Math 101 (A01-A04)
Test 3
Version: C
March 23, 2017
Time: 120 minutes
Instructors: Asad Asaduzzaman, Anthony Cecil, Christopher Eagle

Family name: [redacted] Tutorial section: T10 (Mustafa)

Given name: [redacted] Student ID: V00 870330

Test Score		
Question	**Points**	**Score**
Multiple Choice	18	16
Question 10	6	5
Question 11	6	6
Question 12	5	5
Question 13	5	2
Total	40	34

Instructions:

- Before beginning the test, enter your name and ID number on this cover and on the bubble sheet. Be sure to fill in the bubbles for your ID number.
- The only items you should have with you are writing implements, your OneCard, and your calculator. The only calculators permitted are Sharp EL-510R, Sharp EL-510RN, and Sharp EL-510RNB. No notes or any other aids are permitted. You are responsible for ensuring that you do not have any prohibited items with you during the test.
- Write out your solutions carefully and completely on the question paper provided. Marks will not be awarded for final answers that are not supported by appropriate work. This includes multiple choice problems.
- For multiple choice questions, the exact answer may not appear as one of the options. Solve the question, then select the answer *closest* to your answer. If your answer is exactly equidistant from two options, choose the larger answer.
- If you require extra space, use the backs of test pages or the blank page at the end of the test. Be sure to clearly indicate where you have done so. No outside paper is permitted.
- This test has 14 pages, including this cover and the blank page at the end.
- **Fill in "C" in the "Form" field of the bubble sheet now.**

(). 1. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ whose terms are given by $a_n = \dfrac{9n^2+n-1}{4-3n^2}$. Determine whether the sequence converges or diverges. If it converges, find its limit.

A. -4 B. -3 C. -2 D. -1 E. 0

F. 1 G. 2 H. 3 I. 4 J. The sequence diverges

$\lim_{n\to\infty} \frac{9n^2+n-1}{-3n^2+4} = \frac{9}{-3} = -3$

(2) 2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ whose terms are given by $a_n = \dfrac{n!}{4^n}$. Determine whether the sequence converges or diverges. If it converges, find its limit.

A. 0 B. 1 C. 2 D. 3 E. 4

F. 5 G. 6 H. 7 I. 8 J. The sequence diverges

$\lim_{n\to\infty} \frac{n!}{4^n} \to \infty$ diverges

$\sum_{n=1}^{\infty} \frac{2e^2}{e^{2n}} = \sum_{n=1} 2e^2 \left(\frac{1}{e^2}\right)^n$ $a = 2e^2$, $r = \frac{1}{e^2}$ $\quad \sum 2e^{-2n} e^2 = \sum 2e^2 \left(\frac{1}{e^2}\right)^n$

) 3. Determine whether the series $\sum_{n=1}^{\infty} 2e^{-2n+2}$ converges or diverges. If it converges, find its value.

A. -1.43 B. -0.58 C. -0.16 D. 0 E. 0.12

F. 0.27 G. 1.76 H. 2.31 I. 3.52 J. The series diverges

$\lim_{n\to\infty} \left| \frac{2e^{-2n-2+2}}{2e^{-2n+2}} \right| = \left| \frac{1}{e^2} \right| < 1$ converges

$a = 2e^2 \quad r = \frac{1}{e^2} \qquad S_n = \frac{a}{1-r} = \frac{2e^2}{1-\frac{1}{e^2}} = \frac{2e^2(e^2)}{e^2-1} = \frac{2e^4}{e^2-1} = 17.09$

$\sum 2\left(\frac{1}{e^2}\right)^{n-1} = \frac{2}{1-\frac{1}{e^2}} = 2.313$

(2) 4. Determine whether the series $\sum_{n=1}^{\infty} \sqrt[n]{2}$ converges or diverges. If it converges, find its value.

A. 0.1 B. 0.2 C. 0.3 D. 0.4 E. 0.5

F. 1.0 G. 1.2 H. 1.4 I. 2.0 J. The series diverges

$\sum 2^{\frac{1}{n}} = 2 + \sqrt{2} + \sqrt[3]{2} > 2$

$\lim_{n\to\infty} \sqrt[n]{2} = \lim 2^{\frac{1}{n}} = 2^0 = 1 \neq 0$

1) 5. Suppose that p is a positive constant. For which values of p does the series $\sum_{n=1}^{\infty} \frac{n^3+1}{n^p}$ converge?

A. $p < 1$ B. $p \leq 1$ C. $p > 1$ D. $p \geq 1$ E. $p < 4$

F. $p \leq 4$ G. $p > 4$ H. $p \geq 4$ I. All values of p J. No values of p

$\frac{n^3+1}{n^p} = \frac{1}{n^{p'}}, \ p' > 1$

$p-3 > 1$

$p > 4$

2) 6. Consider the following three series:

a_n fluctuates

$$(\clubsuit) : \sum_{n=1}^{\infty} \frac{(-1)^n}{1-\frac{1}{n}} \quad \text{X} \qquad (\spadesuit) : \sum_{n=1}^{\infty} \frac{(-1)^n}{\sin(n)} \quad \text{X} \qquad (\blacklozenge) : \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \checkmark$$

For which of the above series can we use the alternating series test to show that the series converges?

A. ($\clubsuit$) only B. ($\spadesuit$) only C. ($\blacklozenge$) only

D. ($\blacklozenge$) and ($\spadesuit$), but not ($\clubsuit$) E. ($\spadesuit$) and ($\clubsuit$), but not ($\blacklozenge$) F. ($\blacklozenge$) and ($\clubsuit$), but not ($\spadesuit$)

G. ($\blacklozenge$), ($\spadesuit$), and ($\clubsuit$) H. None of these series

$\sum (-1)^n a_n$

1) all a_n positive

2) a_n [illegible]

3) $a_n \to 0$

2) 7. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3x)^n}{4^{n+1}}$?

A. 0 B. 1/4 C. 3/4 D. 1 E. 4/3

F. 2 G. 3 H. 4 I. 6 J. ∞

$$\lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{(3x)^n} \right| = \lim_{n\to\infty} \left| 3x \cdot \frac{4^n(4)}{4^n(4)^2} \right| = \lim_{n\to\infty} \left| \frac{3x}{4} \right|$$

$$\left| \frac{3x}{4} \right| < 1$$

$$|3x| < 4$$

$$-4 < 3x < 4$$

$$-\frac{4}{3} < x < \frac{4}{3}$$

) 8. The power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$ has radius of convergence $R=1$ (you **do not** need to prove this fact). Which of the following is the interval of convergence of this series?

A. $(-\infty,\infty)$ B. $(-1,1)$ C. $[-1,1)$ D. $(-1,1]$ E. $[-1,1]$

F. $(0,2)$ G. $[0,2)$ H. $(0,2]$ I. $[0,2]$ J. None of the other answers

root test

$$\lim_{n\to\infty}\left|\sqrt[n]{\frac{(x-1)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x-1}{\sqrt[n]{n}}\right| = |x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x \leq 2$$

$x=0$ $\sum\frac{(-1)^{n-1}(-1)^n}{n} = (-1)\sum\frac{1}{n}$ harmonic series diverges

$x=2$ $\sum\frac{(-1)^{n-1}(1)^n}{n} = \sum\frac{(-1)^{n-1}}{n}$ alternating harmonic converges conditionally

2 9. What is the coefficient of x^2 in the Maclaurin series for $f(x) = \sqrt{2x+1}$?

A. $-1/8$ B. $-1/4$ C. $-1/2$ D. -1 E. 0

F. 1 G. $1/2$ H. $1/4$ I. $1/8$ J. $f(x)$ does not have a Maclaurin series

$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$1 + 1x + \frac{(-1)}{2!}x^2$

$f(x) = (2x+1)^{1/2}$

$f'(x) = \frac{1}{\cancel{2}(2x+1)^{1/2}}(\cancel{2})$

$f'(x) = \frac{1}{(2x+1)^{1/2}} = (2x+1)^{-1/2}$

$f''(x) = -\frac{1}{2}(2x+1)^{-3/2}(2)$

$f''(x) = \frac{-1}{(2x+1)^{3/2}}$

10. Determine whether each of the following series converges or diverges. You may use any method you wish, but you must justify your answer.

3) (a) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n^2)}{2n}$

$\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n^2)}{2n} = \sum_{n=3}^{\infty} \frac{\cancel{2}\ln(n)}{2n} = \sum_{n=3}^{\infty} \frac{\ln(n)}{2n} \geq \sum \frac{1}{2n}$ $(\ln(n) > 1$ when $n \geq 3)$

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ harmonic series diverges

so $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n^2)}{2n}$ also diverges by Direct Comparison Test

2

$\displaystyle\int_{3}^{\infty} \frac{\ln(n^2)}{2n} = \frac{1}{4}\int_{\ln 9}^{\infty} u\,du = \infty$

3) (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+n+1}{n^2+3}\right)^{3n}$

root test

$\displaystyle\lim_{n\to\infty} \left|\sqrt[n]{\left(\frac{2n^2+n+1}{n^2+3}\right)^{3n}}\right| = \lim_{n\to\infty} \left|\left(\frac{2n^2+n+1}{n^2+3}\right)^{3}\right| = |(2)^3| = 8 \not< 1$

not less than one, diverges

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+n+1}{n^2+3}\right)^{3n}$ diverges by the root test

3

(6) 11. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ diverges, converges conditionally, or converges absolutely. You may use any method you wish, but you must justify your answer.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(\sqrt{n+1}+\sqrt{n})}$ 1) all positive 2) decreasing 3) $a_n \to 0$ converges by alternating series test

3

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series $p \leq 1$ so $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$ diverges

So by Direct Comparison Test $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges

3

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges conditionally

12.

(1) (a) Suppose that $f(x)$ is an infinitely differentiable function. State the general formula for the Taylor series of $f(x)$ centred at $x = a$.

Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$

1

(1) (b) Suppose that $f(x)$ is an infinitely differentiable function, and N is a positive integer. State the general formula for the order N Taylor polynomial of $f(x)$ centred at $x = a$.

$\sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}$

1

(3) (c) Find the Taylor series of $f(x) = e^{4x+1}$ centred at $x = \pi$. Justify your answer.

$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

$f(x) = e^{4x+1}$ $\quad f(\pi) = e^{4\pi+1}$

$f'(x) = 4e^{4x+1}$ $\quad f'(\pi) = 4e^{4\pi+1}$

$f''(x) = 16e^{4x+1}$ $\quad f''(\pi) = 16e^{4\pi+1}$

$f'''(x) = 64e^{4x+1}$ $\quad f'''(\pi) = 64e^{4\pi+1}$

$e^{4\pi+1} + 4e^{4\pi+1}(x-\pi) + \frac{16e^{4\pi+1}}{2!}(x-\pi)^2 + \frac{(64)}{3!}(x-\pi)^3$

$= \sum_{n=0}^{\infty} \frac{4^n(e^{4\pi+1})(x-\pi)^n}{n!}$

3

13.

(1) (a) Let $f(x) = \sum_{n=0}^{\infty} x^n$. Write a formula for $f(x)$ that is *not* a power series, valid for x in the interval $(-1, 1)$. No justification is required.

$f(x) = \frac{1}{1-x}$

(2) (b) Let $g(x) = \sum_{n=1}^{\infty} nx^n$. Show that $g(x) = \frac{x}{(1-x)^2}$ for all x in $(-1, 1)$. Justify your answer.

$\sum_{n=1}^{\infty} nx^n$ root test $\lim_{n\to\infty} \left|\sqrt[n]{n x^n}\right| = \lim_{n\to\infty} \left|\sqrt[n]{n}\, x\right| = |x| < 1$ interval of convergence $-1 < x < 1$

$g(x)$ as Taylor Series at $(x=0)$

$0 + 1x + \frac{4}{2!} x^2 = \sum_{n=1}^{\infty} n x^n$

$g(x) = \frac{x}{(1-x)^2}$

$g'(x) = \frac{1+x}{(1-x)^3}$

$g''(x) = \frac{4+2x}{(1-x)^4}$

(2) (c) Find the exact value of $\sum_{n=1}^{\infty} \frac{n(-1)^n}{2^n}$. Justify your answer.

$= \sum_{n=1}^{\infty} n\left(\frac{-1}{2}\right)^n$ ratio test

$\lim_{n\to\infty} \left|\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}\right| = \lim_{n\to\infty} \left|\frac{1}{2} \cdot \frac{n+1}{n}\right| = \left|\frac{1}{2}\right| < 1$ converges absolutely

$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n \le \sum_{n=1}^{\infty} n\left(\frac{-1}{2}\right)^n \le \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$

$\frac{3}{2} \le \sum_{n=1}^{\infty} n\left(\frac{-1}{2}\right)^n \le$ X

$\sum_{n=1}^{\infty} \frac{n+1}{2^n} = \frac{3}{2}$ ×