Exercise 5.10

R **Answer (c).**

Let ω_0 denote the fundamental frequency of x. So, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$. We have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \tfrac{1}{4}\int_{-2}^{2} e^{-2|t|}e^{-jk(\pi/2)t}dt \\
&= \tfrac{1}{4}\left[\int_{-2}^{0} e^{2t}e^{-jk(\pi/2)t}dt + \int_{0}^{2} e^{-2t}e^{-jk(\pi/2)t}dt\right] = \tfrac{1}{4}\left[\int_{-2}^{0} e^{(2-jk\pi/2)t}dt + \int_{0}^{2} e^{(-2-jk\pi/2)t}dt\right] \\
&= \tfrac{1}{4}\left(\left[\frac{e^{(2-jk\pi/2)t}}{2-\frac{j\pi}{2}k}\right]\Bigg|_{-2}^{0} + \left[\frac{e^{(-2-jk\pi/2)t}}{-2-\frac{j\pi}{2}k}\right]\Bigg|_{0}^{2}\right) \\
&= \tfrac{1}{4}\left[\left(\frac{1}{2-\frac{j\pi}{2}k}\right)\left(1-e^{(2-jk\pi/2)(-2)}\right) + \left(\frac{1}{-2-\frac{j\pi}{2}k}\right)\left(e^{(-2-jk\pi/2)(2)}-1\right)\right] \\
&= \tfrac{1}{4}\left[\left(\frac{1}{2-\frac{j\pi}{2}k}\right)\left(1-e^{-4+jk\pi}\right) + \left(\frac{1}{-2-\frac{j\pi}{2}k}\right)\left(e^{-4-jk\pi}-1\right)\right] \\
&= \tfrac{1}{4}\left[\frac{1-e^{-4}(-1)^k}{2-\frac{j\pi}{2}k} + \frac{1-e^{-4}(-1)^k}{2+\frac{j\pi}{2}k}\right] = \tfrac{1}{4}\left[1-e^{-4}(-1)^k\right]\left[\frac{1}{2-\frac{j\pi}{2}k} + \frac{1}{2+\frac{j\pi}{2}k}\right] \\
&= \tfrac{1}{4}\left[1-e^{-4}(-1)^k\right]\left[\frac{2+\frac{j\pi}{2}k+2-\frac{j\pi}{2}k}{4+\frac{\pi^2}{4}k^2}\right] = \frac{1-e^{-4}(-1)^k}{4+\frac{\pi^2}{4}k^2} \\
&= \frac{4\left[1-e^{-4}(-1)^k\right]}{16+\pi^2k^2}.
\end{aligned}$$

Exercise 5.11

$\boxed{\text{R}}$ **Answer (b).**

Clearly, x is periodic with period $T = \frac{1}{2}$. So, x has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$$

From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt \\
&= \tfrac{1}{1/2} \int_{-1/4}^{1/4} \delta(t) e^{-jk4\pi t} dt \\
&= 2 \int_{-\infty}^{\infty} \delta(t) e^{-jk4\pi t} dt \\
&= 2 \left[e^{-jk4\pi t} \right] \Big|_{t=0} \\
&= 2.
\end{aligned}$$

Since the system is LTI, we have

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} H(\tfrac{2\pi}{T}k) c_k e^{jk(2\pi/T)t} \\
&= \sum_{k=-\infty}^{\infty} H(4\pi k) c_k e^{j4\pi kt} \\
&= H(-4\pi) c_{-1} e^{j4\pi(-1)t} + H(0) c_0 + H(4\pi) c_1 e^{j4\pi(1)t} \\
&= (1)(2) e^{-j4\pi t} + (1)(2) + (1)(2) e^{j4\pi t} \\
&= 2e^{-j4\pi t} + 2e^{j4\pi t} + 2 \\
&= 2\left(e^{j4\pi t} + e^{-j4\pi t}\right) + 2 \\
&= 4\cos(4\pi t) + 2.
\end{aligned}$$

$$\text{rect}(t) = \begin{cases} 1 & \square \le t < \square \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5.12

R **Answer (b).**

To begin, we observe that the function x satisfies the Dirichlet conditions. Therefore, at each point t_a of discontinuity of x, we have $y(t_a) = \frac{1}{2}[x(t_a^-) + x(t_a^+)]$. Thus, we have

$$\begin{aligned}
y(0) &= \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] \\
&= \tfrac{1}{2}[-25 + 1] \\
&= \tfrac{1}{2}[-24] \\
&= -12 \quad \text{and} \\
y(2) &= \tfrac{1}{2}\left[x(2^-) + x(2^+)\right] \\
&= \tfrac{1}{2}\left[e^2 + (-2^2)\right] \\
&= \tfrac{1}{2}\left[e^2 - 4\right] \\
&= \tfrac{e^2-4}{2}.
\end{aligned}$$

Exercise 5.13

R Answer (b).

We are given

$$c_k = \frac{4jk+4}{(jk-1)^2} \quad \text{and} \quad T = 4.$$

First, we compute the magnitude spectrum of x. We have

$$\begin{aligned}
|c_k| &= \left|\frac{4jk+4}{(jk-1)^2}\right| = \frac{|4jk+4|}{|(jk-1)^2|} = \frac{4\,|jk+1|}{\left(\sqrt{k^2+1}\right)^2} = \frac{4\sqrt{k^2+1}}{\left(\sqrt{k^2+1}\right)^2} \\
&= \frac{4}{\sqrt{k^2+1}}.
\end{aligned}$$

Next, we compute the phase spectrum of x. We have

$$\begin{aligned}
\arg c_k = \arg\left[\frac{4jk+4}{(jk-1)^2}\right] &= \arg(4jk+4) - \arg\left[(jk-1)^2\right] \\
&= \arctan\left(\frac{4k}{4}\right) - 2\left[\arctan\left(\frac{k}{-1}\right) + \pi\right] \\
&= \arctan(k) - 2\arctan(-k) - 2\pi = \arctan(k) + 2\arctan(k) - 2\pi \\
&= 3\arctan(k) - 2\pi.
\end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg c_k = 3\arctan(k) + 2\pi\ell$$

for all integer ℓ.

Exercise 5.14

R Answer (d).

We are given the Fourier series coefficient sequence c, where

$$c_k = j\,\mathrm{sgn}(k)e^{-|3k|}.$$

To begin, we observe that sgn is an odd function. So, we have

$$\begin{aligned} c_{-k} &= j\,\mathrm{sgn}(-k)e^{-|-3k|} \\ &= j[-\mathrm{sgn}(k)]e^{-|3k|} \\ &= -j\,\mathrm{sgn}(k)e^{-|3k|} \\ &= -c_k. \end{aligned}$$

Therefore, c is odd. (Or, alternatively, the sequence c is the product of the odd sequence $v_1(k) = \mathrm{sgn}(k)$ and the even sequence $v_2(k) = je^{-|3k|}$. Since the product of an odd sequence and an even sequence is an odd sequence, c is odd.) Since c is purely imaginary and the conjugate of a purely imaginary number is its negative, c being odd implies that c is conjugate symmetric. Since c is conjugate symmetric, x is real. Therefore, we conclude that x is real and odd.