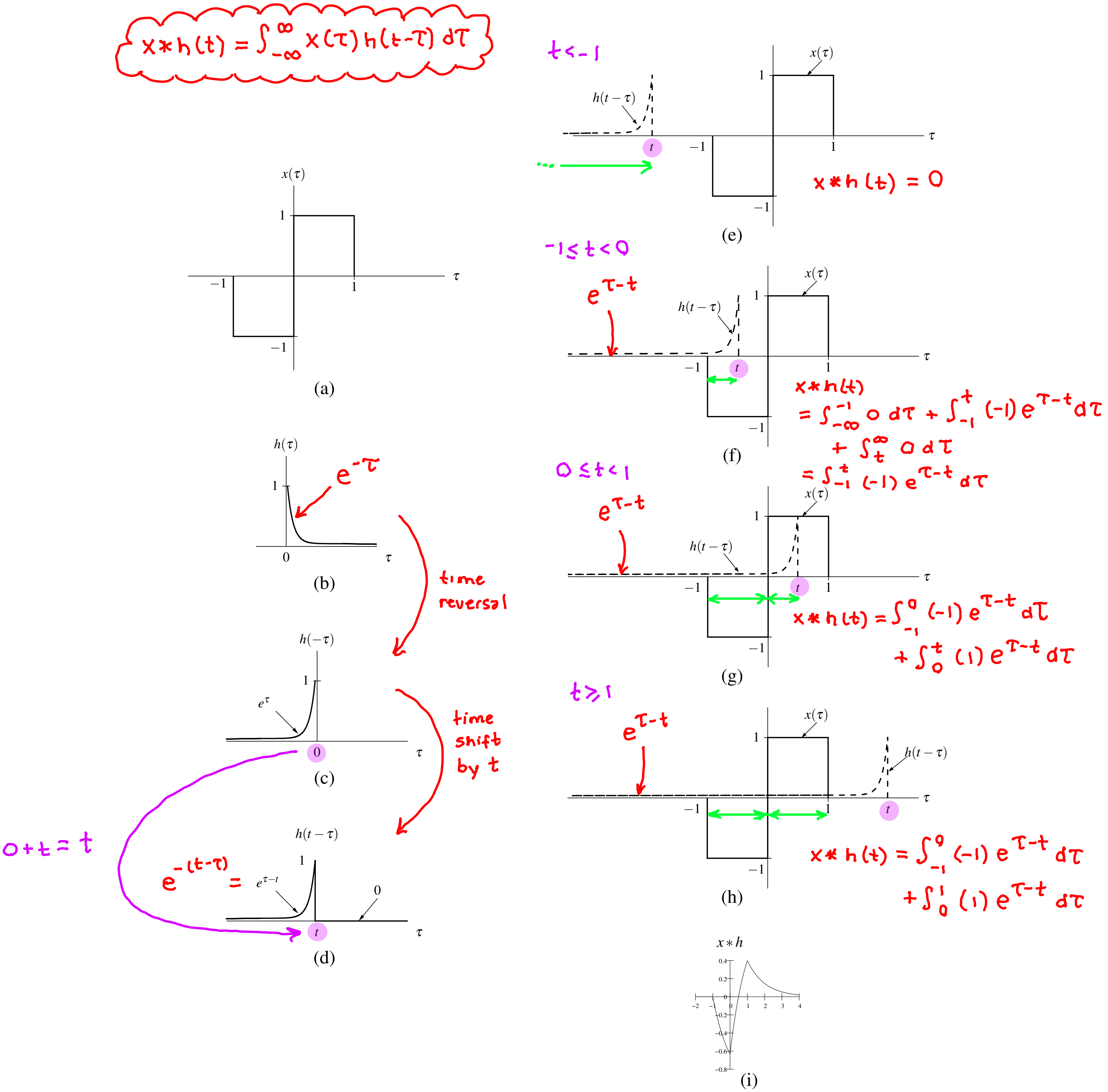


Figure 4.1: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t < -1$, (f) $-1 \le t < 0$, (g) $0 \le t < 1$, and (h) $t \ge 1$; and (i) the convolution result $x*h$.

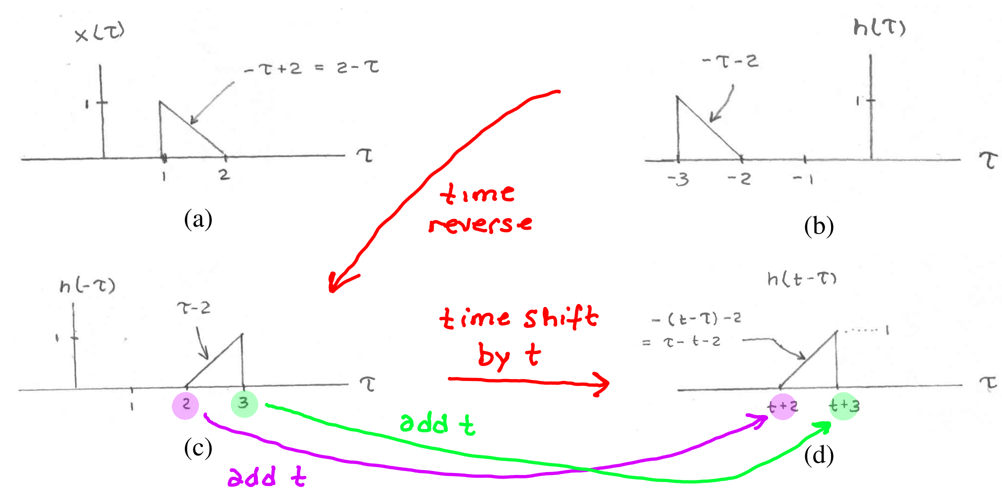
■

Answer (u).

We need to compute $x*h$, where $\boxed{x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}$

$$x(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} -t-2 & -3 \le t < -2 \\ 0 & \text{otherwise.} \end{cases}$$

First, we plot $x(\tau)$ and $h(t-\tau)$ versus τ in Figures (a) and (d), respectively.



(a) (b) (c) (d)

Figure (e): $t<-2$

$x*h(t)=0$

Figure (f): $-2 \le t < -1$

$x*h(t)$

$= \int_1^{t+3} (2-\tau)(\tau-t-2)\,d\tau$

Figure (g): $-1 \le t < 0$

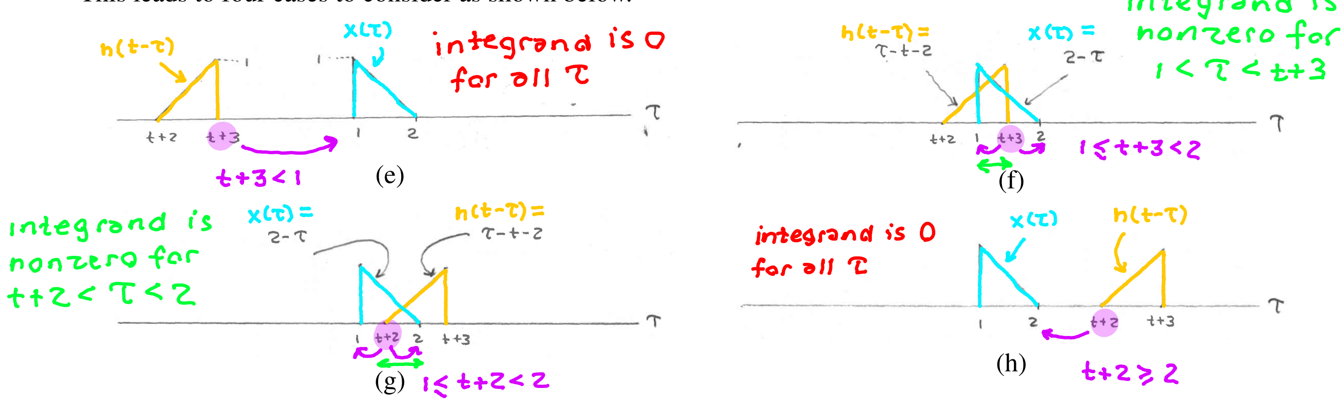
$x*h(t)$

$= \int_{t+2}^{2} (2-\tau)(\tau-t-2)\,d\tau$

Figure (h): $t \ge 0$

$x*h(t)=0$

This leads to four cases to consider as shown below.



(e) (f) (g) (h)

From Figure (e), for $t < -2$ (i.e., $t+3 < 1$), we have

$$x*h(t) = 0.$$

From Figure (f), for $-2 \le t < -1$ (i.e., $1 \le t+3 < 2$), we have

$$x*h(t) = \int_1^{t+3} \underbrace{(2-\tau)}_{x(\tau)}\underbrace{(\tau-t-2)}_{h(t-\tau)}d\tau.$$

From Figure (g), for $-1 \le t < 0$ (i.e., $1 \le t+2 < 2$), we have

$$x*h(t) = \int_{t+2}^{2} \underbrace{(2-\tau)}_{x(\tau)}\underbrace{(\tau-t-2)}_{h(t-\tau)}d\tau.$$

From Figure (h), for $t \ge 0$ (i.e., $t+2 \ge 2$), we have

$$x*h(t) = 0.$$

Simplifying, we obtain

$$x * h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \leq t < -1 \\ -\frac{1}{6}t^3 & -1 \leq t < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1 (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two functions x and h,*

$$x * h = h * x. \tag{4.16}$$

In other words, the result of a convolution is not affected by the order of its operands.

Proof. We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (4.16) as follows:

from definition of convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(\underbrace{t-\tau}_{v})d\tau.$$

$$\boxed{h * x(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau}$$

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) && \text{from change of variable} \\ &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) && \text{infinity dominates sums} \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv && \textstyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv && \text{rearrange factors} \\ &= h * x(t). && \text{definition of convolution} \end{aligned}$$

Remember that changing integration variable changes limits!

(Note that, above, we used the fact that, for any function f, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.) Thus, we have proven that convolution is commutative. ■

