Suppose that we have a complex periodic function x with period T and Fourier series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (5.2). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left.\left[\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\right]\right|_{k=0} \\ &= \frac{1}{T}\int_T x(t)e^{0}dt \\ &= \frac{1}{T}\int_T x(t)dt. \end{aligned}$$

from Fourier series analysis equation

evaluate at $k=0$

$e^0 = 1$

Thus, c_0 is simply the average value of x over a single period.

Example 5.7. The periodic square wave x in Example 5.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) x has the most information.

Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. We have

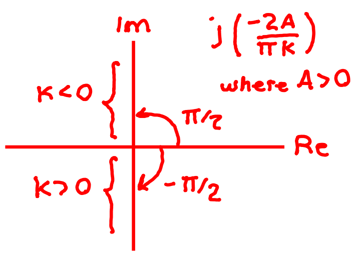
$$\begin{aligned} |c_k| &= \begin{cases} \left|\frac{-j2A}{\pi k}\right| & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases} \end{aligned}$$

$\left|\frac{-j2A}{\pi k}\right| = \frac{|-j2A|}{|\pi k|} = \frac{2A}{\pi|k|}$

(since $|a/b| = |a|/|b|$ and $|ab| = |a||b|$)

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} = -\frac{\pi}{2} \operatorname{sgn} k$, we have

①

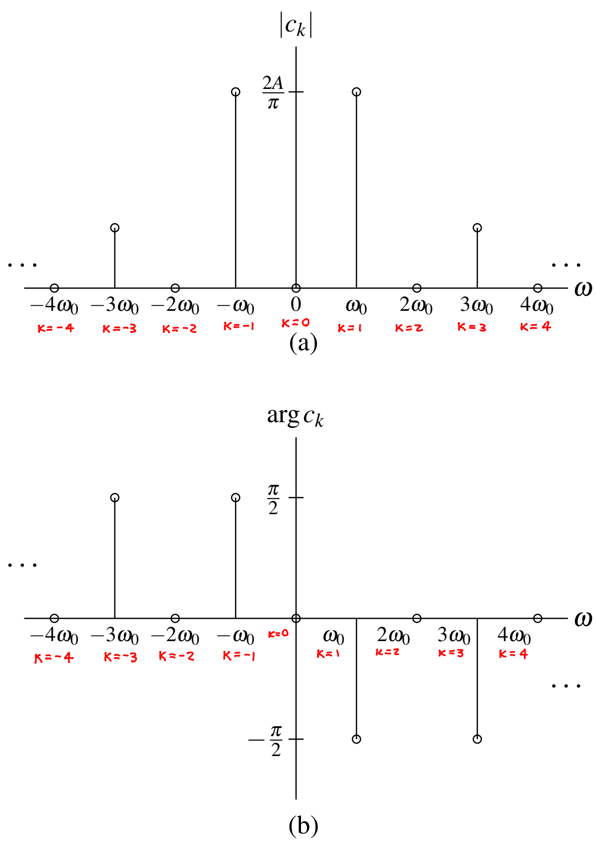


$$\begin{aligned} \arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & k \text{ odd} \\ \arg 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & k \text{ odd}, k<0 \\ -\frac{\pi}{2} & k \text{ odd}, k>0 \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} -\frac{\pi}{2} \operatorname{sgn} k & k \text{ odd} \\ 0 & k \text{ even}. \end{cases} \end{aligned}$$

see ① for $\arg\left[j\left(\frac{-2A}{\pi k}\right)\right]$

$\operatorname{sgn} k = \begin{cases} 1 & k>0 \\ 0 & k=0 \\ -1 & k<0 \end{cases}$

The magnitude and phase spectra of x are plotted in Figures 5.7(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since x is real. Since $|c_k|$ is largest for $k=-1$ and $k=1$, the function x has the most information at frequencies $-\omega_0$ and ω_0. ■



magnitude spectrum

- $|c_k|$ is largest for $k=-1$ $(-\omega_0)$ and $k=1$ (ω_0)
- $|c_k|$ is even since x is real

phase spectrum

- $\arg c_k$ is odd since x is real

Figure 5.7: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

Example 5.9. Consider a LTI system with the frequency response

$$H(\omega) = e^{-j\omega/4}.$$

Find the response y of the system to the input x, where

$$x(t) = \tfrac{1}{2}\cos(2\pi t).$$

Euler $\left[\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta})\right]$

Solution. To begin, we rewrite x as

$$x(t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Fourier series with only two nonzero terms

Thus, we can write

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} \\
&= c_{-1}H(-\omega_0)e^{-j\omega_0 t} + c_1 H(\omega_0) e^{j\omega_0 t} \\
&= \tfrac{1}{4}H(-2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(2\pi)e^{j2\pi t} \\
&= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\
&= \tfrac{1}{4}[e^{-j(2\pi t - \pi/2)} + e^{j(2\pi t - \pi/2)}] \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right).
\end{aligned}$$

from eigenfunction properties of LTI systems

expand summation

substitute for c_{-1}, c_1, w_0

evaluate $H(\ldots)$

combine exponentials

express in terms of cos (Euler)

Observe that $y(t) = x\left(t - \frac{1}{4}\right)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\omega) = e^{-\omega/4}$ is an ideal delay of $\frac{1}{4}$ (i.e., a system that performs a time shift of $\frac{1}{4}$). ■

NOTE: THE APPROACH USED IN THE SOLUTION TO THIS PROBLEM DID NOT REQUIRE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!