0

Yes, There Are Proofs!

"How many dots are there on a pair of dice?" The question once popped out of the box in a game of trivia in which one of us was a player. A long pause and much consternation followed the question. After the correct answer was finally given, the author (a bit smugly) pointed out that the answer "of course" was 6×7, twice the sum of the integers from 1 to 6. "This is because," he declared, "the sum of the integers from 1 to n is $\frac{1}{2}n(n+1)$, so twice this sum is $n(n+1)$ and, in this case, $n = 6$."

"What?" asked one of the players.

At this point, the game was delayed for a considerable period while the author found pencil and paper and made a picture like that in Fig. 0.1. If we imagine the dots on one die to be solid and on the other hollow, then the sum of the dots on two dice is just the number of dots in this picture. There are seven rows of six dots each—42 dots in all. What is more, a similar picture could be drawn for seven-sided dice showing that

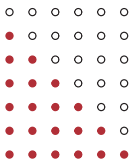


Figure 0.1

$$2(1+2+3+4+5+6+7) = 7 \times 8 = 56$$

and, generally,

$$2(1+2+3+\cdots+n) = n \times (n+1). \qquad (*)$$

Sadly, that last paragraph is fictitious. Everybody was interested in, and most experimented with, the general observation in equation $(*)$, but nobody (except an author) cared why. Everybody wanted to resume the game!

Pause 1 What is the sum $1+2+3+\cdots+100$ of the integers from 1 to 100? ▮

"Are there proofs?" This is one of the first questions students ask when they enter a course in analysis or algebra. Young children continually ask "why" but, for whatever reason, as they grow older, most people only want the facts. We take the view that intellectual curiosity is a hallmark of advanced learning and that the ability to reason logically is an increasingly sought after commodity in the world today. Since sound logical arguments are the essence of mathematics, the subject provides a marvelous training ground for the mind. The expectation that a math course will sharpen the powers of reason and provide instruction in clear thinking surely accounts for the prominence of mathematics in so many university programs today. So yes, **proofs**—**reasons** or **convincing arguments** probably sound less intimidating—will form an integral part of the discussions in this book.

In a scientific context, the term *statement* means an ordinary English statement of fact (subject, verb, and predicate in that order) that can be assigned a *truth value*; that is, it can be classified as being either true or false. We occasionally say *mathematical* statement to emphasize that a statement must have this characteristic property of being true or false. The following are all mathematical statements:

There are 168 primes less than 1000.

Seventeen is an even number.

$\sqrt{3}^{\sqrt{3}}$ is a rational number.

Zero is not negative.

Each statement is certainly either true or false. (It is not necessary to know which.) On the other hand, the following are not mathematical statements:

What are irrational numbers?

Suppose every positive integer is the sum of three squares.

The first is a question and the second a conditional; it would not make sense to classify either as true or false.

0.1 Compound Statements

"And" and "Or"

A *compound statement* is a statement formed from two other statements in one of several ways, for example, by linking them with "and" or "or." Consider

$9 = 3^2$ and $3.14 < \pi$.

This is a compound statement formed from the simpler statements "$9 = 3^2$" and "$3.14 < \pi$." How does the truth of an "and" compound statement depend on the truth of its parts? The rule is

> "p and q" is true if both p and q are true; it is false if either p is false or q is false.

Thus, "$-2^2 = -4$ and $5 < 100$" is true, while "$2^2 + 3^2 = 4^2$ and $3.14 < \pi$" is false.

In the context of mathematics, just as in everyday English, one can build a compound statement by inserting the word "or" between two other statements. In everyday English, "or" can be a bit problematic because sometimes it is used in an inclusive sense, sometimes in an exclusive sense, and sometimes ambiguously, leaving the listener unsure about just what was intended. We illustrate with three sentences.

"To get into that college, you have to have a high school diploma or be over 25." (Both options are allowed.)

"That man is wanted dead or alive." (Here both options are quite impossible.)

"I am positive that either blue or white is in that team's logo." (Might there be both?)

Since mathematics does not tolerate ambiguities, we must decide precisely what "or" should mean. The decision is to make "or" inclusive: "or" always includes the possibility of both.

"p or q" is true if p is true or q is true or both p and q are true; it is false only when both p and q are false.

Thus,

"$7 + 5 = 12$ or 571 is the 125th prime" and "25 is less than or equal to 25"

are both true sentences, while

"5 is an even number or $\sqrt{8} > 3$"

is false.

Implication

Many mathematical statements are *implications*; that is, statements of the form "p implies q," where p and q are statements called, respectively, the *hypothesis* and *conclusion*. The symbol $\rightarrow$ is read *implies*, so

Statement 1: "2 is an even integer $\rightarrow$ 4 is an even integer"

is read "2 is an even integer implies 4 is an even integer."

In Statement 1, "2 is an even number" is the hypothesis and "4 is an even number" is the conclusion.

Implications often appear without the explicit use of the word *implies*. To some ears, Statement 1 might sound better as

"If 2 is an even number, then 4 is an even number."

or

"4 is an even number only if 2 is an even number."

Whatever wording is used, common sense tells us that this implication is true.

Under what conditions will an implication be false? Suppose your parents tell you

Statement 2: If it is sunny tomorrow, you may go swimming.

If it is sunny, but you are not allowed to go swimming, then clearly your parents have said something that is false. However, if it rains tomorrow and you are not allowed to go swimming, it would be unreasonable to accuse them of breaking their word. This example illustrates an important principle.

The implication "$p \rightarrow q$" is false only when the hypothesis p is true and the conclusion q is false. In all other situations, it is true.

In particular, Statement 1 is true since both the hypothesis "2 is an even number" and the conclusion "4 is an even number" are true. Note, however, that an implication is true whenever the hypothesis is false (no matter whether the conclusion is true or false). For example, if it were to rain tomorrow, the implication contained in Statement 2 is true because the hypothesis is false. For the same reason, each of the following implications is true.

If -1 is a positive number, then $2 + 2 = 5$.

If -1 is a positive number, then $2 + 2 = 4$.

Pause 2

Think about the implication, "If $4^2 = 16$, then $-1^2 = 1$." Is it true or false? ▮

The Converse of an Implication

The *converse* of the implication $p \to q$ is the implication $q \to p$. For example, the converse of Statement 1 is

"If 4 is an even number, then 2 is an even number."

Pause 3

Write down the converse of the implication given in PAUSE 2. Is this true or false? ▮

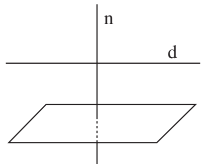
Pause 4

A student was once asked to show that a certain line (in 3-space) was parallel to a certain plane.

Julie answered

If a line d is parallel to a plane and n is a line perpendicular to the plane, then d and n must be perpendicular,

and she proceeded to establish (correctly) that d was perpendicular to n. Was Julie's logic correct? Comment. ▮



Double Implication

Another compound statement that we will use is the double implication $p \leftrightarrow q$, read "p *if and only if* q." As the notation suggests, the statement "$p \leftrightarrow q$" is simply a convenient way to express

"$p \to q$" and "$p \leftarrow q$."

(We would be more likely to write "$q \to p$" than "$p \leftarrow q$.")

Putting together earlier observations, we conclude that

> The double implication "$p \leftrightarrow q$" is true if p and q have the same truth values; it is false if p and q have different truth values.

For example, the statement

"2 is an even number $\leftrightarrow$ 4 is an even number"

is true since both "2 is an even number" and "4 is an even number" are true. However,

"2 is an even number if and only if 5 is an even number"

is false because one side is true while the other is false.

Pause 5

Determine whether each of the following double implications is true or false.

(a) $4^2 = 16 \leftrightarrow -1^2 = -1$.
(b) $4^2 = 16$ if and only if $(-1)^2 = -1$.
(c) $4^2 = 15$ if and only if $-1^2 = -1$.
(d) $4^2 = 15 \leftrightarrow (-1)^2 = -1$. ▮

Negation

The *negation* of the statement p is the statement that asserts that p is not true. We denote the negation of p by "$\neg p$" and say "not p." The negation of "x equals 4" is the statement "x does not equal 4." In mathematical writing, a slash (/) through a symbol is used to express the negation of that symbol. So, for instance, $\neq$ means "not equal." Thus, the negation of "$x = 4$" is "$x \neq 4$." In succeeding chapters, we shall meet other symbols like $\in$, $\subseteq$, and $|$, each of which is negated with a slash, $\notin$, $\not\subseteq$, $\nmid$.

Some rules for forming negations are a bit complicated because it is not enough just to say "not p": We must also understand what is being said! To begin, we suggest that the negation of p be expressed as

"It is not the case that p."

Then we should think for a minute or so about precisely what this means. For example, the negation of "25 is a perfect square" is the statement "It is not the case that 25 is a perfect square," which surely means "25 is not a perfect square." To obtain the negation of

"$n < 10$ or n is odd,"

we begin

"It is not the case that $n < 10$ or n is odd."

A little reflection suggests that this is the same as the more revealing "$n \geq 10$ and n is even."

The negation of an "or" statement is always an "and" statement and the negation of an "and" is always an "or." The precise rules for expressing the negation of compound statements formed with "and" and "or" are due to Augustus De Morgan (whose name we shall see again in Section 1.2).

> The negation of "p and q" is the assertion "$\neg p$ or $\neg q$."
>
> The negation of "p or q" is the assertion "$\neg p$ and $\neg q$."

For example, the negation of "$a^2 + b^2 = c^2$ and $a > 0$" is "Either $a^2 + b^2 \neq c^2$ or $a \leq 0$." The negation of "$x + y = 6$ or $2x + 3y < 7$" is "$x + y \neq 6$ and $2x + 3y \geq 7$."

Pause 6

What is the negation of the implication $p \to q$? ∎

The Contrapositive

The *contrapositive* of the implication "$p \to q$" is the implication "$(\neg q) \to (\neg p)$." For example, the contrapositive of

"If x is an even number, then $x^2 + 3x$ is an even number"

is

"If $x^2 + 3x$ is an odd number, then x is an odd number."

Pause 7

Write down the contrapositive of the implications in PAUSES 2 and 3. In each case, state whether the contrapositive is true or false. How do these truth values compare, respectively, with those of the implications in these Pauses? ∎

Quantifiers

The expressions *there exists* and *for all*, which quantify statements, figure prominently in mathematics. The universal quantifier *for all* (and equivalent expressions such as *for every*, *for any*, and *for each*) says, for example, that a statement is true *for all* integers or *for all* polynomials or *for all* elements of a certain type. The following statements illustrate its use. (Notice how it can be disguised; in particular, note that "for any" and "all" are synonymous with "for all.")

$x^2 + x + 1 > 0$ for all real numbers x.

All polynomials are continuous functions.

For any positive integer n, $2(1+2+3+\cdots+n) = n \times (n+1)$.

$(AB)C = A(BC)$ for all square matrices A, B, and C.

Pause 8

Rewrite "All positive real numbers have real square roots," making explicit use of a universal quantifier. ▮

The existential quantifier *there exists* stipulates the existence of a single element for which a statement is true. Here are some assertions in which it is employed.

There exists a smallest positive integer.

Two sets may have no element in common.

Some polynomials have no real zeros.

Again, we draw attention to the fact that the ideas discussed in this chapter can arise in subtle ways. Did you notice the implicit use of the existential quantifier in the second of the preceding statements?

"There exists a set A and a set B such that A and B have no element in common."

Pause 9

Rewrite "Some polynomials have no real zeros" making use of the existential quantifier. ▮

Here are some statements that employ both types of quantifiers.

There exists a matrix $\mathsf{0}$ with the property that $A + \mathsf{0} = \mathsf{0} + A$ for all matrices A.

For any real number x, there exists an integer n such that $n \le x < n+1$.

Every positive integer is the product of primes.

Every nonempty set of positive integers has a smallest element.

To negate a statement that involves one or more quantifiers in a useful way can be difficult. It's usually helpful to begin with "It is not the case" and then to reflect on what you have written. Consider, for instance, the assertion

"For every real number x, x has a real square root."

A first stab at its negation gives

"It is not the case that every real number x has a real square root."

But what does this really mean? Surely,

"There exists a real number that does not have a real square root."

Notice that the negation of a statement involving the universal quantifier required a statement involving the existential quantifier. This is always the case.

> The negation of "For all something, p" is the statement "There exists something such that $\neg p$."
>
> The negation of "There exists something such that p" is the statement "For all something, $\neg p$."

For example, the negation of

"There exist a and b for which $ab \neq ba$"

is the statement

"For all a and b, $ab = ba$."

0.1.1 REMARK

The symbols $\forall$ and $\exists$ are commonly used for the quantifiers *for all* and *there exists*, respectively. For example, you might encounter the statement

$$\forall x, \exists n \text{ such that } n > x$$

or even, more simply,

$$\forall x, \exists n, n > x$$

in a book in real analysis. We won't use this notation in this book, but it is so common that you should know about it. ❖

What May I Assume?

In our experience, when asked to prove something, students often wonder just what they are allowed to assume. For the rest of this book, the answer is any fact, including the result of any exercise, stated **earlier** in the book. This chapter is somewhat special because we are talking *about* mathematics and endeavoring to use only *familiar* ideas to motivate our discussion. In addition to basic college algebra, here is a list of mathematical definitions and facts that the student is free to assume in this chapter's exercises.

- The product of nonzero real numbers is nonzero.
- The square of a nonzero real number is a positive real number.
- An even integer is one that is of the form $2k$ for some integer k; an odd integer is one that is of the form $2k + 1$ for some integer k.
- The product of two even integers is even; the product of two odd integers is odd; the product of an odd integer and an even integer is even.
- A real number is rational if it is a common fraction, that is, the quotient $\frac{m}{n}$ of integers m and n with $n \neq 0$.
- A real number is irrational if it is not rational. For example, π and $\sqrt[3]{5}$ are irrational numbers.
- An irrational number has a decimal expansion that neither repeats nor terminates.
- A prime is a positive integer $p > 1$ that is divisible evenly only by ± 1 and $\pm p$, for example, 2, 3 and 5.

Answers to Pauses

1. By equation (*), twice the sum of the integers from 1 to 100 is 100×101. So the sum itself is $50 \times 101 = 5050$.
2. This is false. The hypothesis is true, but the conclusion is false: $-1^2 = -1$, not 1.
3. The converse is "If $-1^2 = 1$, then $4^2 = 16$." This is true, because the hypothesis "$-1^2 = 1$" is false.
4. Julie's answer began with the **converse** of what she wanted to show. She intended to say

   If d is perpendicular to n, then the line is parallel to the plane.

   Then, having established that d was perpendicular to n, she would have the result.
5. **(a)** This is true because both statements are true.
   (b) This is false because the two statements have different truth values.
   (c) This is false because the two statements have different truth values.
   (d) This is true because both statements are false.
6. "Not $p \to q$" means $p \to q$ is false. This occurs precisely when p is true and q is false. So $\neg(p \to q)$ is "p and $\neg q$."

7. The contrapositive of the implication in PAUSE 2 is "If $-1^2 \neq 1$, then $4^2 \neq 16$." This is false because the hypothesis is true, but the conclusion is false.

The contrapositive of the implication in PAUSE 3 is "If $4^2 \neq 16$, then $-1^2 \neq 1$." This is true because the hypothesis is false.

These answers are the same as in Pauses 2 and 3. This is always the case, as we shall see in Section 0.2.

8. For all real numbers $x > 0$, x has a real square root.

9. There exists a polynomial with no real zeros.

True/False Questions

(Answers can be found in the back of the book.)

1. "p and q" is false if both p and q are false.
2. If "p and q" is false, then both p and q are false.
3. It is possible for both "p and q" and "p or q" to be false.
4. It is possible for both "p and q" and "p or q" to be true.
5. The implication "If $2^2 = 5$, then $3^2 = 9$" is true.
6. The negation of $a = b = 0$ is $a \neq b \neq 0$.
7. The converse of the implication in Question 5 is true.
8. The double implication "$2^2 = 5$ if and only if $3^2 = 9$" is true.
9. It is possible for both an implication and its converse to be true.
10. The statement "Some frogs have red toes" makes use of the universal quantifier "for all."
11. The negation of an existential quantifier is its own converse.

Exercises

The answers to exercises marked [BB] can be found in the ***B****ack of the* ***B****ook.*

You are urged to read the final paragraph of this section—What may I assume?—before attempting these exercises.

1. Classify each of the following statements as true, false, or not a valid mathematical statement.

(a) [BB] An integer is a rational number.

(b) [BB] Let x denote a real number.

(c) [BB] The square of a real number is a positive number.

(d) Where is Newfoundland?

(e) The product of an integer and an even integer is an even integer.

(f) Suppose this statement is false.

(g) The product of five prime numbers is prime.

2. Classify each of the following statements as true, false, or not a valid mathematical statement. Explain your answers.

(a) [BB] $4 = 2 + 2$ and $7 < \sqrt{50}$.

(b) $4 \neq 2 + 2$ and $7 < \sqrt{50}$.

(c) [BB] 5 is an even number and $16^{-1/4} = \frac{1}{2}$.

(d) 5 is an even number or $16^{-1/4} = \frac{1}{2}$.

(e) [BB] $9 = 3^2$ or $3.14 < \pi$.

(f) $\sqrt{3}^{\sqrt{3}}$ is rational or $(-4)^2 = 16$.

(g) [BB] $4 = 2 + 2 \rightarrow 7 < \sqrt{50}$.

(h) 2 is an even integer $\rightarrow$ 6 is an odd integer.

(i) [BB] Let n denote a positive integer.

(j) $4 = 2 + 2 \leftrightarrow 7 < \sqrt{50}$.

(k) $4 \neq 2 + 2 \rightarrow 7 < \sqrt{50}$.

(l) $4 \neq 2 + 2 \leftrightarrow 7 < \sqrt{50}$.

(m) $4 = 2 + 2 \rightarrow 7 > \sqrt{50}$.

(n) [BB] The area of a circle of radius r is $2\pi r$ or its circumference is πr^2.

(o) $2 + 3 = 5 \rightarrow 5 + 6 = 10$.

(p) [BB] If a and b are integers with $a - b \geq 0$ and $b - a \geq 0$, then $a = b$.

(q) If a and b are integers with $a - b > 0$ and $b - a > 0$, then $a = b$.

3. Rewrite each of the following statements so that it is clear that each is an implication.

(a) [BB] The reciprocal of a positive number is positive.

(b) The product of rational numbers is rational.

(c) A differentiable function is continuous.

(d) [BB] The sum of the degrees of the vertices in a graph is an even number.

(e) [BB] A nonzero matrix is invertible.

(f) The diagonals of a parallelogram bisect each other.

(g) All even integers are negative.

(h) Two orthogonal vectors have dot product 0.

(i) $\frac{n}{n+1}$ is not an integer for any integer n.

(j) $n + 3 > 2$ for every natural number n.

4. Determine whether each of the following implications is true or false.

(a) [BB] If 7 is even, then 25 is odd.

(b) If 7 is odd, then 25 is odd.

(c) [BB] If 7 is even, then 25 is even.

(d) If 7 is odd, then 25 is even.

(e) [BB] If $\sqrt[3]{-8} = -2$, then $\sqrt{4} = \pm 2$.

(f) If $\sqrt[3]{-8} \neq -2$, then $\sqrt{4} = \pm 2$.

(g) [BB] If $\sqrt[3]{-8} = -2$, then $\sqrt{4} \neq \pm 2$.

(h) If $\sqrt[3]{-8} \neq -2$, then $\sqrt{4} \neq \pm 2$.

(i) [BB] If $\sqrt{x^2} = x$, then $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$.

(j) If $\sqrt{x^2} = x$, then $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$.

(k) [BB] If $\sqrt{x^2} \neq x$, then $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$.

(l) If $\sqrt{x^2} \neq x$, then $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$.

5. Write down the negation of each of the following statements in clear and concise English. Do not use the expression "It is not the case that" in your answers.

(a) [BB] Either $a^2 > 0$ or a is not a real number.

(b) x is a real number and $x^2 + 1 = 0$.

(c) [BB] $x = \pm 1$.

(d) Every integer is divisible by a prime.

(e) [BB] For every real number x, there is an integer n such that $n > x$.

(f) There exist a, b, and c such that $(ab)c \neq a(bc)$.

(g) [BB] There exists a planar graph that cannot be colored with at most four colors.

(h) Every Canadian is a fan of the Toronto Maple Leafs or the Montreal Canadiens.

(i) For every $x > 0$, $x^2 + y^2 > 0$ for all y.

(j) $-2 < x < 2$.

(k) [BB] For all integers a and b, there exist integers q and r such that $b = qa + r$.

(l) [BB] There exists an infinite set whose proper subsets are all finite.

(m) There exists a real number x such that, for every integer n, either $n < x$ or $n \geq x + 1$.

(n) If n is an integer, $\frac{n}{n+1}$ is not an integer.

(o) $a \leq x$ and $a \leq y$ and $a \leq z$.

(p) Every vector in the plane is perpendicular to some normal.

6. Write down the converse and the contrapositive of each of the following implications.

(a) [BB] If $\frac{a}{b}$ and $\frac{b}{c}$ are integers, then $\frac{a}{c}$ is an integer.

(b) $x^2 = 1 \rightarrow x = \pm 1$.

(c) If $x^2 = x + 1$, then $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$.

(d) If n is an odd integer, then $n^2 + n - 2$ is an even integer.

(e) [BB] Every Eulerian graph is connected.

(f) $ab = 0 \rightarrow a = 0$ or $b = 0$.

(g) A square is a four-sided figure.

(h) [BB] If $\triangle BAC$ is a right triangle, then $a^2 = b^2 + c^2$.

(i) If $p(x)$ is a polynomial of odd degree, then $p(x)$ has at least one real root.

(j) A linearly independent set of vectors contains at most n vectors.

(k) For any real numbers x and y, if $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$, then f is not one-to-one.

(l) [BB] If there exist real numbers x and y with $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$, then f is not one-to-one.

7. Rewrite each of the following statements using the quantifiers "for all" (or "for every") and "there exists" as appropriate.

(a) [BB] Not all continuous functions are differentiable.

(b) For real x, 2^x is never negative.

(c) [BB] There is no largest real number.

(d) There are infinitely many primes.

(e) [BB] Every positive integer is the product of primes.

(f) All positive real numbers have real square roots.

(g) [BB] There is no smallest integer.

(h) There is no smallest positive real number.

(i) [BB] Not every polynomial has a real root.

(j) Between every two (different) real numbers lies a rational number.

(k) Every polynomial of degree 3 has a real root.

(l) Not every nonzero matrix is invertible.

(m) Some real numbers are nonnegative.

(n) An integer cannot be both even and odd.

(o) If a, b, and c are nonzero integers, $a^3 + b^3 \neq c^3$.

8. Is it possible for both an implication and its converse to be false? Explain your answer.

✍ **9.** On page 4 of the text, we stated as more or less "obvious" the fact that

(∗) $p \leftrightarrow q$ is true if p and q have the same truth values and false if p and q have different truth values.

Using the fact that $p \leftrightarrow q$ means "$p \rightarrow q$ **and** $q \rightarrow p$" together with how the truth values for $x \rightarrow y$ are determined, write a short note (in good English) explaining why (∗) is the case.

0.2 Proofs in Mathematics

Many mathematical theorems are statements that a certain implication is true. A simple result about real numbers says that if x is between 0 and 1 then $x^2 < 1$. In other words, for any choice of a real number between 0 and 1, it is a fact that the square of the number will be less than 1. We are asserting that the implication

Statement 3: "$0 < x < 1 \to x^2 < 1$"

is true. In Section 0.1, the hypothesis and conclusion of an implication could be any two statements, even statements completely unrelated to each other. In the statement of a theorem or a step of a mathematical proof, however, the hypothesis and conclusion will be statements about the same class of objects, and the statement (or step) is the assertion that an implication is always true. The only way for an implication to be false is for the hypothesis to be true and the conclusion false. So the statement of a mathematical theorem or a step in a proof only requires proving that whenever the hypothesis is true the conclusion must also be true.

When the implication "$\mathcal{A} \to \mathcal{B}$" is the statement of a theorem, or one step in a proof, $\mathcal{A}$ is said to be a *sufficient* condition for $\mathcal{B}$ and $\mathcal{B}$ is said to be a *necessary* condition for $\mathcal{A}$. For instance, the implication in Statement 3 can be restated "$0 < x < 1$ is sufficient for $x^2 < 1$" or "$x^2 < 1$ is necessary for $0 < x < 1$."

Pause 10

Rewrite the statement "A matrix with determinant 1 is invertible" so that it becomes apparent that this is an implication. What is the hypothesis? What is the conclusion? This sentence asks which is a necessary condition for what? What is a sufficient condition for what? ▮

To prove that Statement 3 is true, it is **not** enough to take a single example, $x = \frac{1}{2}$ for instance, and claim that the implication is true because $(\frac{1}{2})^2 < 1$. It is **not** better to take ten, or even ten thousand, such examples and verify the implication in each special case. Instead, a **general argument** must be given that works for all x between 0 and 1. Here is such an argument.

Assume that the hypothesis is true; that is, x is a real number with $0 < x < 1$. Since $x > 0$, it must be that $x \cdot x < 1 \cdot x$ because multiplying both sides of an inequality such as $x < 1$ by a positive number such as x preserves the inequality. Hence $x^2 < x$ and, since $x < 1$, $x^2 < 1$ as desired.

Now let us consider the converse of the implication in Statement 3:

Statement 4: "$x^2 < 1 \to 0 < x < 1$."

This is false. For example, when $x = -\frac{1}{2}$, the left-hand side is true, since $(-\frac{1}{2})^2 = \frac{1}{4} < 1$, while the right-hand side is false. So the implication fails when $x = -\frac{1}{2}$. It follows that this implication cannot be used as part of a mathematical proof. The number $x = -\frac{1}{2}$ is called a *counterexample* to Statement 4; that is, a specific example that proves that an implication is false.

There is a very important point to note here. To show that a theorem, or a step in a proof, is false, it is enough to find a single case where the implication does not hold. However, as we saw with Statement 3, to show that a theorem is true, we must give a proof that covers all possible cases.

Pause 11

State the contrapositive of "$0 < x < 1 \to x^2 < 1$." Is this true or false? ▮

Pause 12

Consider the statement "There exists a positive integer n such that $n^2 < n$." Would it make sense to attempt to prove this statement false with a counterexample? Explain. ▮

Theorems in mathematics don't have to be about numbers. For example, a very famous theorem proved in 1976 asserts that if $\mathcal{G}$ is a planar graph then $\mathcal{G}$ can be colored with at most four colors. (The definitions and details are in Chapter 13.) This is an implication of the form "$\mathcal{A} \rightarrow \mathcal{B}$," where the hypothesis $\mathcal{A}$ is the statement that $\mathcal{G}$ is a planar graph and the conclusion $\mathcal{B}$ is the statement that $\mathcal{G}$ can be colored with at most four colors. The Four-Color Theorem states that this implication is true.

Pause 13 (For students who have studied linear algebra.) The statement given in PAUSE 10 is a theorem in linear algebra; that is, the implication is true. State the converse of this theorem. Is this also true? ▮

A common expression in scientific writing is the phrase *if and only if* denoting both an implication and its converse. For example,

Statement 5: "$x^2 + y^2 = 0 \leftrightarrow (x = 0$ and $y = 0)$."

As we saw in Section 0.1, the statement "$\mathcal{A} \leftrightarrow \mathcal{B}$" is a convenient way to express the compound statement

"$\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{A}$."

The sentence "$\mathcal{A}$ is a *necessary and sufficient* condition for $\mathcal{B}$" is another way of saying "$\mathcal{A} \leftrightarrow \mathcal{B}$." The sentence "$\mathcal{A}$ and $\mathcal{B}$ are *necessary and sufficient conditions* for $\mathcal{C}$" is another way of saying

"$(\mathcal{A}$ and $\mathcal{B}) \leftrightarrow \mathcal{C}$."

For example, "a triangle has three equal angles" is a necessary and sufficient condition for "a triangle has three equal sides." We would be more likely to hear "In order for a triangle to have three equal angles, it is necessary and sufficient that it have three equal sides."

To prove that "$\mathcal{A} \leftrightarrow \mathcal{B}$" is true, we must prove separately that "$\mathcal{A} \rightarrow \mathcal{B}$" and "$\mathcal{B} \rightarrow \mathcal{A}$" are both true, using the ideas discussed earlier. In Statement 5, the implication "$(x = 0$ and $y = 0) \rightarrow x^2 + y^2 = 0$" is easy.

Pause 14 Prove that "$x^2 + y^2 = 0 \rightarrow (x = 0$ and $y = 0)$." ▮

As another example, consider

Statement 6: "$0 < x < 1 \leftrightarrow x^2 < 1$."

Is this true? Well, we saw earlier that "$0 < x < 1 \rightarrow x^2 < 1$" is true, but we also noted that its converse, "$x^2 < 1 \rightarrow 0 < x < 1$," is false. It follows that Statement 6 is false.

Pause 15 Determine whether "$-1 < x < 1 \leftrightarrow x^2 < 1$" is true or false. ▮

Sometimes, a theorem in mathematics asserts that three or more statements are *equivalent*, meaning that all possible implications between pairs of statements are true. Thus

"The following are equivalent:

1. $\mathcal{A}$
2. $\mathcal{B}$
3. $\mathcal{C}$"

means that each of the double implications $\mathcal{A} \leftrightarrow \mathcal{B}$, $\mathcal{B} \leftrightarrow \mathcal{C}$, $\mathcal{A} \leftrightarrow \mathcal{C}$ is true. Instead of proving the truth of the six implications here, it is more efficient just to establish the truth, say, of the sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{A}$$

of three implications. It should be clear that if these implications are all true then any implication involving two of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ is also true; for example, the truth of $\mathcal{B} \to \mathcal{A}$ would follow from the truth of $\mathcal{B} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{A}$.

Alternatively, to establish that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are equivalent, we could establish the truth of the sequence

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C} \to \mathcal{B}$$

of implications. Which of the two sequences a person chooses is a matter of preference, but is usually determined by what appears to be the easiest way to argue. Here is an example.

PROBLEM 1. Let x be a real number. Show that the following are equivalent.

(1) $x = \pm 1$.
(2) $x^2 = 1$.
(3) If a is any real number, then $ax = \pm a$.

Solution. To show that these statements are equivalent, it is sufficient to establish the truth of the sequence

$$(2) \to (1) \to (3) \to (2).$$

(2) $\to$ (1): The notation means "assume (2) and prove (1)." Since $x^2 = 1$, $0 = x^2 - 1 = (x+1)(x-1)$. Since the product of real numbers is zero if and only if one of the numbers is zero, either $x + 1 = 0$ or $x - 1 = 0$; hence $x = -1$ or $x = +1$, as required.

(1) $\to$ (3): The notation means "assume (1) and prove (3)." Thus either $x = +1$ or $x = -1$. Let a be a real number. If $x = +1$, then $ax = a \cdot 1 = a$, while if $x = -1$, then $ax = -a$. In every case, $ax = \pm a$ as required.

(3) $\to$ (2): We assume (3) and prove (2). We are given that $ax = \pm a$ for any real number a. With $a = 1$, we obtain $x = \pm 1$ and squaring gives $x^2 = 1$, as desired. ▲

In the index (see *equivalent*), you are directed to other places in this book where we establish the equivalence of a series of statements.

Direct Proof

Most theorems in mathematics are stated as implications: "$\mathcal{A} \to \mathcal{B}$." Sometimes, it is possible to prove such a statement *directly*; that is, by establishing the validity of a sequence of implications:

$$\mathcal{A} \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots \to \mathcal{B}.$$

PROBLEM 2. Prove that for all real numbers x, $x^2 - 4x + 17 \neq 0$.

Solution. We observe that $x^2 - 4x + 17 = (x-2)^2 + 13$ is the sum of 13 and a number, $(x-2)^2$, which is never negative. So $x^2 - 4x + 17 \geq 13$ for any x; in particular, $x^2 - 4x + 17 \neq 0$. ▲

PROBLEM 3. Suppose that x and y are real numbers such that $2x + y = 1$ and $x - y = -4$. Prove that $x = -1$ and $y = 3$.

Solution. $(2x + y = 1 \text{ and } x - y = -4) \to (2x + y) + (x - y) = 1 - 4$

$$\to 3x = -3 \to x = -1.$$

Also,

$$(x = -1 \text{ and } x - y = -4) \to (-1 - y = -4) \to (y = -1 + 4 = 3). \quad \blacktriangle$$

Many of the proofs in this book are direct. In the index (under *direct*), we guide you to several of these.

Proof by Cases

Sometimes a direct argument is made simpler by breaking it into a number of cases, one of which must hold and each of which leads to the desired conclusion.

PROBLEM 4. Let n be an integer. Prove that $9n^2 + 3n - 2$ is even.

Solution. ***Case 1.*** n is even.

Recall that an integer is even if and only if twice another integer. So here we have $n = 2k$ for some integer k. Thus $9n^2 + 3n - 2 = 36k^2 + 6k - 2 = 2(18k^2 + 3k - 1)$, which is even.

Case 2. n is odd.

An integer is odd if and only if it has the form $2k + 1$ for some integer k. So here we write $n = 2k + 1$. Thus $9n^2 + 3n - 2 = 9(4k^2 + 4k + 1) + 3(2k + 1) - 2 = 36k^2 + 42k + 10 = 2(18k^2 + 21k + 5)$, which is even. ▲

In the index, we guide you to other places in this book where we give proofs by cases. (See *cases*.)

Prove the Contrapositive

A very important principle of logic, foreshadowed by PAUSE 7, is summarized in the next theorem.

0.2.1 THEOREM "$\mathcal{A} \to \mathcal{B}$" is true if and only if its contrapositive "$\neg\mathcal{B} \to \neg\mathcal{A}$" is true.

Proof "$\mathcal{A} \to \mathcal{B}$" is false if and only if $\mathcal{A}$ is true and $\mathcal{B}$ is false; that is, if and only if $\neg\mathcal{A}$ is false and $\neg\mathcal{B}$ is true; that is, if and only if "$\neg\mathcal{B} \to \neg\mathcal{A}$" is false. Thus the two statements "$\mathcal{A} \to \mathcal{B}$" and "$\neg\mathcal{B} \to \neg\mathcal{A}$" are false together (and hence true together); that is, they have the same truth values. The result is proved. ●

PROBLEM 5. If the average of four different integers is 10, prove that one of the integers is greater than 11.

Solution. Let $\mathcal{A}$ and $\mathcal{B}$ be the statements

$\mathcal{A}$: "The average of four integers, all different, is 10."

$\mathcal{B}$: "One of the four integers is greater than 11."

We are asked to prove the truth of "$\mathcal{A} \to \mathcal{B}$." Instead, we prove the truth of the contrapositive "$\neg\mathcal{B} \to \neg\mathcal{A}$", from which the result follows by Theorem 0.2.1.

Call the given integers a, b, c, d. If $\mathcal{B}$ is false, then each of these numbers is at most 11 and, since they are all different, the biggest value for $a + b + c + d$ is $11 + 10 + 9 + 8 = 38$. So the biggest possible average would be $\frac{38}{4}$, which is less than 10, so $\mathcal{A}$ is false. ▲

Proof by Contradiction

Sometimes a direct proof of a statement $\mathcal{A}$ seems hopeless: We simply do not know how to begin. In this case, we can sometimes make progress by assuming that the negation of $\mathcal{A}$ is true. If this assumption leads to a statement that is obviously false (an *absurdity*) or to a statement that contradicts something else, then we will have shown that $\neg\mathcal{A}$ is false. So, $\mathcal{A}$ must be true.

PROBLEM 6. Show that there is no largest integer.

Solution. Let $\mathcal{A}$ be the statement "There is no largest integer." If $\mathcal{A}$ is false, then there is a largest integer N. This is absurd, however, because $N+1$ is an integer larger than N. Thus $\neg\mathcal{A}$ is false, so $\mathcal{A}$ is true. ▲

Remember that *rational number* just means common fraction, the quotient $\frac{m}{n}$ of integers m and n with $n \neq 0$. A number that is not rational is called *irrational*.

PROBLEM 7. Suppose that a is a nonzero rational number and that b is an irrational number. Prove that ab is irrational.

Solution. We prove "$\mathcal{A}$: ab is irrational" by contradiction. Assume $\mathcal{A}$ is false. Then ab is rational, so $ab = \frac{m}{n}$ for integers m and n, $n \neq 0$. Now a is given to be rational, so $a = \frac{k}{\ell}$ for integers k and ℓ, $\ell \neq 0$ and $k \neq 0$ (because $a \neq 0$). Thus,

$$b = \frac{m}{na} = \frac{m\ell}{nk}$$

with $nk \neq 0$, so b is rational. This is not true. We have reached a contradiction and hence proved that $\mathcal{A}$ is true. ▲

Here is a well-known but nonetheless beautiful example of a proof by contradiction.

PROBLEM 8. Prove that $\sqrt{2}$ is an irrational number.

Solution. If the statement is false, then there exist integers m and n such that $\sqrt{2} = \frac{m}{n}$. If both m and n are even, we can cancel 2's in numerator and denominator until at least one of them is odd. Thus, *without loss of generality*, we may assume that not both m and n are even.

Squaring both sides of $\sqrt{2} = \frac{m}{n}$, we get $m^2 = 2n^2$, so m^2 is even. Since the square of an odd integer is odd, $m = 2k$ must be even. This gives $4k^2 = 2n^2$, so $2k^2 = n^2$. As before, this implies that n is even, contradicting the fact that not both m and n are even. ▲

Answers to Pauses

10. "A matrix A has determinant $1 \to A$ is invertible."
Hypothesis: A matrix A has determinant 1.
Conclusion: A is invertible.

The invertibility of a matrix is a necessary condition for its determinant being equal to 1; determinant 1 is a sufficient condition for the invertibility of a matrix.

11. The contrapositive is "$x^2 \geq 1 \to (x \leq 0$ or $x \geq 1)$." This is true, and here is a proof.
Assume $x^2 \geq 1$. If $x \leq 0$, we have the desired result, so assume $x > 0$. In this case, if $x < 1$, we have seen that $x^2 < 1$, which is not true, so we must have $x \geq 1$, again as desired.

12. No it would not. To prove the statement false requires proving that $n^2 \geq n$ for **all** positive integers n. A single example is not going to make this point.

13. The converse is "An invertible matrix has determinant 1." This is false: for example, the matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is invertible (the inverse of A is A itself), but $\det A = -1$.

14. Assume that $x^2 + y^2 = 0$. Since the square of a real number cannot be negative and the square of a **nonzero** real number is positive, if either $x^2 \neq 0$ or $y^2 \neq 0$, the sum $x^2 + y^2$ would be positive, which is not true. This means $x^2 = 0$ and $y^2 = 0$, so $x = 0$ and $y = 0$, as desired.

15. This statement is true. To prove it, we must show that two implications are true. ($\longrightarrow$) First assume that $-1 < x < 1$. If $0 < x < 1$, then we saw in the text that $x^2 < 1$ while, if $x = 0$, clearly $x^2 = 0 < 1$. If $-1 < x < 0$, then $0 < -x < 1$ (multiplying an inequality by the negative number -1 reverses it) so, by the argument in the text $(-x)^2 < 1$, that is, $x^2 < 1$. In all cases, we have $x^2 < 1$. Thus $-1 < x < 1 \rightarrow x^2 < 1$ is true.
($\longleftarrow$) Next we prove that $x^2 < 1 \rightarrow -1 < x < 1$ is true. Assume $x^2 < 1$. If $x \geq 1$, then we would also have $x^2 = x \cdot x \geq x \cdot 1 = x \geq 1$, so $x^2 \geq 1$, which is not true. If we had $x \leq -1$, then $-x \geq 1$ and so $x^2 = (-x)^2 \geq 1$, which, again, is not true. We conclude that $-1 < x < 1$, as desired.

True/False Questions

(Answers can be found in the back of the book.)

1. If you want to prove a statement is true, it is enough to find 867 examples where it is true.
2. If you want to prove a statement is false, it is enough to find one example where it is false.
3. The sentence "$\mathcal{A}$ is a sufficient condition for $\mathcal{B}$" is another way of saying "$\mathcal{A} \rightarrow \mathcal{B}$."
4. If $\mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{C} \rightarrow \mathcal{D}$, and $\mathcal{C} \rightarrow \mathcal{A}$ are all true, then $\mathcal{D} \rightarrow \mathcal{B}$ must be true.
5. If $\mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{C} \rightarrow \mathcal{D}$, and $\mathcal{C} \rightarrow \mathcal{A}$ are all true, then $\mathcal{B} \leftrightarrow \mathcal{C}$ must be true.
6. The contrapositive of "$\mathcal{A} \rightarrow \mathcal{B}$" is "$\neg\mathcal{B} \rightarrow \mathcal{A}$."
7. "$\mathcal{A} \rightarrow \mathcal{B}$" is true if and only if its contrapositive is true.
8. π is a rational number.
9. 3.141 is a rational number.
10. If a and b are irrational numbers, then ab must be an irrational number.
11. The statement "Every real number is rational" can be proved false with a counterexample.
12. The statement "There exists an irrational number that is not the square root of an integer" can be proved false with a counterexample.

Exercises

The answers to exercises marked [BB] can be found in the ***B****ack of the* ***B****ook.*

You are urged to read the final paragraph of Section 0.1—What may I assume?—before attempting these exercises.

1. What is the hypothesis and what is the conclusion in each of the following implications?

(a) [BB] The sum of two positive numbers is positive.

(b) The square of the length of the hypotenuse of a right-angled triangle is the sum of the squares of the lengths of the other two sides.

(c) [BB] All primes are even.

(d) Every positive integer bigger than 1 is the product of prime numbers.

(e) The chromatic number of a planar graph is 3.

2. [BB: (a), (c)] In each part of Exercise 1, what condition is necessary for what? What condition is sufficient for what?

3. Exhibit a counterexample to each of the following statements.

(a) [BB] $x^2 = 4 \to x = 2$.

(b) a and b integers and $ab = 1 \to a = b = 1$.

(c) [BB] $(x-3)^2 = x - 3 \to x = 3$.

(d) If the average of four different integers is 10, at least one of the integers must be greater than 12.

(e) The product of two irrational numbers is irrational.

(f) $x - y = y - x$ for all real numbers x and y.

4. Consider the following two statements:

$\mathcal{A}$: The square of every real number is positive.

$\mathcal{B}$: There exists a real number with a square less than -1.

Each of these statements is false. One can easily be proved false with a counterexample. The other requires a (short) direct proof. Which is which? Explain your answer.

5. [BB] Determine whether the following implication is true.

"x is an even integer $\to x + 2$ is an even integer."

6. State the converse of the implication in Exercise 5 and determine whether it is true.

7. Answer Exercise 5 with $\to$ replaced by $\leftrightarrow$. [*Hint*: Exercises 5 and 6.]

8. Consider the statement $\mathcal{A}$: "If n is an integer, $\frac{n}{n+1}$ is not an integer."

(a) Is $\mathcal{A}$ true or false? Either prove true, or give a counterexample to prove false.

(b) Write down the converse, the contrapositive, and the negation of $\mathcal{A}$. Which of these is true? Which is false? Justify each answer with a proof or a counterexample.

9. Let n be an integer greater than 1 and consider the statement "$\mathcal{A}$: $2^n - 1$ prime is necessary for n to be prime."

(a) Write $\mathcal{A}$ as an implication.

(b) Write $\mathcal{A}$ in the form "p is sufficient for q."

(c) Determine whether $\mathcal{A}$ is true or false.

(d) Write the converse of $\mathcal{A}$ as an implication.

(e) Determine whether the converse of $\mathcal{A}$ is true or false. [*Hint*: $(2^a)^b - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1]$.]

10. [BB] A theorem in calculus states that every differentiable function is continuous. State the converse of this theorem.
(For students who have taken calculus) Is the converse true or false? Explain.

11. Let n be an integer, $n \geq 3$. A certain mathematical theorem asserts that n statements $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are equivalent.

(a) A student proves this by showing that $\mathcal{A}_1 \leftrightarrow \mathcal{A}_2$, $\mathcal{A}_2 \leftrightarrow \mathcal{A}_3, \ldots, \mathcal{A}_{n-1} \leftrightarrow \mathcal{A}_n$ are all true. How many implication proofs did the student write down?

(b) Another student proves the truth of $\mathcal{A}_1 \to \mathcal{A}_2, \mathcal{A}_2 \to \mathcal{A}_3, \ldots, \mathcal{A}_{n-1} \to \mathcal{A}_n$, and $\mathcal{A}_n \to \mathcal{A}_1$. How many implication proofs did this student write down?

(c) A third student wishes to find a proof that is different from that in 11(b) but uses the same number of implication proofs as in 11(b). Outline a possible proof for this student.

The next three exercises illustrate that the position of a quantifier is very important.

12. [BB] Consider the assertions

$\mathcal{A}$: "For every real number x, there exists an integer n such that $n \leq x < n + 1$."

$\mathcal{B}$: "There exists an integer n such that $n \leq x < n+1$ for every real number x."

One of these assertions is true. The other is false. Which is which? Explain.

13. Answer Exercise 12 with $\mathcal{A}$ and $\mathcal{B}$ as follows.

$\mathcal{A}$: "There exists a real number y such that $y > x$ for every real number x."

$\mathcal{B}$: "For every real number x, there exists a real number y such that $y > x$."

14. Answer true or false and supply a direct proof or a counterexample to each of the following assertions.

(a) There exists a integer $n \neq 0$ such that nq is an integer for every rational number q.

(b) For every rational number q, there exists an integer $n \neq 0$ such that nq is an integer.

15. (a) Prove that n an even integer $\to n^2 + 3n$ is an even integer.

(b) What is the converse of the implication in (a)? Is the converse true or false? Justify your answer.

16. (a) [BB] Let a be an integer. Show that either a or $a+1$ is even.

(b) [BB] Show that $n^2 + n$ is even for any integer n.

17. Provide a direct proof that $n^2 - n + 5$ is odd for all integers n.

18. [BB] Prove that $2x^2 - 4x + 3 > 0$ for any real number x.

19. Let a and b be integers. By examining the four cases

i. a, b both even,

ii. a, b both odd,

iii. a even, b odd,

iv. a odd, b even,

find a necessary and sufficient condition for $a^2 - b^2$ to be odd.

20. [BB] Let n be an integer. Prove that n^2 is even if and only if n is even.

21. Let x be a real number. Find a necessary and sufficient condition for $x + \dfrac{1}{x} \geq 2$. Prove your answer.

22. [BB] Prove that if n is an odd integer then there is an integer m such that $n = 4m + 1$ or $n = 4m + 3$. [*Hint*: Consider a proof by cases.]

23. Prove that if n is an odd integer, there is an integer m such that $n = 8m + 1$ or $n = 8m + 3$ or $n = 8m + 5$ or $n = 8m + 7$. (You may use the result of Exercise 22.)

24. [BB] Prove that there exists no smallest positive real number. [*Hint*: Find a proof by contradiction.]

25. Let $n = ab$ be the product of positive integers a and b. Prove that either $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

26. [BB] (For students who have studied linear algebra) Suppose 0 is an eigenvalue of a matrix A. Prove that A is not invertible. [*Hint*: There is a short proof by contradiction.]

27. **(a)** Suppose a and b are integers such that $a + b\sqrt{2} = 3 + 5\sqrt{2}$. Prove that $a = 3$ and $b = 5$.

(b) Prove that $3+5\sqrt{2}$ is not the square of a real number of the form $a + b\sqrt{2}$, where a and b are integers.

28. [BB] Suppose a and b are integers such that $a+b+ab = 0$. Prove that $a = b = 0$ or $a = b = -2$. Give a direct proof.

29. Suppose a is an irrational number. Prove that $\dfrac{1}{a}$ is an irrational number.

30. Suppose that a is a rational number and that b is an irrational number. Prove that $a + b$ is irrational.

31. [BB] Prove that the equations

$$\begin{aligned} 2x + 3y - z &= 5 \\ x - 2y + 3z &= 7 \\ x + 5y - 4z &= 0 \end{aligned}$$

have no solution. (Give a proof by contradiction.)

32. Find a proof or exhibit a counterexample to each of the following statements.

(a) [BB] $2x^2 + 3y^2 > 0$ for all real numbers x and y.

(b) a an even integer $\rightarrow \frac{1}{2}a$ an even integer.

(c) [BB] For each real number x, there exists a real number y such that $xy = 1$.

(d) If a and b are real numbers with $a + b$ rational, then a and b are rational.

(e) [BB] a and b real numbers with ab rational $\rightarrow$ a and b rational.

(f) $b^2 - 4ac > 0$ and $a \ne 0 \rightarrow p(x) = ax^2 + bx + c$ has two distinct real roots.

(g) $x^2 \ge x$ for all real numbers x.

(h) $n^2 \ge n$ for all positive integers n.

33. Suppose ABC and $A'B'C'$ are triangles with the same angles. A theorem in Euclidean geometry says that the triangles have pairwise proportional sides. (Such triangles are said to be *similar*.) Does the same property hold for polygons with more than three sides? Give a proof or provide a counterexample.

34. **(a)** [BB] Suppose m and n are integers such that $n^2+1 = 2m$. Prove that m is the sum of the squares of two integers.

(b) [BB] Given that $(4373)^2 + 1 = 2(9{,}561{,}565)$, write 9,561,565 as the sum of two squares.

35. Observe that for any real number x,

$$4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1).$$

(a) Use this identity to express $2^{4n+2} + 1$ (n a positive integer) and $2^{18} + 1$ as the product of two integers each greater than 1.

(b) Express $2^{36} - 1$ as the product of four integers each of which is larger than 1.

36. Suppose $f(n)$ is a polynomial such that $f(n)$ is a prime number for all $n \ge 0$. Show that $f(n) = p$ for some prime p. [*Hint*: Proof by contradiction.]

37. Prove that one of the digits $1, 2, \dots, 9$ occurs infinitely often in the decimal expansion of π.

38. Prove that there exist irrational numbers a and b such that a^b is rational. [*Hint*: Consider $\sqrt{2}^{\sqrt{2}}$.]

Review Exercises for Chapter 0

1. State, with a reason, whether each of the following statements is true or false.

(a) If a and b are integers with $a - b > 0$ and $b - a > 0$, then $a \ne b$.

(b) If a and b are integers with $a - b \ge 0$ and $b - a \ge 0$, then $a \ne b$.

2. Write down the negation of each of the following statements in clear and concise English.

(a) Either x is not a real number or $x > 5$.

(b) There exists a real number x such that $n > x$ for every integer n.

(c) For all positive integers x, y, and z, it is the case that $x^3 + y^3 \ne z^3$.

(d) If a graph has n vertices and $n + 1$ edges, then its chromatic number is at most 3.

(e) For every integer n, there exists a rational number a such that $a = n$.

(f) $a = b = 0$.

3. Write down the converse, the contrapositive and the negation of each of the following implications.

(a) If a and b are integers, then ab is an integer.

(b) If x is an even integer, then x^2 is an even integer.

(c) Every planar graph can be colored with at most four colors.

(d) If a matrix is symmetric, then it equals its transpose.

(e) A spanning set of vectors contains at least n vectors.

(f) If $x^2 + x - 2 < 0$, then $x > -2$ and $x < 1$.

4. Assume a, b, u, v are integers, $u \ne 0$, $v \ne 0$, and consider the statement

A: "If $au + bv = 0$, then $a = b = 0$."

(a) Is A true? Give a proof or exhibit a counterexample.
(b) Write down the converse, negation, and contrapositive of A.
(c) Is the converse of A true? Give a proof or exhibit a counterexample.
(d) Answer 4(c) for the negation and contrapositive.

5. Rewrite each of the following statements using the quantifiers "for all" and "there exists" as appropriate.
(a) Not all countable sets are finite.
(b) 1 is the smallest positive integer.

6. **(a)** Determine whether the following implication is true.

"x is a positive odd integer $\to x+2$ is a positive odd integer."

(b) Repeat (a) with $\to$ replaced by $\leftrightarrow$.

7. Answer true or false and explain.
(a) For real numbers a and b, if $a \le b$ and $b \le a$, then $a = b$.
(b) If $a \le b$ and $b \le a$, then $a = b$ for all real numbers a and b.

8. Examine the following equations

$$6 \times 4 = \frac{100-4}{4};$$
$$13 \times 12 = \frac{625-1}{4};$$
$$27 \times 16 = \frac{1849-121}{4}.$$

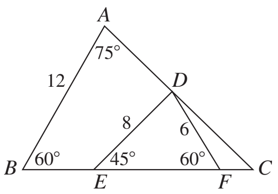
What's going on here? Find a formula for $a \times b$ illustrated by these examples and prove your answer.

9. Let n be an integer. Prove that n^3 is odd if and only if n is odd.

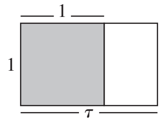
($\longleftarrow$) Assume n is odd. This means that $n = 2x+1$ for some integer x. Then $n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$ is odd.

10. **(a)** Give a direct proof of the fact that $a^2 - 5a + 6$ is even for any integer a.
(b) Suppose a and b are integers and $a^2 - 5b$ is even. Prove that $b^2 - 5a$ is even.

11. Find the length of AC.



12. The number $\tau = \frac{1+\sqrt{5}}{2} \approx 1.618$ is called the *golden mean*. The ancient Greeks felt that a rectangle whose sides were in the ratio 1 to τ was the most esthetically pleasing. Such a rectangle is called *golden*. Hence, for example, each side of the Parthenon, a large rectangular building dating to 435 B.C., is a golden rectangle. Suppose you remove a 1×1 rectangle from a golden rectangle as shown. Show that the rectangle that remains is also golden.



13. Prove, by way of contradiction, that if x is a real number and $x^2 - x - 2 > 0$, then $x < -1$ or $x > 2$.

14. With a proof by contradiction, show that there exists no largest negative rational number.

15. Prove that $\sqrt{3}$ is not a rational number.

16. Prove that between any two positive rational numbers there is another rational number.

17. Consider a standard 8×8 checkerboard that is defective in the sense that two squares in opposite corners have been removed. You are given a box of dominoes (wooden rectangles) each of which covers exactly two squares of the checkerboard. Prove that it is impossible to cover all the squares of the defective board with dominoes. [*Hint*: Squares in opposite corners have the same color. Proof by contradiction.]

18. It is tempting to think that, if a statement involving the natural number n is true for many consecutive values of n, it must be true for all n. In this connection, the following example due to Euler is illustrative.
Let $f(n) = n^2 + n + 41$.
(a) Convince yourself (perhaps with a computer algebra package like Maple or Mathematica) that $f(n)$ is prime for $n = 1, 2, 3, \dots 39$, but that $f(40)$ is not prime.
(b) Show that, for any n of the form $n = k^2 + 40$, $f(n)$ is not prime.

19. Each of the integers 31, 331, 3331, 33331, 333331, 3333331 is prime. Does this imply that any integer comprised of a list of 3's followed by a 1 is prime?

1

Logic

1.1 Truth Tables

In the last chapter, we presented a rather informal introduction to some of the basic concepts of mathematical logic. There are times, however, when a more formal approach can be useful. We begin to look at such an approach now.

Let p and q be statements. For us, remember that *statement* means a statement of fact that is either true or false. The compound statements "p or q" and "p and q," which were introduced in Section 0.1, will henceforth be written "$p \vee q$" and "$p \wedge q$," respectively.

$$p \vee q: \quad p \text{ or } q$$
$$p \wedge q: \quad p \text{ and } q.$$

The way in which the truth values of these compound statements depend on those of p and q can be neatly summarized by tables called *truth tables*. Truth tables for $p \vee q$ and $p \wedge q$ are shown in Fig. 1.1.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1.1 Truth tables for $p \vee q$ (p or q) and $p \wedge q$ (p and q).

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Figure 1.2 Truth tables for $p \to q$ (p implies q) and $\neg p$ (not p).

In each case, the first two columns show all possible truth values for p and q—each is either true (T) or false (F)—and the third column shows the corresponding truth value for the compound statement.

The truth table for the implication $p \to q$, introduced in Section 0.1, is shown on the left in Fig. 1.2. On the right, we show the particularly simple truth table for "$\neg p$," the negation of p

Truth tables for more complicated compound statements can be constructed using the truth tables we have seen so far. For example, the statement "$p \leftrightarrow q$," defined in Section 0.1 as "$(p \to q)$ and $(q \to p)$," is "$(p \to q) \wedge (q \to p)$."

The truth values for $p \to q$ and $q \to p$ are shown in Fig. 1.3. Focusing on columns 3 and 4 and remembering that $r \wedge s$ is true if and only if both r and s are true—see the truth table for "$\wedge$" shown in Fig. 1.1—we obtain the truth table for $(p \to q) \wedge (q \to p)$, that is, for $p \leftrightarrow q$.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 1.3 The truth table for $p \leftrightarrow q$ (p if and only if q).

The first two columns and the last column are the most important, of course, so in future applications we remember $p \leftrightarrow q$ with the simple truth table shown in Fig. 1.4.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 1.4 The truth table for $p \leftrightarrow q$.

Here is another demonstration of how to analyze complex compound statements with truth tables.

EXAMPLE 1 Suppose we want the truth table for $p \to \neg(q \vee p)$.

p	q	$p \vee q$	$\neg(p \vee q)$	$p \to \neg(p \vee q)$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	T
F	F	F	T	T

Although the answer is presented as a single truth table, the procedure is to construct appropriate columns one by one until the answer is reached. Here, columns 1 and 2 are used to form column 3 ($q \vee p$). Then column 4 follows from column 3 and, finally, columns 1 and 4 are used to construct column 5, using the truth table for an implication in Fig. 1.2. ■

When three statements p, q, and r are involved, eight rows are required in a truth table since it is necessary to consider the two possible truth values for r for each of the four possible truth values of p and q.

PROBLEM 2. Construct a truth table for $(p \vee q) \leftrightarrow [((\neg p) \wedge r) \to (q \wedge r)]$.

Solution. The truth table for $p \leftrightarrow q$ shows false exactly when the values of p, q are

T, F. We use this idea here for the last column.

p	q	r	$\neg p$	$(\neg p) \wedge r$	$q \wedge r$	$((\neg p) \wedge r) \rightarrow (q \wedge r)$	$p \vee q$
T	T	T	F	F	T	T	T
T	F	T	F	F	F	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	F	F	F
T	T	F	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	F	T	F	F	T	T
F	F	F	T	F	F	T	F

$(p \vee q) \leftrightarrow [((\neg p) \wedge r) \rightarrow (q \wedge r)]$
T
T
T
T
T
T
T
F

▲

Of course, it is only necessary to construct an entire truth table if a complete analysis of a certain compound statement is desired. We do not need to construct all 32 rows of a truth table to do the next problem.

PROBLEM 3. Find the truth value of

$$[p \rightarrow ((q \wedge (\neg r)) \vee s)] \ \wedge \ [(\neg t) \leftrightarrow (s \wedge r)],$$

where p, q, r, and s are all true, while t is false.

Solution. We evaluate the expression step by step, showing just the relevant row of the truth table.

p	q	r	s	t	$\neg r$	$q \wedge (\neg r)$	$(q \wedge (\neg r)) \vee s$
T	T	T	T	F	F	F	T

$p \rightarrow [(q \wedge (\neg r)) \vee s]$	$\neg t$	$s \wedge r$	$(\neg t) \leftrightarrow (s \wedge r)$
T	T	T	T

$[p \rightarrow [((q \wedge (\neg r)) \vee s)] \wedge [(\neg t) \leftrightarrow (s \wedge r)]$
T

The truth value is true. ▲

A notion that will be important in later sections is that of *logical equivalence*. Formally, statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent if they have identical truth tables.

EXAMPLE 4 In Section 0.1, we defined the *contrapositive* of the statement "$p \rightarrow q$" as the statement "$(\neg q) \rightarrow (\neg p)$." In Theorem 0.2.1, we proved that these implications are logically equivalent without actually introducing the terminology. Here is how to establish the same result using truth tables.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q$	$\neg p$	$(\neg q) \to (\neg p)$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

■

PROBLEM 5. Show that "$\mathcal{A}$: $p \to (\neg q)$" and "$\mathcal{B}$: $\neg(p \wedge q)$" are logically equivalent.

Solution. We simply observe that the final columns of the two truth tables are identical.

p	q	$\neg q$	$p \to (\neg q)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

▲

Pause 1

Let $\mathcal{A}$: $p \to (\neg q)$ and $\mathcal{B}$: $\neg(p \wedge q)$ as in Problem 5. Show that $\mathcal{A} \leftrightarrow \mathcal{B}$ is always true. ▮

1.1.1 DEFINITIONS

A compound statement that is always true, regardless of the truth values assigned to its variables, is a *tautology*. A compound statement that is always false is a *contradiction*. ❖

For example, and as illustrated in PAUSE 1, statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \leftrightarrow \mathcal{B}$ is a tautology. An example of a contradiction is $p \wedge (\neg p)$: This statement is always false.

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

We shall give many other examples of logical equivalences in Section 1.2.

Answer to Pause

1. The truth table in Fig. 1.3 shows that a double implication $p \leftrightarrow q$ is true precisely when both p and q have the same truth values. Looking at the truth tables for $\mathcal{A}$ and $\mathcal{B}$ in Problem 5 we see that $\mathcal{A}$ and $\mathcal{B}$ always have the same value, so $\mathcal{A} \leftrightarrow \mathcal{B}$ is always true.

True/False Questions

(Answers can be found in the back of the book.)

1. "$p \vee q$" means "p and q."
2. A truth table based on four simple statements p, q, r, and s has 16 rows.
3. If $p \wedge q$ is true, then $p \vee q$ is also true.
4. If p and q are both false, the truth value of $(\neg p \vee \neg q) \to (p \leftrightarrow q)$ is also false.
5. If $p \to q$ is false, the truth value of $(\neg p \vee \neg q) \to (p \leftrightarrow q)$ is also false.
6. "$p \to q$" and "$q \to p$" are logically equivalent.
7. A statement and its contrapositive are logically equivalent.
8. "$(p \vee q) \to (p \to q)$" is a tautology.
9. If $\mathcal{B}$ is a tautology and $\mathcal{A}$ is a contradiction, then $(\neg\mathcal{A}) \vee \mathcal{B}$ is a tautology.
10. If $\mathcal{A}$ and $\mathcal{B}$ are both contradictions, then $\mathcal{A} \to \mathcal{B}$ is a tautology.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. Construct a truth table for each of the following compound statements.
   (a) [BB] $p \wedge ((\neg q) \vee p)$
   (b) $(p \wedge q) \vee ((\neg p) \to q)$
   (c) $\neg\,(p \wedge (q \vee p)) \leftrightarrow p$
   (d) [BB] $(\neg\,(p \vee (\neg q))) \wedge ((\neg p) \vee r)$
   (e) $(p \to (q \to r)) \to ((p \wedge q) \vee r)$

2. (a) If $p \to q$ is false, determine the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \to q)$.
   (b) [BB] Is it possible to answer 2(a) if $p \to q$ is true instead of false? Why or why not?

3. [BB] Determine the truth value for

$$[p \to (q \wedge (\neg r))] \;\vee\; [r \leftrightarrow ((\neg s) \vee q)]$$

when p, q, r, and s are all true.

4. Repeat Exercise 3 in the case where p, q, r, and s are all false.

5. (a) [BB] Show that $(p \wedge q) \to (p \vee q)$ is a tautology.
   (b) [BB] Show that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is a contradiction.

6. (a) Show that $q \to (p \to q)$ is a tautology.
   (b) Show that $[p \wedge q] \wedge [(\neg p) \vee (\neg q)]$ is a contradiction.

7. (a) [BB] Show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.
   (b) [BB] Explain in plain English why the answer to 7a makes sense.

8. Show that the statement

$$[p \vee ((\neg r) \to (\neg s))] \;\vee\; \big[\big(s \to ((\neg t) \vee p)\big) \vee \big((\neg q) \to r\big)\big]$$

is neither a tautology nor a contradiction.

9. Given that the compound statement $\mathcal{A}$ is a contradiction, establish each of the following.
   (a) [BB] If $\mathcal{B}$ is any statement, $\mathcal{A} \to \mathcal{B}$ is a tautology.
   (b) If $\mathcal{B}$ is a tautology, $\mathcal{B} \to \mathcal{A}$ is a contradiction.

10. (a) Show that the statement $p \to (q \to r)$ is not logically equivalent to the statement $(p \to q) \to r$.
   (b) What can you conclude from 10a about the compound statement $[p \to (q \to r)] \leftrightarrow [(p \to q) \to r]$?

11. If p and q are statements, then the compound statement $p \veebar q$ (often called the *exclusive or*) is defined to be true if and only if exactly one of p, q is true; that is, either p is true or q is true, but not both p and q are true.
   (a) [BB] Construct a truth table for $p \veebar q$.
   (b) Construct a truth table for $(p \veebar ((\neg p) \wedge q)) \vee q$.
   (c) [BB] Show that $(p \veebar q) \to (p \vee q)$ is a tautology.
   (d) Show that $p \veebar q$ is logically equivalent to $\neg(p \leftrightarrow q)$.

1.2 The Algebra of Propositions

At the conclusion of Section 1.1, we discussed the notion of logical equivalence and noted that statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \leftrightarrow \mathcal{B}$ is a tautology.

We use the notation $\mathcal{A} \iff \mathcal{B}$ to denote the fact that $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent. When this is the case, we often think of statement $\mathcal{B}$ as just a rewording of statement $\mathcal{A}$. Clearly then, it is of interest to be able to determine in an efficient manner when two statements are logically equivalent and when they are not. Truth tables will do this job for us, but, as you may already have noticed, they can become cumbersome rather easily. Another approach is first to gather together some of the fundamental examples of logically equivalent statements and then to analyze more complicated situations by showing how they reduce to these basic examples.

The word *proposition* is a synonym for *(mathematical) statement*. Just as there are rules for addition and multiplication of real numbers—commutativity and associativity, for instance—there are properties of $\wedge$ and $\vee$ that are helpful in recognizing that a given compound statement is logically equivalent to another, often more simple, one.

Some Basic Logical Equivalences

1. **Idempotence:** (i) $(p \vee p) \iff p$
   (ii) $(p \wedge p) \iff p$
2. **Commutativity:** (i) $(p \vee q) \iff (q \vee p)$
   (ii) $(p \wedge q) \iff (q \wedge p)$
3. **Associativity:** (i) $((p \vee q) \vee r) \iff (p \vee (q \vee r))$
   (ii) $((p \wedge q) \wedge r) \iff (p \wedge (q \wedge r))$
4. **Distributivity:** (i) $(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$
   (ii) $(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
5. **Double Negation:** $\neg(\neg p) \iff p$
6. **De Morgan's Laws:** (i) $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$
   (ii) $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q)$

Property 6 was discussed in a less formal manner in Section 0.1.

It is clear that any two tautologies are logically equivalent and that any two contradictions are logically equivalent. Letting **1** denote a tautology and **0** a contradiction, we can add the following properties to our list.

7. (i) $(p \vee \mathbf{1}) \iff \mathbf{1}$
   (ii) $(p \wedge \mathbf{1}) \iff p$
8. (i) $(p \vee \mathbf{0}) \iff p$
   (ii) $(p \wedge \mathbf{0}) \iff \mathbf{0}$
9. (i) $(p \vee (\neg p)) \iff \mathbf{1}$
   (ii) $(p \wedge (\neg p)) \iff \mathbf{0}$
10. (i) $\neg\mathbf{1} \iff \mathbf{0}$
    (ii) $\neg\mathbf{0} \iff \mathbf{1}$

We add three more properties.

11. $(p \to q) \iff [(\neg q) \to (\neg p)]$
12. $(p \to q) \iff [(\neg p) \vee q]$
13. $(p \leftrightarrow q) \iff [(p \to q) \wedge (q \to p)]$

Property 11 simply restates the fact, proved in Theorem 0.2.1, that an implication and its contrapositive are logically equivalent. Property 12 shows that an implication is logically equivalent to a statement that does not use the symbol $\to$. The definition of "$\leftrightarrow$" gives Property 13 immediately.

Pause 2

Given an implication $p \to q$, explain why its *converse*, $q \to p$, and its *inverse*, $[(\neg p) \to (\neg q)]$, are logically equivalent. ∎

Pause 3

Show that $[\neg(p \leftrightarrow q)] \iff [(p \wedge (\neg q)) \vee (q \wedge (\neg p))]$. ▮

PROBLEM 6. Show that $(\neg p) \rightarrow (p \rightarrow q)$ is a tautology.

Solution. Using Property 12, we have

$$\begin{aligned} [(\neg p) \rightarrow (p \rightarrow q)] &\iff [(\neg p) \rightarrow ((\neg p) \vee q)] \\ &\iff [(\neg(\neg p)) \vee ((\neg p) \vee q)] \\ &\iff p \vee [(\neg p) \vee q] \\ &\iff [p \vee (\neg p)] \vee q \iff \mathbf{1} \vee q \iff \mathbf{1}. \end{aligned}$$ ▲

In the exercises, we ask you to verify all the properties of logical equivalence that we have stated. Some are very simple. For example, to see $(p \vee p) \iff p$, we need only observe that p and $p \vee p$ have the same truth tables:

p	$p \vee p$
T	T
F	F

.

Others require more work. To verify the second distributive property, for example, we would construct two truth tables.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	F
F	F	T	T	F
T	T	F	T	T
T	F	F	F	F
F	T	F	T	F
F	F	F	F	F

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	F	T	F	T	T
F	T	T	F	F	F
F	F	T	F	F	F
T	T	F	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	F	F	F	F

PROBLEM 7. Simplify the statement $[\neg(p \vee q)] \vee [(\neg p) \wedge q]$.

Solution. We have

$$\begin{aligned} &\big[[\neg(p \vee q)] \vee [(\neg p) \wedge q]\big] \\ &\quad\iff \big[[(\neg p) \wedge (\neg q)] \vee [(\neg p) \wedge q]\big] && \text{by one of the laws of De Morgan} \\ &\quad\iff (\neg p) \wedge [(\neg q) \vee q] && \text{using a distributive law} \\ &\quad\iff (\neg p) \wedge \mathbf{1} \iff \neg p, \end{aligned}$$

so the given statement is logically equivalent simply to $\neg p$. ▲

In Problems 6 and 7 we used, in a sneaky way, a very important principle of logic, which we now state as a theorem. The fact that we didn't really think about this at the time tells us that the theorem is easily understandable and quite painless to apply in practice.

1.2.1 THEOREM

Suppose $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent statements involving variables $p_1, p_2, \dots, p_n$. Suppose that $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ are statements. If, in $\mathcal{A}$ and $\mathcal{B}$, we replace p_1 by $\mathcal{C}_1$, p_2 by $\mathcal{C}_2$ and so on until we replace p_n by $\mathcal{C}_n$, then the resulting statements will still be logically equivalent.

Pause 4

Explain how Theorem 1.2.1 was used in Problem 7. ▮

PROBLEM 8. Show that $[(p \vee q) \vee ((q \vee (\neg r)) \wedge (p \vee r))] \iff \neg[(\neg p) \wedge (\neg q)]$.

Solution. Using one of the distributive laws, the left-hand side is logically equivalent to

$$[(p \vee q) \vee (q \vee (\neg r))] \wedge [(p \vee q) \vee (p \vee r)].$$

Associativity and idempotence say that the first term here, $[(p \vee q) \vee (q \vee (\neg r))]$, is logically equivalent to

$$\begin{aligned}[p \vee (q \vee (q \vee (\neg r)))] &\iff [p \vee ((q \vee q) \vee (\neg r))] \\ &\iff [p \vee (q \vee (\neg r))] \iff [(p \vee q) \vee (\neg r)],\end{aligned}$$

while the second term, $[(p \vee q) \vee (p \vee r)]$, is logically equivalent to

$$\begin{aligned}[(q \vee p) \vee (p \vee r)] &\iff [q \vee (p \vee (p \vee r))] \\ &\iff [q \vee ((p \vee p) \vee r)] \\ &\iff [q \vee (p \vee r)] \\ &\iff [(q \vee p) \vee r] \iff [(p \vee q) \vee r].\end{aligned}$$

Hence the expression on the left-hand side of the statement we are trying to establish is logically equivalent to

$$\begin{aligned}[((p \vee q) \vee (\neg r)) \wedge ((p \vee q) \vee r)] &\iff [(p \vee q) \vee ((\neg r) \wedge r)] \\ &\iff [(p \vee q) \vee \mathbf{0}] \iff p \vee q.\end{aligned}$$

But this is logically equivalent to the right-hand side of the desired statement by double negation and one of the laws of De Morgan. ▲

The next problem illustrates clearly why employing the basic logical equivalences discussed in this section is often more efficient than working simply with truth tables.

PROBLEM 9. Show that $[s \to (((\neg p) \wedge q) \wedge r)] \iff \neg[(p \vee (\neg (q \wedge r))) \wedge s]$.

Solution.

$$\begin{aligned}[s \to (((\neg p) \wedge q) \wedge r)] &\iff [(\neg s) \vee (((\neg p) \wedge q) \wedge r)] \\ &\iff [(\neg s) \vee ((\neg p) \wedge (q \wedge r))] \\ &\iff [(\neg s) \vee (\neg(p \vee (\neg(q \wedge r))))] \\ &\iff \neg[s \wedge (p \vee (\neg (q \wedge r)))] \\ &\iff \neg[(p \vee (\neg(q \wedge r))) \wedge s].\end{aligned}$$ ▲

A primary application of the work in this section is reducing statements to logically equivalent simpler forms. There are times, however, when a different type of logically equivalent statement is required.

1.2.2 DEFINITION Let $n \geq 1$ be an integer and let $x_1, x_2, \ldots, x_n$ be variables. A *minterm* based on these variables is a compound statement of the form $a_1 \wedge a_2 \wedge \cdots \wedge a_n$, where each a_i is x_i or $\neg x_i$. A compound statement in $x_1, x_2, \ldots, x_n$ is said to be in *disjunctive normal form* if it looks like $y_1 \vee y_2 \vee \cdots \vee y_m$ where the statements $y_1, y_2, \ldots, y_m$ are different minterms. ❖

EXAMPLE 10 $x_1 \wedge \neg x_2 \wedge \neg x_3$ is a minterm (on variables x_1, x_2, x_3) and the statement $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge (\neg x_2) \wedge (\neg x_3))$ is in disjunctive normal form.

$(p \wedge q) \vee (\neg p \wedge \neg q)$ is in disjunctive normal form on variables p, q, but $(p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge q)$ is not (because two minterms are the same). ■

EXAMPLE 11 $p \wedge (q \vee r)$ is not a minterm (because it involves the symbol $\vee$) and the statement $((p \wedge q) \vee r) \wedge ((p \wedge q) \vee (\neg q))$ is not in disjunctive normal form, one reason being that the minterms $(p \wedge q) \vee r$ and $(p \wedge q) \vee (\neg q)$ involve the symbol $\vee$. This statement is logically equivalent to $(p \wedge q) \vee (r \wedge (\neg q))$, which is still not in disjunctive normal form because the minterms, $p \wedge q$ and $r \wedge (\neg q)$, don't contain all the variables. Continuing, however, our statement is logically equivalent to

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \;\vee\; (p \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge r),$$

which is in disjunctive normal form (on the variables p, q, r). ■

As shown in Example 11, when writing a statement in disjunctive normal form, it is very useful to note that

$$x \iff [(x \wedge y) \vee (x \wedge (\neg y))] \tag{1}$$

for any statements x and y. This follows from

$$x \iff (x \wedge \mathbf{1}) \iff [x \wedge (y \vee (\neg y))] \iff [(x \wedge y) \vee (x \wedge (\neg y))].$$

PROBLEM 12. Express $p \to (q \wedge r)$ in disjunctive normal form.

Solution. Method 1: We construct a truth table.

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	F	T
T	T	F	F	F
T	F	F	F	F
F	T	F	F	T
F	F	F	F	T

Now focus attention on the rows for which the statement is true—each of these will contribute a minterm to our answer. For example, in row 1, p, q, and r are all T, so $p \wedge q \wedge r$ agrees with the T in the last column. In row 4, p is F, while q and r are both T. This gives the minterm $(\neg p) \wedge q \wedge r$. In this way, we obtain

$$(p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge (\neg r)) \\ \vee ((\neg p) \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)).$$

Method 2: We have

$$
\begin{aligned}
&[p \to (q \wedge r)] \\
&\quad \Longleftrightarrow \quad [(\neg p) \vee (q \wedge r)] \\
&\quad \Longleftrightarrow \quad [((\neg p) \wedge q) \vee ((\neg p) \wedge (\neg q)) \vee (q \wedge r)] \\
&\quad \Longleftrightarrow \quad [((\neg p) \wedge q \wedge r) \ \vee \ ((\neg p) \wedge q \wedge (\neg r)) \ \vee \ ((\neg p) \wedge (\neg q) \wedge r) \\
&\qquad\qquad \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)) \ \vee \ (p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r)] \\
&\quad \Longleftrightarrow \quad [((\neg p) \wedge q \wedge r) \ \vee \ ((\neg p) \wedge q \wedge (\neg r)) \ \vee \ ((\neg p) \wedge (\neg q) \wedge r) \\
&\qquad\qquad \vee ((\neg p) \wedge (\neg q) \wedge (\neg r)) \ \vee \ (p \wedge q \wedge r)],
\end{aligned}
$$

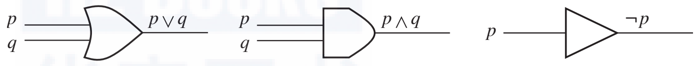
omitting the second occurrence of $(\neg p) \wedge q \wedge r$ at the last step. We leave it to you to decide for yourself which method you prefer. ▲

Application: Three-Way Switches

Disjunctive normal form is useful in applications of logic to computer science, particularly in the construction *logic circuits*.

Most of us can guess how an ordinary light switch works. Flipping the switch up completes a circuit and sends electricity to the light bulb, which then (if it's not broken) goes on. Flipping the switch down cuts the circuit and the light goes off. But how do three-way switches work, you know, two switches that control the same light? You flip one up and the light goes on; you flip the other up and the light goes off, and so on. Exactly how do these switches work?

We illustrate with a *logic circuit*, the key components of which are *OR-gates*, *AND-gates*, and *NOT-gates*. In diagrams, these are depicted with the standard symbols shown in Fig. 1.5. As you might guess, an AND-gate implements the logical connective $\wedge$, an OR-gate implements $\vee$, and a NOT-gate implements $\neg$.



OR-gate AND-gate NOT-gate

Figure 1.5

In Fig. 1.6, we show how these gates can be put together to make three-way switches work as they should. There are two inputs to the circuit, labeled p and q, corresponding to the two switches. We assign these variables the values 0 and 1 to indicate up and down, respectively. Let v denote the current reaching the light bulb, 0 indicating "no current" (so the light is off) and 1 indicating "current" (so the light is on). The table to the left shows how we would like v to depend on p and q. In particular, notice that when both switches are up, the light should be off. Look at the two rows where $v = 1$. Just as in Problem 12, we see that v is logically equivalent to $[(p \wedge (\neg q)] \vee [(\neg p) \wedge q]$, which is in disjunctive normal form. Thus, assuming we can build devices that implement $\wedge$, $\vee$, and $\neg$, we can see easily how to build three-way switches.

p	q	v
0	0	0
1	0	1
0	1	1
1	1	0

Answers to Pauses

2. This is just a restatement of Property 11, writing p instead of q and q instead of p.

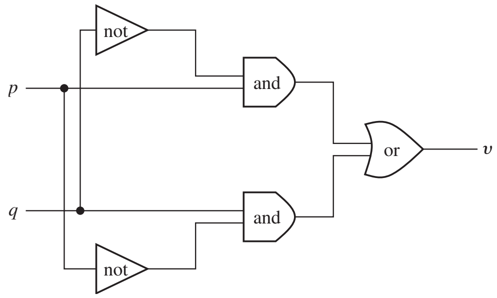


Figure 1.6 Implementation of three-way switches.

3. This could be done with truth tables. Alternatively, we note that

$$\begin{aligned}
\neg\,(p \leftrightarrow q) &\iff [\neg((p \to q) \wedge (q \to p))] \\
&\iff \neg[((\neg p) \vee q) \wedge ((\neg q) \vee p)] \\
&\iff [\neg\,((\neg p) \vee q) \vee (\neg((\neg q) \vee p))] \\
&\iff [(p \wedge (\neg q)) \vee (q \wedge (\neg p))].
\end{aligned}$$

4. In applying the distributive property, we are using $\neg p$, $\neg q$, and q in place of p, q, and r. Also, when applying Property 7, we use $\neg p$ instead of p.

True/False Questions

(Answers can be found in the back of the book.)

1. Two statements $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent precisely when the statement $\mathcal{A} \to \mathcal{B}$ is a tautology.
2. "$\mathcal{A} \iff \mathcal{B}$" and "$\mathcal{A} \leftrightarrow \mathcal{B}$" mean the same thing.
3. $(p \vee p) \iff (p \wedge p)$ for any statement p.
4. $((p \vee q) \wedge r) \iff (p \vee (q \wedge r))$ for any statements p, q, r.
5. $(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$ for any statements p, q, r.
6. $(\neg(p \wedge q)) \iff ((\neg p) \vee (\neg q))$ for any statements p, q.
7. If $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{C}$ is any statement, then $(\mathcal{A} \to \mathcal{C}) \iff (\mathcal{B} \to \mathcal{C})$.
8. $(p \wedge q \wedge (\neg r)) \vee ((\neg p) \wedge (\neg q))$ is in disjunctive normal form.
9. $(p \wedge q \wedge (\neg r)) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$ is in disjunctive normal form.
10. Disjunctive normal form is useful in applications of logic to computer science.

Exercises

The answers to exercises marked [BB] can be found in the ***B****ack of the* ***B****ook.*

1. Verify each of the 13 properties of logical equivalence that appear in this section [BB; 1,3,5,7,9,11,13].
2. **(a)** Show that $p \vee [\neg(p \wedge q)]$ is a tautology.

   (b) What is the negation of the statement in (a)? Show that this negation is a contradiction.
3. Simplify each of the following statements.

   (a) [BB] $(p \wedge q) \vee (\neg((\neg p) \vee q))$

   (b) $(p \vee r) \to [(q \vee (\neg r)) \to ((\neg p) \to r)]$

   (c) $[(p \to q) \vee (q \to r)] \wedge (r \to s)$
4. Using truth tables, verify the following *absorption* properties.

   (a) [BB] $(p \vee (p \wedge q)) \iff p$

   (b) $(p \wedge (p \vee q)) \iff p$

5. Using the properties in the text together with the absorption properties given in Exercise 4, where needed, establish each of the following logical equivalences.

(a) [BB] $[(p \vee q) \wedge (\neg p)] \iff [(\neg p) \wedge q]$

(b) $[p \to (q \to r)] \iff [(p \wedge (\neg r)) \to (\neg q)]$

(c) $[\neg(p \leftrightarrow q)] \iff [p \leftrightarrow (\neg q)]$

(d) [BB] $\neg[(p \leftrightarrow q) \vee (p \wedge (\neg q))] \iff [(p \leftrightarrow (\neg q)) \wedge ((\neg p) \vee q)]$

(e) $[(p \wedge (\neg q)) \wedge ((p \wedge (\neg q)) \vee (q \wedge (\neg r)))] \iff [p \wedge (\neg q)]$

(f) $[p \to (q \vee r)] \iff [p \wedge (\neg q)] \to r]$

(g) $\neg(p \vee q) \vee [(\neg p) \wedge q] \iff \neg p$

6. Prove that the statements $(p \wedge (\neg q)) \to q$ and $(p \wedge (\neg q)) \to \neg p$ are logically equivalent. What simpler statement is logically equivalent to both of them?

7. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are statements with $\mathcal{A}$ and $\mathcal{B}$ logically equivalent.

(a) Show that $\mathcal{A} \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ are logically equivalent.

(b) Show that $\mathcal{A} \wedge \mathcal{C}$ and $\mathcal{B} \wedge \mathcal{C}$ are logically equivalent.

8. In Exercise 11 of Section 1.1 we defined the *exclusive or* "$p \veebar q$" to be true whenever either p or q is true, but not both. For each of the properties discussed in this section (including those of absorption given in Exercise 4) determine whether the property holds with $\veebar$ replacing $\vee$ wherever it occurs [BB; 1,3,7,9,13].

9. Which of the following are in disjunctive normal form (on the appropriate set of variables)?

(a) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$

(b) [BB] $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$

(c) [BB] $p \vee ((\neg p) \wedge q)$

(d) $(p \wedge q) \vee ((\neg p) \wedge (\neg q) \wedge r)$

(e) $(p \wedge q \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$

10. Express each of the following statements in disjunctive normal form.

(a) [BB] $p \wedge q$

(b) [BB] $(p \wedge q) \vee (\neg((\neg p) \vee q))$

(c) $p \to q$

(d) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$

(e) $(p \to q) \wedge (q \wedge r)$

(f) $p \vee [(q \wedge (p \vee (\neg r))]$

✍**11.** Find out what you can about Augustus De Morgan and write a paragraph or two about him, in good English, of course!

1.3 Logical Arguments

Proving a theorem in mathematics involves drawing a conclusion from some given information. The steps required in the proof generally consist of showing that if certain statements are true then the truth of other statements must follow. Taken in its entirety, the proof of a theorem demonstrates that if an initial collection of statements—called *premises* or *hypotheses*—are all true then the conclusion of the theorem is also true.

Different methods of proof were discussed informally in Section 0.2. Now we relate these ideas to some of the more formal concepts introduced in Sections 1.1 and 1.2. First, we define what is meant by a *valid argument*.

1.3.1 DEFINITIONS

An *argument* is a finite collection of statements $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ called *premises* (or *hypotheses*) followed by a statement $\mathcal{B}$ called the *conclusion*. Such an argument is *valid* if, whenever $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are all true, then $\mathcal{B}$ is also true. ❖

It is often convenient to write an elementary argument in column form, like this.

$$\begin{array}{c} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \\ \hline \mathcal{B} \end{array}$$

PROBLEM 13. Show that the argument

$$\begin{array}{c} p \to \neg q \\ r \to q \\ r \\ \hline \neg p \end{array}$$

is valid.

Solution. We construct a truth table.

p	q	r	$\neg q$	$p \to \neg q$	$r \to q$	$\neg p$	
T	T	T	F	F	T	F	
T	F	T	T	T	F	F	
F	T	T	F	T	T	T	$\star$
F	F	T	T	T	F	T	
T	T	F	F	F	T	F	
T	F	F	T	T	T	F	
F	T	F	F	T	T	T	
F	F	F	T	T	T	T	

Observe that row 3—marked with the star ($\star$)—is the only row where the premises $p \to \neg q$, $r \to q$, r are all marked T. In this row, the conclusion $\neg p$ is also T. Thus the argument is valid. ▲

In Problem 13, we were a bit fortunate because there was only one row where all the premises were marked T. In general, to assert that an argument is valid when there are several rows with all premises marked T, it is necessary to check that the conclusion is also T in every such row.

Arguments can be shown to be valid without the construction of a truth table. For example, here is an alternative way to solve Problem 13.

> Assume that all premises are true. In particular, this means that r is true. Since $r \to q$ is also true, q must also be true. Thus $\neg q$ is false and, because $p \to (\neg q)$ is true, p is false. Thus $\neg p$ is true as desired.

PROBLEM 14. Determine whether the following argument is valid.

If I like biology, then I will study it.
Either I study biology or I fail the course.
―――――――――――――――――――――――――――――――――
If I fail the course, then I do not like biology.

Solution. Let p be "I like biology," q be "I study biology," and r be "I fail the course." In symbols, the argument we are to check becomes

$$\begin{array}{c} p \to q \\ q \vee r \\ \hline r \to (\neg p). \end{array}$$

This can be analyzed by a truth table.

p	q	r	$p \to q$	$q \vee r$	$\neg p$	$r \to (\neg p)$	
T	T	T	T	T	F	F	$\star$
T	F	T	F	T	F	F	
F	T	T	T	T	T	T	$\star$
F	F	T	T	T	T	T	$\star$
T	T	F	T	T	F	T	$\star$
T	F	F	F	F	F	T	
F	T	F	T	T	T	T	$\star$
F	F	F	T	F	T	T	

The rows marked $\star$ are those in which the premises are true. In row 1, the premises are true, but the conclusion is F. The argument is not valid. ▲

The theorem that follows relates the idea of a valid argument to the notions introduced in Sections 1.1 and 1.2.

1.3.2 THEOREM

An argument with premises $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and conclusion $\mathcal{B}$ is valid precisely when the compound statement $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{B}$ is a tautology.

Surely, this is not hard to understand. For the implication $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n \to \mathcal{B}$ to be a tautology, it must be the case that whenever $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n$ is true then $\mathcal{B}$ is also true. But $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_n$ is true precisely when each of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ is true, so the result follows from our definition of a valid argument.

In the same spirit as Theorem 1.2.1, we have the following important *substitution theorem*.

1.3.3 THEOREM

Substitution Assume that an argument with premises $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and conclusion $\mathcal{B}$ is valid and that all these statements involve variables $p_1, p_2, \ldots, p_m$. If $p_1, p_2, \ldots, p_m$ are replaced by statements $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$, the resulting argument is still valid.

Rules of Inference

Because of Theorem 1.3.3, some very simple valid arguments that regularly arise in practice are given special names. Here is a list of some of the most common *rules of inference*.

1. **Modus ponens:**
$$\begin{array}{c} p \\ p \to q \\ \hline q \end{array}$$

2. **Modus tollens:**
$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

3. **Disjunctive syllogism:**
$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$$

4. **Chain rule:**
$$\begin{array}{c} p \to q \\ \underline{q \to r} \\ p \to r \end{array}$$

5. **Resolution:**
$$\begin{array}{c} p \vee r \\ \underline{q \vee (\neg r)} \\ p \vee q \end{array}$$

Pause 5 Verify modus tollens. ▮

We illustrate how the rules of inference can be applied.

PROBLEM 15. Show that the following argument is valid.

$$\begin{array}{c} (p \vee q) \to (s \wedge t) \\ \underline{[\neg ((\neg s) \vee (\neg t))] \to [(\neg r) \vee q]} \\ (p \vee q) \to (r \to q) \end{array}$$

Solution. One of the laws of De Morgan and the principle of double negation—see Section 1.2—tell us that

$$[\neg ((\neg s) \vee (\neg t))] \iff [(\neg (\neg s)) \wedge (\neg(\neg t))] \iff (s \wedge t).$$

Property 12 of logical equivalence as given in Section 1.2 says that $(\neg r) \vee q \iff (r \to q)$. Thus the given argument can be rewritten as

$$\begin{array}{c} (p \vee q) \to (s \wedge t) \\ \underline{(s \wedge t) \to (r \to q)} \\ (p \vee q) \to (r \to q) \end{array}$$

The chain rule now tells us that our argument is valid. ▲

Pause 6 If a truth table were used to answer Problem 15, how many rows would be required? ▮

Sometimes, rules of inference need to be combined.

PROBLEM 16. Determine the validity of the following argument.

If I study, then I will pass.
If I do not go to a movie, then I will study.
I failed.

Therefore, I went to a movie.

Solution. Let p, q, and r be the statements

p: "I study."
q: "I pass."
r: "I go to a movie."

The given argument is

$$\begin{array}{c} p \to q \\ (\neg r) \to p \\ \neg q \\ \hline r \end{array}$$

The first two premises imply the truth of $(\neg r) \to q$ by the chain rule. Since $(\neg r) \to q$ and $\neg q$ imply $\neg(\neg r)$ by modus tollens, the validity of the argument follows by the principle of double negation: $\neg(\neg r) \iff r$. ▲

Answers to Pauses

5. While this can be shown with a truth table, we prefer an argument by words. Since $\neg q$ is true, q is false. Since $p \to q$ is true, p must also be false. Hence $\neg p$ is true and we are done.

6. There are five variables, each of which could be T or F, so we would need $2^5 = 32$ rows.

True/False Questions

(Answers can be found in the back of the book.)

1. An argument is valid if, whenever the conclusion is true, then the premises are also true.

2. If the premises of an argument are all contradictions, then the argument is valid.

3. If the premises of an argument are all tautologies and the conclusion is not a tautology, then the argument is not valid.

4. De Morgan's laws are two examples of rules of inference.

5. The chain rule has $p \to q$ and $q \to r$ as its premises.

6. Resolution has $p \wedge q$ as its conclusion.

7. Modus ponens and modus tollens were named after the famous Canadian logician George Modus.

8. To decide whether a given argument is valid, it is always best to use truth tables.

9. To decide whether a given argument is valid, it is never best to use truth tables.

10. We've done enough logic. Let's get on to something different!

Exercises

The answers to exercises marked [BB] can be found in the ***B****ack of the* ***B****ook.*

You are encouraged to use the result of any exercise in this set to assist with the solution of any other.

1. Determine whether or not each of the following arguments is valid.

(a) [BB] $$\begin{array}{c} p \to (q \to r) \\ q \\ \hline p \to r \end{array}$$

(b) [BB] $$\begin{array}{c} p \to q \\ q \vee r \\ \hline r \to (\neg q) \end{array}$$

(c) $$\begin{array}{c} p \to q \\ r \to q \\ \hline r \to p \end{array}$$

(d) $$\begin{array}{c} p \to q \\ (q \vee (\neg r)) \to (p \wedge s) \\ \hline s \to (r \vee q) \end{array}$$

(e) $$\begin{array}{c} p \to (\neg q) \\ r \to q \\ r \\ \hline \neg p \end{array}$$

(f) $$\begin{array}{c} p \to (\neg q) \\ (\neg r) \to p \\ q \\ \hline r \end{array}$$

2. Verify that each of the five rules of inference given in this section is a valid argument.

3. Verify that each of the following arguments is valid.

(a) [BB] $$\begin{array}{c} p \to r \\ q \to r \\ \hline (p \vee q) \to r \end{array}$$

(b) $$\begin{array}{c} p \to r \\ q \to s \\ \hline (p \wedge q) \to (r \wedge s) \end{array}$$

(c)
$$\begin{array}{c} p \vee q \\ (\neg p) \vee r \\ (\neg r) \vee s \\ \hline q \vee s \end{array}$$

(d)
$$\begin{array}{c} p \vee ((\neg q) \wedge r) \\ \neg\,(p \wedge s) \\ \hline \neg\,(s \wedge (q \vee (\neg r))) \end{array}$$

4. Test the validity of each of the following arguments. [*Hint*: You may find Exercise 3(a) helpful.]

(a) [BB]
$$\begin{array}{c} p \rightarrow q \\ (\neg r) \vee (\neg q) \\ r \\ \hline \neg p \end{array}$$

(b)
$$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \rightarrow (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \rightarrow (t \vee s) \\ \hline (p \vee r) \rightarrow (q \vee r) \end{array}$$

(c) [BB]
$$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \rightarrow (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \leftrightarrow (t \vee s) \\ \hline (q \vee r) \rightarrow (p \vee r) \end{array}$$

(d)
$$\begin{array}{c} p \vee (\neg q) \\ (\neg r) \vee (t \vee s) \\ (t \vee s) \rightarrow p \\ \hline (q \vee r) \rightarrow (p \vee r) \end{array}$$

(e)
$$\begin{array}{c} [(p \wedge q) \vee r] \rightarrow (q \wedge r \wedge s) \\ [(\neg p) \wedge (\neg q)] \rightarrow (r \vee p) \\ [p \vee (\neg q) \vee r] \rightarrow (q \wedge s) \\ \hline (p \wedge q) \leftrightarrow [(q \wedge r) \vee s] \end{array}$$

(f) [BB]
$$\begin{array}{c} p \rightarrow q \vee s \\ q \rightarrow r \\ \hline p \rightarrow (r \vee s) \end{array}$$

(g)
$$\begin{array}{c} p \vee (q \rightarrow r) \\ q \vee r \\ r \rightarrow p \\ \hline p \end{array}$$

[*Hint*: part (f)]

(h)
$$\begin{array}{c} p \rightarrow (q \vee r) \\ q \rightarrow s \\ r \rightarrow \neg p \\ \hline p \rightarrow s \end{array}$$

[*Hint*: part (f)]

5. Determine the validity of each of the following arguments. If the argument is one of those listed in the text, name it.

(a) [BB] If I stay up late at night,
then I will be tired in the morning.
I stayed up late last night.
———
I am tired this morning.

(b) [BB] If I stay up late at night,
then I will be tired in the morning.
I am tired this morning.
———
I stayed up late last night.

(c) If I stay up late at night,
then I will be tired in the morning.
I am not tired this morning.
———
I did not stay up late last night.

(d) If I stay up late at night,
then I will be tired in the morning.
I did not stay up late last night.
———
I am not tired this morning.

(e) [BB] Either I wear a red tie or I wear blue socks.
I am wearing pink socks.
———
I am wearing a red tie.

(f) Either I wear a red tie or I wear blue socks.
I am wearing blue socks.
———
I am not wearing a red tie.

(g) [BB] If I work hard, then I earn lots of money.
If I earn lots of money,
then I pay high taxes.
———
If I pay high taxes, then I have worked hard.

(h) If I work hard, then I earn lots of money.
If I earn lots of money,
then I pay high taxes.
———
If I work hard, then I pay high taxes.

(i) If I work hard, then I earn lots of money.
If I earn lots of money, then I pay high taxes.
———
If I do not work hard, then I do not pay high taxes.

(j) If I like mathematics, then I will study.
I will not study.
Either I like mathematics or I like football.
———
I like football.

(k) Either I study or I like football.
If I like football, then I like mathematics.
———
If I don't study, then I like mathematics.

(l) [BB] If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't graduate,
then I didn't pass mathematics.
———
If I graduate, then I studied.

(m) If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't graduate, then I didn't pass mathematics.
———
If I like mathematics, then I will graduate.

6. [BB] Given the premises $p \rightarrow (\neg r)$ and $r \vee q$, either write down a valid conclusion that involves p and q only and is not a tautology or show that no such conclusion is possible.

7. Repeat Exercise 6 with the premises $(\neg p) \to r$ and $r \vee q$.

8. **(a)** [BB] Explain why two premises p and q can always be replaced by the single premise $p \wedge q$, and vice versa.

(b) Using 8(a), verify that this argument is valid:

$$\begin{array}{c} p \wedge q \\ p \to r \\ s \to \neg q \\ \hline (\neg s) \wedge r. \end{array}$$

9. Let n be an integer greater than 1. Show that the following argument is valid.

$$\begin{array}{c} p_1 \to (q_1 \to r_1) \\ p_2 \to (q_2 \to r_2) \\ \vdots \\ p_n \to (q_n \to r_n) \\ q_1 \wedge q_2 \wedge \cdots \wedge q_n \\ \hline (p_1 \to r_1) \wedge (p_2 \to r_2) \wedge \cdots \wedge (p_n \to r_n) \end{array}$$

[*Hint*: Exercise 8(a).]

✍**10.** [BB] What language is being used when we say "modus ponens" or "modus tollens"? Translate these expressions into English and explain.

Key Terms & Ideas

Here are some technical words and phrases that were used in this chapter. Do you know the meaning of each? If you're not sure, check the glossary or index at the back of the book.

argument
conclusion
contradiction
contrapositive
converse
disjunctive normal form
hypothesis
logically equivalent
minterm
negation
premise
tautology
valid argument

Review Exercises for Chapter 1

1. Construct a truth table for the compound statement $[p \wedge (q \to (\neg r))] \to [(\neg q) \vee r]$.

2. Determine the truth value of $[p \vee (q \to ((\neg r) \wedge s))] \leftrightarrow (r \wedge t)$, where p, q, r, s, and t are all true.

3. Determine whether each statement is a tautology, a contradiction, or neither.

(a) $[p \wedge (\neg q)] \wedge [(\neg p) \vee q]$

(b) $(p \to q) \to (p \vee q)$

(c) $p \vee [(p \wedge (\neg q)) \to r]$

(d) $[p \vee q] \leftrightarrow [(\neg q) \wedge r]$

4. Two compound statements $\mathcal{A}$ and $\mathcal{B}$ have the property that $\mathcal{A} \to \mathcal{B}$ is logically equivalent to $\mathcal{B} \to \mathcal{A}$. What can you conclude about $\mathcal{A}$ and $\mathcal{B}$?

5. **(a)** Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are compound statements such that $\mathcal{A} \iff \mathcal{B}$ and $\mathcal{B} \iff \mathcal{C}$. Explain why $\mathcal{A} \iff \mathcal{C}$.

(b) Give a proof of Property 11 that uses the result of Property 12.

6. Establish the logical equivalence of each of the following pairs of statements.

(a) $[(p \to q) \to r]$ and $[(p \vee r) \wedge (\neg(q \wedge (\neg r)))]$

(b) $p \to (q \vee s)$ and $[p \wedge (\neg q)] \to r$

7. Express each of the following statements in disjunctive normal form.

(a) $((p \vee q) \wedge r) \vee ((p \vee q) \wedge (\neg p))$

(b) $[p \vee (q \wedge (\neg r))] \wedge \neg(q \wedge r)$

8. Determine whether each of the following arguments is valid.

(a) $$\begin{array}{c} p \to q \\ \neg p \\ \hline \neg q \end{array}$$

(b) $$\begin{array}{c} \neg((\neg p) \wedge q) \\ \neg(p \wedge r) \\ r \vee s \\ \hline q \to s \end{array}$$

(c) $$\begin{array}{c} p \vee (\neg q) \\ (t \vee s) \to (p \vee r) \\ (\neg r) \vee (t \vee s) \\ p \leftrightarrow (t \vee s) \\ \hline (p \wedge r) \to (q \wedge r) \end{array}$$

9. Discuss the validity of the argument

$$\begin{array}{c} p \wedge q \\ (\neg p) \wedge r \\ \hline \text{Purple toads live on Mars.} \end{array}$$

10. Determine the validity of each of the following arguments. If the argument is one of those listed in the text, name it.

(a) Either I wear a red tie or I wear blue socks.
Either I wear a green hat or I do not wear blue socks.
―――――――――――――――――――――
Either I wear a red tie or I wear a green hat.

(b) If I like mathematics, then I will study.
Either I don't study or I pass mathematics.
If I don't pass mathematics, then I don't graduate.
―――――――――――――――――――――
If I graduate, then I like mathematics.

2 Sets and Relations

2.1 Sets

Any branch of science, like a foreign language, has its own terminology. *Isomorphism*, *cyclotomic*, and *coset* aren't words used except in a mathematical context. On the other hand, quite a number of common English words—field, complex, function—have precise mathematical meanings quite different from their usual ones. Students of French or Spanish know that memory work is a fundamental part of their studies; it is perfectly obvious to them that if they don't know what the words mean their ability to learn grammar and to communicate will be severely hindered. It is, however, not always understood by science students that they too must memorize the terminology of their discipline. Without constant review of the meanings of words, one's understanding of a paragraph of text or the words of a teacher is very limited. We advise readers of this book to maintain and constantly review a mathematical vocabulary list. The authors have included their own such list in a glossary at the back of this book.

What would it be like to delve into a dictionary if you didn't already know the meanings of some of the words in it? Most people, at one time or another, have gone to a dictionary in search of a word only to discover that the definition uses another unfamiliar word. Some reflection indicates that a dictionary can be of no use unless there are some words that are so basic that we can understand them without definitions. Mathematics is the same way. There are a few basic terms that we accept without definitions.

Most of mathematics is based on the single undefined concept of *set*, which we think of as just a collection of things called *elements* or *members*. Primitive humans discovered the set of *natural numbers* with which they learned to count. The set of natural numbers, which is denoted with a capital boldface $\mathbf{N}$ or, in handwriting, with this symbol, $\mathbb{N}$, consists of the numbers $1, 2, 3, \ldots$ (the three dots meaning "and so on").[1] The elements of $\mathbf{N}$ are, of course, just the positive integers. The full set of *integers*, denoted $\mathbf{Z}$ or $\mathbb{Z}$, consists of the natural numbers, their negatives, and 0. We might describe this set by $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Our convention, which is not universal, is that 0 is an integer, but **not** a natural number.

[1] Since the manufacture of boldface symbols such as $\mathbf{N}$ is a luxury not afforded users of chalk or pencil, it has long been traditional to use $\mathbb{N}$ on blackboards or in handwritten work as the symbol for the natural numbers and to call $\mathbb{N}$ a *blackboard bold* symbol.

There are various ways to describe sets. Sometimes it is possible to list the elements of a set within braces.

- {egg1, egg2} is a set containing two elements, egg1 and egg2.
- $\{x\}$ is a set containing one element, x.
- $\mathsf{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.
- $\mathsf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers.

On other occasions, it is convenient to describe a set with *set builder* notation. This has the format

$$\{x \mid x \text{ has certain properties}\},$$

which is read "the set of x such that x has certain properties." We read "such that" at the vertical line, |.

More generally, we see

$$\{\text{some expression} \mid \text{the expression has certain properties}\}.$$

Thus, the set of odd natural numbers could be described as

$$\{n \mid n \text{ is an odd integer}, n > 0\}$$

or as

$$\{2k - 1 \mid k = 1, 2, 3, \ldots\}$$

or as

$$\{2k - 1 \mid k \in \mathsf{N}\}.$$

The expression "$k \in \mathsf{N}$" is read "k belongs to N," the symbol $\in$ denoting set membership. Thus, "$m \in \mathsf{Z}$" simply says that m is an integer. Recall that a slash (/) written over any mathematical symbol negates the meaning of that symbol. So, in the same way that $\pi \neq 3.14$, we have $0 \notin \mathsf{N}$.

The set of common fractions—numbers like $\frac{3}{4}$, $\frac{-2}{17}$, and 5 $(= \frac{5}{1})$, which are ratios of integers with nonzero denominators—is more properly called the set of *rational numbers* and is denoted Q or $\mathbb{Q}$. Formally,

$$\mathsf{Q} = \left\{\tfrac{m}{n} \mid m, n \in \mathsf{Z}, n \neq 0\right\}.$$

The set of all real numbers is denoted R or $\mathbb{R}$. To define the real numbers properly requires considerable mathematical maturity. For our purposes, we think of real numbers as numbers that have decimal expansions of the form $a.a_1a_2\ldots$, where a is an integer and $a_1, a_2, \ldots$ are integers between 0 and 9 inclusive. In addition to the rational numbers, whose decimal expansions terminate or repeat, the real numbers include numbers like $\sqrt{2}$, $\sqrt[3]{17}$, e, π, $\ln 5$, and $\cos\frac{\pi}{6}$ whose decimal expansions neither terminate nor repeat. Such numbers are called *irrational*. An irrational number is a number that cannot be written in the form $\frac{m}{n}$ with m and n both integers. Incidentally, it can be very difficult to decide whether a given real number is irrational. For example, it is unknown whether such numbers as $e + \pi$ or $\frac{e}{\pi}$ are irrational.

The *complex numbers*, denoted C or $\mathbb{C}$, have the form $a + bi$, where a and b are real numbers and $i^2 = -1$; that is,

$$\mathsf{C} = \{a + bi \mid a, b \in \mathsf{R}, i^2 = -1\}.$$

Sometimes people are surprised to discover that a set can be an element of another set. For example, $\{\{a, b\}, c\}$ is a set with two elements, one of which is $\{a, b\}$ and the other c.

Pause 1

Let S denote the set $\{\{a\}, b, c\}$. True or false?

(a) $a \notin S$.
(b) $\{a\} \in S$. ▮

Equality of Sets

2.1.1 DEFINITION

Sets A and B are *equal*, and we write $A = B$, if and only if A and B contain the same elements or neither set contains any element. ❖

EXAMPLE 1

- $\{1, 2, 1\} = \{1, 2\} = \{2, 1\}$;
- $\left\{\frac{1}{2}, \frac{2}{4}, \frac{-3}{-6}, \frac{\pi}{2\pi}\right\} = \left\{\frac{1}{2}\right\}$;
- $\{t \mid t = r - s,\ r, s \in \{0, 1, 2\}\} = \{-2, -1, 0, 1, 2\}$. ■

The Empty Set

One set that arises in a variety of different guises is the set that contains no elements. Consider, for example, the set SMALL of people less than 1 millimeter in height, the set LARGE of people taller than the Eiffel Tower, the set

$$\mathfrak{PECULIAR} = \{n \in \mathsf{N} \mid 5n = 2\},$$

and the set

$$S = \{n \in \mathsf{N} \mid n^2 + 1 = 0\}.$$

These sets are all equal since none of them contains any elements. The unique set that contains no elements is called the *empty set*. Set theorists originally used 0 (zero) to denote this set, but now it is customary to use a 0 with a slash through it, $\emptyset$, to avoid confusion between zero and a capital "Oh."

Pause 2

True or false? $\{\emptyset\} = \emptyset$. ▮

Subsets

2.1.2 DEFINITION

A set A is a *subset* of a set B, and we write $A \subseteq B$, if and only if every element of A is an element of B. If $A \subseteq B$ but $A \neq B$, then A is called a *proper subset* of B and we write $A \subsetneqq B$. ❖

When $A \subseteq B$, it is common to say "A is contained in B" as well as "A is a subset of B." The notation $A \subset B$, which is common, unfortunately means $A \subsetneqq B$ to some people and $A \subseteq B$ to others. For this reason, we avoid it, while reiterating that it is present in a lot of mathematical writing. When you see it, make an effort to discover what the intended meaning is.

We occasionally see "$B \supseteq A$," read "B is a *superset* of A." This is an alternative way to express "$A \subseteq B$," A is a subset of B, just as "$y \geq x$" is an alternative way to express "$x \leq y$." We generally prefer the subset notation.

EXAMPLE 2

- $\{a, b\} \subseteq \{a, b, c\}$
- $\{a, b\} \subsetneqq \{a, b, c\}$
- $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
- $\{a, b\} \in \{a, b, \{a, b\}\}$
- $\mathsf{N} \subsetneqq \mathsf{Z} \subsetneqq \mathsf{Q} \subsetneqq \mathsf{R} \subsetneqq \mathsf{C}$ ■

Note the distinction between $A \subsetneq B$ and $A \not\subseteq B$, the latter expressing the negation of $A \subseteq B$; for example,

$$\{a, b\} \subsetneq \{a, b, c\} \not\subseteq \{a, b, x\}.$$

2.1.3 PROPOSITION For any set A, $A \subseteq A$ and $\emptyset \subseteq A$.

Proof If $a \in A$, then $a \in A$, so $A \subseteq A$. The proof that $\emptyset \subseteq A$ is a classic model of proof by contradiction. If $\emptyset \subseteq A$ is false, then there must exist some $x \in \emptyset$ such that $x \notin A$. This is an absurdity since there is no $x \in \emptyset$. ●

Pause 3 True or false?

(a) $\{\emptyset\} \in \{\{\emptyset\}\}$
(b) $\emptyset \subseteq \{\{\emptyset\}\}$
(c) $\{\emptyset\} \subseteq \{\{\emptyset\}\}$

(As Shakespeare once wrote, "Much ado about nothing.") ▮

The following proposition is an immediate consequence of the definitions of "subset" and "equal sets," and it illustrates the way we prove two sets are equal in practice.

2.1.4 PROPOSITION If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Two assertions are being made here.

($\longrightarrow$) If $A = B$, then A is a subset of B and B is a subset of A.

($\longleftarrow$) If A is a subset of B and B is a subset of A, then $A = B$.

Remember that another way to state Proposition 2.1.4 is to say that, for two sets to be equal, it is **necessary and sufficient** that each be a subset of the other.

Note the distinction between **membership**, $a \in b$, and **subset**, $a \subseteq b$. By the former statement, we understand that a is an element of the set b; by the latter, that a is a set each of whose elements is also in the set b.[2]

EXAMPLE 3 Each of the following assertions is true.

- $\{a\} \in \{x, y, \{a\}\}$
- $\{a\} \subsetneq \{x, y, a\}$
- $\{a\} \not\subseteq \{x, y, \{a\}\}$
- $\{a, b\} \subseteq \{a, b\}$
- $\emptyset \in \{x, y, \emptyset\}$
- $\emptyset \subseteq \{x, y, \emptyset\}$
- $\{\emptyset\} \notin \{x, y, \emptyset\}$ ■

The Power Set

An important example of a set, **all** of whose elements are themselves sets, is the *power set* of a set.

2.1.5 DEFINITION The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$ ❖

[2] Note the use of lowercase letters for sets, which is not common but certainly permissible.

EXAMPLE 4

- If $A = \{a\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$.
- If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
- $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. ■

Answers to Pauses

1. Both statements are true. The set S contains the **set** $\{a\}$ as one of its elements, but not the element a.
2. This statement is false: $\{\emptyset\}$ is not the empty set for it contains one element, the set $\emptyset$.
3. **(a)** True: $\{\{\emptyset\}\}$ is a set that contains the single element $\{\emptyset\}$.
   (b) True: The empty set is a subset of any set.
   (c) False: There is just one element in the set $\{\emptyset\}$, (that is, $\emptyset$), and this is not an **element** of the set $\{\{\emptyset\}\}$, whose only element is $\{\emptyset\}$.

True/False Questions

(Answers can be found in the back of the book.)

1. $5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
2. $-5 \in \{x + 2y \mid x \in \{0, 1, 2\}, y \in \{-2, 0, 2\}\}$
3. If $A = \{a, b\}$, then $b \subseteq A$.
4. If $A = \{a, b\}$, then $\{a\} \in A$.
5. $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$
6. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
7. $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$
8. (Assume A, B, C are sets.) $A \in B, B \in C \to A \subseteq C$.
9. (Assume A, B are sets.) $A \subsetneqq B \to B \not\subseteq A$.
10. If A has two elements, then $\mathcal{P}(\mathcal{P}(A))$ has eight elements.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. List the (distinct) elements in each of the following sets:
   (a) [BB] $\{x \in \mathsf{R} \mid x^2 = 5\}$
   (b) $\{x \in \mathsf{Z} \mid xy = 15 \text{ for some } y \in \mathsf{Z}\}$
   (c) [BB] $\{x \in \mathsf{Q} \mid x(x^2 - 2)(2x + 3) = 0\}$
   (d) $\{x + y \mid x \in \{-1, 0, 1\}, y \in \{0, 1, 2\}\}$
   (e) $\{a \in \mathsf{N} \mid a < -4 \text{ and } a > 4\}$
2. List five elements in each of the following sets:
   (a) [BB] $\{a + bi \mid a, b \in \mathsf{Z}, i^2 = -1\}$
   (b) $\{a + b\sqrt{2} \mid a \in \mathsf{N}, -b \in \{2, 5, 7\}\}$
   (c) $\left\{\frac{x}{y} \mid x, y \in \mathsf{R}, x^2 + y^2 = 25\right\}$
   (d) $\{n \in \mathsf{N} \mid n^2 + n \text{ is a multiple of } 3\}$
3. Let $A = \{1, 2, 3, 4\}$. List all the subsets B of A such that
   (a) [BB] $\{1, 2\} \subseteq B$; **(b)** $B \subseteq \{1, 2\}$;
   (c) $\{1, 2\} \not\subseteq B$; **(d)** $B \not\subseteq \{1, 2\}$;
   (e) $\{1, 2\} \subsetneqq B$; **(f)** $B \subsetneqq \{1, 2\}$.
4. [BB] Let $A = \{\{a, b\}\}$. Are the following statements true or false? Explain your answer.
   (a) $a \in A$. **(b)** $A \in A$. **(c)** $\{a, b\} \in A$.
   (d) There are two elements in A.
5. Determine which of the following are true and which are false. Justify your answers.
   (a) [BB] $3 \in \{1, 3, 5\}$ **(b)** $\{3\} \in \{1, 3, 5\}$
   (c) $\{3\} \subsetneqq \{1, 3, 5\}$
   (d) [BB] $\{3, 5\} \not\subseteq \{1, 3, 5\}$
   (e) $\{1, 3, 5\} \subsetneqq \{1, 3, 5\}$
   (f) $1 \in \{a + 2b \mid a, b \text{ even integers}\}$
   (g) $0 \in \{a + b\sqrt{2} \mid a, b \in \mathsf{Q}, b \neq 0\}$
6. Find the power sets of each of the following sets:
   (a) [BB] $\emptyset$ **(b)** $\{\emptyset\}$ **(c)** $\{\emptyset, \{\emptyset\}\}$
7. Determine whether each of the following statements is true or false. Justify your answers.
   (a) [BB] $\emptyset \subseteq \emptyset$ **(b)** $\emptyset \subseteq \{\emptyset\}$

(c) $\emptyset \in \emptyset$ (d) $\emptyset \in \{\emptyset\}$
(e) [BB] $\{1, 2\} \not\subseteq \{1, 2, 3, \{1, 2, 3\}\}$
(f) $\{1, 2\} \in \{1, 2, 3, \{1, 2, 3\}\}$
(g) $\{1, 2\} \subsetneq \{1, 2, \{\{1, 2\}\}\}$
(h) [BB] $\{1, 2\} \in \{1, 2, \{\{1, 2\}\}\}$
(i) $\{\{1, 2\}\} \subseteq \{1, 2, \{1, 2\}\}$

8. [BB] Let A be a set and suppose $x \in A$. Is $x \subseteq A$ also possible? Explain.

9. **(a)** List all the subsets of the set $\{a, b, c, d\}$ that contain
  i. four elements;
  ii. [BB] three elements;
  iii. two elements;
  iv. one element;
  v. no elements;

(b) How many subsets of $\{a, b, c, d\}$ are there altogether?

10. **(a)** How many elements are in the power set of the power set of the empty set?

(b) Suppose A is a set containing one element. How many elements are in $\mathcal{P}(\mathcal{P}(A))$?

11. **(a)** [BB] If A contains two elements, how many elements are there in the power set of A?

(b) [BB] If A contains three elements, how many elements are there in the power set of A?

(c) [BB] If a set A contains $n \geq 0$ elements, guess how many elements are in the power set of A.

12. Suppose A, B, and C are sets. For each of the following statements, either prove it is true or give a counterexample to show that it is false.
(a) [BB] $A \in B, B \in C \rightarrow A \in C$
(b) $A \subseteq B, B \subseteq C \rightarrow A \subseteq C$
(c) $A \subsetneq B, B \subsetneq C \rightarrow A \subsetneq C$
(d) [BB] $A \in B, B \subseteq C \rightarrow A \in C$
(e) $A \in B, B \subseteq C \rightarrow A \subseteq C$
(f) $A \subseteq B, B \in C \rightarrow A \in C$
(g) $A \subseteq B, B \in C \rightarrow A \subseteq C$

13. Suppose A and B are sets.
(a) Answer true or false and explain: $A \not\subseteq B \rightarrow B \subsetneq A$.
(b) Is the converse of the implication in (a) true or false? Explain.

14. Suppose A, B, and C are sets. Prove or give a counterexample that disproves each of the following assertions.
(a) [BB] $C \in \mathcal{P}(A) \leftrightarrow C \subseteq A$
(b) $A \subseteq B \leftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$
(c) $A = \emptyset \leftrightarrow \mathcal{P}(A) = \emptyset$

2.2 Operations on Sets

In this section, we discuss ways in which two or more sets can be combined to form a new set.

Union and Intersection

2.2.1 DEFINITIONS

The *union* of sets A and B, written $A \cup B$, is the set of elements in A or in B (or in both). The *intersection* of A and B, written $A \cap B$, is the set of elements that belong to both A and B. ❖

EXAMPLE 5

- If $A = \{a, b, c\}$ and $B = \{a, x, y, b\}$, then

$$A \cup B = \{a, b, c, x, y\}, \quad A \cap B = \{a, b\},$$

$$A \cup \{\emptyset\} = \{a, b, c, \emptyset\} \quad \text{and} \quad B \cap \{\emptyset\} = \emptyset.$$

- For any set A, $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$. ■

As with addition and multiplication of real numbers, the union and intersection of sets are *associative* operations. To say that set union is associative is to say that

$$(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$$

for any three sets A_1, A_2, A_3. It follows that the expression

$$A_1 \cup A_2 \cup A_3$$

is unambiguous. The two different interpretations (corresponding to different insertions of parentheses) agree. The union of n sets $A_1, A_2, \ldots, A_n$ is written

$$(1) \qquad A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{i=1}^{n} A_i$$

and represents the set of elements that belong to one or more of the sets A_i. The intersection of $A_1, A_2, \ldots, A_n$ is written

$$(2) \qquad A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \quad \text{or} \quad \bigcap_{i=1}^{n} A_i$$

and denotes the set of elements which belong to all of the sets.

Do not assume from the expression $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ that n is actually greater than 3 since the first part of this expression—$A_1 \cup A_2 \cup A_3$—is present only to make the general pattern clear; a union of sets is being formed. The last term—A_n—indicates that the last set in the union is A_n. If $n = 2$, then $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ means $A_1 \cup A_2$. Similarly, if $n = 1$, the expression $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$ simply means A_1.

While parentheses are not required in expressions like (1) or (2), they are mandatory when both union and intersection are involved. For example, $A \cap (B \cup C)$ and $(A \cap B) \cup C$ are, in general, different sets. This is probably most easily seen by the use of the *Venn diagram* shown in Fig. 2.1.

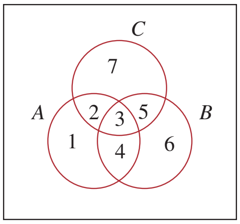


Figure 2.1 A Venn diagram.

The diagram indicates that A consists of the points in the regions labeled 1, 2, 3, and 4; B consists of those points in regions 3, 4, 5, and 6 and C of those in 2, 3, 5, and 7. The set $B \cup C$ consists of points in the regions labeled 3, 4, 5, 6, 2, and 7. Notice that $A \cap (B \cup C)$ consists of the points in regions 2, 3, and 4. The region $A \cap B$ consists of the points in regions 3 and 4; thus, $(A \cap B) \cup C$ is the set of points in the regions labeled 3, 4, 2, 5, and 7. The diagram enables us to see that, in general, $A \cap (B \cup C) \neq (A \cap B) \cup C$ and it shows how we could construct a specific counterexample: We could let A, B, and C be the sets

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6\}, \quad C = \{2, 3, 5, 7\}$$

as suggested by the diagram and then calculate

$$A \cap (B \cup C) = \{2, 3, 4\} \neq \{2, 3, 4, 5, 7\} = (A \cap B) \cup C.$$

There is a way to rewrite $A \cap (B \cup C)$. In Fig. 2.1, we see that $A \cap B$ consists of the points in the regions labeled 3 and 4 and that $A \cap C$ consists of the points in 2 and 3. Thus, the points of $(A \cap B) \cup (A \cap C)$ are those of 2, 3, and 4. These are just the points of $A \cap (B \cup C)$ (as observed previously), so the Venn diagram makes it easy to believe that, in general,

$$(3) \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

While pictures can be helpful in making certain statements seem plausible, they should not be relied on because they can also mislead. For this reason, and because there are situations in which Venn diagrams are difficult or impossible to create, it is important to be able to establish relationships among sets without resorting to a picture.

PROBLEM 6. Let A, B, and C be sets. Verify equation (3) without the aid of a Venn diagram.

Solution. As observed in Proposition 2.1.4, to show that two sets are equal it is sufficient to show that each is a subset of the other. Here this just amounts to expressing the meaning of $\cup$ and $\cap$ in words.

To show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, let $x \in A \cap (B \cup C)$. Then x is in A and also in $B \cup C$. Since $x \in B \cup C$, either $x \in B$ or $x \in C$. This suggests cases.

Case 1: $x \in B$.
In this case, x is in A as well as in B, so it's in $A \cap B$.

Case 2: $x \in C$.
Here x is in A as well as in C, so it's in $A \cap C$.

We have shown that either $x \in A \cap B$ or $x \in A \cap C$. By definition of union, $x \in (A \cap B) \cup (A \cap C)$, completing this half of our proof.

Conversely, we must show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. For this, let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. Thus, x is in both A and B or in both A and C. In either case, $x \in A$. Also, x is in either B or C; thus, $x \in B \cup C$. So x is both in A and in $B \cup C$; that is, $x \in A \cap (B \cup C)$. This completes the proof. ▲

PROBLEM 7. For sets A and B, prove that $A \cap B = A$ if and only if $A \subseteq B$.

Solution. Remember that there are two implications to establish and that we use the symbolism $(\longrightarrow)$ and $(\longleftarrow)$ to mark the start of the proof of each implication.

$(\longrightarrow)$ Here we assume $A \cap B = A$ and must prove $A \subseteq B$. For this, suppose $x \in A$. Then, $x \in A \cap B$ (because we are assuming $A = A \cap B$). Therefore, x is in A and in B, in particular, x is in B. This proves $A \subseteq B$.

$(\longleftarrow)$ Now we assume $A \subseteq B$ and prove $A \cap B = A$. To prove the equality of $A \cap B$ and A, we must prove that each set is a subset of the other. By definition of intersection, $A \cap B$ is a subset of A, so $A \cap B \subseteq A$. On the other hand, suppose $x \in A$. Since $A \subseteq B$, x is in B too; thus, x is in both A and B. Therefore, $A \subseteq A \cap B$. Therefore, $A = A \cap B$. ▲

Pause 4

For sets A and B, prove that $A \cup B = B$ if and only if $A \subseteq B$.

Set Difference

2.2.2 DEFINITIONS

The *set difference* of sets A and B, written $A \setminus B$, is the set of those elements of A that are not in B. The *complement* of a set A is the set $A^c = U \setminus A$, where U is some universal set made clear by the context. ❖

EXAMPLE 8

- $\{a, b, c\} \setminus \{a, b\} = \{c\}$
- $\{a, b, c\} \setminus \{a, x\} = \{b, c\}$
- $\{a, b, \emptyset\} \setminus \emptyset = \{a, b, \emptyset\}$
- $\{a, b, \emptyset\} \setminus \{\emptyset\} = \{a, b\}$
- If A is the set {Monday, Tuesday, Wednesday, Thursday, Friday}, the context suggests that the universal set is the days of the week, so $A^c =$ {Saturday, Sunday}. ■

Notice that $A \setminus B = A \cap B^c$ and also that $(A^c)^c = A$. For example, if $A = \{x \in \mathbf{Z} \mid x^2 > 0\}$, then $A^c = \{0\}$ (it being understood that $U = \mathbf{Z}$) and so

$$(A^c)^c = \{0\}^c = \{x \in \mathbf{Z} \mid x \neq 0\} = A.$$

You may have previously encountered standard notation to describe various types of intervals of real numbers.

2.2.3 DEFINITION

Interval Notation If a and b are real numbers with $a < b$, then

$$[a, b] = \{x \in \mathsf{R} \mid a \le x \le b\} \quad \textbf{closed}$$
$$(a, b) = \{x \in \mathsf{R} \mid a < x < b\} \quad \textbf{open}$$
$$(a, b] = \{x \in \mathsf{R} \mid a < x \le b\} \quad \textbf{half open}$$
$$[a, b) = \{x \in \mathsf{R} \mid a \le x < b\} \quad \textbf{half open}.$$

As indicated, a closed interval is one that includes both endpoints, an open interval includes neither, and a half-open interval includes just one endpoint. A square bracket indicates that the adjacent endpoint is in the interval. To describe infinite intervals, we use the symbol ∞ (which is just a symbol) and make obvious adjustments to our notation. For example,

$$(-\infty, b] = \{x \in \mathsf{R} \mid x \le b\},$$
$$(a, \infty) = \{x \in \mathsf{R} \mid x > a\}.$$

The first interval here is half open; the second is open. ❖

Pause 5

If $A = [-4, 4]$ and $B = [0, 5]$, then $A \setminus B = [-4, 0)$. What is $B \setminus A$? What is A^c? ▮

2.2.4 THE LAWS OF DE MORGAN

The following two laws, of wide applicability, are attributed to Augustus De Morgan (1806–1871), who, together with George Boole (1815–1864), helped to make England a leading center of logic in the nineteenth century.[3]

$$(A \cup B)^c = A^c \cap B^c; \qquad (A \cap B)^c = A^c \cup B^c.$$

Readers should be struck by the obvious connection between these laws and the rules for negating *and* and *or* compound sentences described in Section 0.1. We illustrate by showing the equivalence of the first law of De Morgan and the rule for negating "$\mathcal{A}$ or $\mathcal{B}$."

PROBLEM 9. Prove that $(A \cup B)^c = A^c \cap B^c$ for any sets A, B, and C.

Solution. Let $\mathcal{A}$ be the statement "$x \in A$" and $\mathcal{B}$ be the statement "$x \in B$." Then

$$\begin{aligned} x \in (A \cup B)^c &\leftrightarrow \neg(x \in A \cup B) && \\ &\leftrightarrow \neg(\mathcal{A} \text{ or } \mathcal{B}) && \text{definition of union} \\ &\leftrightarrow \neg\mathcal{A} \text{ and } \neg\mathcal{B} && \text{rule for negating “or”} \\ &\leftrightarrow x \in A^c \text{ and } x \in B^c && \\ &\leftrightarrow x \in A^c \cap B^c && \text{definition of intersection.} \end{aligned}$$

The sets $(A \cup B)^c$ and $A^c \cap B^c$ contain the same elements, so they are the same. ▲

Symmetric Difference

2.2.5 DEFINITION

The *symmetric difference* of two sets A and B is the set $A \oplus B$ of elements that are in A or in B, but not in both. ❖

[3] As pointed out by Rudolf and Gerda Fritsch (*Der Vierfarbensatz*, B. I. Wissenschaftsverlag, Mannheim, 1994 and English translation, *The Four-Color Theorem*, by J. Peschke, Springer-Verlag, 1998), it was in a letter from De Morgan to Sir William Rowan Hamilton that the question giving birth to the famous Four-Color Theorem was first posed. See Section 13.2 for a detailed account of this theorem, whose proof was found relatively recently after over 100 years of effort!

Readers should note that the symbol $\triangle$, as in $A \triangle B$, is also used to denote symmetric difference.

Notice that the symmetric difference of sets can be expressed in terms of previously defined operations. For example,

$$A \oplus B = (A \cup B) \setminus (A \cap B)$$

and

$$A \oplus B = (A \setminus B) \cup (B \setminus A).$$

EXAMPLE 10

- $\{a, b, c\} \oplus \{x, y, a\} = \{b, c, x, y\}$
- $\{a, b, c\} \oplus \emptyset = \{a, b, c\}$
- $\{a, b, c\} \oplus \{\emptyset\} = \{a, b, c, \emptyset\}$ ■

PROBLEM 11. Use a Venn diagram to illustrate the plausibility of the fact that $\oplus$ is an associative operation; that is, use a Venn diagram to illustrate that for any three sets A, B, and C,

$$(A \oplus B) \oplus C = A \oplus (B \oplus C). \tag{4}$$

Solution. With reference to Fig. 2.1 again, $A \oplus B$ consists of the points in the regions labeled 1, 2, 5, and 6 while C consists of the points in the regions 2, 3, 5, and 7. Thus, $(A \oplus B) \oplus C$ is the set of points in the regions 1, 3, 6, and 7. On the other hand, $B \oplus C$ consists of the regions 2, 7, 4, and 6 and A, of regions 1, 2, 3, 4. Thus, $A \oplus (B \oplus C)$ also consists of the points in regions 1, 3, 6, and 7. ▲

As a consequence of (4), the expression $A \oplus B \oplus C$, which conceivably could be interpreted in two ways, is in fact unambiguous. Notice that $A \oplus B \oplus C$ is the set of points in an odd number of the sets A, B, C: Regions 1, 6, and 7 contain the points of just one of the sets while region 3 consists of points in all three. More generally, the symmetric difference $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_n$ of n sets $A_1, A_2, A_3, \ldots, A_n$ is well defined and, as it turns out, is the set of those elements which are members of an odd number of the sets A_i. (See Exercise 20 of Section 5.1.)

The Cartesian Product of Sets

There is yet another way in which two sets can be combined to obtain another.

2.2.6 DEFINITIONS

If A and B are sets, the *Cartesian product* (sometimes also called the *direct product*) of A and B is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

(We say "A cross B" for "$A \times B$.") More generally, the Cartesian product of $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

When all the sets are equal to the same set A, $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$ is written A^n. ❖

The elements of $A \times B$ are called *ordered pairs* because their order is important: $(a, b) \neq (b, a)$ (unless $a = b$). The elements a and b are the *coordinates* of the ordered pair (a, b); the first coordinate is a and the second is b. The elements of A^n are called *n-tuples*.

Elements of $A \times B$ are equal if and only if they have the same first coordinates and the same second coordinates:

$$(a_1, b_1) = (a_2, b_2) \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2.$$

EXAMPLE 12 Let $A = \{a, b\}$ and $B = \{x, y, z\}$. Then

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$$

and

$$B \times A = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}.$$

This example illustrates that, in general, the sets $A \times B$ and $B \times A$ are different. ■

EXAMPLE 13 The Cartesian plane, in which calculus students sketch curves, is a picture of $\mathsf{R} \times \mathsf{R} = \mathsf{R}^2 = \{(x, y) \mid x, y \in \mathsf{R}\}$. The adjective Cartesian is derived from Descartes,[4] as Cartesius was Descartes's name in Latin. ■

PROBLEM 14. Let A, B, and C be sets. Prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Solution. We must prove that any element in $A \times (B \cup C)$ is in $(A \times B) \cup (A \times C)$. Since the elements in $A \times (B \cup C)$ are ordered pairs, we begin by letting $(x, y) \in A \times (B \cup C)$ (this is more helpful than starting with "$x \in A \times B$") and ask ourselves what this means. It means that x, the first coordinate, is in A and y, the second coordinate, is in $B \cup C$. Therefore, y is in either B or C. If y is in B, then, since x is in A, $(x, y) \in A \times B$. If y is in C, then, since x is in A, $(x, y) \in A \times C$. Thus, (x, y) is either in $A \times B$ or in $A \times C$; thus, (x, y) is in $(A \times B) \cup (A \times C)$, which is what we wanted to show. ▲

Pause 6 Let A, B, and C be three sets. Prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. What can you conclude about the sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$? Why? ▮

Pause 7 Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. Is this true if $A = \emptyset$? ▮

Answers to Pauses

4. ($\longrightarrow$) Suppose the first statement, $A \cup B = B$, is true. We show $A \subseteq B$. So let $x \in A$. Then x is certainly in $A \cup B$, by the definition of $\cup$. But $A \cup B = B$, so $x \in B$. Thus, $A \subseteq B$.

($\longleftarrow$) Conversely, suppose the second statement, $A \subseteq B$, is true. We have to show $A \cup B = B$. To prove the sets $A \cup B$ and B are equal, we have to show each is a subset of the other. First, let $x \in A \cup B$. Then x is either in A or in B. If the latter, $x \in B$, and if the former, $x \in B$ because A is a subset of B. In either case, $x \in B$. Thus, $A \cup B \subseteq B$. Second, assume $x \in B$. Then x is in $A \cup B$ by definition of $\cup$. So $B \subseteq A \cup B$ and we have equality, as required.

5. $B \setminus A = (4, 5]$; $A^c = (-\infty, -4) \cup (4, \infty)$.

6. An element of $(A \times B) \cup (A \times C)$ is either in $A \times B$ or in $A \times C$; in either case, it's an ordered pair. So we begin by letting $(x, y) \in (A \times B) \cup (A \times C)$ and noting that either $(x, y) \in A \times B$ or $(x, y) \in A \times C$. In the first case, x is in A and y is in B; in the second case, x is in A and y is in C. In either case, x is in A and y is either in B or in C; so $x \in A$ and $y \in B \cup C$. Therefore, $(x, y) \in A \times (B \cup C)$, establishing the required subset relation. The reverse subset relation was established in Problem 14. We conclude that the two sets in question are equal; that is, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

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[4]René Descartes (1596–1650), together with Pierre de Fermat, the inventor of analytic geometry, introduced the method of plotting points and graphing functions in R^2 with which we are so familiar today.

7. ($\longrightarrow$) Suppose that the statement $A \times B = B \times A$ is true. We prove $A = B$. So suppose $x \in A$. Since $B \neq \emptyset$, we can find some $y \in B$. Thus, $(x, y) \in A \times B$. Since $A \times B = B \times A$, $(x, y) \in B \times A$. So $x \in B$, giving us $A \subseteq B$. Similarly, we show that $B \subseteq A$ and conclude $A = B$.

($\longleftarrow$) On the other hand, if $A = B$ is a true statement, then $A \times B = A \times A = B \times A$.

Finally, if $A = \emptyset$ and B is any nonempty set, then $A \times B = \emptyset = B \times A$, but $A \neq B$. So $A \times B = B \times A$ does not mean $A = B$ in the case $A = \emptyset$.

True/False Questions

(Answers can be found in the back of the book.)

1. If A and B are sets and $A \neq B$, then $A \cap B \subsetneq A \cup B$.

2. If A and B are sets, then $(A \setminus B) \cap (B \setminus A) = \emptyset$.

3. If A and B are sets, then $(A^c \cup B)^c = A \cap B^c$.

4. If A and B are sets, then $A \setminus B \subseteq A \oplus B$.

5. If A and B are sets and $A \neq B$, then $A \oplus B \neq \emptyset$.

6. If A and B are nonempty sets, then $A \times B$ is a nonempty set.

7. The name of Augustus De Morgan appears in both Chapter 1 and Chapter 2 of this text.

8. $(A \subseteq B) \to (B^c \subseteq A^c)$.

9. $(B^c \subseteq A^c) \to (A \subseteq B)$.

10. $((A \setminus B) \subseteq (B \setminus A)) \to (A \subseteq B)$.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. Let $A = \{x \in \mathsf{N} \mid x < 7\}$, $B = \{x \in \mathsf{Z} \mid |x - 2| < 4\}$, and $C = \{x \in \mathsf{R} \mid x^3 - 4x = 0\}$.

(a) [BB] List the elements in each of these sets.

(b) Find $A \cup C$, $B \cap C$, $B \setminus C$, $A \oplus B$, $C \times (B \cap C)$, $(A \setminus B) \setminus C$, $A \setminus (B \setminus C)$, and $(B \cup \emptyset) \cap \{\emptyset\}$.

(c) List the elements in $S = \{(a, b) \in A \times B \mid a = b + 2\}$ and in $T = \{(a, c) \in A \times C \mid a \leq c\}$.

2. Let $S = \{2, 5, \sqrt{2}, 25, \pi, \frac{5}{2}\}$ and $T = \{4, 25, \sqrt{2}, 6, \frac{3}{2}\}$.

(a) [BB] Find $S \cap T$, $S \cup T$, and $T \times (S \cap T)$.

(b) [BB] Find $\mathsf{Z} \cup S$, $\mathsf{Z} \cap S$, $\mathsf{Z} \cup T$, and $\mathsf{Z} \cap T$.

(c) List the elements in each of the sets $\mathsf{Z} \cap (S \cup T)$ and $(\mathsf{Z} \cap S) \cup (\mathsf{Z} \cap T)$. What do you notice?

(d) List the elements of $\mathsf{Z} \cup (S \cap T)$ and list the elements of $(\mathsf{Z} \cup S) \cap (\mathsf{Z} \cup T)$. What do you notice?

3. Let $A = \{(-1, 2), (4, 5), (0, 0), (6, -5), (5, 1), (4, 3)\}$. List the elements in each of the following sets.

(a) [BB] $\{a + b \mid (a, b) \in A\}$

(b) $\{a \mid a > 0 \text{ and } (a, b) \in A \text{ for some } b\}$

(c) $\{b \mid b = k^2 \text{ for some } k \in \mathsf{Z} \text{ and } (a, b) \in A \text{ for some } a\}$

4. List the elements in the sets $A = \{(a, b) \in \mathsf{N} \times \mathsf{N} \mid a \leq b, b \leq 3\}$ and $B = \{\frac{a}{b} \mid a, b \in \{-1, 1, 2\}\}$.

5. For $A = \{a, b, c, \{a, b\}\}$, find

(a) [BB] $A \setminus \{a, b\}$ **(b)** $\{\emptyset\} \setminus \mathcal{P}(A)$

(c) $A \setminus \emptyset$ **(d)** $\emptyset \setminus A$

(e) [BB] $\{a, b, c\} \setminus A$ **(f)** $(\{a, b, c\} \cup \{A\}) \setminus A$

6. Find A^c (with respect to $U = \mathsf{R}$) in each of the following cases.

(a) [BB] $A = (1, \infty) \cup (-\infty, -2]$

(b) $A = (-3, \infty) \cap (-\infty, 4]$

(c) $A = \{x \in \mathsf{R} \mid x^2 \leq -1\}$

7. Let $X = \{1, 2, 3, 4\}$, $Y = \{2, 3, 4, 5\}$, and $Z = \{3, 4, 5, 6\}$. List the elements in the indicated sets. (The universal set is the set of integers.)

(a) $X \oplus (Y \cap Z)$ **(b)** $(X^c \cup Y)^c$

8. Let $n > 3$ and $A = \{1, 2, 3, \ldots, n\}$.

(a) [BB] How many subsets of A contain $\{1, 2\}$?

(b) How many subsets B of A have the property that $B \cap \{1, 2\} = \emptyset$?

(c) How many subsets B of A have the property that $B \cup \{1, 2\} = A$?

Explain your answers.

9. [BB] Let a and b be real numbers with $a < b$. Find $(a, b)^c$, $[a, b)^c$, $(a, \infty)^c$, and $(-\infty, b]^c$.

10. The universal set for this problem is the set of students attending Miskatonic University. Let

- M denote the set of math majors
- CS denote the set of computer science majors
- T denote the set of students who had a test on Friday
- P denote those students who ate pizza last Thursday

Using only the set theoretical notation we have introduced in this chapter, rewrite each of the following assertions.

(a) [BB] Computer science majors had a test on Friday.

(b) [BB] No math major ate pizza last Thursday.

(c) Some math majors did not eat pizza last Thursday.

(d) Those computer science majors who did not have a test on Friday ate pizza on Thursday.

(e) Math or computer science majors who ate pizza on Thursday did not have a test on Friday.

11. Use the set theoretical notation introduced in this chapter to express the negation each of statements (a)–(e) in Exercise 10. Do the same for the converse of any statement that is an implication.

12. Let P denote the set of primes and E the set of even integers. As always, Z and N denote the integers and natural numbers, respectively. Find equivalent formulations of each of the following statements using the notation of set theory that has been introduced in this section.

(a) [BB] There exists an even prime.

(b) 0 is an integer but not a natural number.

(c) Every natural number is an integer.

(d) Not every integer is a natural number.

(e) Every prime except 2 is odd.

(f) 2 is an even prime.

(g) 2 is the only even prime.

13. For $n \in \mathsf{Z}$, let $A_n = \{a \in \mathsf{Z} \mid a \le n\}$. Find each of the following sets.

(a) [BB] $A_3 \cup A_{-3}$ **(b)** $A_3 \cap A_{-3}$

(c) $A_3 \cap (A_{-3})^c$ **(d)** $\bigcap_{i=0}^{4} A_i$

14. [BB] In Fig. 2.1, the region labeled 7 represents the set $C \setminus (A \cup B)$. What set is represented by the region labeled 2? By that labeled 3? By that labeled 4?

15. Let $A = \{1, 2, 4, 5, 6, 9\}$, $B = \{1, 2, 3, 4\}$, and $C = \{5, 6, 7, 8\}$.

(a) Draw a Venn diagram showing the relationship between these sets. Show which elements are in which region.

(b) What are the elements in each of the following sets?

i. $(A \cup B) \cap C$ **ii.** $A \setminus (B \setminus A)$

iii. $(A \cup B) \setminus (A \cap C)$ **iv.** $A \oplus C$

v. $(A \cap C) \times (A \cap B)$

16. **(a)** [BB] Suppose A and B are sets such that $A \cap B = A$. What can you conclude? Why?

(b) Repeat (a) assuming $A \cup B = A$.

17. [BB] Let $n \ge 1$ be a natural number. How many elements are in the set $\{(a, b) \in \mathsf{N} \times \mathsf{N} \mid a \le b \le n\}$? Explain.

18. Suppose A is a subset of $\mathsf{N} \times \mathsf{N}$ with the properties

- $(1, 1) \in A$ and
- if $(a, b) \in A$, then both $(a+1, b)$ and $(a+1, b+1)$ are also in A.

Do you think that $\{(m, n) \in \mathsf{N} \times \mathsf{N} \mid m \ge n\}$ is a subset of A? Explain. [*Hint*: A picture of A in the xy-plane might help.]

19. Let A, B, and C be subsets of some universal set U.

(a) If $A \cap B \subseteq C$ and $A^c \cap B \subseteq C$, prove that $B \subseteq C$.

(b) [BB] Given that $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, does it follow that $B = C$? Justify your answer.

20. Let A, B, and C be sets.

(a) Find a counterexample to the statement $A \cup (B \cap C) = (A \cup B) \cap C$.

(b) Without using Venn diagrams, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

21. Use the first law of De Morgan to prove the second: $(A \cap B)^c = A^c \cup B^c$.

22. [BB] Use the laws of De Morgan and any other set theoretic identities discussed in the text to prove that $(A \setminus B) \setminus C = A \setminus (B \cup C)$ for any sets A, B, and C.

23. Let A, B, C, and D be subsets of a universal set U. Use set theoretic identities discussed in the text to simplify the expression $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c$.

24. Let A, B, and C be subsets of some universal set U. Use set theoretic identities discussed in the text to prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C^c)$.

25. Suppose A, B, and C are subsets of some universal set U.

(a) [BB] Generalize the laws of De Morgan by finding equivalent ways to describe the sets $(A \cup B \cup C)^c$ and $(A \cap B \cap C)^c$.

(b) Find a way to describe the set $(A \cap (B \setminus C))^c \cap A$ without using the symbol ^{c} for set complement.

26. Let A and B be sets.

(a) [BB] Find a necessary and sufficient condition for $A \oplus B = A$.

(b) Find a necessary and sufficient condition for $A \cap B = A \cup B$.

Explain your answers (with Venn diagrams if you wish).

27. Which of the following conditions imply that $B = C$? In each case, either prove or give a counterexample.

(a) [BB] $A \cup B = A \cup C$

(b) $A \cap B = A \cap C$

(c) $A \oplus B = A \oplus C$

(d) $A \times B = A \times C$

28. True or false? In each case, provide a proof or a counterexample.

(a) $A \subseteq C, B \subseteq D \to A \times B \subseteq C \times D$.

(b) $A \not\subseteq B, B \subseteq C \to A \not\subseteq C$.

(c) $A \times B \subseteq C \times D \to A \subseteq C$ and $B \subseteq D$.

(d) $A \subseteq C$ and $B \subseteq D$ if and only if $A \times B \subseteq C \times D$.

(e) [BB] $A \cup B \subseteq A \cap B \to A = B$.

29. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$ for any sets A, B, and C.

30. Let A, B, and C be arbitrary sets. For each of the following, either prove the given statement is true or exhibit a counterexample to prove it is false.
(a) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$
(b) $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$
(c) [BB] $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
(d) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
(e) $(A \setminus B) \times (C \setminus D) = (A \times C) \setminus (B \times D)$

✍**31.** Find out what you can about George Boole and write a paragraph or two about him (in good English, of course).

2.3 Binary Relations

If A and B are sets, remember that the Cartesian product of A and B is the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$. There are occasions when we are interested in a certain subset of $A \times B$. For example, if A is the set of former major league baseball players and $B = \mathsf{N} \cup \{0\}$ is the set of nonnegative integers, then we might naturally be interested in

$$\mathcal{R} = \{(a, b) \mid a \in A, b \in B, \text{player } a \text{ had } b \text{ career home runs}\}.$$

For example, (Hank Aaron, 755) and (Mickey Mantle, 536) are elements of $\mathcal{R}$.

2.3.1 DEFINITIONS

Let A and B denote sets. A *binary relation from A to B* is a subset of $A \times B$. A *binary relation on A* is a subset of $A \times A$. ❖

2.3.2 REMARK

When $\mathcal{R}$ is a binary relation from A to B and the pair (a, b) is in $\mathcal{R}$, we naturally write $(a, b) \in \mathcal{R}$, though the reader should be aware that other authors prefer the notation $a\mathcal{R}b$. ❖

The empty set and the entire Cartesian product $A \times B$ are always binary relations from A to B, although these are generally not as interesting as certain nonempty proper subsets of $A \times B$.

EXAMPLE 15

- If A is the set of students who were registered at the University of Toronto during the Fall 2001 semester and B is the set {History, Mathematics, English, Biology}, then $\mathcal{R} = \{(a, b) \mid a \in A \text{ is enrolled in a course in subject } b\}$ is a binary relation from A to B.
- Let A be the set of surnames of people listed in the Seattle telephone directory. Then $\mathcal{R} = \{(a, n) \mid a \text{ appears on page } n\}$ is a binary relation from A to the set N of natural numbers.
- $\{(a, b) \mid a, b \in \mathsf{N}, \frac{a}{b} \text{ is an integer}\}$ and $\{(a, b) \mid a, b \in \mathsf{N}, a - b = 2\}$ are binary relations on N.
- $\{(x, y) \mid y = x^2\}$ is a binary relation on R whose graph the reader should recognize. ■

Pause 8

What is the common name for this graph? ▮

Our primary intent in this section is to identify special properties of binary relations on a set, so, henceforth, all binary relations will be subsets of $A \times A$ for some set A.

2.3.3 DEFINITION

A binary relation $\mathcal{R}$ on a set A is *reflexive* if and only if $(a, a) \in \mathcal{R}$ for all $a \in A$. ❖

EXAMPLE 16

- $\{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ is a reflexive relation on $\mathbb{R}$ since $x \leq x$ for any $x \in \mathbb{R}$.
- $\{(a, b) \in \mathbb{N}^2 \mid \frac{a}{b} \in \mathbb{N}\}$ is a reflexive relation on $\mathbb{N}$ since $\frac{a}{a}$ is an integer, 1, for any $a \in \mathbb{N}$.
- $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 0\}$ is not a reflexive relation on $\mathbb{R}$ since $(0, 0) \notin \mathcal{R}$. [This example reminds us that a reflexive relation must contain **all** pairs of the form (a, a): **Most** is not enough.] ■

PROBLEM 17. Suppose $\mathcal{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a^2 = b^2\}$. Criticize and then correct the following "proof" that $\mathcal{R}$ is reflexive:

$(a, a) \in \mathcal{R}$ if $a^2 = a^2$.

Solution. The statement "$(a, a) \in \mathcal{R}$ if $a^2 = a^2$" is the implication "$a^2 = a^2 \rightarrow (a, a) \in \mathcal{R}$," which has almost nothing to do with what is required. To prove that $\mathcal{R}$ is reflexive, we must establish an implication of the form "something $\rightarrow$ $\mathcal{R}$ is reflexive." Here is a good argument, in this case.

For any integer a, we have $a^2 = a^2$ and, hence, $(a, a) \in \mathcal{R}$. Therefore, $\mathcal{R}$ is reflexive. ▲

2.3.4 DEFINITION

A binary relation $\mathcal{R}$ on a set A is *symmetric* if and only if

$$\text{if } a, b \in A \text{ and } (a, b) \in \mathcal{R}, \text{ then } (b, a) \in \mathcal{R}.$$ ❖

EXAMPLE 18

- $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a symmetric relation on $\mathbb{R}$ since if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$ too: If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.
- $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is even}\}$ is a symmetric relation on $\mathbb{Z}$ since if $x - y$ is even so is $y - x$.
- $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ is not a symmetric relation on $\mathbb{R}$. For example, $(2, 1) \in \mathcal{R}$ because $2^2 \geq 1$, but $(1, 2) \notin \mathcal{R}$ because $1^2 \ngeq 2$. ■

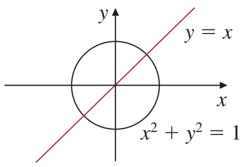
Suppose $\mathcal{R}$ is a binary relation on $A = \mathbb{R}^2$. In this case, the elements of $\mathcal{R}$, being ordered pairs of elements of A, are ordered pairs of elements each of which is an ordered pair of real numbers. Consider, for example,

$$\mathcal{R} = \{(\,(x, y), (u, v)\,) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x^2 + y^2 = u^2 + v^2\}.$$

This is a symmetric relation on $\mathbb{R}^2$ since if $(\,(x, y), (u, v)\,) \in \mathcal{R}$, then $x^2 + y^2 = u^2 + v^2$, so $u^2 + v^2 = x^2 + y^2$, so $(\,(u, v), (x, y)\,) \in \mathcal{R}$.

Pause 9

Is this relation reflexive? ▮



A binary relation on the real numbers (or on any subset of $\mathbb{R}$) is symmetric if, when its points are plotted as usual in the Cartesian plane, the figure is symmetric about the line with equation $y = x$. The set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a symmetric relation because its points are those on the graph of the unit circle centered at the origin, and this circle is certainly symmetric about the line with equation $y = x$.

If a set A has n elements and n is reasonably small, a binary relation on A can be conveniently described by labelling with the elements of A the rows and the columns of an $n \times n$ grid and then inserting some symbol in row a and column b to indicate that (a, b) is in the relation.

EXAMPLE 19 The picture in Fig. 2.2 describes the relation

$$\mathcal{R} = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 2), (3, 3), (4, 4)\}$$

on the set $A = \{1, 2, 3, 4\}$. This relation is reflexive (all points on the *main diagonal*—top left corner to lower right—are present), but not symmetric (the $\times$'s are not symmetrically located with respect to the main diagonal). For example, there is a $\times$ in row 1, column 4, but not in row 4, column 1. ■

	1	2	3	4
1	$\times$	$\times$		$\times$
2	$\times$	$\times$		
3		$\times$	$\times$	
4				$\times$

Figure 2.2

2.3.5 DEFINITION A binary relation $\mathcal{R}$ on a set A is *antisymmetric* if and only if

$$\text{if } a, b \in A \text{ and both } (a, b) \text{ and } (b, a) \text{ are in } \mathcal{R}, \text{ then } a = b.$$ ❖

EXAMPLE 20 $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x \le y\}$ is an antisymmetric relation on R since $x \le y$ and $y \le x$ implies $x = y$; thus, $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$. ■

EXAMPLE 21 If S is a set and $A = \mathcal{P}(S)$ is the power set of S, then $\{(X, Y) \mid X, Y \in \mathcal{P}(S), X \subseteq Y\}$ is antisymmetric since $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$. ■

EXAMPLE 22 $\mathcal{R} = \{(1, 2), (2, 3), (3, 3), (2, 1)\}$ is not antisymmetric on $A = \{1, 2, 3\}$ because $(1, 2) \in \mathcal{R}$ and $(2, 1) \in \mathcal{R}$ but $1 \neq 2$. ■

Note that "antisymmetric" is not the same as "not symmetric." The relation in Example 22 is not symmetric but neither is it antisymmetric.

Pause 10 Why is this relation not symmetric? ▮

Pause 11 Is the relation $\mathcal{R} = \{(\,(x, y), (u, v)\,) \in \mathsf{R}^2 \times \mathsf{R}^2 \mid x^2 + y^2 = u^2 + v^2\}$ antisymmetric? ▮

2.3.6 DEFINITION A binary relation $\mathcal{R}$ on a set A is *transitive* if and only if

$$\text{if } a, b, c \in A, \text{ and both } (a, b) \text{ and } (b, c) \text{ are in } \mathcal{R}, \text{ then } (a, c) \in \mathcal{R}.$$ ❖

EXAMPLE 23 $\mathcal{R} = \left\{(x, y) \in \mathsf{R}^2 \mid x \le y\right\}$ is a transitive relation on R since, if $x \le y$ and $y \le z$, then $x \le z$: if (x, y) and (y, z) are in $\mathcal{R}$, then $(x, z) \in \mathcal{R}$. ■

EXAMPLE 24 $\left\{(a, b) \in \mathsf{Z} \times \mathsf{Z} \mid \frac{a}{b} \text{ is an integer}\right\}$ is a transitive relation on Z since, if $\frac{a}{b}$ and $\frac{b}{c}$ are integers, then so is $\frac{a}{c}$ because $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c}$. ■

EXAMPLE 25

- $\mathcal{R} = \{(x, y), (x, z), (y, u), (x, u)\}$ is a transitive binary relation on the set $\{x, y, z, u\}$ because there is only one pair of the form (a, b), (b, c) belonging to $\mathcal{R}$ (that is, (x, y) and (y, u)) and, for this pair, it is true that $(a, c) = (x, u) \in \mathcal{R}$.
- $\mathcal{R} = \{(a, b), (b, a), (a, a)\}$ is not transitive on $\{a, b\}$ since it contains the pairs (b, a) and (a, b), but not the pair (b, b).
- $\{(a, b) \mid a$ and b are people and a is an ancestor of $b\}$ is a transitive relation since if a is an ancestor of b and b is an ancestor of c, then a is an ancestor of c.
- $\mathcal{R} = \{(x, y) \in \mathsf{R}^2 \mid x^2 \geq y\}$ is not transitive on R because $(3, 4) \in \mathcal{R}$ $(3^2 \geq 4)$ and $(4, 10) \in \mathcal{R}$ $(4^2 \geq 10)$, but $(3, 10) \notin \mathcal{R}$ $(3^2 \not\geq 10)$. ■

Higher-Order Relations

At the beginning of this section, we gave an example showing how the Cartesian product $A \times B$ could be used to store data about former major league baseball players. Such an idea can be extended to more general Cartesian products $A_1 \times A_2 \times \cdots \times A_n$ if we have additional data to store. For example, if a is a major league baseball player, b is his number of career home runs, and c is the number of years he played in the major leagues, then (a, b, c) would be a useful way of presenting this information. For Hank Aaron and Mickey Mantle, we would be talking about (Hank Aaron, 755, 23) and (Mickey Mantle, 536, 18).

2.3.7 DEFINITION

An n-ary relation on sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. ❖

So "binary relation" is just a preferred way of saying "2-ary relation". Similarly, one is more likely to hear the term "ternary relation" than "3-ary relation."

Higher-order relations are used to represent computer databases, and knowledge of their properties is helpful in the development of database management systems. One natural question related to efficiency concerns is how large n needs to be in order to store all desired information.

EXAMPLE 26

Consider three golfers, Bruce, Edgar, and Mike, who played one round of golf at each of two golf courses, Clovelly and Pippy Park, during a week in June. Their success during these rounds can be summarized by a ternary relation on $A \times B \times C$, where $A = \{\text{Bruce, Edgar, Mike}\}$, $B = \{\text{Clovelly, Pippy Park}\}$, and C is the set of integers from 1 to 1000. Such data are summarized as follows:

A	*B*	*C*
Bruce	Clovelly	74
Bruce	Pippy Park	72
Edgar	Clovelly	72
Edgar	Pippy Park	72
Mike	Clovelly	74
Mike	Pippy Park	74

These data can be turned into three binary relations using an operation called *projection*, which simply omits one of the columns. In the current example, we would have three binary relations.

Relation One		Relation Two		Relation Three	
A	*B*	*A*	*C*	*B*	*C*
Bruce	Clovelly	Bruce	74	Clovelly	74
Bruce	Pippy Park	Bruce	72	Pippy Park	72
Edgar	Clovelly	Edgar	72	Clovelly	72
Edgar	Pippy Park	Mike	74	Pippy Park	74
Mike	Clovelly				
Mike	Pippy Park				

However, there is another ternary relation that would give the same set of three binary relations.

A	*B*	*C*
Bruce	Clovelly	72
Bruce	Pippy Park	74
Edgar	Clovelly	72
Edgar	Pippy Park	72
Mike	Clovelly	74
Mike	Pippy Park	74

So it is not possible in this example to retrieve all the information from the three binary relations; hence $n = 3$ is the smallest value of an n-ary relation that will store the given information. ■

Pause 12

In Example 26, suppose Mike shot 120 instead of 74 at Clovelly (and all other scores were unchanged). Is it now possible to retrieve all the information from the three binary relations? ▮

Answers to Pauses

8. Parabola.
9. The answer is yes. For any $(x, y) \in \mathsf{R}^2$, we have $x^2 + y^2 = x^2 + y^2$; in other words, $((x, y), (x, y)) \in \mathcal{R}$ for any $(x, y) \in \mathsf{R}^2$.
10. $(2, 3) \in \mathcal{R}$ but $(3, 2) \notin \mathcal{R}$.
11. No. For example, $((1, 2), (2, 1)) \in \mathcal{R}$ because $1^2+2^2 = 2^2+1^2$ and, similarly, $((2, 1), (1, 2)) \in \mathcal{R}$; however, $(1, 2) \neq (2, 1)$.
12. The second binary relation would now have an extra term, {Mike, 120}, and the third would have the extra term {Clovelly, 120}. No entries would be deleted from any of the three binary relations. Hence, the answer is still **no** because Bruce could still shoot 72 at Clovelly and 74 at Pippy Park, as in the example.

True/False Questions

(Answers can be found in the back of the book.)

1. $\{(x, x + 1) \mid x \in \mathsf{N}\}$ is a binary relation on N, the set of natural numbers.
2. $\{(x, x - 1) \mid x \in \mathsf{N}\}$ is a binary relation on N.

3. Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $\mathcal{R}$ is not reflexive, then we can conclude that $(a, a) \notin \mathcal{R}$.
4. Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $(a, a) \notin \mathcal{R}$, then $\mathcal{R}$ is not reflexive.
5. Let $\mathcal{R}$ be a binary relation on a set A. If $\mathcal{R}$ is not symmetric, then there exist $a, b \in A$ such that $(a, b) \in \mathcal{R}$ but $(b, a) \notin \mathcal{R}$.
6. Let $\mathcal{R}$ be a binary relation on a set A. If $\mathcal{R}$ is antisymmetric, then we can conclude that there exist $a, b \in A$ such that $(a, b) \in \mathcal{R}$ but $(b, a) \notin \mathcal{R}$.
7. Let $\mathcal{R}$ be a binary relation on a set A. If there exist $a, b, c \in A$ such that $(a, b) \in \mathcal{R}$, $(b, c) \in \mathcal{R}$, and $(a, c) \in \mathcal{R}$, then $\mathcal{R}$ must be transitive.
8. If a binary relation $\mathcal{R}$ is antisymmetric, then $\mathcal{R}$ is not symmetric.
9. If a binary relation $\mathcal{R}$ is not symmetric, then it is antisymmetric.
10. Let $\mathcal{R}$ be a binary relation on a set A containing two elements. If $\mathcal{R}$ is reflexive, then $\mathcal{R}$ is transitive.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. [BB] Let B denote the set of books in a college library and S denote the set of students attending that college. Interpret the Cartesian product $S \times B$. Give a sensible example of a binary relation from S to B.
2. Let A denote the set of names of streets in St. John's, Newfoundland, and B denote the names of all the residents of St. John's. Interpret the Cartesian product $A \times B$. Give a sensible example of a binary relation from A to B.
3. Determine which of the properties reflexive, symmetric, transitive apply to the following relations on the set of people.
   (a) [BB] is a father of
   (b) is a friend of
   (c) [BB] is a descendant of
   (d) have the same parents
   (e) is an uncle of
4. With a table like that in Fig. 2.2, illustrate a relation on the set $\{a, b, c, d\}$ that is
   (a) [BB] reflexive and symmetric
   (b) not symmetric and not antisymmetric
   (c) not symmetric but antisymmetric
   (d) transitive
   Include at least six elements in each relation.
5. Let $A = \{1, 2, 3\}$. List the ordered pairs in a relation on A that is
   (a) [BB] not reflexive, not symmetric, and not transitive
   (b) reflexive, but neither symmetric nor transitive
   (c) symmetric, but neither reflexive nor transitive
   (d) transitive, but neither reflexive nor symmetric
   (e) reflexive and symmetric, but not transitive
   (f) reflexive and transitive, but not symmetric
   (g) [BB] symmetric and transitive, but not reflexive
   (h) reflexive, symmetric, and transitive
6. Is it possible for a binary relation to be both symmetric and antisymmetric? If the answer is no, why not? If it is yes, find all such binary relations.
7. [BB] What is wrong with the following argument, which purports to prove that a binary relation that is symmetric and transitive must necessarily be reflexive as well?

   > Suppose $\mathcal{R}$ is a symmetric and transitive relation on a set A and let $a \in A$. Then, for any b with $(a, b) \in \mathcal{R}$, we have also $(b, a) \in \mathcal{R}$ by symmetry. Since now we have both (a, b) and (b, a) in $\mathcal{R}$, we have $(a, a) \in \mathcal{R}$ as well, by transitivity. Thus, $(a, a) \in \mathcal{R}$, so $\mathcal{R}$ is reflexive.

8. Determine whether each of the binary relations $\mathcal{R}$ defined on the given sets A is reflexive, symmetric, antisymmetric, or transitive. If a relation has a certain property, prove this is so; otherwise, provide a counterexample to show that it does not.
   (a) [BB] A is the set of all English words; $(a, b) \in \mathcal{R}$ if and only if a and b have at least one letter in common.
   (b) A is the set of all people. $(a, b) \in \mathcal{R}$ if and only if neither a nor b is currently enrolled at Miskatonic University or else both are enrolled at MU and are taking at least one course together.
9. Answer Exercise 8 for each of the following relations:
   (a) $A = \{1, 2\}$; $\mathcal{R} = \{(1, 2)\}$.
   (b) [BB] $A = \{1, 2, 3, 4\}$; $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (3, 4)\}$.
   (c) [BB] $A = \mathsf{Z}$; $(a, b) \in \mathcal{R}$ if and only if $ab \geq 0$.
   (d) $A = \mathsf{R}$; $(a, b) \in \mathcal{R}$ if and only if $a^2 = b^2$.
   (e) $A = \mathsf{R}$; $(a, b) \in \mathcal{R}$ if and only if $a - b \leq 3$.
   (f) $A = \mathsf{Z} \times \mathsf{Z}$; $((a, b), (c, d)) \in \mathcal{R}$ if and only if $a - c = b - d$.
   (g) $A = \mathsf{N}$; $(a, b) \in \mathcal{R}$ if and only if $a \neq b$.
   (h) $A = \mathsf{Z}$; $\mathcal{R} = \{(x, y) \mid x + y = 10\}$.

(i) [BB] $A = \mathsf{R}^2$; $\mathcal{R} = \{((x, y), (u, v)) \mid x+y \le u+v\}$.
(j) $A = \mathsf{N}$; $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b}$ is an integer.
(k) $A = \mathsf{Z}$; $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b}$ is an integer.

10. Define $\mathcal{R}$ on R by $(x, y) \in \mathcal{R}$ if and only if $1 \le |x| + |y| \le 2$.
(a) Make a sketch in the Cartesian plane showing the region of R^2 defined by $\mathcal{R}$.
(b) Show that $\mathcal{R}$ is neither reflexive nor transitive.
(c) Is $\mathcal{R}$ symmetric? Is it antisymmetric? Explain.

11. Let S be a set that contains at least two elements a and b. Let A be the power set of S. Determine which of the properties—reflexivity, symmetry, antisymmetry, transitivity—each of the following binary relations $\mathcal{R}$ on A possesses. Give a proof or counterexample as appropriate.
(a) [BB] $(X, Y) \in \mathcal{R}$ if and only if $X \subseteq Y$.
(b) $(X, Y) \in \mathcal{R}$ if and only if $X \subsetneq Y$.
(c) $(X, Y) \in \mathcal{R}$ if and only if $X \cap Y = \emptyset$.

12. Let A be the set of books for sale in a certain university bookstore and assume that among these are books with the following properties.

Book	Price	Length
U	\$10	100 pages
W	\$25	125 pages
X	\$20	150 pages
Y	\$10	200 pages
Z	\$5	100 pages

(a) [BB] Suppose $(a, b) \in \mathcal{R}$ if and only if the price of book a is greater than or equal to the price of book b **and** the length of a is greater than or equal to the length of b. Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive?
(b) Suppose $(a, b) \in \mathcal{R}$ if and only if the price of a is greater than or equal to the price of b **or** the length of a is greater than or equal to the length of b. Is $\mathcal{R}$ reflexive? Symmetric? Antisymmetric? Transitive?

13. Returning to Example 26, suppose that Mike shot 120 instead of 74 at Pippy Park (and all other scores remained unchanged). Is it now possible to retrieve all the information from the three binary relations?

2.4 Equivalence Relations

It is useful to think of a binary relation on a set A as establishing relationships between elements of A, the assertion "$(a, b) \in \mathcal{R}$" relating the elements a and b. Such relationships occur everywhere. Two people may be of the same sex, have the same color eyes, live on the same street. These three particular relationships are reflexive, symmetric, and transitive and hence *equivalence relations*.

2.4.1 DEFINITION

An *equivalence relation* on a set A is a binary relation $\mathcal{R}$ on A that is reflexive, symmetric, and transitive. ❖

Suppose A is the set of all people in the world and

$$\mathcal{R} = \{(a, b) \in A \times A \mid a \text{ and } b \text{ have the same parents}\}.$$

This relation is reflexive (every person has the same set of parents as himself/herself), symmetric (if a and b have the same parents, then so do b and a), and transitive (if a and b have the same parents, and b and c have the same parents, then a and c have the same parents) and so $\mathcal{R}$ is an equivalence relation. It may be because of examples like this that it is common to say "a is related to b," rather than "$(a, b) \in \mathcal{R}$," even for an abstract binary relation $\mathcal{R}$.

If $\mathcal{R}$ is a binary relation on a set A and $a, b \in A$, some authors use the notation $a\mathcal{R}b$ to indicate that $(a, b) \in \mathcal{R}$. In this section, we will usually write $a \sim b$ and, in the case of an equivalence relation, say "a is equivalent to b." Thus, to prove that $\mathcal{R}$ is an equivalence relation, we must prove that $\mathcal{R}$ is

reflexive: $a \sim a$ for all $a \in A$,
symmetric: if $a \in A$ and $b \in A$ and $a \sim b$, then $b \sim a$, and
transitive: if $a, b, c \in A$ and both $a \sim b$ and $b \sim c$, then $a \sim c$.

EXAMPLE 27 Let A be the set of students currently registered at the University of Southern California. For $a, b \in A$, call a and b equivalent if their student numbers have the same first two digits. Certainly, $a \sim a$ for every student a because any number has the same first two digits as itself. If $a \sim b$, the student numbers of a and b have the same first two digits, so the student numbers of b and a have the same first two digits; therefore, $b \sim a$. Finally, if $a \sim b$ and $b \sim c$, then the student numbers of a and b have the same first two digits, and the student numbers of b and c have the same first two digits, so the student numbers of a and c have the same first two digits. Since $\mathcal{R}$ is reflexive, symmetric, and transitive, $\mathcal{R}$ is an equivalence relation on A. ■

EXAMPLE 28 Let A be the set of all residents of the United States. Call a and b equivalent if a and b are residents of the same state. The student should mentally confirm that $\sim$ defines an equivalence relation. ■

EXAMPLE 29 (Congruence mod 3)[5] Define $\sim$ on the set $\mathbb{Z}$ of integers by $a \sim b$ if $a - b$ is divisible (evenly) by 3.[6] For any $a \in \mathbb{Z}$, $a - a = 0$ is divisible by 3 and so $a \sim a$. If $a, b \in \mathbb{Z}$ and $a \sim b$, then $a - b$ is divisible by 3, so $b - a$ (the negative of $a - b$) is also divisible by 3. Hence, $b \sim a$. Finally, if $a, b, c \in \mathbb{Z}$ with $a \sim b$ and $b \sim c$, then both $a - b$ and $b - c$ are divisible by 3, so $a - c$, being the sum of $a - b$ and $b - c$, is also divisible by 3. Thus, $\sim$ is an equivalence relation. ■

EXAMPLE 30 The relation $\leq$ on the real numbers—$a \sim b$ if and only if $a \leq b$—is not an equivalence relation on $\mathbb{R}$. While it is reflexive and transitive, it is not symmetric: $4 \leq 5$ but $5 \not\leq 4$. ■

Pause 13 Let A be the set of all people. For $a, b \in A$, define $a \sim b$ if either (i) both a and b are residents of the same state of the United States or (ii) neither a nor b is a resident of any state of the United States. Does $\sim$ define an equivalence relation? ▮

Surely the three most fundamental properties of equality are

reflexivity: $a = a$ for all a;

symmetry: if $a = b$, then $b = a$; and

transitivity: if $a = b$ and $b = c$, then $a = c$.

Thus, equality is an equivalence relation on any set. For this reason, we think of equivalence as a weakening of equality. We have in mind a certain characteristic or property of elements and wish only to consider as different elements that differ with respect to this characteristic. Little children may think of their brothers and sisters as the same and other children as "different." A statistician trying to estimate the percentages of people in the world with different eye colors is only interested in eye color; for her, two people are only "different" if they have different colored eyes. All drop-off points in a given neighborhood of town may be the "same" to the driver of a newspaper truck. An equivalence relation changes our view of the universe (the underlying set A); instead of viewing it as individual elements, attention is directed to certain groups or subsets. The equivalence relation "same parents" groups people into families; "same color eyes" groups people by eye color; "same neighborhood" groups newspaper drop-off points by neighborhood.

The groups into which an equivalence relation divides the underlying set are called *equivalence classes*. The equivalence class of an element is the collection of all things related to it.

[5]This is an example of an important equivalence relation called *congruence* to which we later devote an entire section, Section 4.4.

[6]Within the context of integers, *divisible* always means *divisible evenly*, that is, with remainder 0.

2.4.2 DEFINITION

If $\sim$ denotes an equivalence relation on a set A, the *equivalence class* of an element $a \in A$ is the set $\overline{a} = \{x \in A \mid x \sim a\}$ of all elements equivalent to a.The set of all equivalence classes is called the *quotient set* of A *mod* $\sim$ and denoted $A/\sim$. ❖

Since an equivalence relation is symmetric, it does not matter whether we write $x \sim a$ or $a \sim x$ in the definition of $\overline{a}$. The set of things related to a is the same as the set of things to which a is related.

For the equivalence relation in Example 27, the students who are related to a particular student x are those whose student numbers have the same first two digits as x's student number. For this equivalence relation, an equivalence class is the set of all students whose student numbers begin with the same first two digits. The set of all students has been grouped into smaller sets—the class of 99, for instance (all students whose numbers begin 99), the class of 02 (all students whose numbers begin 02), and so forth. The quotient set is the set of all equivalence classes, so it's

$$A/\sim = \{\text{class of } n \mid n = 05, 04, 03, 02, 01, 00, 99, 98, \dots\}.$$

In Example 28, if x is a resident of some state of the United States, then the people to whom x is related are those people who reside in the same state. The residents of Colorado, for example, form one equivalence class, as do the residents of Rhode Island, the residents of Florida, and so on. The quotient set is the set of all states in the United States.

What are the equivalence classes for the equivalence relation that is congruence mod 3? What is $\overline{0}$, the equivalence class of 0, for instance? If $a \sim 0$, then $a - 0$ is divisible by 3; in other words, a is divisible by 3. Thus, $\overline{0}$ is the set of all integers that are divisible by 3. We shall denote this set $3\mathsf{Z}$. So $\overline{0} = 3\mathsf{Z}$. What is $\overline{1}$? If $a \sim 1$, then $a - 1 = 3k$ for some integer k, so $a = 3k + 1$. Thus, $\overline{1} = \{3k + 1 \mid k \in \mathsf{Z}\}$, a set we denote $3\mathsf{Z} + 1$. Similarly, $\overline{2} = \{3k + 2 \mid k \in \mathsf{Z}\} = 3\mathsf{Z} + 2$ and we have found all the equivalence classes for congruence mod 3.

Pause 14

Why? ▮

Thus the quotient set for congruence mod 3 is

$$\mathsf{Z}/\sim = \{3\mathsf{Z}, 3\mathsf{Z} + 1, 3\mathsf{Z} + 2\}.$$

In general, for given natural numbers n and r, $n\mathsf{Z} + r$ is the set of integers of the form $na + r$ for some $a \in \mathsf{Z}$:

$$n\mathsf{Z} + r = \{na + r \mid a \in \mathsf{Z}\}. \tag{5}$$

Also, we write $n\mathsf{Z}$ instead of $n\mathsf{Z} + 0$:

$$n\mathsf{Z} = \{na \mid a \in \mathsf{Z}\}. \tag{6}$$

The even integers, for instance, can be denoted $2\mathsf{Z}$.

Pause 15

What are the equivalence classes for the equivalence relation described in PAUSE 13? How many elements does the quotient set contain? ▮

PROBLEM 31. For (x, y) and (u, v) in R^2, define $(x, y) \sim (u, v)$ if $x^2+y^2 = u^2+v^2$. Prove that $\sim$ defines an equivalence relation on R^2 and interpret the equivalence classes geometrically.

Solution. If $(x, y) \in \mathsf{R}^2$, then $x^2 + y^2 = x^2 + y^2$, so $(x, y) \sim (x, y)$: The relation is reflexive. If $(x, y) \sim (u, v)$, then $x^2 + y^2 = u^2 + v^2$, so $u^2 + v^2 = x^2 + y^2$ and $(u, v) \sim (x, y)$: The relation is symmetric. Finally, if $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$, then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v^2 = w^2 + z^2$. Thus, $x^2 + y^2 = u^2 + v^2 = w^2 + z^2$. Since $x^2 + y^2 = w^2 + z^2$, $(x, y) \sim (w, z)$, so the relation is transitive.

The equivalence class of (a, b) is

$$\overline{(a, b)} = \{(x, y) \mid (x, y) \sim (a, b)\} = \{(x, y) \mid x^2 + y^2 = a^2 + b^2\}.$$

For example, $\overline{(1, 0)} = \{(x, y) \mid x^2 + y^2 = 1^2 + 0^2 = 1\}$, which we recognize as the graph of a circle in the Cartesian plane with center $(0, 0)$ and radius 1. For general (a, b), letting $c = a^2 + b^2$, the equivalence class $\overline{(a, b)}$ is the set of points (x, y) satisfying $x^2 + y^2 = c$. So this equivalence class is the circle with center $(0, 0)$ and radius $\sqrt{c}$. With one exception, the equivalence classes are circles with center $(0, 0)$. ▲

Pause 16 What is the exception? ▮

2.4.3 PROPOSITION Let $\sim$ denote an equivalence relation on a set A. Let $a \in A$. Then, for any $x \in A$, $x \sim a$ if and only if $\overline{x} = \overline{a}$.

Proof $(\longleftarrow)$ Suppose $\overline{x} = \overline{a}$. We know $x \in \overline{x}$ because $x \sim x$, so $x \in \overline{a}$; thus, $x \sim a$. It is the implication $\rightarrow$ that is the substance of this proposition.

$(\longrightarrow)$ Suppose that $x \sim a$. We must prove that the two sets $\overline{x}$ and $\overline{a}$ are equal. As always, we do this by proving that each set is a subset of the other. First suppose $y \in \overline{x}$. Then $y \sim x$ and $x \sim a$, so $y \sim a$ by transitivity. Therefore, $y \in \overline{a}$, so $\overline{x} \subseteq \overline{a}$. On the other hand, suppose $y \in \overline{a}$. Then $y \sim a$. Since we also have $a \sim x$, we have both $y \sim a$ and $a \sim x$; therefore, by transitivity, $y \sim x$. Thus, $y \in \overline{x}$ and $\overline{a} \subseteq \overline{x}$. Therefore, $\overline{a} = \overline{x}$. ●

In each of the examples of equivalence relations that we have discussed in this section, different equivalence classes never overlapped. If a person is a resident of one state, he or she is not a resident of another. A student number cannot begin with 79 and also with 84. An integer that is a multiple of 3 is not of the form $3k + 1$. These examples are suggestive of a result that is true in general.

2.4.4 PROPOSITION Suppose $\sim$ denotes an equivalence relation on a set A and $a, b \in A$. Then the equivalence classes $\overline{a}$ and $\overline{b}$ are either the same or *disjoint*; that is, $\overline{a} \cap \overline{b} = \emptyset$.

Proof We prove that if $\overline{a} \neq \overline{b}$, then $\overline{a}$ and $\overline{b}$ are disjoint, and we do so by contradiction. Suppose that $\overline{a} \cap \overline{b} \neq \emptyset$. Then there is an element $x \in \overline{a} \cap \overline{b}$. Since $x \in \overline{a}$, $\overline{x} = \overline{a}$ by Proposition 2.4.3. Similarly, since $x \in \overline{b}$, we also have $\overline{x} = \overline{b}$. Thus, $\overline{a} = \overline{b}$, which is a contradiction. ●

If $\sim$ denotes an equivalence relation on A, reflexivity says that every element a in A belongs to some equivalence class, specifically to $\overline{a}$. In conjunction with Proposition 2.4.4, this observation says that the equivalence classes of any equivalence relation divide A into disjoint (that is, nonoverlapping) subsets that cover the entire set, just like the pieces of a jigsaw puzzle. We say that the equivalence classes "partition" A or "form a partition of" A. (The word *partition* is used as both verb and noun.)

2.4.5 DEFINITION

A *partition* of a set A is a collection of disjoint nonempty subsets of A whose union is A. These disjoint sets are called *cells* (or *blocks*). The cells are said to *partition* A. ❖

EXAMPLE 32

- Canada is partitioned into ten provinces and three territories.[7]
- Students are partitioned into groups according to the first two digits of their student numbers.
- The human race is partitioned into groups by eye color.
- A deck of playing cards is partitioned into four suits.
- If $A = \{a, b, c, d, e, f, x\}$, then $\{\{a, b\}, \{c, d, e\}, \{f\}, \{x\}\}$ is a partition of A. So is $\{\{a, x\}, \{b, d, e, f\}, \{c\}\}$. ■

We have seen that the equivalence classes of an equivalence relation on a set A are disjoint sets whose union is A; each element $a \in A$ is in precisely one equivalence class, $\overline{a}$. Thus, we have the following basic theorem about equivalence relations.

2.4.6 THEOREM

The equivalence classes associated with an equivalence relation on a set A form a partition of A.

Not only does an equivalence relation determine a partition, but, conversely, any partition of a set A determines an equivalence relation, specifically, that equivalence relation whose equivalence classes are the cells of the partition. The partition of the integers into "evens" and "odds" corresponds to the equivalence relation that says two integers are equivalent if and only if they are both even or both odd. The partition

$$\{\{a, g\}, \{b, d, e, f\}, \{c\}\}$$

of the set $\{a, b, c, d, e, f, g\}$ corresponds to the equivalence relation whose equivalence classes are $\{a, g\}$, $\{b, d, e, f\}$ and $\{c\}$, that is, to the equivalence relation described in the figure, where a cross in row x and column y is used to indicate $x \sim y$.

	a	g	b	d	e	f	c
a	×	×					
g	×	×					
b			×	×	×	×	
d			×	×	×	×	
e			×	×	×	×	
f			×	×	×	×	
c							×

Pause 17

The suits "heart," "diamond," "club," "spade" partition a standard deck of playing cards. Describe the corresponding equivalence relation on a deck of cards. ▮

The correspondence between equivalence relations and partitions provides a simple way to exhibit equivalence relations on small sets. For example, the equivalence relation defined in Fig. 2.3 can also be described by listing its equivalence classes: $\{a, b\}$ and $\{c\}$.

[7]Nunavut, created from the eastern half of the former Northwest Territories, joined Canada as a third territory on April 1, 1999.

	a	b	c
a	$\times$	$\times$	
b	$\times$	$\times$	
c			$\times$

Figure 2.3 An equivalence relation with two equivalence classes, $\{a, b\}$ and $\{c\}$.

Answers to Pauses

13. It sure does. First, every person is either a resident of the same state in the United States as himself/herself or not a resident of any U.S. state, so $a \sim a$ for all $a \in A$: $\sim$ is reflexive. Second, if $a, b \in A$ and $a \sim b$, then either a and b are residents of the same U.S. state (in which case, so are b and a) or else neither a nor b is a resident of any state in the United States (in which case, neither is b or a). Thus, $b \sim a$: $\sim$ is symmetric. Finally, suppose $a \sim b$ and $b \sim c$. Then either a and b are residents of the same U.S. state or neither is a resident of any U.S. state, and the same holds true for b and c. It follows that either all three of a, b, and c live in the same U.S. state, or none is a resident of a U.S. state. Thus, $a \sim c$: $\sim$ is transitive as well.

14. The equivalence class of $\overline{3}$ is $\{3k + 3 \mid k \in \mathsf{Z}\}$, but this is just the set $3\mathsf{Z}$ of multiples of 3. Thus $\overline{3} = \overline{0}$. The equivalence class of 4 is $\{3k + 4 \mid k \in \mathsf{Z}\}$, but $3k + 4 = 3(k + 1) + 1$, so this set is just $3\mathsf{Z} + 1$, the equivalence class of 1: $\overline{4} = \overline{1}$. In general, the equivalence class of an integer r is $3\mathsf{Z}$ if r is a multiple of 3, $3\mathsf{Z} + 1$ if r is of the form $3a + 1$, and $3\mathsf{Z} + 2$ if r is of the form $3a + 2$. Since every integer r is either a multiple of 3, or $3a + 1$, or $3a + 2$ for some a, the only equivalence classes are $3\mathsf{Z}$, $3\mathsf{Z} + 1$, and $3\mathsf{Z} + 2$.

15. The equivalence class of a consists of those people equivalent to a in the sense of $\sim$. If a does not live in any state of the United States, the equivalence class of a consists of all those people who also live outside any U.S. state. If a does live in a U.S. state, the equivalence class of a consists of those people who live in the same state. The quotient set has 51 elements, consisting of the residents of the 50 U.S. states and the set of people who do not live in any U.S. state.

16. The equivalence class of $(0, 0)$ is the set $\{(0, 0)\}$ whose only element is the single point $(0, 0)$.

17. Two cards are equivalent if and only if they have the same suit. ∎

True/False Questions

(Answers can be found in the back of the book.)

1. "$\iff$" defines an equivalence relation.
2. An equivalence relation on a set A is a binary relation $\mathcal{R}$ on A that is reflexive, symmetric, and transitive.
3. Define a relation $\sim$ on the set of all people in the world by $a \sim b$ if a and b were born in the same year. Then $\sim$ is an equivalence relation.
4. Define a relation $\sim$ on the set of all people in the world by $a \sim b$ if a and b were born within a year of each other. Then $\sim$ is an equivalence relation.
5. If $\sim$ is not an equivalence relation on a set A, then there must exist $a, b \in A$ with $a \sim b$ and $b \not\sim a$.

6. If $\sim$ is an equivalence relation on a set A, the equivalence class of an element $a \in A$ is the set $\overline{a} = \{x \in A \mid a \sim x\}$.
7. $3\mathsf{Z}$ is the set of odd integers.
8. $2\mathsf{Z} + 3$ is the set of odd integers.
9. If $\sim$ is an equivalence relation on a set A and $\overline{x} \neq \overline{a}$, then $x \not\sim a$.
10. If $\sim$ is an equivalence relation on a set A and $a \not\sim b$, then $\overline{a} \cap \overline{b} \neq \emptyset$.
11. The natural numbers can be partitioned into even numbers and odd numbers.

Exercises

*The answers to exercises marked [BB] can be found in the **B**ack of the **B**ook.*

1. Let A be the set of all citizens of New York City. For $a, b \in A$, define $a \sim b$ if and only if
- neither a nor b have a cell phone, or
- both a and b have cell phones in the same exchange (that is, the first three digits of each phone number are the same).

Show that $\sim$ defines an equivalence relation on A and find the corresponding equivalence classes.

2. Explain why each of the following binary relations on $S = \{1, 2, 3\}$ is not an equivalence relation on S.

(a) [BB] $\mathcal{R} = \{(1, 1), (1, 2), (3, 2), (3, 3), (2, 3), (2, 1)\}$

(b) $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 1), (1, 3)\}$

(c)

	1	2	3
1	×	×	×
2	×	×	
3			×

3. [BB] The sets $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ are the equivalence classes for a well-known equivalence relation on the set $S = \{1, 2, 3, 4, 5\}$. What is the usual name for this equivalence relation?

4. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $S = \mathcal{P}(A)$, the power set of A.

(a) For $a, b \in S$, define $a \sim b$ if a and b have the same number of elements. Prove that $\sim$ defines an equivalence relation on S.

(b) How many equivalence classes are there? List one element from each equivalence class.

5. [BB] For $a, b \in \mathsf{R}\setminus\{0\}$, define $a \sim b$ if and only if $\frac{a}{b} \in \mathsf{Q}$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Find the equivalence class of 1.

(c) Show that $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.

6. For natural numbers a and b, define $a \sim b$ if and only if $a^2 + b$ is even. Prove that $\sim$ defines an equivalence relation on N and find the quotient set determined by $\sim$.

7. [BB] For $a, b \in \mathsf{R}$, define $a \sim b$ if and only if $a - b \in \mathsf{Z}$.

(a) Prove that $\sim$ defines an equivalence relation on Z.

(b) What is the equivalence class of 5? What is the equivalence class of $5\frac{1}{2}$?

(c) What is the quotient set determined by this equivalence relation?

8. [BB] For integers a, b, define $a \sim b$ if and only if $2a + 3b = 5n$ for some integer n. Show that $\sim$ defines an equivalence relation on Z.

9. Define $\sim$ on Z by $a \sim b$ if and only if $3a + b$ is a multiple of 4.

(a) Prove that $\sim$ defines an equivalence relation.

(b) Find the equivalence class of 0.

(c) Find the equivalence class of 2.

(d) Make a guess about the quotient set.

10. For integers a and b, define $a \sim b$ if $3a + 4b = 7n$ for some integer n.

(a) Prove that $\sim$ defines an equivalence relation.

(b) Find the equivalence class of 0.

11. [BB] For $a, b \in \mathsf{Z} \setminus \{0\}$, define $a \sim b$ if and only if $ab > 0$.

(a) Prove that $\sim$ defines an equivalence relation on Z.

(b) What is the equivalence class of 5? What's the equivalence class of -5?

(c) What is the partition of $\mathsf{Z} \setminus \{0\}$ determined by this equivalence relation?

12. For $a, b \in \mathsf{Z}$, define $a \sim b$ if and only if $a^2 - b^2$ is divisible by 3.

(a) [BB] Prove that $\sim$ defines an equivalence relation on Z.

(b) What is $\overline{0}$? What is $\overline{1}$?

(c) What is the partition of Z determined by this equivalence relation?

13. Determine, with reasons, whether or not each of the following defines an equivalence relation on the set A.

(a) [BB] A is the set of all triangles in the plane; $a \sim b$ if and only if a and b are congruent.

(b) A is the set of all circles in the plane; $a \sim b$ if and only if a and b have the same center.

(c) A is the set of all straight lines in the plane; $a \sim b$ if and only if a is parallel to b.

(d) A is the set of all lines in the plane; $a \sim b$ if and only if a is perpendicular to b.

14. List the pairs in the equivalence relation associated with each of the following partitions of $A = \{1, 2, 3, 4, 5\}$.

(a) [BB] $\{\{1, 2\}, \{3, 4, 5\}\}$
(b) $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$ **(c)** $\{\{1, 2, 3, 4, 5\}\}$

15. **(a)** List all the equivalence relations on the set $\{a\}$. How many are there altogether?
(b) Repeat (a) for the set $\{a, b\}$.
(c) [BB] Repeat (a) for the set $\{a, b, c\}$.
(d) Repeat (a) for the set $\{a, b, c, d\}$.
(Remark: The number of partitions of a set of n elements grows rather rapidly. There are 52 partitions of a set of five elements, 203 partitions of a set of six elements, and 877 partitions of a set of seven elements.)

16. Define $\sim$ on R^2 by $(x, y) \sim (u, v)$ if and only if $x - y = u - v$.
(a) [BB] Criticize and then correct the following "proof" that $\sim$ is reflexive.

"If $(x, y) \sim (x, y)$, then $x - y = x - y$, which is true."

(b) What is wrong with the following interpretation of symmetry in this situation?

"If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$."

Write a correct statement of the symmetric property (as it applies to the relation $\sim$ in this exercise).

(c) Criticize and then correct the following "proof" that $\sim$ is symmetric.

"$(x, y) \sim (u, v)$ if $x - y = u - v$. Then $u - v = x - y$. So $(u, v) \sim (x, y)$."

(d) Criticize and correct the following "proof" of transitivity.

"$(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$. Then $u - v = w - z$, so if $x - y = u - v$, then $x - y = w - z$. So $(x, y) \sim (w, z)$."

(e) Why does $\sim$ define an equivalence relation on R^2?
(f) Determine the equivalence classes of $(0, 0)$ and $(2, 3)$ and describe these geometrically.

17. [BB] For (x, y) and $(u, v) \in \mathsf{R}^2$ define $(x, y) \sim (u, v)$ if and only if $x^2 - y^2 = u^2 - v^2$. Prove that $\sim$ defines an equivalence relation on R^2. Describe geometrically the equivalence class of $(0, 0)$. Describe geometrically the equivalence class of $(1, 0)$.

18. Determine which of the following define equivalence relations in R^2. For those that do, give a geometrical interpretation of the quotient set.
(a) $(a, b) \sim (c, d)$ if and only if $a + 2b = c + 2d$.
(b) $(a, b) \sim (c, d)$ if and only if $ab = cd$.
(c) $(a, b) \sim (c, d)$ if and only if $a^2 + b = c + d^2$.
(d) $(a, b) \sim (c, d)$ if and only if $a = c$.
(e) $(a, b) \sim (c, d)$ if and only if $ab = c^2$.

19. Let $\overline{a}$ and $\overline{b}$ be two equivalence classes of an equivalence relation. According to Proposition 2.4.4, "if $\overline{a} \neq \overline{b}$, then $\overline{a}$ and $\overline{b}$ are disjoint."
(a) State the converse of the quoted assertion.
(b) Is the converse true? Justify your answer.

20. Let $\sim$ denote an equivalence relation on a set A. Assume $a, b, c, d \in A$ are such that $a \in \overline{b}$, $c \in \overline{d}$, and $d \in \overline{b}$. Prove that $\overline{a} = \overline{c}$.

21. [BB] Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For $a, b \in A$, define $a \sim b$ if and only if ab is a perfect square (that is, the square of an integer).
(a) What are the ordered pairs in this relation?
(b) For each $a \in A$, find $\overline{a} = \{x \in A \mid x \sim a\}$.
(c) Explain why $\sim$ defines an equivalence relation on A.

22. [BB] Let A be the set of all natural numbers and $\sim$ be as in Exercise 21. Show that $\sim$ defines an equivalence relation on A.

23. Repeat Exercise 21 for $A = \{1, 2, 3, 4, 5, 6, 7\}$ and the relation on A defined by $a \sim b$ if and only if $\frac{a}{b}$ is a power of 2, that is, $\frac{a}{b} = 2^t$ for some integer t, positive, negative, or zero.

24. Let A be the set of all natural numbers and $\sim$ be as in Exercise 23. Show that $\sim$ defines an equivalence relation on A.

25. Let $\mathcal{R}$ be an equivalence relation on a set S and let $\{S_1, S_2, \dots, S_t\}$ be a collection of subsets of S with the property that $(a, b) \in \mathcal{R}$ if and only if a and b are elements of the same set S_i, for some i. Suppose that, for each i, $S_i \not\subseteq \bigcup_{j \neq i} S_j$. Prove that $\{S_1, S_2, \dots, S_t\}$ is a partition of S.

2.5 Partial Orders

In the previous section, we defined an equivalence relation as a binary relation that possesses the three fundamental properties of equality—reflexivity, symmetry, transitivity. We mentioned that we view equivalence as a weak form of equality and employed a symbol, $\sim$, suggesting "equals." In an analogous manner, in this section we focus on three fundamental properties of the order relation $\leq$ on the real numbers—reflexivity, antisymmetry, transitivity—and define a binary relation called *partial order*, which can be viewed as a weak form of $\leq$. We shall use the symbol $\preceq$ for a partial order to remind us of its connection with $\leq$ and, for the same reason, say that "a is less than or equal to b" whenever $a \preceq b$.

2.5.1 DEFINITIONS

A *partial order* on a set A is a binary relation that is reflexive, antisymmetric, and transitive. A *partially ordered set*, *poset* for short, is a pair $(A, \preceq)$, where $\preceq$ is a partial order on the set A. ❖

Writing $a \preceq b$ to mean that (a, b) is in the relation, a partial order on A is a binary relation that is

reflexive: $a \preceq a$ for all $a \in A$,

antisymmetric: If $a, b \in A$, $a \preceq b$ and $b \preceq a$, then $a = b$, and

transitive: If $a, b, c \in A$, $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

It is convenient to use the notation $a \prec b$ (and to say "a is less than b") to signify $a \preceq b$, $a \neq b$, just as we use $a < b$ to mean $a \leq b$, $a \neq b$. Similarly, the meanings of $a \succeq b$ and $a \succ b$ should be apparent.

There is little purpose in making a definition unless there is at hand a variety of examples that fit the definition. Here then are a few examples of partial orders.

EXAMPLE 33

- The binary relation $\leq$ on the real numbers (or on any subset of the real numbers) is a partial order because $a \leq a$ for all $a \in \mathsf{R}$ (reflexivity), $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry), and $a \leq b$, $b \leq c$ implies $a \leq c$ (transitivity).
- For any set S, the binary relation $\subseteq$ on the power set $\mathcal{P}(S)$ of S is a partial order because $X \subseteq X$ for any $X \in \mathcal{P}(S)$ (reflexivity), $X \subseteq Y$, $Y \subseteq X$ for $X, Y \in \mathcal{P}(S)$ implies $X = Y$ (antisymmetry), and $X \subseteq Y$, $Y \subseteq Z$ for $X, Y, Z \in \mathcal{P}(S)$ implies $X \subseteq Z$ (transitivity). ■

EXAMPLE 34

(Lexicographic Ordering) Suppose we have some alphabet of symbols (perhaps the English alphabet) that is partially ordered by some relation $\preceq$. By *word* in this context, we mean any string of letters from this alphabet, not necessarily real words. For "words" $a = a_1a_2\cdots a_n$ and $b = b_1b_2\cdots b_m$, define $a \preceq b$ if

- a and b are identical, or
- $a_i \preceq b_i$ in the alphabet at the first position i where the words differ, or
- $n < m$ and $a_i = b_i$ for $i = 1, \ldots, n$. (This is the situation where word a, which is shorter than b, forms the initial sequence of letters in b.)

This ordering of words is called *lexicographic* because, when the basic alphabet is the English alphabet, it is precisely how words are ordered in a dictionary; car $\preceq$ cat $\preceq$ catalog. ■

The adjective *partial*, as in "partial order," draws our attention to the fact that the definition does not require that every pair of elements be *comparable*, in the following sense.

2.5.2 DEFINITION

If $(A, \preceq)$ is a partially ordered set, elements a and b of A are said to be *comparable* if and only if either $a \preceq b$ or $b \preceq a$. ❖

If X and Y are subsets of a set S, it need not be the case that $X \subseteq Y$ or $Y \subseteq X$; for example, $\{a\}$ and $\{b, c\}$ are not comparable.

2.5.3 DEFINITION

If $\preceq$ is a partial order on a set A and every two elements of A are comparable, then $\preceq$ is called a *total order* and the pair $(A, \preceq)$ is called a *totally ordered set*. ❖

The real numbers are totally ordered by $\leq$ because, for every pair a, b of real numbers, either $a \leq b$ or $b \leq a$. On the other hand, the set of sets, $\{\{a\}, \{b\}, \{c\}, \{a, c\}\}$ is not totally ordered by $\subseteq$ since neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$.

Pause 18

Is lexicographic order on a set of words (in the usual sense) a total order? ▮

Partial orders are often pictured by means of a diagram named after Helmut Hasse (1898–1979), for many years professor of mathematics at Göttingen.[8] In the *Hasse diagram* of a partially ordered set $(A, \preceq)$,

- there is a dot (or vertex) associated with each element of A;
- if $a \preceq b$, then the dot for b is positioned higher than the dot for a; and
- if $a \prec b$ and there is no intermediate c with $a \prec c \prec b$, then a line is drawn from a to b. (In this case, we say that the element b *covers* a.)

The effect of the last property here is to remove redundant lines. Two Hasse diagrams are shown in Fig. 2.4. The reader should appreciate that these would be unnecessarily complicated were we to draw all lines from a to b whenever $a \preceq b$ instead of just those lines where b covers a. No knowledge of the partial order is lost by this convention: After all, if $a \preceq b$ and $b \preceq c$, then (by transitivity) $a \preceq c$; so if there is a line from a to b and a line from b to c, then we can correctly infer that $a \preceq c$, from the diagram. For example, in the diagram on the left, we can infer that $1 \preceq 3$ since $1 \preceq 2$ and $2 \preceq 3$. In the diagram on the right, we similarly infer that $\{b\} \preceq \{a, b, c\}$.

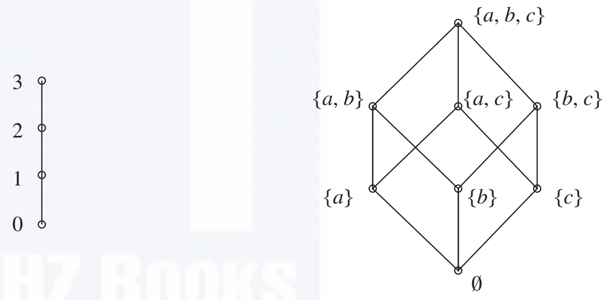


Figure 2.4 The Hasse diagrams for $(\{0, 1, 2, 3\}, \leq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Pause 19

Suppose that in some Hasse diagram a vertex c is "above" another vertex a, but there is no line from a to c. Is it the case that $a \preceq c$? Explain. ▮

2.5.4 DEFINITIONS

An element a of a poset $(A, \preceq)$ is *maximum* if and only if $b \preceq a$ for every $b \in A$ and *minimum* if and only if $a \preceq b$ for every $b \in A$. ❖

In the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$, $\emptyset$ is a minimum element and the set $\{a, b, c\}$ a maximum element. In the poset $\{\, \{a\}, \{b\}, \{c\}, \{a, c\} \,\}$ (with respect to $\subseteq$), there is neither a maximum nor a minimum because, for each of the elements $\{a\}$, $\{b\}$, $\{c\}$, and $\{a, c\}$, there is another of these with which it is not comparable.

If a poset has a maximum element, then this element is unique; similarly, a poset can have at most one minimum. (See Exercise 11.)

[8] There is a fascinating account by S. L. Segal of the ambiguous position in which Hasse found himself during the Nazi period. The article, entitled "Helmut Hasse in 1934," appears in *Historia Mathematica* **7** (1980), 45–56.

One must be careful to distinguish between maximum and **maximal** elements and between minimum and **minimal** elements.

2.5.5 DEFINITIONS

An element a of a poset A is *maximal* if and only if,

$$\text{if } b \in A \text{ and } a \preceq b, \text{ then } b = a$$

and *minimal* if and only if,

$$\text{if } b \in A \text{ and } b \preceq a, \text{ then } b = a.$$ ❖

Thus, a **maximum** element is "bigger" (in the sense of $\preceq$) than every other element in the set, while a **maximal** element is one that is not less than any other. Considering again the poset $\{\{a\}, \{b\}, \{c\}, \{a, c\}\}$, while there is neither a maximum nor a minimum, each of $\{a\}$, $\{b\}$, and $\{c\}$ is minimal, while both $\{b\}$ and $\{a, c\}$ are maximal.

Pause 20

What, if any, are the maximum, minimum, maximal, and minimal elements in the poset whose Hasse diagram appears in Fig. 2.5? ▮

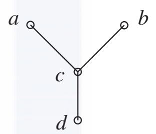


Figure 2.5

2.5.6 DEFINITIONS

Let $(A, \preceq)$ be a poset. An element g is a *greatest lower bound* (abbreviated *glb*) of elements $a, b \in A$ if and only if

1. $g \preceq a$, $g \preceq b$, and
2. if $c \preceq a$ and $c \preceq b$ for some $c \in A$, then $c \preceq g$.

Elements a and b can have at most one glb (see Exercise 14). When this element exists, it is denoted $a \wedge b$, pronounced "a meet b."

An element ℓ is a *least upper bound* (abbreviated *lub*) of a and b if

1. $a \preceq \ell$, $b \preceq \ell$, and
2. if $a \preceq c$, $b \preceq c$ for some $c \in A$, then $\ell \preceq c$.

As with greatest lower bounds, a least upper bound is unique if it exists. The lub of a and b is denoted $a \vee b$, "a join b," if there is such an element. ❖

EXAMPLE 35

- In the poset $(\mathsf{R}, \leq)$, the glb of two real numbers is the smaller of the two and the lub the larger.
- In the poset $(\mathcal{P}(S), \subseteq)$ of subsets of a set S, $A \wedge B = A \cap B$ and $A \vee B = A \cup B$. (See Exercise 12.) Remembering that $\vee$ means $\cup$ and $\wedge$ means $\cap$ in a poset of sets provides a good way to avoid confusing the symbols $\vee$ and $\wedge$ in a general poset. ■

Pause 21

With reference to Fig. 2.5, find $a \vee b$, $a \wedge b$, $b \vee d$, and $b \wedge d$, if these exist. ▮

2.5.7 DEFINITION

A poset $(A, \preceq)$ in which every two elements have a greatest lower bound in A and a least upper bound in A is called a *lattice*. ❖

EXAMPLE 36 The posets described in Examples 35 are both lattices. ■

Answers to Pauses

18. Sure it is; otherwise, it would be awfully hard to use a dictionary.

19. The answer is "not necessarily." We can conclude $a \preceq c$ **only** if there is a sequence of intermediate vertices between a and c with lines between each adjacent pair. Look at the Hasse diagram $(\mathcal{P}(\{a, b, c\}), \subseteq)$ in Fig. 2.4. Here we have $\{b, c\}$ above $\{a\}$, but $\{a\} \not\preceq \{b, c\}$ because these elements are incomparable. On the other hand, $\{a, b, c\}$ is above $\{a\}$ and we can infer that $\{a\} \preceq \{a, b, c\}$ because, for the intermediate vertex $\{a, c\}$, we have upward directed lines from $\{a\}$ to $\{a, c\}$ and from $\{a, c\}$ to $\{a, b, c\}$.

20. There is no maximum, but a and b are maximal; d is both minimal and a minimum.

21. $a \vee b$ does not exist; $a \wedge b = c$; $b \vee d = b$; $b \wedge d = d$.

True/False Questions

(Answers can be found in the back of the book.)

1. A partial order on a set A is a reflexive, antisymmetric, transitive relation on A.
2. The binary relation "$<$" on the set of real numbers is a partial order.
3. The binary relation "$\geq$" on the set of real numbers is a total order.
4. If a set A has more than one element, a total order on A cannot be an equivalence relation.
5. Hasse diagrams are used to identify the equivalence class of a partial order.
6. In a totally ordered set, every maximal element is maximum.
7. If, in a partially ordered set A, every minimal element is minimum, then any two elements of A must be comparable.
8. If a and b are distinct elements of a poset A, then $a \vee b \neq a \wedge b$ (assuming both elements exist).
9. If $a \vee b = a \wedge b$ for elements a, b in a poset A, then $a = b$.
10. The statement in Question 9 is the contrapositive of the statement in Question 8.

Exercises

The answers to exercises marked [BB] can be found in the ***B****ack of the* ***B****ook.*

1. Determine whether each of the following relations is a partial order and state whether each partial order is a total order.

(a) [BB] For $a, b \in \mathsf{R}$, $a \preceq b$ means $a \geq b$.

(b) [BB] For $a, b \in \mathsf{R}$, $a \preceq b$ means $a < b$.

(c) $(\mathsf{R}, \preceq)$, where $a \preceq b$ means $a^2 \leq b^2$.

(d) $(\mathsf{N} \times \mathsf{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$.

(e) $(\mathsf{N} \times \mathsf{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \geq d$.

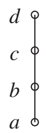
(f) $(\mathcal{W}, \preceq)$, where $\mathcal{W}$ is the set of all strings of letters from the alphabet ("words" real or imaginary), and $w_1 \preceq w_2$ if and only if w_1 has length not exceeding the length of w_2. (Length means number of letters.)

2. (a) [BB] List the elements of the set $\{11, 1010, 100, 1, 101, 111, 110, 1001, 10, 1000\}$ in lexicographic order, given $0 \preceq 1$.

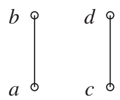
(b) Repeat part (a) assuming $1 \preceq 0$.

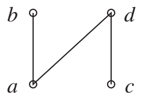
3. [BB; (a), (b)] List all pairs (x, y) with $x \prec y$ in the partial orders described by each of the following Hasse diagrams.

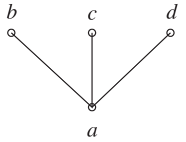
(a)

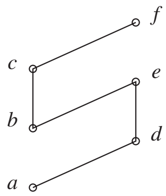


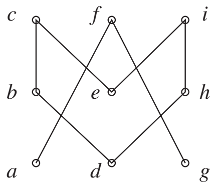
(b)



(c)


(d)


(e)


(f)


4. [BB; (a), (b)] List all minimal, maximal, minimum, and maximum elements for each of the partial orders described in Exercise 3.

5. Draw the Hasse diagrams for each of the following partial orders.

(a) $(\{1, 2, 3, 4, 5, 6\}, \preceq)$

(b) $(\{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, c\}, \{c, d\}\}, \subseteq)$

6. List all minimal, minimum, maximal, and maximum elements for each of the partial orders in Exercise 5.

7. [BB] In the poset $(\mathcal{P}(S), \subseteq)$ of subsets of a set S, under what conditions does one set B cover another set A?

✍ **8.** Learn what you can about Helmut Hasse and write a short biographical note about this person, in good clear English of course!

9. (a) [BB] Prove that any finite (nonempty) poset must contain maximal and minimal elements.

(b) Is the result of (a) true in general for posets of arbitrary size? Explain.

10. (a) Let $A = \mathsf{Z}^2$ and, for $\mathsf{a} = (a_1, a_2)$ and $\mathsf{b} = (b_1, b_2)$ in A, define $\mathsf{a} \preceq \mathsf{b}$ if and only if $a_1 \leq b_1$ and $a_1 + a_2 \leq b_1 + b_2$. Prove that $\preceq$ is a partial order on A. Is this partial order a total order? Justify your answer with a proof or a counterexample.

(b) Generalize the result of part (a) by defining a partial order on the set Z^n of n-tuples of integers. (No proof is required.)

11. (a) [BB] Prove that a poset has at most one maximum element.

(b) Prove that a poset has at most one minimum element.

12. Let S be a nonempty set and let A and B be elements of the power set of S. In the partially ordered set $(\mathcal{P}(S), \subseteq)$,

(a) [BB] prove that $A \wedge B = A \cap B$;

(b) prove that $A \vee B = A \cup B$.

13. Let a and b be two elements of a poset $(A, \preceq)$ with $a \preceq b$.

(a) [BB] Show that $a \vee b$ exists, find this element, and explain your answer.

(b) Show that $a \wedge b$ exists, find this element, and explain your answer.

14. (a) [BB] Prove that a glb of two elements in a poset $(A, \preceq)$ is unique whenever it exists.

(b) Prove that a lub of two elements in a poset $(A, \preceq)$ is unique whenever it exists.

15. (a) If a and b are two elements of a partially ordered set $(A, \preceq)$, the concepts

$$\max(a, b) = \begin{cases} a & \text{if } b \preceq a \\ b & \text{if } a \preceq b \end{cases}$$

and

$$\min(a, b) = \begin{cases} a & \text{if } a \preceq b \\ b & \text{if } b \preceq a \end{cases}$$

do not make sense unless the poset is totally ordered. Explain.

(b) Show that any totally ordered set is a lattice.

16. (a) [BB] Give an example of a partially ordered set that has a maximum and a minimum element but is **not** totally ordered.

(b) Give an example of a totally ordered set that has no maximum or minimum elements.

17. Prove that in a totally ordered set any maximal element is a maximum.

18. Suppose $(A, \preceq)$ is a poset containing a minimum element a.

(a) [BB] Prove that a is minimal.

(b) Prove that a is the only minimal element.

Key Terms & Ideas

Here are some technical words and phrases that were used in this chapter. Do you know the meaning of each? If you're not sure, check the glossary or index at the back of the book.

antisymmetric
binary relation
Cartesian product
cell (of a partition)
comparable
complement
direct product
disjoint
equivalence class
equivalence relation
greatest lower bound
intersection

lattice
least upper bound
maximal
maximum
minimal
minimum
partially ordered set
partial order
partition
poset
power set
proper subset
quotient set
reflexive
set difference
subset
symmetric
symmetric difference
total order
transitive
union

Review Exercises for Chapter 2

1. If $A = \{x \in \mathsf{N} \mid x < 7\}$, $B = \{x \in \mathsf{Z} \mid |x - 5| < 3\}$, and $C = \{2, 3\}$, find $(A \oplus B) \setminus C$.
2. Let $A = \{x \in \mathsf{Z} \mid -1 \le x \le 2\}$, $B = \{2x - 3 \mid x \in A\}$, and $C = \{x \in \mathsf{R} \mid x = \frac{a}{b}, a \in A, b \in B\}$.
   - **(a)** List the elements of A, B, and C.
   - **(b)** List the elements of $(A \cap B) \times B$.
   - **(c)** List the elements of $B \setminus C$.
   - **(d)** List the elements of $A \oplus C$.
3. Let A, B, and C be sets. Are the following statements true or false? In each case, provide a proof or exhibit a counterexample.
   - **(a)** $A \cap B = A$ if and only if $A \subseteq B$.
   - **(b)** $(A \cap B) \cup C = A \cap (B \cup C)$.
   - **(c)** $A \cap B = \emptyset \to A \ne B$.
4. If A, B, and C are sets, $A \ne \emptyset$ and $A \times B = A \times C$, prove that $A = C$.
5. This exercise refers to the Venn diagram shown in Fig. 2.1.
   - **(a)** Use set operations to describe each of the seven regions in terms of A, B, C. For instance, region 1 is $A \setminus (B \cup C)$.
   - **(b)** Use set operations to describe the entire region identified by the labels 2, 3, 4, 5, 7 and the entire region identified by 2, 3, 4.
   - **(c)** Show that $B \setminus (C \setminus A) \ne (B \setminus C) \setminus A$ by listing the regions described by each of these sets.
6. Let $A = \{2, 3, 5, 7, 9\}$, $B = \{3, 5, 6, 8\}$, and $C = \{1, 2, 3, 8, 9\}$.
   - **(a)** Draw a Venn diagram showing the relationship between the sets. Label each element.
   - **(b)** What is
     - **i.** $(A \cup B) \cap C$?
     - **ii.** $A \setminus (B \setminus C)$?
     - **iii.** $A \oplus B$?
     - **iv.** $(A \setminus B) \times (B \cap C)$?
7. Let $A = \{1\}$. Find $\mathcal{P}(\mathcal{P}(A))$.
8. Let A, B, C, and D be sets.
   - **(a)** Give an example showing that the statement "$A \oplus (B \setminus C) = (A \oplus B) \setminus C$" is false in general.
   - **(b)** Prove that the statement "$A \subseteq C, B \subseteq D \to A \times B \subseteq C \times D$" is true.
9. Give an example showing that the statement "$(A \times B) \subseteq (C \times D) \to A \subseteq C$ and $B \subseteq D$" is **false** in general.
10. Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ for sets A, B, and C.
11. Let A be a set.
    - **(a)** What is meant by the term *binary relation on* A?
    - **(b)** Suppose A has 10 elements. How many binary relations are there on A?
12. Let $A = \{x \in \mathsf{R} \mid |x| \le 1\}$ and, for $a, b \in A$, define $(a, b) \in \mathcal{R}$ if and only if $a^2 \le |b|$ and $b^2 \le |a|$. Determine (with reasons) whether $\mathcal{R}$ is reflexive, symmetric, antisymmetric, or transitive.
13. Determine whether $\sim$ is reflexive, symmetric, antisymmetric, transitive, an equivalence relation, or a partial order on A.
    - **(a)** $A = \mathsf{N}$ and, for $a, b \in A$, $a \sim b$ if and only if $a \le 2b$.
    - **(b)** $A = \mathsf{Z}^2$ and, for $(a, b), (c, d) \in A$, $(a, b) \sim (c, d)$ if and only if $a \le b$ and $d \le c$.
14. For $a, b \in \mathsf{N}$, define $a \sim b$ if and only if $a < b$. Determine whether this relation is antisymmetric.
15. Define a relation $\mathcal{R}$ on Z by $a\mathcal{R}b$ if $4a + b$ is a multiple of 5. Show that $\mathcal{R}$ defines an equivalence relation on Z.
16. Define a relation $\mathcal{R}$ on Z by $a\mathcal{R}b$ if $2a + 5b$ is a multiple of 7.
    - **(a)** Prove that $\mathcal{R}$ defines an equivalence relation.
    - **(b)** Is $\mathcal{R}$ a partial order? Explain your answer briefly.
17. Let $\sim$ denote an equivalence relation on a set A. Prove that $x \sim a \leftrightarrow \overline{x} = \overline{a}$ for any $x, a \in A$.
18. Let $\sim$ denote an equivalence relation on a set A. Assume $a, b, c, d \in A$ are such that $a \in \overline{b}$, $d \notin \overline{c}$, and $d \in \overline{b}$. Prove that $\overline{a} \cap \overline{c} = \emptyset$.

19. Let A be a subset of Z^2 and, for $(a, b), (c, d) \in A$, define $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $d \leq b$.

(a) Show that $(A, \preceq)$ is a partially ordered set.

(b) Is $(A, \preceq)$ totally ordered? Explain.

20. Let A be the set of points different from the origin in the Euclidean plane. For $p, q \in A$, define $p \sim q$ if $p = q$ or the line through the distinct points p and q passes through the origin.

(a) Prove that $\sim$ defines an equivalence relation on A.

(b) Find the equivalence classes of $\sim$.

21. Show that $(\mathcal{P}(\mathsf{Z}), \subseteq)$ is a partially ordered set.

22. Let $A = \{1, 2, 4, 6, 8\}$ and, for $a, b \in A$, define $a \preceq b$ if and only if $\frac{b}{a}$ is an integer.

(a) Prove that $\preceq$ defines a partial order on A.

(b) Draw the Hasse diagram for $\preceq$.

(c) List all minimum, minimal, maximum, and maximal elements.

(d) Is $(A, \preceq)$ totally ordered? Explain.

23. Let $(A, \preceq)$ be a poset and $a, b \in A$. Can a and b have two least upper bounds? Explain.