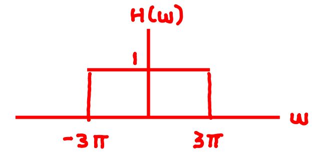
Example 5.10 (Lowpass filtering)**.** Suppose that we have a LTI system with input x, output y, and frequency response H , where

$$H(\omega) = \begin{cases} 1 & |\omega| \le 3\pi \\ 0 & \text{otherwise.} \end{cases}$$



Further, suppose that the input x is the periodic function

$$x(t) = 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t).$$

(a) Find the Fourier series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

don't use Fourier series analysis equation!

Solution. (a) We begin by finding the Fourier series representation of x. Using Euler's formula, we can re-express x as

Euler

simplify and reorder terms

w₀ must be as large as possible

rewrite exponentials as jkw₀

$$\begin{aligned} x(t) &= 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t) \\ &= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\ &= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\ &= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\ &= \underbrace{\tfrac{1}{4}e^{j(-3)(2\pi)t}}_{k=-3} + \underbrace{\tfrac{1}{2}e^{j(-2)(2\pi)t}}_{k=-2} + \underbrace{e^{j(-1)(2\pi)t}}_{k=-1} + \underbrace{e^{j(0)(2\pi)t}}_{k=0} + \underbrace{e^{j(1)(2\pi)t}}_{k=1} + \underbrace{\tfrac{1}{2}e^{j(2)(2\pi)t}}_{k=2} + \underbrace{\tfrac{1}{4}e^{j(3)(2\pi)t}}_{k=3}. \end{aligned}$$

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & k = 0 \\ 1 & k \in \{-1, 1\} \\ \frac{1}{2} & k \in \{-2, 2\} \\ \frac{1}{4} & k \in \{-3, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

(due to eigenfunction properties of LTI systems)

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(k\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$H(k\omega_0) = 1$

$$\begin{aligned} b_0 &= a_0 H([0][2\pi]) = 1(1) = 1, \\ b_1 &= a_1 H([1][2\pi]) = 1(1) = 1, \\ b_{-1} &= a_{-1} H([-1][2\pi]) = 1(1) = 1, \end{aligned}$$

$H(k\omega_0) = 0$

$$\begin{aligned} b_2 &= a_2 H([2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_{-2} &= a_{-2} H([-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_3 &= a_3 H([3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\ b_{-3} &= a_{-3} H([-3][2\pi]) = \tfrac{1}{4}(0) = 0. \end{aligned}$$

we were given

$$H(\omega) = \begin{cases} 1 & \omega \in [-3\pi, 3\pi] \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1, 0, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 5.10(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes.

$b_K = a_K H(k\omega_0)$

$\omega_0 = 2\pi$

$\omega_0 = 2\pi$

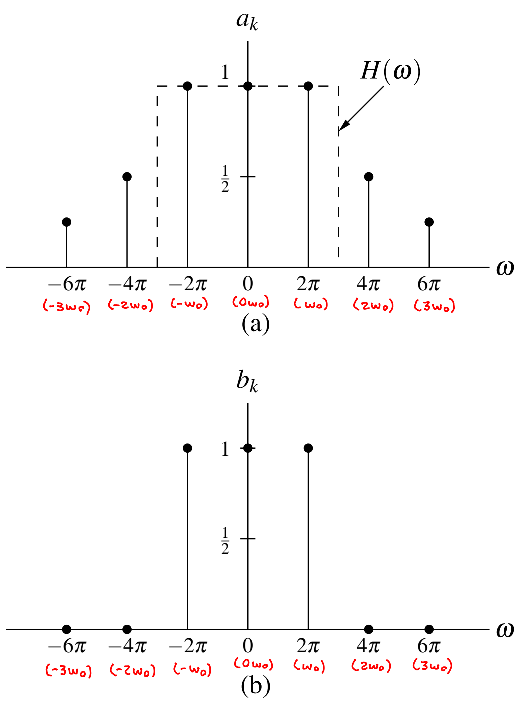


Figure 5.10: Frequency spectra of the (a) input function x and (b) output function y.

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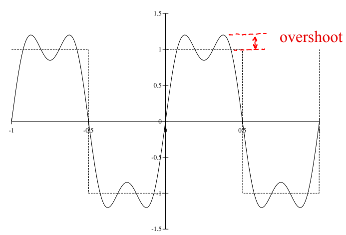
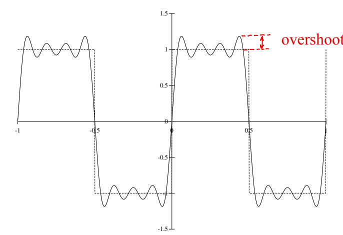
NOTE: THE APPROACH USED TO SOLVE THIS PROBLEM DID NOT INVOLVE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

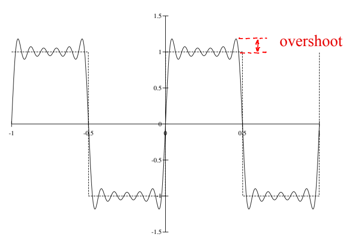
Gibbs Phenomenon: Periodic Square Wave Example



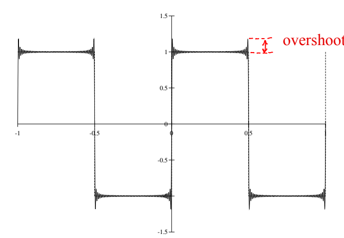
Fourier series truncated after the
3rd harmonic components



Fourier series truncated after the
7th harmonic components



Fourier series truncated after the
11th harmonic components



Fourier series truncated after the
101st harmonic components

Unit:
CT Fourier Transform