Example 5.1 (Fourier series of a periodic square wave)**.** Find the Fourier series representation of the periodic square wave x shown in Figure 5.1.

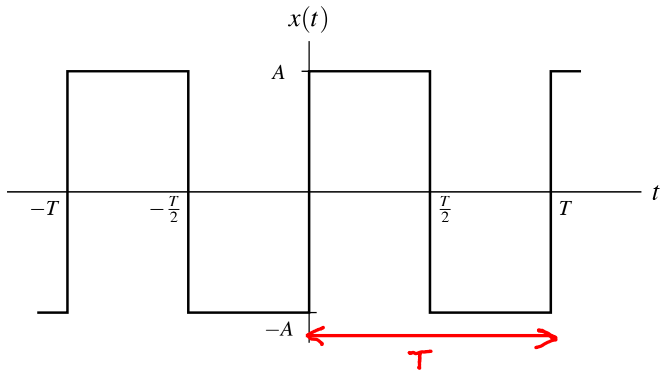

Figure 5.1: Periodic square wave.

Solution. Let us consider the single period of $x(t)$ for $0 \leq t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & 0 \leq t < \frac{T}{2} \\ -A & \frac{T}{2} \leq t < T. \end{cases}$$

Let $\omega_0 = \frac{2\pi}{T}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\left(\int_0^{T/2} Ae^{-jk\omega_0 t}dt + \int_{T/2}^{T}(-A)e^{-jk\omega_0 t}dt\right) \\ &= \begin{cases} \frac{1}{T}\left(\left[\frac{-A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Big|_0^{T/2} + \left[\frac{A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Big|_{T/2}^{T}\right) & k \neq 0 \\ \frac{1}{T}\left([At]|_0^{T/2} + [-At]|_{T/2}^{T}\right) & k = 0. \end{cases} \end{aligned}$$

Fourier series analysis equation

split into 2 integrals and substitute given x

integrate

1

Now, we simplify the expression for c_k for each of the cases $k \neq 0$ and $k = 0$ in turn. First, suppose that $k \neq 0$. We have

$$\begin{aligned} c_k &= \frac{1}{T}\left(\left[\frac{-A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Big|_0^{T/2} + \left[\frac{A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Big|_{T/2}^{T}\right) \\ &= \frac{-A}{j2\pi k}\left(\left[e^{-jk\omega_0 t}\right]\Big|_0^{T/2} - \left[e^{-jk\omega_0 t}\right]\Big|_{T/2}^{T}\right) \\ &= \frac{jA}{2\pi k}\left(\left[e^{-j\pi k} - 1\right] - \left[e^{-j2\pi k} - e^{-j\pi k}\right]\right) \\ &= \frac{jA}{2\pi k}\left[2e^{-j\pi k} - e^{-j2\pi k} - 1\right] \\ &= \frac{jA}{2\pi k}\left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1\right]. \end{aligned}$$

from 1 above

factor out constant and $T\omega_0 = 2\pi$

simplify

②

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

from ②

$$\begin{aligned}
c_k &= \frac{jA}{2\pi k}[2(-1)^k - 1^k - 1] \\
&= \frac{jA}{2\pi k}[2(-1)^k - 2] \\
&= \frac{jA}{\pi k}[(-1)^k - 1] \\
&= \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0. \end{cases}
\end{aligned}$$

simplify

$$(-1)^k - 1 = \begin{cases} -2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Now, suppose that $k = 0$. We have

from ① above

$$\begin{aligned}
c_0 &= \tfrac{1}{T}\left([At]\big|_0^{T/2} + [-At]\big|_{T/2}^{T} \right) \\
&= \tfrac{1}{T}\left[\tfrac{AT}{2} - \tfrac{AT}{2}\right] \\
&= 0.
\end{aligned}$$

simplify

Thus, the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt},$$

where

$$c_k = \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases}$$

■

Example 5.3. Consider the periodic function x with fundamental period $T = 3$ as shown in Figure 5.3. Find the Fourier series representation of x.

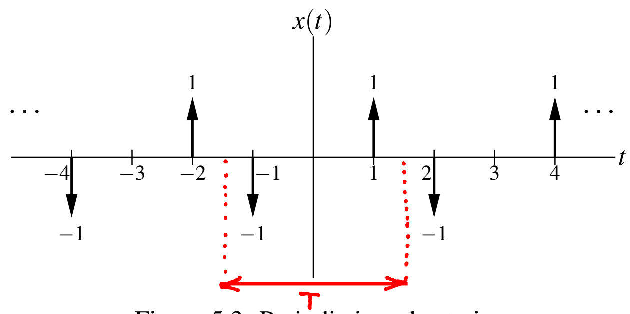

Figure 5.3: Periodic impulse train.

Solution. The function x has the fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \le t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \le t < \frac{3}{2}$). From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt && \text{Fourier series analysis equation} \\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} x(t)e^{-j(2\pi/3)kt}\,dt && \text{consider interval } [-T/2, T/2) \\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} \left[-\delta(t+1)+\delta(t-1)\right]e^{-j(2\pi/3)kt}\,dt && \text{substitute given } x \\
&= \tfrac{1}{3}\left[\int_{-3/2}^{3/2} -\delta(t+1)e^{-j(2\pi/3)kt}\,dt + \int_{-3/2}^{3/2} \delta(t-1)e^{-j(2\pi/3)kt}\,dt\right] && \text{split into 2 integrals} \\
&= \tfrac{1}{3}\left[-e^{-jk(2\pi/3)(-1)} + e^{-jk(2\pi/3)(1)}\right] && \text{extend limits and apply sifting property} \\
&= \tfrac{1}{3}\left[e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}\right] && \text{simplify} \\
&= \tfrac{1}{3}\left[2j\sin\left(-\tfrac{2\pi}{3}k\right)\right] && \text{Euler } \left[\sin\theta = \tfrac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)\right] \\
&= \tfrac{2j}{3}\sin\left(-\tfrac{2\pi}{3}k\right) && \text{simplify} \\
&= -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right). && \text{sin is odd}
\end{aligned}$$

Thus, x has the Fourier series representation

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right)e^{j(2\pi/3)kt}.
\end{aligned}$$

■

Example 5.6. Consider the periodic function x with period $T = 2$ as shown in Figure 5.4. Let $\hat{x}$ denote the Fourier series representation of x (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

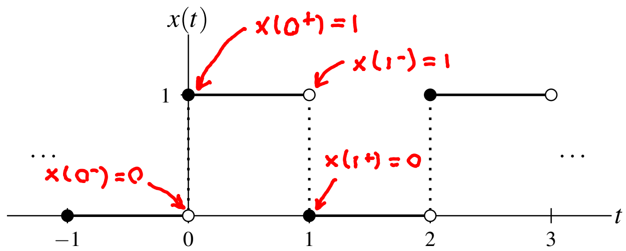

Figure 5.4: Periodic function x.

theorem for function satisfying Dirichlet conditions

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 5.4 applies. Thus, we have that

$$\begin{aligned}
\hat{x}(0) &= \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] \\
&= \tfrac{1}{2}(0+1) \\
&= \tfrac{1}{2} \quad \text{and} \\
\hat{x}(1) &= \tfrac{1}{2}\left[x(1^-) + x(1^+)\right] \\
&= \tfrac{1}{2}(1+0) \\
&= \tfrac{1}{2}.
\end{aligned}$$

average of left and right limits

average of left and right limits

■