ECE 260

EXAM 1
SOLUTIONS

(FALL 2023)

QUESTION 1

$$f(z) = \frac{z^2+2z+1}{(z^4-9z^2)^3} = \frac{(z+1)^2}{[z^2(z^2-9)]^3} = \frac{(z+1)^2}{z^6(z^2-9)^3}$$

$$= \frac{(z+1)^2}{z^6[(z+3)(z-3)]^3} = \frac{(z+1)^2}{z^6(z+3)^3(z-3)^3}$$

f has:

- 2^{nd} order zero at -1
- 6^{th} order pole at 0
- 3^{rd} order poles at -3 and 3

QUESTION 2

$$x(t) = x_1(t) + x_2(t) \quad ①$$

where

$$x_1(t) = \int_t^{\infty} \tau \, \delta(-3\tau - 1) \, d\tau \quad ②$$

$$x_2(t) = \int_{-6}^{6} \tau \cos(\tau) \delta(\tau + 10) \, d\tau \quad ③$$

consider ②

Let $\lambda = -3\tau - 1$
So $\tau = -\frac{1}{3}(\lambda + 1)$ and $d\tau = -\frac{1}{3} d\lambda$

$$\begin{aligned}
x_1(t) &= \int_t^{\infty} \tau \, \delta(-3\tau - 1) \, d\tau \\
&= \int_{-3t-1}^{-\infty} -\tfrac{1}{3}(\lambda+1) \delta(\lambda) \left(-\tfrac{1}{3}\right) d\lambda \\
&= -\int_{-\infty}^{-3t-1} \tfrac{1}{9}(\lambda+1) \delta(\lambda) \, d\lambda \\
&= -\int_{-\infty}^{-3t-1} \tfrac{1}{9}(1) \delta(\lambda) \, d\lambda \\
&= -\tfrac{1}{9} \int_{-\infty}^{-3t-1} \delta(\lambda) \, d\lambda \\
&= \begin{cases} -\frac{1}{9}(1) & -3t-1 \geq 0 \\ -\frac{1}{9}(0) & \text{otherwise} \end{cases}
\end{aligned}$$

$$-3t-1 \geq 0 \Rightarrow -1 \geq 3t \Rightarrow t \leq -\tfrac{1}{3}$$

$$\begin{aligned}
&= \begin{cases} -\frac{1}{9} & t \leq -\frac{1}{3} \\ 0 & \text{otherwise} \end{cases} \\
&= -\tfrac{1}{9} u\left(-t - \tfrac{1}{3}\right) \quad ④
\end{aligned}$$

consider ③

$$\begin{aligned}
x_2(t) &= \int_{-6}^{6} \tau \cos(\tau) \delta(\tau + 10) \, d\tau \\
&= \int_{-6}^{6} 0 \, d\tau \\
&= 0 \quad ⑤
\end{aligned}$$

combining ①, ④, and ⑤

$$\begin{aligned}
x(t) &= x_1(t) + x_2(t) \\
&= -\tfrac{1}{9} u\left(-t - \tfrac{1}{3}\right)
\end{aligned}$$

QUESTION 3

PART (A)

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1 x_1 + a_2 x_2\} = a_1 \mathcal{H} x_1 + a_2 \mathcal{H} x_2$$

PART (B)

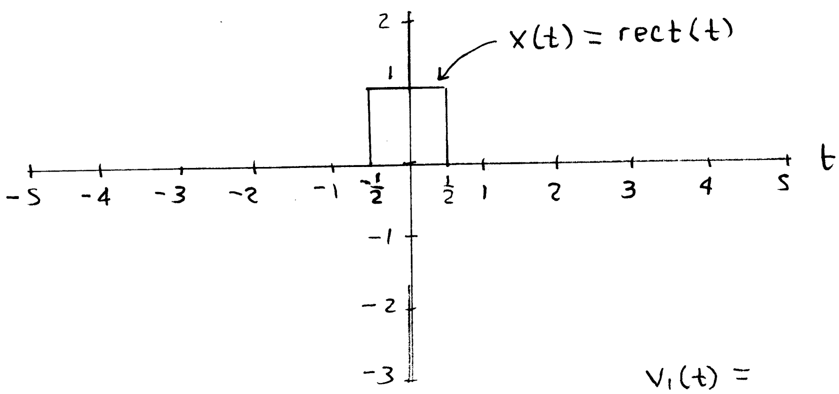
$$\mathcal{H}x(t) = 3x(t) - 1$$

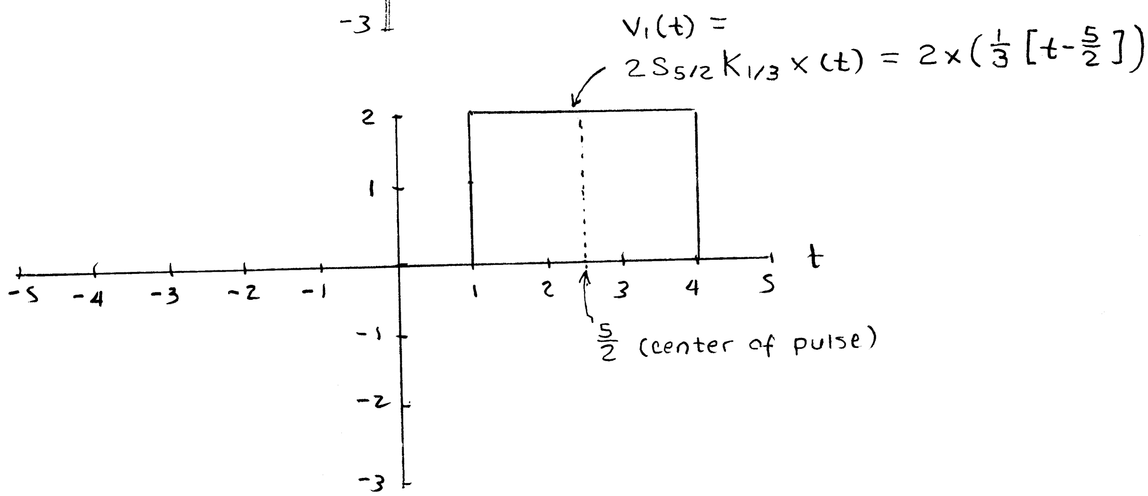
$$\mathcal{H}\{a_1 x_1 + a_2 x_2\}(t) = 3[a_1 x_1(t) + a_2 x_2(t)] - 1$$
$$= 3 a_1 x_1(t) + 3 a_2 x_2(t) - 1$$

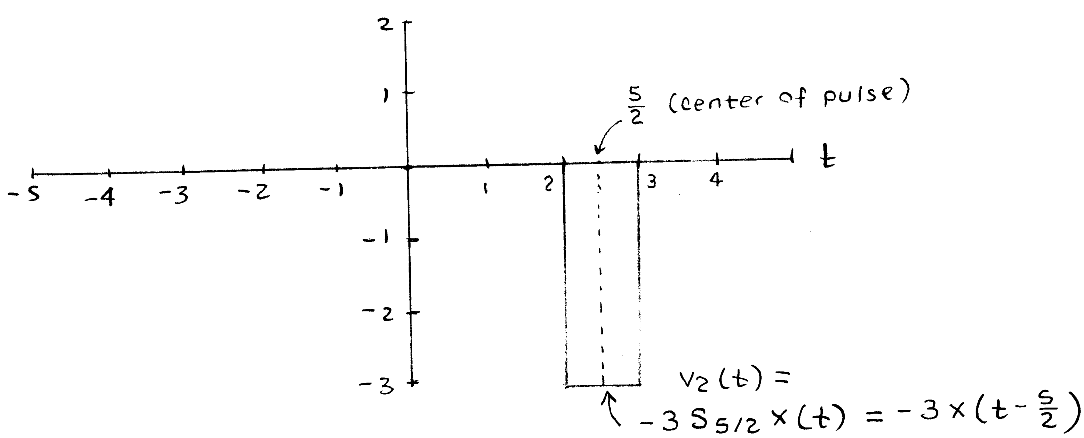
$$a_1 \mathcal{H} x_1(t) + a_2 \mathcal{H} x_2(t) = a_1 [3x_1(t) - 1] + a_2 [3x_2(t) - 1]$$
$$= 3a_1 x_1(t) - a_1 + 3a_2 x_2(t) - a_2$$

Since $\mathcal{H}\{a_1 x_1 + a_2 x_2\} = a_1 \mathcal{H}x_1 + a_2 \mathcal{H}x_2$ does not hold for all x_1 and x_2 and all a_1 and a_2, $\mathcal{H}$ is not linear.

QUESTION 4







$$
\begin{aligned}
y(t) &= v_1(t) + v_2(t) \\
&= 2 S_{5/2} K_{1/3}\, x(t) - 3 S_{5/2}\, x(t) \\
&= 2x\left(\tfrac{1}{3}\left[t-\tfrac{5}{2}\right]\right) - 3x\left(t-\tfrac{5}{2}\right)
\end{aligned}
$$

Note: $S_{t_0} x(t) = x(t-t_0)$ and $K_a x(t) = x(at)$.

QUESTION 5

$$\mathcal{H}x(t) = a x^2(t) + b$$

eigenfunction $x_1(t) = 1$ has eigenvalue $\lambda_1 = -3$

eigenfunction $x_2(t) = -2$ has eigenvalue $\lambda_2 = 3$

$$\mathcal{H}x_1(t) = a x_1^2(t) + b = a(1)^2 + b = a + b \quad ①$$

$$\mathcal{H}x_1(t) = \lambda_1 x_1(t) = -3(1) = -3 \quad ②$$

$$\mathcal{H}x_2(t) = a x_2^2(t) + b = a(-2)^2 + b = 4a + b \quad ③$$

$$\mathcal{H}x_2(t) = \lambda_2 x_2(t) = 3(-2) = -6 \quad ④$$

equating ① and ②, we have

$$a + b = -3 \Rightarrow b = -a - 3 \quad ⑤$$

equating ③ and ④, and substituting ⑤

$$4a + b = -6 \Rightarrow 4a + (-a-3) = -6 \Rightarrow 4a - a - 3 = -6 \Rightarrow$$
$$3a - 3 = -6 \Rightarrow 3a = -3 \Rightarrow a = -1 \quad ⑥$$

from ⑤ and ⑥

$$b = -a - 3 = -(-1) - 3 = 1 - 3 = -2$$

Therefore, $a = -1$ and $b = -2$.

QUESTION 6

The simplest approach is to use element-wise operations, leading to a solution like the following:

```
function B = foo(A)
    B = [A >= 1 & A < 3] .* (A - 1) .^ 2 + [A >= 3] * 4;
end
```

Alternatively, an approach based on iteration can be employed, leading to a much more verbose solution like the following:

```
function B = foo(A)
    for r = 1 : height(A)
        for c = 1 : width(A)
            a = A(r, c);
            if a < 1
                B(r, c) = 0;
            elseif a < 3
                B(r, c) = (a - 1) ^ 2;
            else
                B(r, c) = 4;
            end
        end
    end
end
```