Exercise 6.107

L Answer (a).

We are given the T-periodic function x, where

$$x(t) = \tfrac{A}{T}t \quad \text{for } t \in [0,T).$$

Let x_T denote a function equal to x on the interval $[0,T)$ and zero elsewhere. Let X_T denote the Fourier transform of x_T. Recalling the formula for the Fourier transform of a T-periodic function (expressed in terms of the Fourier transform of a single period of the function), we have

$$X(\omega) = \sum_{k=-\infty}^{\infty} \tfrac{2\pi}{T} X_T\left(\tfrac{2\pi}{T}k\right) \delta\left(\omega - \tfrac{2\pi}{T}k\right).$$

From the Fourier-transform analysis equation, we have

$$\begin{aligned} X_T(\omega) &= \int_0^T x_T(t) e^{-j\omega t} dt \\ &= \int_0^T \tfrac{A}{T} t e^{-j\omega t} dt \\ &= \tfrac{A}{T} \int_0^T t e^{-j\omega t} dt. \end{aligned}$$

To compute the integral in the preceding equation, there are two cases to consider: $\omega = 0$ and $\omega \neq 0$.

First, consider the case that $\omega \neq 0$. From (F.1), we have

$$\begin{aligned} X_T(\omega) &= \tfrac{A}{T} \left[\tfrac{1}{(-j\omega)^2} e^{-j\omega t}(-j\omega t - 1) \right] \Big|_0^T \\ &= \tfrac{A}{T} \left(\tfrac{1}{\omega^2}\right) \left[e^{-j\omega t}(j\omega t + 1) \right] \Big|_0^T \\ &= \tfrac{A}{T\omega^2} \left[e^{-j\omega T}(j\omega T + 1) - 1 \right]. \end{aligned}$$

Evaluating X_T at $\frac{2\pi}{T}k$ (where $k \neq 0$), we have

$$\begin{aligned} X_T\left(\tfrac{2\pi}{T}k\right) &= \tfrac{A}{T(2\pi k/T)^2} \left[e^{-j2\pi k}(j2\pi k + 1) - 1 \right] \\ &= \tfrac{AT}{4\pi^2 k^2}(j2\pi k) \\ &= \tfrac{jAT}{2\pi k} \quad \text{for } k \neq 0. \end{aligned}$$

Now, consider the case that $\omega = 0$. We have

$$\begin{aligned} X_T(\omega) &= \tfrac{A}{T} \int_0^T t\,dt \\ &= \tfrac{A}{T} \left[\tfrac{1}{2}t^2\right] \Big|_0^T \\ &= \tfrac{A}{T} \left(\tfrac{1}{2}T^2\right) \\ &= \tfrac{AT}{2}. \end{aligned}$$

Evaluating X_T at $\frac{2\pi}{T}k$ (where $k = 0$), we have

$$X_T\left(\tfrac{2\pi}{T}k\right) = \tfrac{AT}{2} \quad \text{for } k = 0.$$

Combining the above results, we have

$$X_T\left(\tfrac{2\pi}{T}k\right) = \begin{cases} \frac{AT}{2} & k = 0 \\ \frac{jAT}{2\pi k} & k \neq 0. \end{cases}$$

Using the formula for the Fourier transform of a periodic function from above, we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \tfrac{2\pi}{T} X_T\left(\tfrac{2\pi}{T}k\right) \delta\left(\omega - \tfrac{2\pi}{T}k\right) \\ &= \left(\tfrac{2\pi}{T}\right)\left(\tfrac{AT}{2}\right)\delta(\omega) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\tfrac{2\pi}{T}\right)\left(\tfrac{jAT}{2\pi k}\right)\delta\left(\omega - \tfrac{2\pi}{T}k\right) \\ &= \pi A\delta(\omega) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \tfrac{jA}{k}\delta\left(\omega - \tfrac{2\pi}{T}k\right). \end{aligned}$$

(Note that the “\” symbol denotes set subtraction.)