Chapter 3

Continuous-Time Fourier Series (Chapter 4)

4.1 Find the Fourier series representation (in complex exponential form) of each of the signals given below. In each case, explicitly identify the fundamental period and Fourier series coefficient sequence c_k.
(a) $x(t) = 1 + \cos \pi t + \sin^2 \pi t$;
(b) $x(t) = [\cos 4t][\sin t]$; and
(c) $x(t) = |\sin 2\pi t|$. [Hint: $\int e^{ax} \sin bx dx = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2+b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

Solution.

(a) We can rewrite $x(t)$ in the form of a Fourier series by simple algebraic manipulation as follows:

$$\begin{aligned} x(t) &= 1 + \cos \pi t + \sin^2 \pi t \\ &= 1 + \tfrac{1}{2}[e^{j\pi t} + e^{-j\pi t}] + \left[\tfrac{1}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right]\right]^2 \\ &= 1 + \tfrac{1}{2}e^{j\pi t} + \tfrac{1}{2}e^{-j\pi t} - \tfrac{1}{4}[e^{j2\pi t} - 2 + e^{-j2\pi t}] \\ &= -\tfrac{1}{4}e^{-j2\pi t} + \tfrac{1}{2}e^{-j\pi t} + \tfrac{3}{2} + \tfrac{1}{2}e^{j\pi t} - \tfrac{1}{4}e^{j2\pi t}. \end{aligned}$$

Thus, we have that

$$x(t) = \sum_{k=\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = \pi$ (i.e., $T = 2$) and

$$c_k = \begin{cases} \frac{3}{2} & \text{for } k = 0 \\ \frac{1}{2} & \text{for } k = \pm 1 \\ -\frac{1}{4} & \text{for } k = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

(c) The signal $x(t)$ is periodic with period $T = \frac{1}{2}$ and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1/2} = 4\pi$. From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{1/2}\int_0^{1/2} |\sin 2\pi t|\, e^{-jk4\pi t}dt \\
&= 2\int_0^{1/2} e^{-j4\pi kt}\sin 2\pi t dt \\
&= 2\left[\frac{e^{-j4\pi kt}[-j4\pi k\sin 2\pi t - 2\pi\cos 2\pi t]}{(-j4\pi k)^2+(2\pi)^2}\right]\Bigg|_0^{1/2} \\
&= \tfrac{2(2\pi)}{-16\pi^2k^2+4\pi^2}\left[e^{-j4\pi kt}\left[-j2k\sin 2\pi t-\cos 2\pi t\right]\right]\Big|_0^{1/2} \\
&= \tfrac{1}{\pi(1-4k^2)}\left[e^{-j4\pi k/2}[-j2k\sin 2\pi/2-\cos 2\pi/2]-[-\cos 0]\right] \\
&= \tfrac{1}{\pi(1-4k^2)}[e^{-j2\pi k}[-j2k\sin\pi-\cos\pi]+\cos 0] \\
&= \tfrac{1}{\pi(1-4k^2)}[2] \\
&= \frac{2}{\pi(1-4k^2)}.
\end{aligned}$$

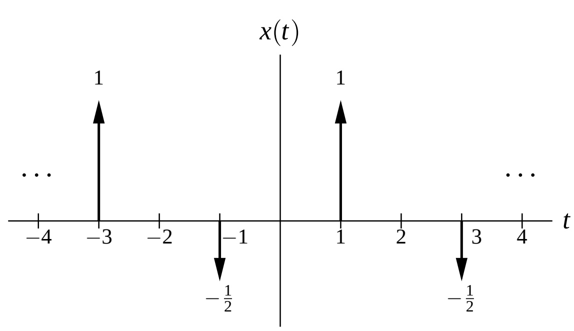
Since the integral table entry given (i.e., for the integral $\int e^{ax}\sin bx dx$) is valid for the case of $a=0$, we did not need to assume that $k \neq 0$ in the above integration. Therefore, the above expression is valid for all k. Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

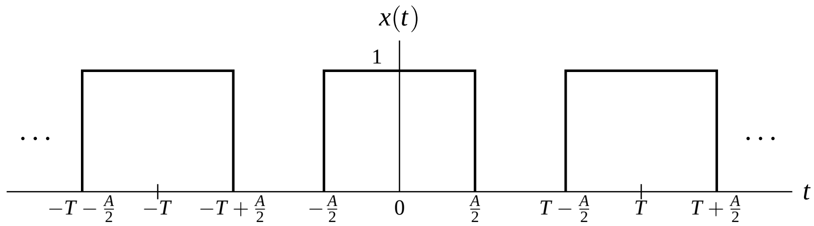
where $\omega_0 = 4\pi$ and

$$c_k = \frac{2}{\pi(1-4k^2)}.$$

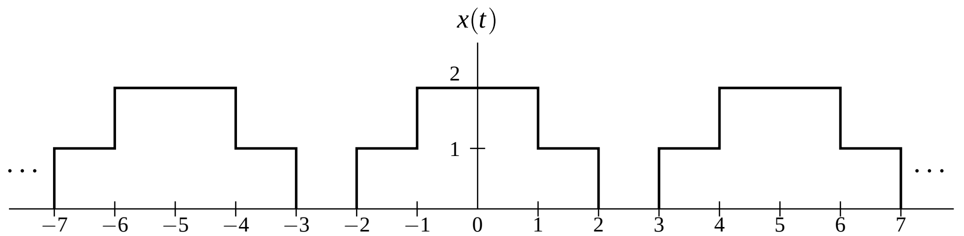
4.2 For each of the signals shown in the figure below, find the corresponding Fourier series coefficient sequence.



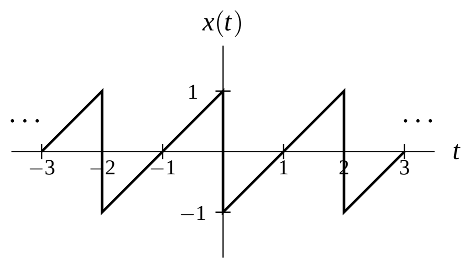
(a)



(b)



(c)



(d)

Solution.

(a) We calculate the fundamental frequency ω_0 as

$$\omega_0 = \tfrac{2\pi}{T} = \tfrac{2\pi}{4} = \tfrac{\pi}{2}.$$

(So, we have $T = \frac{2\pi}{\pi/2} = 4$.) From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{4}\int_{-2}^{2}[\delta(t-1) - \tfrac{1}{2}\delta(t+1)]e^{-j\pi kt/2}dt \\
&= \tfrac{1}{4}\left[\int_{-2}^{2}\delta(t-1)e^{-j\pi kt/2}dt - \tfrac{1}{2}\int_{-2}^{2}\delta(t+1)e^{-j\pi kt/2}dt\right] \\
&= \tfrac{1}{4}[e^{-j\pi k/2} - \tfrac{1}{2}e^{j\pi k/2}] \\
&= \tfrac{1}{4}e^{-j\pi k/2} - \tfrac{1}{8}e^{j\pi k/2}.
\end{aligned}$$

(c) The signal $x(t)$ is periodic with period $T = 5$ and frequency $\omega_0 = \frac{2\pi}{5}$. From the Fourier series analysis equation, we can write

$$
\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{5}\int_{-5/2}^{5/2} x(t)e^{-j2\pi kt/5}dt \\
&= \tfrac{1}{5}\left[\int_{-2}^{-1} e^{-j2\pi kt/5}dt + \int_{-1}^{1} 2e^{-j2\pi kt/5}dt + \int_{1}^{2} e^{-j2\pi kt/5}dt\right] \\
&= \tfrac{1}{5}\left[\int_{-2}^{2} e^{-j2\pi kt/5}dt + \int_{-1}^{1} e^{-j2\pi kt/5}\right] \\
&= \tfrac{1}{5}\left[\left[\tfrac{1}{-j2\pi k/5}e^{-j2\pi kt/5}\right]\Big|_{-2}^{2} + \left[\tfrac{1}{-j2\pi k/5}e^{-j2\pi kt/5}\right]\Big|_{-1}^{1}\right] \\
&= \tfrac{1}{-j2\pi k}\left[e^{-j2\pi kt/5}\Big|_{-2}^{2} + e^{-j2\pi kt/5}\Big|_{-1}^{1}\right] \\
&= \tfrac{1}{-j2\pi k}\left[e^{-j4\pi k/5} - e^{j4\pi k/5} + e^{-j2\pi k/5} - e^{j2\pi k/5}\right] \\
&= \tfrac{1}{-j2\pi k}[-2j\sin 4\pi k/5 - 2j\sin 2\pi k/5] \\
&= \tfrac{1}{\pi k}[\sin 4\pi k/5 + \sin 2\pi k/5] \\
&= \frac{\sin 4\pi k/5}{\pi k} + \frac{\sin 2\pi k/5}{\pi k} \\
&= \tfrac{4}{5}\operatorname{sinc} 4\pi k/5 + \tfrac{2}{5}\operatorname{sinc} 2\pi k/5.
\end{aligned}
$$

In the above derivation, we assumed that $k \neq 0$. So, now we must consider the case of $k = 0$. From the Fourier series analysis equation, we have

$$
\begin{aligned}
c_0 &= \tfrac{1}{T}\int_T x(t)dt \\
&= \tfrac{1}{5}\int_{-5/2}^{5/2} x(t)dt \\
&= \tfrac{1}{5}\left[\int_{-2}^{-1} dt + \int_{-1}^{1} 2dt + \int_{1}^{2} dt\right] \\
&= \tfrac{1}{5}[1+4+1] \\
&= \tfrac{6}{5}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
c_k &= \begin{cases} \tfrac{6}{5} & \text{for } k = 0 \\ \tfrac{4}{5}\operatorname{sinc} 4\pi k/5 + \tfrac{2}{5}\operatorname{sinc} 2\pi k/5 & \text{otherwise} \end{cases} \\
&= \tfrac{4}{5}\operatorname{sinc} 4\pi k/5 + \tfrac{2}{5}\operatorname{sinc} 2\pi k/5.
\end{aligned}
$$

The first few coefficients are approximately as follows:

$$c_0 = 1.2, \quad c_1 = c_{-1} \approx 0.489828, \quad \text{and} \quad c_2 = c_{-2} \approx -0.057816.$$

4.6 A periodic signal $x(t)$ with period T and Fourier series coefficient sequence c_k is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if $x(t)$ is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all t, then $x(t)$ is odd harmonic.

Solution.

(b) From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \int_{T/2}^{T} x(t)e^{-jk\omega_0 t}dt\right].
\end{aligned}$$

Now, we employ a change a variable in the second integral. Let $\lambda = t + T/2$ so that $t = \lambda - T/2$ and $d\lambda = dt$. Applying this change of variable, we obtain

$$\begin{aligned}
c_k &= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \int_{T}^{3T/2} x(\lambda - \tfrac{T}{2})e^{-jk\omega_0(\lambda - T/2)}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \int_{T}^{3T/2} -x(\lambda)e^{-jk\omega_0(\lambda - T/2)}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - \int_{T}^{3T/2} x(\lambda)e^{jk\omega_0 T/2}e^{-jk\omega_0\lambda}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - e^{jk\pi}\int_{T}^{3T/2} x(\lambda)e^{-jk\omega_0\lambda}d\lambda\right].
\end{aligned}$$

Now, we rename the dummy variable of integration in the second integral from λ to t. This yields

$$\begin{aligned}
c_k &= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - e^{j\pi k}\int_{T}^{3T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - (-1)^k\int_{0}^{T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \tfrac{1}{T}\left[(1-(-1)^k)\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \begin{cases} \tfrac{2}{T}\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt & \text{for odd } k \\ 0 & \text{for even } k. \end{cases}
\end{aligned}$$

Therefore, $c_k = 0$ for even k.

ALTERNATIVE SOLUTION. From the Fourier series analysis equation, we have

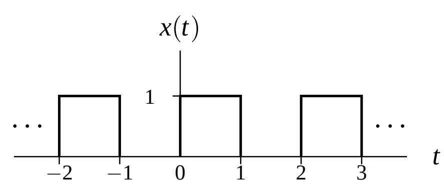
$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{T}\int_T (-x(t-T/2))e^{-jk\omega_0 t}dt \\
&= -\tfrac{1}{T}\int_T x(t-T/2)e^{-jk\omega_0 t}dt \\
&= -\tfrac{1}{T}\int_\alpha^{\alpha+T} x(t-T/2)e^{-jk\omega_0 t}dt.
\end{aligned}$$

Now, we employ a change of variable. Let $v = t - T/2$ so that $t = v + T/2$ and $dv = dt$. Applying the change of variable, we obtain

$$
\begin{aligned}
c_k &= -\tfrac{1}{T}\int_{\alpha-T/2}^{\alpha+T/2} x(v)e^{-jk\omega_0(v+T/2)}dv \\
&= -\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}e^{-jk\omega_0 T/2}dv \\
&= -\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}e^{-jk(2\pi/2)}dv \\
&= (-1)^k\left(-\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}dv\right) \\
&= (-1)^k(-c_k) \\
&= (-1)^{k+1}c_k.
\end{aligned}
$$

So, we have that $c_k = (-1)^{k+1}c_k$. If k is even, then $c_k = -c_k$. This implies, however, that $c_k = 0$. Therefore, for even k, we have that $c_k = 0$.

4.8 Find the Fourier series coefficient sequence c_k of the periodic signal $x(t)$ shown in the figure below. Plot the frequency spectrum of this signal including the first five harmonics.



Solution.

The signal $x(t)$ is periodic with period $T = 2$, and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$. From the Fourier series analysis equation, we have

$$
\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{2}\int_0^2 x(t)e^{-j\pi kt}dt \\
&= \tfrac{1}{2}\int_0^1 e^{-j\pi kt}dt \\
&= \tfrac{1}{2}\left[\tfrac{1}{-j\pi k}e^{-j\pi kt}\right]\Big|_0^1 \\
&= \tfrac{-1}{j2\pi k}\left[e^{-j\pi kt}\right]\Big|_0^1 \\
&= \tfrac{1}{j2\pi k}\left[1-e^{-j\pi k}\right] \\
&= \tfrac{1}{j2\pi k}\left[1-(-1)^k\right] \\
&= \begin{cases} -\frac{j}{\pi k} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}
\end{aligned}
$$

Since we assumed that $k \neq 0$ in the derivation above, we must now consider the case of $k = 0$. From the Fourier series analysis equation, we have

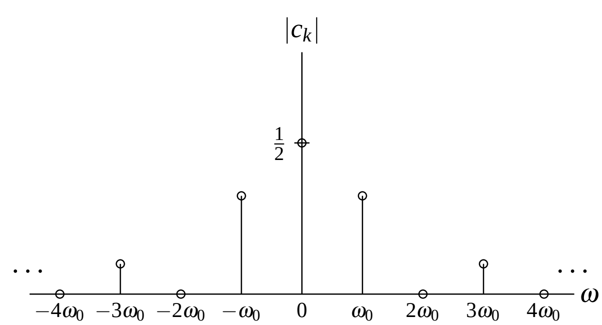
$$
\begin{aligned}
c_0 &= \tfrac{1}{T}\int_T x(t)\,dt \\
&= \tfrac{1}{2}\int_0^2 x(t)\,dt \\
&= \tfrac{1}{2}\int_0^1 dt \\
&= \tfrac{1}{2}[t]\big|_0^1 \\
&= \tfrac{1}{2}.
\end{aligned}
$$

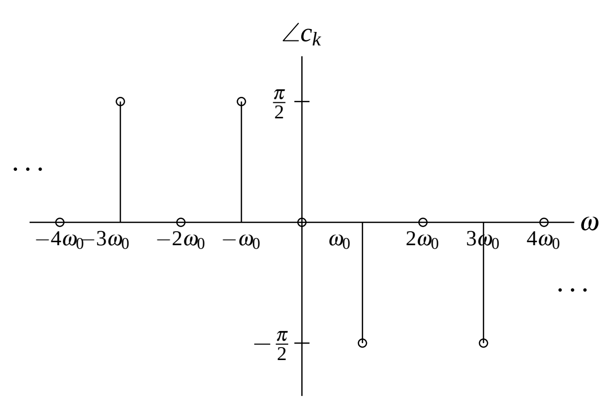
Thus, we have

$$
c_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ -\frac{j}{\pi k} & \text{for odd } k \\ 0 & \text{for even } k,\ k \neq 0 \end{cases}
$$

Calculating the first several Fourier series coefficients yields the following:

k	$\lvert c_k \rvert$	$\arg c_k$
0	$\frac{1}{2}$	0
1	$\frac{1}{\pi}$	$-\frac{\pi}{2}$
2	0	0
3	$\frac{1}{3\pi}$	$-\frac{\pi}{2}$
4	0	0
5	$\frac{1}{5\pi}$	$-\frac{\pi}{2}$





4.9 Suppose that we have a LTI system with frequency response

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output $y(t)$ of the system if the input $x(t)$ is given by

$$x(t) = 1 + 2\cos 2t + 2\cos 4t + \tfrac{1}{2}\cos 6t.$$

Solution.

We begin by finding the Fourier series representation of $x(t)$. Using Euler's relation, we can rewrite $x(t)$ as

$$\begin{aligned} x(t) &= 1 + 2\cos 2t + 2\cos 4t + \tfrac{1}{2}\cos 6t \\ &= 1 + 2\left[\tfrac{1}{2}(e^{j2t} + e^{-j2t})\right] + 2\left[\tfrac{1}{2}(e^{j4t} + e^{-j4t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6t} + e^{-j6t})\right] \\ &= 1 + e^{j2t} + e^{-j2t} + e^{j4t} + e^{-j4t} + \tfrac{1}{4}e^{j6t} + \tfrac{1}{4}e^{-j6t}. \end{aligned}$$

Thus, we have that the Fourier series representation of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2$ and

$$a_k = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = \pm 1 \\ 1 & \text{for } k = \pm 2 \\ \frac{1}{4} & \text{for } k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Since the system is LTI, we know that the output $y(t)$ has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $b_k = a_k H(jk\omega_0)$. Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned} b_0 &= a_0 H(j[0][2]) = 0, \\ b_1 &= a_1 H(j[1][2]) = 0, \\ b_{-1} &= a_{-1} H(j[-1][2]) = 0, \\ b_2 &= a_2 H(j[2][2]) = 0, \\ b_{-2} &= a_{-2} H(j[-2][2]) = 0, \\ b_3 &= a_3 H(j[3][2]) = \tfrac{1}{4}(1) = \tfrac{1}{4}, \quad \text{and} \\ b_{-3} &= a_{-3} H(j[-3][2]) = \tfrac{1}{4}(1) = \tfrac{1}{4}. \end{aligned}$$

Thus, we have

$$b_k = \begin{cases} \frac{1}{4} & \text{for } k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the output $y(t)$ is given by

$$\begin{aligned} y(t) &= \tfrac{1}{4}e^{-j6t} + \tfrac{1}{4}e^{j6t} \\ &= \tfrac{1}{4}[e^{-j6t} + e^{j6t}] \\ &= \tfrac{1}{4}[2\cos 6t] \\ &= \tfrac{1}{2}\cos 6t. \end{aligned}$$

4.101 Consider the periodic signal $x(t)$ shown in Figure B of Problem 4.2 where $T = 1$ and $A = \frac{1}{2}$. We can show that this signal $x(t)$ has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2}\operatorname{sinc}\frac{\pi k}{2}$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N(t)$ converges to $x(t)$. [**HINT:** You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Please note that the MATLAB `sinc` function is NOT defined in the same way as in the lecture notes. The MATLAB `sinc` function is defined as $\operatorname{sinc} x = (\sin(\pi x))/(\pi x)$. So, it might be wise to avoid the use of this MATLAB function altogether.]
(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N(t)$ converge to $x(t)$ uniformly (i.e., equally fast) everywhere? If not, where is the rate of convergence slower?
(c) The signal $x(t)$ is not continuous everywhere. For example, the signal has a discontinuity at $t = \frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N(t)$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

Solution.

(a) The graphs necessary in this problem can be generated using the code given below.

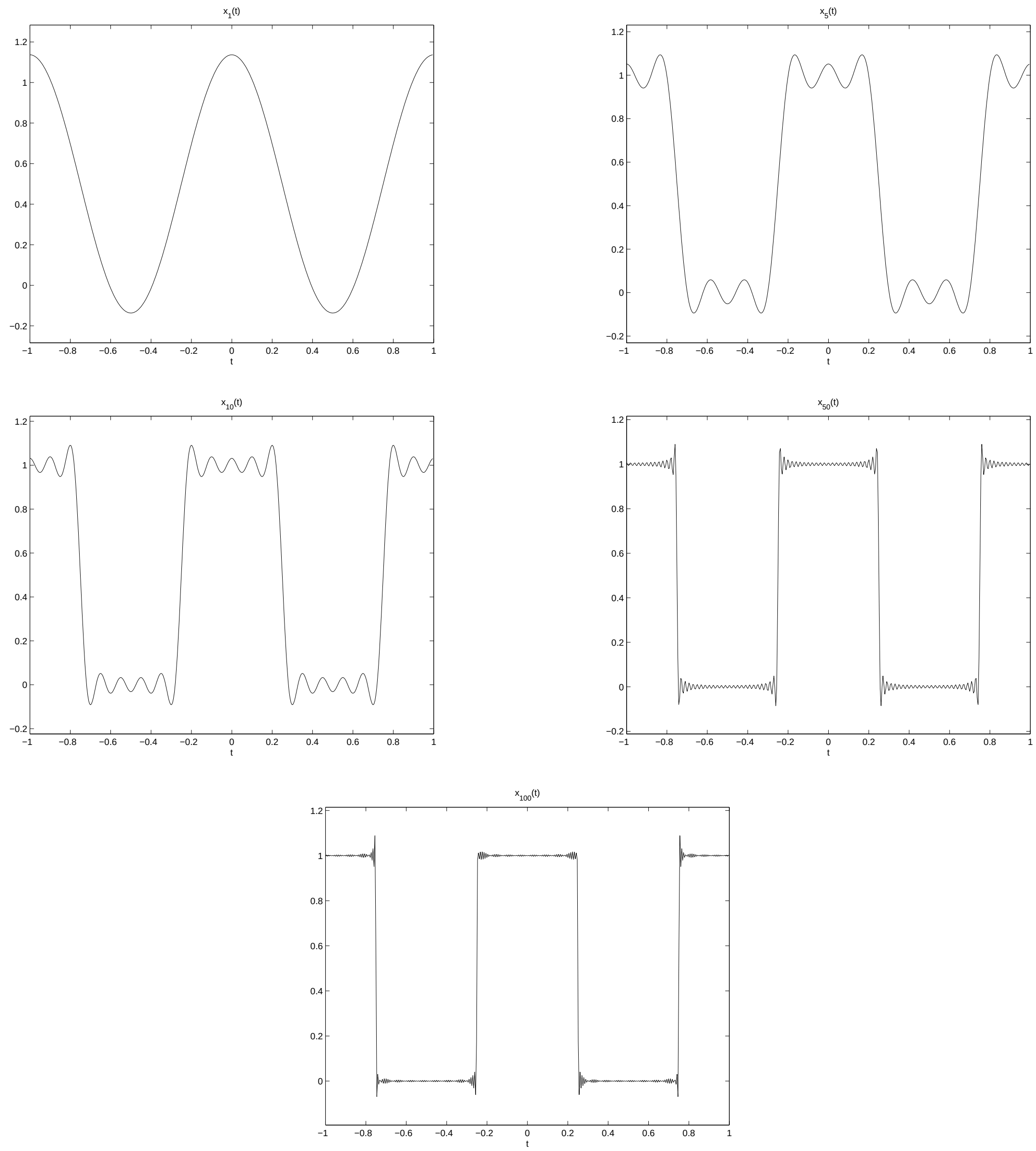
Listing 3.1: main.m

```
clear all

for n = [1 5 10 50 100]
    % Sum the appropriate number of terms.
    f = symsum(sym('0.5 * mysinc(pi * k / 2) * exp(j * k * w * t)'), ...
      'k', -n, n);
    % Plot the result.
    ezplot(subs(f, 'w', 2 * pi), [-1 1]);
    title(sprintf('x_{%d}(t)', n));
    % Pause for a moment so that the graph does not disappear too quickly.
    pause(1);
    % Print the graph to a file.
    eval(sprintf('print -dps data/sqrwav_%d.ps', n));
end
```

Listing 3.2: mysinc.m

```
function y = mysinc(x)
% mysinc - Compute the sinc function (as defined in the lecture notes)
% sinc(x) returns a matrix whose elements are the sinc of the
% elements of x

% Initialize the output array to all ones.
y = ones(size(x));
% Determine the indices of all nonzero elements in the input array.
i = find(x);
% Compute the sinc function for all nonzero elements.
% The zero elements are already covered, since the output
% array was initialized to all ones above.
y(i) = sin(x(i)) ./ (x(i));
return
```

Using the above code, we obtain the graphs given below.



(b) The function $\hat{x}_N(t)$ does not converge to $x(t)$ uniformly (i.e., at the same rate everywhere). The rate of convergence is (relatively) lower at/near the points of discontinuity of $x(t)$.

(c) At the point of discontinuity of $x(t)$ located at $t = \frac{1}{4}$, the function $\hat{x}_N(t)$ appears to converge to the average of the left and right limits of $x(t)$ at that point, namely the value of $\frac{1}{2}$.

