Math 110 - Homework 10
Topic: Eigenvalues and eigenvectors

Due at 6:00pm (Pacific) on Friday, November 26, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 5.1 of the online textbook, as well as the supplemental questions about complex numbers posted on Brightspace.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` is a square matrix then the command `eig(A)` returns a list of the eigenvalues of A, with each one repeated according to its algebraic multiplicity.
- If `A` is a square matrix and you run the command `[P,D] = eig(A)` then D will be a diagonal matrix whose entries are the eigenvalues of A (repeated according to algebraic multiplicity) and P will be a matrix whose columns are eigenvectors ordered so that the jth column of P is an eigenvector for the eigenvalue appearing in position (j, j) of D. If it is possible, P will be made to be invertible.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. Find all of the eigenvalues of A, as well as the algebraic and geometric multiplicities of each eigenvalue.

Solution: We start by finding the characteristic polynomial (students should show the steps of calculating the determinant):

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 2 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 4 & 4 & 4-\lambda & 4 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = \lambda(\lambda-6)(\lambda-3)^2.$$

We therefore have that the eigenvalues, with their algebraic multiplicities, are 0 ($\mathrm{alg}(0) = 1$), 6 ($\mathrm{alg}(6) = 1$) and 3 ($\mathrm{alg}(3) = 2$).

Now we turn to the geometric multiplicities. For the eigenvalue 0 we have (by a result from class) $1 \leq \mathrm{geo}(0) \leq \mathrm{alg}(0) = 1$, so $\mathrm{geo}(0) = 1$. Similarly $\mathrm{geo}(6) = 1$. We could also find these geometric multiplicities by finding bases for the eigenspaces.

For the eigenvalue 3 the information we have so far only tells us that $\mathrm{geo}(3)$ is either 1 or 2. To find out which it is we will need to find a basis for $E_3 = \mathrm{null}(A - 3I)$. We row reduce (again, students should show the steps here):

$$A - 3I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 9/4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From here we see that a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in E_3 if and only if $x_1 - 2x_3 = 0$ and $x_2 + (9/4)x_3 + x_4 = 0$.

That gives us:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -(9/4)x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -9/4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\left\{ \begin{bmatrix} 2 \\ -9/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for E_3, so $\dim(E_3) = 2$, and therefore $\mathrm{geo}(3) = 2$.

2. Let L be the line through the origin in $\mathbb{R}^2$ with slope 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto L, that is, $T(\overrightarrow{v}) = \mathrm{proj}_{\overrightarrow{d}}(\overrightarrow{v})$, where $\overrightarrow{d}$ is a direction vector for L. You may use, without proof, the fact that T is a linear transformation.

(a) Find the matrix $[T]$.

Solution: We need to calculate $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$. To do that, we first note that a direction vector for L is $\overrightarrow{d} = \begin{bmatrix}1\\3\end{bmatrix}$. Therefore

$$\begin{aligned}
T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) &= \mathrm{proj}_{\overrightarrow{d}}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&= \left(\frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}{\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix}}\right)\begin{bmatrix}1\\3\end{bmatrix}\\
&= \begin{bmatrix}1/10\\3/10\end{bmatrix}
\end{aligned}$$

Similarly,

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3/10\\9/10\end{bmatrix}.$$

Therefore the matrix of T is

$$A = [T] = \begin{bmatrix}1/10 & 3/10\\3/10 & 9/10\end{bmatrix}.$$

(b) Find the eigenvalues of $[T]$.

Solution: The eigenvalues of T are the same as the eigenvalues of the matrix A we found in part (a), so we calculate using A.

$$\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix}1/10-\lambda & 3/10\\3/10 & 9/10-\lambda\end{bmatrix}\right)\\
&= (1/10-\lambda)(9/10-\lambda)-(3/10)(3/10)\\
&= \lambda^2-\lambda\\
&= \lambda(\lambda-1)
\end{aligned}$$

Therefore the eigenvalues are 0 and 1.

(c) Find a basis for each eigenspace of $[T]$.

Solution: For $\lambda = 0$, we row-reduce

$$A - 0I \to \begin{bmatrix}1 & 3\\0 & 0\end{bmatrix}.$$

We then see that $\begin{bmatrix}x\\y\end{bmatrix}$ is in E_0 if and only if $x+3y=0$, i.e., $y = -\frac{1}{3}x$. Therefore a basis for E_0 is

$$\left\{\begin{bmatrix}1\\-1/3\end{bmatrix}\right\}.$$

For $\lambda = 1$, we row-reduce

$$A - I = \begin{bmatrix} -9/10 & 3/10 \\ 3/10 & -1/10 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore $\begin{bmatrix} x \\ y \end{bmatrix}$ is in E_1 if and only if $y = 3x$, so a basis for E_1 is

$$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}.$$

(d) Briefly explain what your results from (b) and (c) mean geometrically.

Solution: The vectors in E_1 are not moved at all by T, and we see from (b) that $\vec{v}$ is in E_1 if and only if $\vec{v}$ is in the same direction as the line L. On the other hand, vectors in E_0 are sent to the vector $\vec{0}$, and we see from (b) that the vectors in E_0 are exactly those vectors in the direction orthogonal to L. In summary, T does not change vectors lying along L, and it sends vectors orthogonal to L to $\vec{0}$.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. In this question, as usual, $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are the standard basis vectors for $\mathbb{R}^3$ (that is, $\vec{e_j}$ has a 1 in the jth position, and has 0 everywhere else).

(a) Suppose that D is a 3×3 diagonal matrix. Show that $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are eigenvectors of D.

Solution: Suppose that $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Then

$$D\vec{e}_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a\vec{e}_1.$$

A similar calculation shows that $D\vec{e}_2 = b\vec{e}_2$ and $D\vec{e}_3 = c\vec{e}_3$.

(b) Suppose that A is a 3×3 matrix, and that $\vec{e_1}, \vec{e_2}$, and $\vec{e_3}$ are eigenvectors of A. Is it true that A must be a diagonal matrix? If so, explain why. If not, give a specific example of a non-diagonal matrix A for which $\vec{e_1}, \vec{e_2}$, and $\vec{e_3}$ are eigenvectors.

Solution: We show that A must be diagonal. The key observation is that $A\vec{e}_i$ is the ith column of A.

Let $\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3$ be the columns of A, and let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues corresponding to the eigenvectors $\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3$. Then on the one hand $A\overrightarrow{e}_1 = \lambda_1\overrightarrow{e}_1$, but on the other hand the observation noted above tells us that $A\overrightarrow{e}_1 = \overrightarrow{a}_1$. We therefore have $\overrightarrow{a}_1 = \lambda_1\overrightarrow{e}_1$, so the first column of A is $\begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix}$. Similarly $\overrightarrow{a}_2 = \lambda_2\overrightarrow{e}_2$ and $\overrightarrow{a_3} = \lambda_3\overrightarrow{e_3}$, so

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

That is, A is diagonal.

2. Let $B = \begin{bmatrix} -2 & -14 & 0 & -6 & -12 & -6 \\ 21 & 17 & 25 & -8 & 16 & 8 \\ -1 & 3 & 5 & -2 & 4 & 2 \\ -8 & 4 & -20 & 4 & 12 & 16 \\ -26 & -2 & -80 & 8 & 4 & 2 \\ 4 & -2 & 40 & -12 & 4 & 12 \end{bmatrix}$.
Explain why every eigenvector of B must have at least one entry that is not a real number.

Solution: Using the MATLAB command `eig` we find that the eigenvalues of B are $10i, -10i, 10+2i, 10-2i, 10+10i, 10-10i$. In particular, we notice that none of the eigenvalues are real. Now if $\overrightarrow{0} \neq \overrightarrow{v}$ is any vector with all real entries then, since the entries of B are all real, $B\overrightarrow{v}$ will have all real entries. On the other hand, if λ is any of the eigenvalues listed above, then if the entries of $\overrightarrow{v}$ are all real we will have that the entries of $\lambda\overrightarrow{v}$ are not real, making $B\overrightarrow{v} = \lambda\overrightarrow{v}$ impossible. Thus any non-zero vector with all real entries cannot be an eigenvector of B, so each eigenvector must have at least one non-real entry.