Solution. Consider an input of the form $x(t) = 2\pi k$ where k is an arbitrary integer. The response $\mathcal{H}x$ to such an input is given by

$$\begin{aligned}\mathcal{H}x(t) &= \sin[x(t)] \\ &= \sin(2\pi k) \\ &= 0.\end{aligned}$$

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$) that all result in the same output. Therefore, the system is not invertible. ■

Example 3.26. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = 3x(3t+3).$$

Solution. Let $y = \mathcal{H}x$. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} & y(t) = 3x(3t+3) \\ \Rightarrow \quad & y\left(\tfrac{1}{3}t - 1\right) = 3x(t) \\ \Rightarrow \quad & x(t) = \tfrac{1}{3}y\left(\tfrac{1}{3}t - 1\right).\end{aligned}$$

In other words, $\mathcal{H}^{-1}$ is given by $\mathcal{H}^{-1}y(t) = \frac{1}{3}y\left(\frac{1}{3}t - 1\right)$. Since we have just found $\mathcal{H}^{-1}$, $\mathcal{H}^{-1}$ exists. Therefore, the system $\mathcal{H}$ is invertible. ■

Example 3.27. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the response $\mathcal{H}x$ of the system to an input x of the form

$$x(t) = \alpha,$$

where α is a real constant. We have that

$$\begin{aligned}\mathcal{H}x(t) &= \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0.\end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible. ■

3.8.4 Bounded-Input Bounded-Output (BIBO) Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system $\mathcal{H}$ is **BIBO stable** if, for every bounded function x, $\mathcal{H}x$ is also bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t). In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output). Often, the triangle inequality (F.16) can be quite helpful when trying to show that a system is BIBO stable.

In practical terms, a BIBO stable system is well behaved in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time. Usually, a system that is not BIBO stable will have serious safety issues. For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely one big lawsuit as well).

Example 3.28 (Squarer)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = x^2(t).$$

Solution. Suppose that the input x is bounded such that (for all t)

$$|x(t)| \leq A,$$

where A is a finite real constant. Squaring both sides of the inequality, we obtain

$$|x(t)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations on the left-hand side of the inequality, we have

$$\left|x^2(t)\right| \leq A^2.$$

Using the fact that $\mathcal{H}x(t) = x^2(t)$, we can write

$$|\mathcal{H}x(t)| \leq A^2.$$

Since A is finite, A^2 is also finite. Thus, we have that $\mathcal{H}x$ is bounded (i.e., $|\mathcal{H}x(t)| \leq A^2 < \infty$ for all t). Therefore, the system is BIBO stable. ■

Example 3.29 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

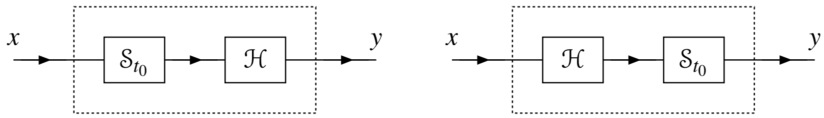
Solution. Suppose that we choose the input $x = u$ (where u denotes the unit-step function). Clearly, u is bounded (i.e., $|u(t)| \leq 1$ for all t). Calculating the response $\mathcal{H}x$ to this input, we have

$$\begin{aligned}\mathcal{H}x(t) &= \int_{-\infty}^{t} u(\tau)d\tau \\ &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t.\end{aligned}$$

From this result, however, we can see that as $t \to \infty$, $\mathcal{H}x(t) \to \infty$. Thus, the output $\mathcal{H}x$ is unbounded for the bounded input x. Therefore, the system is not BIBO stable. ■

Example 3.30. Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$



(a) (b)

Figure 3.33: Systems that are equivalent if $\mathcal{H}$ is time invariant (i.e., $\mathcal{H}$ commutes with $\mathcal{S}_{t_0}$). (a) A system that first time shifts by t_0 and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}\mathcal{S}_{t_0}x$); and (b) a system that first applies $\mathcal{H}$ and then time shifts by t_0 (i.e., $y = \mathcal{S}_{t_0}\mathcal{H}x$).

Solution. Suppose that x is bounded. Then, $x(-t)$ is also bounded. Since the difference of two bounded functions is bounded, $x(t) - x(-t)$ is bounded. Multiplication of a bounded function by a finite constant yields a bounded result. So, the function $\frac{1}{2}[x(t) - x(-t)]$ is bounded. Thus, $\mathcal{H}x(t)$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable. ■

Example 3.31 (Ideal differentiator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \mathcal{D}x(t)$$

and $\mathcal{D}$ denotes the derivative operator.

Solution. Consider the input $x(t) = \sin\left(t^2\right)$. Clearly, x is bounded, since the sine function is bounded. In particular, $|x(t)| \leq 1$ for all real t. Now, consider the response of the system to the input x. We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}x(t) \\ &= \mathcal{D}\left\{\sin(t^2)\right\}(t) \\ &= 2t\cos(t^2).\end{aligned}$$

Clearly, $\mathcal{H}x$ is unbounded, since $|\mathcal{H}x(t)|$ grows without bound as $|t| \to \infty$. Thus, the output is not bounded for some bounded input. Therefore, the system is not BIBO stable. ■

3.8.5 Time Invariance

A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function x and every real number t_0, the following condition holds:

$$\mathcal{H}x(t - t_0) = \mathcal{H}x'(t) \text{ for all } t \quad \text{where} \quad x'(t) = x(t - t_0)$$

(i.e., $\mathcal{H}$ *commutes with time shifts*). In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input function results in an identical time shift in the output function. A system that is not time invariant is said to be **time varying** (or **shift varying**). In effect, time invariance means that the two systems shown in Figure 3.33 are equivalent, where $\mathcal{S}_{t_0}$ denotes an operator that applies a time shift of t_0 to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t - t_0)$).

In simple terms, a time invariant system is a system whose behavior does not change with respect to time. Practically speaking, compared to time-varying systems, time-invariant systems are much easier to design and analyze, since their behavior does not change with respect to time.

Example 3.32. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(t-t_0) &= \sin[x(t-t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \sin[x'(t)] \\ &= \sin[x(t-t_0)]. \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ∎

Example 3.33. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = tx(t).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(t-t_0) &= (t-t_0)x(t-t_0) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= tx(t-t_0). \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant (i.e., the system is time varying). ∎

Example 3.34. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sum_{k=-10}^{10} x(t-k).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(t-t_0) &= \sum_{k=-10}^{10} x(t-t_0-k) \quad \text{and} \\ \mathcal{H}x'(t) &= \sum_{k=-10}^{10} x'(t-k) \\ &= \sum_{k=-10}^{10} x(t-k-t_0) \\ &= \sum_{k=-10}^{10} x(t-t_0-k). \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ∎

Example 3.35. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

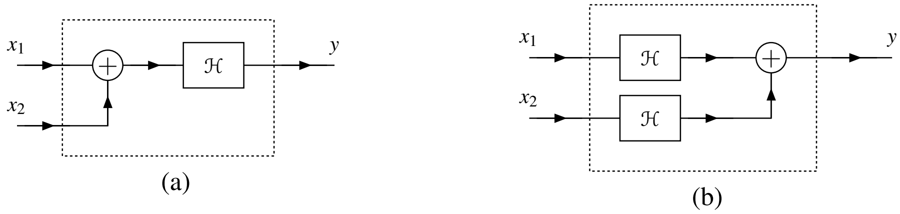


Figure 3.34: Systems that are equivalent if $\mathcal{H}$ is additive (i.e., $\mathcal{H}$ commutes with addition). (a) A system that first performs addition and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(x_1 + x_2)$); and (b) a system that first applies $\mathcal{H}$ and then performs addition (i.e., $y = \mathcal{H}x_1 + \mathcal{H}x_2$).

Figure 3.35: Systems that are equivalent if $\mathcal{H}$ is homogeneous (i.e., $\mathcal{H}$ commutes with scalar multiplication). (a) A system that first performs scalar multiplication and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(ax)$); and (b) a system that first applies $\mathcal{H}$ and then performs scalar multiplication (i.e., $y = a\mathcal{H}x$).

Solution. Let $x'(t) = x(t - t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x(t-t_0) &= \tfrac{1}{2}[x(t-t_0) - x(-(t-t_0))] \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t+t_0)] \quad \text{and} \\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t-t_0)].
\end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant. ■

3.8.6 Linearity

Two of the most and frequently-occurring mathematical operations are addition and scalar multiplication. For this reason, it is often extremely helpful to know if these operations commute with the operation performed by a given system. The system properties to be introduced next relate to this particular issue.

A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

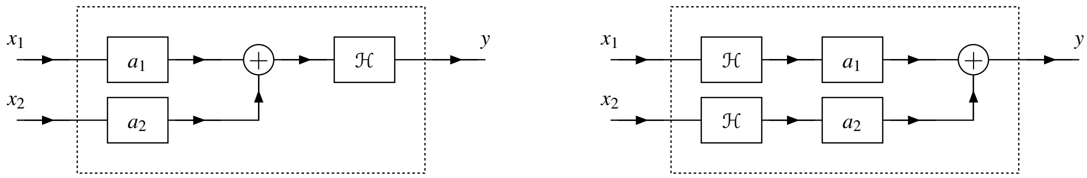
(i.e., $\mathcal{H}$ *commutes with addition*). Essentially, a system $\mathcal{H}$ being additive means that the two systems shown in Figure 3.34 are equivalent.

A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

(i.e., $\mathcal{H}$ *commutes with scalar multiplication*). Essentially, a system $\mathcal{H}$ being homogeneous means that the two systems shown in Figure 3.35 are equivalent.

The additivity and homogeneity properties can be combined into a single property known as superposition. In particular, a system $\mathcal{H}$ is said to have the **superposition** property, if for all functions x_1 and x_2 and all complex



(a) (b)

Figure 3.36: Systems that are equivalent if $\mathcal{H}$ is linear (i.e., $\mathcal{H}$ commutes with linear combinations). (a) A system that first computes a linear combination and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(a_1x_1 + a_2x_2)$); and (b) a system that first applies $\mathcal{H}$ and then computes a linear combination (i.e., $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$).

constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with linear combinations*). A system that is both additive and homogeneous (or equivalently satisfies superposition) is said to be **linear**. Essentially, a system $\mathcal{H}$ being linear means that the two systems shown in Figure 3.36 are equivalent. To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds. Practically speaking, linear systems are much easier to design and analyze than nonlinear systems.

Example 3.36. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = tx(t).$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1tx_1(t) + a_2tx_2(t) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= t\left[a_1x_1(t) + a_2x_2(t)\right] \\ &= a_1tx_1(t) + a_2tx_2(t). \end{aligned}$$

Since $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

Example 3.37. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|.$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1\left|x_1(t)\right| + a_2\left|x_2(t)\right| \quad \text{and} \\ \mathcal{H}x'(t) &= \left|x'(t)\right| \\ &= \left|a_1x_1(t) + a_2x_2(t)\right|. \end{aligned}$$

At this point, we recall the triangle inequality (F.16). Thus, $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t)=-1 \quad \text{and} \quad \mathcal{H}x'(t)=1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

Example 3.38. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t)=\operatorname{Odd}(x)(t)=\tfrac{1}{2}\left[x(t)-x(-t)\right].$$

Solution. Let $x'(t)=a_1x_1(t)+a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) &= \tfrac{1}{2}a_1[x_1(t)-x_1(-t)]+\tfrac{1}{2}a_2[x_2(t)-x_2(-t)] \quad \text{and}\\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t)-x'(-t)]\\
&= \tfrac{1}{2}[a_1x_1(t)+a_2x_2(t)-[a_1x_1(-t)+a_2x_2(-t)]]\\
&= \tfrac{1}{2}[a_1x_1(t)-a_1x_1(-t)+a_2x_2(t)-a_2x_2(-t)]\\
&= \tfrac{1}{2}a_1[x_1(t)-x_1(-t)]+\tfrac{1}{2}a_2[x_2(t)-x_2(-t)].
\end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2)=a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the system is linear. ■

Example 3.39. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t)=x(t)x(t-1).$$

Solution. Let $x'(t)=a_1x_1(t)+a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) &= a_1x_1(t)x_1(t-1)+a_2x_2(t)x_2(t-1) \quad \text{and}\\
\mathcal{H}x'(t) &= x'(t)x'(t-1)\\
&= [a_1x_1(t)+a_2x_2(t)][a_1x_1(t-1)+a_2x_2(t-1)]\\
&= a_1^2x_1(t)x_1(t-1)+a_1a_2x_1(t)x_2(t-1)+a_1a_2x_1(t-1)x_2(t)+a_2^2x_2(t)x_2(t-1).
\end{aligned}$$

Clearly, the expressions for $\mathcal{H}(a_1x_1+a_2x_2)$ and $a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ are quite different. Consequently, these expressions are not equal for many choices of a_1, a_2, x_1, and x_2 (e.g., $a_1=2$, $a_2=0$, $x_1(t)=1$, and $x_2(t)=0$). Therefore, the superposition property does not hold and the system is not linear. ■

Example 3.40 (Ideal integrator)**.** A system $\mathcal{H}$ is defined by the equation

$$\mathcal{H}x(t)=\int_{-\infty}^{t}x(\tau)d\tau.$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we consider the additivity property. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x_1(t)+\mathcal{H}x_2(t) &= \int_{-\infty}^{t}x_1(\tau)d\tau+\int_{-\infty}^{t}x_2(\tau)d\tau \quad \text{and}\\
\mathcal{H}(x_1+x_2)(t) &= \int_{-\infty}^{t}(x_1+x_2)(\tau)d\tau\\
&= \int_{-\infty}^{t}[x_1(\tau)+x_2(\tau)]d\tau\\
&= \int_{-\infty}^{t}x_1(\tau)d\tau+\int_{-\infty}^{t}x_2(\tau)d\tau.
\end{aligned}$$

Since $\mathcal{H}(x_1+x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we consider the homogeneity property. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a\mathcal{H}x(t) &= a\int_{-\infty}^{t} x(\tau)d\tau \quad \text{and} \\ \mathcal{H}(ax)(t) &= \int_{-\infty}^{t} (ax)(\tau)d\tau \\ &= \int_{-\infty}^{t} ax(\tau)d\tau \\ &= a\int_{-\infty}^{t} x(\tau)d\tau. \end{aligned}$$

Since $\mathcal{H}(ax) = a\mathcal{H}x$ for all x and a, the system is homogeneous.

Lastly, we consider the linearity property. Since the system is both additive and homogeneous, it is linear. ■

Example 3.41. A system $\mathcal{H}$ is given by

$$\mathcal{H}x(t) = \operatorname{Re}[x(t)].$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)] \quad \text{and} \\ \mathcal{H}(x_1+x_2)(t) &= \operatorname{Re}[(x_1+x_2)(t)] \\ &= \operatorname{Re}[x_1(t)+x_2(t)] \\ &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)]. \end{aligned}$$

Since $\mathcal{H}(x_1+x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we check if the homogeneity property is satisfied. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a\mathcal{H}x(t) &= a\operatorname{Re}x(t) \quad \text{and} \\ \mathcal{H}(ax)(t) &= \operatorname{Re}[(ax)(t)] \\ &= \operatorname{Re}[ax(t)]. \end{aligned}$$

In order for $\mathcal{H}$ to be homogeneous, $a\mathcal{H}x(t) = \mathcal{H}(ax)(t)$ must hold for all x and all complex a. Suppose that $a = j$ and x is not identically zero (i.e., x is not the function $x(t) = 0$). In this case, we have

$$\begin{aligned} a\mathcal{H}x(t) &= j\operatorname{Re}[x(t)] \quad \text{and} \\ \mathcal{H}(ax)(t) &= \operatorname{Re}[(jx)(t)] \\ &= \operatorname{Re}[jx(t)] \\ &= \operatorname{Re}[j(\operatorname{Re}[x(t)] + j\operatorname{Im}[x(t)])] \\ &= \operatorname{Re}(-\operatorname{Im}[x(t)] + j\operatorname{Re}[x(t)]) \\ &= -\operatorname{Im}[x(t)]. \end{aligned}$$

Thus, the quantities $\mathcal{H}(ax)$ and $a\mathcal{H}x$ are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

3.8.7 Eigenfunctions

An **eigenfunction** of a system $\mathcal{H}$ is a function x that satisfies

$$\mathcal{H}x = \lambda x,$$

for some complex constant λ, which is called an **eigenvalue**. In other words, if x is an eigenfunction of $\mathcal{H}$, $\mathcal{H}x$ is a *scalar multiple* of x (i.e., a constant times x). Essentially, a system behaves as an ideal amplifier (i.e., performs amplitude scaling) when presented with one of its eigenfunctions as input. The significance of the eigenfunction property cannot be overstated. No matter how complicated a system might be, it exhibits extremely simple behavior for its eigenfunctions. We can often exploit this simplicity to reduce the complexity of solving many types of problems involving systems. In fact, as we will see later, eigenfunctions essentially form the basis for many of the mathematical tools that we use for studying systems.

Example 3.42. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t),$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos(2t)$; and
(b) $x(t) = t^3$.

Solution. (a) Consider the case that $x(t) = \cos(2t)$. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{\cos(2t)\}(t) \\ &= \mathcal{D}\{-2\sin(2t)\}(t) \\ &= -4\cos(2t) \\ &= -4x(t).\end{aligned}$$

Thus, $\mathcal{H}x$ is a scalar multiple of x (with the scalar multiple being -4). Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) Consider $x(t) = t^3$. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \tfrac{6}{t^2}x(t).\end{aligned}$$

Since $\mathcal{H}x$ is not a scalar multiple of x, x is not an eigenfunction of $\mathcal{H}$. ■

Example 3.43 (Ideal amplifier)**.** Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x(t) = ax(t),$$

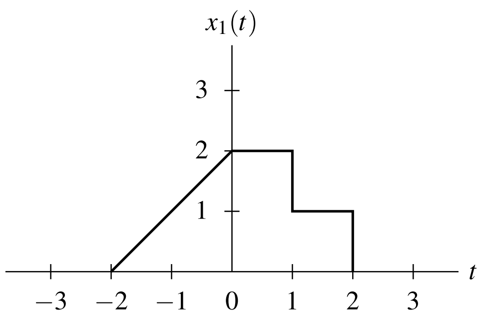
where a is a complex constant. Clearly, every function is an eigenfunction of $\mathcal{H}$ with eigenvalue a. ■

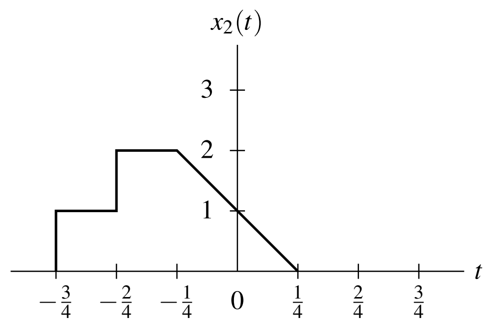
3.9 Exercises

3.9.1 Exercises Without Answer Key

3.1 Identify the independent- and dependent-variable transformations that must be applied to the function x in order to obtain each function y given below. Choose the transformations such that time shifting precedes time scaling and amplitude scaling precedes amplitude shifting. Be sure to clearly indicate the order in which the transformations are to be applied.
(a) $y(t) = x(2t-1)$;
(b) $y(t) = x(\frac{1}{2}t+1)$;
(c) $y(t) = 2x(-\frac{1}{2}t+1)+3$;
(d) $y(t) = -\frac{1}{2}x(-t+1)-1$; and
(e) $y(t) = -3x(2[t-1])-1$;
(f) $y(t) = x(7[t+3])$.

3.2 Given the functions x_1 and x_2 shown in the figures below, express x_2 in terms of x_1.



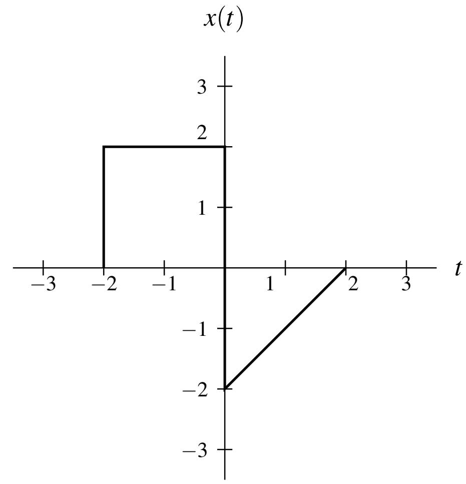


3.3 Suppose that we have two functions x and y related as

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that y can be formed by first time shifting x by b and then time scaling the result by a.
(b) Show that y can also be formed by first time scaling x by a and then time shifting the result by $\frac{b}{a}$.

3.4 Given the function x shown in the figure below, plot and label each of the following functions:
(a) $x(t-1)$;
(b) $x(2t)$;
(c) $x(-t)$;
(d) $x(2t+1)$; and
(e) $\frac{1}{4}x(-\frac{1}{2}t+1)-\frac{1}{2}$.



3.5 Given the function

$$x(t) = u(t+2) + u(t+1) + u(t) - 2u(t-1) - u(t-2),$$

find and sketch $y(t) = x(-4t-1)$.

3.6 Determine if each function x given below is periodic, and if it is, find its fundamental period.
(a) $x(t) = \cos(2\pi t) + \sin(5t)$;
(b) $x(t) = [\cos(4t - \frac{\pi}{3})]^2$;
(c) $x(t) = e^{j2\pi t} + e^{j3\pi t}$;
(d) $x(t) = 1 + \cos(2t) + e^{j5t}$;
(e) $x(t) = \cos(14t-1) + \cos(77t-3)$;
(f) $x(t) = \cos(et) + \sin(42t)$; and
(g) $x(t) = |\sin(\pi t)|$.

3.7 If the function x is T-periodic, show that the function y is also T-periodic in the case that:
(a) $y(t) = cx(t)$, where c is a complex constant;
(b) $y(t) = x(t) + c$, where c is a complex constant; and
(c) $y(t) = x(t-c)$, where c is a real constant.

3.8 Let y be the function given by

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - Tk),$$

where x is any arbitrary function and T is a strictly positive real constant. Show that y is T periodic.

3.9 Determine whether each function x given below is even, odd, or neither even nor odd.
(a) $x(t) = t^3$;
(b) $x(t) = t^3|t|$;
(c) $x(t) = |t^3|$;
(d) $x(t) = \cos(2\pi t)\sin(2\pi t)$;
(e) $x(t) = e^{j2\pi t}$; and
(f) $x(t) = \frac{1}{2}[e^t + e^{-t}]$.

3.10 Prove each of the following assertions:
(a) The sum of two even functions is even.
(b) The sum of two odd functions is odd.
(c) The sum of an even function and an odd function, where neither function is identically zero, is neither even nor odd.
(d) The product of two even functions is even.
(e) The product of two odd functions is even.
(f) The product of an even function and an odd function is odd.

3.11 Show that, if x is an odd function, then

$$\int_{-A}^{A} x(t)dt = 0,$$

where A is a positive real constant.

3.12 Show that, for any function x,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_\mathrm{e}^2(t)dt + \int_{-\infty}^{\infty} x_\mathrm{o}^2(t)dt,$$

where x_e and x_o denote the even and odd parts of x, respectively.

3.13 Let x denote a function with derivative y.
(a) Show that if x is even then y is odd.
(b) Show that if x is odd then y is even.

3.14 For an arbitrary function x with even and odd parts x_e and x_o, respectively, show that:
(a) $\int_{-\infty}^{\infty} x_\mathrm{e}(t)x_\mathrm{o}(t)dt = 0$; and
(b) $\int_{-\infty}^{\infty} x(t)dt = \int_{-\infty}^{\infty} x_\mathrm{e}(t)dt$.

3.15 Show that:
(a) if a function x is T-periodic and even, then $\int_0^T x(t)dt = 2\int_0^{T/2} x(t)dt$.
(b) if a function x is T-periodic and odd, then $\int_0^T x(t)dt = 0$.

3.16 Show that the only function that is both even and odd is the zero function (i.e., the function x satisfying $x(t) = 0$ for all t).

3.17 For each case below, for the function x (of a real variable) having the properties stated, find $x(t)$ for all t.
(a) The function x is such that:

- $x(t) = \begin{cases} t-1 & 1 \leq t \leq 2 \\ 3-t & 2 < t \leq 3; \end{cases}$
- the function v is causal, where $v(t) = x(t-1)$; and
- the function w is odd, where $w(t) = x(t+1)$.

(b) The function x is such that:

- $x(t) = t-1$ for $0 \leq t \leq 1$;
- the function v is causal, where $v(t) = x(t-1)$; and
- the function w is odd, where $w(t) = x(t)+1$.

(c) The function x is such that:

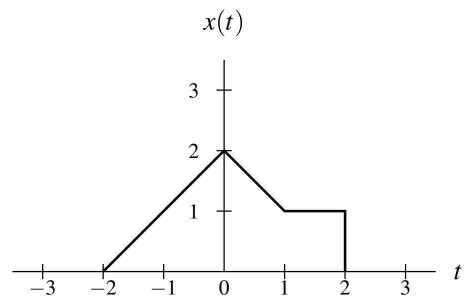
- x is causal; and
- the function $x_e = \mathrm{Even}(x)$ is given by $x_e(t) = t[u(t) - u(t-1)] + u(t-1)$ for $t \geq 0$.

3.18 Let Ex denote the energy of the function x. Show that:
(a) $E\{ax\} = a^2 Ex$, where a is a real constant;
(b) $Ex' = Ex$, where $x'(t) = x(t - t_0)$ and t_0 is a real constant; and
(c) $Ex' = \frac{1}{|a|} Ex$, where $x'(t) = x(at)$ and a is a nonzero real constant.

3.19 Show that, if a function x is conjugate symmetric, then:
(a) $\mathrm{Re}\, x(t) = \mathrm{Even}\, x(t)$ for all t; and
(b) $\mathrm{Im}\, x(t) = \frac{1}{j} \mathrm{Odd}\, x(t)$ for all t.

3.20 Fully simplify each of the expressions below.
(a) $\int_{-\infty}^{\infty} \sin\left(2t + \frac{\pi}{4}\right) \delta(t) dt$;
(b) $\int_{-\infty}^{t} \cos(\tau) \delta(\tau + \pi) d\tau$;
(c) $\int_{-\infty}^{\infty} x(t) \delta(at - b) dt$, where a and b are real constants and $a \neq 0$;
(d) $\int_{0}^{2} e^{j2t} \delta(t - 1) dt$;
(e) $\int_{-\infty}^{t} \delta(\tau) d\tau$; and
(f) $\int_{0}^{\infty} \tau^2 \cos(\tau) \delta(\tau + 42) d\tau$.

3.21 Suppose that we have the function x shown in the figure below. Use unit-step functions to find a single expression for $x(t)$ that is valid for all t.



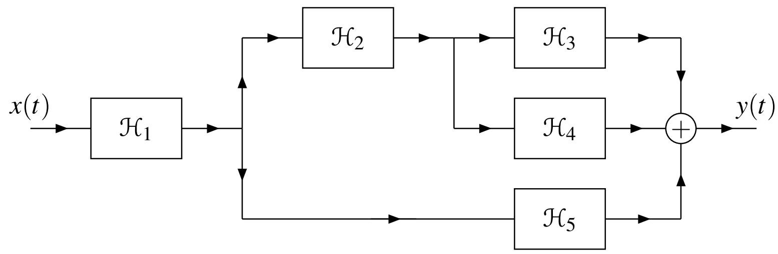
3.22 For each function x given below, find a single expression for x (i.e., an expression that does not involve multiple cases). Group similar unit-step function terms together in the expression for x.

(a) $x(t) = \begin{cases} -t-3 & -3 \leq t < -2 \\ -1 & -2 \leq t < -1 \\ t^3 & -1 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ -t+3 & 2 \leq t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = \begin{cases} -1 & t < -1 \\ t & -1 \leq t < 1 \\ 1 & t \geq 1; \quad \text{and} \end{cases}$

(c) $x(t) = \begin{cases} 4t+4 & -1 \le t < -\frac{1}{2} \\ 4t^2 & -\frac{1}{2} \le t < \frac{1}{2} \\ -4t+4 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$

3.23 For the system shown in the figure below, express the output y in terms of the input x and the transformations $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_5$.



3.24 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = x(t-1)+1$;
(d) $\mathcal{H}x(t) = \int_{t}^{\infty} x(\tau)d\tau$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$;
(f) $\mathcal{H}x(t) = tx(t)$; and
(g) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$.

3.25 Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = x(t-1)+1$;
(d) $\mathcal{H}x(t) = \int_{t}^{\infty} x(\tau)d\tau$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$; and
(f) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$.

3.26 For each system $\mathcal{H}$ given below, determine if $\mathcal{H}$ is invertible, and if it is, specify its inverse.
(a) $\mathcal{H}x(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $\mathcal{H}x(t) = e^{x(t)}$, where x is a real function;
(c) $\mathcal{H}x(t) = \operatorname{Even}(x)(t) - \operatorname{Odd}(x)(t)$;
(d) $\mathcal{H}x(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator; and
(e) $\mathcal{H}x(t) = x^2(t)$.

3.27 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(t) = \int_{t}^{t+1} x(\tau)d\tau$ [Hint: For any function f, $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$.];
(b) $\mathcal{H}x(t) = \frac{1}{2}x^2(t) + x(t)$;
(c) $\mathcal{H}x(t) = 1/x(t)$;
(d) $\mathcal{H}x(t) = e^{-|t|}x(t)$; and
(e) $\mathcal{H}x(t) = \left(\frac{1}{t-1}\right)x(t)$.

3.28 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(t) = \mathcal{D}x(t)$; where $\mathcal{D}$ denotes the derivative operator;
(b) $\mathcal{H}x(t) = \mathrm{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = \int_t^{t+1} x(\tau - \alpha)d\tau$, where α is a constant;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, where h is an arbitrary (but fixed) function;
(e) $\mathcal{H}x(t) = x(-t)$; and
(f) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

3.29 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = e^{x(t)}$;
(c) $\mathcal{H}x(t) = \mathrm{Even}(x)(t)$;
(d) $\mathcal{H}x(t) = x^2(t)$; and
(e) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, where h is an arbitrary (but fixed) function.

3.30 Determine whether each system $\mathcal{H}$ given below is additive and/or homogeneous.
(a) $\mathcal{H}x(t) = x^*(t)$; and
(b) $\mathcal{H}x(t) = \mathrm{Im}[x(t)]$.

3.31 Show that if a system $\mathcal{H}$ is either additive or homogeneous, it has the property that, for a function x that is identically zero (i.e., $x(t) = 0$ for all t), $\mathcal{H}x$ is identically zero (i.e., $\mathcal{H}x(t) = 0$ for all t).

3.32 Let z denote a function that is identically zero (i.e., $z(t) = 0$ for all t). Show that a linear system $\mathcal{H}$ is invertible if and only if $\mathcal{H}x = z$ implies that $x = z$ (i.e., the only input x that produces an output $y = z$ is $x = z$).

3.33 For each system $\mathcal{H}$ and the functions $\{x_k\}$ given below, determine if each of the x_k is an eigenfunction of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(t) = x^2(t)$, $x_1(t) = a$, $x_2(t) = e^{-at}$, and $x_3(t) = \cos t$, where a is a complex constant;
(b) $\mathcal{H}x(t) = \mathcal{D}x(t)$, $x_1(t) = e^{at}$, $x_2(t) = e^{at^2}$, and $x_3(t) = 42$, where $\mathcal{D}$ denotes the derivative operator and a is a real constant;
(c) $\mathcal{H}x(t) = \int_{t-1}^{t} x(\tau)d\tau$, $x_1(t) = e^{at}$, $x_2(t) = t$, and $x_3(t) = \sin t$, where a is a nonzero complex constant; and
(d) $\mathcal{H}x(t) = |x(t)|$, $x_1(t) = a$, $x_2(t) = t$, $x_3(t) = t^2$, where a is a strictly positive real constant.

3.9.2 Exercises With Answer Key

3.101 For each case below, where the function y is generated from the function x by applying the (ordered) sequence of transformations given (starting from y), express y in terms of x.
(a) time shift by 3; time scale by 2; and
(b) time scale by 2; time shift by 3.

Short Answer. (a) $y(t) = x(2t-3)$; (b) $y(t) = x(2t-6)$

3.102 For each function x given below, determine whether x is periodic, and if it is, find its fundamental period T.
(a) $x(t) = 3\cos(\sqrt{2}t) + 7\cos(2t)$;
(b) $x(t) = [3\cos(2t)]^3$; and
(c) $x(t) = 7\cos(35t+3) + 5\sin(15t-2)$.

Short Answer. (a) not periodic; (b) π-periodic; (c) $\left(\frac{2\pi}{5}\right)$-periodic

3.103 For each case below, for the function x (of a real variable) having the properties stated, find $x(t)$ for all t.
(a) The function x is such that:

- $x(t) = 1-t$ for $0 \leq t \leq 1$;
- the function w is even, where $w(t) = x(t+1)$; and
- the function v is causal, where $v(t) = x(t) - 1$.

(b) The function x is such that:

- $x(t) = e^{t+4}$ for $t < -4$;
- $x(t) = a$ for $-4 \leq t \leq -2$, where a is a real constant; and
- the function $v(t) = x(t-3)$ is odd.

(c) The function x is such that:

- the function $v(t) = x(t)+1$ is causal;
- the function $w(t) = x(-t)-1$ is causal; and
- $x(0) = 0$.

(d) The function x is such that:

- the function $x(t) = 1$ for $2 < t \leq 3$;
- the function $v_1(t) = x(t+1)$ is causal;
- the function $v_2(t) = x(t+3)$ is anticausal; and
- the function $v_3(t) = x(t+2)$ is odd.

(e) The function x is such that:

- the function $x(t) = t-1$ for $1 \leq t \leq 2$;
- x is causal;
- the function $v_1(t) = x(t+2)$ is anticausal; and
- the function $v_2(t) = x(t+1)$ is even.

(f) The function x is such that:

- $\operatorname{Even} x(t) = t$ for $t \leq 0$; and
- $\operatorname{Odd} x(t) = t^2$ for $t > 0$.

(g) The function x is such that:

- the function $v(t) = x(t+2)$ is conjugate symmetric; and
- $x(t) = j[u(t-3) - u(t-5)]$ for $t \geq 2$.

(h) The function x is such that:

- the function $v(t) = x(t) - 1$ is causal; and
- x is odd.

(i) The function x is such that:

- $\operatorname{Re} x(t) = t$ for $t \geq 0$;
- $\operatorname{Im} x(t) = t^2$ for $t < 0$; and
- x is conjugate symmetric.

(j) The function x is such that:

- $x(t) = 2-t$ for $0 \leq t < 1$;
- the function $v(t) = x(t) - 2$ is causal; and
- the function $w(t) = x(t+1)$ is odd.

Short Answer.
(a) $x(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ t-1 & 1 < t \le 2 \\ 1 & \text{otherwise;} \end{cases}$
(b) $x(t) = \begin{cases} e^{t+4} & t < -4 \\ 0 & -4 \le t \le -2 \\ -e^{-t-2} & t > -2; \end{cases}$
(c) $x(t) = \operatorname{sgn}(t)$;
(d) $x(t) = \begin{cases} -1 & 1 < t < 2 \\ 1 & 2 < t < 3 \\ 0 & \text{otherwise;} \end{cases}$
(e) $x(t) = \begin{cases} 1-t & 0 \le t < 1 \\ t-1 & 1 \le t \le 2 \\ 0 & \text{otherwise;} \end{cases}$
(f) $x(t) = (t^2 - t)\operatorname{sgn}(t)$;
(g) $x(t) = \begin{cases} -j & -1 < t \le 1 \\ j & 3 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$
(h) $x(t) = -\operatorname{sgn}(t)$;
(i) $x(t) = (t - jt^2)\operatorname{sgn}(t)$;
(j) $x(t) = \begin{cases} 2 & t < 0 \\ 2-t & 0 \le t < 1 \\ 0 & t = 1 \\ -t & 1 < t \le 2 \\ -2 & t > 2 \end{cases}$

3.104 Determine whether each function x given below is even, odd, or neither even nor odd.
(a) $x(t) = e^{-|t|}\sin(t)$; and
(b) $x(t) = e^{-t^2}\cos(t)$.

Short Answer. (a) odd; (b) even

3.105 Simplify each of the following expressions:
(a) $\dfrac{(\omega^2+1)\delta(\omega-1)}{\omega^2+9}$;
(b) $\dfrac{\sin(k\omega)\delta(\omega)}{\omega}$;
(c) $\int_{-\infty}^{\infty} e^{t-1}\cos[\frac{\pi}{2}(t-5)]\delta(t-3)dt$;
(d) $\int_{-\infty}^{\infty} \delta(2t-3)\sin(\pi t)dt$;
(e) $\int_{t}^{\infty} (\tau^2+1)\delta(\tau-2)d\tau$;
(f) $\int_{-4}^{4} e^{-\tau}\cos(\tau)\delta\left(\tau-\frac{\pi}{3}\right)d\tau + \int_{-2}^{2} \tau^2\cos(\tau)\delta(\tau-\pi)d\tau$; and
(g) $(t^2+1)^4 e^{-t}\operatorname{sinc}(t)\delta(t-\pi)$.

Short Answer. (a) $\frac{1}{5}\delta(\omega-1)$; (b) $k\delta(\omega)$; (c) $-e^2$; (d) $-\frac{1}{2}$; (e) $5u(2-t)$; (f) $\frac{1}{2}e^{-\pi/3}$; (g) 0

3.106 For each function x given below, find a single expression for x (i.e., an expression that does not involve multiple cases). If the expression for x consists of a finite number of terms, group similar unit-step function terms together in the expression for x.
(a) $x(t) = 1 - t^2$ for $-1 \leq t < 1$ and $x(t) = x(t-2)$ for all t;
(b) $x(t) = \begin{cases} -e^{t+1} & t < -1 \\ t & -1 \leq t < 1 \\ (t-2)^2 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(c) $x(t) = \begin{cases} (t/\pi + 1)^2 & t < -\pi \\ \cos(t/2) & -\pi \leq t \leq \pi \\ (t/\pi - 1)^2 & t > \pi; \end{cases}$; and
(d) $x(t) = \begin{cases} t+2 & -2 \leq t < -1 \\ t^2 & -1 \leq t \leq 1 \\ 1 & t > 1 \\ 0 & \text{otherwise.} \end{cases}$.

Short Answer.
(a) $x(t) = \sum_{k=-\infty}^{\infty} (-t^2 + 4kt - 4k^2 + 1)[u(t-2k+1) - u(t-2k-1)]$;
(b) $x(t) = -e^{t+1} + (t + e^{t+1})u(t+1) + (t-1)(t-4)u(t-1) - (t-2)^2 u(t-2)$;
(c) $x(t) = (t/\pi + 1)^2 + [\cos(t/2) - (t/\pi + 1)^2]u(t+\pi) + [(t/\pi - 1)^2 - \cos(t/2)]u(t-\pi)$;
(d) $x(t) = (t+2)u(t+2) + [t^2 - t - 2]u(t+1) + [1 - t^2]u(t-1)$

3.107 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(t) = u[x(t)]$;
(b) $\mathcal{H}x(t) = x[u(t)]$;
(c) $\mathcal{H}x(t) = 42$;
(d) $\mathcal{H}x(t) = x(t^2)$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} 2x(\tau)\delta(\tau - t)d\tau$;
(f) $\mathcal{H}x(t) = [x(t+1)]^{-1}$;
(g) $\mathcal{H}x(t) = x(-t)$; and
(h) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)u(\tau - t - 2)d\tau$.

Short Answer. (a) memoryless; (b) has memory; (c) memoryless; (d) has memory; (e) memoryless; (f) has memory; (g) has memory; (h) has memory

3.108 Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(t) = x(at)$, where a is a nonzero real constant;
(b) $\mathcal{H}x(t) = tu(t)x(t)$;
(c) $\mathcal{H}x(t) = x(t-a)$, where a is a strictly negative real constant;
(d) $\mathcal{H}x(t) = [x(t+1)]^{-1}$;
(e) $\mathcal{H}x(t) = x(-t)$;
(f) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau - 1)d\tau$; and
(g) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)u(\tau - t - 3)d\tau$.

Short Answer. (a) causal if and only if $a = 1$; (b) causal; (c) not causal; (d) not causal; (e) not causal; (f) causal; (g) not causal

3.109 Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(t) = \cos[x(t)]$;
(b) $\mathcal{H}x(t) = x * x(t)$, where $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$;
(c) $\mathcal{H}x(t) = \operatorname{Even} x(t)$;
(d) $\mathcal{H}x(t) = \operatorname{Re} x(t)$;
(e) $\mathcal{H}x(t) = |x(t)|$;
(f) $\mathcal{H}x(t) = 2x(t) + 3$; and
(g) $\mathcal{H}x(t) = x(t) + x(t-1)$.

Short Answer. (a) not invertible; (b) not invertible; (c) not invertible; (d) not invertible; (e) not invertible; (f) invertible; (g) not invertible

3.110 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(t) = u(t)x(t)$;
(b) $\mathcal{H}x(t) = \ln x(t)$;
(c) $\mathcal{H}x(t) = e^{x(t)}$;
(d) $\mathcal{H}x(t) = e^{t}x(t)$;
(e) $\mathcal{H}x(t) = \cos[x(t)]$;
(f) $\mathcal{H}x(t) = x * x(t)$, where $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$;
(g) $\mathcal{H}x(t) = 3x(3t+3)$; and
(h) $\mathcal{H}x(t) = 2x(t) + 1$.

Short Answer. (a) BIBO stable; (b) not BIBO stable; (c) BIBO stable; (d) not BIBO stable; (e) BIBO stable (if x is real valued or complex valued); (f) not BIBO stable; (g) BIBO stable; (h) BIBO stable

3.111 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(t) = \int_{-4}^{4} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \mathcal{D}\{x^2\}(t)$, where $\mathcal{D}$ denotes the derivative operator;
(c) $\mathcal{H}x(t) = tu(t)x(t)$;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$; and
(e) $\mathcal{H}x(t) = 3x(3t+3)$.

Short Answer. (a) not time invariant; (b) time invariant; (c) not time invariant; (d) not time invariant; (e) not time invariant

3.112 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(t) = \frac{1}{3}[x(t) - 2]$;
(b) $\mathcal{H}x(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator;
(c) $\mathcal{H}x(t) = tu(t)x(t)$;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$;
(e) $\mathcal{H}x(t) = t^2\mathcal{D}^2x(t) + t\mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator;
(f) $\mathcal{H}x(t) = 3x(3t+3)$;
(g) $\mathcal{H}x(t) = \int_{-1}^{1} x(\tau)d\tau$;
(h) $\mathcal{H}x(t) = |t|\,x(t)$;
(i) $\mathcal{H}x(t) = (t+1)^2x(t)$;
(j) $\mathcal{H}x(t) = 42$;
(k) $\mathcal{H}x(t) = 0$; and
(l) $\mathcal{H}x(t) = [x(t)]^{-1}$.

Short Answer. (a) not linear; (b) linear; (c) linear; (d) not linear; (e) linear; (f) linear; (g) linear; (h) linear; (i) linear; (j) not linear; (k) linear; (l) not linear

3.113 Determine whether each system $\mathcal{H}$ given below is additive and/or homogeneous.
(a) $\mathcal{H}x(t) = \frac{x^2(t)}{\mathcal{D}x(t)}$, where $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) homogeneous but not additive

3.114 For each system $\mathcal{H}$ and the functions $\{x_k\}$ given below, determine if each of the x_k is an eigenfunction of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(t) = \mathcal{D}^2 x(t)$, $x_1(t) = \cos t$, $x_2(t) = \sin t$, and $x_3(t) = 42$, where $\mathcal{D}$ denotes the derivative operator;
(b) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau$, $x_1(t) = e^{2t}$, and $x_2(t) = e^t u(-t)$;
(c) $\mathcal{H}x(t) = t^2\mathcal{D}^2 x(t) + t\mathcal{D}x(t)$ and $x_1(t) = t^k$, where k is an integer constant such that $k \geq 2$, and $\mathcal{D}$ denotes the derivative operator;
(d) $\mathcal{H}x(t) = u(t)x(t)$, $x_1(t) = 0$, $x_2(t) = 1$, $x_3(t) = u(t+1)$, and $x_4(t) = u(t-1)$; and
(e) $\mathcal{H}x(t) = t\mathcal{D}x(t)$, $x_1(t) = 3t^2$, $x_2(t) = \pi$, where $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) x_1 is an eigenfunction with eigenvalue -1, x_2 is an eigenfunction with eigenvalue -1, x_3 is an eigenfunction with eigenvalue 0; (b) x_1 is an eigenfunction with eigenvalue $\frac{1}{2}$, x_2 is not an eigenfunction; (c) x_1 is an eigenfunction with eigenvalue k^2; (d) x_1 is an eigenfunction with eigenvalue 0; x_2 is not an eigenfunction; x_3 is not an eigenfunction; x_4 is an eigenfunction with eigenvalue 1; (e) x_1 is an eigenfunction with eigenvalue 2; x_2 is an eigenfunction with eigenvalue 0

Chapter 4

Continuous-Time Linear Time-Invariant Systems

4.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were linearity and time invariance. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant (LTI)** systems.

4.2 Continuous-Time Convolution

In the context of LTI systems, a mathematical operation known as convolution turns out to be particularly important. The **convolution** of the functions x and h, denoted $x*h$, is defined as the function

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{4.1}$$

Herein, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

Notationally, $x*h$ denotes a function, namely the function that results from convolving x and h. In contrast, $x*h(t)$ denotes the function $x*h$ evaluated at t. Although we could equivalently write $x*h(t)$ with an extra pair of brackets as $(x*h)(t)$, we usually omit this extra pair of brackets, since doing so does not introduce any ambiguity and leads to more compact notation. That is, there is only one sensible way to group operations in the expression $x*h(t)$. The grouping $x*[h(t)]$ would not make sense since a convolution requires two functions as operands and $h(t)$ is not a function, but rather the value of h evaluated at t. Thus, the only sensible way to interpret the expression $x*h(t)$ is as $(x*h)(t)$.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolution integral. Suppose that, for the given functions x and h, we wish to compute

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Of course, we could naively attempt to compute $x*h$ by evaluating $x*h(t)$ as a separate integral for each possible value of t. This approach, however, is not feasible, as t can assume an infinite number of values, and therefore, an infinite number of integrals would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the integrand in terms of the intermediate function w_t where

$$w_t(\tau) = x(\tau)h(t-\tau).$$

(Note that $w_t(\tau)$ is implicitly a function of t.) This means that we need to compute

$$x * h(t) = \int_{-\infty}^{\infty} w_t(\tau) d\tau.$$

Now, we observe that, for most functions x and h of practical interest, the form of $w_t(\tau)$ typically remains fixed over particular ranges of t. Thus, we can compute the convolution result $x * h$ by first identifying each of the distinct expressions for $w_t(\tau)$ and the range over which each expression is valid. Then, for each range, we evaluate an integral. In this way, we typically only need to compute a small number of integrals instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for computing a convolution:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $w_t(\tau)$.
4. Increase t gradually until the expression for $w_t(\tau)$ changes form. Record the interval over which the expression for $w_t(\tau)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. For each of the intervals identified above, integrate $w_t(\tau)$ in order to find an expression for $x * h(t)$. This will yield an expression for $x * h(t)$ for each interval.
7. The results for the various intervals can be combined in order to obtain the convolution result $x * h(t)$ for all t.

Example 4.1. Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \le t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t} d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau} d\tau \\ &= -e^{-t}[e^{\tau}]\big|_{-1}^{t} \\ &= -e^{-t}[e^{t} - e^{-1}] \\ &= e^{-t-1} - 1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \le t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\
&= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\
&= e^{-t}[e^{-1}-1+e^{t}-1] \\
&= 1+(e^{-1}-2)e^{-t}.
\end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \ge 1$. From Figure 4.1(h), we can see that

$$\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\
&= e^{-t}[e^{-1}-1+e-1] \\
&= (e-2+e^{-1})e^{-t}.
\end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1}-1 & -1 \le t < 0 \\ (e^{-1}-2)e^{-t}+1 & 0 \le t < 1 \\ (e-2+e^{-1})e^{-t} & 1 \le t. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.1(i). ■

Example 4.2. Compute the convolution $x * h$, where

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.2(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.2(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.2(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.2(e) to (h).

First, we consider the case of $t < 0$. From Figure 4.2(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.6}$$

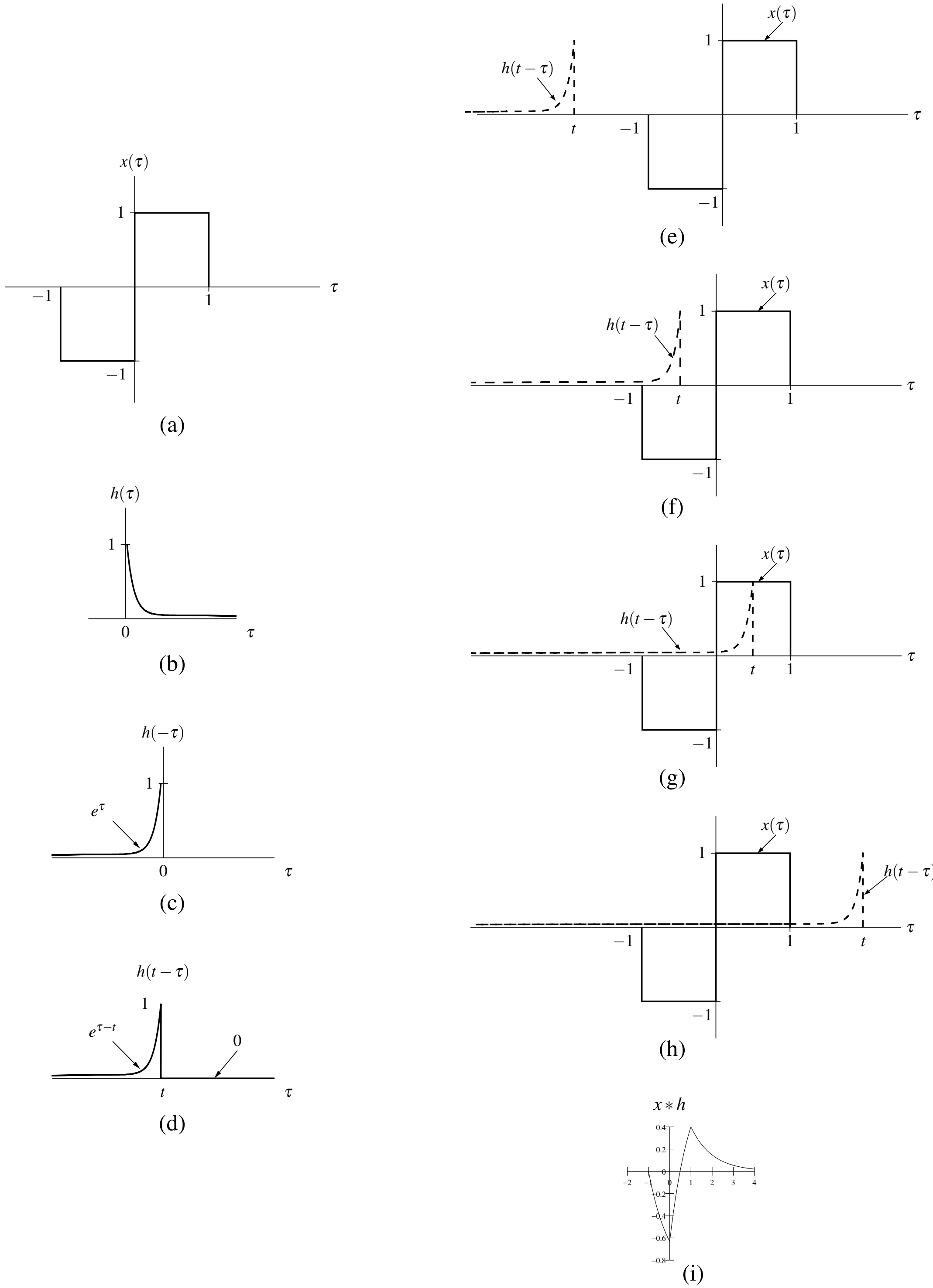


Figure 4.1: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<-1$, (f) $-1 \leq t < 0$, (g) $0 \leq t < 1$, and (h) $t \geq 1$; and (i) the convolution result $x*h$.

Second, we consider the case of $0 \leq t < 1$. From Figure 4.2(f), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_0^t (t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]|_0^t \\ &= t^2 - \tfrac{1}{2}t^2 \\ &= \tfrac{1}{2}t^2. \end{aligned} \tag{4.7}$$

Third, we consider the case of $1 \leq t < 2$. From Figure 4.2(g), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} (t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]_{t-1}^{1} \\ &= t - \tfrac{1}{2}(1)^2 - [t(t-1) - \tfrac{1}{2}(t-1)^2] \\ &= t - \tfrac{1}{2} - [t^2 - t - \tfrac{1}{2}(t^2 - 2t + 1)] \\ &= -\tfrac{1}{2}t^2 + t. \end{aligned} \tag{4.8}$$

Fourth, we consider the case of $t \geq 2$. From Figure 4.2(h), we can see that

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.9}$$

Combining the results of (4.6), (4.7), (4.8), and (4.9), we have that

$$x*h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 1 \\ -\frac{1}{2}t^2 + t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$

The convolution result $x*h$ is plotted in Figure 4.2(i). ■

Example 4.3. Compute the quantity $x*h$, where

$$x(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ -t+2 & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \quad \text{and} \quad h(t) = u(t) - u(t-1).$$

Solution. Due to the somewhat ugly nature of the expressions for $x(t)$ and $h(t)$, this problem can be more easily solved if we use the graphical interpretation of convolution to guide us. We begin by plotting the functions x and h, as shown in Figures 4.3(a) and (b), respectively.

Next, we need to determine $h(t-\tau)$, the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.3(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 4.3(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.3(e) to (i).

