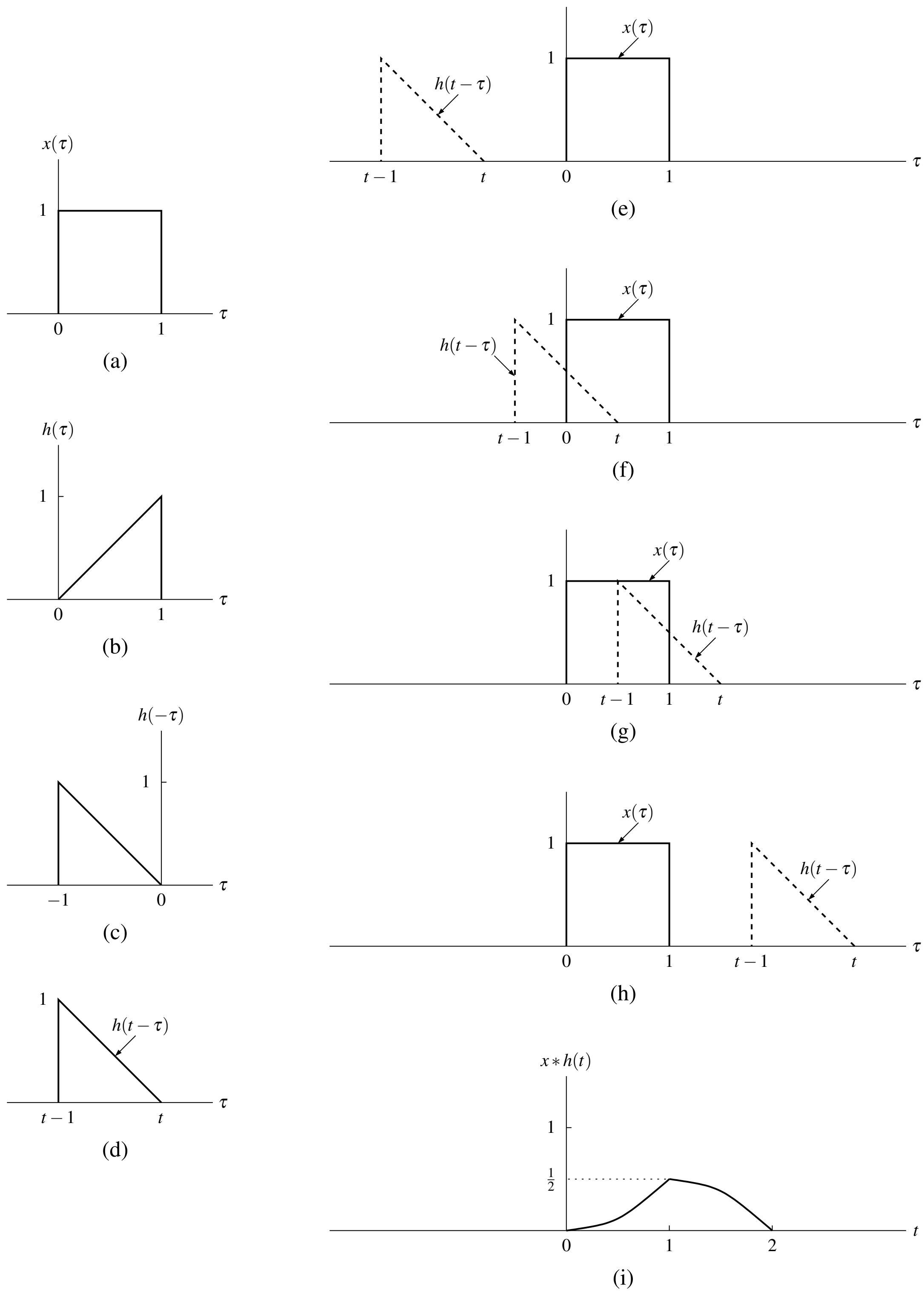

Figure 4.2: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$; and (i) the convolution result $x*h$.

First, we consider the case of $t < 0$. From Figure 4.3(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.10}$$

Second, we consider the case of $0 \leq t < 1$. From Figure 4.3(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t} \tau d\tau \\ &= [\tfrac{1}{2}\tau^2]_0^t \\ &= \tfrac{1}{2}t^2. \end{aligned} \tag{4.11}$$

Third, we consider the case of $1 \leq t < 2$. From Figure 4.3(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} \tau d\tau + \int_{1}^{t} (-\tau+2)d\tau \\ &= [\tfrac{1}{2}\tau^2]_{t-1}^{1} + [-\tfrac{1}{2}\tau^2 + 2\tau]_1^t \\ &= \tfrac{1}{2} - [\tfrac{1}{2}(t-1)^2] - \tfrac{1}{2}t^2 + 2t - [-\tfrac{1}{2} + 2] \\ &= -t^2 + 3t - \tfrac{3}{2}. \end{aligned} \tag{4.12}$$

Fourth, we consider the case of $2 \leq t < 3$. From Figure 4.3(h), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{2} (-\tau+2)d\tau \\ &= [-\tfrac{1}{2}\tau^2 + 2\tau]_{t-1}^{2} \\ &= 2 - [-\tfrac{1}{2}t^2 + 3t - \tfrac{5}{2}] \\ &= \tfrac{1}{2}t^2 - 3t + \tfrac{9}{2}. \end{aligned} \tag{4.13}$$

Lastly, we consider the case of $t \geq 3$. From Figure 4.3(i), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.14}$$

Combining the results of (4.10), (4.11), (4.12), (4.13), and (4.14) together, we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 1 \\ -t^2 + 3t - \frac{3}{2} & 1 \leq t < 2 \\ \frac{1}{2}t^2 - 3t + \frac{9}{2} & 2 \leq t < 3 \\ 0 & t \geq 3. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.3(j). ■

Example 4.4. Compute the convolution $x * h$, where

$$x(t) = e^{-at}u(t), \quad h(t) = u(t),$$

and a is a strictly positive real constant.

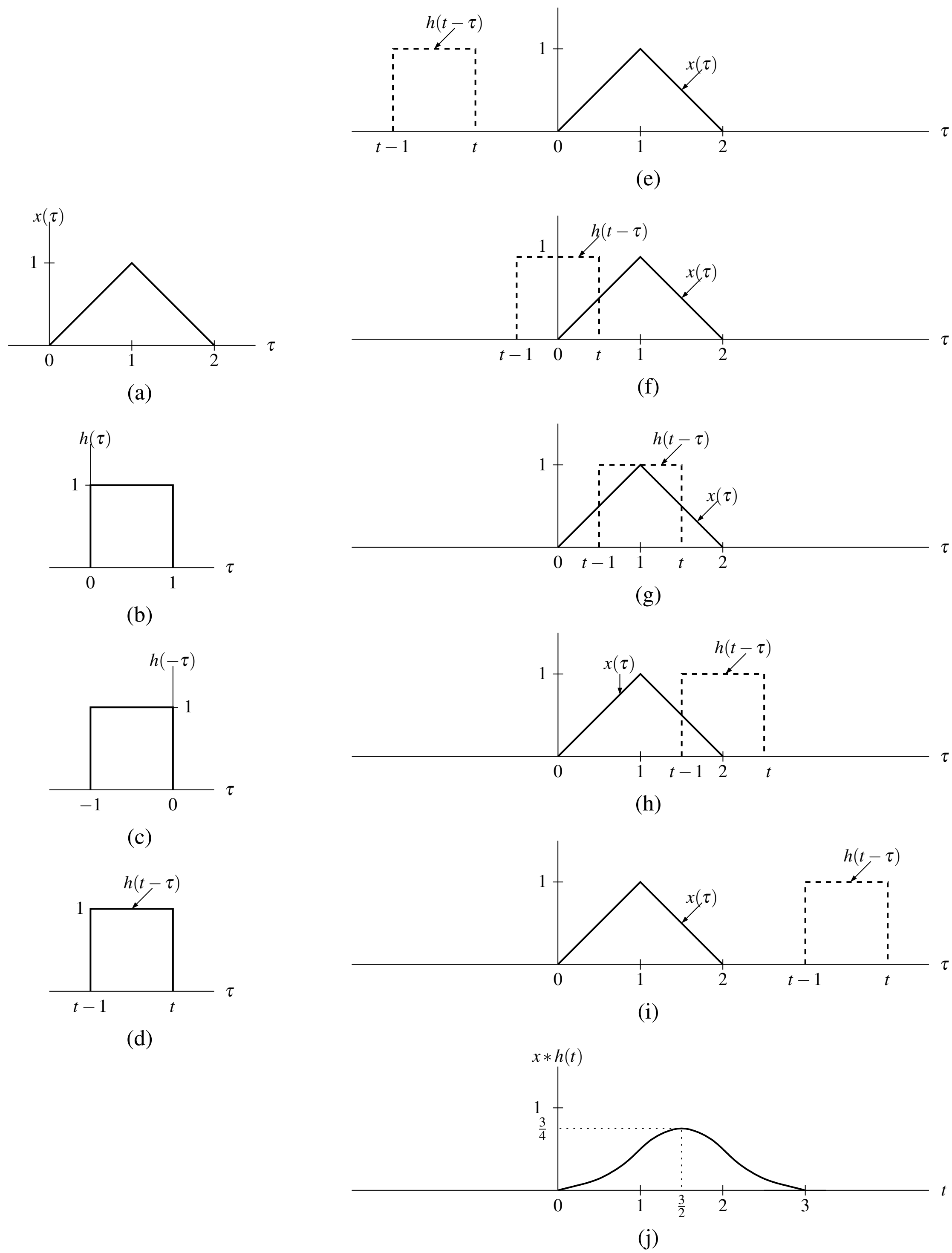

Figure 4.3: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, (h) $2 \le t < 3$, and (i) $t \ge 3$; and (j) the convolution result $x*h$.

Solution. Since x and h are relatively simple functions, we will solve this problem without the aid of graphs. Our objective in this example is twofold. First, we want to show that it is possible, if one is very careful, to perform simple convolutions without using graphs as aids. Second, we would like to show that this is actually somewhat tricky to do correctly, and probably it would have been better to draw graphs for guidance in this example in order to reduce the likelihood of errors.

From the definition of convolution, we have

$$\begin{aligned} x * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau. \end{aligned} \tag{4.15}$$

Since

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0 \leq \tau \text{ and } \tau \leq t \\ 0 & \text{otherwise,} \end{cases}$$

the integrand can only be nonzero if $0 \leq \tau$ and $\tau \leq t$ (which necessarily requires that $t \geq 0$). So, if $t < 0$, the integrand will be zero, and $x * h(t) = 0$. Now, let us consider the case of $t > 0$. From (4.15), we have

$$\begin{aligned} x * h(t) &= \int_0^t e^{-a\tau} d\tau \\ &= [-\tfrac{1}{a}e^{-a\tau}]_0^t \\ &= \tfrac{1}{a}(1-e^{-at}). \end{aligned}$$

Thus, we have

$$\begin{aligned} x * h(t) &= \begin{cases} \frac{1}{a}(1-e^{-at}) & t > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \tfrac{1}{a}(1-e^{-at})u(t). \end{aligned}$$

Note that, as the above solution illustrates, computing a convolution without graphs as aids can be somewhat tricky to do correctly, even when the functions being convolved are relatively simple like the ones in this example. If some steps in the above solution are unclear, it would be helpful to sketch graphs to assist in the convolution computation. For example, the use of graphs, like those shown in Figure 4.4, would likely make the above convolution much easier to compute correctly. ∎

4.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties.

Theorem 4.1 (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two functions x and h,*

$$x * h = h * x. \tag{4.16}$$

In other words, the result of a convolution is not affected by the order of its operands.

Proof. We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (4.16) as follows:

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

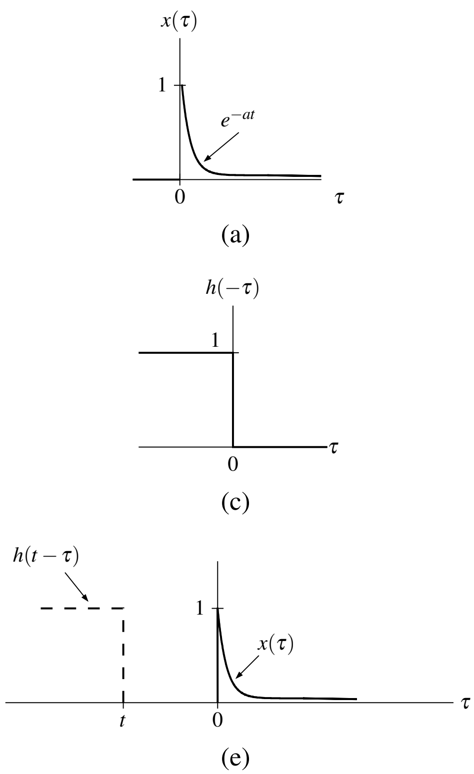

Figure 4.4: Evaluation of the convolution $x * h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; and the functions associated with the product in the convolution integral for (e) $t < 0$ and (f) $t > 0$.

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) \\ &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv \\ &= h * x(t). \end{aligned}$$

(Note that, above, we used the fact that, for any function f, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.) Thus, we have proven that convolution is commutative. ■

Theorem 4.2 (Associativity of convolution)**.** *Convolution is associative. That is, for any three functions x, h_1, and h_2,*

$$(x * h_1) * h_2 = x * (h_1 * h_2). \tag{4.17}$$

In other words, the final result of multiple convolutions does not depend on how the convolution operations are grouped.

Proof. To begin, we use the definition of the convolution operation to expand the left-hand side of (4.17) as follows:

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} [x * h_1(v)] h_2(t-v)dv \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)d\tau \right) h_2(t-v)dv. \end{aligned}$$

Now, we change the order of integration to obtain

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)h_2(t-v)dv d\tau.$$

Pulling the factor of $x(\tau)$ out of the inner integral yields

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau)h_2(t-v)dv d\tau.$$

Next, we perform a change of variable. Let $\lambda = v - \tau$ which implies that $v = \lambda + \tau$ and $d\lambda = dv$. Using this change of variable, we can write

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty-\tau}^{\infty-\tau} h_1(\lambda)h_2(t-\lambda-\tau)d\lambda d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda)h_2(t-\lambda-\tau)d\lambda d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(\lambda)h_2([t-\tau]-\lambda)d\lambda \right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left[h_1 * h_2(t-\tau) \right] d\tau \\ &= (x * [h_1 * h_2])(t). \end{aligned}$$

Thus, we have proven that convolution is associative. ■

Theorem 4.3 (Distributivity of convolution)**.** *Convolution is distributive. That is, for any three functions x, h_1, and h_2,*

$$x*(h_1+h_2) = x*h_1 + x*h_2. \tag{4.18}$$

In other words, convolution can be distributed across addition.

Proof. The proof of this property is relatively simple. Expanding the left-hand side of (4.18), we have

$$\begin{aligned}(x*[h_1+h_2])(t) &= \int_{-\infty}^{\infty} x(\tau)[h_1(t-\tau)+h_2(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau \\ &= x*h_1(t) + x*h_2(t).\end{aligned}$$

Thus, we have shown that convolution is distributive. ■

The identity for an operation defined on elements of a set is often extremely helpful to know. Consider the operations of addition and multiplication as defined for real numbers. For any real number a, $a+0=a$. Since adding zero to a has no effect (i.e., the result is a), we call 0 the **additive identity**. For any real number a, $1\cdot a = a$. Since multiplying a by 1 has no effect (i.e., the result is a), we call 1 the **multiplicative identity**. Imagine for a moment how difficult arithmetic would be if we did not know that $a+0=a$ or $1\cdot a = a$. For this reason, identity values are clearly of fundamental importance.

Earlier, we were introduced to a new operation known as convolution. So, in light of the above, it is natural to wonder if there is a convolutional identity. In fact, there is, as given by the theorem below.

Theorem 4.4 (Convolutional identity)**.** *For any function x,*

$$x*\delta = x. \tag{4.19}$$

In other words, δ is the convolutional identity (i.e., convolving any function x with δ simply yields x).

Proof. Suppose that we have an arbitrary function x. From the definition of convolution, we can write

$$x*\delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Now, let us employ a change of variable. Let $\lambda = -\tau$ so that $\tau = -\lambda$ and $d\tau = -d\lambda$. Applying the change of variable, we obtain

$$\begin{aligned}x*\delta(t) &= \int_{-(-\infty)}^{-(\infty)} x(-\lambda)\delta(t+\lambda)(-1)d\lambda \\ &= \int_{\infty}^{-\infty} x(-\lambda)\delta(t+\lambda)(-1)d\lambda \\ &= \int_{-\infty}^{\infty} x(-\lambda)\delta(\lambda+t)d\lambda.\end{aligned} \tag{4.20}$$

From the equivalence property of δ, we can rewrite the preceding equation as

$$\begin{aligned}x*\delta(t) &= \int_{-\infty}^{\infty} x(-[-t])\delta(\lambda+t)d\lambda \\ &= \int_{-\infty}^{\infty} x(t)\delta(\lambda+t)d\lambda.\end{aligned}$$

Factoring $x(t)$ out of the integral, we obtain

$$x*\delta(t) = x(t)\int_{-\infty}^{\infty} \delta(\lambda+t)d\lambda.$$

Since $\int_{-\infty}^{\infty}\delta(\lambda)d\lambda = 1$ implies that $\int_{-\infty}^{\infty}\delta(\lambda+t)d\lambda = 1$, we have

$$x*\delta(t) = x(t).$$

Thus, δ is the convolutional identity (i.e., $x*\delta = x$). (Alternatively, we could have directly applied the sifting property to (4.20) to show the desired result.) ■

4.4 Periodic Convolution

The convolution of two periodic functions is usually not well defined. This motivates an alternative notion of convolution for periodic signals known as periodic convolution. The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

where $\int_T$ denotes integration over an interval of length T. The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

4.5 Characterizing LTI Systems and Convolution

As a matter of terminology, the **impulse response** h of a system $\mathcal{H}$ is defined as

$$h = \mathcal{H}\delta.$$

In other words, the impulse response of a system is the output that it produces when presented with δ as an input. As it turns out, a LTI system has a very special relationship between its input, output, and impulse response, as given by the theorem below.

Theorem 4.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. To begin, we assume that $\mathcal{H}$ is LTI (i.e., $\mathcal{H}$ is both linear and time invariant). Using the fact that δ is the convolutional identity, we can write

$$\mathcal{H}x = \mathcal{H}\{x * \delta\}.$$

From the definition of convolution, we have

$$\mathcal{H}x = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot - \tau)d\tau\right\}.$$

Since $\mathcal{H}$ is linear, we can pull the integral and $x(\tau)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

$$\mathcal{H}x = \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(\cdot - \tau)\}d\tau. \tag{4.21}$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by τ. That is, we have

$$\mathcal{H}\{\delta(\cdot - \tau)\} = h(\cdot - \tau).$$

Thus, we can rewrite (4.21) as

$$\begin{aligned}\mathcal{H}x &= \int_{-\infty}^{\infty} x(\tau)h(\cdot - \tau)d\tau \\ &= x * h.\end{aligned}$$

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■

By Theorem 4.5 above, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input. Consequently, the impulse response provides a very powerful tool for the study of LTI systems.

Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.22}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.23}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t).$$

Substituting (4.22) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) \\
&= x * u(t) \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t^+}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)d\tau.
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.22) is, in fact, the ideal integrator given by (4.23). ■

Example 4.6. Consider a LTI system $\mathcal{H}$ with impulse response h, where

$$h(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response y of the system to the input x given by

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Plots of x and h are given in Figures 4.5(a) and (b), respectively. Since the system is LTI, we know that

$$y(t) = x * h(t).$$

Thus, in order to find the response y of the system to the input x, we simply need to compute the convolution $x * h$.

We begin by plotting the functions x and h as shown in Figures 4.5(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.5(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 4.5(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x

and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.5(e) to (h).

First, we consider the case of $t < 0$. From Figure 4.5(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.24}$$

Second, we consider the case of $0 \leq t < 1$. From Figure 4.5(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t. \end{aligned} \tag{4.25}$$

Third, we consider the case of $1 \leq t < 2$. From Figure 4.5(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} d\tau \\ &= [\tau]|_{t-1}^{1} \\ &= 1-(t-1) \\ &= 2-t. \end{aligned} \tag{4.26}$$

Fourth, we consider the case of $t \geq 2$. From Figure 4.5(h), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.27}$$

Combining the results of (4.24), (4.25), (4.26), and (4.27), we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.5(i). The response y of the system to the specified input is simply $x * h$. ■

4.6 Step Response of LTI Systems

The **step response** s of a system $\mathcal{H}$ is defined as

$$s = \mathcal{H}u$$

(i.e., the step response of a system is the output it produces for a unit-step function input). In the case of a LTI system, it turns out that the step response is closely related to the impulse response, as given by the theorem below.

Theorem 4.6. *The step response s and impulse response h of a LTI system are related as*

$$h(t) = \frac{ds(t)}{dt} \quad \text{and} \quad s(t) = \int_{-\infty}^{t} h(\tau)d\tau.$$

That is, the impulse response h is the derivative of the step response s.

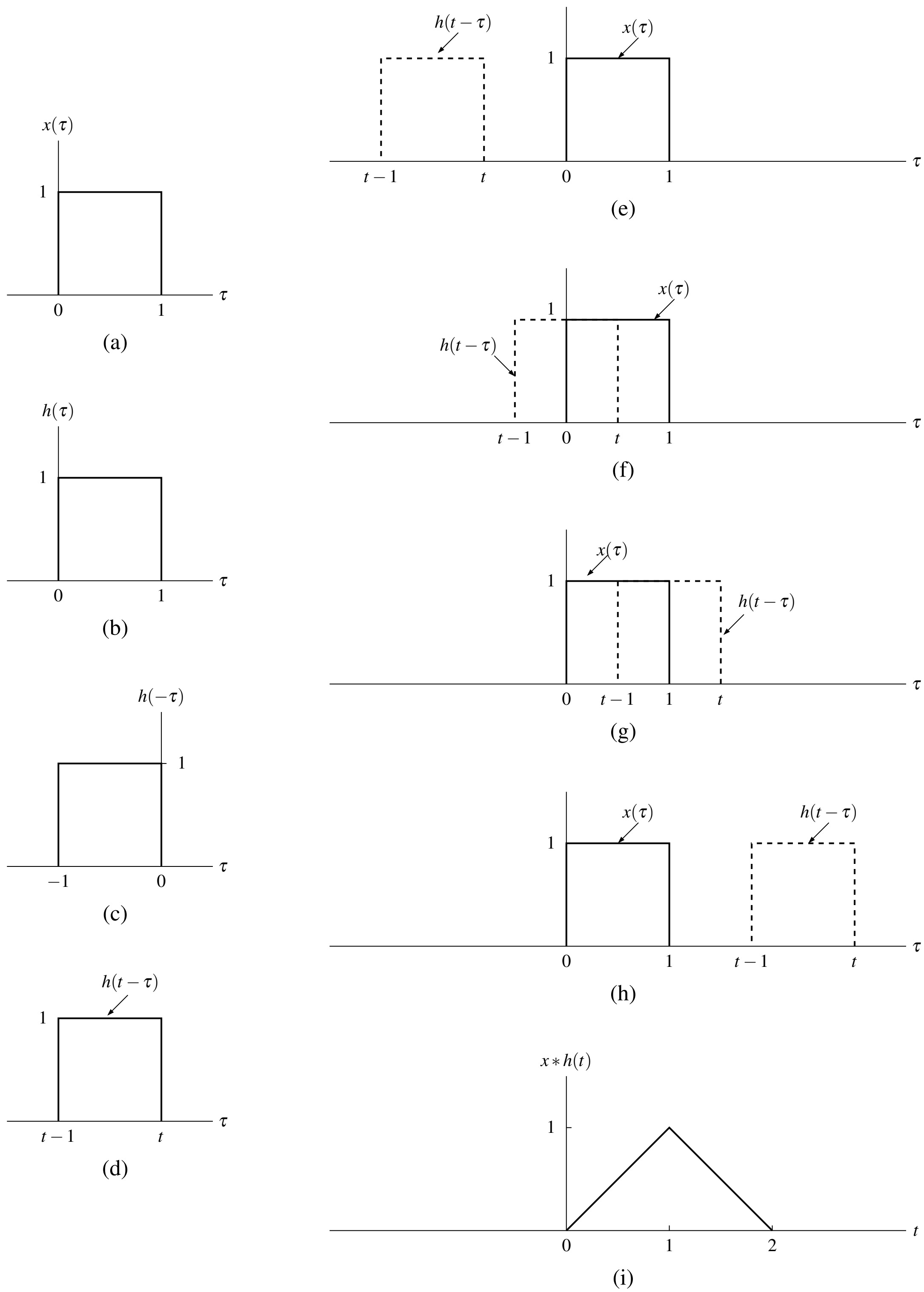

Figure 4.5: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$; and (i) the convolution result $x*h$.

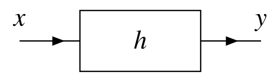

Figure 4.6: Block diagram representation of continuous-time LTI system with input x, output y, and impulse response h.

Proof. Using the fact that $s = u * h$, we can write

$$\begin{aligned} s(t) &= u * h(t) \\ &= h * u(t) \\ &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} h(\tau)d\tau. \end{aligned}$$

Thus, s can be obtained by integrating h. Taking the derivative of s, we obtain

$$\begin{aligned} \frac{ds(t)}{dt} &= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{-\infty}^{t+\Delta t} h(\tau)d\tau - \int_{-\infty}^{t} h(\tau)d\tau\right] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{t}^{t+\Delta t} h(\tau)d\tau \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}(h(t)\Delta t) \\ &= h(t). \end{aligned}$$

Thus, h is the derivative of s. ■

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. In particular, the impulse response can be determined from the step response via differentiation. From a practical point of view, the step response is more useful for characterizing a system based on experimental measurements. Obviously, we cannot directly measure the impulse response of a system because we cannot (in the real world) generate a unit-impulse function or an accurate approximation thereof. We can, however, produce a reasonably good approximation of the unit-step function in the real world. Thus, we can measure the step response and from it determine the impulse response.

4.7 Block Diagram Representation of Continuous-Time LTI Systems

Frequently, it is convenient to represent continuous-time LTI systems in block diagram form. Since a LTI system is completely characterized by its impulse response, we often label such a system with its impulse response in a block diagram. That is, we represent a LTI system with input x, output y, and impulse response h, as shown in Figure 4.6.

4.8 Interconnection of Continuous-Time LTI Systems

Suppose that we have a LTI system with input x, output y, and impulse response h. We know that x and y are related as $y = x * h$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

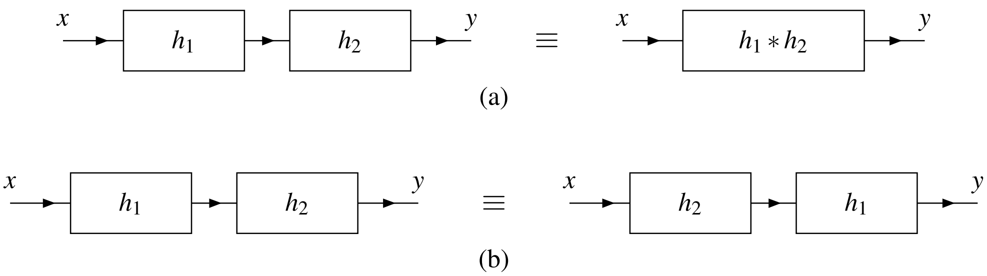

Figure 4.7: Equivalences for the series interconnection of continuous-time LTI systems. The (a) first and (b) second equivalences.

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 4.7(a). From the block diagram on the left side of Figure 4.7(a), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity of convolution, however, this is equivalent to

$$y = x * (h_1 * h_2).$$

Thus, the series interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 * h_2$. In other words, we have the equivalence shown in Figure 4.7(a).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 4.7(b). From the block diagram on the left side of Figure 4.7(b), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity and commutativity of convolution, this is equivalent to

$$\begin{aligned} y &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1. \end{aligned}$$

Thus, interchanging the two LTI systems does not change the behavior of the overall system with input x and output y. In other words, we have the equivalence shown in Figure 4.7(b).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a parallel configuration, as shown on the left-side of Figure 4.8. From the block diagram on the left side of Figure 4.8, we have

$$y = x * h_1 + x * h_2.$$

Due to convolution being distributive, however, this equation can be rewritten as

$$y = x * (h_1 + h_2).$$

Thus, the parallel interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 + h_2$. In other words, we have the equivalence shown in Figure 4.8.

Example 4.7. Consider the system with input x, output y, and impulse response h as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

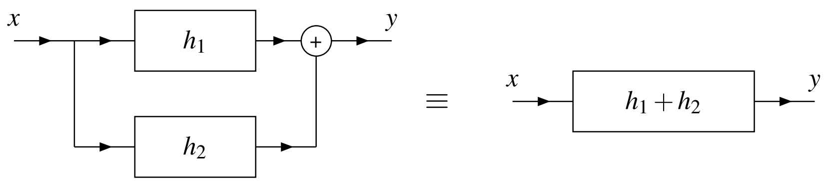

Figure 4.8: Equivalence for the parallel interconnection of continuous-time LTI systems.

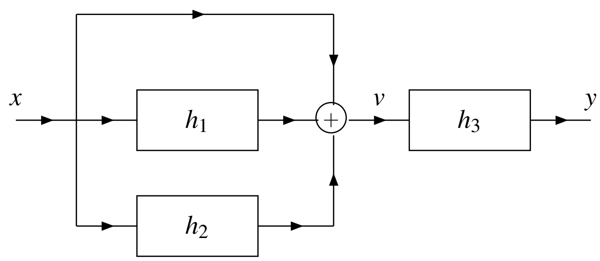

Figure 4.9: System interconnection example.

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t).$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [h_3 + h_1 * h_3 + h_2 * h_3](t). \end{aligned}$$

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t).$$ ■

4.9 Properties of Continuous-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, BIBO stability, and invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

4.9.1 Memory

The first system property to be considered is memory.

Theorem 4.7 (Memorylessness of LTI system)**.** *A LTI system with impulse response h is memoryless if and only if*

$$h(t) = 0 \textit{ for all } t \neq 0.$$

Proof. Recall that a system is memoryless if its output y at any arbitrary time depends only on the value of its input x at that same time. Suppose now that we have a LTI system with input x, output y, and impulse response h. The output y at some arbitrary time t_0 is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= h * x(t_0) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t_0-\tau)d\tau. \end{aligned}$$

Consider the integral in the above equation. In order for the system to be memoryless, the result of the integration is allowed to depend on $x(t)$ only for $t = t_0$. This, however, is only possible if

$$h(t) = 0 \quad \text{for all } t \neq 0. \qquad \blacksquare$$

From the preceding theorem, it follows that a memoryless LTI system must have an impulse response h of the form

$$h(t) = K\delta(t) \tag{4.28}$$

where K is a complex constant. As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y(t) &= x * (K\delta)(t) \\ &= K(x * \delta)(t) \\ &= Kx(t). \end{aligned}$$

In other words, a memoryless LTI system must be an ideal amplifier (i.e., a system that simply performs amplitude scaling).

Example 4.8. Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

Example 4.9. Consider the LTI system with the impulse response h given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse function δ. Therefore, the system is memoryless (i.e., does not have memory). ■

4.9.2 Causality

The next system property to be considered is causality.

Theorem 4.8 (Causality of LTI system)**.** *A LTI system with impulse response h is causal if and only if*

$$h(t) = 0 \textit{ for all } t < 0.$$

(i.e., h is causal).

Proof. Recall that a system is causal if its output y at any arbitrary time t_0 does not depend on its input x at a time later than t_0. Suppose that we have the LTI system with input x, output y, and impulse response h. The value of the output y at t_0 is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t_0-\tau)d\tau \\ &= \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau + \int_{t_0^+}^{\infty} x(\tau)h(t_0-\tau)d\tau. \end{aligned} \tag{4.29}$$

In order for the expression for $y(t_0)$ in (4.29) not to depend on $x(t)$ for $t > t_0$, we must have that

$$h(t) = 0 \quad \text{for } t < 0 \tag{4.30}$$

(i.e., h is causal). In this case, (4.29) simplifies to

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau.$$

Clearly, the result of this integration does not depend on $x(t)$ for $t > t_0$ (since τ varies from $-\infty$ to t_0). Therefore, a LTI system is causal if its impulse response h satisfies (4.30). ■

Example 4.10. Consider the LTI system with impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. ■

Example 4.11. Consider the LTI system with impulse response h given by

$$h(t) = \delta(t + t_0),$$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. ■

4.9.3 Invertibility

The next system property to be considered is invertibility.

Theorem 4.9 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with impulse response h. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has an impulse response h_{inv} that satisfies*

$$h * h_{\mathsf{inv}} = \delta.$$

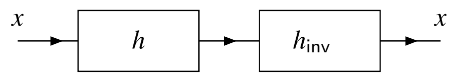

Figure 4.10: System in cascade with its inverse.

Proof. To begin, we need to show that the inverse of a LTI system, if it exists, must also be LTI. This part of the proof, however, is left as an exercise for the reader in Exercise 4.10. (The general approach to take for this problem is to show that: 1) the inverse of a linear system, if it exists, is linear; and 2) the inverse of a time-invariant system, if it exists, is time invariant.) We assume that this part of the proof has been demonstrated and proceed.

Suppose now that the inverse system $\mathcal{H}^{-1}$ exists. We have that

$$\mathcal{H}x = x * h \quad \text{and} \quad \mathcal{H}^{-1}x = x * h_{\mathsf{inv}}.$$

From the definition of an inverse system, we have that, for every function x,

$$\mathcal{H}^{-1}\mathcal{H}x = x.$$

Expanding the left-hand side of the preceding equation, we obtain

$$\begin{aligned} &\mathcal{H}^{-1}[x * h] = x \\ \Leftrightarrow \quad & x * h * h_{\mathsf{inv}} = x. \end{aligned} \tag{4.31}$$

This relationship is expressed diagrammatically in Figure 4.10. Since the unit-impulse function is the convolutional identity, we can equivalently rewrite (4.31) as

$$x * h * h_{\mathsf{inv}} = x * \delta.$$

This equation, however, must hold for arbitrary x. Thus, by comparing the left- and right-hand sides of this equation, we conclude

$$h * h_{\mathsf{inv}} = \delta. \tag{4.32}$$

Therefore, if $\mathcal{H}^{-1}$ exists, it must have an impulse response h_{inv} that satisfies (4.32). This completes the proof. ■

From the preceding theorem, we have the following result:

Theorem 4.10 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with impulse response h is invertible if and only if there exists a function h_{inv} satisfying*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. The proof follows immediately from the result of Theorem 4.9 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. ■

Example 4.12. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathsf{inv}} = \delta. \tag{4.33}$$

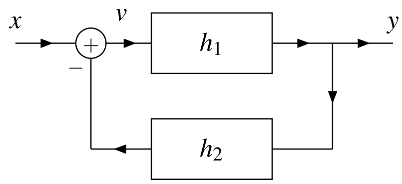

Figure 4.11: Feedback system with input x and output y.

So, let us attempt to solve this equation for h_{inv}. Substituting the given function h into (4.33) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
& h * h_{\mathsf{inv}}(t) = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} h(\tau) h_{\mathsf{inv}}(t-\tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h_{\mathsf{inv}}(t-\tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} \delta(\tau - t_0) h_{\mathsf{inv}}(t-\tau) d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$$\begin{aligned}
& h_{\mathsf{inv}}(t-\tau)\big|_{\tau=t_0} = \tfrac{1}{A}\delta(t) \\
\Rightarrow \quad & h_{\mathsf{inv}}(t-t_0) = \tfrac{1}{A}\delta(t).
\end{aligned} \tag{4.34}$$

Substituting $t + t_0$ for t in the preceding equation yields

$$\begin{aligned}
h_{\mathsf{inv}}([t+t_0]-t_0) = \tfrac{1}{A}\delta(t+t_0) \quad & \Leftrightarrow \\
h_{\mathsf{inv}}(t) = \tfrac{1}{A}\delta(t+t_0). &
\end{aligned}$$

Since $A \neq 0$, the function h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

Example 4.13. Consider the system with the input x and output y as shown in Figure 4.11. Each subsystem in the block diagram is LTI and labelled with its impulse response. Use the notion of an inverse system in order to express y in terms of x.

Solution. From Figure 4.11, we can write:

$$v = x - y * h_2 \quad \text{and} \tag{4.35}$$

$$y = v * h_1. \tag{4.36}$$

Substituting (4.35) into (4.36), and simplifying we obtain

$$\begin{aligned}
& y = [x - y * h_2] * h_1 \\
\Rightarrow \quad & y = x * h_1 - y * h_2 * h_1 \\
\Rightarrow \quad & y + y * h_2 * h_1 = x * h_1 \\
\Rightarrow \quad & y * \delta + y * h_2 * h_1 = x * h_1 \\
\Rightarrow \quad & y * [\delta + h_2 * h_1] = x * h_1.
\end{aligned} \tag{4.37}$$

For convenience, we now define the function g as

$$g = \delta + h_2 * h_1. \tag{4.38}$$

So, we can rewrite (4.37) as

$$y * g = x * h_1. \tag{4.39}$$

Thus, we have almost solved for y in terms of x. To complete the solution, we need to eliminate g from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response g. This inverse system has an impulse response g_{inv} given by

$$g * g_{\mathsf{inv}} = \delta. \tag{4.40}$$

This relationship follows from the definition of an inverse system. Now, we use g_{inv} in order to simplify (4.39) as follows:

$$\begin{aligned}
& y * g = x * h_1 \\
\Rightarrow \quad & y * g * g_{\mathsf{inv}} = x * h_1 * g_{\mathsf{inv}} \\
\Rightarrow \quad & y * \delta = x * h_1 * g_{\mathsf{inv}} \\
\Rightarrow \quad & y = x * h_1 * g_{\mathsf{inv}}.
\end{aligned}$$

Thus, we can express the output y in terms of the input x as

$$y = x * h_1 * g_{\mathsf{inv}},$$

where g_{inv} is given by (4.40) and g is given by (4.38). ■

4.9.4 BIBO Stability

The last system property to be considered is BIBO stability.

Theorem 4.11 (BIBO stability of LTI system)**.** *A LTI system with the function*[1] *h as its impulse response is BIBO stable if and only if*

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{4.41}$$

(i.e., h is absolutely integrable).

Proof. Recall that a system is BIBO stable if, for every bounded input, the system produces a bounded output. Suppose that we have a LTI system with input x, output y, and impulse response h.

First, we consider the sufficiency of (4.41) for BIBO stability. Assume that $|x(t)| \le A < \infty$ for all t (i.e., x is bounded). We can write

$$\begin{aligned}
y(t) &= x * h(t) \\
&= h * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau) x(t - \tau)\, d\tau.
\end{aligned}$$

[1] Strictly speaking, this theorem requires h to be Lebesgue-measurable and locally integrable [17]. So, this theorem does not apply, for example, if $h = \delta$ (since δ is not measurable in the Lebesgue sense). This comment is relegated to a footnote, however, since topics like measure theory are beyond the scope of this book.

By taking the magnitude of both sides of the preceding equation, we obtain

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|. \tag{4.42}$$

One can show, for any two functions f_1 and f_2, that

$$\left| \int_{-\infty}^{\infty} f_1(t)f_2(t)dt \right| \leq \int_{-\infty}^{\infty} |f_1(t)f_2(t)|\,dt.$$

Using this inequality, we can rewrite (4.42) as

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau = \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau.$$

We know (by assumption) that $|x(t)| \leq A$ for all t, so we can replace $|x(t)|$ by its bound A in the above inequality to obtain

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau \leq \int_{-\infty}^{\infty} A\,|h(\tau)|\,d\tau = A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{4.43}$$

Thus, we have

$$|y(t)| \leq A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{4.44}$$

Since A is finite, we can deduce from (4.44) that y is bounded if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \tag{4.45}$$

(i.e., h is absolutely integrable). Thus, the absolute integrability of the impulse response h is a sufficient condition for BIBO stability.

Now, we consider the necessity of (4.41) for BIBO stability. Suppose that h is not absolutely integrable. That is, suppose that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input x given by

$$x(t) = e^{-j\arg[h(-t)]}.$$

Since $\left|e^{j\theta}\right| = 1$ for all real θ, x is bounded (i.e., $|x(t)| \leq 1$ for all t). The output y is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-j\arg[h(-\tau)]}h(t-\tau)d\tau. \end{aligned} \tag{4.46}$$

Now, let us consider the output value $y(t)$ at $t = 0$. From (4.46), we have

$$y(0) = \int_{-\infty}^{\infty} e^{-j\arg[h(-\tau)]}h(-\tau)d\tau. \tag{4.47}$$

Since $e^{-j\arg z}z = |z|$ for all complex z, $e^{-j\arg[h(-\tau)]}h(-\tau) = |h(-\tau)|$, and we can simplify (4.47) to obtain

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} |h(-\tau)|\,d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)|\,d\tau \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input x will result in an unbounded output y (where $y(t)$ is unbounded for $t = 0$). Thus, the absolute integrability of h is also necessary for BIBO stability. This completes the proof. ■

Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt \\
&= \int_{0}^{\infty} e^{at}\,dt \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}$$

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}$$

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\
&= -\tfrac{1}{a}.
\end{aligned}$$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 4.15. Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau) d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{-\infty}^{\infty} |u(t)|\, dt \\ &= \int_{0}^{\infty} 1\, dt \\ &= \infty. \end{aligned}$$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable. ■

4.10 Eigenfunctions of Continuous-Time LTI Systems

Earlier, in Section 3.8.7, we were introduced to notion of eigenfunctions of systems. Given that eigenfunctions have the potential to simplify the mathematics associated with systems, it is natural to wonder what eigenfunctions LTI systems might have. In this regard, the following theorem is enlightening.

Theorem 4.12 (Eigenfunctions of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{st}$, where s is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st}, \tag{4.48a}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau. \tag{4.48b}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(s)$. Note that (4.48a) only holds for values of s for which $H(s)$ converges (i.e., values of s in the region of convergence of H).

Proof. To begin, we observe that a system $\mathcal{H}$ is LTI if and only if it computes a convolution (i.e., $\mathcal{H}x = x * h$ for some h). We have

$$\begin{aligned} \mathcal{H}x(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau \\ &= H(s)e^{st}. \end{aligned}$$

■

As a matter of terminology, the function H that appears in the preceding theorem (i.e., Theorem 4.12) is referred to as the **system function** (or **transfer function**) of the system $\mathcal{H}$. The system function completely characterizes the behavior of a LTI system. Consequently, system functions are often useful when working with LTI systems. As it turns out, an integral of the form appearing in (4.48b) is of great importance, as it defines what is called the Laplace transform. We will study the Laplace transform in great depth later in Chapter 7.

Note that a LTI system can have eigenfunctions other than complex exponentials. For example, the system in Example 3.43 is LTI and has every function as an eigenfunction. Also, a system that has every complex exponential function as an eigenfunction is not necessarily LTI. This is easily demonstrated by the example below.

Example 4.16. Let S denote the set of all complex exponential functions (i.e., S is the set of all functions x of the form $x(t) = ae^{st}$ where $a, s \in \mathbb{C}$). Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x = \begin{cases} x & x \in S \\ 1 & \text{otherwise.} \end{cases}$$

For any function $x \in S$, we have $\mathcal{H}x = x$, implying that x is an eigenfunction of $\mathcal{H}$ with eigenvalue 1. Therefore, every complex exponential function is an eigenfunction of $\mathcal{H}$.

Now, we show that $\mathcal{H}$ is not linear. In what follows, let a denote an arbitrary complex constant. Consider the function $x(t) = t$. Clearly, $x \notin S$. Since $x \notin S$, we have $\mathcal{H}x = 1$, which implies that

$$a\mathcal{H}x = a.$$

Next, consider the function $ax(t) = at$. Since $ax \notin S$, we have

$$\mathcal{H}(ax) = 1.$$

From the above equations, however, we conclude that $\mathcal{H}(ax) = a\mathcal{H}x$ only in the case that $a = 1$. Therefore, $\mathcal{H}$ is not homogeneous and consequently not linear. So, $\mathcal{H}$ is an example of a system that has every complex exponential as an eigenfunction, but is not LTI. ■

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances.

Suppose that we have a LTI system $\mathcal{H}$ with input x, output y, impulse response h, and system function H. Suppose now that we can express some arbitrary input signal x as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}.$$

(As it turns out, many functions can be expressed in this way.) From the eigenfunction properties of LTI systems, the response of the system to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) \\ &= \mathcal{H}\left\{\sum_k a_k e^{s_k t}\right\}(t) \\ &= \sum_k a_k \mathcal{H}\{e^{s_k t}\}(t) \\ &= \sum_k a_k H(s_k) e^{s_k t}. \end{aligned}$$

Thus, we have that

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{4.49}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. Furthermore, observe that the relationship between the input $x(t) = \sum_k a_k e^{s_k t}$ and output y in (4.49) does not involve convolution (such as in the equation $y = x * h$). In fact, the formula for y is identical to that for x except for the insertion of a constant multiplicative factor $H(s_k)$. In effect, we have used eigenfunctions to replace convolution with the much simpler operation of multiplication by a constant.

Example 4.17. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.48b). Substituting the given function h into (4.48b), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t-1) e^{-st} dt \\ &= \left[e^{-st}\right]\Big|_{t=1} \\ &= e^{-s}. \end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned} x(t) &= e^t \cos(\pi t) \\ &= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] \\ &= \tfrac{1}{2} e^{(1+j\pi)t} + \tfrac{1}{2} e^{(1-j\pi)t}. \end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \text{ for } k \in \{0,1\} \quad \text{and} \quad s_k = \begin{cases} 1 + j\pi & k = 0 \\ 1 - j\pi & k = 1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

$$\begin{aligned} y(t) &= \sum_{k=0}^{1} a_k H(s_k) e^{s_k t} \\ &= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t} \\ &= \tfrac{1}{2} H(1+j\pi) e^{(1+j\pi)t} + \tfrac{1}{2} H(1-j\pi) e^{(1-j\pi)t} \\ &= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)t} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)t} \\ &= \tfrac{1}{2} e^{t-1+j\pi t-j\pi} + \tfrac{1}{2} e^{t-1-j\pi t+j\pi} \\ &= \tfrac{1}{2} e^{t-1} e^{j\pi(t-1)} + \tfrac{1}{2} e^{t-1} e^{-j\pi(t-1)} \\ &= e^{t-1} \left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] \\ &= e^{t-1} \cos[\pi(t-1)]. \end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

