Example 7.38 (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.24 with input v_1 and output v_2. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function H of this system. (b) Determine whether the system is BIBO stable. (c) Determine the step response of the system.

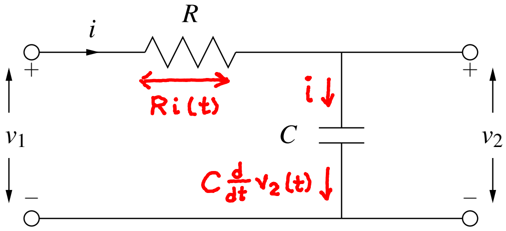


Figure 7.24: Simple RC network.

Solution. (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \tag{7.14a}$$
$$i(t) = C\tfrac{d}{dt}v_2(t). \tag{7.14b}$$

taking LT

Taking the Laplace transform of (7.14) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \tag{7.15a}$$
$$I(s) = CsV_2(s). \tag{7.15b}$$

substitute (7.15b) into (7.15a) to eliminate I

Substituting (7.15b) into (7.15a) and rearranging, we obtain

$$\begin{aligned}
& V_1(s) = R[CsV_2(s)] + V_2(s) \\
\Rightarrow\ & V_1(s) = RCsV_2(s) + V_2(s) \\
\Rightarrow\ & V_1(s) = [1 + RCs]V_2(s) \\
\Rightarrow\ & \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + RCs}.
\end{aligned}$$

factor

rearrange

Thus, we have that the system function H is given by

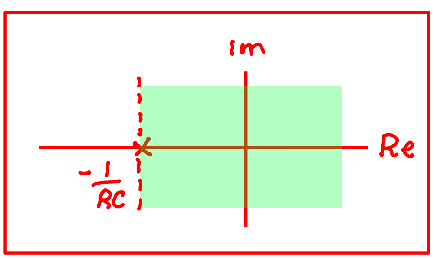
$V_2(s) = V_1(s)H(s) \Longrightarrow$
$H(s) = \dfrac{V_2(s)}{V_1(s)}$

$$\begin{aligned}
H(s) &= \frac{1}{1 + RCs} \\
&= \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \\
&= \frac{\frac{1}{RC}}{s - (-\frac{1}{RC})}.
\end{aligned}$$

divide numerator and denominator by RC

explicitly show poles

(*)



Since the system can be physically realized, it must be causal. Therefore, the ROC of H must be a right-half plane. Thus, we may infer that the ROC of H is $\operatorname{Re}(s) > -\frac{1}{RC}$. So, we have

see (*)

$$H(s) = \frac{1}{1 + RCs} \quad \text{for } \operatorname{Re}(s) > -\tfrac{1}{RC}.$$

(b) Since resistance and capacitance are (strictly) positive quantities, $R > 0$ and $C > 0$. Thus, $-\frac{1}{RC} < 0$. Consequently, the ROC contains the imaginary axis and the system is stable.

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(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{1}{1+RCs}\right)V_1(s). \end{aligned}$$

Since system is LTI

substitute for H

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1 = u$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for V_1 into the above expression for V_2, we have

$$\begin{aligned} V_2(s) &= \left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right) \\ &= \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}. \end{aligned}$$

$v_1(t) = u(t) \overset{LT}{\longleftrightarrow} V_1(s) = \frac{1}{s}$

divide numerator and denominator by RC

Now, we need to compute the inverse Laplace transform of V_2 in order to determine v_2. To simplify this task, we find the partial fraction expansion for V_2. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+\tfrac{1}{RC})V_2(s)|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that V_2 has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

LT table

$$v_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}(t).$$

RHP

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

inverse LT

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= \left(1 - e^{-t/(RC)}\right)u(t). \end{aligned}$$ ■

$u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s}$ for $Re(s) > 0$

$e^{-at}u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s+a}$ for $Re(s) > -a$

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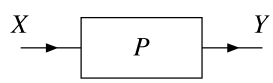
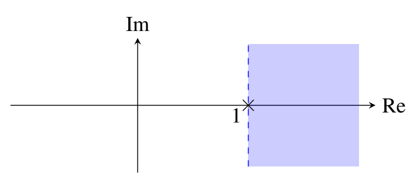
Stabilization Example: Unstable Plant

- causal LTI plant:



$$P(s) = \frac{10}{s-1}$$ has pole at 1

- ROC of P:



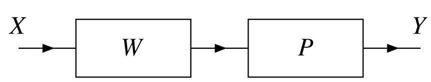
ROC is RHP since system is causal

system is not BIBO stable since ROC does not contain imaginary axis

- system is not BIBO stable

Stabilization Example: Using Pole-Zero Cancellation

- system formed by series interconnection of plant and causal LTI compensator:



$$P(s) = \frac{10}{s-1}, \quad W(s) = \frac{s-1}{10(s+1)}$$

- system function H of overall system:

pole-zero cancellation

$$H(s) = W(s)P(s) = \left(\frac{s-1}{10(s+1)}\right)\left(\frac{10}{s-1}\right) = \frac{1}{s+1}$$

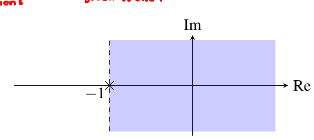
has pole at -1

connecting systems in series multiplies system functions

substitute given W and P

multiply

- ROC of H:



ROC is RHP since system is causal

system is BIBO stable since ROC contains imaginary axis

- overall system is BIBO stable