Example 6.9 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

$= v(t - [-\frac{\theta}{w_0}])$ table of FT pairs

Solution. Let $v(t) = A\cos(\omega_0 t)$ (1) so that $x(t) = v(t + \frac{\theta}{\omega_0})$ (2). Also, let $V = \mathcal{F}v$. From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad (3)$$

Using this transform pair and the linearity property of the Fourier transform, we have that

linearity

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) && \text{from FT of (1)} \\ (4) \quad &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{from FT pair (3) (and linearity)} \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0}V(\omega) && \text{from FT of (2) using time-domain shifting property } [e^{-jw(-\theta/w_0)}] \\ &= e^{j\omega\theta/\omega_0}A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{substituting expression for } V(w) \text{ from (4)} \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \overset{\text{CTFT}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \quad \blacksquare$$

Example 6.10 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = \cos(\omega_0 t)\cos(20\pi t),$$

where ω_0 is a real constant.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

cos(20πt)

$$\begin{aligned}
X(\omega) &= \left(\mathcal{F}\{\cos(\omega_0 t)\overbrace{(\tfrac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})}^{\cos(20\pi t)}\}\right)(\omega) \\
&= \left(\mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t) + \tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega) \\
&= \tfrac{1}{2}\left(\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}\right)(\omega) + \tfrac{1}{2}\left(\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega).
\end{aligned}$$

distribute

linearity property

table of FT pairs

From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad (1)$$

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

frequency domain shifting property

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega - 20\pi) + \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega + 20\pi) \\
&= \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\big|_{v=\omega+20\pi} \\
&= \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]\right) \\
&= \tfrac{\pi}{2}\left[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)\right].
\end{aligned}$$

FT pair (1)

substitute

■

Example 6.11 (Time scaling property of the Fourier transform)**.** Using the Fourier transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right), \quad ①$$

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let ② $v(t) = \operatorname{rect} t$ so that ③ $x(t) = v(at)$. Also, let $V = \mathcal{F}v$. From the given transform pair, we know that

$$V(\omega) = (\mathcal{F}\{\operatorname{rect} t\})(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right). \tag{6.9}$$

← from FT of ② using FT pair ①

From the definition of v and the time-scaling property of the Fourier transform, we have

$$④ \rightarrow \quad X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

← from FT of ③ using time scaling property

Substituting the expression for V in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

substituting (6.9) into ④

Thus, we have shown that

$$\operatorname{rect}(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right). \qquad \blacksquare$$

Example 6.12 (Fourier transform of a real function)**.** Let X denote the Fourier transform of the function x. Show that, if x is real, then X is conjugate symmetric (i.e., $X(\omega) = X^*(-\omega)$ for all ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(t)\}(\omega) = X^*(-\omega). \quad \leftarrow \text{from conjugation property}$$

$x^* = x$ since x is real

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\omega) = X^*(-\omega),$$

$\mathcal{F}x = X$ (by definition)

or equivalently

$$X(\omega) = X^*(-\omega).$$

■