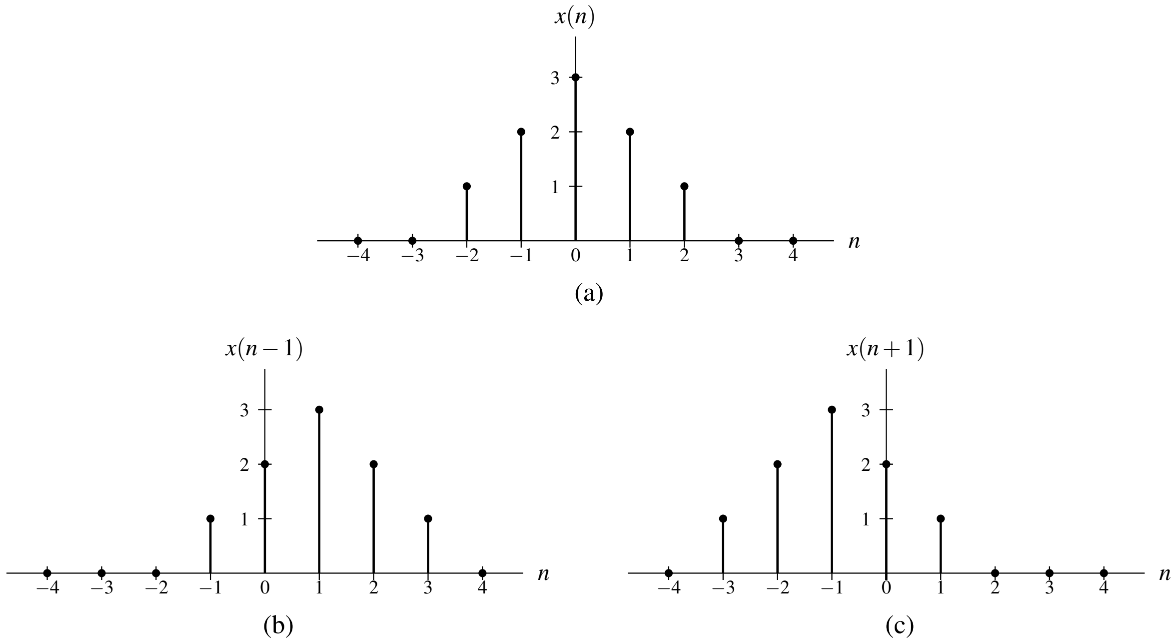

Figure 8.1: Example of time shifting.

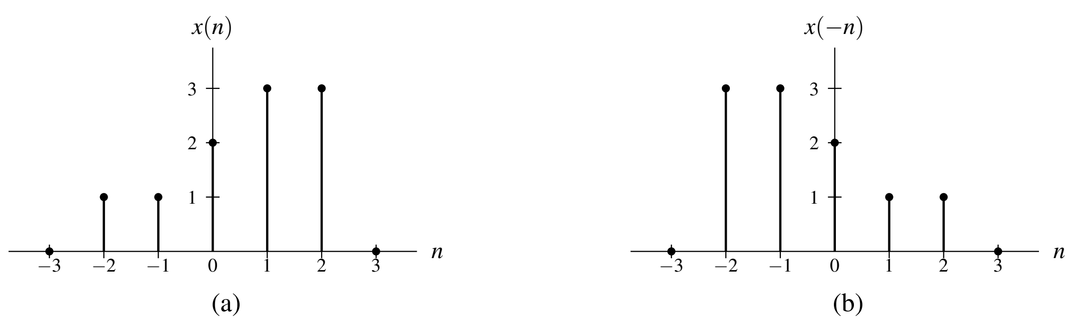

Figure 8.2: Example of time reversal.

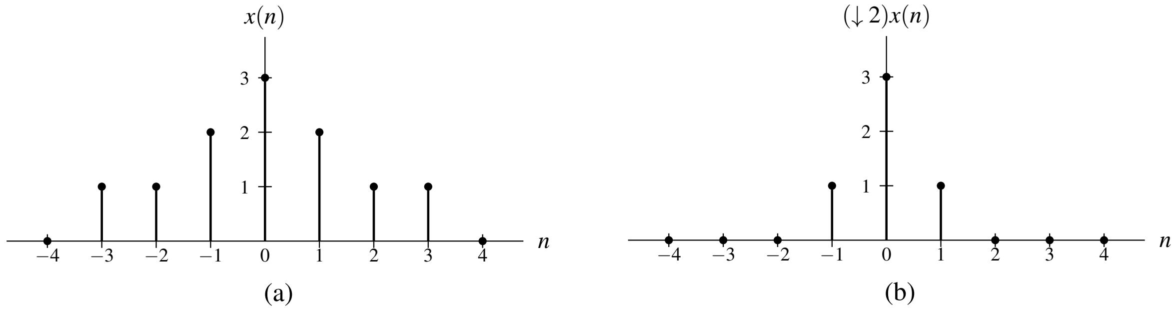

Figure 8.3: Downsampling example. (a) Original sequence x. (b) Result obtained by 2-fold downsampling of x.

8.2.3 Downsampling

The next transformation to be considered is called downsampling. **Downsampling** maps a sequence x to the sequence y given by

$$y(n) = (\downarrow a)x(n) = x(an), \tag{8.3}$$

where a is a (strictly) positive integer. The constant a is referred to as the **downsampling factor**. Downsampling with a downsampling factor of a is referred to as ***a*-fold downsampling**. In simple terms, a-fold downsampling keeps every ath sample from the original sequence and discards the others. Thus, downsampling reduces the sampling rate/density by a factor of a.

To illustrate the effects of downsampling, an example is provided in Figure 8.3. Applying 2-fold downsampling to the sequence x shown in Figure 8.3(a) yields the sequence shown in Figure 8.3(b).

8.2.4 Upsampling (Time Expansion)

The next transformation to be considered is called upsampling. **Upsampling** (also known as **time expansion**) maps a sequence x to the sequence y given by

$$y(n) = (\uparrow a)x(n) = \begin{cases} x(n/a) & n/a \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

where a is a (strictly) positive integer. The constant a is referred to as the **upsampling factor**. Upsampling with a upsampling factor of a is referred to as ***a*-fold upsampling**. In simple terms, a-fold upsampling results in the insertion of $a-1$ zeros between the samples of the original input sequence. Thus, upsampling increases the sampling rate/density by a factor of a.

To illustrate the effects of upsampling, an example is provided in Figure 8.4. Applying 2-fold upsampling to the sequence x in Figure 8.4(a) yields the sequence in Figure 8.4(b).

8.2.5 Combined Independent-Variable Transformations

Some independent-variable transformations commute, while other do not. The issue of commutativity is important, for example, when trying to simplify or manipulate expressions involving combined transformations. A time reversal operation commutes with a downsampling or upsampling operation. A time shift operation (with a nonzero shift) does not commute with a time-reversal, downsampling, or upsampling operation. A downsampling operation does not commute with an upsampling operation unless the upsampling and downsampling factors are coprime (i.e., have no common factors).

Consider a transformation that maps the input sequence x to the output sequence y given by

$$y(n) = x(an-b), \tag{8.4}$$

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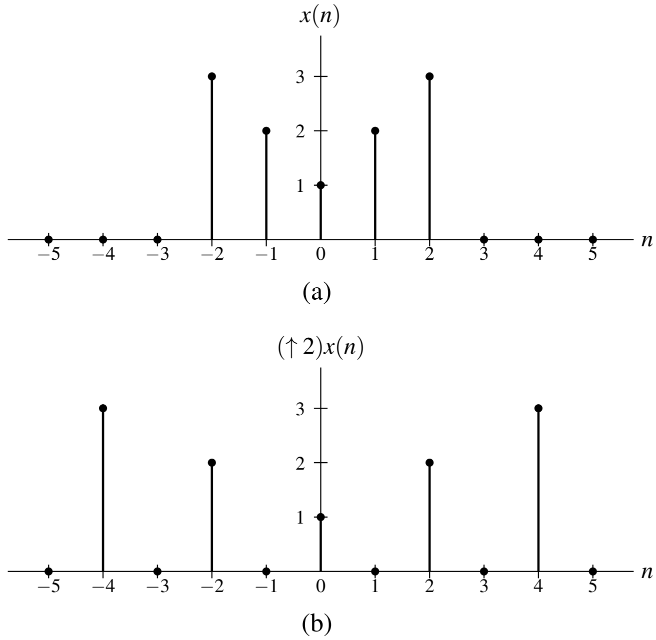

Figure 8.4: Upsampling example. (a) Original sequence x. (b) Result obtained by 2-fold upsampling of x.

where a and b are integers and $a \neq 0$. Such a transformation is a combination of time shifting, downsampling, and time-reversal operations.

The transformation (8.4) is equivalent to:

1. first, time shifting x by b;
2. then, downsampling the result by $|a|$ and, if $a < 0$, time reversing as well.

If $\frac{b}{a}$ is an integer, the transformation (8.4) is also equivalent to:

1. first, downsampling x by $|a|$ and, if $a < 0$, time reversing;
2. then, time shifting the result by $\frac{b}{a}$.

Observe that the shifting amount differs in these two interpretations (i.e., b versus $\frac{b}{a}$). This is due to the fact that a time shift operation does not commute with either a downsampling or time-reversal operation. The proof that the above two interpretations are valid is left as an exercise for the reader in Exercise 8.2.

8.2.6 Two Perspectives on Independent-Variable Transformations

A transformation of the independent variable can be viewed in terms of:

1. the effect that the transformation has on the *sequence*; or
2. the effect that the transformation has on the *horizontal axis*.

This distinction is important because such a transformation has *opposite* effects on the sequence and horizontal axis. For example, the (time-shifting) transformation that replaces n by $n-b$ (where b is an integer) in the expression for $x(n)$ can be viewed as a transformation that:

1. shifts the sequence x *right* by b units; or
2. shifts the horizontal axis *left* by b units.

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In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *sequence*. If one is not careful to consider that we are interested in the sequence perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

8.3 Properties of Sequences

Sequences can possess a number of interesting properties. In what follows, we consider the properties of symmetry and periodicity (introduced earlier) in more detail. Also, we present several other sequence properties. The properties considered are frequently useful in the analysis of signals and systems.

8.3.1 Remarks on Symmetry

At this point, we make some additional comments about even and odd sequences (introduced earlier). Since sequences are often summed or multiplied, one might wonder what happens to the even/odd symmetry properties of sequences under these operations. In what follows, we introduce a few results in this regard.

Sums involving even and odd sequences have the following properties:

- The sum of two even sequences is even.
- The sum of two odd sequences is odd.
- The sum of an even sequence and odd sequence is neither even nor odd, provided that the sequences are not identically zero.

Products involving even and odd sequences have the following properties:

- The product of two even sequences is even.
- The product of two odd sequences is even.
- The product of an even sequences and an odd sequences is odd.

(The proofs of the above properties involving sums and products of even and odd sequences is left as an exercise for the reader in Exercise 8.9.)

As it turns out, any arbitrary sequence can be expressed as the sum of an even and odd sequence, as elaborated upon by the theorem below.

Theorem 8.1 (Decomposition of sequence into even and odd parts)**.** *Any arbitrary sequence x can be uniquely represented as the sum of the form*

$$x(n) = x_\mathrm{e}(n) + x_\mathrm{o}(n), \tag{8.5}$$

where x_e and x_o are even and odd, respectively, and given by

$$x_\mathrm{e}(n) = \tfrac{1}{2}\left[x(n) + x(-n)\right] \quad \text{and} \tag{8.6}$$

$$x_\mathrm{o}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right]. \tag{8.7}$$

*As a matter of terminology, x_e is called the **even part** of x and is denoted $\mathrm{Even}\{x\}$, and x_o is called the **odd part** of x and is denoted $\mathrm{Odd}\{x\}$.*

Proof. From (8.6) and (8.7), we can easily confirm that $x_\mathrm{e} + x_\mathrm{o} = x$ as follows:

$$\begin{aligned} x_\mathrm{e}(n) + x_\mathrm{o}(n) &= \tfrac{1}{2}(x(n) + x(-n)) + \tfrac{1}{2}(x(n) - x(-n)) \\ &= \tfrac{1}{2}x(n) + \tfrac{1}{2}x(-n) + \tfrac{1}{2}x(n) - \tfrac{1}{2}x(-n) \\ &= x(n). \end{aligned}$$

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Furthermore, we can easily verify that x_e is even and x_o is odd. From the definition of x_e in (8.6), we have

$$\begin{aligned} x_\mathrm{e}(-n) &= \tfrac{1}{2}(x(-n)+x(-(-n))) \\ &= \tfrac{1}{2}(x(n)+x(-n)) \\ &= x_\mathrm{e}(n). \end{aligned}$$

Thus, x_e is even. From the definition of x_o in (8.7), we have

$$\begin{aligned} x_\mathrm{o}(-n) &= \tfrac{1}{2}(x(-n)-x(n)) \\ &= \tfrac{1}{2}(-x(n)+x(-n)) \\ &= -x_\mathrm{o}(n). \end{aligned}$$

Thus, x_o is odd.

Lastly, we show that the decomposition of x into the sum of an even sequence and odd sequence is unique. Suppose that x can be written as the sum of an even sequence and odd sequence in two ways as

$$x(n) = f_\mathrm{e}(n)+f_\mathrm{o}(n) \quad \text{and} \tag{8.8a}$$
$$x(n) = g_\mathrm{e}(n)+g_\mathrm{o}(n), \tag{8.8b}$$

where f_e and g_e are even and f_o and g_o are odd. Equating these two expressions for x, we have

$$f_\mathrm{e}(n)+f_\mathrm{o}(n) = g_\mathrm{e}(n)+g_\mathrm{o}(n).$$

Rearranging this equation, we have

$$f_\mathrm{e}(n)-g_\mathrm{e}(n) = g_\mathrm{o}(n)-f_\mathrm{o}(n).$$

Now, we consider the preceding equation more carefully. Since the sum of even sequences is even and the sum of odd sequences is odd, we have that the left- and right-hand sides of the preceding equation correspond to even and odd sequences, respectively. Thus, we have that the even sequence $f_\mathrm{e}(n)-g_\mathrm{e}(n)$ is equal to the odd sequence $g_\mathrm{o}(n)-f_\mathrm{o}(n)$. The only sequence, however, that is both even and odd is the zero sequence. (A proof of this fact is left as an exercise for the reader in Exercise 8.11.) Therefore, we have that

$$f_\mathrm{e}(n)-g_\mathrm{e}(n) = g_\mathrm{o}(n)-f_\mathrm{o}(n) = 0.$$

In other words, we have that

$$f_\mathrm{e}(n) = g_\mathrm{e}(n) \quad \text{and} \quad f_\mathrm{o}(n) = g_\mathrm{o}(n).$$

This implies that the two decompositions of x given by (8.8a) and (8.8b) must be the same decomposition (i.e., they cannot be distinct). Thus, the decomposition of x into the sum of an even sequence and odd sequence is unique. ∎

8.3.2 Remarks on Periodicity

Since we often add sequences, it is helpful to know if the sum of periodic sequences is also periodic. We will consider this issue next, but before doing so we first must introduce the notion of a least common multiple and greatest common divisor.

The **least common multiple (LCM)** of two nonzero integers a and b, denoted $\operatorname{lcm}(a,b)$, is the smallest positive integer that is divisible by both a and b. From this definition, it immediately follows that, if a and b are coprime, then $\operatorname{lcm}(a,b) = |ab|$. The quantity $\operatorname{lcm}(a,b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the highest power for each prime factor appearing in these factorizations.

The **greatest common divisor (GCD)** of two integers a and b, denoted $\gcd(a,b)$, is the largest positive integer that divides both a and b, where at least one of a and b is nonzero. In the case that a and b are both zero, we define $\gcd(0,0) = 0$. Since $0/a = 0 \in \mathbb{Z}$ (i.e., a divides 0) and $a/a = 1 \in \mathbb{Z}$ (i.e., a divides a), $\gcd(a,0) = \gcd(0,a) = |a|$. Since, the signs of integers do not affect divisibility, $\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b)$. From the definition of the GCD, it immediately follows that, if a and b are coprime (i.e., have no common factors), then $\gcd(a,b) = 1$. The quantity $\gcd(a,b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the lowest power for each prime factor appearing in these factorizations.

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Example 8.1 (Least common multiple)**.** Find the LCM of each pair of integers given below.
(a) 20 and 6;
(b) 54 and 24; and
(c) 24 and 90.

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the LCM, we take the highest power of each prime factor in these two factorizations. Thus, we have 2^2 from the factorization of 20, 3^1 from the factorization of 6, and 5^1 from the factorization of 20. Therefore, we conclude

$$\begin{aligned}\operatorname{lcm}(20,6) &= 2^2 \cdot 3^1 \cdot 5^1 \\ &= 60.\end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned}\operatorname{lcm}(54,24) &= \operatorname{lcm}(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^3 \cdot 3^3 \\ &= 216.\end{aligned}$$

(c) Again, using a similar process as above, we have

$$\begin{aligned}\operatorname{lcm}(24,90) &= \operatorname{lcm}(2^3 \cdot 3^1, 2^1 \cdot 3^2 \cdot 5^1) \\ &= 2^3 \cdot 3^2 \cdot 5^1 \\ &= 360.\end{aligned}$$

■

Example 8.2 (Greatest common divisor)**.** Find the GCD of each pair of integers given below.
(a) 20 and 6;
(b) 54 and 24; and
(c) 24 and 90.

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the GCD, we take the lowest power of each prime factor in these two factorizations. Thus, we have 2^1 from the factorization of 6, 3^0 from the factorization of 20, and 5^0 from the factorization of 6. Therefore, we conclude

$$\begin{aligned}\gcd(20,6) &= 2^1 \cdot 3^0 \cdot 5^0 \\ &= 2.\end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned}\gcd(54,24) &= \gcd(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^1 \cdot 3^1 \\ &= 6.\end{aligned}$$

(c) Again, using a similar process as above, we have

$$\begin{aligned}\gcd(24,90) &= \gcd(2^3 \cdot 3^1, 2^1 \cdot 3^2 \cdot 5^1) \\ &= 2^1 \cdot 3^1 \cdot 5^0 \\ &= 6.\end{aligned}$$

■

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Having introduced the LCM, we know consider whether the sum of two periodic sequences is periodic. In this regard, the theorem below is enlightening.

Theorem 8.2 (Sum of periodic sequences)**.** *For any two periodic sequences x_1 and x_2 with periods N_1 and N_2, respectively, the sequence $x = x_1 + x_2$ is periodic with period $N = \operatorname{lcm}(N_1, N_2)$.*

Proof. Since N is an integer multiple of both N_1 and N_2, we can write $N = k_1 N_1$ and $N = k_2 N_2$ for some positive integers k_1 and k_2. So, we can write

$$\begin{aligned} x(n+N) &= x_1(n+N) + x_2(n+N) \\ &= x_1(n+k_1N_1) + x_2(n+k_2N_2) \\ &= x_1(n) + x_2(n) \\ &= x(n). \end{aligned}$$

Thus, x is periodic with period N. ■

Unlike in the case of the sum of periodic functions, the sum of periodic sequences is always periodic.

Example 8.3. The sequences $x_1(n) = \cos\left(\frac{\pi}{6}n\right)$ and $x_2(n) = \sin\left(\frac{2\pi}{45}n\right)$ have fundamental periods $N_1 = 12$ and $N_2 = 45$, respectively. Find the fundamental period N of the sequence $y = x_1 + x_2$.

Solution. We have

$$\begin{aligned} N &= \operatorname{lcm}(N_1, N_2) \\ &= \operatorname{lcm}(12, 45) \\ &= \operatorname{lcm}(2^2 \cdot 3, 3^2 \cdot 5) \\ &= 2^2 \cdot 3^2 \cdot 5 \\ &= 180. \end{aligned}$$

The sequences x_1, x_2, and $x_1 + x_2$ are plotted in Figure 8.5. ■

Sometimes a sequence may be both periodic and possess even or odd symmetry. In this situation, the result of the following theorem is sometimes useful.

Theorem 8.3. *Let x be an arbitrary N-periodic sequence x. Then, the following assertions hold:*

1. *if x is even, then $x(n) = x(N-n)$ for all $n \in \mathbb{Z}$;*
2. *if x is odd, then $x(n) = -x(N-n)$ for all $n \in \mathbb{Z}$; and*
3. *if x is odd, then $x(0) = 0$ for both even and odd N, and $x\left(\frac{N}{2}\right) = 0$ for even N.*

Proof. A proof of this theorem is left as an exercise for the reader in Exercise 8.5. ■

8.3.3 Support of Sequences

We can classify sequences based on the interval over which their value is nonzero. This is sometimes referred to as the support of a sequence. In what follows, we introduce some terminology related to the support of sequences.

A sequence x is said to be **left sided** if, for some finite constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n > n_0.$$

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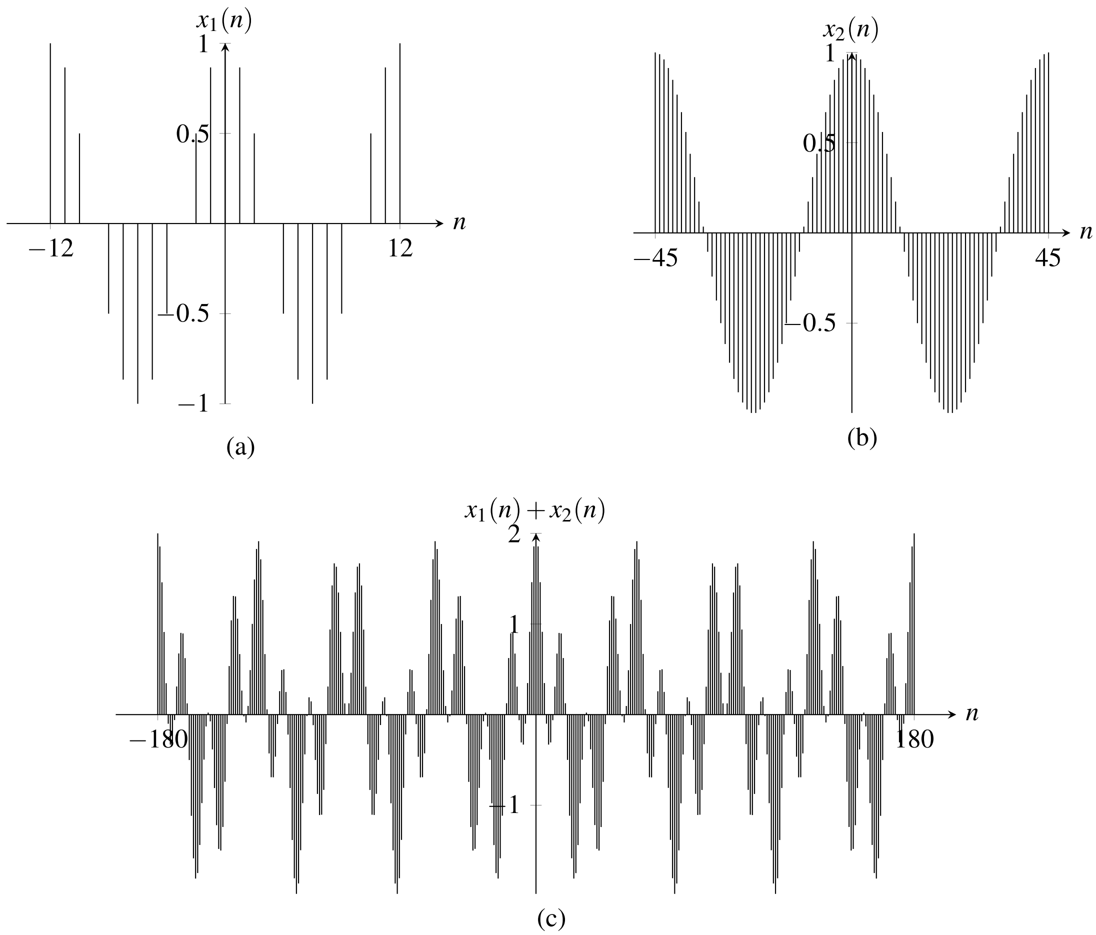

Figure 8.5: Sequences for Example 8.3.

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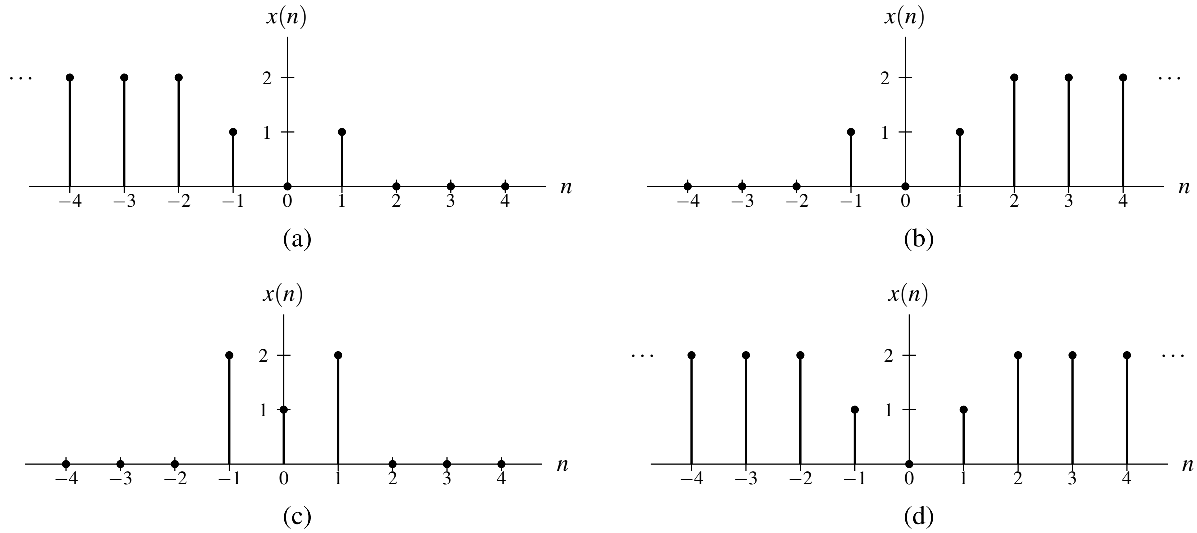

Figure 8.6: Examples of sequences with various sidedness properties. A sequence that is (a) left sided but not right sided, (b) right sided but not left sided, (c) finite duration, and (d) two sided.

In other words, the sequence is only potentially nonzero to the left of some point. A sequence x is said to be **right sided** if, for some finite constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n < n_0.$$

In other words, the sequence is only potentially nonzero to the right of some point. A sequence that is both left sided and right sided is said to be **time limited** or **finite duration**. A sequence that is neither left sided nor right sided is said to be **two sided**. Note that every sequence is exactly one of: left sided but not right sided, right sided but not left sided, finite duration, or two sided. Examples of left-sided (but not right-sided), right-sided (but not left-sided), finite-duration, and two-sided sequences are shown in Figure 8.6.

A sequence x is said to be **causal** if

$$x(n) = 0 \quad \text{for all } n < 0.$$

A causal sequence is a special case of a right-sided sequence. Similarly, a sequence x is said to be **anticausal** if

$$x(n) = 0 \quad \text{for all } n > 0.$$

An anticausal sequence is a special case of a left-sided sequence. Note that the qualifiers “causal” and “anticausal”, when applied to sequences, have nothing to do with cause and effect. In this sense, this choice of terminology is arguably not the best.

8.3.4 Bounded Sequences

A sequence x is said to be **bounded** if there exists some (finite) nonnegative real constant A such that

$$|x(n)| \leq A \quad \text{for all } n$$

(i.e., $x(n)$ is finite for all n). For example, the sequence $x(n) = (-1)^n$ bounded, since

$$|(-1)^n| \leq 1 \text{ for all } n.$$

In contrast, the sequence $x(n) = n$ is not bounded, since

$$\lim_{n \to \infty} |n| = \infty.$$

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8.3.5 Signal Energy

The **energy** E contained in the sequence x is given by

$$E = \sum_{k=-\infty}^{\infty} |x(n)|^2 .$$

As a matter of terminology, a signal x with finite energy is said to be an **energy signal**.

8.3.6 Examples

Example 8.4. Let x be a sequence with the following properties:
- $v(n) = x(n-3)$ is causal; and
- x is odd.

Determine for what values of n the quantity $x(n)$ must be zero.

Solution. Since v is causal, we have

$$\begin{aligned} & v(n) = 0 \text{ for } n < 0 \\ \Rightarrow\quad & x(n-3) = 0 \text{ for } n < 0 \\ \Rightarrow\quad & x(n) = 0 \text{ for } n+3 < 0 \\ \Rightarrow\quad & x(n) = 0 \text{ for } n < -3. \end{aligned}$$

Since x is odd, $x(0) = 0$ and $x(n) = 0$ for $n > 3$. Therefore, $x(n) = 0$ for all $n \notin \{-3,-2,-1,1,2,3\}$. ■

Example 8.5. A sequence x has the following properties:
- $x(n) = n+2$ for $-1 \leq n \leq 1$;
- $v_1(n) = x(n-1)$ is causal; and
- $v_2(n) = x(n+1)$ is even.

Find $x(n)$ for all n.

Solution. Since $v_1(n) = x(n-1)$ is causal, we have

$$\begin{aligned} & x(n-1) = 0 \ \text{ for } n < 0 \\ \Rightarrow\quad & x([n+1]-1) = 0 \ \text{ for } (n+1) < 0 \\ \Rightarrow\quad & x(n) = 0 \ \text{ for } n < -1. \end{aligned}$$

From this and the fact that $x(n) = n+2$ for $-1 \leq n \leq 1$, we have

$$x(n) = \begin{cases} n+2 & -1 \leq n \leq 1 \\ 0 & n \leq -2. \end{cases} \tag{8.9}$$

So, we only need to determine $x(n)$ for $n \geq 2$. Since $v_2(n) = x(n+1)$ is even, we have

$$\begin{aligned} & v_2(n) = v_2(-n) \\ \Rightarrow\quad & x(n+1) = x(-n+1) \\ \Rightarrow\quad & x([n-1]+1) = x(-[n-1]+1) \\ \Rightarrow\quad & x(n) = x(-n+2) \\ \Rightarrow\quad & x(n) = x(2-n). \end{aligned}$$

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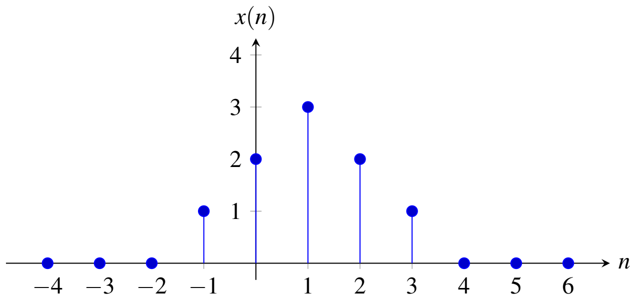

Figure 8.7: The sequence x from Example 8.5.

Using this with (8.9), we obtain

$$\begin{aligned}
x(n) &= x(2-n) \\
&= \begin{cases} (2-n)+2 & -1 \leq 2-n \leq 1 \\ 0 & 2-n \leq -2 \end{cases} \\
&= \begin{cases} 4-n & -3 \leq -n \leq -1 \\ 0 & -n \leq -4 \end{cases} \\
&= \begin{cases} 4-n & 1 \leq n \leq 3 \\ 0 & n \geq 4. \end{cases}
\end{aligned}$$

Therefore, we conclude

$$x(n) = \begin{cases} 0 & n \leq -2 \\ 2+n & n \in \{-1,0\} \\ 4-n & n \in \{1,2,3\} \\ 0 & n \geq 4. \end{cases}$$

A plot of x is given in Figure 8.7. ■

8.4 Elementary Sequences

A number of elementary sequences are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

8.4.1 Real Sinusoidal Sequences

One important class of sequences is the real sinusoids. A **real sinusoidal sequence** is a sequence of the form

$$x(n) = A\cos(\Omega n + \theta), \tag{8.10}$$

where A, Ω, and θ are real constants. For all integer k,

$$x_k(n) = A\cos[(\Omega + 2\pi k)n + \theta] \tag{8.11}$$

is the same sequence (due to the fact that the cos function is 2π-periodic).

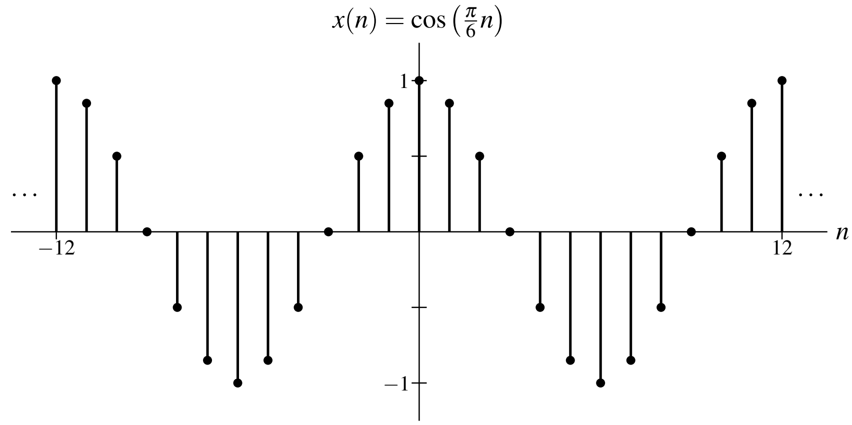

Figure 8.8: Example of a real-sinusoidal sequence.

The real sinusoidal sequence x in (8.10) is periodic if and only if $\frac{\Omega}{2\pi}$ is a rational number, in which case the fundamental period is the smallest integer of the form $\frac{2\pi k}{|\Omega|}$ where k is a (strictly) positive integer. In particular, if $\Omega = \frac{2\pi\ell}{m}$ where ℓ and m are integers, x can be shown to have the fundamental period

$$N = \frac{m}{\gcd(\ell, m)}.$$

In the case that ℓ and m are coprime (i.e., have no common factors), $N = \frac{m}{\gcd(\ell,m)} = \frac{m}{1} = m$. (A proof of this is left as an exercise for the reader in Exercise 8.6.)

An example of a periodic real sinusoid with fundamental period 12 is shown plotted in Figure 8.8.

In the case of periodic real sinusoidal sequences, the frequency of the sequence is often treated as a signed quantity. In other words, we often employ the notion of signed frequency, as discussed in Section 2.10.2. Assuming that the real sinusoidal sequence x given by (8.10) is periodic, x has a signed frequency of Ω. In most cases, we simply refer to the signed frequency as the "frequency". Normally, this does not cause any confusion, as it is usually clear from the context whether the frequency is being treated as a signed or unsigned quantity.

Unlike their continuous-time counterparts, real sinusoidal sequences have an upper bound on the rate at which they can oscillate. This is evident from (8.11) above, since every sinusoid with a frequency Ω outside the range $[0, 2\pi)$ is identical to a sinusoid whose frequency is contained in this range. Moreover, for frequencies $\Omega \in [0, 2\pi)$, the highest rate of oscillation would correspond to π, since π is the point that is equally far from both 0 and 2π (which correspond to the lowest rate of oscillation). As the frequency of a real sinusoidal sequence is increased from 0, its rate of oscillation will increase until a frequency of π is reached and then start to decrease until a frequency of 2π is hit. This behavior then repeats periodically for each subsequent interval of length 2π. Thus, more generally, a real sinusoidal sequence achieves the highest rate of oscillation when its frequency is an odd integer multiple of π (i.e., $(2k+1)\pi$ for integer k) and achieves the lowest rate of oscillation when its frequency is an even integer multiple of π (i.e., $(2k)\pi$ for integer k). Figure 8.9 shows the effect of increasing the frequency of a real sinusoidal sequence from 0 to 2π. The sequence with frequency 0 has no oscillation (i.e., is constant) while the sequence with frequency π oscillates most rapidly.

Example 8.6 (Fundamental period of real sinusoid)**.** Determine if each sequence x given below is periodic, and if it is, find its fundamental period.
(a) $x(n) = \cos(42n)$; and
(b) $x(n) = \sin\left(\frac{4\pi}{11}n\right)$.

Solution. (a) Since $\frac{2\pi}{42} = \frac{\pi}{21}$ is not rational, x is not periodic.

(b) Since

$$(2\pi)/\left(\frac{4\pi}{11}\right) = (2\pi)\left(\frac{11}{4\pi}\right) = \frac{11}{2}$$

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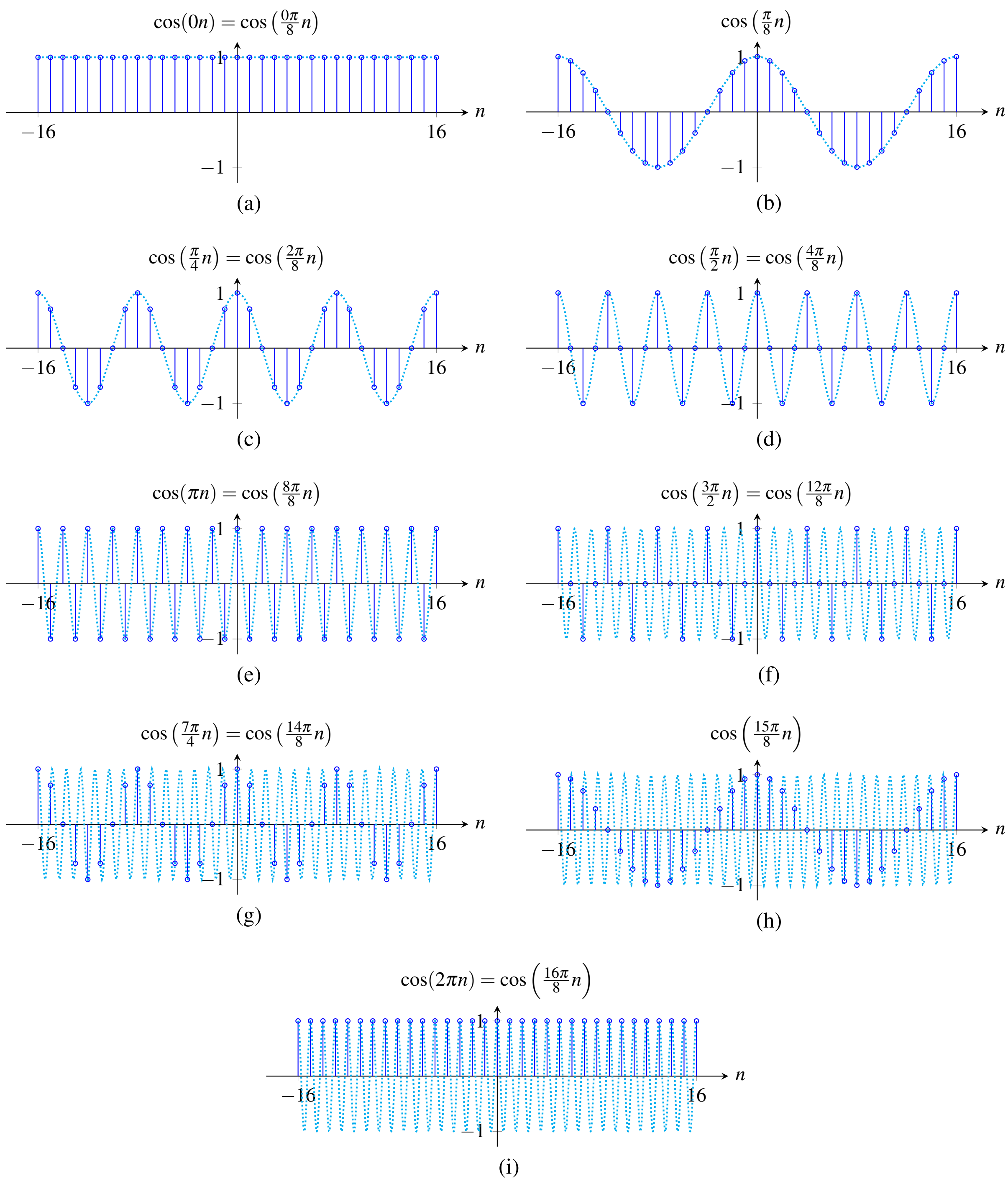

Figure 8.9: The effect of increasing the frequency of a real sinusoidal sequence. A plot of $x(n) = \cos(\Omega n)$ for Ω having each of the values (a) $\frac{0\pi}{8} = 0$, (b) $\frac{1\pi}{8} = \frac{\pi}{8}$, (c) $\frac{2\pi}{8} = \frac{\pi}{4}$, (d) $\frac{4\pi}{8} = \frac{\pi}{2}$, (e) $\frac{8\pi}{8} = \pi$, (f) $\frac{12\pi}{8} = \frac{3\pi}{2}$, (g) $\frac{14\pi}{8} = \frac{7\pi}{4}$, (h) $\frac{15\pi}{8}$, and (i) $\frac{16\pi}{8} = 2\pi$.

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is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{11}{2}k$, where k is a strictly positive integer. Thus, $N = 11$ (corresponding to $k = 2$). ■

8.4.2 Complex Exponential Sequences

Another important class of sequences is the complex exponentials. A **complex exponential sequence** is a sequence of the form

$$x(n) = ca^n, \tag{8.12}$$

where c and a are complex constants. Such a sequence can also be equivalently expressed in the form

$$x(n) = ce^{bn},$$

where b is a complex constant chosen as $b = \ln a$. (This form is more similar to that presented for complex exponential functions). A complex exponential can exhibit one of a number of distinct modes of behavior, depending on the values of the parameters c and a. For example, as special cases, complex exponentials include real exponentials and complex sinusoids. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

8.4.2.1 Real Exponential Sequences

The first special case of the complex exponentials to be considered is the real exponentials. In the case of a **real exponential sequence**, we restrict c and a in (8.12) to be real. A real exponential can exhibit one of several distinct modes of behavior, depending on the magnitude and sign of a, as illustrated in Figure 8.10. If $|a| > 1$, the magnitude of $x(n)$ increases exponentially as n increases (i.e., a growing exponential). If $|a| < 1$, the magnitude of $x(n)$ decreases exponentially as n increases (i.e., a decaying exponential). If $|a| = 1$, the magnitude of $x(n)$ is a constant, independent of n. If $a > 0$, $x(n)$ has the same sign for all n. If $a < 0$, $x(n)$ alternates in sign as n increases/decreases.

8.4.2.2 Complex Sinusoidal Sequences

The second special case of the complex exponentials that we shall consider is the complex sinusoids. In the case of a **complex sinusoidal sequence**, the parameters in (8.12) are such that c and a are complex and $|a| = 1$ (i.e., a is of the form $e^{j\Omega}$ where Ω is real). That is, a **complex sinusoidal sequence** is a sequence of the form

$$x(n) = ce^{j\Omega n}, \tag{8.13}$$

where c is complex and Ω is real. Using Euler's relation, we can rewrite $x(n)$ as

$$x(n) = \underbrace{|c|\cos(\Omega n + \arg c)}_{\operatorname{Re}\{x(n)\}} + j\underbrace{|c|\sin(\Omega n + \arg c)}_{\operatorname{Im}\{x(n)\}}.$$

Thus, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are real sinusoids. To illustrate the form of a complex sinusoid, the real and imaginary parts of a particular complex sinusoid are plotted in Figure 8.11.

The complex sinusoidal sequence x in (8.13) is periodic if and only if $\frac{\Omega}{2\pi}$ is a rational number, in which case the fundamental period is the smallest integer of the form $\frac{2\pi k}{|\Omega|}$, where k is a (strictly) positive integer. In particular, if $\Omega = \frac{2\pi\ell}{m}$ where ℓ and m are integers, x can be shown to have the fundamental period

$$N = \frac{m}{\gcd(\ell, m)}.$$

In the case that ℓ and m are coprime (i.e., have no common factors), $N = \frac{m}{\gcd(\ell,m)} = \frac{m}{1} = m$. (A proof of this is left as an exercise for the reader in Exercise 8.6.)

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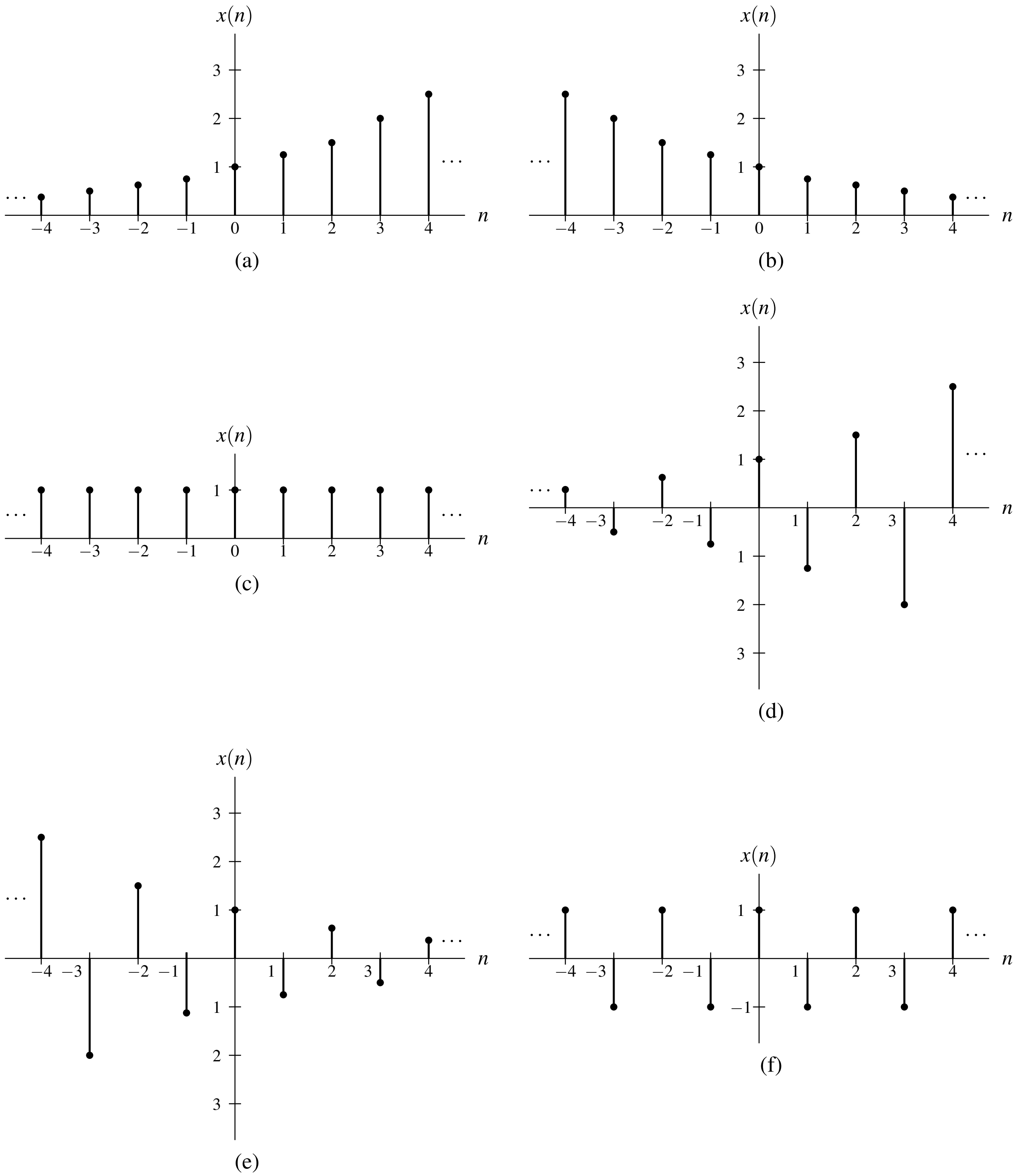

Figure 8.10: Examples of real exponential sequences. (a) $|a| > 1$, $a > 0$ [$a = \frac{5}{4}$; $c = 1$]; (b) $|a| < 1$, $a > 0$ [$a = \frac{4}{5}$; $c = 1$]; (c) $|a| = 1$, $a > 0$ [$a = 1$; $c = 1$]; (d) $|a| > 1$, $a < 0$ [$a = -\frac{5}{4}$; $c = 1$]; (e) $|a| < 1$, $a < 0$ [$a = -\frac{4}{5}$; $c = 1$]; and (f) $|a| = 1$, $a < 0$ [$a = -1$; $c = 1$].

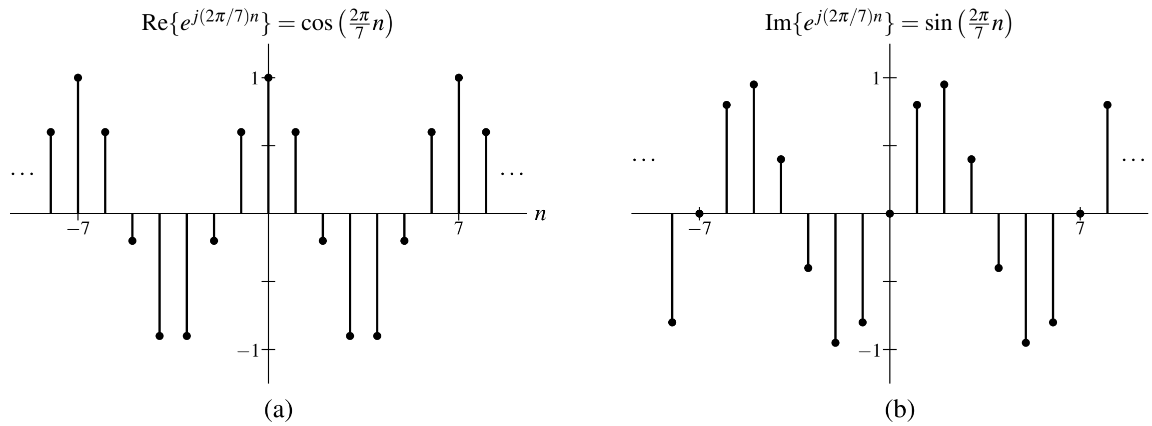

Figure 8.11: Example of complex sinusoidal sequence $x(n) = e^{j(2\pi/7)n}$. The (a) real and (b) imaginary parts of x.

In the case of periodic complex sinusoidal sequences, the frequency of the sequence is often treated as a signed quantity. In other words, we often employ the notion of signed frequency, as discussed in Section 2.10.2. Assuming that the complex sinusoidal sequence x given by (8.13) is periodic, x has a signed frequency of Ω. In most cases, we simply refer to the signed frequency as the "frequency". Normally, this does not cause any confusion, as it is usually clear from the context whether the frequency is being treated as a signed or unsigned quantity.

Unlike their continuous-time counterparts, complex sinusoidal sequences have an upper bound on the rate at which they can oscillate. This is due to similar reasoning as in the case of real sinusoidal sequences (discussed earlier in Section 8.4.1). For frequencies $\Omega \in [0, 2\pi)$, the oscillation rate of complex sinusoidal sequences is greatest at a frequency of π and lowest at a frequency of 0. As the frequency Ω increases from 0 to π, the oscillation rate increases, and as the frequency increases from π to 2π, the oscillation rate decreases.

Example 8.7 (Fundamental period of complex sinusoid)**.** Determine if each sequence x given below is periodic, and if it is, find its fundamental period.
(a) $x(n) = e^{j42n}$;
(b) $x(n) = e^{j(4\pi/11)n}$; and
(c) $x(n) = e^{j(\pi/3)n}$.

Solution. (a) Since $\frac{2\pi}{42} = \frac{\pi}{21}$ is not rational, x is not periodic.

(b) Since

$$(2\pi)/\left(\tfrac{4\pi}{11}\right) = (2\pi)\left(\tfrac{11}{4\pi}\right) = \tfrac{11}{2}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{11}{2}k$, where k is a strictly positive integer. Thus, $N = 11$ (corresponding to $k = 2$). Alternatively, the fundamental period N of $x(n) = e^{j(2\pi[2/11])n}$ is given by

$$N = \frac{11}{\gcd(11,2)} = \frac{11}{\gcd(11^1, 2^1)} = \frac{11}{1} = 11.$$

(c) Since

$$(2\pi)/\left(\tfrac{\pi}{3}\right) = (2\pi)\left(\tfrac{3}{\pi}\right) = \tfrac{6}{1}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{6}{1}k$, where k is a strictly positive integer. Thus, $N = 6$ (corresponding to $k = 1$). Alternatively, the fundamental period N of $x(n) = e^{j(\pi/3)n} = e^{j(2\pi[1]/6)n}$ is given by

$$N = \frac{6}{\gcd(6,1)} = \frac{6}{\gcd(2^1 \cdot 3^1, 1)} = \frac{6}{1} = 6. \qquad \blacksquare$$

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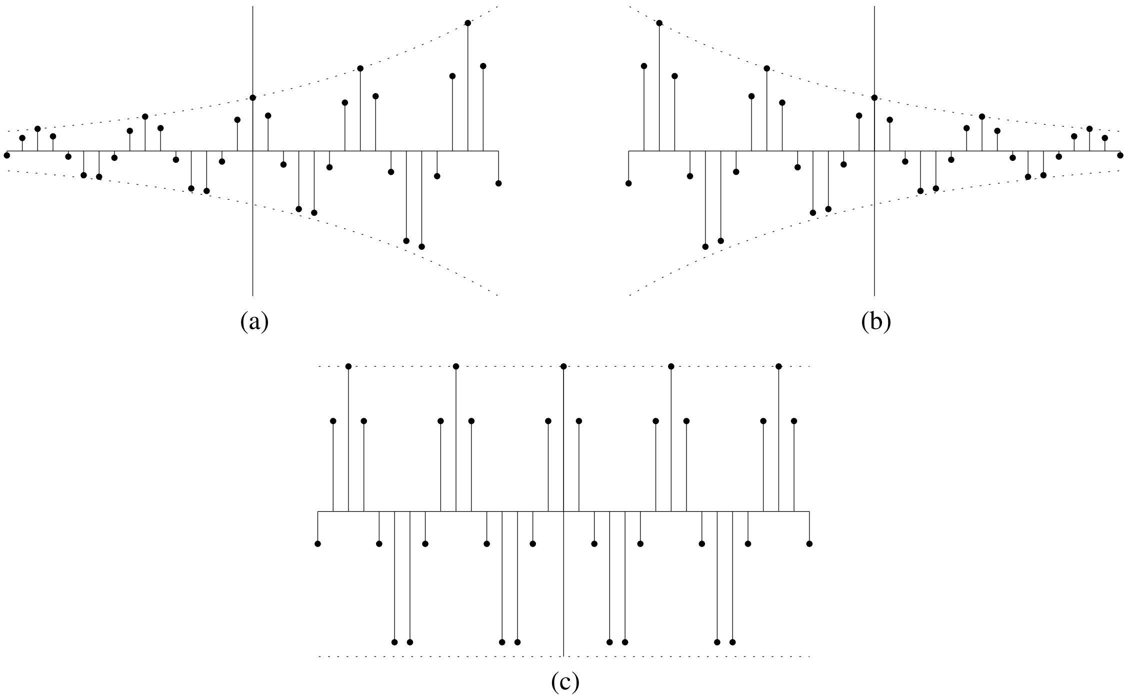

Figure 8.12: Various mode of behavior for the real and imaginary parts of a complex exponential sequence. (a) $|a| > 1$; (b) $|a| < 1$; and (c) $|a| = 1$.

8.4.2.3 General Complex Exponential Sequences

Lastly, we consider general complex exponential sequences. That is, we consider the general case of (8.12) where c and a are both complex. Letting $c = |c|\,e^{j\theta}$ and $a = |a|\,e^{j\Omega}$ where θ and Ω are real, and using Euler's relation, we can rewrite $x(n)$ as

$$x(n) = \underbrace{|c|\,|a|^n \cos(\Omega n + \theta)}_{\operatorname{Re}\{x(n)\}} + j\underbrace{|c|\,|a|^n \sin(\Omega n + \theta)}_{\operatorname{Im}\{x(n)\}}.$$

Thus, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the product of a real exponential and real sinusoid. One of several distinct modes of behavior is exhibited by x, depending on the value of a, as illustrated in Figure 8.12. If $|a| = 1$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are real sinusoids. If $|a| > 1$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the product of a real sinusoid and a growing real exponential. If $|a| < 1$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the product of a real sinusoid and a decaying real exponential.

8.4.3 Relationship Between Complex Exponentials and Real Sinusoids

From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$ce^{j\Omega n} = c\cos(\Omega n) + jc\sin(\Omega n).$$

Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$c\cos(\Omega n + \theta) = \tfrac{c}{2}\left[e^{j(\Omega n+\theta)} + e^{-j(\Omega n+\theta)}\right] \quad \text{and}$$
$$c\sin(\Omega n + \theta) = \tfrac{c}{2j}\left[e^{j(\Omega n+\theta)} - e^{-j(\Omega n+\theta)}\right].$$

This result follows from Euler's relation and is simply a restatement of (A.8).

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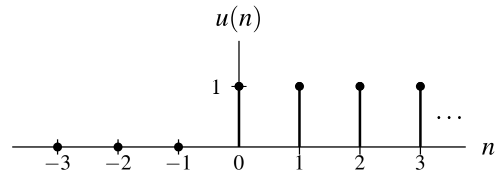

Figure 8.13: The unit-step sequence.

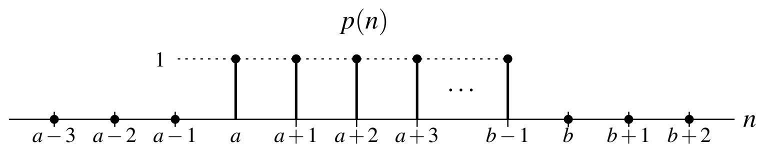

Figure 8.14: The rectangular sequence.

8.4.4 Unit-Step Sequence

Another elementary sequence often used in systems theory is the unit-step sequence. The **unit-step sequence**, denoted u, is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this sequence is shown in Figure 8.13.

8.4.5 Unit-Rectangular Pulse

A family of sequences that is often useful is unit-rectangular pulses. A **unit rectangular pulse** is a sequence of the form

$$p(n) = \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise,} \end{cases} \tag{8.14}$$

where a and b are integer constants satisfying $a < b$. The graph of a unit rectangular pulse has the general form shown in Figure 8.14. As is formally shown in Example 8.8 below, p can be expressed in terms of the unit-step sequence u as

$$p(n) = u(n-a) - u(n-b). \tag{8.15}$$

This particular way of expressing rectangular-pulse sequences is often extremely useful.

Example 8.8. Show that the unit rectangular pulse sequence p given by (8.14) can be written as specified in (8.15).

Solution. Recall that u is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

From this definition, we can write

$$u(n-a) = \begin{cases} 1 & n-a \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & n \geq a \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

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Similarly, we can write

$$u(n-b) = \begin{cases} 1 & n-b \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & n \geq b \\ 0 & \text{otherwise.} \end{cases} \tag{8.17}$$

From (8.16) and (8.17), we can write

$$\begin{aligned} u(n-a) - u(n-b) &= \begin{cases} 0-0 & n < a \\ 1-0 & a \leq n < b \\ 1-1 & n \geq b. \end{cases} \\ &= \begin{cases} 0 & n < a \\ 1 & a \leq n < b \\ 0 & n \geq b. \end{cases} \\ &= \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise.} \end{cases} \\ &= p(n). \end{aligned}$$

Thus, we have shown that

$$p(n) = u(n-a) - u(n-b).$$ ■

8.4.6 Unit-Impulse Sequence

In systems theory, one elementary sequence of fundamental importance is the unit-impulse sequence. The **unit-impulse sequence** (also known as the **delta sequence**), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8.18}$$

(Note that the unit-impulse sequence is a very different thing from the unit-impulse function.) The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

The accumulation (i.e., running sum) of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

A plot of δ is shown in Figures 8.15.

The unit-impulse sequence has two important properties that follow from its definition in (8.18). These properties are given by the theorems below.

Theorem 8.4 (Equivalence property)**.** *For any sequence x and any integer constant n_0, the following identity holds:*

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0). \tag{8.19}$$

Proof. The proof essentially follows immediately from the fact that the unit-impulse sequence is only nonzero at a single point. ■

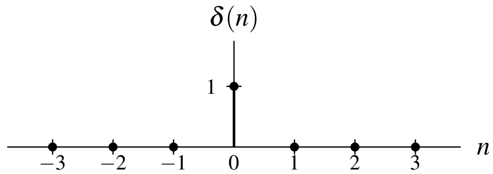

Figure 8.15: The unit-impulse sequence.

Theorem 8.5 (Sifting property). *For any sequence x and any integer constant n_0, the following identity holds:*

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$

Proof. From the definition of the delta sequence, we have

$$\sum_{n=-\infty}^{\infty} \delta(n) = 1.$$

This implies that, for any integer constant n_0,

$$\sum_{n=-\infty}^{\infty} \delta(n-n_0) = 1.$$

Multiplying both sides of the equation by $x(n_0)$, we obtain

$$\sum_{n=-\infty}^{\infty} x(n_0)\delta(n-n_0) = x(n_0).$$

Then, using the equivalence property in (8.19), we have

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0). \qquad \blacksquare$$

Trivially, the sequence δ is also even.

Example 8.9. Evaluate the summation

$$\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1).$$

Solution. Using the sifting property of the unit impulse sequence, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1) &= \sin\left(\tfrac{\pi}{2}n\right)\Big|_{n=1} \\ &= \sin\left(\tfrac{\pi}{2}\right) \\ &= 1.\end{aligned} \qquad \blacksquare$$

8.5 Representing Arbitrary Sequences Using Elementary Sequences

In the earlier sections, we introduced a number of elementary sequences. Often in signal analysis, it is convenient to represent arbitrary sequences in terms of elementary sequences. Here, we consider how the unit-step sequence can be exploited in order to obtain alternative representations of sequences.

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Example 8.10. Consider the piecewise-linear sequence x given by

$$x(n) = \begin{cases} n+7 & -6 \leq n \leq -4 \\ 4 & -3 \leq n \leq 2 \\ 6-n & 3 \leq n \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(n)$ (involving unit-step sequences) that is valid for all n.

Solution. A plot of x is shown in Figure 8.16(a). We consider each segment of the piecewise-linear sequence separately. The first segment (i.e., for n) can be expressed as

$$v_1(n) = (n+7)[u(n+6) - u(n+3)].$$

This sequence is plotted in Figure 8.16(b). The second segment (i.e., for n) can be expressed as

$$v_2(n) = 4[u(n+3) - u(n-3)].$$

This sequence is plotted in Figure 8.16(c). The third segment (i.e., for n) can be expressed as

$$v_3(n) = (6-n)[u(n-3) - u(n-6)].$$

This sequence is plotted in Figure 8.16(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(n) &= v_1(n) + v_2(n) + v_3(n) \\ &= (n+7)[u(n+6) - u(n+3)] + 4[u(n+3) - u(n-3)] + (6-n)[u(n-3) - u(n-6)] \\ &= (n+7)u(n+6) - (n+7)u(n+3) + 4u(n+3) - 4u(n-3) + (6-n)u(n-3) - (6-n)u(n-6) \\ &= (n+7)u(n+6) + (-n-3)u(n+3) + (2-n)u(n-3) + (n-6)u(n-6). \end{aligned}$$

Thus, we have found a single expression for $x(n)$ that is valid for all n. ■

8.6 Discrete-Time Systems

Suppose that we have a system with input x and output y. Such a system can be described mathematically by the equation

$$y = \mathcal{H}x, \tag{8.20}$$

where $\mathcal{H}$ denotes an operator (i.e., transformation). The operator $\mathcal{H}$ simply maps the input sequence x to the output sequence y. Such an operator might be associated with a system of difference equations, for example.

Alternatively, we sometimes express the relationship (8.20) using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

Furthermore, if clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \rightarrow y.$$

Note that the symbols "$\rightarrow$" and "=" have very different meanings. For example, the notation $x \rightarrow y$ does not in any way imply that $x = y$. The symbol "$\rightarrow$" should be read as "produces" (not as "equals"). That is, "$x \rightarrow y$" should be read as "the input x produces the output y".

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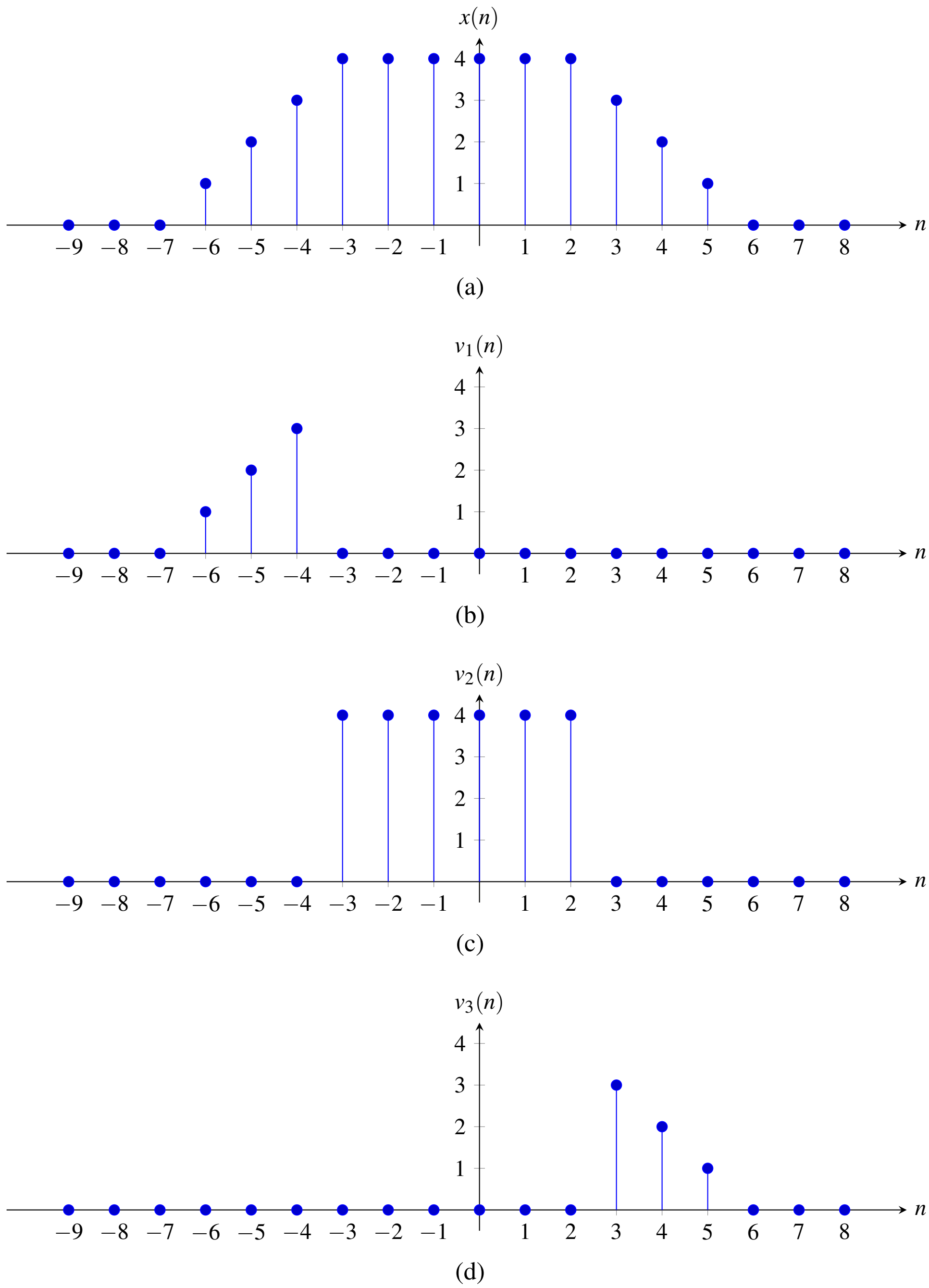

Figure 8.16: Representing a piecewise-linear sequence using unit-step sequences. (a) The sequence x. (b), (c), and (d) Three sequences whose sum is x.

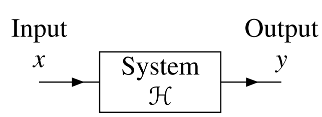

Figure 8.17: Block diagram of system.

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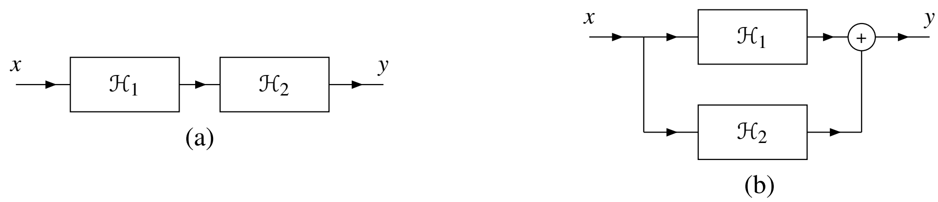

Figure 8.18: Interconnection of systems. The (a) series interconnection and (b) parallel interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$.

8.6.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathcal{H}$ and having the input x and output y. Often, we represent such a system using a block diagram as shown in Figure 8.17.

8.6.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 8.18. The first type of connection, as shown in Figure 8.18(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_2\mathcal{H}_1 x. \tag{8.21}$$

The second type of connection, as shown in Figure 8.18(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x. \tag{8.22}$$

The system equations in (8.21) and (8.22) cannot be simplified further unless the definitions of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are known.

8.7 Properties of Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

8.7.1 Memory

A system $\mathcal{H}$ is said to be **memoryless** if, for every integer n_0, $\mathcal{H}x(n_0)$ does not depend on $x(n)$ for some $n \neq n_0$. In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time. A system that is not memoryless is said to have **memory**. Although simple, a memoryless system is not very flexible, since its current output value cannot rely on past or future values of the input.

Example 8.11 (Ideal amplifier)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = Ax(n)$$

and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = Ax(n_0).$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ only for $n = n_0$. Therefore, the system is memoryless. ■

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Example 8.12 (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = \sum_{k=-\infty}^{n_0} x(n).$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ for $-\infty < n \leq n_0$. So, $\mathcal{H}x(n_0)$ is dependent on $x(n)$ for some $n \neq n_0$ (e.g., $n_0 - 1$). Therefore, the system has memory (i.e., is not memoryless). ■

Example 8.13. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = e^{x(n)}.$$

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = e^{x(n_0)}.$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ only for $n = n_0$. Therefore, the system is memoryless. ■

Example 8.14. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. For any x and any integer n_0, we have that $\mathcal{H}x(n_0)$ depends on $x(n)$ for $n = n_0$ and $n = -n_0$. Since $\mathcal{H}x(n_0)$ depends on $x(n)$ for $n \neq n_0$, the system has memory (i.e., the system is not memoryless). ■

8.7.2 Causality

A system $\mathcal{H}$ is said to be **causal** if, for every integer n_0, $\mathcal{H}x(n_0)$ does not depend on $x(n)$ for some $n > n_0$. In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier* points in time (i.e., *not later* points in time). A memoryless system is always causal, although the converse is not necessarily true.

If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 8.15 (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(n).$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = \sum_{k=-\infty}^{n_0} x(n).$$

Thus, we can see that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $-\infty < n \leq n_0$. Since all of the values in this interval are less than or equal to n_0, the system is causal. ■

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Example 8.16. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \sum_{k=n-1}^{n+1} x(k).$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = \sum_{k=n_0-1}^{n_0+1} x(k).$$

Thus, we can see that $\mathcal{H}x(n_0)$ only depends on $x(n)$ for $n_0 - 1 \leq n \leq n_0 + 1$. Since at least one value in this interval is greater than n_0 (i.e., $n_0 + 1$), the system is not causal. ■

Example 8.17. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = (n+1)e^{x(n-1)}.$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = (n_0+1)e^{x(n_0-1)}.$$

Thus, we can see that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $n = n_0 - 1$. Since $n_0 - 1 \leq n_0$, the system is causal. ■

Example 8.18. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. For any x and any integer constant n_0, we have that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $n = n_0$ and $n = -n_0$. Suppose that $n_0 = -1$. In this case, we have that $\mathcal{H}x(n_0)$ (i.e., $\mathcal{H}x(-1)$) depends on $x(n)$ for $n = 1$ but $n = 1 > n_0$. Therefore, the system is not causal. ■

8.7.3 Invertibility

The **inverse** of a system $\mathcal{H}$ (if such an inverse exists) is a system $\mathcal{G}$ such that, for every sequence x,

$$\mathcal{G}\mathcal{H}x = x$$

(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{G}$ is a system whose input and output are equal). As a matter of notation, the inverse of $\mathcal{H}$ is denoted $\mathcal{H}^{-1}$. The relationship between a system and its inverse is illustrated in Figure 8.19. The two systems in this figure must be equivalent, due to the relationship between $\mathcal{H}$ and $\mathcal{H}^{-1}$ (i.e., $\mathcal{H}^{-1}$ cancels $\mathcal{H}$).

A system $\mathcal{H}$ is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists). An invertible system must be such that its input x can always be uniquely determined from its output $\mathcal{H}x$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

To show that a system is invertible, we simply find the inverse system. To show that a system is not invertible, it is sufficient to find two distinct inputs to that system that result in identical outputs. In practical terms, invertible systems are nice in the sense that their effects can be undone.

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