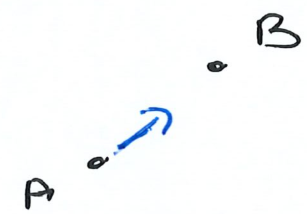
Gravitational Potential Energy.

Near Earth's surface $PE_g = mgz + C$

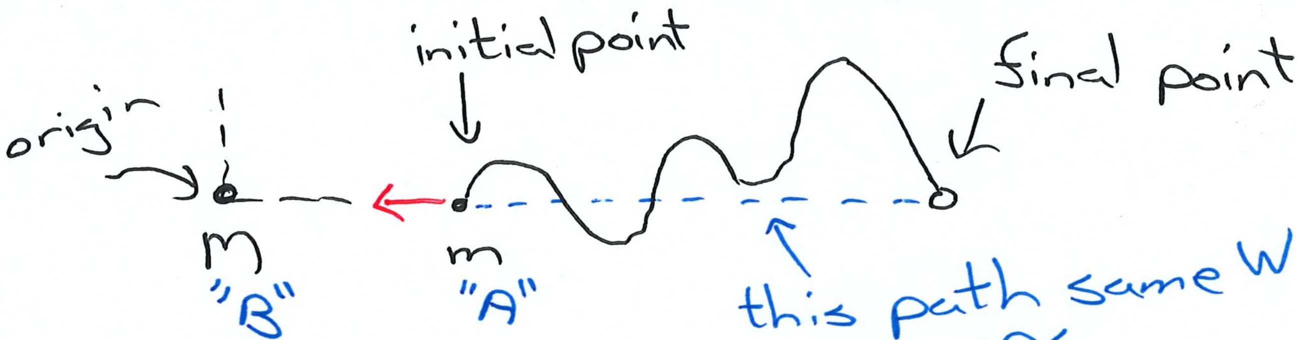
Derived from : $\vec{F}_g = -mg\hat{k}$

Know



$$\vec{F}_{\text{on A by B}} = -G \frac{m_a m_b}{|\vec{r}_A - \vec{r}_B|^2} \frac{\vec{r}_A - \vec{r}_B}{|\vec{r}_A - \vec{r}_B|}$$

What is PE for this?



this path same W
same ΔPE

$\vec{r}(s) = s\hat{\imath}$

$r_i < s < r_f$

initial & final separations

$$d\vec{r} = \left(\frac{d\vec{r}(s)}{ds}\right) ds$$

$$= ds\,\hat{\imath}$$

$$\vec{F}(s) = -G \frac{mM}{|s\hat{i} - 0|^2} \frac{s\hat{i} - 0}{|s\hat{i} - 0|}$$

$$= -G \frac{mM}{s^2} \frac{\cancel{s}\hat{i}}{\cancel{s}}$$

$$\vec{F}(s) \cdot d\vec{r} = \left(-G \frac{mM}{s^2} \hat{i}\right) \cdot (ds\,\hat{i})$$

$$= -G\, mM \frac{ds}{s^2}$$

$$W_g = \int_{r_i}^{r_f} -GM_m \frac{ds}{s^2}$$

$$= -GM_m \int_{r_i}^{r_f} s^{-2}\, ds$$

$$= (-GM_m) \frac{1}{-2+1} s^{-2+1} \Big|_{r_i}^{r_f}$$

$$= +GM_m\, s^{-1} \Big|_{r_i}^{r_f}$$

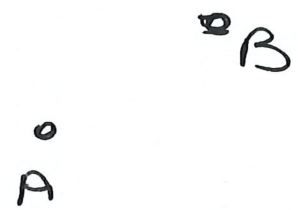
$$= \frac{GM_m}{r_f} - \frac{GM_m}{r_i}$$

$$\int x^n\, ds = \frac{1}{n+1} x^{n+1} + C$$

$$\Delta PE_g = -W_g = -\frac{GMm}{r_f} - \left(-\frac{GMm}{r_i}\right)$$

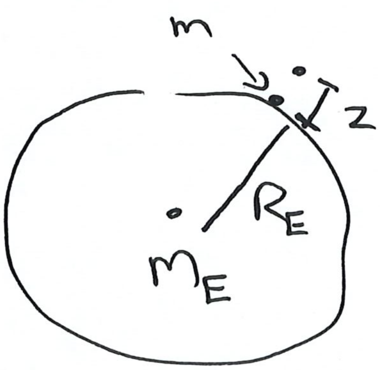
$$PE_f - PE_i$$

$$PE_g = -\frac{GMm}{r} + C$$



$$PE = -\frac{G m_A m_B}{|\vec{r}_A - \vec{r}_B|} + C$$

Does this match with $PE_g = mgz$?



Going surface → surf $+ z\hat{k}$

ΔPE using $mgz + C$

is $\Delta PE = mgz$

ΔPE using $-\dfrac{G m_A m_B}{|\vec{r}_A - \vec{r}_B|} + C$

$$PE_f = -\frac{G M_E m}{(R_E + z)} + C$$

$$PE_i = -\frac{G M_E m}{R_E} + C$$

$$\Delta PE = \left(-\frac{G M_E m}{R_E + z} + \cancel{C}\right) - \left(-\frac{G M_E m}{R_E} + \cancel{C}\right)$$

Reminder: linearizing approximating functions
(Taylor polynomials)

$$f(x) \approx f(x_0) + f'(x_0)[x - x_0] + \frac{f''(x_0)}{2}[x - x_0]^2$$

$$+ \frac{f'''(x_0)}{3!}[x - x_0]^3 + \ldots$$

$\dfrac{1}{1+x}$ has expansion${}_{-1}$ around $x = 0$

$$f(x) = f(0) + f'(0)[x - 0] + \frac{f''(0)}{2}[x - 0]^2 + \ldots$$

$f(0) = \dfrac{1}{1+0}$ $\quad f'(x) = \dfrac{-1}{(1+x)^2}$ $\quad f''(x) = \dfrac{2}{(1+x)^3}$

$$\frac{1}{1+x} \approx 1 - x + x^2 - \dots,$$

$$\Delta PE = -\frac{GM_E m}{R_E} \frac{1}{(1+\underbrace{z/R_E}_{\text{call it } x})} + \frac{GM_E m}{R_E}$$

$$\approx -\frac{GM_E m}{R_E}\left(1 - \frac{z}{R_E} + \left(\frac{z}{R_E}\right)^2 - \dots\right) + \frac{GM_E m}{R_E}$$

$$\approx \underbrace{\frac{GM_E m}{R_E}\frac{z}{R_E}}_{g} - \frac{GM_E m}{R_E}\frac{z^2}{R_E^2} + \dots$$

$$\approx gmz - \cancel{\text{small}}$$

10-7-Example-PE3

$1\times10^{6}\,kg$

Potential energy - III

A rocket is launched straight up from the surface of a planet which has a mass of $1.0 \times 10^{24} kg$ and a radius of $1.6 \times 10^{6} m$. The planet has no atmosphere. The rocket has an initial velocity (straight up) of $8000 \frac{m}{s}$.

- What is the change in the rocket's potential energy as it moves to $2.4 \times 10^{6} m$ from the center of the planet?
- What is its speed at this distance from the planet?
- What is the rocket's maximum distance from the planet?

$$PE_g = -\frac{G m_A m_B}{|\vec{r}_A - \vec{r}_B|} + \cancel{C}^{\,0}$$

PE = 0 at ∞ separation

$$\Delta PE = PE_f - PE_i$$

$$= -\frac{G m_A m_B}{2.4\times10^{6} m} - \left(\frac{-G m_A m_B}{1.6\times10^{6} m}\right)$$

$$= \left(6.67\times10^{-11} \frac{Nm^2}{kg^2}\right)\left(1\times10^{24} kg\right)\left(1\times10^{6} kg\right)\left[\frac{-1}{2.4\times10^{6} m} + \frac{1}{1.6\times10^{6} m}\right]$$

$$= 6.67\times10^{19} Nm^2 \left[-\frac{1}{2.4\times10^{6} m} + \frac{1}{1.6\times10^{6} m}\right]$$

$$= 6.67\times10^{13} J \left(-\frac{1}{2.4} + \frac{1}{1.6}\right)$$

$$= 1.39\times10^{13} J$$

$$\Delta KE = W_{net}$$

$$\Delta KE + \Delta PE = \cancelto{0}{W_{NC}}$$

$$\frac{1}{2}\cancel{m}|\vec{v}_f|^2 - \frac{1}{2}\cancel{m}|\vec{v}_i|^2 + \cancel{\Delta PE} - \frac{GM\cancel{m}}{r_f} - \left(-\frac{GM\cancel{m}}{r_i}\right) = 0$$

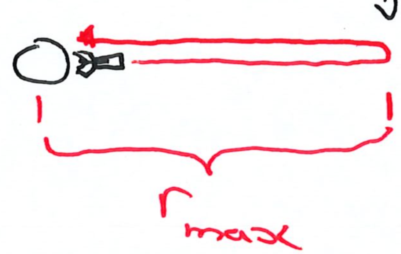
$$\frac{1}{2}|\vec{v}_f|^2 - \frac{1}{2}(8000\,m/s)^2 + \frac{1.39\times10^{13}\,J}{1\times10^{6}\,kg} = 0$$

$$\frac{1}{2}|\vec{v}_f|^2 - \underbrace{\frac{1}{2}(8000\,m/s)^2}_{3.2\times10^{7}\,m^2/s^2} + 1.39\times10^{7}\,m^2/s^2 = 0$$

$$\frac{1}{2}|\vec{v}_f|^2 = 1.81\times10^{7}\,m^2/s^2$$

$$|\vec{v}_f| = 6.02\times10^{3}\,m/s$$

Max distance



instant farthest

$|\vec{v}_f| = 0$

$$\Delta KE + \Delta PE = 0$$

$$\cancelto{0}{\frac{1}{2}m|\vec{v}_f|^2} - \frac{1}{2}\cancel{m}|\vec{v}_i|^2 - \frac{GM\cancel{m}}{r_{max}} + \frac{GM\cancel{m}}{r_i} = 0$$

$$\frac{1}{r_{max}} = \frac{1}{r_i} - \frac{1}{2GM}|\vec{v}_i|^2$$

$$\frac{1}{r_{max}} = 6.25\times10^{-7}\,m^{-1} - \underbrace{\frac{1}{2\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)\left(1\times10^{24}\,kg\right)}\left(8000\,m/s\right)^2}_{4.80\times10^{-7}\,m^{-1}}$$

$$= 1.45\times10^{-7}\,m^{-1}$$

$$r_{max} = 6.89\times10^{6}\,m$$

if big enough

$\frac{1}{r_{max}} < 0$

means object escapes to ∞

$\frac{1}{r_{max}} = 0$

the $|\vec{v}_i|$ for this "escape velocity"