CSC 225

Algorithms and Data Structures I

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ECS 516

Fundamental Principles of Counting

The Rule of Sum

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

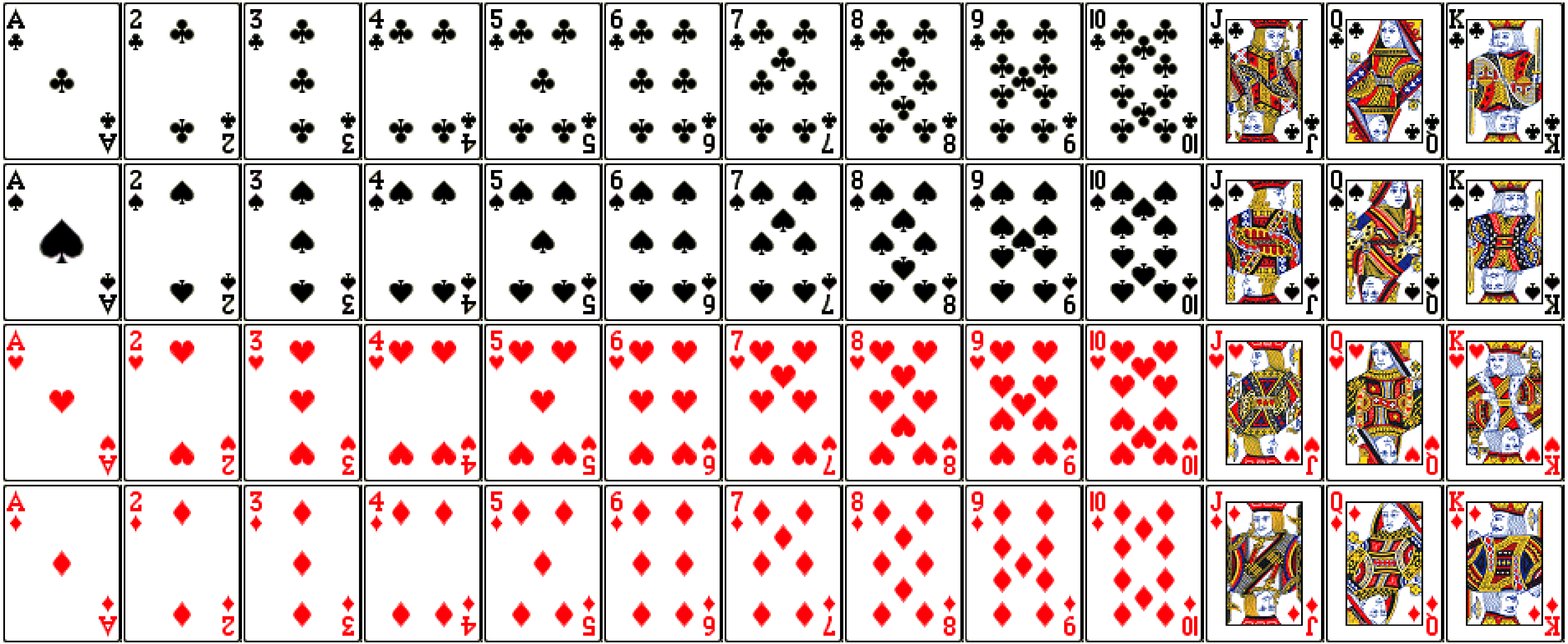
Fundamental Principles of Counting

The Rule of Product

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks can be performed simultaneously, then performing both tasks can be accomplished in any one of $m \cdot n$ ways.

Consider a standard deck of cards

- 4 suits – 2 black (clubs, spades), 2 red (hearts, diamonds)
- 13 “ranks” in each – Ace, 1, …, 10, Jack, Queen, King



Example 1

I have 5 distinct red cards and 4 distinct black cards

$m=5$, $n=4$

a) How many ways can I choose 1 card?

$m+n=9$

b) How many ways can I choose 1 red card then 1 black card?

$m \cdot n = 20$

c) How many ways can I choose 2 cards?

1st

$$\underbrace{(5}_{\text{red cards}}+\underbrace{4)}_{\text{Black cards}}(\underbrace{5+4}_{\text{red + black cards}}\underbrace{-1)}_{\text{already chose 1 card}} = 9 \cdot 8 = 72$$

Fundamental Principles of Counting

Permutation

Application of the Rule of Products when counting linear arrangements of distinct objects.

Example 2

I have 5 distinct red cards and 4 distinct black cards

a) How many arrangements of the red cards?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{5!} = 120$

✓ understandable

b) How many arrangements of the black cards?

$4! = 24$

✓ same

c) How many ways of arranging all the red followed by all the black cards?

$5! \cdot 4! = 120 \cdot 24 = 2880$

happening in stages

d) How many arrangements of all the cards?

$(5+4)! \leftarrow 9! = 362{,}880 \longrightarrow (5+4)!$

Example 3a

Consider a full standard deck of 52 distinct cards

a) How many arrangements of a standard deck of cards are there? That is, what is the total number of possible shuffles?

$52! = 8.065817\ldots \times 10^{67}$

Example 3b

Consider a full standard deck of 52 distinct cards

b) How many ways can I arrange 5 cards from the deck? That is, how many permutations of 5 cards from 52? $\rightarrow \frac{52!}{(52-5)!}$

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311,875,200$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47!} = \frac{52!}{47!}$$

$$\frac{52!}{(52-5)!}$$

Fundamental Principles of Counting

Permutations

In general, the number of permutations of size r from n distinct objects, where $0 \leq r \leq n$, is given by

${}_nP_r$

P^n_r

${}_nP_R$

$$P(n,r) = \frac{n!}{(n-r)!}$$

- Note: $P(n,0) = \frac{n!}{n!} = 1$ and $P(n,n) = \frac{n!}{0!} = n!$

Example 4

a) How many permutations of the letters in the word COMPUTER?

$n = 8$

$$P(8,8) = 8! = 40,320$$

b) What if we permute only 5 letters from COMPUTER?

$r = 5$

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{40,320}{6} = 6,720$$

c) How many permutations of BALL?

$n = 4$

Example 4 continued

c) Note that in practice we cannot distinguish between the two L's as say L_1 and L_2

Table 1.1

A	B	L	L
A	L	B	L
A	L	L	B
B	A	L	L
B	L	A	L
B	L	L	A
L	A	B	L
L	A	L	B
L	B	A	L
L	B	L	A
L	L	A	B
L	L	B	A

(a)

A	B	L_1	L_2	A	B	L_2	L_1
A	L_1	B	L_2	A	L_2	B	L_1
A	L_1	L_2	B	A	L_2	L_1	B
B	A	L_1	L_2	B	A	L_2	L_1
B	L_1	A	L_2	B	L_2	A	L_1
B	L_1	L_2	A	B	L_2	L_1	A
L_1	A	B	L_2	L_2	A	B	L_1
L_1	A	L_2	B	L_2	A	L_1	B
L_1	B	A	L_2	L_2	B	A	L_1
L_1	B	L_2	A	L_2	B	L_1	A
L_1	L_2	A	B	L_2	L_1	A	B
L_1	L_2	B	A	L_2	L_1	B	A

(b)

$4!$

L's $\overline{}$ $2!$

$$\frac{4!}{1!1!2!} = 12$$

d) Now consider the permutations of DATABASES.

$n = 9$, 3 A's, 2 S's

$$\frac{9!}{1!1!1!1!3!2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2} = 30{,}240$$

Fundamental Principles of Counting

In general, the number of linear arrangements of n objects is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

where there are n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, …, and n_r indistinguishable objects of an rth type and $n_1 + n_2 + \cdots + n_r = n$

Example 3 Revisited

Consider a full standard deck of 52 distinct cards

c) How many distinct poker hands exist? That is, how many ways can you pull 5 cards from 52 where order doesn't matter?

$$P(52,5) = \frac{52!}{47!} = \frac{311{,}855{,}200}{120} = \boxed{2{,}598{,}960}$$

$n = 52, \; r = 5$

Permutation: $\frac{n!}{(n-r)!}$

Combination: $\frac{n!}{r!(n-r)!}$

$$\frac{P(n,r)}{r!} = C(n,r)$$

Fundamental Principles of Counting

Combinations

In general, the number of combinations of r objects from n distinct objects, where $0 \leq r \leq n$, is given by

$$\binom{n}{r} = C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- Note: $C(n, 0) = \frac{n!}{0!\,n!} = 1$ and $C(n, n) = \frac{n!}{n!\,0!} = 1$

Poker Hand Rankings

1.
2.
3.
4.

13 Ranks
+
2 3 4 5
6 7 8 9
10
A K Q J

4 Suits
+
Clubs
Hearts
Spades
Diamonds

Example 3 Revisited

Consider a full standard deck of 52 distinct cards

d) How many royal flushes exist? 4

e) How many straight flushes? $9 \cdot 4 = 36$

f) 4 of a kind? $\binom{13}{1}\binom{48}{1} = 624$

g) Full house?

h) …

$$\underbrace{\underbrace{\binom{13}{1}\binom{4}{3}}_{\text{3 of kind}}\underbrace{\binom{12}{1}\binom{4}{2}}_{\text{2 of kind}}}_{\text{permutation}} = 3744$$

~~permutati~~ permutation

Example 5

7 people go for lunch. Each person has an option of one of a cheeseburger (c), hot dog (h), taco (t), or fish sandwich (f). How many possible orders are there?

- Possibilities:
  1. ccccccc
  2. chhttff
  3. hhhffff
  4. …

Fundamental Principles of Counting

Combinations with Repetition

In general, taking n distinct objects, with repetition, taken r at a time can be done in

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

ways.

Example 6

n

A donut shop has 20 distinct types of donuts with at least 12 of each type in the store. How many ways can you select a dozen (12) donuts?

$n = 20, \; r = 12$

$$\binom{20+12-1}{12} = \binom{31}{12} = 141{,}120{,}525$$

Example 7

Determine all the integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7$$

$n = 4$

$r = 7$

x|xx|xxx|xx

where $x_i \geq 0$ for all $i = 1,2,3,4$.

$\binom{10}{4}$ $\binom{9}{3}$ $\binom{10}{7} = 120$ $\binom{11}{4}$ $\binom{7}{4}$

$\binom{10}{3} = \frac{10!}{3!\,7!}$ ←

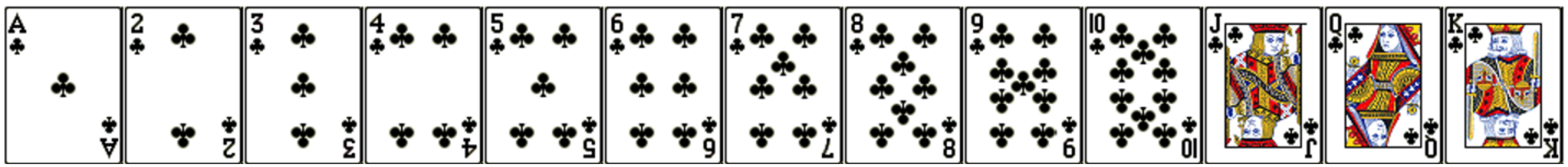
Pigeonhole Principle

The Pigeonhole Principle

If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.

Example 8

If I draw 14 cards from a standard deck of 52, will there be a pair?



ranks: $n = 13$ ← pigeonhole

cards: $m = 14$ ← pigeons

∴: by PHP we must have at least one pair

Example 9

Given a text file with 500,000 words of size 4 or less, is it possible that the words are distinct?

$\#\text{words} = 26^4 + 26^3 + 26^2 + 26^1$

$= 475{,}254 = n$ = possible words formed w/ choices

$m = 500{,}000$

m = words that exist in the file

Since $m > n$, there at least two words that are the same

Example 10

While on a 4-week vacation I will drink at least 1 beer a day but no more than 40 total. Will there be a span of consecutive days in which I drink exactly 15 beers?