Example 7.42 (Unilateral Laplace transform of second-order derivative)**.** Find the unilateral Laplace transform Y of y in terms of the unilateral Laplace transform X of x, where

$$y(t) = x''(t)$$

and the prime symbol denotes derivative (e.g., x'' is the second derivative of x)

Solution. Define the function

$$v(t) = x'(t) \tag{7.17}$$

so that

$$y(t) = v'(t). \tag{7.18}$$

Let V denote the unilateral Laplace transform of v. Taking the unilateral Laplace transform of (7.17) (using the time-domain differentiation property), we have

$$\begin{aligned} V(s) &= \mathcal{L}_{\mathrm{u}}\left\{x'\right\}(s) \\ &= sX(s) - x(0^-). \end{aligned} \tag{7.19}$$

time-domain differentiation property

Taking the unilateral Laplace transform of (7.18) (using the time-domain differentiation property), we have

$$\begin{aligned} Y(s) &= \mathcal{L}_{\mathrm{u}}\left\{v'\right\}(s) \\ &= sV(s) - v(0^-). \end{aligned} \tag{7.20}$$

time-domain differentiation property

Substituting (7.19) into (7.20), we have

substituting (7.19) into (7.20)

$$\begin{aligned} Y(s) &= s\left[sX(s) - x(0^-)\right] - v(0^-) \\ &= s^2X(s) - sx(0^-) - x'(0^-). \end{aligned}$$

$v = x'$ and multiply

Thus, we have that

$$Y(s) = s^2X(s) - sx(0^-) - x'(0^-). \qquad \blacksquare$$

Example 7.43. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y''(t)+3y'(t)+2y(t)=x(t),$$

where the prime symbol denotes derivative. If $x(t)=5u(t)$, $y(0^-)=1$, and $y'(0^-)=-1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
&\mathcal{L}_\mathrm{u}\left\{y''+3y'+2y\right\}(s)=\mathcal{L}_\mathrm{u}x(s) \\
\Rightarrow\quad &\mathcal{L}_\mathrm{u}\left\{y''\right\}(s)+3\mathcal{L}_\mathrm{u}\left\{y'\right\}(s)+2\mathcal{L}_\mathrm{u}y(s)=\mathcal{L}_\mathrm{u}x(s) \\
\Rightarrow\quad &\left[s^2Y(s)-sy(0^-)-y'(0^-)\right]+3\left[sY(s)-y(0^-)\right]+2Y(s)=X(s) \\
\Rightarrow\quad &s^2Y(s)-sy(0^-)-y'(0^-)+3sY(s)-3y(0^-)+2Y(s)=X(s) \\
\Rightarrow\quad &\left[s^2+3s+2\right]Y(s)=X(s)+sy(0^-)+y'(0^-)+3y(0^-) \\
\Rightarrow\quad &Y(s)=\frac{X(s)+sy(0^-)+y'(0^-)+3y(0^-)}{s^2+3s+2}. \quad ①
\end{aligned}$$

linearity

take ULT

multiply

move terms not containing Y to right-hand side and factor out Y on left-hand side

divide both sides by s^2+3s+2

Since $x(t)=5u(t)$ ③, we have

take ULT of ③ ULT table

$$X(s)=\mathcal{L}_\mathrm{u}\{5u(t)\}(s)=\tfrac{5}{s}. \quad ②$$

Substituting this expression for X and the given initial conditions into the above equation yields

$$Y(s)=\frac{\left(\frac{5}{s}\right)+s-1+3}{s^2+3s+2}=\frac{s^2+2s+5}{s(s+1)(s+2)}.$$

substituting ② into ①

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(s)=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1&=sY(s)\big|_{s=0} && \text{from formula for simple pole case}\\
&=\left.\frac{s^2+2s+5}{(s+1)(s+2)}\right|_{s=0}\\
&=\tfrac{5}{2},\\
A_2&=(s+1)Y(s)\big|_{s=-1} && \text{from formula for simple pole case}\\
&=\left.\frac{s^2+2s+5}{s(s+2)}\right|_{s=-1}\\
&=-4,\quad\text{and}\\
A_3&=(s+2)Y(s)\big|_{s=-2} && \text{from formula for simple pole case}\\
&=\left.\frac{s^2+2s+5}{s(s+1)}\right|_{s=-2}\\
&=\tfrac{5}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s)=\frac{5/2}{s}-\frac{4}{s+1}+\frac{5/2}{s+2}.$$

recall:

$Y(s) = \frac{5}{2}\left(\frac{1}{s}\right) - 4\left(\frac{1}{s+1}\right) + \frac{5}{2}\left(\frac{1}{s+2}\right)$

Taking the inverse unilateral Laplace transform of Y yields

taking inverse ULT

$$\begin{aligned} y(t) &= \mathcal{L}_{\text{u}}^{-1}Y(t) \\ &= \tfrac{5}{2}\mathcal{L}_{\text{u}}^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_{\text{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{5}{2}\mathcal{L}_{\text{u}}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &= \tfrac{5}{2} - 4e^{-t} + \tfrac{5}{2}e^{-2t} \quad \text{for } t \geq 0. \end{aligned}$$

linearity

from ULT table

■

$1 \overset{\text{ULT}}{\longleftrightarrow} \frac{1}{s}$; $e^{-at} \overset{\text{ULT}}{\longleftrightarrow} \frac{1}{s+a}$