Frequency-Domain Shifting (Modulation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$

  where ω_0 is an arbitrary real constant.
- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

Example 6.10

Time- and Frequency-Domain Scaling (Dilation)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

- This is known as the **dilation (or time/frequency-domain scaling) property** of the Fourier transform.

Conjugation

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$x^*(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$
- This is known as the **conjugation property** of the Fourier transform.

Example 6.12

Duality

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$
- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by
$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta)e^{-j\theta\lambda}d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)e^{j\theta\lambda}d\theta.$$
- That is, the forward and inverse Fourier transform equations are identical except for a ***factor of*** 2π and ***different sign*** in the parameter for the exponential function.
- Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively ***double*** the number of Fourier transform pairs that we know.

Example 6.13

Time-Domain Convolution

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Example 6.14

Time-Domain Multiplication

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi} X_1 * X_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\theta) X_2(\omega - \theta) d\theta.$$

- This is known as the **(time-domain) multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

Example 6.15

Time-Domain Differentiation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the **(time-domain) differentiation property** of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

Example 6.16

Frequency-Domain Differentiation

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

Time-Domain Integration

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the **(time-domain) integration property** of the Fourier transform.
- Whereas differentiation in the time domain corresponds to ***multiplication*** by $j\omega$ in the frequency domain, integration in the time domain is associated with ***division*** by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

Example 6.18

Parseval's Relation

- Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$.
- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

(i.e., the energy of x and energy of X are equal up to a factor of 2π).
- This relationship is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

Example 6.19

Even/Odd Symmetry

- For a function x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

Real Functions

- A function x is ***real*** if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

  (i.e., X is ***conjugate symmetric***).
- Thus, for a real-valued function, the portion of the graph of $X(\omega)$ for $\omega < 0$ is ***completely redundant***, as it is determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

  (i.e., $|X(\omega)|$ is ***even*** and $\arg X(\omega)$ is ***odd***).
- Note that x being real does ***not*** necessarily imply that X is real.

More Fourier Transforms

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Section 6.4

Fourier Transform of Periodic Functions

Fourier Transform of Periodic Functions

- The Fourier transform can be generalized to also handle periodic functions.
- Consider a periodic function x with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).
- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \frac{1}{T}X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

Fourier Transform of Periodic Functions (Continued)

- The Fourier transform X of a periodic function is a series of impulses that occur at integer multiples of the fundamental frequency ω_0 (i.e., $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$).
- Due to the preceding fact, the Fourier transform of a periodic function can only be nonzero at integer multiples of the fundamental frequency.
- The Fourier series coefficient sequence a is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$ (i.e., $a_k = \frac{1}{T} X_T(k\omega_0)$).

Example 6.20 Example 6.21 Example 6.24

Section 6.5

Fourier Transform and Frequency Spectra of Functions

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on functions.
- That is, instead of viewing a function as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a function as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform of a function x provides a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the ***frequency spectrum*** of the function.

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in ***polar form*** as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t + \arg X(\omega)]}d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency ω contributes to the integration result x.
- The quantity $\arg X(\omega)$ determines how the complex sinusoid at frequency ω is shifted related to complex sinusoids at other frequencies.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the areas of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

- Expressing the integral (from the previous slide) as the ***limit of a sum***, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \left|X(\omega)\right| e^{j[\omega t + \arg X(\omega)]},$$

  where $\omega = k\Delta\omega$.
- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega = k\Delta\omega$ that has had its ***amplitude scaled*** by a factor of $|X(\omega)|$ and has been ***time shifted*** by an amount that depends on $\arg X(\omega)$.
- For a given $\omega = k\Delta\omega$ (which is associated with the kth term in the summation), the ***larger*** $|X(\omega)|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega t}$ will be, and therefore the ***larger the contribution*** the kth term will make to the overall summation.
- In this way, we can use $|X(\omega)|$ as a ***measure*** of how much information a function x has at the frequency ω.

Fourier Transform and Frequency Spectra (Continued 2)

- The Fourier transform X of the function x is referred to as the **frequency spectrum** of x.
- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.
- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.
- Since the Fourier transform is a function of a real variable, a function can potentially have information at any real frequency.
- Since the Fourier transform X of a periodic function x with fundamental frequency ω_0 and the Fourier series coefficient sequence a is given by $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, the Fourier transform and Fourier series give consistent results for the frequency spectrum of a periodic function.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

Example 6.30

Frequency Spectra of Real Functions

- Recall that, for a real function x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is ***conjugate symmetric***), which is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a real function is always ***even***.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a real function is always ***odd***.
- Due to the symmetry in the frequency spectra of real functions, we typically ***ignore negative frequencies*** when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Magnitude and Phase Distortion in Audio

SKIP SLIDE

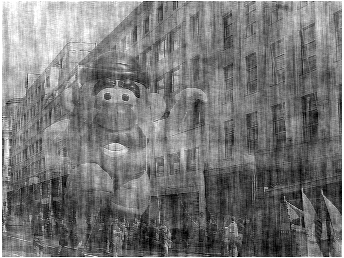
- The relative importance of the magnitude spectrum and phase spectrum is highly dependent on the particular application of interest.
- Consider the case of the human auditory system (i.e., human hearing).
- The human auditory system tends to be quite sensitive to changes in the magnitude spectrum of a signal.
- That is, a significant change in the magnitude spectrum of an audio signal is very likely to lead to a noticable difference in the perceived sound.
- On the other hand, the human auditory system tends to be much less sensitive to changes in the phase spectrum of a signal.
- In other words, changes to the phase spectrum of an audio signal are often only barely perceptible or not perceptible at all.
- For the above reasons, in applications involving the human auditory system, magnitude distortion (i.e., distortion of the magnitude spectrum) often tends to be more of a concern than phase distortion (i.e., distortion of the phase spectrum).

Magnitude and Phase Distortion in Images

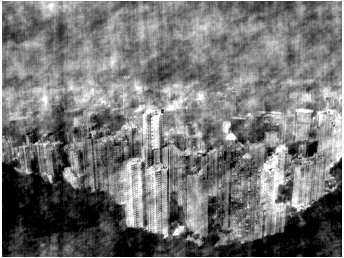
SKIP SLIDE

Image A

Image B



Magnitude Spectrum from Image B and
Phase Spectrum from Image A



Magnitude Spectrum from Image A and
Phase Spectrum from Image B

SKIP SLIDE

Image A

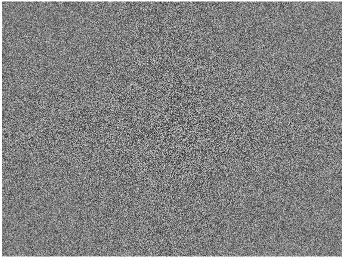
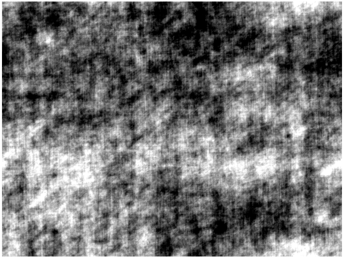


Image B (White Noise)

Magnitude Spectrum from Image B and *Phase Spectrum from Image A*



Magnitude Spectrum from Image A and *Phase Spectrum from Image B*