ECE 260

EXAM 2
SOLUTIONS

(FALL 2023)

QUESTION 1

$$\mathcal{H}x(t) = e^{-at}\int_{-\infty}^{t} x(v)\, e^{av}\, dv, \quad a \in \mathbb{R}$$

PART (A)

$$\begin{aligned}
h(t) &= \mathcal{H}\delta(t) \\
&= e^{-at}\int_{-\infty}^{t} \delta(v)\, e^{av}\, dv \\
&= \begin{cases} e^{-at}(1) & t \geq 0 \\ e^{-at}(0) & \text{otherwise} \end{cases} \\
&= \begin{cases} e^{-at} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
&= e^{-at}u(t)
\end{aligned}$$

PART (B)

A LTI system with impulse response h is causal if and only if $h(t) = 0$ for all $t < 0$.

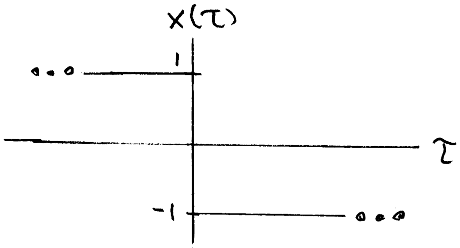
For all a, we have

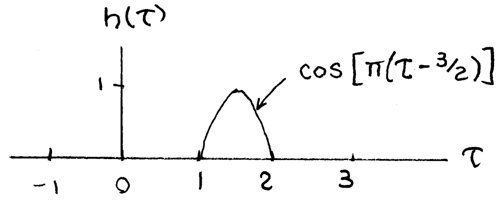
$$e^{-at}u(t) = 0 \quad \text{for all } t < 0$$

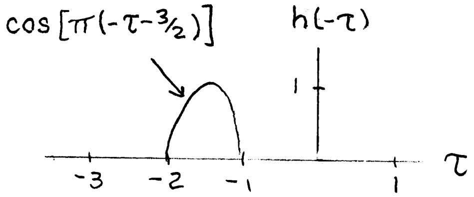
since $u(t) = 0$ for all $t < 0$.

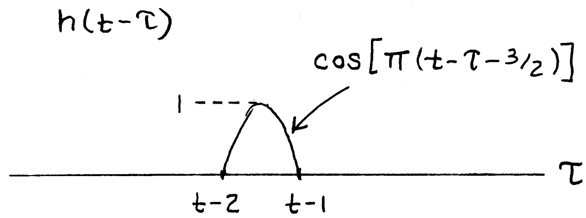
Therefore, $\mathcal{H}$ is causal for all $a \in \mathbb{R}$.

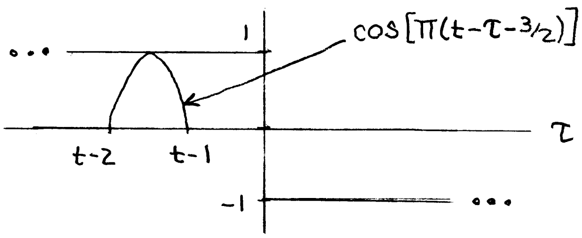
QUESTION 2





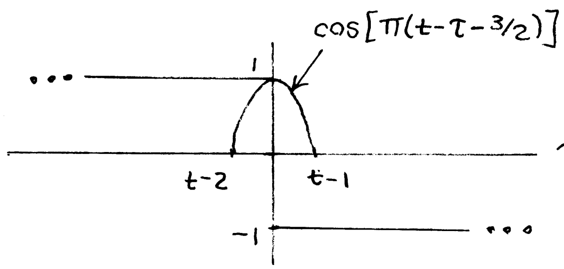






$t-1<0 \Rightarrow t<1$

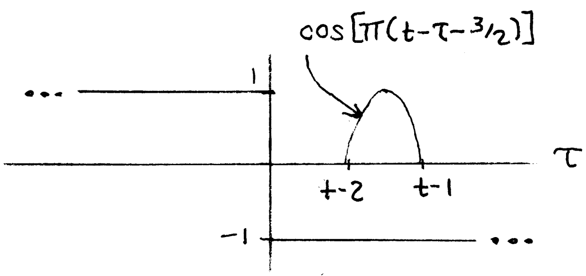
$x*h(t) = \int_{t-2}^{t-1} \cos[\pi(t-\tau-3/2)]\,d\tau$



$t-2<0\leq t-1 \Rightarrow t<2 \cap t\geq 1 \Rightarrow$

$1\leq t<2$

$$x*h(t) = \int_{t-2}^{0} \cos[\pi(t-\tau-3/2)]\,d\tau + \int_{0}^{t-1} -\cos[\pi(t-\tau-3/2)]\,d\tau$$



$t-2\geq 0 \Rightarrow t\geq 2$

$x*h(t) = \int_{t-2}^{t-1} -\cos[\pi(t-\tau-3/2)]\,d\tau$

$$x*h(t) = \begin{cases} \int_{t-2}^{t-1} \cos[\pi(t-\tau-3/2)\,d\tau & t<1 \\ \int_{t-2}^{0} \cos[\pi(t-\tau-3/2)\,d\tau + \int_{0}^{t-1} -\cos[\pi(t-\tau-3/2)]\,d\tau & 1\leq t<2 \\ \int_{t-2}^{t-1} -\cos[\pi(t-\tau-3/2)]\,d\tau & t\geq 2 \end{cases}$$

QUESTION 3

PART (A)

A LTI system with impulse response h is BIBO stable if and only if $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

PART (B)

$h(t) = e^{-a|t|}, \quad a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left|e^{-a|t|}\right| dt$$

$$= \int_{-\infty}^{\infty} e^{-a|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, dt + \int_{0}^{\infty} e^{-at}\, dt$$

assuming $a \neq 0$, we have

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \left[\frac{1}{a} e^{at}\right]\Big|_{-\infty}^{0} + \left[\frac{-1}{a} e^{-at}\right]\Big|_{0}^{\infty}$$

$$= \frac{1}{a}\left[e^{at}\right]\Big|_{-\infty}^{0} - \frac{1}{a}\left[e^{-at}\right]\Big|_{0}^{\infty}$$

$$= \begin{cases} \frac{1}{a}[1-0] - \frac{1}{a}[0-1] & a > 0 \\ \infty & a < 0 \end{cases}$$

$$= \begin{cases} \frac{2}{a} & a > 0 \\ \infty & a < 0 \end{cases}$$

if $a = 0$, $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} 1\, dt = \infty$

Therefore, $\mathcal{H}$ is BIBO stable if and only if $a > 0$.

QUESTION 4

System $\mathcal{H}$ is LTI ; system function $H(s) = e^{2s}$ for all $s \in \mathbb{C}$

$x(t) = e + \sin(3t)$

$$
\begin{aligned}
\mathcal{H}x(t) &= \mathcal{H}\{e + \sin(3\bullet)\}(t) \\
&= \mathcal{H}\left\{e\, e^{j0\bullet} + \tfrac{1}{j2}\left[e^{j3\bullet} - e^{-j3\bullet}\right]\right\}(t) && \text{Euler} \\
&= \mathcal{H}\left\{e\, e^{j0\bullet} + \tfrac{1}{j2} e^{j3\bullet} - \tfrac{1}{j2} e^{-j3\bullet}\right\}(t) && \text{linearity} \\
&= e\,\mathcal{H}\{e^{j0\bullet}\}(t) + \tfrac{1}{j2}\mathcal{H}\{e^{j3\bullet}\}(t) - \tfrac{1}{j2}\mathcal{H}\{e^{-j3\bullet}\}(t) \\
&= e\,H(0)\,e^{j0t} + \tfrac{1}{j2} H(j3)\, e^{j3t} - \tfrac{1}{j2} H(-j3)\, e^{-j3t} && \text{eigenfunction} \\
&= e(1)(1) + \tfrac{1}{j2}\left(e^{2(j3)}\right) e^{j3t} - \tfrac{1}{j2}\left(e^{2(-j3)}\right) e^{-j3t} \\
&= e + \tfrac{1}{j2} e^{j6} e^{j3t} - \tfrac{1}{j2} e^{-j6} e^{-j3t} \\
&= e + \tfrac{1}{j2} e^{j(3t+6)} - \tfrac{1}{j2} e^{-j(3t+6)} \\
&= e + \tfrac{1}{j2}\left[e^{j(3t+6)} - e^{-j(3t+6)}\right] && \text{Euler} \\
&= e + \sin(3t+6) \\
&= e + \sin[3(t+2)]
\end{aligned}
$$