Example 6.35 (Frequency response to differential equation)**.** A LTI system with input x and output y has the frequency response

$$H(\omega) = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Since system is LTI,
$Y(w) = X(w)H(w) \Rightarrow$
$H(w) = \dfrac{Y(w)}{X(w)}$

Find the differential equation that characterizes this system.

Solution. From the given frequency response H, we have

$$\frac{Y(\omega)}{X(\omega)} = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Multiplying both sides by $(-5\omega^2 + 2)X(\omega)$, we have

$$-5\omega^2 Y(\omega) + 2Y(\omega) = -7\omega^2 X(\omega) + 11j\omega X(\omega) + 3X(\omega).$$

write with powers of jw

Applying some simple algebraic manipulation yields

$$5(j\omega)^2 Y(\omega) + 2Y(\omega) = 7(j\omega)^2 X(\omega) + 11(j\omega)X(\omega) + 3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$\left(\frac{d}{dt}\right)^n x(t) \overset{FT}{\longleftrightarrow} (jw)^n X(w)$$

$$5y''(t) + 2y(t) = 7x''(t) + 11x'(t) + 3x(t).$$ ■

Example 6.38 (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(t) = \tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t).$$

from FT table:

$1 \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega)$

Using frequency-domain methods, find the response y of the system to the input

$$x(t) = -1 + 2\cos(2t) + \cos(4t) - \cos(6t).$$

$\cos(\omega_0 t) \overset{FT}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

Solution. Taking the Fourier transform of x, we have

taking FT

$$\begin{aligned} X(\omega) &= -2\pi\delta(\omega) + 2(\pi[\delta(\omega-2)+\delta(\omega+2)]) + \pi[\delta(\omega-4)+\delta(\omega+4)] - \pi[\delta(\omega-6)+\delta(\omega+6)] \\ &= -\pi\delta(\omega+6) + \pi\delta(\omega+4) + 2\pi\delta(\omega+2) - 2\pi\delta(\omega) + 2\pi\delta(\omega-2) + \pi\delta(\omega-4) - \pi\delta(\omega-6). \end{aligned}$$

The frequency spectrum X is shown in Figure 6.22(a). Now, we compute the frequency response H of the system. Using the results of Example 6.36, we can determine H to be

Example 6.36 found the FT pair

$\frac{2\omega_b}{\pi}\operatorname{sinc}(\omega_b t)\cos(\omega_a t) \overset{FT}{\longleftrightarrow} \operatorname{rect}\left(\frac{\omega-\omega_a}{2\omega_b}\right) + \operatorname{rect}\left(\frac{\omega+\omega_a}{2\omega_b}\right)$

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t)\}(\omega) \\ &= \operatorname{rect}\left(\tfrac{\omega-4}{2}\right) + \operatorname{rect}\left(\tfrac{\omega+4}{2}\right) \\ &= \begin{cases} 1 & 3 \le |\omega| \le 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

using result from Example 6.36 with $\omega_b = 1$, $\omega_a = 4$

definition of rect function

The frequency response H is shown in Figure 6.22(b). The frequency spectrum Y of the output is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi\delta(\omega+4) + \pi\delta(\omega-4). \end{aligned}$$

only two shifted delta functions are nonzero when $H(\omega) \neq 0$ [see Figures 6.22(a) and (b).]

Taking the inverse Fourier transform, we obtain

taking inverse FT

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\pi\delta(\omega+4) + \pi\delta(\omega-4)\}(t) \\ &= \mathcal{F}^{-1}\{\pi[\delta(\omega+4) + \delta(\omega-4)]\}(t) \\ &= \cos(4t). \end{aligned}$$

from table of FT pairs

$\cos(\omega_0 t) \overset{FT}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

■

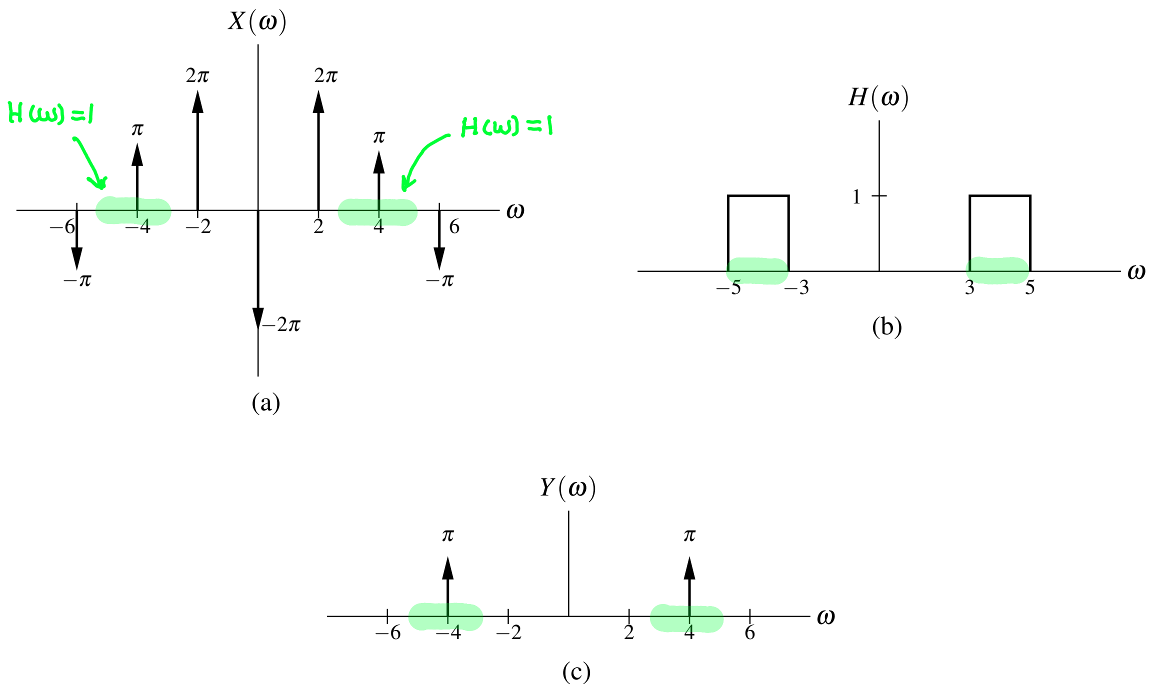

Figure 6.22: Frequency spectra for bandpass filtering example. (a) Frequency spectrum of the input x. (b) Frequency response of the system. (c) Frequency spectrum of the output y.

Example 6.40 (Simple RL network)**.** Consider the resistor-inductor (RL) network shown in Figure 6.26 with input v_1 and output v_2. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response H of the system. (b) Find the response v_2 of the system to the input $v_1(t) = \operatorname{sgn} t$.

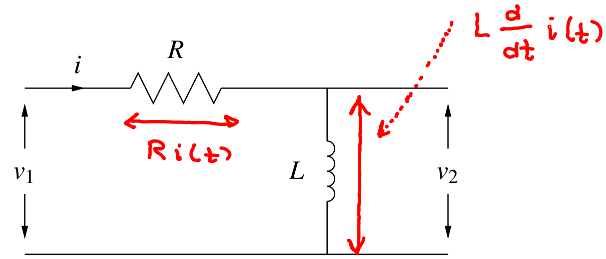

Figure 6.26: Simple RL network.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{6.35}$$

$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{6.36}$$

(Recall that the voltage v across an inductor L is related to the current i through the inductor as $v(t) = L\frac{d}{dt}i(t)$.) Taking the Fourier transform of (6.35) and (6.36) yields

using time-domain differentiation property
$\frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega X(\omega)$

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \tag{6.37}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{6.38}$$

From (6.37) and (6.38), we have

(*) Since System is LTI,
$V_2(\omega) = V_1(\omega) H(\omega) \Rightarrow$
$H(\omega) = \frac{V_2(\omega)}{V_1(\omega)}$

$$\begin{aligned} H(\omega) &= \overset{(*)}{\frac{V_2(\omega)}{V_1(\omega)}} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{6.39}$$

substitute (6.38) in numerator and (6.37) in denominator

cancel I's

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \operatorname{sgn} t$ (as given). Taking the Fourier transform of the input v_1 (with the aid of Table 6.2), we have

$= F\{\operatorname{sgn} t\}(\omega)$ from FT table

$$V_1(\omega) = \frac{2}{j\omega}. \tag{6.40}$$

From the definition of the system, we know

from (*)

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{6.41}$$

Substituting (6.40) and (6.39) into (6.41), we obtain

$V_1(\omega)$ $H(\omega)$

substitute

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

cancel factors of $j\omega$