Unit:
Laplace Transform

Relationship Between the Laplace and Fourier Transforms

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Recall the definition of the Laplace transform in (7.2). Consider now the special case of (7.2) where $s = j\omega$ and ω is real (i.e., $\mathrm{Re}(s) = 0$). In this case, (7.2) becomes

$$\begin{aligned} X(j\omega) &= \left[\int_{-\infty}^{\infty} x(t) e^{-st} dt \right] \Bigg|_{s=j\omega} && \text{from definition of LT} \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt && \text{substitute jw for s} \\ &= \mathcal{F}x(\omega). && \text{from definition of FT} \end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at $s = j\omega$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(j\omega) = \mathcal{F}x(\omega). \tag{7.4}$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(j\omega)$. When this notation is used, the function X actually corresponds to the Laplace transform of x rather than its Fourier transform (i.e., the expression $X(j\omega)$ corresponds to the Laplace transform evaluated at points on the imaginary axis).

Relationship Between the Laplace and Fourier Transforms (General Case)

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Now, consider the general case of an arbitrary complex value for s in (7.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (7.2), we obtain

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt \\ &= \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \end{aligned}$$

substituting $\sigma + j\omega$ for s in LT definition

split exponential in two

definition of FT

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \tag{7.5}$$

Thus, the Laplace transform of x can be viewed as the (CT) Fourier transform of $x'(t) = e^{-\sigma t} x(t)$ (i.e., x weighted by a real exponential function).

Example 7.3. Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{0}^{\infty}. \end{aligned}$$

definition of LT

combine exponentials and use u to change limits

integrate

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{0}^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{0}^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)\infty}e^{-j\omega\infty} - 1\right]. \end{aligned}$$

real exponential
$\begin{cases} 0 & \sigma+a>0 \\ \infty & \sigma+a<0 \end{cases}$

complex sinusoid
finite but limit does not exist

factor and split exponentials

take difference

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\operatorname{Re}(s) > -a$). In this case, we have that

$$\begin{aligned} X(s) &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)[0-1] \\ &= \left(\tfrac{-1}{s+a}\right)(-1) \\ &= \frac{1}{s+a}. \end{aligned}$$

if Re(s) > -a

rewrite in terms of s ($s=\sigma+j\omega$)

simplify

Thus, we have that

$$e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) > -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.

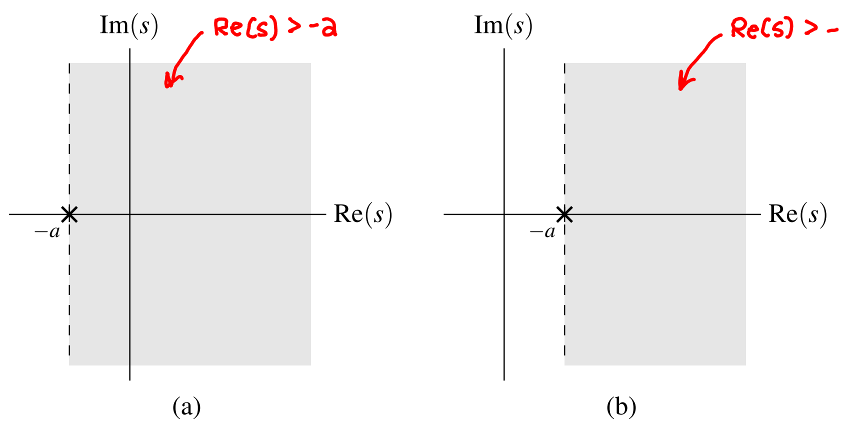

Figure 7.2: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

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