Solution. Due to the form of the formula for x, x is clearly N-periodic. Recalling the Fourier-series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Choosing to take the summation over $[0\,..\,N-1]$ and substituting the given formula for x, we have

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) e^{-j(2\pi/N)kn}.$$

From the sifting property of the δ sequence, we have

$$\begin{aligned} c_k &= \tfrac{1}{N} e^0 \\ &= \tfrac{1}{N}. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

where

$$c_k = \tfrac{1}{N} \quad \text{for all } k \in \mathbb{Z}.$$ ■

Example 10.3. Find the Fourier-series representation of the 5-periodic sequence x given by

$$x(n) = \begin{cases} -\frac{1}{2} & n = -1 \\ 1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & n \in \{-2, 0, 2\}. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Choosing to take the summation over $[-2\,..\,2]$ and substituting the given formula for x, we have

$$\begin{aligned} c_k &= \tfrac{1}{5} \sum_{n=-2}^{2} x(n) e^{-j(2\pi/5)kn} \\ &= \tfrac{1}{5} \left(-\tfrac{1}{2} e^{-j(2\pi/5)(-1)k} + e^{-j(2\pi/5)(0)k} + \tfrac{1}{2} e^{-j(2\pi/5)(1)k} \right). \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} c_k &= \tfrac{1}{5} \left(-\tfrac{1}{2} e^{j(2\pi/5)k} + 1 + \tfrac{1}{2} e^{-j(2\pi/5)k} \right) \\ &= \tfrac{1}{5} \left(1 - \tfrac{1}{2} (e^{j(2\pi/5)k} - e^{-j(2\pi/5)k}) \right) \\ &= \tfrac{1}{5} \left(1 - \tfrac{1}{2} \left[2j \sin\left(\tfrac{2\pi}{5} k\right) \right] \right) \\ &= \tfrac{1}{5} \left[1 - j \sin\left(\tfrac{2\pi}{5} k\right) \right]. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 5 \rangle} c_k e^{j(2\pi/5)kn},$$

where

$$c_k = \tfrac{1}{5}\left[1 - j\sin\left(\tfrac{2\pi}{5}k\right)\right] \quad \text{for all } k \in \mathbb{Z}. \qquad \blacksquare$$

Example 10.4. Find the Fourier-series representation of the 8-periodic sequence x given by

$$x(n) = \begin{cases} 1 & n \in [0\,..\,3] \\ 0 & n \in [4\,..\,7]. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Taking the summation over $[0\,..\,7]$, we have

$$\begin{aligned} c_k &= \tfrac{1}{8} \sum_{n=0}^{7} x(n) e^{-j(2\pi/8)kn} \\ &= \tfrac{1}{8} \sum_{n=0}^{3} e^{-j(\pi/4)kn} = \tfrac{1}{8} \sum_{n=0}^{3} \left(e^{-j(\pi/4)n}\right)^k. \end{aligned}$$

Using the formula for the sum of a geometric sequence, we have

$$c_k = \begin{cases} \tfrac{1}{8}\left(\dfrac{(e^{-j(\pi/4)k})^4 - 1}{e^{-j(\pi/4)k} - 1}\right) & k \neq 0 \\ \tfrac{1}{8} \displaystyle\sum_{k=0}^{3} 1 & k = 0. \end{cases}$$

Now, we must simplify this expression for c_k for the cases of $k \neq 0$ and $k = 0$. First, we consider the case of $k \neq 0$. We have

$$\begin{aligned} c_k &= \tfrac{1}{8}\left(\frac{(e^{-j(\pi/4)k})^4 - 1}{e^{-j(\pi/4)k} - 1}\right) = \tfrac{1}{8}\left(\frac{e^{-j\pi k} - 1}{e^{-j(\pi/4)k} - 1}\right) \\ &= \tfrac{1}{8}\left(\frac{e^{-j(\pi/2)k}\left[e^{-j(\pi/2)k} - e^{j(\pi/2)k}\right]}{e^{-j(\pi/8)k}\left[e^{-j(\pi/8)k} - e^{j(\pi/8)k}\right]}\right) = \tfrac{1}{8}\left(\frac{e^{-j(\pi/2)k}[2j\sin\left(-\frac{\pi}{2}k\right)]}{e^{-j(\pi/8)k}[2j\sin\left(-\frac{\pi}{8}k\right)]}\right) \\ &= \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}. \end{aligned} \tag{10.5}$$

Now, we consider the case of $k = 0$. We have

$$\begin{aligned} c_0 &= \tfrac{1}{8} \sum_{k=0}^{3} 1 = \tfrac{4}{8} \\ &= \tfrac{1}{2}. \end{aligned}$$

Furthermore, we observe that using (10.5) to compute c_k for $k = 0$ yields

$$\begin{aligned}
\left.\frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}\right|_{k=0} &= \left.\frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}k\right)}{8\left[\frac{3j\pi}{8}e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)+\frac{\pi}{8}\cos\left(\frac{\pi}{8}k\right)e^{j(3\pi/8)k}\right]}\right|_{k=0} \\
&= \frac{\frac{\pi}{2}}{8[0+\frac{\pi}{8}]} = \frac{\frac{\pi}{2}}{\pi} \\
&= \tfrac{1}{2}.
\end{aligned}$$

So, as it turns out, (10.5) happens to yield the correct result for $k = 0$. Consequently, c_k is given by (10.5) for all $k \in \mathbb{Z}$. Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 8\rangle} c_k e^{j(\pi/4)kn},$$

where

$$c_k = \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)} \quad \text{for all } k \in \mathbb{Z}. \qquad \blacksquare$$

Example 10.5. Find the Fourier-series representation of the 8-periodic sequence x given by

$$x(n) = \begin{cases} n & n \in [-2\,..\,2] \\ 0 & n = -3 \text{ or } n \in [3\,..\,4]. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn}.$$

Performing the summation over $[-3\,..\,4]$, we have

$$\begin{aligned}
c_k &= \tfrac{1}{8}\sum_{n=-3}^{4} x(n)e^{-j(2\pi/8)kn} \\
&= \tfrac{1}{8}\sum_{n=-2}^{2} ne^{-j(\pi/4)kn} \\
&= \tfrac{1}{8}\left[-2e^{-j(\pi/4)(-2)k} - e^{-j(\pi/4)(-1)k} + e^{-j(\pi/4)(1)k} + 2e^{-j(\pi/4)(2)k}\right] \\
&= \tfrac{1}{8}\left[-2e^{j(\pi/2)k} - e^{j(\pi/4)k} + e^{-j(\pi/4)k} + 2e^{-j(\pi/2)k}\right] \\
&= \tfrac{1}{8}\left[2e^{-j(\pi/2)k} - 2e^{j(\pi/2)k} + e^{-j(\pi/4)k} - e^{j(\pi/4)k}\right] \\
&= \tfrac{1}{8}\left[2\left(e^{-j(\pi/2)k} - e^{j(\pi/2)k}\right) + \left(e^{-j(\pi/4)k} - e^{j(\pi/4)k}\right)\right] \\
&= \tfrac{1}{8}\left[4j\sin\left(-\tfrac{\pi}{2}k\right) + 2j\sin\left(-\tfrac{\pi}{4}k\right)\right] \\
&= \tfrac{1}{8}\left[-4j\sin\left(\tfrac{\pi}{2}k\right) - 2j\sin\left(\tfrac{\pi}{4}k\right)\right] \\
&= -\tfrac{j}{2}\sin\left(\tfrac{\pi}{2}k\right) - \tfrac{j}{4}\sin\left(\tfrac{\pi}{4}k\right) \\
&= -j\left[\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}k\right) + \tfrac{1}{4}\sin\left(\tfrac{\pi}{4}k\right)\right].
\end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 8\rangle} c_k e^{j(\pi/4)kn},$$

where

$$c_k = -j\left[\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}k\right) + \tfrac{1}{4}\sin\left(\tfrac{\pi}{4}k\right)\right] \quad \text{for all } k \in \mathbb{Z}. \qquad \blacksquare$$

Example 10.6 (Fourier series of an even real sequence)**.** Let x be an even real N-periodic sequence with the Fourier-series coefficient sequence c. Show that

- c is real (i.e., $\operatorname{Im}\{c_k\} = 0$ for all $k \in \mathbb{Z}$);
- c is even (i.e., $c_k = c_{-k}$ for all $k \in \mathbb{Z}$); and
- $c_0 = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)$.

Solution. From the Fourier-series analysis equation (10.2) and using Euler's relation, we can write

$$\begin{aligned} c_k &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \\ &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(-\tfrac{2\pi}{N}kn\right) + j\sin\left(-\tfrac{2\pi}{N}kn\right)\right] \\ &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]. \end{aligned}$$

At this point, we split the further manipulation of the preceding equation into two cases: 1) N is even and 2) N is odd.

First, we consider the case of even N. We have

$$c_k = \tfrac{1}{N}\sum_{n=-(N/2-1)}^{N/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into four parts (i.e., for n in $\left[-(\frac{N}{2}-1)\,..\,-1\right]$, $\{0\}$, $\left[1\,..\,\frac{N}{2}-1\right]$, and $\left\{\frac{N}{2}\right\}$), we obtain

$$\begin{aligned} c_k &= \tfrac{1}{N}\Bigg[x(0)[\cos(0) - j\sin(0)] + \sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] \\ &\qquad + x\left(\tfrac{N}{2}\right)[\cos(\pi k) - j\sin(\pi k)] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\Bigg] \\ &= \tfrac{1}{N}\Bigg[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + \sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\Bigg]. \end{aligned}$$

Performing a change of variable in the first summation and using the fact that x and cos are even and sin is odd, we have

$$c_k = \tfrac{1}{N}\Bigg[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\Bigg].$$

Combining the two summations and simplifying, we obtain

$$c_k = \tfrac{1}{N}\left[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + 2\sum_{n=1}^{N/2-1} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right].$$

Next, we consider the case of odd N. We have

$$c_k = \tfrac{1}{N}\sum_{n=-(N-1)/2}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into three parts (i.e., for n in $\left[-\frac{N-1}{2} \,..\, -1\right]$, $\{0\}$, and $\left[1 \,..\, \frac{N-1}{2}\right]$), we obtain

$$c_k = \frac{1}{N}\left[x(0) + \sum_{n=-(N-1)/2}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Performing a change of variable in the first summation and using the fact that x and cos are even and sin is odd, we have

$$c_k = \frac{1}{N}\left[x(0) + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[x(0) + 2\sum_{n=1}^{(N-1)/2} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right].$$

Combining the results for the cases of N even and N odd, we have

$$c_k = \begin{cases} \frac{1}{N}\left[x(0) + (-1)^k x\left(\frac{N}{2}\right) + 2\sum_{n=1}^{N/2-1} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right] & N \text{ even} \\ \frac{1}{N}\left[x(0) + 2\sum_{n=1}^{(N-1)/2} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right] & N \text{ odd.} \end{cases}$$

Since x and cos are real, the quantity c_k must also be real. Thus, we have that $\operatorname{Im}(c_k) = 0$. Also, since $(-1)^k$ and $\cos\left(\frac{2\pi}{N}kn\right)$ are even, c is even.

Consider now the quantity c_0. Substituting $k = 0$ into the Fourier-series analysis equation, we have

$$\begin{aligned} c_0 &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{j(2\pi/N)kn}\Big|_{k=0} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n). \end{aligned}$$

Thus, c_0 has the stated value. ■

Example 10.7 (Fourier series of an odd real sequence)**.** Let x be an odd real N-periodic sequence with the Fourier-series coefficient sequence c. Show that

- c is purely imaginary (i.e., $\operatorname{Re}\{c_k\} = 0$ for all $k \in \mathbb{Z}$);
- c is odd (i.e., $c_k = -c_{-k}$ for all $k \in \mathbb{Z}$); and
- $c_0 = 0$.

Solution. From the Fourier-series analysis equation (10.2) and using Euler's relation, we can write

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(-\tfrac{2\pi}{N}kn\right) + j\sin\left(-\tfrac{2\pi}{N}kn\right)\right] \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]. \end{aligned}$$

At this point, we split the further manipulation of the preceding equation into two cases: 1) N is even and 2) N is odd.

First, we consider the case of even N. We have

$$c_k = \frac{1}{N} \sum_{n=-(N/2-1)}^{N/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into four parts (i.e., for n in $\left[-(\frac{N}{2}-1) \mathinner{..} -1\right]$, $\{0\}$, $\left[1 \mathinner{..} \frac{N}{2}-1\right]$, and $\left\{\frac{N}{2}\right\}$) and using the fact that $x(0)=0$ and $x\left(\frac{N}{2}\right)=0$, we obtain

$$\begin{aligned} c_k &= \frac{1}{N}\Bigg[x(0)[\cos 0 - j\sin 0] + \sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] \\ &\qquad x\left(\tfrac{N}{2}\right)[\cos(\pi k) - j\sin(\pi k)] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\Bigg] \\ &= \frac{1}{N}\left[\sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right]. \end{aligned}$$

(Note that $x(0) = x\left(\frac{N}{2}\right) = 0$ due to Theorem 8.3.) Performing a change of variable in the first summation and using the fact that x and sin are odd and cos is even, we have

$$c_k = \frac{1}{N}\left[-\sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[-2j\sum_{n=1}^{N/2-1} x(n)\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Next, we consider the case of odd N. We have

$$c_k = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into three parts (i.e., for n in $\left[-\frac{(N-1)}{2} \mathinner{..} -1\right]$, $\{0\}$, and $\left[1 \mathinner{..} \frac{N-1}{2}\right]$) and using the fact that $x(0)=0$, we obtain

$$c_k = \frac{1}{N}\left[\sum_{n=-(N-1)/2}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Performing a change of variable in the first summation and using the fact that x and sin are odd and cos is even, we have

$$c_k = \frac{1}{N}\left[-\sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[-2j\sum_{n=1}^{(N-1)/2} x(n)\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Combining the above results, we have

$$c_k = \begin{cases} -\frac{2j}{N} \displaystyle\sum_{n=1}^{N/2-1} x(n)\sin\left(\frac{2\pi}{N}kn\right) & N \text{ even} \\ -\frac{2j}{N} \displaystyle\sum_{n=1}^{(N-1)/2} x(n)\sin\left(\frac{2\pi}{N}kn\right) & N \text{ odd.} \end{cases}$$

Since x and sin are real, the quantity c_k must be imaginary. Thus, we have that $\operatorname{Re}(c_k) = 0$. Since sin is odd, c is odd.

Consider now the quantity c_0. Substituting $k = 0$ into the above formula for c_k, we have

$$c_0 = 0.$$

Thus, c_0 has the stated value. ■

10.4 Comments on Convergence of Discrete-Time Fourier Series

Since the analysis and synthesis equations for (DT) Fourier series involve only finite sums (as opposed to infinite series), convergence is not a significant concern. If an N-periodic sequence is bounded (i.e., is finite in value), its Fourier-series coefficient sequence will exist and be bounded and the Fourier-series analysis and synthesis equations must converge.

10.5 Properties of Discrete-Time Fourier Series

Fourier-series representations possess a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also summarized later in Table 10.1 (on page 422).

10.5.1 Linearity

Arguably, the most important property of Fourier series is linearity, as introduced below.

Theorem 10.2 (Linearity)**.** *Let x and y denote two N-periodic sequences. If*

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k,$$

then

$$Ax(n) + By(n) \overset{\text{DTFS}}{\longleftrightarrow} Aa_k + Bb_k,$$

where A and B are complex constants. In other words, a linear combination of sequences produces the same linear combination of their Fourier-series coefficients.

Proof. To prove the above property, we proceed as follows. First, we express x and y in terms of their corresponding Fourier series as

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \quad \text{and} \quad y(n) = \sum_{k=\langle N\rangle} b_k e^{jk(2\pi/N)n}.$$

Now, we determine the Fourier series of $Ax + By$. Using the Fourier-series representations of x and y, we have

$$\begin{aligned} Ax(n) + By(n) &= A\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} + B\sum_{k=\langle N\rangle} b_k e^{jk(2\pi/N)n} \\ &= \sum_{k=\langle N\rangle} Aa_k e^{jk(2\pi/N)n} + \sum_{k=\langle N\rangle} Bb_k e^{jk(2\pi/N)n} \\ &= \sum_{k=\langle N\rangle} (Aa_k + Bb_k) e^{jk(2\pi/N)n}. \end{aligned}$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with coefficient sequence $c'_k = Aa_k + Bb_k$. Therefore, we have that $Ax(n) + By(n) \overset{\text{DTFS}}{\longleftrightarrow} Aa_k + Bb_k$. ∎

10.5.2 Translation (Time Shifting)

The next property of Fourier series to be introduced is the translation (i.e., time-shifting) property, as given below.

Theorem 10.3 (Translation (i.e., time shifting))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$x(n-n_0) \overset{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} a_k,$$

where n_0 is an integer constant.

Proof. To prove the translation property, we proceed as follows. The Fourier series of x is given by

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

Substituting $n - n_0$ for n in this equation, we have

$$\begin{aligned} x(n-n_0) &= \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)(n-n_0)} \\ &= \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} e^{-jk(2\pi/N)n_0} \\ &= \sum_{k=\langle N\rangle} \left(a_k e^{-jk(2\pi/N)n_0}\right) e^{jk(2\pi/N)n}. \end{aligned}$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with coefficient sequence $a'_k = a_k e^{-jk(2\pi/N)n_0}$. Therefore, we have that $x(n-n_0) \overset{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} a_k$. ∎

From the above theorem, we can see that time shifting a periodic sequence does not change the magnitude of its Fourier-series coefficients (since $\left|e^{j\theta}\right| = 1$ for all real θ).

10.5.3 Modulation (Frequency Shifting)

The next property of Fourier series to be introduced is the modulation (i.e., frequency-shifting) property, as given below.

Theorem 10.4 (Modulation (i.e., frequency shifting))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$e^{jM(2\pi/N)n} x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_{k-M},$$

where M is an integer constant.

Proof. To prove the modulation property, we proceed as follows. From the definition of Fourier series, we have

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)kn}.$$

Multiplying both sides of this equation by $e^{j(2\pi/N)Mn}$, we can write

$$\begin{aligned} e^{j(2\pi/N)Mn}x(n) &= e^{j(2\pi/N)Mn} \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)Mn} e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)(k+M)n}. \end{aligned}$$

Now, we employ a change of variable. Let $\ell = k+M$ so that $k = \ell - M$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta+N-1]$ becomes a summation over $\ell \in [-(\eta+N-1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying the change of variable, we obtain

$$e^{j(2\pi/N)Mn}x(n) = \sum_{\ell=\langle N\rangle} a_{\ell-M} e^{j(2\pi/N)\ell n}.$$

Now, we observe that the right-hand side of this equation is a Fourier series with the coefficient sequence $a'_\ell = a_{\ell-M}$. So, the Fourier-series coefficient sequence for $e^{j(2\pi/N)Mn}x(n)$ (i.e., the left-hand side of the equation) is $a'_k = a_{k-M}$. Therefore, we have that $e^{jM(2\pi/N)n}x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_{k-M}$. ■

10.5.4 Reflection (Time Reversal)

The next property of Fourier series to be introduced is the reflection (i.e., time-reversal) property, as given below.

Theorem 10.5 (Reflection (i.e., time reversal))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$x(-n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_{-k}.$$

Proof. To prove the time-reversal property, we proceed in the following manner. The Fourier series of x is given by

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

Substituting $-n$ for n yields

$$x(-n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)(-n)}.$$

Now, we employ a change of variable. Let $\ell = -k$ so that $k = -\ell$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta+N-1]$ becomes a summation over $\ell \in [-(\eta+N-1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying this change of variable, we have

$$\begin{aligned} x(-n) &= \sum_{\ell=\langle N\rangle} a_{-\ell} e^{j(-\ell)(2\pi/N)(-n)} \\ &= \sum_{\ell=\langle N\rangle} a_{-\ell} e^{j\ell(2\pi/N)n}. \end{aligned}$$

Now, we observe that the right-hand side of this equation is a Fourier series with the coefficient sequence $a'_\ell = a_{-\ell}$. Therefore, we have that $x(-n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_{-k}$. ■

In other words, the above theorem states that time reversing a sequence time reverses the corresponding sequence of Fourier-series coefficients.

10.5.5 Conjugation

The next property of Fourier series to be introduced is the conjugation property, as given below.

Theorem 10.6 (Conjugation)**.** *For an N-periodic sequence x with Fourier-series coefficient sequence c,*

$$x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}.$$

Proof. From the definition of Fourier series, we have

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{jk(2\pi/N)n}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(n) &= \left(\sum_{k=\langle N \rangle} c_k e^{jk(2\pi/N)n} \right)^* \\ &= \sum_{k=\langle N \rangle} \left(c_k e^{jk(2\pi/N)n} \right)^* \\ &= \sum_{k=\langle N \rangle} c^*_k e^{-jk(2\pi/N)n}. \end{aligned}$$

Now, we employ a change of variable. Let $\ell = -k$ so that $k = -\ell$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta + N - 1]$ becomes a summation over $\ell \in [-(\eta + N - 1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying the change of variable, we obtain

$$x^*(n) = \sum_{\ell=\langle N \rangle} c^*_{-\ell} e^{j\ell(2\pi/N)n}.$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with the coefficient sequence $c'_\ell = c^*_{-\ell}$. So, the Fourier-series coefficient sequence c' of x^* (i.e., the left-hand side of the equation) is $c'_k = c^*_{-k}$. Therefore, we have shown that $x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}$. ■

In other words, the above theorem states that conjugating a sequence has the effect of time reversing and conjugating the corresponding Fourier-series coefficient sequence.

10.5.6 Duality

The next property of Fourier series to be introduced is the duality property, as given below.

Theorem 10.7 (Duality)**.** *Let x be an N-periodic sequence with the corresponding Fourier-series coefficient sequence a. Then,*

$$a_n \overset{\text{DTFS}}{\longleftrightarrow} \tfrac{1}{N} x(-k).$$

Proof. From the definition of Fourier series, we have

$$x(k) = \sum_{\ell=\langle N \rangle} a_\ell e^{j(2\pi/N)\ell k}.$$

Substituting $-k$ for k, we have

$$\begin{aligned} x(-k) &= \sum_{\ell=\langle N \rangle} a_\ell e^{j(2\pi/N)\ell(-k)} \\ &= \sum_{\ell=\langle N \rangle} a_\ell e^{-j(2\pi/N)\ell k}. \end{aligned}$$

Multiplying both sides of the equation by $\frac{1}{N}$, we obtain

$$\tfrac{1}{N}x(-k) = \tfrac{1}{N} \sum_{\ell=\langle N \rangle} a_\ell e^{-j(2\pi/N)\ell k}.$$

Now, we observe that the right-hand side is the (Fourier-series analysis) formula for computing the kth Fourier-series coefficient of a. Thus, the duality property holds. ■

The duality property stated in the preceding theorem follows from the high degree of similarity in the equations for the Fourier-series analysis and synthesis equations, given by (10.2) and (10.1), respectively. To make this similarity more obvious, we can rewrite the Fourier-series analysis and synthesis equations, respectively, as

$$X(m) = \tfrac{1}{N} \sum_{\ell=\langle N \rangle} x(\ell) e^{-j(2\pi/N)\ell m} \quad \text{and} \quad x(m) = \sum_{\ell=\langle N \rangle} X(\ell) e^{j(2\pi/N)\ell m}.$$

Observe that these two equations are identical except for: 1) a factor of N; and 2) a different sign in the parameter for the exponential function. Consequently, if we were to accidentally use one equation in place of the other, we would obtain an almost correct result. In fact, this almost correct result could be made to be correct by compensating for the above two differences (i.e., the factor of N and the sign difference in the exponential function). This is, in effect, what the duality property states.

Although the relationship $x(n) \overset{\text{DTFS}}{\longleftrightarrow} X(k)$ only directly provides us with the Fourier-series coefficient sequence X of the sequence x, the duality property allows us to indirectly infer the Fourier-series coefficient sequence of X. Consequently, the duality property can be used to effectively double the number of Fourier-series relationships that we know.

10.5.7 Periodic Convolution

The next property of Fourier series to be introduced is the periodic-convolution property, as given below.

Theorem 10.8 (Periodic convolution). *Let x and y be N-periodic sequences with their respective Fourier-series representations given by*

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn} \quad \textit{and} \quad y(n) = \sum_{k=\langle N \rangle} b_k e^{j(2\pi/N)kn}.$$

Let $z(n) = x \circledast y(n)$, where z has the Fourier-series representation

$$z(n) = \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}.$$

The sequences a, b, and c are related as

$$c_k = N a_k b_k.$$

Proof. From the definition of periodic convolution, we have

$$x \circledast y(n) = \sum_{m=\langle N \rangle} x(m) y(n-m).$$

Expanding x and y in terms of their Fourier-series representations, we have

$$x \circledast y(n) = \sum_{m=\langle N \rangle} \left(\sum_{\ell=\langle N \rangle} a_\ell e^{j(2\pi/N)\ell m} \right) \left(\sum_{k=\langle N \rangle} b_k e^{j(2\pi/N)k(n-m)} \right).$$

Changing the order of summations and rearranging, we have

$$\begin{aligned} x \circledast y(n) &= \sum_{m=\langle N \rangle} \sum_{k=\langle N \rangle} \sum_{\ell=\langle N \rangle} a_\ell b_k e^{j(2\pi/N)\ell m} e^{j(2\pi/N)k(n-m)} \\ &= \sum_{m=\langle N \rangle} \sum_{k=\langle N \rangle} \sum_{\ell=\langle N \rangle} a_\ell b_k e^{j(2\pi/N)kn} e^{j(2\pi/N)(\ell-k)m} \\ &= \sum_{k=\langle N \rangle} \sum_{\ell=\langle N \rangle} a_\ell b_k e^{j(2\pi/N)kn} \sum_{m=\langle N \rangle} e^{j(2\pi/N)(\ell-k)m}. \end{aligned}$$

Taking the two outermost summations over $[0\,..\,N-1]$, we have

$$x \circledast y(n) = \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} a_\ell b_k e^{j(2\pi/N)kn} \sum_{m=\langle N \rangle} e^{j(2\pi/N)(\ell-k)m}.$$

Now, we observe that

$$\sum_{m=\langle N \rangle} e^{j(2\pi/N)(\ell-k)m} = \begin{cases} N & (\ell-k)/N \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

(The proof of this fact is left as an exercise for the reader in Exercise A.9.) Moreover, since $\ell - k \in [-(N-1)\,..\,N-1]$, $(\ell-k)/N \in \mathbb{Z}$ implies that $\ell - k = 0$ (i.e., $\ell = k$). Using these facts, we can simplify the above expression for $x \circledast y$ to obtain

$$\begin{aligned} x \circledast y(n) &= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} a_\ell b_k e^{j(2\pi/N)kn} N\delta(\ell-k) \\ &= \sum_{k=0}^{N-1} a_k b_k e^{j(2\pi/N)kn} N \\ &= \sum_{k=0}^{N-1} N a_k b_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N \rangle} N a_k b_k e^{j(2\pi/N)kn}. \end{aligned}$$

Now, we simply observe that the right-hand side of the preceding equation is a Fourier series with the coefficient sequence $c_k = Na_k b_k$. Therefore, the Fourier-series coefficient sequence c of $x \circledast y$ (i.e., the left-hand side of the equation) is given by $c_k = Na_k b_k$. ■

10.5.8 Multiplication

The next property of Fourier series to be considered is the multiplication property, as given below.

Theorem 10.9 (Multiplication)**.** *Let x and y be N-periodic sequences given by*

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn} \quad \text{and} \quad y(n) = \sum_{k=\langle N \rangle} b_k e^{j(2\pi/N)kn}.$$

Let $z(n) = x(n)y(n)$, where z has the Fourier-series representation

$$z(n) = \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}.$$

The sequences a, b, and c are related as

$$c_k = \sum_{n=\langle N \rangle} a_n b_{k-n} \quad (\text{i.e., } c = a \circledast b).$$

Proof. From the Fourier-series analysis equation, we can write

$$c_k = \frac{1}{N} \sum_{\ell=\langle N \rangle} x(\ell) y(\ell) e^{-j(2\pi/N)k\ell}.$$

Replacing x by its Fourier-series representation, we obtain

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{\ell=\langle N \rangle} \left(\sum_{n=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} \right) y(\ell) e^{-j(2\pi/N)k\ell} \\ &= \frac{1}{N} \sum_{\ell=\langle N \rangle} \sum_{n=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} y(\ell) e^{-j(2\pi/N)k\ell}. \end{aligned}$$

Reversing the order of the two summations and rearranging, we have

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{n=\langle N \rangle} \sum_{\ell=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} y(\ell) e^{-j(2\pi/N)k\ell} \\ &= \sum_{n=\langle N \rangle} a_n \left(\frac{1}{N} \sum_{\ell=\langle N \rangle} y(\ell) e^{-j(2\pi/N)(k-n)\ell} \right). \end{aligned}$$

Observing that the expression on the preceding line in the large pair of parenthesis is simply the formula for computing the $(k-n)$th Fourier-series coefficient of y, we conclude

$$c_k = \sum_{n=\langle N \rangle} a_n b_{k-n}. \qquad \blacksquare$$

10.5.9 Parseval's Relation

Another important property of Fourier series relates to the energy of sequences, as given by the theorem below.

Theorem 10.10 (Parseval's relation)**.** *An N-periodic sequence x and its corresponding Fourier-series coefficient sequence c satisfy the relationship*

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |c_k|^2$$

(i.e., the energy in x and the energy in c are equal up to a scale factor).

Proof. Let x, y, and z denote N-periodic sequences with the Fourier series given by

$$\begin{aligned} x(n) &= \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn}, \\ y(n) &= \sum_{k=\langle N \rangle} b_k e^{j(2\pi/N)kn}, \quad \text{and} \\ z(n) = x(n)y(n) &= \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}. \end{aligned}$$

From the multiplication property of Fourier series (i.e., Theorem 10.9), we have

$$c_k = \sum_{n=\langle N \rangle} a_n b_{k-n}. \tag{10.6}$$

Now, let $y(n) = x^*(n)$ so that $z(n) = x(n)x^*(n) = |x(n)|^2$. From the conjugation property of Fourier series (i.e., Theorem 10.6), since $y(n) = x^*(n)$, we know

$$b_k = a^*_{-k}.$$

So, we can rewrite (10.6) as

$$\begin{aligned} c_k &= \sum_{n=\langle N \rangle} a_n a^*_{-(k-n)} \\ &= \sum_{n=\langle N \rangle} a_n a^*_{n-k}. \end{aligned} \tag{10.7}$$

From the Fourier-series analysis equation, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 e^{-j(2\pi/N)kn}. \tag{10.8}$$

Equating (10.7) and (10.8), we obtain

$$\tfrac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 e^{-j(2\pi/N)kn} = \sum_{n=\langle N \rangle} a_n a^*_{n-k}.$$

Letting $k = 0$ in the preceding equation yields

$$\tfrac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{n=\langle N \rangle} a_n a^*_n = \sum_{n=\langle N \rangle} |a_n|^2. \qquad \blacksquare$$

The above theorem is simply stating that the amount of energy in x and the amount of energy in the Fourier-series coefficient sequence c are equal up to a scale factor. In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

10.5.10 Even/Odd Symmetry

Next, we consider the relationship between Fourier series and even/odd symmetry. As it turns out, Fourier series preserve signal symmetry. In other words, we have the result below.

Theorem 10.11 (Even/odd symmetry)**.** *For an N-periodic sequence x with Fourier-series coefficient sequence c, the following properties hold:*

$$\begin{gathered} x \text{ is even if and only if } c \text{ is even}; \quad \text{and} \\ x \text{ is odd if and only if } c \text{ is odd}. \end{gathered}$$

Proof. The proof is left as an exercise for the reader in Exercise 10.3. ■

In other words, the above theorem states that the even/odd symmetry properties of x and c always match (i.e., Fourier series preserve symmetry).

10.5.11 Real Sequences

Consider the Fourier-series representation of the periodic sequence x given by (10.1). In the most general case, x is a complex-valued sequence, but let us now suppose that x is real-valued. In the case of real-valued sequences, an important relationship exists between the Fourier-series coefficients c_k and c_{-k} as given by the theorem below.

Theorem 10.12 (Fourier series of a real-valued sequence)**.** *Let x be a periodic sequence with Fourier-series coefficient sequence c. The sequence x is real-valued if and only if*

$$c_k = c^*_{-k} \text{ for all } k \in \mathbb{Z} \tag{10.9}$$

(i.e., c is conjugate symmetric).

Proof. First, we show that x being real-valued implies that c is conjugate symmetric. Assume that x is real-valued. From the Fourier series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Substituting $-k$ for k in the preceding equation and taking the conjugate of both sides, we obtain

$$\begin{aligned} c_{-k}^* &= \left(\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)(-k)n} \right)^* \\ &= \left(\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{j(2\pi/N)kn} \right)^* \\ &= \frac{1}{N} \sum_{n=\langle N \rangle} x^*(n) e^{-j(2\pi/N)kn}. \end{aligned}$$

Since x is real-valued (i.e., $x^* = x$), we have

$$\begin{aligned} c_{-k}^* &= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} \\ &= c_k. \end{aligned}$$

Thus, c is conjugate symmetric. Therefore, x being real-valued implies that c is conjugate symmetric.

Now, we show that c being conjugate symmetric implies that x is real-valued. Assume that c is conjugate symmetric. From the Fourier series synthesis equation, we have

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(n) &= \left(\sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn} \right)^* \\ &= \sum_{k=\langle N \rangle} \left(c_k e^{j(2\pi/N)kn} \right)^* \\ &= \sum_{k=\langle N \rangle} c_k^* e^{-j(2\pi/N)kn} \\ &= \sum_{k=N_0}^{N_0+N-1} c_k^* e^{-j(2\pi/N)kn}. \end{aligned}$$

Now, we employ a change of variable. Let $k' = -k$ so that $k = -k'$. Applying the change of variable and dropping the primes, we obtain

$$\begin{aligned} x^*(n) &= \sum_{k=-N_0-N+1}^{-N_0} c_{-k}^* e^{-j(2\pi/N)(-k)n} \\ &= \sum_{k=-N_0-N+1}^{-N_0} c_{-k}^* e^{j(2\pi/N)kn}. \end{aligned}$$

Since each term in the summation on the right-hand side of the preceding equation is N-periodic in k, the summation can be taken over any N consecutive integers. Thus, we have

$$x^*(n) = \sum_{k=\langle N \rangle} c_{-k}^* e^{j(2\pi/N)kn}.$$

Since $c_k = c^*_{-k}$ for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} x^*(n) &= \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn} \\ &= x(n). \end{aligned}$$

Thus, x is real-valued. Therefore, c being conjugate symmetric implies that x is real-valued. This completes the proof. ■

Using the relationship in (10.9), we can derive an alternative form of the Fourier series for the case of real-valued sequences. In particular, the Fourier series of a sequence x can be expressed as

$$x(n) = \begin{cases} \alpha_0 + \displaystyle\sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\tfrac{2\pi}{N}kn\right) + \beta_k \sin\left(\tfrac{2\pi}{N}kn\right)\right] + \alpha_{N/2}\cos(\pi n) & N \text{ even} \\ \alpha_0 + \displaystyle\sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\tfrac{2\pi}{N}kn\right) + \beta_k \sin\left(\tfrac{2\pi}{N}kn\right)\right] & N \text{ odd,} \end{cases}$$

where

$$\alpha_0 = a_0, \quad \alpha_{N/2} = a_{N/2}, \quad \alpha_k = 2\operatorname{Re} a_k, \quad \text{and} \quad \beta_k = -2\operatorname{Im} a_k.$$

This is known as the **trigonometric form** of a Fourier series. Note that the trigonometric form of a Fourier series only involves real quantities, whereas the exponential form involves some complex quantities. For this reason, the trigonometric form may sometimes be preferred when dealing with Fourier series of real-valued sequences.

As noted earlier in Theorem 10.12, the Fourier series of a real-valued sequence has a special structure. In particular, a sequence x is real-valued if and only if its Fourier-series coefficient sequence c is conjugate symmetric (i.e., $c_k = c^*_{-k}$ for all k). From properties of complex numbers, one can show that

$$c_k = c^*_{-k} \text{ for all } k$$

is equivalent to

$$|c_k| = |c_{-k}| \text{ for all } k \quad \text{and} \quad \arg c_k = -\arg c_{-k} \text{ for all } k$$

(i.e., $|c_k|$ is even and $\arg c_k$ is odd). Note that x being real-valued does *not* necessarily imply that c is real-valued.

As it turns out, the Fourier-series coefficient sequence corresponding to a real-valued sequence has a number of other special properties, as given by the theorem below.

Theorem 10.13 (Fourier series of a real-valued sequence)**.** *Let x denote a real-valued N-periodic sequence with the corresponding Fourier-series coefficient sequence c. Then, the following assertions can be made about c_k for the single period of c corresponding to $k \in [0\,..\,N-1]$:*

1. *$c_k = c^*_{N-k}$ for $k \in [1\,..\,N-1]$;*
2. *of the N coefficients c_k for $k \in [0\,..\,N-1]$, only $\left\lfloor \frac{N}{2} \right\rfloor + 1$ coefficients are independent; for example, c_k for $k \in \left[0\,..\,\left\lfloor \frac{N}{2} \right\rfloor\right]$ completely determines c_k for all $k \in [0\,..\,N-1]$;*
3. *c_0 is real; and*
4. *if N is even, $c_{N/2}$ is real.*

Proof. From Theorem 10.12, we have

$$c_k = c^*_{-k} \quad \text{for all } k \in \mathbb{Z}. \tag{10.10}$$

Since c is N-periodic, $c_{-k} = c_{N-k}$ and we can rewrite the above equation as

$$c_k = c^*_{N-k} \quad \text{for all } k \in \mathbb{Z}. \tag{10.11}$$

This proves assertion 1 of the theorem.

Evaluating (10.10) at $k = 0$, we obtain

$$c_0 = c^*_0.$$

Thus, c_0 must be real. This proves assertion 3 of the theorem.

Evaluating (10.11) at $k = \frac{N}{2}$ where $\frac{N}{2} \in \mathbb{Z}$, we obtain

$$\begin{aligned} c_{N/2} &= c^*_{N-N/2}. \\ &= c^*_{N/2}. \end{aligned}$$

Thus, $c_{N/2}$ must be real. This proves assertion 4 of the theorem.

Evaluating (10.11) for $k \in \left[\left\lfloor \frac{N}{2} \right\rfloor + 1 \,.\,. N-1\right]$, we have

$$\begin{aligned} c_{\lfloor N/2 \rfloor + 1} &= c^*_{N-(\lfloor N/2 \rfloor+1)} = c^*_{N-\lfloor N/2 \rfloor - 1} = c^*_{N+\lceil -N/2 \rceil - 1} = c^*_{\lceil N-1-N/2 \rceil} \\ &= c^*_{\lceil N/2-1 \rceil} = c^*_{\lceil N/2 \rceil - 1} = c^*_{\lfloor (N-1)/2 \rfloor + 1 - 1} \\ &= c^*_{\lfloor (N-1)/2 \rfloor}, \\ &\vdots \\ c_{N-3} &= c^*_{N-(N-3)} \\ &= c^*_3, \\ c_{N-2} &= c^*_{N-(N-2)} \\ &= c^*_2, \quad \text{and} \\ c_{N-1} &= c^*_{N-(N-1)} \\ &= c^*_1. \end{aligned}$$

(Note that, in the simplification of the expression for $c_{\lfloor N/2 \rfloor + 1}$ above, we used some properties of the floor and ceiling functions introduced in Section 3.5.10.) Thus, c_k for $k \in \left[\left\lfloor \frac{N}{2} \right\rfloor + 1 \,.\,. N-1\right]$ is completely determined from c_k for $k \in \left[0 \,.\,. \left\lfloor \frac{N-1}{2} \right\rfloor\right]$. Therefore, only $\left\lfloor \frac{N}{2} \right\rfloor + 1$ of the N coefficients c_k for $k \in [0 \,.\,. N-1]$ are independent. This proves assertion 2 of the theorem. ■

The above theorem (i.e., Theorem 10.13) shows that the Fourier-series coefficient sequence c corresponding to a real-valued sequence has a high degree of redundancy. In particular, approximately half of the coefficients of c over a single period of c are redundant.

10.6 Discrete Fourier Transform (DFT)

Closely related to DT Fourier series is the discrete Fourier transform (DFT) (not to be confused with the discrete-time Fourier transform, which is introduced later in Chapter 11). The DFT is essentially a slightly modified definition of (DT) Fourier series that is typically useful in the context of the DT Fourier transform (to be discussed later). In what follows, we briefly introduce the DFT.

Consider the Fourier-series synthesis and analysis equations given by

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{jk(2\pi/N)n} \quad \text{and} \quad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Table 10.1: Properties of DT Fourier series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n) + \beta y(n)$	$\alpha a_k + \beta b_k$
Translation	$x(n - n_0)$	$e^{-jk(2\pi/N)n_0} a_k$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N} x(-k)$
Periodic Convolution	$x \circledast y(n)$	$N a_k b_k$
Multiplication	$x(n)y(n)$	$a \circledast b_k$

Property	
Parseval's Relation	$\frac{1}{N} \sum\limits_{n=\langle N \rangle} \lvert x(n) \rvert^2 = \sum\limits_{k=\langle N \rangle} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Now let us define a sequence a' as $a'_k = Na_k$. In other words, a' is just a renormalized (i.e., scaled) version of the original Fourier-series coefficient sequence a. Rewriting the Fourier-series synthesis and analysis equations from above in terms of a' (instead of a) with both of the summations taken over $[0\,..\,N-1]$, we obtain

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{and} \quad a'_k = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}.$$

Since x and a' are both N-periodic, each of these sequences is completely characterized by its N samples over a single period. If we only consider the behavior of x and a' over a single period, this leads to the equations

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{for } n \in [0\,..\,N-1] \quad \text{and}$$
$$a'_k = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

As it turns out, the above two equations define what is known as the discrete Fourier transform (DFT).

The **discrete Fourier transform (DFT)** X of the sequence x is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

The preceding equation is known as the **DFT analysis equation**. The **inverse DFT** x of the sequence X is given by

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)kn} \quad \text{for } n \in [0\,..\,N-1].$$

The preceding equation is known as the **DFT synthesis equation**. The DFT maps a finite-length sequence of N elements to another finite-length sequence of N elements. The DFT will be considered in more detail later in Chapter 11.

Since the DFT is essentially a renormalized (i.e., rescaled) version of DT Fourier series, the properties of the DFT and Fourier series are very similar. The only differences (aside from the sequence indices being taken modulo N in the DFT case) are some differing scale factors appearing in the formulas associated with a few properties. The properties of the DFT are summarized in Table 10.2. As can be seen from this table, most of the properties of the DFT are similar to their counterparts listed earlier for Fourier series in Table 10.1. The only notable differences are the duality, periodic convolution, and multiplication properties and Parseval's relation. Some of the constants appearing in the formulas associated with these properties differ by a factor of N relative to their Fourier-series counterparts.

10.7 Fourier Series and Frequency Spectra

The Fourier series represents a (periodic) sequence in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a sequence. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a sequence is concentrated. For example, if only the Fourier-series coefficients for the low-order harmonics have large magnitudes, then the sequence is mostly associated with low frequencies. On the other hand, if a function has many large magnitude coefficients for high-order harmonics, then the sequence has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier-series representation provides a means for measuring the frequency content of a sequence. The distribution of the energy/information in a sequence over different frequencies is referred to as the **frequency spectrum** of the sequence.

To gain further insight into the role played by the Fourier-series coefficients c_k in the context of the frequency

Table 10.2: Properties of the Discrete Fourier Transform

Property	Time Domain	Fourier Domain
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(k) + a_2 X_2(k)$
Translation	$x(n - n_0)$	$e^{-jk(2\pi/N)n_0} X(k)$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	$X(k - k_0)$
Reflection	$x(-n)$	$X(-k)$
Conjugation	$x^*(n)$	$X^*(-k)$
Duality	$X(n)$	$Nx(-k)$
Periodic Convolution	$x_1 \circledast x_2(n)$	$X_1(k) X_2(k)$
Multiplication	$x_1(n) x_2(n)$	$\frac{1}{N} X_1 \circledast X_2(k)$

Property	
Parseval's Relation	$\sum_{n=0}^{N-1} \lvert x(n) \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X(k) \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

spectrum of the sequence x, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$\begin{aligned} x(n) &= \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} |c_k| e^{j\arg c_k} e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} |c_k| e^{j((2\pi/N)kn+\arg c_k)}. \end{aligned}$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with the frequency $\frac{2\pi}{N}k$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j(2\pi/N)kn}$, and therefore the larger the contribution the kth term (which is associated with frequency $\frac{2\pi}{N}k$) will make to the overall summation. Consequently, $|c_k|$ can be used to quantify how much information a sequence x has at the frequency $\frac{2\pi}{N}k$. The quantity $\arg c_k$ is also important because its affects how much different complex sinusoids in the Fourier series are time-shifted relative to one another before being added together.

To formalize the notion of frequency spectrum, the **frequency spectrum** of a sequence x is essentially its corresponding Fourier-series coefficients c_k. Due to the above interpretation of a Fourier series in terms of the polar form, we are often interested in $|c_k|$ and $\arg c_k$. As a matter of terminology, we refer to $|c_k|$ as the **magnitude spectrum** of x and $\arg c_k$ as the **phase spectrum** of x.

Since the graphical presentation of information is often helpful for visualization purposes, we often want to plot frequency spectra of sequences. Since three-dimensional plots are usually more difficult to generate (especially by hand) than two-dimensional ones and can often be more difficult to interpret accurately, we usually present frequency spectra in graphical form using only two-dimensional plots. In the case that the frequency spectrum is either purely real or purely imaginary, we typically plot the frequency spectrum directly on a single pair of axes. Most often, however, the frequency spectrum will be complex (but neither purely real nor purely imaginary), in which case we plot the frequency spectrum in polar form by using two plots, one showing the magnitude spectrum and one showing the phase spectrum. When plotting frequency spectra (including magnitude and phase spectra), the horizontal axis is labelled with the frequency corresponding to Fourier-series coefficient indices, rather than the indices themselves. Frequency is used for the quantity corresponding to the horizontal axis since frequency has a direct physical meaning (i.e., the rate at which something is oscillating), whereas the Fourier-series coefficient index is just an integer that has no direct physical meaning.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real sequence x, the Fourier-series coefficient sequence c is conjugate symmetric (i.e., $c_k = c^*_{-k}$ for all k). This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real sequence is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real sequence is always odd.

Example 10.8. The 8-periodic sequence x in Example 10.4 has the Fourier-series coefficient sequence c given by

$$c_k = \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}.$$

Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) in $(-\pi,\pi]$ the sequence x has the most information.

Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. Taking the magnitude of c_k, we have

$$\begin{aligned}
|c_k| &= \left|\frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}\right| \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{\left|8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)\right|} \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{8\left|e^{j(3\pi/8)k}\right|\left|\sin\left(\frac{\pi}{8}k\right)\right|} \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{8\left|\sin\left(\frac{\pi}{8}k\right)\right|}.
\end{aligned}$$

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Taking the argument of c_k, We have

$$\begin{aligned}
\arg c_k &= \arg\left(\frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}\right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \arg\left[8e^{j(3\pi/8)k}\sin\left(\tfrac{\pi}{8}k\right)\right] \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \left(\arg 8e^{j(3\pi/8)k} + \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right]\right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \left(\tfrac{3\pi}{8}k + \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right]\right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right] - \tfrac{3\pi}{8}k.
\end{aligned}$$

The magnitude and phase spectra of x are plotted in Figures 10.1(a) and (b), respectively. Note that the magnitude spectrum has even symmetry, while the phase spectrum has odd symmetry. This is what we should expect, since x is real. Considering only frequencies in $(-\pi,\pi]$, the sequence x has the most information at a frequency of 0 as this corresponds to the Fourier-series coefficient with the largest magnitude (i.e., $\frac{1}{2}$). ∎

Example 10.9. Earlier, in Example 10.2, we saw that the N-periodic sequence

$$x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n - N\ell)$$

has the Fourier-series coefficient sequence

$$c_k = \frac{1}{N}.$$

For the case that $N = 10$, find the magnitude and phase spectra of x, and plot the frequency spectrum of x.

Solution. The magnitude and phase spectra of x are given by the magnitude and argument of c_k, respectively. Taking the magnitude of c_k, we have

$$|c_k| = \left|\tfrac{1}{10}\right| = \tfrac{1}{10}.$$

Similarly, taking the argument of c_k, we have

$$\arg c_k = \arg \tfrac{1}{10} = 0.$$

Since all of the c_k are real, we can plot the frequency spectrum of x directly using a single two-dimensional plot (instead of using two two-dimensional plots with one for each of the magnitude and phase spectra). The frequency spectrum of x is plotted in Figure 10.2. ∎

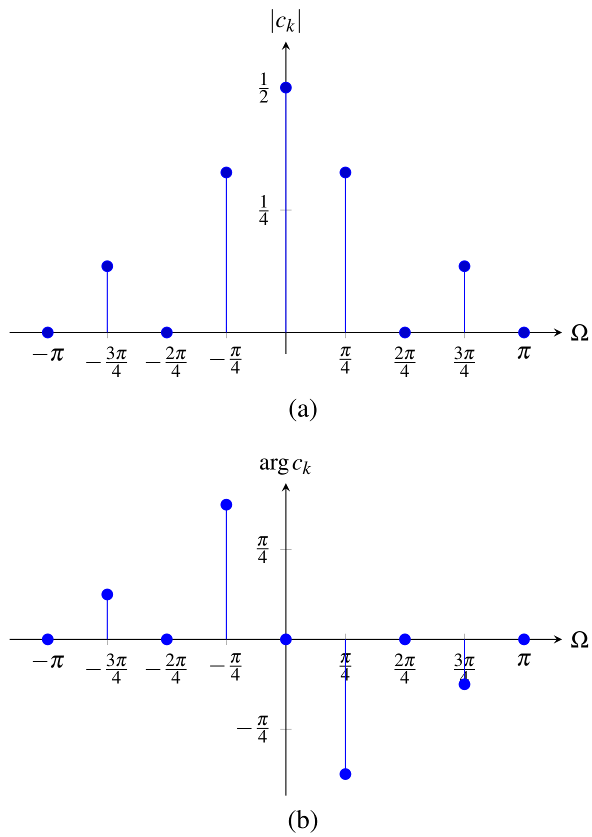

Figure 10.1: Frequency spectrum of x. (a) Magnitude spectrum and (b) phase spectrum.

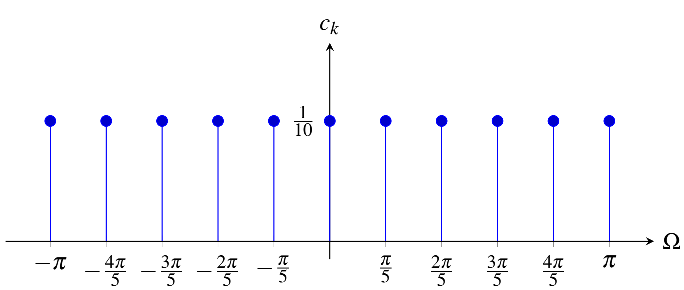

Figure 10.2: Frequency spectrum of x.

10.8 Fourier Series and LTI Systems

From earlier, in Theorem 9.12, we know that complex exponentials are eigensequences of LTI systems. Since complex sinusoids are a special case of complex exponentials, it follows that complex sinusoids are also eigensequences of LTI systems. In particular, we have the result below.

Corollary 10.1. *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a sequence of the form $x(n) = e^{j\Omega n}$, where Ω is an arbitrary real constant (i.e., x is an arbitrary complex sinusoid), the following holds:*

$$\mathcal{H}x(n) = H(\Omega)e^{j\Omega n},$$

where

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}. \tag{10.12}$$

That is, x is an eigensequence of $\mathcal{H}$ with the corresponding eigenvalue $H(\Omega)$.

The preceding result (i.e., Corollary 10.1) is simply a special case of Theorem 9.12 for $z = e^{j\Omega}$. Note that, in order to obtain more convenient notation, the function H in Corollary 10.1 is defined differently from the function H in Theorem 9.12. In particular, letting H_F and H_Z denote the function H that appears in each of Corollary 10.1 and Theorem 9.12, respectively, we have the relationship $H_\mathsf{F}(\Omega) = H_\mathsf{Z}(e^{j\Omega})$.

As a matter of terminology, the function H in (10.12) is referred to as the **frequency response** of the system $\mathcal{H}$. The frequency response completely characterizes the behavior of a LTI system. Consequently, the frequency response is often useful when working with LTI systems. As it turns out, (10.12) is of fundamental importance, as it defines what is called the (DT) Fourier transform. We will study the (DT) Fourier transform in great depth later in Chapter 11.

Let us now consider an application of eigensequences. Since convolution can often be quite painful to handle at the best of times, let us exploit eigensequences in order to devise a means to avoid having to deal with convolution directly in certain circumstances. Suppose that we have an N-periodic sequence x represented in terms of a Fourier series as

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}.$$

Using (10.12) and the superposition property, we can determine the system response y to the input x as follows:

$$\begin{aligned}
y(n) &= \mathcal{H}x(n) \\
&= \mathcal{H}\left\{ \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn} \right\}(n) \\
&= \sum_{k=\langle N \rangle} \mathcal{H}\left\{ c_k e^{j(2\pi/N)kn} \right\}(n) \\
&= \sum_{k=\langle N \rangle} c_k \mathcal{H}\left\{ e^{j(2\pi/N)kn} \right\}(n) \\
&= \sum_{k=\langle N \rangle} c_k H\left(\tfrac{2\pi}{N}k\right) e^{j(2\pi/N)kn}.
\end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier-series coefficient corresponds. In other words, if

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k$$

then

$$y(n) \overset{\text{DTFS}}{\longleftrightarrow} H\left(\tfrac{2\pi}{N}k\right) c_k.$$

Example 10.10. A LTI system has the impulse response

$$h(n) = \left(\tfrac{1}{2}\right)^{n+1} u(n).$$

Find the response y of this system to the input x, where

$$x(n) = 1 + \tfrac{1}{2}\cos\left(\tfrac{\pi}{2}n\right) + \tfrac{1}{8}\cos(\pi n).$$

Solution. To begin, we find the frequency response H of the system. Recalling (10.12), we have

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}.$$

Substituting the given expression for h, we obtain

$$\begin{aligned}
H(\Omega) &= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} u(n)e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{1}{2}e^{-j\Omega}\right)^{n}.
\end{aligned}$$

Using the formula for the sum of an infinite geometric sequence, we have

$$\begin{aligned}
H(\Omega) &= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}e^{-j\Omega}}\right) \\
&= \frac{1}{2-e^{-j\Omega}}.
\end{aligned}$$

Now, we find the Fourier-series representation of x. The fundamental period N of x is $\operatorname{lcm}(4,2) = 4$. We can rewrite x as

$$\begin{aligned}
x(n) &= 1 + \tfrac{1}{2}\left[\tfrac{1}{2}\left(e^{j(\pi/2)n} + e^{-j(\pi/2)n}\right)\right] + \tfrac{1}{8}\left[\tfrac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\right] \\
&= 1 + \tfrac{1}{4}e^{j(\pi/2)n} + \tfrac{1}{4}e^{-j(\pi/2)n} + \tfrac{1}{8}\left(\tfrac{1}{2}\right)(2)e^{j\pi n} \\
&= \tfrac{1}{4}e^{-j(\pi/2)n} + 1 + \tfrac{1}{4}e^{j(\pi/2)n} + \tfrac{1}{8}e^{j\pi n} \\
&= \tfrac{1}{4}e^{j(2\pi/4)(-1)n} + 1 + \tfrac{1}{4}e^{j(2\pi/4)(1)n} + \tfrac{1}{8}e^{j(2\pi/4)(2)n}.
\end{aligned}$$

Thus, the Fourier series for x is given by

$$x(n) = \sum_{k=-1}^{2} a_k e^{j(2\pi/4)kn},$$

where

$$a_k = \begin{cases} \frac{1}{4} & k=-1 \\ 1 & k=0 \\ \frac{1}{4} & k=1 \\ \frac{1}{8} & k=2. \end{cases}$$

The magnitude spectrum of x is plotted in Figure 10.3 along with the magnitude response $|H(\cdot)|$.