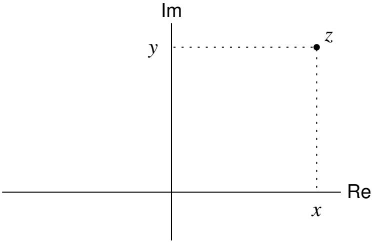
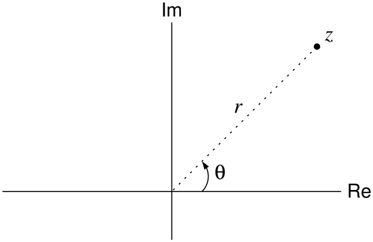




Cartesian form:

$z = x + jy$

where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$

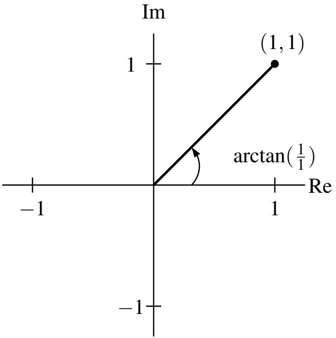
Polar form:

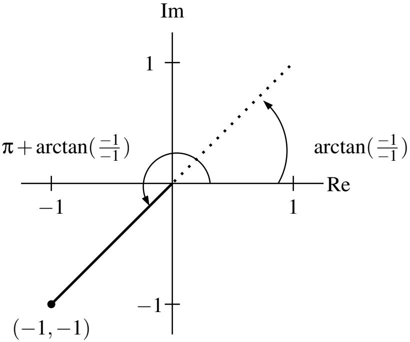
$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$

where $r = |z|$ and $\theta = \arg z$

The arctan Function

- The range of the arctan function is $-\pi/2$ (exclusive) to $\pi/2$ (exclusive).
- Consequently, the arctan function always yields an angle in either the first or fourth quadrant.





The atan2 Function

- The angle θ that a vector from the origin to the point (x,y) makes with the positive x axis is given by $\theta = \operatorname{atan2}(y,x)$, where

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases}$$

- The range of the atan2 function is from $-\pi$ (exclusive) to π (inclusive).
- For the complex number z expressed in Cartesian form $x + jy$, $\operatorname{Arg} z = \operatorname{atan2}(y,x)$.
- Although the atan2 function is quite useful for computing the principal argument (or argument) of a complex number, it is ***not advisable to memorize*** the definition of this function. It is better to simply understand what this function is doing (namely, intelligently applying the arctan function).

- Let z be a complex number with the Cartesian and polar form representations given respectively by

$$z = x + jy \quad \text{and} \quad z = re^{j\theta}.$$

- To convert from ***polar to Cartesian*** form, we use the following identities:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- To convert from ***Cartesian to polar*** form, we use the following identities:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{atan2}(y, x) + 2\pi k,$$

  where k is an arbitrary integer.

- Since the $\operatorname{atan2}$ function simply amounts to the intelligent application of the $\arctan$ function, instead of memorizing the definition of the $\operatorname{atan2}$ function, one should simply ***understand*** how to use the $\arctan$ function to achieve the same result.

Properties of Complex Numbers

- For complex numbers, addition and multiplication are ***commutative***. That is, for any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$
$$z_1 z_2 = z_2 z_1.$$

- For complex numbers, addition and multiplication are ***associative***. That is, for any three complex numbers z_1, z_2, and z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- For complex numbers, the ***distributive*** property holds. That is, for any three complex numbers z_1, z_2, and z_3,

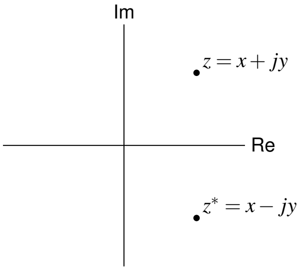
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Conjugation

- The **conjugate** of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

- Geometrically, the conjugation operation reflects a point in the complex plane about the real axis.
- The geometric interpretation of the conjugate is illustrated below.



Properties of Conjugation

- For every complex number z, the following identities hold:

$$\begin{aligned} \left|z^*\right| &= |z|, \\ \arg z^* &= -\arg z, \\ zz^* &= |z|^2, \\ \operatorname{Re} z &= \tfrac{1}{2}(z+z^*), \quad \text{and} \\ \operatorname{Im} z &= \tfrac{1}{2j}(z-z^*). \end{aligned}$$

- For all complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned} (z_1+z_2)^* &= z_1^* + z_2^*, \\ (z_1 z_2)^* &= z_1^* z_2^*, \quad \text{and} \\ (z_1/z_2)^* &= z_1^*/z_2^*. \end{aligned}$$

Addition

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

- That is, to add complex numbers expressed in Cartesian form, we simply add their real parts and add their imaginary parts.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + jr_1 \sin\theta_1) + (r_2 \cos\theta_2 + jr_2 \sin\theta_2) \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

- That is, to add complex numbers expressed in polar form, we first rewrite them in Cartesian form, and then add their real parts and add their imaginary parts.
- For the purposes of addition, it is easier to work with complex numbers expressed in Cartesian form.

Multiplication

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

- That is, to multiply two complex numbers expressed in Cartesian form, we use the distributive law along with the fact that $j^2 = -1$.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}.$$

- That is, to multiply two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of multiplication, it is easier to work with complex numbers expressed in polar form.

- ***Cartesian form:*** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$
$$= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}.$$

- That is, to compute the quotient of two complex numbers expressed in Cartesian form, we convert the problem into one of division by a real number.
- ***Polar form:*** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of division, it is easier to work with complex numbers expressed in polar form.

- For any complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned}
|z_1 z_2| &= |z_1|\,|z_2|, \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0, \\
\arg z_1 z_2 &= \arg z_1 + \arg z_2, \quad \text{and} \\
\arg\left(\frac{z_1}{z_2}\right) &= \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.
\end{aligned}$$

- The above properties trivially follow from the polar representation of complex numbers.

Euler's Relation and De Moivre's Theorem

- **Euler's relation.** For all real θ,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

- From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \quad \text{and}$$
$$\sin\theta = \tfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right).$$

- **De Moivre's theorem.** For all real θ and all ***integer*** n,

$$e^{jn\theta} = \left(e^{j\theta}\right)^n.$$

[Note: This relationship does not necessarily hold for ***real*** n.]

Roots of Complex Numbers

- Every complex number $z = re^{j\theta}$ (where $r = |z|$ and $\theta = \arg z$) has n distinct ***nth roots*** given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

- For example, 1 has the two distinct square roots 1 and -1.

Quadratic Formula

- Consider the equation

$$az^2 + bz + c = 0,$$

where a, b, and c are real, z is complex, and $a \neq 0$.

- The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- This formula is often useful in factoring quadratic polynomials.
- The quadratic $az^2 + bz + c$ can be factored as $a(z - z_0)(z - z_1)$, where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

- A **complex function** maps complex numbers to complex numbers. For example, the function $F(z) = z^2 + 2z + 1$, where z is complex, is a complex function.
- A complex **polynomial function** is a mapping of the form

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where z, $a_0, a_1, \ldots, a_n$ are complex.
- A complex **rational function** is a mapping of the form

$$F(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex.
- Observe that a polynomial function is a special case of a rational function.
- Herein, we will mostly focus our attention on polynomial and rational functions.

Continuity

- A function F is said to be **continuous at a point** z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z).$$

- A function that is continuous at every point in its domain is said to be **continuous**.
- Polynomial functions are continuous everywhere.
- Rational functions are continuous everywhere except at points where the denominator polynomial becomes zero.

Differentiability

- A function F is said to be **differentiable at a point** $z = z_0$ if the limit

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

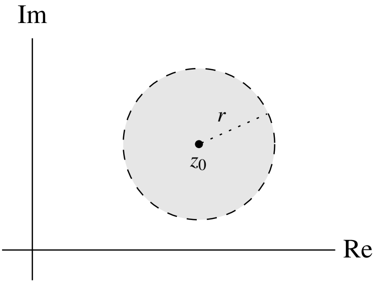
  exists. This limit is called the **derivative** of F at the point $z = z_0$.
- A function is said to be **differentiable** if it is differentiable at every point in its domain.
- The rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then
  1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;
  2. $(F+G)'(z_0) = F'(z_0) + G'(z_0)$;
  3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;
  4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and
  5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).
- A polynomial function is differentiable everywhere.
- A rational function is differentiable everywhere except at the points where its denominator polynomial becomes zero.

Open Disks

- An **open disk** in the complex plane with center z_0 and radius r is the set of complex numbers z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real number.
- A plot of an open disk is shown below.



Analyticity

- A function is said to be **analytic at a point** z_0 if it is differentiable at every point in an open disk about z_0.
- A function is said to be **analytic** if it is analytic at every point in its domain.
- A polynomial function is analytic everywhere.
- A rational function is analytic everywhere, except at the points where its denominator polynomial becomes zero.

Example A.10 Example A.11

Zeros and Singularities

- If a function F is zero at the point z_0 (i.e., $F(z_0) = 0$), F is said to have a **zero** at z_0.
- If a function F is such that $F(z_0) = 0, F^{(1)}(z_0) = 0, \ldots, F^{(n-1)}(z_0) = 0$ (where $F^{(k)}$ denotes the kth order derivative of F), F is said to have an **nth order zero** at z_0.
- A point at which a function fails to be analytic is called a **singularity**.
- Polynomials do not have singularities.
- Rational functions can have a type of singularity called a pole.
- If a function F is such that $G(z) = 1/F(z)$ has an nth order zero at z_0, F is said to have an **nth order pole** at z_0.
- A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

Zeros and Poles of a Rational Function

- Given a rational function F, we can always express F in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

  where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_M$ and $\beta_1, \beta_2, \ldots, \beta_N$ are strictly positive integers.
- One can show that F has ***poles*** at $b_1, b_2, \ldots, b_N$ and ***zeros*** at $a_1, a_2, \ldots, a_M$.
- Furthermore, the kth pole (i.e., b_k) is of ***order*** β_k, and the kth zero (i.e., a_k) is of ***order*** α_k.
- When plotting zeros and poles in the complex plane, the symbols “o” and “x” are used to denote zeros and poles, respectively.

Part 9

Partial Fraction Expansions (PFEs)

Motivation for PFEs

- Sometimes it is beneficial to be able to express a rational function as a sum of ***lower-order*** rational functions.
- This can be accomplished using a type of decomposition known as a partial fraction expansion.
- Partial fraction expansions are often useful in the calculation of inverse Laplace transforms, inverse z transforms, and inverse CT/DT Fourier transforms.

Strictly-Proper Rational Functions

- Consider a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

- The function F is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial).
- Through polynomial long division, any rational function can be written as the sum of a polynomial and a strictly-proper rational function.
- A ***strictly-proper*** rational function can be expressed as a sum of lower-order rational functions, with such an expression being called a partial fraction expansion.

Section 9.1

PFEs for First Form of Rational Functions