Example 12.42 (Savings withdrawal schedule)**.** Consider a savings account that earns compounding interest at a fixed rate of 6% annually and is compounded monthly. The balance of the account is initially \$10000. Starting in the first month of the second year, monthly withdrawals of \$100 are started. Determine how long it will take in order for the account to become overdrawn (i.e., have a negative balance).

Solution. Let $y(n)$ denote the balance of the account at the start of the nth compounding period, and let $x(n)$ denote the amount withdrawn at the start of the same compounding period. Let y be a sequence such that $y(n)$ is the account balance at the start of the nth compounding period. Then, y satisfies the difference equation

$$y(n) = \left(1 + \tfrac{\alpha r}{100}\right) y(n-1) + x(n),$$

where $x(n)$ is the amount added to the account at the start of the nth compounding period, r is the annual interest rate in percent, and α is the fraction of a year in the compounding period (e.g., $\alpha = \frac{1}{12}$ for monthly compounding). Substituting the given values of $r = 6$ and $\alpha = \frac{1}{12}$, we obtain

$$y(n) = \left(\tfrac{201}{200}\right) y(n-1) + x(n), \tag{12.20}$$

The initial condition $y(-1)$ corresponds to the initial account balance. Thus, we have

$$y(-1) = 10000. \tag{12.21}$$

The time index n is in units of compounding periods, with $n = -1$ corresponding to the starting time (i.e., time "zero"). Withdrawals commence at the beginning of the second year, which corresponds to $n = -1 + 12 = 11$. So, we have

$$x(n) = -100u(n-11). \tag{12.22}$$

Taking the unilateral z transform of the causal sequence x (in (12.22)), we have

$$X(z) = \frac{-100z^{-11}}{1-z^{-1}}.$$

Taking the unilateral z transform of the difference equation (12.20), we have

$$\begin{aligned}
& Y(z) = \tfrac{201}{200}\left(z^{-1}Y(z) + y(-1)\right) + X(z) \\
\Rightarrow\quad & Y(z) = \tfrac{201}{200}z^{-1}Y(z) + \tfrac{201}{200}y(-1) + X(z) \\
\Rightarrow\quad & Y(z) = \tfrac{201}{200}z^{-1}Y(z) + 10050 - \frac{100z^{-11}}{1-z^{-1}}.
\end{aligned}$$

Rearranging the above equation to solve for Y, we have

$$\begin{aligned}
& \left(1 - \tfrac{201}{200}z^{-1}\right)Y(z) = 10050 - \frac{100z^{-11}}{1-z^{-1}} \\
\Rightarrow\quad & Y(z) = \frac{10050}{1 - \frac{201}{200}z^{-1}} - \frac{100z^{-11}}{\left(1-z^{-1}\right)\left(1 - \frac{201}{200}z^{-1}\right)}.
\end{aligned}$$

Now, we find a partial fraction expansion of the second term in the preceding equation for Y excluding the z^{-11} factor. This expansion has the form

$$\frac{100}{\left(1-z^{-1}\right)\left(1 - \frac{201}{200}z^{-1}\right)} = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1 - \frac{201}{200}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1-z^{-1}\right)Y(z)\big|_{z=1} \\
&= \left.\frac{1}{1-\frac{201}{200}z^{-1}}\right|_{z=1} \\
&= \frac{1}{-1/200} = -20000 \quad \text{and} \\
A_2 &= \left(1-\tfrac{201}{200}z^{-1}\right)Y(z)\big|_{z=201/200} \\
&= \left.\frac{100}{1-z^{-1}}\right|_{z=201/200} \\
&= \frac{100}{1/201} = 20100.
\end{aligned}$$

Substituting the above partial fraction expansion into the above expression for $Y(z)$, we have

$$\begin{aligned}
Y(z) &= \frac{10050}{1-\frac{201}{200}z^{-1}} - z^{-11}\left[\frac{-20000}{1-z^{-1}} + \frac{20100}{1-\frac{201}{200}z^{-1}}\right] \\
&= 10050\frac{1}{1-\frac{201}{200}z^{-1}} + 20000z^{-11}\frac{1}{1-z^{-1}} - 20100z^{-11}\frac{1}{1-\frac{201}{200}z^{-1}}.
\end{aligned}$$

Taking the inverse z transform of Y, we have

$$y(n) = 10050\left(\tfrac{201}{200}\right)^n u(n) + 20000u(n-11) - 20100\left(\tfrac{201}{200}\right)^{n-11} u(n-11) \ \text{ for } n \geq 0.$$

The sequence y starts monotonically decreasing for $n > 11$ and $y(160) \approx 61.20 \geq 0$ and $y(161) \approx -38.50 < 0$. Therefore, the account becomes overdrawn at $n = 161$, which corresponds to $161-(-1) = 162$ months from the time of the initial balance. ■

Example 12.43 (Fibonacci sequence)**.** The Fibonacci sequence f is defined as

$$f_n = \begin{cases} f_{n-1} + f_{n-2} & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0, \end{cases}$$

where n is a nonnegative integer. The first few elements of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

Use the z transform to find a closed-form expression for the nth Fibonacci number f_n.

Solution. The Fibonacci sequence can be expressed in terms of a linear difference equation with constant coefficients. In particular, the Fibonacci sequence is given by the solution x to the difference equation

$$x(n) = x(n-1) + x(n-2) + \delta(n-1), \tag{12.23a}$$

with the initial conditions

$$x(-1) = x(-2) = 0. \tag{12.23b}$$

We can easily confirm that this equation is correct. Substituting $n = 0$ and $n = 1$ into (12.23a) yields

$$\begin{aligned} x(0) &= x(-1) + x(-2) + \delta(-1) \\ &= 0 + 0 + 0 \\ &= 0 \\ &= f_0 \quad \text{and} \\ x(1) &= x(0) + x(-1) + \delta(0) \\ &= 0 + 0 + 1 \\ &= 1 \\ &= f_1. \end{aligned}$$

Substituting $n \geq 2$ into (12.23a) yields

$$\begin{aligned} x(n) &= x(n-1) + x(n-2) + \delta(n-1) \\ &= x(n-1) + x(n-2) + 0 \\ &= x(n-1) + x(n-2) \\ &= f_n. \end{aligned}$$

Thus, the Fibonacci sequence is, in fact, the solution of (12.23a).

Now, we employ the z transform to solve the above difference equation. Taking the unilateral z transform of the difference equation, we obtain

$$\begin{aligned} & \mathcal{Z}_\mathrm{u}x(z) = z^{-1}\mathcal{Z}_\mathrm{u}x(z) + x(-1) + z^{-1}\left[z^{-1}\mathcal{Z}_\mathrm{u}x(z) + x(-1)\right] + x(-2) + z^{-1}\mathcal{Z}_\mathrm{u}\delta(z) + \delta(-1). \\ \Rightarrow\quad & X(z) = z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2) + z^{-1}\mathcal{Z}_\mathrm{u}\delta(z) + \delta(-1). \\ \Rightarrow\quad & X(z) = z^{-1}X(z) + z^{-2}X(z) + z^{-1}. \\ \Rightarrow\quad & (1 - z^{-1} - z^{-2})X(z) = z^{-1} \\ \Rightarrow\quad & X(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}. \end{aligned}$$

(Note that, since all of the initial conditions are zero, we could have simply employed the bilateral z transform.) So, we have

$$X(z) = \frac{z}{z^2 - z - 1}.$$

We factor the denominator polynomial in order to determine the poles of X. Using the quadratic formula, we find the roots of $z^2 - z - 1$ to be $\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} = \frac{1\pm\sqrt{5}}{2}$. Thus, we have

$$X(z) = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}.$$

Now, we find a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{1}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}.$$

The expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z - \frac{1+\sqrt{5}}{2}} + \frac{A_2}{z - \frac{1-\sqrt{5}}{2}}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned}
A_1 &= \left(z - \tfrac{1+\sqrt{5}}{2}\right)\left(\tfrac{X(z)}{z}\right)\Big|_{z=\frac{1+\sqrt{5}}{2}} \\
&= \left(\frac{1}{z - \frac{1-\sqrt{5}}{2}}\right)\Bigg|_{z=\frac{1+\sqrt{5}}{2}} \\
&= \tfrac{1}{\sqrt{5}} \quad \text{and} \\
A_2 &= \left(z - \tfrac{1-\sqrt{5}}{2}\right)\left(\tfrac{X(z)}{z}\right)\Big|_{z=\frac{1-\sqrt{5}}{2}} \\
&= \left(\frac{1}{z - \frac{1+\sqrt{5}}{2}}\right)\Bigg|_{z=\frac{1-\sqrt{5}}{2}} \\
&= -\tfrac{1}{\sqrt{5}}.
\end{aligned}$$

Thus, we have

$$X(z) = \tfrac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1+\sqrt{5}}{2}}\right) - \tfrac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1-\sqrt{5}}{2}}\right).$$

Taking the inverse z transform, we obtain

$$x(n) = \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^n \quad \text{for } n \geq 0.$$

■

Example 12.44 (First-order difference equation)**.** Consider the causal system with input x and output y characterized by the difference equation

$$2y(n) + y(n-1) = x(n).$$

If $x(n) = \left(\frac{1}{4}\right)^n u(n)$ and $y(-1) = 2$, find y.

Solution. Taking the unilateral z transform of x, we have

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}.$$

Taking the unilateral z transform of both sides of the given difference equation yields

$$2Y(z) + z^{-1}Y(z) + y(-1) = X(z).$$

Substituting the above expression for X and the given initial condition into this equation yields

$$\begin{aligned}
& 2Y(z) + z^{-1}Y(z) + 2 = \frac{1}{1 - \frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & \left(2 + z^{-1}\right)Y(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} - 2 \\
\Leftrightarrow \quad & 2\left(1 + \tfrac{1}{2}z^{-1}\right)Y(z) = \frac{1 - 2 + \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & 2\left(1 + \tfrac{1}{2}z^{-1}\right)Y(z) = \frac{-1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & Y(z) = \frac{-\frac{1}{2} + \frac{1}{4}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}.
\end{aligned}$$

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1-\tfrac{1}{4}z^{-1}\right)Y(z)\Big|_{z=1/4} \\
&= \left.\frac{-\frac{1}{2}+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right|_{z=1/4} = \tfrac{1/2}{3} \\
&= \tfrac{1}{6} \quad \text{and} \\
A_2 &= (1+\tfrac{1}{2}z^{-1})Y(z)\Big|_{z=-1/2} \\
&= \left.\frac{-\frac{1}{2}+\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}\right|_{z=-1/2} = \tfrac{-1}{3/2} \\
&= -\tfrac{2}{3}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(z) = \tfrac{1}{6}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) - \tfrac{2}{3}\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right).$$

Taking the inverse unilateral z transform of Y yields

$$y(n) = \tfrac{1}{6}\left(\tfrac{1}{4}\right)^n - \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^n \quad \text{for } n \geq 0.$$ ■

Example 12.45 (Second-order difference equation)**.** Consider the causal system with input x and output y characterized by the difference equation

$$10y(n) + 3y(n-1) - y(n-2) = x(n).$$

If $x(n) = 6u(n)$, $y(-1) = 3$, and $y(-2) = 6$, find y.

Solution. Taking the unilateral z transform of x, we obtain

$$X(z) = \frac{6}{1-z^{-1}}.$$

Taking the unilateral z transform of both sides of the given difference equation yields

$$\begin{aligned}
& 10Y(z) + 3\left[z^{-1}Y(z) + y(-1)\right] - \left(z^{-1}\left[z^{-1}Y(z) + y(-1)\right] + y(-2)\right) = X(z) \\
\Rightarrow\quad & 10Y(z) + 3z^{-1}Y(z) + 3y(-1) - \left(z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right) = X(z) \\
\Rightarrow\quad & 10Y(z) + 3z^{-1}Y(z) + 3y(-1) - z^{-2}Y(z) - z^{-1}y(-1) - y(-2) = X(z) \\
\Rightarrow\quad & 10Y(z) + 3z^{-1}Y(z) - z^{-2}Y(z) + 3y(-1) - y(-2) - z^{-1}y(-1) = X(z).
\end{aligned}$$

Rearranging in order to solve for Y, we have

$$\begin{aligned}
& \left(10 + 3z^{-1} - z^{-2}\right)Y(z) = -3y(-1) + y(-2) + z^{-1}y(-1) + X(z) \\
\Rightarrow\quad & Y(z) = \frac{-3y(-1) + y(-2) + z^{-1}y(-1) + X(z)}{10 + 3z^{-1} - z^{-2}}.
\end{aligned}$$

Now, we factor the denominator of Y. Solving for the roots of $10+3z^{-1}-z^{-2} = z^{-2}(10z^2+3z-1) = 0$, we obtain $\frac{-3\pm\sqrt{3^2-4(10)(-1)}}{2(10)} = \frac{3\pm7}{20}$ So, the roots of $10+3z^{-1}-z^{-2}=0$ are $\frac{1}{5}$ and $-\frac{1}{2}$. Thus, we have

$$Y(z) = \frac{-3y(-1)+y(-2)+z^{-1}y(-1)+X(z)}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.$$

Substituting the above expression for X and the given initial conditions into the above equation for Y, we obtain

$$\begin{aligned}
Y(z) &= \frac{-3(3)+6+3z^{-1}+\frac{6}{1-z^{-1}}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{-3+3z^{-1}+\frac{6}{1-z^{-1}}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{(-3+3z^{-1})(1-z^{-1})+6}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{-3+6z^{-1}-3z^{-2}+6}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{3+6z^{-1}-3z^{-2}}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.
\end{aligned}$$

Now, we find a partial fraction expansion for Y. Such an expansion has the form

$$Y(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}} + \frac{A_3}{1-\frac{1}{5}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1-z^{-1}\right)Y(z)\big|_{z=1} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}\right|_{z=1} = \frac{6}{10(\frac{3}{2})(\frac{4}{5})} \\
&= \tfrac{1}{2}, \\
A_2 &= \left(1+\tfrac{1}{2}z^{-1}\right)Y(z)\big|_{z=-1/2} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10(1-z^{-1})\left(1-\frac{1}{5}z^{-1}\right)}\right|_{z=-1/2} = \frac{3-12-12}{10(3)\left(\frac{7}{5}\right)} = \tfrac{-21}{42} \\
&= -\tfrac{1}{2}, \quad \text{and} \\
A_3 &= \left(1-\tfrac{1}{5}z^{-1}\right)Y(z)\big|_{z=1/5} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)}\right|_{z=1/5} = \tfrac{3+30-75}{10(-4)(7/2)} = \tfrac{42}{140} \\
&= \tfrac{3}{10}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(z) = \tfrac{1}{2}\left(\frac{1}{1-z^{-1}}\right) - \tfrac{1}{2}\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + \tfrac{3}{10}\left(\frac{1}{1-\frac{1}{5}z^{-1}}\right).$$

Taking the inverse z transform of Y, we obtain

$$y(n) = \tfrac{1}{2} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n + \tfrac{3}{10}\left(\tfrac{1}{5}\right)^n \quad \text{for } n \geq 0.$$ ■

12.18 Exercises

12.18.1 Exercises Without Answer Key

12.1 Using the definition of the z transform, find the z transform X of each sequence x given below.
(a) $x(n) = nu(n)$;
(b) $x(n) = na^n u(n)$, where a is a complex constant; and
(c) $x(n) = \cos(an)u(n)$, where a is a nonzero real constant.

12.2 Using properties of the z transform and a table of z transform pairs, find the z transform X of each sequence x given below.
(a) $x(n) = n\left(\frac{1}{2}\right)^{|n|}$;
(b) $x(n) = n\left(\frac{1}{3}\right)^{|n-1|}$;
(c) $x(n) = \left(\frac{2}{3}\right)^{n-1} u(n-1)$;
(d) $x(n) = 2^n u(-n)$;
(e) $x(n) = \delta(n+5) - \delta(n-5)$;
(f) $x(n) = 3^n u(-n-2)$;
(g) $x(n) = x_1 * x_2(n)$, where $x_1(n) = 2^{n-1}u(n)$ and $x_2(n) = \cos(\frac{\pi}{6}n + \frac{\pi}{3})u(n)$;
(h) $x(n) = n\sin\left(\frac{\pi}{2}n\right)u(n)$;
(i) $x(n) = u * x_1(n)$, where $x_1(n) = u(n-1)$; and
(j) $x(n) = c^n \cos(an+b)u(-n-1)$, where $a, b \in \mathbb{R}$, $c \in \mathbb{C}$, $a \neq 0$, and $|c| > 1$.

12.3 For each z transform algebraic expression X given below, determine all possible ROCs of X.
(a) $X(z) = \dfrac{3}{1 + \frac{1}{3}z^{-1}}$;
(b) $X(z) = \dfrac{z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 3z^{-1}\right)}$;
(c) $X(z) = 1 + z^{-1}$;
(d) $X(z) = \dfrac{z}{(z^2 + \frac{1}{4})(z^2 - \frac{1}{4})}$; and
(e) $X(z) = \dfrac{z}{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{5}\right)}$.

12.4 Determine whether the sequence x with each z transform X given below is finite duration, right sided but not left sided, left sided but not right sided, or two sided.
(a) $X(z) = \dfrac{5}{1 + \frac{1}{3}z^{-1}}$ for $|z| > \frac{1}{3}$;
(b) $X(z) = \dfrac{z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$ for $\frac{1}{4} < |z| < \frac{1}{2}$;
(c) $X(z) = \dfrac{z^{-1}}{1 - z^{-1}}$ for $|z| < 1$;
(d) $X(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})$ for $|z| > 0$; and
(e) $X(z) = \frac{1}{3}(z^2 + z + 1)$ for all finite z.

12.5 Find the inverse z transform x of each function X given below. If a partial fraction expansion is used, it must be chosen so as to avoid unnecessary time shifts in the final answer for x.
(a) $X(z) = \dfrac{10z^2 - 15z + 3}{(z-3)\left(z - \frac{1}{3}\right)}$ for $|z| > 3$;

(b) $X(z) = \dfrac{5z^2 - 8z + 2}{(z-2)\left(z-\frac{1}{2}\right)}$ for $\frac{1}{2} < |z| < 2$;
(c) $X(z) = \dfrac{1 - z^{-9}}{z-1}$ for $|z| > 1$ [Hint: A PFE is not necessary.];
(d) $X(z) = \dfrac{2z^3}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ for $|z| > \frac{3}{4}$;
(e) $X(z) = \dfrac{1}{(z-1)\left(z+\frac{1}{2}\right)}$ for $|z| > 1$;
(f) $X(z) = \dfrac{2z}{1-z} + 1 + z^{-1}$ for $|z| > 1$; and
(g) $X(z) = \dfrac{\frac{5}{6}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{9}z^{-1}\right)}$ for $|z| < \frac{2}{9}$.

12.6 For each algebraic expression X given below for the z transform of x, find all possible x.
(a) $X(z) = \dfrac{2 - z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{2}z^{-1}\right)}$;
(b) $X(z) = \dfrac{5}{1+\frac{1}{3}z^{-1}}$; and
(c) $X(z) = \dfrac{-z^2 + \frac{1}{4}z}{(z-1)\left(z-\frac{3}{4}\right)}$.

12.7 Find the inverse z transform x of each function X given below.
(a) $X(z) = e^{a/z}$ for $|z| > 0$, where a is a complex constant [Hint: Use (F.12).];
(b) $X(z) = -\ln(1 - az)$ for $|z| < |a|^{-1}$ [Hint: Use (F.13).];
(c) $X(z) = -\ln(1 - a^{-1}z^{-1})$ for $|z| > |a|^{-1}$ [Hint: Use (F.13).]; and
(d) $X(z) = \sin\left(z^{-1}\right)$ for $|z| > 0$. [Hint: Use (F.10).].

12.8 For the causal LTI system with input x and output y characterized by each difference equation given below, find the system function H of the system.
(a) $y(n) - \frac{2}{3}y(n-1) + \frac{1}{9}y(n-2) = x(n) + \frac{1}{2}x(n-1)$; and
(b) $y(n+2) - \frac{1}{4}y(n+1) - \frac{1}{4}y(n) + \frac{1}{16}y(n-1) = x(n+2) - x(n+1)$.

12.9 For each LTI system whose system function H is given below, find the difference equation that characterizes the system.
(a) $H(z) = \dfrac{1 - z^{-1} - \frac{2}{9}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{16}z^{-3}}$ for $|z| > \frac{1}{2}$; and
(b) $H(z) = \dfrac{z^2 + \frac{1}{4}}{z^3 - \frac{17}{12}z^2 + \frac{5}{8}z - \frac{1}{12}}$ for $|z| > \frac{2}{3}$.

12.10 Determine whether the LTI system having each system function H given below is causal.
(a) $H(z) = \dfrac{z^2 + 3z + 2}{z-1}$ for $|z| > 1$;
(b) $H(z) = \dfrac{1 + 3z^{-1}}{1 - z^{-1}}$ for $|z| < 1$; and
(c) $H(z) = \dfrac{1 + \frac{1}{2}z^{-1}}{z - \frac{1}{2}}$ for $|z| > \frac{1}{2}$.

12.11 For the LTI system $\mathcal{H}$ with each system function H given below, find all inverses of $\mathcal{H}$ (specified in terms of system functions). Comment on the causality and BIBO stability of each inverse system.

(a) $H(z) = \dfrac{z-\frac{1}{4}}{z-\frac{1}{2}}$ for $|z| > \frac{1}{2}$;

(b) $H(z) = \dfrac{1}{(z^2-\frac{1}{4})(z^2-\frac{1}{9})}$ for $|z| > \frac{1}{2}$; and

(c) $H(z) = \dfrac{\left(z^2-\frac{1}{4}\right)\left(z^2+4\right)}{\left(z^2-\frac{1}{9}\right)\left(z^2+\frac{1}{9}\right)}$ for $|z| > \frac{1}{3}$.

12.12 For each LTI system whose system function H is given below, determine the ROC of H if the system is BIBO stable (if such a ROC exists).

(a) $H(z) = \dfrac{1+z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$;

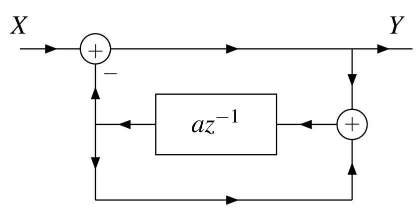
(b) $H(z) = \dfrac{1-\frac{1}{3}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$;

(c) $H(z) = \dfrac{1+z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$;

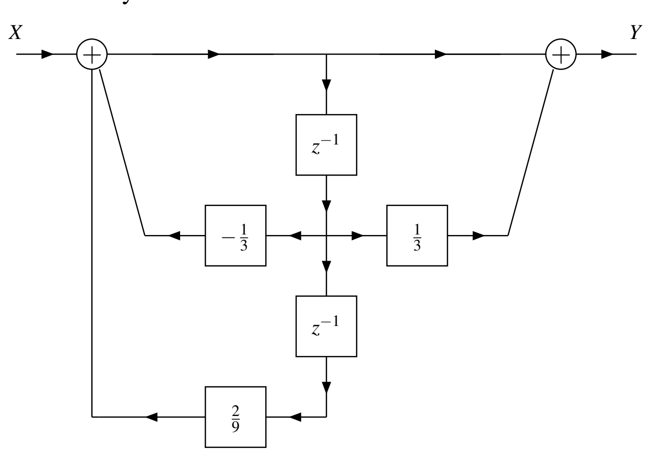
(d) $H(z) = \dfrac{1}{1-5z^{-1}+6z^{-2}}$; and

(e) $H(z) = \dfrac{z^2-1}{z^2+1}$.

12.13 Consider the system with input z transform X and output z transform Y as shown in the figure, where each causal LTI subsystem is labelled with its system function and a is an arbitrary real constant. (a) Find the system function H of the overall system. (b) Determine for what values of a the system is BIBO stable.



12.14 Consider the system $\mathcal{H}$ with input z transform X and output z transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine whether the system $\mathcal{H}$ is BIBO stable.



12.15 For the system with input x and output y characterized by each difference equation given below, find y for the given x and initial conditions.
(a) $y(n) - \frac{1}{2}y(n-1) = x(n)$, where $x(n) = 2u(n)$ and $y(-1) = 2$;
(b) $y(n) + \frac{1}{4}y(n-2) = x(n)$, where $x(n) = 0$, $y(-1) = 0$, and $y(-2) = 4$; and
(c) $y(n) + 3y(n-1) = x(n)$, where $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and $y(-1) = 1$.

12.18.2 Exercises With Answer Key

12.101 Find the inverse z transform x of each function X given below.
(a) $X(z) = \dfrac{z\left(10z^2 - \frac{27}{2}z + \frac{15}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)}$ for $|z| > 1$.

Short Answer. (a) $x(n) = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u(n)$

12.19 MATLAB Exercises

Currently, there are no MATLAB exercises.

Part III

Appendices

Appendix A

Complex Analysis

A.1 Introduction

Complex analysis is an essential tool in the study of signals and systems. For this reason, a brief review of complex analysis is provided in this appendix.

A.2 Complex Numbers

A **complex number** is a number of the form

$$z = x + jy,$$

where x and y are real and j is the constant defined by

$$j^2 = -1$$

(i.e., $j = \sqrt{-1}$). The **real part**, **imaginary part**, **magnitude**, and **argument** of the complex number z are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, $|z|$, and $\arg z$, respectively, and defined as

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y,$$
$$|z| = \sqrt{x^2 + y^2}, \quad \text{and} \quad \arg z = \operatorname{atan2}(y,x) + 2\pi k,$$

where k is an arbitrary integer, and

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases} \tag{A.1}$$

(The notation $\angle z$ is sometimes also used to denote the quantity $\arg z$.) The complex number z can be represented by a point (x,y) in the complex plane, as illustrated in Figure A.1. This figure also shows the relationship between the real part, imaginary part, magnitude, and argument of a complex number.

For any given complex number z, the quantity $\arg z$ is not unique. This follows from the fact that, for any integer k, the quantities θ and $\theta + 2\pi k$ physically represent the same overall angular displacement. The value θ of $\arg z$ that lies in the range $-\pi < \theta \leq \pi$ is called the **principal argument** of z and is denoted as $\operatorname{Arg} z$. For a particular nonzero z, this quantity is uniquely specified. In particular, $\operatorname{Arg} z = \operatorname{atan2}(y,x)$.

As an aside, we note that the function $\operatorname{atan2}(y,x)$ computes the angle that the directed line segment from $(0,0)$ to (x,y) forms with the real axis, and is defined such that $-\pi < \operatorname{atan2}(y,x) \leq \pi$.

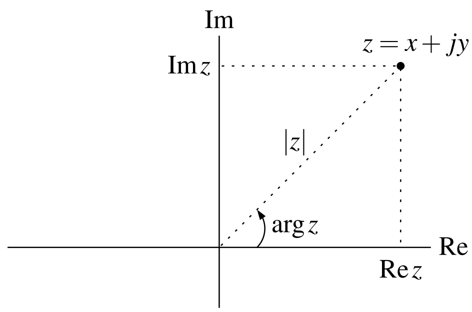


Figure A.1: Graphical representation of a complex number.

Example A.1. Compute the real part, imaginary part, magnitude, and principal argument of each complex number z below.
(a) $z = \frac{\sqrt{3}}{2} + j\frac{1}{2}$; and
(b) $z = 1 - j$.

Solution. (a) We have

$$\operatorname{Re} z = \tfrac{\sqrt{3}}{2}, \quad \operatorname{Im} z = \tfrac{1}{2},$$
$$|z| = \sqrt{\left(\tfrac{\sqrt{3}}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = 1, \quad \text{and}$$
$$\operatorname{Arg} z = \arctan\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{\pi}{6}.$$

(b) We have

$$\operatorname{Re} z = 1, \quad \operatorname{Im} z = -1,$$
$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \text{and}$$
$$\operatorname{Arg} z = \arctan\left(\tfrac{-1}{1}\right) = -\tfrac{\pi}{4}.$$

■

A.3 Representations of Complex Numbers

Two different representations are commonly used for complex numbers, namely, the Cartesian and polar forms. The Cartesian form is also sometimes referred to as rectangular form. Depending on the particular situation, one form may be more convenient to use than the other.

In the case of the **Cartesian form**, a complex number z is represented as

$$z = x + jy,$$

where x and y are real. That is, z is expressed directly in terms of its real and imaginary parts. The quantity z can also be treated as a point (x, y) in a Cartesian coordinate system as shown in Figure A.2(a).

In the case of the **polar form**, a complex number z is represented as

$$z = r(\cos\theta + j\sin\theta),$$

where r and θ are real and $r \geq 0$. One can show through simple geometry that $r = |z|$ and $\theta = \arg z$. Thus, in the polar case, a complex number is expressed directly in terms of its magnitude and argument. In this way, we

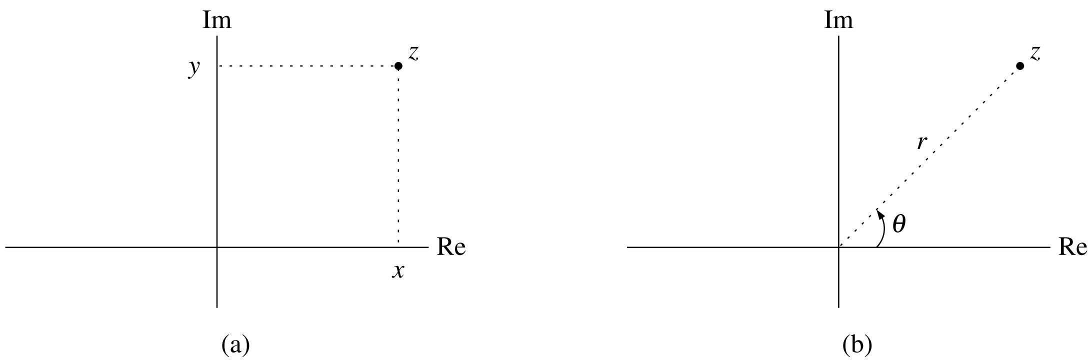


Figure A.2: Representations of complex numbers. The (a) Cartesian and (b) polar forms.

can treat the quantity z as a point (r,θ) in a polar coordinate system, as shown in Figure A.2(b). As we note later, $\cos\theta + j\sin\theta = e^{j\theta}$. Therefore, the polar form can equivalently be expressed as

$$z = re^{j\theta}.$$

This exponential notation is often used, due to its relative compactness.

A.4 Arithmetic Operations

In what follows, we consider a few basic arithmetic operations on complex numbers.

A.4.1 Conjugation

The **conjugate** of the complex number $z = x + jy$ (where x and y are real) is denoted as z^* and defined as

$$z^* = x - jy.$$

Geometrically, the conjugation operation reflects a point in the complex plane about the real axis, as illustrated in Figure A.3. One can easily verify that for any complex numbers z, z_1, and z_2, the following identities hold:

$$\begin{aligned} zz^* &= |z|^2; \\ \operatorname{Re} z &= \tfrac{1}{2}(z+z^*); \\ \operatorname{Im} z &= \tfrac{1}{2j}(z-z^*); \\ (z_1+z_2)^* &= z_1^*+z_2^*; \\ (z_1z_2)^* &= z_1^*z_2^*; \quad \text{and} \\ (z_1/z_2)^* &= z_1^*/z_2^* \text{ for } z_2 \neq 0. \end{aligned}$$

Trivially, we also have that

$$\begin{aligned} |z^*| &= |z| \quad \text{and} \\ \arg z^* &= -\arg z. \end{aligned}$$

A.4.2 Addition

Consider the addition of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are expressed in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

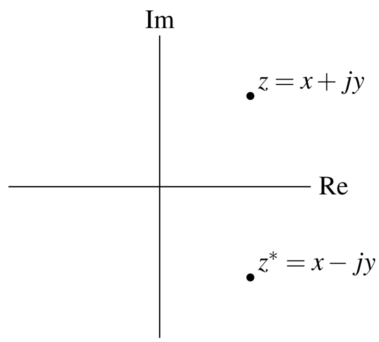


Figure A.3: Conjugate of complex number.

Then, the sum of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

Suppose that z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the sum of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

Example A.2 (Addition with the Cartesian form)**.** Given that $z_1 = 3 + j4$ and $z_2 = -2 - j3$, compute $z_1 + z_2$.

Solution. We have

$$z_1 + z_2 = (3 + j4) + (-2 - j3) = 1 + j. \qquad \blacksquare$$

Example A.3 (Addition with the polar form)**.** Given that $z_1 = 3 + j4$ and $z_2 = \sqrt{2}e^{j5\pi/4}$, compute $z_1 + z_2$.

Solution. In order to compute this sum, we first convert z_2 to Cartesian form to obtain

$$\begin{aligned} \operatorname{Re} z_2 &= \sqrt{2}\cos\left(\tfrac{5\pi}{4}\right) = -1 \quad \text{and} \\ \operatorname{Im} z_2 &= \sqrt{2}\sin\left(\tfrac{5\pi}{4}\right) = -1. \end{aligned}$$

Then, we have

$$z_1 + z_2 = (3 + j4) + (-1 - j) = 2 + j3. \qquad \blacksquare$$

A.4.3 Multiplication

Consider the multiplication of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the product of z_1 and z_2 can be computed as

$$\begin{aligned} z_1z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1x_2 + jx_1y_2 + jx_2y_1 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1). \end{aligned}$$

Suppose that z_1 and z_2 are represented in polar form as

$$z_1 = r_1e^{j\theta_1} \quad \text{and} \quad z_2 = r_2e^{j\theta_2}.$$

Then, the product of z_1 and z_2 can be computed as

$$\begin{aligned} z_1z_2 &= \left(r_1e^{j\theta_1}\right)\left(r_2e^{j\theta_2}\right) \\ &= r_1r_2e^{j(\theta_1+\theta_2)}. \end{aligned}$$

Example A.4 (Multiplication with the Cartesian form)**.** Given that $z_1 = 2 + j3$ and $z_2 = 3 - j4$, compute z_1z_2.

Solution. Using straightforward algebraic manipulation, we have

$$z_1z_2 = (2 + j3)(3 - j4) = 6 - j8 + j9 + 12 = 18 + j. \qquad \blacksquare$$

Example A.5 (Multiplication with the polar form)**.** Given that $z_1 = \sqrt{2}e^{j\pi/4}$ and $z_2 = 3e^{j\pi/6}$, compute z_1z_2.

Solution. Using straightforward algebraic manipulation, we obtain

$$z_1z_2 = (\sqrt{2}e^{j\pi/4})(3e^{j\pi/6}) = 3\sqrt{2}e^{j5\pi/12}. \qquad \blacksquare$$

A.4.4 Division

Consider the division of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the quotient of z_1 and z_2 can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \left(\frac{z_1}{z_2}\right)\left(\frac{z_2^*}{z_2^*}\right) = \frac{z_1z_2^*}{z_2z_2^*} = \frac{z_1z_2^*}{|z_2|^2} \\ &= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 - jx_1y_2 + jx_2y_1 + y_1y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}. \end{aligned}$$

Suppose that z_1 and z_2 are represented in polar form as

$$z_1 = r_1e^{j\theta_1} \quad \text{and} \quad z_2 = r_2e^{j\theta_2}.$$

Then, the quotient of z_1 and z_2 can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} \\ &= \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}. \end{aligned}$$

Example A.6 (Division with the Cartesian form)**.** Given that $z_1 = 1 + j$ and $z_2 = 2 - j$, compute z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \left(\frac{1+j}{2-j}\right)\left(\frac{2+j}{2+j}\right) = \frac{2+2j+j-1}{2^2+1^2} = \frac{1+3j}{5}. \qquad \blacksquare$$

Example A.7 (Division with the polar form)**.** Given that $z_1 = 2e^{j\pi/3}$ and $z_2 = 3e^{j\pi/4}$, compute z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \frac{2e^{j\pi/3}}{3e^{j\pi/4}} = \tfrac{2}{3}e^{j\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = \tfrac{2}{3}e^{j\pi/12}. \qquad \blacksquare$$

A.4.5 Properties of the Magnitude and Argument

For arbitrary complex numbers z_1 and z_2, the following identities hold:

$$|z_1 z_2| = |z_1|\,|z_2|; \tag{A.2a}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0; \tag{A.2b}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2; \quad \text{and} \tag{A.2c}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0. \tag{A.2d}$$

The preceding identities trivially follow from the polar representation of complex numbers. The identities (A.2a) and (A.2c) imply two additional identities as follows. For all complex numbers z and all integers n, the following identities hold:

$$\begin{aligned} |z^n| &= |z|^n; \quad \text{and} \\ \arg(z^n) &= n \arg z. \end{aligned}$$

A.5 Arithmetic Properties of Complex Numbers

In what follows, we consider some of the properties of arithmetic over the complex numbers.

A.5.1 Commutative Property

For complex numbers, addition and multiplication are commutative. That is, for any two complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \quad \text{and} \\ z_1 z_2 &= z_2 z_1. \end{aligned}$$

A.5.2 Associative Property

For complex numbers, addition and multiplication are associative. That is, for any three complex numbers z_1, z_2, and z_3, the following identities hold:

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \quad \text{and} \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3). \end{aligned}$$

A.5.3 Distributive Property

The distributive property also holds for complex numbers. That is, for any three complex numbers z_1, z_2, and z_3, the following identity holds:

$$z_1(z_2+z_3) = z_1z_2+z_1z_3.$$

A.6 Roots of Complex Numbers

Every complex number z has n distinct nth roots in the complex plane. In particular, the nth roots of $z = re^{j\theta}$, where $r = |z|$ and $\theta = \arg z$, are given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k \in [0\,.\,.\,n-1]. \tag{A.3}$$

Example A.8. Find the four fourth roots of 2.

Solution. Let z_k for $k \in [0\,.\,.\,3]$ denote each of the four fourth roots of 2. From (A.3), we have

$$z_k = \sqrt[4]{2}e^{j(0+2\pi k)/4} = \sqrt[4]{2}e^{jk\pi/2}.$$

Thus, we obtain

$$\begin{aligned}
z_0 &= \sqrt[4]{2}e^{j0} = \sqrt[4]{2}, \\
z_1 &= \sqrt[4]{2}e^{j\pi/2} = j\sqrt[4]{2}, \\
z_2 &= \sqrt[4]{2}e^{j\pi} = -\sqrt[4]{2}, \quad \text{and} \\
z_3 &= \sqrt[4]{2}e^{j3\pi/2} = -j\sqrt[4]{2}.
\end{aligned}$$

So, we conclude that the four fourth roots of 2 are

$$\sqrt[4]{2}, \quad j\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \text{and} \quad -j\sqrt[4]{2}.$$ ■

A.7 Euler's Relation and De Moivre's Theorem

An important relationship exists between exponentials and sinusoids as given by the theorem below.

Theorem A.1 (Euler's relation)**.** *For any real θ, the following identity holds:*

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{A.4}$$

*This identity is known as **Euler's relation**.*

Proof. The preceding theorem can be proven as follows. Recall that the Maclaurin series expansions of e^x, $\cos x$, and $\sin x$ are given by

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots, \tag{A.5}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots, \quad \text{and} \tag{A.6}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots. \tag{A.7}$$

Using (A.5), we can write $e^{j\theta}$ as the series

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots.$$

By regrouping terms and using (A.6) and (A.7), we obtain

$$\begin{aligned} e^{j\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + j\sin\theta. \end{aligned}$$

■

From Euler's relation, we can deduce the following additional identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and} \tag{A.8a}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}). \tag{A.8b}$$

Another important result involving exponential functions is given by the theorem below.

Theorem A.2 (De Moivre's theorem)**.** *For all real θ and all integer n, the following identity holds:*

$$e^{jn\theta} = \left(e^{j\theta}\right)^n. \tag{A.9}$$

*This result is known as **De Moivre's theorem**.*

Proof. The result of the above theorem can be proven by induction, and is left as an exercise for the reader. (See Exercise A.8.) ■

Note that, in the preceding theorem, n must be an integer. The identity (A.9) does not necessarily hold if n is not an integer. For example, consider $\theta = -\pi$ and $n = \frac{1}{2}$ (where n is clearly not an integer). We have that

$$\begin{aligned} e^{jn\theta} &= e^{j(1/2)(-\pi)} = e^{-j\pi/2} = -j \quad \text{and} \\ \left(e^{j\theta}\right)^n &= \left(e^{-j\pi}\right)^{1/2} = (-1)^{1/2} = j. \end{aligned}$$

Clearly, in this case, $e^{jn\theta} \neq \left(e^{j\theta}\right)^n$.

A.8 Conversion Between Cartesian and Polar Form

Suppose that we have a complex number $z = x + jy = re^{j\theta}$. Using Euler's relation, we can derive the following expressions for converting from polar to Cartesian form:

$$x = r\cos\theta \quad \text{and} \tag{A.10a}$$
$$y = r\sin\theta. \tag{A.10b}$$

Similarly, we can deduce the following expressions for converting from Cartesian to polar form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \tag{A.11a}$$
$$\theta = \operatorname{atan2}(y, x), \tag{A.11b}$$

where the atan2 function is as defined in (A.1).

If we choose to use the arctan function directly in order to compute θ (instead of using the atan2 function), we must be careful to consider the quadrant in which the point (x, y) lies. This complication is due to the fact that the arctan function is defined such that $-\frac{\pi}{2} < \arctan\theta < \frac{\pi}{2}$. Consequently, if the point does not lie in the first or fourth quadrant of the complex plane, the arctan function will not yield the desired angle.

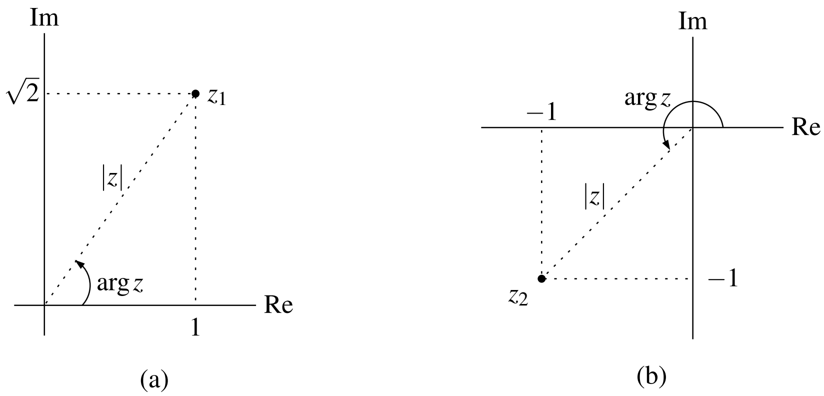


Figure A.4: Example of converting complex numbers from Cartesian to polar form. The case of the (a) first and (b) second part of the example.

Example A.9. Express each complex number z given below in polar form.
(a) $z = 1 + j\sqrt{2}$; and
(b) $z = -1 - j$.

Solution. (a) The magnitude and argument of z are given by

$$|z| = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{3} \quad \text{and}$$
$$\arg z = \arctan\left(\tfrac{\sqrt{2}}{1}\right) = \arctan\sqrt{2}.$$

Thus, we have

$$z = \sqrt{3}e^{j(\arctan\sqrt{2})}.$$

The result is plotted in Figure A.4(a).

(b) The magnitude and argument of z are given by

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \quad \text{and}$$
$$\arg z = \arctan\left(\tfrac{-1}{-1}\right) - \pi = \arctan(1) - \pi = -\tfrac{3\pi}{4}.$$

Thus, we have

$$z = \sqrt{2}e^{-j3\pi/4}.$$

The result is plotted in Figure A.4(b). ■

A.9 Complex Functions

A **complex function** maps complex numbers to complex numbers. For example, the function

$$f(z) = z^2 + 2z + 1,$$

where z is complex, is a complex function.

A **complex polynomial function** is mapping of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where a_0, a_1, ..., a_n, and z are complex. A **complex rational function** is a mapping of the form

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex. In the context of systems theory, polynomial and rational functions play a particularly important role.

Given any complex function f, we can always write f in the form $f(z) = \operatorname{Re}[f(z)] + j\operatorname{Im}[f(z)]$. Writing z in Cartesian form as $z = x + jy$, we have that $f(z) = \operatorname{Re}[f(x+jy)] + j\operatorname{Im}[f(x+jy)]$. Now, we can express $\operatorname{Re}[f(x+jy)]$ as a real-valued function v of the two real variables x and y. Similarly, we can express $\operatorname{Im}[f(x+jy)]$ as a real-valued function w of the two real variables x and y. Thus, we can always express a complex function f in the form

$$f(z) = f(x+jy) = v(x,y) + jw(x,y), \tag{A.12}$$

where v and w are each real-valued functions of the two real variables x and y (and $z = x + jy$).

A.10 Circles, Disks, and Annuli

A **circle** in the complex plane with center z_0 and radius r is the set of points z satisfying

$$|z - z_0| = r,$$

where r is a strictly positive real constant. A plot of a circle is shown in Figure A.5.

A **disk** is the set of points inside of a circle, possibly including the points on the circle itself. If the points on the circle are not included in the set, the disk is said to be open; otherwise, the disk is said to be closed. More formally, an **open disk** with center z_0 and radius r is the set of points z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real constant. A plot of an open disk is shown in Figure A.6.

An **annulus** (i.e., a ring) is the set of points between two concentric circles, possibly including the points on one or both circles. If an annulus does not include the points on its two defining circles, it is said to be open. More formally, an **open annulus** with center z_0, inner radius r_1, and outer radius r_2 is the set of points z satisfying

$$r_1 < |z - z_0| < r_2,$$

where r_1 and r_2 are strictly positive real constants. A plot of an annulus is shown in Figure A.7.

A.11 Limit

Let f be a complex function and z_0 a complex number. We want to define the limit of $f(z)$ as z approaches z_0. Unlike in the case of real functions, the value z_0 can be approached from infinitely many directions in the complex plane. In order for the limit to be useful, however, we want it to be defined in such a way that it is independent of the direction from which z_0 is approached. With this in mind, we define the notion of a limit below.

A function f evaluated at z is said to have the limit L as z approaches z_0 if

1. f is defined in some open disk about z_0, except possibly at the point z_0; and
2. for every positive real number ε, there exists a positive real number δ such that $|f(z) - L| < \varepsilon$ for all values of z in the disk $|z - z_0| < \delta$ except $z = z_0$.

This limit can be expressed as

$$\lim_{z \to z_0} f(z) = L.$$

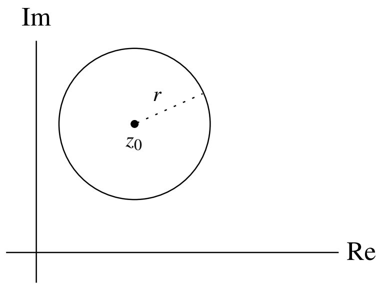


Figure A.5: Circle about z_0 with radius r.

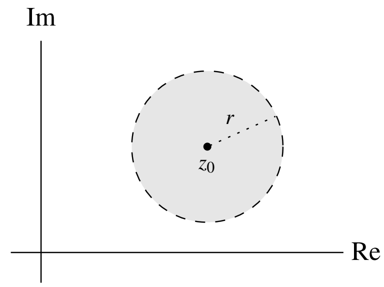


Figure A.6: Open disk of radius r about z_0.

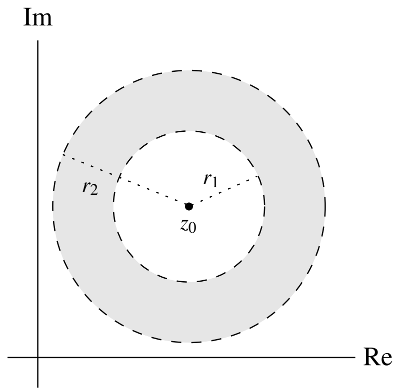


Figure A.7: Open annulus about z_0 with inner radius r_1 and outer radius r_2.

A.12 Continuity

A function f is said to be **continuous at a point** z_0 if $f(z_0)$ is defined and given by

$$f(z_0) = \lim_{z \to z_0} f(z).$$

A function is said to be **continuous** if it is continuous at every point in its domain. Polynomial functions are continuous everywhere. For example, the function $f(z) = 3z^3 + z^2 - z + 1$ is continuous for all complex z. Rational functions (i.e., quotients of polynomials) are continuous everywhere except at points where the denominator polynomial becomes zero. For example, the function $f(z) = \frac{(z+j)(z-j)}{(z-1)(z+1)}$ is continuous for all complex z except $z = 1$ and $z = -1$.

A.13 Differentiability

A function f is said to be **differentiable at a point** $z = z_0$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is called the **derivative** of f at the point z_0. A function is said to be **differentiable** if it is differentiable at every point in its domain.

In general, the rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. Let f and g be functions and let a be a scalar constant. Let the prime symbol denote a derivative. If $f'(z_0)$ and $g'(z_0)$ exist, then

1. $(af)'(z_0) = af'(z_0)$ for any complex constant a;

2. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$;
3. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$;
4. $(f/g)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$; and
5. if $z_0 = g(w_0)$ and $g'(w_0)$ exists, then the derivative of $f(g(z))$ at w_0 is $f'(z_0)g'(w_0)$ (i.e., the chain rule).

Polynomial functions are differentiable everywhere. Rational functions are differentiable everywhere except at points where the denominator polynomial becomes zero.

A.14 Analyticity

A function f is said to be **analytic at a point** z_0 if it is differentiable at every point in some open disk about z_0. A function f is said to be **analytic** if it is analytic at every point in its domain.

One extremely useful test for the analyticity of a function is given by the theorem below.

Theorem A.3 (Cauchy-Riemann equations)**.** *Let f be a complex function expressed in the form of* (A.12). *So, we have*

$$f(z) = v(x,y) + jw(x,y),$$

where $z = x + jy$. The function f is analytic in S if and only if v and w satisfy the following conditions at all points in S:

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

*These equations are known as the **Cauchy-Riemann equations**.*

Proof. A proof of this theorem is somewhat tedious and therefore omitted here. ∎

Polynomial functions are both continuous and differentiable everywhere. Therefore, such functions are analytic everywhere. Rational functions are both continuous and differentiable everywhere, except at points where the denominator polynomial becomes zero. Consequently, rational functions are analytic at all but these points.

Example A.10. Determine for what values of z the function $f(z) = z^2$ is analytic.

Solution. First, we observe that f is a polynomial function. Then, we recall that polynomial functions are analytic everywhere. Therefore, f is analytic everywhere.

Alternate Solution. We can demonstrate the analyticity of f using Theorem A.3. We express z in Cartesian form as $z = x + jy$. We rewrite f in the form of $f(x,y) = v(x,y) + jw(x,y)$ as follows:

$$f(z) = f(x+jy) = (x+jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j(2xy).$$

Thus, we have that $f(z) = v(x,y) + jw(x,y)$, where

$$v(x,y) = x^2 - y^2 \quad \text{and} \quad w(x,y) = 2xy.$$

Now, computing the partial derivatives, we obtain

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2x, \quad \frac{\partial v}{\partial y} = -2y, \quad \text{and} \quad \frac{\partial w}{\partial x} = 2y.$$

From this, we can see that

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

Therefore, the Cauchy-Riemann equations are satisfied for all complex $z = x + jy$. Therefore, f is analytic everywhere. ∎