Math 110 - Homework 7
Topic: Matrix algebra

Due at 6:00pm (Pacific) on Friday, October 29, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 4.1-4.3 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- If `A` and `B` are matrices of the same size, then to add them use `A + B`.
- If `A` is a matrix then $2A$ is computed by `2 * A`.
- If `A` is a matrix then A^t is computed by `A'`.
- If `A` and `B` are matrices of appropriate sizes so that AB is defined, it is computed by `A * B`.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Let $A = \begin{bmatrix} 1 & -3 & 5 & 2 \\ 0 & -2 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. For each of the following parts, either perform the indicated calculation or explain why doing so is impossible.

   (a) $A + B$

   Solution: Not defined, because the sizes of A and B are not the same.

   (b) BA

   Solution: $BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 & 2 \\ 0 & -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 & 2 \\ 0 & 2 & -1 & 1 \\ 1 & -9 & 8 & -1 \\ 1 & -5 & 6 & 1 \end{bmatrix}$.

   (c) BA^t

   Solution: Not defined, because B is 4×2 and A^t is also 4×2, so the number of columns of B does not match the number of rows of A^t.

   (d) CA^t

   Solution: $CA^t = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & -2 \\ 5 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -7 \\ 2 & -1 \end{bmatrix}$.

   (e) $AB + CB$

   Solution: $AB + CB = (A + C)B = \begin{bmatrix} 3 & 0 & 5 & 3 \\ 0 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 1 & 5 \end{bmatrix}$.

   (f) $A^2 B^2$

   Solution: Not defined, because A is not square, so A^2 is not defined.

   (g) $(AB)^2$

Solution: $(AB)^2 = \left(\begin{bmatrix} 1 & -3 & 5 & 2 \\ 0 & -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} 8 & 20 \\ 0 & 4 \end{bmatrix}^2 = \begin{bmatrix} 64 & 240 \\ 0 & 16 \end{bmatrix}$.

~~2. Let $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$. Find 2×2 matrices B and C such that $AB = AC$ but $B \neq C$.~~

~~**Solution:** Many examples are possible. One way that we might find such an example is to notice that the second column of A is $-2/3$ times the first column of A. Thus taking twice the first column plus three times the second column will give us the zero vector. That is, using the definition of multiplying a matrix times a vector, we have $\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now the matrix AB will be obtained by multiplying A times each of the columns of B and putting the results as the columns of AB. If $B = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ we therefore have $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Choosing $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we also get $AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but we can see that $B \neq C$.~~

3. Suppose that A and B are 3×3 symmetric matrices.

(a) Find an example where AB is not symmetric.

Solution: We could approach this systematically, by assigning variables to the entries of A and B and then carrying out the multiplication. However, we can also get away with just doing a guess and check. One possible solution (among infinitely many possibilities!) is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

We can see visually that $A = A^t$ and $B = B^t$, but

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix},$$

so $(AB)^t \neq AB$, and hence AB is not symmetric.

(b) Now suppose that we also know that $AB = BA$. Show that in this case AB is symmetric.

Solution: We have $A = A^t$, $B = B^t$, and $AB = BA$, so we have:

$$(AB)^t = B^t A^t = BA = AB.$$

Therefore in this case AB is symmetric.

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_4 \\ x_1 + x_2 + 3x_3 \\ x_1 - x_5 \\ 3x_4 + x_5 \\ x_2 + 2x_3 \end{bmatrix},$$

and let $S : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 - x_3 \\ 3x_1 + 2x_3 \\ 2x_2 - 5x_3 \\ 4x_1 + 5x_2 - 6x_3 \\ 2x_1 - x_2 + x_3 \end{bmatrix}.$$

Is there a vector $\overrightarrow{v}$ in $\mathbb{R}^3$ such that $(T \circ S)(\overrightarrow{v}) = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$? If so, find one. If not, explain why not.

Solution: There are several ways to approach this question; we outline a solution that lets MATLAB do the annoying calculations.

By reading off coefficients, or by using the definition, we have

$$[T] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix},$$

and

$$[S] = \begin{bmatrix} 1 & -1 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -5 \\ 4 & 5 & -6 \\ 2 & -1 & 1 \end{bmatrix}.$$

Let $\overrightarrow{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$. Let $A = [T \circ S]$. Then we know that a vector $\overrightarrow{v}$ such that $(T \circ S)(\overrightarrow{v}) = \overrightarrow{b}$ is the same thing as a solution to the linear system represented by the augmented matrix $[A|\overrightarrow{b}]$, and we also know that $A = [T][S]$. We therefore ask MATLAB to compute `rref`$([[T]*[S]\ \vec{b}])$. It finds that the reduced row echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The fourth row tells us that there are no solutions, so there is no vector $\vec{v}$ that makes $(T \circ S)(\vec{v}) = \vec{b}$.