Theorem 3.2 (Sum of periodic functions)**.** *Let x_1 and x_2 be (continuous) periodic functions with periods T_1 and T_2, respectively. Then, the function $x = x_1 + x_2$ is a periodic if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers). Suppose now that x is periodic. Let $T_1/T_2 = q/r$ where q and r are integers and coprime (i.e., have no common factors). Then, x is periodic with period $T = \operatorname{lcm}(T_1, T_2) = rT_1 = qT_2$.*

Proof. We provide only a partial proof. Assuming that x is periodic, we show that it must be periodic with period T. Since x is periodic, $T = \operatorname{lcm}(T_1, T_2)$ must exist. Since T is an integer multiple of both T_1 and T_2, we can write $T = k_1T_1$ and $T = k_2T_2$ for some positive integers k_1 and k_2. Thus, we have

$$\begin{aligned}
x(t+T) &= x_1(t+T) + x_2(t+T) \\
&= x_1(t+k_1T_1) + x_2(t+k_2T_2) \\
&= x_1(t) + x_2(t) \\
&= x(t).
\end{aligned}$$

Thus, x is periodic with period T. ■

In passing, we note that the above result can be extended to the more general case of the sum of N periodic functions. The sum of N periodic functions $x_1, x_2, \ldots, x_N$ with periods $T_1, T_2, \ldots, T_N$, respectively, is periodic if and only if the ratios of the periods are rational numbers (i.e., T_1/T_k is rational for $k = 2, 3, \ldots, N$). If the sum is periodic, then the fundamental period is simply $\operatorname{lcm}\{T_1, T_2, \ldots, T_N\}$.

Example 3.2. Let $x_1(t) = \sin(\pi t)$ and $x_2(t) = \sin t$. Determine whether the function $y = x_1 + x_2$ is periodic.

Solution. Denote the fundamental periods of x_1 and x_2 as T_1 and T_2, respectively. We then have

$$T_1 = \tfrac{2\pi}{\pi} = 2 \quad \text{and} \quad T_2 = \tfrac{2\pi}{1} = 2\pi.$$

Here, we used the fact that the fundamental period of $\sin(\alpha t)$ is $\frac{2\pi}{|\alpha|}$. Thus, we have

$$\tfrac{T_1}{T_2} = \tfrac{2}{2\pi} = \tfrac{1}{\pi}.$$

Since π is an irrational number, $\frac{T_1}{T_2}$ is not rational. Therefore, y is not periodic. ■

Example 3.3. Let $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$ and $x_2(t) = \sin(7\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. Thus, we have

$$T_1 = \tfrac{2\pi}{2\pi} = 1 \quad \text{and} \quad T_2 = \tfrac{2\pi}{7\pi} = \tfrac{2}{7}.$$

Taking the ratio of T_1 to T_2, we have

$$\tfrac{T_1}{T_2} = \tfrac{7}{2}.$$

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 7 and 2 are coprime,

$$T = 2T_1 = 7T_2 = 2.$$ ■

Example 3.4. Let $x_1(t) = \cos(6\pi t)$ and $x_2(t) = \sin(30\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. We have

$$T_1 = \tfrac{2\pi}{6\pi} = \tfrac{1}{3} \quad \text{and} \quad T_2 = \tfrac{2\pi}{30\pi} = \tfrac{1}{15}.$$

Thus, we have

$$\tfrac{T_1}{T_2} = (\tfrac{1}{3})/(\tfrac{1}{15}) = \tfrac{15}{3} = \tfrac{5}{1}.$$

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 5 and 1 are coprime, we have

$$T = 1T_1 = 5T_2 = \tfrac{1}{3}.$$ ∎

3.4.3 Support of Functions

We can classify functions based on the interval over which their function value is nonzero. This is sometimes referred to as the support of a function. In what follows, we introduce some terminology related to the support of functions.

A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0.$$

In other words, the value of the function is always zero to the right of some point. A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0.$$

In other words, the value of the function is always zero to the left of some point. A function that is both left sided and right sided is said to be **time limited** or **finite duration**. A function that is neither left sided nor right sided is said to be **two sided**. Note that every function is exactly one of: left sided but not right sided, right sided but not left sided, finite duration, or two sided. Examples of left-sided (but not right-sided), right-sided (but not left-sided), finite-duration, and two-sided functions are shown in Figure 3.8.

A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

A causal function is a special case of a right-sided function. Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

An anticausal function is a special case of a left-sided function. Note that the qualifiers "causal" and "anticausal", when applied to functions, have nothing to do with cause and effect. In this sense, this choice of terminology is arguably not the best.

3.4.4 Bounded Functions

A function x is said to be **bounded** if there exists some (finite) nonnegative real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is finite for all t). For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \to \infty} |p(t)| = \infty.$$

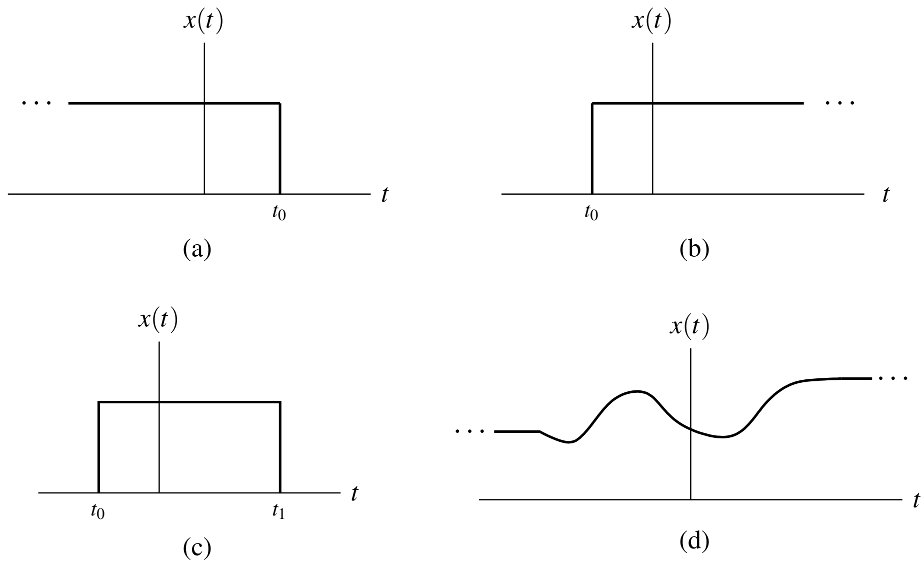


Figure 3.8: Examples of functions with various sidedness properties. A function that is (a) left sided but not right sided, (b) right sided but not left sided, (c) finite duration, and (d) two sided.

3.4.5 Signal Energy and Power

The **energy** E contained in the function x is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

As a matter of terminology, a function x with finite energy is said to be an **energy signal**. The **average power** P contained in the function x is given by

$$P = \lim_{T \to \infty} \tfrac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

As a matter of terminology, a function x with (nonzero) finite average power is said to be a **power signal**.

3.4.6 Examples

Below, we consider several examples that use signal properties in various ways.

Example 3.5. Let x be a function with the following properties:

$$v(t) = x(t-3) \text{ is causal; and}$$
$$x \text{ is odd.}$$

Determine for what values of t the quantity $x(t)$ must be zero.

Solution. Since v is causal, we know that $v(t) = 0$ for all $t < 0$. Since $v(t) = x(t-3)$, this implies that

$$x(t) = 0 \quad \text{for } t < -3. \tag{3.11}$$

Since x is odd, we know that

$$x(t) = -x(-t) \quad \text{for all } t. \tag{3.12}$$

From (3.11) and (3.12), we have $-x(-t) = 0$ for $t < -3$ which implies that

$$x(t) = 0 \quad \text{for } t > 3. \tag{3.13}$$

Substituting $t = 0$ into (3.12) yields $x(0) = -x(0)$ which implies that

$$x(0) = 0. \tag{3.14}$$

Combining (3.11), (3.13), and (3.14), we conclude that $x(t)$ must be zero for

$$t < -3, \quad t > 3, \quad \text{or} \quad t = 0.$$

■

Example 3.6. Consider a function x with the following properties:

- $x(t) = t+5$ for $-5 \leq t \leq -3$;
- $v_1(t) = x(t-5)$ is causal; and
- $v_2(t) = x(t-3)$ is even.

Find $x(t)$ for all t.

Solution. Since $v_1(t) = x(t-5)$ is causal, we have that

$$\begin{aligned}
& v_1(t) = 0 \quad \text{for } t < 0 \\
\Rightarrow\quad & x(t-5) = 0 \quad \text{for } t < 0 \\
\Rightarrow\quad & x([t+5]-5) = 0 \quad \text{for } (t+5) < 0 \\
\Rightarrow\quad & x(t) = 0 \quad \text{for } t < -5.
\end{aligned}$$

From this and the fact that $x(t) = t+5$ for $-5 \leq t \leq -3$, we have

$$x(t) = \begin{cases} t+5 & -5 \leq t \leq -3 \\ 0 & t < -5. \end{cases} \tag{3.15}$$

So, we only need to determine $x(t)$ for $t > -3$. Since $v_2(t) = x(t-3)$ is even, we have

$$\begin{aligned}
& v_2(t) = v_2(-t) \\
\Rightarrow\quad & x(t-3) = x(-t-3) \\
\Rightarrow\quad & x([t+3]-3) = x(-[t+3]-3) \\
\Rightarrow\quad & x(t) = x(-t-6).
\end{aligned}$$

Using this with (3.15), we obtain

$$\begin{aligned}
x(t) &= x(-t-6) \\
&= \begin{cases} (-t-6)+5 & -5 \leq -t-6 \leq -3 \\ 0 & -t-6 < -5 \end{cases} \\
&= \begin{cases} -t-1 & 1 \leq -t \leq 3 \\ 0 & -t < 1 \end{cases} \\
&= \begin{cases} -t-1 & -3 \leq t \leq -1 \\ 0 & t > -1. \end{cases}
\end{aligned}$$

Therefore, we conclude that

$$x(t) = \begin{cases} 0 & t < -5 \\ t+5 & -5 \leq t < -3 \\ -t-1 & -3 \leq t < -1 \\ 0 & t \geq -1. \end{cases}$$

A plot of x is given in Figure 3.9. ■

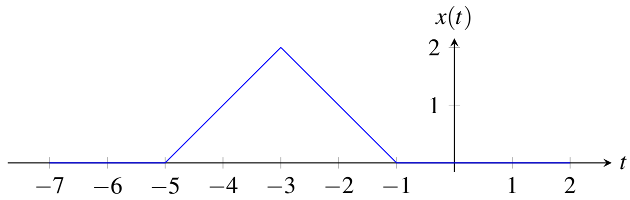


Figure 3.9: The function x from Example 3.6.

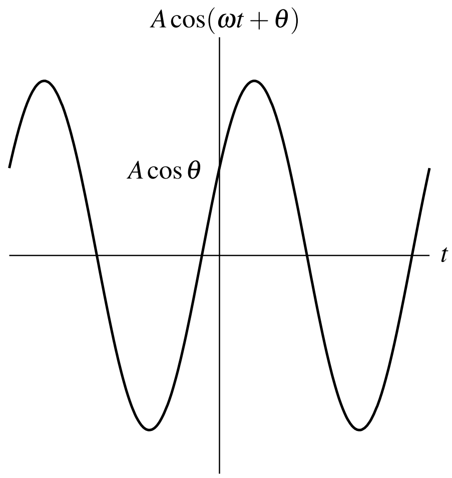


Figure 3.10: Real sinusoidal function.

3.5 Elementary Functions

A number of elementary signals are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

3.5.1 Real Sinusoidal Functions

One important class of functions is the real sinusoids. A **real sinusoidal function** x has the general form

$$x(t) = A\cos(\omega t + \theta),$$

where A, ω, and θ are real constants. Such a function is periodic with fundamental period $T = \frac{2\pi}{|\omega|}$, and has a plot resembling that shown in Figure 3.10.

3.5.2 Complex Exponential Functions

Another important class of functions is the complex exponentials. A **complex exponential function** x has the general form

$$x(t) = Ae^{\lambda t}, \tag{3.16}$$

where A and λ are complex constants. Complex exponentials are of fundamental importance to systems theory, and also provide a convenient means for representing a number of other classes of functions. A complex exponential can exhibit one of a number of distinctive modes of behavior, depending on the values of its parameters A and λ. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

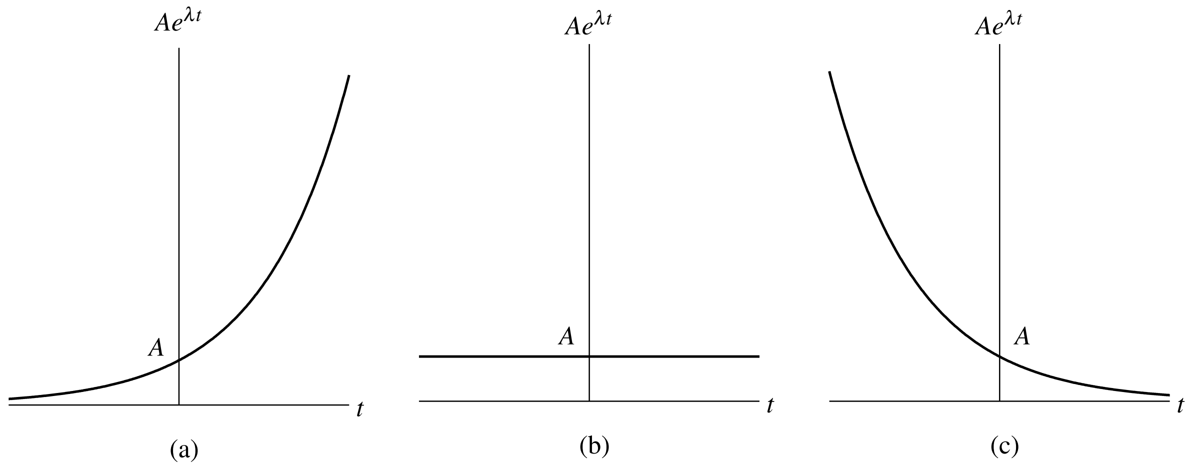


Figure 3.11: Real exponential function for (a) $\lambda > 0$, (b) $\lambda = 0$, and (c) $\lambda < 0$.

3.5.2.1 Real Exponential Functions

The first special case of the complex exponentials to be considered is the real exponentials. In the case of a **real exponential function**, we restrict A and λ in (3.16) to be real. A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated in Figure 3.11. If $\lambda > 0$, $x(t)$ increases exponentially as t increases (i.e., a growing exponential). If $\lambda < 0$, $x(t)$ decreases exponentially as t increases (i.e., a decaying or damped exponential). If $\lambda = 0$, $x(t)$ simply equals the constant A for all t.

3.5.2.2 Complex Sinusoidal Functions

The second special case of the complex exponentials that we shall consider is the complex sinusoids. In the case of a **complex sinusoidal function**, the parameters in (3.16)) are such that A is complex and λ is purely imaginary (i.e., $\operatorname{Re}(\lambda) = 0$). For convenience, let us re-express A in polar form and λ in Cartesian form as follows:

$$A = |A|\, e^{j\theta} \quad \text{and} \quad \lambda = j\omega,$$

where θ and ω are real constants. Using Euler's relation (A.4), we can rewrite (3.16) as

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A|\, e^{j\theta} e^{j\omega t} \\ &= |A|\, e^{j(\omega t + \theta)} \\ &= |A| \cos(\omega t + \theta) + j\, |A| \sin(\omega t + \theta). \end{aligned}$$

From the above equation, we can see that each of the real and imaginary parts of x is periodic with fundamental period $\frac{2\pi}{|\omega|}$. Furthermore, from this, we can deduce that x is also periodic with fundamental period $\frac{2\pi}{|\omega|}$. To illustrate the form of a complex sinusoid, we plot its real and imaginary parts in Figure 3.12. The real and imaginary parts are the same except for a phase difference.

The complex sinusoidal function $x(t) = e^{j\omega t}$ is plotted for the cases of ω equal to 2π and -2π in Figures 3.13(a) and (b), respectively. From these figures, one can see that the curve traced by $x(t)$ as t increases is a helix. If the curve is viewed by looking straight down the t axis in the direction of $-\infty$, the helix appears to turn in a counterclockwise direction if $\omega > 0$ (i.e., a right-handed helix) and a clockwise direction if $\omega < 0$ (i.e., a left-handed helix). Note that, although both of these complex sinusoids oscillate at the same rate, they are not translated (i.e., time-shifted) and/or reflected (i.e., time-reversed) versions of one another. Left- and right-handed helices are fundamentally different shapes. One cannot be made into the other by translation (i.e., time shifting) and/or reflection (i.e., time reversal).

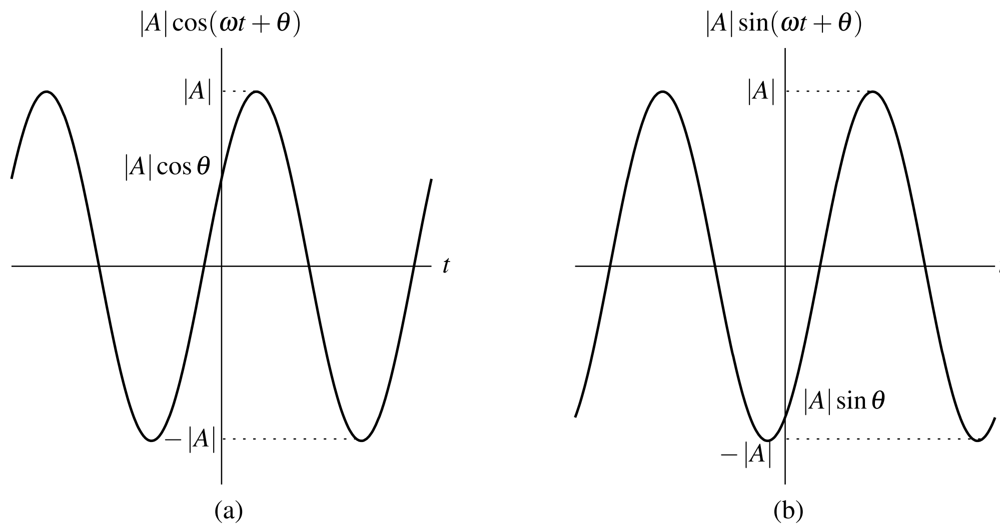


Figure 3.12: Complex sinusoidal function. (a) Real and (b) imaginary parts.

As demonstrated above in Figure 3.13, the complex sinusoid $x(t) = e^{j\omega t}$ behaves fundamentally differently for $\omega > 0$ and $\omega < 0$. For this reason, signed frequency (as introduced earlier in Section 2.10.2) is often used when working with complex sinusoids. That is, we often refer to ω as the frequency of the complex sinusoid. This is an example of the use of signed frequency (as ω is clearly a signed quantity). The positive direction of rotation (i.e., $\omega > 0$) corresponds to a right-handed helix, while the negative direction of rotation (i.e., $\omega < 0$) corresponds to a left-handed helix.

3.5.2.3 General Complex Exponential Functions

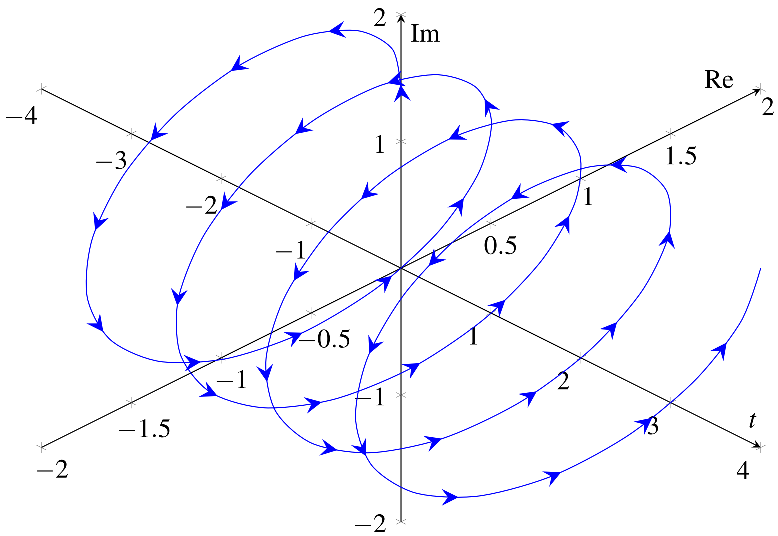
Lastly, we consider general complex exponential functions. That is, we consider the general case of (3.16) where A and λ are both complex. For convenience, let us re-express A in polar form and λ in Cartesian form as

$$A = |A|\,e^{j\theta} \quad \text{and} \quad \lambda = \sigma + j\omega,$$

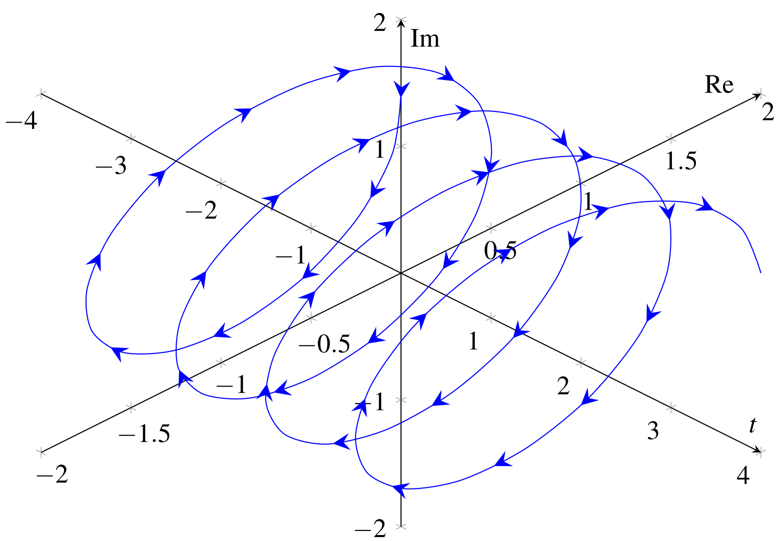
where θ, σ, and ω are real constants. Substituting these expressions for A and λ into (3.16), we obtain

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A|\,e^{j\theta}e^{(\sigma+j\omega)t} \\ &= |A|\,e^{\sigma t}e^{j(\omega t+\theta)} \\ &= |A|\,e^{\sigma t}\cos(\omega t+\theta) + j\,|A|\,e^{\sigma t}\sin(\omega t+\theta). \end{aligned}$$

We can see that $\mathrm{Re}[x(t)]$ and $\mathrm{Im}[x(t)]$ have a similar form. Each is the product of a real exponential and real sinusoid. One of three distinct modes of behavior is exhibited by x, depending on the value of σ. If $\sigma = 0$, $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are real sinusoids. If $\sigma > 0$, $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are each the product of a real sinusoid and a growing real exponential. If $\sigma < 0$, $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are each the product of a real sinusoid and a decaying real exponential. These three cases are illustrated for $\mathrm{Re}(x)$ in Figure 3.14.



(a)



(b)

Figure 3.13: The complex sinusoidal function $x(t) = e^{j\omega t}$ for (a) $\omega = 2\pi$ and (b) $\omega = -2\pi$.

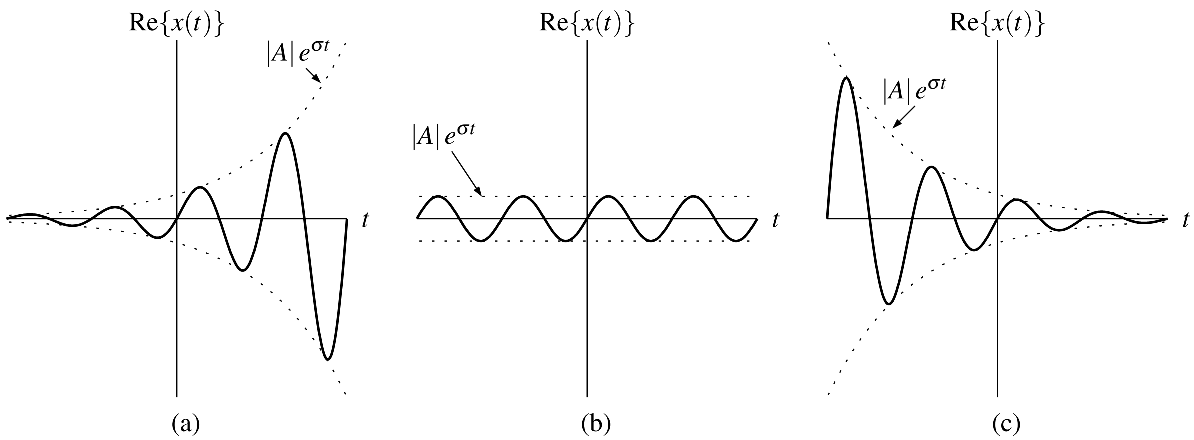


Figure 3.14: Real part of a general complex exponential function for (a) $\sigma > 0$, (b) $\sigma = 0$, and (c) $\sigma < 0$.

3.5.3 Relationship Between Complex Exponential and Real Sinusoidal Functions

A real sinusoid can be expressed as the sum of two complex sinusoids using the identity

$$A\cos(\omega t + \theta) = \frac{A}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right) \quad \text{and} \tag{3.17}$$

$$A\sin(\omega t + \theta) = \frac{A}{2j}\left(e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right). \tag{3.18}$$

This result follows from Euler's relation and is simply a restatement of (A.8).

3.5.4 Unit-Step Function

Another elementary function often used in systems theory is the unit-step function. The **unit-step function** (also known as the **Heaviside function**) is denoted as u and defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.15.

Clearly, u is discontinuous at the origin. At this point of discontinuity, we have chosen to define u such that its value is 1 (i.e., $u(0) = 1$). As it turns out, however, this choice is somewhat arbitrary. That is, for most practical purposes, due to technical reasons beyond the scope of this book, what is important is that $u(0)$ is finite, not its specific value. For this reason, some authors choose to leave the value of $u(0)$ as an unspecified (but finite) constant, while others choose to assign a specific value to $u(0)$. In cases where a specific value is chosen for $u(0)$, the values most commonly used are 0, $\frac{1}{2}$, and 1. Obviously, in the case of this book, the author has chosen to use $u(0) = 1$.

3.5.5 Signum Function

Another function closely related to the unit-step function is the so called signum function. The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \tag{3.19}$$

A plot of this function is given in Figure 3.16. From (3.19), one can see that the signum function simply computes the sign of a number.

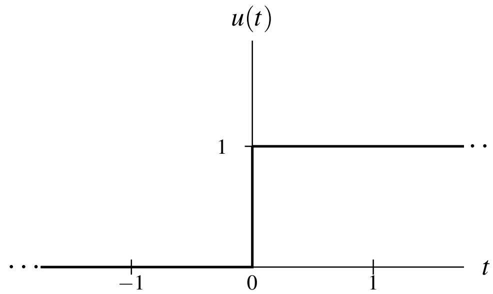


Figure 3.15: Unit-step function.

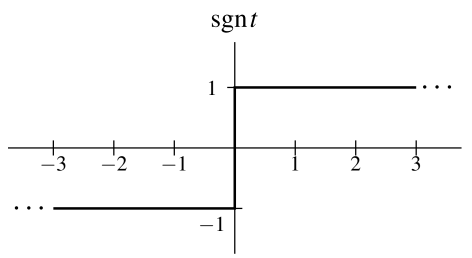


Figure 3.16: Signum function.

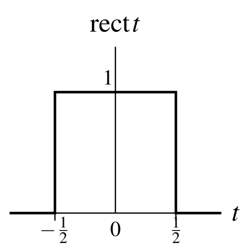


Figure 3.17: Rectangular function.

3.5.6 Rectangular Function

Another useful function is the rectangular function. The **rectangular function** (also known as the **unit-rectangular pulse function**) is denoted as rect and is defined as

$$\operatorname{rect} t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is shown in Figure 3.17.

Example 3.7 (Extracting part of a function with a rectangular pulse)**.** Use the rectangular function to extract one period of the function x shown in Figure 3.18(a).

Solution. Let us choose to extract the period of $x(t)$ for $-\frac{T}{2} < t \leq \frac{T}{2}$. In order to extract this period, we want to multiply x by a function that is one over this interval and zero elsewhere. Such a function is simply $v(t) = \operatorname{rect}\left(\frac{1}{T}t\right)$ as shown in Figure 3.18(b). Multiplying v and x results in the function shown in Figure 3.18(c). ■

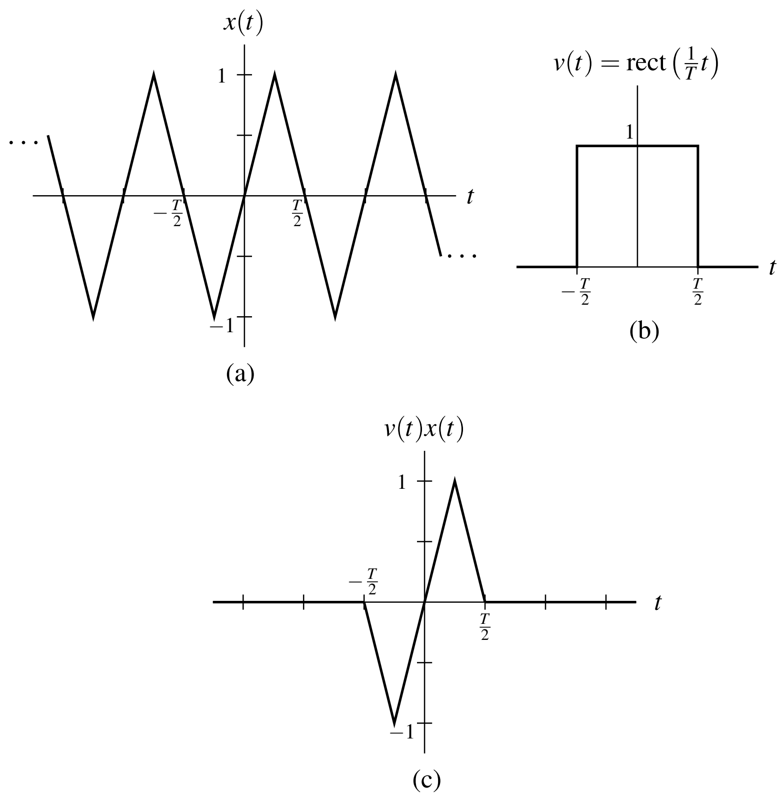


Figure 3.18: Using the rectangular function to extract one period of a periodic function x. (a) The function x. (b) A time-scaled rectangular function v. (c) The product of x and v.

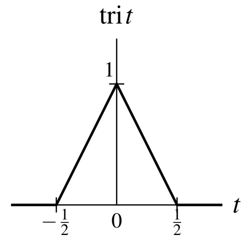


Figure 3.19: Triangular function.

3.5.7 Indicator Function

Functions and sequences that are one over some subset of their domain and zero elsewhere appear very frequently in engineering (e.g., the unit-step function, rectangular function, the unit-step sequence appearing later in Chapter 8, and the delta sequence appearing later in Chapter 8). Indicator function notation provides a concise way to denote such functions and sequences. The **indicator function** of a subset S of a set A, denoted χ_S, is defined as

$$\chi_S(t) = \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.8. A rectangular pulse (defined on $\mathbb{R}$) having an amplitude of 1, a leading edge at a, and falling edge at b is $\chi_{[a,b]}$. The unit-step function (defined on $\mathbb{R}$) is $\chi_{[0,\infty)}$. The unit-rectangular pulse (defined on $\mathbb{R}$) is $\chi_{[-1/2,1/2)}$. ■

3.5.8 Triangular Function

Another useful elementary function is the **triangular function** (also known as the **unit triangular pulse function**), which is denoted as tri and defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2\,|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.19.

3.5.9 Cardinal Sine Function

In the study of signals and systems, a function of the form $x(t) = \frac{\sin t}{t}$ frequently appears. Therefore, as a matter of convenience, this particular function is given a special name, the cardinal sine function. More formally, the **cardinal sine function** (also known as the **sinc function**) is denoted sinc and defined as

$$\operatorname{sinc} t = \frac{\sin t}{t}. \tag{3.20}$$

The name "sinc" is simply a contraction of the function's full Latin name "sinus cardinalis" (cardinal sine). By using l'Hopital's rule, one can confirm that $\operatorname{sinc} t$ is well defined for $t = 0$. That is, $\operatorname{sinc} 0 = 1$.

It is worthwhile to note that a definition of the sinc function different from the one above is also sometimes used in the literature. In particular, the sinc function is sometimes defined as

$$x(t) = \frac{\sin(\pi t)}{\pi t}. \tag{3.21}$$

In order to avoid any possible confusion, we will refer to the function x defined by (3.21) as the **normalized sinc function**. As some examples of the practices followed by others, the definition of the sinc function in (3.20) is used by [8, 9, 14], while the definition in (3.21) is employed by [7, 13, 15] as well as the MATLAB software.

3.5.10 Rounding-Related Functions

Rounding can sometimes be an operation of interest. In what follows, we introduce several functions that are related to rounding.

The **floor function**, denoted $\lfloor \cdot \rfloor$, is a function that maps a real number x to the largest integer not more than x. In other words, the floor function rounds a real number to the nearest integer in the direction of negative infinity. For example,

$$\left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \quad \left\lfloor \tfrac{1}{2} \right\rfloor = 0, \quad \text{and} \quad \lfloor 1 \rfloor = 1.$$

The **ceiling function**, denoted $\lceil \cdot \rceil$, is a function that maps a real number x to the smallest integer not less than x. In other words, the ceiling function rounds a real number to the nearest integer in the direction of positive infinity. For example,

$$\left\lceil -\tfrac{1}{2} \right\rceil = 0, \quad \left\lceil \tfrac{1}{2} \right\rceil = 1, \quad \text{and} \quad \lceil 1 \rceil = 1.$$

Some identities involving the floor and ceiling functions that are often useful include the following:

$$\begin{aligned}
\lfloor x+n \rfloor &= \lfloor x \rfloor + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}; \\
\lceil x+n \rceil &= \lceil x \rceil + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}; \\
\lceil x \rceil &= -\lfloor -x \rfloor \quad \text{for } x \in \mathbb{R}; \\
\lfloor x \rfloor &= -\lceil -x \rceil \quad \text{for } x \in \mathbb{R}; \\
\left\lceil \frac{m}{n} \right\rceil &= \left\lfloor \frac{m+n-1}{n} \right\rfloor = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0; \quad \text{and} \\
\left\lfloor \frac{m}{n} \right\rfloor &= \left\lceil \frac{m-n+1}{n} \right\rceil = \left\lceil \frac{m+1}{n} \right\rceil - 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0.
\end{aligned}$$

The **remainder function**, denoted mod, is a function that takes two integers m and n and yields the remainder obtained after dividing m by n, where the remainder is defined to always be nonnegative. For example, $\operatorname{mod}(10,3) = 1$, $\operatorname{mod}(-3,2) = 1$, and $\operatorname{mod}(8,2) = 0$. The mod function can be expressed in terms of the floor function as

$$\operatorname{mod}(m,n) = m - n\lfloor m/n \rfloor \quad \text{for } n,m \in \mathbb{Z} \text{ and } n \neq 0.$$

3.5.11 Delta Function

In systems theory, one elementary function of fundamental importance is the delta function. Instead of defining this function explicitly, it is defined in terms of its properties. In particular, the **delta function** (also known as the **Dirac delta function** or **unit-impulse function**) is denoted as δ and defined as the function with the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and} \tag{3.22a}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{3.22b}$$

From these properties, we can see that the function is zero everywhere, except at the origin where it is undefined. Indeed, this is an unusual function. Although it is zero everywhere except at a single point, it has a nonzero integral. Technically, the delta function is not a function in the ordinary sense. Rather, it is what is known as a **generalized function**.

Graphically, we represent the delta function as shown in Figure 3.20. Since the function assumes an infinite value at the origin, we cannot plot the true value of the function. Instead, we use a vertical arrow to represent this infinite value. To show the strength of the impulse, its weight is also indicated. In Figure 3.21, we plot a scaled and time-shifted version of the delta function.

As it turns out, the δ function can be viewed as the limit of a sequence of functions, where the functions in this sequence are pulses that become progressively narrower and taller and all have an integral of one. By viewing δ in

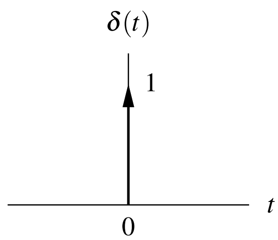


Figure 3.20: Delta function.

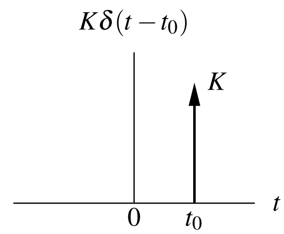


Figure 3.21: Scaled and time-shifted delta function.

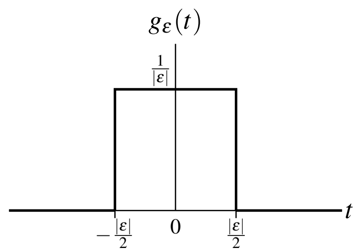


Figure 3.22: The rectangular-pulse function g_ε.

this way, we can gain deeper insight into this function. In what follows, we will consider expressing the δ function as the limit of a sequence of rectangular pulses.

Consider the family of rectangular-pulse functions consisting of functions of the form

$$g_\varepsilon(t) = \begin{cases} \frac{1}{|\varepsilon|} & |t| < \frac{|\varepsilon|}{2} \\ 0 & \text{otherwise,} \end{cases}$$

where ε is a real constant. A plot of g_ε is shown in Figure 3.22. Clearly, g_ε is a rectangular pulse of width $|\varepsilon|$ and height $\frac{1}{|\varepsilon|}$ that is centered at the origin. Observe that, as $|\varepsilon|$ decreases, the width of the pulse decreases and the height of the pulse increases. Also, observe that, regardless of how ε is chosen, the pulse has an area of one (i.e., $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.) Thus, the δ function can be viewed as the following limit involving g_ε:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

To assist in the visualization of this limiting process, Figure 3.24 shows g_ε plotted for several values of ε. From this figure, we can visually confirm that $g_\varepsilon \to \delta$ as $\varepsilon \to 0$. Thus, δ can be viewed as the limiting case of a rectangular pulse where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Informally, one can also think of the delta function δ as the derivative of the unit-step function u. Strictly speaking, however, the derivative of u does not exist in the ordinary sense, since u is not continuous at 0. To be more precise, δ is what is called the **generalized derivative** of u. The generalized derivative is essentially an extension of the notion of (ordinary) derivative, which can be well defined even for functions with discontinuities.

The delta function has several important properties that follow from its definition in (3.22). These properties are given by the theorems below.

Theorem 3.3 (Equivalence property)**.** *For any function x that is continuous at the point t_0,*

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \quad \textit{for all } t. \tag{3.23}$$

This result is known as the ***equivalence property****. This property is illustrated graphically in Figure 3.25.*

Proof. The proof follows immediately from the fact that the delta function is only nonzero at a single point. ■

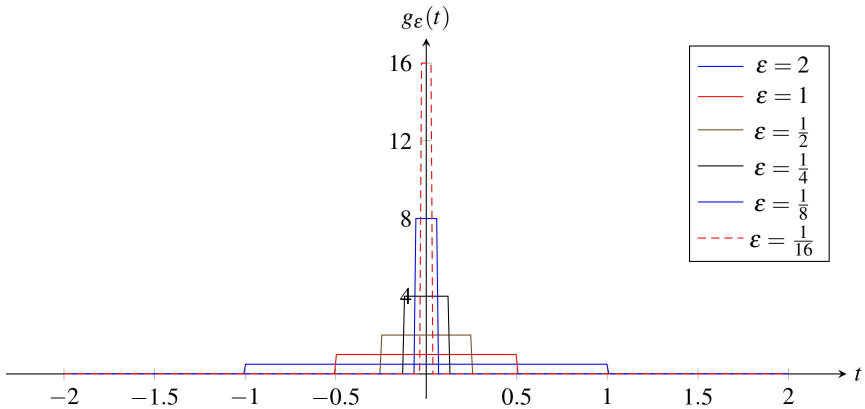


Figure 3.23: Plot of g_ε for several values of ε.

Theorem 3.4 (Sifting property)**.** *For any function x that is continuous at the point t_0,*

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \tag{3.24}$$

This result is known as the ***sifting property****.*

Proof. From (3.22b), we can write

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1.$$

Multiplying both sides of the preceding equation by $x(t_0)$ yields

$$\int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0).$$

Then, by using the equivalence property in (3.23), we can write

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$ ∎

Theorem 3.5 (Scaling property)**.** *For all nonzero real a, the following identity holds:*

$$\delta(at) = \frac{1}{|a|}\delta(t). \tag{3.25}$$

This identity is sometimes referred to as the ***scaling property*** *of the delta function.*

Proof. Consider the family of (Gaussian) functions consisting of functions of the form

$$d_\varepsilon(t) = \frac{1}{|\varepsilon|\sqrt{\pi}} e^{-(t/\varepsilon)^2}.$$

We can observe several properties of d_ε that are true regardless of the value of ε. First, d_ε has a maximum of $\frac{1}{|\varepsilon|\sqrt{\pi}}$ at the origin and decays at a quadratic-exponential rate as the distance from the origin increases. Second, as $|\varepsilon|$ decreases, the interval over which d_ε is not extremely small in value decreases in length. Third, d_ε is such that

$$\int_{-\infty}^{\infty} d_\varepsilon(t)dt = 1.$$

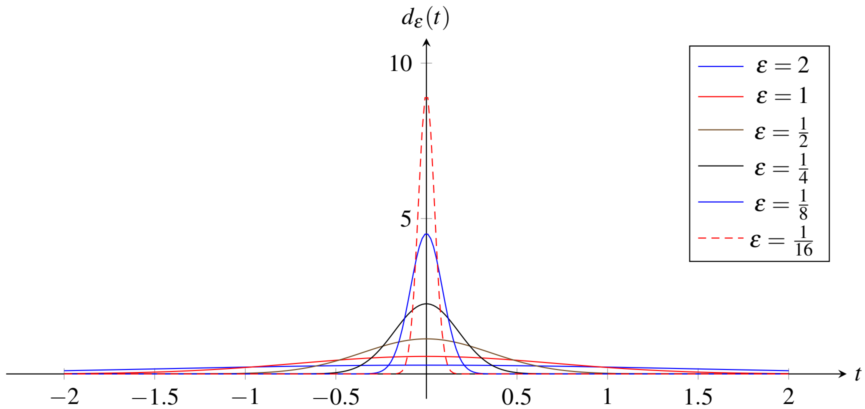


Figure 3.24: Plot of d_ε for several values of ε.

Thus, the δ function can be viewed as a limiting case of d_ε. In particular, we have that

$$\begin{aligned}\delta(t) &= \lim_{\varepsilon \to 0} d_\varepsilon(t) \\ &= \lim_{\varepsilon \to 0} \frac{1}{|\varepsilon|\sqrt{\pi}} e^{-(t/\varepsilon)^2}.\end{aligned}$$

For visualization purposes, Figure 3.24 shows d_ε plotted for several values of ε. From this figure, we can see that $d_\varepsilon \to \delta$ as $\varepsilon \to 0$. Substituting at for t in the preceding equation, we obtain

$$\delta(at) = \lim_{\varepsilon \to 0} \frac{1}{|\varepsilon|\sqrt{\pi}} e^{-(at/\varepsilon)^2}.$$

Now, we apply a change of variable. Let $\varepsilon' = \varepsilon/a$ so that $\varepsilon = \varepsilon' a$ and $a = \varepsilon/\varepsilon'$. Applying the change of variable, we obtain

$$\begin{aligned}\delta(at) &= \lim_{\varepsilon' \to 0} \frac{1}{|\varepsilon' a|\sqrt{\pi}} e^{-(at/[\varepsilon' a])^2} \\ &= \lim_{\varepsilon' \to 0} \frac{1}{|\varepsilon'|\,|a|\sqrt{\pi}} e^{-(t/\varepsilon')^2} \\ &= \lim_{\varepsilon' \to 0} \frac{1}{|a|} \frac{1}{|\varepsilon'|\sqrt{\pi}} e^{-(t/\varepsilon')^2} \\ &= \frac{1}{|a|} \lim_{\varepsilon' \to 0} \frac{1}{|\varepsilon'|\sqrt{\pi}} e^{-(t/\varepsilon')^2} \\ &= \frac{1}{|a|}\delta(t).\end{aligned}$$

Thus, we have shown that (3.25) holds. ■

Theorem 3.6 (Even property)**.** *The δ function is such that*

$$\delta(t) = \delta(-t).$$

Proof. The proof immediately follows from the definition of the δ function. ■

As we shall see later, the equivalence, sifting, scaling, and even properties of δ are extremely helpful. Lastly, note that it follows from the definition of δ that integrating this function over any interval not containing the origin will result in the value of zero.

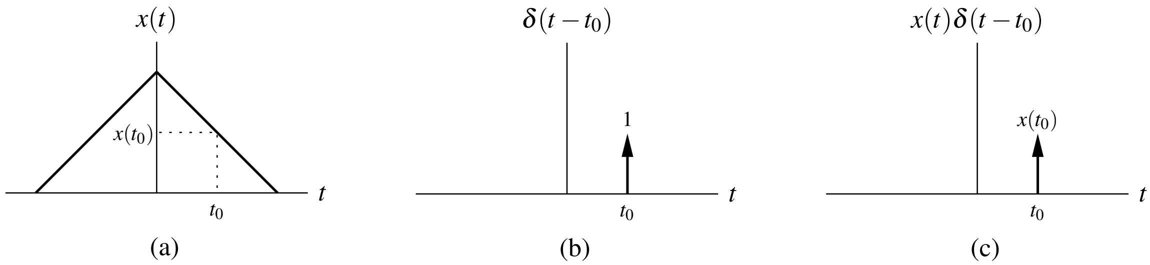


Figure 3.25: Graphical interpretation of equivalence property. (a) A function x; (b) a time-shifted delta function; and (c) the product of the these two functions.

Example 3.9 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} \sin(t)\delta\left(t-\tfrac{\pi}{4}\right)dt.$$

Solution. Using the sifting property of the delta function, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(t)\delta\left(t-\tfrac{\pi}{4}\right)dt &= \sin\left(\tfrac{\pi}{4}\right)\\ &= \tfrac{1}{\sqrt{2}}.\end{aligned}$$

■

Example 3.10 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt.$$

Solution. First, we observe that the integral to be evaluated does not quite have the same form as (3.24). So, we need to perform a change of variable. Let $\tau = 4t$ so that $t = \frac{1}{4}\tau$ and $dt = \frac{1}{4}d\tau$. Performing the change of variable, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt &= \int_{-\infty}^{\infty} \tfrac{1}{4}\sin\left[2\pi\left(\tfrac{1}{4}\tau\right)\right]\delta(\tau-1)d\tau\\ &= \int_{-\infty}^{\infty} \tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\tau\right)\delta(\tau-1)d\tau.\end{aligned}$$

Now the integral has the desired form, and we can use the sifting property of the delta function to write

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt &= \left[\tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\tau\right)\right]\Big|_{\tau=1}\\ &= \tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\right)\\ &= \tfrac{1}{4}.\end{aligned}$$

■

Example 3.11. Evaluate the integral $\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau$.

Solution. Using the equivalence property of the delta function given by (3.23), we can write

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \int_{-\infty}^{t}(2^2+1)\delta(\tau-2)d\tau\\ &= 5\int_{-\infty}^{t}\delta(\tau-2)d\tau.\end{aligned}$$

Using the defining properties of the delta function given by (3.22), we have that

$$\begin{aligned}\int_{-\infty}^{t} \delta(\tau-2)d\tau &= \begin{cases}1 & t \geq 2\\ 0 & t < 2\end{cases}\\ &= u(t-2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\int_{-\infty}^{t} (\tau^2+1)\delta(\tau-2)d\tau &= \begin{cases}5 & t \geq 2\\ 0 & t < 2\end{cases}\\ &= 5u(t-2).\end{aligned}$$

■

3.6 Representation of Arbitrary Functions Using Elementary Functions

In the earlier sections, we introduced a number of elementary functions. Often in signal analysis, it is convenient to represent arbitrary functions in terms of elementary functions. Here, we consider how the unit-step function can be exploited in order to obtain alternative representations of functions.

Example 3.12 (Rectangular function)**.** Show that the rect function can be expressed in terms of u as

$$\operatorname{rect} t = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right).$$

Solution. Using the definition of u and time-shift transformations, we have

$$u\left(t+\tfrac{1}{2}\right) = \begin{cases}1 & t \geq -\frac{1}{2}\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad u\left(t-\tfrac{1}{2}\right) = \begin{cases}1 & t \geq \frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we have

$$\begin{aligned}u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right) &= \begin{cases}0-0 & t < -\frac{1}{2}\\ 1-0 & -\frac{1}{2} \leq t < \frac{1}{2}\\ 1-1 & t \geq \frac{1}{2}\end{cases}\\ &= \begin{cases}0 & t < -\frac{1}{2}\\ 1 & -\frac{1}{2} \leq t < \frac{1}{2}\\ 0 & t \geq \frac{1}{2}\end{cases}\\ &= \begin{cases}1 & -\frac{1}{2} \leq t < \frac{1}{2}\\ 0 & \text{otherwise}\end{cases}\\ &= \operatorname{rect} t.\end{aligned}$$

Graphically, we have the scenario depicted in Figure 3.26. ■

The result in the preceding example can be generalized. For a rectangular pulse x of height 1 with a rising edge at a and falling edge at b, one can show that

$$\begin{aligned}x(t) &= u(t-a) - u(t-b)\\ &= \begin{cases}1 & a \leq t < b\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

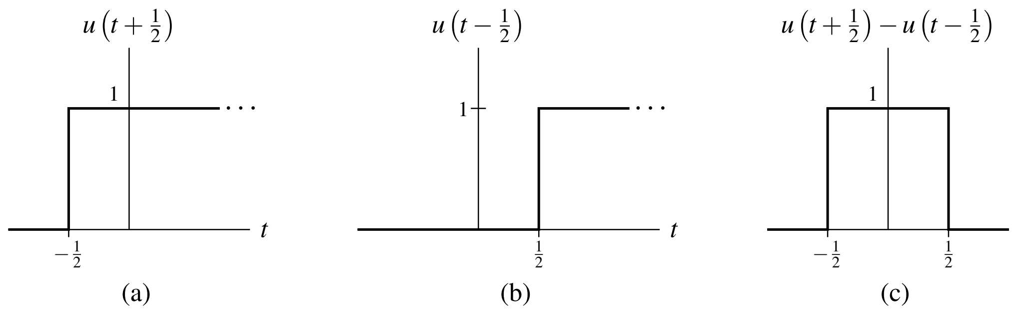


Figure 3.26: Representing the rectangular function using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their difference (which is the rectangular function).

Example 3.13 (Piecewise-linear function)**.** Consider the piecewise-linear function x given by

$$x(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 3-t & 2 \leq t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. A plot of x is shown in Figure 3.27(a). We consider each segment of the piecewise-linear function separately. The first segment (i.e., for $0 \leq t < 1$) can be expressed as

$$v_1(t) = t[u(t) - u(t-1)].$$

This function is plotted in Figure 3.27(b). The second segment (i.e., for $1 \leq t < 2$) can be expressed as

$$v_2(t) = u(t-1) - u(t-2).$$

This function is plotted in Figure 3.27(c). The third segment (i.e., for $2 \leq t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

This function is plotted in Figure 3.27(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t. ■

Example 3.14 (Piecewise-polynomial function)**.** Consider the piecewise-polynomial function x given by

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ (t-2)^2 & 1 \leq t < 3 \\ 4-t & 3 \leq t < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

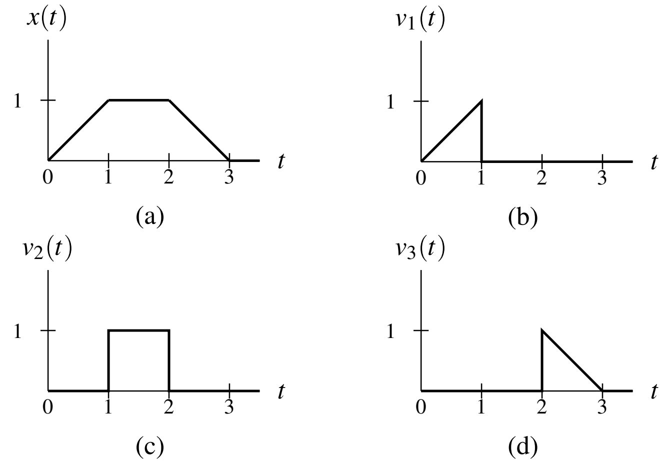


Figure 3.27: Representing a piecewise-linear function using unit-step functions. (a) The function x. (b), (c), and (d) Three functions whose sum is x.

Solution. A plot of x is shown in Figure 3.28(a). We consider each segment of the piecewise-polynomial function separately. The first segment (i.e., for $0 \leq t < 1$) can be written as

$$v_1(t) = u(t) - u(t-1).$$

This function is plotted in Figure 3.28(b). The second segment (i.e., for $1 \leq t < 3$) can be written as

$$v_2(t) = (t-2)^2[u(t-1) - u(t-3)] = (t^2 - 4t + 4)[u(t-1) - u(t-3)].$$

This function is plotted in Figure 3.28(c). The third segment (i.e., for $3 \leq t < 4$) can be written as

$$v_3(t) = (4-t)[u(t-3) - u(t-4)].$$

This function is plotted in Figure 3.28(d). Now, we observe that $x = v_1 + v_2 + v_3$. So, we have

$$\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= [u(t) - u(t-1)] + (t^2 - 4t + 4)[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)] \\
&= u(t) + (t^2 - 4t + 4 - 1)u(t-1) + (4 - t - [t^2 - 4t + 4])u(t-3) - (4-t)u(t-4) \\
&= u(t) + (t^2 - 4t + 3)u(t-1) + (-t^2 + 3t)u(t-3) + (t-4)u(t-4).
\end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t. ■

Example 3.15 (Periodic function)**.** Consider the periodic function x shown in Figure 3.29(a). Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. We begin by finding an expression for a single period of x. Let us denote this expression as v. We can then write:

$$v(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}).$$

This function is plotted in Figure 3.29(b). In order to obtain the periodic function x, we must repeat v every 2 units (since the period of x is 2). This can be accomplished by adding an infinite number of shifted copies of v as given by

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} v(t-2k) \\
&= \sum_{k=-\infty}^{\infty} [u(t + \tfrac{1}{2} - 2k) - u(t - \tfrac{1}{2} - 2k)].
\end{aligned}$$

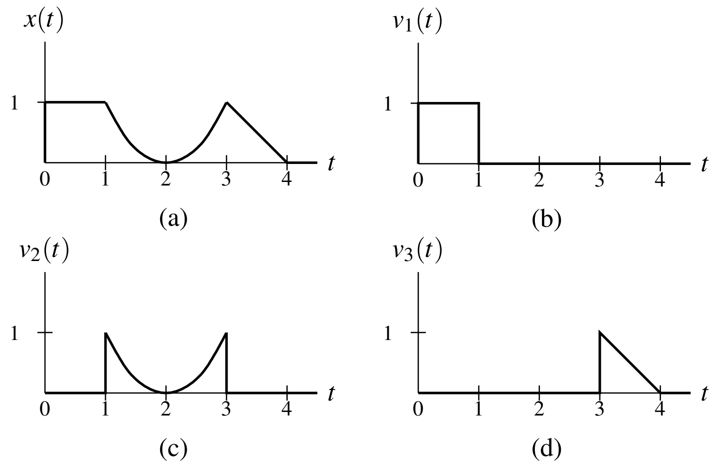


Figure 3.28: Representing a piecewise-polynomial function using unit-step functions. (a) The function x; and (b), (c), and (d) three functions whose sum is x.

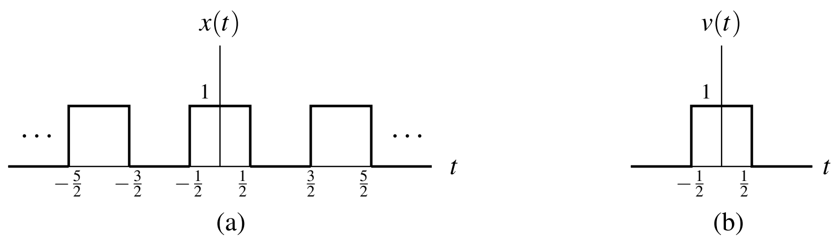


Figure 3.29: Representing a periodic function using unit-step functions. (a) The periodic function x; and (b) a function v that consists of a single period of x.

Thus, we have found a single expression for x that is valid for all t. ■

3.7 Continuous-Time Systems

Suppose that we have a system with input x and output y. Such a system can be described mathematically by the equation

$$y = \mathcal{H}x, \tag{3.26}$$

where $\mathcal{H}$ is an operator (i.e., transformation) representing the system. The operator $\mathcal{H}$ simply maps the input function x to the output function y. Alternatively, we sometimes express the relationship (3.26) using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

Furthermore, if clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \to y.$$

Note that the symbols "$\to$" and "=" have very different meanings. For example, the notation $x \to y$ does not in any way imply that $x = y$. The symbol "$\to$" should be read as "produces" (not as "equals"). That is, "$x \to y$" should be read as "the input x produces the output y".

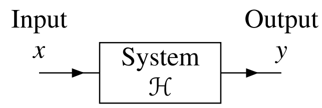


Figure 3.30: Block diagram of system.

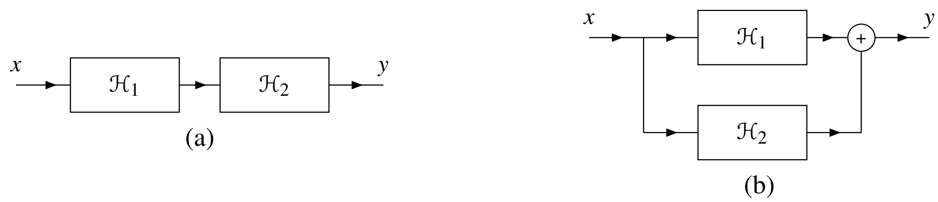


Figure 3.31: Interconnection of systems. (a) Series interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$. (b) Parallel interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$.

3.7.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathcal{H}$ with input x and output y. Often, we represent such a system using a block diagram as shown in Figure 3.30.

3.7.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 3.31. The first type of connection, as shown in Figure 3.31(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_2\mathcal{H}_1 x. \tag{3.27}$$

The second type of connection, as shown in Figure 3.31(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x. \tag{3.28}$$

The system equations in (3.27) and (3.28) cannot be simplified further unless the definitions of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are known.

3.8 Properties of Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

3.8.1 Memory

A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t \neq t_0$. In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time. A system that is not memoryless is said to have **memory**. Although simple, a memoryless system is not very flexible, since its current output value cannot rely on past or future values of the input.

Example 3.16 (Ideal amplifier)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ■

Example 3.17 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ for $-\infty < t \leq t_0$. So, $\mathcal{H}x(t_0)$ is dependent on $x(t)$ for some $t \neq t_0$ (e.g., $t_0 - 1$). Therefore, the system has memory (i.e., is not memoryless). ■

Example 3.18. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = e^{x(t)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = e^{x(t_0)}.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ■

Example 3.19. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \tfrac{1}{2}\left[x(t_0) - x(-t_0)\right].$$

Thus, for any x and any real t_0, we have that $\mathcal{H}x(t_0)$ depends on $x(t)$ for $t = t_0$ and $t = -t_0$. Since $\mathcal{H}x(t_0)$ depends on $x(t)$ for some $t \neq t_0$, the system has memory (i.e., the system is not memoryless). ■

3.8.2 Causality

A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t > t_0$. In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier* points in time (i.e., *not later* points in time). A memoryless system is always causal, although the converse is not necessarily true.

If the independent variable represents time, a system must be causal in order to be physically realizable (i.e., capable of being built). Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 3.20 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$. Since all of the values in this interval are less than or equal to t_0, the system is causal. ■

Example 3.21. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{t_0-1}^{t_0+1} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ only depends on $x(t)$ for $t_0 - 1 \leq t \leq t_0 + 1$. Since some of the values in this interval are greater than t_0 (e.g., $t_0 + 1$), the system is not causal. ■

Example 3.22. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = (t+1)e^{x(t-1)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

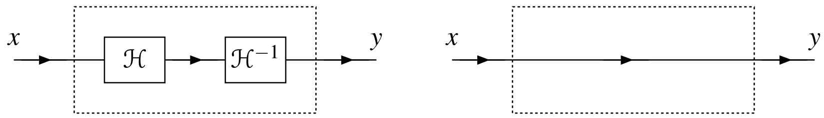
$$\mathcal{H}x(t_0) = (t_0+1)e^{x(t_0-1)}.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0 - 1$. Since $t_0 - 1 \leq t_0$, the system is causal. ■

Example 3.23. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. For any x and any real constant t_0, we have that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0$ and $t = -t_0$. Suppose that $t_0 = -1$. In this case, we have that $\mathcal{H}x(t_0)$ (i.e., $\mathcal{H}x(-1)$) depends on $x(t)$ for $t = 1$ but $t = 1 > t_0$. Therefore, the system is not causal. ■



(a) (b)

Figure 3.32: Systems that are equivalent (assuming $\mathcal{H}^{-1}$ exists). (a) First and (b) second system.

3.8.3 Invertibility

The **inverse** of a system $\mathcal{H}$ is a system $\mathcal{G}$ such that, for every function x,

$$\mathcal{G}\mathcal{H}x = x$$

(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{G}$ is a system whose input and output are equal). In other words, the effect of $\mathcal{H}$ is cancelled by $\mathcal{G}$. As a matter of notation, the inverse of $\mathcal{H}$ is denoted $\mathcal{H}^{-1}$. The relationship between a system and its inverse is illustrated in Figure 3.32. The two systems in this figure must be equivalent, due to the relationship between $\mathcal{H}$ and $\mathcal{H}^{-1}$ (i.e., $\mathcal{H}^{-1}$ cancels $\mathcal{H}$).

A system $\mathcal{H}$ is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists). An invertible system must be such that its input x can always be uniquely determined from its output $\mathcal{H}x$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

To show that a system is invertible, we simply find the inverse system. To show that a system is not invertible, it is sufficient to find two distinct inputs to that system that result in identical outputs. In practical terms, invertible systems are nice in the sense that their effects can be undone.

Example 3.24. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = x(t - t_0)$$

and t_0 is a real constant.

Solution. Let $y = \mathcal{H}x$. By substituting $t + t_0$ for t in $y(t) = x(t - t_0)$, we obtain

$$\begin{aligned} y(t + t_0) &= x(t + t_0 - t_0) \\ &= x(t). \end{aligned}$$

Thus, we have shown that

$$x(t) = y(t + t_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(t) = \mathcal{H}^{-1}y(t)$$

where

$$\mathcal{H}^{-1}y(t) = y(t + t_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system $\mathcal{H}$ is invertible. ■

Example 3.25. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

