Proof. In what follows, let δ denote the delta sequence (i.e., $\delta(n)$ is 1 for $n = 0$ and 0 elsewhere). From the definition of periodic convolution, we have

$$\begin{aligned}
x \circledast y(t) &= \int_T x(\tau)y(t-\tau)d\tau \\
&= \int_T \left(\sum_{\ell=-\infty}^{\infty} a_\ell e^{j\omega_0\ell\tau}\right)\left(\sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k(t-\tau)}\right)d\tau \\
&= \int_T \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0\ell\tau}e^{j\omega_0 k(t-\tau)}d\tau \\
&= \int_T \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt}e^{j\omega_0(\ell-k)\tau}d\tau \\
&= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt}\int_T e^{j\omega_0(\ell-k)\tau}d\tau \\
&= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt}T\delta(\ell-k) \\
&= \sum_{k=-\infty}^{\infty} a_k b_k e^{j\omega_0 kt}T \\
&= \sum_{k=-\infty}^{\infty} Ta_k b_k e^{j\omega_0 kt}. \qquad (5.13)
\end{aligned}$$

In the above simplification, we used the fact that

$$\begin{aligned}
\int_T e^{j(2\pi/T)kt}dt &= \begin{cases} T & k=0 \\ 0 & \text{otherwise} \end{cases} \\
&= T\delta(k).
\end{aligned}$$

Now, we simply observe that the right-hand side of (5.13) is a Fourier series. Therefore, the Fourier series coefficient sequence c of $x \circledast y$ is given by $c_k = Ta_k b_k$. ■

5.5.7 Multiplication

The next property of Fourier series to be considered is the multiplication property, as given below.

Theorem 5.11 (Multiplication)**.** *Let x and y be T-periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \textit{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$*. Let* $z(t) = x(t)y(t)$*, where*

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences a, b, and c are related as

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

(i.e., c is the DT convolution of a and b).

Proof. From the Fourier series analysis equation, we can write

$$c_k = \frac{1}{T}\int_T x(t)y(t)e^{-j(2\pi/T)kt}\,dt.$$

Replacing x by its Fourier series representation, we obtain

$$\begin{aligned} c_k &= \frac{1}{T}\int_T \left(\sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt}\right) y(t)e^{-j(2\pi/T)kt}\,dt \\ &= \frac{1}{T}\int_T \sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt} y(t)e^{-j(2\pi/T)kt}\,dt. \end{aligned}$$

Reversing the order of the summation and integration, we have

$$\begin{aligned} c_k &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_T a_n e^{j(2\pi/T)nt} y(t)e^{-j(2\pi/T)kt}\,dt \\ &= \sum_{n=-\infty}^{\infty} a_n \left(\frac{1}{T}\int_T y(t)e^{-j(2\pi/T)(k-n)t}\,dt\right). \end{aligned}$$

Observing that the expression on the preceding line in the large pair of parenthesis is simply the formula for computing the $(k-n)$th Fourier series coefficient of y, we conclude

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}. \qquad \blacksquare$$

5.5.8 Parseval's Relation

Another important property of Fourier series relates to the energy of functions and sequences, as given by the theorem below.

Theorem 5.12 (Parseval's relation)**.** *A periodic function x and its Fourier series coefficient sequence c satisfy the relationship*

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

(i.e., the energy in x and the energy in c are equal).

Proof. Let x, y, and z denote T-periodic functions with the Fourier series given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \\ y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad \text{and} \\ z(t) = x(t)y(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \end{aligned}$$

From the multiplication property of Fourier series (i.e., Theorem 5.11), we have

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}. \tag{5.14}$$

Now, let $y(t) = x^*(t)$ so that $z(t) = x(t)x^*(t) = |x(t)|^2$. From the conjugation property of Fourier series (i.e., Theorem 5.9), since $y(t) = x^*(t)$, we know

$$b_k = a^*_{-k}.$$

So, we can rewrite (5.14) as

$$\begin{aligned} c_k &= \sum_{n=-\infty}^{\infty} a_n a^*_{-(k-n)} \\ &= \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}. \end{aligned} \tag{5.15}$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt. \tag{5.16}$$

Equating (5.15) and (5.16), we obtain

$$\frac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt = \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}.$$

Letting $k = 0$ in the preceding equation yields

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} a_n a^*_n = \sum_{n=-\infty}^{\infty} |a_n|^2. \qquad \blacksquare$$

The above theorem is simply stating that the amount of energy in x (i.e., $\frac{1}{T}\int_T |x(t)|^2 dt$) and the amount of energy in the Fourier series coefficient sequence c (i.e., $\sum_{k=-\infty}^{\infty} |c_k|^2$) are equal. In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

5.5.9 Even and Odd Symmetry

Fourier series preserves signal symmetry. In other words, we have the result below.

Theorem 5.13 (Even/odd symmetry)**.** *For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:*

x is even if and only if c is even; and
x is odd if and only if c is odd.

Proof. The proof is left as an exercise for the reader in Exercise 5.4. $\blacksquare$

In other words, the above theorem states that the even/odd symmetry properties of x and c always match (i.e., Fourier series preserve symmetry).

5.5.10 Real Functions

Consider the Fourier series representation of the periodic function x given by (5.1). In the most general case, x is a complex-valued function, but let us now suppose that x is real valued. In the case of real-valued functions, an important relationship exists between the Fourier series coefficients c_k and c_{-k} as given by the theorem below.

Theorem 5.14 (Fourier series of real-valued function)**.** *Let x be a periodic function with Fourier series coefficient sequence c. The function x is real valued if and only if*

$$c_k = c^*_{-k} \text{ for all } k \tag{5.17}$$

(i.e., c is conjugate symmetric).

Proof. Suppose that we can represent x in the form of a Fourier series, as given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{5.18}$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing k by $-k$ in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}. \tag{5.19}$$

Suppose now that x is real valued. Then, $x^* = x$ and the right-hand sides of (5.18) and (5.19) must be equal, implying that $c_k = c_{-k}^*$ for all k.

Suppose now that $c_k = c_{-k}^*$ for all k. Then, the right-hand sides of (5.18) and (5.19) must be equal, implying that $x^* = x$ (i.e., x is real valued). ■

Using the relationship in (5.17), we can derive two alternative forms of the Fourier series for the case of real-valued functions. We begin by rewriting (5.1) in a slightly different form. In particular, we rearrange the summation to obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right].$$

Substituting $c_k = c_{-k}^*$ from (5.17), we obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right].$$

Now, we observe that the two terms inside the summation are complex conjugates of each other. So, we can rewrite the equation as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}(c_k e^{jk\omega_0 t}). \tag{5.20}$$

Let us now rewrite c_k in polar form as

$$c_k = |c_k| e^{j\theta_k},$$

where θ_k is real (i.e., $\theta_k = \arg c_k$). Substituting this expression for c_k into (5.20) yields

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[|c_k| e^{j(k\omega_0 t + \theta_k)} \right] \\ &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left(|c_k| \left[\cos(k\omega_0 t + \theta_k) + j\sin(k\omega_0 t + \theta_k) \right] \right) \\ &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[|c_k| \cos(k\omega_0 t + \theta_k) + j\,|c_k| \sin(k\omega_0 t + \theta_k) \right]. \end{aligned}$$

Finally, further simplification yields

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k)$$

(where $\theta_k = \arg c_k$). This is known as the **combined trigonometric form** of a Fourier series.

A second alternative form of the Fourier series can be obtained by expressing c_k in Cartesian form as

$$c_k = \tfrac{1}{2}(a_k - jb_k).$$

where a_k and b_k are real. Substituting this expression for c_k into (5.20) from earlier yields

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[\tfrac{1}{2}(a_k - jb_k)e^{jk\omega_0 t}\right] \\ &= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left[(a_k - jb_k)\left(\cos[k\omega_0 t] + j\sin[k\omega_0 t]\right)\right] \\ &= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left[a_k\cos(k\omega_0 t) + ja_k\sin(k\omega_0 t) - jb_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]. \end{aligned}$$

Further simplification yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]$$

(where $a_k = \operatorname{Re}(2c_k)$ and $b_k = -\operatorname{Im}(2c_k)$). This is known as the **trigonometric form** of a Fourier series.

By comparing the various forms of the Fourier series introduced above, we can see that the quantities c_k, a_k, b_k, and θ_k are related as

$$2c_k = a_k - jb_k \quad \text{and} \quad c_k = |c_k|e^{j\theta_k}.$$

(Recall that a_k, b_k, and θ_k are real and c_k is complex.) Note that each of the trigonometric and combined-trigonometric forms of Fourier series only involve real quantities, whereas the the exponential form involves some complex quantities. For this reason, the trigonometric and combined-trigonometric forms may sometimes be preferred when dealing with Fourier series of real-valued functions.

As noted earlier in Theorem 5.14, the Fourier series of a real-valued function has a special structure. In particular, a function x is real valued if and only if its Fourier series coefficient sequence c satisfies $c_k = c_{-k}^*$ for all k (i.e., c is conjugate symmetric). Thus, for a real-valued function, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients. From properties of complex numbers, one can show that

$$c_k = c_{-k}^* \text{ for all } k$$

is equivalent to

$$|c_k| = |c_{-k}| \text{ for all } k \quad \text{and} \quad \arg c_k = -\arg c_{-k} \text{ for all } k$$

(i.e., $|c_k|$ is *even* and $\arg c_k$ is *odd*). Note that x being real valued does *not* necessarily imply that c is real.

5.6 Fourier Series and Frequency Spectra

The Fourier series represents a function in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a function. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a function is concentrated. For example, if only the Fourier series

Table 5.1: Properties of CT Fourier series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t)+\beta y(t)$	$\alpha a_k+\beta b_k$
Translation	$x(t-t_0)$	$e^{-jk(2\pi/T)t_0}a_k$
Modulation	$e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Periodic Convolution	$x \circledast y(t)$	Ta_kb_k
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_nb_{k-n}$

Property	
Parseval's relation	$\frac{1}{T}\int_T \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$
Even symmetry	x is even $\Leftrightarrow$ a even
Odd symmetry	x is odd $\Leftrightarrow$ a odd
Real	x is real $\Leftrightarrow$ a is conjugate symmetric

coefficients for the low-order harmonics have large magnitudes, then the function is mostly associated with low frequencies. On the other hand, if a function has many large magnitude coefficients for high-order harmonics, then the function has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier series representation provides a means for measuring the frequency content of a function. The distribution of the energy/information in a function over different frequencies is referred to as the **frequency spectrum** of the function.

To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the function x, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j\arg c_k} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}. \end{aligned}$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the larger the contribution the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation. In this way, we can use $|c_k|$ as a measure of how much information a function x has at the frequency $k\omega_0$.

Various ways exist to illustrate the frequency spectrum of a function. Typically, we plot the Fourier series coefficients as a function of frequency. Since, in general, the Fourier series coefficients are complex valued, we usually display this information using two plots. One plot shows the magnitude of the coefficients as a function of frequency. This is called the **magnitude spectrum**. The other plot shows the arguments of the coefficients as a function of frequency. In this context, the argument is referred to as the phase, and the plot is called the **phase spectrum**.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real function x, the Fourier series coefficient sequence c is conjugate symmetric (i.e., $c_k = c_{-k}^*$ for all k). This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real function is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real function is always odd.

Example 5.7. Consider the 1-periodic function x with the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt},$$

where

$$c_k = \begin{cases} \frac{j}{20} & k=-7 \\ \frac{2j}{20} & k=-5 \\ \frac{4j}{20} & k=-3 \\ \frac{13j}{20} & k=-1 \\ -\frac{13j}{20} & k=1 \\ -\frac{4j}{20} & k=3 \\ -\frac{2j}{20} & k=5 \\ -\frac{j}{20} & k=7 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have

$$\begin{aligned} x(t) &= \tfrac{j}{20}e^{-j14\pi t} + \tfrac{2j}{20}e^{-j10\pi t} + \tfrac{4j}{20}e^{-j6\pi t} + \tfrac{13j}{20}e^{-j2\pi t} \\ &\quad - \tfrac{13j}{20}e^{j2\pi t} - \tfrac{4j}{20}e^{j6\pi t} - \tfrac{2j}{20}e^{j10\pi t} - \tfrac{j}{20}e^{j14\pi t}. \end{aligned}$$

Clearly, this Fourier series has 8 (nonzero) terms (i.e., 8 nonzero Fourier-series coefficients). Moreover, since c is conjugate symmetric, x is real. In particular, from Euler's relation, we have

$$x(t) = \tfrac{13}{10}\sin(2\pi t) + \tfrac{4}{10}\sin(6\pi t) + \tfrac{2}{10}\sin(10\pi t) + \tfrac{1}{10}\sin(14\pi t).$$

A plot of x is shown in Figure 5.7(a).

Suppose that we want to approximate x by keeping only 4 of the 8 terms in the Fourier series. Furthermore, we want to do this in a manner that yields the most faithful reproduction of x (i.e., minimizes error). Since terms with larger magnitude Fourier series coefficients make a more significant contribution to the Fourier series sum, it makes sense intuitively that these are the terms that we should select. Thus, the best approximation would be obtained by choosing the terms for $k \in \{-3,-1,1,3\}$. A plot of the resulting approximation is shown in Figure 5.7(b). Clearly, this approximation faithfully reproduces the general trends in the original function x. In fact, it can be shown that this choice is the one that minimizes mean-squared error.

Suppose instead that we chose the 4 terms with the smallest magnitude (nonzero) Fourier series coefficients. Intuitively, this choice should be a very poor one, since it keeps the terms that make the least significant contribution to the Fourier-series sum. The terms with the smallest magnitude (nonzero) coefficients correspond to the terms for $k \in \{-7,-5,5,7\}$. If these terms are kept, we obtain the approximation shown in Figure 5.7(c). Clearly, this approximation is a very poor one, failing to capture even the general trends in the original function x.

This example helps to illustrate that the most dominant terms in the Fourier series sum are the ones with the largest magnitude coefficients. In this sense, the Fourier series coefficient magnitudes can be used to quantify how much each term in the Fourier series contributes to the overall Fourier series sum. ■

Example 5.8. The periodic square wave x in Example 5.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even,} \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) x has the most information.

Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. We have

$$\begin{aligned} |c_k| &= \begin{cases} \left|\frac{-j2A}{\pi k}\right| & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} = -\frac{\pi}{2}\operatorname{sgn} k$, we have

$$\begin{aligned} \arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & k \text{ odd} \\ \arg 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & k \text{ odd}, k < 0 \\ -\frac{\pi}{2} & k \text{ odd}, k > 0 \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} -\frac{\pi}{2}\operatorname{sgn} k & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

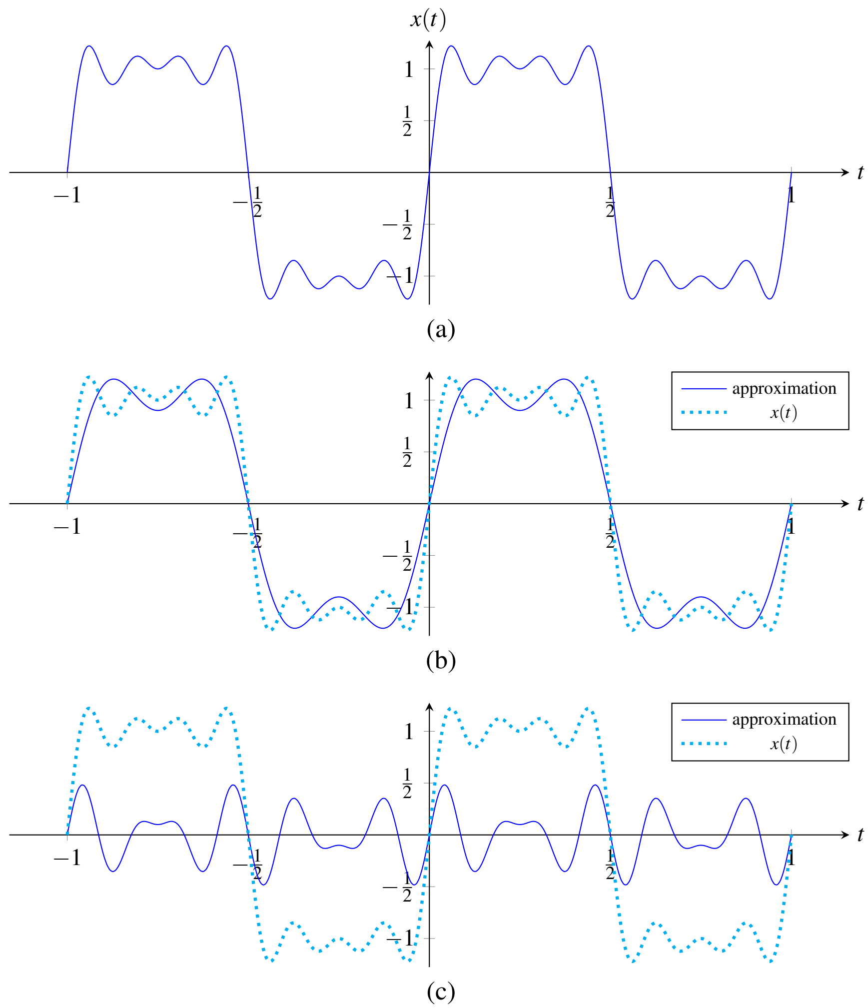


Figure 5.7: Approximation of the Fourier series for the function x. (a) The function x. (b) The approximation obtained by taking the 4 terms in the Fourier series with the largest magnitude coefficients. (c) The approximation obtained by taking the 4 terms in the Fourier series with the smallest magnitude (nonzero) coefficients.

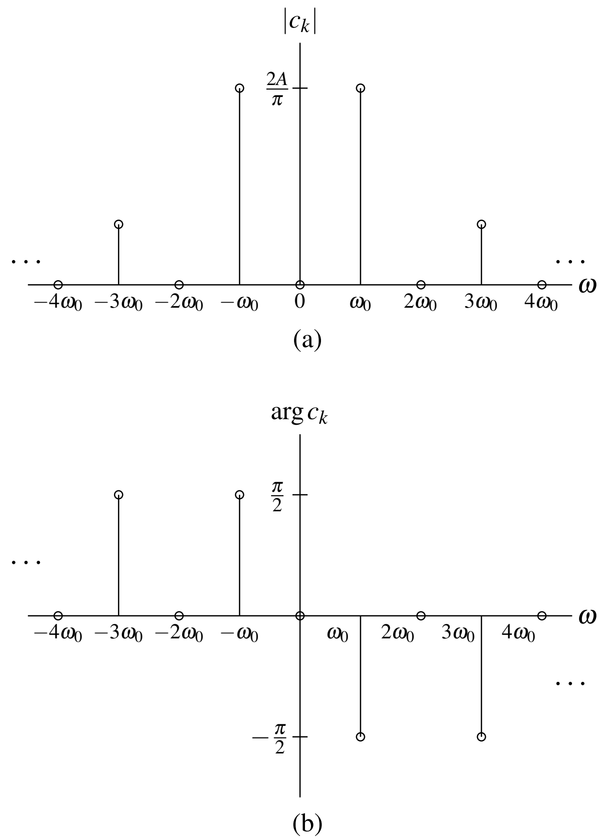


Figure 5.8: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

The magnitude and phase spectra of x are plotted in Figures 5.8(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since x is real. Since $|c_k|$ is largest for $k = -1$ and $k = 1$, the function x has the most information at frequencies $-\omega_0$ and ω_0. ■

Example 5.9. The periodic impulse train x in Example 5.2 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence c given by

$$c_k = \frac{A}{T},$$

where A is a positive real constant. Find and plot the magnitude and phase spectra of x.

Solution. We have $|c_k| = \frac{A}{T}$ and $\arg c_k = 0$. The magnitude and phase spectra of x are plotted in Figures 5.9(a) and (b), respectively. ■

5.7 Fourier Series and LTI Systems

From earlier, in Theorem 4.12, we know that complex exponentials are eigenfunctions of LTI systems. Since complex sinusoids are a special case of complex exponentials, it follows that complex sinusoids are also eigenfunctions of LTI systems. In other words, we have the result below.

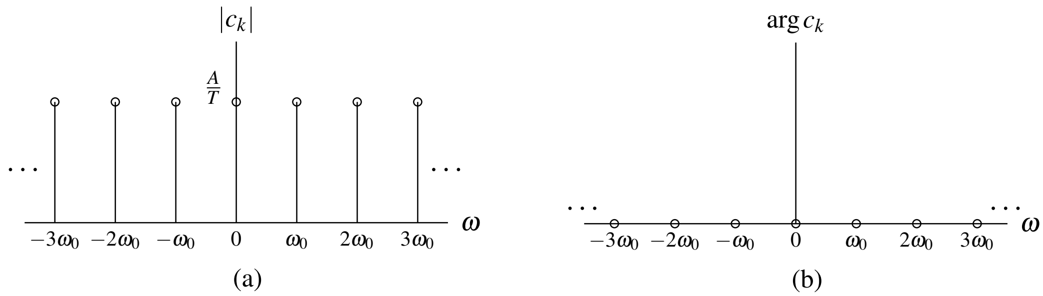


Figure 5.9: Frequency spectrum for the periodic impulse train. (a) Magnitude spectrum and (b) phase spectrum.

Corollary 5.1. *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{j\omega t}$, where ω is an arbitrary real constant (i.e., x is an arbitrary complex sinusoid), the following holds:*

$$\mathcal{H}x(t) = H(\omega)e^{j\omega t},$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt. \tag{5.21}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(\omega)$.

The preceding result (i.e., Corollary 5.1) is simply a special case of Theorem 4.12 for $s = j\omega$. Note that, in order to obtain more convenient notation, the function H in Corollary 5.1 is defined differently from the function H in Theorem 4.12. In particular, letting H_{F} and H_{L} denote the function H that appears in each of Corollary 5.1 and Theorem 4.12, respectively, we have the relationship $H_{\mathsf{F}}(\omega) = H_{\mathsf{L}}(j\omega)$.

As a matter of terminology, the function H in (5.21) is referred to as the **frequency response** of the system $\mathcal{H}$. The frequency response completely characterizes the behavior of a LTI system. Consequently, the frequency response is often useful when working with LTI systems. As it turns out, an integral of the form appearing on the right-hand side of (5.21) is of great importance, as it defines what is called the (CT) Fourier transform. We will study the (CT) Fourier transform in great depth later in Chapter 6.

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances. Suppose now that we have a periodic function x represented in terms of a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using (5.21) and the superposition property, we can determine the system response y to the input x as follows:

$$\begin{aligned}
y(t) &= \mathcal{H}x(t) \\
&= \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} \mathcal{H}\left\{c_k e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} c_k \mathcal{H}\left\{e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}.
\end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier series coefficient corresponds. In other words, if

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$$

then

$$y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(k\omega_0)c_k.$$

Example 5.10. Consider a LTI system with the frequency response

$$H(\omega) = e^{-j\omega/4}.$$

Find the response y of the system to the input x, where

$$x(t) = \tfrac{1}{2}\cos(2\pi t).$$

Solution. To begin, we rewrite x as

$$x(t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can write

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} \\
&= c_{-1}H(-\omega_0)e^{-j\omega_0 t} + c_1 H(\omega_0) e^{j\omega_0 t} \\
&= \tfrac{1}{4}H(-2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(2\pi)e^{j2\pi t} \\
&= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\
&= \tfrac{1}{4}[e^{-j(2\pi t-\pi/2)} + e^{j(2\pi t-\pi/2)}] \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right).
\end{aligned}$$

Observe that $y(t) = x\left(t - \frac{1}{4}\right)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\omega) = e^{-j\omega/4}$ is an ideal delay of $\frac{1}{4}$ (i.e., a system that performs a time shift of $\frac{1}{4}$). ■

5.8 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. Frequency-selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency greater in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise}, \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 5.10(a).

The **ideal highpass filter** eliminates all frequency components with a frequency lesser in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise}, \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 5.10(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise}, \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is shown in Figure 5.10(c).

Example 5.11 (Lowpass filtering)**.** Suppose that we have a LTI system with input x, output y, and frequency response H, where

$$H(\omega) = \begin{cases} 1 & |\omega| \leq 3\pi \\ 0 & \text{otherwise}. \end{cases}$$

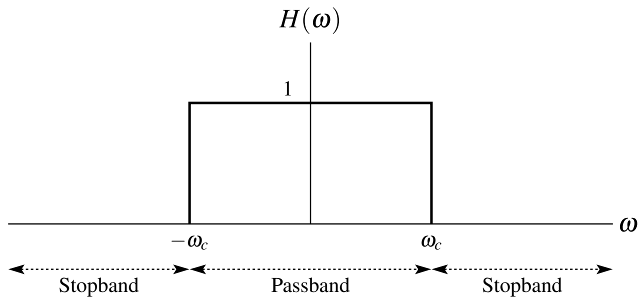
Further, suppose that the input x is the periodic function

$$x(t) = 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t).$$

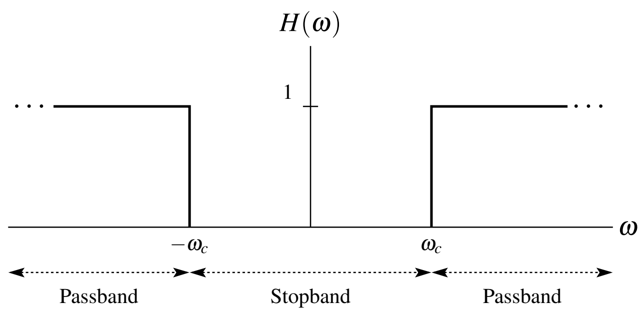
(a) Find the Fourier series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

Solution. (a) We begin by finding the Fourier series representation of x. Using Euler's formula, we can re-express x as

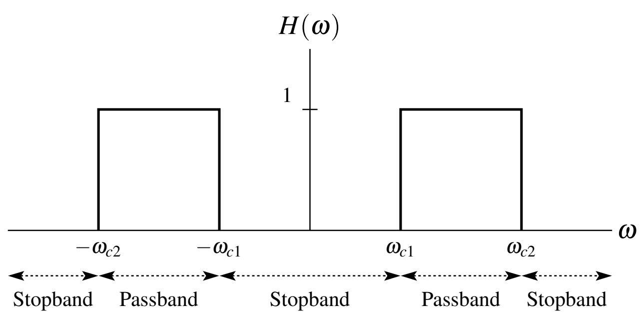
$$\begin{aligned}
x(t) &= 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t) \\
&= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\
&= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\
&= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\
&= \tfrac{1}{4}e^{j(-3)(2\pi)t} + \tfrac{1}{2}e^{j(-2)(2\pi)t} + e^{j(-1)(2\pi)t} + e^{j(0)(2\pi)t} + e^{j(1)(2\pi)t} + \tfrac{1}{2}e^{j(2)(2\pi)t} + \tfrac{1}{4}e^{j(3)(2\pi)t}.
\end{aligned}$$



(a)



(b)



(c)

Figure 5.10: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ \frac{1}{2} & k \in \{-2,2\} \\ \frac{1}{4} & k \in \{-3,3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(k\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned}
b_0 &= a_0 H([0][2\pi]) = 1(1) = 1, \\
b_1 &= a_1 H([1][2\pi]) = 1(1) = 1, \\
b_{-1} &= a_{-1} H([-1][2\pi]) = 1(1) = 1, \\
b_2 &= a_2 H([2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_{-2} &= a_{-2} H([-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_3 &= a_3 H([3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\
b_{-3} &= a_{-3} H([-3][2\pi]) = \tfrac{1}{4}(0) = 0.
\end{aligned}$$

Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 5.11(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes. ■

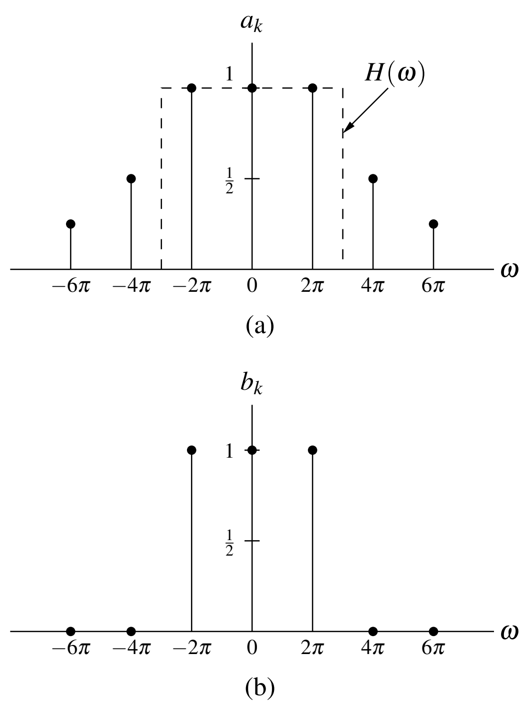


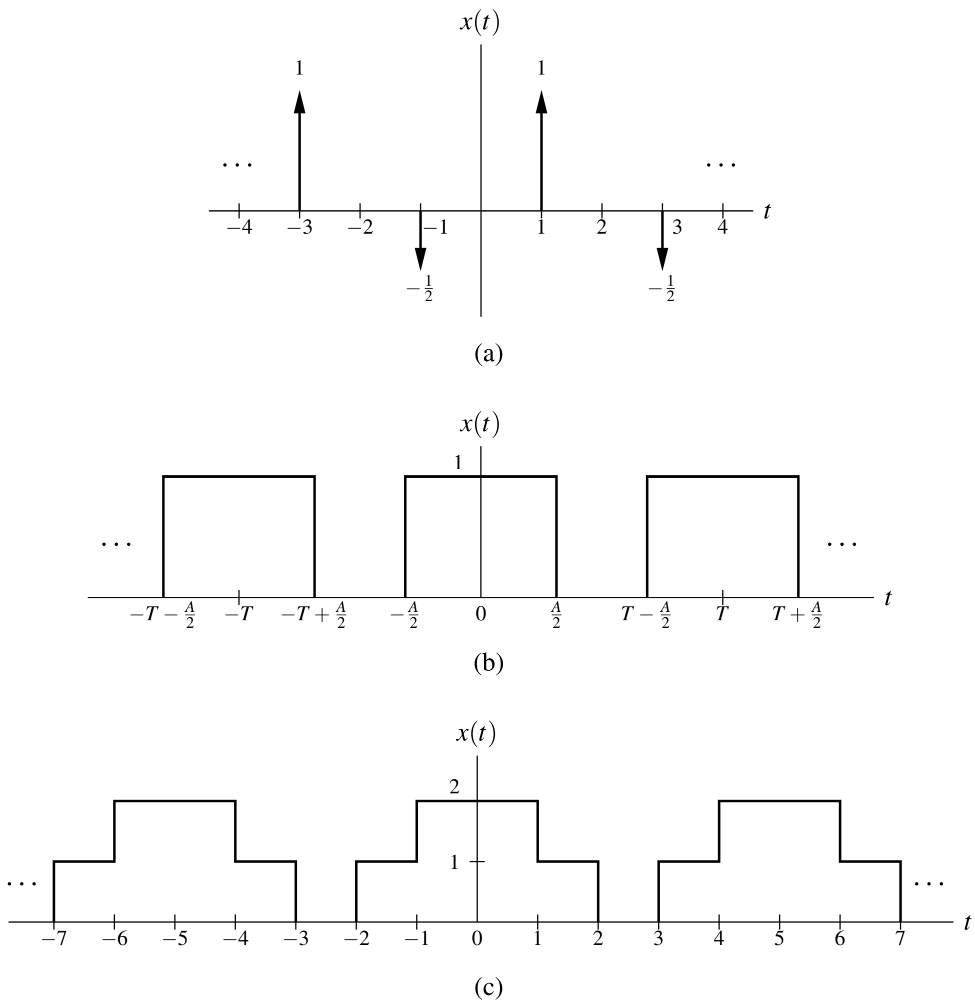
Figure 5.11: Frequency spectra of the (a) input function x and (b) output function y.

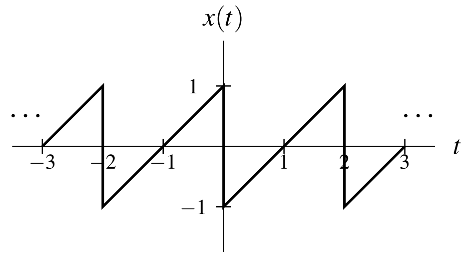
5.9 Exercises

5.9.1 Exercises Without Answer Key

5.1 For each case below, find the Fourier series representation (in complex exponential form) of the function x, explicitly identifying the fundamental period of x and the Fourier series coefficient sequence c.
(a) $x(t) = 1 + \cos(\pi t) + \sin^2(\pi t)$;
(b) $x(t) = \cos(4t)\sin(t)$; and
(c) $x(t) = |\sin(2\pi t)|$. [Hint: $\int e^{ax}\sin(bx)dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2+b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

5.2 For each of the periodic functions shown in the figures below, find the corresponding Fourier series coefficient sequence.





(d)

5.3 Find the Fourier series coefficient sequence c of each periodic function x given below with fundamental period T.
(a) $x(t) = 2\delta(t-3)+2\delta(t-5)+\delta(t-7)-\delta(t-9)+3\delta(t-12)$ and $T = 16$; express c in terms of sin and cos to whatever extent is possible.

5.4 Show that, for a complex-valued periodic function x with the Fourier series coefficient sequence c:
(a) x is even if and only if c is even; and
(b) x is odd if and only if c is odd.

5.5 Let x be a T-periodic function x with the Fourier-series coefficient sequence c. Determine the Fourier series coefficient sequence d of the function $y(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator.

5.6 Let x be a periodic function with the Fourier series coefficient sequence c given by

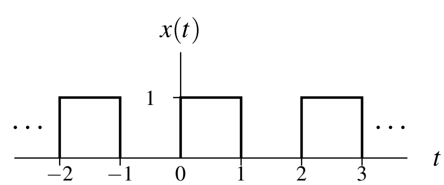
$$c_k = \begin{cases} 1 & k = 0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{otherwise.} \end{cases}$$

Using Fourier series properties as appropriate, determine if each of the following assertions is true:
(a) x is real;
(b) x is even;
(c) $\frac{d}{dt}x(t)$ is even. [Hint: Consider Exercise 5.5 first.]

5.7 A periodic function x with period T and Fourier series coefficient sequence c is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if x is odd harmonic, then $x(t) = -x(t-\frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t-\frac{T}{2})$ for all t, then x is odd harmonic.

5.8 Let x be a periodic function with fundamental period T and Fourier series coefficient sequence c. Find the Fourier series coefficient sequence c' of each of the following functions x' in terms of c:
(a) $x' = \mathrm{Even}(x)$
(b) $x' = \mathrm{Re}(x)$.

5.9 Find the Fourier series coefficient sequence c of the periodic function x shown in the figure below. Plot the frequency spectrum of x, including the first five harmonics.



5.10 Consider a LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output y of the system if the input x is given by

$$x(t) = 1 + 2\cos(2t) + 2\cos(4t) + \tfrac{1}{2}\cos(6t).$$

5.9.2 Exercises With Answer Key

5.101 For each function x given below, determine whether x has a Fourier series representation, and if it does, find the fundamental period T and coefficient sequence c of the representation.
(a) $x(t) = 2 + \cos(10\pi t) + \sin(5t)$;
(b) $x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \frac{1}{8}\sin(4\pi t)$; and
(c) $x(t) = 10 + 4\cos\left(6\pi t - \frac{\pi}{3}\right) - 2\sin\left(15\pi t - \frac{\pi}{6}\right)$.

Short Answer. (a) does not have Fourier series; (b) $T = 1$ and nonzero coefficients are $(c_{-2}, c_{-1}, c_0, c_1, c_2) = (\frac{j}{16}, \frac{1}{4}, 1, \frac{1}{4}, -\frac{j}{16})$; (c) $T = \frac{2}{3}$ and nonzero coefficients are $(c_{-5}, c_{-2}, c_0, c_2, c_5) = (-je^{j\pi/6}, 2e^{j\pi/3}, 10, 2e^{-j\pi/3}, je^{-j\pi/6})$

5.102 Find the Fourier series coefficient sequence c of each periodic function x given below with fundamental period T.
(a) $x(t) = e^{-t}$ for $-1 \leq t < 1$ and $T = 2$;
(b) $x(t) = \operatorname{rect}(t - \frac{3}{2}) - \operatorname{rect}(t + \frac{3}{2})$ for $-\frac{5}{2} \leq t < \frac{5}{2}$ and $T = 5$;
(c) $x(t) = e^{-2|t|}$ for $-2 \leq t < 2$ and $T = 4$;
(d) $x(t) = -\delta(t+1) + \delta(t) + \delta(t-1)$ for $-2 \leq t < 2$ and $T = 4$;
(e) $x(t) = 5e^{3t}$ for $0 \leq t < 5$ and $T = 5$;
(f) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$ for $-2 \leq t < 2$ and $T = 4$;
(g) $x(t) = t^2$ for $-1 \leq t < 1$ and $T = 2$ [Hint: See (F.2).];
(h) $x(t) = \sin\left(\frac{\pi}{2}t\right)[u(t-1) - u(t-2)]$ for $0 \leq t < 2$ and $T = 2$ [Hint: See (F.4).];
(i) $x(t) = t[u(t) - u(t-1)]$ for $0 \leq t < 2$ and $T = 2$ [Hint: See (F.1).];
(j) $x(t) = |t|[u(t+1) - u(t-1)]$ for $-2 \leq t < 2$ and $T = 4$ [Hint: See (F.5).];
(k) $x(t) = 4\delta(t-1) - 4\delta(t-2) + 6\delta(t-3) + 6\delta(t-5)$ for $0 \leq t < 8$ and $T = 8$;
(l) $x(t) = 2\delta(t) + \delta(t-1) + \delta(t-2)$ for $0 \leq t < 4$ and $T = 4$; and
(m) $x(t) = \sum_{k=-\infty}^{\infty} 3\delta(t - 4k)$ (where T is implicit in the definition of x).

Short Answer.
(a) $c_k = \dfrac{(-1)^k(e - e^{-1})}{j2\pi k + 2}$;
(b) $c_k = \begin{cases} \frac{1}{j\pi k}\left(\cos\left(\frac{2\pi k}{5}\right) - \cos\left(\frac{4\pi k}{5}\right)\right) & k \neq 0 \\ 0 & k = 0; \end{cases}$
(c) $c_k = \dfrac{4\left[1 - e^{-4}(-1)^k\right]}{16 + \pi^2 k^2}$;
(d) $c_k = -\frac{j}{2}\sin(\frac{\pi}{2}k) + \frac{1}{4}$;
(e) $c_k = \dfrac{5(e^{15} - 1)}{15 - j2\pi k}$;
(f) $c_k = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}k\right)$;

(g) $c_k = \begin{cases} \dfrac{(-1)^k 2}{\pi^2 k^2} & k \neq 0 \\ \frac{1}{3} & k = 0; \end{cases}$

(h) $c_k = \dfrac{1+(-1)^k j2k}{4\pi\left(\frac{1}{4}-k^2\right)}$;

(i) $c_k = \begin{cases} \dfrac{(-1)^k(j\pi k+1)-1}{2\pi^2 k^2} & k \neq 0 \\ \frac{1}{4} & k = 0; \end{cases}$

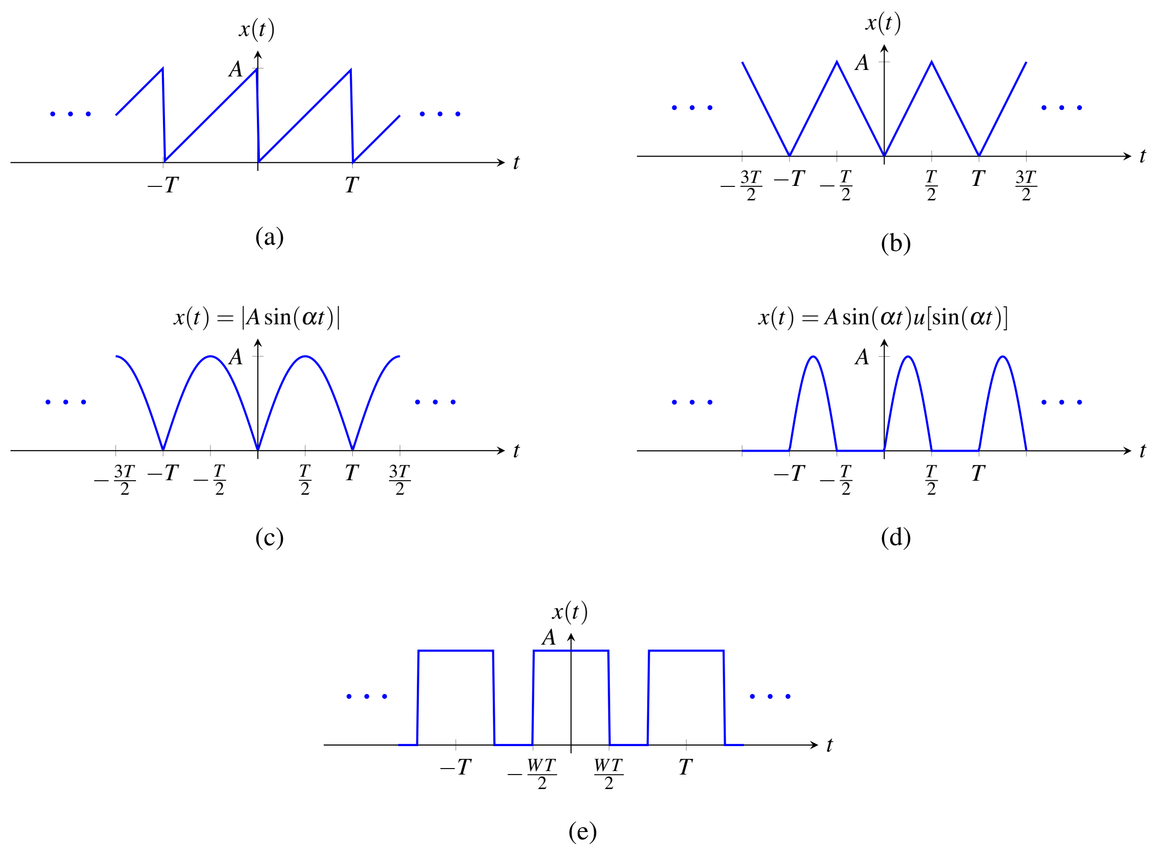
(j) $c_k = \begin{cases} \dfrac{\pi k \sin\left(\frac{\pi}{2}k\right)+2\cos\left(\frac{\pi}{2}k\right)-2}{\pi^2 k^2} & k \neq 0 \\ \frac{1}{4} & k = 0; \end{cases}$

(k) $c_k = je^{-j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)+\frac{3}{2}(-1)^k\cos\left(\frac{\pi}{4}k\right)$;

(l) $c_k = \frac{1}{2}\left[1+e^{-j(3\pi/4)k}\cos\left(\frac{\pi}{4}k\right)\right]$;

(m) $c_k = \frac{3}{4}$

5.103 For each T-periodic function x shown in the figures below, find the corresponding Fourier series coefficient sequence c. (Some of the integrals listed in Section F.4 may be helpful for some parts of this exercise.)



Short Answer.

(a) $c_k = \begin{cases} \frac{A}{2} & k = 0 \\ \frac{jA}{2\pi k} & \text{otherwise;} \end{cases}$

(b) $c_k = \begin{cases} \frac{A}{2} & k = 0 \\ 0 & k \text{ even and } k \neq 0 \\ \frac{-2A}{(\pi k)^2} & k \text{ odd;} \end{cases}$

(c) $c_k = \frac{2A}{\pi(1-4k^2)}$;

(d) $c_k = \begin{cases} \frac{A}{\pi(1-k^2)} & k \text{ even} \\ \frac{-jAk}{4} & k \in \{-1,1\} \\ 0 & \text{otherwise;} \end{cases}$

(e) $c_k = AW \operatorname{sinc}(\pi W k)$

5.104 For each case below, where the function x has the Fourier series $y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$ (where T denotes the fundamental period of x), find $y(t)$ for the specified values of t.

(a) $x(t) = \begin{cases} e^{t+2} & -2 \leq t < -1 \\ 1 & -1 \leq t < 1 \\ e^{-t+3} - e & 1 \leq t < 2 \end{cases}$ and $x(t) = x(t+4)$, $t \in \{-1, 2\}$;

(b) $x(t) = \begin{cases} e^t & 0 \leq t < 2 \\ -t^2 & 2 \leq t < 5 \end{cases}$ and $x(t) = x(t+5)$, $t \in \{0, 2\}$;

(c) $x(t) = \begin{cases} 1 + e^t & -1 < t < 0 \\ e^{-2t} & 0 \leq t \leq 1 \end{cases}$ and $x(t) = x(t+2)$, $t \in \{0, 1\}$;

(d) $x(t) = \begin{cases} t^2 + 2t + 1 & -2 \leq t < 0 \\ -t^2 + 2t - \pi & 0 \leq t < 2 \end{cases}$ and $x(t) = x(t+4)$, $t \in \{0, 1\}$; and

(e) $x(t) = \begin{cases} e^{-t} & 0 \leq t < 1 \\ t - 1 & 1 \leq t < 2 \\ e^{t-3} & 2 \leq t < 3 \end{cases}$ and $x(t) = x(t+3)$, $t \in \{1, 2\}$.

Short Answer. (a) $y(-1) = \frac{e+1}{2}$ and $y(2) = \frac{1}{2}$; (b) $y(0) = -12$ and $y(2) = \frac{e^2-4}{2}$; (c) $y(0) = \frac{3}{2}$ and $y(1) = \frac{e^2+e+1}{2e^2}$; (d) $y(0) = \frac{1-\pi}{2}$ and $y(1) = 1 - \pi$; (e) $y(1) = \frac{1}{2e}$ and $y(2) = \frac{e+1}{2e}$

5.105 For each case below, where the T-periodic function x has the Fourier series coefficient sequence c, find the magnitude and phase spectra of x.

(a) $c_k = \dfrac{jk-1}{jk+1}$ and $T = 2\pi$;

(b) $c_k = \dfrac{4jk+4}{(jk-1)^2}$ and $T = 4$;

(c) $c_k = \dfrac{-1}{(2+j\pi k)^2}$ and $T = 2$;

(d) $c_k = \left(\dfrac{e^{j3k}}{j2k-1}\right)^2$ and $T = 2$; and

(e) $c_k = \dfrac{j4\pi^2 k^2}{(j2\pi k - 1)^{10}}$ and $T = 1$.

Short Answer. (a) $|c_k| = 1$ and $\arg c_k = -2\arctan(k) + (2\ell+1)\pi$ (where $\ell \in \mathbb{Z}$); (b) $|c_k| = \frac{4}{\sqrt{k^2+1}}$ and $\arg c_k = 3\arctan(k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$); (c) $|c_k| = \frac{1}{4+\pi^2 k^2}$ and $\arg c_k = -2\arctan\left(\frac{\pi}{2}k\right) + (2\ell+1)\pi$ (where $\ell \in \mathbb{Z}$); (d) $|c_k| = \frac{1}{4k^2+1}$ and $\arg c_k = 6k + 2\arctan(2k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$); (e) $|c_k| = \frac{4\pi^2 k^2}{(4\pi^2 k^2+1)^5}$ and $\arg(c_k) = \frac{\pi}{2} + 10\arctan(2\pi k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$)

5.106 For each case below, where the periodic function x has the Fourier series coefficient sequence c, determine whether x is each of the following: real, even, odd.
(a) $c_k = e^{-|k|}$;
(b) $c_k = \dfrac{e^{-j3k}}{k^2}$ if $k \neq 0$ and $c_0 = 0$;
(c) $c_k = \operatorname{sgn}(k)e^{-|k|}$;
(d) $c_k = j\operatorname{sgn}(k)e^{-|3k|}$;
(e) $c_k = j|k|e^{-k^2}$;
(f) $c_k = \dfrac{1}{k+j}$;
(g) $c_k = \begin{cases} j\sin\left(\frac{\pi}{2}k\right) & k \in [-32\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$
(h) $c_k = \begin{cases} \cos(\pi k) & k \in [-32\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$
(i) $c_k = \begin{cases} 2^{-k} & k \in [0\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$ and
(j) $c_k = \begin{cases} k^3 & k \in [-8\,..\,8] \\ 0 & \text{otherwise.} \end{cases}$

Short Answer. (a) real and even; (b) real but not even/odd; (c) odd but not real; (d) real and odd; (e) even but not real; (f) not real and not even/odd; (g) real and odd; (h) real and even; (i) not real and not even/odd; (j) odd but not real

5.107 For each case below, find the response y of the LTI system with frequency response H to the input x.
(a) $H(\omega) = \begin{cases} -3 & \omega < -1 \\ 0 & -1 \leq \omega \leq 0 \\ 3 & \omega > 0 \end{cases}$ and $x(t) = 4\cos(t) + 2\cos(2t)$;
(b) $H(\omega) = \operatorname{rect}\left(\frac{1}{10\pi}\omega\right)$ and $x(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{1}{2}k\right)$;
(c) $H(\omega) = \operatorname{sgn}(\omega)$ and $x(t) = 4 + 3\cos(t) + 2\cos(3t)$;
(d) $H(\omega) = \begin{cases} 2 & 4 \leq |\omega| \leq 11 \\ 0 & \text{otherwise} \end{cases}$ and $x(t) = 1 + \frac{1}{2}\sin(5t) + \frac{1}{4}\cos(10t) + \frac{1}{8}\sin(15t)$;
(e) $H(\omega) = \frac{5}{j\omega}$ and $x(t) = 8\sin(2t) + 6\cos(3t)$; and
(f) $H(\omega) = \frac{1}{4+j\omega}$ and $x(t) = 8 + \cos(3t)$.

Short Answer. (a) $y(t) = 6e^{jt} + 6j\sin(2t)$; (b) $y(t) = 4\cos(4\pi t) + 2$; (c) $y(t) = 3j\sin(t) + 2j\sin(3t)$; (d) $y(t) = \sin(5t) + \frac{1}{2}\cos(10t)$; (e) $y(t) = 10\sin(3t) - 20\cos(2t)$; (f) $y(t) = 2 + \frac{1}{5}\cos\left[3t - \arctan\left(\frac{3}{4}\right)\right]$

5.108 For each case below, find the response y of the LTI system with impulse response h to the input x.
(a) $h(t) = e^{-3t}u(t)$ and $x(t) = 10 + 4\cos(4t) + 2\cos(6t)$ [Note that $\int_0^\infty e^{-at}e^{-jbt}dt = \frac{1}{a+jb}$ for all (strictly) positive real a and all real b.]; and
(b) $h(t) = e^{t}u(-t)$ and $x(t) = 4\cos(t) + 2\cos(2t)$ [Note that $\int_{-\infty}^0 e^{at}e^{-jbt}dt = \frac{1}{a-jb}$ for all (strictly) positive real a and all real b.].

Short Answer. (a) $y(t) = \frac{10}{3} + \frac{2}{3\sqrt{5}}\cos[6t - \arctan(2)] + \frac{4}{5}\cos\left[4t - \arctan\left(\frac{4}{3}\right)\right]$; (b) $y(t) = \frac{2}{\sqrt{5}}\cos[2t + \arctan(2)] + 2\sqrt{2}\cos\left(t + \frac{\pi}{4}\right)$

5.10 MATLAB Exercises

5.201 Consider the periodic function x shown in Figure B of Exercise 5.2, where $T = 1$ and $A = \frac{1}{2}$. We can show that x has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2}\operatorname{sinc}\left(\frac{\pi k}{2}\right)$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N$ converges to x. [Hint: You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Note that the MATLAB `sinc` function does not compute the sinc function as defined herein. Instead, the MATLAB `sinc` function computes the normalized sinc function as defined by (3.21).]
(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N$ converge to x uniformly (i.e., at the same rate everywhere)? If not, where is the rate of convergence slower?
(c) The function x is not continuous everywhere. For example, x has a discontinuity at $\frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

Chapter 6

Continuous-Time Fourier Transform

6.1 Introduction

The (CT) Fourier series provides an extremely useful representation for periodic functions. Often, however, we need to deal with functions that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) functions, known as the Fourier transform.

6.2 Development of the Continuous-Time Fourier Transform for Aperiodic Functions

As demonstrated earlier, the Fourier series is an extremely useful function representation. Unfortunately, this representation can only be used for periodic functions, since a Fourier series is inherently periodic. Many functions, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic functions. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic function can be viewed as a periodic function with a period of infinity. By viewing an aperiodic function as this limiting case of a periodic function where the period is infinite, we can use the Fourier series to develop a more general function representation that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the function being considered exists.)

Suppose that we have an aperiodic function x. From x, let us define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

In other words, $x_T(t)$ is identical to $x(t)$ over the interval $-\frac{T}{2} \leq t < \frac{T}{2}$ and zero otherwise. Let us now repeat the portion of $x_T(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$ to form a T-periodic function $\tilde{x}$. That is, we define $\tilde{x}$ as

$$\tilde{x}(t) = x_T(t) \;\text{ for } -\tfrac{T}{2} \leq t < \tfrac{T}{2} \quad \text{and} \quad \tilde{x}(t) = \tilde{x}(t+T).$$

In Figures 6.1 and 6.2, we provide illustrative examples of the functions x, x_T, and $\tilde{x}$.

Before proceeding further, we make two important observations that we will use later. First, from the definition of x_T, we have

$$\lim_{T\to\infty} x_T(t) = x(t). \tag{6.2}$$

Second, from the definition of x_T and $\tilde{x}$, we have

$$\lim_{T\to\infty} \tilde{x}(t) = x(t). \tag{6.3}$$

Now, let us consider the function $\tilde{x}$. Since $\tilde{x}$ is periodic, we can represent it using a Fourier series as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{6.4}$$

where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\,dt \tag{6.5}$$

and $\omega_0 = \frac{2\pi}{T}$. Since $x_T(t) = \tilde{x}(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$, we can rewrite (6.5) as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-jk\omega_0 t}\,dt.$$

Furthermore, since $x_T(t) = 0$ for $t < -\frac{T}{2}$ and $t \geq \frac{T}{2}$, we can rewrite the preceding expression for a_k as

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t}\,dt.$$

Substituting this expression for a_k into (6.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}$:

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T}\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau}\,d\tau\right] e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{\omega_0}{2\pi}\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau}\,d\tau\right] e^{jk\omega_0 t} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau}\,d\tau\right] e^{jk\omega_0 t}\omega_0.
\end{aligned}$$

Substituting the above expression for $\tilde{x}$ into (6.3), we obtain

$$x(t) = \lim_{T\to\infty} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau}\,d\tau\right] e^{jk\omega_0 t}\omega_0. \tag{6.6}$$

Now, we must evaluate the above limit. As $T \to \infty$, we have that $\omega_0 \to 0$. Thus, in the limit above, $k\omega_0$ becomes a continuous variable which we denote as ω, ω_0 becomes the infinitesimal $d\omega$, and the summation becomes an integral. This is illustrated in Figure 6.3. Also, as $T \to \infty$, we have that $x_T \to x$. Combining these results, we can rewrite (6.6) to obtain

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\,d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt.$$

Thus, we have found a representation of the aperiodic function x in terms of complex sinusoids at all frequencies. We call this the Fourier transform representation of the function x.

6.3 Generalized Fourier Transform

In the previous section, we used a limiting process involving the analysis and synthesis equations for Fourier series in order to develop a new mathematical tool known as the Fourier transform. As it turns out, many functions of

