202201 Math 122 [A01] Quiz #4

March 10th, 2022

Name: Solutions #V00: ____________

This test has 2 pages and 5 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except where explicitly noted, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Let $X = \{x_1, x_2, \ldots, x_{15}\}$. Fill in the blanks. You do not need to simplify your answer. No justification is needed.

   (a) The number of subsets of X that contain x_1 but not x_{15} equals 2^{13}.

   (b) The number of proper subsets of X that contain x_2 equals $2^{14}-1$.

   (c) The number of nonempty, proper subsets of X equals $2^{15}-2$.

   (d) The number of subsets of X that contain x_1 or x_2 (or both) equals $3 \cdot 2^{13}$. $= 2^{13}+2^{13}+2^{13}$ $= 2^{14}+2^{14}-2^{13}$

2. Consider the sequence $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$.

   (a) [2] Write a recursive definition for the sequence if it is $-7, -2, 3, 8, 13, 18, \ldots$.
   $\underbrace{-7,-2}_{+5}, \underbrace{-2,3}_{+5}, \underbrace{3,8}_{+5}$

   $a_0 = -7$

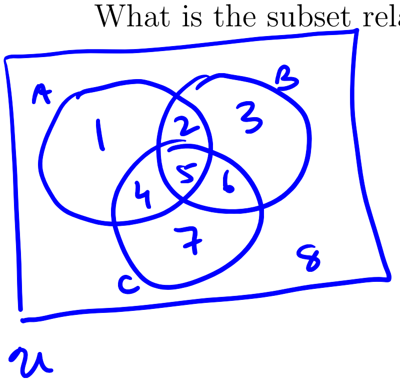
   $a_n = a_{n-1} + 5, \quad n \geq 1$

   (b) [1] Using your recursive definition in part (a) and assuming that $a_{50} = 243$, show how you would calculate a_{52}.

   $a_{51} = a_{50} + 5 = 243 + 5 = 248$

   $a_{52} = a_{51} + 5 = 248 + 5 = 253$

3. [3] Let A, B, and C be sets. Use a Venn diagram to create a counterexample to show that the statement $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$ is false.
   What is the subset relationship that is suggested here?



$u = \{1,2,3,4,5,6,7,8\}$

$A = \{1,2,4,5\}, \quad B = \{2,3,5,6\}, \quad C = \{4,5,6,7\}$

$A \setminus (B \cap C) = A \setminus \{5,6\} = \{1,2,4\}$

$(A \setminus B) \cap (A \setminus C) = \{1,4\} \cap \{1,2\} = \{1\}$

with this counterexample we see $A \setminus (B \cap C) \neq (A \setminus B) \cap (A \setminus C)$

suggessted subset relationship: $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cap C)$

4. [5] Use the principle of mathematical induction to prove that

$$2 + 4 + 6 + \cdots + 2n = n(n+1) \quad \text{for } n \geq 1.$$

$S(n)$: $2+4+\cdots+2n = n(n+1)$

Basis: $n=1$ LHS $=2$ RHS $= 1(2) = 2$

Induction Hypothesis: Assume $S(n)$ is true for some $n=k$, where $k \geq 1$.

That is, assume $2+4+6+\cdots+2k = k(k+1)$ for some $k \geq 1$.

Induction Step: Look at $n=k+1$.

$$\begin{aligned} \text{LHS} &= 2+4+6+\cdots+2(k+1) \\ &= \underbrace{2+4+6+\cdots+2k}+2(k+1) \\ &= k(k+1) + 2(k+1) \quad \text{by Ind Hyp.} \\ &= (k+1)(k+2) = \text{RHS} \end{aligned}$$

Conclusion:

By PMI, $2+4+6+\cdots+2n = n(n+1)$ for all $n \geq 1$. ∎

5. [2] Indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

F (a) For all sets A and B with $|A| = 17$ and $|B| = 10$, then $|A \cup B| = 27$. $|A \cup B| = |A| + |B| - |A \cap B|$ so only if $|A \cap B| = 0$

T (b) For the sequence recursively defined by $a_0 = 2$, $a_1 = 3$ and $a_n = 5a_{n-1} - 2a_{n-2}$ for $n \geq 2$, we have that $a_3 = 49$. $a_2 = 5a_1 - 2a_0 = 5(3) - 2(2) = 11$ $a_3 = 5a_2 - 2a_1 = 5(11) - 2(3) = 49$

F (c) If an open statement $S(n)$ is true for every $n \in \{1, 2, \ldots, k\}$, then $S(n)$ is true when $n = k+1$.

T (d) Suppose we're trying to show by induction that $S(n)$ is true for all $n \geq 1$ and the induction step where $n = k+1$ requires that $S(k-2)$ be known to be true. Then in the basis step we must check the truth of $S(n)$ for at least the values of $n = 1$, $n = 2$, and $n = 3$.

c) Only if you can show the induction step where $S(k) \Rightarrow S(k+1)$

d) Need 3 steps back, so need at least 3 cases in basis $(n=1, n=2, n=3)$