Exercise 5.102

L Answer (l).

We are given

$$x(t) = 2\delta(t) + \delta(t-1) + \delta(t-2) \text{ for } 0 \le t < 4 \quad \text{and} \quad x(t) = x(t+4).$$

From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \frac{1}{T}\int_0^T x(t)e^{-j(2\pi/T)kt}dt && \text{Fourier series analysis equation} \\
&= \tfrac{1}{4}\int_0^{4^-} \left[2\delta(t) + \delta(t-1) + \delta(t-2)\right] e^{-j(2\pi/4)kt}dt && \text{substitute given } x \\
&= \tfrac{1}{4}\int_{-\infty}^{\infty} \left[2\delta(t) + \delta(t-1) + \delta(t-2)\right] e^{-j(\pi/2)kt}dt && \text{change limits; simplify exponent} \\
&= \tfrac{1}{4}\left[\int_{-\infty}^{\infty} 2\delta(t)e^{-j(\pi/2)kt}dt + \int_{-\infty}^{\infty} \delta(t-1)e^{-j(\pi/2)kt}dt + \int_{-\infty}^{\infty} \delta(t-2)e^{-j(\pi/2)kt}dt\right] && \text{integrate each term separately} \\
&= \tfrac{1}{4}\left[2e^{-j(\pi/2)k(0)} + e^{-j(\pi/2)k(1)} + e^{-j(\pi/2)k(2)}\right] && \text{sifting property} \\
&= \tfrac{1}{4}\left[2 + e^{-j(\pi/2)k} + e^{-j\pi k}\right] && \text{simplify exponents} \\
&= \tfrac{1}{2} + \tfrac{1}{4}\left[e^{-j(\pi/2)k} + e^{-j\pi k}\right] && \text{factor out 1/4} \\
&= \tfrac{1}{2} + \tfrac{1}{4}e^{-j(3\pi/4)k}\left[e^{j(\pi/4)k} + e^{-j(\pi/4)k}\right] && \text{factor out average exponent} \\
&= \tfrac{1}{2} + \tfrac{1}{4}e^{-j(3\pi/4)k}\left[2\cos\left(\tfrac{\pi}{4}k\right)\right] && \text{Euler} \\
&= \tfrac{1}{2} + \tfrac{1}{2}e^{-j(3\pi/4)k}\cos\left(\tfrac{\pi}{4}k\right) && \text{simplify} \\
&= \tfrac{1}{2}\left[1 + e^{-j(3\pi/4)k}\cos\left(\tfrac{\pi}{4}k\right)\right].
\end{aligned}$$

Thus, we have

$$c_k = \tfrac{1}{2}\left[1 + e^{-j(3\pi/4)k}\cos\left(\tfrac{\pi}{4}k\right)\right].$$

