Section 3.1

Independent- and Dependent-Variable Transformations

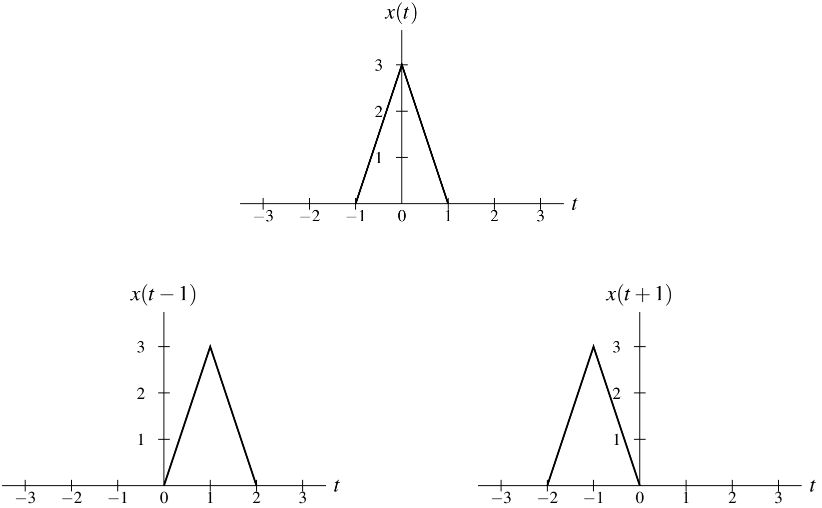
Time Shifting (Translation)

- **Time shifting** (also called **translation**) maps the input function x to the output function y as given by

$$y(t) = x(t-b),$$

  where b is a real number.
- Such a transformation shifts the function (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

Time Shifting (Translation): Example

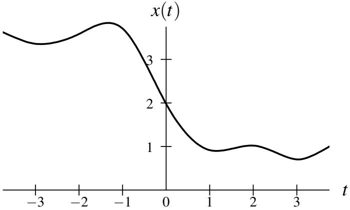


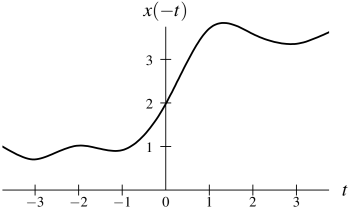
Time Reversal (Reflection)

- **Time reversal** (also known as **reflection**) maps the input function x to the output function y as given by

$$y(t) = x(-t).$$

- Geometrically, the output function y is a reflection of the input function x about the (vertical) line $t = 0$.





Time Compression/Expansion (Dilation)

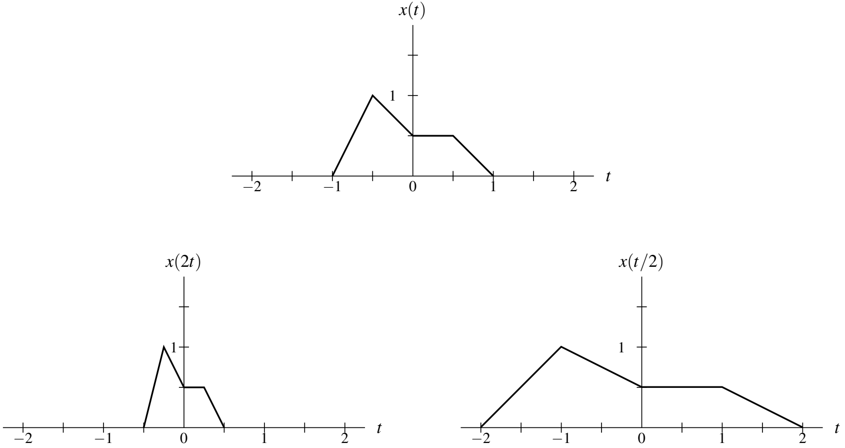
- **Time compression/expansion** (also called **dilation**) maps the input function x to the output function y as given by

$$y(t) = x(at),$$

where a is a ***strictly positive*** real number.

- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, y is ***compressed*** along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, y is ***expanded*** (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

Time Compression/Expansion (Dilation): Example



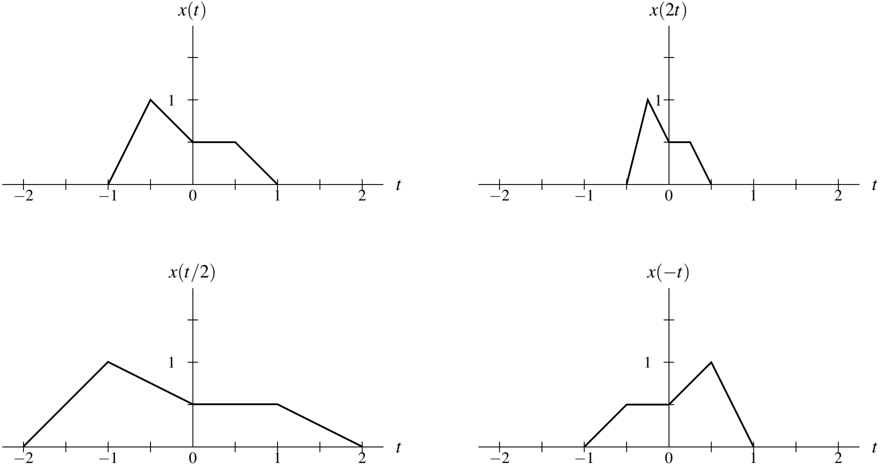
Time Scaling (Dilation/Reflection)

- **Time scaling** maps the input function x to the output function y as given by

$$y(t) = x(at),$$

  where a is a ***nonzero*** real number.
- Such a transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or time reversal.
- If $|a| > 1$, the function is ***compressed*** along the time axis by a factor of $|a|$.
- If $|a| < 1$, the function is ***expanded*** (i.e., stretched) along the time axis by a factor of $\left|\frac{1}{a}\right|$.
- If $|a| = 1$, the function is neither expanded nor compressed.
- If $a < 0$, the function is also time reversed.
- Dilation (i.e., expansion/compression) and time reversal ***commute***.
- Time reversal is a special case of time scaling with $a = -1$; and time compression/expansion is a special case of time scaling with $a > 0$.

Time Scaling (Dilation/Reflection): Example



Combined Time Scaling and Time Shifting

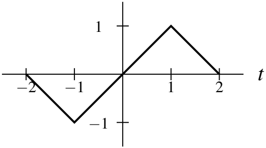
- Consider a transformation that maps the input function x to the output function y as given by
$$y(t) = x(at - b),$$
where a and b are real numbers and $a \neq 0$.
- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting ***do not commute***, we must be particularly careful about the order in which these transformations are applied.
- The above transformation has two distinct but equivalent interpretations:
  1. first, time shifting x by b, and then time scaling the result by a;
  2. first, time scaling x by a, and then time shifting the result by b/a.
- Note that the time shift is not by the same amount in both cases.
- In particular, note that when time scaling is applied first followed by time shifting, the time shift is by b/a, not b.

Exercise 3.3

Combined Time Scaling and Time Shifting: Example

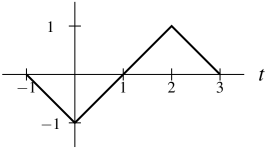
Given x as shown below, find $y(t) = x(2t-1)$.

$x(t)$

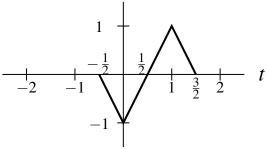


time shift by 1 and then time scale by 2

$p(t) = x(t-1)$

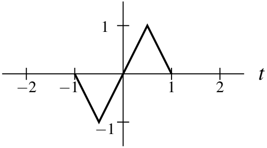


$y(t) = p(2t)$

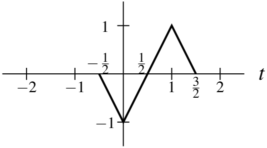


time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$



$y(t) = q(t-1/2)$



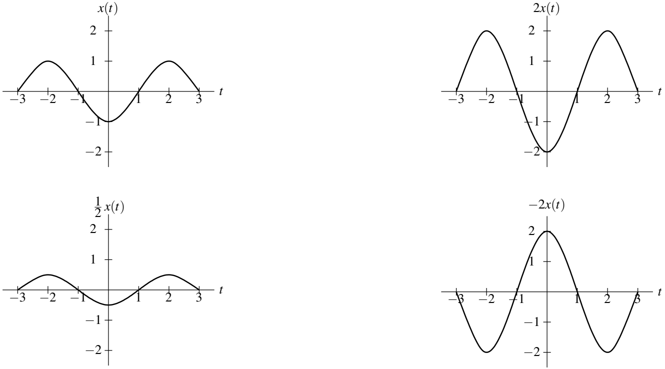
- A transformation of the independent variable can be viewed in terms of
  1. the effect that the transformation has on the ***function***; or
  2. the effect that the transformation has on the ***horizontal axis***.
- This distinction is important because such a transformation has ***opposite*** effects on the function and horizontal axis.
- For example, the (time-shifting) transformation that replaces t by $t-b$ (where b is a real number) in $x(t)$ can be viewed as a transformation that
  1. shifts the function x ***right*** by b units; or
  2. shifts the horizontal axis ***left*** by b units.
- In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the ***function***.
- If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

Amplitude Scaling

- **Amplitude scaling** maps the input function x to the output function y as given by

$$y(t) = ax(t),$$

  where a is a real number.
- Geometrically, the output function y is ***expanded/compressed*** in amplitude and/or ***reflected*** about the horizontal axis.

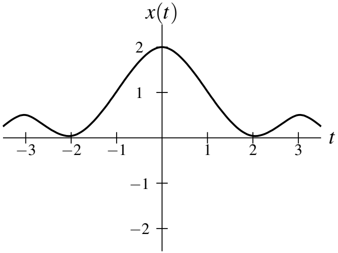


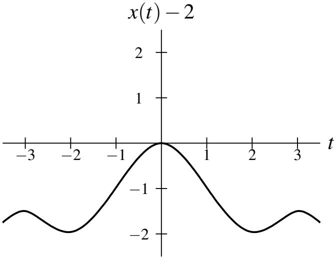
- **Amplitude shifting** maps the input function x to the output function y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to x.





Combined Amplitude Scaling and Amplitude Shifting

- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input function x to the output function y, as given by

$$y(t) = ax(t) + b,$$

  where a and b are real numbers.
- Equivalently, the above transformation can be expressed as

$$y(t) = a\left[x(t) + \tfrac{b}{a}\right].$$

- The above transformation is equivalent to:
  1. first amplitude scaling x by a, and then amplitude shifting the resulting function by b; or
  2. first amplitude shifting x by b/a, and then amplitude scaling the resulting function by a.

Section 3.2

Properties of Functions

Symmetry and Addition/Multiplication

- Sums involving even and odd functions have the following properties:
  - The sum of two even functions is even.
  - The sum of two odd functions is odd.
  - The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.
- That is, the ***sum*** of functions with the ***same type of symmetry*** also has the ***same type of symmetry***.
- Products involving even and odd functions have the following properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
- That is, the ***product*** of functions with the ***same type of symmetry*** is ***even***, while the ***product*** of functions with ***opposite types of symmetry*** is ***odd***.

Decomposition of a Function into Even and Odd Parts

- Every function x has a ***unique*** representation of the form

$$x(t) = x_e(t) + x_o(t),$$

where the functions x_e and x_o are ***even*** and ***odd***, respectively.

- In particular, the functions x_e and x_o are given by

$$x_e(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_o(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

- The functions x_e and x_o are called the **even part** and **odd part** of x, respectively.
- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

Theorem 3.1

Sum of Periodic Functions

- **Sum of periodic functions.** For two periodic functions x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:
  1. The sum y is periodic if and only if the ratio T_1/T_2 is a ***rational number*** (i.e., the quotient of two integers).
  2. If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and ***coprime*** (i.e., have no common factors). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)
- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N-1$ times.

Example 3.2 Example 3.4

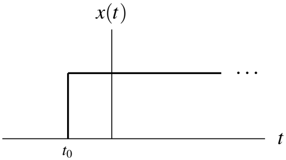
Right-Sided Functions

- A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e., x is ***only potentially nonzero to the right of*** t_0).

- An example of a right-sided function is shown below.



- A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal function is a ***special case*** of a right-sided function.
- A causal function is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

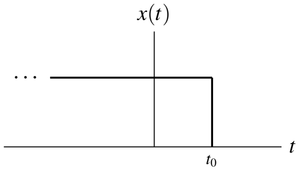
Left-Sided Functions

- A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is ***only potentially nonzero to the left of*** t_0).

- An example of a left-sided function is shown below.



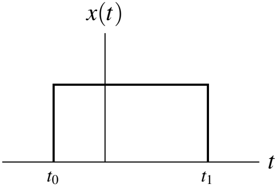
- Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

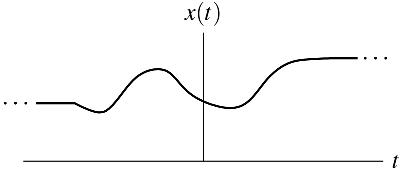
- An anticausal function is a ***special case*** of a left-sided function.
- An anticausal function is not to be confused with an anticausal system. In these two contexts, the word “anticausal” has very different meanings.

Finite-Duration and Two-Sided Functions

- A function that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration function is shown below.



- A function that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided function is shown below.



Bounded Functions

- A function x is said to be **bounded** if there exists some (***finite***) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is ***finite*** for all t).

- For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

- In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \to \infty} |p(t)| = \infty.$$

Energy and Power of a Function

- The **energy** E contained in the function x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the function x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

- A signal with (nonzero) finite average power is said to be a **power signal**.

Section 3.3

Elementary Functions

- A **real sinusoidal function** is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

  where A, ω, and θ are ***real*** constants.
- Such a function is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.
- A real sinusoid has a plot resembling that shown below.