Example 6.17 (Frequency-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = t\cos(\omega_0 t),$$

where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for x yields

$$X(\omega) = \mathcal{F}\{t\cos(\omega_0 t)\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{t\cos(\omega_0 t)\}(\omega) && \text{from definition of } x \\ &= j\left(\mathcal{D}\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega), && \text{frequency-domain differentiation property} \end{aligned}$$

where $\mathcal{D}$ denotes the derivative operator. Evaluating the Fourier transform on the right-hand side using Table 6.2, we obtain

table of FT pairs

$$\begin{aligned} X(\omega) &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] && \text{from FT pair (1)} \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] && \text{factor out } \pi \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). && \text{derivative operator is linear} \end{aligned}$$

■

$$\cos(\omega_0 t) \overset{\text{FT}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \quad (1)$$

Example 6.18 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform X of the function $x = u$.

Solution. We begin by observing that x can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau. \quad ①$$

Now, we consider the Fourier transform of x. We have

from ①

$$X(\omega) = \left(\mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)d\tau\right\}\right)(\omega).$$

time-domain integration property

From the time-domain integration property, we can write

$$X(\omega) = \tfrac{1}{j\omega}\mathcal{F}\delta(\omega) + \pi\mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$F\delta(\omega) = 1$

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \tfrac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

drop 1's

Thus, we have shown that $u(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. ■

Example 6.19 (Energy of the sinc function)**.** Consider the function $x(t) = \operatorname{sinc}\left(\frac{1}{2}t\right)$, which has the Fourier transform X given by $X(\omega) = 2\pi \operatorname{rect} \omega$. Compute the energy of x.

①

Solution. We could directly compute the energy of x as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt. = \int_{-\infty}^{\infty} \left| \frac{\sin t/2}{t/2} \right|^2 dt \rightarrow ☹ \end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega \\ &= 2\pi \int_{-1/2}^{1/2} d\omega \\ &= 2\pi[\omega]|_{-1/2}^{1/2} \\ &= 2\pi[\tfrac{1}{2} + \tfrac{1}{2}] \\ &= 2\pi. \end{aligned}$$

from given X in ①

rect $t = 1$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$ and zero otherwise

cancel one 2π factor

integrate

Thus, we have

$$E = \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt = 2\pi.$$

■

Answer (g).

We are asked to find the Fourier transform Y of

$$y(t) = \left[te^{-j5t}x(t)\right]^*.$$

In what follows, we use the prime symbol to denote the derivative (i.e., f' denotes the derivative of f). To begin, we have

$$\begin{aligned} y(t) &= \left[te^{-j5t}x(t)\right]^* \\ &= \left[e^{-j5t}\underbrace{tx(t)}_{v_1(t) = tx(t)\ \ 1}\right]^*. \end{aligned}$$

Letting $v_1(t) = tx(t)$, we have

$$y(t) = \left[\underbrace{e^{-j5t}v_1(t)}_{v_2(t) = e^{-j5t}v_1(t)\ \ 2}\right]^*.$$

Letting $v_2(t) = e^{-j5t}v_1(t)$, we have

$$y(t) = v_2^*(t). \quad 3$$

Thus, we have written $y(t)$ as

$$3 \longrightarrow \quad y(t) = v_2^*(t)$$

where

$$\begin{aligned} 1 &\longrightarrow \quad v_1(t) = tx(t) \quad \text{and} \\ 2 &\longrightarrow \quad v_2(t) = e^{-j5t}v_1(t). \end{aligned}$$

Taking the Fourier transforms of the preceding equations, we obtain

$$\begin{aligned} 4 \quad && V_1(\omega) &= jX'(\omega), && \longleftarrow \text{FT of } 1 \text{ using frequency-domain differentiation property} \\ 5 \quad && V_2(\omega) &= V_1(\omega+5), \quad \text{and} && \longleftarrow \text{FT of } 2 \text{ using frequency-domain shifting property} \\ 6 \quad && Y(\omega) &= V_2^*(-\omega). && \longleftarrow \text{FT of } 3 \text{ using conjugation property} \end{aligned}$$

Combining the above equations, we have

$$\begin{aligned} Y(\omega) &= V_2^*(-\omega) && \longleftarrow 6 \\ &= [V_1(-\omega+5)]^* && \text{substitute } 5 \\ &= \left[jX'(-\omega+5)\right]^* && \text{substitute } 4 \\ &= -jX'^*(-\omega+5). && (ab)^* = a^*b^* \end{aligned}$$