Motivation Behind the Laplace Transform (Continued)

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the Laplace transform of h.
- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the Laplace transform has many additional uses.

Section 7.1

Laplace Transform

(Bilateral) Laplace Transform

- The (bilateral) **Laplace transform** of the function x, denoted $\mathcal{L}x$ or X, is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

- The **inverse Laplace transform** of X, denoted $\mathcal{L}^{-1}X$ or x, is then given by

$$\mathcal{L}^{-1}X(t) = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$$

  where $\mathrm{Re}(s) = \sigma$ is in the ROC of X. (Note that this is a ***contour integration***, since s is complex.)

- We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Laplace Transforms

- Two different versions of the Laplace transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) Laplace transform; and
  2. the ***unilateral*** (or ***one-sided***) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the ***lower limit of integration***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0 (i.e., $\int_{-\infty}^{\infty} x(t)e^{-st}dt$ versus $\int_{0^-}^{\infty} x(t)e^{-st}dt$).
- For the most part, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean ***bilateral*** Laplace transform.

Remarks on Operator Notation

- For a function x, the Laplace transform of x is denoted using operator notation as $\mathcal{L}x$.
- The Laplace transform of x evaluated at s is denoted $\mathcal{L}x(s)$.
- Note that $\mathcal{L}x$ is a function, whereas $\mathcal{L}x(s)$ is a number.
- Similarly, for a function X, the inverse Laplace transform of X is denoted using operator notation as $\mathcal{L}^{-1}X$.
- The inverse Laplace transform of X evaluated at t is denoted $\mathcal{L}^{-1}X(t)$.
- Note that $\mathcal{L}^{-1}X$ is a function, whereas $\mathcal{L}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

Remarks on Dot Notation

- Often, we would like to write an expression for the Laplace transform of a function without explicitly naming the function.
- For example, consider writing an expression for the Laplace transform of the function $v(t) = x(5t-3)$ but without using the name "v".
- It would be incorrect to write "$\mathcal{L}x(5t-3)$" as this is the function $\mathcal{L}x$ evaluated at $5t-3$, which is not the meaning that we wish to convey.
- Also, strictly speaking, it would be incorrect to write "$\mathcal{L}\{x(5t-3)\}$" as the operand of the Laplace transform operator must be a function, and $x(5t-3)$ is a number (i.e., the function x evaluated at $5t-3$).
- Using dot notation, we can write the following strictly-correct expression for the desired Laplace transform: $\mathcal{L}\{x(5\cdot-3)\}$.
- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

Remarks on Notational Conventions

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
  1. in the expression for the operand of a Laplace transform operator, the ***independent variable is assumed to be the variable named "t"*** unless otherwise indicated (i.e., in terms of dot notation, each "t" is treated as if it were a "$\cdot$")
  2. in the expression for the operand of the inverse Laplace transform operator, the ***independent variable is assumed to be the variable named "s"*** unless otherwise indicated (i.e., in terms of dot notation, each "s" is treated as if it were a "$\cdot$").
- For example, with these conventions:
  - "$\mathcal{L}\{(t-\tau)u(t-\tau)\}$" denotes the function that is the Laplace transform of the function $v(t) = (t-\tau)u(t-\tau)$ (not the Laplace transform of the function $v(\tau) = (t-\tau)u(t-\tau)$).
  - "$\mathcal{L}^{-1}\{\frac{1}{s^2-\lambda}\}$" denotes the function that is the inverse Laplace transform of the function $V(s) = \{\frac{1}{s^2-\lambda}\}$ (not the inverse Laplace transform of the function $V(\lambda) = \{\frac{1}{s^2-\lambda}\}$).

Relationship Between Laplace and Fourier Transforms

- Let X and X_F denote the Laplace and (CT) Fourier transforms of x, respectively.
- The function X evaluated at $j\omega$ (where ω is real) yields $X_F(\omega)$. That is,

$$X(j\omega) = X_F(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.
- The function X evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \operatorname{Re}(s)$ and $\omega = \operatorname{Im}(s)$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(\sigma + j\omega) = X'_F(\omega),$$

  where X'_F is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.
- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

Proof for Special Case

Proof for General Case

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Section 7.2

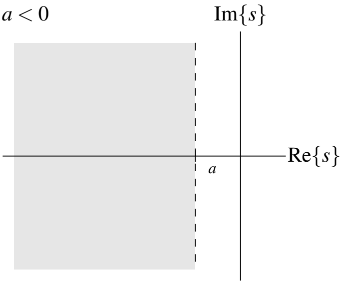
Region of Convergence (ROC)

Left-Half Plane (LHP)

- The set R of all complex numbers s satisfying

$$\operatorname{Re}(s) < a$$

  for some real constant a is said to be a **left-half plane (LHP)**.
- Some examples of LHPs are shown below.



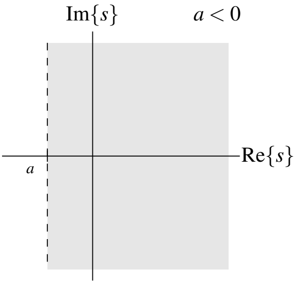
Right-Half Plane (RHP)

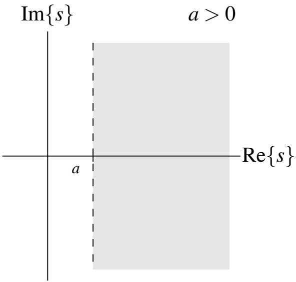
- The set R of all complex numbers s satisfying

$$\operatorname{Re}(s) > a$$

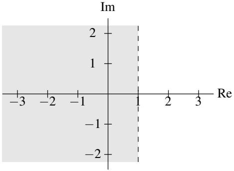
for some real constant a is said to be a **right-half plane (RHP)**.

- Some examples of RHPs are shown below.

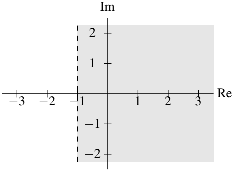




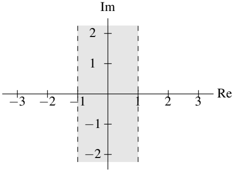
- For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



R_1



R_2



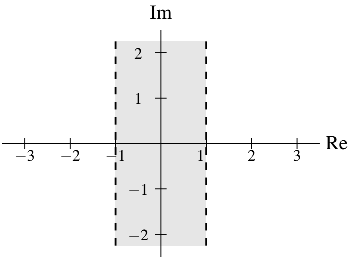
$R_1 \cap R_2$

Adding a Scalar to a Set

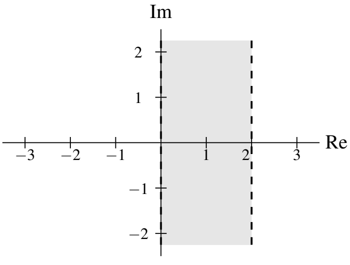
- For a set S and a scalar constant a, $S+a$ denotes the set given by

$$S+a = \{z+a : z \in S\}$$

  (i.e., $S+a$ is the set formed by adding a to each element of S).
- Effectively, adding a scalar to a set applies a translation (i.e., shift) to the region associated with the set.
- An illustrative example is given below.



R



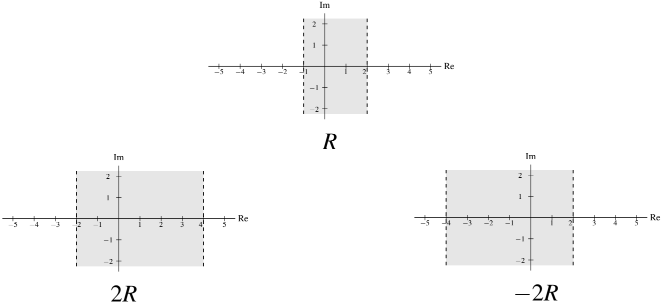
$R+1$

Multiplying a Set by a Scalar

- For a set S and a scalar constant a, aS denotes the set given by

$$aS = \{az : z \in S\}$$

  (i..e, aS is the set formed by multiplying each element of S by a).
- Multiplying z by a effects z by: scaling by $|a|$ and rotating about the origin by $\arg a$.
- So, effectively, multiplying a set by a scalar applies a scaling and/or rotation to the region associated with the set.
- An illustrative example is given below.

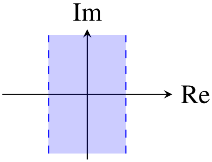


Region of Convergence (ROC)

- As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different functions can have the same algebraic expressions for X.
- On the slides that follow, we will examine a number of key properties of the ROC of the Laplace transform.

ROC Property 1: General Form

- The ROC of a Laplace transform consists of ***strips parallel to the imaginary axis*** in the complex plane.
- That is, if a point s_0 is in the ROC, then the vertical line through s_0 (i.e., $\mathrm{Re}(s) = \mathrm{Re}(s_0)$) is also in the ROC.
- Some examples of sets that would be either valid or invalid as ROCs are shown below.



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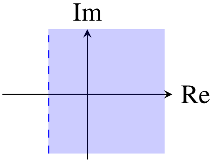

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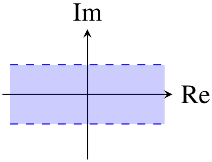

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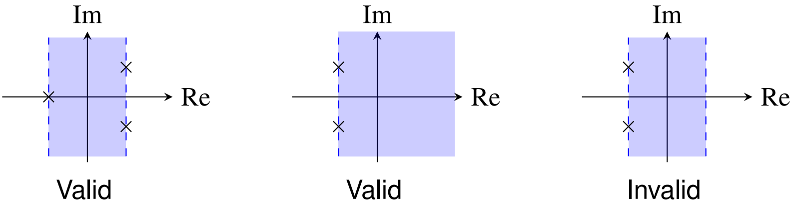
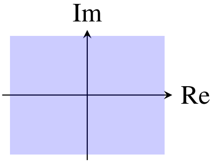
ROC Property 2: Rational Laplace Transforms

- If a Laplace transform X is a ***rational*** function, the ROC of X ***does not contain any poles*** and is ***bounded by poles or extends to infinity***.
- Some examples of sets that would be either valid or invalid as ROCs of rational Laplace transforms are shown below.



- If a function x is ***finite duration*** and its Laplace transform X converges for at least one point, then X converges for ***all*** points in the complex plane (i.e., the ROC is the entire complex plane).
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown below.



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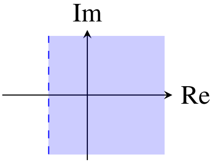

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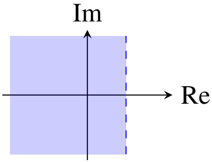

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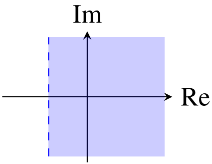
ROC Property 4: Right-Sided Functions

- If a function x is ***right sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC is a ***RHP*** including $\mathrm{Re}(s) = \sigma_0$).
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is right sided, are shown below.



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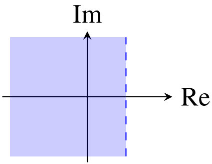

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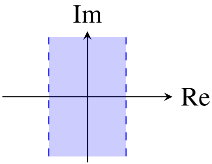

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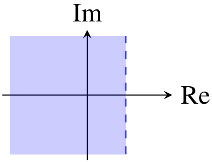
ROC Property 5: Left-Sided Functions

- If a function x is ***left sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC is a ***LHP*** including $\mathrm{Re}(s) = \sigma_0$).
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is left sided, are shown below.



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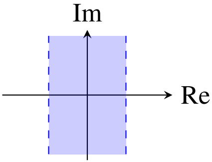

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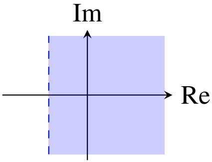

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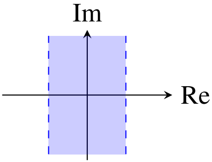
ROC Property 6: Two-Sided Functions

- If a function x is ***two sided*** and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then the ROC will consist of a ***strip*** in the complex plane that includes the line $\operatorname{Re}(s) = \sigma_0$.
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown below.



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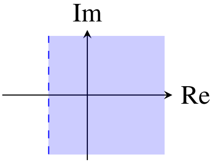

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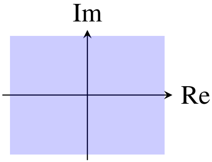

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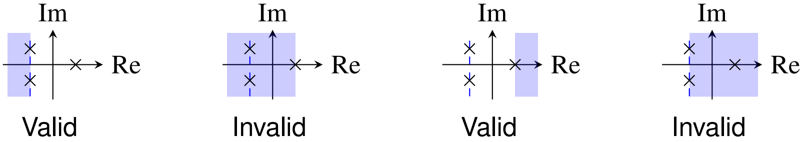
ROC Property 7: More on Rational Laplace Transforms

- If the Laplace transform X of a function x is ***rational*** (with at least one pole), then:
  1. If x is ***right sided***, the ROC of X is to the right of the rightmost pole of X (i.e., the ***RHP*** to the ***right of the rightmost pole***).
  2. If x is ***left sided***, the ROC of X is to the left of the leftmost pole of X (i.e., the ***LHP*** to the ***left of the leftmost pole***).
- This property is implied by properties 1, 2, 4, and 5.
- Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given below.

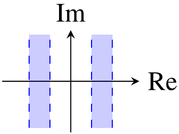


General Form of the ROC

- To summarize the results of properties 3, 4, 5, and 6, if the Laplace transform X of the function x exists, the ROC of X depends on the left- and right-sidedness of x as follows:

x		ROC of X
left sided	right sided	
no	no	strip
no	yes	RHP
yes	no	LHP
yes	yes	everywhere

- Thus, we can infer that, if X exists, its ROC can only be of the form of a LHP, a RHP, a vertical strip, or the entire complex plane.
- For example, the sets shown below would not be valid as ROCs.



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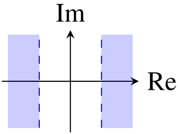

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Section 7.3

Properties of the Laplace Transform