11.18 Relationship Between DT and CT Fourier Transforms

Earlier, in Section 6.20, we were introduced to impulse sampling. For a bandlimited function x, a relationship exists between the Fourier transform of the function obtained from impulse sampling x and the Fourier transform of the sequence obtained from sampling x. This relationship is given by the theorem below.

Theorem 11.21. *Let x be a bandlimited function and let T denote a sampling period for x that satisfies the Nyquist condition. Let $\tilde{y}$ be the function obtained by impulse sampling x with sampling period T. That is,*

$$\tilde{y}(t) = \sum_{n=-\infty}^{\infty} x(Tn)\delta(t-Tn).$$

Let y denote the sequence obtaining by sampling x with sampling period T. That is,

$$y(n) = x(Tn).$$

Let $\tilde{Y}$ denote the (CT) Fourier transform of $\tilde{y}$ and let Y denote the (DT) Fourier transform of y. Then, the following relationship holds:

$$Y(\Omega) = \tilde{Y}\left(\tfrac{\Omega}{T}\right) \quad \textit{for all } \Omega \in \mathbb{R}.$$

Proof. We take the (CT) Fourier transform of $\tilde{y}$ to obtain

$$\begin{aligned}
\tilde{Y}(\omega) &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x(Tn)\delta(\cdot - Tn)\right\}(\omega) \\
&= \sum_{n=-\infty}^{\infty} x(Tn)\mathcal{F}\{\delta(\cdot - Tn)\}(\omega) \\
&= \sum_{n=-\infty}^{\infty} y(n)e^{-jTn\omega} \\
&= \sum_{n=-\infty}^{\infty} y(n)e^{-j(T\omega)n} \\
&= Y(T\omega).
\end{aligned}$$

Thus, we have shown that

$$Y(T\omega) = \tilde{Y}(\omega).$$

Letting $\omega = \frac{\Omega}{T}$ in the preceding equation yields

$$Y(\Omega) = \tilde{Y}\left(\tfrac{\Omega}{T}\right). \qquad \blacksquare$$

11.19 Relationship Between DT Fourier Transform and DFT

As it turns out, an important relationship exists between the DT Fourier transform and the DFT, as given by the theorem below.

Theorem 11.22. *Let x be a finite-duration sequence such that $x(n) = 0$ for all $n \notin [0\,..\,M-1]$ and let X denote the (DT) Fourier transform of x. Let $\tilde{X}$ denote the N-point DFT of X. That is,*

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn} \quad \textit{for } k \in [0\,..\,N-1].$$

Suppose now that $N \geq M$. Then, the following relationship holds:

$$X\left(\tfrac{2\pi}{N}k\right) = \tilde{X}(k) \quad \textit{for } k \in [0\,..\,N-1].$$

In other words, the elements of the sequence $\tilde{X}$ correspond to uniformly-spaced samples of the function X.

Proof. From the definition of the DFT, we have

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}.$$

Since $x(n) = 0$ for all $n \notin [0\,..\,M-1]$ and $N \geq M$, it must be the case that $x(n) = 0$ for all $n \notin [0\,..\,N-1]$. Therefore, we can equivalently rewrite the summation on the right-hand side of the above equation to be taken over all integers, yielding

$$\begin{aligned}\tilde{X}(k) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(2\pi/N)kn} \\ &= \sum_{k=-\infty}^{\infty} x(n) e^{-j\Omega n}\bigg|_{\Omega=(2\pi/N)k} \\ &= X\left(\tfrac{2\pi}{N}k\right).\end{aligned}$$

■

The above theorem (i.e., Theorem 11.22) has extremely important practical implications. Most systems that we build these days have a digital computer at their core. Digital computers, being discrete-time systems, are quite adept at processing and manipulating finite-length sequences, which can be represented as a finite-size array of numbers. In contrast, digital computers have a much more difficult time algebraically (i.e., symbolically) manipulating functions. Consequently, it is much easier for a computer to calculate a DFT than a DT Fourier transform. So, in practice, the DT Fourier transform is often evaluated using the DFT. The DFT is used to obtain uniformly-spaced samples of the DT Fourier transform, and then interpolation techniques can be used to assign values to the Fourier transform at points between these samples. When computing an N-point DFT, we are free to choose N however we like. Therefore, we can choose N so that the Fourier transform can be determined on a fine enough grid that interpolation will yield a sufficiently accurate approximation of the Fourier transform for the application at hand.

Example 11.41. Consider the sequence

$$x(n) = u(n) - u(n-4).$$

This sequence can be shown to have the Fourier transform

$$X(\Omega) = e^{-j(3/2)\Omega}\left[\frac{\sin(2\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right].$$

Clearly, $x(n) = 0$ for all $n \notin [0\,..\,3]$. Therefore, we can determine samples of X using an N-point DFT, where $N \geq 4$. The N-point DFT of x is shown plotted in Figures 11.20, 11.21, 11.22, and 11.23 for N chosen as 4, 8, 16, and 64, respectively. To show relationship between the DFT and Fourier transform, the Fourier transform is also superimposed on each plot. We have used the 2π-periodicity of X in order to plot X over the interval $(-\pi, \pi]$ (instead of $[0, 2\pi)$). Examining the plots, we can see that the sampled frequency spectrum obtained from the DFT represents the spectrum more faithfully as N increases. Of course, the downside to increasing N is greater computational and memory costs.

■

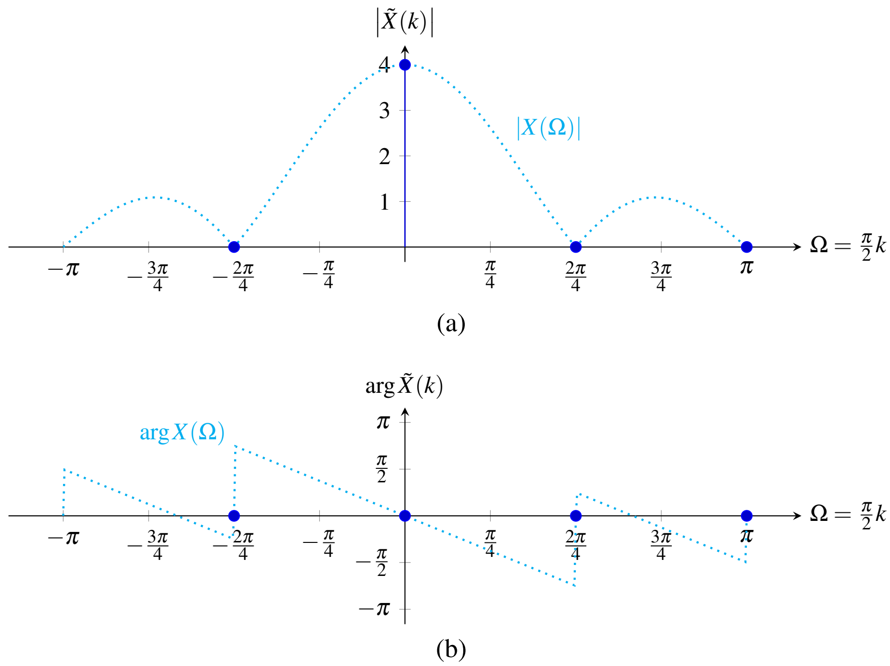


Figure 11.20: The sampled DT Fourier transform obtained from the DFT when $N = 4$. (a) Magnitude spectrum. (b) Phase spectrum.

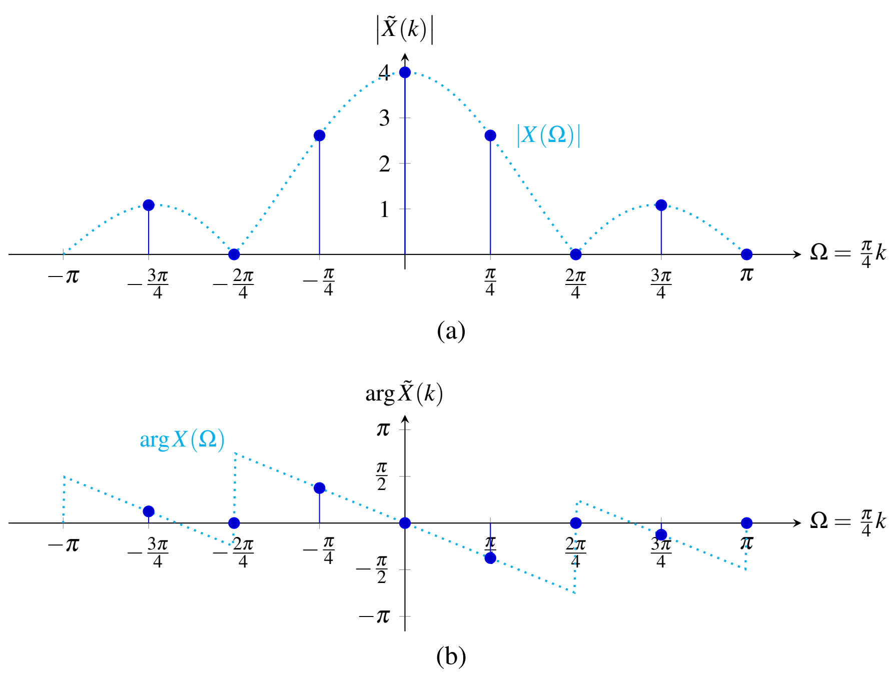


Figure 11.21: The sampled DT Fourier transform obtained from the DFT when $N = 8$. (a) Magnitude spectrum. (b) Phase spectrum.

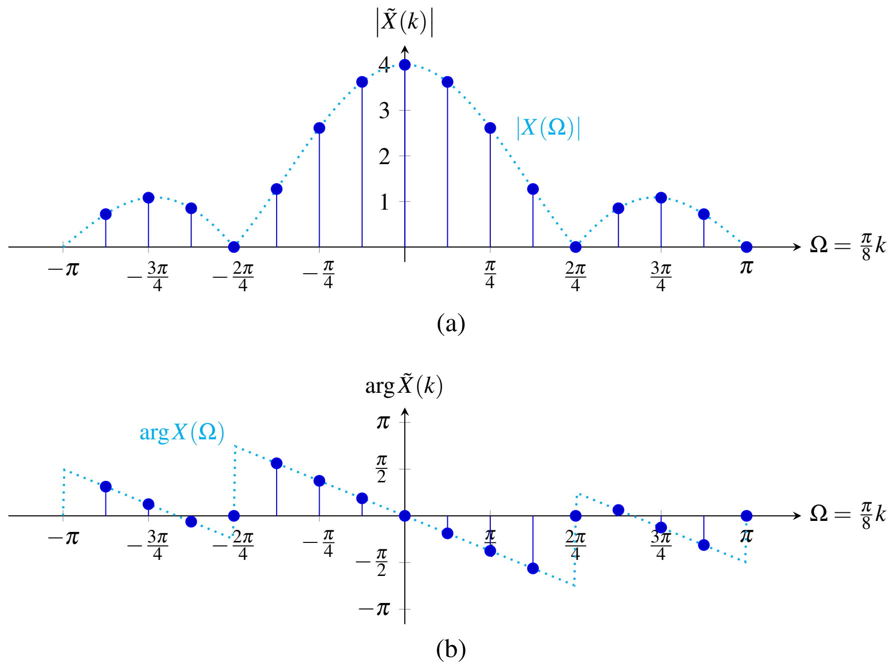


Figure 11.22: The sampled DT Fourier transform obtained from the DFT when $N = 16$. (a) Magnitude spectrum. (b) Phase spectrum.

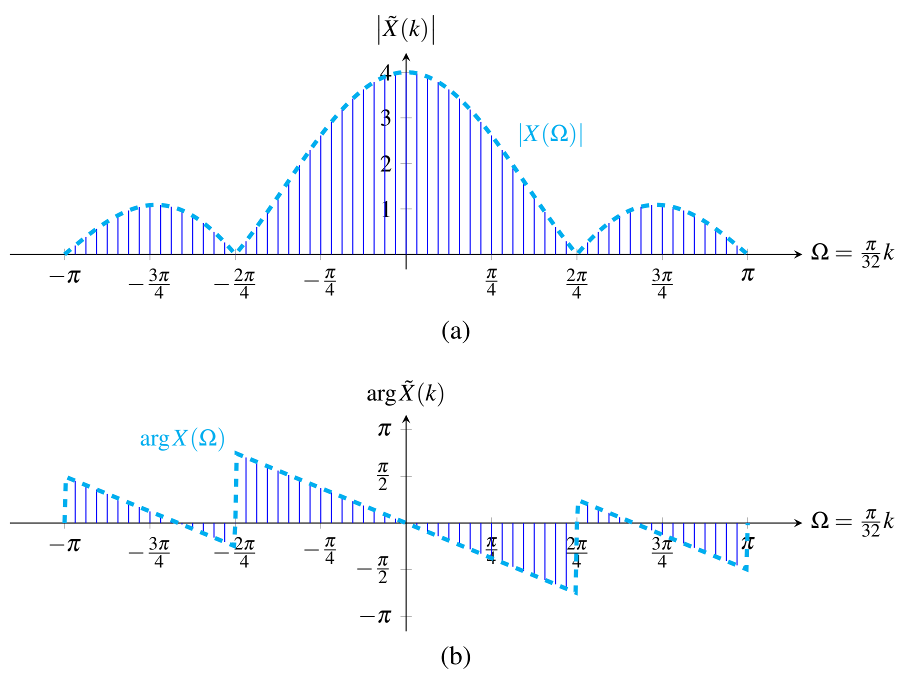


Figure 11.23: The sampled DT Fourier transform obtained from the DFT when $N = 64$. (a) Magnitude spectrum. (b) Phase spectrum.

11.20 Exercises

11.20.1 Exercises Without Answer Key

11.1 Using the Fourier transform analysis equation, find the Fourier transform X of each sequence x given below.
(a) $x(n) = n[u(n) - u(n-N)]$, where N is an integer constant [Hint: Use (F.14)];
(b) $x(n) = |n|[u(n+M) - u(n-M-1)]$, where M is an integer constant [Hint: Use (F.14)];
(c) $x(n) = a^n[u(n) - n(n-M)]$, where a is a complex constant satisfying $a \neq 1$, and M is an integer constant;
(d) $x(n) = a^{|2n|}$, where a is a complex constant satisfying $|a| < 1$;
(e) $x(n) = a^n u(n-1)$, where a is a complex constant and $|a| < 1$;
(f) $x(n) = \delta(n+a) + \delta(n-a)$, where a is an integer; and
(g) $x(n) = \delta(n+a) - \delta(n-a)$, where a is an integer.

11.2 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = \cos\left(\frac{2\pi}{7}n - \frac{\pi}{3}\right)$;
(b) $x(n) = u(n+5) - u(n-5)$;
(c) $x(n) = x_1 * x_2(n)$, where $x_1(n) = \delta(n) - \frac{1}{3}\operatorname{sinc}\left(\frac{\pi}{3}n\right)$ and $x_2(n) = \frac{2}{3}\operatorname{sinc}\left(\frac{2\pi}{3}n\right)$;
(d) $x(n) = \cos^2\left(\frac{\pi}{5}n\right)$;
(e) $x(n) = (\uparrow 3)x_1(n)$, where $x_1(n) = \frac{1}{12}\operatorname{sinc}\left(\frac{\pi}{12}n\right)$;
(f) $x(n) = \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{7}n\right) u(n)$;
(g) $x(n) = \operatorname{sinc}\left(\frac{\pi}{6}n\right)\cos\left(\frac{\pi}{2}n\right)$;
(h) $x(n) = \left(\frac{3}{4}\right)^n u(n-2)$;
(i) $x(n) = (n-2)\left(\frac{1}{3}\right)^{n-2} u(n-2)$;
(j) $x(n) = n\left(\frac{2}{3}\right)^{n+1} u(n+1)$;
(k) $x(n) = (n+1)\left(\frac{3}{4}\right)^n u(n)$;
(l) $x(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$;
(m) $x(n) = \left(\frac{1}{3}\right)^{n+1} u(n)$;
(n) $x(n) = (-1)^n$;
(o) $x(n) = (-1)^n u(n)$; and
(p) $x(n) = (-1)^n u(-n-1)$.

11.3 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = (n-m)a^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(b) $x(n) = na^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(c) $x(n) = (n-m)a^n u(n)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(d) $x(n) = a^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(e) $x(n) = a^{n-m}u(n)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$; and
(f) $x(n) = a^n u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$.

11.4 For each sequence y given below, find the Fourier transform Y of y in terms of the Fourier transform X of the sequence x.
(a) $y(n) = nx(n-3)$;
(b) $y(n) = n^2 x(n+1)$;
(c) $y(n) = x(1-n) + x(n-1)$;

(d) $y(n) = \frac{1}{2}[x^*(n) + x(n)]$;
(e) $y(n) = (n+1)x(n+1)$;
(f) $y(n) = x^*(-n)$;
(g) $y(n) = x(n+m) + x(n-m)$, where $m \in \mathbb{Z}$; and
(h) $y(n) = x(n+m) - x(n-m)$, where $m \in \mathbb{Z}$.

11.5 Using the Fourier transform synthesis equation, find the inverse Fourier transform x of each function X given below.
(a) $X(\Omega) = \begin{cases} 1 & |\Omega| \leq B \\ 0 & B < |\Omega| \leq \pi, \end{cases}$ where B is a positive real constant;
(b) $X(\Omega) = \begin{cases} -2j & \Omega \in (-\pi, 0] \\ 2j & \Omega \in (0, \pi]; \end{cases}$ and
(c) $X(\Omega) = \cos(k\Omega)$ for $\Omega \in [-\pi, \pi]$, where k is an integer constant.

11.6 Using a table of Fourier transform pairs and properties of the Fourier transform, find the inverse Fourier transform x of each function X given below.
(a) $X(\Omega) = e^{-j10\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{3}}\right]$;
(b) $X(\Omega) = \frac{3}{5 - 4\cos\left(\Omega - \frac{\pi}{3}\right)}$;
(c) $X(\Omega) = \frac{e^{j2\Omega}}{\left(e^{j\Omega} - \frac{1}{3}\right)^2}$;
(d) $X(\Omega) = 4\cos^2(\Omega) + 4\sin^2(3\Omega)$ [Hint: Consider the form of X.]; and
(e) $X(\Omega) = \frac{-\frac{5}{6}e^{-j\Omega} + 5}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}$ [Hint: Use a partial-fraction expansion.].

11.7 Using properties of the Fourier transform and the Fourier transform pair

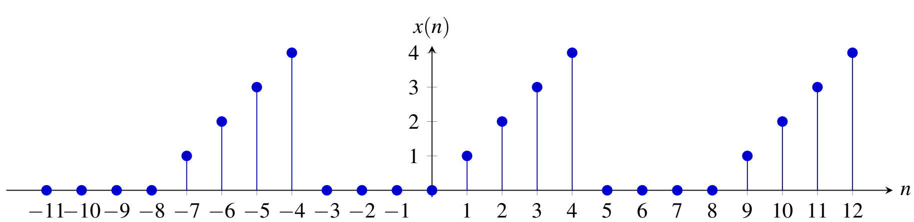
$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\Omega}},$$

show by induction that the inverse Fourier transform of the function X_k is the sequence x_k, where

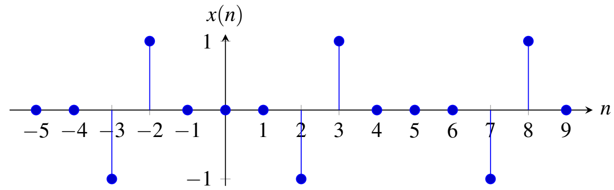
$$X_k(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^k}, \quad x_k(n) = \frac{(n+k-1)!}{n!(k-1)!}a^n u(n),$$

and k is a (strictly) positive integer. [Hint: $x_{k+1}(n) = \frac{n+k}{k}x_k(n)$.]

11.8 Find the Fourier transform X of each periodic sequence x shown below. The number of samples appearing in each plot is an integer multiple of the period.



(a)



(b)

11.9 Determine whether each function X given below is a valid (DT) Fourier transform.
(a) $X(\Omega) = \pi + j$;
(b) $X(\Omega) = \frac{1}{2}\Omega$;
(c) $X(\Omega) = \cos(5\Omega)$;
(d) $X(\Omega) = \cos\left(\frac{1}{5}\Omega\right)$;
(e) $X(\Omega) = \dfrac{\sin(4\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}$; and
(f) $X(\Omega) = \delta(\Omega)$.

11.10 Let x be a sequence with the Fourier transform X. Show that:
(a) if x is even, $X(\Omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos(n\Omega)$; and
(b) if x is odd, $X(\Omega) = -2j\sum_{k=1}^{\infty} x(n)\sin(n\Omega)$.

11.11 Let x be a sequence with even and odd parts x_e and x_o, respectively. Let X, X_e, and X_o denote the Fourier transforms of x, x_e, and x_o, respectively. Show that, for real x, the following relationships hold:
(a) $X_e(\Omega) = \operatorname{Re}[X(\Omega)]$; and
(b) $X_o(\Omega) = j\operatorname{Im}[X(\Omega)]$.

11.12 For each pair of sequences x_1 and x_2 given below, use the convolution property of the Fourier transform to compute $x = x_1 * x_2$.
(a) $x_1(n) = \frac{1}{4}\operatorname{sinc}\left(\frac{\pi}{4}n\right)$ and $x_2(n) = \frac{1}{7}\operatorname{sinc}\left(\frac{\pi}{7}n\right)$; and
(b) $x_1(n) = \frac{1}{5}\operatorname{sinc}\left(\frac{\pi}{5}n\right)$ and $x_2(n) = \cos\left(\frac{\pi}{2}n\right)$.

11.13 Compute the energy contained in each sequence x given below.
(a) $x(n) = \operatorname{sinc}\left(\frac{\pi}{11}n\right)$; and
(b) $x(n) = \frac{2}{5}\operatorname{sinc}\left(\frac{\pi}{5}n\right)\cos\left(\frac{\pi}{2}n\right)$.

11.14 Show that, for a real sequence x, the (DT) Fourier-transform synthesis equation can be expressed as

$$x(n) = \frac{1}{\pi}\int_0^{\pi} |X(\Omega)|\cos(\Omega n + \arg[X(\Omega)])d\Omega.$$

11.15 For each sequence x given below, compute the frequency spectrum X of x, and find and plot the corresponding magnitude and phase spectra.
(a) $x(n) = u(n-2) - u(n-7)$;
(b) $x(n) = \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1)$;
(c) $x(n) = \frac{1}{2}\delta(n) - \frac{1}{2}\delta(n-1)$; and
(d) $x(n) = (-1)^n[u(n+10) + u(n-11)]$.

11.16 For each difference equation below that defines a LTI system with input x and output y, find the frequency response H of the system.
(a) $y(n) - \frac{1}{2}y(n-1) = x(n)$;
(b) $y(n) - y(n-1) - y(n-2) = x(n-1)$;
(c) $10y(n) + 3y(n-1) - y(n-2) = x(n)$;
(d) $y(n+2) - \frac{1}{4}y(n+1) - \frac{1}{4}y(n) + \frac{1}{16}y(n-1) = x(n+2) - x(n+1)$; and
(e) $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$.

11.17 For each frequency response H given below for a LTI system with input x and output y, find the difference equation that characterizes the system.
(a) $H(\Omega) = \dfrac{e^{j\Omega} + \frac{8}{25}}{e^{j2\Omega} + e^{j\Omega} + \frac{4}{25}}$;
(b) $H(\Omega) = \dfrac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}$;
(c) $H(\Omega) = -e^{j3\Omega} + 3e^{j2\Omega} + 3e^{j\Omega} - 1$; and
(d) $H(\Omega) = \dfrac{e^{j2\Omega}}{e^{j2\Omega} - \frac{1}{2}e^{j\Omega} + \frac{1}{4}}$.

11.18 Consider a LTI system with the input x, output y, and frequency response H that is characterized by the difference equation

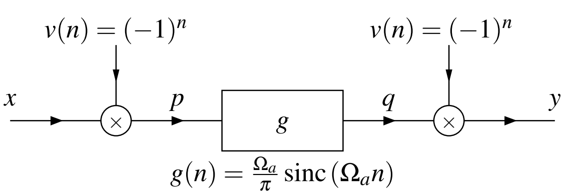
$$y(n) - ay(n-1) = bx(n) + x(n-1),$$

where a and b are real constants and $|a| < 1$. Suppose now that the system is allpass (i.e., $|H(\Omega)| = 1$ for all $\Omega \in \mathbb{R}$). Find an expression for b in terms of a.

11.19 Let h be the impulse response of a LTI system with input x and output y. For each case below, find y.
(a) $h(n) = \left(\frac{1}{2}\right)^n u(n)$ and $x(n) = (-1)^n$;
(b) $h(n) = \frac{1}{2}\operatorname{sinc}\left(\frac{\pi}{2}n\right)$ and $x(n) = \frac{1}{2} + \frac{1}{5}\sin\left(\frac{\pi}{3}n\right) + \frac{1}{6}\cos(\pi n)$; and
(c) $h(n) = \frac{1}{3}\operatorname{sinc}\left(\frac{\pi}{3}n\right)$ and $x(n) = \sum_{k=-\infty}^{\infty}\delta(n-8k)$.

11.20 Let H be the frequency response of a LTI system with input x and output y. For each case below, find y.
(a) $H(\Omega) = e^{-j2\Omega}\operatorname{rect}\left(\frac{2}{\pi}\Omega\right)$ for $\Omega \in (-\pi, \pi]$ and $x(n) = \operatorname{sinc}\left(\frac{\pi}{2}n\right)$.

11.21 Consider the system $\mathcal{H}$ with input x and output y as shown in the figure below. The block labelled by g is a LTI system with impulse response g, where $\Omega_a \in (0, \pi)$. (Note that, the subsystem corresponding to each multiplier is not TI.) Let X, Y, V, P, Q, and G denote the Fourier transforms of x, y, v, p, q, and g, respectively. Find an expression for Y in terms of X. Determine if $\mathcal{H}$ is LTI, and if it is, find its frequency response H.



11.22 Using the Fourier transform pair

$$a^{|n|} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2} \quad \text{for } |a|<1$$

and the duality relationship between the DT Fourier transform and CT Fourier series, find the (CT) Fourier series coefficients of the 1-periodic function

$$x(t) = \frac{1}{5-4\cos(2\pi t)}.$$

11.20.2 Exercises With Answer Key

11.101 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = \cos\left(\frac{\pi}{4}n - \frac{\pi}{12}\right)$; and
(b) $x(n) = u(n+10) - u(n+2)$.

Short Answer. (a) $X(\Omega) = \frac{1}{2}e^{-j\pi/12}\delta\left(\Omega-\frac{\pi}{4}\right) + \frac{1}{2}e^{j\pi/12}\delta\left(\Omega+\frac{\pi}{4}\right)$; (b) $X(\Omega) = e^{j(13/2)\Omega}\frac{\sin(2\Omega)}{\sin(\Omega/2)}$

11.21 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 12

z Transform

12.1 Introduction

In this chapter, we introduce an important mathematical tool in the study of discrete-time signals and systems known as the z transform. The z transform can (more or less) be viewed as a generalization of the classical discrete-time Fourier transform. Due to its more general nature, the z transform has a number of advantages over the classical Fourier transform. First, the z transform representation exists for some sequences that do not have Fourier transform representations. So, we can handle some sequences with the z transform that cannot be handled with the Fourier transform. Second, since the z transform is (in some sense) a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

12.2 Motivation Behind the z Transform

In Section 9.10, we showed that complex exponentials are eigensequences of discrete-time LTI systems. Suppose now that we have a discrete-time LTI system with impulse response h. Due to the eigensequence property, the response y of the system to the complex exponential input $x(n) = z^n$ (where z is a complex constant) is

$$y(n) = H(z)z^n,$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h(n)z^{-n}. \tag{12.1}$$

Previously, we referred to H as the system function. In this chapter, we will learn that H is, in fact, the z transform of h. That is, the summation in (12.1) is simply the definition of the z transform. In the case that $z = e^{j\Omega}$ where Ω is real (i.e., z is a point on the unit circle), (12.1) becomes the discrete-time Fourier transform summation (studied in Chapter 11). Since (12.1) includes the Fourier transform as a special case, the z transform can be viewed as a generalization of the (classical) Fourier transform.

12.3 Definition of z Transform

The (bilateral) **z transform** of the sequence x is denoted as $\mathcal{Z}x$ or X and is defined as

$$\mathcal{Z}x(z) = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \tag{12.2}$$

Similarly, the **inverse z transform** of X is denoted $\mathcal{Z}^{-1}X$ or x and is given by

$$\mathcal{Z}^{-1}X(n) = x(n) = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz, \tag{12.3}$$

where Γ is a counterclockwise closed circular contour centered at the origin and with radius r. We refer to x and X as a z transform pair and denote this relationship as

$$x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z).$$

As is evident from (12.3), the computation of the inverse z transform requires a contour integration. In particular, we must integrate along a circular contour in the complex plane. Often, such a contour integration can be tedious to compute. Consequently, in practice, we do not usually compute the inverse z transform using (12.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the z transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral z transform is most often used to solve difference equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the unilateral and bilateral z transforms is in the lower limit of summation. In the bilateral case, the lower limit is $-\infty$, while in the unilateral case, the lower limit is zero. In the remainder of this chapter, we will focus our attention primarily on the the bilateral z transform. We will, however, briefly introduce the unilateral z transform as a tool for solving difference equations. Unless otherwise noted, all subsequent references to the z transform should be understood to mean the bilateral z transform.

12.4 Remarks on Notation Involving the z Transform

The z transform operator $\mathcal{Z}$ and inverse z transform operator $\mathcal{Z}^{-1}$ map a sequence to a function and a function to a sequence, respectively. Consequently, the operand for each of these operators must be a function/sequence (not a number). Consider the unnamed sequence that maps n to $e^{-|n/10|}$ as shown in Figure 12.1. Suppose that we would like to write an expression that denotes the z transform of this sequence. At first, we might be inclined to write "$\mathcal{Z}\{e^{-|n/10|}\}$". Strictly speaking, however, this notation is not correct, since the z transform operator requires a sequence as an operand and "$e^{-|n/10|}$" (strictly speaking) denotes a number (i.e., the value of the sequence in the figure evaluated at n). Essentially, the cause of our problems here is that the sequence in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a sequence x using the equation $x(n) = e^{-|n/10|}$ and then write the z transform as "$\mathcal{Z}x$". Unfortunately, introducing a new sequence name just for the sake of strictly correct notation is often undesirable as it frequently leads to highly verbose writing.

One way to avoid overly verbose writing when referring to sequences without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the sequence from Figure 12.1 that maps n to $e^{-|n/10|}$. Using strictly correct notation, we could write the z transform of this sequence as "$\mathcal{Z}\left\{e^{-|\cdot/10|}\right\}$". In other words, we can indicate that an expression refers to a sequence (as opposed to the value of sequence) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 12.1. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 12.1 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the z transform of the sequence x defined by the equation $x(n) = n^2e^{-3n}u(n)$ (without the need to introduce the named sequence x), we can write: $\mathcal{Z}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}$.

2. To denote the z transform of the sequence x defined by the equation $x(n) = n^2e^{-3n}u(n)$ evaluated at $3z$ (without the need to introduce the named sequence x), we can write: $\mathcal{Z}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}(3z)$.

3. To denote the inverse z transform of the function X defined by the equation $X(z) = z^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{(\cdot)^{-1}\right\}$.

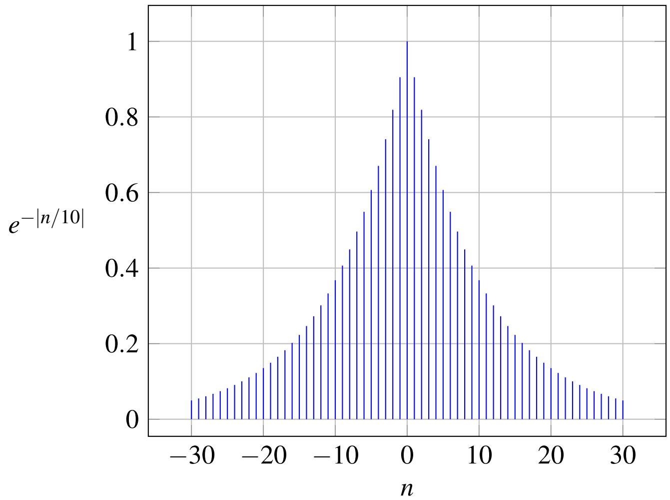


Figure 12.1: A plot of $e^{-|n/10|}$ versus n.

4. To denote the inverse z transform of the function X defined by the equation $X(z) = z^{-1}$ evaluated at $n-1$ (without the need to introduce the named sequence X), we can write: $\mathcal{Z}^{-1}\left\{(\cdot)^{-1}\right\}(n-1)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the z transform operator $\mathcal{Z}$, the variable "n" is assumed to be the independent variable for the sequence to which the z transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "n" were a "$\cdot$");

- unless indicated otherwise, in an expression for the operand of the inverse z transform operator $\mathcal{Z}^{-1}$, the variable "z" is assumed to be the independent variable for the function to which the inverse z transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "z" were a "$\cdot$").

Some examples of using these book-sanctioned notational conventions can be found below in Example 12.2. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 12.2 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the z transform of the sequence x defined by the equation $x(n) = n^2 e^{-3n} u(n)$ (without the need to introduce the named function x), we can write: $\mathcal{Z}\left\{n^2 e^{-3n} u(n)\right\}$.

2. To denote the z transform of the sequence x defined by the equation $x(n) = n^2 e^{-3n} u(n)$ evaluated at $3z$ (without the need to introduce the named function x), we can write: $\mathcal{Z}\left\{n^2 e^{-3n} u(n)\right\}(3z)$.

3. To denote the inverse z transform of the function X defined by the equation $X(z) = 1 + z^{-1} + z^{-2}$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{1 + z^{-1} + z^{-2}\right\}$.

4. To denote the inverse z transform of the function X defined by the equation $X(z) = 1 + z^{-1} + z^{-2}$ evaluated at $n-1$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{1 + z^{-1} + z^{-2}\right\}(n-1)$. ■

Since applying the z transform operator to a sequence yields a function, we can evaluate this function at some value. Similarly, since applying the inverse z transform operator to a function yields a sequence, we can evaluate this sequence at some value. Again, consider the sequence from Figure 12.1 that maps n to $e^{-|n/10|}$. To denote the value of the z transform of this sequence evaluated at $3z$, we would write "$\mathcal{Z}\{e^{-|\cdot/10|}\}(3z)$" using dot notation or "$\mathcal{Z}\{e^{-|n/10|}\}(3z)$" using the book-sanctioned notational conventions described above.

12.5 Relationship Between z Transform and Discrete-Time Fourier Transform

In Section 12.3 of this chapter, we introduced the z transform, and in Chapter 11, we studied the (DT) Fourier transform. As it turns out, the z transform and (DT) Fourier transform are very closely related. Recall the definition of the z transform in (12.2). Consider now the special case of (12.2) where $z = e^{j\Omega}$ and Ω is real (i.e., z is on the unit circle). In this case, (12.2) becomes

$$\begin{aligned} X(e^{j\Omega}) &= \left[\sum_{n=-\infty}^{\infty} x(n) z^{-n} \right] \Bigg|_{z=e^{j\Omega}} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= \mathcal{F}x(\Omega) \end{aligned}$$

Thus, the Fourier transform is simply the z transform evaluated at $z = e^{j\Omega}$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(e^{j\Omega}) = \mathcal{F}x(\Omega).$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(e^{j\Omega})$. When this notation is used, the function X actually corresponds to the z transform of x rather than its Fourier transform (i.e., the expression $X(e^{j\Omega})$ corresponds to the z transform evaluated at points on the unit circle).

Now, consider the general case of an arbitrary complex value for z in (12.2). Let us express z in polar form as $z = re^{j\Omega}$, where $r = |z|$ and $\Omega = \arg z$. Substituting $z = re^{j\Omega}$ into (12.2), we obtain

$$\begin{aligned} X(re^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x(n)(re^{j\Omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \left[r^{-n} x(n) \right] e^{-j\Omega n} \\ &= \mathcal{F}\{r^{-n} x(n)\}(\Omega). \end{aligned}$$

Thus, we have shown

$$X(re^{j\Omega}) = \mathcal{F}\{r^{-n} x(n)\}(\Omega). \tag{12.4}$$

For this reason, the z transform of x can be viewed as the (DT) Fourier transform of $x'(n) = r^{-n} x(n)$ (i.e., x weighted by a real exponential sequence). As a consequence of multiplying by the real exponential r^{-n}, the z transform of a sequence may exist when the Fourier transform of the same sequence does not.

By using the above relationship, we can derive the formula for the inverse z transform given in (12.3). Let X denote the z transform of x, and let $z = re^{j\Omega}$, where r and Ω are real and $r \geq 0$. From the relationship between the Fourier and z transforms in (12.4), we have

$$X(re^{j\Omega}) = \mathcal{F}\{r^{-n} x(n)\}(\Omega),$$

where r is chosen so that $X(z)$ converges for $z = re^{j\Omega}$. Taking the inverse Fourier transform of both sides of the preceding equation, we obtain

$$r^{-n}x(n) = \mathcal{F}^{-1}\{X(re^{j\Omega})\}(n).$$

Multiplying both sides of this equation by r^n, we have

$$x(n) = r^n \mathcal{F}^{-1}\{X(re^{j\Omega})\}(n).$$

Using the formula for the inverse Fourier transform, we can write

$$x(n) = r^n \tfrac{1}{2\pi} \int_{2\pi} X(re^{j\Omega}) e^{j\Omega n} d\Omega.$$

Moving r^n inside integral yields

$$x(n) = \tfrac{1}{2\pi} \int_{2\pi} X(re^{j\Omega})(re^{j\Omega})^n d\Omega.$$

Now, we employ a change of variable. Let $z = re^{j\Omega}$ so that $dz = jre^{j\Omega} d\Omega = jz d\Omega$ and $d\Omega = \frac{1}{j} z^{-1} dz$. Applying the change of variable, we obtain

$$x(n) = \tfrac{1}{2\pi} \oint X(z) z^n \left(\tfrac{1}{j}\right) z^{-1} dz.$$

As Ω goes from 0 to 2π, z traces in a counter-clockwise direction a closed circular contour centered at the origin and with radius r. Thus, we have

$$x(n) = \tfrac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz,$$

where Γ denotes a counter-clockwise closed circular contour centered at the origin with radius r. In other words, we have shown that (12.3) holds.

12.6 z Transform Examples

In this section, we calculate the z transform of several relatively simple functions. In the process, we gain some important insights into the z transform.

Example 12.3. Find the z transform X of the sequence

$$x(n) = a^n u(n),$$

where a is a real constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n. \end{aligned}$$

In order to simplify the preceding summation, we recall the formula for the sum of an infinite geometric sequence (given by (F.9)). The summation converges for $\left|az^{-1}\right| < 1$. Rearranging this inequality, we obtain

$$\begin{aligned} & \left|az^{-1}\right| < 1 \\ \Rightarrow\quad & \left|a^{-1}z\right| > 1 \\ \Rightarrow\quad & \left|a^{-1}\right| |z| > 1 \\ \Rightarrow\quad & |z| > |a|. \end{aligned}$$

From the formula for the sum of an infinite geometric sequence, we can write

$$\begin{aligned} X(z) &= \frac{1}{1-az^{-1}} \\ &= \frac{z}{z-a} \quad \text{for } |z| > |a|. \end{aligned}$$

Thus, we have

$$a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad \text{for } |z| > |a|. \qquad \blacksquare$$

Example 12.4. Find the z transform X of the sequence

$$x(n) = -a^n u(-n-1),$$

where a is a real constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n} \\ &= -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n \\ &= -\sum_{n=1}^{\infty} (a^{-1}z)^n. \end{aligned}$$

Recalling the formula for the sum of an infinite geometric sequence (as given by (F.9)), we deduce that $X(z)$ only converges for $\left|a^{-1}z\right| < 1$. Rearranging this inequality, we obtain

$$\begin{aligned} & \left|a^{-1}z\right| < 1 \\ \Rightarrow\quad & \left|az^{-1}\right| > 1 \\ \Rightarrow\quad & |a| \left|z^{-1}\right| > 1 \\ \Rightarrow\quad & |z| < |a|. \end{aligned}$$

From the formula for the sum of an infinite geometric sequence, we can write

$$\begin{aligned} X(z) &= -\frac{a^{-1}z}{1-a^{-1}z} \\ &= \frac{1}{1-az^{-1}} \\ &= \frac{z}{z-a} \quad \text{for } |z| < |a|. \end{aligned}$$

Thus, we have

$$-a^n u(-n-1) \xleftrightarrow{\text{ZT}} \frac{1}{1-az^{-1}} \quad \text{for } |z| < |a|. \qquad \blacksquare$$

At this point, we compare the results of Examples 12.3 and 12.4, and make an important observation. Notice that the same algebraic expression for X was obtained in both of these examples (i.e., $X(z) = \frac{z}{z-a}$). The only difference is in the convergence properties of X. In one case, $X(z)$ converges for $|z| > |a|$, while in the other it converges for $|z| < |a|$. As it turns out, one must specify both the algebraic expression for X and its region of convergence in order to uniquely determine $x = \mathcal{Z}^{-1}X$ from X.

Example 12.5. Find the z transform X of the sequence

$$x(n) = \delta(n-n_0),$$

where n_0 is an integer constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \delta(n-n_0)z^{-n} \\ &= z^{-n_0}. \end{aligned}$$

If $n_0 > 0$, X has a pole at 0, and therefore the ROC of X is all nonzero (finite) complex numbers as well as ∞. If $n_0 < 0$, X has a pole at ∞, and therefore the ROC of X is all (finite) complex numbers. If $n_0 = 0$, X has no poles, and therefore the ROC of X is all (finite) complex numbers as well as ∞. Thus, the ROC is all nonzero (finite) complex numbers as well as possibly 0 and/or ∞. Thus, we have

$$\delta(n-n_0) \xleftrightarrow{\text{ZT}} z^{-n_0} \text{ for all nonzero (finite) complex } z \text{ as well as possibly } z=0 \text{ and/or } z=\infty. \qquad \blacksquare$$

Example 12.6. Find the z transform X of the sequence

$$x(n) = \tfrac{1}{n!}a^n u(n),$$

where a is a real constant.

Solution. From the definition of the z transform, we have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \tfrac{1}{n!}a^n u(n)z^{-n} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!}a^n z^{-n} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(a/z)^n. \end{aligned}$$

Observing from (F.12) that $e^z = \sum_{n=0}^{\infty} \frac{1}{n!}z^n$ for all complex z, we can rewrite the above equation for X to obtain

$$X(z) = e^{a/z}.$$

(Note that a number of useful series, including the one used above for the exponential function, can be found in Section F.6.) Now, we consider the ROC of X. The only way that the expression for $X(z)$ can fail to converge is if $z = 0$, in which case division by zero occurs. Thus, the ROC of X is $|z| > 0$ (i.e., $z \neq 0$). So, in conclusion, we have

$$X(z) = e^{a/z} \text{ for } |z| > 0. \qquad \blacksquare$$

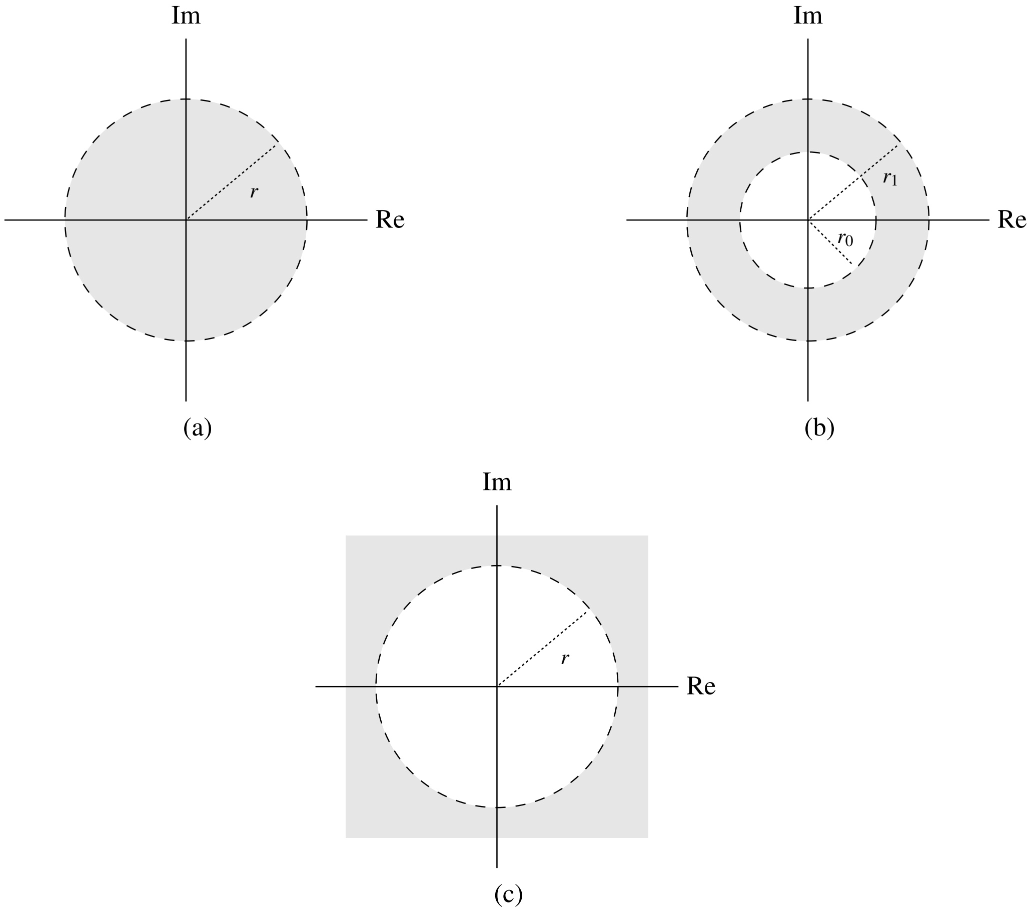


Figure 12.2: Examples of various types of sets. (a) A disk with center 0 and radius r; (b) an annulus with center 0, inner radius r_0, and outer radius r_1; and (c) an exterior of a circle with center 0 and radius r.

12.7 Region of Convergence for the z Transform

Before discussing the region of convergence (ROC) of the z transform in detail, we need to introduce some terminology involving sets in the complex plane. A **disk** (or more specifically an open disk) with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| < r,$$

where r is a real constant and $r > 0$. An **annulus** with center 0, inner radius r_0, and outer radius r_1 is the set of all complex numbers z satisfying

$$r_0 < |z| < r_1,$$

where r_0 and r_1 are real constants and $0 < r_0 < r_1$. The **exterior of a circle** with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| > r,$$

where r is a real constant and $r > 0$. Examples of a disk, an annulus, and an exterior of a circle are given in Figure 12.2.

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B. An illustrative example of set intersection is shown in Figure 12.3.

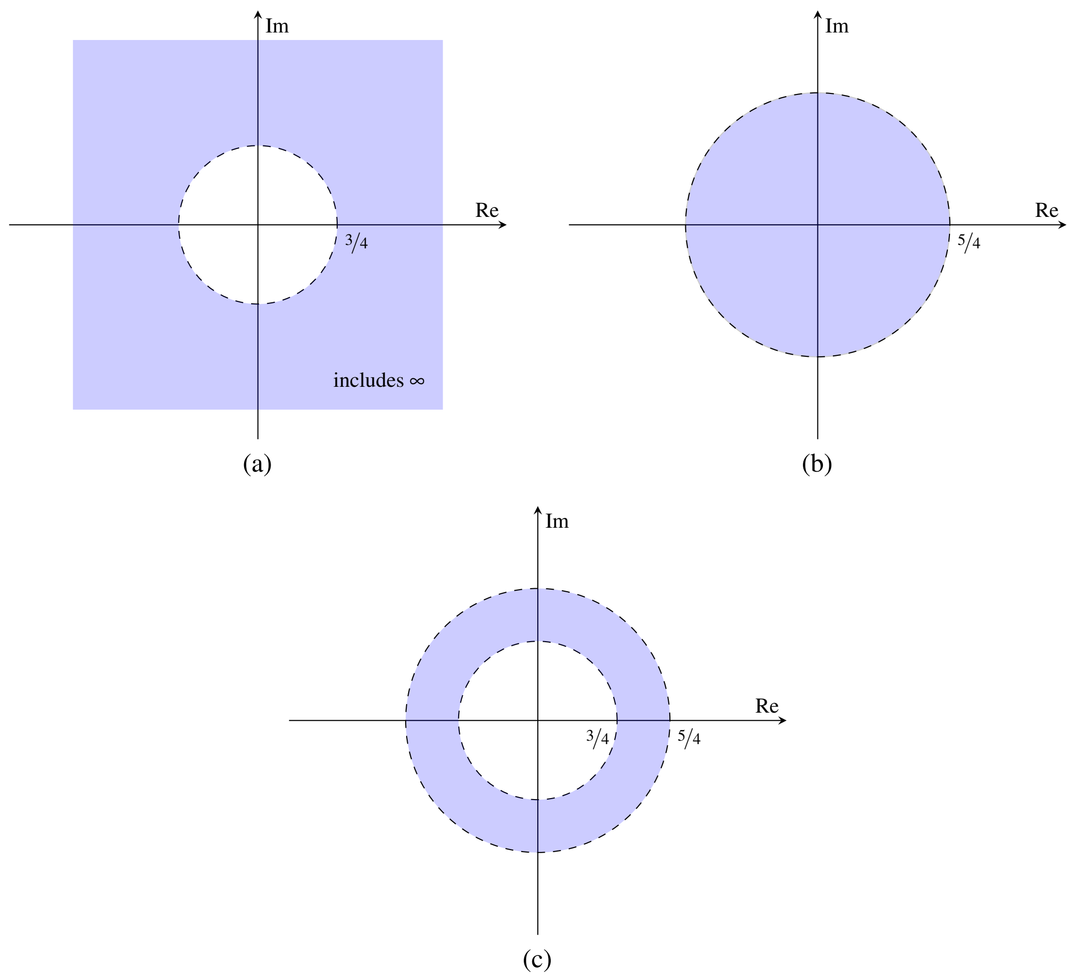


Figure 12.3: Example of set intersection. The sets (a) R_1 and (b) R_2; and (c) their intersection $R_1 \cap R_2$.

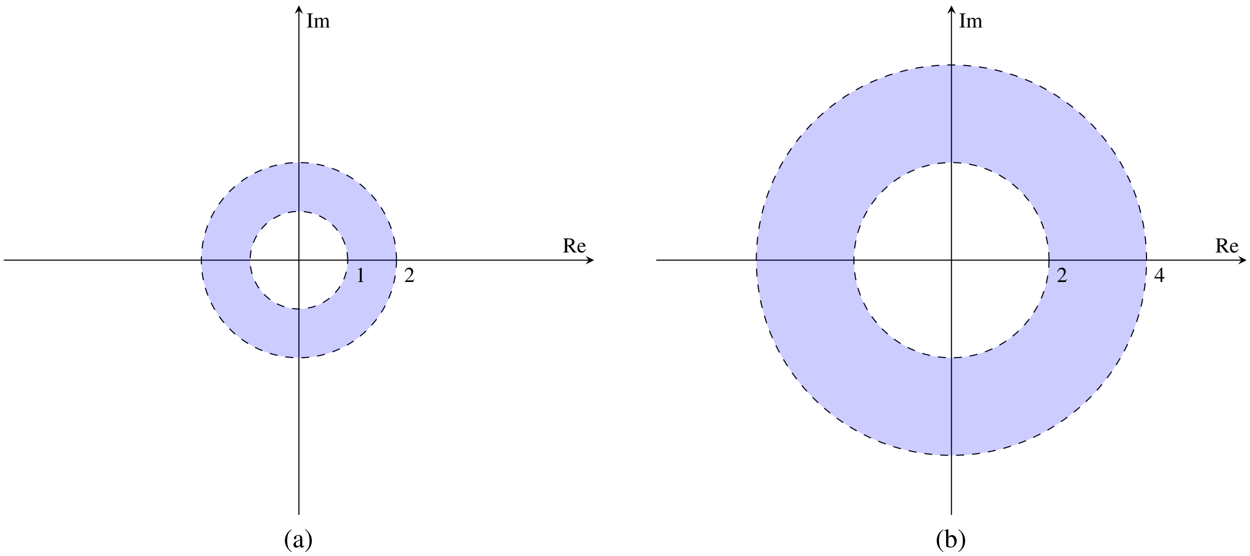


Figure 12.4: Example of multiplying a set by a scalar. (a) The set R. (b) The set $2R$

For a set S and a complex constant a, aS denotes the set given by

$$aS = \{az : z \in S\}.$$

That is, aS denotes the set formed by multiplying each element of S by a. In passing, we note that there is a simple geometric relationship between the sets S and aS. In particular, the region in the complex plane corresponding to the set aS is obtained by rotating the region corresponding to S about the origin by $\arg a$ and then scaling the result by a factor of $|a|$. (The order of the rotation and scaling can be interchanged, since these operations commute.) For example, suppose that R is the set of complex numbers z satisfying

$$1 < |z| < 2,$$

as shown in Figure 12.4(a). Then, $2R$ is the set of complex numbers z satisfying

$$2 < |z| < 4,$$

as shown in Figure 12.4(b). Furthermore, since R is an annulus centered at the origin, $R = aR$ for all complex a satisfying $|a| = 1$. That is, rotating the set R about the origin by any angle yields the same set R.

For a set S, S^{-1} denotes the set given by

$$S^{-1} = \{z^{-1} : z \in S\}.$$

That is, S^{-1} denotes the set formed by taking the reciprocal of each element of S. For example, suppose that R is the set of complex numbers z satisfying

$$|z| > \tfrac{3}{4},$$

as shown in Figure 12.5(a). Then, R^{-1} is the set of complex numbers z satisfying

$$|z| < \tfrac{4}{3},$$

as shown in Figure 12.5(b).

As we saw earlier, for a sequence x, the complete specification of its z transform X requires not only an algebraic expression for X, but also the ROC associated with X. Two distinct sequences can have the same algebraic expression for their z transform. In what follows, we examine some of the constraints on the ROC (of the z transform) for various classes of sequences.

One can show that the ROC of the z transform has the following properties:

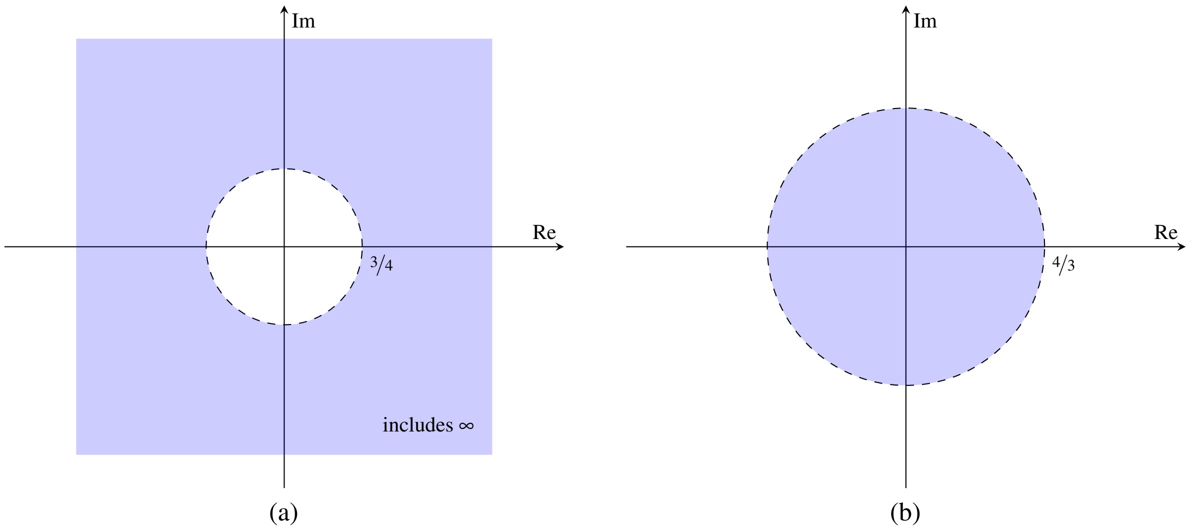


Figure 12.5: Example of the reciprocal of a set. (a) The set R; and its reciprocal R^{-1}.

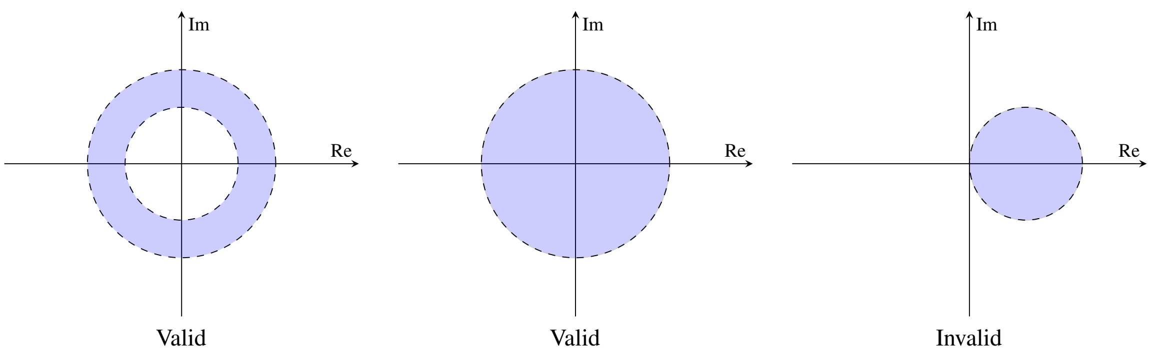


Figure 12.6: Examples of sets that would be either valid or invalid as the ROC of a z transform.

1. The ROC of the z transform X consists of concentric circles centered at the origin in the complex plane. That is, if a point z_0 is in the ROC, then the circle centered at the origin passing through z_0 (i.e., $|z| = |z_0|$) is also in the ROC.

   Justification: The z transform X of the sequence x is simply the (DT) Fourier transform of $x'(n) = |z|^{-n}x(n)$. Thus, X converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on $|z|$, the convergence of the z transform only depends on $|z|$.

   Some examples of sets that would be either valid or invalid as ROCs are shown in Figure 12.6.

2. If a z transform X is rational, the ROC of X does not contain any poles and is bounded by poles or extends to infinity.

   Partial justification: Since X is rational, its value becomes infinite at a pole. So obviously, X does not converge at a pole. Therefore, it follows that the ROC of X cannot contain a pole.

   Some examples of sets that would be either valid or invalid as ROCs of rational z transforms are shown in Figure 12.7.

3. If a sequence x is finite duration and its z transform X converges for at least one point, then X converges for all points in the complex plane, except possibly 0 and/or ∞ (i.e., the ROC is a disk which may exclude 0 or include

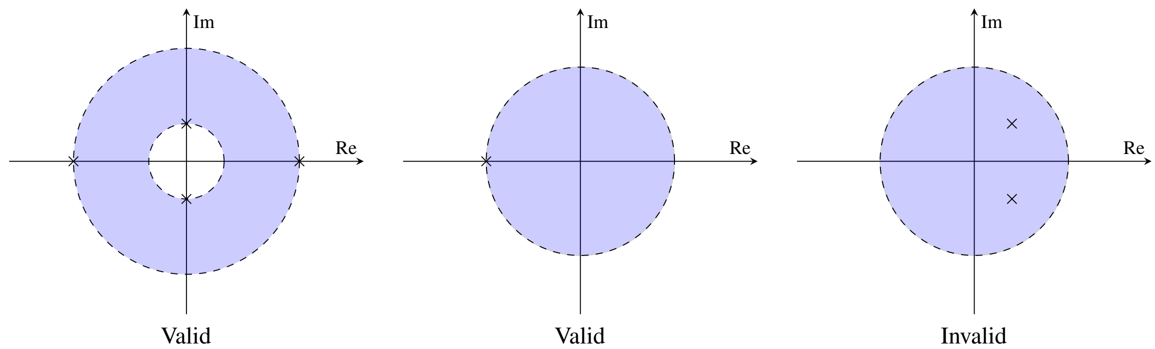


Figure 12.7: Examples of sets that would be either valid or invalid as the ROC of a rational z transform.

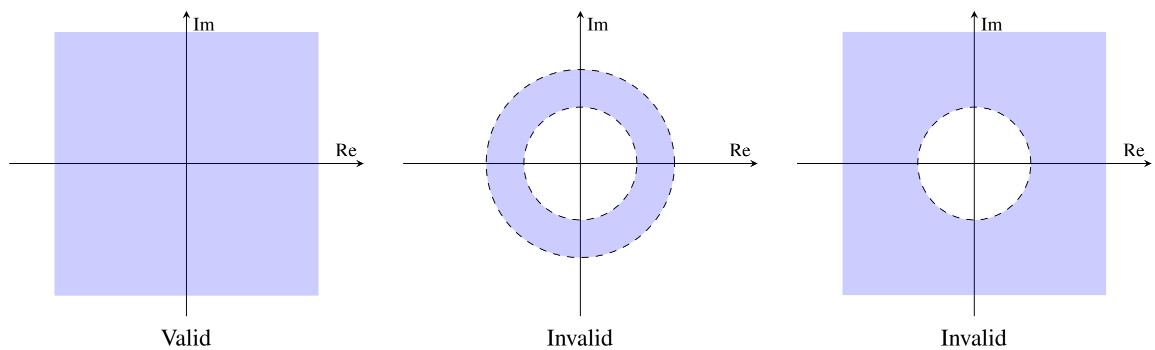


Figure 12.8: Examples of sets that would be either valid or invalid as the ROC of the z transform of a finite-duration sequence.

∞). The ROC includes 0 if $x(n) = 0$ for all $n > 0$ (i.e., $X(z)$ has no negative powers of z). The ROC includes ∞ if x is causal (i.e., $X(z)$ has no positive powers of z). Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown in Figure 12.8.

4. If a sequence x is right sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all (finite) values of z for which $|z| > r_0$ will also be in the ROC of X (i.e., the ROC contains the exterior of a circle centered at 0, possibly including ∞). The ROC includes ∞ if x is causal (i.e., $X(z)$ has no positive powers of z). Moreover, if x is right sided but not left sided, then the ROC of X is the exterior of a circle centered at 0, possibly including ∞. Examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown in Figure 12.9.

5. If a sequence x is left sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all values of z for which $0 < |z| < r_0$ will also be in the ROC of X (i.e., the ROC contains a disk centered at 0, possibly excluding 0). The ROC includes 0 if $x(n) = 0$ for all $n > 0$ (i.e., $X(z)$ has no negative powers of z). Moreover, if x is left sided but not right sided, the ROC is a disk centered at 0, possibly excluding 0. Examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown in Figure 12.10.

6. If a sequence x is two sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then the ROC of X will consist of a ring in the complex plane that contains the circle $|z| = r_0$ (i.e., the ROC is an annulus centered at 0). Examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown in Figure 12.11.

7. If a sequence x has a rational z transform X (with at least one pole), then:

   (a) If x is right sided, the ROC of X is the region in the complex plane outside the circle of radius equal to the largest magnitude of the poles of X (i.e., the region outside the outermost pole), possibly including ∞.

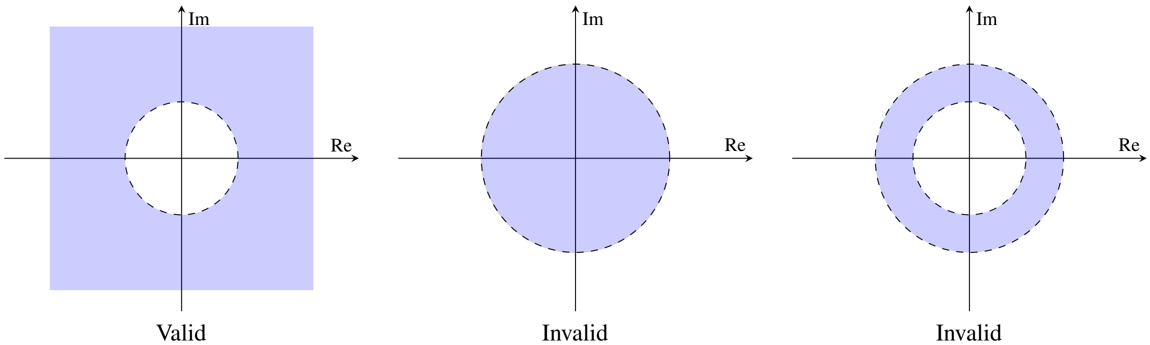


Figure 12.9: Examples of sets that would be either valid or invalid as the ROC of the z transform of a sequence that is right sided but not left sided.

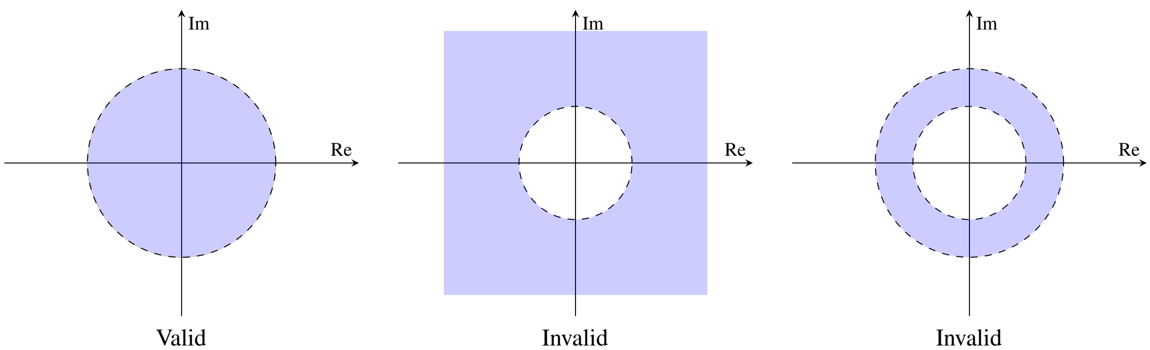


Figure 12.10: Examples of sets that would be either valid or invalid as the ROC of the z transform of a sequence that is left sided but not right sided.

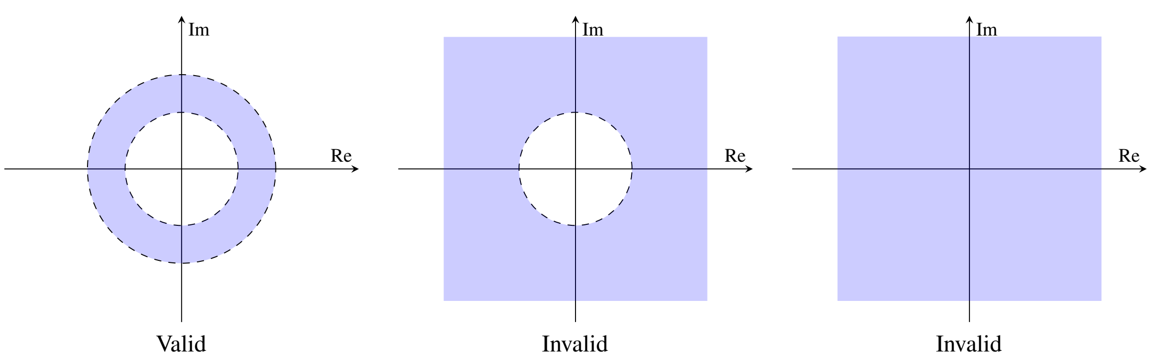


Figure 12.11: Examples of sets that would be either valid or invalid as the ROC of the z transform of a two-sided sequence.

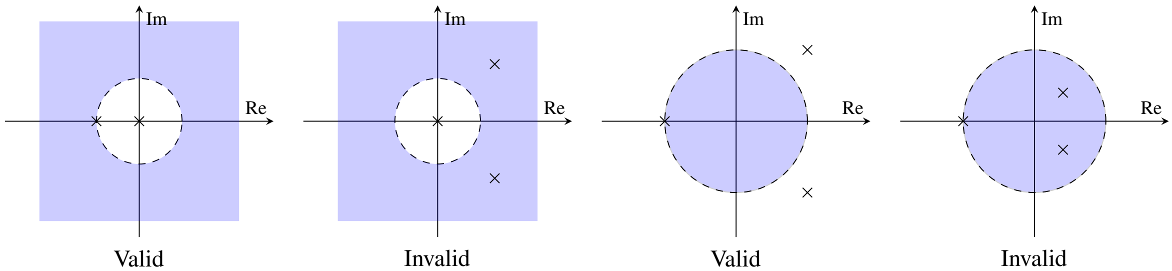


Figure 12.12: Examples of sets that would be either valid or invalid as the ROC of a rational z transform of a left/right-sided sequence.

Table 12.1: Relationship between the sidedness properties of x and the ROC of $X = \mathcal{Z}x$

x		
left sided	right sided	ROC of X
yes	yes	everywhere, except possibly 0 and/or ∞
no	yes	exterior of circle centered at origin, possibly including ∞
yes	no	disk centered at origin, possibly excluding 0
no	no	annulus centered at origin

(b) If x is left sided, the ROC of X is the region in the complex plane inside the circle of radius equal to the smallest magnitude of the nonzero poles of X and extending inward to, and possibly including, 0 (i.e., the region inside the innermost nonzero pole).

Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are shown in Figure 12.12.

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every sequence can be classified as exactly one of left sided but not right sided, right sided but not left sided, two sided (i.e, neither left nor right sided), or finite duration (i.e., both left and right sided), we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of

- the entire complex plane, except possibly 0 and/or ∞;
- the exterior of a circle centered at the origin, possibly including ∞;
- a disk centered at the origin, possibly excluding 0;
- an annulus centered at the origin; or
- the empty set.

In particular, the ROC of X depends on the left- and right-sidedness of x as shown in Table 12.1. Thus, the ROC must be a connected set. (A set S is said to be connected, if for every two elements a and b in S, there exists a path from a to b that is contained in S.) For example, the sets shown in Figure 12.13 would not be valid as ROCs.

Example 12.7. The z transform X of the sequence x has the algebraic expression

$$X(z) = \frac{1}{(z^2-1)(z^2+4)}.$$

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding sequence x is left sided but not right sided, right sided but not left sided, two sided, or finite duration.

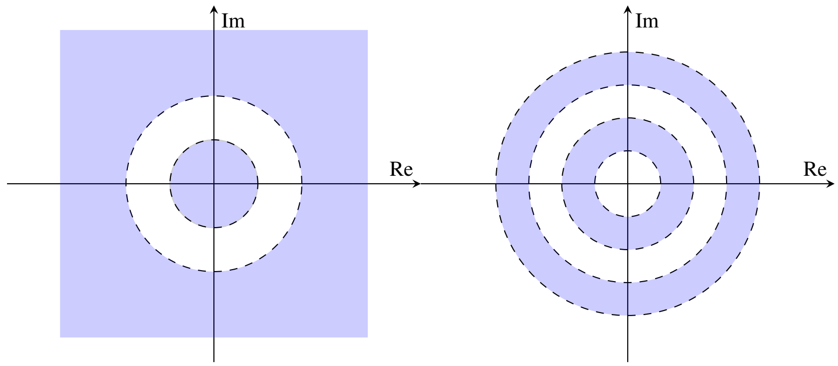


Figure 12.13: Examples of sets that would not be a valid ROC of a z transform.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(z) = \frac{1}{(z+1)(z-1)(z+j2)(z-j2)}.$$

Thus, X has poles at -1, 1, $-2j$, and $2j$. Since these poles only have two distinct magnitudes (namely, 1 and 2), there are three possible ROCs:

i) $|z| < 1$,
ii) $1 < |z| < 2$, and
iii) $|z| > 2$.

These ROCs are plotted in Figures 12.14(a), (b), and (c), respectively. The first ROC is a disk, so the corresponding x must be left sided but not right sided. The second ROC is an annulus, so the corresponding x must be two sided. The third ROC is the exterior of a circle, so the corresponding x must be right sided but not left sided. ■

12.8 Properties of the z Transform

The z transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 12.2 (on page 551).

12.8.1 Linearity

Arguably, the most important property of the z transform is linearity, as introduced below.

Theorem 12.1 (Linearity)**.** *If $x_1(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)$ with ROC R_1 and $x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_2(z)$ with ROC R_2, then*

$$a_1x_1(n) + a_2x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2,$$

where a_1 and a_2 are arbitrary complex constants. This is known as the ***linearity property*** *of the z transform.*

Proof. Let $y(n) = a_1x_1(n) + a_2x_2(n)$, and let Y denote the z transform of y. Using the definition of the z transform

