R **7.105** For each case below, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x, where the ROCs of X and Y are R_X and R_Y, respectively.
(a) $y(t) = t(x * x)(t)$;
(b) $y(t) = x * h\left(\frac{1}{3}t - 1\right)$, where h is an arbitrary function whose Laplace transform is H with ROC R_H;
(c) $y(t) = (t+1)^{100}x(t+1)$.
(d) $y(t) = t^{100}x(t+1)$; and
(e) $y(t) = (t+1)^{100}x(t)$.

Short Answer.
(a) $Y(s) = -2X(s)\frac{d}{ds}[X(s)]$ for $s \in R_X$;
(b) $Y(s) = 3e^{-3s}X(3s)H(3s)$ for $s \in \frac{1}{3}(R_X \cap R_H)$;
(c) $Y(s) = e^s\left(\frac{d}{ds}\right)^{100}X(s)$ for $s \in R_X$;
(d) $Y(s) = \left(\frac{d}{ds}\right)^{100}[e^sX(s)]$ for $s \in R_X$;
(e) $Y(s) = e^s\left(\frac{d}{ds}\right)^{100}[e^{-s}X(s)]$ for $s \in R_X$

Exercise 7.105

R Answer (e).

We are given the function $y(t) = (t+1)^{100}x(t)$. We can rewrite y as

$$\begin{aligned} y(t) &= (t+1)^{100}x(t) \\ &= (t+1)^{100}x([t+1]-1). \end{aligned}$$

Defining v_1 as

$$v_1(t) = t^{100}x(t-1).$$

we can rewrite the above formula for y as

$$y(t) = v_1(t+1).$$

Defining v_2 as

$$v_2(t) = x(t-1),$$

we can rewrite the above formula for v_1 as

$$v_1(t) = t^{100}v_2(t).$$

Taking the Laplace transform of the preceding equations, we have

$$\begin{aligned} Y(s) &= e^s V_1(s) \quad \text{for } s \in R_Y = R_{V_1}; \\ V_1(s) = (-1)^{100}\left(\tfrac{d}{ds}\right)^{100} V_2(s) &= \left(\tfrac{d}{ds}\right)^{100} V_2(s) \quad \text{for } s \in R_{V_1} = R_{V_2}; \quad \text{and} \\ V_2(s) &= e^{-s}X(s) \quad \text{for } s \in R_{V_2} = R_X. \end{aligned}$$

Combining the preceding equations, we have

$$\begin{aligned} Y(s) &= e^s \left(\tfrac{d}{ds}\right)^{100} V_2(s) \\ &= e^s \left(\tfrac{d}{ds}\right)^{100} \left[e^{-s}X(s)\right]. \end{aligned}$$

Moreover, we additionally have $R_Y = R_{V_1} = R_{V_2} = R_X$. Thus, we conclude

$$Y(s) = e^s \left(\tfrac{d}{ds}\right)^{100} \left[e^{-s}X(s)\right] \quad \text{for } s \in R_X.$$

R **7.110** For the causal LTI system with input x and output y that is characterized by each differential equation given below, find the system function H of the system.
(a) $\mathcal{D}^2y(t)+3\mathcal{D}y(t)+2y(t)=5\mathcal{D}x(t)+7x(t)$; and
(b) $\mathcal{D}^2y(t)-5\mathcal{D}y(t)+6y(t)=\mathcal{D}x(t)+7x(t)$.

Short Answer. (a) $H(s)=\dfrac{5s+7}{(s+1)(s+2)}$ for $\operatorname{Re}(s)>-1$; (b) $H(s)=\dfrac{s+7}{(s-2)(s-3)}$ for $\operatorname{Re}(s)>3$

Exercise 7.110

R Answer (a).

We are given a causal LTI system with input x and output y that is characterized by the differential equation

$$\mathcal{D}^2 y(t) + 3\mathcal{D}y(t) + 2y(t) = 5\mathcal{D}x(t) + 7x(t).$$

Let X and Y denote the Laplace transforms of x and y, respectively. Taking the Laplace transform of the given differential equation, we obtain

$$s^2 Y(s) + 3sY(s) + 2Y(s) = 5sX(s) + 7X(s).$$

Rearranging, we have

$$\begin{aligned} & (s^2+3s+2)Y(s) = (5s+7)X(s) \\ \Rightarrow\quad & \frac{Y(s)}{X(s)} = \frac{5s+7}{s^2+3s+2} = \frac{5s+7}{(s+1)(s+2)}. \end{aligned}$$

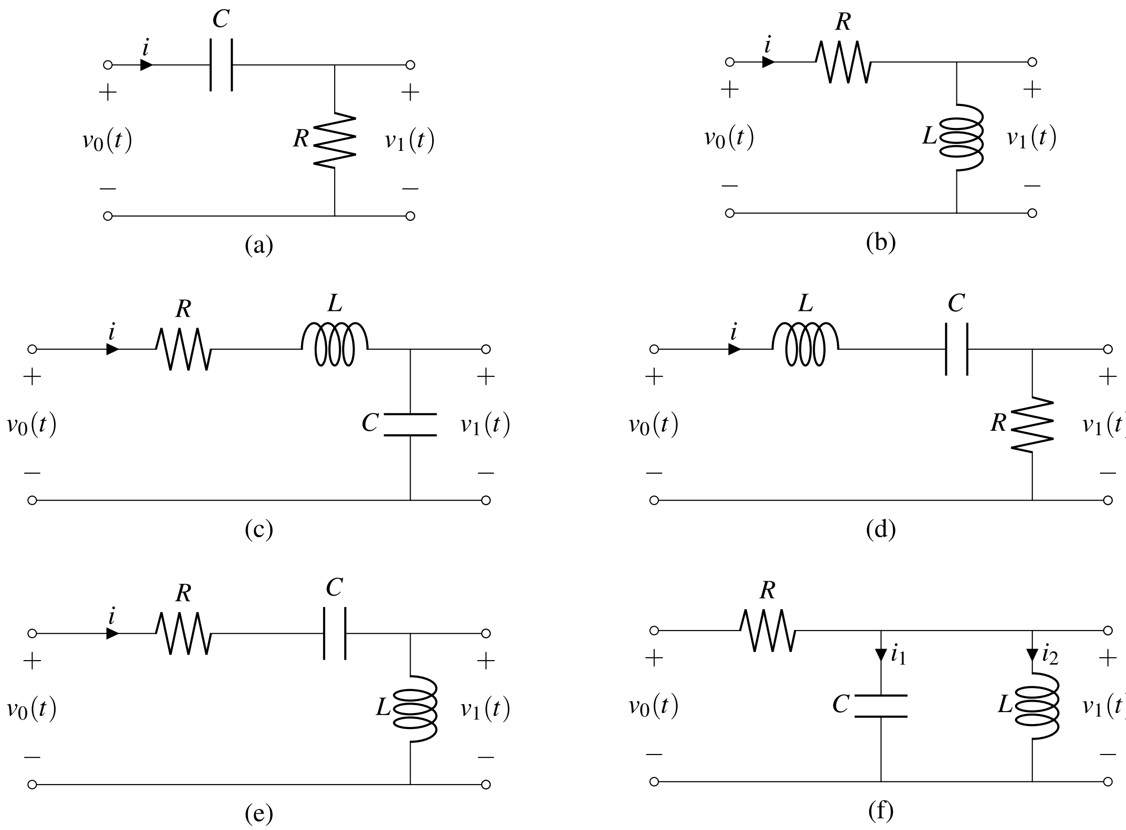
Since $H(s) = \frac{Y(s)}{X(s)}$, we have

$$H(s) = \frac{5s+7}{(s+1)(s+2)}.$$

Clearly, H has poles at -1 and -2. Since the system is causal and H is rational, the ROC of H is the right-half plane to the right of the rightmost pole, namely, $\operatorname{Re}(s) > -1$. Therefore, we conclude

$$H(s) = \frac{5s+7}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

R 7.113 For each of the LTI circuits with input v_0 and output v_1 shown in the figures below:
(i) Find the differential equation that characterizes the circuit.
(ii) Find the system function H of the circuit.
(iii) Determine whether the circuit is BIBO stable.
(iv) Determine the type of ideal frequency-selective filter that the circuit best approximates.



Short Answer.
(a) $\mathcal{D}v_0(t) = \mathcal{D}v_1(t) + \frac{1}{RC}v_1(t)$; $H(s) = \frac{RCs}{RCs+1}$ for $\operatorname{Re}(s) > -\frac{1}{RC}$; BIBO stable; highpass filter;
(b) $v_1(t) = \frac{L}{R}\mathcal{D}v_0(t) - \frac{L}{R}\mathcal{D}v_1(t)$; $H(s) = \frac{Ls}{Ls+R}$ for $\operatorname{Re}(s) > -\frac{R}{L}$; BIBO stable; highpass filter;
(c) $v_0(t) = LC\mathcal{D}^2v_1(t) + RC\mathcal{D}v_1(t) + v_1(t)$; $H(s) = \frac{1}{LCs^2+RCs+1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}]}{2LC}\right)$; BIBO stable; lowpass filter;
(d) $\mathcal{D}v_0(t) = \frac{L}{R}\mathcal{D}^2v_1(t) + \mathcal{D}v_1(t) + \frac{1}{RC}v_1(t)$; $H(s) = \frac{RCs}{LCs^2+RCs+1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}\right)$; BIBO stable; bandpass filter with passband center at $\pm\frac{1}{\sqrt{LC}}$;
(e) $\mathcal{D}^2v_0(t) = \mathcal{D}^2v_1(t) + \frac{R}{L}\mathcal{D}v_1(t) + \frac{1}{LC}v_1(t)$; $H(s) = \frac{LCs^2}{LCs^2+RCs+1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}\right)$; BIBO stable; highpass filter;
(f) $\mathcal{D}v_0(t) = RC\mathcal{D}^2v_1(t) + \mathcal{D}v_1(t) + \frac{R}{L}v_1(t)$; $H(s) = \frac{Ls}{RLCs^2+Ls+R}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-L+\sqrt{L^2-4R^2LC}}{2RLC}\right)$; BIBO stable; bandpass filter with passband center $\pm\frac{1}{\sqrt{LC}}$

Exercise 7.113

R Answer (f).

(i) From basic circuit analysis, we have

$$\begin{aligned} v_0(t) &= R[i_1(t)+i_2(t)]+v_1(t), \\ i_1(t) &= C\tfrac{d}{dt}v_1(t), \quad \text{and} \\ i_2(t) &= \tfrac{1}{L}\int_{-\infty}^{t} v_1(\tau)d\tau. \end{aligned}$$

Combining the preceding equations, we have

$$\begin{aligned} v_0(t) &= R[C\tfrac{d}{dt}v_1(t)+\tfrac{1}{L}\int_{-\infty}^{t} v_1(\tau)d\tau]+v_1(t) \quad \Rightarrow \\ v_0(t) &= RC\tfrac{d}{dt}v_1(t)+\tfrac{R}{L}\int_{-\infty}^{t} v_1(\tau)d\tau+v_1(t). \end{aligned}$$

Taking the derivative of both sides of this equation, we obtain

$$\mathcal{D}v_0(t) = RC\mathcal{D}^2v_1(t)+\mathcal{D}v_1(t)+\tfrac{R}{L}v_1(t).$$

(ii) Taking the Laplace transform of the differential equation that characterizes the system, we obtain

$$\begin{aligned} sV_0(s) &= s^2RCV_1(s)+sV_1(s)+\tfrac{R}{L}V_1(s) \quad \Rightarrow \\ sV_0(s) &= \left[RCs^2+s+\tfrac{R}{L}\right]V_1(s) \quad \Rightarrow \\ \frac{V_1(s)}{V_0(s)} &= \frac{s}{RCs^2+s+\frac{R}{L}} = \frac{Ls}{RLCs^2+Ls+R}. \end{aligned}$$

Since the system is LTI, $H(s) = \frac{V_1(s)}{V_0(s)}$. So, we have

$$H(s) = \frac{Ls}{RLCs^2+Ls+R} \quad \text{for } \operatorname{Re}(s) > \operatorname{Re}(b),$$

where $b = \frac{-L+\sqrt{L^2-4R^2LC}}{2RLC}$ (i.e., b is the rightmost root of $RLCs^2+Ls+R=0$).

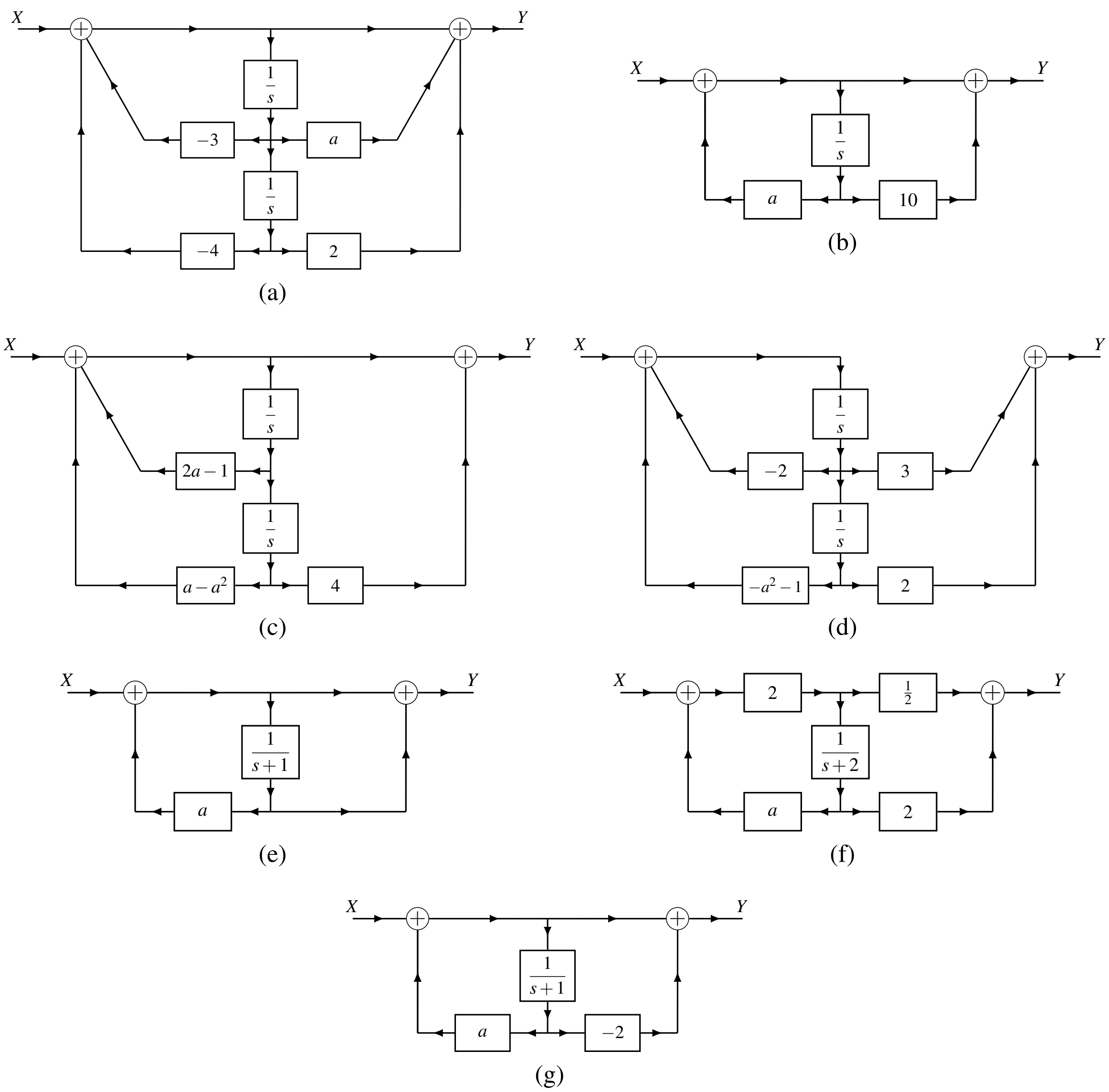
(iii) Since R, L, and C are all strictly positive, $\operatorname{Re}(b) < 0$. In particular, if $L^2-4R^2LC < 0$, then $\sqrt{L^2-4R^2LC}$ is imaginary and $\operatorname{Re}(b) = -\frac{L}{2RLC} = -\frac{1}{2RC} < 0$; otherwise, $\sqrt{L^2-4R^2LC} < L$ and $\operatorname{Re}(b) < 0$. Since $\operatorname{Re}(b) < 0$, the ROC of H contains the imaginary axis and consequently the system is BIBO stable.

(iv) Let H_F denote the frequency response of the system. Since the system is BIBO stable, $H_F(\omega) = H(j\omega)$. So, $H_F(\omega) = \frac{jL\omega}{-RLC\omega^2+jL\omega+R}$. Consequently, we have

$$\begin{aligned} |H_F(\omega)| &= \left|\frac{jL\omega}{-RLC\omega^2+jL\omega+R}\right| \\ &= \frac{|jL\omega|}{|-RLC\omega^2+jL\omega+R|} \\ &= \frac{L|\omega|}{\sqrt{(-RLC\omega^2+R)^2+(L\omega)^2}} \\ &= \frac{L|\omega|}{\sqrt{R^2(1-LC\omega^2)^2+(L\omega)^2}}. \end{aligned}$$

By differentiation, we can confirm that $|H_F(\omega)|$ has a maximum of 1 at $\omega = \pm\frac{1}{\sqrt{LC}}$ (which corresponds to $1-LC\omega^2=0$). Also, $|H_F(0)|=0$ and $\lim_{|\omega|\to\infty}|H_F(\omega)|=0$. Therefore, the system best approximates an ideal bandpass filter with the passband centered at $\pm\frac{1}{\sqrt{LC}}$.

R **7.116** Each figure below shows a system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y. Each subsystem in the figure is LTI and causal and labelled with its system function, and a is a real constant. (i) Find the system function H of the system $\mathcal{H}$. (ii) Determine whether the system $\mathcal{H}$ is BIBO stable.



Short Answer.

(a.i) $H(s) = \dfrac{s^2+as+2}{s^2+3s+4}$ for $\operatorname{Re}(s) > -\frac{3}{2}$; (a.ii) BIBO stable for all a;

(b.i) $H(s) = \dfrac{s+10}{s-a}$ for $\operatorname{Re}(s) > a$ (except when $a = -10$); (b.ii) BIBO stable if and only if $a < 0$;

(c.i) $H(s) = \dfrac{s^2+4}{s^2+(1-2a)s+a^2-a} = \dfrac{(s+2j)(s-2j)}{(s-a+1)(s-a)}$ for $\operatorname{Re}(s) > a$; (c.ii) BIBO stable if and only if $a < 0$;

(d.i) $H(s) = \dfrac{3s+2}{s^2+2s+a^2+1} = \dfrac{3(s+\frac{2}{3})}{(s+1+ja)(s+1-ja)}$ for $\operatorname{Re}(s) > -1$; (d.ii) BIBO stable for all a;

(e.i) $H(s) = \frac{s+2}{s-a+1}$ for $\operatorname{Re}(s) > a-1$ (except when $a=-1$); (e.ii) BIBO stable if and only if $a<1$;
(f.i) $H(s) = \frac{s+6}{s-2a+2}$ for $\operatorname{Re}(s) > 2a-2$ (except when $a=-2$); (f.ii) BIBO stable if and only if $a<1$;
(g.i) $H(s) = \frac{s-1}{s-a+1} = \frac{s-1}{s-(a-1)}$ for $\operatorname{Re}(s) > a-1$ (except when $a=2$); (g.ii) BIBO stable if and only if $a<1$ or $a=2$

Exercise 7.116

R **Answer (d).**

(i) In what follows, let V denote the output of the leftmost adder in the block diagram. From the block diagram, we have

$$V(s) = X(s) - \left(\frac{2}{s}\right)V(s) - \left(\frac{a^2+1}{s^2}\right)V(s) \quad \Rightarrow \quad X(s) = \left(1 + \frac{2}{s} + \frac{a^2+1}{s^2}\right)V(s) \quad \text{and}$$
$$Y(s) = \left(\frac{3}{s}\right)V(s) + \left(\frac{2}{s^2}\right)V(s) = \left(\frac{3}{s} + \frac{2}{s^2}\right)V(s).$$

Combining the preceding two equations, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\left(\frac{3}{s} + \frac{2}{s^2}\right)V(s)}{\left(1 + \frac{2}{s} + \frac{a^2+1}{s^2}\right)V(s)} = \frac{3s+2}{s^2+2s+a^2+1}.$$

Rewriting H in fully factored form, we have

$$H(s) = \frac{3s+2}{(s-p_1)(s-p_2)},$$

where p_1 and p_2 are to be determined. From the quadratic formula, we have

$$p_k = \frac{-2 \pm \sqrt{2^2 - 4(a^2+1)}}{2} = -1 \pm \tfrac{1}{2}\sqrt{4 - 4a^2 - 4} = -1 \pm \tfrac{1}{2}\sqrt{-4a^2} = -1 \pm \tfrac{1}{2}j2a = -1 \pm ja.$$

So, we have

$$p_1 = -1 - ja \quad \text{and} \quad p_2 = p_1^* = -1 + ja.$$

Furthermore, $\operatorname{Re}(p_1) = \operatorname{Re}(p_2) = -1$. Thus, we have

$$H(s) = \frac{3(s+\frac{2}{3})}{(s+1+ja)(s+1-ja)}.$$

Note that pole-zero cancellation cannot occur in H, since the zero of H is real and the poles of H are not real. Since the system is causal, the ROC R_H of H must be the RHP to the right of the rightmost pole of H. Consequently, R_H is given by

$$R_H = \{\operatorname{Re}(s) > -1\}.$$

(ii) The system is BIBO stable if R_H contains the imaginary axis. Clearly, R_H contains the imaginary axis for all a. Therefore, the system is BIBO stable for all a.

R **7.118** For each differential equation given below that characterizes a causal (incrementally-linear TI) system with input x and output y, find y for the case of the given x and initial conditions.
(a) $\mathcal{D}^2y(t)+4\mathcal{D}y(t)+3y(t)=x(t)$, where $x(t)=u(t)$ and $y(0^-)=0$ and $\mathcal{D}y(0^-)=1$; and
(b) $\mathcal{D}^2y(t)+5\mathcal{D}y(t)+6y(t)=x(t)$, where $x(t)=\delta(t)$ and $y(0^-)=1$ and $\mathcal{D}y(0^-)=-1$.

Short Answer. (a) $y(t)=\frac{1}{3}-\frac{1}{3}e^{-3t}$ for $t\geq 0$; (b) $y(t)=3e^{-2t}-2e^{-3t}$ for $t\geq 0$

Exercise 7.118

R **Answer (b).**

We are asked to solve for y in the equation

$$\mathcal{D}^2y(t)+5\mathcal{D}y(t)+6y(t)=x(t)$$

in the case that

$$x(t)=\delta(t),\quad y(0^-)=1,\quad \text{and}\quad \mathcal{D}y(0^-)=-1.$$

Taking the unilateral Laplace transform of the equation, we have

$$\begin{aligned}
s^2Y(s)-sy(0^-)-\mathcal{D}y(0^-)+5[sY(s)-y(0^-)]+6Y(s)=1 \quad &\Rightarrow \\
s^2Y(s)-sy(0^-)-\mathcal{D}y(0^-)+5sY(s)-5y(0^-)+6Y(s)=1 \quad &\Rightarrow \\
[s^2+5s+6]Y(s)=1+sy(0^-)+\mathcal{D}y(0^-)+5y(0^-).&
\end{aligned}$$

So, we have

$$\begin{aligned}
Y(s)&=\frac{1+sy(0^-)+\mathcal{D}y(0^-)+5y(0^-)}{s^2+5s+6}\\
&=\frac{1+s-1+5}{s^2+5s+6}\\
&=\frac{s+5}{(s+2)(s+3)}.
\end{aligned}$$

Now, we find a partial fraction expansion for Y. Such an expansion is of the form

$$Y(s)=\frac{A_1}{s+2}+\frac{A_2}{s+3}.$$

Computing the expansion coefficients, we obtain

$$\begin{aligned}
A_1&=[(s+2)Y(s)]|_{s=-2}=\left.\left[\frac{s+5}{s+3}\right]\right|_{s=-2}=\frac{3}{1}=3 \quad \text{and}\\
A_2&=[(s+3)Y(s)]|_{s=-3}=\left.\left[\frac{s+5}{s+2}\right]\right|_{s=-3}=\frac{2}{-1}=-2.
\end{aligned}$$

Thus, Y has the partial fraction expansion

$$Y(s)=\frac{3}{s+2}-\frac{2}{s+3}.$$

Taking the inverse Laplace transform of Y, we obtain

$$\begin{aligned}
y(t)&=3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)-2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)\\
&=3e^{-2t}-2e^{-3t}\quad \text{for } t\geq 0.
\end{aligned}$$

