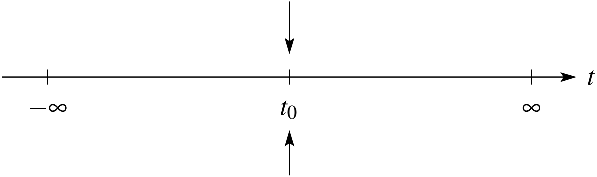
Section 3.5

Properties of (CT) Systems

Memory

- A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t \neq t_0$.
- In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same*** point in time.
- A system that is not memoryless is said to have **memory**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.

If the system $\mathcal{H}$ is memoryless,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at t_0.



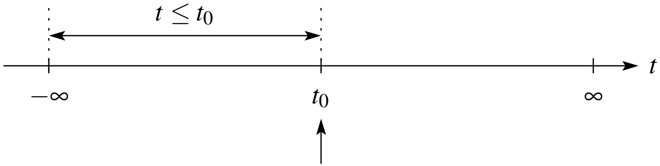
Consider the calculation of the
output $\mathcal{H}x$ at t_0.

Causality

- A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same or earlier points*** in time (i.e., ***not later points in time***).
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time*** (e.g., the independent variable might represent position).
- A memoryless system is always causal, although the converse is not necessarily true.

Causality (Continued)

If the system $\mathcal{H}$ is causal,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at points $t \le t_0$.

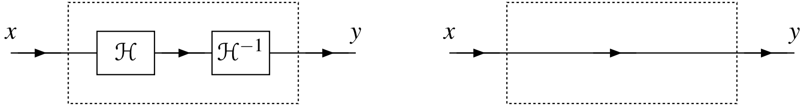


Consider the calculation of the
output $\mathcal{H}x$ at t_0.

- The **inverse** of a system $\mathcal{H}$ (if it exists) is another system $\mathcal{H}^{-1}$ such that, for every function x,
$$\mathcal{H}^{-1}\mathcal{H}x = x$$
(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose input and output are equal).
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input can always be ***uniquely*** determined from its output.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs*** (i.e., $x_1 \neq x_2 \Rightarrow \mathcal{H}x_1 \neq \mathcal{H}x_2$).
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs*** (i.e., $x_1 \neq x_2$ and $\mathcal{H}x_1 = \mathcal{H}x_2$).
- In practical terms, invertible systems are “nice” in the sense that their ***effects can be undone***.

Invertibility (Continued)

- A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ ***is an identity***):



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$

System 2: $y = x$

Bounded-Input Bounded-Output (BIBO) Stability

- A system $\mathcal{H}$ is said to be **bounded-input bounded-output (BIBO) stable** if, for every bounded function x, $\mathcal{H}x$ is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t).
- In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input is finite everywhere (in its domain), the output will also be finite everywhere.
- Usually, a system that is not BIBO stable will have ***serious safety issues***.
- For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely a big lawsuit as well).

Example 3.27 Example 3.28

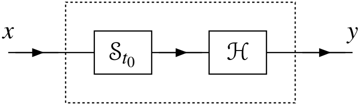
Time Invariance (TI)

- A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t, \quad \text{where} \quad x'(t) = x(t-t_0)$$

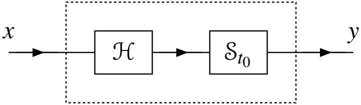
  (i.e., $\mathcal{H}$ ***commutes with time shifts***).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

- Let $\mathcal{S}_{t_0}$ denote an operator that applies a ***time shift of t_0*** to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t-t_0)$).
- A system $\mathcal{H}$ is ***time invariant*** if and only if the following two systems are equivalent (i.e., ***$\mathcal{H}$ commutes with $\mathcal{S}_{t_0}$***):



System 1: $y = \mathcal{H}\mathcal{S}_{t_0}x$

$$\begin{bmatrix} y(t) = \mathcal{H}x'(t) \\ x'(t) = \mathcal{S}_{t_0}x(t) = x(t-t_0) \end{bmatrix}$$



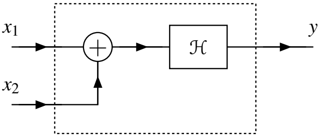
System 2: $y = \mathcal{S}_{t_0}\mathcal{H}x$

$$\left[y(t) = \mathcal{H}x(t-t_0)\right]$$

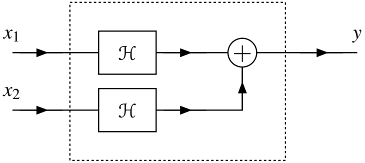
Example 3.32 Example 3.33

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:
$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$
(i.e., ***$\mathcal{H}$ commutes with addition***).
- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:
$$\mathcal{H}(ax) = a\mathcal{H}x$$
(i.e., ***$\mathcal{H}$ commutes with scalar multiplication***).
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is ***linear***, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:
$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$
(i.e., ***$\mathcal{H}$ commutes with linear combinations***).
- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems.

- The system $\mathcal{H}$ is ***additive*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with addition***):

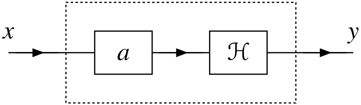


System 1: $y = \mathcal{H}(x_1 + x_2)$

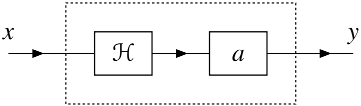


System 2: $y = \mathcal{H}x_1 + \mathcal{H}x_2$

- The system $\mathcal{H}$ is ***homogeneous*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with scalar multiplication***):

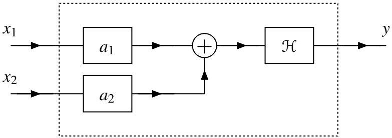


System 1: $y = \mathcal{H}(ax)$

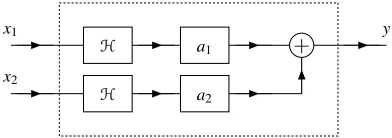


System 2: $y = a\mathcal{H}x$

- The system $\mathcal{H}$ is ***linear*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with linear combinations***):



System 1: $y = \mathcal{H}(a_1x_1 + a_2x_2)$



System 2: $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$

Example 3.35 Example 3.36

Eigenfunctions of Systems

- A function x is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if

$$\mathcal{H}x = \lambda x,$$

  where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Many of the mathematical tools developed for the study of CT systems have eigenfunctions as their basis.

Example 3.41

Part 4

Continuous-Time Linear Time-Invariant (LTI) Systems

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 4.1

Convolution

CT Convolution

- The (CT) **convolution** of the functions x and h, denoted $x * h$, is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

Example X.4.1

Practical Convolution Computation

- To compute the convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

we proceed as follows:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(t)$.
4. Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

Example 4.1 Exercise 4.18(u)

Properties of Convolution

- The convolution operation is ***commutative***. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any functions x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any functions x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

Theorem 4.1

Representation of Functions Using Impulses

- For any function x,

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t).$$

- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any function x,

$$x * \delta = x.$$

Periodic Convolution

SKIP SLIDE

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic functions known as periodic convolution.
- The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

  where $\int_T$ denotes integration over an interval of length T.
- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

  (i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

Section 4.2

Convolution and LTI Systems

Impulse Response

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

Step Response

- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a LTI system are related as

$$h(t) = \frac{ds(t)}{dt}.$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.