Exercise 5.108

L Answer (b).

We are given the LTI system with impulse response h and input x, where

$$h(t) = e^t u(-t) \quad \text{and} \quad x(t) = 4\cos(t) + 2\cos(2t).$$

First, we find the frequency response H of the system. We have

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt && \text{formula for frequency response} \\
&= \int_{-\infty}^{\infty} e^t u(-t) e^{-j\omega t} dt && \text{substitute given } h \\
&= \int_{-\infty}^{0} e^t e^{-j\omega t} dt && \text{use unit step to change limits} \\
&= \frac{1}{1 - j\omega} && \text{use integral given in problem statement} \\
&= \frac{1 + j\omega}{(1 - j\omega)(1 + j\omega)} && \text{force denominator to be real} \\
&= \frac{1 + j\omega}{\omega^2 + 1}. && \text{multiply}
\end{aligned}$$

(Note that the integration step above was evaluated using the given integral-table entry.) Now, we express $H(\omega)$ in polar form, in order to greatly simplify some later steps in the solution (where it is extremely helpful to have $H(\omega)$ expressed in terms of an exponential function). Taking the magnitude and argument of $H(\omega)$, we obtain

$$|H(\omega)| = \left| \frac{1 + j\omega}{\omega^2 + 1} \right| \overset{\text{mag. of quotient}}{=} \frac{|1 + j\omega|}{|\omega^2 + 1|} \overset{\text{mag.}}{=} \frac{\sqrt{1 + \omega^2}}{1 + \omega^2} \overset{\text{divide}}{=} \frac{1}{\sqrt{1 + \omega^2}} \quad \text{and}$$

$$\arg[H(\omega)] = \arg\left(\frac{1 + j\omega}{\omega^2 + 1}\right) \underset{\text{arg. of quotient}}{=} \arg(1 + j\omega) - \arg(\omega^2 + 1) \underset{\omega^2+1>0}{=} \arg(1 + j\omega) \underset{\text{positive real part}}{=} \arctan(\omega).$$

So, rewriting $H(\omega)$ in polar form yields

$$H(\omega) = \frac{1}{\sqrt{\omega^2 + 1}} e^{j\arctan(\omega)}. \qquad *$$

Let T and ω_0 denote the fundamental period and frequency of x, respectively. We have $\omega_0 = 1$ and $T = \frac{2\pi}{\omega_0} = 2\pi$. Expressing x as a Fourier series, we have

$$\begin{aligned}
x(t) &= 4\cos(t) + 2\cos(2t) && \text{Euler} \\
&= 4\left[\tfrac{1}{2}\left(e^{jt} + e^{-jt}\right)\right] + 2\left[\tfrac{1}{2}\left(e^{j2t} + e^{-j2t}\right)\right] && \text{multiply} \\
&= 2e^{jt} + 2e^{-jt} + e^{j2t} + e^{-j2t} && \text{reorder terms} \\
&= e^{-j2t} + 2e^{-jt} + 2e^{jt} + e^{j2t}.
\end{aligned}$$

Thus, we have $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$ where

$$c_k = \begin{cases} 1 & k \in \{-2, 2\} \\ 2 & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Due to the eigenfunction properties of LTI systems, we have

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} H\left(\tfrac{2\pi}{T}k\right) c_k e^{j(2\pi/T)kt} && \text{eigenfunctions and linearity} \\
&= \sum_{k=-\infty}^{\infty} H(k) c_k e^{jkt} && \tfrac{2\pi}{T} = 1 \\
&= H(-2)c_{-2}e^{-j2t} + H(-1)c_{-1}e^{-jt} + H(1)c_1 e^{jt} + H(2)c_2 e^{j2t} && \text{drop zero terms} \\
&= \left(\tfrac{1}{\sqrt{5}}\right)(1)e^{j\arctan(-2)}e^{-j2t} + \left(\tfrac{1}{\sqrt{2}}\right)(2)e^{j(-\pi/4)}e^{-jt} + \left(\tfrac{1}{\sqrt{2}}\right)(2)e^{j(\pi/4)}e^{jt} + \left(\tfrac{1}{\sqrt{5}}\right)(1)e^{j\arctan(2)}e^{j2t} && \text{evaluate } H(\ldots) \text{ using } (*) \\
&= \tfrac{1}{\sqrt{5}}e^{j\arctan(-2)}e^{-j2t} + \sqrt{2}e^{-j\pi/4}e^{-jt} + \sqrt{2}e^{j\pi/4}e^{jt} + \tfrac{1}{\sqrt{5}}e^{j\arctan(2)}e^{j2t} && \text{multiply} \\
&= \tfrac{1}{\sqrt{5}}\left(e^{j\arctan(2)}e^{j2t} + e^{-j\arctan(2)}e^{-j2t}\right) + \sqrt{2}\left(e^{j\pi/4}e^{jt} + e^{-j\pi/4}e^{-jt}\right) && \text{group terms and factor} \\
&= \tfrac{1}{\sqrt{5}}\left(e^{j[2t+\arctan(2)]} + e^{-j[2t+\arctan(2)]}\right) + \sqrt{2}\left(e^{j[t+\pi/4]} + e^{-j[t+\pi/4]}\right) && \text{combine exponentials} \\
&= \tfrac{1}{\sqrt{5}}\left(2\cos\left[2t+\arctan(2)\right]\right) + \sqrt{2}\left[2\cos\left(t+\tfrac{\pi}{4}\right)\right] && \text{Euler} \\
&= \tfrac{2}{\sqrt{5}}\cos\left[2t+\arctan(2)\right] + 2\sqrt{2}\cos\left(t+\tfrac{\pi}{4}\right). && \text{multiply}
\end{aligned}
$$

Therefore, we conclude

$$y(t) = \tfrac{2}{\sqrt{5}}\cos\left[2t+\arctan(2)\right] + 2\sqrt{2}\cos\left(t+\tfrac{\pi}{4}\right).$$