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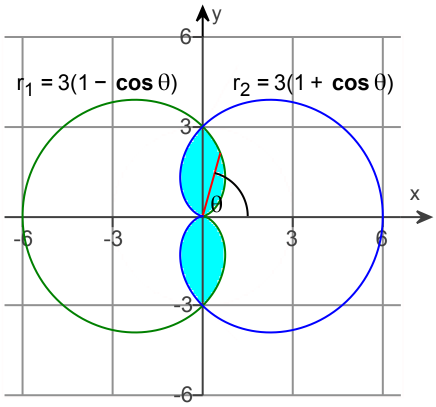
Find the area of the region shared by the cardioids $r = 3(1 + \cos\theta)$ and $r = 3(1 - \cos\theta)$.

Sketch the region to determine its boundaries and find the limits of integration. The region shared by the two cardioids is the shaded region shown to the right.




Notice that $r_1 = 3(1 - \cos(-\theta)) = 3(1 - \cos\theta)$ and $r_2 = 3(1 + \cos(-\theta)) = 3(1 + \cos\theta)$. Thus, there is symmetry with respect to the x-axis. The area of the entire shaded region is twice the area of the shaded region above the x-axis.

Notice from the graph that the two cardioids of the region above the x-axis intersect at (0,0) and (0,3). Thus, the shaded region above the x-axis can be divided into the region in the first-quadrant limited by the graph of $r_1$ and the coordinate axes, and the region in the second-quadrant limited by the graph of $r_2$ and the coordinate axes.

The area of the entire shaded region can be written as $A = 2\left(\int_\alpha^\beta \frac{1}{2}r_1^2\,d\theta + \int_\beta^\gamma \frac{1}{2}r_2^2\,d\theta\right)$, where $\alpha$ is the lower limit of $\theta$ on $r_1$, $\beta$ is the upper limit of $\theta$ on $r_1$ and the lower limit of $\theta$ on $r_2$, and $\gamma$ is the upper limit of $\theta$ on $r_2$.

The area in the first-quadrant covered by r, which lies within the cardioid $r_1 = 3(1 - \cos\theta)$, corresponds to $\theta$ between the positive x-axis $(\theta = 0)$ and the positive y-axis $\left(\theta = \frac{\pi}{2}\right)$. Thus, $\alpha = 0$ and $\beta = \frac{\pi}{2}$ radians. The area in the second-quadrant covered by r, which lies within the cardioid $r_2 = 3(1 + \cos\theta)$, corresponds to $\theta$ between the positive y-axis $\left(\theta = \frac{\pi}{2}\right)$ and the negative x-axis $(\theta = \pi)$. Thus, $\beta = \frac{\pi}{2}$ and $\gamma = \pi$ radians.

Substituting for $\alpha$, $\beta$ and $\gamma$, and taking the factor of 2 inside the integrals, the area of the shaded region is given by

$$\int_0^{\pi/2} r_1^2\,d\theta + \int_{\pi/2}^{\pi} r_2^2\,d\theta.$$

Substitute for $r_1$ and $r_2$ in the integral formula for the area and expand the terms.

$$\begin{aligned} A &= \int_0^{\pi/2} r_1^2\,d\theta + \int_{\pi/2}^{\pi} r_2^2\,d\theta \\ &= \int_0^{\pi/2} 3^2(1 - \cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} 3^2(1 + \cos\theta)^2\,d\theta \\ &= \int_0^{\pi/2} 9\left(1 - 2\cos\theta + \cos^2\theta\right)\,d\theta + \int_{\pi/2}^{\pi} 9\left(1 + 2\cos\theta + \cos^2\theta\right)\,d\theta \end{aligned}$$

In order to integrate the term in $\cos^2\theta$, substitute $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$.

$$A = \int_0^{\pi/2} 9\left(1 - 2\cos\theta + \cos^2\theta\right)\, d\theta + \int_{\pi/2}^{\pi} 9\left(1 + 2\cos\theta + \cos^2\theta\right)\, d\theta$$

$$= \int_0^{\pi/2} 9\left(1 - 2\cos\theta + \left(\frac{1 + \cos 2\theta}{2}\right)\right) d\theta + \int_{\pi/2}^{\pi} 9\left(1 + 2\cos\theta + \left(\frac{1 + \cos 2\theta}{2}\right)\right) d\theta$$

Simplify the integrand and integrate term by term. Use $\frac{d}{d\theta}(\sin n\theta) = n \cos n\theta$.

$$A = \int_0^{\pi/2} 9\left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta + \int_{\pi/2}^{\pi} 9\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{\pi/2} 9\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta + \int_{\pi/2}^{\pi} 9\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 9\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} + 9\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_{\pi/2}^{\pi}$$

Group like terms together and evaluate term by term.

$$A = 9\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} + 9\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_{\pi/2}^{\pi}$$

$$= 9\left[\frac{3}{2}\theta\right]_0^{\pi/2} - 9[2\sin\theta]_0^{\pi/2} + 9\left[\frac{\sin 2\theta}{4}\right]_0^{\pi/2} + 9\left[\frac{3}{2}\theta\right]_{\pi/2}^{\pi} + 9[2\sin\theta]_{\pi/2}^{\pi} + 9\left[\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

Evaluate the first term.

$$9\left[\frac{3}{2}\theta\right]_0^{\pi/2} = 9\left(\frac{3\pi}{4} - 0\right) = \frac{27\pi}{4}$$

Evaluate the second term, using $\sin\frac{\pi}{2} = 1$ and $\sin 0 = 0$.

$$9[2\sin\theta]_0^{\pi/2} = 9\left(2\sin\frac{\pi}{2} - 2\sin 0\right) = 9((2\times 1) - (2\times 0)). = 18$$

Evaluate the third term, using $\sin\pi = 0$ and $\sin 0 = 0$.

$$9\left[\frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 9\left(\frac{\sin\pi}{4} - \frac{\sin 0}{4}\right) = 9(0 - 0) = 0$$

Similarly, evaluate the remaining terms $9\left[\frac{3}{2}\theta\right]_{\pi/2}^{\pi}$, $9[2\sin\theta]_{\pi/2}^{\pi}$, and $9\left[\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$.

$$9\left[\frac{3}{2}\theta\right]_{\pi/2}^{\pi} = \frac{27\pi}{4} \qquad 9[2\sin\theta]_{\pi/2}^{\pi} = -18 \qquad 9\left[\frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi} = 0$$

Simplify to obtain the area inside the shaded region.

$$A = 9\left[\frac{3}{2}\theta\right]_0^{\pi/2} - 9[2\,\mathbf{sin}\,\theta]_0^{\pi/2} + 9\left[\frac{\mathbf{sin}\,2\theta}{4}\right]_0^{\pi/2} + 9\left[\frac{3}{2}\theta\right]_{\pi/2}^{\pi} + 9[2\,\mathbf{sin}\,\theta]_{\pi/2}^{\pi} + 9\left[\frac{\mathbf{sin}\,2\theta}{4}\right]_{\pi/2}^{\pi}$$

$$= \frac{27\pi}{4} - 18 + 0 + \frac{27\pi}{4} - 18 + 0$$

$$= \frac{27\pi}{2} - 36.$$