R **4.101** Using the graphical method, compute $x*h$ for each pair of functions x and h given below. (Do not compute $x*h$ indirectly by instead computing $h*x$ and using the commutative property of convolution.)

(a) $x(t) = 2\operatorname{rect}(t-\frac{1}{2})$ and $h(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = u(t-1)$ and $h(t) = \begin{cases} t+1 & -1 \le t < 0 \\ t-1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x(t) = \begin{cases} t-2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \operatorname{rect}\left[\frac{1}{2}(t+2)\right]$;

(d) $x(t) = \operatorname{rect}\left[\frac{1}{3}\left(t-\frac{3}{2}\right)\right]$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x(t) = \begin{cases} \frac{1}{4}(t-1)^2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x(t) = \begin{cases} 2\cos\left(\frac{\pi}{4}t\right) & 0 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left[\frac{1}{2}(t-2)\right]$;

(h) $x(t) = \begin{cases} \frac{1}{2}t-\frac{1}{2} & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-1 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{tri}\left[\frac{1}{2}(t-3)\right]$;

(j) $x(t) = \begin{cases} \frac{1}{4}t-\frac{1}{4} & 1 \le t < 5 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} \frac{3}{2}-\frac{1}{2}t & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x(t) = \begin{cases} 1-\frac{1}{100}t & 0 \le t < 100 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = e^{-t}u(t-1)$;

(m) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} 1-(t-2)^2 & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x(t) = e^{-t}u(t)$ and $h(t) = e^{-3t}u(t-2)$;

(o) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left(t-\frac{3}{2}\right)$;

(p) $x(t) = e^{-2t}u(t)$ and $h(t) = \operatorname{rect}\left(t-\frac{5}{2}\right)$;

(q) $x(t) = u(t-1)$ and $h(t) = \begin{cases} \sin[\pi(t-1)] & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(r) $x(t) = u(t)$ and $h(t) = \operatorname{rect}\left(\frac{1}{4}[t-4]\right)$;

(s) $x(t) = e^{-t}u(t)$ and $h(t) = e^{2-2t}u(t-1)$;

(t) $x(t) = e^{-3t}u(t)$ and $h(t) = u(t+1)$; and

(u) $x(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-2 & -3 \le t < -2 \\ 0 & \text{otherwise.} \end{cases}$

Short Answer.

(a) $x*h(t) = \begin{cases} \int_0^{t+1} -2d\tau & -1 \le t < 0 \\ \int_0^t 2d\tau + \int_t^1 -2d\tau & 0 \le t < 1 \\ \int_{t-1}^1 2d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x*h(t) = \begin{cases} \int_1^{t+1}(-\tau+t+1)d\tau & 0 \le t < 1 \\ \int_1^t(-\tau+t-1)d\tau + \int_t^{t+1}(-\tau+t+1)d\tau & 1 \le t < 2 \\ \int_{t-1}^t(-\tau+t-1)d\tau + \int_t^{t+1}(-\tau+t+1)d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x*h(t) = \begin{cases} \int_1^{t+3}(\tau-2)d\tau & -2 \le t < 0 \\ \int_{t+1}^3(\tau-2)d\tau & 0 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(d) $x*h(t) = \begin{cases} \int_0^{t-1}(t-\tau-1)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1}(t-\tau-1)d\tau & 2 \le t < 4 \\ \int_{t-2}^3(t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x*h(t) = \begin{cases} \int_1^{t-1}\frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1}\frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 3 \le t < 4 \\ \int_{t-2}^3\frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x*h(t) = \begin{cases} \int_0^{t-1}2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1}2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 2 \le t < 3 \\ \int_{t-2}^2 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 3 \le t < 4 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x*h(t) = \begin{cases} \int_{t-3}^{t-1}e^\tau d\tau & t < 1 \\ \int_{t-3}^0 e^\tau d\tau + \int_0^{t-1}e^{-\tau}d\tau & 1 \le t < 3 \\ \int_{t-3}^{t-1}e^{-\tau}d\tau & t \ge 3; \end{cases}$

(h) $x*h(t) = \begin{cases} \int_1^{t+2}\left(\frac{1}{2}\tau-\frac{1}{2}\right)(\tau-t-1)d\tau & -1 \le t < 0 \\ \int_{t+1}^{t+2}\left(\frac{1}{2}\tau-\frac{1}{2}\right)(\tau-t-1)d\tau & 0 \le t < 1 \\ \int_{t+1}^3\left(\frac{1}{2}\tau-\frac{1}{2}\right)(\tau-t-1)d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x*h(t) = \begin{cases} \int_{t-4}^{t-3}e^\tau(\tau-t+4)d\tau + \int_{t-3}^{t-2}e^\tau(t-\tau-2)d\tau & t < 2 \\ \int_{t-4}^{t-3}e^\tau(\tau-t+4)d\tau + \int_{t-3}^0 e^\tau(t-\tau-2)d\tau + \int_0^{t-2}e^{-\tau}(t-\tau-2)d\tau & 2 \le t < 3 \\ \int_{t-4}^0 e^\tau(\tau-t+4)d\tau + \int_0^{t-3}e^{-\tau}(t-\tau+4)d\tau + \int_{t-3}^{t-2}e^{-\tau}(t-\tau-2)d\tau & 3 \le t < 4 \\ \int_{t-4}^{t-3}e^{-\tau}(\tau-t+4)d\tau + \int_{t-3}^{t-2}e^{-\tau}(t-\tau-2)d\tau & t \ge 4; \end{cases}$

(j) $x*h(t) = \begin{cases} \int_1^{t-1}\left(\frac{1}{4}\tau-\frac{1}{4}\right)\left(\frac{1}{2}\tau-\frac{1}{2}t+\frac{3}{2}\right)d\tau & 2 \le t < 4 \\ \int_{t-3}^{t-1}\left(\frac{1}{4}\tau-\frac{1}{4}\right)\left(\frac{1}{2}\tau-\frac{1}{2}t+\frac{3}{2}\right)d\tau & 4 \le t < 6 \\ \int_{t-3}^5\left(\frac{1}{4}\tau-\frac{1}{4}\right)\left(\frac{1}{2}\tau-\frac{1}{2}t+\frac{3}{2}\right)d\tau & 6 \le t < 8 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x*h(t) = \begin{cases} \int_{-10}^{t-1}(t-\tau-1)d\tau & -9 \le t < -8 \\ \int_{t-2}^{t-1}(t-\tau-1)d\tau & -8 \le t < 11 \\ \int_{t-2}^{10}(t-\tau-1)d\tau & 11 \le t < 12 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x*h(t) = \begin{cases} 0 & t < 1 \\ \int_0^{t-1}\left(1-\frac{1}{100}\tau\right)e^{\tau-t}d\tau & 1 \le t < 101 \\ \int_0^{100}\left(1-\frac{1}{100}\tau\right)e^{\tau-t}d\tau & t \ge 101; \end{cases}$

(m) $x * h(t) = \begin{cases} \int_{-10}^{t-1} [1-(t-\tau-2)^2] d\tau & -9 \le t < -7 \\ \int_{t-3}^{t-1} [1-(t-\tau-2)^2] d\tau & -7 \le t < 11 \\ \int_{t-3}^{10} [1-(t-\tau-2)^2] d\tau & 11 \le t < 13 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x * h(t) = \begin{cases} \int_0^{t-2} e^{-\tau} e^{3\tau-3t} d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(o) $x * h(t) = \begin{cases} \int_{t-2}^{t-1} e^{\tau} d\tau & t < 1 \\ \int_{t-2}^{0} e^{\tau} d\tau + \int_0^{t-1} e^{-\tau} d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} e^{-\tau} d\tau & t \ge 2; \end{cases}$

(p) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} e^{-2\tau} d\tau & 2 \le t < 3 \\ \int_{t-3}^{t-2} e^{-2\tau} d\tau & t \ge 3; \end{cases}$

(q) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_1^{t-1} \sin(\pi t - \pi\tau - \pi) d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1} \sin(\pi t - \pi\tau - \pi) d\tau & t \ge 3; \end{cases}$

(r) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} 1 d\tau & 2 \le t < 6 \\ \int_{t-6}^{t-2} 1 d\tau & t \ge 6; \end{cases}$

(s) $x * h(t) = \begin{cases} \int_0^{t-1} e^{\tau-2t+2} d\tau & t \ge 1 \\ 0 & \text{otherwise;} \end{cases}$

(t) $x * h(t) = \begin{cases} \int_0^{t+1} e^{-3\tau} d\tau & t \ge -1 \\ 0 & \text{otherwise;} \end{cases}$

(u) $x * h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \le t < -1 \\ -\frac{1}{6}t^3 & -1 \le t < 0 \\ 0 & \text{otherwise} \end{cases}$

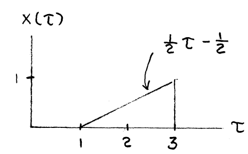
Exercise 4.101

R Answer (h).

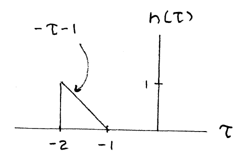
We need to compute $x * h$, where

$$x(t) = \begin{cases} \frac{1}{2}t - \frac{1}{2} & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} -t-1 & -2 \le t < -1 \\ 0 & \text{otherwise.} \end{cases}$$

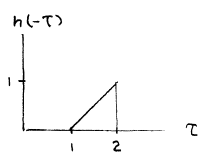
To assist in the convolution computation, we first plot $x(\tau)$ and $h(t-\tau)$ versus τ as shown below in Figures (a) and (d), respectively. (Figures (b) and (c) show intermediate results obtained in the determination of Figure (d).)



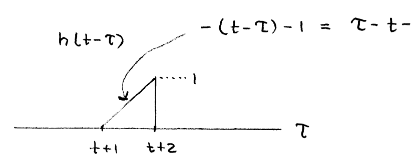
(a)



(b)

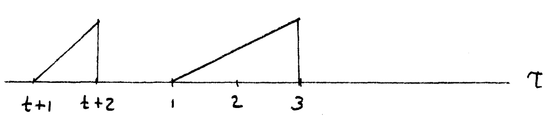


(c)

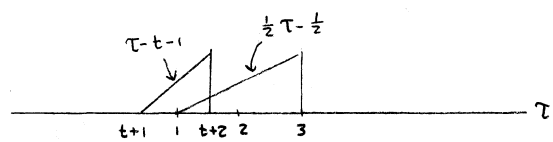


(d)

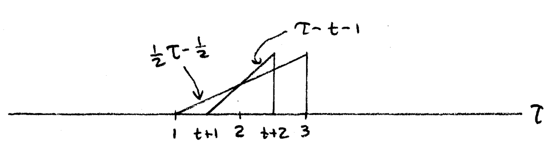
From the above plots, we can deduce that there are five cases (i.e., intervals of t) to be considered, which correspond to the scenarios shown in the graphs below.



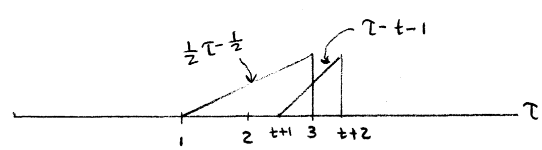
(e)



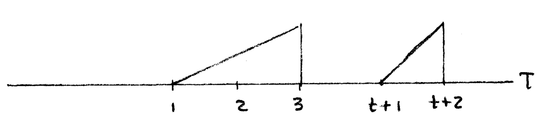
(f)



(g)



(h)



(i)

From Figure (e), for $t < -1$ (i.e., $t+2 < 1$), we have $x*h(t) = 0$. From Figure (f), for $-1 \leq t < 0$ (i.e., $t+2 \geq 1$ and $t+1 < 1$), we have

$$x*h(t) = \int_1^{t+2} \left(\tfrac{1}{2}\tau - \tfrac{1}{2}\right)(\tau - t - 1)d\tau.$$

From Figure (g), for $0 \leq t < 1$ (i.e., $t+1 \geq 1$ and $t+2 < 3$), we have

$$x*h(t) = \int_{t+1}^{t+2} \left(\tfrac{1}{2}\tau - \tfrac{1}{2}\right)(\tau - t - 1)d\tau.$$

From Figure (h), for $1 \leq t < 2$ (i.e., $t+2 \geq 3$ and $t+1 < 3$), we have

$$x*h(t) = \int_{t+1}^{3} \left(\tfrac{1}{2}\tau - \tfrac{1}{2}\right)(\tau - t - 1)d\tau.$$

From Figure (i), for $t \geq 2$ (i.e., $t+1 \geq 3$), we have $x*h(t) = 0$. Combining the above results, we have that

$$x*h(t) = \begin{cases} \int_1^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & -1 \leq t < 0 \\ \int_{t+1}^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & 0 \leq t < 1 \\ \int_{t+1}^{3} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & 1 \leq t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

R **4.106** Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.
(a) $\mathcal{H}x(t) = \int_{t}^{\infty} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} e^{-|\tau|}x(t-\tau)d\tau$;
(c) $\mathcal{H}x(t) = \int_{t-5}^{t-4} x(\tau)d\tau$; and
(d) $\mathcal{H}x(t) = x(t) + x(t-1)$.

Short Answer. (a) $h(t) = u(-t)$; (b) $h(t) = e^{-|t|}$; (c) $h(t) = u(t-4) - u(t-5)$; (d) $h(t) = \delta(t) + \delta(t-1)$

Exercise 4.106

R Answer (b).

We are given the LTI system $\mathcal{H}$ with impulse response h that is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{\infty} e^{-|\tau|} x(t-\tau) d\tau.$$

From the definition of the impulse response, we have

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} e^{-|\tau|} \delta(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} e^{-|\tau|} \delta(\tau-t) d\tau \\
&= \left[e^{-|\tau|} \right] \Big|_{\tau=t} \\
&= e^{-|t|}.
\end{aligned}$$

R **4.107** Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(t) = u(t+1) - u(t-1)$;
(b) $h(t) = e^{-5t}u(t-1)$;
(c) $h(t) = (t^2-1)\sin(t)\delta(t+1)$;
(d) $h(t) = \pi\delta(t+42)$;
(e) $h(t) = (t^2+4)[u(t+5) - u(t+3)]$; and
(f) $h(t) = \cos(t)\delta(t+\frac{\pi}{2}) + 5\delta(t)$.

Short Answer. (a) has memory, not causal; (b) has memory, causal; (c) memoryless, causal; (d) has memory, not causal; (e) has memory, not causal; (f) memoryless, causal

Exercise 4.107

R **Answer (a).**

The impulse response h can be equivalently expressed as

$$h(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $t = -\frac{1}{2}$), the system has memory. Since $h(t) \neq 0$ for some $t < 0$ (e.g., $t = -\frac{1}{2}$), the system is not causal.

R **4.108** Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = u(t-1) - u(t-2)$;
(b) $h(t) = e^{-2t^2}$ [Hint: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$];
(c) $h(t) = t^{-2}u(-t-1)$;
(d) $h(t) = e^{-t}\sin(t)u(t)$;
(e) $h(t) = e^{-t}u(-t)$;
(f) $h(t) = te^{-3t}u(t-1)$ [Hint: See (F.1).]; and
(g) $h(t) = te^{-2t}u(1-t)$ [Hint: See (F.1).].

Short Answer. (a) BIBO stable ($\int_{-\infty}^{\infty} |h(t)|\,dt = 1$); (b) BIBO stable ($\int_{-\infty}^{\infty} |h(t)|\,dt = \sqrt{\frac{\pi}{2}}$); (c) BIBO stable ($\int_{-\infty}^{\infty} |h(t)|\,dt = 1$); (d) BIBO stable ($\int_{-\infty}^{\infty} |h(t)|\,dt \leq 1$); (e) not BIBO stable; (f) BIBO stable ($\int_{-\infty}^{\infty} |h(t)|\,dt = \frac{4}{9e^3}$); (g) not BIBO stable

Exercise 4.108

R Answer (b).

We are given a LTI system with impulse response

$$h(t) = e^{-2t^2}.$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{-2t^2}\right| dt \\ &= \int_{-\infty}^{\infty} e^{-2t^2} dt \\ &= \int_{-\infty}^{\infty} e^{-\left(\sqrt{2}t\right)^2} dt.\end{aligned}$$

Now, we employ a change of variable. Let $\tau = \sqrt{2}t$ so that $t = \frac{1}{\sqrt{2}}\tau$ and $dt = \frac{1}{\sqrt{2}}d\tau$. Applying the change of variable, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} e^{-\tau^2}\left(\tfrac{1}{\sqrt{2}}\right) d\tau \\ &= \tfrac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau \\ &= \tfrac{1}{\sqrt{2}}\sqrt{\pi} \\ &= \sqrt{\tfrac{\pi}{2}}.\end{aligned}$$

Since $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$, the system is BIBO stable.

R **4.109** For each case below, find the response y of the LTI system with system function H to the input x.

(a) $H(s) = \dfrac{1}{(s+1)(s+2)}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -1$; and $x(t) = 1 + \frac{1}{2}e^{-t/2} + \frac{1}{3}e^{-t/3}$;

(b) $H(s) = s$ for all $s \in \mathbb{C}$; and $x(t) = 1 + 2e^{-t/2} + 3e^{-t/3}$;

(c) $H(s) = \dfrac{1}{s+1}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -1$; and $x(t) = 2\cos(t)$;

(d) $H(s) = se^{-s}$ for all $s \in \mathbb{C}$; and $x(t) = 4\cos(t) + 2\sin(3t)$;

(e) $H(s) = \dfrac{1}{e^s(s+4)}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -4$; and $x(t) = 11 + 7e^{-2t} + 5e^{-3t}$; and

(f) $H(s) = s^2$ for all $s \in \mathbb{C}$; and $x(t) = 7 + e^{-5t} + 4\cos(3t)$.

Short Answer. (a) $y(t) = \frac{1}{2} + \frac{2}{3}e^{-t/2} + \frac{3}{10}e^{-t/3}$; (b) $y(t) = -e^{-t/2} - e^{-t/3}$; (c) $y(t) = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$; (d) $y(t) = 6\cos(3t-3) - 4\sin(t-1)$; (e) $y(t) = \frac{11}{4} + \frac{7}{2}e^{2(1-t)} + 5e^{3(1-t)}$; (f) $y(t) = 25e^{-5t} - 36\cos(3t)$

Exercise 4.109

R Answer (a).

We are given a LTI system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } s \in \mathbb{C} \text{ such that } \mathrm{Re}(s) > -1.$$

Furthermore, we are given

$$x(t) = 1 + \tfrac{1}{2}e^{-t/2} + \tfrac{1}{3}e^{-t/3}.$$

Since the system is LTI, the response y of the system to the input x is given by

$$\begin{aligned}
y(t) &= H(0)[1] + H\left(-\tfrac{1}{2}\right)\left[\tfrac{1}{2}e^{-t/2}\right] + H\left(-\tfrac{1}{3}\right)\left[\tfrac{1}{3}e^{-t/3}\right] \\
&= \tfrac{1}{2}[1] + \tfrac{1}{(1/2)(3/2)}\left[\tfrac{1}{2}e^{-t/2}\right] + \tfrac{1}{(2/3)(5/3)}\left[\tfrac{1}{3}e^{-t/3}\right] \\
&= \tfrac{1}{2} + \tfrac{2}{3}e^{-t/2} + \tfrac{3}{10}e^{-t/3}.
\end{aligned}$$

