Set 15 - The Normal Distribution

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Stat 260 Lecture Notes
Sets 15 - The Normal Distribution

In this section we will look at a specific continuous distribution called the *normal distribution*. The pdf for the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

try graphing $\mu = 0$, $\sigma = 1$ $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

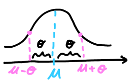
all just numbers

where $-\infty < x < \infty$, and $\mu =$ the mean, and $\sigma =$ the standard deviation.

As we saw in Sets 13 and 14, we will be using the cdf to calculate probabilities for a continuous random variable (not the pdf), so we will not need this $f(x)$ function. Instead we will only use a cdf table to calculate our probabilities here.

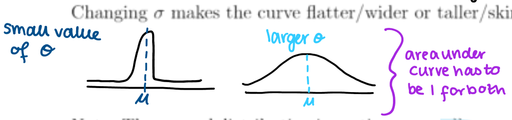
→ don't have to integrate

How to picture the normal distribution:



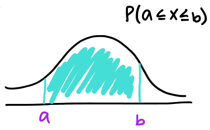
- the bell curve
- symmetric about μ
- inflection points at one standard deviation away from the mean (at $x = \mu \pm \sigma$)

The exact shape of the curve depends on the values of μ and σ.
Changing μ moves the picture left or right. (changes where peak is)
Changing σ makes the curve flatter/wider or taller/skinnier. (changes where inflection points are)



Note: The normal distribution is continuous so all our rules from Set 13 and 14 apply (the area under the curve is 1, $f(x) \geq 0$ for all x, $P(a \leq X \leq b)$ is the area under the $f(x)$ curve from $x = a$ to $x = b$).

$P(a \leq X \leq b) = \text{area}$



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→ online assignment 2 (can do most questions already)

Test 2 : sets 10 - 17

need calculator

given formula sheet and distribution tables

8 short answer, 1 long answer

practice midterms

For a continuous random variable X, to calculate $P(X \leq x)$ by hand using the pdf $f(x)$ we would need calculus. Notice that

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

we never use this

is a very difficult integral - we would need numerical methods to evaluate this.

Instead we will use our cdf table to find $P(X \leq x)$. (In other words, you will never have to write down the $f(x)$ function to solve a question with the normal distribution!)

When we had cdf tables for the binomial distribution we had many tables for different combinations of the values of n and p. We could do the same for the normal distribution but then we would need many, many, many tables for all the different combinations of values of μ and σ (remember, μ and σ do not have to be whole numbers!). This would result in needing a massive book of tables to solve our questions.

Instead, we "standardize" the particular distribution we are working with and then we only need one table.

shift & scale

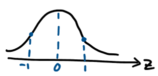
transform question into standardized then use the tables

If Z is a normal random variable with $\mu = 0$ and $\sigma = 1$ then it is called a standard normal random variable. (Here we use Z instead of X to highlight the fact that the variable has been standardized. We will use X for our original distribution and Z for the standardized distribution.)

Z indicates it's the standard form

We have a cdf table to calculate $P(Z \leq z)$ for a standard normal random variable Z.

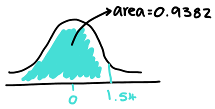
Z :



Example 1: Find the following probabilities for the standard normal random variable Z.

(a) $P(Z \leq 1.54)$. $= 0.9382$

$\downarrow 1.5 \quad \rightarrow 0.04$



$P(Z \geq 1.54) = 1 - P(Z \leq 1.54)$

$= 1 - 0.9382$

(b) $P(Z \leq -1.69)$. $= 0.0455$

How do we find $P(X \leq x)$ for a normally distributed random variable X with $\mu \neq 0$ or $\sigma \neq 1$?

We standardize by substituting

$$Z = \frac{X - \mu}{\sigma}$$

moves

scales

NOT on formula sheet

- -3.49 is smallest value
  - ↳ if value off table small then 0
  - ↳ off table big then 1

Example 2: Suppose X is a normal random variable with $\mu = 3$ and $\sigma = 2$.

(a) Find $P(X \leq 5)$.

round to 2 decimal places

$P(x \leq 5) = P\left(\frac{x-\mu}{\sigma} \leq \frac{5-3}{2}\right) = P(z \leq 1) = P(z \leq 1.00) = 0.8413$

(b) Find $P(X < 5)$. $= P(x \leq 5) = 0.8413$

Example 3: Suppose X is a normal random variable with $\mu = 15$ and $\sigma = 7$. *information you need to be given

(a) Find $P(X \geq 25)$. → can do complement / standardizing in any order.

round to 2 decimal places

$= P\left(\frac{x-\mu}{\sigma} \geq \frac{25-15}{7}\right) = P(z \geq 1.43) = 1 - P(z \leq 1.43)$

$= 1 - 0.9236$

$= 0.0764$

can have ≥ because it's a continuous random variable

b/c tables are only accurate to 2 decimal places

(b) Find $P(6 \leq X \leq 27)$. $= P\left(\frac{6-15}{7} \leq \frac{x-\mu}{\sigma} \leq \frac{27-15}{7}\right)$

$= P(-1.29 \leq z \leq 1.71)$

$= P(z \leq 1.71) - P(z \leq -1.29)$

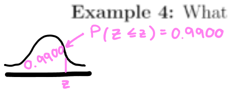
$= 0.9564 - 0.0985$

$= 0.8579$

rounding negatives

We can use the Z table (the cdf table for the normal distribution) in reverse.

Example 4: What is the value of z so that $P(Z \leq z) = 0.9900$?



$P(z \leq 2.33) = 0.9901$

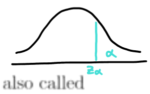
$P(z \leq z) = 0.9900$

→ almost never get exact value you're looking for

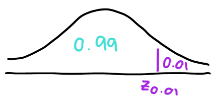
→ find closest value on table

→ find probability closest

Notation: z_α is the z value so that the area to the right is α. That is, z_α is the value so that $P(Z \geq z_\alpha) = \alpha$. The z_α value is also called a critical value.



Example 5: Find $z_{0.01}$.



$P(Z \geq z_{0.01}) = 0.01$

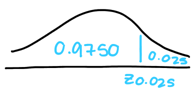
$P(z \leq z_{0.01}) = 0.9900$

In example 4 we saw that

$P(z \leq 2.33) = 0.9901$

So $z_{0.01} = 2.33$

Example 6: Find $z_{0.025}$.

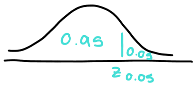


lookup prob. 0.9750

$z_{0.025} = 1.96$

* General rule: pick probability closest to ours on the table, if our value is exactly between two choices then we pick the halfway point.

Example 7: Find $z_{0.05}$.



look up the prob. 0.9500

Table has: 0.9495 and 0.9505

1.64   1.645   1.65

exactly halfway between

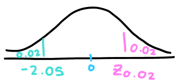
on table: $P(z \leq 1.64) = 0.9495$

$P(z \leq 1.65) = 0.9505$

So $z_{0.05} = 1.645$.

Another way to find z_α:

Example 8: Find $z_{0.02}$.



look up z value for the prob. 0.02.

on table $P(z \leq -2.05) = 0.02$

using symmetry, we have

$P(z \geq 2.05) = 0.02$

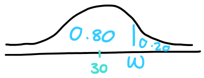
So $z_{0.02} = +2.05$

Notation:

"is distributed as"

- $X \sim N(\mu, \sigma)$
- $X \sim N(\mu, \sigma^2)$
- $X \sim \text{binomial}(n, p)$
- $X \sim \text{Poisson}(\lambda)$

Example 9: Suppose X is a normal random variable with mean 30 and standard deviation 17. Find the cutoff value w so that 20% of the distribution is above w.



look up the prob. 0.8000 on stat table.

$Z = 0.84$

$$z = \frac{x - \mu}{\sigma}$$

$$0.84 = \frac{w - 30}{17}$$

$$w = 17(0.84) + 30$$
$$= 44.28$$