Properties of the Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathrm{Re}(s) > 0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) < -a$
10	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
11	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
12	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$
13	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$

Linearity

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$a_1x_1(t)+a_2x_2(t) \overset{\text{LT}}{\longleftrightarrow} a_1X_1(s)+a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger (in the case that pole-zero cancellation occurs).

Example 7.8 Example 7.9

Time-Domain Shifting

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(t - t_0) \overset{\text{LT}}{\longleftrightarrow} e^{-st_0}X(s) \text{ with ROC } R,$$

  where t_0 is an arbitrary real constant.
- This is known as the **time-domain shifting property** of the Laplace transform.

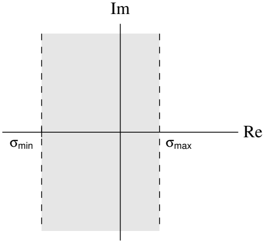
Example 7.10

Laplace-Domain Shifting

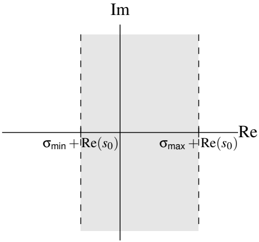
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t}x(t) \overset{\text{LT}}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R' = R + \operatorname{Re}(s_0),$$

  where s_0 is an arbitrary complex constant.
- This is known as the **Laplace-domain shifting property** of the Laplace transform.
- As illustrated below, the ROC R is ***shifted*** right by $\operatorname{Re}(s_0)$.

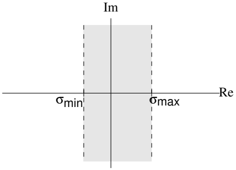


R

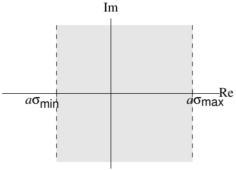


$R' = R + \operatorname{Re}(s_0)$

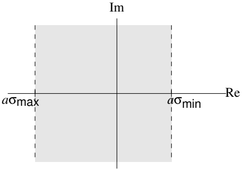
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then
$$x(at) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC } R' = aR,$$
where a is a nonzero real constant.
- This is known as the **(time-domain/Laplace-domain) scaling property** of the Laplace transform.
- As illustrated below, the ROC R is ***scaled*** and ***possibly flipped*** left to right.



R



$R' = aR,\ a > 0$



$R' = aR,\ a < 0$

Example 7.12

Conjugation

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x^*(t) \overset{\text{LT}}{\longleftrightarrow} X^*(s^*) \text{ with ROC } R.$$

- This is known as the **conjugation property** of the Laplace transform.

Example 7.13

Time-Domain Convolution

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$x_1 * x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2.$$

- This is known as the **time-domain convolution property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger than this intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the Laplace domain.
- Consequently, it is often much easier to work with LTI systems in the Laplace domain, rather than the time domain.

Time-Domain Differentiation

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC } R' \text{ containing } R.$$

- This is known as the **time-domain differentiation property** of the Laplace transform.
- The ROC R' always contains R but can be larger than R (if pole-zero cancellation occurs).
- Differentiation in the time domain becomes ***multiplication by*** s in the Laplace domain.
- Consequently, it can often be much easier to work with differential equations in the Laplace domain, rather than the time domain.

- If $x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$-tx(t) \stackrel{\text{LT}}{\longleftrightarrow} \frac{dX(s)}{ds} \text{ with ROC } R.$$

- This is known as the **Laplace-domain differentiation property** of the Laplace transform.

Time-Domain Integration

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s}X(s) \text{ with ROC } R' \text{ containing } R \cap \{\operatorname{Re}(s) > 0\}.$$

- This is known as the **time-domain integration property** of the Laplace transform.
- The ROC R' always contains at least $R \cap \{\operatorname{Re}(s) > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes ***division by*** s in the Laplace domain.
- Consequently, it is often much easier to work with integral equations in the Laplace domain, rather than the time domain.

Example 7.17

Initial Value Theorem

- For a function x with Laplace transform X, if x is ***causal*** and contains ***no impulses or higher order singularities at the origin***, then

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

  where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.
- This result is known as the **initial value theorem**.
- In situations where X is known but x is not, the initial value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $x(0^+)$.
- In practice, the values of functions at the origin are frequently of interest, as such values often convey information about the initial state of systems.
- The initial value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

Final Value Theorem

- For a function x with Laplace transform X, if x is ***causal*** and $x(t)$ has a ***finite limit*** as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

- This result is known as the **final value theorem**.
- In situations where X is known but x is not, the final value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $\lim_{t\to\infty} x(t)$.
- In practice, the values of functions at infinity are frequently of interest, as such values often convey information about the steady-state behavior of systems.
- The final value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

Example 7.18

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Section 7.4

Determination of Inverse Laplace Transform

Finding Inverse Laplace Transform

- Recall that the inverse Laplace transform x of X is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds,$$

  where $\mathrm{Re}(s) = \sigma$ is in the ROC of X.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in tables.

Example 7.27 Example 7.28

Section 7.5

Laplace Transform and LTI Systems

System Function of LTI Systems

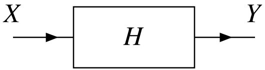
- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the Laplace transform of the impulse response).
- A LTI system is ***completely characterized*** by its system function H.
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the ***frequency response*** of the LTI system.

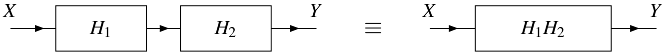
Block Diagram Representations of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.

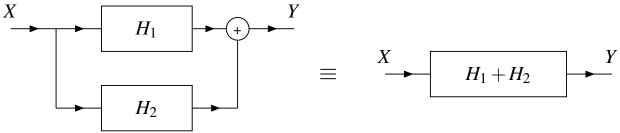


- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

- The ***series*** interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H_1 H_2$. That is, we have the equivalence shown below.



- The ***parallel*** interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with the system function $H_1 + H_2$. That is, we have the equivalence shown below.



Causality

- If a LTI system is ***causal***, its impulse response is causal, and therefore ***right sided***. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a ***causal*** LTI system is a ***RHP*** or the ***entire complex plane***.
- In general, the ***converse*** of the above theorem is ***not necessarily true***. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is ***rational***, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a ***rational*** system function H, ***causality*** of the system is ***equivalent*** to the ROC of H being the ***RHP to the right of the rightmost pole*** or, if H has no poles, the entire complex plane.

BIBO Stability

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function H contains the ***imaginary axis*** (i.e., $\mathrm{Re}(s) = 0$).
- **Theorem.** A ***causal*** LTI system with a (proper) ***rational*** system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have ***negative real parts***).

Example 7.32 Example 7.33 Example 7.34

Invertibility

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

LTI Systems and Differential Equations

- Many LTI systems of practical interest can be represented using an ***Nth-order linear differential equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\frac{d}{dt}\right)^k x(t),$$

  where the a_k and b_k are complex constants and $M \leq N$.
- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always ***rational***.

Example 7.36 Example 7.37

Section 7.6

Application: Circuit Analysis