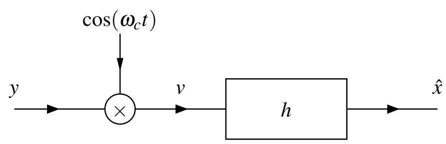
6.20 Let x be a real function with Fourier transform X satisfying $X(\omega) = 0$ for $|\omega| > \omega_b$. We use amplitude modulation to produce the function $y(t) = x(t)\sin(\omega_c t)$. Note that $\omega_c \gg \omega_b$. In order to recover the original function x, it is proposed that the system shown in the figure below be used. This system contains a LTI subsystem that is labelled with its impulse response h. Let Y, V, $\hat{X}$, and H denote the Fourier transforms of y, v, $\hat{x}$, and h, respectively. The system is such that

$$H(\omega) = \begin{cases} 2 & |\omega| < \omega_b \\ 0 & \text{otherwise.} \end{cases}$$

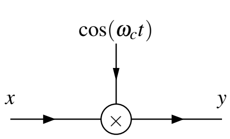


(a) Find an expression for Y in terms of X. Find an expression for $\hat{X}$ in terms of V. Find an expression for $\hat{X}$ in terms of X.
(b) Compare $\hat{x}$ and x. Comment on the utility of the proposed system.

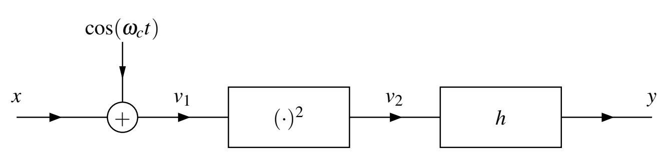
6.21 In this exercise, we consider the effect of a loss of phase synchronization between the carrier signals used in the transmitter and receiver of a DSB-SC AM system. In what follows, we assume that the signal to be transmitted is always bandlimited to frequencies in $[-\omega_M, \omega_M]$. The transmitter with input x and output y is characterized by the equation $y(t) = \cos(\omega_c t)x(t)$. The receiver with input y and output $\tilde{x}$ is characterized by the equation $\tilde{x}(t) = v * h(t)$, where $v(t) = \cos(\omega_c t - \theta)y(t)$ and h is the impulse response of an ideal lowpass filter with a passband gain of 2 and a cutoff frequency ω_{c0} satisfying $\omega_M < \omega_{c0} < 2\omega_c - \omega_M$. (In other words, we have a DSB/SC system similar to the one in Section 6.19.2, except that the receiver uses a carrier signal that has been phase shifted by θ.) Find $\tilde{x}$ in terms of x.

Short Answer. $\tilde{x}(t) = \cos(\theta)x(t)$

6.22 When discussing DSB/SC amplitude modulation, we saw that a system of the form shown below in Figure A is often useful. In practice, however, the multiplier unit needed by this system is not always easy to implement. For this reason, we sometimes employ a system like that shown below in Figure B. In this second system, we sum the sinusoidal carrier and modulating signal x and then pass the result through a nonlinear squaring device (i.e., $v_2(t) = v_1^2(t)$). This system also contains a LTI subsystem with impulse response h.



(a)



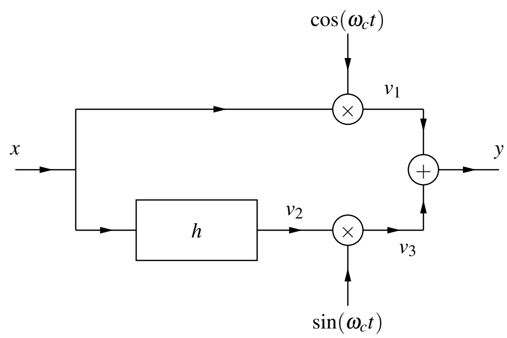
(b)

Let X, V_1, V_2, and H denote the Fourier transforms of x, v_1, v_2, and h, respectively. Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$ (i.e., x is bandlimited).
(a) Find an expression for v_1, v_2, and V_2. (Hint: If $X(\omega) = 0$ for $|\omega| > \omega_b$, then using the time-domain convolution property of the Fourier transform, we can deduce that the Fourier transform of x^2 is zero for $|\omega| > 2\omega_b$.)
(b) Determine the frequency response H required for the system shown in Figure B to be equivalent to the system in Figure A. State any assumptions made with regard to the relationship between ω_c and ω_b. (Hint: It might be helpful to sketch X and V_2 for the case of some simple X. Then, compare V_2 to X in order to deduce your answer.)

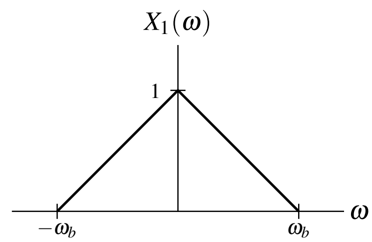
6.23 Consider the system with input x and output y as shown in Figure A below. The impulse response h is that of an ideal Hilbert transformer, whose frequency response H is given by

$$H(\omega) = -j \operatorname{sgn} \omega.$$

Let X, Y, V_1, V_2, and V_3 denote the Fourier transforms of x, y, v_1, v_2, and v_3, respectively.



(a)



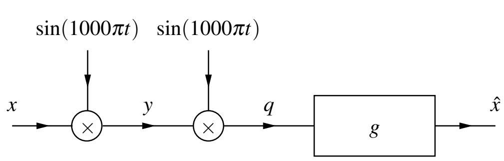
(b)

(a) Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$, where $\omega_b \ll \omega_c$. Find expressions for V_1, V_2, V_3, and Y in terms of X.
(b) Suppose that $X = X_1$, where X_1 is as shown in Figure B. Sketch V_1, V_2, V_3, and Y in this case.
(c) Draw the block diagram of a system that could be used to recover x from y.

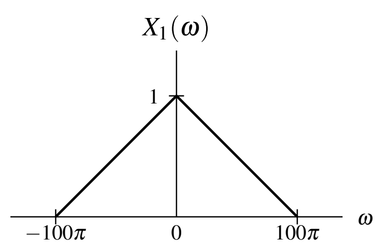
6.24 Consider the system shown below in Figure A with input x and output $\hat{x}$, where this system contains a LTI subsystem with impulse response g. The Fourier transform G of g is given by

$$G(\omega) = \begin{cases} 2 & |\omega| \leq 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let X, $\hat{X}$, Y, and Q denote the Fourier transforms of x, $\hat{x}$, y, and q, respectively.



(a)



(b)

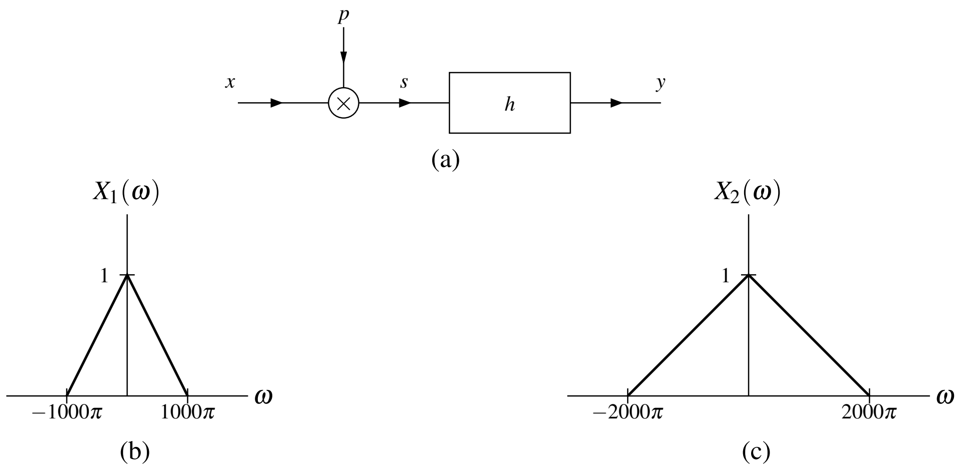
(a) Suppose that $X(\omega) = 0$ for $|\omega| > 100\pi$. Find expressions for Y, Q, and $\hat{X}$ in terms of X.
(b) If $X = X_1$ where X_1 is as shown in Figure B, sketch Y, Q, and $\hat{X}$.

6.25 The bandwidth of a LTI system is most simply defined as the bandwidth of the system's frequency response H. Explain why a (LTI) communication channel with (finite) bandwidth B cannot be used to (reliably) transmit a signal with bandwidth greater than B.

6.26 Consider the system shown below in Figure A with input x and output y. Let X, P, S, H, and Y denote the Fourier transforms of x, p, s, h, and y, respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - \tfrac{n}{1000}\right) \quad \text{and} \quad H(\omega) = \tfrac{1}{1000}\operatorname{rect}\left(\tfrac{\omega}{2000\pi}\right).$$

(a) Derive an expression for S in terms of X. Derive an expression for Y in terms of S and H.
(b) Suppose that $X = X_1$, where X_1 is as shown in Figure B. Using the results of part (a), plot S and Y. Indicate the relationship (if any) between the input x and output y of the system.
(c) Suppose that $X = X_2$, where X_2 is as shown in Figure C. Using the results of part (a), plot S and Y. Indicate the relationship (if any) between the input x and output y of the system.



6.27 A function x is bandlimited to 22 kHz (i.e., only has spectral content for frequencies f in the range $[-22000, 22000]$). Due to excessive noise, the portion of the spectrum that corresponds to frequencies f satisfying $|f| > 20000$ has been badly corrupted and rendered useless. (a) Determine the minimum sampling rate for x that would allow the uncorrupted part of the spectrum to be recovered. (b) Suppose now that the corrupted part of the spectrum were eliminated by filtering prior to sampling. In this case, determine the minimum sampling rate for x.

6.28 A function x is bandlimited for frequencies in the range $[-B, B]$. Find the lowest rate at which the function $y(t) = x^2(t)$ can be sampled such that aliasing does not occur.

6.29 Let y_1 and y_2 be functions bandlimited to frequencies in the range $[-\omega_b, \omega_b]$. Suppose that these functions are sampled at a frequency ω_s satisfying the Nyquist condition to produce the sequences

$$x_1(n) = y_1(Tn) \quad \text{and} \quad x_2(n) = y_2(Tn),$$

where $T = \frac{2\pi}{\omega_s}$. Now, consider the function $y = y_1 * y_2$. Suppose that y is also sampled with period T to produce the sequence

$$x(n) = y(Tn).$$

(a) Show that y is bandlimited to frequencies in the range $[-\omega_b, \omega_b]$, meaning that it must be possible to recover y exactly from its samples.
(b) Show that the samples of y can be computed by

$$x(n) = \frac{2\pi}{\omega_s} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k).$$

(c) Explain how we might use the above results to compute the (continuous-time) convolution of bandlimited functions using a (discrete-time) computer.

6.30 Suppose that we want to transmit a binary sequence y (where $y(n)$ is either zero or one) over a continuous-time communication channel. To do this, we choose to represent each binary sample of y with a continuous-time pulse. Using the sampling theorem, show that it is possible to transmit the bits of y at a rate of $2B$ bits per second over an ideal (i.e., noise-free) channel of bandwidth B. As it turns out, this is the theoretical upper bound on the data transmission rate, assuming that each pulse is used to represent only a single bit of data. (Hint: According to the sampling theorem, a continuous-time function of bandwidth B can be constructed from $2B$ pieces of information.)

6.21.2 Exercises With Answer Key

6.101 Using the Fourier transform analysis equation, find the Fourier transform X of each function x below.
(a) $x(t) = \delta(t+a) + \delta(t-a)$, where a is a real constant;
(b) $x(t) = \operatorname{rect}\left(\frac{t}{T}\right)$, where T is a nonzero real constant. (Hint: Be careful to correctly consider the case that $T < 0$.)

Short Answer. (a) $X(\omega) = 2\cos(a\omega)$; (b) $X(\omega) = |T| \operatorname{sinc}\left(\frac{T\omega}{2}\right)$

6.102 Using the Fourier transform synthesis equation, find the inverse Fourier transform x of the function

$$X(\omega) = \operatorname{rect}\left(\frac{\omega}{2B}\right),$$

where B is a nonzero real constant. (Hint: Be careful to correctly consider the case that $B < 0$.)

Short Answer. $x(t) = \frac{|B|}{\pi} \operatorname{sinc}(Bt)$

6.103 Using properties of the Fourier transform and a table of Fourier transform pairs, find the Fourier transform X of each function x given below.
(a) $x(t) = \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right]$;
(b) $x(t) = e^{-j2t} \operatorname{sgn}(-t-1)$.
(c) $x(t) = e^{-j2t} \frac{1}{3t+1}$;
(d) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1) d\tau$;
(e) $x(t) = (t+1)\sin(5t-3)$;
(f) $x(t) = \sin(2\pi t)\delta(t-\frac{\pi}{2})$;
(g) $x(t) = e^{-jt} \frac{1}{3t-2}$;
(h) $x(t) = e^{j5t}\cos(2t)u(t)$;
(i) $x(t) = \operatorname{sinc}^2(at)$, where a is a nonzero real constant; and
(j) $x(t) = x_1 * x_2(t)$, where $x_1(t) = t^2 e^{-t} u(t)$ and $x_2(t) = (t-1)e^{-(t-1)} u(t-1)$.

Short Answer.
(a) $X(\omega) = u(\omega)$;
(b) $X(\omega) = e^{j(\omega+2)} \frac{j2}{\omega+2}$;
(c) $X(\omega) = -\frac{j\pi}{3} e^{j(\omega+2)/3} \operatorname{sgn}(\omega+2)$;
(d) $X(\omega) = \frac{1}{e^2}\left[\left(\frac{5}{j5\omega-\omega^2}\right) e^{-j\omega/5} + \pi\delta(\omega)\right]$;

(e) $X(\omega) = j\pi\left[e^{j3}\delta(\omega+5) - e^{-j3}\delta(\omega-5)\right] + \pi\frac{d}{d\omega}\left[e^{-j3}\delta(\omega-5) - e^{j3}\delta(\omega+5)\right]$;
(f) $X(\omega) = \sin(\pi^2)e^{-j\pi\omega/2}$;
(g) $X(\omega) = -\frac{j\pi}{3}e^{-j2(\omega+1)/3}\operatorname{sgn}(\omega+1)$;
(h) $X(\omega) = \frac{1}{2}\left(\pi\delta(\omega-7) + \frac{1}{j(\omega-7)} + \pi\delta(\omega-3) + \frac{1}{j(\omega-3)}\right)$;
(i) $X(\omega) = \frac{\pi}{|a|}\operatorname{tri}\left(\frac{1}{4a}\omega\right)$;
(j) $X(\omega) = e^{-j\omega}\frac{2}{(1+j\omega)^5}$

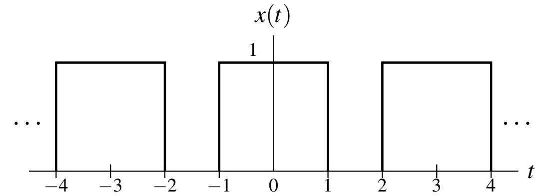
6.104 Using properties of the Fourier transform and the given Fourier transform pair, find the Fourier transform X of each function x.
(a) $x(t) = \operatorname{tri}\left(\frac{1}{T}t\right)$, where T is a nonzero real constant; $\operatorname{rect}\left(\frac{1}{T}t\right) \overset{\text{CTFT}}{\longleftrightarrow} |T|\operatorname{sinc}\left(\frac{T}{2}\omega\right)$ [Hint: $\operatorname{tri}\left(\frac{1}{2}t\right) = \operatorname{rect} * \operatorname{rect}(t)$]; and
(b) $x(t) = \frac{2a}{t^2+a^2}$; $e^{-a|t|} \overset{\text{CTFT}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$.

Short Answer. (a) $X(\omega) = \frac{|T|}{2}\operatorname{sinc}^2\left(\frac{T}{4}\omega\right)$; (b) $X(\omega) = 2\pi e^{-a|\omega|}$

6.105 Using properties of the Fourier transform and a table of Fourier transform pairs, find the Fourier transform Y of each function y given below in terms of the Fourier transform X of the function x.
(a) $y(t) = t\cos(3t)x(t)$;
(b) $y(t) = (t-1)^{100}x^*(t-1)$;
(c) $y(t) = \int_{-\infty}^{3t} x^*(\tau-1)d\tau$;
(d) $y(t) = \cos(3t-1)x(t)$;
(e) $y(t) = \mathcal{D}x(t)\sin(t-2)$, where $\mathcal{D}$ denotes the derivative operator;
(f) $y(t) = tx(t)\sin(3t)$;
(g) $y(t) = e^{j7t}[x * x(t-1)]$;
(h) $y(t) = tx(-t)$; and
(i) $y(t) = tx(t-3)$.

Short Answer.
(a) $Y(\omega) = \frac{j}{2}X'(\omega-3) + \frac{j}{2}X'(\omega+3)$ (where the prime symbol denotes the first derivative);
(b) $Y(\omega) = e^{-j\omega}(\frac{d}{d\omega})^{100}[X^*(-\omega)]$;
(c) $Y(\omega) = \frac{1}{3}\left(\frac{3}{j\omega}(e^{-j\omega/3}X^*(-\omega/3))^* + \pi X^*(0)\delta(\omega/3)\right)$;
(d) $Y(\omega) = \frac{1}{2}e^{-j}X(\omega-3) + \frac{1}{2}e^{j}X(\omega+3)$;
(e) $Y(\omega) = \frac{1}{2}\left[e^{-j2}(\omega-1)X(\omega-1) - e^{j2}(\omega+1)X(\omega+1)\right]$;
(f) $Y(\omega) = \frac{1}{2}\frac{d}{d\omega}[X(\omega-3) - X(\omega+3)]$;
(g) $Y(\omega) = e^{j7}e^{-j\omega}X^2(\omega-7)$;
(h) $Y(\omega) = -jX'(-\omega)$;
(i) $Y(\omega) = 3e^{-j3\omega}X(\omega) + je^{-j3\omega}X'(\omega)$

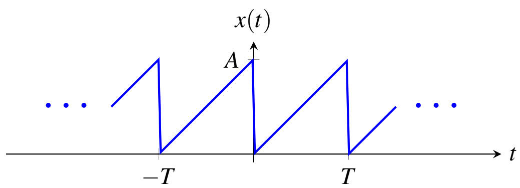
6.106 Find the Fourier transform X of each periodic function x shown below.



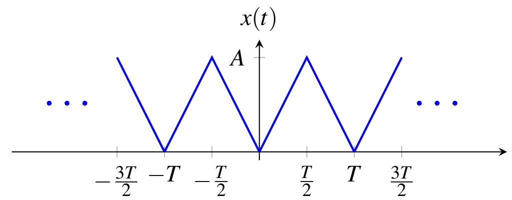
(a)

Short Answer. (a) $X(\omega) = \frac{4\pi}{3} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{2\pi}{3}k\right) \delta(\omega - \frac{2\pi}{3}k)$

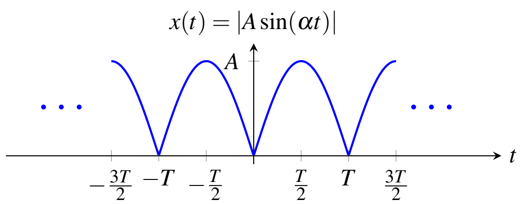
6.107 For each T-periodic function x shown in the figures below, find the Fourier transform X of x. (Some of the integrals listed in Section F.4 may be helpful for some parts of this exercise.)



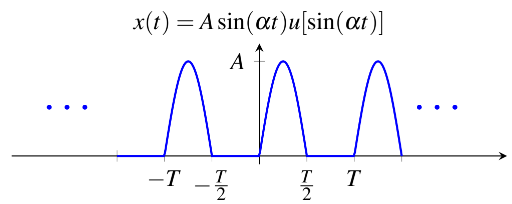
(a)



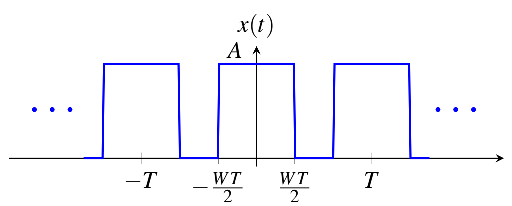
(b)



(c)



(d)



(e)

Short Answer.
(a) $X(\omega) = \pi A \delta(\omega) + \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{jA}{k} \delta\left(\omega - \frac{2\pi}{T}k\right)$;
(b) $X(\omega) = \pi A \delta(\omega) - \sum_{\substack{k\in\mathbb{Z}\\k \text{ odd}}} \frac{4A}{\pi k^2} \delta\left(\omega - \frac{2\pi}{T}k\right)$;
(c) $X(\omega) = \sum_{k=-\infty}^{\infty} \frac{4A}{1-4k^2} \delta\left(\omega - \frac{2\pi}{T}k\right)$;
(d) $X(\omega) = \sum_{k\in\{-1,1\}} \frac{-j\pi Ak}{2} \delta\left(\omega - \frac{2\pi}{T}k\right) + \sum_{\substack{k\in\mathbb{Z}\\k \text{ even}}} \frac{2A}{1-k^2} \delta\left(\omega - \frac{2\pi}{T}k\right)$;
(e) $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi AW \operatorname{sinc}(\pi Wk) \delta\left(\omega - \frac{2\pi}{T}k\right)$

6.108 Find the Fourier transform X of each T-periodic function x given below.
(a) $x(t) = e^{-t}$ for $0 \leq t < 1$; $T = 1$.

Short Answer. (a) $X(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi(e-1)}{e(1+j2\pi k)} \delta(\omega - 2\pi k)$

6.109 Using properties of the Fourier transform and a table of Fourier transform pairs, find the Fourier transform Y of each function y given below in terms of the Fourier transform X of the function x.
(a) $y(t) = r(t)x(t)$, where $r(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}(50t - 5k)$.

Short Answer. (a) $Y(\omega) = \frac{1}{5}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi}{5}k\right) X(\omega - 20\pi k)$

6.110 Using properties of the Fourier transform and a table of Fourier transform pairs, compute the convolution $y = x_1 * x_2$ for the functions x_1 and x_2 given in each case below.
(a) $x_1(t) = \operatorname{sinc}(at - b_1)$ and $x_2(t) = \operatorname{sinc}(at - b_2)$, where a, b_1, and b_2 are real constants with $a \neq 0$;
(b) $x_1(t) = \operatorname{sinc}(at)$ and $x_2(t) = \operatorname{sinc}(bt)$, where a and b are (strictly) positive real constants;
(c) $x_1(t) = e^{-t}u(t)$ and $x_2(t) = e^{-t-1}u(t-1)$; and
(d) $x_1(t) = te^{-3t}u(t)$ and $x_2(t) = e^{-3t}u(t)$.

Short Answer.
(a) $y(t) = \frac{\pi}{|a|}\operatorname{sinc}(at - b_1 - b_2)$;
(b) $y(t) = \frac{\pi}{\max(a,b)}\operatorname{sinc}[\min(a,b)t]$;
(c) $y(t) = e^{-2}(t-1)e^{1-t}u(t-1)$;
(d) $y(t) = \frac{1}{2}t^2e^{-3t}u(t)$

6.111 For each case below, where the function x has the Fourier transform X and the Fourier transform representation $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, find $y(t)$ at the specified values of t.

(a) $x(t) = \begin{cases} 8t^2+1 & 0 \le t < \frac{1}{2} \\ t - \frac{3}{2} & \frac{1}{2} \le t < \frac{3}{2} \\ \pi & \frac{3}{2} \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $t \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$; and

(b) $x(t) = \begin{cases} e^{-t} & -1 \le t < 0 \\ t + \frac{1}{2} & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$ and $t \in \{-1, 0\}$.

Short Answer. (a) $y\left(\frac{1}{2}\right) = 1$ and $y\left(\frac{3}{2}\right) = \frac{\pi}{2}$; (b) $y(-1) = \frac{e}{2}$ and $y(0) = \frac{3}{4}$

6.112 Using Parseval's relation, evaluate the integral $\int_{-\infty}^{\infty} \operatorname{sinc}^2(kt)dt$, where k is a nonzero real constant.

Short Answer. π/k

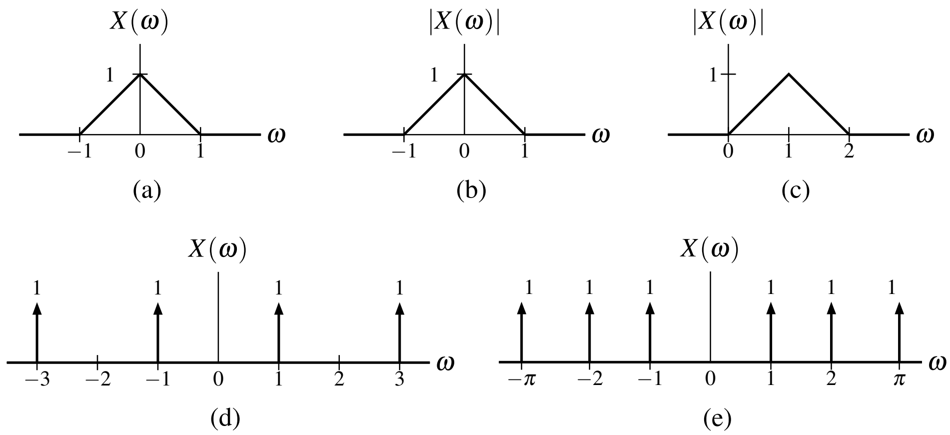
6.113 For each pair of functions M and P given below, find the function x having magnitude spectrum M and phase spectrum P.
(a) $M(\omega) = 1$ and $P(\omega) = \omega$; and
(b) $M(\omega) = \operatorname{rect}\left(\frac{\omega}{3}\right)$ and $P(\omega) = 5\omega$.

Short Answer. (a) $x(t) = \delta(t+1)$; (b) $x(t) = \frac{3}{2\pi}\operatorname{sinc}\left[\frac{3}{2}(t+5)\right]$.

6.114 For each function x given below, compute the frequency spectrum X of x, and find the corresponding magnitude and phase spectra.
(a) $x(t) = \mathcal{D}[e^{-4t}u(t)](t)$, where $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) $X(\omega) = \dfrac{j\omega}{j\omega + 4}$, $|X(\omega)| = \dfrac{|\omega|}{\sqrt{\omega^2+16}}$, and $\arg X(\omega) = \frac{\pi}{2}\operatorname{sgn}(\omega) - \arctan(\omega/4)$

6.115 Let x denote a function with the Fourier transform X. For each frequency/magnitude spectrum given below, determine (if possible) whether x has each of the following properties: real, even/odd, periodic, finite duration, and finite energy.



Short Answer. (a) real, even, aperiodic, not finite duration, finite energy; (b) aperiodic, finite energy, not finite duration; (c) not real, aperiodic, not finite duration, finite energy, not even/odd; (d) real, even, periodic (fundamental period $\frac{2\pi}{1} = 2\pi$), not finite duration, not finite energy; (e) real, even, aperiodic, not finite duration, not finite energy

6.116 For each differential/integral equation below that characterizes a LTI system with input x and output y, find the frequency response H of the system. Note that $\mathcal{I}$ denotes the integration operator $\mathcal{I}x(t) = \int_{-\infty}^{t} x(\tau)d\tau$ and $\mathcal{D}$ denotes the derivative operator.
(a) $\mathcal{D}y(t) + 3y(t) = x(t)$;
(b) $\mathcal{D}^2y(t) + 4\mathcal{D}y(t) + 3y(t) = \mathcal{D}x(t) + 2x(t)$;
(c) $\mathcal{D}y(t) + 3y(t) + 2\mathcal{I}y(t) = \mathcal{D}x(t) + 5x(t)$; and
(d) $5\mathcal{D}y(t) - 2y(t) + 7\mathcal{I}y(t) = 3\mathcal{I}x(t) - x(t)$.

Short Answer. (a) $H(\omega) = \dfrac{1}{j\omega+3}$; (b) $H(\omega) = \dfrac{j\omega+2}{-\omega^2+4j\omega+3}$; (c) $H(\omega) = \dfrac{\omega^2-5j\omega}{\omega^2-3j\omega-2}$; (d) $H(\omega) = \dfrac{3-j\omega}{-5\omega^2-2j\omega+7}$

6.117 For each frequency response H given below for a LTI system with input x and output y, find the differential equation that characterizes the system.
(a) $H(\omega) = \dfrac{j\omega-1}{\omega^2+4}$;
(b) $H(\omega) = \dfrac{5j\omega+3}{7j\omega^3-2j\omega^2+11}$; and
(c) $H(\omega) = \dfrac{1}{\left(j\omega+\frac{1}{3}\right)^2}$.

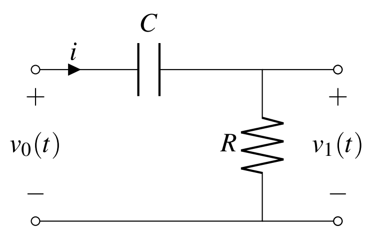
Short Answer. (a) $-\mathcal{D}^2y(t) + 4y(t) = \mathcal{D}x(t) - x(t)$; (b) $-7\mathcal{D}^3y(t) + 2j\mathcal{D}^2y(t) + 11y(t) = 5\mathcal{D}x(t) + 3x(t)$; (c) $\mathcal{D}^2y(t) + \frac{2}{3}\mathcal{D}y(t) + \frac{1}{9}y(t) = x(t)$

6.118 For each case below, use frequency-domain methods to find the response y of the LTI system with impulse response h and frequency response H to the input x.

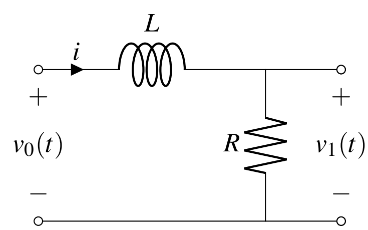
(a) $H(\omega) = j\omega$ and $x(t) = 1 + \frac{1}{4}\cos(2t) + \frac{1}{9}\sin(3t)$;
(b) $H(\omega) = -\mathcal{D}\delta(t)$ where $\mathcal{D}$ denotes the derivative operator and $x(t) = 10 + \cos(2t) + \sin(6t)$;
(c) $h(t) = (\pi t)^{-1}$ and $x(t) = 1 - \frac{1}{2}\cos(2t) + \frac{1}{3}\sin(3t)$; and
(d) $h(t) = e^{-3(t-1)}u(t-1)$ and $x(t) = t^2 e^{-3t}u(t)$.

Short Answer. (a) $y(t) = -\frac{1}{2}\sin(2t) + \frac{1}{3}\cos(3t)$; (b) $y(t) = 2\sin(2t) - 6\cos(6t)$; (c) $y(t) = -\frac{1}{2}\sin(2t) - \frac{1}{3}\cos(3t)$; (d) $y(t) = \frac{1}{3}(t-1)^3 e^{-3(t-1)}u(t-1)$

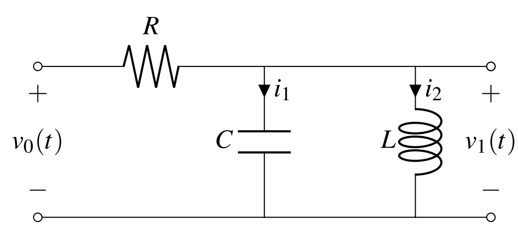
6.119 For each LTI circuit with input v_0 and output v_1, find the differential equation that characterizes the circuit, find the frequency response H of the circuit, and determine which type of frequency-selective filter this circuit best approximates.



(a)



(b)



(c)

Short Answer.
(a) $\mathcal{D}v_0(t) = \mathcal{D}v_1(t) + \frac{1}{RC}v_1(t)$; $H(\omega) = \dfrac{j\omega RC}{j\omega RC + 1}$; highpass;
(b) $v_0(t) = \frac{L}{R}\mathcal{D}v_1(t) + v_1(t)$; $H(\omega) = \dfrac{R}{j\omega L + R}$; lowpass;
(c) $\mathcal{D}v_0(t) = RC\mathcal{D}^2 v_1(t) + \mathcal{D}v_1(t) + \frac{R}{L}v_1(t)$; $H(\omega) = \dfrac{j\omega L}{-RLC\omega^2 + j\omega L + R}$; bandpass

6.120 In this exercise, we consider the effect of a loss of frequency synchronization between the carrier signals used in the transmitter and receiver of a DSB-SC AM system. In what follows, we assume that the signal to be transmitted is always bandlimited to frequencies in $[-\omega_M, \omega_M]$. The transmitter with input x and output y is characterized by the equation $y(t) = \cos(\omega_c t)x(t)$. The receiver with input y and output $\tilde{x}$ is characterized by the equation $\tilde{x}(t) = v * h(t)$, where $v(t) = \cos[(\omega_c + \Delta\omega)t]y(t)$ and h is the impulse response of an ideal lowpass filter with a passband gain of 2 and a cutoff frequency ω_{c0} satisfying $\omega_M + \Delta\omega < \omega_{c0} < 2\omega_c + \Delta\omega - \omega_M$. (In other words, we have a DSB/SC system similar to the one in Section 6.19.2, except that the receiver uses a carrier signal with a frequency of $\omega_c + \Delta\omega$ instead of ω_c.) Find $\tilde{x}$ in terms of x.

Short Answer. $\tilde{x}(t) = \cos(\Delta\omega t)x(t)$

6.121 For each function x below, by direct application of the Nyquist sampling theorem, determine the lowest sampling rate ω_s at which x can be sampled such that it can be exactly reconstructed from its samples.
(a) $x(t) = \sin(15t)$;
(b) $x(t) = 10 + 4\sin(15t) + 2\cos(20t)$;
(c) $x(t) = \operatorname{sinc}(5t-3)$;
(d) $x(t) = \operatorname{sinc}^2(20t)$;
(e) $x(t) = \cos(10t)\operatorname{sinc}(30t)$; and
(f) $x(t) = x_1 * x_2(t)$, where $x_1(t) = e^{-t}u(t)$ and $x_2(t) = \operatorname{sinc}(10t)$.

Short Answer. (a) 30; (b) 40; (c) 10; (d) 80; (e) 80; (f) 20

6.122 A real sinusoidal function x having frequency ω_0 is ideally sampled with a sampling rate ω_s, yielding the sequence v. Bandlimited interpolation is then applied to v to produce the function y. For each case given below, determine the frequencies present in the spectrum of y.
(a) $\omega_0 = 50$, $\omega_s = 90$;
(b) $\omega_0 = 50$, $\omega_s = 110$;
(c) $\omega_0 = 100$, $\omega_s = 50$; and
(d) $\omega_0 = 179$, $\omega_s = 60$.

Short Answer. (a) ± 40; (b) ± 50; (c) 0; (d) ± 1

6.123 An audio signal x consists of two perfect sinusoidal tones at 440 Hz and 880 Hz. The signal x is sampled at a rate of f_s Hz and then played back on a loudspeaker. Determine how many tones will be heard on the loudspeaker and their frequencies, if f_s equals (a) 500 Hz and (b) 2000 Hz. (Assume that a human can hear frequencies from 20 Hz to 20 kHz.)

Short Answer. (a) Two tones are heard with frequencies 60 Hz and 120 Hz; (b) Two tones are heard at 440 Hz and 880 Hz

6.124 Consider a bandlimited function x that has been sampled at a frequency ω_s (in radians) satisfying the Nyquist condition to produce the sequence y. Find an expression for the Fourier transform X of the function x in terms of the sequence y of samples and ω_s. [Hint: Recall that, from the sampling theorem, $x(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}(\frac{\omega_s}{2}t - \pi n)$.]

Short Answer. $X(\omega) = \frac{2\pi}{\omega_s}\sum_{n=-\infty}^{\infty} y(n)e^{-j2\pi n\omega/\omega_s}\operatorname{rect}\left(\frac{\omega}{\omega_s}\right)$

6.22 MATLAB Exercises

6.201 (a) Consider a frequency response H of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k\omega^k}{\sum_{k=0}^{N-1} b_k\omega^k},$$

where a_k and b_k are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments:

1) a vector containing the a_k coefficients;
2) a vector containing the b_k coefficients; and
3) a vector containing the values of ω for which to evaluate $H(\omega)$.

The function should generate two return values:

1) a vector of function values; and
2) a vector of points at which the function was evaluated.

If the function is called with no output arguments (i.e., the `nargout` variable is zero), then the function should plot the magnitude and phase responses before returning. (Hint: The `polyval` function may be helpful.)
(b) Use the function developed in part (a) to plot the magnitude and phase responses of the system with the frequency response

$$H(\omega) = \frac{16.0000}{1.0000\omega^4 - j5.2263\omega^3 - 13.6569\omega^2 + j20.9050\omega + 16.0000}.$$

For each of the plots, use the frequency range $[-5, 5]$.
(c) What type of ideal frequency-selective filter does this system approximate?

6.202 Consider the filter associated with each of the frequency responses given below. In each case, plot the magnitude and phase responses of the filter, and indicate what type of ideal frequency-selective filter it best approximates.
(a) $H(\omega) = \dfrac{\omega_b^3}{(j\omega)^3 + 2\omega_b(j\omega)^2 + 2\omega_b(j\omega) + \omega_b^3}$ where $\omega_b = 1$;
(b) $H(\omega) = \dfrac{(j\omega)^5}{(j\omega)^5 + 17.527635(j\omega)^4 + 146.32995(j\omega)^3 + 845.73205(j\omega)^2 + 2661.6442(j\omega) + 7631.0209}$; and
(c) $H(\omega) = \dfrac{13.104406(j\omega)^3}{(j\omega)^6 + 3.8776228(j\omega)^5 + 34.517979(j\omega)^4 + 75.146371(j\omega)^3 + 276.14383(j\omega)^2 + 248.16786(j\omega) + 512}$.
(Hint: Use the `freqs` function with $s = j\omega$ to compute the frequency response. The `abs`, `angle`, `linspace`, `plot`, `xlabel`, `ylabel`, and `print` functions may also prove useful for this problem.)

6.203 (a) Use the `butter` and `besself` functions to design a tenth-order Butterworth lowpass filter and tenth-order Bessel lowpass filter, each with a cutoff frequency of 10 rad/s.
(b) For each of the filters designed in part (a), plot the magnitude and phase responses using a linear scale for the frequency axis. In the case of the phase response, plot the unwrapped phase (as this will be helpful later in part (d) of this problem). (Hint: The `freqs` and `unwrap` functions may be helpful.)
(c) Consider the magnitude responses for each of the filters. Recall that an ideal lowpass filter has a magnitude response that is constant in the passband. Which of the two filters more closely approximates this ideal behavior?
(d) Consider the phase responses for each of the filters. An ideal lowpass filter has a phase response that is a linear function. Which of the two filters has a phase response that best approximates a linear (i.e., straight line) function in the passband?

Chapter 7

Laplace Transform

7.1 Introduction

In this chapter, we introduce another important mathematical tool in the study of signals and systems known as the Laplace transform. The Laplace transform can be viewed as a generalization of the (classical) Fourier transform. Due to its more general nature, the Laplace transform has a number of advantages over the Fourier transform. First, the Laplace transform representation exists for some functions that do not have a (classical) Fourier transform representation. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform. Second, since the Laplace transform is a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

7.2 Motivation Behind the Laplace Transform

In Section 4.10, we showed that complex exponentials are eigenfunctions of LTI systems. Suppose that we have a LTI system with impulse response h. This eigenfunction property leads to the result that the response y of the system to the complex exponential input $x(t) = e^{st}$ (where s is a complex constant) is

$$y(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt. \tag{7.1}$$

Previously, we referred to H as the system function. In this chapter, we will learn that H is, in fact, the Laplace transform of h. That is, the integral in (7.1) is simply the definition of the Laplace transform. In the case that $s = j\omega$ where ω is real (i.e., s is purely imaginary), (7.1) becomes the Fourier transform integral (studied in Chapter 6). From our earlier reading, we know that $H(j\omega)$ is the frequency response of the LTI system (i.e., the Fourier transform of h). Since (7.1) includes the Fourier transform as a special case, the Laplace transform can be thought of as a generalization of the (classical) Fourier transform.

7.3 Definition of the Laplace Transform

The (bilateral) **Laplace transform** of the function x is denoted as $\mathcal{L}x$ or X and is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt. \tag{7.2}$$

Similarly, the **inverse Laplace transform** of X is denoted $\mathcal{L}^{-1}X$ or x and is given by

$$\mathcal{L}^{-1}X(t) = x(t) = \tfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds, \tag{7.3}$$

where $\sigma = \operatorname{Re}(s)$. We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s).$$

As we can see from (7.3), the calculation of the inverse Laplace transform requires a contour integration (since s is a complex variable). In particular, we must integrate along the vertical line $s = \sigma$ in the complex plane. Such a contour integration is often not so easy to compute. Therefore, in practice, we do not usually compute the inverse Laplace transform using (7.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the Laplace transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the lower limit of integration. In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0. In the remainder of this chapter, we will focus our attention primarily on the bilateral Laplace transform. We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations. Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean bilateral Laplace transform.

7.4 Remarks on Notation Involving the Laplace Transform

Each of the Laplace transform operator $\mathcal{L}$ and inverse Laplace transform operator $\mathcal{L}^{-1}$ map a function to a function. Consequently, the operand for each of these operators must be a function (not a number). Consider the unnamed function that maps t to $e^{-|t|}$ as shown in Figure 7.1. Suppose that we would like to write an expression that denotes the Laplace transform of this function. At first, we might be inclined to write "$\mathcal{L}\{e^{-|t|}\}$". Strictly speaking, however, this notation is not correct, since the Laplace transform operator requires a function as an operand and "$e^{-|t|}$" (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at t). Essentially, the cause of our problems here is that the function in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a function x using the equation $x(t) = e^{-|t|}$ and then write the Laplace transform as "$\mathcal{L}x$". Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to highly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 7.1 that maps t to $e^{-|t|}$. Using strictly correct notation, we could write the Laplace transform of this function as "$\mathcal{L}\left\{e^{-|\cdot|}\right\}$". In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 7.1. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 7.1 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}$.
2. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ evaluated at $s-5$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}(s-5)$.
3. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{(\cdot)^{-1}\right\}$.

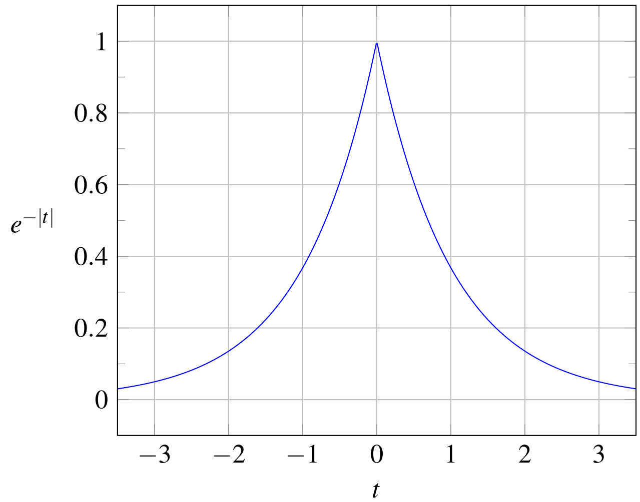


Figure 7.1: A plot of $e^{-|t|}$ versus t.

4. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{(\cdot)^{-1}\right\}(t-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Laplace transform operator $\mathcal{L}$, the variable "t" is assumed to be the independent variable for the function to which the Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "t" were a "$\cdot$");
- unless indicated otherwise, in an expression for the operand of the inverse Laplace transform operator $\mathcal{L}^{-1}$, the variable "s" is assumed to be the independent variable for the function to which the inverse Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "s" were a "$\cdot$").

Some examples of using these book-sanctioned notational conventions can be found below in Example 7.2. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 7.2 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{t^2e^{-3t}u(t)\right\}$.
2. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ evaluated at $s-5$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{t^2e^{-3t}u(t)\right\}(s-5)$.
3. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{s^{-1}\right\}$.
4. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{s^{-1}\right\}(t-3)$. ■

Since applying the Laplace transform operator or inverse Laplace transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 7.1 that maps t to $e^{-|t|}$. To denote the value of the Laplace transform of this function evaluated at $s-1$, we would write "$\mathcal{L}\{e^{-|\cdot|}\}(s-1)$" using dot notation or "$\mathcal{L}\{e^{-|t|}\}(s-1)$" using the book-sanctioned notational conventions described above.

7.5 Relationship Between Laplace Transform and Continuous-Time Fourier Transform

In Section 7.3 of this chapter, we introduced the Laplace transform, and in the previous chapter, we studied the (CT) Fourier transform. As it turns out, the Laplace transform and (CT) Fourier transform are very closely related. Recall the definition of the Laplace transform in (7.2). Consider now the special case of (7.2) where $s = j\omega$ and ω is real (i.e., $\operatorname{Re}(s) = 0$). In this case, (7.2) becomes

$$\begin{aligned}
X(j\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]\Bigg|_{s=j\omega} \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \mathcal{F}x(\omega).
\end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at $s = j\omega$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(j\omega) = \mathcal{F}x(\omega). \tag{7.4}$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(j\omega)$. When this notation is used, the function X actually corresponds to the Laplace transform of x rather than its Fourier transform (i.e., the expression $X(j\omega)$ corresponds to the Laplace transform evaluated at points on the imaginary axis).

Now, consider the general case of an arbitrary complex value for s in (7.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (7.2), we obtain

$$\begin{aligned}
X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\
&= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \\
&= \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega).
\end{aligned}$$

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega). \tag{7.5}$$

Therefore, the Laplace transform of x can be viewed as the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$ (i.e., x weighted by a real exponential function). As a consequence of multiplying by the real exponential $e^{-\sigma t}$, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

By using the above relationship, we can derive the formula for the inverse Laplace transform given in (7.3). Let X denote the Laplace transform of x, and let $s = \sigma + j\omega$, where σ and ω are real. From the relationship between the Fourier and Laplace transforms in (7.5), we have

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega),$$

where σ is chosen so that $X(s)$ converges for $s = \sigma + j\omega$. Taking the inverse Fourier transform of both sides of the preceding equation yields

$$\mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t) = e^{-\sigma t}x(t).$$

Multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = e^{\sigma t}\mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t).$$

From the definition of the inverse Fourier transform, we have

$$\begin{aligned} x(t) &= e^{\sigma t}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega\right] \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega. \end{aligned}$$

Since $s = \sigma + j\omega$, we have that $ds = jd\omega$, and consequently,

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\left(\tfrac{1}{j}\right)ds \\ &= \tfrac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds. \end{aligned}$$

Thus, we have just derived the inverse Laplace transform formula in (7.3).

7.6 Laplace Transform Examples

In this section, we calculate the Laplace transform of several relatively simple functions. In the process, we gain some important insights into the Laplace transform.

Example 7.3. Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{0}^{\infty}. \end{aligned}$$

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{0}^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{0}^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)\infty}e^{-j\omega\infty} - 1\right]. \end{aligned}$$

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\operatorname{Re}(s) > -a$). In this case, we have that

$$\begin{aligned} X(s) &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)[0 - 1] \\ &= \left(\tfrac{-1}{s+a}\right)(-1) \\ &= \frac{1}{s+a}. \end{aligned}$$

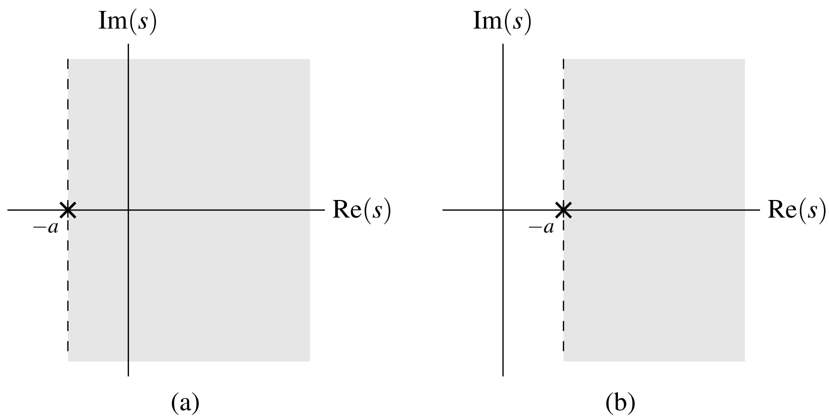


Figure 7.2: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

Thus, we have that

$$e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) > -a.$$

The region of convergence for X is illustrated in Figures 7.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively. ■

Example 7.4. Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we can write

$$\begin{aligned}
X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\
&= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt \\
&= \int_{-\infty}^{0} -e^{-at}e^{-st}dt \\
&= \int_{-\infty}^{0} -e^{-(s+a)t}dt \\
&= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}.
\end{aligned}$$

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

$$\begin{aligned}
X(s) &= \left[\left(\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} \\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0} \\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[1 - e^{(\sigma+a)\infty}e^{j\omega\infty}\right].
\end{aligned}$$

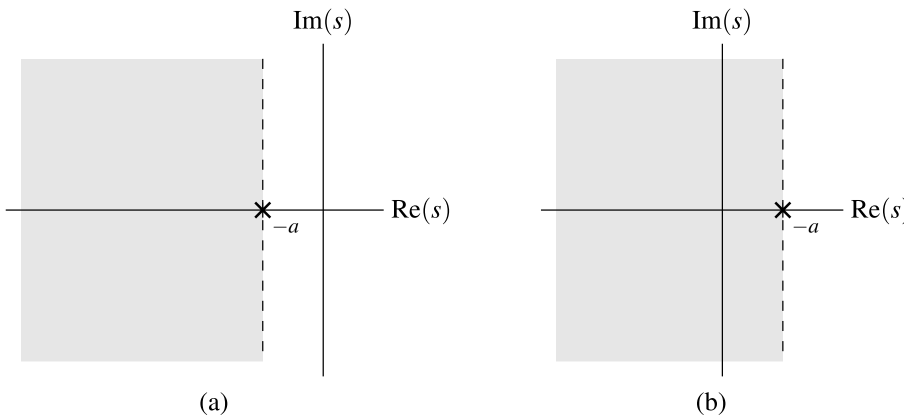


Figure 7.3: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\operatorname{Re}(s) < -a$). In this case, we have

$$\begin{aligned} X(s) &= \left(\tfrac{1}{\sigma+a+j\omega}\right)[1-0] \\ &= \frac{1}{s+a}. \end{aligned}$$

Thus, we have that

$$-e^{-at}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) < -a.$$

The region of convergence for X is illustrated in Figures 7.3(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively. ■

At this point, we compare the results of Examples 7.3 and 7.4, and make an important observation. Notice that the same algebraic expression for X was obtained in both of these examples (i.e., $X(s) = \frac{1}{s+a}$). The only difference is in the convergence properties of X. In one case, $X(s)$ converges for $\operatorname{Re}(s) > -a$ while in the other it converges for $\operatorname{Re}(s) < -a$. As it turns out, one must specify both the algebraic expression for X and its region of convergence in order to uniquely determine $x = \mathcal{L}^{-1}X$ from X.

Example 7.5 (Laplace transform of the unit-step function)**.** Find the Laplace transform X of the function

$$x(t) = u(t).$$

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}u(s) \\ &= \int_{-\infty}^{\infty} u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-st}dt \\ &= \left[\left(-\tfrac{1}{s}\right)e^{-st}\right]\Big|_0^\infty. \end{aligned}$$

At this point, we substitute $s = \sigma + j\omega$ in order to more clearly see the region of convergence for this expression. This yields

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma+j\omega}\right)e^{-(\sigma+j\omega)t}\right]\Big|_0^\infty \\ &= \left[\left(-\tfrac{1}{\sigma+j\omega}\right)e^{-\sigma t}e^{-j\omega t}\right]\Big|_0^\infty. \end{aligned}$$

Thus, we can see that the above expression converges only for $\sigma > 0$ (i.e., $\mathrm{Re}(s) > 0$). In this case, we have

$$\begin{aligned} X(s) &= \left(-\tfrac{1}{\sigma+j\omega}\right)[0-1] \\ &= \left(-\tfrac{1}{s}\right)(-1) \\ &= \frac{1}{s}. \end{aligned}$$

Thus, we have that

$$u(t) \xleftrightarrow{\text{LT}} \frac{1}{s} \quad \text{for } \mathrm{Re}(s) > 0. \qquad \blacksquare$$

Example 7.6 (Laplace transform of the delta function)**.** Find the Laplace transform X of the function

$$x(t) = A\delta(t-t_0),$$

where A and t_0 are arbitrary real constants.

Solution. From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{A\delta(t-t_0)\}(s) \\ &= \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-st}dt \\ &= A\int_{-\infty}^{\infty} \delta(t-t_0)e^{-st}dt. \end{aligned}$$

Using the sifting property of the delta function, we can simplify this result to obtain

$$X(s) = Ae^{-st_0}.$$

Thus, we have shown that

$$A\delta(t-t_0) \xleftrightarrow{\text{LT}} Ae^{-st_0} \quad \text{for all } s. \qquad \blacksquare$$

7.7 Region of Convergence for the Laplace Transform

Before discussing the region of convergence (ROC) of the Laplace transform in detail, we need to introduce some terminology involving sets in the complex plane. Let R denote a set in the complex plane. A set R comprised of all complex numbers s such that

$$\mathrm{Re}(s) < a,$$

for some real constant a, is said to be a **left-half plane** (LHP). A set R comprised of all complex numbers s such that

$$\mathrm{Re}(s) > a,$$

for some real constant a, is said to be a **right-half plane** (RHP). Examples of LHPs and RHPs are given in Figure 7.4.

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B. An illustrative example of set intersection is shown in Figure 7.5.

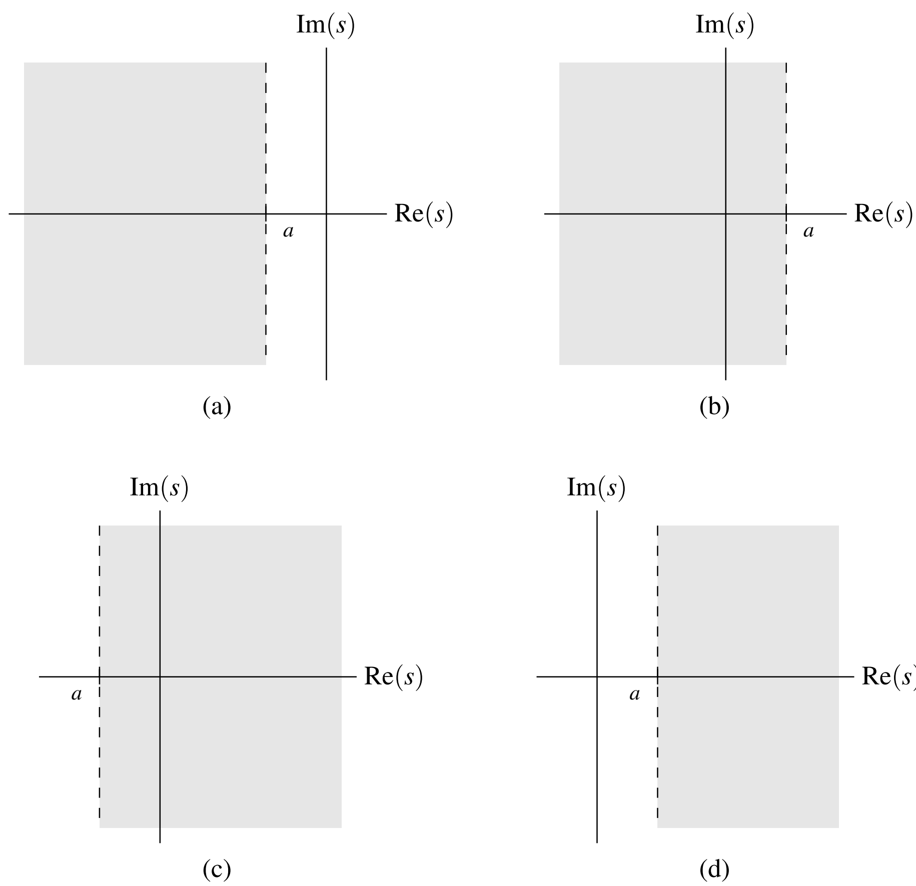


Figure 7.4: Examples of LHPs and RHPs. An example of a LHP in the case that (a) $a < 0$ and (b) $a > 0$. An example of a RHP in the case that (c) $a < 0$ and (d) $a > 0$.

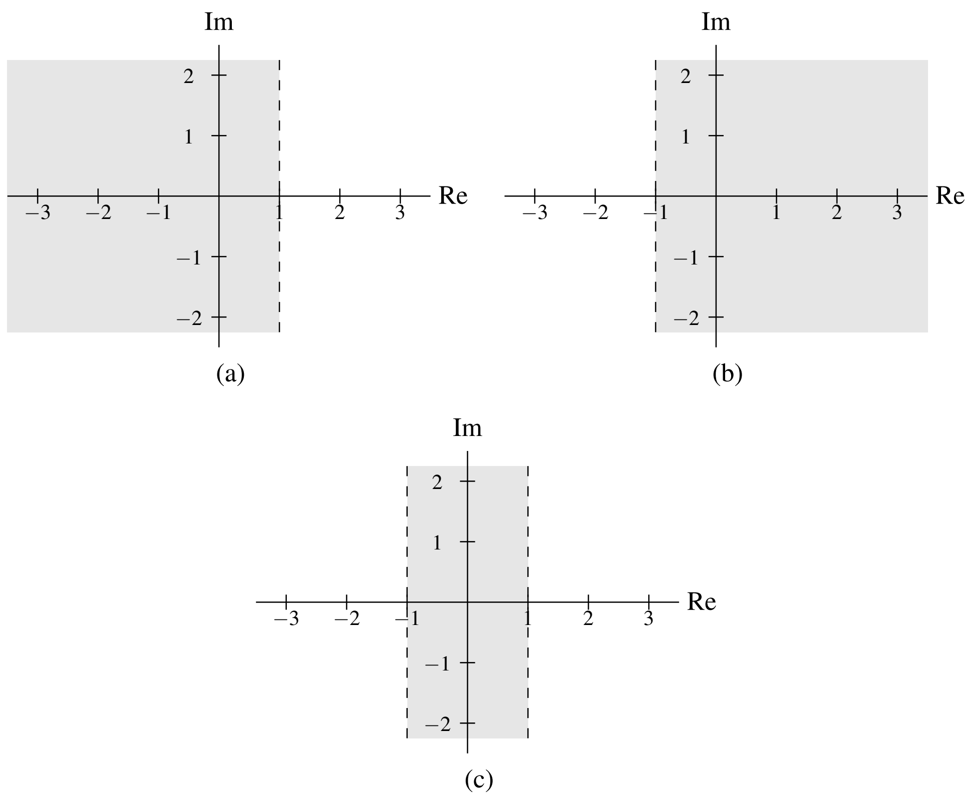


Figure 7.5: Example of set intersection. The sets (a) R_1 and (b) R_2; and (c) their intersection $R_1 \cap R_2$.

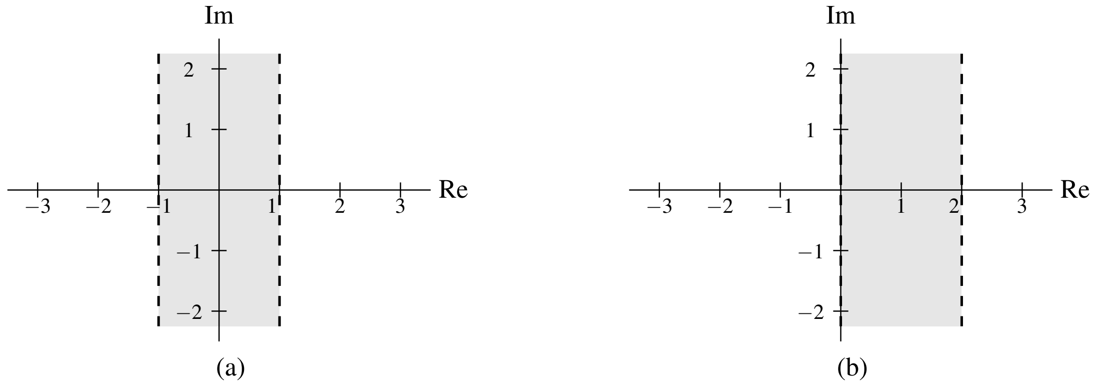


Figure 7.6: Example of adding a scalar to a set. (a) The set R. (b) The set $R+1$.

For a set S and a scalar constant a, $S+a$ denotes the set given by

$$S+a = \{z+a : z \in S\}.$$

That is, $S+a$ denotes the set formed by adding a to each element of S. For example, suppose that R is the set of complex numbers s satisfying

$$-1 < \mathrm{Re}(s) < 1,$$

as shown in Figure 7.6(a). Then, $R+1$ is the set of complex numbers s satisfying

$$0 < \mathrm{Re}(s) < 2,$$

as shown in Figure 7.6(b).

For a set S and a scalar constant a, aS denotes the set given by

$$aS = \{az : z \in S\}.$$

That is, aS denotes the set formed by multiplying each element of S by a. For example, suppose that R is the set of complex numbers s satisfying

$$-1 < \mathrm{Re}(s) < 2,$$

as shown in Figure 7.7(a). Then, $2R$ is the set of complex numbers s satisfying

$$-2 < \mathrm{Re}(s) < 4,$$

as shown in Figure 7.7(b); and $-2R$ is the set of complex numbers s satisfying

$$-4 < \mathrm{Re}(s) < 2,$$

as shown in Figure 7.7(c).

As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X. Two distinct functions can have the same algebraic expression for their Laplace transform. In what follows, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of functions.

One can show that the ROC of the Laplace transform has the following properties:

1. The ROC of the Laplace transform X consists of strips parallel to the imaginary axis in the complex plane. That is, if s is in the ROC, then $s+j\omega$ is in the ROC for all real ω. Some examples of sets that would be either valid or invalid as ROCs are shown in Figure 7.8.

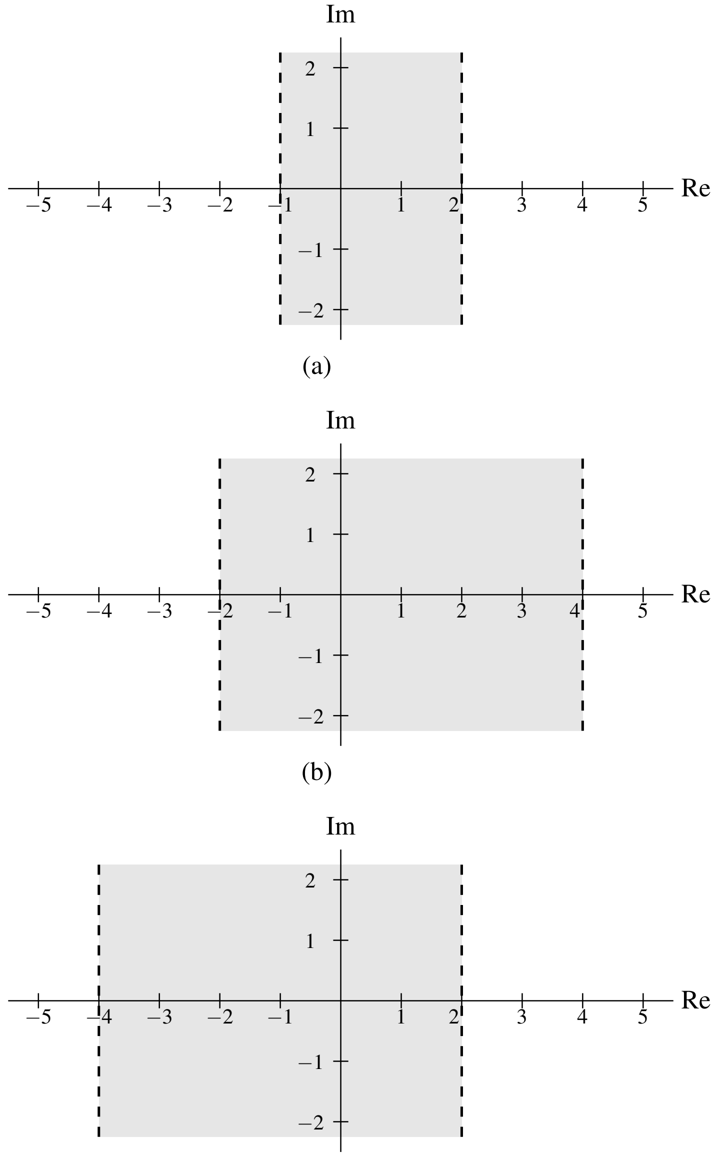


Figure 7.7: Example of multiplying a set by a scalar. (a) The set R. The sets (b) $2R$ and (c) $-2R$.

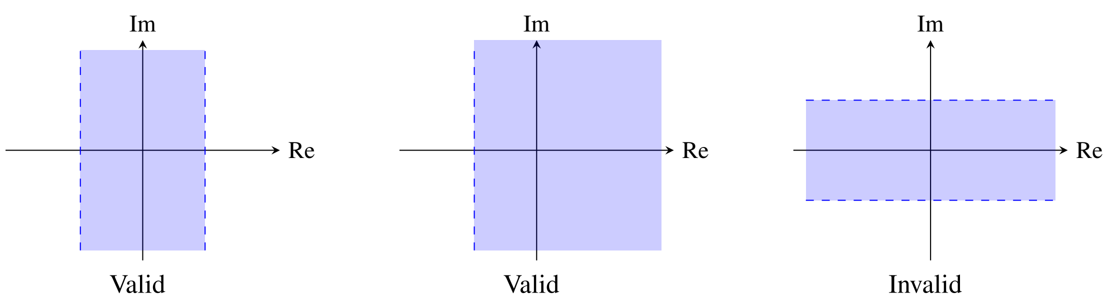


Figure 7.8: Examples of sets that would be either valid or invalid as the ROC of a Laplace transform.

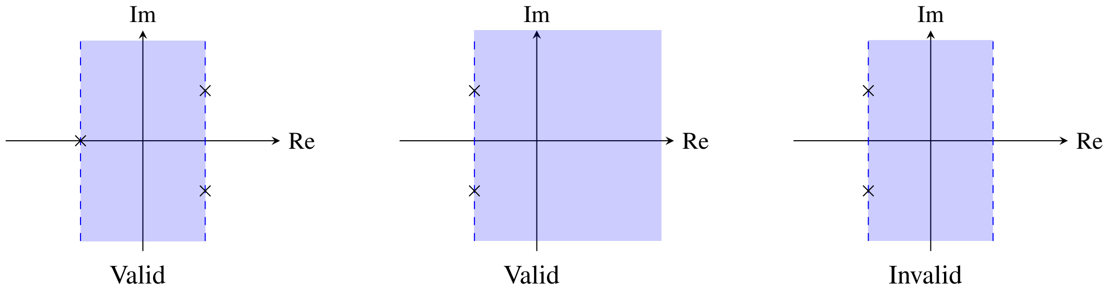


Figure 7.9: Examples of sets that would be either valid or invalid as the ROC of a rational Laplace transform.

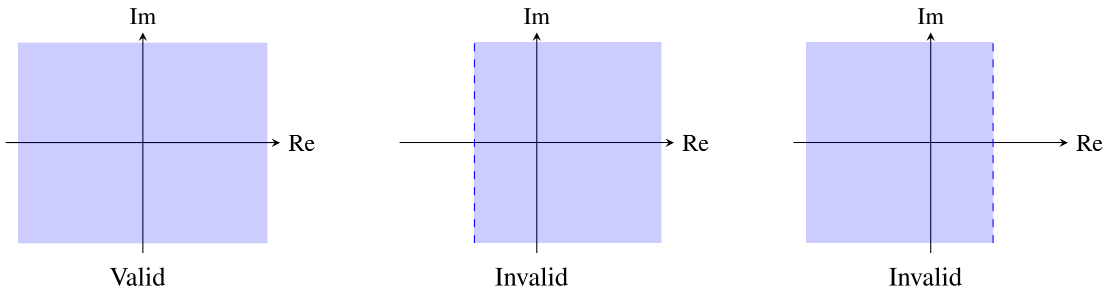


Figure 7.10: Examples of sets that would be either valid or invalid as the ROC of a Laplace transform of a finite-duration function.

Justification: The Laplace transform X of the function x is simply the (CT) Fourier transform of $x'(t) = x(t)e^{-\operatorname{Re}(s)t}$. Thus, X converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on $\operatorname{Re}(s)$, the convergence of the Laplace transform only depends on $\operatorname{Re}(s)$.

2. If the Laplace transform X is a rational function, the ROC does not contain any poles, and the ROC is bounded by poles or extends to infinity. Some examples of sets that would be either valid or invalid as ROCs of rational Laplace transforms are shown in Figure 7.9.

   Partial justification: Since X is rational, its value becomes infinite at a pole. So obviously, X does not converge at a pole. Therefore, it follows that the ROC cannot contain a pole.

3. If a function x is finite duration and its Laplace transform X converges for some value of s, then X converges for all values of s (i.e., the ROC is the entire complex plane). Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown in Figure 7.10.

4. If a function x is right sided and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then all values of s for which $\operatorname{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC includes a right-half plane containing $\operatorname{Re}(s) = \sigma_0$). Moreover, if x is right sided but not left sided, the ROC of X is a right-half plane. Some examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown in Figure 7.11.

5. If a function x is left sided and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then all values of s for which $\operatorname{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC includes a left-half plane containing $\operatorname{Re}(s) = \sigma_0$). Moreover, if x is left sided but not right sided, the ROC of X is a left-half plane. Some examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown in Figure 7.12.

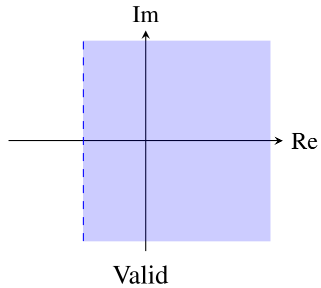


Figure 7.11: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a function that is right sided but not left sided.

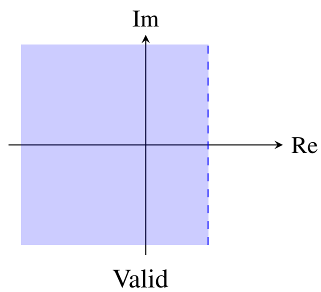


Figure 7.12: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a function that is left sided but not right sided.

6. If a function x is two sided and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then the ROC will consist of a strip in the complex plane that includes the line $\mathrm{Re}(s) = \sigma_0$. Some examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown in Figure 7.13.

7. If the Laplace transform X of a function x is rational, then:

   (a) If x is right sided, the ROC of X is to the right of the rightmost pole of X (i.e., the right-half plane to the right of the rightmost pole).

   (b) If x is left sided, the ROC of X is to the left of the leftmost pole of X (i.e., the left-half plane to the left of the leftmost pole).

   Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given in Figure 7.14.

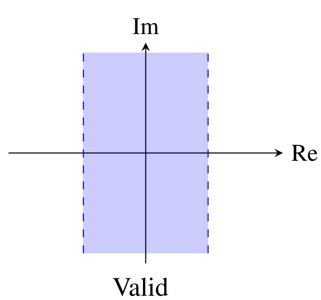


Figure 7.13: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a two-sided function.