11-3-Theory
-Equal Area

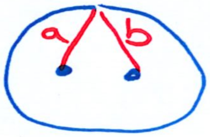
Kepler's Laws

For objects orbitting large, central mass under influence of gravity

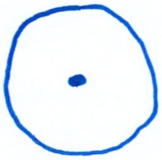
3 facts about orbits

1) Object orbits in ellipse with central mass at one of ellipse's foci (Proof: PHYS 321A)

Ellipse is ~ a circle



$a+b = \text{const}$



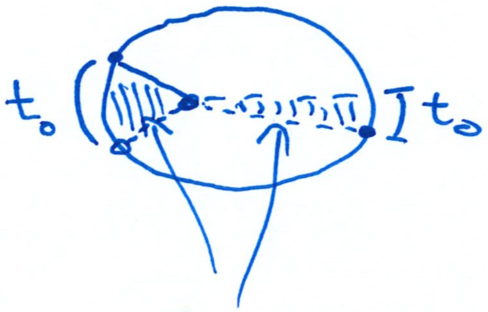
two foci same point → circle

2) "Harmonic law"

$$\underset{\substack{\uparrow \\ \text{period}}}{T^2} \propto \underset{\substack{\nwarrow \\ \text{orbital radius}}}{r^3}$$

Showed for circular orbit earlier

3) "Equal Area law"



Same areas

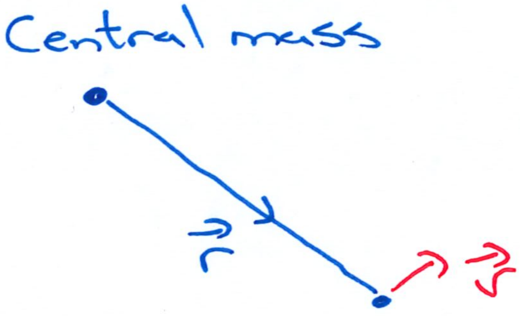
Plausible: closer in → goes faster

Central mass at origin

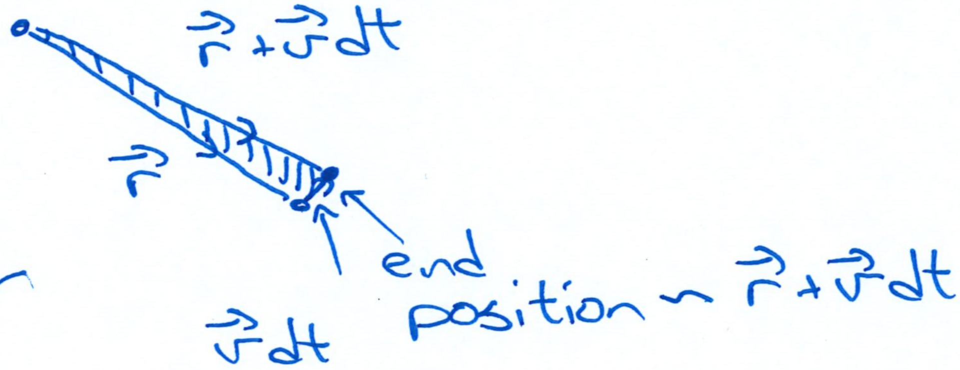
$$\vec{\tau}_{\left(\substack{\text{on orbiting}\\ \text{thing}}\right)} = \underbrace{\vec{r} \times \vec{F}_g}_{= 0}$$

in $-\vec{r}$

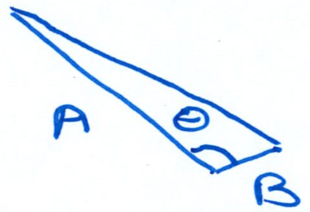
$$\hookrightarrow \quad \frac{d}{dt}\vec{L} = 0$$



What is area swept out in time dt?



For triangle



Area of triangle $\frac{1}{2} AB \sin\theta$

area $\frac{1}{2}|\vec r||\vec v dt| \underbrace{\sin\theta}_{\text{angle between } \vec r \,\&\, \vec v}$

area $\frac{1}{2} dt\, |\vec r \times \underset{\overset{\uparrow}{m}}{\vec v}|$

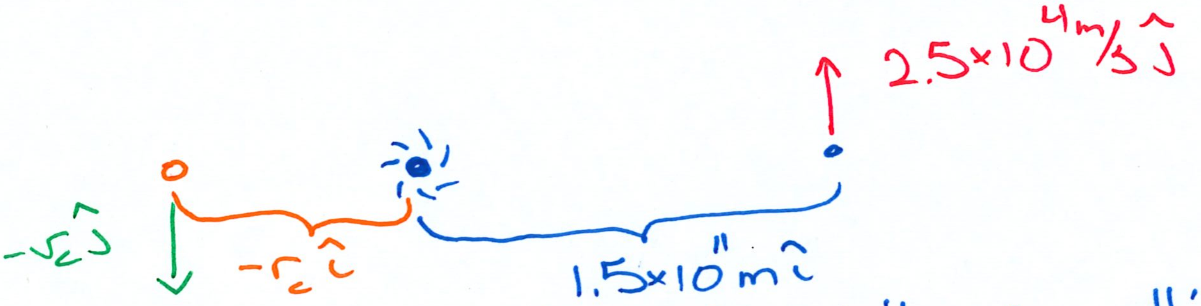
area $\propto \underset{\overset{\uparrow}{\text{constant}}}{|\vec L|}\, dt$

11-4-Example-Synthesis2

Synthesis - II

A planet orbits a star of mass $2.0 \times 10^{3}0 kg$. At a particular instant the planet is $1.5 \times 10^{11} m$ away from the star and the planet travels at $2.5 \times 10^{4} \frac{m}{s}$. The velocity vector is at 90° to the vector from the star to the planet.

What is the closest that the planet gets to the star?



Know : - Planet orbits in ellipse

- Only $\vec{F}$ from star
  - KE + PE conserved
  - $\vec{L}$ conserved

Closest - ~~Same location~~

- ~~Same $\vec{v}$~~
- $\vec{r}$ & $\vec{p}$ at 90°

$\vec{L}$ conserved ⇒

$$\vec{r}\times\vec{p}_i = m(1.5\times10^{11} m\,\hat{i})\times(2.5\times10^{4} m/s\,\hat{j})$$

$$= m(3.75\times10^{15} m^2/s\,\hat{k})$$

$$\vec{L} = \vec{r}\times\vec{p}_s = m(-r_c\hat{i})\times(-v_c\hat{j})$$

$$= m\,r_c v_c\,\hat{k}$$

$$\boxed{r_c v_c = 3.75\times10^{15}\ \text{m}^2/\text{s}}$$

At start

$$KE+PE = \tfrac{1}{2}m\left(2.5\times10^{4}\ \text{m/s}\right)^2 + \left(-\frac{G\,2\times10^{30}\ \text{kg}\ m}{1.5\times10^{11}\ \text{m}}\right)$$

$$= m\left(3.125\times10^{8}\ \text{m}^2/\text{s}^2 - 8.893\times10^{8}\ \text{m}^2/\text{s}^2\right)$$

$$= m\left(-5.768\times10^{8}\ \text{m}^2/\text{s}^2\right)$$

$$= \quad KE+PE = \tfrac{1}{2}m v_c^2 + \left(-\frac{G\,2\times10^{39}\ \text{kg}\ m}{r_c}\right)$$

$$\boxed{-5.768\times10^{8}\ \text{m}^2/\text{s}^2 = \tfrac{1}{2}v_c^2 - \frac{1.334\times10^{20}\ \text{m}^3/\text{s}^2}{r_c}}$$

$$\frac{1}{r_c} = \frac{v_c}{3.75\times10^{15}\ \text{m}^2/\text{s}}$$

$$-5.768\times10^{8}\ \text{m}^2/\text{s}^2 = \tfrac{1}{2}v_c^2 - 3.557\overset{\times10^4}{\downarrow}\text{m/s}\ v_c$$

$$v_c = \frac{3.557\times10^{4}\ \text{m/s} \pm \sqrt{\left(3.557\times10^{4}\right)^2 - 4\,\tfrac{1}{2}\left(5.768\times10^{8}\ \text{m}^2/\text{s}^2\right)}}{(2)\left(\tfrac{1}{2}\right)}$$

$$= 3.557\times10^{4}\,\text{m/s} \pm \sqrt{1.118\times10^{8}\,\text{m}^2/\text{s}^2}$$

$$= 3.557\times10^{4}\,\text{m/s} \pm 1.057\times10^{4}\,\text{m/s}$$

$$\cancel{2.5\times10^{4}}\ \text{m/s}$$

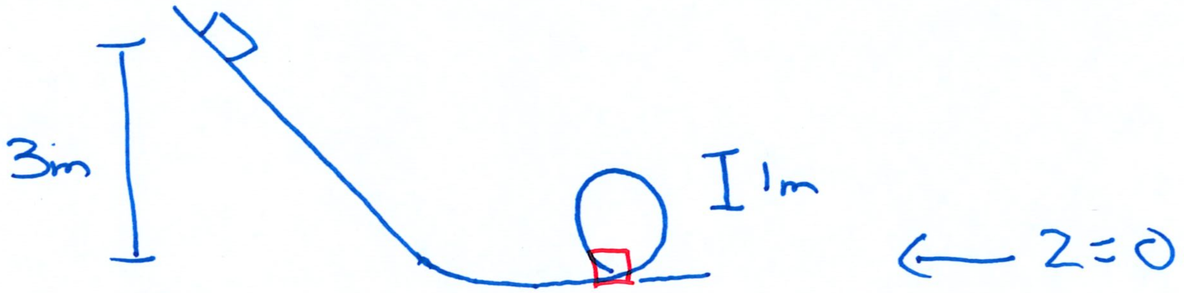
$$4.61\times10^{4}\,\text{m/s}$$

$$r_c = \frac{3.75\times10^{15}\,\text{m}^2/\text{s}}{4.61\times10^{4}\,\text{m/s}}$$

$$= 8.1\times10^{10}\,\text{m}$$

Transitions:

If there's an instant when we go from one situation to another then, can use info from first to help with second.



Mass m slides down frictionless ramp, goes through R= loop.
If m=2kg, what is $|\vec{F}_N|$ when mass at bottom of loop?

- Use conservation of energy to find speed at bottom
- Use speed + what know re circular motion ⇒ $|\vec{F}_N|$

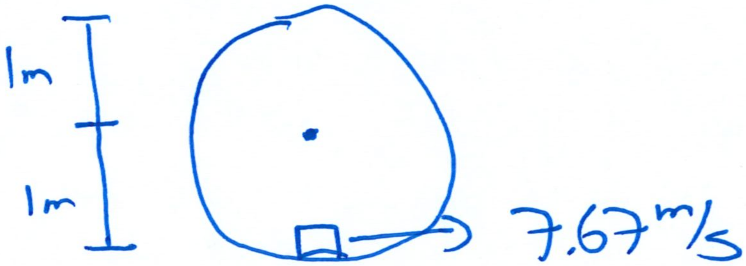
$$\Delta KE + \Delta PE = 0$$

$$KE_f - KE_i + PE_f - PE_i = 0$$

$$\frac{1}{2}m|\vec{v}_f|^2 - \frac{1}{2}m(0\,\text{m/s})^2 + mg0\text{m} - mg3\text{m} = 0$$

$$|\vec{v}_f|^2 = 2g3\text{m}$$

$$|\vec{v}_f| = 7.67\,\text{m/s}$$



$$\vec{a} = \frac{|\vec{v}|^2}{R}\ (\text{to center})$$

$$\vec{a} = \frac{|\vec{v}|^2}{R}\hat{k}$$

$$\frac{\vec{F}_{net}}{m} = \vec{a} = \frac{1}{m}\left(\vec{F}_g + |\vec{F}_N|\hat{k}\right)$$

$$\frac{|\vec{v}|^2}{R}\hat{k} = \frac{1}{m}\left(-mg\hat{k} + |\vec{F}_N|\hat{k}\right)$$

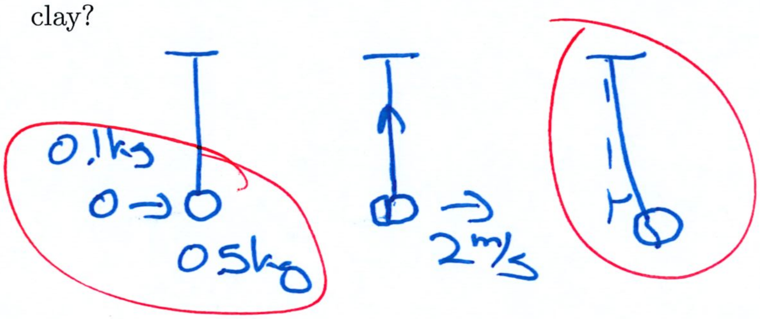
$$|\vec{F}_N| = \frac{m|\vec{v}|^2}{R} + mg \Rightarrow 137.2\,\text{N}$$

11-6-Example-Synthesis3

Synthesis - III

A clay lump with a mass of $0.50kg$ is suspended by a rope which is $0.8m$ long. A ball of mass $0.10kg$ is thrown at the lump of clay, and sticks to it. Just prior to the ball hitting the clay, the ball travelled horizontally with a speed of $12\frac{m}{s}$.

What is the tension in the rope just after the ball hits and sticks to the clay?



going in circle

Find speed of combo using $\vec{p}$

$$\vec{P}_i = \vec{P}_f$$

$$(0.1kg)(12\,m/s\,\hat{i}) + (0.5kg)(0\,m/s) = (0.6kg)\vec{v}$$

$$\vec{v} = 2\,m/s\,\hat{i}$$

Going in circle

$$\vec{a} = \frac{|\vec{v}|^2}{R}(\text{to center}) + \cancel{\frac{d|\vec{v}|}{dt}(\text{along } \vec{v})}$$

$$\vec{a} = \frac{(2\,m/s)^2}{0.8m}\hat{k}$$

$$= 5\,m/s^2\,\hat{k}$$

$$\frac{\vec{F}_{net}}{m} = \frac{1}{m}\left(\vec{F}_{rope} + \vec{F}_g\right)$$

$$= \frac{1}{m}\left(T\hat{k} - mg\hat{k}\right)$$

$$5\,m/s^2\,\hat{k} = \frac{1}{0.6kg}\left(T\hat{k} - (0.6kg)(9.8\,N/kg)\hat{k}\right)$$

$$3N\hat{k} = T\hat{k} - 5.88N\hat{k}$$

$$T = 8.88N$$