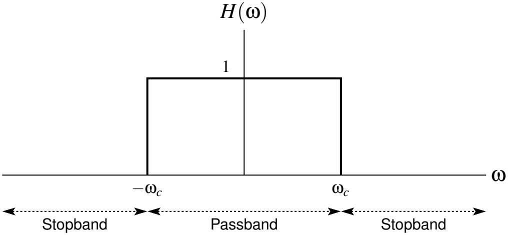
Ideal Lowpass Filter

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.



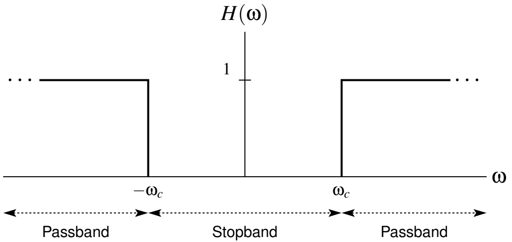
Ideal Highpass Filter

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.

- A plot of this frequency response is given below.

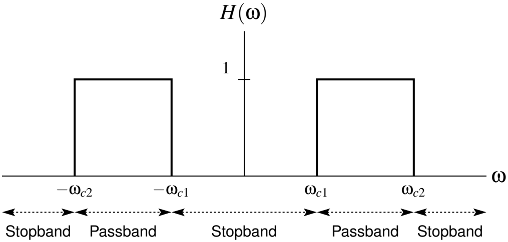


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



Part 6

Continuous-Time Fourier Transform (CTFT)

Motivation for the Fourier Transform

- The (CT) Fourier series provide an extremely useful representation for periodic functions.
- Often, however, we need to deal with functions that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (CT) Fourier transform can be used to represent both periodic and aperiodic functions.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 6.1

Fourier Transform

Development of the Fourier Transform [Aperiodic Case]

- The Fourier series is an extremely useful function representation.
- Unfortunately, this function representation can only be used for periodic functions, since a Fourier series is inherently periodic.
- Many functions are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic functions.
- By viewing an aperiodic function as the limiting case of a periodic function with period T where $T \to \infty$, we can use the Fourier series to develop a function representation that can be used for aperiodic functions, known as the Fourier transform.

- Recall that the Fourier series representation of a T-periodic function x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt \right)}_{c_k} e^{jk(2\pi/T)t}.$$

- In the above representation, if we take the limit as $T \to \infty$, we obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right)}_{X(\omega)} e^{j\omega t} d\omega$$

(i.e., as $T \to \infty$, the outer summation becomes an integral, $\frac{1}{T} = \frac{\omega_0}{2\pi}$ becomes $\frac{1}{2\pi} d\omega$, and $k(2\pi/T) = k\omega_0$ becomes ω).

- This representation for aperiodic functions is known as the Fourier transform representation.

Generalized Fourier Transform

- The classical Fourier transform for aperiodic functions does not exist (i.e., $\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ fails to converge) for some functions of great practical interest, such as:
  - a nonzero constant function;
  - a periodic function (e.g., a real or complex sinusoid);
  - the unit-step function (i.e., u); and
  - the signum function (i.e., sgn).
- Fortunately, the Fourier transform can be extended to handle such functions, resulting in what is known as the **generalized Fourier transform**.
- For our purposes, we can think of the classical and generalized Fourier transforms as being defined by the same formulas.
- Therefore, in what follows, we will not typically make a distinction between the classical and generalized Fourier transforms.

CT Fourier Transform (CTFT)

- The (CT) **Fourier transform** of the function x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a function x has the Fourier transform X, we write $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$.
- A function x and its Fourier transform X constitute what is called a **Fourier transform pair**.

Example 6.1 Example 6.3

Remarks on Operator Notation

SKIP SLIDE

- For a function x, the Fourier transform of x is denoted using operator notation as $\mathcal{F}x$.
- The Fourier transform of x evaluated at ω is denoted $\mathcal{F}x(\omega)$.
- Note that $\mathcal{F}x$ is a function, whereas $\mathcal{F}x(\omega)$ is a number.
- Similarly, for a function X, the inverse Fourier transform of X is denoted using operator notation as $\mathcal{F}^{-1}X$.
- The inverse Fourier transform of X evaluated at t is denoted $\mathcal{F}^{-1}X(t)$.
- Note that $\mathcal{F}^{-1}X$ is a function, whereas $\mathcal{F}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

Remarks on Dot Notation

SKIP SLIDE

- Often, we would like to write an expression for the Fourier transform of a function without explicitly naming the function.
- For example, consider writing an expression for the Fourier transform of the function $v(t) = x(5t-3)$ but without using the name "v".
- It would be incorrect to write "$\mathcal{F}x(5t-3)$" as this is the function $\mathcal{F}x$ evaluated at $5t-3$, which is not the meaning that we wish to convey.
- Also, strictly speaking, it would be incorrect to write "$\mathcal{F}\{x(5t-3)\}$" as the operand of the Fourier transform operator must be a function, and $x(5t-3)$ is a number (i.e., the function x evaluated at $5t-3$).
- Using dot notation, we can write the following strictly-correct expression for the desired Fourier transform: $\mathcal{F}x(5\cdot-3)$.
- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

Remarks on Notational Conventions

SKIP SLIDE

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
  1. in the expression for the operand of a Fourier transform operator, the ***independent variable is assumed to be the variable named "t"*** unless otherwise indicated (i.e., in terms of dot notation, each "t" is treated as if it were a "$\cdot$")
  2. in the expression for the operand of the inverse Fourier transform operator, the ***independent variable is assumed to be the variable named "ω"*** unless otherwise indicated (i.e., in terms of dot notation, each "ω" is treated as if it were a "$\cdot$").
- For example, with these conventions:
  - "$\mathcal{F}\{\cos(t-\tau)\}$" denotes the function that is the Fourier transform of the function $v(t) = \cos(t-\tau)$ (not the Fourier transform of the function $v(\tau) = \cos(t-\tau)$).
  - "$\mathcal{F}^{-1}\{\delta(3\omega-\lambda)\}$" denotes the function that is the inverse Fourier transform of the function $V(\omega) = \delta(3\omega-\lambda)$ (not the inverse Fourier transform of the function $V(\lambda) = \delta(3\omega-\lambda)$).

Section 6.2

Convergence Properties of the Fourier Transform

Convergence of the Fourier Transform

- Consider an arbitrary function x.
- The function x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

SKIP SLIDE

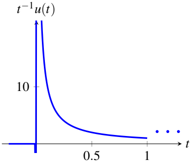
- If a function x is ***continuous*** and ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)|\,d\omega < \infty$), then the Fourier transform representation of x converges ***pointwise*** (i.e., $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]e^{j\omega t}d\omega$ for all t).
- Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.

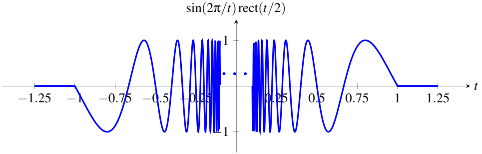
- If a function x is of ***finite energy*** (i.e., $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$), then its Fourier transform representation converges in the ***MSE sense***.
- In other words, if x is of finite energy, then the energy E in the difference function $\tilde{x} - x$ is zero; that is,

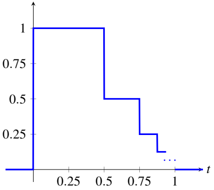
$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2 \, dt = 0.$$

- Since, in situations of practical interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.
- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

- The **Dirichlet conditions** for the function x are as follows:
  1. the function x is ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$);
  2. on any finite interval, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***); and
  3. on any finite interval, x has a ***finite number of discontinuities*** and each discontinuity is itself ***finite***.
- Examples of functions violating the Dirichlet conditions are shown below.







Convergence of the Fourier Transform: Dirichlet Case

- If a function x satisfies the ***Dirichlet conditions***, then:
  1. the Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x; and
  2. at each point t_a of discontinuity of x, the Fourier transform representation $\tilde{x}$ converges to

  $$\tilde{x}(t_a) = \frac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

  where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.

- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Example 6.6

Section 6.3

Properties of the Fourier Transform

Properties of the (CT) Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 d\omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

(CT) Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert\operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}(Bt)$	$\operatorname{rect}\left(\frac{\omega}{2B}\right)$
10	$e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

Linearity

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1x_1(t) + a_2x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Fourier transform.

Example 6.7

Time-Domain Shifting (Translation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t-t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.
- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

Example 6.9