Example 4.1. Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \le t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t}-e^{-1}] \\ &= e^{-t-1}-1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \le t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\ &= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\ &= e^{-t}[e^{-1}-1+e^{t}-1] \\ &= 1+(e^{-1}-2)e^{-t}. \end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \ge 1$. From Figure 4.1(h), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\ &= e^{-t}[e^{-1}-1+e-1] \\ &= (e-2+e^{-1})e^{-t}. \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \leq t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \leq t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \leq t. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.1(i).

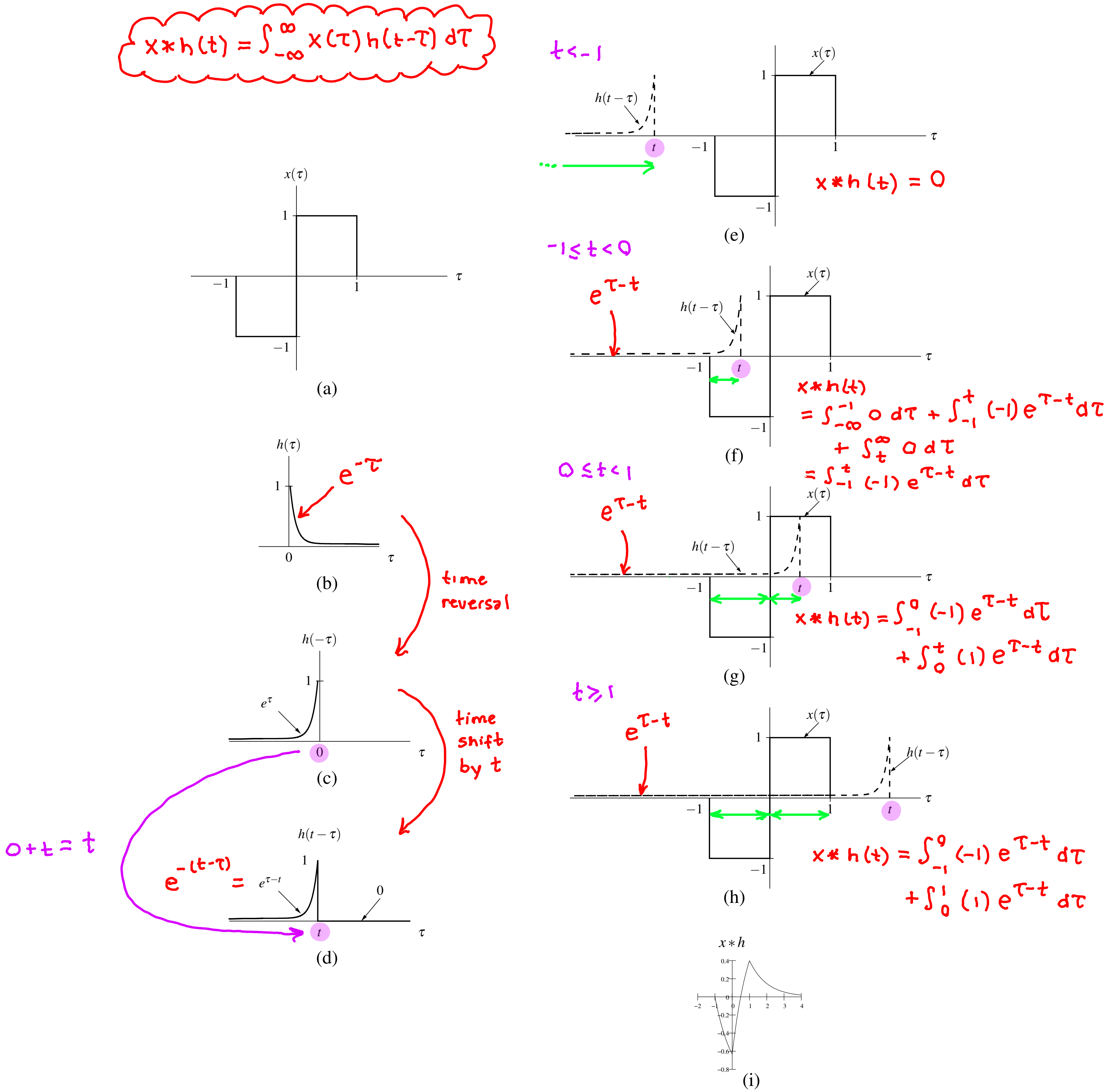


Figure 4.1: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<-1$, (f) $-1 \le t < 0$, (g) $0 \le t < 1$, and (h) $t \ge 1$; and (i) the convolution result $x*h$.

■

Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.23}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.24}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t). \qquad (1)$$

Substituting (4.23) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{from (1)} \\
&= x * u(t) && \text{substitute given function } h \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau && \text{definition of convolution} \\
&= \int_{-\infty}^{t} x(\tau)\underbrace{u(t-\tau)}_{1}d\tau + \int_{t^+}^{\infty} x(\tau)\underbrace{u(t-\tau)}_{0}d\tau && \text{split into two integrals} \\
&= \int_{-\infty}^{t} x(\tau)d\tau. && \text{second integral is 0}
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.23) is, in fact, the ideal integrator given by (4.24). ■

Example 4.7. Consider the system with input x, output y, and impulse response h as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

To begin, we label all signals in Figure 4.9.

(1)

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

δ is convolutional identity

distributive property

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t). \quad (2)$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [\underbrace{h_3 + h_1 * h_3 + h_2 * h_3}_{h}](t). \end{aligned}$$

from (2)

substituting (1) for v

distributive and associative properties and convolutional identity

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t).$$

■

Recall that, for any LTI system with input x, output y, and impulse response h, y = x * h.

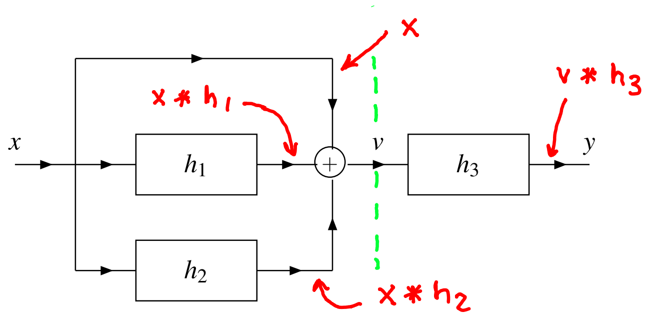


Figure 4.9: System interconnection example.

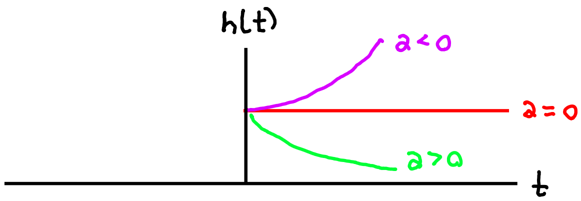
Example 4.8. Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

condition for memorylessness violated



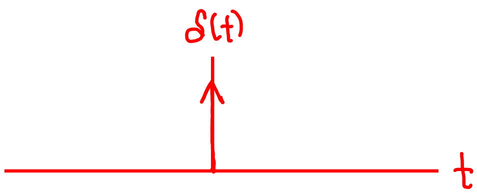
memoryless $\Leftrightarrow$ $h(t)=0$ for all $t \neq 0$

Example 4.9. Consider the LTI system with the impulse response h given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse function δ. Therefore, the system is memoryless (i.e., does not have memory). ■



memoryless $\Longleftrightarrow$ $h(t) = 0$ for all $t \neq 0$

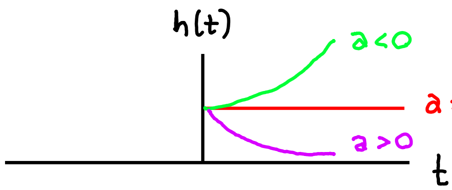
Example 4.10. Consider the LTI system with impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. ■

this is true regardless of a



causal $\iff$ $h(t)=0$ for all $t<0$

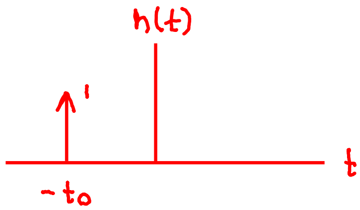
Example 4.11. Consider the LTI system with impulse response h given by

$$h(t) = \delta(t+t_0),$$

$t_0 > 0$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. ■



causal : $h(t) = 0$ for all $t<0$

Example 4.12. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\text{inv}} = \delta. \tag{4.34}$$

$\mathcal{H}$ is invertible if and only if a solution for h_{inv} exists

So, let us attempt to solve this equation for h_{inv}. Substituting the given function h into (4.34) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
& h * h_{\text{inv}}(t) = \delta(t) \\
\Rightarrow\ & \int_{-\infty}^{\infty} h(\tau) h_{\text{inv}}(t-\tau)\, d\tau = \delta(t) \\
\Rightarrow\ & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h_{\text{inv}}(t-\tau)\, d\tau = \delta(t) \\
\Rightarrow\ & \int_{-\infty}^{\infty} \delta(\tau - t_0) \underline{h_{\text{inv}}(t-\tau)}\, d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

definition of convolution

substitute given function h

divide both sides by $A \neq 0$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$h_{\text{inv}}(t-\tau)\big|_{\tau = t_0} = \frac{1}{A}\delta(t)$ sifting property

$$h_{\text{inv}}(t - t_0) = \tfrac{1}{A}\delta(t). \tag{4.35}$$

Substituting $t + t_0$ for t in the preceding equation yields

$$\begin{aligned}
h_{\text{inv}}([t + t_0] - t_0) = \tfrac{1}{A}\delta(t + t_0) \quad &\Leftrightarrow \\
h_{\text{inv}}(t) = \tfrac{1}{A}\delta(t + t_0).
\end{aligned}$$

impulse response of inverse system

Since $A \neq 0$, the function h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt && \text{split integration interval and use fact that } u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \int_{0}^{\infty} e^{at}\,dt && \text{drop zero integral} \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} && \text{identify two cases for integration} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases} && \text{integrate}
\end{aligned}$$

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right). && \text{what is } e^{a\infty}\text{?}
\end{aligned}$$

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} && \text{assuming } a<0 \\
&= -\tfrac{1}{a}.
\end{aligned}$$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases} \qquad \text{combining above results}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 4.15. Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \quad (1)$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\,d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

using (1) and $h = \mathcal{H}\delta$

integral is 1 if integration interval includes origin

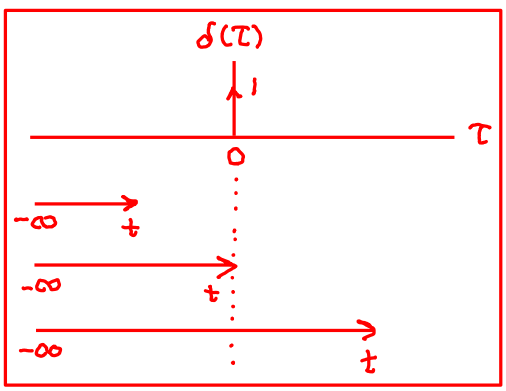
definition of unit-step function

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} 1\,dt \\ &= \infty. \end{aligned}$$

$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable. ■



Example 4.16. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.49). Substituting the given function h into (4.49), we obtain

$$\begin{aligned}
H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt && \text{(4.49)} \\
&= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt && \text{substitute given } h \\
&= \left[e^{-st}\right]\Big|_{t=1} && \text{sifting property} \\
&= e^{-s}.
\end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned}
x(t) &= e^t \cos(\pi t) && \text{Euler} \\
&= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] && \text{exponent rules} \\
&= \tfrac{1}{2}e^{(1+j\pi)t} + \tfrac{1}{2}e^{(1-j\pi)t}.
\end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \quad \text{and} \quad s_k = \begin{cases} 1 + j\pi & k = 0 \\ 1 - j\pi & k = 1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

$$\mathcal{H}\{a_k e^{s_k t}\}(t) = a_k H(s_k) e^{s_k t}$$

$$\begin{aligned}
y(t) &= \sum_{k=0}^{1} a_k H(s_k) e^{s_k t} && \text{expand summation} \\
&= a_0 H(s_0)e^{s_0 t} + a_1 H(s_1)e^{s_1 t} && \text{substitute for } a_k, s_k \\
&= \tfrac{1}{2}H(1+j\pi)e^{(1+j\pi)t} + \tfrac{1}{2}H(1-j\pi)e^{(1-j\pi)t} && \text{evaluate } H(\ldots) \\
&= \tfrac{1}{2}e^{-(1+j\pi)}e^{(1+j\pi)t} + \tfrac{1}{2}e^{-(1-j\pi)}e^{(1-j\pi)t} \\
&= \tfrac{1}{2}e^{t-1+j\pi t-j\pi} + \tfrac{1}{2}e^{t-1-j\pi t+j\pi} && \text{rearrange} \\
&= \tfrac{1}{2}e^{t-1}e^{j\pi(t-1)} + \tfrac{1}{2}e^{t-1}e^{-j\pi(t-1)} \\
&= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] && \text{Euler} \\
&= e^{t-1}\cos\left[\pi(t-1)\right].
\end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

NOTE: THIS SOLUTION DID NOT REQUIRE THE COMPUTATION OF A CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!