Sol^{n} Key

MATH 100, Fall, 2021
Tutorial #3
Asymptotes, Continuity & Derivatives

The following three questions concern asymptotes and graphs. For each function given, find the:

- domain of the function and all axis intercepts.
- all horizontal and vertical asymptotes, with computational justification for each one.
- all oblique (a.k.a. slant) asymptotes, if any. Again, provide computational justification.
- a sketch of the graph of the function that incorporates all the information obtained above. Your sketch may not be completely correct, but it must be consistent with your analysis above to get any marks.

Q1. $f(x) = \dfrac{(x-14)(x+2)(x-1)}{x^3}$

Q2. $g(x) = \dfrac{(x-3)(x+4)(x+2)}{(x^2-1)}$

Q3. $h(x) = \dfrac{x^2-4}{x^4-1}$

Q4

1. Show from the *definition of derivative* that (a) $\frac{d}{dx}x = 1$, (b) $\frac{d}{dx}x^2 = 2x$, and (c) $\frac{d}{dx}x^3 = 3x^2$. Although these are well-known to you, we want you to trace through the derivations in preparation for the next step. Your answers must evaluate limits of derivative quotients to get any marks.
2. From part (1) we see a pattern arising. Use this to handle the general case, obtaining the result $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$. A solution will start by writing out the derivative quotient for x^n at x and then simplifying to evaluate the limit. No marks for solution by any method that does not use the limit of a derivative quotient.

Q5 Use the Intermediate Value Theorem to show that the graph of $f(x) = x^3 - 8x + 1$ has at least three roots on the interval $[-3, 3]$. Hint: Evaluate f at integer points on this interval and apply the theorem where appropriate. Explain *each* use of the theorem in detail.

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$$f(x) = \frac{(x-14)(x+2)(x-1)}{x^3}$$

① Domain: $x \neq 0$, ie $(-\infty, 0) \cup (0, \infty)$

No: y-axis intercept

Roots: $x = 14, -2, 1$

② Vertical asymptote: $x = 0$ (denom $\to 0$, numerator $\to \neq 0$)

Horizontal:
$$\frac{(x-14)(x+2)(x-1)}{x^3} = \frac{\frac{1}{x^3}(x-14)(x+2)(x-1)}{\frac{1}{x^3}\, x^3}$$

$$= \frac{\left(1-\frac{14}{x}\right)\left(1+\frac{2}{x}\right)\left(1-\frac{1}{x}\right)}{1} = f(x),\ x \neq 0$$

$\lim_{x\to\infty} f(x) = \frac{1\cdot 1\cdot 1}{1} = 1$ $\lim_{x\to-\infty} f(x) = 1.$ $\boxed{y = 1}$ horiz. Asymp.

③ No slant asymptotes since $\lim_{x\to\pm\infty} f(x) = 1$

④ Need $\lim_{x\to 0^+} f(x)$: $\frac{(x-14)(x+2)(x-1)}{x^3} = \underbrace{(x-14)(x+2)(x-1)}_{\to 28}\ \underbrace{\frac{1}{x^3}}_{\to +\infty}$

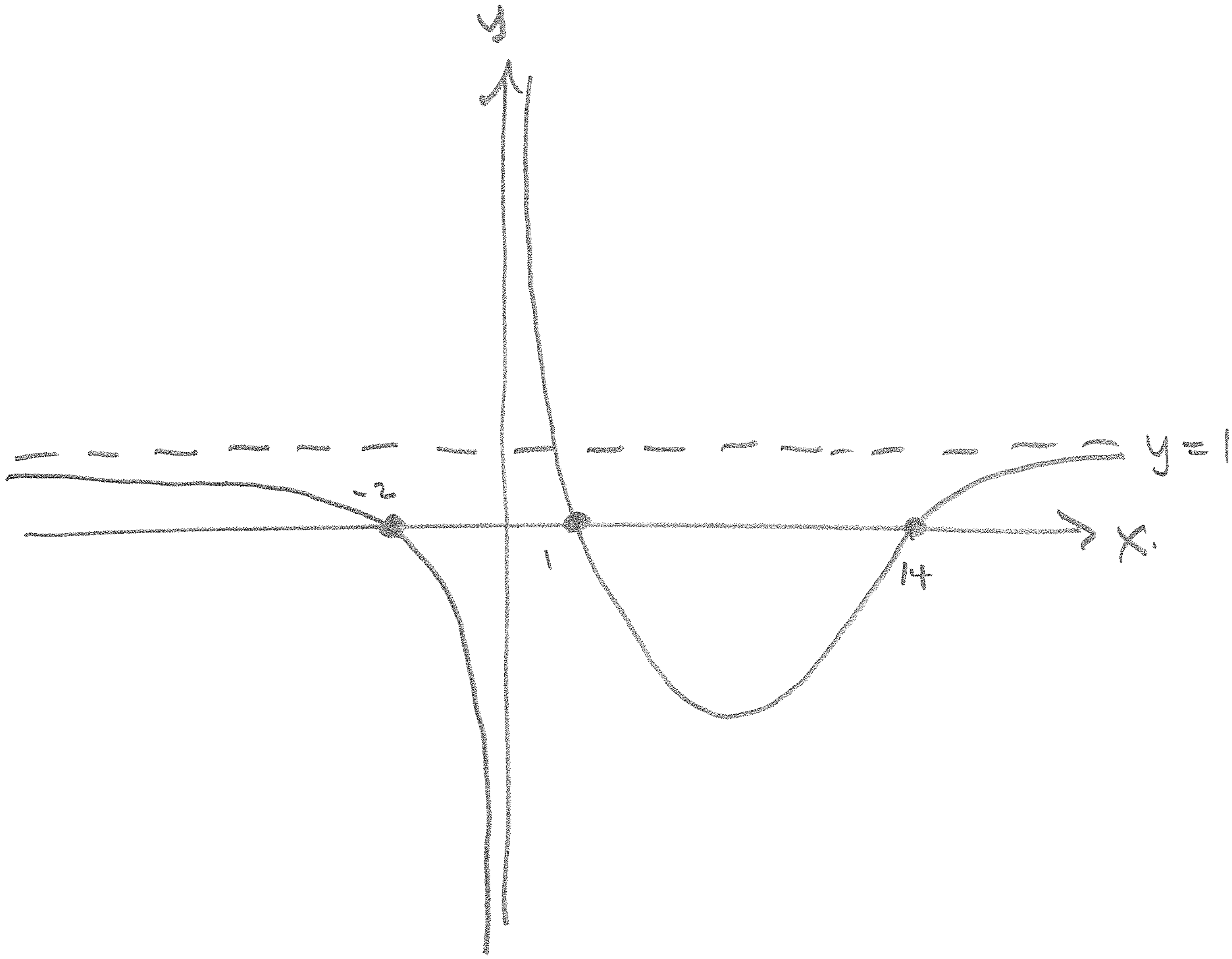
$$\Rightarrow \lim_{x\to 0^+} f(x) = +\infty$$

$\lim_{x\to 0^-} f(x)$. same as above except $\frac{1}{x^3} \to -\infty$.

$$\Rightarrow \lim_{x\to 0^-} f(x) = -\infty$$

Graph ↓ are

Q1 Graph



$$f(x) = \frac{(x-14)(x+2)(x-1)}{x^3}$$

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$g(x) = \dfrac{(x-3)(x+4)(x+2)}{x^2-1}$ ← cubic ← quadratic

1. <u>y-intercept</u> $(x=0)$ $g(0) = 24$. Domain: $x \neq 1, -1$

<u>roots</u> $x = 3, -4, -2$.

2. $\lim\limits_{x\to -1^+} g(x) = \lim\limits_{x\to -1^+} \dfrac{(x-3)(x+4)(x+2)}{x-1} \cdot \dfrac{1}{x+1} = +\infty$

$\dfrac{(x-3)(x+4)(x+2)}{x-1} \to 6$, $\dfrac{1}{x+1} \to +\infty$

$\lim\limits_{x\to -1^-} g(x) =$ — similar — $= -\infty$

$-\infty$

$\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} \dfrac{(x-3)(x+4)(x+2)}{x+1} \cdot \dfrac{1}{x-1} = -\infty$

-15, $+\infty$

$\lim\limits_{x\to 1^-} g(x) =$ ~~~~ similar. $\dfrac{1}{x-1} = +\infty$

$-\infty$

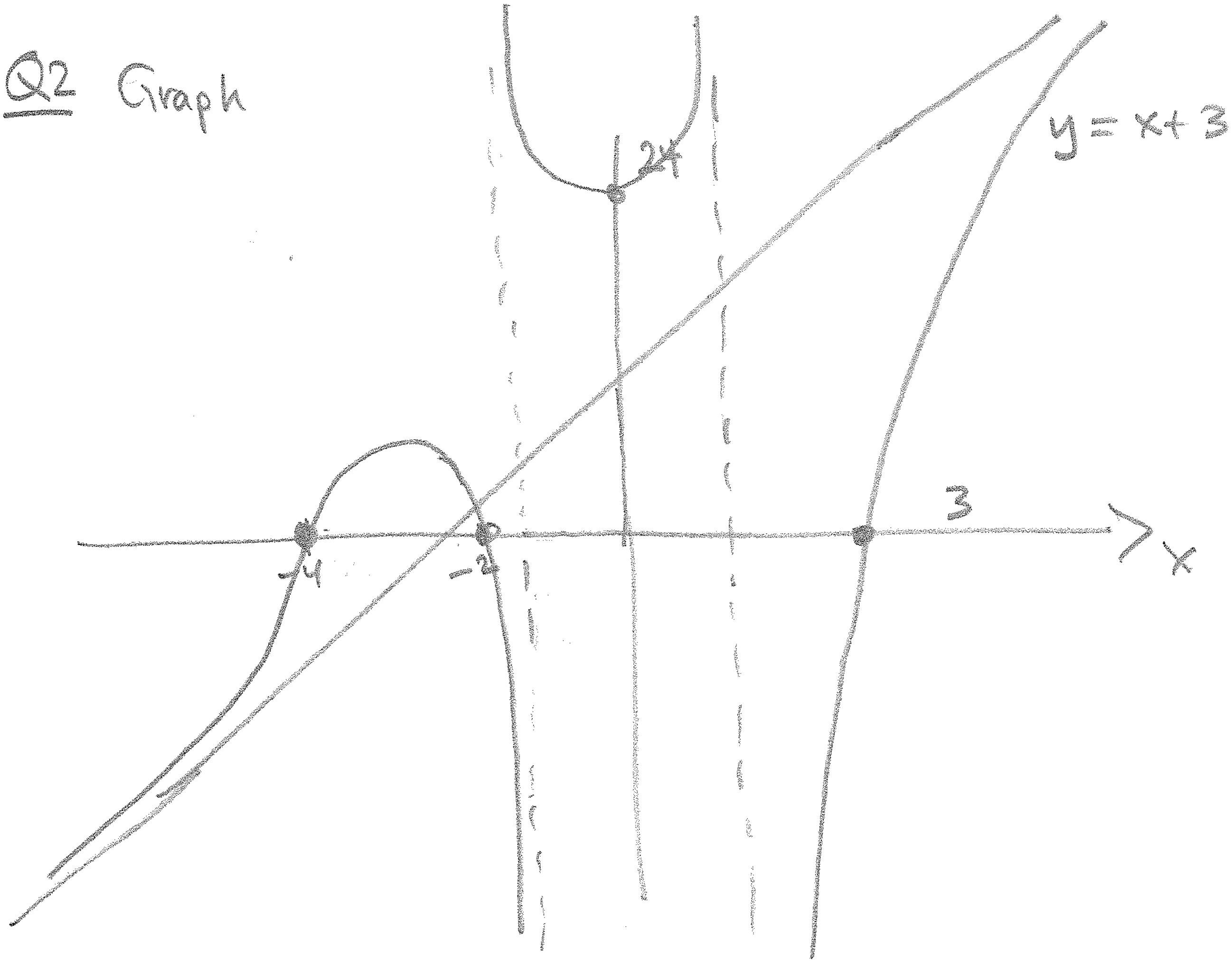
<u>Vertical Asymptotes</u> $x = \pm 1$.

3. <u>Slant</u> $g(x) = \dfrac{x^3+3x^2-10x-24}{x^2-1} \overset{\text{long div.}}{=} \underline{x+3} - \dfrac{9x+21}{x^2-1}$

$y = x+3$ = slant asymptote

4. Graph → over

Q2 Graph



$$g(x) \stackrel{?}{=} \frac{(x-3)(x+4)(x+2)}{x^2-1}$$

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$h(x) = \dfrac{x^2-4}{x^4-1}$ ① Domain: $x \neq \pm 1$. (roots of x^4-1)

y-intercept $(x=0)$ at $\frac{-4}{-1} = 4$

roots $x^2-4=0 \iff x=\pm 2$.

② $\lim\limits_{x\to\pm\infty} \dfrac{x^2-4}{x^4-1} = \lim\limits_{x\to\pm\infty} \dfrac{\frac{1}{x^2}-\frac{4}{x^4}}{1-\frac{1}{x^4}} = \frac{0}{1}$ in both directions

$\lim\limits_{x\to 1^+} h(x) = \lim\limits_{x\to 1^+} \underbrace{\dfrac{x^2-4}{x^3+x^2+x+1}}_{-3/4} \cdot \underbrace{\dfrac{1}{x-1}}_{+\infty} = -\infty$.

$\lim\limits_{x\to 1^-} h(x) =$ same $\underbrace{\dfrac{1}{x-1}}_{-\infty} = +\infty$

$\lim\limits_{x\to -1^+} h(x) = \underbrace{\dfrac{x^2-4}{x^3-x^2+x-1}}_{\to \frac{-3}{-4} = 3/4} \underbrace{\dfrac{1}{x+1}}_{+\infty} = +\infty$

$\lim\limits_{x\to -1^-} h(x) =$ Same $\underbrace{\dfrac{1}{x+1}}_{-\infty} = -\infty$.

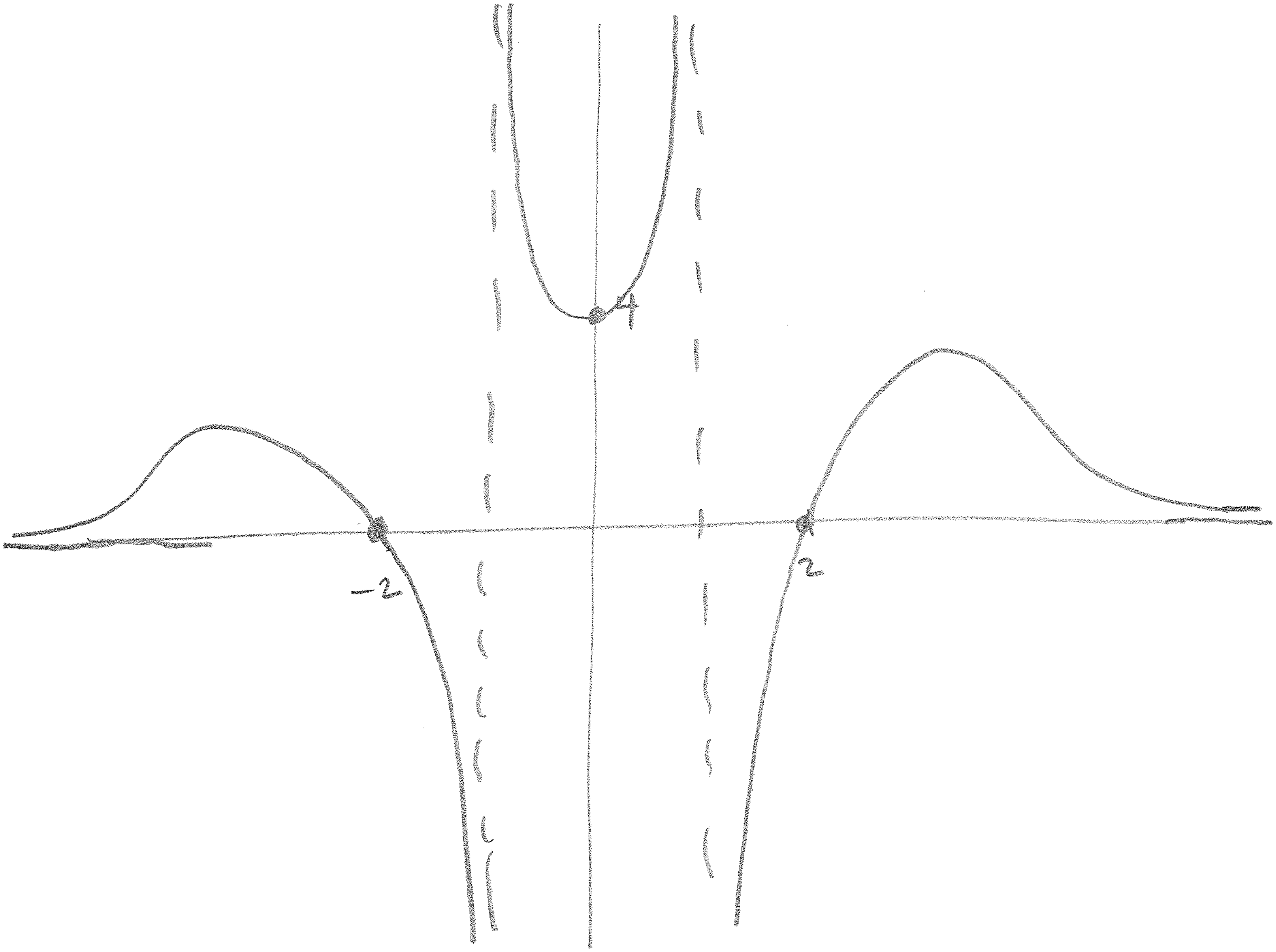
Horizontal Asymp $y=0$ as $x\to\pm\infty$
Vert asymp. $x=1$, $x=-1$.

③ No slant asymptotes.

④ Graph → over

Q3 Graph

Note symmetry for $h(x)$: $h(x) = h(-x)$



$y = h(x)$

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$n=1$: $\lim\limits_{h\to 0} \dfrac{(x+h)-x}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1$ ✓

$n=2$: $\lim\limits_{h\to 0} \dfrac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0} \dfrac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$

$= \lim\limits_{h\to 0} \dfrac{2x+h}{1} = 2x$ ✓

$n=3$: $\lim\limits_{h\to 0} \dfrac{(x+h)^3-x^3}{h} = \lim\limits_{h\to 0} \dfrac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2+3xh+h}{1} = 3x^2$ ✓

General n :

$\lim\limits_{h\to 0} \dfrac{(x+h)^n-x^n}{h}$

$\overset{*}{=} \lim\limits_{h\to 0} \dfrac{[\cancel{x^n}+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+h^n]-\cancel{x^n}}{h}$

$= \lim\limits_{h\to 0} \left[nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+h^{n-1}\right]$

$= nx^{n-1}$ ✓

(*) look up: binomial expansion / binomial theorem.

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$x=$	-3	-2	-1	0	1	2	3
$f=$	-2	9	8	1	-6	-7	4

sign ch.   sign ch.   sign ch.

① On $[-3,-2]$ f is continuous and $f(-3) < \underset{c}{0} < f(-2)$
IV theorem $\Rightarrow$ there is (at least one) r_1 in $[-3,-2]$
so $f(r_1) = 0$

② On $[0, 1]$ f is continuous and $f(0) > 0$, $f(1) < 0$
IVTh $\Rightarrow$ there is r_2 in $[0,1]$, $f(r_2) = 0$

③ On $[2, 3]$ f is continuous and $f(2) < 0$, $f(3) > 0$.
IV theorem $\Rightarrow$ there is r_3 in $[2,3]$ so $f(r_3) = 0$

Roots $= -3 < r_1 < r_2 < r_3 < 3$