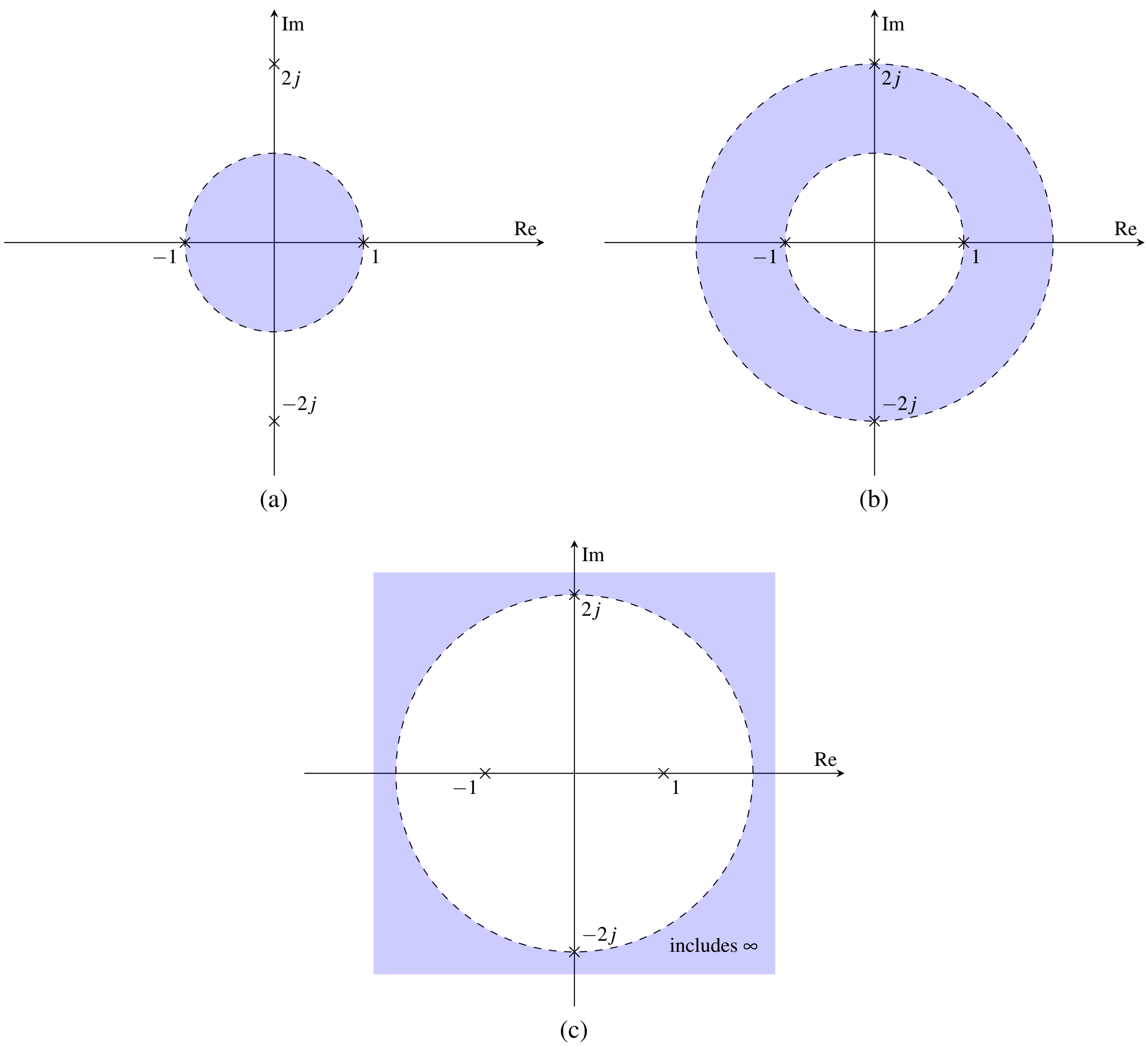

Figure 12.14: ROCs for example. The (a) first, (b) second, and (c) third possible ROCs for X.

and straightforward algebraic manipulation, we have

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n} \\
&= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} \\
&= a_1 X_1(z) + a_2 X_2(z).
\end{aligned}$$

The ROC R can be deduced as follows. If X_1 and X_2 both converge at some point z, say $z = \lambda$, then any linear combination of these functions must also converge at $z = \lambda$. Therefore, the ROC R must contain the intersection of R_1 and R_2. Thus, we have shown that the linearity property holds. ■

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $a_1 X_1(z) + a_2 X_2(z)$.

Example 12.8 (Linearity property without pole-zero cancellation)**.** Find the z transform X of the sequence

$$x(n) = a^{|n|},$$

where a is a complex constant satisfying $|a| < 1$.

Solution. To begin, we observe that x can be written as $x = x_1 + x_2$, where

$$x_1(n) = a^{-n} u(-n-1) \quad \text{and} \quad x_2(n) = a^n u(n).$$

From Table 12.3, we know that

$$-a^n u(-n-1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \text{ for } |z| < |a| \quad \text{and} \quad a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \text{ for } |z| > |a|.$$

Thus, we have

$$X_1(z) = -\frac{z}{z-a^{-1}} \text{ for } |z| < \left|a^{-1}\right| \quad \text{and} \quad X_2(z) = \frac{z}{z-a} \text{ for } |z| > |a|.$$

From the linearity property of the z transform, we have

$$\begin{aligned}
X(z) &= X_1(z) + X_2(z) \\
&= -\frac{z}{z-a^{-1}} + \frac{z}{z-a} \\
&= \frac{z^2 - a^{-1}z - (z^2 - az)}{(z-a)(z-a^{-1})} \\
&= \frac{z^2 - a^{-1}z - z^2 + az}{(z-a)(z-a^{-1})} \\
&= \frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}.
\end{aligned}$$

Now, we must determine the ROC R of X. Let R_1 and R_2 denote the ROCs of X_1 and X_2, respectively. We know that R must contain $R_1 \cap R_2$, where

$$\begin{aligned}
R_1 \cap R_2 &= \{|z| < \left|a^{-1}\right|\} \cap \{|z| > |a|\} \\
&= \{|a| < |z| < \left|a^{-1}\right|\}.
\end{aligned}$$

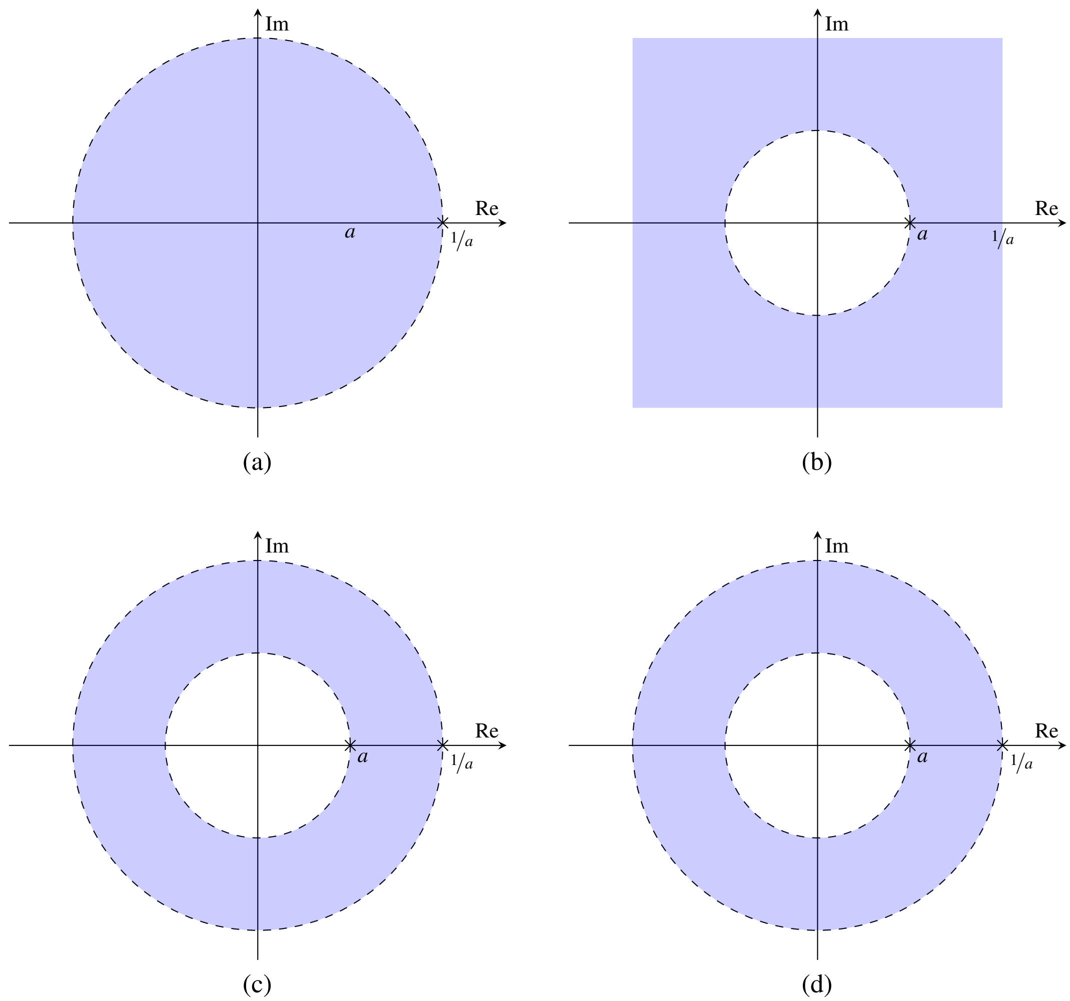

Figure 12.15: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Furthermore, R cannot be larger than this intersection, since X has a poles at a and a^{-1}. Therefore, $R = R_1 \cap R_2$. The various ROCs are illustrated in Figure 12.15. So, in conclusion, we have

$$X(z) = \frac{(a-a^{-1})z}{(z-a)(z-a^{-1})} \quad \text{for } |a| < |z| < \left|a^{-1}\right|. \qquad \blacksquare$$

Example 12.9 (Linearity property with pole-zero cancellation)**.** Using the linearity property of the z transform and the z transform pairs

$$u(n+1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z^2}{z-1} \quad \text{for } |z| > 1 \quad \text{and} \quad u(n-2) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{z(z-1)} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = u(n+1) - u(n-2).$$

Solution. To begin, let

$$x_1(n) = u(n+1) \quad \text{and} \quad x_2(n) = u(n-2),$$

and let X_1 and X_2 denote the z transforms of x_1 and x_2, respectively. Also, let R_X, R_{X_1}, and R_{X_2} denote the ROCs of X, X_1, and X_2, respectively. From the linearity property of the z transform, we have

$$\begin{aligned} X(z) &= \mathcal{Z}\{x_1 - x_2\}(z) \\ &= X_1(z) - X_2(z). \end{aligned}$$

Using the given z transform pairs, we can rewrite this as

$$\begin{aligned} X(z) &= \frac{z^2}{z-1} - \frac{1}{z(z-1)} \\ &= \frac{z^3-1}{z(z-1)} \\ &= \frac{(z-1)(z^2+z+1)}{z(z-1)} \\ &= \frac{z^2+z+1}{z}. \end{aligned}$$

From the given z transform pairs, we know

$$R_{X_1} = R_{X_2} = \{|z| > 1\}.$$

Now, we must determine R_X. From the linearity property, we know that R_X must at least contain $R_{X_1} \cap R_{X_2}$. Thus, R_X contains

$$R_{X_1} \cap R_{X_2} = R_{X_1} \cap R_{X_1} = R_{X_1} = \{|z| > 1\}.$$

We still must determine, however, if R_X is larger than this intersection. Since the functions X_1 and X_2 are rational, R_X can only be larger than this intersection if pole-zero cancellation occurs. The poles and ROCs of X_1 and X_2 are shown in Figures 12.16(a) and (b), respectively, and the set $R_{X_1} \cap R_{X_2}$ is shown in Figure 12.16(c) along with the poles of X. Clearly, a pole at 1 was cancelled in the computation of X (i.e., X does not have a pole at 1 whereas X_1 and X_2 do have such a pole). Now, we consider whether the ROC could be larger than the one shown in Figure 12.16(c). Since X is rational, we know that the ROC must be bounded by poles or extend outwards towards infinity and inwards towards zero if not bounded by a pole. Clearly, the region in Figure 12.16(c) is not bounded by a pole on its inner boundary. Therefore, R_X must be larger than $R_{X_1} \cap R_{X_2}$. In particular, to obtain R_X, we must extend the region inwards until just before it reaches the pole at the origin. Therefore, we have that

$$R_X = \{|z| > 0\}.$$

The poles of X and R_X are shown in Figure 12.16(d). So, in conclusion, we have

$$X(z) = \frac{z^2+z+1}{z} \text{ for } |z| > 0. \qquad \blacksquare$$

12.8.2 Translation (Time Shifting)

The next property of the z transform to be introduced is the translation (i.e., time-domain shifting) property, as given below.

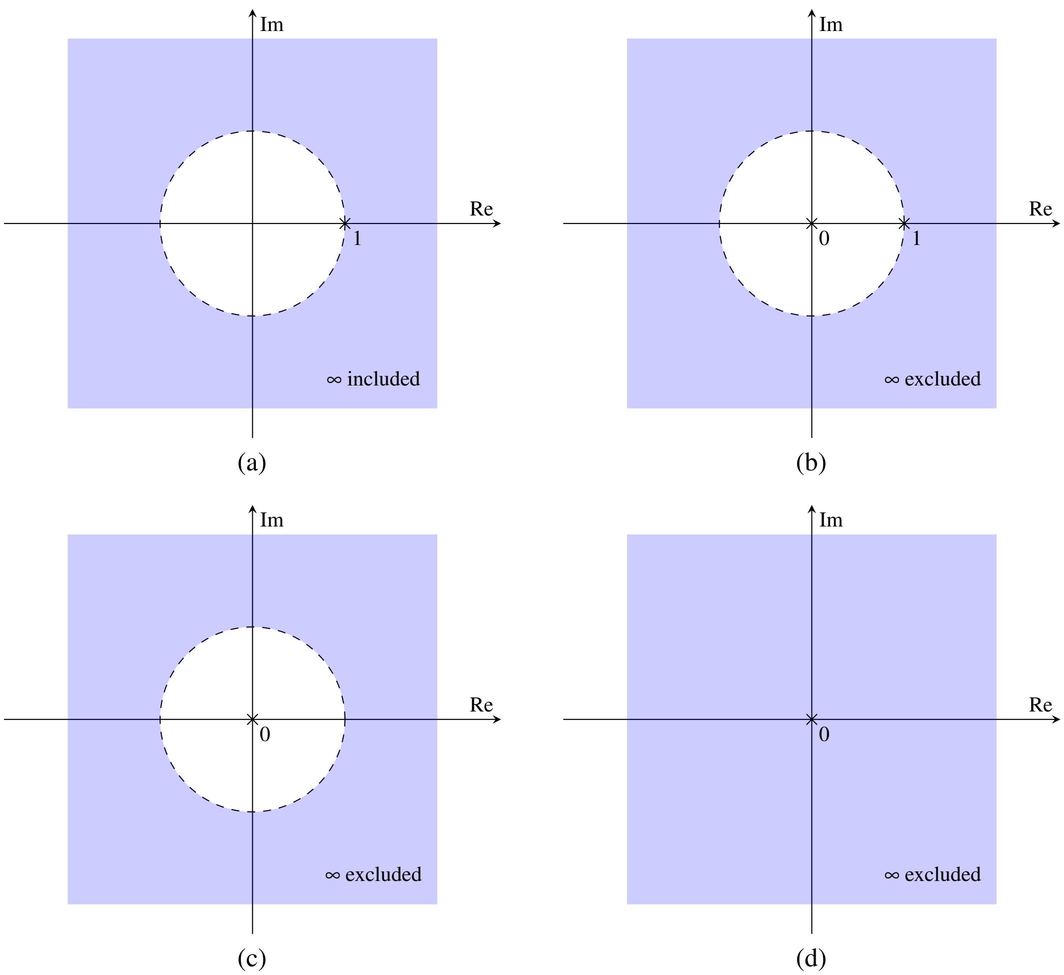

Figure 12.16: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Theorem 12.2 (Translation (i.e., time shifting))**.** *If $x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$x(n-n_0) \stackrel{\text{ZT}}{\longleftrightarrow} z^{-n_0} \quad \textit{with ROC } R \textit{ except for the possible addition/deletion of zero or infinity},$$

where n_0 is an integer constant. This is known as the ***translation property*** *(or* ***time-shifting property****) of the z transform.*

Proof. Let $y(n) = x(n-n_0)$, and let Y denote the z transform of y. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n}.$$

Now, we employ a change of variable. Let $k = n-n_0$ so that $n = k+n_0$. Applying the change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x(k)z^{-(k+n_0)} \\ &= z^{-n_0}\sum_{k=-\infty}^{\infty} x(k)z^{-k} \\ &= z^{-n_0}X(z). \end{aligned}$$

Now, we must determine the ROC R' of Y. If $X(z)$ converges for some z, then $Y(z)$ can only fail to converge due to the pole associated with z^{-n_0}. This pole is at 0 or ∞ for $n_0 > 0$ and $n_0 < 0$, respectively. So, R' must be the same as R with the possible exception of 0 and ∞. Thus, we have shown that the translation property holds. ■

Example 12.10 (Translation property)**.** Find the z transform X of the sequence

$$x(n) = u(n-n_0),$$

where n_0 is an integer constant.

Solution. From Table 12.3, we know that $u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1}$ for $|z| > 1$. Using this fact and the time-shifting property of the z transform, we can write

$$\begin{aligned} X(z) &= z^{-n_0}\mathcal{Z}u(z) \\ &= z^{-n_0}\frac{z}{z-1} \\ &= \frac{z^{1-n_0}}{z-1} \quad \text{for } |z| > 1. \end{aligned}$$

Thus, we have

$$u(n-n_0) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z^{1-n_0}}{z-1} \quad \text{for } |z| > 1.$$

■

Example 12.11 (Rectangular pulse)**.** Using properties of the z transform and the transform pair

$$u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = \begin{cases} 1 & n \in [n_0 \,..\, n_1) \\ 0 & \text{otherwise}, \end{cases}$$

where n_0 and n_1 are (finite) integer constants and $n_0 < n_1$.

Solution. To begin, we observe that x can be equivalently written as

$$x(n) = u(n-n_0) - u(n-n_1).$$

Let $v_1(n) = u(n-n_0)$ and $v_2(n) = u(n-n_1)$ so that $x(n) = v_1(n) - v_2(n)$. Taking the z transform of v_1 using the translation property of the z transform, we have

$$V_1(z) = z^{-n_0} \mathcal{Z}u(z).$$

Using the given z transform pair, we obtain

$$V_1(z) = z^{-n_0} \left(\frac{z}{z-1} \right).$$

Taking the z transform of v_2 using the translation property of the z transform, we have

$$V_2(z) = z^{-n_1} \mathcal{Z}u(z).$$

Using the given z transform pair, we obtain

$$V_2(z) = z^{-n_1} \left(\frac{z}{z-1} \right).$$

Taking the z transform of x using the linearity property of the z transform, we have

$$X(z) = V_1(z) - V_2(z).$$

Substituting the formulas for V_1 and V_2 into the formula for X, we obtain

$$\begin{aligned} X(z) &= z^{-n_0} \left(\frac{z}{z-1} \right) - z^{-n_1} \left(\frac{z}{z-1} \right) \\ &= (z^{-n_0} - z^{-n_1}) \left(\frac{z}{z-1} \right). \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane, except possibly for 0. Whether 0 is included in the ROC depends on the specific values of n_0 and n_1. For example, the ROC includes 0 if $n_1 \leq 1$. ■

12.8.3 Complex Modulation (z-Domain Scaling)

The next property of the z transform to be introduced is the complex modulation (i.e., z-domain scaling) property, as given below.

Theorem 12.3 (Complex modulation (i.e., z-domain scaling))**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$a^n x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z/a) \quad \text{with ROC } R' = |a| R,$$

where a is a nonzero complex constant. This is known as the ***complex modulation property*** *(or z-domain scaling property) of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = a^n x(n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \\ &= X(z/a). \end{aligned}$$

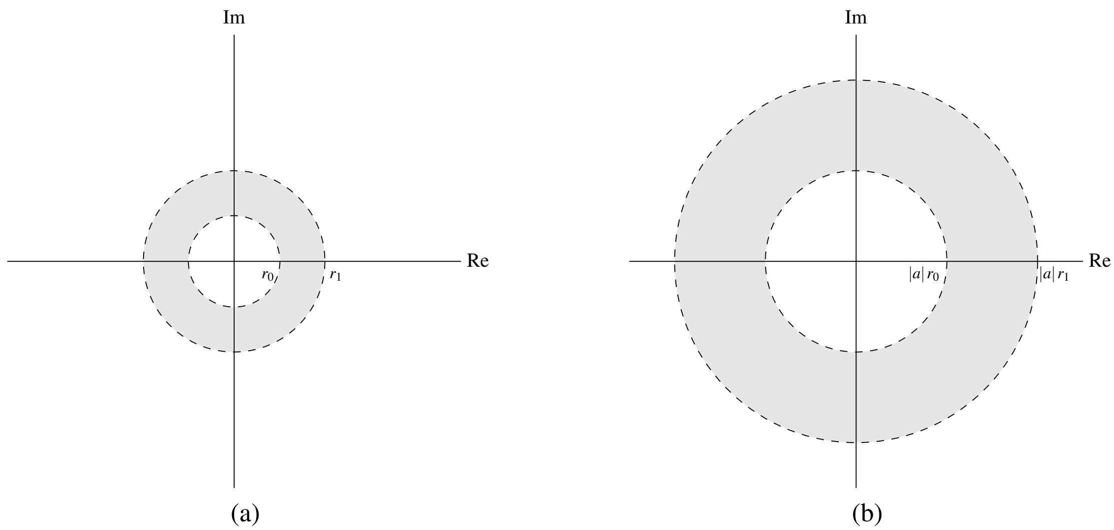

Figure 12.17: ROCs for complex modulation. The ROC of the z transform of the sequence (a) before and (b) after scaling.

Now, we must determine the ROC R' of Y. Since $X(z) = Y(az)$, Y must converge at az if X converges at z. Thus, $R' = aR$. Expressing a in polar form, we have

$$R' = |a|\,e^{j\arg a}R.$$

Due to the fact that R must always consist of concentric circles centered at the origin, $R = e^{j\arg\theta}R$ for any real constant θ. Thus, we can rewrite the above expression for R' as

$$R' = |a|\,R.$$

Thus, we have shown that the complex modulation property holds. ■

The effect of complex modulation on the ROC of the z transform is illustrated in Figure 12.17. Suppose that the ROC of the z transform of a sequence x is as shown in Figure 12.17(a). Then, the ROC of the z transform of the sequence $y(n) = a^n x(n)$ is as shown in Figure 12.17(b).

Example 12.12 (Complex modulation property)**.** Using properties of the z transform and the z transform pair

$$u(n) \xleftrightarrow{\text{ZT}} \tfrac{z}{z-1} \;\text{ for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = a^n u(n).$$

Solution. Using the z-domain scaling property and the given z transform pair, we can write

$$\begin{aligned} X(z) &= \frac{z/a}{z/a-1} \\ &= \frac{z}{z-a}. \end{aligned}$$

For the ROC R_X of X, we have

$$|z/a| > 1 \Rightarrow |z|\,/\,|a| > 1 \Rightarrow |z| > |a|\,.$$

Thus, we have

$$a^n u(n) \xleftrightarrow{\text{ZT}} \frac{z}{z-a} \quad \text{for } |z| > |a|.$$ ■

12.8.4 Conjugation

The next property of the z transform to be introduced is the conjugation property, as given below.

Theorem 12.4 (Conjugation)**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC R, then*

$$x^*(n) \overset{\text{ZT}}{\longleftrightarrow} X^*(z^*) \quad \textit{with ROC } R.$$

*This is known as the **conjugation property** of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x^*(n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} \\ &= \left(\sum_{n=-\infty}^{\infty} x^*(n) z^{-n} \right)^{**} \\ &= \left(\sum_{n=-\infty}^{\infty} x(n) (z^{-n})^* \right)^{*} \\ &= \left(\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n} \right)^{*} \\ &= X^*(z^*). \end{aligned}$$

Now, we consider the ROC of Y. Since $Y(z) = X^*(z^*)$, the expression $Y(z)$ converges if and only if $X(z^*)$ converges. In turn, $X(z^*)$ converges if and only if $z^* \in R$. Since $z^* \in R$ if and only if $z \in R$, the ROC of Y is R. Thus, we have shown that the conjugation property holds. ∎

Example 12.13 (Conjugation property)**.** Let x and y be two sequences related by

$$y(n) = \operatorname{Re}[x(n)].$$

Let X and Y denote the z transforms of x and y, respectively. Find Y in terms of X.

Solution. From properties of complex numbers, we know that $\operatorname{Re}\alpha = \frac{1}{2}(\alpha + \alpha^*)$. So, we have

$$\begin{aligned} Y(z) &= \mathcal{Z}\left\{ \tfrac{1}{2}[x(n) + x^*(n)] \right\}(z) \\ &= \tfrac{1}{2}\mathcal{Z}x(n) + \tfrac{1}{2}\mathcal{Z}\{x^*(n)\}(z) \\ &= \tfrac{1}{2}X(z) + \tfrac{1}{2}X^*(z^*). \end{aligned}$$

The ROC of Y is the same as the ROC of X. ∎

12.8.5 Time Reversal

The next property of the z transform to be introduced is the time-reversal property, as given below.

Theorem 12.5 (Time reversal)**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC R, then*

$$x(-n) \overset{\text{ZT}}{\longleftrightarrow} X(z^{-1}) \quad \textit{with ROC } R' = R^{-1}.$$

*This is known as the **time-reversal property** of the z transform.*

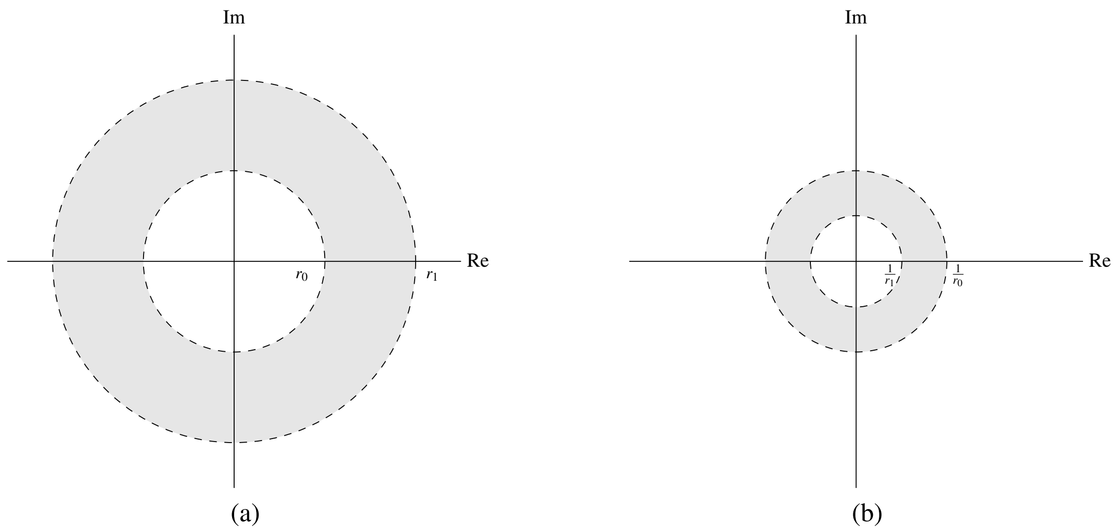

Figure 12.18: ROCs for time reversal. The ROC of the z transform of the sequence (a) before and (b) after time reversal.

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(-n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}.$$

Now, we employ a change of variable. Let $k = -n$ so that $n = -k$. Applying the change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x(k)z^{k} \\ &= \sum_{k=-\infty}^{\infty} x(k)(z^{-1})^{-k} \\ &= X(z^{-1}). \end{aligned}$$

Since $Y(z) = X(z^{-1})$, the expression $Y(z)$ must converge if $z^{-1} \in R$, or equivalently, $z \in R^{-1}$. So, the ROC of Y is R^{-1}. Thus, we have shown that the time-reversal property holds. ∎

The effect of time reversal on the ROC of the z transform is illustrated in Figure 12.18. Suppose that the ROC of the z transform of a sequence x is as shown in Figure 12.18(a). Then, the ROC of the z transform of the sequence $y(n) = x(-n)$ is as shown in Figure 12.18(b).

Example 12.14 (Time-reversal property)**.** Using properties of the z transform and the z transform pair

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = u(-n).$$

Solution. Using the given z transform pair and the time-reversal property of the z transform, we can write

$$X(z) = \frac{z^{-1}}{z^{-1}-1} \quad \text{for } \left|z^{-1}\right| > 1.$$

Simplifying the algebraic expression, we obtain

$$X(z) = \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}.$$

Simplifying the expression for the ROC, we obtain

$$\left|z^{-1}\right| > 1 \Rightarrow |z| < 1.$$

Thus, we have

$$X(z) = \frac{1}{1-z} \quad \text{for } |z| < 1.$$

■

Example 12.15. Let x and y be two sequences related by

$$y(n) = \text{Even}\{x\}(n) = \tfrac{1}{2}[x(n) + x(-n)].$$

Let X and Y denote the z transforms of x and y, respectively. Find Y in terms of X.

Solution. Let $v(n) = x(-n)$ and let V denote the z transform of v. Let R_X and R_Y denote the ROCs of X and Y, respectively. From the time-reversal property of the z transform, we have

$$V(z) = X\left(z^{-1}\right) \text{ for } z \in R_X^{-1}.$$

From the linearity property of the z transform, we have

$$\begin{aligned} Y(z) &= \tfrac{1}{2}[X(z) + V(z)] \\ &= \tfrac{1}{2}[X(z) + X(z^{-1})] \quad \text{and} \\ R_Y &\supset R_X \cap R_X^{-1}. \end{aligned}$$

(Note that "$\supset$" means "contains".) We cannot simplify further without additional knowledge about x. ■

12.8.6 Upsampling (Time Expansion)

In Section 8.2.4, the upsampling operation for sequences was introduced. One might wonder what effect upsampling has on the z transform of a sequence. The answer to this question is given by the theorem below.

Theorem 12.6 (Upsampling (i.e., time expansion))**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$(\uparrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z^M) \quad \textit{with ROC } R' = R^{1/M},$$

*where $R^{1/M}$ denotes the set formed from the Mth roots of the elements in R. This is known as the **upsampling property** (or **time-expansion property**) of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = (\uparrow M)x(n)$. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}.$$

Since $y(n) = 0$ if n/M is not an integer, we can rewrite this summation as

$$Y(z) = \sum_{\substack{n \in \mathbb{Z}: \\ M \text{ divides } n}} y(n) z^{-n}.$$

Now, we employ a change of variable. Let $\lambda = n/M$ so that $n = M\lambda$. (Note that λ will always be an integer since the above summation is only taken over terms where n/M is an integer.) Applying the change of variable, we obtain

$$Y(z) = \sum_{\lambda=-\infty}^{\infty} y(M\lambda)z^{-M\lambda}.$$

Using the fact that $x(n) = y(Mn)$, we can rewrite the above equation as

$$\begin{aligned} Y(z) &= \sum_{\lambda=-\infty}^{\infty} x(\lambda)(z^M)^{-\lambda} \\ &= X(z^M). \end{aligned}$$

Now, we consider the ROC R' of Y. Since $Y(z) = X(z^M)$, Y converges at z if and only if X converges at z^M, or equivalently, Y converges at $z^{1/M}$ if and only if X converges at z. This implies that $R' = R^{1/M}$. Thus, we have shown that the upsampling property holds. ■

Example 12.16 (Upsampling property)**.** Find the z transform X of the sequence

$$x(n) = \begin{cases} 1 & n \geq 0 \text{ and } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Solution. To begin, we observe that $x = (\uparrow 2)u$. From Table 12.3, we know that

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{z}{z-1} \text{ for } |z| > 1.$$

Using the upsampling property of the z transform, we have

$$\begin{aligned} X(z) &= \mathcal{Z}u(z^2) \\ &= \frac{z^2}{z^2-1}. \end{aligned}$$

Let R denote the set $|z| > 1$ (i.e., the ROC of the z transform of u). Since $R^{1/2} = R$, the ROC of X is R. Thus, we have

$$(\uparrow 2)u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z^2}{z^2-1} \text{ for } |z| > 1.$$ ■

12.8.7 Downsampling

In Section 8.2.3, the downsampling operation for sequences was introduced. One might wonder what effect downsampling has on the z transform of a sequence. The answer to this question is given by the theorem below.

Theorem 12.7 (Downsampling)**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$(\downarrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right) \quad \textit{with ROC } R' = R^M,$$

*where R^M denotes the set formed from the Mth powers of the elements in R. This is known as the **downsampling property** of the z transform.*

Proof. To prove this theorem, we proceed as follows. Let $y = (\downarrow M)x$, and let Y denote the z transform of y. The z transform Y can be written as

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(Mn) z^{-n}. \end{aligned}$$

Now, we define the sequence

$$v(n) = \begin{cases} x(n) & \text{if } M \text{ divides } n \text{ (i.e., } n/M \text{ is an integer)} \\ 0 & \text{otherwise.} \end{cases}$$

Using this definition (which implies that $v(Mn) = x(Mn)$ for all integer n), we have

$$Y(z) = \sum_{n=-\infty}^{\infty} v(Mn) z^{-n}.$$

Now, we employ a change of variable. Let $n' = Mn$ so that $n = n'/M$. Applying the change of variable and dropping the primes, we obtain

$$Y(z) = \sum_{\substack{n \in \mathbb{Z}: \\ M \text{ divides } n}} v(n) z^{-n/M}.$$

Since $v(n)$ is zero when M does not divide n, the constraint on the summation index that M divides n can be removed to yield

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} v(n) z^{-n/M} \\ &= V(z^{1/M}). \end{aligned} \tag{12.5}$$

To complete the proof, we express V in terms of X. We observe that v can be written as

$$v(n) = c(n)x(n) \quad \text{where} \quad c(n) = \begin{cases} 1 & \text{if } M \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

The M-periodic sequence c has the Fourier series representation

$$c(n) = \tfrac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi kn/M}.$$

Thus, we can compute the z transform V of v as follows:

$$\begin{aligned} V(z) &= \mathcal{Z}\{c(n)x(n)\}(z) \\ &= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi kn/M} x(n) \right) z^{-n} \\ &= \tfrac{1}{M} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{M-1} e^{j2\pi kn/M} x(n) z^{-n} \\ &= \tfrac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n) \left(z e^{-j2\pi k/M} \right)^{-n} \\ &= \tfrac{1}{M} \sum_{k=0}^{M-1} X(z e^{-j2\pi k/M}). \end{aligned} \tag{12.6}$$

Combining (12.5) and (12.6), we have

$$\begin{aligned} Y(z) &= V(z^{1/M}) \\ &= \tfrac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j2\pi k/M}). \end{aligned}$$

Now, we consider the ROC R' of Y. From the preceding equation, Y converges at z if X converges at $z^{1/M}$, or equivalently, Y converges at z^M if X converges at z. Thus, $R' = R^M$. So, we conclude that

$$Y(z) = \tfrac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j2\pi k/M}) \text{ for } z \in R^M.$$

This completes the proof. ■

Example 12.17 (Downsampling property)**.** Let x and y be two sequences related by $y(n) = (\downarrow 2)x(n)$, and let X and Y denote the z transforms of x and y, respectively. Suppose that

$$X(z) = \frac{z^2-1}{z^2-\frac{1}{4}} \text{ for } |z| > \tfrac{1}{2}.$$

Find Y.

Solution. From the downsampling property of the z transform, we have

$$\begin{aligned} Y(z) &= \tfrac{1}{2}\sum_{k=0}^{1} X(e^{-j\pi k}z^{1/2}) \\ &= \tfrac{1}{2}\sum_{k=0}^{1} X\left[(-1)^k z^{1/2}\right] \\ &= \tfrac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right] \\ &= \tfrac{1}{2}\left(\frac{(z^{1/2})^2-1}{(z^{1/2})^2-\frac{1}{4}} + \frac{(-z^{1/2})^2-1}{(-z^{1/2})^2-\frac{1}{4}}\right) \\ &= \tfrac{1}{2}\left(\frac{z-1}{z-\frac{1}{4}} + \frac{z-1}{z-\frac{1}{4}}\right) \\ &= \frac{z-1}{z-\frac{1}{4}}. \end{aligned}$$

The ROC of Y is $|z| > \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

In passing, we note that the inverse z transforms of X and Y are given by

$$\begin{aligned} x(n) &= 4\delta(n) - \tfrac{3}{2}\left([1+(-1)^n]\left(\tfrac{1}{2}\right)^n\right)u(n) \quad \text{and} \\ y(n) &= 4\delta(n) - 3\left(\tfrac{1}{4}\right)^n u(n). \end{aligned}$$

As we would expect, $y(n) = x(2n)$. ■

12.8.8 Convolution

The next property of the z transform to be introduced is the convolution property, as given below.

Theorem 12.8 (Convolution)**.** *If* $x_1(n) \overset{\text{ZT}}{\longleftrightarrow} X_1(z)$ *with ROC* R_1 *and* $x_2(n) \overset{\text{ZT}}{\longleftrightarrow} X_2(z)$ *with ROC* R_2*, then*

$$x_1 * x_2(n) \overset{\text{ZT}}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2.$$

This is known as the ***time-domain convolution property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. From the definition of the z transform, we have

$$\mathcal{Z}\{x_1 * x_2\}(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}.$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying the change of variable, we obtain

$$\begin{aligned}\mathcal{Z}\{x_1 * x_2\}(z) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)z^{-(\lambda+k)} \\ &= \sum_{k=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} x_1(k)x_2(\lambda)z^{-\lambda}z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} \sum_{\lambda=-\infty}^{\infty} x_2(\lambda)z^{-\lambda} \\ &= X_1(z)X_2(z).\end{aligned}$$

The expression $X_1(z)X_2(z)$ converges if z is such that $X_1(z)$ and $X_2(z)$ both converge (i.e., $z \in R_1 \cap R_2$). So, the ROC R must contain $R_1 \cap R_2$. Thus, we have shown that the convolution property holds. ■

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $X_1(z)X_2(z)$.

The time-domain convolution property of the z transform has important practical implications. Since the z transform effectively converts a convolution into a multiplication, the z transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when working with (DT) LTI systems, for example, since such systems fundamentally involve convolution.

Example 12.18 (Convolution property)**.** Find the z transform X of the sequence

$$x(n) = u * u(n).$$

Solution. From Table 12.3, we have the z transform pair

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \text{ for } |z| > 1.$$

Taking the z transform of x with the help of the above z transform pair, we have

$$\begin{aligned}X(z) &= \mathcal{Z}\{u * u\}(z) \\ &= \mathcal{Z}u(z)\mathcal{Z}u(z) \\ &= \left(\frac{z}{z-1}\right)\left(\frac{z}{z-1}\right) \\ &= \frac{z^2}{(z-1)^2}.\end{aligned}$$

Let R_X denote the ROC of X and let $R = \{|z| > 1\}$ (i.e., R is the ROC of the z transform of u). We have that R_X must contain $R \cap R = R$. Since the terms being added to obtain X are rational and no pole-zero cancellation occurs, the ROC cannot be larger than R. Thus, $R_X = \{|z| > 1\}$. Therefore, we conclude

$$X(z) = \frac{z^2}{(z-1)^2} \text{ for } |z| > 1.$$ ■

12.8.9 z-Domain Differentiation

The next property of the z transform to be introduced is the z-domain differentiation property, as given below.

Theorem 12.9 (z-domain differentiation)**.** *If* $x(n) \xleftrightarrow{\text{ZT}} X(z)$ *with ROC R, then*

$$nx(n) \xleftrightarrow{\text{ZT}} -z\tfrac{d}{dz}X(z) \quad \textit{with ROC } R.$$

This is known as the ***z-domain differentiation property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. From the definition of the z transform, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}.$$

Taking the derivative of both sides of the preceding equation, we obtain

$$\begin{aligned}
\tfrac{d}{dz}X(z) &= \tfrac{d}{dz}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x(n)\tfrac{d}{dz}(z^{-n}) \\
&= \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1} \\
&= -z^{-1}\sum_{n=-\infty}^{\infty} nx(n)z^{-n}.
\end{aligned}$$

Multiplying both sides of the preceding equation by $-z$, we have

$$\begin{aligned}
-z\tfrac{d}{dz}X(z) &= \sum_{n=-\infty}^{\infty} nx(n)z^{-n} \\
&= \mathcal{Z}\{nx(n)\}(z).
\end{aligned}$$

Thus, we have shown that the z-domain differentiation property holds. ■

Example 12.19 (z-domain differentiation property)**.** Using properties of the z transform and the z transform pair

$$\tfrac{1}{n!}a^n u(n) \xleftrightarrow{\text{ZT}} e^{a/z} \text{ for } |z| > 0,$$

find the z transform X of the sequence

$$x(n) = \tfrac{1}{n!}na^n u(n).$$

Solution. Let $v(n) = \frac{1}{n!}a^n u(n)$ (i.e., v is the sequence appearing in the given z transform pair) so that $x(n) = nv(n)$. Taking the z transform of v, we have

$$V(z) = e^{a/z} \text{ for } |z| > 0.$$

Taking the z transform of x using the z-domain differentiation property of the z transform, we obtain

$$X(z) = -z\tfrac{d}{dz}V(z).$$

Substituting the formula for V into the formula for X, we have

$$\begin{aligned}
X(z) &= -z\tfrac{d}{dz}e^{a/z} \\
&= -ze^{a/z}\tfrac{d}{dz}(az^{-1}) \\
&= -ze^{a/z}(-az^{-2}) \\
&= az^{-1}e^{a/z}.
\end{aligned}$$

The ROC of X is the same as the ROC of the given z transform pair. Thus, we have

$$X(z) = az^{-1}e^{a/z} \text{ for } |z| > 0. \qquad \blacksquare$$

12.8.10 Differencing

The next property of the z transform to be introduced is the differencing property, as given below.

Theorem 12.10 (Differencing)**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC R, then*

$$x(n) - x(n-1) \overset{\text{ZT}}{\longleftrightarrow} (1 - z^{-1})X(z) \text{ with ROC } R' \text{ containing } R \cap \{|z| > 0\}.$$

This is known as the ***differencing property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(n) - x(n-1)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} [x(n) - x(n-1)]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x(n)z^{-n} - \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \\
&= X(z) - \sum_{n=-\infty}^{\infty} x(n-1)z^{-n}.
\end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n - 1$ so that $n = \lambda + 1$. Applying the change of variable, we obtain

$$\begin{aligned}
Y(z) &= X(z) - \sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-(\lambda+1)} \\
&= X(z) - z^{-1}\sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-\lambda} \\
&= X(z) - z^{-1}X(z) \\
&= (1 - z^{-1})X(z).
\end{aligned}$$

Now, we consider the ROC R' of Y. Suppose that X is rational. Since $Y(z) = (1 - z^{-1})X(z) = \frac{z-1}{z}X(z)$, unless the pole at 0 introduced by the $\frac{z-1}{z}$ factor is cancelled, Y has a pole at 0. In this case, the ROC R' cannot contain the origin. Hence, the restriction that $|z| > 0$. Thus, we have shown that the differencing property holds. $\blacksquare$

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{|z| > 0\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $(1 - z^{-1})X(z) = \frac{z-1}{z}X(z)$.

Example 12.20 (Differencing property)**.** Find the z transform Y of the sequence

$$y(n) = x(n) - x(n-1),$$

where

$$x(n) = a^n u(n)$$

and a is a complex constant.

Solution. From Table 12.3, we have the z transform pair

$$a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \quad \text{for } |z| > |a|.$$

Using the differencing property of the z transform and the above z transform pair, we have

$$\begin{aligned} Y(z) &= (1-z^{-1})\left(\frac{z}{z-a}\right) \\ &= \left(\frac{z-1}{z}\right)\left(\frac{z}{z-a}\right) \\ &= \frac{z-1}{z-a}. \end{aligned}$$

The ROC R of Y is $|z| > |a|$, unless $a = 1$, in which case R is the entire complex plane. So, we have

$$Y(z) = \frac{z-1}{z-a} \quad \text{for } z \in R \quad \text{where} \quad R = \begin{cases} |z| > |a| & a \neq 1 \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

■

12.8.11 Accumulation

The next property of the z transform to be introduced is the accumulation property, as given below.

Theorem 12.11 (Accumulation)**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC* R*, then*

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} X(z) \ \text{ with ROC } R' \text{ containing } R \cap \{|z| > 1\}.$$

This is known as the ***accumulation property*** *of the z transform.*

Proof. To prove this theorem, we proceed as follows. Let $y(n) = \sum_{k=-\infty}^{n} x(k)$ and let Y denote the z transform of y. From the definition of the z transform, we can write

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} x(k) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} x(k) u(n-k) z^{-n}. \end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying this change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) u(\lambda) z^{-(\lambda+k)} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{\lambda=-\infty}^{\infty} u(\lambda) z^{-\lambda} \\ &= X(z) \sum_{\lambda=0}^{\infty} z^{-\lambda}. \end{aligned}$$

Assuming that $|z| > 1$ (so that the infinite geometric series converges), we can simplify the infinite geometric series above (using (F.9)) to yield

$$Y(z) = \tfrac{z}{z-1}X(z).$$

The ROC R' of Y must satisfy $|z| > 1$ for all $z \in R'$ in order to ensure that the infinite series encountered above converges. Suppose that X is rational. Assuming that no pole-zero cancellation occurs, Y has all of the same poles as X plus a new pole added at 1. Therefore, $R' = R \cap \{|z| > 1\}$. Thus, we have shown that the accumulation property holds. ■

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{|z| > 1\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $\tfrac{z}{z-1}X(z)$.

The accumulation property of the z transform has important practical implications. Since the z transform effectively converts accumulation into multiplication (by $\tfrac{z}{z-1}$), the z transform can be used as a means to avoid directly dealing with accumulation operations. This can often be beneficial when working with equations involving accumulation.

Example 12.21 (Accumulation property)**.** Using the z transform pair

$$\delta(n) \overset{\text{ZT}}{\longleftrightarrow} 1$$

and the accumulation property of the z transform, find the z transform X of the sequence

$$x(n) = u(n).$$

Solution. To begin, we observe that

$$x(n) = \sum_{k=-\infty}^{n} \delta(k).$$

Taking the z transform of x using the accumulation property of the z transform, we have

$$X(z) = \tfrac{z}{z-1}\mathcal{Z}\delta(z).$$

Using the given z transform pair, we have

$$\begin{aligned} X(z) &= \tfrac{z}{z-1}(1) \\ &= \tfrac{z}{z-1}. \end{aligned}$$

Since $\mathcal{Z}\delta$ is rational and no pole-cancellation occurs in the multiplication of 1 and $\tfrac{z}{z-1}$, the ROC R of X is

$$R = \{|z| > 1\}.$$

Thus, we conclude that

$$X(z) = \frac{z}{z-1} \quad \text{for } |z| > 1.$$ ■

12.8.12 Initial and Final Value Theorems

The next properties of the z transform to be introduced are known as the initial and final value theorems, as given below.

Theorem 12.12 (Initial-value theorem)**.** *Let x be a sequence with z transform X. If x is causal, then*

$$x(0) = \lim_{z\to\infty} X(z). \tag{12.7}$$

This result is known as the ***initial-value theorem****.*

Proof. To prove the above result, we proceed as follows. We start from the limit on the right-hand side of (12.7). From the definition of the z transform, we have

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=-\infty}^{\infty} x(n)z^{-n}.$$

Since x is causal, the preceding equation can be rewritten as

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n)z^{-n}.$$

Interchanging the order of the summation and limit on the right-hand side, we have

$$\begin{aligned}\lim_{z\to\infty} X(z) &= \sum_{n=0}^{\infty} x(n) \lim_{z\to\infty} z^{-n} \\ &= \sum_{n=0}^{\infty} x(n)\delta(n) \\ &= x(0).\end{aligned}$$

Thus, we have shown that the initial-value theorem holds. ∎

Theorem 12.13 (Final-value theorem)**.** *Let x be a sequence with the z transform X. If x is causal and $\lim_{n\to\infty} x(n)$ exists, then*

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)]. \tag{12.8}$$

This result is known as the ***final-value theorem****.*

Proof. To prove the above property, we proceed as follows. We consider the right-hand side of (12.8). From the time-domain shifting property, we know that

$$\mathcal{Z}\{x(n+1)-x(n)\}(z) = (z-1)X(z).$$

Moreover, we also have that

$$\mathcal{Z}\{x(n+1)-x(n)\}(z) = \lim_{n\to\infty} \sum_{k=-n}^{n} [x(k+1)-x(k)]z^{-k}.$$

Using this fact and that fact that x is causal, we can write the right-hand side of (12.8) as

$$\begin{aligned}&\lim_{z\to 1}\lim_{n\to\infty} \sum_{k=-n}^{n} [x(k+1)-x(k)]z^{-k} \\ &= \lim_{z\to 1}\lim_{n\to\infty} \big(x(0)z^{1} + [x(1)-x(0)]z^{0} + [x(2)-x(1)]z^{-1} + \ldots + [x(n)-x(n-1)]z^{-n+1} + \\ &\qquad [x(n+1)-x(n)]z^{-n}\big) \\ &= \lim_{z\to 1}\lim_{n\to\infty} \big(x(0)(z^{1}-1) + x(1)(1-z^{-1}) + \ldots + x(n)(z^{-n+1}-z^{-n}) + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty}\lim_{z\to 1} \big(x(0)(z^{1}-1) + x(1)(1-z^{-1}) + \ldots + x(n)(z^{-n+1}-z^{-n}) + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty}\lim_{z\to 1} \big(x(0)(z-1) + x(1)\tfrac{z-1}{z} + \ldots + x(n)\tfrac{z-1}{z^{n}} + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty} x(n+1) \\ &= \lim_{n\to\infty} x(n).\end{aligned}$$

Thus, we have shown that the final-value theorem holds. ■

Example 12.22 (Initial/final value theorem)**.** A causal sequence x with a (well-defined) limit at ∞ has the z transform

$$X(z) = \frac{4z^2 - 3z}{2z^2 - 3z + 1}.$$

Find $x(0)$ and $\lim_{n\to\infty} x(n)$.

Solution. From the initial value theorem, we have

$$\begin{aligned} x(0) &= \lim_{z\to\infty} \frac{4z^2 - 3z}{2z^2 - 3z + 1} \\ &= \tfrac{4}{2} \\ &= 2. \end{aligned}$$

From the final value theorem, we have

$$\begin{aligned} \lim_{n\to\infty} x(n) &= \lim_{z\to 1} \left[(z-1)\frac{4z^2 - 3z}{2z^2 - 3z + 1} \right] \\ &= \lim_{z\to 1} \left[(z-1)\frac{4z^2 - 3z}{2(z-\frac{1}{2})(z-1)} \right] \\ &= \lim_{z\to 1} \left[\frac{4z^2 - 3z}{2(z-\frac{1}{2})} \right] \\ &= \tfrac{1}{1} \\ &= 1. \end{aligned}$$

In passing, we note that the inverse z transform of X is given by

$$x(n) = \left[\left(\tfrac{1}{2}\right)^n + 1 \right] u(n).$$

As we would expect, the values computed above for $x(0)$ and $x(\infty)$ are consistent with this equation for x. ■

Amongst other things, the initial and final value theorems can be quite useful in checking for errors in z transform calculations. For example, suppose that we are asked to compute the z transform X of the sequence x. If we were to make a mistake in this computation, the values obtained for $x(0)$ and $\lim_{n\to\infty} x(n)$ using X with the initial and final value theorems and using x directly would most likely disagree. In this manner, we can relatively easily detect some types of errors in z transform calculations.

12.9 More z Transform Examples

Earlier in this chapter, we derived a number of z transform pairs. Some of these and other important transform pairs are listed in Table 12.3. Using the various z transform properties listed in Table 12.2 and the z transform pairs listed in Table 12.3, we can more easily determine the z transform of more complicated functions.

Example 12.23. Using properties of the z transform and the z transform pair

$$u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = \sum_{k=-\infty}^{n} [u(k+5) - u(k)].$$

Table 12.2: Properties of the (bilateral) z transform

Property	Time Domain	z Domain	ROC
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$	At least $R_1 \cap R_2$
Translation	$x(n-n_0)$	$z^{-n_0}X(z)$	R, except for possible addition/deletion of 0
Complex Modulation	$a^n x(n)$	$X(z/a)$	$\lvert a\rvert R$
Conjugation	$x^*(n)$	$X^*(z^*)$	R
Time Reversal	$x(-n)$	$X(1/z)$	R^{-1}
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$	$R^{1/M}$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$	R^M
Convolution	$x_1 * x_2(n)$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
z-Domain Differentiation	$nx(n)$	$-z\frac{d}{dz}X(z)$	R
Differencing	$x(n)-x(n-1)$	$\frac{z-1}{z}X(z)=(1-z^{-1})X(z)$	At least $R \cap \lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{z}{z-1}X(z)=\frac{1}{1-z^{-1}}X(z)$	At least $R \cap \lvert z\rvert > 1$

Property	
Initial Value Theorem	$x(0)=\lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n)=\lim_{z\to 1}[(z-1)X(z)]$

Table 12.3: Transform pairs for the (bilateral) z transform

Pair	$x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{z}{z-1}=\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
3	$-u(-n-1)$	$\frac{z}{z-1}=\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
4	$nu(n)$	$\frac{z}{(z-1)^2}=\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert>1$
5	$-nu(-n-1)$	$\frac{z}{(z-1)^2}=\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert<1$
6	$a^nu(n)$	$\frac{z}{z-a}=\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
7	$-a^nu(-n-1)$	$\frac{z}{z-a}=\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$
8	$na^nu(n)$	$\frac{az}{(z-a)^2}=\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>\lvert a\rvert$
9	$-na^nu(-n-1)$	$\frac{az}{(z-a)^2}=\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<\lvert a\rvert$
10	$\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!}a^nu(n)$	$\frac{z^m}{(z-a)^m}=\frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert>\lvert a\rvert$
11	$-\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!}a^nu(-n-1)$	$\frac{z^m}{(z-a)^m}=\frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert<\lvert a\rvert$
12	$\cos(\Omega_0n)u(n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}=\frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
13	$-\cos(\Omega_0n)u(-n-1)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}=\frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert<1$
14	$\sin(\Omega_0n)u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}=\frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
15	$-\sin(\Omega_0n)u(-n-1)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}=\frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert<1$
16	$a^n\cos(\Omega_0n)u(n)$	$\frac{z(z-a\cos\Omega_0)}{z^2-2az\cos\Omega_0+a^2}=\frac{1-(a\cos\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert>\lvert a\rvert$
17	$a^n\sin(\Omega_0n)u(n)$	$\frac{az\sin\Omega_0}{z^2-2az\cos\Omega_0+a^2}=\frac{(a\sin\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert>\lvert a\rvert$
18	$u(n)-u(n-M), M>0$	$\frac{z(1-z^{-M})}{z-1}=\frac{1-z^{-M}}{1-z^{-1}}$	$\lvert z\rvert>0$
19	$a^{\lvert n\rvert}, \lvert a\rvert<1$	$\frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}$	$\lvert a\rvert<\lvert z\rvert<\lvert a^{-1}\rvert$

Solution. Let $v_1(n) = u(n+5)$ so that

$$x(n) = \sum_{k=-\infty}^{n} [v_1(k) - u(k)].$$

Let $v_2(n) = v_1(n) - u(n)$ so that

$$x(n) = \sum_{k=-\infty}^{n} v_2(k).$$

Taking the z transform of the various sequences, we obtain

$$\begin{aligned} X(z) &= \frac{z}{z-1} V_2(z), \\ V_2(z) &= V_1(z) - \frac{z}{z-1}, \quad \text{and} \\ V_1(z) &= z^5 \left(\frac{z}{z-1} \right). \end{aligned}$$

By substitution, we have

$$\begin{aligned} X(z) &= \frac{z}{z-1} V_2(z) \\ &= \frac{z}{z-1} \left(V_1(z) - \frac{z}{z-1} \right) \\ &= \frac{z}{z-1} \left(z^5 \frac{z}{z-1} - \frac{z}{z-1} \right) \\ &= \frac{z}{z-1} \left(\frac{z^6}{z-1} - \frac{z}{z-1} \right) \\ &= \frac{z}{z-1} \left(\frac{z^6 - z}{z-1} \right) \\ &= \frac{z^2(z^5-1)}{(z-1)^2}. \end{aligned}$$

Since $x(n) = 0$ for all $n \leq -6$, x is right sided. Therefore, the ROC of X must be outside the outermost pole at 1. Thus, we have

$$X(z) = \frac{z^2(z^5-1)}{(z-1)^2} \text{ for } |z| > 1.$$

■

Example 12.24. Find the z transform X of the sequence

$$x(n) = nu(n-1).$$

Solution. Let $v_1(n) = u(n-1)$ so that

$$x(n) = nv_1(n).$$

Taking the z transforms of the various sequences, we have

$$\begin{aligned} V_1(z) &= z^{-1} \left(\frac{z}{z-1} \right) = \frac{1}{z-1} \quad \text{and} \\ X(z) &= -z \frac{d}{dz} V_1(z). \end{aligned}$$

Substituting, we have

$$\begin{aligned} X(z) &= -z\tfrac{d}{dz}\left[(z-1)^{-1}\right] \\ &= -z\left[-(z-1)^{-2}\right] \\ &= \frac{z}{(z-1)^2}. \end{aligned}$$

Since $x(n) = 0$ for all $n < 1$, x is right sided. Therefore, the ROC of X is outside the outermost pole at 1. Thus, we have

$$X(z) = \frac{z}{(z-1)^2} \quad \text{for } |z| > 1. \qquad \blacksquare$$

12.10 Determination of the Inverse z Transform

As suggested earlier, in practice, we rarely use (12.3) directly in order to compute the inverse z transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a variety of other techniques such as partial fraction expansions, power series expansions, and polynomial long division. We will consider each of these approaches in the sections that follow.

12.10.1 Partial Fraction Expansions

To find the inverse z transform of a rational function, partial fraction expansions are often employed. With this approach, in order to find the inverse z transform of a rational function, we begin by finding a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions (corresponding to the terms in the partial fraction expansion) for which we can usually find the inverse z transforms in a table (e.g., such as Table 12.3). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions is provided in Appendix B for those who might not be acquainted with such expansions.

Consider the computation of the inverse z transform x of a rational function X. Suppose that X has the P (distinct) poles $p_1, p_2, \ldots, p_P$, where the order of p_k is denoted q_k. Assuming that X is strictly proper, it has a partial fraction expansion (in the variable z) of the form

$$X(z) = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A_{k,\ell}\frac{1}{(z-p_k)^\ell}.$$

To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{1}{(z-a)^m}$ for some complex constant a and some positive integer m. Unfortunately, Table 12.3 does not directly contain the entries necessary to perform such inverse transform calculations. Although we could use the entries in the table in conjunction with the translation (i.e., time shifting) property of the z transform in order to perform the inverse transform calculation, this would lead to an inverse z transform expression that is complicated by numerous time shifting operations (which can often be undesirable). For this reason, we instead compute a partial fraction expansion of $X(z)/z$. This partial fraction has the form

$$\frac{X(z)}{z} = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A'_{k,\ell}\frac{1}{(z-p_k)^\ell}. \tag{12.9}$$

From this partial fraction expansion, we can rewrite X as

$$X(z) = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A'_{k,\ell}\frac{z}{(z-p_k)^\ell}.$$