Unit:
CT LTI Systems

Example X.4.1

Let x and h denote functions, and let t denote a real number.

x * h

This expression denotes the function resulting from convolving the function x with the function h.

(x * h)(t)

x * h(t)

Both of these expressions denote the number resulting from convolving the function x with the function h and then evaluating the resulting function at the point t.

(x + h)(t)

x(t) + h(t)

These expressions have slightly different meanings (i.e., the former is adding functions while the latter is adding numbers), but they are both valid mathematical expressions and, by definition, they are always equal since the addition of functions is defined pointwise (i.e., (x+h)(t) = x(t) + h(t)).

x(t) * h(t)

Strictly speaking, this expression is not mathematically valid, as it is attempting to convolve the number x(t) with the number h(t). Both operands of a convolution operation, however, must be functions. Convolution cannot be defined in a pointwise manner. In other words, (x*h)(t) does not equal x(t) * h(t) because the latter expression is not even mathematically valid. Sadly, many engineering textbooks abuse notation in this way, and this often leads to confusion for students. Sometimes this abused notation x(t) * h(t) is intended to mean x * h; sometimes it might mean x * h(t); and yet other times it may mean something else entirely (and the reader is simply forced to guess the intended meaning).

Example 4.1. Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \le t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t} - e^{-1}] \\ &= e^{-t-1} - 1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \le t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\ &= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\ &= e^{-t}[e^{-1} - 1 + e^{t} - 1] \\ &= 1 + (e^{-1} - 2)e^{-t}. \end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \ge 1$. From Figure 4.1(h), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\ &= e^{-t}[e^{-1} - 1 + e - 1] \\ &= (e - 2 + e^{-1})e^{-t}. \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x*h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \leq t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \leq t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \leq t. \end{cases}$$

The convolution result $x*h$ is plotted in Figure 4.1(i).

