\hat{r}_{5\ to\ 3}$ for the mass configuration from 7.8.

This means that for this calculation

$$r = |\vec{r}_{5\ to\ 3}| = \sqrt{\vec{r}_{5\ to\ 3} \cdot \vec{r}_{5\ to\ 3}} = \sqrt{(-1m)^2 + (1m)^2} = 1.41m \qquad (7.18)$$

and so

$$\hat{r}_{5\ to\ 3} = \frac{\vec{r}_{5\ to\ 3}}{r} = -0.71\hat{\imath} + 0.71\hat{\jmath} \qquad (7.19)$$

and finally that

$$\begin{aligned} \vec{F}_{5\ on\ 3} &= -G\frac{m_3 m_5}{r^2}\hat{r}_{5\ to\ 3} \\ &= -\left(6.67 \times 10^{-11}\frac{Nm^2}{kg^2}\right)\frac{3kg\ 5kg}{(1.41m)^2}\hat{r}_{5\ to\ 3} \\ &= -5.0 \times 10^{-10}N\,(-0.71\hat{\imath} + 0.71\hat{\jmath}) \\ &= 3.5 \times 10^{-10}N\hat{\imath} - 3.5 \times 10^{-10}N\hat{\jmath} \qquad (7.20) \end{aligned}$$

To get the total force on the $3kg$ mass, we add (*as vectors*) the forces that we calculated in equations 7.16 and 7.20.

$$\begin{aligned} \vec{F}_{net\ on\ 3kg} &= \vec{F}_{4\ on\ 3} + \vec{F}_{5\ on\ 3} \\ &= \left(-2.5 \times 10^{-11}N\hat{\imath} + 7.6 \times 10^{-11}N\hat{\jmath}\right) + \\ &\qquad \left(3.5 \times 10^{-10}N\hat{\imath} - 3.5 \times 10^{-10}N\hat{\jmath}\right) \\ &= 3.2 \times 10^{-10}N\hat{\imath} - 2.7 \times 10^{-10}N\hat{\jmath} \qquad (7.21) \end{aligned}$$

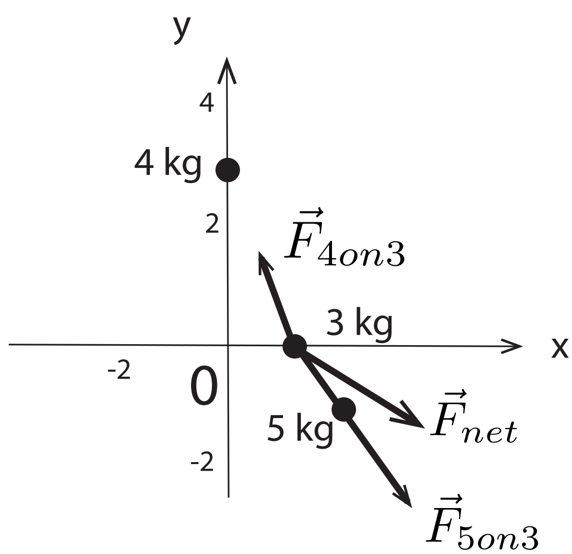


Figure 7.10: The net force and the forces from each individual mass for the mass configuration from 7.8.

This net force, as a sum of the two component forces, is shown in figure 7.10.

Things you should have noticed Something cultural that you should see: when we use r in this context, we mean the *magnitude* of the vector from one center of mass (or charge) to the other. We particularly emphasized how you calculate the vector from one location to another. This is a key skill you'll need to practice.

Student Exercises The electrostatic force also gives you practice withthe $\frac{1}{r^2}$ forces:

Suppose that a charge of $3nC$ is at $1m\hat{\imath}$, a charge of $4nC$ is at $1m\hat{\imath} - 3m\hat{\jmath}$, and a charge of $-6nC$ is at $1m\hat{\imath} - 3m\hat{\jmath} + 2m\hat{k}$.

- What is the force on the $3nC$ charge? *It is* $1.6\times10^{-9}N\hat{\jmath}+6.9\times10^{-9}N\hat{k}$.
- What is the force on the $3nC$ charge if the $-6nC$ charge is $6nC$? *It is* $2.24 \times 10^{-8}N\hat{\jmath} - 0.69 \times 10^{-8}N\hat{k}$.
- What is the force on the $3nC$ charge if the $4nC$ charge is $6nC$? *It is* $7.6 \times 10^{-9}N\hat{\jmath} + 6.910^{-9}N\hat{k}$.

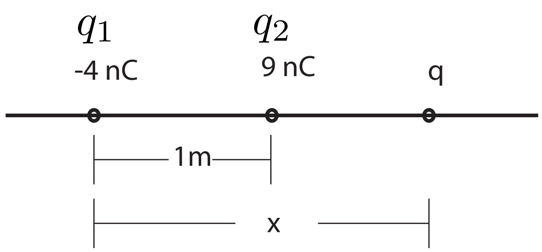


Figure 7.11: Three charges are located along the x-axis. A q_1 charge is at the origin, a q_2 charge is at $x_0\hat{\imath}$ and a charge q is at an unknown position $x\hat{\imath}$. As drawn, $x > x_0$.

7.4 Electric Force and Equilibrium

Example A charge of q_1 is at the origin, and a charge of q_2 is at the location $x_0\hat{\imath}$. If $q_1 = -4nC$, $q_2 = 9nC$, and $x_0 = 1m$, where along the x-axis should a charge q be located so that it experiences no net force?

Worked Solution This is an example of how you can use a model to find an algebraic solution to a problem. We start by drawing a sketch of the situation, as in figure 7.11. To figure out the force on charge q, we can use the technique from the previous worked problem; the difference is that we have to make use of the electric force, rather than the gravitational force.

We can recall that the expression for the electric force is

$$\vec{F}_{A\ on\ B} = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{r^2}\hat{r}_{A\ to\ B}. \tag{7.22}$$

Just as before, we have to find the vectors from the two other charges to our unknown charge. We parametrized the location of charge q as being at $x\hat{\imath}$, but we do not know what x is. We want to determine the required x such that the net force is 0.

On our charge q, due to the first charge, we can calculate:

$$\vec{r}_{1\ to\ q} = \vec{x}_q - \vec{x}_1 = (x\hat{\imath}) - 0 = x\hat{\imath} \tag{7.23}$$

Now, we have to calculate $\hat{r}_{1\ to\ q}$, and we run into a problem:

$$\hat{r}_{1\ to\ q} = \frac{\vec{r}_{1\ to\ q}}{|\vec{r}_{1\ to\ q}|} = \frac{x\hat{\imath}}{\sqrt{x^2}} \tag{7.24}$$

Notice that $\sqrt{x^2}$ is necessarily positive, so this is $\frac{x}{|x|}\hat{\imath}$; in other words, the direction changes depending on whether q is to the right or left of the origin. This means that from charge q_1 the force is

$$\vec{F}_{1\ on\ q} = \frac{1}{4\pi\epsilon_0}\frac{qq_1}{r^2}\hat{r}_{1\ to\ q} = \frac{1}{4\pi\epsilon_0}\frac{qq_1}{x^2}\frac{x\hat{\imath}}{|x|} \tag{7.25}$$

Similarly we can find that $\vec{r}_{2\ to\ q} = (x - x_0)\,\hat{\imath}$, so the force from chargre q_2 is

$$\vec{F}_{2\ on\ q} = \frac{1}{4\pi\epsilon_0}\frac{qq_2}{r^2}\hat{r}_{2\ to\ q} = \frac{1}{4\pi\epsilon_0}\frac{qq_2}{(x-x_0)^2}\frac{(x-x_0)\hat{\imath}}{|x-x_0|} \tag{7.26}$$

In general, it is hard to do algebraic manipulations with absolute value signs. We can get rid of these absolute signs at the expense of a bit of extra work. We separate our problem into three cases: $x < 0$, $0 < x < x_0$, and $x_0 < x$ (this works provided $x_0 > 0$, which we will assume). Then the condition $\vec{F}_{net} = 0$ can be written:

$$\begin{aligned}
0 &= \frac{1}{4\pi\epsilon_0}\frac{qq_1}{x^2}\hat{\imath} + \frac{1}{4\pi\epsilon_0}\frac{qq_2}{(x-x_0)^2}\hat{\imath} && x > x_0 \\
0 &= \frac{1}{4\pi\epsilon_0}\frac{qq_1}{x^2}\hat{\imath} - \frac{1}{4\pi\epsilon_0}\frac{qq_2}{(x-x_0)^2}\hat{\imath} && 0 < x < x_0 \\
0 &= -\frac{1}{4\pi\epsilon_0}\frac{qq_1}{x^2}\hat{\imath} - \frac{1}{4\pi\epsilon_0}\frac{qq_2}{(x-x_0)^2}\hat{\imath} && x < 0 \qquad (7.27)
\end{aligned}$$

Dividing out common factors and noting that each vector only has a component in the x-direction we end up with

$$0 = \frac{q_1}{x^2} + \frac{q_2}{(x-x_0)^2} \qquad x > x_0 \tag{7.28}$$

$$0 = \frac{q_1}{x^2} - \frac{q_2}{(x-x_0)^2} \qquad 0 < x < x_0 \tag{7.29}$$

$$0 = \frac{q_1}{x^2} + \frac{q_2}{(x-x_0)^2} \qquad x < 0 \tag{7.30}$$

We will solve equations 7.28 and 7.30, which are identical, for the values of q_1 and q_2 that are given. For this case, we have

$$\begin{aligned}
0 &= \frac{-4nC}{x^2} + \frac{9nC}{(x-1m)^2} \\
\frac{4}{x^2} &= \frac{9}{(x-1m)^2} \\
4(x-1m)^2 &= 9x^2 \\
0 &= 5x^2 + 8mx - 4m^2 \qquad (7.31)
\end{aligned}$$

We can solve for x using the quadratic formula, and get

$$x = \frac{-(8m) \pm \sqrt{(8m)^2 - 4(5)(4m^2)}}{2(5)} = \frac{-8m \pm 12m}{10} \tag{7.32}$$

The two possible solutions here are $x = -2m$ and $x = 0.4m$. We have to discard the $x = 0.4m$ solution because it is outside the domain of validity for equations 7.28 and 7.30: These two equations were only good provided $x < 0$ or $x > x_0$ (and $x_0 = 1m$ in our example).

We will check similarly for 7.29 and find

$$\begin{aligned} 0 &= \frac{-4nC}{x^2} - \frac{9nC}{(x-1m)^2} \\ -4(x-1m)^2 &= 9x^2 \\ 0 &= 13x^2 - 8mx + 4m^2 \end{aligned} \tag{7.33}$$

and for this, the quadratic formula gives

$$x = \frac{-(-8m) \pm \sqrt{(-8m)^2 - 4(13)(4m^2)}}{2(13)} = \frac{8m \pm \sqrt{-40m^2}}{26} \tag{7.34}$$

In other words, there are no real solutions (in the mathematical 'non-complex' sense) to this in the range between $x = 0$ and $x = 1m$ for these charges.

So, the only point where the net force is 0 is at $-2m\hat{\imath}$.

The big point: In this, the key thing to notice was that the location of the test charge gave us information about the direction (vector) of the electric force. We had to take that into account when we were solving our individual equations. Also, notice that as we solved this there were different equations in different domains: to choose the proper physical solution you have to make sure the numerical answer you get falls into the range that your equation 'works.'

Student Exercises

- A charge $3nC$ is at $1m\hat{\jmath}$, and a charge $-30nC$ is at $-1m\hat{\jmath}$. Find a location on the y-axis were there is no net force on a charge q. *There is no force on q at $y = 1.92m$.*
- Repeat the previous problem with a charge of $-3nC$ at $-1m\hat{\jmath}$. Is there any solution? *No, there isn't.*

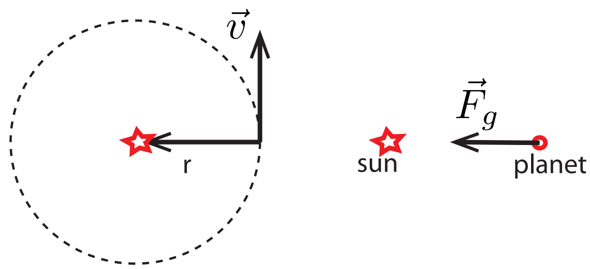


Figure 7.12: A planet orbits a star in a circle at a radius of r. The planet's orbital velocity $\vec{v}$ is perpendicular to the vector from the planet to the star.

- Imagine that two identical charges of $q = 1.6 \times 10^{-19}C$ are held together by a spring with unstretched length 0 and spring constant $k = 1.5 \times 10^{15}\frac{N}{m}$. What is their equilibrium separation? (This kind of consideration can be used to model (badly) the size/structure of an atom - it is the origin of string theory.) *It turns out that this case has a separation of about* $5.4 \times 10^{-15}m$.

7.5 Circular motion

7.5.1 Gravity and planetary orbits

Example A planet of mass m travels around a star of mass M in a circular orbit of radius r. If the planet goes around once every 224 days at an orbital radius of $1.08 \times 10^{11}m$ (just like Venus, what a coincidence), what is the mass of the star?

Worked Solution The key to this is some understanding of circular motion, together with an application of Newtonian Gravity. We can start by observing that there is only one force on the planet: the force of gravity from the star. The magnitude of this force is $G\frac{Mm}{r^2}$. The direction is in towards the center of the circle, as illustrated in figure 7.12.

The net force (ie the force of gravitation) must supply the centripetal force. Since the direction is towards the center of the circle of the orbit, we know that the magnitude of that force of gravity on the planet must be $m\frac{v^2}{r}$

where v is the planet's orbital speed ($v = |\vec{v}|$). This tells us that

$$\begin{aligned}
\vec{F}_{net} &= \vec{F}_g \\
\left|\vec{F}_{net}\right| &= \left|\vec{F}_g\right| \\
m\frac{v^2}{r} &= G\frac{Mm}{r^2}
\end{aligned} \tag{7.35}$$

There are two unknown quantities: v and r, but we know that they are related, because the distance around the circle ($2\pi r$) is going to be equal to the distance travelled by the planet at constant speed v in a time T. In this, T is the orbital period – the 'year'. This implies another relationship: $2\pi r = vT$. When we substitute this into our previous work we obtain

$$\begin{aligned}
m\frac{v^2}{r} &= G\frac{Mm}{r^2} \\
v^2 &= G\frac{M}{r} \\
\left(\frac{2\pi r}{T}\right)^2 &= G\frac{M}{r} \\
\frac{4\pi^2}{G}\frac{r^3}{T^2} &= M
\end{aligned} \tag{7.36}$$

We know that $T = 224days = 224days\left(24\frac{hours}{day}\right)\left(3600\frac{s}{hour}\right)$, so we get

$$M = \frac{4\pi^2}{6.67 \times 10^{-11}\frac{Nm^2}{kg^2}}\frac{(1.08 \times 10^{11}m)^3}{(1.93 \times 10^7 s)^2} = 2.0 \times 10^{30}kg \tag{7.37}$$

Notice the key idea that the *net* force supplies the centripetal acceleration. Since gravity supplies the net force and we know its magnitude, they can be related.

Student exercises

- Mercury orbits at a distance of $5.8 \times 10^{10}m$ from the Sun (the mass of which we just found) How long is the year on Mercury? *It is* $7.6 \times 10^6 s$, *or approximately* $88days$. *Note that we calculated this, we did not just look it up.*

- Suppose some cosmic accident resulted in a charge of $2 \times 10^{17}C$ on the Sun and $-2 \times 10^{17}C$ on the Earth. Assuming that the radius of the orbit doesn't change (and neither does the masses of the bodies), how long would Earth's year be? What if the two charges were the same? *Having opposite charges would result in a period of* $2.62 \times 10^7 s$*, if the charges were the same there would be less net force inwards so assuming a circular orbit the period would be* $4.28 \times 10^7 s$*.*

7.5.2 Motion in a Magnetic Field

Example An electron with charge $q = -1.6 \times 10^{-19} C$ and mass $m = 9.1 \times 10^{-31} kg$ is travelling initially at $\vec{v} = 2.0 \times 10^5 \frac{m}{s} \hat{\imath}$ in a magnetic field $\vec{B} = 0.5T\hat{k}$. At that instant, what is the vector from the electron to the center of the circle in which it is travelling?

Worked Solution The statement of the question essentially gave most of the game away: we expect the electron to travel in a circle. We will check that, and carry on.

For a charged particle moving in a magnetic field, the force it experiences is $\vec{F}_B = q\vec{v} \times \vec{B}$. We can calculate this for the given quantities, and we find that the net force is

$$\begin{aligned}
\vec{F}_B &= q\vec{v} \times \vec{B} \\
&= q\left(2.0 \times 10^5 \frac{m}{s}\hat{\imath}\right) \times \left(\vec{B} = 0.5T\hat{k}\right) \\
&= q\left(-1.0 \times 10^5 \frac{N}{C}\hat{\jmath}\right) \\
&= \left(-1.6 \times 10^{-19} C\right)\left(-1.0 \times 10^5 \frac{N}{C}\hat{\jmath}\right) \\
&= 1.6 \times 10^{-14} N\hat{\jmath} \qquad (7.38)
\end{aligned}$$

If you need to remind yourself how to calculate the cross product, now is a good time to look back at the sections on torque (chapter 3) and vectors (chapter 1).

Now, to check that the particle is moving in a circle, we have to look at the relation between the force and the velocity: Since the force is at 90° to the velocity at the time we have calculated, what will happen is that the direction of the motion will change, but the speed will remain constant. After a tiny bit, the electron will be moving at $2.0 \times 10^5 \frac{m}{s}$ along some line in the x-y plane. Since that is still at 90° to the magnetic field (which is in the z-direction) the magnitude of the force will not change, but since the force involves the cross-product of the velocity and the magnetic field, the force will be perpendicular to the velocity, and will have hence changed directions.

The ingredients we have here: changing direction of motion, constant speed of motion, net force perpendicular to direction of motion, constant magnitude but changing direction of net force, all point toward circular motion.

We know that the magnitude of the net force is related to the speed and radius when moving in circular motion:

$$\begin{aligned}
\left|\vec{F}_{net}\right| &= \left|\vec{F}_B\right| \\
m\frac{v^2}{r} &= 1.6 \times 10^{-14} N \\
r &= \frac{mv^2}{1.6 \times 10^{-14} N} \\
&= \frac{9.1 \times 10^{-31} kg (2.0 \times 10^5 \frac{m}{s})^2}{1.6 \times 10^{-14} N} \\
&= 2.3 \times 10^{-6} m
\end{aligned} \tag{7.39}$$

At the instant we calculate the force, the acceleration was in $\hat{\jmath}$ (i.e. in the y-direction) which means that the center of the circle was $2.3 \times 10^{-6} m\hat{\jmath}$ away at that time.

Some comments We are reminding you (again) of the important thing about circular motion: there is a force of constant magnitude but varying direction pointing in to the center of a circle. The magnitude is related to the object's speed, its mass, and the radius of the circle.

Note that if the electron had a z-component of velocity, there would be no z-component of force; it would travel in a circle in the x-y plane but with a constant z-component of velocity. This means that it would travel in a 'corkscrew-like' shape called a 'helix'.

Student Exercises Charged particles encounter magnetic fields in a variety of real-world applications, from thing like aurora to mass spectrometers and cathode-ray-tube based televisions.

- Look at the situation sketched in figure 7.13. In it, a particle of mass m and charge q is initially travelling with a velocity $v_0\hat{\imath}$ and it enters a region where $\vec{B} = -B_0\hat{\jmath}$. The particle travels in a curved path. What is the distance d between where the particle enters the 'Mass

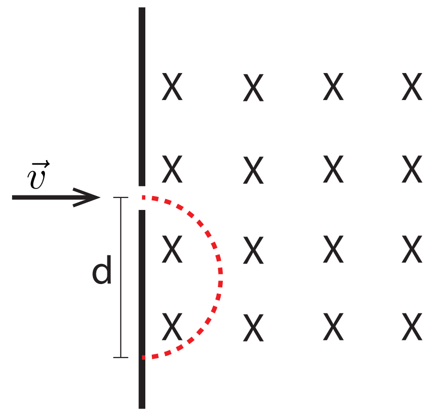


Figure 7.13: A particle of charge q enters a region where the magnetic field is $-B_0\hat{\jmath}$ with a speed $v_0\hat{\imath}$.

Spectrometer' and where it collides with the side (as shown)? d should depend only on q, m, v_0 and B_0.

Given $q = 1.6 \times 10^{-19}C$, $m = 3.82 \times 10^{-26}kg$, $v_0 = 5.38 \times 10^4 \frac{m}{s}$, and $B = 0.25T$, what is the numerical value for d? *We do not give the formula, but for the numbers given, $d = 0.103m$.*

- For the situation described in the previous question, how can you distinguish different *ions*? (That is, atoms with more or less electrons than a neutral atom has.)

- A particle of charge q travels with velocity $\vec{v} = 300\frac{m}{s}\hat{\jmath}$ in a region where the magnetic field is $\vec{B} = 1.5T\hat{k}$. What electric field is required for there to be zero net force? *We find that an electric field $\vec{E} = -450\frac{N}{C}\hat{\imath}$ is required.*

- A particle of charge q travels with velocity $\vec{v} = 400\frac{m}{s}\hat{k}$ in a region with an electric field $\vec{E} = 200\hat{\imath}$. What magnetic field is required for there to be zero net force? *We find that a magnetic field of $\vec{B} = 0.5T\hat{\jmath}$ is required.*

7.6 Questions

- A $6kg$ mass is at the origin, a $4kg$ mass is at $2m\hat{\imath}$, and an $8kg$ mass is at $4m\hat{\jmath}$. Find the net gravitational force on each.
- A mass M is at the origin, a mass m_1 is a distance r_1 away along the positive x-axis, and another mass m_2 is a distance r_2 away from the origin along a line that makes an angle θ with the x-axis. Find the magnitude of the gravitational field at the origin in terms of m_1, m_2, r_1, r_2, G, and θ. *Hint:* It might be easier for you if you assume that the angle θ is measured counterclockwise in the x-y plane.
- If q_1 is at the origin, and $q_2 = aq_1$ is a distance r away, is there a condition on a and r that tells you if there is a solution?
- A $5000kg$ mass is at $10m\hat{\imath}$ and a $4000kg$ mass is at the origin. Where along the x-axis will another mass experience no net force?

7.7 Answers

- The net forces are:

$$\begin{aligned}
\vec{F}_{net,6} &= 4.0 \times 10^{-10} N\hat{\imath} + 2.0 \times 10^{-10} N\hat{\jmath} \\
\vec{F}_{net,4} &= -4.48 \times 10^{-10} N\hat{\imath} + 0.95 \times 10^{-10} N\hat{\jmath} \\
\vec{F}_{net,8} &= 0.48 \times 10^{-10} N\hat{\imath} - 2.95 \times 10^{-10} N\hat{\jmath}
\end{aligned} \tag{7.40}$$

  In passing, note that the sum of all the net forces is 0, as required by Newton's third law.

- We aren't going to present a *formula* for this, but the way that you check is by seeing if, once you convert things into appropriate angles, you can reproduce the magnitudes from the previous question.
- If $a > 0$ there is an equilibrium point between the two charges. If $a < -1$ there is an equilibrum point; q_1 is between the equilibrium point and q_2. If $-1 < a < 0$ then there is an equilibrium point, and q_2 is between the equilibrium point and q_1. If $a = -1$ there is no solution.
- At $4.72m\hat{\imath}$.

Chapter 8

Integral Calculus

8.1 Summary

The eighth chapter of the text discusses integral calculus. We can use integral calculus to help us do two kinds of problems. The first is to 'add up' a large number of small things, which could be vectors, or numbers, or pieces of a larger object whose mass, moment of inertia, or similar we wish to obtain. The second is to solve differential equations, convenient because the laws of physics are expressible in terms of differential equations.

Some key points to remember are:

- The fundamental theorem of calculus says that if the function $g(x)$ is the derivative of $f(x)$ (i.e. that $\frac{d}{dx}f(x) = g(x)$) then

$$\int_a^b g(x)dx = f(b) - f(a) \tag{8.1}$$

- An indefinite integral 'undoes' a differentiation. If $\frac{d}{dx}f(x) = g(x)$ then

$$\int g(x)dx = f(x) + C \tag{8.2}$$

  where C is an *arbitrary* constant.

- The constant of integration C only appears in indefinite integrals.

- There are a number of very useful integrals that often appear in physics:

$$
\begin{aligned}
\int x^n dx &= \frac{1}{n+1} x^{n+1} + C \quad \text{for } n \neq -1 \\
\int x^{-1} dx &= \ln x + C \\
\int \cos ax dx &= \frac{1}{a} \sin ax + C \\
\int \sin ax dx &= -\frac{1}{a} \cos ax + C \\
\int e^{ax} dx &= \frac{1}{a} e^{ax} + C \qquad (8.3)
\end{aligned}
$$

8.2 Area in a region bounded by curves

Example Find the area of the region enclosed by the curves in the xy plane $y = f(x)$ and $y = g(x)$ for $f(x) = x$ and $g(x) = 2x^2 - 4$.

Worked Solution The first job is to identify the shape we care about. When you draw $y = f(x)$ you get a straight line on the xy plane, and when you draw $y = g(x)$ you get a parabola opening upwards. The region we care about is that enclosed by the two curves, with $f(x)$ above and $g(x)$ below.

These two curves meet at two different values of x, specifically $x = \frac{1 \pm \sqrt{33}}{4} \approx -1.186$ or 1.686. If we imagine breaking the region up into a number of small rectangles of width dx, and then adding their areas, we will need to know the height of the rectangles. Since in this region we know that $f(x) \geq g(x)$, the rectangle's height is $f(x) - g(x)$.

The area enclosed is then

$$
\begin{aligned}
\text{Area} &= \int_{lower\ x}^{upper\ x} [f(x) - g(x)]\, dx \\
&= \int_{-1.186}^{1.686} \left[x - \left(2x^2 - 4\right)\right] dx \\
&= \int_{-1.186}^{1.686} \left[-2x^2 + x + 4\right] dx \\
&= -\frac{2}{3}x^3 + \frac{1}{2}x^2 + 4x \Big|_{-1.186}^{1.686} \qquad (8.4)
\end{aligned}
$$

Substituting in the values of the upper and lower bound of integration gives us an area of 7.899.

You should have noticed that this is a straightforward integral. The mildly tricky parts were finding the bounds of integration and the height of the rectangles we were adding together in the regular Riemann-sum way. The integral itself was a polynomial.

Student Exercises Here are a few area examples:

- Find the area bounded by the line $y = \sin(2x)$, $y = 0$, $x = 0$, and $x = \frac{\pi}{2}$. *The area is* 1*, since the antiderivative of* $\sin(2x)$ *is* $-\frac{1}{2}\cos(2x) + C$ *and* $\cos(0) = 1$ *while* $\cos(\pi) = -1$.
- Find the area bounded by the lines $y = 1$, $y = \frac{1}{x}$, $x = 1$, and $x = 2$. *The area is* $1 - \ln 2 \approx 0.307$*. The antiderivative of the appropriate integrand is* $x - \ln x + C$*, and* $\ln 1 = 0$.
- Find the area bounded by the lines $x = 1 + \cos 2y$ and $x = -1 + e^{-y}$, $y = 2$ and $y = 4$. *In this case the natural way to proceed is to let* y *be the independent variable and* x *be the dependent variable, so the integral is over* y *rather than* x*. The integrand would be* $2 + \cos(2y) - e^{-y}$ *which has an antiderivative of* $2y + \frac{1}{2}\sin(2y) + e^{-y} + C$ *and between* 2 *and* 4 *the area is approximately* 4.756.

8.3 Arc Length

8.3.1 A straight line

Example Suppose that a curve is defined as $y = f(x)$ in the xy plane with $f(x) = 2x - 3$. What is the distance along the curve from $x = 0$ to $x = 4$?

Worked Solution This is an example of a *parametric* curve. The way we will approach it is to write both the x and y components of the position vector as functions of another variable, perhaps s. We then write $\vec{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath}$. After we have done this we find the displacement $d\vec{r}(s) = \left[\frac{d}{ds}\vec{r}(s)\right] ds$. This tells us the displacement of the point as the parameter changes from s to $s + ds$. We find the *magnitude* of $d\vec{r}$, and then add it up by integrating.

In this case we have

$$\vec{r}(s) = s\hat{\imath} + (2s - 3)\,\hat{\jmath} \tag{8.5}$$

so the functions are $x(s) = s$ and $y(s) = 2s - 3$. Note that $y(s) = 2x(s) - 3$ by direct substitution. The range of s we care about is between $s = 0$ and

$s = 4$ because x was between 0 and 4. Knowing how to differentiate we get that

$$\frac{d}{ds}\vec{r}(s) = 1\hat{\imath} + 2\hat{\jmath} \tag{8.6}$$

so we have that

$$d\vec{r}(s) = \left(\frac{d}{ds}\vec{r}(s)\right) ds = ds\hat{\imath} + 2ds\hat{\jmath} \tag{8.7}$$

and this has a magnitude

$$|d\vec{r}(s)| = |ds\hat{\imath} + 2ds\hat{\jmath}| = \sqrt{5}ds \tag{8.8}$$

Now that we know the value of $|d\vec{r}|$ we can calculate the arc length straightforwardly as

$$\begin{aligned} \text{Length} &= \int_{start}^{stop} |d\vec{r}| \\ &= \int_{s=0}^{4} \sqrt{5}ds \\ &= 4\sqrt{5} \approx 8.94 \end{aligned} \tag{8.9}$$

You might be wondering why we set up this problem and did it in such a convoluted way. You could have solved this using *geometry* and saying that we wanted the length of a straight line from $0\hat{\imath} - 3\hat{\jmath}$ to $4\hat{\imath} + 5\hat{\jmath}$ (the two endpoints), which you can immediately check is 8.94. The reason we did it is so you can see that the technique we outline work well in a case where you *know* the answer from other considerations.

Student Exercises

- Consider the curve given by $\vec{r}(s) = 3\cos s\hat{\imath} + 3\sin s\hat{\jmath}$. What is the length of the arc swept out between $s = \pi$ and $s = 2\pi$? Between $s = 0$ and $s = 3\pi$? *In this, s is essentially an angle, in radians. The arc length in the first case is $3\pi \approx 9.42$, and the arc length in the second case is $9\pi \approx 28.3$. The second case is analogous to going around a circle one and a half times. This example is relevant to the case of circular motion.*

- What is the length of the curve $y = \sqrt{9 - x^2}$ for the interval between $x = 0$ and $x = 3$? Hint: the easiest way is to make a substitution so that y and x are both functions of the same parameter; note that $y^2 + x^2 = 9$. *The arc length in this case is $\frac{3}{2}\pi \approx 4.71$.*

8.3.2 A curve

Exercise Determine the arc length of the curve $y = f(x)$ with $f(x) = 2x^2 - 3$ between $x = 1$ and $x = 3$.

Worked Solution We proceed in exactly the same way as in the previous example. We make the parametrization

$$\vec{r}(s) = x(s)\hat{\imath} + y(s)\hat{\jmath} \tag{8.10}$$

with the identification $x(s) = s$, and $y(s) = 2s^2 - 3$. Then we calculate both $d\vec{r}(s)$ and $|d\vec{r}(s)|$: we find

$$\begin{aligned} d\vec{r}(s) &= ds\hat{\imath} + (4s)\,ds\hat{\jmath} && (8.11)\\ |d\vec{r}(s)| &= \sqrt{1 + (4s)^2}ds && (8.12) \end{aligned}$$

Note that the $(4s)^2$ term in the above is $\left(\frac{d}{ds}y(s)\right)^2$, so equation 8.12 can be reexpressed as

$$|d\vec{r}(s)| = \sqrt{1 + \left(\frac{d}{ds}y(s)\right)^2}ds \tag{8.13}$$

which is likely to be the equation you are told when your math class teaches you to calculate arc length. The key that makes that equation appropriate is the parametrization: when you write $y = f(x)$ you are implicitly identifying the parameter s with x exactly as we did above.

The arc length is then

$$\text{Length} = \int_{s=1}^{3} \sqrt{1 + (4s)^2}ds \tag{8.14}$$

This is not an integral which can be solved by the simple techniques outlined in the text chapter. In subsequent Math classes you will learn techniques such as *substitution* which will allow you to evaluate integrals like this directly. For now, notice that the derivative of

$$g(s) = 4\left(\frac{s\sqrt{s^2 + \frac{1}{16}}}{2} + \frac{1}{32}\ln\left(s + \sqrt{s^2 + \frac{1}{16}}\right)\right) \tag{8.15}$$

is exactly the integrand. Substituting the bounds of integration in we get a length of approximately 16.14.

The main point of this example is not the actual integral but rather the calculation of the general formula for the arc length of a curve.

Student exercise

- A particle moves with a position as a function of time given by

$$\vec{r}(t) = 5\frac{m}{s}t\hat{\imath} + 3m\cos\left(2\frac{1}{s}t\right)\hat{\jmath} + 3m\sin\left(2\frac{1}{s}t\right)\hat{k} \tag{8.16}$$

  How far does it move between $t = 2s$ and $t = 3s$? *Calculate the arc length just like we did for two dimensions, but instead for three. You will find that it has gone along a pat* $7.81m$ *long.*

8.4 Velocity and displacement

Problem A particle has a velocity which is given as a function of time as

$$\vec{v}(t) = 5\frac{m}{s}\hat{\imath} + 2\frac{m}{s}e^{-0.2\frac{1}{s}t}\hat{\jmath} \tag{8.17}$$

What is the particle's displacement between $t = 2s$ and $t = 3s$?

Worked solution The differential equation that relates position and velocity is $\frac{d}{dt}\vec{r}(t) = \vec{v}(t)$. Using the fundamental theorem of calculus this then tells us that

$$\int_{t_1}^{t_2} \vec{v}(t)dt = \vec{r}(t_2) - \vec{r}(t_1) \tag{8.18}$$

This is exactly the quantity (the displacement) that we are asked for.

We need to integrate. The integrand is a *vector* but that is not a problem because we can use the fact that vectors add together linearly and unit vectors are essentially just constants. So, we have

$$\begin{aligned}
\text{Displacement} &= \int_{2s}^{3s} \vec{v}(t)dt \\
&= \int_{2s}^{3s} \left(5\frac{m}{s}\hat{\imath} + 2\frac{m}{s}e^{-0.2\frac{1}{s}t}\hat{\jmath}\right)dt \\
&= \left(5\frac{m}{s}\hat{\imath}\right)\int_{2s}^{3s} dt + \left(2\frac{m}{s}\hat{\jmath}\right)\int_{2s}^{3s} e^{-0.2\frac{1}{s}t}dt \\
&= \left(5\frac{m}{s}\hat{\imath}\right)(3s - 2s) + \left(2\frac{m}{s}\hat{\jmath}\right)\left(\frac{1}{-0.2s^{-1}}\right)\left(e^{-0.6} - e^{-0.4}\right) \\
&= 5m\hat{\imath} + 1.21m\hat{\jmath}
\end{aligned} \tag{8.19}$$

Notice how we substituted in the values for t at the ends of the integration region.

A key point in all this is that the integral of a *vector* is just a number of individual integrals. One for each component. This would not be true if the integral is that of a *scalar* such as you get in arc length or when you take a dot product.

Student exercises

- Suppose that a mass experiences a *constant* acceleration of $\vec{a} = 2\frac{m}{s^2}\hat{\imath} + 3\frac{m}{s^2}\hat{\jmath}$.
  - By how much does the velocity change between $t = 2s$ and $t = 4s$? *It changes by* $4\frac{m}{s}\hat{\imath} + 6\frac{m}{s}\hat{\jmath}$.
  - If the velocity were $4\frac{m}{s}\hat{\imath}$ at $t = 0s$ what is it at $t = 3s$? *It is* $10\frac{m}{s}\hat{\imath} + 9\frac{m}{s}\hat{\jmath}$.
  - How much does the position change between $t = 2s$ and $t = 4s$? Can you solve this if you don't assume the velocity at a particular instant? What if you assume that the velocity were $4\frac{m}{s}\hat{\imath}$ at $t = 0s$? *You can't determine this without knowing the velocity at an instant. The velocity at a particular instant is essentially the integration constant that appears. If you knew that* $f(x) = x + C$ *you couldn't work out* $\int_1^2 f(x)dx$ *without knowing* C. *Once the initial condition is given the displacement can be worked out as* $\Delta r = 20m\hat{\imath} + 18m\hat{\jmath}$.
  - Where is the mass at $t = 4s$? *We don't know without specifiying another constant of integration. This is why the expression for position as a function of time for constant acceleration is* $\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$. $\vec{r}_0$ *and* $\vec{v}_0$ *at these constants.*

- A mass has a velocity of

$$\vec{v}(t) = 4\frac{m}{s}\cos\left(2\frac{1}{s}t\right)\hat{\imath} + 3\frac{m}{s}\sin\left(3\frac{1}{s}t\right)\hat{\jmath} \tag{8.20}$$

  What is the mass's displacement between $t = 0s$ and $t = 2s$? *The displacement is about* $-1.51m\hat{\imath} + 0.04m\hat{\jmath}$.

Chapter 9

Momentum

9.1 Summary

Read the ninth chapter of the text, which introduces momentum. The reason we discuss momentum is as an alternative formulation of Newton's second law: a net force results in a change in momentum. The particular usefulness of this new idea is that in the absence of *external* forces, an interacting system will have a constant net momentum. Momentum is an example of a 'conserved' quantity.

- The momentum of a point particle is $\vec{p} = m\vec{v}$. Momentum is a *vector* quantity, and all calculuations using momentum must treat it that way.

- Newton's second law can be formulated in terms of momentum. A net force on a point particle results in a change its momentum: $\vec{F}_{net} = \frac{d}{dt}\vec{p}$.

- A system of particles, in the absence of external forces, will have a constant *net* momentum. This is an expression of Newton's third law: A pair of interacting particles will have their momentum change in opposite ways because the forces they exert on each other are equal in magnitude and opposite in direction.

- The forces that the colliding objects exert on each other are typically much larger than the other forces they are subjected to, so the objects can be treated as *approximately* isolated. We analyze collisions based on the assumption that momentum is conserved (constant) in the interaction. The momentum that is conserved is the sum of all the momenta of the individual particles.

- For a system of particles, the motion of their *center of mass* can be explained by considering only the *external* forces.

- The position of the center of mass is the mass-weighted average position. If there are n particles then

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots m_n} \tag{9.1}$$

  If the mass is continuously distributed with a density $\rho(\vec{r})$ then the position of the center of mass is

$$\vec{r}_{CM} = \frac{\int \vec{r}\rho(\vec{r})d^3\vec{r}}{\int \rho(\vec{r})d^3\vec{r}} \tag{9.2}$$

  The quantity $d^3\vec{r}$ is the little element of volume.

- The change in a particle's momentum (often called the *impulse*, and sometimes denoted $\vec{J}$) on a particle between times t_1 and t_2 is

$$\Delta\vec{p} = \vec{J} = \int_{t_1}^{t_2} \vec{F}dt = \vec{F}_{avg}\Delta t \tag{9.3}$$

  This relationship *defines* the average force exerted between the two times; $\Delta t = t_2 - t_1$.

- The idea of 'impulse' is often useful in situations when we do not know the *exact* details of the force being exerted.

9.2 A ball hitting a bat

Problem A $0.25kg$ ball, initially travelling horizontally at $20\frac{m}{s}$ is struck by a bat. After the collision it is travelling back in the opposite direction that it was initially travelling at an angle of $30°$ above the horizontal, and at a speed of $35\frac{m}{s}$. If the collision lasted for $1.50\times10^{-3}s$, what is the magnitude of the average force on the ball during the collision?

Worked Solution If you have ever seen a slow-motion picture of a baseball colliding with a bat (or a golf ball being hit by a club, or a tennis ball being hit by a racket, or a football being kicked, or anything like that) you will have noticed that the ball itself is *deformed* during the collision – it changes shape. The fact that it chages shape (and then returns to the original shape) suggests that it will act something like a spring: there is a force which is related to how much it is deformed. Our assumption with springs was that the force was *proportional* to the deformation, but we do not know enough about how balls are constructed to be sure that is a reasonable assumption. We are going to ignore this subtlety and simply calculate the *average* force, since we do not know the details of the actual forces exerted.

The fact we will use is that the *impulse*, or change in momentum is related to the average force. In fact:

$$\begin{aligned}
\vec{F}_{avg}\Delta t &= \Delta\vec{p} \\
&= \vec{p}_f - \vec{p}_i \\
&= m\vec{v}_f - m\vec{v}_i \\
\vec{F}_{avg} &= \frac{m\vec{v}_f - m\vec{v}_i}{\Delta t} \\
\left|\vec{F}_{avg}\right| &= \frac{m}{\Delta t}\left|\vec{v}_f - \vec{v}_i\right| \qquad (9.4)
\end{aligned}$$

In this question, Δt is the duration of the collision, m is the mass, and $\vec{v}_i$ and $\vec{v}_f$ are the initial and final velocities respectively.

From the statement of the question we have:

$$\begin{aligned}
\vec{v}_i &= 20\frac{m}{s}\hat{\imath} \\
\vec{v}_f &= 35\frac{m}{s}\left(-\cos 30^\circ\hat{\imath} + \sin 30^\circ\hat{k}\right) \\
m &= 0.25kg \\
\Delta t &= 1.5\times 10^{-3}s \qquad (9.5)
\end{aligned}$$

This means that $\vec{v}_f - \vec{v}_i = -50.3\frac{m}{s} + 17.5\frac{m}{s}\hat{k}$ and hence that $|\vec{v}_f - \vec{v}_i| = 53.3\frac{m}{s}$, which gives that $\left|\vec{F}_{avg}\right| = 8.88\times 10^3N$. Note that the direction of this force was 19° above the horizontal and 'back' in the direction the ball was originally going.

We talked about this because needed an excuse to calculate a change in momentum. Those calculations are very similar to changes in any vector quantity, so emphasizing that it is a vector is worthwhile.

When you are looking at the *average* force exerted, the relation we derived (equation 9.4) tells us that (all things being equal) a bigger change in velocity requires a bigger force, a shorter contact requires a bigger force, and if the mass is larger the force required for a given change in velocity will be larger.

Student Exercises

- Why are airbags so effective in cars? Consider an $80kg$ man travelling in a car horizontally in the x-direction at $30\frac{m}{s}$. (This is about $110\frac{km}{h}$.) Suppose the car collides with something, comes to a stop, and exerts a force on the man to stop him. What is the magnitude of the average force exerted on him if:
  - The thing exerting the force is the steering wheel column, and it stops him in $0.006s$. *In this case the average force is* $4.0 \times 10^5 N$.
  - The thing exerting the force is an airbag which stops him in $0.12s$. *In this case the magnitude of the average force is* $2.0 \times 10^4 N$.

9.3 An inelastic collision

Example A particle of mass m_1 travels at velocity $\vec{v}_1$ and collides with a second particle of mass m_2 travelling at initial velocity $\vec{v}_2$. The two masses stick together. Immediately after the collision, what is the velocity (direction and speed) of the wreckage?

Obtain a numerical answer is $m_1 = 2kg$, $m_2 = 1kg$, $\vec{v}_1 = 3\frac{m}{s}\hat{\imath}$, and $\vec{v}_2 = 8\frac{m}{s}\hat{\jmath}$, which is illustrated in figure 9.1.

What is the change of momentum of each of the individual masses in the collision?

Worked Solution This is a key type of problem because it illustrates the idea of *conservation of momentum*. When we say that momentum is conserved in a collision we usually mean that for the duration of the collision, the forces that the two particles exert on each other are so much larger in magnitude than the other forces they experience that *for all intents and purposes* the only forces are ones between the two colliding objects. In this situation, the momentum *immediately* before the collision is the same as the momentum immediately after.

We can write this in mathematical language, knowing that the two masses have combined after the collision:

$$\begin{aligned}
\vec{p}_{before} &= \vec{p}_{after} \\
\vec{p}_{1,before} + \vec{p}_{2,before} &= \vec{p}_{after} \\
m_1\vec{v}_1 + m_2\vec{v}_2 &= m_{combined}\vec{v}_{combined} \\
\vec{v}_{combined} &= \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_{combined}} \qquad (9.6)
\end{aligned}$$

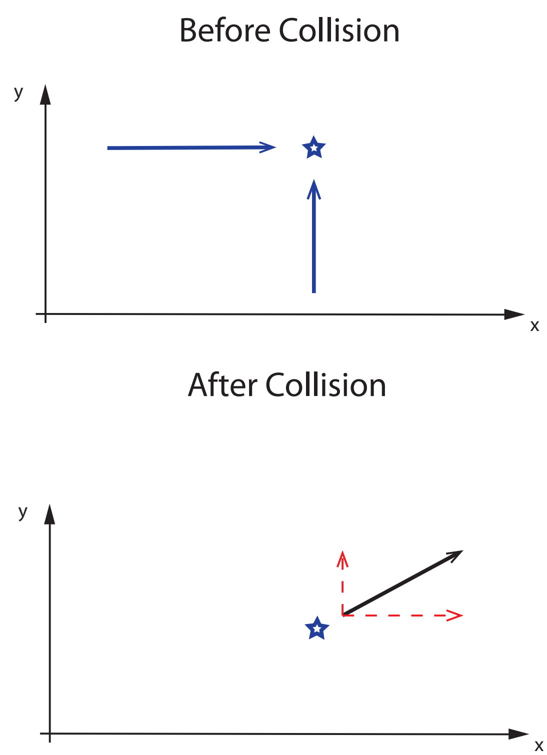


Figure 9.1: The figure shows the momentum vectors before a collison of two particles. The star indicates the collision point. After the collision, the particles have merged and there is one final momentum vector.

For the values we have:

$$\begin{aligned}\vec{p}_{1,before} = m_1\vec{v}_1 &= 6kg\frac{m}{s}\hat{\imath}\\ \vec{p}_{2,before} = m_2\vec{v}_2 &= 8kg\frac{m}{s}\hat{\jmath}\end{aligned}$$

so we have that $\vec{v}_{combined} = 2\frac{m}{s}\hat{\imath} + 2.7\frac{m}{s}\hat{\jmath}$. This velocity has a magnitude of $3.3\frac{m}{s}$ and it makes an angle of $53°$ counterclockwise from the positive x-axis.

To find the change in momentum for the $2kg$ object (which after the collision travels locked together with the $1kg$ object) we have to take the difference between $\vec{p}_{final}$ and $\vec{p}_{initial}$. The initial momentum is $m_1\vec{v}_1$, and the final momentum *of this mass* is $m_1\vec{v}_{combined} = 2kg\left(2\frac{m}{s}\hat{\imath} + 2.7\frac{m}{s}\hat{\jmath}\right)$. Then

$$\begin{aligned}\text{2kg Mass}: \quad \Delta\vec{p}_1 &= \vec{p}_{final} - \vec{p}_{initial}\\ &= \left(4kg\frac{m}{s}\hat{\imath} + 5.3kg\frac{m}{s}\hat{\jmath}\right) - 6kg\frac{m}{s}\hat{\imath}\\ &= -2kg\frac{m}{s} + 5.3kg\frac{m}{s}\hat{\jmath} \qquad (9.7)\end{aligned}$$

Similarly, for the second mass the change in momentum is the difference between its $\vec{p}_{final}$ and $\vec{p}_{initial}$. These are $m_2\vec{v}_2$ and $m_2\vec{v}_{combined}$ respectively. This means

$$\begin{aligned}\text{1kg Mass}: \quad \Delta\vec{p}_2 &= \vec{p}_{final} - \vec{p}_{initial}\\ &= \left(2kg\frac{m}{s}\hat{\imath} + 2.7kg\frac{m}{s}\hat{\jmath}\right) - 8kg\frac{m}{s}\hat{\jmath}\\ &= 2kg\frac{m}{s}\hat{\imath} - 5.3kg\frac{m}{s}\hat{\jmath}\end{aligned} \qquad (9.8)$$

Note that for the numbers provided

$$\Delta\vec{p}_2 = -\Delta\vec{p}_1 \qquad (9.9)$$

This effect is generic: when two object which are otherwise isolated interact the change in one's momentum is compensated by the change in the other's momentum. The total momentum is unchanged.

We have a numerical answer for the change in momentum of the $2kg$ piece and the $1kg$ piece. They are equal in magnitude and opposite in direction.

Some things to notice: The thing that should be jumping out at you is how relatively straightforward calculations involving momentum are. That is why physicists like the concept - it is straightforward and easy to apply.

In this case, by 'straightforward' we really mean 'linear'. You just have to add and subtract vectors, which we have done a lot.

The second thing that is important to notice (but not surprising) is that the two changes in momentum were opposite. This means that the total momentum was the same before and after the collision. It is a reflection of Newton's third law, together with the fact that $\vec{F}_{net} = \frac{d}{dt}\vec{p}$.

Student Exercises: Suppose that an object of mass m_1 travels with speed $|\vec{v}_1|$ and collides with an object of mass m_2 travelling with an initial speed $|\vec{v}_2|$. The two masses stick together after the collision. The angle between $\vec{v}_1$ and $\vec{v}_2$ is θ.

- Find the final speed of the combined mass if $m_1 = 4kg$, $m_2 = 2kg$, $|\vec{v}_1| = 6\frac{m}{s}$, $|\vec{v}_2| = 8\frac{m}{s}$ and the angle between the masses' velocities is $\theta = 53°$. *The combined speed is* $5.99\frac{m}{s}$*.*

- Find the angle the combined mass's velocity makes with the vector defined by the velocity of mass m_1 for the numbers provided above. *The angle is* $20.9°$*.*

- Symbollically, find the final speed of the combined mass in terms of m_1, m_2, $|\vec{v}_1|$, $|\vec{v}_2|$, and θ. Find the angle ϕ that the final velocity of the combined mass makes with initial velocity $\vec{v}_1$ in terms of the same variables. *It is probably going to make your life easier if you assume that* $\vec{v}_1$ *travels along the x-axis; make sure your answer matches the result from equation 9.6 if* $\theta = 90°$*, and make sure you reproduce the results above.*

9.4 An explosion

Example A particle initially at rest explodes into three pieces: A $3kg$ piece which travels along the x-axis at $5\frac{m}{s}$, a $2kg$ piece which travels along a line making an angle of $135°$ with the positive x-axis in the x-y plane at $8\frac{m}{s}$, and a $1kg$ piece. What is the velocity of the $1kg$ piece after the explosion?

This is illustrated in figure 9.2.

Worked Solution This is *another* application of conservation of momentum. The explosion produces forces act between the three fragments. After

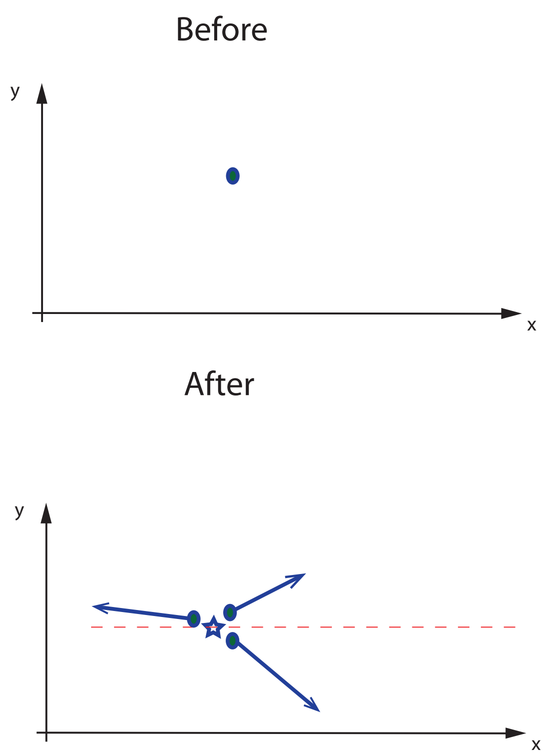


Figure 9.2: The figure shows a single stationary particle in the before part. The particle explodes into three pieices.

the explosion, the momenta are

$$\begin{aligned}
1\text{kg}: \ \vec{p}_1 &= m_1\vec{v}_1 = 1kg\vec{v}_1 \quad \text{We want } \vec{v}_1 \\
2\text{kg}: \ \vec{p}_2 &= m_2\vec{v}_2 = 2kg\left(-5.66\frac{m}{s}\hat{\imath} + 5.66\frac{m}{s}\hat{\jmath}\right) \\
3\text{kg}: \ \vec{p}_3 &= m_3\vec{v}_3 = 3kg\left(5\frac{m}{s}\hat{\imath}\right) \qquad (9.10)
\end{aligned}$$

Using conservation of momentum, together with the fact that the particle is initially at rest, we have

$$\begin{aligned}
\vec{p}_{initial} &= \vec{p}_{final} \\
0 &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \\
\vec{p}_1 &= -\vec{p}_2 - \vec{p}_3 \\
\vec{v}_1 &= \frac{-\vec{p}_2 - \vec{p}_3}{m_1} \qquad (9.11)
\end{aligned}$$

and with the values we have for $\vec{p}_2$ and $\vec{p}_3$ we have

$$\vec{v}_1 = -3.7\frac{m}{s}\hat{\imath} - 11.3\frac{m}{s}\hat{\jmath} \qquad (9.12)$$

What you should have noticed: Once again, we had to apply conservation of momentum. That is really the theme of this chapter: Momentum is a vector, and it is conserved in isolated interactions.

Student Exercise This question is intended to get you to understand how rocket engines work.

A rocket has total mass $3000kg$, of which $2000kg$ is fuel. When the rocket is turned on, the gas goes out at a rate of $1.0\frac{kg}{s}$ at a speed of $1500\frac{m}{s}$ relative to the rocket. If the rocket is initially at rest, what is the rocket's speed after 30 minutes?

To solve this follow the following procedure:

- Suppose a spaceship of total mass M_{tot} is travelling at velocity $v\hat{\imath}$ as shown in the top half of figure 9.3. Now suppose there is an explosion and it ejects a small amount of mass Δm so that it is travelling at velocity $(v - v_{ex})\hat{\imath}$. What is the *change* in speed of the rest of the spaceship? (in this set-up, v_{ex} is the speed at which 'gas' is being ejected from the spaceship.) Note, ignore terms where two Δ quantities are multiplied together.

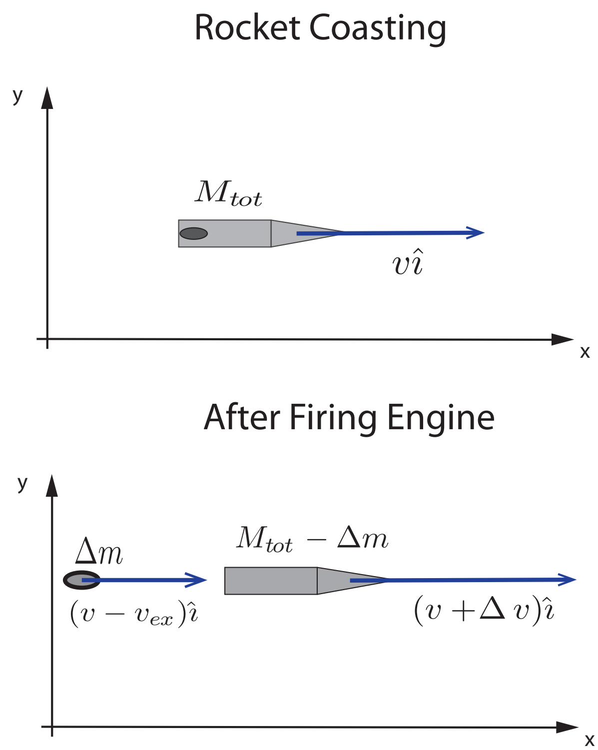


Figure 9.3: The upper half of the figure shows a rocket of mass M_{tot} moving with velocity $v\hat{\imath}$. The lower half of the figures shows the situation after a mass Δm has been expelled from the rocket motor with a velocity v_{ex}.

- If the mass Δm was ejected in a short amount of time Δt, what is the *average* magnitude of acceleration of the remainder of the spaceship during that time? Express this in terms of v_{ex}, M, Δm, and Δt.

- Now, imagine that M_{tot} is the mass at some instant of time, t, then we can see that M_{tot} is $M(t)$ the (changing) mass of the spaceship. Explain why $\frac{\Delta m}{\Delta t} \approx -\frac{d}{dt}M(t)$ using the definition of the derivative,

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{M(t+\Delta t) - M(t)}{\Delta t} = \frac{M(t) - \Delta m - M(t)}{\Delta t} \tag{9.13}$$

  Note that the mass of the spaceship is changing because it is exhausting gas out the back - it is the same principle as how a bottle full of water and compressed air works: some water is shot out the back, reducing the mass but pushing the rest in the opposite direction.

- If Δt is very short, explain why the expression for acceleration magnitude can be expressed as $|\vec{a}| = -v_{ex}\frac{1}{M(t)}\frac{dM}{dt}$

- If the rocket starts from rest at mass M_1, exhausts all its fuel at a constant speed v_{ex} and when its fuel is exhausted has a mass M_2 show that its final speed is $v_{ex} \ln\left(\frac{M_1}{M_2}\right)$ by integrating.

After all that work, you should get a speed of $1374\frac{m}{s}$.

9.5 Center of Mass and Projectile Motion

Problem Suppose a ball is thrown, and part-way through its flight, it explodes into two pieces. Argue convincingly that the center of mass of these two pieces will follow the same path the ball would have if it had not met with an unfortunate demise.

Worked Solution We have analyzed constant acceleration motion (such as by gravity) in chapter 5. At the instant before the ball explodes it is at a location $\vec{r}_i$, and travelling at a velocity $\vec{v}_i$. It experiences a constant acceleration $\vec{g}$, so it follows the path:

$$\vec{r}(t) = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{g}t^2 \tag{9.14}$$

assuming that we call the time at which it explodes $t = 0s$.

The explosion happens fast, and breaks the ball into two pieces. At this instant, we apply conservation of momentum:

$$\begin{aligned}\vec{p}_{initial} &= \vec{p}_1 + \vec{p}_2 \\ (m_1 + m_2)\vec{v}_i &= m_1\vec{v}_1 + m_2\vec{v}_2 \qquad (9.15)\end{aligned}$$

The two masses will travel immediately after the explosion with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. This means that the trajectories they will follow are:

$$\begin{aligned}\vec{r}_1(t) &= \vec{r}_i + \vec{v}_1 t + \frac{1}{2}\vec{g}t^2 \\ \vec{r}_2(t) &= \vec{r}_i + \vec{v}_2 t + \frac{1}{2}\vec{g}t^2 \qquad (9.16)\end{aligned}$$

Both have their initial position at $\vec{r}_i$ because the two fragements came from the same ball. The general expression for the center of mass of two objects is

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (9.17)$$

Applying this to our case:

$$\begin{aligned}\vec{r}_{cm}(t) &= \frac{m_1\vec{r}_1(t) + m_2\vec{r}_2(t)}{m_1 + m_2} \\ &= \frac{m_1\left(\vec{r}_i + \vec{v}_1 t + \frac{1}{2}\vec{g}t^2\right)}{m_1 + m_2} + \\ &\qquad \frac{m_2\left(\vec{r}_i + \vec{v}_2 t + \frac{1}{2}\vec{g}t^2\right)}{m_1 + m_2} \\ &= \vec{r}_i + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}t + \frac{1}{2}\vec{g}t^2 \\ &= \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{g}t^2 \qquad (9.18)\end{aligned}$$

for the last equality, we are using the result from equation 9.15, and we notice that this expression is the same as equation 9.14, so the center of mass travels in the same path that the ball was going to (until some of it hits something, which would exert a force on at least one fragment.)

What you should notice: we did this example primarily to highlight how you calculate the center of mass for a set of objects that are moving in time. We also wanted to show that you can use momentum ideas even if there is a net force (as there is for projectile motion in the presence of gravity.)

Student Exercise Three particles are located at the following positions: mass one is 3 kg and located at $\vec{r}_1 = 4m\hat{\imath}+2m\hat{\jmath}$, mass two is 1 kg and located at $\vec{r}_2 = -4m\hat{\imath}-2m\hat{\jmath}$, and mass three is 3kg and located at $\vec{r}_3 = -1m\hat{\imath}+3m\hat{\jmath}$. Find the location of the center of mass of this set of three particles. *The center of mass can be shown to be at* $0.71m\hat{\imath} + 1.86m\hat{\jmath}$.

9.6 Questions

- What average force is required to change the velocity of a $1.5kg$ ball from $20\frac{m}{s}\hat{\imath} + 10\frac{m}{s}\hat{\jmath}$ to $-10\frac{m}{s}\hat{\imath} + 20\frac{m}{s}\hat{\jmath}$ in $3.0s$?

- A ball of mass m_1 travelling with velocity $\vec{v}_{1,i}$ hits a ball of mass m_2 which is initially at rest. The two balls have masses that do not change during collision. The first ball travels with velocity $\vec{v}_{1,f}$ after the collision. This situation is illustrated in figure 9.4. Find the velocity (speed and direction) of the second ball after the collision if $m_1 = 5kg$, $m_2 = 15kg$, $\vec{v}_{1,i} = 20\frac{m}{s}\hat{\imath}$, and $\vec{v}_{1,f} = 5\frac{m}{s}\hat{\imath} + 15\frac{m}{s}\hat{\jmath}$.

9.7 Answers

- If this case the average force required is $-15N\hat{\imath} + 5N\hat{\jmath}$.

- The final speed of the ball is $|\vec{v}| = 7.07\frac{m}{s}$, along the unit vector $0.71\hat{\imath} - 0.71\hat{\jmath}$.

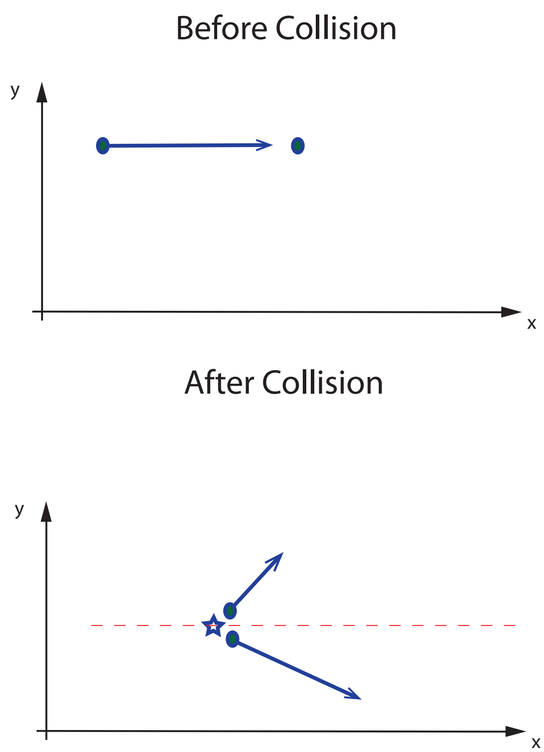


Figure 9.4: The figure shows the momentum vectors before a collison of a moving particle with a stationary particle. The particles retain their original masses after the collision.

Chapter 10

Angular Momentum

10.1 Summary

Read the tenth chapter of the text. In previous chapters, we have learned about how objects in static equilibrium experience no net force as well as no net torque. Here we will learn how objects change their rotational state in response to non-zero torques. We will find that a quantity called 'angular momentum' changes in response to non-zero net torques the same way that momentum changes in response to net forces.

Some of the key points are:

- For point at distance r from a pivot, when the angle from the pivot to that point changes by θ (measured in *radians* from a fixed axis) then the point has moved an arc length $r\theta$.

- The tangential speed of a point which is at a fixed distance from a axis of rotation is $v = \left|\frac{d}{dt}r\theta\right| = r\left|\frac{d}{dt}\theta\right|$. The quantity $\frac{d}{dt}\theta \equiv \omega$ is often called the angular speed. Note that $v = r\omega$.

- The tangential acceleration of a point which it at a fixed distance r from an axis of rotation is $a_{tan} = \frac{d}{dt}v = r\frac{d}{dt}\left|\omega\right|$. If this quantity is positive, the rotation is speeding up, and if it is negative the rotation is slowing down.

- The angular momentum of a particle around a point is $\vec{L} = \vec{r} \times \vec{p}$. The vector $\vec{r}$ is the vector from the point around which you are measuring $\vec{L}$ to the particle.

- The angular momentum of an object rotating along an axis can be found by imagining breaking the object up into little pieces, calculating

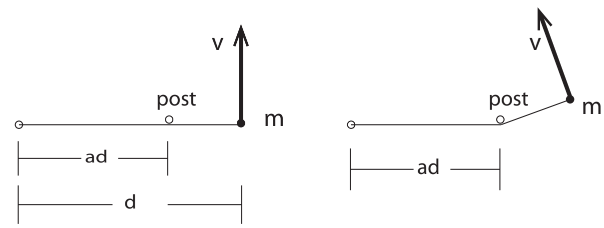


Figure 10.1: A ball of mass m is moving in a horizontal circle of radius d until the rope to which it is attached hits a post at a distance ad from the original center of the circle.

the angular momentum of each, and adding them together. Since the speed of each piece is related to the angular speed, it turns out that the magnitude of the angular momentum is $\left|\vec{L}\right| = I\omega$.

- The direction of the angular momentum of a rotating rigid object is – in the cases we study in this course – along the axis of rotation. The magnitude is given by $I\omega$ and the direction is given by the right-hand rule.

- I is a measure of the distribution of the mass of an object. You calculate I by adding, for each 'piece' of an object, the mass of that piece multiplied by the square of how far it is from the rotation axis.

$$I = \int_{all\ the\ object} r^2 dm = \int |\vec{r}|^2 \rho(\vec{r}) d^3\vec{r} \tag{10.1}$$

10.2 A ball swinging on a rope

Problem You are swinging a ball of mass m in a horizontal circle of radius d at speed v. The ball is attached by a rope which will break if it is subjected to a tension bigger than T. There is a post a distance ad from you which will get in the way of the rope, as shown in figure 10.1

- If $m = 1kg$, $v = 3\frac{m}{s}$, $d = 2m$, and the maximum tension before the rope breaks is $T = 10N$, what is the biggest possible value of a such that the rope does not immediately break?

- If the post had circumference $c = 2cm$ and the post was placed at $a = \frac{1}{2}$, how many times would the rope wind around the post before it broke?

Worked Solution Before we start to work through the problem, we have to have a vision for what the issue will be. We know that the ball is moving in a circle, and then something is in the way of the rope. This means that the rope will wrap around the post it hits, and this in turn means that the rope will then be holding the ball in a smaller circle. In fact, a smaller circle of radius $(1 - a)d$. A smaller radius means, all other things being equal, a larger tension on the rope because the circle is smaller with the same speed, so the centrepetal acceleration is larger. When the acceleration gets too large, so the required tension is too large, the rope will break.

The first part of this question does not actually require the concept of angular momentum, but we will use it to set up the second part.

The ball travels, immediately after the rope hits the post, in a circle of radius $(1-a)d$. This means that the tension in the rope must be enough to supply the net force of magnitude (recall the material on circular motion) $m\frac{v^2}{r}$. This means that

$$
\begin{aligned}
T &= \left|\vec{F}_{net}\right| < T_{max} \\
\frac{mv^2}{r} &< T_{max} \\
\frac{mv^2}{(1-a)d} &< T_{max} \\
\frac{mv^2}{dT_{max}} &< (1-a) \qquad (10.2)
\end{aligned}
$$

For the given values of m, v, d, and T_{max}, $1 - a < 0.45$, so the largest possible $a = 0.55$, and that means that if the post is more than $1.1m$ away from the place you are swinging the ball, the rope will break immediately upon contact with the post.

The interesting part here was the critical condition that $m\frac{v^2}{r} < T_{max}$. Now that we have seen this, we will look at what happens when the rope starts to wrap around the pole. Obviously, the 'effective' length of the rope will get shorter, and in the condition we have, that means r gets smaller, and we get closer to the rope breaking. We haven't worked out what happens to v (the speed) as the radius of the circle r gets smaller. In the first part, we treated v as though it did not change, because we were trying to figure out what would happen immediately upon the rope hitting the post.

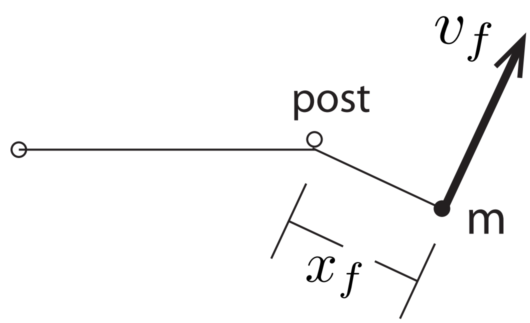


Figure 10.2: The geometry of the ball from figure 10.1 as it wraps around.

We want to figure out the angular momentum of the ball as it goes around the post: As we see in figure 10.2, initially the angular momentum of the ball *around the post* is

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ &= [(1-a)d\hat{\imath}] \times (mv\hat{\jmath}) \\ &= (1-a)dmv\hat{k} \end{aligned} \tag{10.3}$$

Since the radius vector (from the post to the ball) and the velocity vector are perpendicular to each other, we could have gotten this from the way of calculating cross products by obtaining the magnitude from $\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta$, and gotten the direction using the right hand rule.

Now, we look at figure 10.2, and determine out the angular momentum after the rope has wrapped a bit. In this case, the mass is x_f from the pole, and it is travelling at speed v_f. The two vectors of which these are the magnitudes are still perpendicular. Calculating the angular momentum using the right-hand rule (and assuming that the paper is the x-y plane, as we did before) we get $\vec{L} = x_f m v_f \hat{k}$.

Now for the key part: we equate the two values we got for angular momentum. Why? Because there is no torque *around the pole* exerted by the rope on the mass because the force is in towards the pole, and the radius vector which we care about because $\vec{\tau} = \vec{r} \times \vec{F}$ is in the opposite direction to the force, so the angle between them is 180°, and so the torque has 0 magnitude. This tells us that

$$
\begin{aligned}
\vec{L}_{after} &= \vec{L}_{original} \\
(1-a)\, dmv\hat{k} &= x_f m v_f \hat{k} \\
(1-a)dv &= x_f v_f \\
v_f &= \frac{(1-a)d}{x_f} v
\end{aligned}
\tag{10.4}
$$

In other words, as the ball wraps (i.e. as x_f gets smaller) the speed of the ball gets larger. Note that $(1-a)$, d, and v are quantities that are known.

Now, we can find out how much wrapping will happen: At the instant the rope breaks

$$m\frac{v^2}{r} = T_{max} \tag{10.5}$$

We have called the speed of the ball once it is wrapping v_f, and the distance from the pole to the ball is x_f, so the point we are looking for is when the required force is the same as the maximum force

$$m\frac{v_f^2}{x_f} = T_{max} \tag{10.6}$$

but finally we have a relation between v_f and x_f:

$$
\begin{aligned}
m\frac{v_f^2}{x_f} &= T_{max} \\
m\frac{\left(\frac{(1-a)dv}{x_f}\right)^2}{x_f} &= T_{max} \\
m\frac{(1-a)^2 d^2 v^2}{x_f^3} &= T_{max} \\
x_f &= \left(\frac{m(1-a)^2 d^2 v^2}{T_{max}}\right)^{1/3}
\end{aligned}
\tag{10.7}
$$

Putting in the known values: $m = 1kg$, $a = 0.5$, $d = 2m$, $v = 3\frac{m}{s}$, and $T_{max} = 10N$, we get $x_f = 0.965m$. Since the circumference of the rod was $0.02m$, this means it will wind about 1 and two thirds time around before it breaks.

Some things to notice The really big points of this example were:

- In the case of a central force (one directed inwards towards the center of rotation) the angular momentum is unchanged.
- If you have a point particle, and it has constant angular momentum, as it moves closer to the axis of rotation, it will speed up.
- If you thought about this instead as being a rigid body with all the mass at one end and rotating about the other, the change (reduction) of the moment of inertia as the rope winds (so the ball gets closer) would mean a corresponding increase in the angular speed.

Student Exercises Suppose a dumbbell can be modelled as a pair of balls, each with mass $m = 0.5kg$, and the two masses are joined by a rigid rod of length $L = 0.4m$. The two balls rotate around their *center of mass* along an axis perpendicular to the rod at an angular speed $\omega = 10\frac{1}{s}$.

- What is the speed of each mass? *It is* $2.0\frac{m}{s}$.
- What is the magnitude of the angular momentum of one of the balls around the center of mass? *It is* $0.2\frac{kg\ m^2}{s}$.
- What is the moment of inertia of the dumbbell around its axis of rotation? *The moment of inertia is* $0.04kg\ m^2$.
- Does the magnitude of the angular momentum calculated from the moment of inertia and the angular speed match the total angular momentum by adding up the angular momentum of the two balls? *It should be clear that it does.*

10.3 Constant Acceleration Merry-Go-Round

Example A merry-go-round (essentially, a uniform disk in the horizontal plane that is free to rotate around a vertical axis) has a radius of r_0. A small block of mass m is placed a distance ar_0 from the center of the merry go round. It has a coefficient of static friction μ_s with the surface on which it rests.

The merry-go-round starts rotating from rest at a constant angular acceleration α. If $r_0 = 2m$, $a = 0.333$, $m = 5kg$, $\mu_s = 0.5$, and $\alpha = 0.1\frac{1}{s^2}$ how long after the start of rotation does the mass start to slide?

Worked Solution The first thing we have to do is think through why the mass is going to slide. We know that there is a constant angular acceleration α, which means that the angular speed ω is increasing. If ω is increasing, that means that the speed v is also increasing. Since the speed is increasing, this means that the force required to give the centripetal acceleration will also increase. At some point, the required force will be bigger than the force that friction can provide; then it will start to slide.

We can start by looking at a free-body diagram for the mass. There are three forces being exerted on it: the downwards force of gravity, the upwards normal force, and the force of static friction. The force of static friction acts horizontally, and this force is what must provide the acceleration of the mass: $\vec{F}_{net} = m\vec{a} = \vec{F}_s + \vec{F}_n + \vec{F}_g$. As usual for a horizontal surface, the magnitude of $\vec{F}_n$ is mg, so the maximum possible magnitude of the force of friction is $\mu_s mg$.

Now, what does this force have to do? It must supply the change (increase) in speed, and it must also supply the centripetal acceleration. The increase in speed is supplied by the *tangential* acceleration; we know that the magnitude of this is $r\alpha$. Similarly, we know that the *centripetal* acceleration has magnitude $\frac{v^2}{r}$, and the speed $v = r\omega$, so the centripetal acceleration could be expressed as $\frac{(r\omega)^2}{r} = r\omega^2$. This can be seen in figure 10.3. The magnitude of the acceleration is

$$\begin{aligned} |\vec{a}| &= \sqrt{a_{tan}^2 + a_{cen}^2} \\ &= \sqrt{(ar_0\alpha)^2 + (ar_0\omega^2)^2} \\ &= ar_0\sqrt{\alpha^2 + \omega^4} \end{aligned} \tag{10.8}$$

but since the rotation started from rest, we have that $\omega = \alpha t$. Also, the maximum force magnitude is $\mu_s mg$, so the maximum acceleration it provides is $\mu_s g$, and so we have to find the largest value of t that

$$ar_0\sqrt{\alpha^2 + (\alpha t)^4} \leq \mu_s g \tag{10.9}$$

and this gives us

$$\begin{aligned} \alpha^2 + (\alpha t)^4 &\leq \left(\frac{\mu_s g}{ar_0}\right)^2 \\ \frac{1}{\alpha^2} + t^4 &\leq \left(\frac{\mu_s g}{\alpha^2 ar_0}\right)^2 \\ t^4 &\leq \left(\frac{\mu_s g}{\alpha^2 ar_0}\right)^2 - \frac{1}{\alpha^2} \end{aligned} \tag{10.10}$$

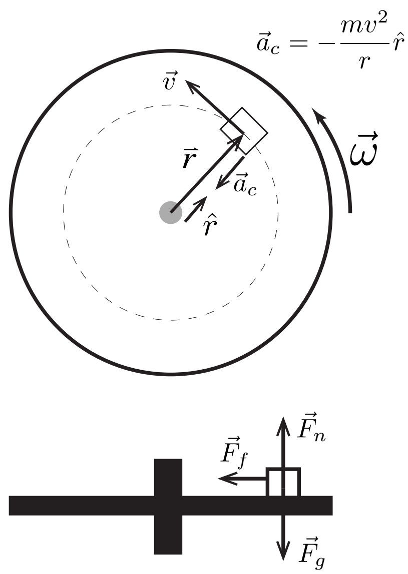


Figure 10.3: A mass is a distance $|\vec{r}|$ from the center of a merry-go-round. The force of friction provides the centripetal acceleration as well as any change in speed.

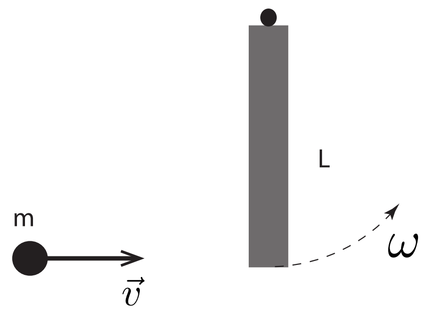


Figure 10.4: A ball of mass m and velocity $\vec{v}$ is heading horizontally towards the bottom end of a ballistic pendulum of length L which is hanging vertically. The ball will stick to the bottom of the pendulum, which will then swing with angular speed ω as shown.

so the largest t that satisfies this inequality ($t^4 \leq 63580 s^4$ for the numbers given) is $15.9s$.

More things to note: The things that are especially critical here are the relation between the angular and linear kinematics. Knowing the angular acceleration told us about the tangential component of acceleration; knowing the angular speed told us the speed of the rotating object, which in turn told us the centripetal component of acceleration. It was very important that these two components of acceleration had to be treated together to give us the magnitude of the overall acceleration. Since we knew the relation between angular speed and angular acceleration, we were able to find the magnitude of the overall acceleration in terms of given quantities and t, the unknown quantity.

Student Exercise

- A particular ballistic pendulum consists of a rigid rod of length L suspended vertically from one end. This rod has moment of inertia I around its suspension point. A ball of mass m is shot horizontally at speed v, as shown in figure 10.4. The ball collides and sticks to the rod.

  Suppose that $v = 15\frac{m}{s}$, $m = 0.1kg$, and the moment of inertia (*of just the rod*) $I = 0.75kg\ m^2$, and $L = 0.3m$. What is the angular speed of

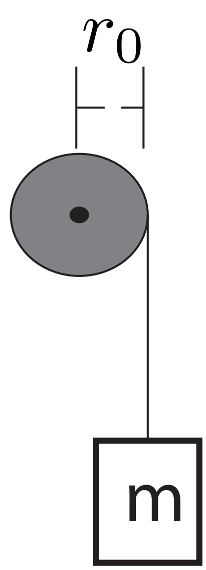


Figure 10.5: A cylinder of radius r has moment of inertia I_0 around its axis of symmetry, about which it is free to rotate. A massless, inextensible rope is attached to a ball of mass m and wound around the cylinder.

this ball and rod combination around the pin at the top immediately after the collision? *The angular speed after the collision is* $\omega = 0.593\frac{1}{s}$.

- A ball of mass $4.0kg$ is held up by a rope of length $L = 0.5m$ and is at rest in equilibrium. A ball of mass $1.0kg$ is travelling at $10\frac{m}{s}\hat{\imath}$ and hits and bounces off the first ball. After the collision the $1.0kg$ ball is travelling with velocity $-2.0\frac{m}{s}\hat{\imath}$. Find the tension in the rope immediately after the collision. *It is* $111N$.

10.4 An Atwood Machine

Example A mass m is attached to a rope which is wound around a cylinder of radius r_0 and which has a moment of inertia I_0. This is shown in figure 10.5.

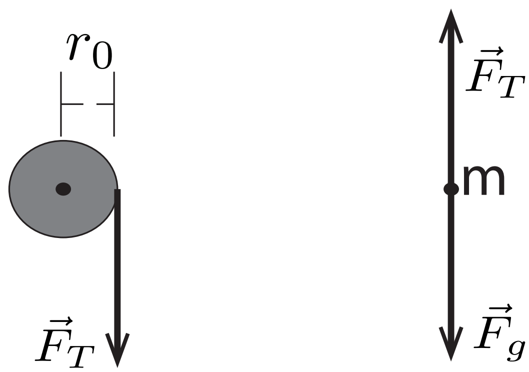


Figure 10.6: Sketches of the free-body diagrams for both the cylinder and the suspended mass from 10.5.

If $m = 0.5kg$, $r_0 = 0.4m$, and $I_0 = 50kg\ m^2$, what is the acceleration of the mass?

Worked Solution The purpose of this question is to illustrate why it was so important to have a massless frictionless pulley when we talked about Atwood Machines before.

The big picture of the question is the same as what we had for Atwood machines. We will apply Newton's 2nd law to the suspended mass to get its acceleration; the acceleration will be related to the tension in the rope (unknown) and the mass's weight (known in terms of m). We will apply the rule that $\vec{\tau} = \frac{d}{dt}\vec{L}$ (that torque changes angular momentum) to find the rate of angular acceleration of the cylinder; this will relate the angular acceleration to the tension in the rope (since the tension is what provides the torque). Finally, we will relate the angular acceleration to the mass's acceleration. There will be three relationships between three things we don't know: $|\vec{a}|$, α, and the tension T. Three relationships between three unknown things will give us the chance to solve for each.

We will give this a shot! First, we look at the free-body diagrams, shown in figure 10.6.

The case of the mass, applying Newton's laws are easy:

$$\begin{aligned}
\vec{F}_{net,m} &= \vec{F}_T + \vec{F}_g \\
m\vec{a} &= T\hat{k} + \left(-mg\hat{k}\right) \\
m\left(-a\hat{k}\right) &= (T - mg)\,\hat{k} \\
a &= g - \frac{T}{m}
\end{aligned} \tag{10.11}$$

In this, we have set it so that a is the vertical component of the acceleration, and we chose a particular sign convention (see the $-$ sign in the 3rd line) that if $a > 0$ the mass is falling down. This was not essential, but it will make our algebra easier.

Now, we need to find out what is happening with the cylinder. There is going to be an upwards force on it from the axle. This will keep it in translational equilibrium, but the cylinder is subject to forces exerted at different points, so it will change its rotational state. We calculate the torque around the axle (axis of rotation of the cylinder) and we will use the fact that the torque and the angular momentum both have only components along the line perpendicular to the paper. (If you don't believe this, check it yourself.)

$$\begin{aligned}
\vec{\tau}_{net} &= \frac{d}{dt}\vec{L} \\
|\vec{\tau}| &= \left|\frac{d}{dt}\vec{L}\right| \\
r_0 T &= I\left|\frac{d}{dt}\omega\right| \\
r_0 T &= I\alpha
\end{aligned} \tag{10.12}$$

In this, we have gone ahead and calculated the magnitude of the torque using the method of simply obtaining the magnitude. The direction is into the paper (in the positive y-direction if 'up' the page is the z-direction).

The final relationship is between α and a. The rope does not stretch, so as the distance the mass goes down corresponds to the amount the rope 'unwinds' from the cylinder. This means that the amount the mass goes down is $d = r_0\theta$ when the cylinder unwinds by an angle θ. Differentiating the distance with respect to time twice, we find that $a = r_0\frac{d^2}{dt^2}\theta \to a = r_0\alpha$. Note that something is implied here: the mass's displacement is $-d\hat{k}$, so the second time derivative of that is $-\frac{d^2}{dt^2}d\hat{k}$. If you compare to how we wrote the acceleration this says that $a = \frac{d^2}{dt^2}d$, so we have taken care of the signs appropriately.

There are three relations:

$$\begin{aligned} a &= g - \frac{T}{m} \\ r_0 T &= I\alpha \\ a &= r_0\alpha \end{aligned} \tag{10.13}$$

substituting the third into the second we then have

$$\begin{aligned} a &= g - \frac{T}{m} \\ r_0 T &= I\frac{a}{r_0} \end{aligned} \tag{10.14}$$

Solving the second equation for T in terms of a and substituting gives

$$\begin{aligned} a &= g - \frac{1}{m}\frac{Ia}{r_0^2} \\ a\left(1 + \frac{I}{mr_0^2}\right) &= g \\ a &= \frac{g}{1 + \frac{I}{mr_0^2}} \end{aligned} \tag{10.15}$$

When we plug in the numbers given in the statement of the question, we get $a = 1.6 \times 10^{-2}\frac{m}{s^2}$.

You should have noticed how similar this was to the previous Atwood machine problems: we used considerations that let us find the acceleration or angular acceleration, and we also used a relation between the motion of the two objects to find a relation between the angular and linear accelerations. This set of linear equations allowed us to solve for the unknown quantities. Notice that we used, almost without second thought, the relations between linear and angular quantities.

Student Exercises

- For the problem done above, using the given values for m, r_0, and I, find the tension T in the rope. *The tension works out to* $4.892N$.

- Use the tension you calculated in the previous question to calculate the time rate of change of angular momentum (about the axis of the cylinder) for both the disk and the falling mass. *We find that for the disk* $\left|\frac{d\vec{L}}{dt}\right| = 1.9578\frac{kg\ m^2}{s^2}$ *and that for the mass* $\left|\frac{d\vec{L}}{dt}\right| = 0.0032\frac{kg\ m^2}{s^2}$. *Note that the sum of these two components is the same as the net torque on this system. (calculated around the cylinder's axis)*

10.5 Questions

- Two masses $m = 0.5kg$ are attached to a rod of variable length. The masses rotate attached to the rod and initially spin around their center of mass at a rate $\omega = 10\frac{1}{s}$. The length of the rod is initally $0.4m$, but it changes to $0.14m$.
  - What is the new moment of inertia I_{new} of the dumbbell?
  - What is the new angular speed of the rotating dumbbell?
- Consider the Atwood machine illustrated in figure 10.7. Assume that $m_1 = 1kg$, $m_2 = 2kg$, $I = 10kgm^2$, $R = 0.3m$, and then
  - Find the magnitude of the mass's acceleration.
  - Find the magnitude of the angular acceleration.
  - Find the two tensions T_1 and T_2 in the rope holding the masses up. Are they the same?
- Consider the situation illustrated in figure 10.8. If the stick is released from rest, and the ball is not attached to the stick, what is the ratio of the accelerations of the ball and the end of the stick? (Does the ball stay at the end of the stick as they both fall?)

10.6 Answers

- For the change in configuration of the dumbbell we find
  - The new moment of inertia is $I_{new} = 4.9 \times 10^{-3} kg\ m^2$.
  - The new angular speed is $\omega_{new} = 81.6\frac{1}{s}$.
- For the Atwood machine illustrated we find:
  - The magnitude of the mass's acceleration is $8.59 \times 10^{-2}\frac{m}{s^2}$.

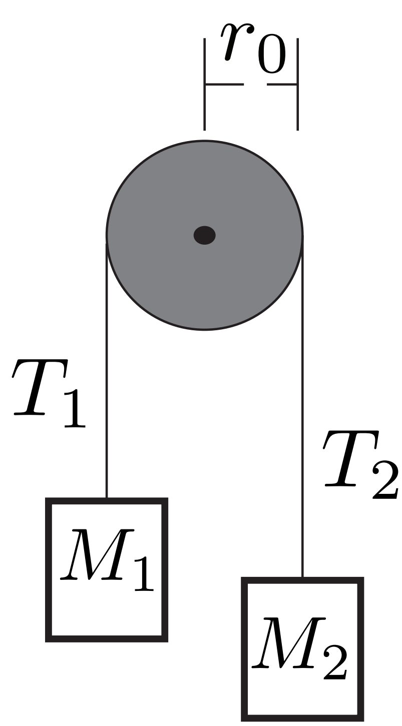


Figure 10.7: A mass m_1 and another mass m_2 are attached to each other via a rope which passes over a disk of moment of inertia I and radius R.

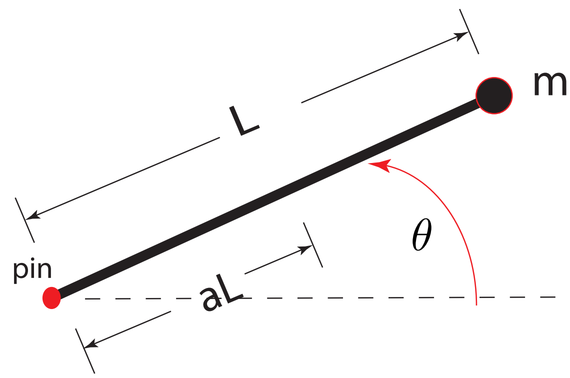


Figure 10.8: A ball of mass m is placed on the end of a stick of length L. The other end of the stick is a pivot point. The moment of inertia of the stick is $I = \frac{1}{3}ML^2$, and the mass is M.

  - The magnitude of the disk's angular acceleration is $0.286\frac{1}{s^2}$.
  - The two tensions are $T_1 = 9.88N$ and $T_2 = 19.4N$. Note that they cannot be the same, otherwise there would be no torque on the disk.

- The angular acceleration of the end of the stick (assuming it is horizontal) is $\frac{3}{2}\frac{g}{L}$. This means that the end of the stick will fall faster than the ball.

Chapter 11

Work and Kinetic Energy

11.1 Summary

Read the eleventh chapter of the text. What we have focussed on so far is the application of laws relating forces: what the requirements for rotational and translational equilibrium are; what accelerations (angular or linear) are produced by net torques and net forces; the mechanics of the forces produced by various mechanisms; and how motion responds to different forces. What we are going to do now is try and develop a way of evaluating motion where the details of exactly what force is exerted when does not matter, or is not what we are interested in. In some ways, this is similar to momentum, in that the details of the force are not what is important; rather the 'net' effect, as determined by an integral.

Some of the key points are:

- The work done by a constant force on an object that undergoes a displacement $\Delta\vec{r}$ is defined by

$$W = \vec{F} \cdot \Delta\vec{r} \tag{11.1}$$

  W can be positive, negative, or zero. The units of W are the Joule [J]; $1[J] = 1\left[kg\frac{m^2}{s^2}\right]$.

- The work done by a non-constant force on an object that undergoes a displacement can be calculated by breaking up the path the object follows into a large number of little displacements (which we call $d\vec{r}$) in such a way that we can *approximate* the force as constant over that tiny displacement. We add up the word done in each displacement to get the total work; this is the same as saying that $W = \int_{starting}^{end} \vec{F} \cdot d\vec{r}$.

- Newton's second law, combined with what we know about the relationship between acceleration and velocity, tells us that if work is done the speed of an object will change.

- The relation between net work and speed is given by $W_{net} = KE_{final} - KE_{initial}$.

- The quantity denoted KE is called 'Kinetic Energy' and defined, for a point particle, as $KE = \frac{1}{2}m\,|\vec{v}|^2$.

- For an extended object, we can imagine dividing it into a large number of small pieces and finding the Kinetic Energy for each piece and then adding the energies together. For an extended object that is not rotating, this will just give $\frac{1}{2}M\,|\vec{v}_{cm}|^2$ where $\vec{v}_{cm}$ is the center of mass velocity.

- For a rotating extended object the kinetic energy is $KE = \frac{1}{2}I\omega^2$, where ω is the rate of rotation and I is the moment of inertia. This is strictly true only for the rotation around one of the 'principle axes', however in this course we will always be considering only that case.

- Forces for which the work done being does not depend on the path taken are called 'conservative' forces, while forces for which the work done does depend on the path taken between the end-points are called 'non-conservative' forces.

- The reason to use work and energy considerations is to simplify the calculations; sometimes the details of how the force is applied, or what the path taken are not relevant, or would lead to things that are calculationally harder.

- In 'elastic' collisions the final kinetic energy is the same as the initial kinetic energy.

11.2 Work done along different paths

Example A mass m is moved over a horizontal surface with which it has a coefficient of kinetic friction μ_k. Assuming that the mass is $m = 10kg$ and $\mu_k = 0.5$, what is the work done by friction in the following cases:

- The mass is moved from $1m\hat{\imath}$ to $4m\hat{\imath}$ along a straight line.

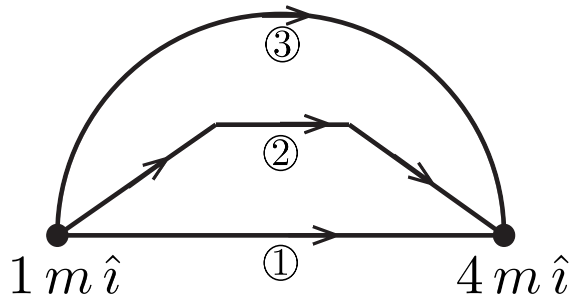


Figure 11.1: Three different paths from $1m\hat{\imath}$ to $4m\hat{\imath}$.

- The mass is moved from $1m\hat{\imath}$ to $2m\hat{\imath} + 1m\hat{\jmath}$ and then to $3m\hat{\imath} + 1m\hat{\jmath}$ and from there to $4m\hat{\imath}$.

- The mass is moved from $1m\hat{\imath}$ to $4m\hat{\imath}$ along the path given by

$$\vec{r}(t) = \left(2.5m - 1.5m\cos(\pi s^{-1}t\right)\hat{\imath} + 1.5m\sin(\pi s^{-1}t)\hat{\jmath} \tag{11.2}$$

  in the time between $t = 0s$ and $t = 1s$.

These three paths are shown in figure 11.1. Note that the paths all have the same starting and ending points.

Worked Solution The critical part of this is to think through what the work done by the friction force means. The force of kinetic friction is always in the opposite direction to the motion. We will use that here.

The first thing to figure out is the magnitude of the force of friction. We could draw a free-body diagram, but we have done this particular case enough before. A box is on a horizontal surface, and is not accelerating in the vertical direction. This means that the downward force of gravity is exactly counteracted by the upwards normal force, so the magnitude of the normal force is $\left|\vec{F}_n\right| = mg$, and hence the magnitude of the force of friction is $\mu_k \left|\vec{F}_n\right| = \mu_k mg$.

Now, we are in a position to calculate the work done along the straight-line path from $1m\hat{\imath}$ to $4m\hat{\imath}$. Since the force is constant we know that the work done is

$$
\begin{aligned}
W_{fric} &= \vec{F}_f \cdot \Delta\vec{r} \\
&= \vec{F}_f \cdot (\vec{r}_f - \vec{r}_i) \\
&= \vec{F}_f \cdot (4m\hat{\imath} - 1m\hat{\imath}) \qquad (11.3)
\end{aligned}
$$

We can find the *direction* of the force of friction by finding the unit vector in the opposite direction to that piece of displacement. In this case, the vector describing the displacement is $\Delta\vec{r} = 3m\hat{\imath}$, and the unit vector in this direction is $\frac{\Delta\vec{r}}{|\Delta\vec{r}|} = \frac{3m\hat{\imath}}{|3m\hat{\imath}|} = \frac{3m\hat{\imath}}{3m} = \hat{\imath}$. To get the unit vector in the *opposite* direction, you multiply by -1, since negative signs tell us about direction. For this displacement, $\vec{F}_f = (\mu_k mg)(-\hat{\imath})$, so we can write the work as

$$
\begin{aligned}
W_{frice} &= \vec{F}_f \cdot \Delta\vec{r} \\
&= \left(\left|\vec{F}_f\right|\right)\left(-\frac{\Delta\vec{r}}{|\Delta\vec{r}|}\right) \cdot \Delta\vec{r} \\
&= -\mu_k mg\,|\Delta\vec{r}| \\
&= -(0.5)(10kg)\left(9.8\frac{N}{kg}\right)(3m) = -147J \qquad (11.4)
\end{aligned}
$$

The fundamental result that we got here: that kinetic friction will do work equal to $-\mu_k mg\,|\Delta\vec{r}|$ for straight-line displacements over a horizontal rough surface. This will be useful in subsequent calculations for other parts of this problem.

Now, we calculate the word done on the second path. The key thing to remember here is that the definition we have of work as $W = \vec{F} \cdot \Delta\vec{r}$ only works for a constant force. Constant means both direction and magnitude, so there are three constant forces at work in the second path; one for each of the three segments.

We find the displacements. For $1m\hat{\imath}$ to $2m\hat{\imath} + 1m\hat{\jmath}$, $\Delta\vec{r} = 1m\hat{\imath} + 1m\hat{\jmath}$, and $|\Delta\vec{r}| = \sqrt{2}m$; for the $2m\hat{\imath} + 1m\hat{\jmath}$ to $3m\hat{\imath} + 1m\hat{\jmath}$ segment, $\Delta\vec{r} = 1m\hat{\imath}$, so $|\Delta\vec{r}| = 1m$; and for the $3m\hat{\imath} + 1m\hat{\jmath}$ to $4m\hat{\imath}$ segment, $\Delta\vec{r} = 1m\hat{\imath} - 1m\hat{\jmath}$, and $|\Delta\vec{r}| = \sqrt{2}m$.

We can now calculate the work done by friction in the three steps as

$$\begin{aligned}
W_{fric} &= W_{step\ 1} + W_{step\ 2} + W_{step\ 3} \\
&= -\mu_k mg\,|\Delta\vec{r}|_{step\ 1} - \mu_k mg\,|\Delta\vec{r}|_{step\ 2} - \mu_k mg\,|\Delta\vec{r}|_{step\ 3} \\
&= -\mu_k mg\left(|\Delta\vec{r}|_{step\ 1} + |\Delta\vec{r}|_{step\ 2} + |\Delta\vec{r}|_{step\ 3}\right) \\
&= -(0.5)(10kg)\left(9.8\frac{N}{kg}\right)(1.41m + 1m + 1.41m) = -188J
\end{aligned} \tag{11.5}$$

Finally, we will calculate the work done following third (curved) path. This is a bit harder, since the path does not obviously split up into discrete chunks. What we do is take the path we know, break it up into a bunch of tiny segments, calculate the work done by friction in each of them, and add them up. The key idea is that if the displacement is small, the force of friction is *approximately* constant in direction. We know that the position is given by $\vec{r}(t)$, and that in the time between t and $t + dt$ the object will have a displacement of $\vec{v}dt$ (remember that this is the same as saying that $\frac{d}{dt}\vec{r} = \vec{v}$). Then, using the rule that the work done is $-\mu_k mg\,|\Delta r|$ in a small step, this means that dW (the small amount of work done in a small time interval) is going to be $-\mu_k mg\,|\vec{v}dt| = -\mu_k mg\,|\vec{v}|\,dt$. Now, we just have to calculate what $|\vec{v}|$ is:

$$\begin{aligned}
\vec{v} &= \frac{d}{dt}\vec{r}(t) \\
&= \frac{d}{dt}\left(\left(2.5m - 1.5m\cos(\pi s^{-1}t)\right)\hat{\imath} + 1.5m\sin(\pi s^{-1}t)\hat{\jmath}\right) \\
&= 1.5\pi\frac{m}{s}\sin(\pi s^{-1}t)\hat{\imath} - 1.5\pi\frac{m}{s}\cos(\pi s^{-1}t)\hat{\jmath} \\
|\vec{v}| &= \left|1.5\pi\frac{m}{s}\sin(\pi s^{-1}t)\hat{\imath} - 1.5\pi\frac{m}{s}\cos(\pi s^{-1}t)\hat{\jmath}\right| \\
&= \sqrt{\left(1.5\pi\frac{m}{s}\sin(\pi s^{-1}t)\right)^2 + \left(-1.5\pi\frac{m}{s}\cos(\pi s^{-1}t)\right)^2} \\
&= 1.5\pi\frac{m}{s}
\end{aligned} \tag{11.6}$$

This tells us that $dW = -\mu_k mg\left(1.5\pi\frac{m}{s}\right)dt$. This is the amount of work done in time interval dt. We have to add up all the appropriate dWs (the ones that make up part of our time interval for going between the two endpoints).

$$\begin{aligned} W &= \int_{start}^{stop} dW \\ &= \int_{0s}^{1s} -\mu_k mg \left(1.5\pi \frac{m}{s}\right) dt \\ &= -\mu_k mg 1.5\pi \frac{m}{s} \int_{0s}^{1s} dt \\ &= -(0.5)(10kg)\left(9.8\frac{N}{m}\right) 1.5\pi \frac{m}{s}\,(1s) = -231J \end{aligned} \tag{11.7}$$

The different paths resulted in different amounts of work done. This tells us that friction is not a conservative force.

Things you should notice here

- The work done by friction depended on the path taken. Since the work done depended on the path taken, this tells us that kinetic friction is a non-conservative force.
- We calculated work done for a non-straight-line path by adding up the work done along each little straight bit that made up the path.
- In the case of a curve which was 'parametrized' we ended up calculating its length by finding the length of each 'tiny bit' that occurred between some time t and another time $t + dt$.

Student Exercises

- Find the work done by gravity moving a box of mass $m = 3kg$ from $1m\hat{\imath}$ to $4m\hat{\imath} + 3m\hat{k}$ along the paths specified
  - A straight line from $1m\hat{\imath}$ to $4m\hat{\imath}+3m\hat{k}$. *The work done is* $-88.2J$.
  - A horizontal line from $1m\hat{\imath}$ to $4m\hat{\imath}$, and then a vertical line from $4m\hat{\imath}$ to $4m\hat{\imath} + 3m\hat{k}$. *The work done in the first step is* $0J$, *and in the second step it is* $-88.2J$, *so the total is* $-88.2J$.
  - Along the path $x\hat{\imath} + \left(x - 1m + 2m\sin(x\pi m^{-1})\right)\hat{k}$ from $x = 1m$ to $x = 4m$. *There are three terms you have to integrate. The* $x\hat{\imath}$ *term the integrand* $\vec{F} \cdot d\vec{r}$ *vanishes. The* $(x - 1m)\hat{k}$ *term will give an integral of* $-88.2J$, *and the* sin *term has a non-zero indefinite integral, but when you substitute in the bounds of integration the definite integral turns out to be* $0J$. *The sum is* $-88.2J$.

Are the results the same or different? What does this tell you about whether the force of gravity is a conservative force? *The results are all the same. Gravity is conservative.*

- Find the work done by the electric force on a charge q which is moved in a region with a constant electric field $\vec{E}$:
  - If $q = 1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved in a straight line from the origin to $1m\hat{\imath} + 2m\hat{\jmath}$.
  - If $q = 1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved from the origin to $2m\hat{\jmath}$, and then in a straight line to $1m\hat{\imath} + 2m\hat{\jmath}$.
  - If $q = -1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved from the origin to $2m\hat{\jmath}$, and then in a straight line to $1m\hat{\imath} + 2m\hat{\jmath}$.

  Can you tell if the electric force is conservative or not from this?

11.3 Block sliding on a rough slope

Example A block of mass m is sliding with initial speed v_i along a rough surface with which it has a coefficient of kinetic friction of μ_k. The surface makes an angle of θ with respect to the horizontal. This is shown in figure 11.2. If $m = 3kg$, $\mu_k = 0.4$, $\theta = 20°$, $v_i = 5\frac{m}{s}$, how far along the surface will the mass have gone when it has a speed of $v_f = 2\frac{m}{s}$? In other words, for what d does $v_f = 2\frac{m}{s}$?

Worked Solution We are going to attack this problem with the work-energy theorem. This tells us that

$$\begin{aligned} \Delta KE &= W_{net} \\ \frac{1}{2}m\left|\vec{v}_f\right|^2 - \frac{1}{2}m\left|\vec{v}_i\right|^2 &= \vec{F}_{net} \cdot \Delta\vec{r} \end{aligned} \tag{11.8}$$

Now, what we have to do is figure out the things that make up the net force, and express the displacement as an appropriate vector. In this step, we have used a piece of intution - that all forces on the block are constant. We would have to calculate the work as an integral if the forces were varying.

The free-body diagram for this mass shown in figure 11.3. There are a total of three forces: gravity, the normal force, and the friction force. Using the $\hat{p}$, $\hat{n}$ coordinate system illustrated in figure 11.3 we can express

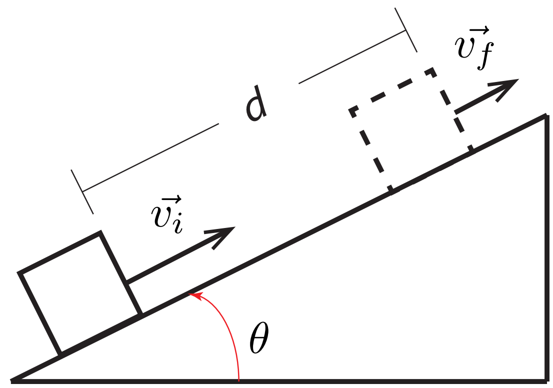


Figure 11.2: A block of mass m is sliding along a rough surface which makes an angle of θ with the horizontal. The initial state, and the final state where the block is travelling at speed v_f while being a distance d farther along the slope are illustrated.

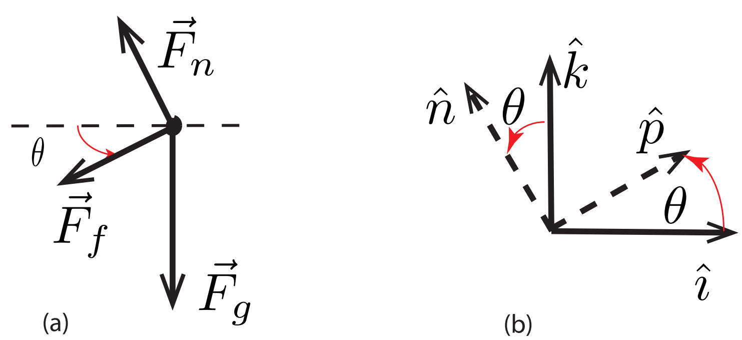


Figure 11.3: Part (a) shows the free-body diagram for the mass moving as desribed in figure 11.2. Part (b) shows the $\hat{p}$, $\hat{n}$ coordinate system we are using for simplicity here.

the individual forces as

$$\begin{aligned}
\vec{F}_n &= \left|\vec{F}_n\right| \hat{n} \\
\vec{F}_f &= -\left|\vec{F}_f\right| \hat{p} \\
\vec{F}_g &= -mg\cos\theta\hat{n} - mg\sin\theta\hat{p} \\
\vec{F}_{net} &= \vec{F}_n + \vec{F}_f + \vec{F}_g \\
&= \left(\left|\vec{F}_n\right| - mg\cos\theta\right)\hat{n} + \left(-mg\sin\theta - \left|\vec{F}_f\right|\right)\hat{p} \qquad (11.9)
\end{aligned}$$

As usual, the condition that the mass is not accelerating in the $\hat{n}$ direction tells us that $\left|\vec{F}_n\right| = mg\cos\theta$. The net force simplifies as $\vec{F}_{net} = -\left(mg\sin\theta + \mu_k mg\cos\theta\right)\hat{p}$. We have used the usual relationship between the normal force and the force of kinetic friction here.

The mass's displacement up the slope is $d\hat{p}$.

This means that the total work is

$$\begin{aligned}
\frac{1}{2}m\left|\vec{v}_f\right|^2 - \frac{1}{2}m\left|\vec{v}_i\right|^2 &= \vec{F}_{net}\cdot\Delta\vec{r} \\
&= \left(-\left(mg\sin\theta + \mu_k mg\cos\theta\right)\hat{p}\right)\cdot\left(d\hat{p}\right) \\
&= -mgd\left(\sin\theta + \mu_k\cos\theta\right) \\
v_f^2 - v_i^2 &= -2gd\left(\sin\theta + \mu_k\cos\theta\right) \qquad (11.10)
\end{aligned}$$

Here we have used the notation $|\vec{v}| = v$ (the magnitude of the velocity is the speed).

Using the known values of $v_f = 2\frac{m}{s}$, $v_i = 5\frac{m}{s}$, $\mu_k = 20°$, and $\theta = 20°$, we find that $d = 1.5m$.

Things to notice in this problem: Here, we could have gotten the work done by each individual force (note that the work would have been 0 for the normal force, and would have been negative for gravity and for friction). Even had we done this, we still would have needed to use the second law analysis to get the magnitude of the force of friction. Also note that this result tells us what would happen for something going horizontally on a rough surface (so $\theta = 0$) or straight up (so that $\theta = 90°$)

Student Exercises The problem of blocks sliding on a plane is a rich one. Here are some similar questions:

- Consider the system described in figure 11.2. If $\mu_k = 0$, $\theta = 30°$, and $v_i = 4\frac{m}{s}$, what is the maximum distance along the slope the block will

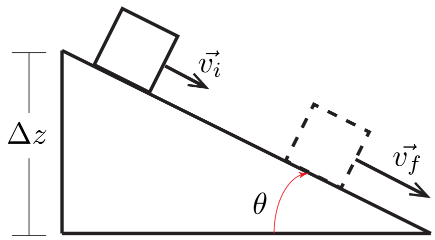


Figure 11.4: A block of mass m slides along a rough surface with which it has a coefficient of kinetic friction of μ_k. The surface makes an angle of θ below the horizontal.

travel? *It will travel along the slope until it stops, when its velocity will vanish. The maximum distance is* $1.63m$.

- Consider the system shown in figure 11.4. If $m = 3kg$, $v_i = 4\frac{m}{s}$, $\mu_k = 0.2$, and $\theta = 30°$, what is the speed of the block when it has reduced its height above the ground by $2m$? *It will be travelling at* $6.45\frac{m}{s}$.

- In the system shown in figure 11.4, for $m = 3kg$, and $\theta = 30°$, what value of μ_k is such that the speed of the block remains constant as it slides down? *The required coefficient of kinetic friction if* $\mu_k = 0.577$.

11.4 A mass in a loop-the-loop

Example A ball of mass m slides down a frictionless path and enters a loop of radius R. If the initial height of the ball was h, and the ball starts from rest, what is the normal force on the ball by the track when the angle from the center of the circle to the ball is θ? This is illustrated in figure 11.5.

If $m = 0.1kg$, $R = 0.2m$, $h = 0.6m$, and $\theta = 30°$, find the numerical value of the normal force.

Worked Solution Our strategy for this example is to use what we know about the work-energy theorem to find the speed of the ball at the point specified. From this, we can find the needed magnitude of the perpendicular component of the total force. This total force is supplied by the normal force and by the force of gravity; we can find the centripetal component of the

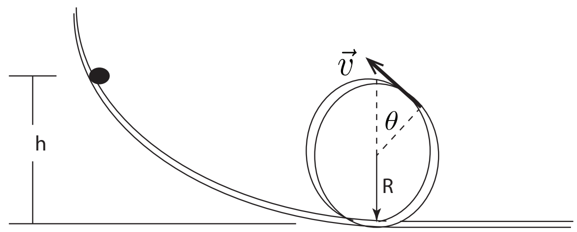


Figure 11.5: A ball slides down from initial height h and enters a circle of radius R along a frictionless track.

force of gravity, and the rest is going to have to be supplied by the normal force.

We need to do a little geometry first. By looking at figure 11.5 we see that the z-component of the initial position of the ball is h, and the z-component of the final position of the ball is $R + R\cos\theta$. The calculation that you did in a previous example should have convinced you that the work done by the force of gravity is easy to calculate. We find that

$$\begin{aligned}
\Delta KE &= W_{net} \\
\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= W_{normal} + W_{grav} \\
&= 0 + \vec{F}_g \cdot \Delta\vec{r} \\
&= \left(-mg\hat{k}\right) \cdot \left(\text{doesn't matter } \hat{\imath} + (R + R\cos\theta - h)\,\hat{k}\right) \\
&= -mg\,(R(1+\cos\theta) - h) \qquad (11.11)
\end{aligned}$$

Assuming that the initial speed is zero, we get that

$$v_f^2 = 2mgh - 2mgR\,(1 + \cos\theta) \qquad (11.12)$$

It's worth talking about how we got the two works that went into the net work: The work done by gravity is the simple one since gravity was a constant force. For a constant force we know that $W = \vec{F}\cdot\Delta r$; we didn't care about the x-component of Δr because we knew (or can easily verify) that changing the x-component doesn't change the value of the work done. The work done by the normal force must, in principle, be calculated by an integral because the direction of the normal force changes: $W = \int \vec{F} \cdot d\vec{r}$ *however* this can be simplified a lot: remember that this integral really means 'break

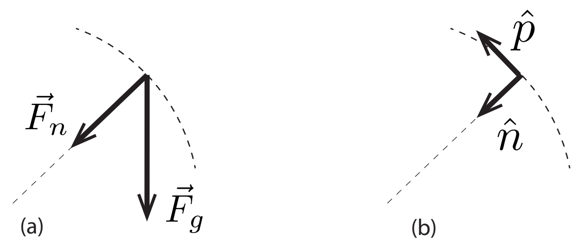


Figure 11.6: In part (a) there is a free body diagram for the ball on the frictionless track from 11.5. In part (b) we see a convenient coordinate system: $\hat{n}$ points to the center of the circle, perpendicular to the track, and $\hat{p}$ point along the direction of motion of the ball.

the path into a whole bunch of tiny steps, each with a constant force, and add up the tiny amount of work for each tiny step.' This saves us, because the normal force is *for a small step* perpendicular to that small step, so $\vec{F} \cdot d\vec{r} = 0$ for each $d\vec{r}$; and so $W = 0$ since we are adding up a whole lot of 0s.

From the relationship in equation 11.11 we can figure out the speed that the ball is moving at this point. We will draw a free-body diagram of the ball at this point, displaced by θ from the top in figure 11.6. We can analyze the free-body diagram and find that

$$\begin{aligned} \vec{F}_{net} &= \vec{F}_g + \vec{F}_n \\ m\frac{v^2}{R}\hat{n} + \text{something } \hat{p} &= [mg\cos(\theta)\hat{n} - mg\sin(\theta)\hat{p}] + \left|\vec{F}_n\right|\hat{n} \quad (11.13) \end{aligned}$$

The $\hat{n}$ component of the *net* force *must* be $m\frac{v^2}{R}$ since the ball is moving in a circle. The $\hat{p}$ component of the net force tells us something about what is happening to the speed; we do not care for the purposes of this question what it is, since we only want to obtain the magnitude of the normal force, but notice that it's negative. The ball is slowing down because the ball is going up which should seem reasonable.

The $\hat{n}$ component of equation 11.13 is

$$m\frac{v^2}{R} = \left(mg\cos\theta + \left|\vec{F}_n\right|\right) \tag{11.14}$$

and into this we can put the result of equation 11.12 combined the identification between v and v_f. This means that

$$\begin{aligned}
m\frac{v^2}{R} &= \left(mg\cos\theta + \left|\vec{F}_n\right|\right) \\
-\frac{2mg\left(R(1+\cos\theta)-h\right)}{R} &= \left(mg\cos\theta + \left|\vec{F}_n\right|\right) \\
\left|\vec{F}_n\right| &= -2mg(1+\cos\theta) + \frac{2mgh}{R} - mg\cos\theta \\
&= -2mg - 3mg\cos\theta + \frac{2mgh}{R} \qquad (11.15)
\end{aligned}$$

Using the given numbers $m = 0.1kg$, $R = 0.2m$, $h = 0.6m$, and $\theta = 30°$ we can calculate that $\left|\vec{F}_n\right| = 1.37N$.

What you should have picked up: This is another problem where we have to put together a number of ideas to get a result. A couple pieces of culture you should be noticing are that:

- We do things in terms of variables, not numbers, as much as reasonably possible.
- We typically string together a number of concepts in a single question.

Another thing, critical in this case, is that the we use the work-energy theorem in a place where constant acceleration motion isn't appropriate. The forces are not all constant; we do not have uniform circular motion either. We can use Work to account for the net effect of all the existing forces on the speed of the mass.

Student Exercises

- Consider the system shown in figure 11.5 and described above.
  - Find the initial height (in terms of R) so that the normal force is exactly zero when $\theta = 0$. *This is a classic result, that the required h (assuming that the mass is sliding, and that the system is frictionless) is $h = \frac{5}{2}R$.*
  - What would the magnitude of the normal force have to be if h was smaller? What would that mean? *If h were smaller the normal force would have to be 'negative'; really what would happen is that the* component *of the force in the $\hat{n}$ direction would be negative, which would mean that the force was directed into the track.*

11.5 A position-varying force

Example An ideal spring with spring constant k is initially stretched a distance l_i from its equilibrium length. How much work will the spring do while it is stretched more so that it ends up stretched l_f from the equilibrium length?

Worked Solution We start by setting up the problem in figure 11.7. In this figure we see the spring stretched by an amount x from equilibrium so that the force the spring is exerting is $-kx\hat{\imath}$ (it is pulling back towards its equilibrium length). Suppose that we were going to stretch the spring a tiny bit more, so that the end is moved by $dx\hat{\imath}$ (dx is very small, so that the force is approximately constant.) In this case, the tiny amount of work done $dW = \vec{F} \cdot \Delta\vec{r} = (-kx\hat{\imath}) \cdot (dx\hat{\imath}) = -kxdx$.

Suppose now that (as in the question) we want to find out how much work the spring does going from some initial length l_1 to some final length l_2. We start by breaking up the displacement up into a whole bunch of tiny steps, each of length dx. Then, if the spring is at some intermediate length (say, x) we just figured out how much work is done in that tiny step: $dW = -kxdx$ (how convenient that we figured it out before). Now, we need to *add up* the dW for all values of x:

$$\begin{aligned} W_{net} &= \int_{starting}^{ending} dW \\ &= \int_{x=l_1}^{l_2} dW \\ &= \int_{x=l_1}^{l_2} (-kxdx) \\ &= -k \int_{x=l_1}^{x_2} xdx \\ &= -\frac{k}{2}\, x^2\Big|_{l_1}^{l_2} \\ &= -\frac{k}{2}l_2^2 + \frac{k}{2}l_1^2 \end{aligned} \tag{11.16}$$

and there we have it: the work is related to the change in the square of how much the spring is compressed or stretched.

What you should notice: The key thing we did in this particular example was really break down how you get the work done by a changing force:

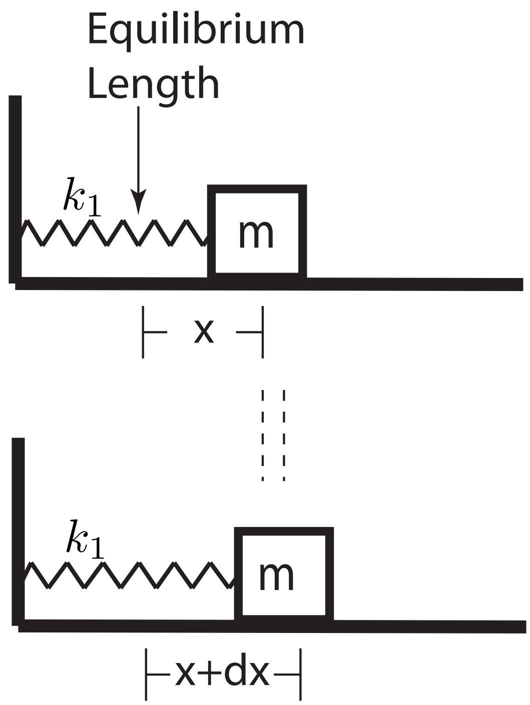


Figure 11.7: A spring of spring constant k being stretched. We show the spring where x measure the distance of the end of the spring from its equilibrium length, if $x > 0$ the spring is stretched, and if $x < 0$ the spring is compressed. We have chosen the $\hat{\imath}$ direction as along the spring. We also show the length dx of a small amount the spring is stretched.

you find the work done in a small step, and add it up for the different small steps that make the large displacement we actually care about. In this case, what we had to do is find the amount of work for the step between being stretched by x and $x + dx$ and then added this up for all values of x that were appropriate.

Student Exercises

- Suppose the spring is initially compressed by l_1 and is further compressed by l_2 (with $l_2 > l_1$). How much work is done in this process? *The answer is* $-\frac{k}{2}l_2^2 + \frac{k}{2}l_1^2$.

- How much work is done by an ideal spring of spring constant k during a process where it starts at its equilibrium length and is compressed so that it is Δx shorter than the equilibruim length? Find a numerical answer for $k = 40\frac{N}{m}$ and $\Delta x = 0.2m$. *In this case the spring has done* $-0.8J$ *of work.*

11.6 Falling onto a spring

Example A ball of mass m is released from rest a distance h above a spring with spring constant k. When the ball hits the spring, the spring compresses. What is the maximum compression of the spring if $m = 1.5kg$, $d = 3m$, and $k = 500\frac{N}{m}$? The situation is illustrated in figure 11.8.

Worked Solution Our approach for this is going to be to use the work-energy theorem (again!). Our idea is that the ball starts with zero kinetic energy. As it falls gravity does work (positive) on it (making it speed up) until it hits the top of the spring. As the spring compresses the spring does (negative) work on the ball (which makes it slow down) and gravity does more work (still positive) making it speed up. The net effect of the spring and gravity is to slow the ball down finally to a stop.

We will look at a diagram of the situation in figure 11.9. We express the work-energy theorem for the whole process of the ball falling:

$$\begin{aligned} \Delta KE &= W_{net} \\ \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= W_g + W_{spring\ before\ hit} + W_{spring\ after\ hit} \end{aligned} \tag{11.17}$$

In this, since for our case the ball started and ended at rest, we know that $v_i = 0$ and that $v_f = 0$. Since the ball has fallen, we know that the total

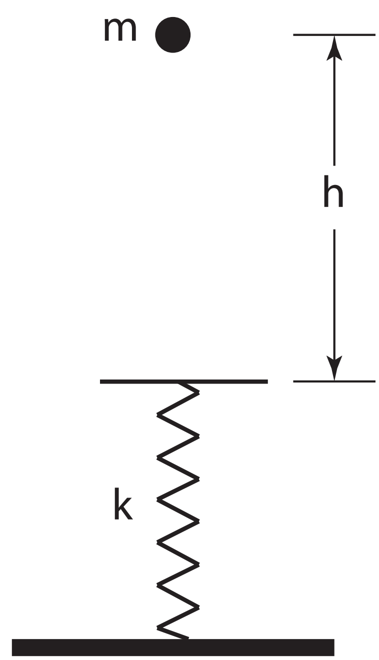


Figure 11.8: A ball of mass m is dropped from height h onto a spring of spring constant k.

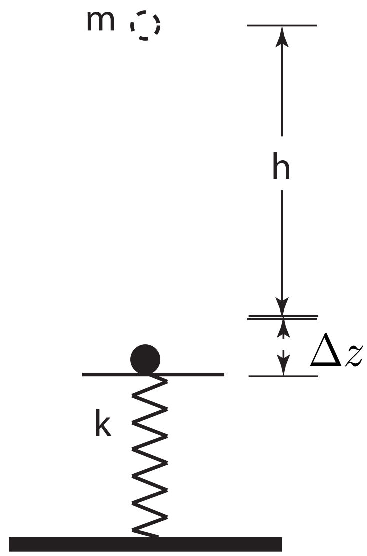


Figure 11.9: The distance h that the ball falls prior to hitting the top of the spring, and Δz is the amount that the spring is compressed.

displacement of it is $\Delta\vec{r} = -(h + \Delta z)\hat{k}$. We also know that the spring has been compressed by Δz – fortunately, we did the calculation just above to figure out how much work was done by a spring in being compressed.

$$\begin{aligned}
\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= W_g + W_{spring\ before\ hit} + W_{spring\ after\ hit} \\
0 - 0 &= \vec{F}_g \cdot \Delta\vec{r} + 0 + W_{spring} \\
0 &= \left(-mg\hat{k}\right) \cdot \left(-(h + \Delta z)\hat{k}\right) + \left(-\frac{1}{2}k\Delta z^2\right) \\
0 &= mgh + mg\Delta z - \frac{1}{2}k\Delta z^2 \qquad (11.18)
\end{aligned}$$

The work done by the spring before the ball hit it is zero because before the ball hit the spring the end of the spring had a displacement of 0. The work done after the spring hit was the result from one of the student exercises (with $l_1 = 0$ and $l_2 = \Delta z$).

The result in 11.18 is a quadratic expression for Δz, and we can solve it and find that

$$\begin{aligned}
\Delta z &= \frac{-mg \pm \sqrt{(mg)^2 - 4(mgh)\left(-\frac{1}{2}k\right)}}{2(\left(-\frac{1}{2}k\right)} \\
&= \frac{mg}{k} \mp \sqrt{\left(\frac{mg}{k}\right)^2 + 2\frac{mgh}{k}} \qquad (11.19)
\end{aligned}$$

Putting in the numbers, we have $m = 1.5kg$, $d = 3m$, and $k = 500\frac{N}{m}$. This means that $\Delta z = 4.5 \times 10^{-1}m$ or $-3.9 \times 10^{-1}m$. We discard the negative solution because that would mean that the spring had stretched up towards the ball as it was falling – it's non-physical. Had the ball *attached* itself to the end of the spring what we would see is the ball oscillate between those two positions.

Some things to notice about the solution The first thing to notice here is that we have dealt with the motion using the work-energy theorem. We had to be careful about the total work done by gravity and the work done by the spring. The other part is that our result, we had two numerically acceptable answers, however only one was what we wanted. This is a common situation: you have to interpret the solution to see if the *number* you get makes sense.

Student Exercises

- A block of mass $m = 1.0kg$ slides along a rough horizontal surface with which it has coefficient of kinetic friction $\mu_k = 0.2$, until it hits a spring of spring constant $k = 500\frac{N}{m}$. If the block initially had speed $v = 5.0\frac{m}{s}$ when it was $d = 2.0m$ from the end of the spring, by how much is the spring compressed when the block comes to a stop? *The block stops when the spring is compressed by* $0.18m$.

- Assuming that $\mu_s = \mu_k$, does the mass stay at rest when it stops? *Work out the net force on it at that compression. It cannot be in equilibrium.*

- What changes to the set-up above would result in the mass staying at rest? *We will not spell out the answer, but think about what effect the changes would have on the work done and the final compression of the spring.*

11.7 An elastic collision

Example A ball of mass m_1 initially travels at velocity $\vec{v}_i$ and hits a second ball of mass m_2 which is initially at rest. After the elastic collision, the ball of mass m travels at $90°$ to its original direction. What is the angle the velocity of the second ball makes with the direction defined by $\vec{v}_i$?

If $v_i = 30\frac{m}{s}$, $m_1 = 2kg$, $m_2 = 5kg$ find the numerical value of the angle.

Worked Solution Problems involving elastic collisions are *conceptually* simple, but sometimes they have involved algebra associated with them. The central idea is that there is a set of linear conditions (associated with conservation of momentum) that tell us about the various components of the total momentum (and hence the initial or final momentum) as well as a quadratic condition relating the speeds, which comes from the fact that the kinetic energy is the same before and after the interaction.

We call the two masses 1 and 2 respectively. The two conditions we have are that:

$$\begin{aligned} \vec{p}_{1,i} + \vec{p}_{2,i} &= \vec{p}_{1,f} + \vec{p}_{2,f} \\ KE_{1,i} + KE_{2,i} &= KE_{1,f} + KE_{2,f} \end{aligned} \tag{11.20}$$

Now, we know a few things: mass 1 initially travels at speed v_i, and we might as well call the direction it travels the x-direction, so the initial velocity of mass 1 is $\vec{v}_{1,i} = v_i\hat{\imath}$. This tells us that $KE_{1,i} = \frac{1}{2}m_1v_i^2$, and that $\vec{p}_{1,i} = m_1v_i\hat{\imath}$.

We were told that the second ball is initially at rest, so $\vec{v}_{2,i} = 0$ and hence $KE_{2,i} = 0$ as well as $\vec{p}_{2,i} = 0$. We were also told that after the collision mass 1 travels at 90° to the original direction - there is not much to go on, but we can call that the y-direction, and call the speed it travels after the collision as v_f, so we have $\vec{v}_{1,f} = v_f\hat{\jmath}$ and $\vec{p}_{1,f} = m_1 v_f \hat{\jmath}$ with $KE_{1,f} = \frac{1}{2}m_1 v_f^2$.

When we combine these observations into the relations in 11.20 we get

$$\begin{aligned} m_1 v_i \hat{\imath} + 0 &= m_1 v_f \hat{\jmath} + m_2 \vec{v}_{2,f} \\ \frac{1}{2}m_1 v_i^2 + 0 &= \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 \left|\vec{v}_{2,f}\right|^2 \end{aligned} \tag{11.21}$$

There are three unknown quantities here: the x and y components of $\vec{v}_{2,f}$, and the speed of mass 1 after the collision v_f. Note that v_i, the initial speed, isn't given as an *number* but it is, in principle, a known quantity.

When we re-arrange the relations we had for the momentum that

$$\begin{aligned} m_1\left(v_i\hat{\imath} - v_f\hat{\jmath}\right) &= m_2\vec{v}_{2,f} \\ \vec{v}_{2,f} &= \frac{m_1}{m_2}\left(v_i\hat{\imath} - v_f\hat{\jmath}\right) \\ \left|\vec{v}_{2,f}\right|^2 &= \left(\frac{m_1}{m_2}\right)^2\left(v_i^2 + v_f^2\right) \end{aligned}$$

We got $\vec{v}_{2,f}$ explicitly in terms of v_i and v_f (the second of which is still unknown) and then we got the magnitude of $\vec{v}_{2,f}$ which will be critical when we find the Kinetic Energy.

Now, we can substitute this into the relations for kinetic energy:

$$\frac{1}{2}m_1 v_i^2 - \frac{1}{2}m_1 v_f^2 = \frac{1}{2}m_2\left|\vec{v}_{2,f}\right|^2 \tag{11.22}$$

$$\begin{aligned} \frac{1}{2}m_1\left(v_i^2 - v_f^2\right) &= \frac{1}{2}m_2\left(\frac{m_1}{m_2}\right)^2\left(v_i^2 + v_f^2\right) \\ \left(v_i^2 - v_f^2\right) &= \frac{m_1}{m_2}\left(v_i^2 + v_f^2\right) \end{aligned}$$

$$v_i^2\left(1 - \frac{m_1}{m_2}\right) = v_f^2\left(1 + \frac{m_1}{m_2}\right) \tag{11.23}$$

At the start, it looks like an application of the work-energy theorem: The reduction in KE for mass 1 supplies the *increase* in KE for the second mass. By the end, we have a relation for v_f in terms only of v_i and the ratio of the masses. We find that $v_f = v_i\sqrt{\frac{m_2 - m_1}{m_2 + m_1}}$.

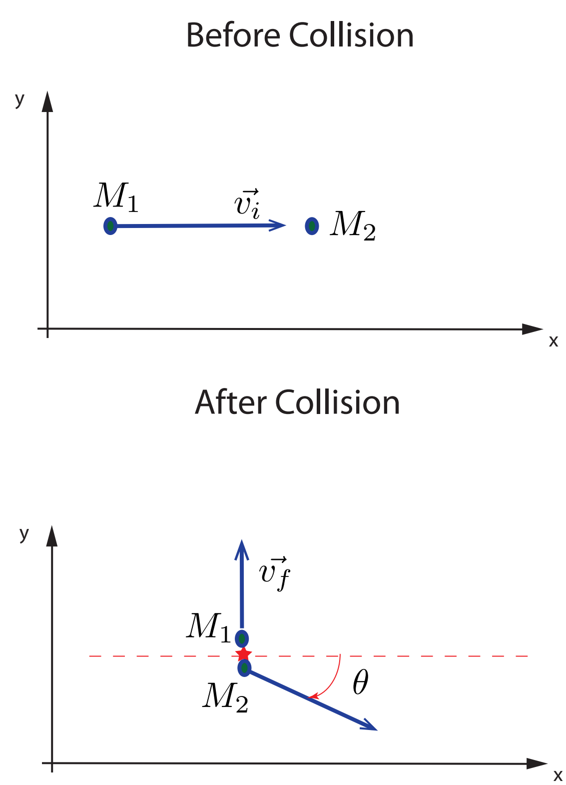


Figure 11.10: The geometry of the velocity $\vec{v}_{2,f}$ including the angle θ with the original direction.

Notice that the math spits out something interesting: this can't be solved if $m_1 > m_2$, which means that we will never have a heavy mass collide with a lighter one and be deflected by 90°.

Finally, we can write the velocity of the second mass:

$$\begin{aligned} m_2\vec{v}_{2,f} &= m_1 v_i \hat{\imath} - m_1 v_f \hat{\jmath} \\ \vec{v}_{2,f} &= \frac{m_1}{m_2}\left(v_i\hat{\imath} - v_i\sqrt{\frac{m_2-m_1}{m_2+m_1}}\hat{\jmath}\right) \end{aligned} \tag{11.24}$$

As can be seen in figure 11.10 the angle θ that $\vec{v}_{2,f}$ naturally makes with the x-axis satisfies $\tan\theta = \sqrt{\frac{m_2-m_1}{m_2+m_1}}$.

For the values we have: $v_i = 30\frac{m}{s}$, $m_1 = 2kg$, $m_2 = 5kg$ this means that $\theta = 23.2°$.

The reason we talked about this: This exercise has a couple of good points: first we had to set up and name variables that we were going to use – this is a valuable skill that shows up in all sorts of other problems. Secondly, it shows the general form of problems with elastic collisions: you use momentum to solve reduce the number of variables, the kinetic energy relation lets you get a numerical value for one, and then you can substitute back to get the other unknown quantities, if needed. There was also a trigonometry reminder hidden in finding the scattering angle.

Student Exercises

- A ball of mass m_1 travelling at $v_1\hat{\imath}$ collides elastically with a ball of mass m_2 travelling at $-v_2\hat{\imath}$. After the collision, the masses move along the x-axis. If $m_1 = 3kg$, $m_2 = 5kg$, $v_1 = 4\frac{m}{s}$, and $v_2 = 1\frac{m}{s}$, find the values of those velocities. *After the collision ball 1 travels at* $-2.25\frac{m}{s}\hat{\imath}$ *and ball 2 travels at* $2.75\frac{m}{s}\hat{\imath}$.

- The two balls in the previous question collide and stick together. What is the fraction of kinetic energy lost in the collision? Find the answer for the numerical values of velocity and mass given above. *After the collision there is* 94.2% *less kinetic energy.*

11.8 Questions

- Find the work done by the electric force on a charge q which is moved in a region with a constant electric field $\vec{E}$:
  - If $q = 1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved in a straight line from the origin to $1m\hat{\imath} + 2m\hat{\jmath}$.
  - If $q = 1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved from the origin to $2m\hat{\jmath}$, and then in a straight line to $1m\hat{\imath} + 2m\hat{\jmath}$.
  - If $q = -1.6 \times 10^{-19}C$, $\vec{E} = 5000\frac{N}{C}\hat{\imath}$, and the charge is moved from the origin to $2m\hat{\jmath}$, and then in a straight line to $1m\hat{\imath} + 2m\hat{\jmath}$.

  Can you tell if the electric force is conservative or not from this?

- Consider the system described in figure 11.2. If $\mu_k = 0$ and $v_i = 6\frac{m}{s}$, what is the velocity of the block when $d = 0m$? *Hints: Are there more than one answer, more than one time this can occur? Can you relate this to constant acceleration motion?*

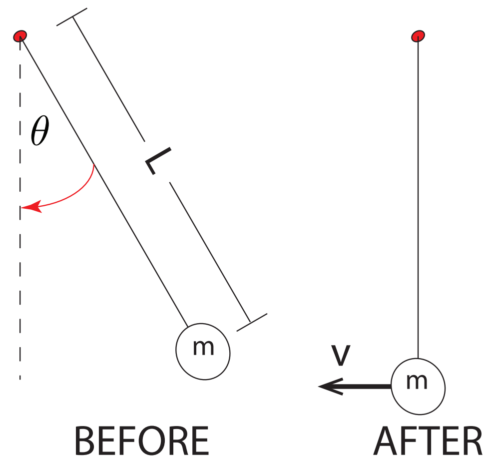


Figure 11.11: A simple pendulum with a bob of mass m and a string of length L.

- Consider a ball of mass m which is held by a string of length L. It is pulled up from the equilibrium point (hanging straight down) and released from rest when the string makes an angle of θ with the vertical, as shown in figure 11.11
  - If $m = 0.4kg$, $\theta = 30°$, and $L = 1.5m$, how much work was done by gravity on the mass between when it was released and when it reached the bottom of the swing?
  - If $m = 0.4kg$, $\theta = 30°$, and $L = 1.5m$, how fast is it going at the bottom of the swing?
  - If $m = 0.4kg$, $\theta = 30°$, and $L = 1.5m$, what is the tension at the bottom of the swing?
- Suppose some exotic material exerted a restoring force if it was compressed or stretched which had magnitude $\left|\vec{F}\right| = a(\Delta x)^3$, where Δx is the amount it is compressed or stretched, and a is a constant. If $a = 1000\frac{N}{m^3}$, and a ball of mass $m = 1kg$ is placed next to the end of this material:
  - What is the magnitude of the acceleration of the ball initially if it was released from rest when $\Delta x = 0.2m$?

- What is the maximum kinetic energy of the ball if it is accelerated from rest by this exotic material from an initial displacement of $\Delta x = 0.2m$?

- A mass $m = 0.5kg$ is resting loosely on top of a spring of spring constant $k = 100\frac{N}{m}$ which is initially compressed by $\delta x = 0.4m$. The ball is released from rest. How high is it when the ball has half the speed it had when the bompression of the spring became 0?

11.9 Answers

- For the constant electic fields and charges listed we find:
  - The work done is $8.0 \times 10^{-16}J$.
  - The work done in the two stages is $0J$ and $8.0 \times 10^{-16}J$.
  - The work done is the two stages is $0J$ and $-8.0 \times 10^{-16}J$.

  The work done (where the charge was the same) was the same for a similar displacement. The electric force is also conservative.

- We find that the *speed* of the block must be $6\frac{m}{s}$ in this frictionless case. The velocity can be either up or down the slope. This is like throwing a ball and finding that the speed is the same when it is at a certain level irrespective of whether it is going up or down.

- For the pendulum shown in figure 11.11 we find
  - The work done by gravity during the swing described is $0.788J$.
  - The mass is moving at $1.98\frac{m}{s}$ at the bottom of the swing.
  - The tension in the rope is $4.97N$ at the bottom of the swing.

- For this question you will need to set up and do an integra.
  - The magnetude of the acceleration for the case given is $8.0\frac{m}{s^2}$.
  - The maximum kinetic energy when accelerated from rest as descrbed is $0.40J$.

- The mass has a speed of $4.9\frac{m}{s}$ at compression $0m$, at which time it detatched from the spring. It is at a height of $0.92m$ when it has half that speed.

Chapter 12

Potential Energy

12.1 Summary

Read the twelfth chapter of the text. It introduces the concept of potential energy in the context (the only one, really) of conservative forces.

- A force is conservative if the work it does only depends on the beginning and ending points of the motion. In most situations we consider, this is the same as saying: 'if the force is only a function of position then the force is conservative.'

- For any conservative force we define the potential energy using the relationship $\Delta PE = -W_c$; the change in potential energy is *the negative* of the work done by that force.

- Even though potential energy is *related to* forces, since it is defined in terms of work done, it is a *scalar*, and NOT a vector.

- Potential energies have an arbitrary offset: the zero point can be defined, and can't be determined unambiguously from the work done by the force.

- The work-energy theorem can be expressed using potential energy: $\Delta KE + \Delta PE = W_{nc}$ where W_{nc} is the work done by non-conservative forces such as friction.

- Dealing with potential energy and the Work done by conservative forces are equivalent; you do one or the other, not both.

- The potential energy of two point masses separated by a distance r is $PE = -G\frac{m_1 m_2}{r}$.
- The potential energy of a mass a distance h from the surface of the Earth is mgh (In this approximation, h is at most a few kilometers)
- The potential energy of an ideal spring which has had its end compressed ($\Delta l < 0$) or stretched ($\Delta l > 0$) from equilibrium length is $\frac{1}{2}k(\Delta l)^2$.
- The potential energy of two point charges separated by a distance r is $\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$.

12.2 Potential Energy for Newtonian Gravity

Example A mass m_1 is held stationary at the origin. Another mass m_2 is originally at $\vec{r}_i$, and is moved to $\vec{r}_f$. What is the change in this mass's potential energy? Find the numerical value using $m_1 = 2 \times 10^{20} kg$, $m_2 = 3 \times 10^{10} kg$, $\vec{r}_i = 5 \times 10^6 m\hat{\imath}$, and $\vec{r}_f = 7 \times 10^6 m\hat{\jmath}$.

Worked Solution Our strategy is going to be to make use of the definition of potential energy: that $\Delta PE = -W_c$. The hard-ish part is going to be to calculate the work done by gravity as an object is moved – the force of gravity changes both direction and magnitude as the mass m_2 is moved. This means that we are going to have to use our techniques for calculating the work done by a varying force in this case, and combined with what we know about the change in potential energy, we can see that:

$$\begin{aligned} \Delta PE &= -W_g \\ &= -\int_{starting\ point}^{ending\ point} \vec{F}_g \cdot d\vec{r} \end{aligned} \tag{12.1}$$

To figure out how to calculate the work using equation 12.1 we need to, as usual, sketch out the problem. This is done in figure 12.1. The tricky part is to determine how to handle the change in the direction of the force of gravity.

In the figure 12.1 we can see the small change in position $d\vec{r}$ broken up into two parts (remember that vectors can always be broken up into orthogonal parts). We write this vector $d\vec{r} = (dr)\hat{r} + rd\theta\hat{\theta}$. In this, dr is the change in the distance of the mass from the origin, measured along a straight line, $d\theta$ is the change in the angle so $rd\theta$ is the change in distance

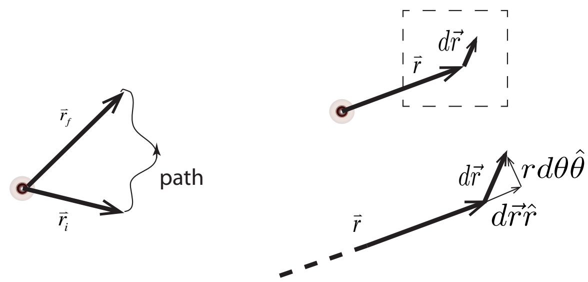


Figure 12.1: On the left we show the starting and ending location of the mass, and a potential path between them. On the right we show and the small change in position $d\vec{r}$ broken up into a radial part and a tangential part.

along a direction at 90° to the line between the two masses, $\hat{r}$ is the unit vector from the origin (mass m_1) to the second mass (m_2), and $\hat{\theta}$ is a unit vector at 90° to $\hat{r}$.

We can find the small amount of work done as the position changes by $d\vec{r}$. At this moment, the distance between m_1 and m_2 is r, which is somewhere between $|\vec{r}_i|$ and $|\vec{r}_f|$. At this point, the magnitude of the force of gravity on mass m_2 is $G\frac{m_1 m_2}{r^2}$ and the direction is $-\hat{r}$ where $\hat{r}$ is the unit vector from m_1 to m_2. Since m_1 is at the origin, we can easily express $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$. This means that

$$
\begin{aligned}
\vec{F} &= -G\frac{m_1 m_2}{r^2}\hat{r} \\
&= -G\frac{m_1 m_2}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|} \\
dW &= \vec{F}\cdot d\vec{r} \\
&= -G\frac{m_1 m_2}{r^2}\hat{r}\cdot\left(dr\hat{r} + rd\theta\hat{\theta}\right) \\
&= -G\frac{m_1 m_2}{r^2}dr \qquad (12.2)
\end{aligned}
$$

The last equality can be seen from what we know about $\hat{r}$ and $\hat{\theta}$: they are unit vectors which are perpendicular to each other.

The remarkable thing about this result is that the motion in the θ direction does not contribute to the work at all. This means that the *only* thing that matters when we calculate the work done by this gravity is the change in radial distance, not the change in angle or the change in the direction of the force between them.

Now we have got:

$$\begin{aligned}
\Delta PE &= -W_g \\
&= -\int_{starting}^{ending} \vec{F} \cdot d\vec{r} \\
&= -\int_{starting}^{ending} -G\frac{m_1 m_2}{r^2} dr \\
&= Gm_1 m_2 \int_{starting}^{ending} \frac{1}{r^2} dr \\
&= Gm_1 m_2 \int_{|\vec{r}_i|}^{|\vec{r}_f|} \frac{1}{r^2} dr \\
&= Gm_1 m_2 \left(-\frac{1}{r}\right)\Bigg|_{|\vec{r}_i|}^{|\vec{r}_f|} \\
&= \left(-G\frac{m_1 m_2}{|\vec{r}_f|}\right) - \left(-G\frac{m_1 m_2}{|\vec{r}_i|}\right) \qquad (12.3)
\end{aligned}$$

Since $\Delta PE = PE_f - PE_i$, this tells us that

$$PE_{grav} = -G\frac{m_1 m_2}{r} + C \qquad (12.4)$$

where r is the separation between the centers of mass of the two objects. The constant is conventionally chosen so that the potential energy is 0 as the separation gets infinite.

Finally, we can put numbers into our expression for the change in potential energy, and find in this case that with $m_1 = 2\times 10^{20}kg$, $m_2 = 3\times 10^{10}kg$, $\vec{r}_i = 5\times 10^6 m\hat{\imath}$, and $\vec{r}_f = 7\times 10^6 m\hat{\jmath}$ we have $\Delta PE = 2.29\times 10^{13}J$.

Why did we do this? The *KEY* thing here was the demonstration of how the calculation of the change in potential energy works. If you understand this calculation, you'll be able to do other calculations of other potential energies.

Student Exercises In this section, we will ask you to calculate the change in potential energy in a number of situations: what you should do here is work through the calculation using the definition of $\Delta PE = -W_c$. Just getting the (numerical) answer is not the desired outcome; understanding how you got it is.

- A charge $q_1 = 6.0 \times 10^{-6}C$ is held stationary at the origin, while a second charge $q_2 = 4.0 \times 10^{-6}C$ is moved from position $\vec{r}_i = 5m\hat{\imath}$ to $\vec{r}_f = 5.66m\hat{\imath} - 5.66m\hat{k}$. What is the change in the potential energy of the second charge? Make sure you understand the difference in negative signs between this and the Newtonian gravity question. *The change in potential energy is* $-1.62 \times 10^{-2}J$*; the potential energy has decreased becasue the electric force will be pushing the second charge away, and so will do* positive *work in this situation. This means that the potential energy will decrease.*

- A spring of spring constant $k = 8000\frac{N}{m}$ is initially compressed from its equilibrium length by $l_1 = 2.0cm$. It is forced to stretch until it is $l_2 = 4.0cm$ longer than its equilibrium length. What is the change in the spring's potential energy during this process? *The change in potential energy is* $4.8J$*. The spring is* farther *away from its equilibrium length and it has exerted a force to resist this, which has done negative work. This corresponds to the increase in potential energy.*

- A ball of mass $m = 3.0kg$ near the surface of the earth is moved from the origin to position $\vec{r} = 8m\hat{\imath} + 5m\hat{k}$. What is the ball's change in gravitational potential energy? Make sure you understand why this only depends on the z-component of the final position. *Gravity has done negative work. The change in potential energy is* $147J$*.*

- A point charge $q = 3.0 \times 10^{-4}C$ is moved from position $\vec{r}_i = 4m\hat{\imath}$ to position $\vec{r}_f = 7m\hat{\imath}$ in a region where the electric field is constant $\vec{E} = 4000\frac{N}{C}\hat{\imath} + 3000\frac{N}{C}\hat{\jmath}$. What is the change in the charge's potential energy? *The charge has its potential energy change by* $-3.6J$*.*

12.3 Gravity near the Earth's surface

Example Find the change in the gravitational potential energy of a ball of mass m which is lifted from the Earth's surface to a height h above the surface (with $h \ll R_E$, where R_E is the Earth's radius.) Does this match the value that you would get using the expression for change in potential energy by lifting the ball a distance h up in a region of constant gravitational force?

Worked Solution Something should be bothering you: we have calculated two different expressions for the potential energy due to gravity. Using Newton's $G\frac{m_1 m_2}{r^2}$ as the force magnitude gives us $PE = -G\frac{m_1 m_2}{r}$, and using mg as the magnitude of the force of gravity gives us $PE = mgz$. They are different, but they are supposed to explain the same thing. Can we reconcile the two expressions?

The way we will do this is to ask what happens to the potential energy of an object of mass m as it is moved from the surface of the earth, to a height h above the surface of the earth. It is easy to calculate that the change in potential energy will be

$$\begin{aligned} \Delta PE &= PE_f - PE_i \\ &= \left(-G\frac{M_E m}{R_E + h}\right) - \left(-G\frac{M_E m}{R_E}\right) \end{aligned} \tag{12.5}$$

This is *exact*, but it is clearly not in the form we recognize. To get that, we're going to have to do an *approximation*. The approximation that we will use is that h is really small compared to R_E, and so we can write

$$\begin{aligned} \Delta PE &= \left(-G\frac{M_E m}{R_E + h}\right) - \left(-G\frac{M_E m}{R_E}\right) \\ &= \left(-G\frac{M_E m}{R_E}\frac{1}{\left(1 + \frac{h}{R_E}\right)}\right) - \left(-G\frac{M_E m}{R_E}\right) \end{aligned} \tag{12.6}$$

This probably does not look like an improvement, but now we can use a piece of calculus: The Taylor polynomial.

A quick reminder: The derivatives of a function help you approximate the value of the function close to a point. For example, the derivative tells you the *rate* at which a function changes. We can use this idea to approximate, say, $(1.01)^2$. If we do not have a calculator, we could do the following: We identify that the 'base' function here is $f(x) = x^2$, and as long as δx is small, we have that

$$f(x) = f(x_0 + \delta x) \approx f(x_0) + f'(x_0)\delta x \tag{12.7}$$

(in this, $f'(x)$ is the first derivative). We have $x_0 = 1$, and $\delta x = 0.01$, and using what we know about derivatives, $f'(x) = 2x$. Putting all this in, we have that $f(1.01) \approx f(1) + f'(1)0.01 = 1^2 + 2(1)0.01 = 1.02$, which is very close to the value we get from our calculator of 1.0201.

Why did we remind you of this? Well, we have a term in our expression for the change in potential energy of $\frac{1}{1+\frac{h}{R_E}}$. Since $h \ll R_E$, this has the form of $\frac{1}{1+small}$. We will try to adapt this to what we just said about Taylor approximation. Consider $f(x) = \frac{1}{x}$, then $f(1+small) = \frac{1}{1+small}$, the quantity we want to estimate. We now identify $x_0 = 1$, and $\delta x = \frac{h}{R_E}$ and use the result from equation 12.7 for the lowest order Taylor approximation. We calculate the derivative of $\frac{1}{x}$ and substitute x_0 in, giving

$$\frac{1}{x} \approx \frac{1}{x_0} + \left(-\frac{1}{x_0^2}\right)\delta x \tag{12.8}$$

and so

$$\frac{1}{\left(1+\frac{h}{R_E}\right)} \approx 1 + (-1)\frac{h}{R_E} \tag{12.9}$$

Putting this into our expression for the change in the potential energy, we have

$$\begin{aligned} \Delta PE &= \left(-G\frac{M_E m}{R_E}\frac{1}{\left(1+\frac{h}{R_E}\right)}\right) - \left(-G\frac{M_E m}{R_E}\right) \\ &\approx \left(-G\frac{M_E m}{R_E}\right)\left(1+(-1)\frac{h}{R_E}\right) - \left(-G\frac{M_E m}{R_E}\right) \\ &= G\frac{M_E}{R_E^2}mh \end{aligned} \tag{12.10}$$

This is exactly the same change in potential energy you get using $\Delta PE = mg\Delta z$ and $g = \frac{GM_E}{R_E^2}$ (because Δz is the change in height, which we have called h), so the two expressions are equivalent within the approximation that we're close to the Earth's surface.

Key things this is supposed to illustrate:

- How you apply the idea of change in potential energy.
- That the $-G\frac{m_1 m_2}{r}$ and mgh expressions for gravitational potential energy give the same changes in potential energy close to the Earth's surface – this means the two are effectively just a constant away from each other.
- An example of the very common practice (in physics) of using calculus to make an approximation to simplify a more complicated expression.

Student Exercises Suppose there are masses $500kg$ at each of $1m\hat{\imath}$ and $-1m\hat{\imath}$, and another mass $5kg$ is at $15m\hat{\imath}$. Note that $15m \gg 1m$.

- By how much does the gravitational potential energy of the $5kg$ mass differ from what it would be if there was, instead, a mass $1000kg$ at the origin? *The difference between the true value of the potential energy and the value for the 'all at the origin' approximation is* $-9.9 \times 10^{-11} J$.

- Taylor approximate the $\frac{1}{15m+1m} + \frac{1}{15m-1m}$ terms that appear in the expression for the exact potential energy to the lowest order where you get something different than $\frac{2}{15m}$ (this order will be *quadratic*). What is the value of this term in the approximation? *It is* $-9.88 \times 10^{-11} J$. *What this shows is that the Taylor polynomial to second order has almost completely accounted for the difference between the exact potential energy value and the 'all at the origin' approximation. This* shows *why we use polynomial expansions so much. They work and simplify our calculations.*

12.4 Central forces

Example At its closest approach to the Sun, a particular comet is a distance r_i from the Sun and travelling at speed v_i. Some time later, it is at a distance r_f from the sun. At that time, what is the angle between its velocity and the vector from it to the Sun?

Find the numerical answer for the case of $r_i = 5.0 \times 10^{10} m$ (roughly the distance from the Sun to Mercury), $v_i = 7.0 \times 10^4 \frac{m}{s}$ (roughly 1.4 times Mercury's orbital speed), and $r_f = 1.0 \times 10^{11} m$ (roughly the distance from the Sun to Venus). The mass of the Sun is approximately $2.0 \times 10^{30} kg$.

Worked Solution There are a couple parts in this: one thing we have to do is figure out the speed of the comet when it is at the distance r_f away from the Sun. This is easy. We apply the work-energy theorem. The next part is a bit harder, we have to figure out the *angle* between $\vec{r}$ and $\vec{v}$ (and we know both $|\vec{r}|$ and $|\vec{v}|$). The conserved quantity we can use for this is *angular momentum*. Since the force exerted by the Sun (through gravity) is in the same direction as $\vec{r}$, it will exert no torque, and that means that the angular momentum $\vec{L}$ will be constant, which means that the *magnitude* of the angular momentum will be constant too.

We sketched the problem in figure 12.2. First, the easier part: finding

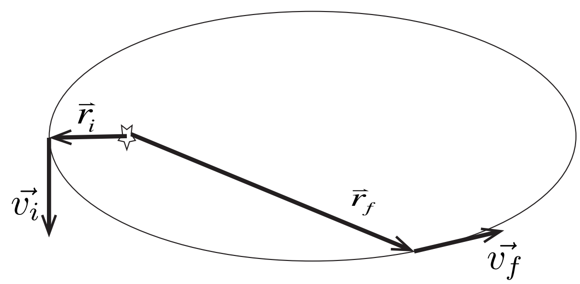


Figure 12.2: A comet orbiting the sun, with the closest point to the sun and another, arbitrary point highlighted.

the speed of the comet when it is at a distance r_f from the Sun. We know that

$$\begin{aligned}
W_{nc} + W_c &= \Delta KE \\
W_{nc} &= \Delta KE - W_c \\
W_{nc} &= \Delta KE + \Delta PE \\
0 &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 + \left(-G\frac{M_{sun}m}{r_f}\right) - \left(-G\frac{M_{sun}m}{r_i}\right) \\
v_f^2 &= v_i^2 + 2G\frac{M_{sun}}{r_f} - 2G\frac{M_{sun}}{r_i} \qquad (12.11)
\end{aligned}$$

and thus we have the final speed of the comet in terms of known quantities. For the values given: $v_i = 7.0 \times 10^4 \frac{m}{s}$, $r_i = 5 \times 10^{10} m$, $r_f = 1.0 \times 10^{11} m$, and $M_{sun} = 2.0 \times 10^{30} kg$, we obtain $v_f = 4.72 \times 10^4 \frac{m}{s}$.

For the angular momentum, we recall that $\vec{L} = \vec{r} \times \vec{p} \to \left|\vec{L}\right| = m\left|\vec{r}\right|\left|\vec{v}\right|\sin\theta$ where θ is the angle between $\vec{r}$ and $\vec{v}$. At the closest point to the sun, the angle between $\vec{r}$ and $\vec{v}$ is 90°; which means that $\left|\vec{L}\right| = mr_iv_i$, and it is also $\left|\vec{L}\right| = mr_fv_f\sin\theta$ with θ the quantity we want. Comparing these, we have

$$mr_iv_i = mr_fv_f\sin\theta \qquad (12.12)$$

which can be solved for θ now that we have all the relevant variables. In fact with r_i, v_i, r_f, and v_f as above, we obtain $\sin\theta = 0.741 \to \theta = 47.9°$.

What you should notice: In this, the key idea is the application of two different conservation laws: the total energy (kinetic and potential) stayed constant because there were no non-conservative forces; the angular momentum stayed constant because there was no torque.

Student Exercises

- One idea for a way to launch things into space from Earth (without using a rocket) is to have a giant 'cannon' which takes a mass and accelerates it over some distance. Assuming that a ball of mass $m = 3000kg$ was sitting on the surface of the Earth, and then placed into this 'cannon'. The cannon would exert a constant force of magnitude F along its length d. What force magnitude (F) would be needed to have the ball 'escape' from Earth neglecting air resistance? Answer in terms of G, d, R_E, M_E. If $d = 10km$, what would the speed of the ball be at the end of the cannon? What if $d = 100km$? $d = 1000km$? *We do not provide the formula you should derive, but for the three different ds specified, the required force magnitudes are* $1.88 \times 10^7 N$, $1.88 \times 10^6 N$, *and* $1.88 \times 10^5 N$ *respectively. Their speeds upon escaping the cannon's mouth are* $1.12 \times 10^4 \frac{m}{s}$, $1.11 \times 10^4 \frac{m}{s}$, *and* $1.04 \times 10^4 \frac{m}{s}$ *respectively.*

12.5 Collision of ions

Example An ion of mass m_1 and charge q_1 is travelling at velocity $\vec{v}_{1,i}$ towards another ion of mass m_2 and charge q_2 which is initially at rest. Assuming that after the collision the first ion travels in a direction parallel to its original velocity:

- What is the velocity of charge q_2 after they interact?
- What is the closest the two charges get to each other?

Get a numerical answer for $m_1 = 1kg$, $m_2 = 2kg$, $\vec{v}_{1,i} = 5\frac{m}{s}\hat{\imath}$, $q_1 = 4 \times 10^{-6}C$, and $q_2 = 6 \times 10^{-6}C$. These aren't really 'ion' sized numbers, but we will use them. This is illustrated in figure 12.3.

Worked Solution This is a collision problem, obviously. For the first part, we do a standard momentum analysis.

We know that since the electric force is a conservative force, this will be an elastic collision. That means that the masses have the same kinetic energy

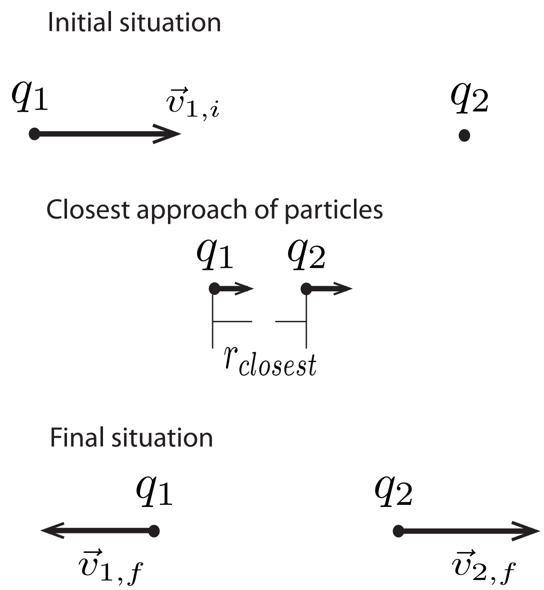


Figure 12.3: Two masses, with charge q_1 and q_2 respectively interact. Initially q_1 is travelling at $\vec{v}_{1,i}$ towards a second stationary charge. After some time they are at their closest separation, and since a force has been acting, they are both moving. Finally, the two charges are separated by a long distance, so that their force on each other is negligible, but they are moving at $\vec{v}_{1,f}$ and $\vec{v}_{2,f}$.

after they have finished interacting as they did before they started. We will go ahead and write out what we know about conservation of momentum.

$$\begin{aligned} \vec{p}_{1,i} + \vec{p}_{2,i} &= \vec{p}_{1,f} + \vec{p}_{2,f} \\ m_1\vec{v}_{1,i} + 0 &= m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f} \\ m_1 v_{1,i}\hat{\imath} &= m_1 v_{1,f}\hat{\imath} + m_2 v_{2,f}\hat{\imath} \end{aligned} \tag{12.13}$$

It is important notice a couple things we did there: we put in that the second mass started at rest, so $\vec{p}_{2,i} = 0$. We decided to call the initial direction of mass 1 the $\hat{\imath}$ direction (x-direction), so we wrote the initial velocity as 'speed' multiplied by 'direction'. In the statement of the problem we know that after the collision, mass 1 travels parallel to the original direction, so its direction is also along $\hat{\imath}$; $v_{1,f}$ could be positive or negative in equation 12.13 which would correspond to moving to the right or the left. Finally, we wrote the velocity of the second mass after the collision as $v_{2,f}\hat{\imath}$. In doing this we were using *intuition* about conservation of momentum: The initial and final momentum of mass 1 is in the x-direction, and the initial momentum of mass 2 is zero. This means that the final momentum of mass 2 must be in the x-direction – the y and z components must be zero.

Similarly, using the fact it is an elastic collision,

$$\begin{aligned} KE_{1,i} + KE_{2,i} &= KE_{1,f} + KE_{2,f} \\ \frac{1}{2}m_1 v_{1,i}^2 + 0 &= \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 \end{aligned} \tag{12.14}$$

In equations 12.14 and 12.13 there are two unknowns: $v_{1,f}$ and $v_{2,f}$. There are two relations, so we have some hope of solving them. We do not care about $v_{1,f}$ for our final answer, so we eliminate it from expression 12.14 using the relation from 12.13: $v_{1,f} = v_{1,i} - \frac{m_2}{m_1}v_{2,f}$. This means that

$$\begin{aligned} \frac{1}{2}m_1 v_{1,i}^2 &= \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 \\ &= \frac{1}{2}m_1\left(v_{1,i} - \frac{m_2}{m_1}v_{2,f}\right)^2 + \frac{1}{2}m_2 v_{2,f}^2 \\ &= \frac{1}{2}m_1 v_{1,i}^2 - m_2 v_{1,i}v_{2,f} + \frac{1}{2}\frac{m_2^2}{m_1}v_{2,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 \\ 0 &= -m_2 v_{1,i}v_{2,f} + \frac{1}{2}\frac{m_2^1 + m_1 m_2}{m_1}v_{2,f}^2 \end{aligned} \tag{12.15}$$

which gives two solutions for $v_{2,f}$:

$$v_{2,f} = 0, \text{ or } v_{2,f} = \frac{2m_1}{m_1 + m_2}v_{1,i} \tag{12.16}$$

As usual when there are two solutions we have to think a bit about what they mean. The solution $v_{2,f} = 0$ corresponds to mass two not moving so it is the other solution we want.

Now, to find the closest approach. Again, we need to use a bit of intuition. The charged particles will get closer to each other, eventually reach a closest point, and then separate. We need to some calculus to be really precise about what we mean. The vector from one mass to the other is $\vec{r}_2 - \vec{r}_1$, so the distance between them is $|\vec{r}_2 - \vec{r}_1|$. The rate at which this changes is $\frac{d}{dt}|\vec{r}_2 - \vec{r}_1|$ which we can analyze as follows:

$$\begin{aligned}
\frac{d}{dt}|\vec{r}_2 - \vec{r}_1| &= \sqrt{(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)} \\
&= \frac{1}{2}\frac{1}{\sqrt{(\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)}} 2(\vec{r}_2 - \vec{r}_1)\cdot\left(\frac{d}{dt}\vec{r}_2 - \frac{d}{dt}\vec{r}_1\right) \\
&= \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}\cdot(\vec{v}_2 - \vec{v}_1) \qquad (12.17)
\end{aligned}$$

At the closest point, the rate at which the distance between them changes should vanish (it was getting smaller, and soon it will be getting bigger). This means that if $\vec{v}_2 = \vec{v}_1$, this condition is satisfied. At the instant of closest approach the masses are moving at the same velocity.

Now that we know this, it should be simple to get the distance of closest appoach. The two masses are moving with the same velocity, which means we can find their total kinetic energy. This will be less than the total kinetic energy before the collision started, so we can infer that something has done *work* on the masses, and that this work has changed the kinetic energy. The only thing available to do this work is a conservative force: the electric force.

We express the work-energy theorem:

$$\begin{aligned}
W_c + W_{nc} &= \Delta KE \\
W_c + 0 &= KE_{1,f} + KE_{2,f} - (KE_{1,i} + KE_{2,i}) \\
-\Delta PE &= KE_{1,f} + KE_{2,f} - KE_{1,i} \\
0 &= KE_{1,f} + KE_{2,f} - KE_{1,i} + \Delta PE \qquad (12.18)
\end{aligned}$$

When we talk about the final KE here, we mean the KE when the two masses are closest together. The two masses will have the same speed, which we can find by applying conservation of momentum:

$$\begin{aligned}
m_1 v_{1,i}\hat{\imath} &= m_1 v_{1,f}\hat{\imath} + m_2 v_{2,f}\hat{\imath} \\
m_1 v_{1,i} &= (m_1 + m_2)\, v_{common} \qquad (12.19)
\end{aligned}$$

Since the two masses have the same velocity.

We return to the work-energy theorem again:

$$\begin{aligned}
0 &= KE_{1,f} + KE_{2,f} - KE_{1,i} + \Delta PE \\
0 &= \frac{1}{2}m_1 v_{common}^2 + \frac{1}{2}m_2 v_{common}^2 - \frac{1}{2}m_1 v_{1,i}^2 + \Delta PE \\
&= \frac{1}{2}\frac{m_1^2}{m_1+m_2}v_{1,i}^2 - \frac{1}{2}m_1 v_{1,i}^2 + \Delta PE \qquad (12.20)
\end{aligned}$$

We can use this to get a numerical value for ΔPE, since we know m_1, m_2, $v_{1,i}$. Knowing the number for ΔPE, we can find their closest approach assuming that they started very (infinitely) far away:

$$\Delta PE = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{closest}} - \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{starting}} \qquad (12.21)$$

and with $r_{starting} \to \infty$, we have

$$\Delta PE = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{closest}} \qquad (12.22)$$

and we know everything except $r_{closest}$.

Numerically, using $m_1 = 1kg$, $m_2 = 2kg$, $\vec{v}_{1,i} = 5\frac{m}{s}\hat{\imath}$, the speed of mass 2 after the collision is $v_{2,f} = 3.3\frac{m}{s}$. We find that at closest approach $\Delta PE = 8.3J$, and with $q_1 = 4 \times 10^{-6}C$, and $q_2 = 6 \times 10^{-6}C$ we have $r_{closest} = 2.6 \times 10^{-2}m$.

Some things you should notice: This problem was a collision problem, despite the masses never actually coming into contact with each other. This is a common feature of conservative forces: things do not have to touch. We worked through a conservation of momentum question, and it went as they always do: there were unknowns (the final components of velocity) that were related by an equal number of relations. The equations we get are linear, from momentum conservation, and quadratic from the fact it was an elastic collision. Since there were enough relations we were able to solve them.

Another thing was how we found out the closest appoach: we used that when the masses are at closest approach, we have their relative speed vanishes. More correctly, $(\vec{r}_2 - \vec{r}_1) \cdot (\vec{v}_2 - \vec{v}_1) = 0$. Another possible solution is their relative velocity is perpendicular to the vector between them. Since we know that there is only motion in the direction we choose to call the x-direction, that is not possible. We used the common speed to figure out the kinetic energy at the time of closest approach, and used the difference

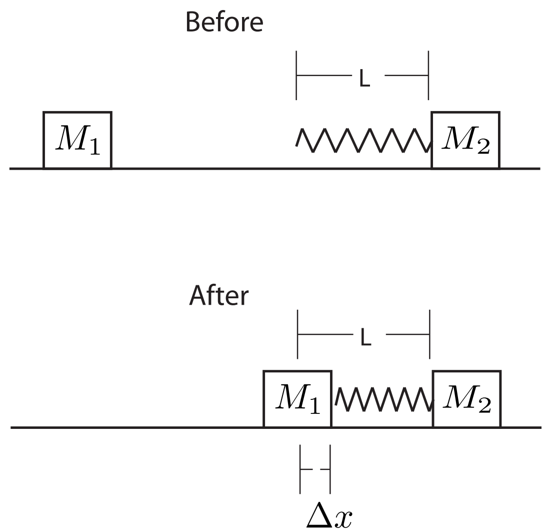


Figure 12.4: A mass m_1 travels towards a second mass m_2 which has a spring on one end.

to tell us how much work had been done by the electric force, and hence what the change in potential energy was, and so what the distance of closest approach was.

Student Exercises

- A box of mass m_1 and initial velocity $v_{1,i}\hat{\imath}$ travels towards a second box of mass m_2 and initial velocity $-v_{2,i}\hat{\imath}$. The second box has a spring of unstretched length l and spring constant k attached to the front as shown in 12.4. Assume that the two boxes only move along the x-axis after the collision. For the numerical values, take $m_1 = 2kg$, $m_2 = 3kg$, $v_{1,i} = 6\frac{m}{s}$, $v_{2,i} = 4\frac{m}{s}$, $k = 10000\frac{N}{m}$, and $l = 0.5m$.
  - What is the maximum compression of the spring? *The maximum compression is* $0.1095m$.
  - What is the velocity of the second box after the collision? *The velocity of the second box after the collision is* $4.0\frac{m}{s}\hat{\imath}$.

  Note that in this case $\vec{p}_{total} = 0$ *so that at closest approach the boxes are at rest.*

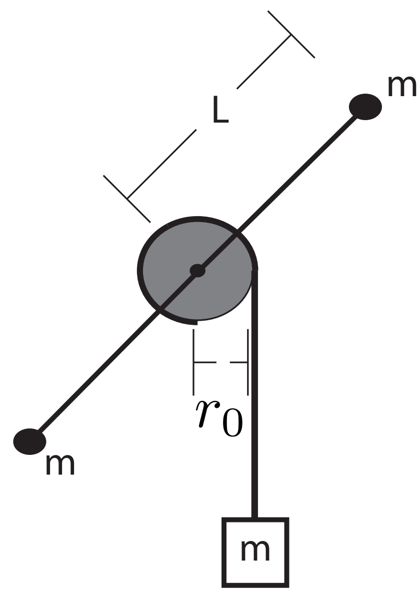


Figure 12.5: A rod of length $2L$ is attached to a cylinder of radius r and has masses m attached to each end. A mass m is attached to a rope which wraps around the cylider.

12.6 Atwood Machines

Example An Atwood Machine consists of a mass m suspended by a rope wrapped around a massless cylinder of radius r_0 which rotates on a frictionless pivot. The cylinder is also attached to a massless rod of length $2L$ which has masses m on each end. This situation is depicted in figure 12.5. The mass is allowed to fall from rest. When it has fallen a distance d, what is the magnitude of its velocity?

Worked Solution We could solve this using the techniques that relate torque and angular acceleration like we did in previous examples. We will not, however, do this. We have a new concept, and we can show that will give us the same answer.

The key thing we will have to do is relate the speed of the mass m to the speed of the rotating masses. When we know that, we can figure out the kinetic energy, and just use the work-energy theorem to find the speed.

As we discussed before, when something of radius r is rotating at angular speed ω, then its linear speed is $v = r\omega$. This means that the speed of the fallen mass $v = r_0\omega$ becasue the rope is unwinding without slipping. Since we know r_0, in principle, there are only two unknown things. The speed of the masses attached to the end of the rod are each $v_{masses} = L\omega = v\frac{L}{r_0}$ with the last equality obtained by substituting for ω.

Once the mass has fallen and is travelling at v, the total kinetic energy is

$$\begin{aligned} KE_{net} &= KE_{falling\ mass} + KE_{rotating\ masses} \\ &= \frac{1}{2}mv^2 + 2\frac{1}{2}mv_{masses}^2 \\ &= \frac{1}{2}m\left(1 + 2\left(\frac{L}{r_0}\right)^2\right)v^2 \end{aligned} \tag{12.23}$$

Now, we can use the work-energy theorem:

$$\begin{aligned} W_{nc} &= \Delta KE + \Delta PE \\ 0 &= KE_f - KE_i + \Delta PE \\ &= KE_f - 0 + mg\Delta z \\ &= \frac{1}{2}m\left(1 + 2\left(\frac{L}{r_0}\right)^2\right)v^2 + mg(-h) \end{aligned} \tag{12.24}$$

This means that the final speed is

$$v = \sqrt{\frac{2gh}{1 + 2\frac{L^2}{r_0^2}}} \tag{12.25}$$

Identifying that in this case, the moment of inertia of the cylinder-mass combination that was rotating was $2mL^2$, we could re-write this for a more general object as

$$v = \sqrt{\frac{2mgh}{m + \frac{I}{r_0^2}}} \tag{12.26}$$

Some comments: The other way to do this problem would have been to do the typical Atwood machine set-up. You would have had three relations, one between the tension in the rope and the acceleration of the mass, one between the torque provided by the tension in the rope and the angular acceleration, and one between the angular acceleration and the mass's acceleration. Three equations in three unknowns are easy to solve, so you find the acceleration. Then, use the relation between acceleration and displacement to get the final speed.

You can do this, and get the same answer. Why? It is the same physics, just a different path through the calculation. There is an overarching idea in this class: that there are plenty of ways to understand the same problem. In some cases, one is more calculationally convenient than the other, but overall the same physics gets you to the same predictions about what will happen in the world.

Student Exercise A mass $m = 3kg$ is attached to a cylinder which has radius $r_0 = 0.1m$ by a rope. The cylinder rotates on a horizontal, frictionless axis. The mass is allowed to fall from rest, and when it has fallen by $d = 2m$, it has a speed of $v = 0.5\frac{m}{s}$. What is the moment of inertia of the cylinder? *The moment of inertia is* $I = 4.67kg\ m^2$.

12.7 Questions

- A mass $m = 2kg$ is suspended from the ceiling of a room by a spring which has unstretched length 0 and spring constant $k = 500\frac{N}{m}$.
  - How far below the ceiling is the equilibrium length of the spring?
  - What is the potential energy of the spring-mass system at the equilibrium position if the potential energy is 0 when the ball is at the ceiling?
  - What is the potential energy when the mass is Δz away from that equilibrium position if the potential energy is 0 at the ceiling?
- A charge q_1 is held fixed at the origin, and a charge q_2 on a small ion of mass m is initially at $\vec{r}_i$ travelling at velocity $\vec{v}_i$. What is the closest that the ball with charge q_2 gets to the origin, and what is its speed at that point? Assume that $m = 2.5 \times 10^{-25}kg$, $q_1 = 3.2 \times 10^{-16}C$, $q_2 = -4.8 \times 10^{-19}C$, $\vec{v}_i = -2.0\frac{m}{s}\hat{\imath}$, and $\vec{r}_i = 10m\hat{\imath} + 1m\hat{\jmath}$. *Hint: the angular momentum around the origin is constant, and the condition for closest approach is* $\vec{r} \cdot \vec{v} = 0$.
- A box of mass m_1 and a box of mass m_2 are initially at rest on a horizontal frictionless surface. There is a spring of spring constant k which is compressed by Δx shorter than its equilibrum length, and it

is not attached to either. The two boxes are released from rest. What is the final speed of box 1? For a numerical answer, assume $m_1 = 5kg$, $m_2 = 3kg$, $k = 500\frac{N}{m}$, and $\Delta x = 0.1m$.

- An Atwood machine is formed by connecting a $5kg$ mass to a $6kg$ mass by stretching a light rope over a massless and frictionless pulley. The two masses are released from rest. When the $6kg$ mass has fallen by $1m$ how fast is it going?

12.8 Answers

- For the mass hanging from the spring below we have
  - The equilibrium length is $3.92cm$ below the ceiling.
  - The potential energy is $-0.384J$ if the potential energy is 0 for the unstretched spring and 0 for gravity when the mass is at the ceiling.
  - The potential energy when displaced vertically by Δz from equilibrium is $-0.384J + 250\frac{N}{m}\Delta z^2$.
- The speed at closest approach is $6.01\frac{m}{s}$ and the separation is $0.33m$.
- Box 1 will travel with a speed of $0.612\frac{m}{s}$.
- The $6kg$ mass is going down at $1.33\frac{m}{s}$ and the $5kg$ mass is rising at the same rate. Make sure you used energy considerations to get this value.

Chapter 13

Electricity

13.1 Overview

In this workbook section we discuss electricity. The reason that this fits here is that we have talked about how forces produce acceleration, and how *conservative* forces produce changes in potential energy. The current is a bulk movement of charged particles; it would be difficult to follow the motion of one individually. This motion of charged particles can do useful work.

You will notice that this material is not in the textbook. This is because you learned this in your BC Physics 12 (or equivalent) course. In that sense it is review, but it is thematically appropriate here because it is an application of the idea of a conservative force.

The key points you need to know are:

- If there is a difference in electric potential between two locations, there will be an electric field between those two points. This electric field can exert a force on *mobile* charged particles (such as electrons) which causes them to move. This bulk motion of charged particles is called a current.

- For a large class of materials (called *Ohmic Resistors*) there is a linear relationship between the magnitude of the applied electric field (and hence the potential difference) and the resulting current. $\left|\vec{E}\right| = \rho I$. In this, ρ is a number which depends on the size, shape, and material of the resistive element.

- This Ohmic relationship is often written as $\Delta V = IR$ where R is the resistance across an element, and ΔV is the difference in electric potential between the two ends.

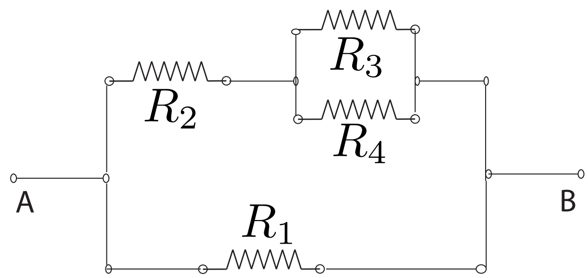


Figure 13.1: Four resistors are joined by wires on a path from point A to point B. R_1 is in parallel with the a combination which has R_2 in series with a parallel combination of R_3 and R_4.

- The rate at which energy is dissipated in a resistive element is $I\,|\Delta V| = I^2R = \frac{|\Delta V|^2}{R}$.
- Circuits can be analyzed using Kirchoff's laws:
  - The sum of all currents into a point is equal to the sum of all currents out of a junction. This is an expression of the fact that the total charge in the universe remains constant.
  - The sum of all potential (voltage) changes around a closed loop is 0. This is an expression of the fact that the electric force is conservative.
- It is often possible to analyze circuits by replacing resistive elements with *equivalent resistors.*
  - Two resistors R_1 and R_2 in *series* have an equivalent resistance of $R_{eq} = R_1 + R_2$.
  - Two resistors R_1 and R_2 in *parallel* have an equivalent resistance which satisifes $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$.
- Since we often model wires as something with *effectively* no resistance, it is possible to re-draw circuits to make them clearer.

13.2 A simple circuit

Example Four resistors are connected by wires as shown in figure 13.1.

If $R_1 = 40\Omega$, $R_2 = 30\Omega$, $R_3 = 20\Omega$, and $R_4 = 10\Omega$, and the potential at point A is $3V$ higher than that at B:

- What is the equivalent resistance for this circuit?
- What is the total current from A to B.
- What is the rate of energy dissipation in each resistor?

Worked Solution To find the answer to the posed question (including the rate of energy dissipation) we are going to have to eventually find the current through each of the resistors.

We choose to do the equivalent resistance first: The general strategy for finding an equivalent reistance is to break the problem down into a sequence of steps where you know how to do each one. For example, we notice that R_1 is in parallel with the rest of the circuit, so if we can find the equivalent resistance of the rest of the resistors, we can then use our knowledge of how parallel resistances work to get the total resistance. When we look at the smaller R_2, R_3, R_4 combination, R_2 is in series with the other two, which are in parallel with each other. We are going to find the equivalent resistance of R_3 and R_4 in parallel, then find the equivalent resistance of that, in series with R_2, and then use that total equivalent resistance to get the overall equivalent resistance. This strategy is sketched in figure 13.2.

Generally, have been following a rule to keep things in a calculation as symbolic as we can, but here, a strict application of this principle would require some complicated algebraic expressions that (in this case) doesn't really increase clarity, so we will substitute in the numbers immediately: The equivalent resistance of R_3 and R_4 in parallel is:

$$\frac{1}{R_{eq34}} = \frac{1}{R_3} + \frac{1}{R_4} \rightarrow R_{eq34} = 6.67\Omega \tag{13.1}$$

We then find the equivalent resistance of R_2 in series with this equivalent resistance R_{eq34}:

$$R_{eq234} = R_2 + R_{eq34} \rightarrow R_{eq234} = 36.67\Omega \tag{13.2}$$

Finally, R_1 is in parallel with this resistor which replaces R_2, R_3, and R_4:

$$\frac{1}{R_{eq1-4}} = \frac{1}{R_1} + \frac{1}{R_{eq234}} \rightarrow R_{eq1-4} = 19.13\Omega \tag{13.3}$$

and this is the overall equivalent resistance.

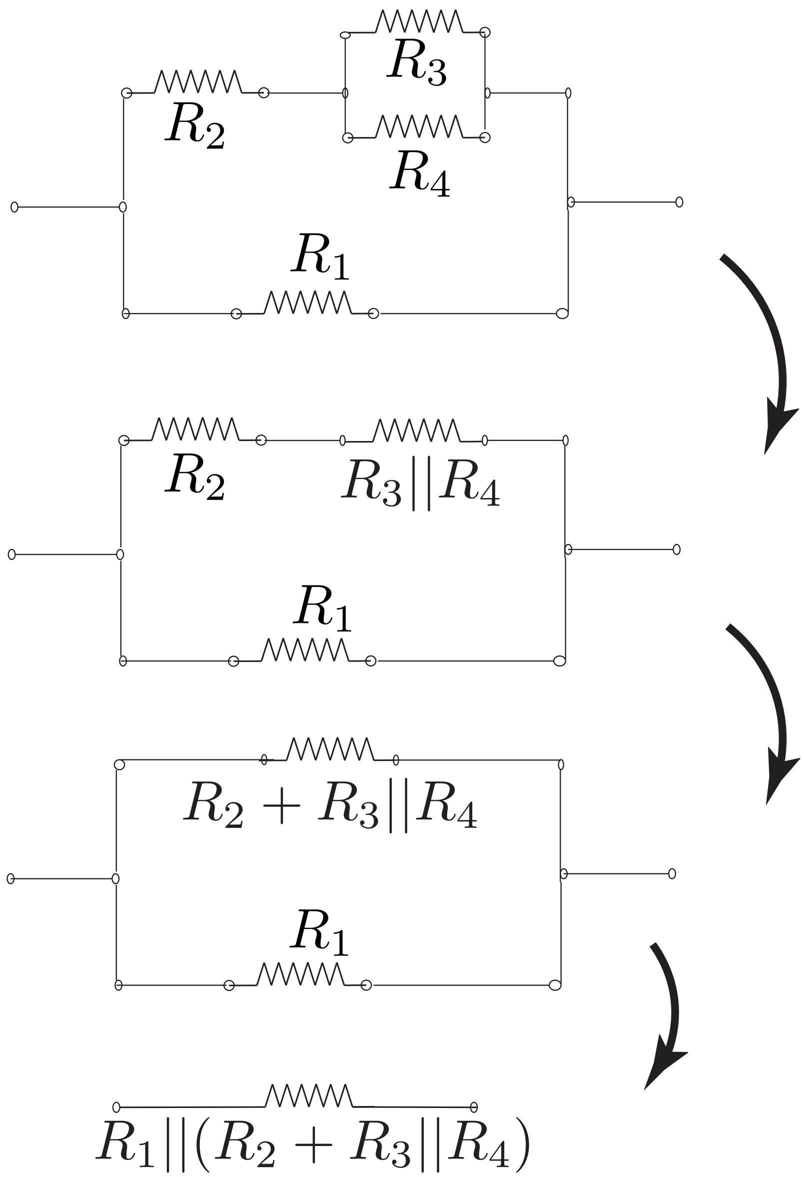


Figure 13.2: The step-by-step replacement of resistors with equivalent resistors for the problem posed in figure 13.1.

This means that the total current through the circuit, if $V_{AB} = 3V$, is $0.157A$.

Now, we find the current through each of the individual resistors: Across resistor R_1, the potential difference is $3V$, so using Ohm's law $\Delta V = I_1 R_1 \rightarrow I_1 = 0.075A$. Applying Ohm's law to the equivalent resistance R_{eq234} we find that the current through that set of resistors is $0.082A$, and we note with happiness that this means that the currents through R_1 and the others add up to the total current we found before. Since we know the current through the '234' equivalent resistance, we know the current through R_2 must be the same ($0.082A$). This leaves us with the problem of getting the currents through R_3 and R_4: the secret to this is to find the voltage across them – we know that the *total* voltage difference across the 234 equvialent resistor is $3V$, and all the current goes through R_2, so applying Ohm's law again, this means that $\Delta V_2 = I_2 R_2 = 0.082A30\Omega = 2.45V$, which means that $0.55V$ is the potential difference across R_3 and R_4, giving currents of $0.027A$ and $0.055A$ respectively (note that the total is $0.082A$, exactly the same as what was going through R_2.)

We can now calculate the power dissipated in each resistor:

$$\begin{aligned} R_1 = 40\Omega \qquad & I_1 = 0.075A \qquad & P_1 = I_1^2 R_1 = 0.225W \\ R_2 = 30\Omega \qquad & I_2 = 0.082A \qquad & P_2 = 0.202W \\ R_3 = 20\Omega \qquad & I_3 = 0.027A \qquad & P_3 = 0.015W \\ R_4 = 10\Omega \qquad & I_4 = 0.055A \qquad & P_4 = 0.030W \\ R_{eq1-4} = 19.13\Omega \qquad & I_{total} 0.157A \qquad & P_{total} = 0.472W \end{aligned} \tag{13.4}$$

A couple of things to notice:

- The way we work on getting the equivalent resistance was by finding little pieces we could get the equivalent resistance for and using that to simplify our calculations step by step.
- For two resistors in parallel the voltage change across them is the same.
- For two resistors in series, the current through them is the same.
- If you add up the power output for all the individual resistors in a circuit, the number you get is the same as if you had used the total current and the equivalent resistance.

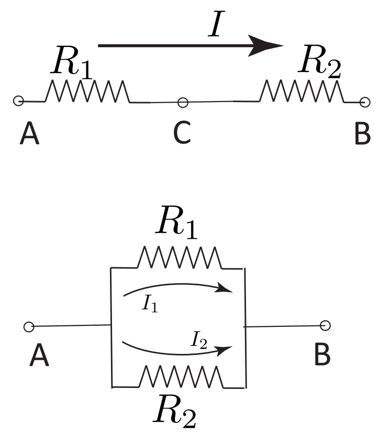


Figure 13.3: In one configuration resistors R_1 and R_2 are in series, in the other resistors R_1 and R_2 are in series.

Student Exercises

- Derive the series and parallel equivalent resistances:
  - Consider two resistors in series as shown in figure 13.3. What is the change in voltage from A to B in terms of I and the equivalent resistance? What is the change in voltage from A to B in terms of the change in voltage across R_1 and R_2? Equate these two expressions for the same thing. *When you divide out the common I, you should get $R_{eq} = R_1 + R_2$.*
  - Consider the two resistors in parallel as shown in figure 13.3. What is the relation between the total current I, and the currents I_1 and I_2 through R_1 and R_2. Express I, I_1, and I_2 in terms of ΔV, R_{eq}, R_1 and R_2. Substitute them in. *When you divide out the common ΔV and you should have $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$.*
- Find the equivalent resistance of R_1, R_2, and R_3 all connected in parallel. *The equivalent resistence is $R_{eq} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$.*
- Find the equivalent resistance of R_1, R_2, and R_3 all connected in series. *The equivalent resistance is $R_{eq} = R_1 + R_2 + R_3$.*

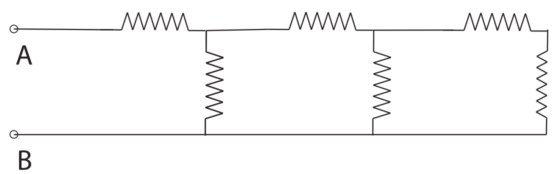


Figure 13.4: Six resistors, each with resistance R are connected between points A and B.

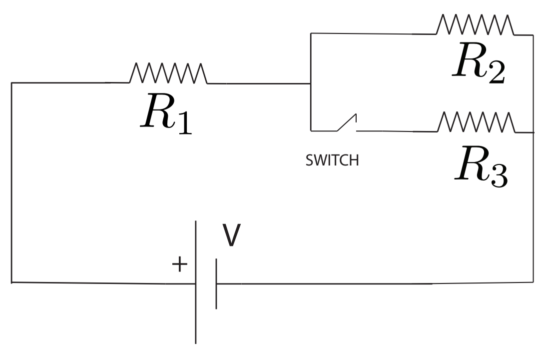


Figure 13.5: R_1 is in series with the combination of R_2 and R_3 in parallel. These are connected to a battery of potential difference between the terminals V.

- Find the equivalent resistance between A and B of six resistors connected as shown in figure 13.4. *The equivalent resistance is* $1.625R$*, assuming the resistance of each individual resistor is* R*.*

13.3 Kirchoff's Laws and Equivalent Resistors

Problem Three resistors are connected by wires as shown in figure 13.5. The three resistors are connected to a battery of voltage V. The switch is closed and therefore electric current can flow through it.

If $R_1 = 10\Omega$, $R_2 = 15\Omega$, $R_3 = 20\Omega$, and $V = 10V$, what is the current in each resistor?

Worked Solution We can approach this by two different methods. The first is to make use of the concept of equivalent resistance; the second is to use Kirchoff's laws. Both will give the same final answer.

First, equivalent resistance: We can conceptually replace the two parallel resistors R_2 and R_3 with a single equivalent resistor $R_{eq,23} = \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1}$. This is in series with the resistor R_1, so the overall equivalent resistance $R_{eq,total} = R_1 + R_{eq,23}$. Knowing that, we can put in the numbers $R_1 = 10\Omega$, $R_2 = 15\Omega$, $R_3 = 20\Omega$, and get $R_{eq,total} = 18.57\Omega$.

For the overall circuit, $\Delta V = IR$, and with a source voltage of $10V$, we have that $I_{total} = 0.538A$. This is the current through the first resistor R_1, and for it $\Delta V = IR \rightarrow \Delta V_{across\ 1} = I_{through\ 1} R_1 = 5.38V$. This means that, since the potential has increased by $10V$ across the battery, and then decreased by $5.38V$ across R_1 that the potential difference across R_2 and R_3 is each $4.62V$. Applying Ohm's law to each, we find that across R_2, we have $\Delta V = IR \rightarrow \Delta V_{across\ 2} = I_{through\ 2} R_2$ and since $\Delta V_{across\ 2} = 4.62V$, and $R_2 = 15\Omega$, then $I_{through\ 2} = 0.307A$. Similarly, $\Delta V_{across\ 3} = I_{through\ 3} R_3$ so with $R_3 = 20\Omega$, we have $I_{through\ 3} = 0.231A$. Note that $I_{through\ 2} + I_{through\ 3} = I_{through\ 1}$. This is because the current had two different ways to go once it had been through the first resistor; it is an expression of Kirchoff's laws.

Now, let us do the same problem using Kirchoff's laws. We label the currents through each of the resistors as I_1, I_2, and I_3, and the current through the battery as I_b respectively. This is shown in figure 13.6 There are three possible loops in this circuit, shown in figure 13.7 We can use Kirchoff's loop law to derive a relationship for each of these loops:

$$\begin{aligned} \text{From loop 1}: \ 0 &= V - I_1 R_1 - I_2 R_2 \\ \text{From loop 2}: \ 0 &= V - I_1 R_1 - I_3 R_3 \\ \text{From loop 3}: \ 0 &= -I_2 R_2 + I_3 R_3 \end{aligned} \tag{13.5}$$

In this, note the mnemonic we used: In the loop going over a voltage source from negative to positive *raises* the potential by whatever the voltage is; going over a resistor in the direction of current *lowers* the potential by an amount given by Ohm's law, and going over a resistor in the *opposite* direction to the current *raises* the potential. We had to use this rule in the expression we got from loop 3.

At this point, you might think we can go ahead and solve for the Is, since we have three relations and three unknown quantities. The problem is that this set of three equations is *secretly* only two! There is an interesting reason why: The third expression tells us that $I_2 R_2 = I_3 R_3$. Knowing this,

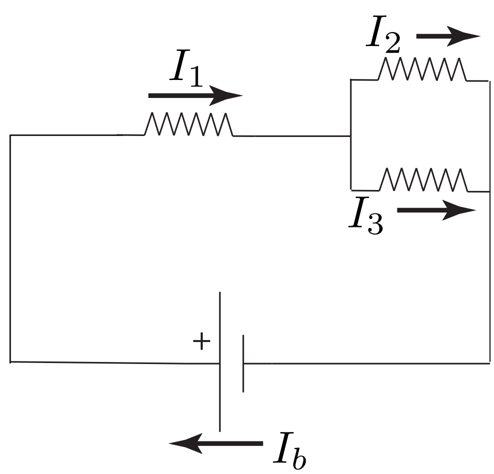


Figure 13.6: The currents through each of the resistors and battery in 13.5 are labelled.

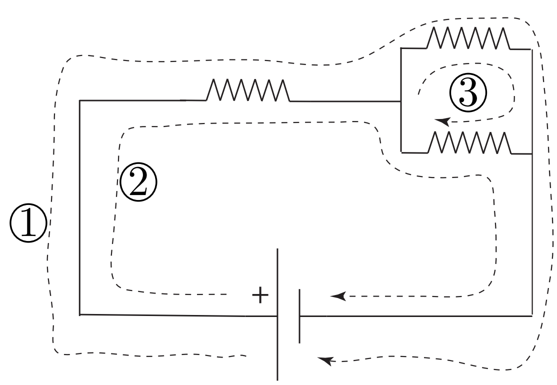


Figure 13.7: Three closed loops in the circuit from 13.5.

if we substituted for either I_2 or I_3 in the remaining two expressions, we'd get two copies of the *same* equation. (Try this yourself; it's true. If you take a linear algebra course like UVic's MATH 110 or 211, you'll encounter this sort of thing as a 'degenerate' set of equations, and the *matrix* you construct to represent this set of equations has a determinant of zero.)

This means we need to find more relationships. The obvious candidate is Kirchoff's junction rule which says that the current into a point is equal to the current out of a point. (Strictly, this is only true in direct current circuits, but those are what we are studying.) There are three places we can apply this: The current that flows out of the battery flows into R_1; the current that flows out of R_1 splits into R_2 and R_3, and the current that flows out of R_2 and R_3 combines and flows into the battery.

$$\begin{aligned} I_b &= I_1 \\ I_1 &= I_2 + I_3 \\ I_2 + I_3 &= I_b \end{aligned} \tag{13.6}$$

Substituting that $I_b = I_1$, there's one more relation here: that $I_1 = I_2 + I_3$, so we have (finally) the three-equation set:

$$\begin{aligned} 0 &= V - I_1 R_1 - I_2 R_2 \\ 0 &= V - I_1 R_1 - I_3 R_3 \\ I_1 &= I_2 + I_3 \end{aligned} \tag{13.7}$$

We substitute the expression for I_1 into the other two and get:

$$\begin{aligned} 0 &= V - (I_2 + I_3) R_1 - I_2 R_2 \\ 0 &= V - (I_2 + I_3) R_1 - I_3 R_3 \end{aligned} \tag{13.8}$$

and the second equation there says that $I_3 = \frac{V - I_2 R_1}{R_1 + R_3}$, which we substitute in to get

$$\begin{aligned} 0 &= V - I_2 (R_1 + R_2) - R_1 \frac{V - I_2 R_1}{R_1 + R_3} \\ \rightarrow & \quad V \left(1 - \frac{R_1}{R_1 + R_3}\right) = I_2 \left(R_1 + R_2 - \frac{R_1^2}{R_1 + R_3}\right) \end{aligned} \tag{13.9}$$

and putting in the known values for V, R_1, R_2, and R_3 gives $I_2 = 0.307A$ (which was what we got previously). Using this value for I_2 in

$$0 = V - (I_2 + I_3) R_1 - I_3 R_3 \tag{13.10}$$

we get that $I_3 = 0.231A$ and so I_1 is the sum of these: $I_1 = 0.538A$. These two ways of analyzing gave us the same results.

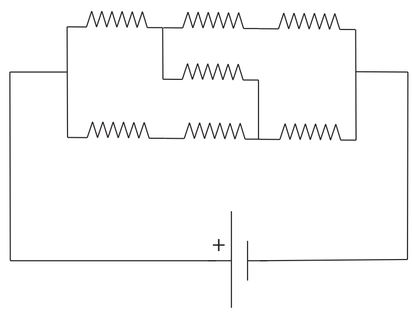


Figure 13.8: A battery with terminal voltage V_b connected to seven identical resistors R.

You should notice two major points:

- The 'equivalent resistance' method gave us the same currents as the 'Kirchoff's laws' method. The two ways use the same physics.
- When you are applying Kirchoff's laws, you always have to use some equations generated by the loop law and some by the junction law.

Student Exercises

- Repeat the analysis above only using the relations

$$\begin{aligned} 0 &= V - I_1R_1 - I_2R_2 \\ 0 &= -I_2R_3 + I_3R_3 \\ I_1 &= I_2 + I_3 \end{aligned} \tag{13.11}$$

  Make sure you get the same numerical values we did. *You do!*

- What happens when the switch is open?
- Consider the circuit in figure 13.8. If all the resistors have the same resistance $R = 10\Omega$, and the battery is a constant voltage source with $V_b = 6V$, what is the power dissipated in the central resistor?

The overall equivalent resistence is 14Ω. *The current through the middle resistor is* $0.0857A$, *and the power dissipated is* $7.34 \times 10^{-2}W$.

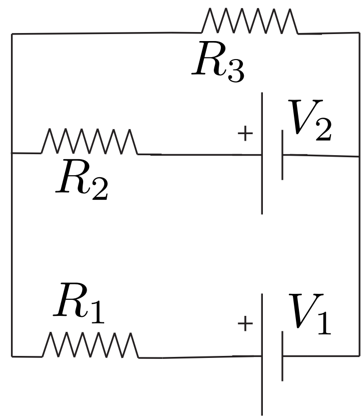


Figure 13.9: Two batteries V_1 and V_2 are each connected in series with a resistor, R_1 and R_2 respectively, and these two battery/resistor combinations are connected in parallel with each other and connected to a third resistor R_3.

13.4 Multiple voltage sources

Example Consider the circuit in figure 13.9 Find the current through the resistor R_2 if

- $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, $V_1 = 20V$, and $V_2 = 5V$.
- $R_1 = 30\Omega$, $R_2 = 20\Omega$, $R_3 = 10\Omega$, $V_1 = 20V$, and $V_2 = 5V$.
- $R_1 = 30\Omega$, $R_2 = 20\Omega$, $R_1 = 30\Omega$, $V_1 = 20V$, and $V_2 = 10V$.

Worked Solution This time, there is very little that we could do with equivalent resistances: there's no set of resistors that's 'isolated' so that we could replace them with one. This means we have to appoach this with the slightly more general Kirchoff's laws approach.

We start by labelling currents and the three different loops, as seen in figure 13.10. One thing we should note: We really did not know which way the current in any of the resistors was going to go. We will do the work with the directions we have assumed, and check our final answers later. (This should remind you of things we did in the section on momentum. When we did not know which direction something moved after a collision

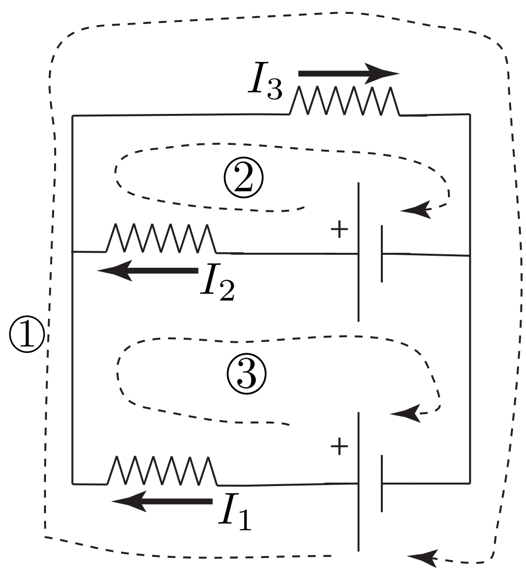


Figure 13.10: Currents I_1, I_2, and I_3 are put on the circuit from 13.9 together with labelling the three possible loops.

we made and assumption. We then did the work. Afterwards we checked if the assumption held up.)

Looking at the figure, we have three relations from the loop law:

$$\begin{aligned}
\text{From loop 1}: \ 0 &= V_1 - I_1 R_1 - I_3 R_3 \\
\text{From loop 2}: \ 0 &= V_2 - I_2 R_2 - I_3 R_3 \\
\text{From loop 3}: \ 0 &= V_1 - I_1 R_1 + I_2 R_2 - V_2
\end{aligned} \tag{13.12}$$

As before, this set of three expressions is only *really* two relations between the Is, so we need something from the junction rule:

$$I_1 + I_2 = I_3 \tag{13.13}$$

which gives us a full set of three equations:

$$\begin{aligned}
0 &= V_1 - I_1 R_1 - I_3 R_3 \\
0 &= V_2 - I_2 R_2 - I_3 R_3 \\
I_1 + I_2 &= I_3
\end{aligned} \tag{13.14}$$

Taking the third of this set of equations and substituting it in to the first and second gives

$$\begin{aligned}
0 &= V_1 - I_1 (R_1 + R_3) - I_2 R_3 \\
0 &= V_2 - I_2 (R_2 + R_3) - I_1 R_3
\end{aligned} \tag{13.15}$$

The first of these two equations tells us that $I_1 = \frac{V_1 - I_2 R_3}{R_1 + R_3}$, so substituting this into the second gives

$$\begin{aligned}
0 &= V_2 - I_2 (R_2 + R_3) - \frac{V_1 - I_2 R_3}{R_1 + R_3} R_3 \\
&= \left(V_2 - V_1 \frac{R_3}{R_1 + R_3} \right) - I_2 \left(R_2 + R_3 + \frac{R_3^2}{R_1 + R_3} \right) \\
\left(V_2 - V_1 \frac{R_3}{R_1 + R_3} \right) &= I_2 \left(R_2 + R_3 \frac{R_1 + R_3}{R_1 + R_3} + \frac{R_3^2}{R_1 + R_3} \right) \\
\left(V_2 - V_1 \frac{R_3}{R_1 + R_3} \right) &= I_2 \left(R_2 + \frac{R_1 R_3}{R_1 + R_3} \right)
\end{aligned} \tag{13.16}$$

This gives us what we want to know (I_2) so we can just substitute the numbers for the different cases:

- The combination $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, $V_1 = 20V$, and $V_2 = 5V$, gives $I_2 = -0.364A$

- The combination $R_1 = 30\Omega$, $R_2 = 20\Omega$, $R_3 = 10\Omega$, $V_1 = 20V$, and $V_2 = 5V$, gives $I_2 = 0A$
- The combination $R_1 = 30\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, $V_1 = 20V$, and $V_2 = 10V$, gives $I_2 = 0.182A$.

We need to quickly check: what does the negative sign mean? Recall that we did not know which way the currents were going. We drew them in, and analyzed the circuit as it went. The negative sign means that we drew the current the *wrong* way. In the first case, this means that the current is *actually* flowing from left to right in I_2, not the other way, and flowing *backwards* through the battery. In the second case there is no current.

We did this question because there are a couple lessons that are very useful for circuits that are contained here:

- It is not always obvious which way current flows through a resistor.
- The direction could depend on the other elements (resistors, voltage sources) in the circuit.
- When applying Kirchoff's laws, you can *choose* one direction, and then do the algebra. If you are wrong, there will be a negative sign somewhere as a clue.

Student Exercises:

- If V_1, V_2, and R_1 are known, what value of R_3 will give no current through R_2? (answer in terms of the known quantities) If you have a larger resistance R_3 than the value you just found, what direction will the current go? *If the resistance is* $R_3 = -\frac{V_2 R_1}{V_2 - V_1}$ *there will be no current. If* R_3 *is bigger than this, as in the first case, the current through* R_2 *will be towards battery 2. If it is less then the current will be as drawn.*
- Imagine that the battery V_2 in figure 13.9 was reversed. Find the currents through each resistor if $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, $V_1 = 20V$, and $V_2 = 5V$. Is there any ambiguity about the direction of the current in resistor 2? *We find* $I_1 = 1.045A$, $I_2 = -0.727A$, *and* $I_3 = 0.318A$. *The current* I_2 *is always in the opposite direction to that shown in figure 13.10. Depending on the values of* V_1, V_2, R_1, *or* R_2 *the current* I_3 *may be in either direction.*