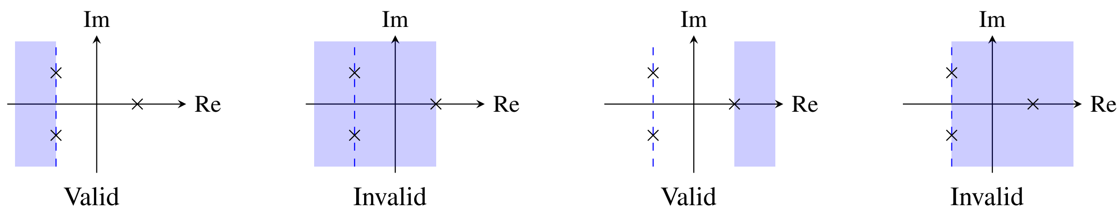

Figure 7.14: Examples of sets that would be either valid or invalid as the ROC of a rational Laplace transform of a left/right-sided function.

Figure 7.15: Relationship between the sidedness properties of x and the ROC of $X = \mathcal{L}x$

x		
left sided	right sided	ROC of X
no	no	strip
no	yes	RHP
yes	no	LHP
yes	yes	everywhere

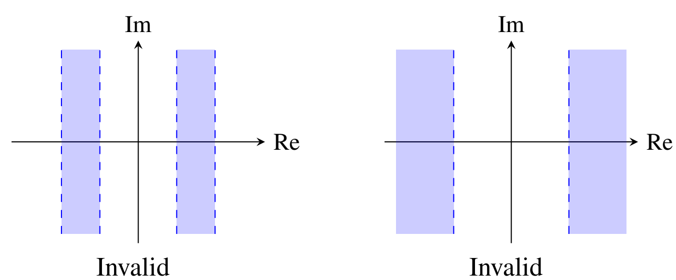

Figure 7.16: Examples of sets that would not be a valid ROC of a Laplace transform.

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every function can be classified as one of left sided but not right sided, right sided but not left sided, two sided (i.e., neither left nor right sided), or finite duration (i.e., both left and right sided), we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a left-half plane, a right-half plane, a (single) vertical strip, the entire complex plane, or the empty set. In particular, the ROC of X depends on the left- and right-sidedness of x as shown in Table 7.15. Thus, the ROC must be a connected region. (A set S is said to be connected, if for every two elements a and b in S, there exists a path from a to b that is contained in S.) That is, the ROC cannot consist of multiple (unconnected) vertical strips. For example, the sets shown in Figure 7.16 would not be valid as ROCs.

Example 7.7. The Laplace transform X of the function x has the algebraic expression

$$X(s) = \frac{s+\frac{1}{2}}{(s^2+2s+2)(s^2+s-2)}.$$

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding function x is left sided but not right sided, right sided but not left sided, two sided, or finite duration.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)(s-1)}.$$

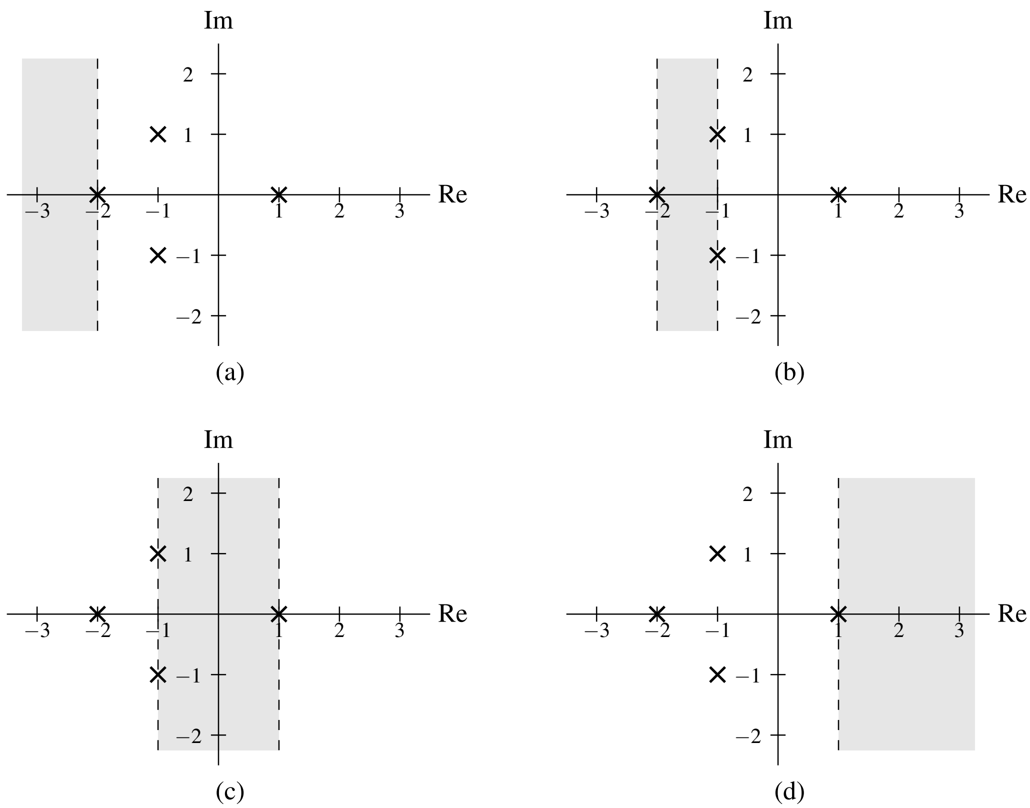

Figure 7.17: ROCs for example. The (a) first, (b) second, (c) third, and (d) fourth possible ROCs for X.

Thus, X has poles at -2, $-1-j$, $-1+j$, and 1. Since these poles only have three distinct real parts (namely, -2, -1, and 1), there are four possible ROCs:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$,
iii) $-1 < \operatorname{Re}(s) < 1$, and
iv) $\operatorname{Re}(s) > 1$.

These ROCs are plotted in Figures 7.17(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding x must be left sided but not right sided. The second ROC is a vertical strip, so the corresponding x must be two sided. The third ROC is a vertical strip, so the corresponding x must be two sided. The fourth ROC is a right-half plane, so the corresponding x must be right sided but not left sided. ■

7.8 Properties of the Laplace Transform

The Laplace transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 7.1 (on page 273). Also, for convenience, several Laplace-transform pairs are given later in Table 7.2 (on page 274). In what follows, we will sometimes refer to transform pairs in this table.

7.8.1 Linearity

Arguably, the most important property of the Laplace transform is linearity, as introduced below.

Theorem 7.1 (Linearity)**.** *If* $x_1(t) \overset{LT}{\longleftrightarrow} X_1(s)$ *with ROC* R_1 *and* $x_2(t) \overset{LT}{\longleftrightarrow} X_2(s)$ *with ROC* R_2*, then*

$$a_1x_1(t) + a_2x_2(t) \overset{LT}{\longleftrightarrow} a_1X_1(s) + a_2X_2(s) \;\text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

where a_1 *and* a_2 *are arbitrary complex constants.*

Proof. Let $y(t) = a_1x_1(t) + a_2x_2(t)$, and let Y denote the Laplace transform of y. Using the definition of the Laplace transform and straightforward algebraic manipulation, we have

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-st}dt \\ &= \int_{-\infty}^{\infty} a_1x_1(t)e^{-st}dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-st}dt \\ &= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}dt \\ &= a_1X_1(s) + a_2X_2(s). \end{aligned}$$

The ROC R can be deduced as follows. If X_1 and X_2 both converge at some point s, say $s = \lambda$, then any linear combination of these functions must also converge at $s = \lambda$. Therefore, the ROC R must contain the intersection of R_1 and R_2. Thus, we have shown that the linearity property holds. ■

In the preceding theorem, note that the ROC of the result can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $a_1X_1(s) + a_2X_2(s)$.

Example 7.8 (Linearity property of the Laplace transform)**.** Find the Laplace transform X of the function

$$x = x_1 + x_2,$$

where

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

Solution. Using Laplace transform pairs from Table 7.2, we have

$$\begin{aligned} X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) \\ &= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\ X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) \\ &= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) \\ &= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \\ &= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

So, from the definition of X, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 + x_2\}(s) \\ &= X_1(s) + X_2(s) \\ &= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2+1}{(s+1)(s+2)} \\ &= \frac{s+3}{(s+1)(s+2)}. \end{aligned}$$

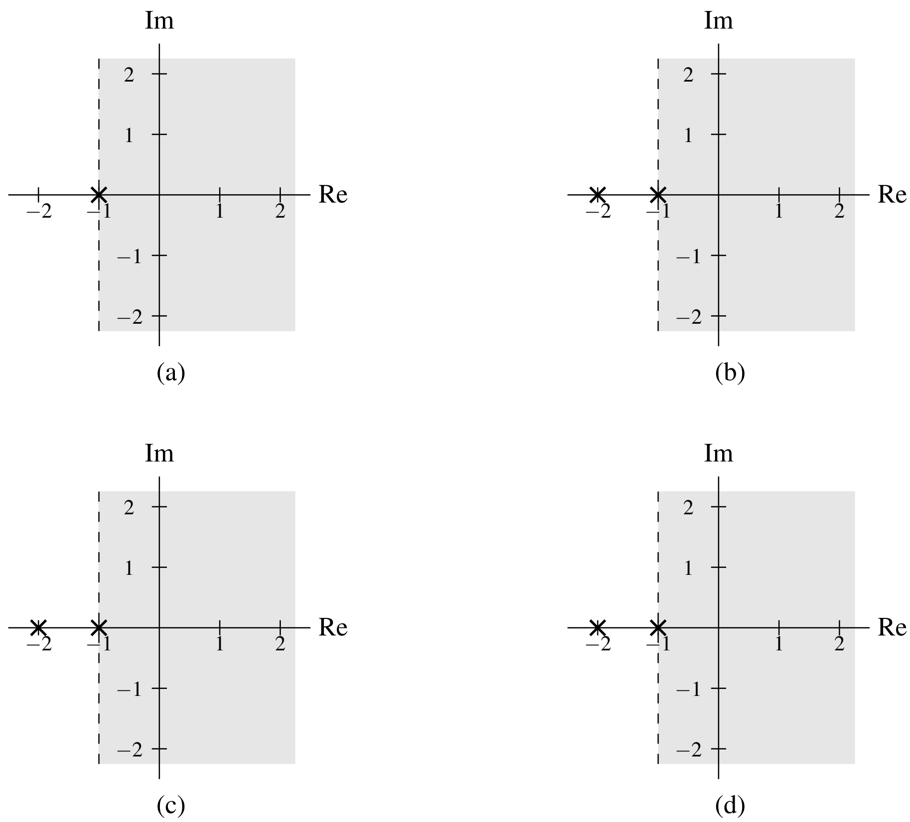

Figure 7.18: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Now, we must determine the ROC of X. We know that the ROC of X must contain the intersection of the ROCs of X_1 and X_2. So, the ROC must contain $\mathrm{Re}(s) > -1$. Furthermore, the ROC cannot be larger than this intersection, since X has a pole at -1. Therefore, the ROC of X is $\mathrm{Re}(s) > -1$. The various ROCs are illustrated in Figure 7.18. So, in conclusion, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}(s) > -1. \qquad \blacksquare$$

Example 7.9 (Linearity property of the Laplace transform and pole-zero cancellation)**.** Find the Laplace transform X of the function

$$x = x_1 - x_2,$$

where x_1 and x_2 are as defined in the previous example.

Solution. From the previous example, we know that

$$X_1(s) = \frac{1}{s+1} \quad \text{for } \mathrm{Re}(s) > -1 \quad \text{and}$$
$$X_2(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}(s) > -1.$$

From the definition of X, we have

$$\begin{aligned}X(s) &= \mathcal{L}\{x_1 - x_2\}(s) \\ &= X_1(s) - X_2(s) \\ &= \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2-1}{(s+1)(s+2)} \\ &= \frac{s+1}{(s+1)(s+2)} \\ &= \frac{1}{s+2}.\end{aligned}$$

Now, we must determine the ROC of X. We know that the ROC of X must at least contain the intersection of the ROCs of X_1 and X_2. Therefore, the ROC must contain $\mathrm{Re}(s) > -1$. Since X is rational, we also know that the ROC must be bounded by poles or extend to infinity. Since X has only one pole and this pole is at -2, the ROC must also include $-2 < \mathrm{Re}(s) < -1$. Therefore, the ROC of X is $\mathrm{Re}(s) > -2$. In effect, the pole at -1 has been cancelled by a zero at the same location. As a result, the ROC of X is larger than the intersection of the ROCs of X_1 and X_2. The various ROCs are illustrated in Figure 7.19. So, in conclusion, we have

$$X(s) = \frac{1}{s+2} \quad \text{for } \mathrm{Re}(s) > -2.$$ ■

7.8.2 Time-Domain Shifting

The next property of the Laplace transform to be introduced is the time-domain shifting property, as given below.

Theorem 7.2 (Time-domain shifting)**.** *If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then*

$$x(t-t_0) \overset{\text{LT}}{\longleftrightarrow} e^{-st_0}X(s) \;\; \textit{with ROC } R,$$

where t_0 is an arbitrary real constant.

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(t-t_0)$, and let Y denote the Laplace transform of y. From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt.$$

Now, we perform a change of variable. Let $\tau = t - t_0$ so that $t = \tau + t_0$ and $d\tau = dt$. Applying this change of variable, we obtain

$$\begin{aligned}Y(s) &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau \\ &= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau \\ &= e^{-st_0}X(s).\end{aligned}$$

The ROC of Y is the same as the ROC of X, since Y and X differ only by a finite constant factor (i.e., e^{-st_0}). Thus, we have proven that the time-domain shifting property holds. ■

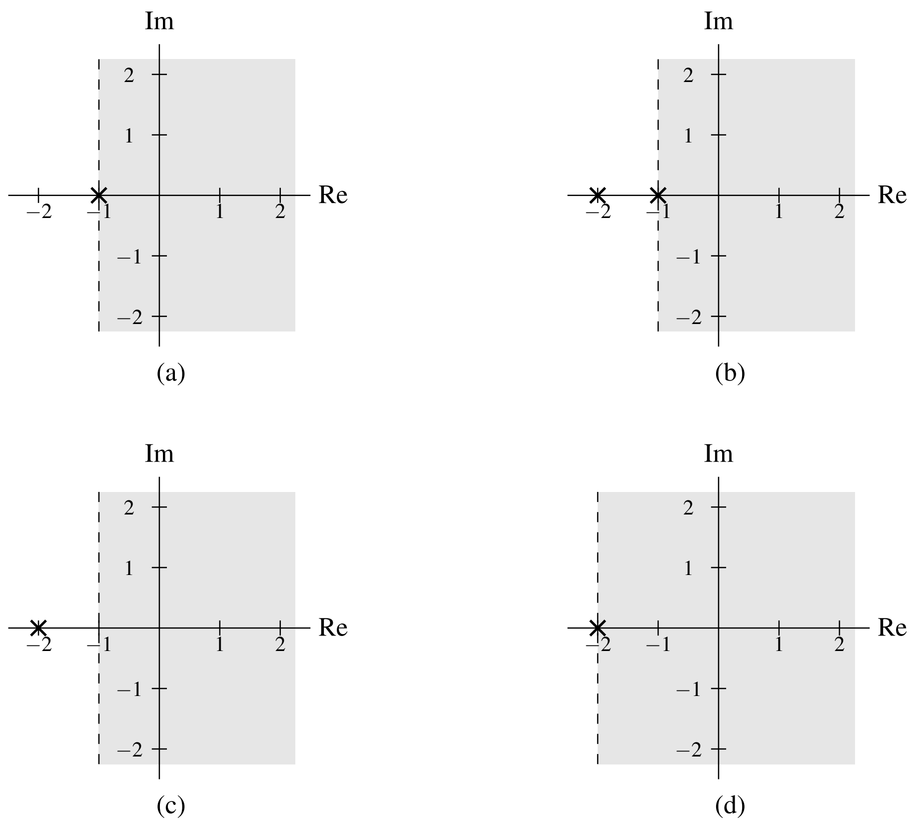

Figure 7.19: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Example 7.10 (Time-domain shifting property)**.** Find the Laplace transform X of

$$x(t) = u(t-1).$$

Solution. From Table 7.2, we know that

$$u(t) \overset{\text{LT}}{\longleftrightarrow} 1/s \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain shifting property, we can deduce

$$x(t) = u(t-1) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = e^{-s}\left(\tfrac{1}{s}\right) \text{ for } \operatorname{Re}(s) > 0.$$

Therefore, we have

$$X(s) = \frac{e^{-s}}{s} \text{ for } \operatorname{Re}(s) > 0. \qquad \blacksquare$$

7.8.3 Laplace-Domain Shifting

The next property of the Laplace transform to be introduced is the Laplace-domain shifting property, as given below.

Theorem 7.3 (Laplace-domain shifting)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$e^{s_0 t}x(t) \overset{LT}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R + \operatorname{Re}(s_0),$$

where s_0 is an arbitrary complex constant. The ROCs are illustrated in Figure 7.20.

Proof. To prove the above property, we proceed as follows. Let $y(t) = e^{s_0 t}x(t)$, and let Y denote the Laplace transform of y. Using the definition of the Laplace transform and straightforward manipulation, we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} e^{s_0 t}x(t)e^{-st}\,dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t}\,dt \\ &= X(s-s_0). \end{aligned}$$

Since $Y(s+s_0) = X(s)$, Y converges at $\lambda + s_0$ if and only if X converges at λ. Since the convergence properties of a Laplace transform only depend on the real part of the s parameter, Y converges at $\lambda + \operatorname{Re}(s_0)$ if and only if X converges at λ. Consequently, the ROC of Y is simply the ROC of X shifted by $\operatorname{Re}(s_0)$. Thus, we have shown that the Laplace-domain shifting property holds. $\blacksquare$

Example 7.11 (Laplace-domain shifting property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of

$$x(t) = e^{5t}e^{-|t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \text{ for } -1 < \operatorname{Re}(s) < 1.$$

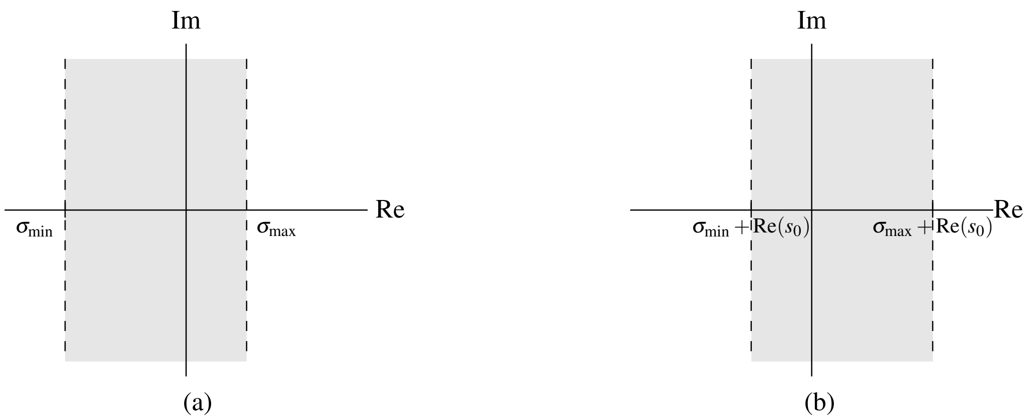

Figure 7.20: Regions of convergence for Laplace-domain shifting. (a) Before shift. (b) After shift.

Using the Laplace-domain shifting property, we can deduce

$$x(t) = e^{5t}e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{2}{1-(s-5)^2} \text{ for } -1+5 < \mathrm{Re}(s) < 1+5,$$

Thus, we have

$$X(s) = \frac{2}{1-(s-5)^2} \text{ for } 4 < \mathrm{Re}(s) < 6.$$

Rewriting X in factored form, we have

$$X(s) = \frac{2}{1-(s-5)^2} = \frac{2}{1-(s^2-10s+25)} = \frac{2}{-s^2+10s-24} = \frac{-2}{s^2-10s+24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

$$X(s) = \frac{-2}{(s-4)(s-6)} \text{ for } 4 < \mathrm{Re}(s) < 6. \qquad \blacksquare$$

7.8.4 Time-Domain/Laplace-Domain Scaling

The next property of the Laplace transform to be introduced is the time-domain/Laplace-domain scaling property, as given below.

Theorem 7.4 (Time-domain/Laplace-domain scaling)**.** *If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then*

$$x(at) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right) \text{ with ROC } R_1 = aR,$$

where a is a nonzero real constant.

Proof. To prove the above property, we proceed as below. Let $y(t) = x(at)$, and let Y denote that Laplace transform of y. From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(at)e^{-st}dt.$$

Now, we perform a change of variable. Let $\tau = at$ so that $t = \tau/a$ and $d\tau = adt$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$Y(s) = \begin{cases} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a > 0 \\ \int_{\infty}^{-\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a < 0 \end{cases}$$
$$= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a < 0. \end{cases}$$

Combining the two cases for a (i.e., $a > 0$ and $a < 0$), we obtain

$$Y(s) = \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau$$
$$= \frac{1}{|a|} X\left(\frac{s}{a}\right).$$

Since $|a|Y(as) = X(s)$, Y converges at $a\lambda$ if and only if X converges at λ. Thus, the ROC of Y is aR. Thus, we have proven that the scaling property holds. ■

The effect of time-domain scaling on the ROC of the Laplace transform is illustrated in Figure 7.21. Suppose that the ROC of the Laplace transform of a function x is as shown in Figure 7.21(a). Then, the ROC of the Laplace transform of the function $y(t) = x(at)$ is as shown in Figure 7.21(b) for the case that $a > 0$ and Figure 7.21(c) for the case that $a < 0$.

Example 7.12 (Time-domain scaling property)**.** Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of the function

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \text{ for } -1 < \operatorname{Re}(s) < 1.$$

Using the time-domain scaling property, we can deduce

$$x(t) = e^{-|3t|} \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{|3|} \frac{2}{1-(\frac{s}{3})^2} \quad \text{for } 3(-1) < \operatorname{Re}(s) < 3(1).$$

Thus, we have

$$X(s) = \frac{2}{3\left[1-(\frac{s}{3})^2\right]} \text{ for } -3 < \operatorname{Re}(s) < 3.$$

Simplifying, we have

$$X(s) = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \text{ for } -3 < \operatorname{Re}(s) < 3.$$ ■

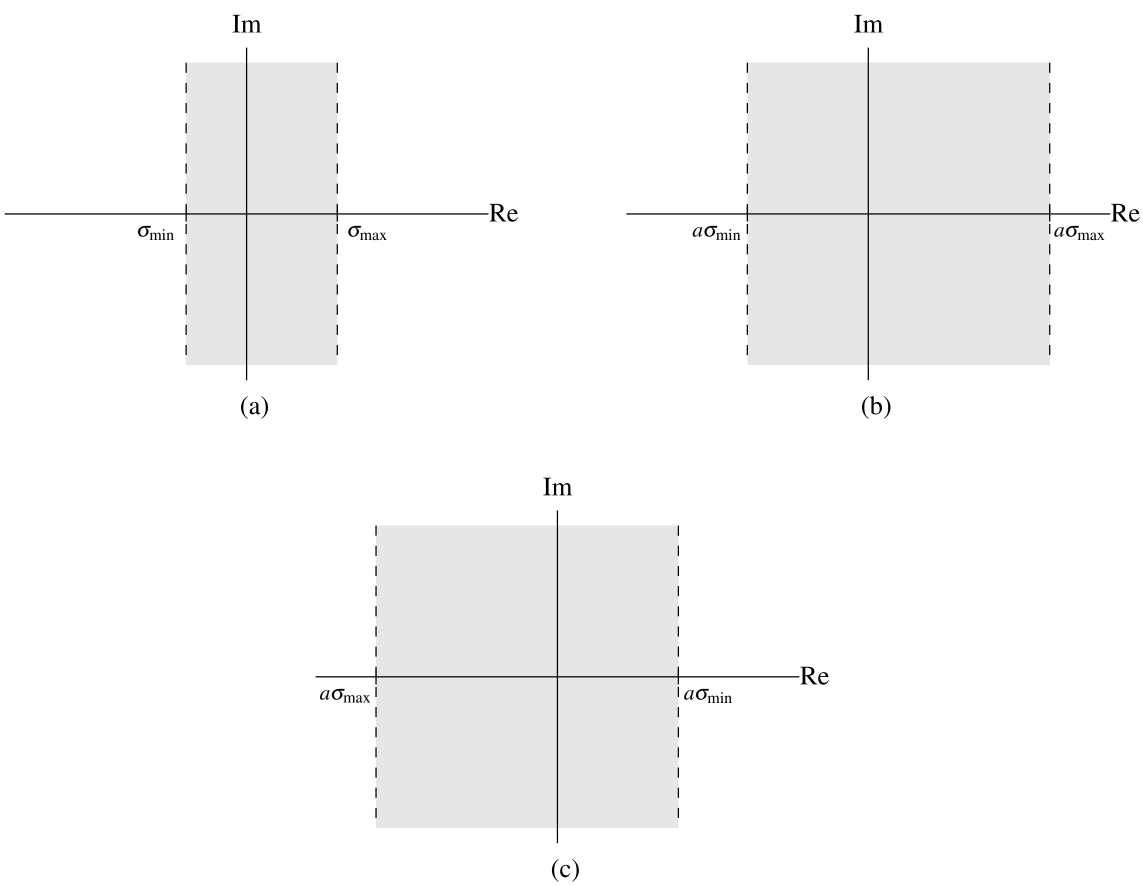

Figure 7.21: Regions of convergence for time-domain/Laplace-domain scaling. (a) Before scaling. After scaling for (b) $a > 0$ and (c) $a < 0$.

7.8.5 Conjugation

The next property of the Laplace transform to be introduced is the conjugation property, as given below.

Theorem 7.5 (Conjugation)**.** *If* $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ *with ROC R, then*

$$x^*(t) \overset{\text{LT}}{\longleftrightarrow} X^*(s^*) \quad \text{with ROC } R.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = x^*(t)$, let Y denote the Laplace transform of y, and let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform and the properties of conjugation, we can write

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty} x^*(t)e^{-st}dt \\
&= \left[\left(\int_{-\infty}^{\infty} x^*(t)e^{-st}dt\right)^*\right]^* \\
&= \left[\int_{-\infty}^{\infty} [x^*(t)]^* (e^{-st})^* dt\right]^* \\
&= \left[\int_{-\infty}^{\infty} x(t)(e^{-st})^* dt\right]^*.
\end{aligned}$$

Now, we observe that $(e^{-st})^* = e^{-s^*t}$. Thus, we can write

$$\begin{aligned}
Y(s) &= \left[\int_{-\infty}^{\infty} x(t)e^{-s^*t}dt\right]^* \\
&= X^*(s^*).
\end{aligned}$$

We determine the ROC of Y as follows. First, we observe that $X(s) = Y^*(s^*)$. Since $Y^*(s^*) = X(s)$, Y converges at λ if and only if X converges at λ^*. We know, however, that convergence only depends on the real part of λ. So, Y converges at λ if and only if X converges at λ. From these results, we have that the ROC of Y must be identical to the ROC of X. Thus, we have shown that the conjugation property holds. ∎

Example 7.13 (Conjugation property)**.** Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1+j} \quad \text{for } \operatorname{Re}(s) > -1,$$

find the Laplace transform X of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. To begin, let $v(t) = e^{(-1-j)t}u(t)$ (i.e., v is the function whose Laplace transform is given in the Laplace-transform pair above) and let V denote the Laplace transform of v. First, we determine the relationship between x and v. We have

$$\begin{aligned}
x(t) &= \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^* \\
&= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^* \\
&= \left[e^{(-1-j)t}u(t)\right]^* \\
&= v^*(t).
\end{aligned}$$

Thus, $x = v^*$. Next, we find the Laplace transform of x. We are given

$$v(t) = e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} V(s) = \frac{1}{s+1+j} \quad \text{for } \operatorname{Re}(s) > -1.$$

Using the conjugation property, we can deduce

$$x(t) = e^{(-1+j)t}u(t) \quad \stackrel{\text{LT}}{\longleftrightarrow} \quad X(s) = \left(\frac{1}{s^*+1+j}\right)^* \quad \text{for } \operatorname{Re}(s) > -1.$$

Simplifying the algebraic expression for X, we have

$$X(s) = \left(\frac{1}{s^*+1+j}\right)^* = \frac{1^*}{(s^*+1+j)^*} = \frac{1}{s+1-j}.$$

Therefore, we can conclude

$$X(s) = \frac{1}{s+1-j} \quad \text{for } \operatorname{Re}(s) > -1. \qquad \blacksquare$$

7.8.6 Time-Domain Convolution

The next property of the Laplace transform to be introduced is the time-domain convolution property, as given below.

Theorem 7.6 (Time-domain convolution)**.** *If* $x_1(t) \stackrel{LT}{\longleftrightarrow} X_1(s)$ *with ROC* R_1 *and* $x_2(t) \stackrel{LT}{\longleftrightarrow} X_2(s)$ *with ROC* R_2*, then*

$$x_1 * x_2(t) \stackrel{LT}{\longleftrightarrow} X_1(s)X_2(s) \quad \textit{with ROC } R \textit{ containing } R_1 \cap R_2.$$

Proof. To prove the above property, we proceed as below. Let $y(t) = x_1 * x_2(t)$, and let Y denote the Laplace transform of y. From the definition of the Laplace transform and convolution, we have

$$\begin{aligned}
Y(s) &= \mathcal{L}\left\{\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right\}(s) \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-st}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}d\tau dt.
\end{aligned}$$

Changing the order of integration, we have

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}dtd\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}dt\right]x_1(\tau)d\tau.
\end{aligned}$$

Now, we perform a change of variable. Let $v = t-\tau$ so that $t = v+\tau$ and $dv = dt$. Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-s(v+\tau)}dv\right]x_1(\tau)d\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]e^{-s\tau}x_1(\tau)d\tau \\
&= \left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]\left[\int_{-\infty}^{\infty} e^{-s\tau}x_1(\tau)d\tau\right] \\
&= X_1(s)X_2(s).
\end{aligned}$$

Now, we consider the ROC R of Y. If X_1 and X_2 both converge at some λ, then $Y(s) = X_1(s)X_2(s)$ must also converge at λ. Therefore, R must contain the intersection of R_1 and R_2. Thus, we have shown that the time-domain convolution property holds. $\blacksquare$

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $X_1(s)X_2(s)$.

The time-domain convolution property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts a convolution into a multiplication, the Laplace transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when working with (CT) LTI systems, for example, since such systems fundamentally involve convolution.

Example 7.14 (Time-domain convolution property)**.** Find the Laplace transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

Solution. From Table 7.2, we have that

$$x_1(t) = \sin(3t)u(t) \stackrel{\text{LT}}{\longleftrightarrow} X_1(s) = \frac{3}{s^2+9} \text{ for } \operatorname{Re}(s) > 0 \quad \text{and}$$
$$x_2(t) = tu(t) \stackrel{\text{LT}}{\longleftrightarrow} X_2(s) = \frac{1}{s^2} \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain convolution property, we have

$$x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) \text{ for } \{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of X is $\{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation occurs. Simplifying the expression for X, we conclude

$$X(s) = \frac{3}{s^2(s^2+9)} \text{ for } \operatorname{Re}(s) > 0.$$

■

7.8.7 Time-Domain Differentiation

The next property of the Laplace transform to be introduced is the time-domain differentiation property, as given below.

Theorem 7.7 (Time-domain differentiation)**.** *If $x(t) \stackrel{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$\frac{dx(t)}{dt} \stackrel{LT}{\longleftrightarrow} sX(s) \text{ with ROC } R' \text{ containing } R.$$

Proof. To prove the above property, we proceed as follows. Let $\mathcal{D}$ denote that derivative operator, let $y = \mathcal{D}x$, and let Y denote the Laplace transform of y. From the definition of the inverse Laplace transform, we have

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds.$$

Differentiating both sides of this equation with respect to t, we have

$$y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}\,ds.$$

Observing that the right-hand side of the above equation is simply the inverse Laplace transform of $sX(s)$, we can write

$$y(t) = \mathcal{L}^{-1}\{sX(s)\}(t).$$

Taking the Laplace transform of both sides yields

$$Y(s) = sX(s).$$

Now, we consider the ROC R' of $Y(s) = sX(s)$. Clearly, Y must converge at λ if X converges at λ. Since multiplication by s has the potential to cancel a pole in X, it is possible that R' may be larger than R (i.e., the ROC of X). Consequently, R' must at least contain R. Thus, we have shown that the time-domain differentiation property holds. ■

In the preceding theorem, note that the ROC R' can be larger than R. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $sX(s)$.

The time-domain differentiation property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts differentiation into multiplication (by s), the Laplace transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations, for example.

Example 7.15 (Time-domain differentiation property)**.** Find the Laplace transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. From Table 7.2, we have that

$$\delta(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad 1 \;\text{ for all } s.$$

Using the time-domain differentiation property, we can deduce

$$x(t) = \tfrac{d}{dt}\delta(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = s(1) \;\text{ for all } s.$$

Therefore, we have

$$X(s) = s \quad \text{for all } s.$$

■

7.8.8 Laplace-Domain Differentiation

The next property of the Laplace transform to be introduced is the Laplace-domain differentiation property, as given below.

Theorem 7.8 (Laplace-domain differentiation)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$-tx(t) \overset{LT}{\longleftrightarrow} \tfrac{d}{ds}X(s) \;\text{ with ROC } R.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = -tx(t)$ and let Y denote the Laplace transform of y. From the definition of the Laplace transform, we have

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

Differentiating both sides of the above equation with respect to s, we obtain

$$\begin{aligned} \tfrac{d}{ds}X(s) &= \int_{-\infty}^{\infty} -tx(t)e^{-st}dt \\ &= Y(s). \end{aligned}$$

Thus, we have shown that the Laplace-domain differentiation property holds. ■

Example 7.16 (Laplace-domain differentiation property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform X of the function

$$x(t) = te^{-2t}u(t).$$

Solution. We are given

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

$$x(t) = te^{-2t}u(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) \quad \text{for } \operatorname{Re}(s) > -2.$$

Simplifying the algebraic expression for X, we have

$$X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) = -\tfrac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

7.8.9 Time-Domain Integration

The next property of the Laplace transform to be introduced is the time-domain integration property, as given below.

Theorem 7.9 (Time-domain integration)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{LT}{\longleftrightarrow} \frac{1}{s}X(s) \;\; \textit{with ROC } R' \textit{ containing } R \cap \{\operatorname{Re}(s) > 0\}.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, and let Y and U denote the Laplace transforms of y and u, respectively. First, we observe that

$$y(t) = x * u(t).$$

Taking the Laplace transform of both sides of this equation, we have

$$Y(s) = \mathcal{L}\{x * u\}(s).$$

From the time-domain convolution property of the Laplace transform, we have

$$\begin{aligned} Y(s) &= \mathcal{L}x(s)\mathcal{L}u(s) \\ &= X(s)U(s). \end{aligned}$$

From Example 7.5, we know that $U(s) = \frac{1}{s}$ for $\operatorname{Re}(s) > 0$. So,

$$Y(s) = \frac{1}{s}X(s).$$

Now, we need to consider the ROC R' of Y. Clearly, Y must converge at λ if X and U both converge at λ. Consequently, R' must contain the intersection of the ROCs of X and U. Since U converges for $\operatorname{Re}(s) > 0$, R' must contain $R \cap (\operatorname{Re}(s) > 0)$. Thus, we have shown that the time-domain integration property holds. $\blacksquare$

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{\mathrm{Re}(s) > 0\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $\frac{1}{s}X(s)$.

The time-domain integration property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts integration into division (by s), the Laplace transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations, for example.

Example 7.17 (Time-domain integration property)**.** Find the Laplace transform X of the function

$$x(t) = \int_{-\infty}^{t} e^{-2\tau} \sin(\tau) u(\tau) d\tau.$$

Solution. From Table 7.2, we have that

$$e^{-2t} \sin(t) u(t) \stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{(s+2)^2+1} \quad \text{for } \mathrm{Re}(s) > -2.$$

Using the time-domain integration property, we can deduce

$$x(t) = \int_{-\infty}^{t} e^{-2\tau} \sin(\tau) u(\tau) d\tau \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{s}\left[\frac{1}{(s+2)^2+1}\right] \quad \text{for } \{\mathrm{Re}(s) > -2\} \cap \{\mathrm{Re}(s) > 0\}.$$

The ROC of X is $\{\mathrm{Re}(s) > -2\} \cap \{\mathrm{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for X, we have

$$X(s) = \frac{1}{s}\left[\frac{1}{(s+2)^2+1}\right] = \frac{1}{s}\left(\frac{1}{s^2+4s+4+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+5}\right).$$

Therefore, we have

$$X(s) = \frac{1}{s(s^2+4s+5)} \quad \text{for } \mathrm{Re}(s) > 0.$$

[Note: $s^2+4s+5 = (s+2-j)(s+2+j)$.] ■

7.8.10 Initial and Final Value Theorems

The next properties of the Laplace transform to be introduced are known as the initial and final value theorems, as given below.

Theorem 7.10 (Initial value theorem)**.** *Let x be a function with the Laplace transform X. If x is causal and contains no impulses or higher order singularities at the origin, then*

$$x(0^+) = \lim_{s \to \infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.

Proof. To prove the above property, we proceed as below. First, we expand x as a Taylor series at 0^+.

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)t + \ldots + x^{(n)}(0^+)\frac{t^n}{n!} + \ldots\right] u(t),$$

where $x^{(n)}$ denotes the nth derivative of x. Taking the Laplace transform of the above equation, and using the fact that $\mathcal{L}\{t^n u(t)\}(s) = \frac{n!}{s^{n+1}}$, we can write

$$X(s) = x(0^+)\frac{1}{s} + x^{(1)}(0^+)\frac{1}{s^2} + \ldots + x^{(n)}(0^+)\frac{1}{s^{n+1}} + \ldots.$$

Multiplying both sides of this equation by s, we obtain

$$sX(s) = x(0^+) + x^{(1)}(0^+)\frac{1}{s} + \ldots + x^{(n)}(0^+)\frac{1}{s^n} + \ldots.$$

Taking the limit as $s \to \infty$, we have

$$\lim_{s\to\infty} sX(s) = x(0^+).$$

Thus, the initial value theorem holds. ■

Theorem 7.11 (Final value theorem)**.** *Let x be a function with the Laplace transform X. If x is causal and $x(t)$ has a finite limit as $t \to \infty$, then*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

Proof. Let x' denote the derivative of x. The differentiation property of the unilateral Laplace transform (introduced later in Section 7.17) states that $s\mathcal{L}_u x(s) - x(0^-) = \mathcal{L}_u x'(s)$. Furthermore, since x is causal, this property can be rewritten as $s\mathcal{L}x(s) - x(0^-) = \mathcal{L}_u x'(s)$. Thus, we have that

$$sX(s) - x(0^-) = \int_{0^-}^{\infty} x'(t)e^{-st}dt.$$

Taking the limit of both sides of this equation as $s \to 0$, we obtain

$$\begin{aligned}
\lim_{s\to 0}\left[sX(s) - x(0^-)\right] &= \lim_{s\to 0}\int_{0^-}^{\infty} x'(t)e^{-st}dt \\
&= \int_{0^-}^{\infty} x'(t)dt \\
&= x(t)|_{0^-}^{\infty} \\
&= \lim_{t\to\infty} x(t) - x(0^-).
\end{aligned}$$

Thus, we have

$$\lim_{s\to 0} sX(s) - x(0^-) = \lim_{t\to\infty} x(t) - x(0^-).$$

Adding $x(0^-)$ to both sides of the equation yields

$$\lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t).$$ ■

Example 7.18 (Initial and final value theorems)**.** A bounded causal function x with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s} \quad \text{for } \mathrm{Re}(s) > 0.$$

Determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Solution. Since x is causal and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) \\
&= \lim_{s\to\infty} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] \\
&= \lim_{s\to\infty} \frac{2s^2+3s+2}{s^2+2s+2} \\
&= 2.
\end{aligned}$$

Since x is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\ &= \lim_{s\to 0} s\left(\frac{2s^2+3s+2}{s^3+2s^2+2s}\right) \\ &= \left.\frac{2s^2+3s+2}{s^2+2s+2}\right|_{s=0} \\ &= 1.\end{aligned}$$

In passing, we note that the inverse Laplace transform x of X can be shown to be

$$x(t) = [1+e^{-t}\cos t]u(t).$$

As we would expect, the values calculated above for $x(0^+)$ and $\lim_{t\to\infty} x(t)$ are consistent with this formula for x. ■

Amongst other things, the initial and final value theorems can be quite useful in checking for errors in Laplace transform calculations. For example, suppose that we are asked to compute the Laplace transform X of the function x. If we were to make a mistake in this computation, the values obtained for $x(0)$ and $\lim_{t\to\infty} x(t)$ using X with the initial and final value theorems and using x directly would most likely disagree. In this manner, we can relatively easily detect some types of errors in Laplace transform calculations.

7.9 More Laplace Transform Examples

Earlier in this chapter, we derived a number of Laplace transform pairs. Some of these and other important transform pairs are listed in Table 7.2. Using the various Laplace transform properties listed in Table 7.1 and the Laplace transform pairs listed in Table 7.2, we can more easily determine the Laplace transform of more complicated functions.

Example 7.19. Using properties of the Laplace transform and the Laplace transform pair

$$e^{-a|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{-2a}{(s+a)(s-a)} \quad \text{for } -a < \operatorname{Re}(s) < a,$$

find the Laplace transform X of the function

$$x(t) = e^{-5|3t-7|}.$$

Solution. We begin by re-expressing x in terms of the following equations:

$$\begin{aligned} v_1(t) &= e^{-5|t|}, \\ v_2(t) &= v_1(t-7), \quad \text{and} \\ x(t) &= v_2(3t). \end{aligned}$$

In what follows, let R_{V_1}, R_{V_2}, and R_X denote the ROCs of V_1, V_2, and X, respectively. Taking the Laplace transform of the above three equations, we obtain

$$\begin{aligned} V_1(s) &= \frac{-10}{(s+5)(s-5)}, \quad R_{V_1} = (-5 < \operatorname{Re}(s) < 5), \\ V_2(s) &= e^{-7s}V_1(s), \quad R_{V_2} = R_{V_1}, \\ X(s) &= \tfrac{1}{3}V_2(s/3), \quad \text{and} \quad R_X = 3R_{V_2}. \end{aligned}$$

Table 7.1: Properties of the (bilateral) Laplace transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\operatorname{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\operatorname{Re}(s) > 0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Table 7.2: Transform pairs for the (bilateral) Laplace transform

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) < -a$
10	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
11	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
12	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$
13	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$

Combining the above equations, we have

$$\begin{aligned}
X(s) &= \tfrac{1}{3}V_2(s/3) \\
&= \tfrac{1}{3}e^{-7(s/3)}V_1(s/3) \\
&= \tfrac{1}{3}e^{-7s/3}V_1(s/3) \\
&= \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
R_X &= 3R_{V_2} \\
&= 3R_{V_1} \\
&= 3\{-5 < \operatorname{Re}(s) < 5\} \\
&= \{-15 < \operatorname{Re}(s) < 15\}.
\end{aligned}$$

Thus, we have shown that

$$X(s) = \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \text{ for } -15 < \operatorname{Re}(s) < 15. \qquad \blacksquare$$

Example 7.20. Find the Laplace transform X of the function

$$x(t) = [e^{-t} + e^{-2t}]u(t).$$

Solution. We can calculate X as

$$\begin{aligned}
X(s) &= \mathcal{L}\{[e^{-t}+e^{-2t}]u(t)\}(s) \\
&= \mathcal{L}\{e^{-t}u(t)\}(s) + \mathcal{L}\{e^{-2t}u(t)\}(s) \\
&= \frac{1}{s+1} + \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \cap \operatorname{Re}(s) > -2 \\
&= \frac{s+2+s+1}{(s+1)(s+2)} \\
&= \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.
\end{aligned}$$

Thus, we have that

$$X(s) = \frac{2s+3}{(s+1)(s+2)} \text{ for } \operatorname{Re}(s) > -1. \qquad \blacksquare$$

Example 7.21. Find the Laplace transform X of the function

$$x(t) = [e^{-2t} + e^{-3t}]u(t-1).$$

Solution. To begin, we rewrite x as

$$x(t) = [e^{-2t} + e^{-3t}]v_1(t)$$

where

$$v_1(t) = u(t-1).$$

Taking the Laplace transform of the preceding equations, we have

$$\begin{aligned} V_1(s) &= \mathcal{L}\{u(t-1)\}(s) \\ &= e^{-s}\left(\tfrac{1}{s}\right) \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

and

$$\begin{aligned} X(s) &= \mathcal{L}\{[e^{-2t}+e^{-3t}]v_1(t)\}(s) \\ &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s). \end{aligned}$$

Now, we focus on simplifying the preceding expression for X. Let R_{V_1} denote the ROC of V_1. Then, we have

$$\begin{aligned} \mathcal{L}\{e^{-2t}v_1(t)\}(s) &= V_1(s+2) \quad \text{for } s \in R_{V_1} - 2 \quad \text{and} \\ \mathcal{L}\{e^{-3t}v_1(t)\}(s) &= V_1(s+3) \quad \text{for } s \in R_{V_1} - 3. \end{aligned}$$

Thus, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s) \\ &= V_1(s+2) + V_1(s+3) \quad \text{for } (R_{V_1}-2) \cap (R_{V_1}-3). \end{aligned}$$

Substituting the earlier expression for V_1 into the above equation yields

$$\begin{aligned} X(s) &= e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } (\operatorname{Re}(s) > -2) \cap (\operatorname{Re}(s) > -3) \\ &= e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } \operatorname{Re}(s) > -2. \end{aligned}$$

Thus, we have that

$$X(s) = e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \ \text{ for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

Example 7.22. Find the Laplace transform X of the function

$$x(t) = \delta(t) + u(t).$$

Solution. Taking the Laplace transform of x, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{\delta + u\}(s) \\ &= \mathcal{L}\delta(s) + \mathcal{L}u(s). \end{aligned}$$

From Table 7.2, we have

$$\begin{aligned} \delta(t) &\overset{\text{LT}}{\longleftrightarrow} 1 \quad \text{for all } s \quad \text{and} \\ u(t) &\overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

Substituting the above results, we obtain

$$\begin{aligned} X(s) &= 1 + \tfrac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0 \\ &= \frac{s+1}{s}. \end{aligned}$$

Thus, we have that

$$X(s) = \frac{s+1}{s} \ \text{ for } \operatorname{Re}(s) > 0. \qquad \blacksquare$$

Example 7.23. Find the Laplace transform X of the function

$$x(t) = te^{-3|t|}.$$

Solution. To begin, we rewrite x as

$$x(t) = te^{3t}u(-t) + te^{-3t}u(t).$$

Taking the Laplace transform of x, we have

$$\begin{aligned}
X(s) &= \mathcal{L}\{te^{3t}u(-t) + te^{-3t}u(t)\}(s) \\
&= \mathcal{L}\{te^{3t}u(-t)\}(s) + \mathcal{L}\{te^{-3t}u(t)\}(s) \\
&= \frac{-(1!)}{(s-3)^2} + \frac{1!}{(s+3)^2} \quad \text{for } \operatorname{Re}(s) > -3 \cap \operatorname{Re}(s) < 3 \\
&= \frac{-(s+3)^2 + (s-3)^2}{(s+3)^2(s-3)^2} \\
&= \frac{-(s^2+6s+9) + s^2 - 6s + 9}{(s+3)^2(s-3)^2} \\
&= \frac{-12s}{(s+3)^2(s-3)^2}.
\end{aligned}$$

Thus, we have that

$$X(s) = \frac{-12s}{(s+3)^2(s-3)^2} \quad \text{for } -3 < \operatorname{Re}(s) < 3.$$

■

Example 7.24. Consider the function

$$y(t) = e^{-t}[x * x(t)].$$

Let X and Y denote the Laplace transforms of x and y, respectively. Find an expression for Y in terms of X.

Solution. To begin, we rewrite y as

$$y(t) = e^{-t}v_1(t)$$

where

$$v_1(t) = x * x(t).$$

Let V_1 denote the Laplace transform of v. Let R_X, R_Y, and R_V denote the ROCs of X, Y, and V, respectively. Taking the Laplace transform of the above equations yields

$$\begin{aligned}
V_1(s) &= \mathcal{L}v_1(s) \\
&= \mathcal{L}\{x * x\}(s) \\
&= \mathcal{L}x(s)\mathcal{L}x(s) \\
&= X^2(s) \quad \text{for } s \in R_X \quad \text{and} \\
Y(s) &= \mathcal{L}\{e^{-t}v_1(t)\}(s) \\
&= V_1(s+1) \quad \text{for } s \in R_{V_1} - 1.
\end{aligned}$$

Substituting the above expression for V_1 into the above formula for Y, we have

$$\begin{aligned}
Y(s) &= V_1(s+1) \\
&= X^2(s+1) \quad \text{for } s \in R_X - 1.
\end{aligned}$$

■

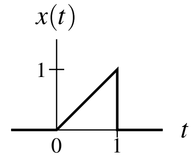

Figure 7.22: Function for the Laplace transform example.

Example 7.25. Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform X of the function x shown in Figure 7.22.

First solution (which incurs more work due to differentiation). First, we express x using unit-step functions. We have

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{tu(t-1)\}(s).$$

We have

$$\mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2} \quad \text{and}$$

$$\begin{aligned} \mathcal{L}\{tu(t-1)\}(s) = -\mathcal{L}\{-tu(t-1)\}(s) &= -\tfrac{d}{ds}\left(\frac{e^{-s}}{s}\right) \\ &= -\left(\tfrac{d}{ds}e^{-s}s^{-1}\right) \\ &= -(-e^{-s}s^{-1} - s^{-2}e^{-s}) \\ &= \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}. \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \left[\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \\ &= \frac{1 - se^{-s} - e^{-s}}{s^2}. \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane.

Second solution (which incurs less work by avoiding differentiation). First, we express x using unit-step functions to yield

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

To simplify the subsequent Laplace transform calculation, we choose to rewrite x as

$$\begin{aligned} x(t) &= tu(t) - tu(t-1) + u(t-1) - u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1). \end{aligned}$$

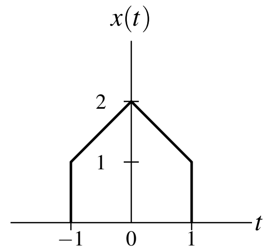

Figure 7.23: Function for the Laplace transform example.

(This is motivated by a preference to compute the Laplace transform of $(t-1)u(t-1)$ instead of $tu(t-1)$.) Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{(t-1)u(t-1)\}(s) - \mathcal{L}\{u(t-1)\}(s).$$

We have

$$\begin{aligned}
\mathcal{L}\{tu(t)\}(s) &= \frac{1}{s^2}, \\
\mathcal{L}\{(t-1)u(t-1)\}(s) &= e^{-s}\mathcal{L}\{tu(t)\}(s) \\
&= e^{-s}\left(\frac{1}{s^2}\right) \\
&= \frac{e^{-s}}{s^2}, \quad \text{and} \\
\mathcal{L}\{u(t-1)\}(s) &= e^{-s}\mathcal{L}\{u(t)\}(s) \\
&= e^{-s}\left(\frac{1}{s}\right) \\
&= \frac{e^{-s}}{s}.
\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}
X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
&= \frac{1 - e^{-s} - se^{-s}}{s^2}.
\end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane. ■

Example 7.26. Find the Laplace transform X of the function x shown in Figure 7.23.

First solution (which incurs more work due to differentiation). First, we express x using unit-step functions. We have

$$\begin{aligned}
x(t) &= (t+2)[u(t+1) - u(t)] + (-t+2)[u(t) - u(t-1)] \\
&= tu(t+1) - tu(t) + 2u(t+1) - 2u(t) - tu(t) + tu(t-1) + 2u(t) - 2u(t-1) \\
&= tu(t+1) + 2u(t+1) - 2tu(t) + tu(t-1) - 2u(t-1).
\end{aligned}$$

