Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt \\
&= \int_{0}^{\infty} e^{at}\,dt \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}$$

split integration interval and use fact that $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

drop zero integral

identify two cases for integration

integrate

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}$$

what is $e^{a\infty}$?

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\
&= -\tfrac{1}{a}.
\end{aligned}$$

assuming $a<0$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

combining above results

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ∎

Example 4.15. Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \quad (1)$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

using (1) and $h = \mathcal{H}\delta$

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\,d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

integral is 1 if integration interval includes origin

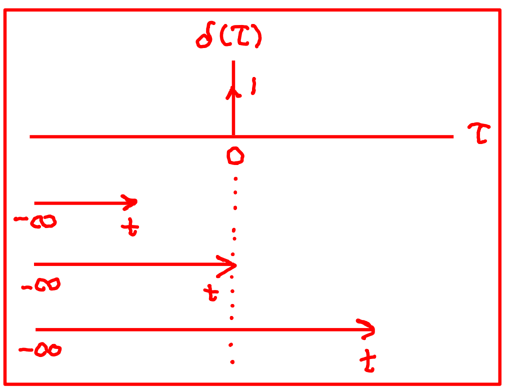
definition of unit-step function

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} 1\,dt \\ &= \infty. \end{aligned}$$

$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable. ■



Theorem 4.12 (Eigenfunctions of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{st}$, where s is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{4.49}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(s)$.

Proof. We have

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{commutative property of convolution} \\
&= h * x(t) && \text{definition of convolution} \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau && \text{substitute given function } x \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau && \text{factor out } e^{st} \\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\
&= H(s)e^{st}. && \blacksquare
\end{aligned}$$

call this $H(s)$

Suppose that we have a LTI system $\mathcal{H}$ with input x, output y, impulse response h, and system function H. Suppose now that we can express some arbitrary input signal x as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}. \quad 1$$

(As it turns out, many functions can be expressed in this way.) From the eigenfunction properties of LTI systems, the response of the system to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) && \text{substitute 1 for x} \\ &= \mathcal{H}\left\{\sum_k a_k e^{s_k t}\right\}(t) && \text{linearity of } \mathcal{H} \\ &= \sum_k a_k \mathcal{H}\{e^{s_k t}\}(t) && \text{complex exponentials are eigenfunctions of LTI systems} \\ &= \sum_k a_k H(s_k) e^{s_k t}. \end{aligned}$$

Thus, we have that

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{4.48}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. Furthermore, observe that the relationship between the input $x(t) = \sum_k a_k e^{s_k t}$ and output y in (4.48) does not involve convolution (such as in the equation $y = x * h$). In fact, the formula for y is identical to that for x except for the insertion of a constant multiplicative factor $H(s_k)$. In effect, we have used eigenfunctions to replace convolution with the much simpler operation of multiplication by a constant.