Exercise 5.107

L Answer (a).

We are given the LTI system $\mathcal{H}$ with input x, output y, and frequency response H, where

$$H(\omega) = \begin{cases} -3 & \omega < -1 \\ 0 & -1 \leq \omega \leq 0 \\ 3 & \omega > 0 \end{cases} \quad \text{and} \quad x(t) = 4\cos(t) + 2\cos(2t).$$

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Since $\mathcal{H}$ is LTI, every complex exponential function is an eigenfunction of $\mathcal{H}$. We rewrite x as a linear combination of eigenfunctions of $\mathcal{H}$ to obtain

$$\begin{aligned} x(t) &= 4\left[\tfrac{1}{2}(e^{jt} + e^{-jt})\right] + 2\left[\tfrac{1}{2}(e^{j2t} + e^{-j2t})\right] && \text{from 1 using Euler} \\ &= 2e^{jt} + 2e^{-jt} + e^{j2t} + e^{-j2t}. && \text{expand} \end{aligned}$$

From the linearity of $\mathcal{H}$, we have

$$\begin{aligned} y(t) &= \mathcal{H}x(t) && \text{from definitions of } x \text{ and } y \\ &= \mathcal{H}\{2e^{j(\cdot)} + 2e^{-j(\cdot)} + e^{j2(\cdot)} + e^{-j2(\cdot)}\}(t) && \text{substitute 2 for } x \\ &= \mathcal{H}\{2e^{j(\cdot)}\}(t) + \mathcal{H}\{2e^{-j(\cdot)}\}(t) + \mathcal{H}\{e^{j2(\cdot)}\}(t) + \mathcal{H}\{e^{-j2(\cdot)}\}(t) && \text{additivity of } \mathcal{H} \\ &= 2\mathcal{H}\{e^{j(\cdot)}\}(t) + 2\mathcal{H}\{e^{-j(\cdot)}\}(t) + \mathcal{H}\{e^{j2(\cdot)}\}(t) + \mathcal{H}\{e^{-j2(\cdot)}\}(t). && \text{homogenity of } \mathcal{H} \end{aligned}$$

From the eigenfunction properties of $\mathcal{H}$, we have

$$\begin{aligned} y(t) &= 2H(1)e^{jt} + 2H(-1)e^{-jt} + H(2)e^{j2t} + H(-2)e^{-j2t} && \text{substitute into 1} \\ &= 2(3)e^{jt} + 2(0)e^{-jt} + 3e^{j2t} + (-3)e^{-j2t} && \text{multiply} \\ &= 6e^{jt} + 3e^{j2t} - 3e^{-j2t} && \text{factor} \\ &= 6e^{jt} + 3(e^{j2t} - e^{-j2t}) && \text{Euler} \\ &= 6e^{jt} + 3[2j\sin(2t)] && \text{multiply} \\ &= 6e^{jt} + 6j\sin(2t). \end{aligned}$$