# Partial Fraction Expansions (PFEs) [CT and DT Contexts]

- Any rational function $F$ can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_0}{v^n + b_{n-1} v^{n-1} + \ldots + b_0}.$$

- Furthermore, the denominator polynomial $D(v) = v^n + b_{n-1} v^{n-1} + \ldots + b_0$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_n)^{q_n},$$

  where the $p_k$ are distinct and the $q_k$ are integers.
- If $F$ has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that $F$ is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of $F$, there are ***two cases*** to consider:
  1. $F$ has ***only simple poles***; and
  2. $F$ has ***at least one repeated pole***.

- Suppose that the (rational) function $F$ has only simple poles.
- Then, the denominator polynomial $D$ for $F$ is of the form

$$D(v) = (v - p_1)(v - p_2)\cdots(v - p_n),$$

where the $p_k$ are distinct.

- In this case, $F$ has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{v - p_1} + \frac{A_2}{v - p_2} + \ldots + \frac{A_{n-1}}{v - p_{n-1}} + \frac{A_n}{v - p_n},$$

where

$$A_k = (v - p_k)F(v)|_{v=p_k}.$$

- Note that the (simple) pole $p_k$ contributes a single term to the partial fraction expansion.

# Repeated-Pole Case [CT and DT Contexts]

- Suppose that the (rational) function $F$ has at least one repeated pole.
- In this case, $F$ has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{v-p_1} + \frac{A_{1,2}}{(v-p_1)^2} + \ldots + \frac{A_{1,q_1}}{(v-p_1)^{q_1}}\right] + \left[\frac{A_{2,1}}{v-p_2} + \ldots + \frac{A_{2,q_2}}{(v-p_2)^{q_2}}\right] + \ldots + \left[\frac{A_{P,1}}{v-p_P} + \ldots + \frac{A_{P,q_P}}{(v-p_P)^{q_P}}\right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k-\ell)!}\left[\left[\tfrac{d}{dv}\right]^{q_k-\ell}\left[(v-p_k)^{q_k}F(v)\right]\right]\Big|_{v=p_k}.$$

- Note that the $q_k$th-order pole $p_k$ contributes $q_k$ terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Section 9.2

# PFEs for Second Form of Rational Functions

- Any rational function $F$ can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_1 v + a_0}{b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1}.$$

- Furthermore, the denominator polynomial $D(v) = b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (1 - p_1^{-1} v)^{q_1} (1 - p_2^{-1} v)^{q_2} \cdots (1 - p_n^{-1} v)^{q_n},$$

where the $p_k$ are distinct and the $q_k$ are integers.

- If $F$ has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that $F$ is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of $F$, there are ***two cases*** to consider:
  1. $F$ has ***only simple poles***; and
  2. $F$ has ***at least one repeated pole***.

- Suppose that the (rational) function $F$ has only simple poles.
- Then, the denominator polynomial $D$ for $F$ is of the form

$$D(v) = (1 - p_1^{-1}v)(1 - p_2^{-1}v) \cdots (1 - p_n^{-1}v),$$

  where the $p_k$ are distinct.
- In this case, $F$ has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{1 - p_1^{-1}v} + \frac{A_2}{1 - p_2^{-1}v} + \ldots + \frac{A_{n-1}}{1 - p_{n-1}^{-1}v} + \frac{A_n}{1 - p_n^{-1}v},$$

  where

$$A_k = (1 - p_k^{-1}v)F(v)\Big|_{v=p_k}.$$

- Note that the (simple) pole $p_k$ contributes a single term to the partial fraction expansion.

- Suppose that the (rational) function $F$ has at least one repeated pole.
- In this case, $F$ has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{1-p_1^{-1}v} + \frac{A_{1,2}}{(1-p_1^{-1}v)^2} + \ldots + \frac{A_{1,q_1}}{(1-p_1^{-1}v)^{q_1}}\right] + \left[\frac{A_{2,1}}{1-p_2^{-1}v} + \ldots + \frac{A_{2,q_2}}{(1-p_2^{-1}v)^{q_2}}\right] + \ldots + \left[\frac{A_{P,1}}{1-p_P^{-1}v} + \ldots + \frac{A_{P,q_P}}{(1-p_P^{-1}v)^{q_P}}\right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k-\ell)!}(-p_k)^{q_k-\ell}\left[\left[\tfrac{d}{dv}\right]^{q_k-\ell}\left[(1-p_k^{-1}v)^{q_k}F(v)\right]\right]\Big|_{v=p_k}.$$

- Note that the $q_k$th-order pole $p_k$ contributes $q_k$ terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Part 10

# Miscellany

- The sum of the arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$ is given by

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n[2a+d(n-1)]}{2}.$$

- The sum of the geometric sequence $a, ra, r^2a, \ldots, r^{n-1}a$ is given by

$$\sum_{k=0}^{n-1} r^k a = a\frac{r^n-1}{r-1} \quad \text{for } r \neq 1.$$

- The sum of the infinite geometric sequence $a, ra, r^2a, \ldots$ is given by

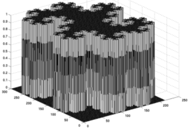
$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1-r} \quad \text{for } |r| < 1.$$

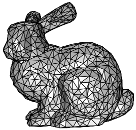
Part 11

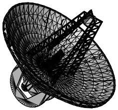
# Epilogue

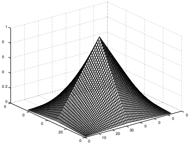
# Other Courses Offered by the Author of These Lecture Slides

- If you did not suffer permanent emotional scarring as a result of using these lecture slides and you happen to be a student at the University of Victoria, you might wish to consider taking another one of the courses developed by the author of these lecture slides:
  - ***ECE 486: Multiresolution Signal and Geometry Processing with C++***
  - ***SENG 475: Advanced Programming Techniques for Robust Efficient Computing***
- For further information about the above courses (including the URLs for web sites of these courses), please refer to the slides that follow.









## ECE 486/586: Multiresolution Signal and Geometry Processing with C++

- normally offered in Summer (May-August) term; only prerequisite ECE 310
- subdivision surfaces and subdivision wavelets
  - 3D computer graphics, animation, gaming (Toy Story, Blender software)
  - geometric modelling, visualization, computer-aided design
- multirate signal processing and wavelet systems
  - sampling rate conversion (audio processing, video transcoding)
  - signal compression (JPEG 2000, FBI fingerprint compression)
  - communication systems (transmultiplexers for CDMA, FDMA, TDMA)
- C++ (classes, templates, standard library), OpenGL, GLUT, CGAL
- software applications (using C++)
- for more information, visit course web page: `http://www.ece.uvic.ca/~mdadams/courses/wavelets`

## SEENG 475: Advanced Programming Techniques for Robust Efficient Computing (With C++)

- advanced programming techniques for robust efficient computing explored in context of C++ programming language
- topics covered may include:
  - concurrency, multithreading, transactional memory, parallelism, vectorization; cache-efficient coding; compile-time versus run-time computation; compile-time versus run-time polymorphism; generic programming techniques; resource/memory management; copy and move semantics; exception-safe coding
- applications areas considered may include:
  - geometry processing, computer graphics, signal processing, and numerical analysis
- open to any student with necessary prerequisites, which are:
  - SENG 265 or CENG 255 or CSC 230 or CSC 349A or ECE 255 or permission of Department
- for more information, see course web site:

  `http://www.ece.uvic.ca/~mdadams/courses/cpp`

Part 12

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