**Example 7.25.** Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform $X$ of the function $x$ shown in Figure 7.13.

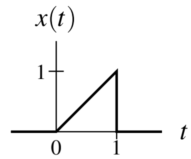



Figure 7.13: Function for the Laplace transform example.

*Second solution (which incurs less work by avoiding differentiation).* First, we express $x$ using unit-step functions to yield

$$\begin{aligned} x(t) &= t[u(t)-u(t-1)] \\ &= tu(t)-tu(t-1). \end{aligned}$$

add and subtract $u(t-1)$

To simplify the subsequent Laplace transform calculation, we choose to rewrite $x$ as

$$\begin{aligned} x(t) &= tu(t)-tu(t-1)+u(t-1)-u(t-1) \\ &= tu(t)-(t-1)u(t-1)-u(t-1). \end{aligned}$$

group two middle terms together

taking LT

(This is motivated by a preference to compute the Laplace transform of $(t-1)u(t-1)$ instead of $tu(t-1)$.) Taking the Laplace transform of both sides of the preceding equation, we obtain

$tu(t)$ time shifted by 1

$u(t)$ time shifted by 1 and then multiplied by $t$ (requires differentiation)

$$X(s) = \underbrace{\mathcal{L}\{tu(t)\}(s)}_{1} - \underbrace{\mathcal{L}\{(t-1)u(t-1)\}(s)}_{2} - \underbrace{\mathcal{L}\{u(t-1)\}(s)}_{3}. \quad (*)$$

We have

$$1 \quad \mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2}, \quad \text{from LT table}$$

$$\begin{aligned} 2 \quad \mathcal{L}\{(t-1)u(t-1)\}(s) &\overset{\text{time shifting}}{=} e^{-s}\mathcal{L}\{tu(t)\}(s) \\ &\overset{\text{LT table}}{=} e^{-s}\left(\frac{1}{s^2}\right) \\ &\overset{\text{multiply}}{=} \frac{e^{-s}}{s^2}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} 3 \quad \mathcal{L}\{u(t-1)\}(s) &\overset{\text{time shifting}}{=} e^{-s}\mathcal{L}\{u(t)\}(s) \\ &\overset{\text{LT table}}{=} e^{-s}\left(\frac{1}{s}\right) \\ &\overset{\text{multiply}}{=} \frac{e^{-s}}{s}. \end{aligned}$$

Combining the above results, we have

substituting 1, 2, and 3 into $(*)$

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ &= \frac{1-e^{-s}-se^{-s}}{s^2}. \end{aligned}$$

Since $x$ is finite duration, the ROC of $X$ is the entire complex plane. ■

**Example 7.27.** Find the inverse Laplace transform $x$ of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \operatorname{Re}(s) < 2.$$

*Solution.* We begin by rewriting $X$ in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

Strictly proper with 1st order poles at $-1$ and $2$

Then, we find a partial fraction expansion of $X$. We know that $X$ has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned}
A_1 &= (s+1)X(s)|_{s=-1} \\
&= \left.\frac{2}{s-2}\right|_{s=-1} \\
&= -\frac{2}{3} \quad \text{and} \\
A_2 &= (s-2)X(s)|_{s=2} \\
&= \left.\frac{2}{s+1}\right|_{s=2} \\
&= \frac{2}{3}.
\end{aligned}$$

So, $X$ has the expansion

$-1 < \operatorname{Re}(s) < 2$

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

$$x(t) = \underbrace{\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t)}_{①} - \underbrace{\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)}_{②}. \tag{7.6}$$

LT table

Using Table 7.2 and the given ROC, we have

$-1 < \operatorname{Re}(s) < 2$

$$\begin{aligned}
① \quad -e^{2t}u(-t) &\overset{\text{LT}}{\longleftrightarrow} \frac{1}{s-2} \quad \text{for } \operatorname{Re}(s) < 2 \quad \text{and} \\
② \quad e^{-t}u(t) &\overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1.
\end{aligned}$$

ROC must contain $-1 < \operatorname{Re}(s) < 2$ (see Ⓐ and Ⓑ)

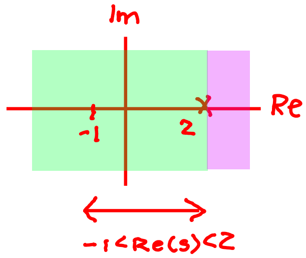
Substituting these results into (7.6), we obtain

$$\begin{aligned}
x(t) &= \tfrac{2}{3}[-e^{2t}u(-t)] - \tfrac{2}{3}[e^{-t}u(t)] \\
&= -\tfrac{2}{3}e^{2t}u(-t) - \tfrac{2}{3}e^{-t}u(t).
\end{aligned}$$

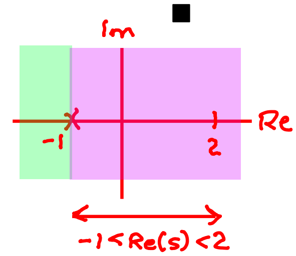
substituting the inverse LTs from ① and ②

■

Ⓐ possible ROCs for $\frac{1}{s-2}$




Ⓑ possible ROCs for $\frac{1}{s+1}$



**Example 7.28** (Rational function with a repeated pole). Find the inverse Laplace transform $x$ of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

strictly proper with 1st order pole at −2 and 2nd order pole at −1

*Solution.* To begin, we find a partial fraction expansion of $X$. We know that $X$ has an expansion of the form

terms from pole at −1 — term from pole at −2

$$X(s) = \boxed{\frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2}} + \boxed{\frac{A_{21}}{s+2}}.$$

Calculating the coefficients of the expansion, we obtain

formula for repeated pole case

coefficient index

pole order

$$A_{11} = \frac{1}{(2-1)!}\left[\left(\frac{d}{ds}\right)^{2-1}[(s+1)^2 X(s)]\right]\bigg|_{s=-1} = \frac{1}{1!}\left[\frac{d}{ds}[(s+1)^2 X(s)]\right]\big|_{s=-1} = \left[\frac{d}{ds}\left(\frac{2s+1}{s+2}\right)\right]\bigg|_{s=-1}$$
$$= \left[\frac{(s+2)(2)-(2s+1)(1)}{(s+2)^2}\right]\bigg|_{s=-1} = \left[\frac{2s+4-2s-1}{(s+2)^2}\right]\bigg|_{s=-1} = \left[\frac{3}{(s+2)^2}\right]\bigg|_{s=-1} = 3,$$
$$A_{12} = \frac{1}{(2-2)!}\left[\left(\frac{d}{ds}\right)^{2-2}[(s+1)^2 X(s)]\right]\bigg|_{s=-1} = \frac{1}{0!}\left[(s+1)^2 X(s)\right]\big|_{s=-1} = \frac{2s+1}{s+2}\bigg|_{s=-1} = \frac{-1}{1} = -1, \quad \text{and}$$

formula for simple pole case

$$A_{21} = (s+2)X(s)|_{s=-2} = \frac{2s+1}{(s+1)^2}\bigg|_{s=-2} = \frac{-3}{1} = -3.$$

Thus, $X$ has the expansion

Re(s) > −1

$$X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.7}$$

Re(s) > −1 — Re(s) > −1 — Re(s) > −2

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the three inverse Laplace transforms appearing in (7.7), we must associate a ROC with each of the three expressions $\frac{1}{s+1}$, $\frac{1}{(s+1)^2}$, and $\frac{1}{s+2}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.16. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of $X$ (i.e., $\operatorname{Re}(s) > -1$).

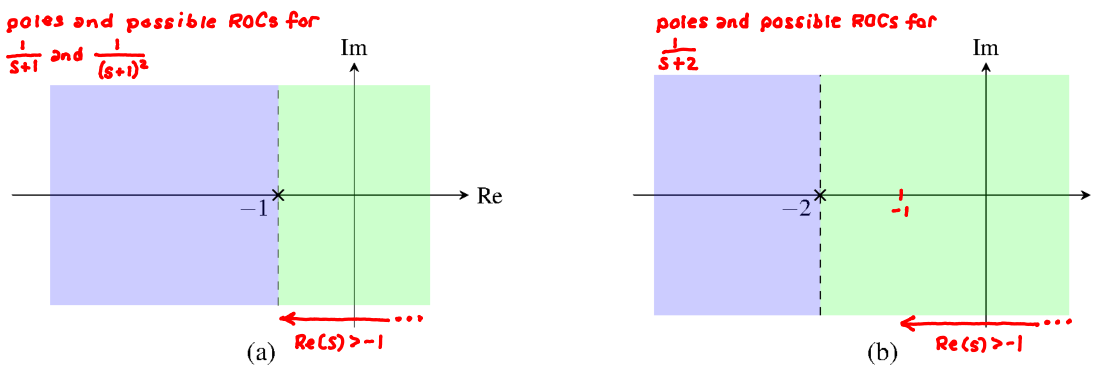



Figure 7.16: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s+1}$ and $\frac{1}{(s+1)^2}$; and (b) $\frac{1}{s+2}$.

LT table

From Table 7.2, we have

$$x(t) = 3L^{-1}\left\{\frac{1}{s+1}\right\}(t) - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3L^{-1}\left\{\frac{1}{s+2}\right\}(t) \quad (7.7)$$

Re(s) > -1 Re(s) > -1 Re(s) > -2

$$\begin{aligned}
&① \quad e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1, \\
&② \quad te^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{(s+1)^2} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and} \\
&③ \quad e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.
\end{aligned}$$

Substituting these results into (7.7), we obtain

Substituting ①, ②, and ③ into (7.7)

$$\begin{aligned}
x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t) \\
&= \left(3e^{-t} - te^{-t} - 3e^{-2t}\right)u(t).
\end{aligned}$$

■