Math 110 - Homework 1
Topic: Systems of linear equations

Due at 6:00pm (Pacific) on Friday, September 17, submitted through Crowdmark.

Practice

Before beginning the graded portion of this worksheet, we **strongly** recommend that you practice the basic techniques related to this week's material. Mastering the techniques used in these questions is essential for completing the rest of the worksheet, as well as for success on the tests and exam. The relevant questions this week are from Section 1.1 and 1.2 of the online textbook.

MATLAB

Each week we will provide you with a list of new MATLAB commands relevant to the material on the worksheet. You are welcome, and in fact encouraged, to use MATLAB for the calculations in Part II of the worksheet. On Part I you must do the calculations by hand and show your work.

Here are the new commands you will likely find useful for this week's problems:

- To declare the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ use `A = [1 2 3; 4 5 6; 7 8 9]`. Note that MATLAB does not distinguish between augmented and non-augmented matrices, so it will be up to you to remember how to interpret the columns of the matrix.
- To calculate the reduced row echelon form of a matrix A, use `rref(A)`.
- If you have two matrices A and B with the same number of rows, you can use the command `[A B]` to create a new matrix with the columns of A followed by the columns of B. This is particularly useful when A is the coefficient matrix of a system and B is the single column of constants appearing on the other side of the equality sign.
- The command `A\b` attempts to produce a solution to the system $[A|b]$, but its behaviour can be surprising if the system is inconsistent or has more than one solution. In this course you will probably be best to use `rref([A b])` instead, and then interpret the result yourself.

Graded questions

The questions on the following page are the ones to be submitted for grading. You are permitted to discuss these questions with other students, your tutorial TA, or your instructors - however, the final product that you submit must be written in your own words, and reflect your own understanding. You are **not** permitted to post these questions anywhere on the internet. Your final solutions should be understandable by a student who has been keeping up with this course but does not have any knowledge of the material beyond what we have seen in class - in particular, if you have seen techniques from matrix algebra that have not yet been discussed in the course, do not use them in your solutions.

Part I: Calculation by hand

For all questions in this section you must show all of the details of your calculations. Credit will be given only if you show the steps by which you obtain your final answer.

1. Find all solutions to the following system of linear equations by reducing an appropriate augmented matrix to reduced row echelon form.

$$3x - y + z = 2$$
$$x + z = 1$$
$$y - 3z = 0.$$

Solution: We write the augmented matrix corresponding to this system, then perform row operations.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & 0 \end{array}\right] \to_{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \\ 0 & 1 & -3 & 0 \end{array}\right]$$
$$\to_{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & -3 & 0 \end{array}\right]$$
$$\to_{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & -3 & 0 \end{array}\right]$$
$$\to_{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -5 & -1 \end{array}\right]$$
$$\to_{-\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$
$$\to_{R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$
$$\to_{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$

Translating back to equations, the reduced row echelon form we obtained say $x = 4/5$, $y = 3/5$, $z = 1/5$. Therefore this is the only solution to the given system of equations.

2. Let $A = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ -1 & 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 1 & 4 & 5 & 6 \\ 1 & 9 & 8 & 8 \end{bmatrix}$.
   Either find a sequence of row operations that transforms A into B or explain why no such sequence exists.

Solution: We could try to transform A into B directly, but with no guarantee that we choose the right operations, we don't know if we will succeed. Instead, we calculate RREF(A) and RREF(B). If they are the same, then we know that A can be transformed into B (and we get a sequence of row operations that does the job). If they are different then we know that A cannot be transformed into B. First, we find RREF(A).

$$\begin{aligned}
\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ -1 & 2 & 1 & -3 \end{bmatrix} &\to_{R_3+R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 3 & 2 \end{bmatrix} \\
&\to_{R_1-3R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 3 & 2 \end{bmatrix} \\
&\to_{R_3-5R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -12 & -3 \end{bmatrix} \\
&\to_{-\frac{1}{12}R_3} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1/4 \end{bmatrix} \\
&\to_{R_2-3R_3} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/4 \end{bmatrix} \\
&\to_{R_1+7R_3} \begin{bmatrix} 1 & 0 & 0 & 15/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}
\end{aligned}$$

Now we find RREF(B).

$$\begin{aligned}
\begin{bmatrix} 1 & 3 & 2 & 5 \\ 1 & 4 & 5 & 6 \\ 1 & 9 & 8 & 8 \end{bmatrix} &\to_{R_2-R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 1 & 9 & 8 & 8 \end{bmatrix} \\
&\to_{R_3-R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 6 & 6 & 3 \end{bmatrix} \\
&\to_{R_1-3R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 6 & 6 & 3 \end{bmatrix} \\
&\to_{R_3-6R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -12 & -3 \end{bmatrix}
\end{aligned}$$

We're not at RREF(B) yet, but we have reached a matrix that we saw on the path from A to RREF(A), so at this point we know that if we continue we will find that RREF(A) = RREF(B). We also have a way of getting a sequence of operations to take us from A to B: We start along the path taking A to RREF(A), but then when we reach the last matrix displayed above we start reversing the row operations to get to B. Here's the full calculation:

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ -1 & 2 & 1 & -3 \end{bmatrix} \\
&\to_{R_3+R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 3 & 2 \end{bmatrix} \\
&\to_{R_1-3R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 3 & 2 \end{bmatrix} \\
&\to_{R_3-5R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -12 & -3 \end{bmatrix} \\
&\to_{R_3+6R_2} \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 6 & 6 & 3 \end{bmatrix} \\
&\to_{R_1+3R_2} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 0 & 6 & 6 & 3 \end{bmatrix} \\
&\to_{R_3+R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 1 & 3 & 1 \\ 1 & 9 & 8 & 8 \end{bmatrix} \\
&\to_{R_2+R_1} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 1 & 4 & 5 & 6 \\ 1 & 9 & 8 & 8 \end{bmatrix} \\
&= B
\end{aligned}
$$

Part II: Concepts and connections

In this section you are permitted to use MATLAB to carry out any necessary computations. Almost all of the grades in this section will be awarded for your explanations of *why* you calculated what you did, and what it means. If you use MATLAB to do a calculation, be sure to tell us that you've done so, and also write down both what commands you used and what the output was. Note that there may be questions that can be solved without doing any calculations, or where MATLAB is not helpful for the calculations you need to do; in such cases, do the calculations by hand. If you do use MATLAB for any calculations and it gives you a decimal answer, then give your answers rounded to 2 decimal places.

1. Find all values of k for which the following system of linear equations has no solutions, or show that no such values of k exist:

$$x - y + kz = 1$$

$$kx + y + kz = 0$$

$$x + y + z = 3.$$

Solution: We set up an augmented matrix and row-reduce. MATLAB can do this if we declare a symbolic variable k, but this requires an extra MATLAB package, and considerable care - MATLAB will freely divide by things that depend on k, and those could sometimes be zero! So we do the row reducing by hand this time.

$$\left[\begin{array}{ccc|c} 1 & -1 & k & 1 \\ k & 1 & k & 0 \\ 1 & 1 & 1 & 3 \end{array}\right] \to_{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & k & 1 \\ 1 & 1 & 1 & 3 \\ k & 1 & k & 0 \end{array}\right]$$

$$\to_{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & k & 1 \\ k & 1 & k & 0 \end{array}\right]$$

$$\to_{\substack{R_3 - kR_1 \\ R_2 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & k-1 & -2 \\ 0 & 1-k & 0 & -3k \end{array}\right]$$

$$\to_{\substack{-R_2 \\ \frac{1}{1-k}R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 1-k & 2 \\ 0 & 1 & 0 & -3k/(1-k) \end{array}\right]$$

$$\to_{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -3k/(1-k) \\ 0 & 2 & 1-k & 2 \end{array}\right]$$

$$\to_{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -3k/(1-k) \\ 0 & 0 & 1-k & 2 + (6k/(1-k)) \end{array}\right]$$

$$\to_{\frac{1}{1-k}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -3k/(1-k) \\ 0 & 0 & 1 & (2(1-k) + 6k)/(1-k)^2 \end{array}\right]$$

We've reached row echelon form, which is far enough to tell us how many solutions there will be. It looks like there will always be a unique solution, but that isn't quite the whole story: The process that we have taken is legitimate as long as $1 - k \neq 0$, because of the times when we divided by $1 - k$. So in fact, what we have shown is that *as long as* $1 - k \neq 0$ there is a unique solution. We still don't know what happens if $1 - k = 0$, because our calculations don't handle that case. That is, we still need to examine the special case when $k = 1$. In that case, we have:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \end{array}\right] \to_{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right].$$

The bottom row of this matrix represents the equation $0x + 0y + 0z = 3$, which is impossible. We have therefore shown that the given system of equations has no solutions precisely if $k = 1$.

2. The "Erdős number" of a mathematician is a measure of the collaboration distance from that mathematician to the famous combinatorialist Paul Erdős (this is the mathematician's version of the six degrees of Kevin Bacon game - Google it!). Chris has found a three-digit number with the following properties:

- The sum of the digits is 15.
- The hundreds digit plus the ones digit is twice the tens digit.

- If the digits of the number are reversed and then added to the original number, the answer is 1110.
- The tens digit plus the ones digit equals 8.
- The hundreds digit minus the tens digit is Chris' Erdős number.

Turn the information above into a system of linear equations, then solve the system to find Chris' Erdős number.

Solution: Suppose the number Chris found has digits abc (that is, the number is $100a+10b+c$). Then the first four conditions tell us the following:

$$a+b+c=15$$

$$a+c=2b$$

$$b+c=8$$

$$100a+10b+c+100c+10b+a=1110$$

Cleaning this up a bit, we get the following system of equations:

$$a+b+c=15$$

$$a-2b+c=0$$

$$b+c=8$$

$$101a+20b+101c=1110$$

We need to solve this system. If we enter the following MATLAB commands we can quickly get the answer:

$$\texttt{A} = [\texttt{1 1 1 15}; \texttt{1} \ -\texttt{2 1 0}; \texttt{0 1 1 8}; \texttt{101 20 101 1110}];$$

$$\texttt{rref(A)}$$

The answer is reported as

$$\texttt{ans} = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This tells us that $a=7$, $b=5$, and $c=3$. Thus Chris' number was 753. Now the last bit of given information tells us that Chris' Erdős number is $7-5=2$.
In case you're curious about the path, Chris published the paper "Distribution of the number of encryptions in revocation schemes for stateless receivers" with Bruce Richmond in 2008, and Bruce Richmond published the paper "On graphical partitions" with Paul Erdős in 1993.