Exercise 5.108

Answer (a).

We are given the LTI system with impulse response h and input x, where

$$h(t) = e^{-3t}u(t) \quad \text{and} \quad x(t) = 10 + 4\cos(4t) + 2\cos(6t).$$

First, we find the frequency response H of the system. We have

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} e^{-3t}u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-3t}e^{-j\omega t}\,dt \\
&= \frac{1}{3+j\omega} = \frac{3-j\omega}{(3+j\omega)(3-j\omega)} = \frac{3-j\omega}{\omega^2+9} = \frac{\sqrt{\omega^2+9}}{\omega^2+9}e^{j\arctan(-\omega/3)}. \\
&= \frac{1}{\sqrt{\omega^2+9}}e^{-j\arctan(\omega/3)}.
\end{aligned}$$

Let T and ω_0 denote the fundamental period and frequency of x, respectively. We have $\omega_0 = 2$ and $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$. Expressing x as a Fourier series, we have

$$\begin{aligned}
x(t) &= 10 + 4\cos(4t) + 2\cos(6t) \\
&= 10 + 4\left[\tfrac{1}{2}\left(e^{j4t} + e^{-j4t}\right)\right] + 2\left[\tfrac{1}{2}\left(e^{j6t} + e^{-j6t}\right)\right] \\
&= 10 + 2e^{j4t} + 2e^{-j4t} + e^{j6t} + e^{-j6t} \\
&= e^{-j6t} + 2e^{-j4t} + 10 + 2e^{j4t} + e^{j6t}.
\end{aligned}$$

Thus, we have $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$ where

$$c_k = \begin{cases} 1 & k \in \{-3,3\} \\ 2 & k \in \{-2,2\} \\ 10 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Due to the eigenfunction properties of LTI systems, we have

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H\left(\tfrac{2\pi}{T}k\right) e^{j(2\pi/T)kt} \\
&= \sum_{k=-\infty}^{\infty} c_k H(2k) e^{j2kt} \\
&= c_{-3}H(-6)e^{-j6t} + c_{-2}H(-4)e^{-j4t} + c_0H(0) + c_2H(4)e^{j4t} + c_3H(6)e^{j6t} \\
&= (1)\left(\tfrac{1}{\sqrt{45}}\right)e^{-j\arctan(-6/3)}e^{-j6t} + (2)\left(\tfrac{1}{5}\right)e^{-j\arctan(-4/3)}e^{-j4t} + (10)\left(\tfrac{1}{3}\right) + (2)\left(\tfrac{1}{5}\right)e^{-j\arctan(4/3)}e^{j4t} \\
&\quad + (1)\left(\tfrac{1}{\sqrt{45}}\right)e^{-j\arctan(6/3)}e^{j6t} \\
&= \tfrac{1}{\sqrt{45}}e^{j\arctan(2)}e^{-j6t} + \tfrac{2}{5}e^{j\arctan(4/3)}e^{-j4t} + \tfrac{10}{3} + \tfrac{2}{5}e^{-j\arctan(4/3)}e^{j4t} + \tfrac{1}{\sqrt{45}}e^{-j\arctan(2)}e^{j6t} \\
&= \tfrac{10}{3} + \tfrac{1}{\sqrt{45}}\left(e^{-j\arctan(2)}e^{j6t} + e^{j\arctan(2)}e^{-j6t}\right) + \tfrac{2}{5}\left(e^{-j\arctan(4/3)}e^{j4t} + e^{j\arctan\left(\frac{4}{3}\right)}e^{-j4t}\right) \\
&= \tfrac{10}{3} + \tfrac{1}{\sqrt{45}}\left(2\cos\left[6t - \arctan(2)\right]\right) + \tfrac{2}{5}\left(2\cos\left[4t - \arctan\left(\tfrac{4}{3}\right)\right]\right) \\
&= \tfrac{10}{3} + \tfrac{2}{3\sqrt{5}}\cos\left[6t - \arctan(2)\right] + \tfrac{4}{5}\cos\left[4t - \arctan\left(\tfrac{4}{3}\right)\right].
\end{aligned}$$

Therefore, we conclude

$$y(t) = \tfrac{10}{3} + \tfrac{2}{3\sqrt{5}} \cos\left[6t - \arctan(2)\right] + \tfrac{4}{5} \cos\left[4t - \arctan\left(\tfrac{4}{3}\right)\right].$$

