202201 Math 122 [A01] Quiz #6

April 7th, 2022

Name: Solutions #V00: ______

This test has 2 pages and 6 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except where explicitly noted, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

   F (a) For positive integers a and b, if there exist integers x and y such that $ax + by = 3$, then $\gcd(a, b) = 3$. Can only say that $\gcd(a,b) \mid 3$, so $\gcd(a,b) = 1$ or 3.

   F (b) For integers a, b, and c, if a and b are relatively prime, and b and c are relatively prime, then a and c are relatively prime. 2 and 3, 3 and 4 are relatively prime, but 2 and 4 are not

   T (c) The last digit of 109^{100} is the same as the last digit of 1001^{122}. $109^{100} \equiv (-1)^{100} \equiv 1 \pmod{10}$ $1001^{122} \equiv 1^{122} \equiv 1 \pmod{10}$

   T (d) If $k \equiv 3 \pmod 8$, then $9k^2 - 5k + 4 \equiv 6 \pmod 8$.

   $9k^2 - 5k + 4 \equiv 81 - 15 + 4$
   $\equiv 1 - 7 + 4$
   $\equiv -2 \equiv 6 \pmod 8$

2. [3] Determine if $(14265)_7$ is divisible by 6. (Hint: modular arithmetic.)

$$\begin{aligned}(14265)_7 &= 1\times 7^4 + 4\times 7^3 + 2\times 7^2 + 6\times 7 + 5\\ &\equiv 1\times 1^4 + 4\times 1^3 + 2\times 1^2 + 6\times 1 + 5 \pmod 6\\ &\equiv 1+4+2+6+5 \pmod 6\\ &\equiv 18 \pmod 6\\ &\equiv 0 \pmod 6\end{aligned}$$

since $(14265)_7 \equiv 0 \pmod 6$ we know $(14265)_7$ is divisible by 6.

3. [2] Suppose $\mathcal{R}$ is an equivalence relation defined on the set $A = \{1, 2, 3, 4, 5, 6\}$, and the equivalence classes of $\mathcal{R}$ are $[1] = \{1, 4, 5\}$, $[2] = \{2, 6\}$, and $[3] = \{3\}$. Write the relation $\mathcal{R}$ as a list of ordered pairs.

$\mathcal{R} = \{(1,1), (4,4), (5,5), (1,4), (4,1), (1,5), (5,1), (4,5), (5,4), (2,2), (6,6), (2,6), (6,2), (3,3)\}$.

4. [2] Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{3x-4}{7}$ is one-to-one.

Suppose
$$\begin{aligned} f(x_1) &= f(x_2)\\ \frac{3x_1-4}{7} &= \frac{3x_2-4}{7}\\ 3x_1 - 4 &= 3x_2 - 4\\ 3x_1 &= 3x_2\\ x_1 &= x_2 \end{aligned}$$
$\therefore$ f is one-to-one.

5. Let $\mathcal{R}$ be the relation on the set $A = \{1, 2, \ldots, 10\}$ defined by $\mathcal{R} = \{(a, b) : a - b \geq 0\}$. Determine each of the following and provide a proof or counterexample as an explanation.

(a) [1] Is $\mathcal{R}$ reflexive?

Yes. Since $x - x = 0 \geq 0$ we have that $(x, x) \in \mathcal{R}$ for all $x \in A$.

(b) [1] Is $\mathcal{R}$ symmetric?

No. $(2,1) \in \mathcal{R}$ since $2 - 1 = 1 \geq 0$
but $(1,2) \notin \mathcal{R}$ since $1 - 2 = -1 \ngeq 0$

(c) [2] Is $\mathcal{R}$ transitive?

Yes. Suppose $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.
Then $x - y \geq 0$ and $y - z \geq 0$.
Now $x - z = x - y + y - z \geq 0$ since $x - y \geq 0$ and $y - z \geq 0$.
Therefore $(x, z) \in \mathcal{R}$.

6. [2] Indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

F (a) For a set A with at least two elements, the relation $A \times A$ is anti-symmetric.

F (b) If $\mathcal{R}$ is a transitive relation on $\{1, 2, 3\}$ such that $(1, 2) \in \mathcal{R}$ and $(2, 3) \notin \mathcal{R}$, then $(1, 3) \notin \mathcal{R}$.

T (c) Consider the relation $\mathcal{R}$ on the set $\mathbb{Z}$ defined by $\mathcal{R} = \{(a, b) : a \text{ and } b \text{ have the same last digit}\}$. Then $(-1327)\mathcal{R}(34167)$.

F (d) The set $\{(1, x), (2, z), (1, y)\}$ defines a function from $\{1, 2, 3\}$ to $\{x, y\}$.
1 is in more than one ordered pair.