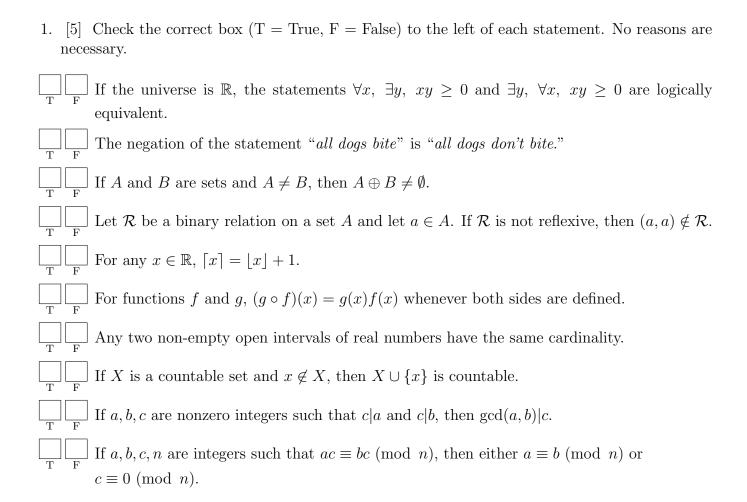
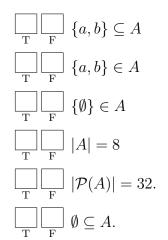
UNIVERSITY OF VICTORIA DECEMBER EXAMINATIONS 2013

MATH 122: Logic and Foundations

Instructor and section (check one	□ K. Mynhardt	[A01] CRN 12132 [A02] CRN 12133	
NAME:			
V00#:			_
Duration: 3 Hours			
Answers should be written on t	the exam paper.		
The exam consists of 20 questi	ons, for a total of 80 m	arks.	
Please show all of your work ar	nd justify your answers	when appropriate.	
There are 10 pages (numbered)), not including covers.		
Count the pages before beginning	ng and report any disc	repancy immediately to	the invigilator.
The only calculator permitted	is the Sharp EL 510-R	or EL-510RNB	
Total:			=
		80	40



2. [3] Let $A = \{a, b, \{a, b, c\}, \{\emptyset, \{\emptyset\}\}, \{c\}\}$. Check the correct box to the left of each statement. No reasons are necessary.



3. [2] Using basic known equivalences, show that $(\neg p \land q) \lor \neg (p \lor q)$ is logically equivalent to $\neg p$.

4. [4] Prove the following logical argument, giving a list of statements and reasons.

$$\begin{array}{c}
p \lor q \\
\neg p \lor r \\
r \to s \\
\hline
\vdots \quad q \lor s
\end{array}$$

#	statement	reason
1.	$p \lor q$	premise
2.		premise
3.	$r \to s$	premise
$\stackrel{4}{\downarrow}$.		
*		

- 5. Let A, B and C be sets.
 - (a) [3] Prove that $A \setminus (B \cap C^c) \subseteq (A \cap B^c) \cup (A \cap C)$.

(b) [2] Use a Venn diagram to investigate whether these sets may, in fact, be equal. Make a conjecture. Do not prove it.

6. [3] Let $a, b, c \in \mathbb{Z}$. Prove that if a|b and b|c then a|c.

- 7. Let A and B be nonempty sets. Consider the statement: if $A \times B = B \times A$ then A = B.
 - (a) [1] Write the contrapositive of the given statement.
 - (b) [3] Prove the statement in (a).

- (c) [1] What does the result in part (b) tell you about the original statement?
- (d) [1] Does the truth value of the original statement change if $A = \emptyset$? Explain.

8. [4] Prove that the set of rational numbers is countable. Use a diagram to illustrate your proof.

- 9. [4] Consider the relation \mathcal{R} defined on the set \mathbb{Z} of integers by $(a,b) \in \mathcal{R}$ if and only if $a-b \leq 5$. Consider the statements below. If a statement is true, prove it. If it is false, give a counterexample.
 - (a) \mathcal{R} is reflexive.
 - (b) \mathcal{R} is symmetric.
 - (c) \mathcal{R} is antisymmetric.
 - (d) \mathcal{R} is transitive.
- 10. [3] Let $f:A\to B$ and $g:B\to C$ be functions. Prove that if $g\circ f$ is one-to-one then f is one-to-one.

- 11. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by f(x) = 4 + |2x + 3|.
 - (a) [1] Determine $\operatorname{rng} f$.
 - (b) [2] Give reasons why f is neither one-to-one nor onto.

(c) [1] Explain how to replace the target \mathbb{R} of f with a set $B \subseteq \mathbb{R}$ so that the function $g: \mathbb{R} \to B$, defined by g(x) = f(x) for all $x \in \mathbb{R}$, is onto.

(d) [1] Explain how to replace the domain \mathbb{R} of g with a set $A \subseteq \mathbb{R}$ so that the function $h: A \to B$, defined by h(x) = g(x) for all $x \in A$, is one-to-one and onto.

(e) [2] Find a formula for h^{-1} .

12. [3] Let a and b be integers and let p be a prime such that $gcd(a, p^2) = p$ and $gcd(b, p^3) = p^2$. Determine $gcd(ab, p^4)$.

13. [2] Use the Fundamental Theorem of Arithmetic to prove that every integer $n \geq 2$ is divisible by a prime number.

14. [3] Determine the last digit of 33^{66} .

15. [3] Find the positive integer b if $(122)_b = (203)_7$.

16. [5] Let a_n be the sequence recursively defined by $a_0 = 1$, $a_1 = -3$, $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \ge 2$. Use strong induction to show that $a_n = (-3)^n$ for all integers $n \ge 0$.

17. (a) [2] Assume that $1+2+\cdots+k=\frac{(k+(1/2))^2}{2}$ for some $k\geq 1$. Use this hypothesis to prove that

$$1 + 2 + \dots + (k+1) = \frac{((k+1) + (1/2))^2}{2}.$$

(b) [2] Is the statement $1+2+\cdots+n=\frac{(n+(1/2))^2}{2}$ true for all integers $n\geq 1$? Explain.

18. [2] Let a_1, a_2, a_3, \ldots be the sequence recursively defined by $a_1 = 1$ and, for n > 1, $a_n = 3a_{n-1} + 1$. Find the first 4 terms of the sequence and conjecture a formula for a_n . Do not prove it.

19. Let $A = \{a, b, c\}$ and $B = \{u, x, y, z\}$. Answer the following questions. No reasons are necessary.

- (a) [1] There are functions from A to B.
- (b) [1] There are 1-1 functions from A to B.
- (c) [2] There are functions f from A to B such that f(a) = x or f(a) = y.
- 20. Let $S = \{1, 2, ..., 1000\}.$
 - (a) [2] Explain why the number of integers in S divisible by 11 is $\lfloor 1000/11 \rfloor = 90$. State a general result in which 11 is replaced by an arbitrary positive integer b.

(b) [3] How many integers in S are divisible by at least one of the numbers 3, 5, 11?

(c) [2] How many integers in S relatively prime to 165?