A plot of the frequency spectrum $X$ is shown in Figure 6.22(a). Using the results of Example 6.37, we can determine the frequency response $H$ of the system to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\{300\operatorname{sinc}(300\pi t)\}(\omega) \\ &= \operatorname{rect}\left(\tfrac{\omega}{2[300\pi]}\right) \\ &= \begin{cases} 1 & |\omega| \leq 300\pi \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response $H$ is shown in Figure 6.22(b). The frequency spectrum $Y$ of the output can be computed as

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi). \end{aligned}$$

The frequency spectrum $Y$ is shown in Figure 6.22(c). Taking the inverse Fourier transform of $Y$ yields

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{\tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi)\right\}(t) \\ &= \pi\mathcal{F}^{-1}\delta(t) + \tfrac{3}{4}\mathcal{F}^{-1}\{\pi[\delta(\omega+200\pi)+\delta(\omega-200\pi)]\}(t) \\ &= \pi(\tfrac{1}{2\pi}) + \tfrac{3}{4}\cos(200\pi t) \\ &= \tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t). \end{aligned}$$

■

**Example 6.39** (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(t) = \tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t).$$

Using frequency-domain methods, find the response $y$ of the system to the input

$$x(t) = -1 + 2\cos(2t) + \cos(4t) - \cos(6t).$$

*Solution.* Taking the Fourier transform of $x$, we have

$$\begin{aligned} X(\omega) &= -1\mathcal{F}\{1\}(\omega) + 2\mathcal{F}\{\cos(2t)\}(\omega) + \mathcal{F}\{\cos(4t)\}(\omega) - \mathcal{F}\{\cos(6t)\}(\omega) \\ &= -2\pi\delta(\omega) + 2(\pi[\delta(\omega-2)+\delta(\omega+2)]) + \pi[\delta(\omega-4)+\delta(\omega+4)] - \pi[\delta(\omega-6)+\delta(\omega+6)] \\ &= -\pi\delta(\omega+6) + \pi\delta(\omega+4) + 2\pi\delta(\omega+2) - 2\pi\delta(\omega) + 2\pi\delta(\omega-2) + \pi\delta(\omega-4) - \pi\delta(\omega-6). \end{aligned}$$

The frequency spectrum $X$ is shown in Figure 6.23(a). Now, we compute the frequency response $H$ of the system. Using the results of Example 6.37, we can determine $H$ to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t)\}(\omega) \\ &= \operatorname{rect}\left(\tfrac{\omega-4}{2}\right) + \operatorname{rect}\left(\tfrac{\omega+4}{2}\right) \\ &= \begin{cases} 1 & 3 \leq |\omega| \leq 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response $H$ is shown in Figure 6.23(b). The frequency spectrum $Y$ of the output is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi\delta(\omega+4) + \pi\delta(\omega-4). \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\pi\delta(\omega+4) + \pi\delta(\omega-4)\}(t) \\ &= \mathcal{F}^{-1}\{\pi[\delta(\omega+4)+\delta(\omega-4)]\}(t) \\ &= \cos(4t). \end{aligned}$$

■

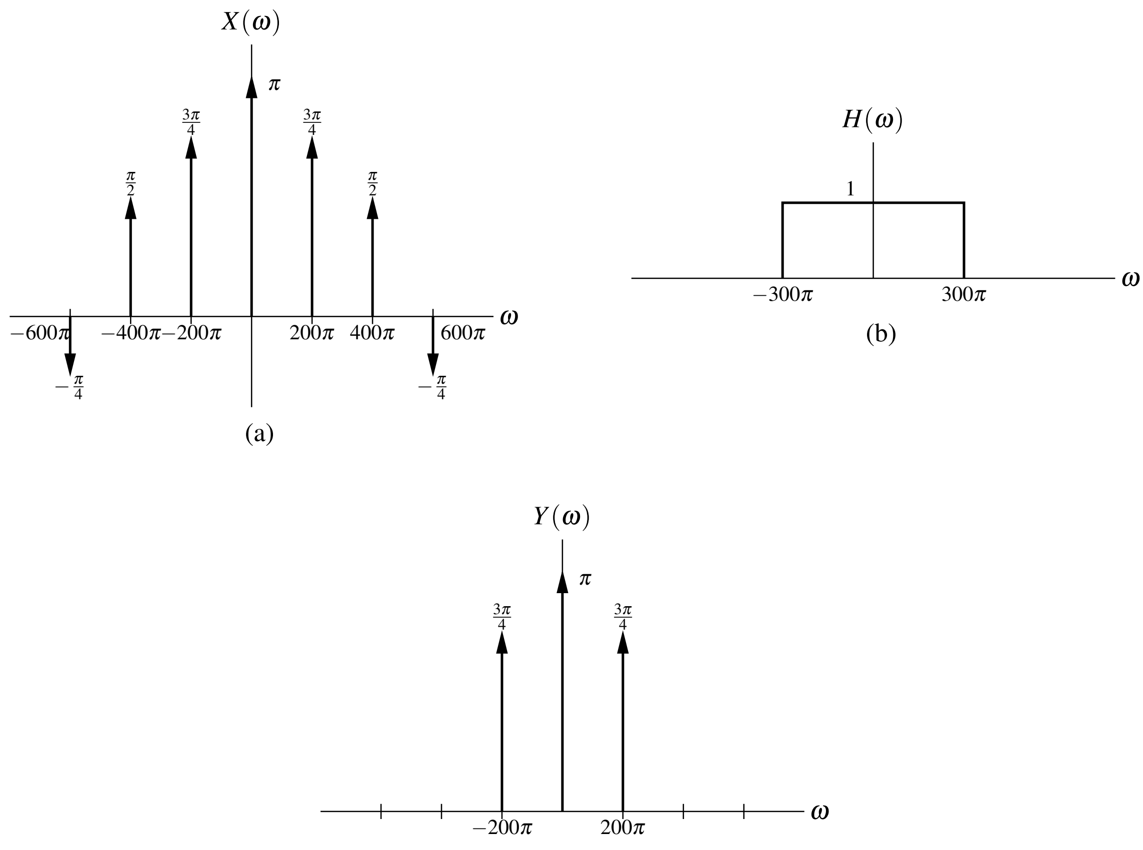



Figure 6.22: Frequency spectra for the lowpass filtering example. (a) Frequency spectrum of the input $x$. (b) Frequency response of the system. (c) Frequency spectrum of the output $y$.

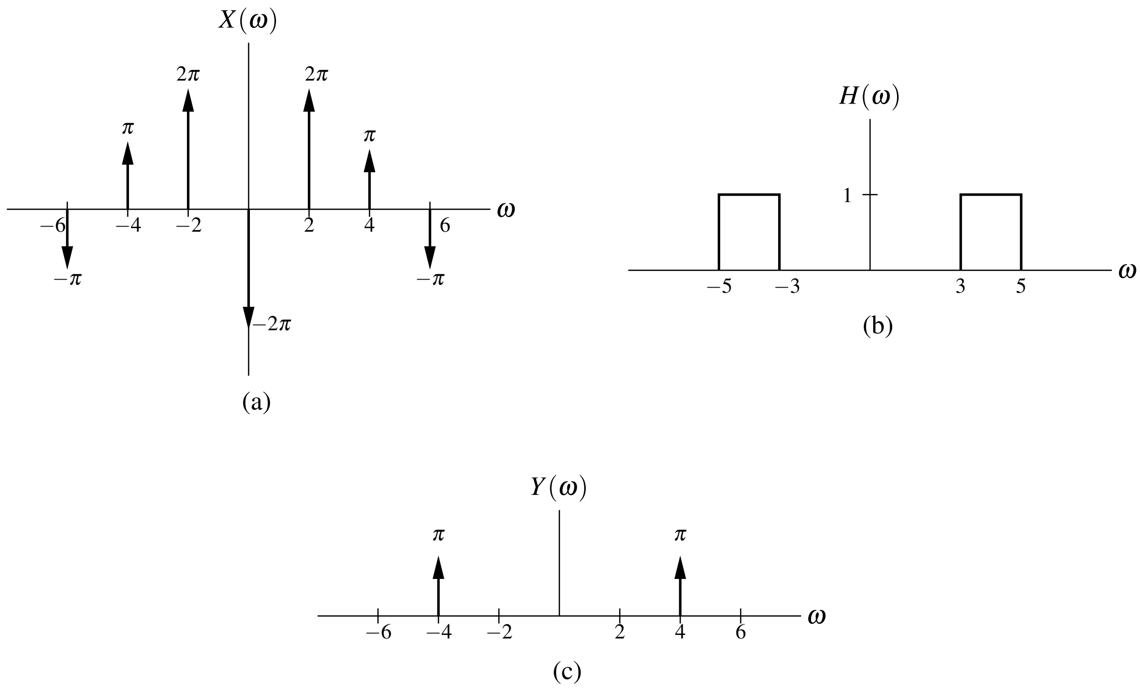



Figure 6.23: Frequency spectra for bandpass filtering example. (a) Frequency spectrum of the input $x$. (b) Frequency response of the system. (c) Frequency spectrum of the output $y$.

## 6.17 Equalization

Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand. As a result, we would like to change the frequency response of the system to be something more desirable. This process of modifying the frequency response in this way is referred to as **equalization**. Essentially, equalization is just a filtering operation, where the filtering is applied with the specific goal of obtaining a more desirable frequency response.

Let us now examine the mathematics behind equalization. Consider the LTI system with impulse response $h_{\mathsf{orig}}$ as shown in Figure 6.24(a). Let $H_{\mathsf{orig}}$ denote the Fourier transform of $h_{\mathsf{orig}}$. Suppose that the frequency response $H_{\mathsf{orig}}$ is undesirable for some reason (i.e., the system does not behave in a way that is good for the application at hand). Consequently, we would instead like to have a system with frequency response $H_{\mathsf{d}}$. In effect, we would like to somehow change the frequency response $H_{\mathsf{orig}}$ of the original system to $H_{\mathsf{d}}$. This can be accomplished by using another system called an **equalizer**. More specifically, consider the new system shown in Figure 6.24(b), which consists of a LTI equalizer with impulse response $h_{\mathsf{eq}}$ connected in series with the original system having impulse response $h_{\mathsf{orig}}$. Let $H_{\mathsf{eq}}$ denote the Fourier transform of $h_{\mathsf{eq}}$. From the block diagram, we have

$$Y(\omega) = H(\omega)X(\omega),$$

where $H(\omega) = H_{\mathsf{orig}}(\omega)H_{\mathsf{eq}}(\omega)$. In effect, we want to force $H$ to be equal to $H_{\mathsf{d}}$ so that the overall (i.e., series-interconnected) system has the frequency response desired. So, we choose the equalizer to be such that

$$H_{\mathsf{eq}}(\omega) = \frac{H_{\mathsf{d}}(\omega)}{H_{\mathsf{orig}}(\omega)}.$$

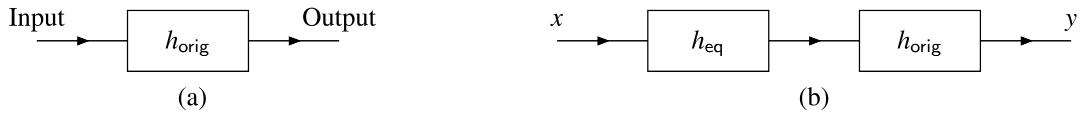



Figure 6.24: Equalization example. (a) Original system. (b) New system with equalization.

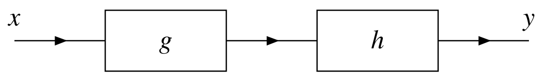



Figure 6.25: System from example that employs equalization.

Then, we have

$$\begin{aligned} H(\omega) &= H_{\text{orig}}(\omega)H_{\text{eq}}(\omega) \\ &= H_{\text{orig}}(\omega)\left[\frac{H_d(\omega)}{H_{\text{orig}}(\omega)}\right] \\ &= H_{\text{d}}(\omega). \end{aligned}$$

Thus, the system in Figure 6.24(b) has the frequency response $H_{\text{d}}$ as desired.

Equalization is used in many applications. In real-world communication systems, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel. In audio applications, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

**Example 6.40** (Communication channel equalization)**.** Consider a LTI communication channel with frequency response

$$H(\omega) = \tfrac{1}{3+j\omega}.$$

Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response $G$ of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 6.25, where $g$ and $h$ denote the inverse Fourier transforms of $G$ and $H$, respectively.

*Solution.* An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want $H(\omega)G(\omega) = 1$ or equivalently $G(\omega) = 1/H(\omega)$. Thus, we conclude that

$$\begin{aligned} G(\omega) &= \frac{1}{H(\omega)} \\ &= \frac{1}{\left(\frac{1}{3+j\omega}\right)} \\ &= 3+j\omega. \end{aligned}$$

■

## 6.18 Circuit Analysis

One application of the Fourier transform is circuit analysis. In this section, we consider this particular application.

The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

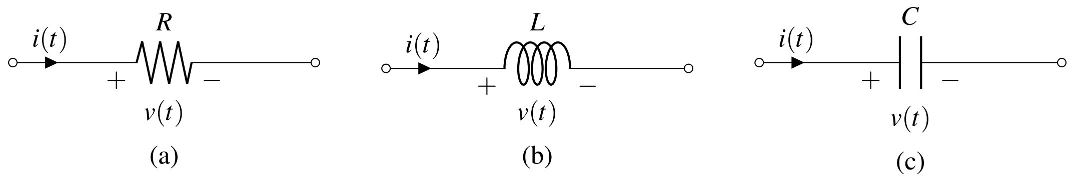



Figure 6.26: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 6.26(a), is governed by the relationship

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

where $R$, $v$ and $i$ denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance $R$ is a nonnegative quantity (i.e., $R \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \quad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega)\right),$$

where $V$ and $I$ denote the Fourier transforms of $v$ and $i$, respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 6.26(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

where $L$, $v$, and $i$ denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance $L$ is a nonnegative quantity (i.e., $L \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega LI(\omega) \quad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{j\omega L}V(\omega)\right),$$

where $V$ and $I$ denote the Fourier transforms of $v$ and $i$, respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 6.26(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\right),$$

where $C$, $v$, and $i$ denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance $C$ is a nonnegative quantity (i.e., $C \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = \tfrac{1}{j\omega C}I(\omega) \quad \left(\text{or equivalently, } I(\omega) = j\omega CV(\omega)\right),$$

where $V$ and $I$ denote the Fourier transforms of $v$ and $i$, respectively.

Observe that, in the case of inductors and capacitors, the equations that characterize these circuit elements are arguably much simpler to express in the Fourier domain than in the time domain. Consequently, the use of the Fourier transform has the potential to greatly simplify the process of analyzing circuits containing inductors and capacitors.

**Example 6.41** (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 6.27 with input $v_0$ and output $v_1$. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the differential equation that characterizes this system. (b) Find the frequency response $H$ of the system. (c) Determine the type of frequency-selective filter that this system best approximates. (d) Find $v_1$ in the case that $v_0(t) = e^{-t/(RC)}u(t)$.

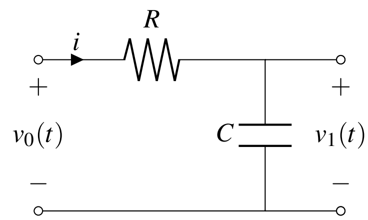



Figure 6.27: Simple RC network.

*Solution.* (a) From basic circuit analysis, we can write

$$v_0(t) = Ri(t) + v_1(t) \quad \text{and}$$
$$i(t) = C\tfrac{d}{dt}v_1(t).$$

(Recall that the current $i$ through a capacitor $C$ is related to the voltage $v$ across the capacitor as $v(t) = C\frac{d}{dt}i(t)$.) Combining the preceding equations, we obtain

$$v_0(t) = RC\tfrac{d}{dt}v_1(t) + v_1(t).$$

Thus, we have found the differential equation that characterizes the system.

(b) Taking the Fourier transform of the differential equation that characterizes the system, we have

$$\begin{aligned} V_0(\omega) &= j\omega RCV_1(\omega) + V_1(\omega) \quad \Rightarrow \\ V_0(\omega) &= (jRC\omega + 1)V_1(\omega) \quad \Rightarrow \\ \frac{V_1(\omega)}{V_0(\omega)} &= \frac{1}{jRC\omega + 1}. \end{aligned}$$

Since the system is LTI, $H(\omega) = \frac{V_1(\omega)}{V_0(\omega)}$. So, we have

$$H(\omega) = \frac{1}{jRC\omega + 1}. \tag{6.36}$$

Thus, we have found the frequency response of the system.

(c) Taking the magnitude of $H(\omega)$, we obtain

$$\begin{aligned} |H(\omega)| &= \left|\frac{1}{jRC\omega + 1}\right| \\ &= \frac{1}{\sqrt{(RC\omega)^2 + 1}}. \end{aligned}$$

Since $|H(0)| = 1$ and $\lim_{|\omega|\to\infty} |H(\omega)| = 0$, the system best approximates a lowpass filter.

(d) Now, suppose that $v_0(t) = e^{-t/(RC)}u(t)$ as given. Taking the Fourier transform of the input $v_0$ (with the aid of Table 6.2), we have

$$\begin{aligned} V_0(\omega) &= \frac{1}{\frac{1}{RC} + j\omega} \\ &= \frac{RC}{jRC\omega + 1}. \end{aligned} \tag{6.37}$$

Since the system is LTI, we know

$$V_1(\omega) = H(\omega)V_0(\omega). \tag{6.38}$$

Substituting (6.37) and (6.36) into (6.38), we obtain

$$\begin{aligned} V_1(\omega) &= \left(\frac{1}{jRC\omega+1}\right)\left(\frac{RC}{jRC\omega+1}\right) \\ &= \frac{RC}{(jRC\omega+1)^2}. \end{aligned}$$

Taking the inverse Fourier transform of both sides of this equation, we obtain

$$\begin{aligned} v_1(t) &= \mathcal{F}^{-1}\left\{\frac{RC}{(jRC\omega+1)^2}\right\}(t) \\ &= \mathcal{F}^{-1}\left\{\frac{RC}{[(RC)(j\omega+\frac{1}{RC})]^2}\right\}(t) \\ &= \mathcal{F}^{-1}\left\{\frac{1}{RC(j\omega+\frac{1}{RC})^2}\right\}(t) \\ &= \tfrac{1}{RC}\mathcal{F}^{-1}\left\{\frac{1}{(j\omega+\frac{1}{RC})^2}\right\}(t). \end{aligned}$$

Using Table 6.2, we can simplify to obtain

$$v_1(t) = \tfrac{1}{RC}te^{-t/(RC)}u(t).$$

Thus, we have found $v_1$ in the case of the given $v_0$. ■

## 6.19 Amplitude Modulation

In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal. For example, voice/audio signals typically have information in the range of 0 to 22 kHz. Often, it is not practical to transmit such a signal using its original frequency range. Two potential problems with such an approach are: 1) interference and 2) constraints on antenna length. Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference. Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals. For the preceding reasons, we often need to change the frequency range associated with a signal before transmission. In what follows, we consider one possible scheme for accomplishing this. This scheme is known as amplitude modulation.

Amplitude modulation (AM) is used in many communication systems. Numerous variations on amplitude modulation are possible. Here, we consider two of the simplest variations: double-side-band/suppressed-carrier (DSB/SC) and single-side-band/suppressed-carrier (SSB/SC).

### 6.19.1 Modulation With a Complex Sinusoid

Consider the communication system shown in Figure 6.28. In what follows, we will analyze the behavior of this system in detail.

First, let us consider the transmitter in Figure 6.28(a). The transmitter is the system with the input $x$ and output $y$ that is characterized by the equation

$$y(t) = c_1(t)x(t), \tag{6.39}$$

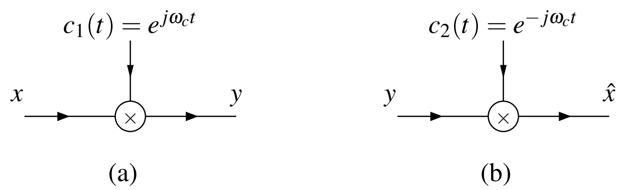



Figure 6.28: Simple communication system. (a) Transmitter and (b) receiver.

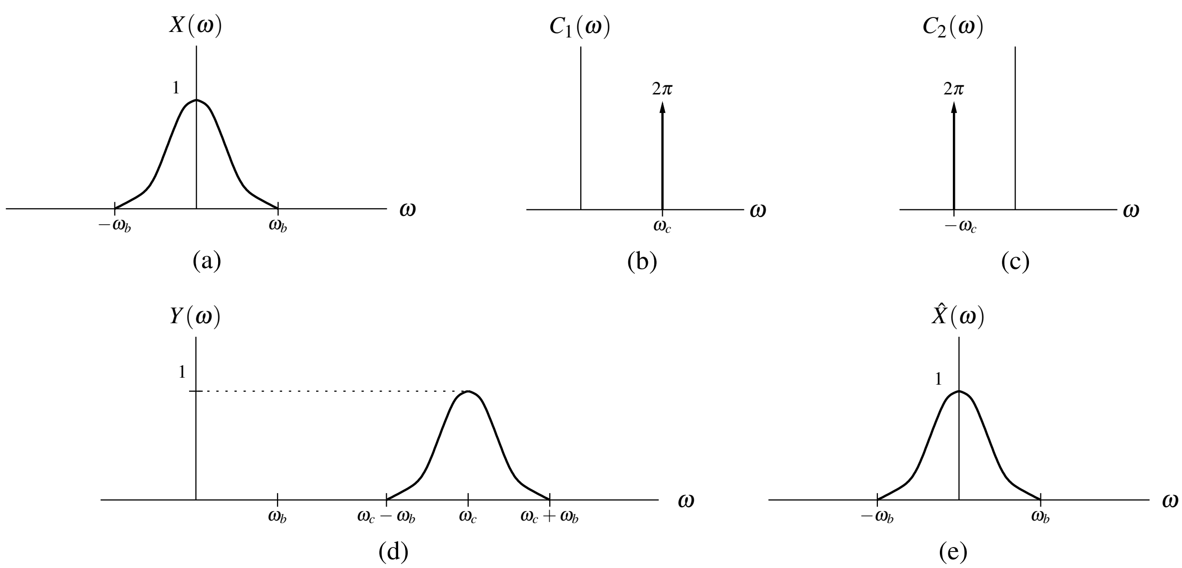



Figure 6.29: Frequency spectra for modulation with a complex sinusoid. (a) Spectrum of the transmitter input. (b) Spectrum of the complex sinusoid used in the transmitter. (c) Spectrum of the complex sinusoid used in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the receiver output.

where

$$c_1(t) = e^{j\omega_c t}.$$

Let $X$, $Y$, and $C_1$ denote the Fourier transforms of $x$, $y$, and $c_1$, respectively. Taking the Fourier transform of both sides of (6.39), we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{c_1 x\}(\omega) \\ &= \mathcal{F}\left\{e^{j\omega_c t}x(t)\right\}(\omega) \\ &= X(\omega - \omega_c). \end{aligned} \tag{6.40}$$

Thus, the frequency spectrum of the (transmitter) output is simply the frequency spectrum of the (transmitter) input shifted by $\omega_c$. The relationship between the frequency spectra of the input and output is illustrated in Figure 6.29. Clearly, the spectrum of the output has been shifted to a different frequency range as desired. Next, we need to determine whether the receiver can recover the original signal $x$ from the transmitted signal $y$.

Consider the receiver shown in Figure 6.28(b). The receiver is a system with the input $y$ and output $\hat{x}$ that is characterized by the equation

$$\hat{x}(t) = c_2(t)y(t), \tag{6.41}$$

where

$$c_2(t) = e^{-j\omega_c t}.$$

In order for the communication system to be useful, we need for the received signal $\hat{x}$ to be equal to the original signal $x$ from the transmitter. Let $Y$, $\hat{X}$, and $C_2$ denote the Fourier transform of $y$, $\hat{x}$, and $c_2$, respectively. Taking the Fourier transform of both sides of (6.41), we obtain

$$\begin{aligned} \hat{X}(\omega) &= \mathcal{F}\{c_2 y\}(\omega) \\ &= \mathcal{F}\{e^{-j\omega_c t}y(t)\}(\omega) \\ &= Y(\omega + \omega_c). \end{aligned}$$

Substituting the expression for $Y$ in (6.40) into this equation, we obtain

$$\begin{aligned} \hat{X}(\omega) &= X([\omega + \omega_c] - \omega_c) \\ &= X(\omega). \end{aligned}$$

Since $\hat{X} = X$, we have that the received signal $\hat{x}$ is equal to the original signal $x$ from the transmitter. Thus, the communication system has the desired behavior. The relationship between the frequency spectra of the various signals in the AM system is illustrated in Figure 6.29.

Although the above result is quite interesting mathematically, it does not have direct practical application. The difficulty here is that $c_1$, $c_2$, and $y$ are complex-valued signals, and we cannot realize complex-valued signals in the physical world. This communication system is not completely without value, however, as it inspires the development of the practically useful system that we consider next.

### 6.19.2 DSB/SC Amplitude Modulation

Now, let us consider the communication system shown in Figure 6.30. This system is known as a double-side-band/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. The receiver in Figure 6.30(b) contains a LTI subsystem that is labelled with its impulse response $h$. The DSB/SC AM system is very similar to the one considered earlier in Figure 6.28. In the new system, however, multiplication by a complex sinusoid has been replaced by

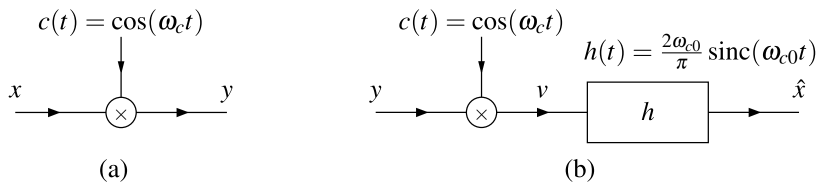



Figure 6.30: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

multiplication by a real sinusoid. The new system also requires that the input signal $x$ be bandlimited to frequencies in the interval $[-\omega_b, \omega_b]$ and that

$$\omega_b < \omega_{c0} < 2\omega_c - \omega_b. \tag{6.42}$$

The reasons for this restriction will become clear after having studied this system in more detail.

Consider the transmitter shown in Figure 6.30(a). The transmitter is a system with input $x$ and output $y$ that is characterized by the equation

$$y(t) = c(t)x(t),$$

where

$$c(t) = \cos(\omega_c t).$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{cx\}(\omega) \\
&= \mathcal{F}\{\cos(\omega_c t)x(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)\right] \\
&= \tfrac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right].
\end{aligned} \tag{6.43}$$

(Note that, above, we used the fact that $\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})$.) Thus, the frequency spectrum of the (transmitter) output is the average of two shifted versions of the frequency spectrum of the (transmitter) input. The relationship between the frequency spectra of the input and output can be seen through Figures 6.31(a) and (d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Next, we must determine whether the receiver can recover the original signal $x$.

Consider the receiver shown in Figure 6.30(b). The receiver is a system with input $y$ and output $\hat{x}$ that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \tag{6.44a}$$

$$\hat{x}(t) = v * h(t), \tag{6.44b}$$

where $c$ is as defined earlier and

$$h(t) = \frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t). \tag{6.44c}$$

Let $H$, $Y$, $V$, and $\hat{X}$ denote the Fourier transforms of $h$, $y$, $v$ and $\hat{x}$, respectively. Taking the Fourier transform of $\hat{X}$ (in (6.44b)), we have

$$\hat{X}(\omega) = H(\omega)V(\omega). \tag{6.45}$$

Taking the Fourier transform of $h$ (in (6.44c)) with the assistance of Table 6.2, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\tfrac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t)\right\}(\omega) \\ &= 2\operatorname{rect}\left(\tfrac{\omega}{2\omega_{c0}}\right) \\ &= \begin{cases} 2 & |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Taking the Fourier transform of $v$ (in (6.44a)) yields

$$\begin{aligned} V(\omega) &= \mathcal{F}\{cy\}(\omega) \\ &= \mathcal{F}\{\cos(\omega_c t)y(t)\}(\omega) \\ &= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t}+e^{-j\omega_c t}\right)y(t)\right\}(\omega) \\ &= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t}y(t)\right\}(\omega)+\mathcal{F}\left\{e^{-j\omega_c t}y(t)\right\}(\omega)\right] \\ &= \tfrac{1}{2}[Y(\omega-\omega_c)+Y(\omega+\omega_c)]. \end{aligned}$$

Substituting the expression for $Y$ in (6.43) into this equation, we obtain

$$\begin{aligned} V(\omega) &= \tfrac{1}{2}\left[\tfrac{1}{2}[X([\omega-\omega_c]-\omega_c)+X([\omega-\omega_c]+\omega_c)]+\tfrac{1}{2}[X([\omega+\omega_c]-\omega_c)+X([\omega+\omega_c]+\omega_c)]\right] \\ &= \tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c). \end{aligned} \tag{6.46}$$

The relationship between $V$ and $X$ can be seen via Figures 6.31(a) and (e). Substituting the above expression for $V$ into (6.45) and simplifying, we obtain

$$\begin{aligned} \hat{X}(\omega) &= H(\omega)V(\omega) \\ &= H(\omega)\left[\tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c)\right] \\ &= \tfrac{1}{2}H(\omega)X(\omega)+\tfrac{1}{4}H(\omega)X(\omega-2\omega_c)+\tfrac{1}{4}H(\omega)X(\omega+2\omega_c) \\ &= \tfrac{1}{2}[2X(\omega)]+\tfrac{1}{4}(0)+\tfrac{1}{4}(0) \\ &= X(\omega). \end{aligned}$$

In the above simplification, since $H(\omega) = 2\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right)$ and condition (6.42) holds, we were able to deduce that $H(\omega)X(\omega) = 2X(\omega)$, $H(\omega)X(\omega-2\omega_c) = 0$, and $H(\omega)X(\omega+2\omega_c) = 0$. The relationship between $\hat{X}$ and $X$ can be seen from Figures 6.31(a) and (f). Thus, we have that $\hat{X} = X$, implying $\hat{x} = x$. So, we have recovered the original signal $x$ at the receiver. This system has managed to shift $x$ into a different frequency range before transmission and then recover $x$ at the receiver. This is exactly what we wanted to accomplish.

### 6.19.3 SSB/SC Amplitude Modulation

By making a minor modification to the DSB/SC AM system, we can reduce the bandwidth requirements of the system by half. The resulting system is referred to as a single-side-band/suppressed-carrier (SSB/SC) AM system. This modified system is illustrated in Figure 6.32. The transmitter in Figure 6.32(a) contains a LTI subsystem that is labelled with its impulse response $g$. Similarly, the receiver in Figure 6.32(b) contains a LTI subsystem that is labelled with its impulse response $h$. Let $X$, $Y$, $Q$, $V$, $\hat{X}$, $C$, $G$, and $H$ denote the Fourier transforms of $x$, $y$, $q$, $v$, $\hat{x}$, $c$, $g$, and $h$, respectively.

The transmitter is a system with input $x$ and output $y$ that is characterized by the equations

$$\begin{aligned} q(t) &= c(t)x(t) \quad \text{and} \\ y(t) &= q * g(t), \end{aligned}$$

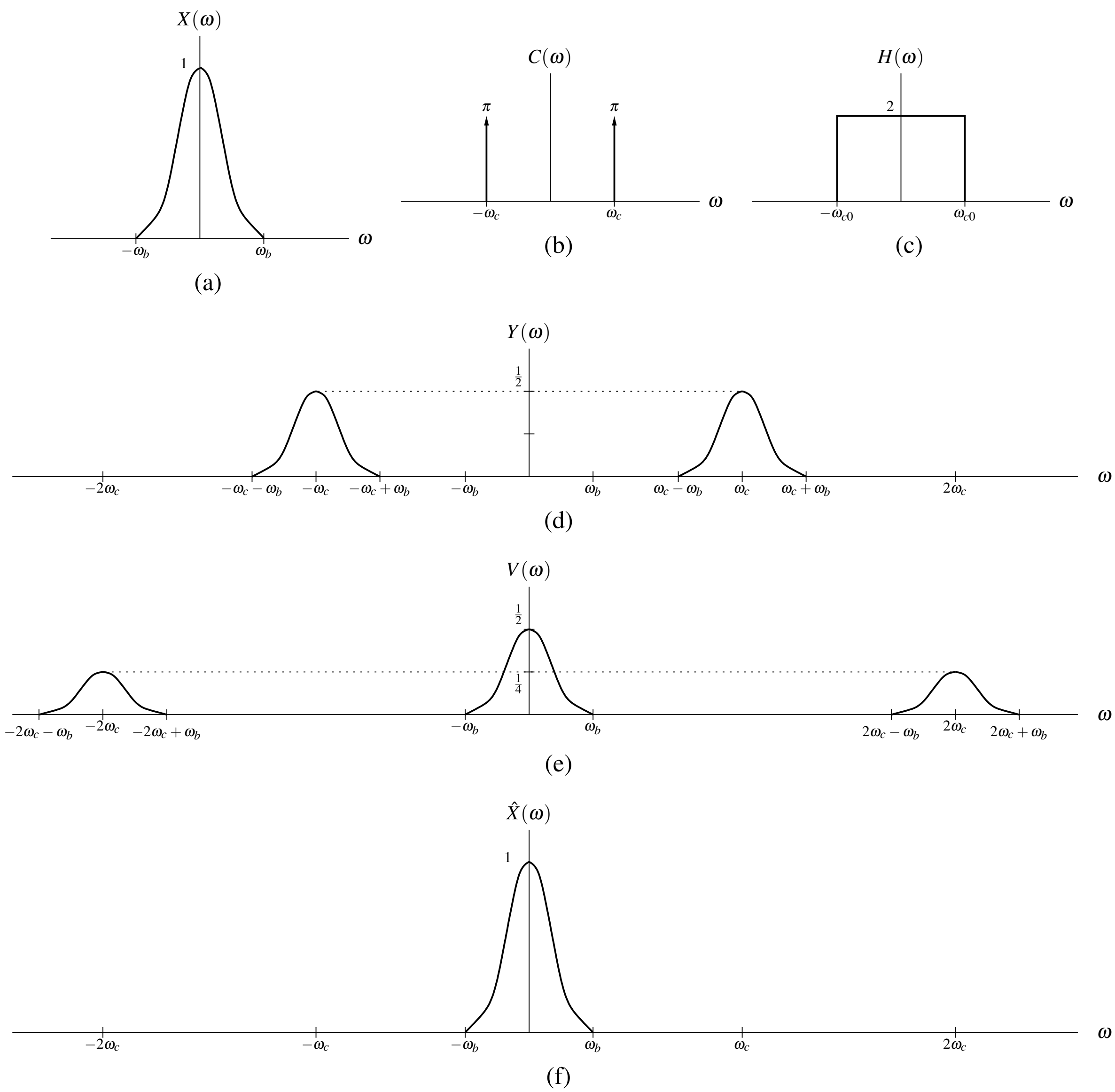



Figure 6.31: Signal spectra for DSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoidal function used in the transmitter and receiver. (c) Frequency response of the filter in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the multiplier output in the receiver. (f) Spectrum of the receiver output.

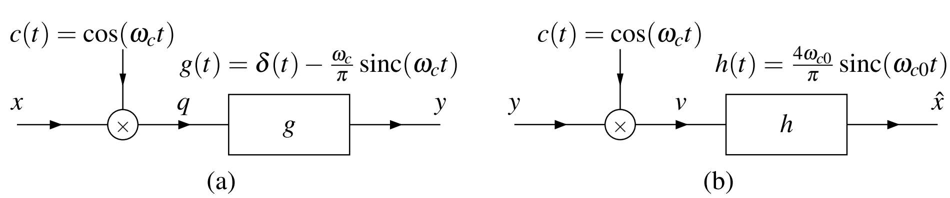



Figure 6.32: SSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

where

$$c(t) = \cos(\omega_c t) \quad \text{and}$$
$$g(t) = \delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t).$$

Taking the Fourier transform of $g$, we obtain

$$G(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise}, \end{cases} \quad \text{and}$$

The receiver is a system with input $y$ and output $\hat{x}$ that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and}$$
$$\hat{x}(t) = v * h(t)$$

where $c$ is as defined earlier and

$$h(t) = \tfrac{4\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t).$$

Taking the Fourier transform of $h$, we obtain

$$H(\omega) = \begin{cases} 4 & |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise}. \end{cases}$$

Figure 6.33 depicts the transformations that the signal undergoes as it passes through the system. Again, the output from the receiver is equal to the input to the transmitter. A detailed analysis of this communication system is left as an exercise for the reader.

## 6.20 Sampling and Interpolation

Often, we encounter situations in which we would like to process a continuous-time signal in the discrete-time domain or vice versa. For example, we might have a continuous-time audio signal that we would like to process using a digital computer (which is a discrete-time system), or we might have a discrete-time audio signal that we wish to play on a loudspeaker (which is a continuous-time system). Clearly, some means is needed to link the continuous- and discrete-time domains. This connection is established through processes known as sampling and interpolation. In what follows, we will formally introduce these processes and study them in some detail.

Sampling allows us to create sequence (i.e., a discrete-time signal) from a function (i.e., a continuous-time signal). Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**. With this scheme, a sequence $y$ of samples is obtained from a function $x$ as given by

$$y(n) = x(Tn) \quad \text{for all integer } n, \tag{6.47}$$

where $T$ is a positive real constant. As a matter of terminology, $T$ is referred to as the **sampling period**, and $\omega_s = \frac{2\pi}{T}$ is referred to as the (angular) **sampling frequency**. A system such as that described by (6.47) is known as an **ideal continuous-to-discrete-time (C/D) converter**, and is shown diagrammatically in Figure 6.34. An example of periodic sampling is shown in Figure 6.35. Figure 6.35(a) shows a function $x$ to be sampled, and Figure 6.35(b) shows the sequence $y$ obtained by sampling $x$ with the sampling period $T = 10$.

Interpolation allows us to construct a function (i.e., a continuous-time signal) from a sequence (i.e., a discrete-time signal). In effect, for a given sequence, this process constructs a function that would produce the given sequence when sampled, typically with some additional constraints imposed. More formally, for a given sequence $y$ associated with a sampling period $T$, interpolation produces a function $\hat{x}$ as given by

$$\hat{x} = \mathcal{H}y,$$

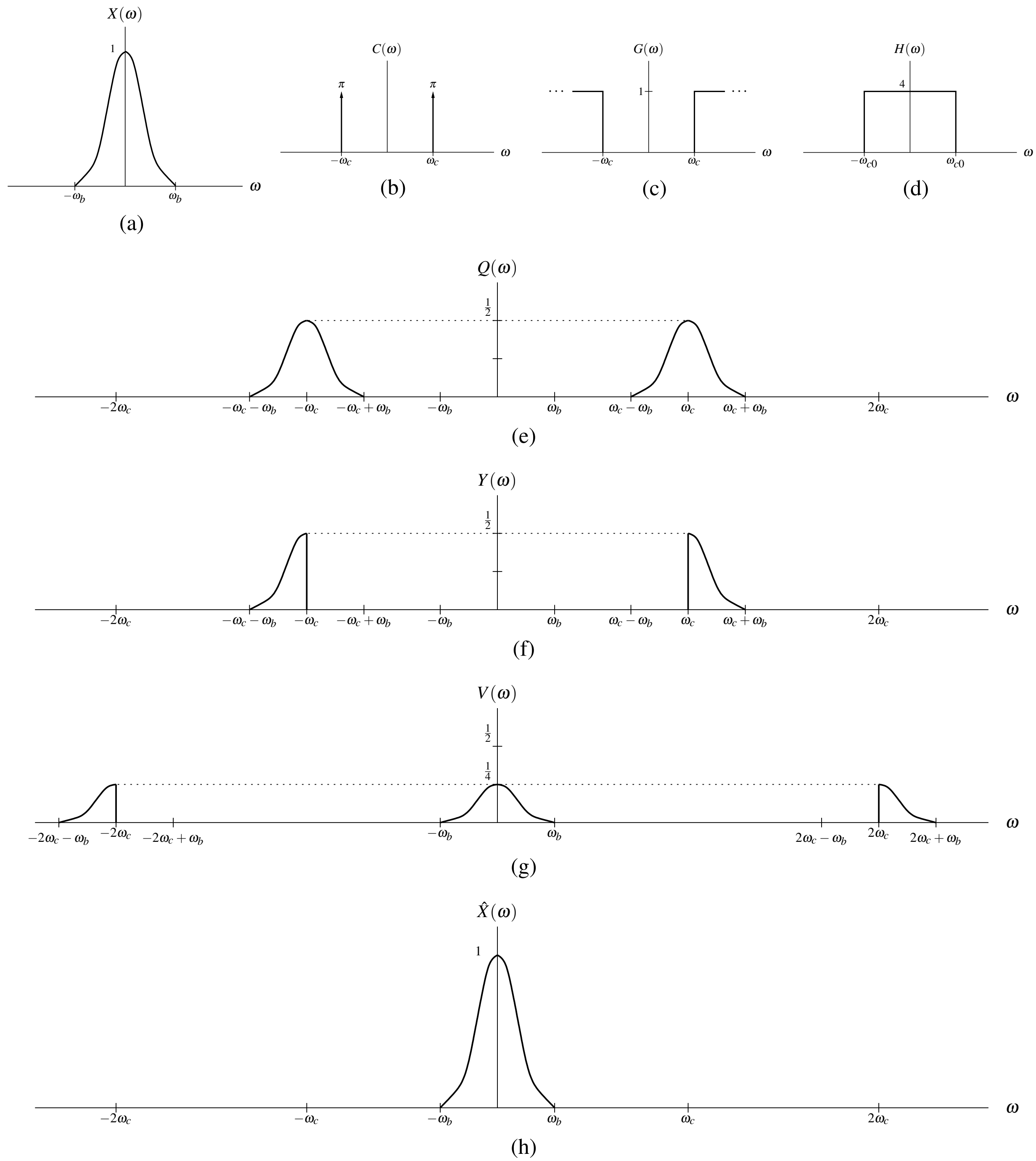



Figure 6.33: Signal spectra for SSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoid used in the transmitter and receiver. (c) Frequency response of the filter in the transmitter. (d) Frequency response of the filter in the receiver. (e) Spectrum of the multiplier output in the transmitter. (f) Spectrum of the transmitted signal. (g) Spectrum of the multiplier output in the receiver. (h) Spectrum of the receiver output.

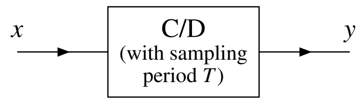



Figure 6.34: Ideal C/D converter with input function $x$ and output sequence $y$.

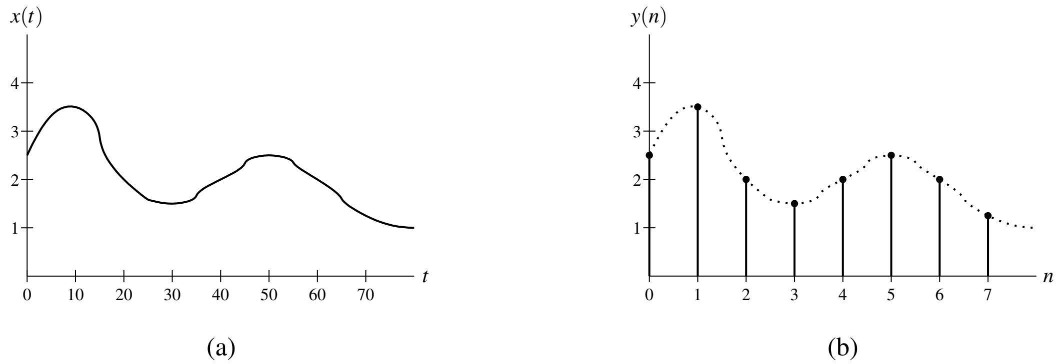



Figure 6.35: Example of periodic sampling. (a) The function $x$ to be sampled and (b) the sequence $y$ produced by sampling $x$ with a sampling period of 10.

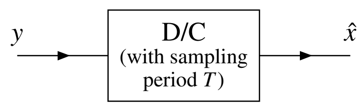



Figure 6.36: Ideal D/C converter with input sequence $y$ and output function $\hat{x}$.

subject to the constraint that

$$\hat{x}(Tn) = y(n) \quad \text{for all integer } n,$$

where $\mathcal{H}$ is an operator that maps a sequence to a function. The precise form of $\mathcal{H}$ depends on the particular interpolation scheme employed. Although there are many different ways in which to perform interpolation, we will focus our attention in subsequent sections on one particular scheme known as bandlimited interpolation. The interpolation process is performed by a system known as an **ideal discrete-to-continuous-time (D/C) converter**, as shown in Figure 6.36.

In the absence of any constraints on the sampling process, a function cannot be uniquely determined from a sequence of its equally-spaced samples. In other words, the sampling process is not generally invertible. Consider, for example, the functions $x_1$ and $x_2$ given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

If we sample each of these functions with the sampling period $T = 1$, we obtain the respective sequences

$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and} \quad y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

Thus, $y_1 = y_2$ in spite of the fact that $x_1 \neq x_2$. This example trivially shows that if no constraints are placed upon a function, then the function cannot be uniquely determined from its samples.

Fortunately, under certain circumstances, a function can be recovered exactly from its samples. In particular, in the case that the function being sampled is bandlimited, we can show that a sequence of its equally-spaced samples uniquely determines the function if the sampling period is sufficiently small. This result, known as the sampling theorem, is of paramount importance.

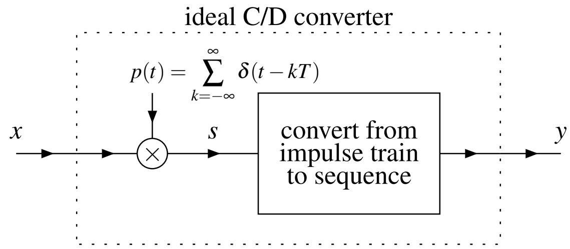



Figure 6.37: Model of ideal C/D converter with input function $x$ and output sequence $y$.

### 6.20.1 Sampling

In order to gain some insight into sampling, we need a way in which to mathematically model this process. To this end, we employ the simple model for the ideal C/D converter shown in Figure 6.37. In short, we may view the process of sampling as impulse train modulation followed by conversion of an impulse train to a sequence of sample values. More specifically, to sample a function $x$ with sampling period $T$, we first multiply $x$ by the periodic impulse train $p$ to obtain

$$s(t) = x(t)p(t), \tag{6.48}$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT).$$

Then, we take the weights of successive impulses in $s$ to form a sequence $y$ of samples. The sampling frequency is given by $\omega_s = \frac{2\pi}{T}$. As a matter of terminology, $p$ is referred to as a sampling function. From the diagram, we can see that the signals $s$ and $y$, although very closely related, have some key differences. The impulse train $s$ is a function (i.e., continuous-time signal) that is zero everywhere except at integer multiples of $T$ (i.e., at sample points), while $y$ is a sequence (i.e., discrete-time signal), defined only on the integers with its values corresponding to the weights of successive impulses in $s$. The various signals involved in sampling are illustrated in Figure 6.38.

In passing, we note that the above model of sampling is only a mathematical convenience. That is, the model provides us with a relatively simple way in which to study the mathematical behavior of sampling. The above model, however, is not directly useful as a means for actually realizing sampling in a real-world system. Obviously, the impulse train employed in the above model poses some insurmountable problems as far as implementation is concerned.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original function $x$ and its impulse-train sampled version $s$. In what follows, let $X$, $Y$, $P$, and $S$ denote the Fourier transforms of $x$, $y$, $p$, and $s$, respectively. Since $p$ is $T$-periodic, it can be represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{6.49}$$

Using the Fourier series analysis equation, we calculate the coefficients $c_k$ to be

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T}\int_{-\infty}^{\infty} \delta(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T} \\ &= \tfrac{\omega_s}{2\pi}. \end{aligned} \tag{6.50}$$

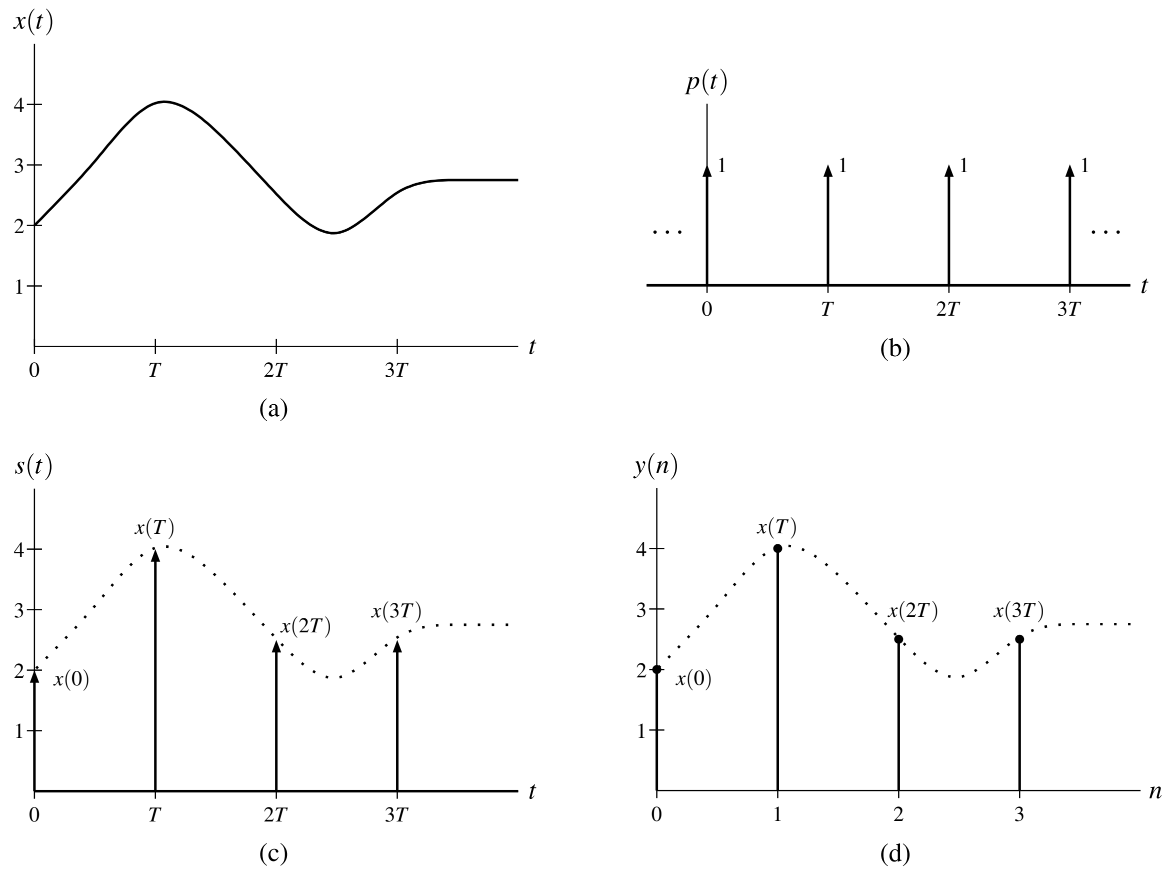



Figure 6.38: An example of the various signals involved in the sampling process for a sampling period of $T$. (a) The function $x$ to be sampled. (b) The sampling function $p$. (c) The impulse-modulated function $s$. (d) The sequence $y$ produced by sampling.

Substituting (6.49) and (6.50) into (6.48), we obtain

$$\begin{aligned} s(t) &= x(t) \sum_{k=-\infty}^{\infty} \tfrac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}. \end{aligned}$$

Taking the Fourier transform of $s$ yields

$$S(\omega) = \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{6.51}$$

Thus, the spectrum of the impulse-train sampled function $s$ is a scaled sum of an infinite number of shifted copies of the spectrum of the original function $x$.

Now, we consider a simple example to further illustrate the behavior of the sampling process in the frequency domain. Suppose that we have a function $x$ with the Fourier transform $X$ where $|X(\omega)| = 0$ for $|\omega| > \omega_m$ (i.e., $x$ is bandlimited). To simplify the visualization process, we will assume $X$ has the particular form shown in Figure 6.39(a). In what follows, however, we only actually rely on the bandlimited nature of $x$ and not its specific definition. So, the results that we derive in what follows generally apply to any bandlimited function. From (6.51), we know that $S$ is formed by the superposition of an infinite number of shifted copies of $X$. Upon more careful consideration, we can see that two distinct situations can arise. That is, the nonzero portions of the shifted copies of $X$ used to form $S$ can either: 1) overlap or, 2) not overlap. These two cases are illustrated in Figures 6.39(c) and 6.39(b), respectively. From these graphs, we can see that the nonzero portions of the shifted copies of $X$ will not overlap if

$$\omega_m < \omega_s - \omega_m \quad \text{and} \quad -\omega_m > -\omega_s + \omega_m$$

or equivalently

$$\omega_s > 2\omega_m.$$

Consider the case in which the copies of the original spectrum $X$ in $S$ do not overlap, as depicted in Figure 6.39(b). In this situation, the spectrum $X$ of the original function is clearly discernible in the spectrum $S$. In fact, one can see that the original spectrum $X$ can be obtained directly from $S$ through a lowpass filtering operation. Thus, the original function $x$ can be exactly recovered from $s$.

Now, consider the case in which copies of the original spectrum $X$ in $S$ do overlap. In this situation, multiple frequencies in the spectrum $X$ of the original function are mapped to the same frequency in $S$. This phenomenon is referred to as **aliasing**. Clearly, aliasing leads to individual periods of $S$ having a different shape than the original spectrum $X$. When aliasing occurs, the shape of the original spectrum $X$ is no longer discernible from $S$. Consequently, we are unable to recover the original function $x$ from $s$ in this case.

### 6.20.2 Interpolation and Reconstruction of a Function From Its Samples

Interpolation allows us to construct a function (i.e., continuous-time signal) from a sequence (i.e., discrete-time signal). This process is essentially responsible for determining the value of a function between sample points. Except in very special circumstances, it is not generally possible to exactly reproduce a function from its samples. Although many interpolation schemes exist, we shall focus our attention shortly on one particular scheme. The interpolation process can be modeled with the simple ideal D/C converter system, shown in Figure 6.40. This particular type of interpolation is known as **bandlimited interpolation**.

Recall the ideal C/D converter of Figure 6.37. Since the process of converting an impulse train to a sequence is invertible, we can reconstruct the original function $x$ from a sequence $y$ of its samples if we can somehow recover $x$ from $s$. Let us suppose now that $x$ is bandlimited. As we saw in the previous section, we can recover $x$ from $s$ provided that $x$ is bandlimited and sampled at a sufficiently high rate so as to avoid aliasing. In the case that aliasing does not occur, we can reconstruct the original function $x$ from $y$ using the ideal D/C converter shown in Figure 6.40. In what

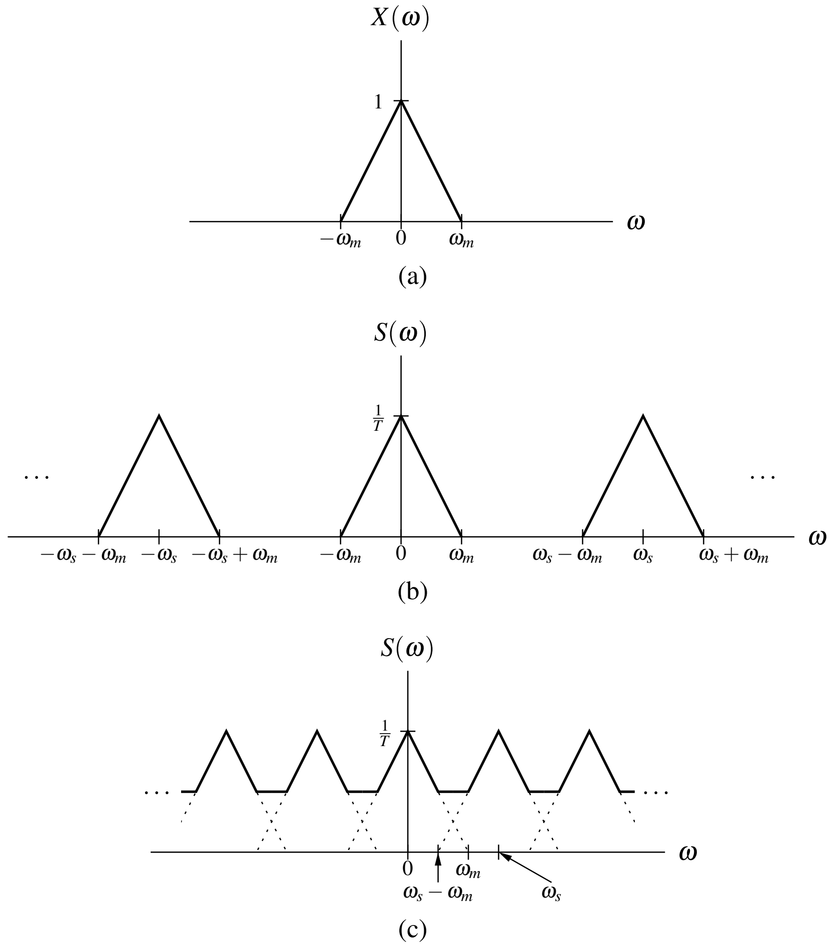



Figure 6.39: Effect of impulse-train sampling on the frequency spectrum. (a) Spectrum of the function $x$ being sampled. (b) Spectrum of $s$ in the absence of aliasing. (c) Spectrum of $s$ in the presence of aliasing.

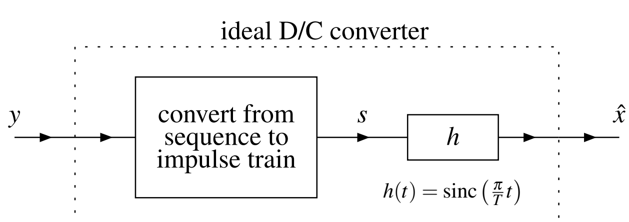



Figure 6.40: Model of ideal D/C converter with input sequence $y$ and output function $\hat{x}$.

follows, we will derive a formula for computing the original function $\hat{x}$ from its samples $y$. Consider the model of the D/C converter. We have a lowpass filter with impulse response

$$h(t) = \operatorname{sinc}\left(\tfrac{\pi t}{T}\right) = \operatorname{sinc}\left(\tfrac{\omega_s t}{2}\right)$$

and frequency response

$$H(\omega) = T \operatorname{rect}\left(\tfrac{T\omega}{2\pi}\right) = \tfrac{2\pi}{\omega_s} \operatorname{rect}\left(\tfrac{\omega}{\omega_s}\right) = \begin{cases} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise.} \end{cases}$$

First, we convert the sequence $y$ to the impulse train $s$ to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn).$$

Then, we filter the resulting function $s$ with the lowpass filter having impulse response $h$, yielding

$$\begin{aligned}
\hat{x}(t) &= s * h(t) \\
&= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(t-\tau) \sum_{n=-\infty}^{\infty} y(n)\delta(\tau - Tn)d\tau \\
&= \sum_{n=-\infty}^{\infty} y(n) \int_{-\infty}^{\infty} h(t-\tau)\delta(\tau - Tn)d\tau \\
&= \sum_{n=-\infty}^{\infty} y(n)h(t - Tn) \\
&= \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right].
\end{aligned}$$

If $x$ is bandlimited and aliasing is avoided, $\hat{x} = x$ and we have a formula for exactly reproducing $x$ from its samples $y$.

### 6.20.3 Sampling Theorem

In the preceding sections, we have established the important result given by the theorem below.

**Theorem 6.19** (Sampling Theorem)**.** *Let $x$ be a function with the Fourier transform $X$, and let $y$ be the sequence resulting from the periodic sampling of $x$ with the sampling period $T$ (i.e., $y(n) = x(Tn)$). Suppose that $|X(\omega)| = 0$ for all $|\omega| > \omega_M$ (i.e., $x$ is bandlimited to the interval $[-\omega_M, \omega_M]$). Then, $x$ is uniquely determined by $y$ if*

$$\omega_s > 2\omega_M, \tag{6.52}$$

*where $\omega_s = \frac{2\pi}{T}$. In particular, if (6.52) is satisfied, we have that*

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right],$$

*or equivalently (i.e., rewritten in terms of $\omega_s$ instead of $T$),*

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left(\tfrac{\omega_s}{2}t - \pi n\right).$$

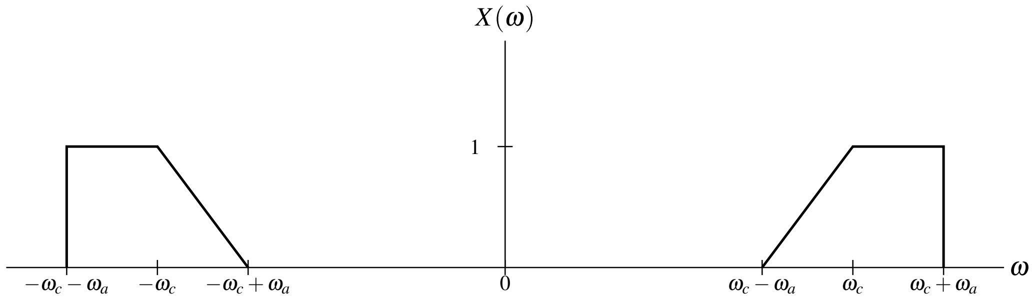



Figure 6.41: Frequency spectrum of the function $x$.

As a matter of terminology, we refer to (6.52) as the **Nyquist condition** (or Nyquist criterion). Also, we call $\frac{\omega_s}{2}$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**. It is important to note that the Nyquist condition is a strict inequality. Therefore, to ensure aliasing does not occur in the most general case, one must choose the sampling rate larger than the Nyquist rate. One can show, however, that if the frequency spectrum does not have impulses at the Nyquist frequency, it is sufficient to sample at exactly the Nyquist rate.

**Example 6.42.** Let $x$ denote a continuous-time audio signal with Fourier transform $X$. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period $T$ with which $x$ can be sampled that will allow $x$ to be exactly recovered from its samples.

*Solution.* The function $x$ is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

$$\begin{aligned}\omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi.\end{aligned}$$

Thus, the largest permissible sampling period is given by

$$\begin{aligned}T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{88200\pi} \\ &= \tfrac{1}{44100}.\end{aligned}$$

■

Although the sampling theorem provides an upper bound on the sampling rate that holds in the case of arbitrary bandlimited functions, in some special cases it may be possible to employ an even smaller sampling rate. This point is further illustrated by way of the example below.

**Example 6.43.** Consider the function $x$ with the Fourier transform $X$ shown in Figure 6.41 (where $\omega_c \gg \omega_a$). (a) Using the sampling theorem directly, determine the largest permissible sampling period $T$ that will allow $x$ to be exactly reconstructed from its samples. (b) Explain how one can exploit the fact that $X(\omega) = 0$ for a large portion of the interval $[-\omega_c - \omega_a, \omega_c + \omega_a]$ in order to reduce the rate at which $x$ must be sampled.

*Solution.* (a) The function $x$ is bandlimited to $[-\omega_m, \omega_m]$, where $\omega_m = \omega_c + \omega_a$. Thus, the minimum sampling rate required is given by

$$\begin{aligned}\omega_s &> 2\omega_m \\ &= 2(\omega_c + \omega_a) \\ &= 2\omega_c + 2\omega_a.\end{aligned}$$

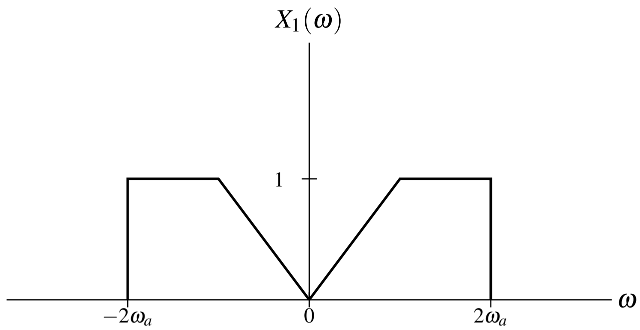



Figure 6.42: Frequency spectrum of the function $x_1$.

and the maximum sampling period is calculated as

$$\begin{aligned} T &< \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{2\omega_c+2\omega_a} \\ &= \tfrac{\pi}{\omega_c+\omega_a}. \end{aligned}$$

(b) We can modulate and lowpass filter $x$ in order to compress all of its spectral information into the frequency range $[-2\omega_a, 2\omega_a]$, yielding the function $x_1$. That is, we have

$$x_1(t) = \{x(t)\cos[(\omega_c - \omega_a)t]\} * h(t)$$

where

$$h(t) = \tfrac{4\omega_a}{\pi}\operatorname{sinc}(2\omega_a t) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad H(\omega) = 2\operatorname{rect}\left(\tfrac{\omega}{4\omega_a}\right).$$

This process can be inverted (by modulation and filtering) to obtain $x$ from $x_1$. In particular, we have that

$$x(t) = \{x_1(t)\cos[(\omega_c - \omega_a)t]\} * h_2(t)$$

where

$$h_2(t) = \delta(t) - \tfrac{2(\omega_c-\omega_a)}{\pi}\operatorname{sinc}[(\omega_c - \omega_a)t] \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad H_2(\omega) = 2 - 2\operatorname{rect}\left[\tfrac{\omega}{4(\omega_c-\omega_a)}\right].$$

Let $X_1$ denote the Fourier transform of $x_1$. The Fourier transform $X_1$ is as shown in Figure 6.42. Applying the sampling theorem to $x_1$ we find that the minimum sampling rate is given by

$$\begin{aligned} \omega_s &> 2(2\omega_a) \\ &= 4\omega_a \end{aligned}$$

and the largest sampling period is given by

$$\begin{aligned} T &< \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{4\omega_a} \\ &= \tfrac{\pi}{2\omega_a}. \end{aligned}$$

Since $\omega_c \gg \omega_a$ (by assumption), this new sampling period is larger than the one computed in part (a) of this example. ■

# 6.21 Exercises

## 6.21.1 Exercises Without Answer Key

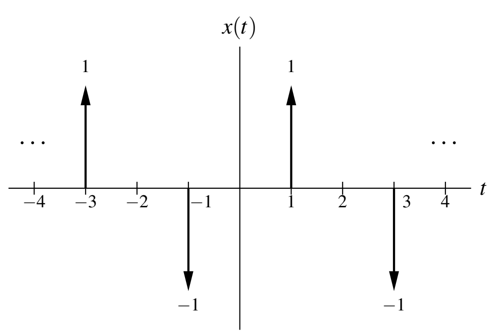
**6.1** Using the Fourier transform analysis equation, find the Fourier transform $X$ of each function $x$ below.
(a) $x(t) = \operatorname{rect}(t - t_0)$, where $t_0$ is a constant;
(b) $x(t) = e^{-4t}u(t-1)$;
(c) $x(t) = 3[u(t) - u(t-2)]$; and
(d) $x(t) = e^{-|t|}$.

**6.2** Let $y(t) = x(t+a) + x(t-a)$, where $a$ is a real constant, and let $X$ and $Y$ denote the Fourier transforms of $x$ and $y$, respectively. Show that $Y(\omega) = 2X(\omega)\cos(a\omega)$.

**6.3** Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform $X$ of each function $x$ below.
(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t}u(t+2)$;
(c) $x(t) = \cos(t)u(t)$;
(d) $x(t) = 6[u(t) - u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\operatorname{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = x_1 * x_2(t)$, where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = te^{-3t}u(t)$; and
(j) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT)$, where $a$ is a complex constant satisfying $|a| < 1$ and $T$ is a strictly-positive real constant; (Hint: Recall the formula for the sum of an infinite geometric sequence (i.e., (F.9)).)

**6.4** For each function $y$ given below, find the Fourier transform $Y$ of $y$ in terms of the Fourier transform $X$ of $x$.
(a) $y(t) = x(at - b)$, where $a$ and $b$ are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \mathcal{D}(x*x)(t)$, where $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = tx(2t-1)$;
(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$; and
(h) $y(t) = (\mathcal{D}x) * x_1(t)$, where $x_1(t) = e^{-jt}x(t)$ and $\mathcal{D}$ denotes the derivative operator.

**6.5** Find the Fourier transform $X$ of each periodic function $x$ shown below.



(a)

**6.6** Let $X$ denote the Fourier transform of $x$. Show that:
(a) if $x$ is even, then $X(\omega) = 2\int_0^\infty x(t)\cos(\omega t)dt$; and
(b) if $x$ is odd, then $X(\omega) = -2j\int_0^\infty x(t)\sin(\omega t)dt$.

**6.7** Let $x$ be a real function with even part $x_\mathrm{e}$ and odd part $x_\mathrm{o}$. Let $X$, $X_\mathrm{e}$, and $X_\mathrm{o}$ denote the Fourier transforms of $x$, $x_\mathrm{e}$, and $x_\mathrm{o}$, respectively. Show that:
(a) $X_\mathrm{e}(\omega) = \mathrm{Re}[X(\omega)]$; and
(b) $X_\mathrm{o}(\omega) = j\,\mathrm{Im}[X(\omega)]$.

**6.8** Using the time-domain convolution property of the Fourier transform, compute the convolution $h = h_1 * h_2$, where

$$h_1(t) = 2000\,\mathrm{sinc}(2000\pi t) \quad \text{and} \quad h_2(t) = \delta(t) - 1000\,\mathrm{sinc}(1000\pi t).$$

**6.9** Compute the energy contained in the function $x(t) = 200\,\mathrm{sinc}(200\pi t)$.

**6.10** For each function $x$ given below, compute the frequency spectrum of $x$, and find and plot the corresponding magnitude and phase spectra.
(a) $x(t) = e^{-at}u(t)$, where $a$ is a positive real constant; and
(b) $x(t) = \mathrm{sinc}\left(\frac{1}{200}t - \frac{1}{200}\right)$.

**6.11** Show that, if a function $x$ has bandwidth $B$, then $y(t) = x^n(t)$ has bandwidth $nB$.

**6.12** Show that a function cannot be both timelimited and bandlimited. [Hint: Let $X$ denote the Fourier transform of $x$. Suppose that $x$ is timelimited and also bandlimited such that $X(\omega) = 0$ for $|\omega| \geq B$. Due to the bandlimited nature of $x$, we have that $X(\omega) = X(\omega)\,\mathrm{rect}\left(\frac{\omega}{2B'}\right)$ for $B' > B$. Then, show that the inverse Fourier transform of the preceding equation leads to a contradiction. To do this, you will need to observe that the convolution of a timelimited function with a non-timelimited function must be a non-timelimited function.]

**6.13** The ideal Hilbert transformer is a LTI system with the frequency response $H(\omega) = -j\,\mathrm{sgn}\,\omega$. This type of system is useful in a variety of signal processing applications (e.g., SSB/SC amplitude modulation). By using the duality property of the Fourier transform, find the impulse response $h$ of this system.

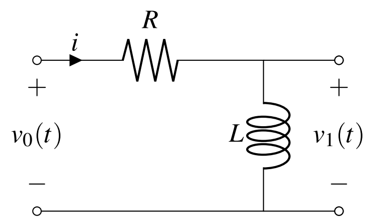
**6.14** For each differential/integral equation below that defines a LTI system with input $x$ and output $y$, find the frequency response $H$ of the system. (Note that the prime symbol denotes differentiation.)
(a) $y''(t) + 5y'(t) + y(t) + 3x'(t) - x(t) = 0$;
(b) $y'(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5x'(t) - x(t) = 0$; and
(c) $y''(t) + 5y'(t) + 6y(t) = x'(t) + 11x(t)$.

**6.15** For each frequency response $H$ given below for a LTI system with input $x$ and output $y$, find the differential equation that characterizes the system.
(a) $H(\omega) = \dfrac{j\omega}{1+j\omega}$; and
(b) $H(\omega) = \dfrac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$.

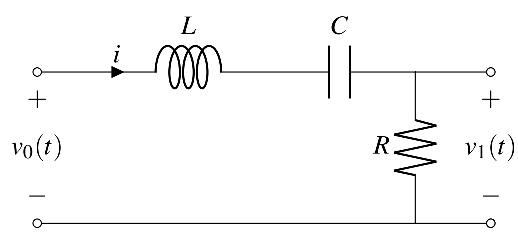
**6.16** For each case below, use frequency-domain methods to find the response $y$ of the LTI system with impulse response $h$ and frequency response $H$ to the input $x$.
(a) $h(t) = \delta(t) - 300\,\mathrm{sinc}(300\pi t)$ and $x(t) = \frac{1}{2} + \frac{3}{4}\cos(200\pi t) + \frac{1}{2}\cos(400\pi t) - \frac{1}{4}\cos(600\pi t)$.

**6.17** Consider the LTI resistor-inductor (RL) network with input $v_0$ and output $v_1$ as shown in the figure below.



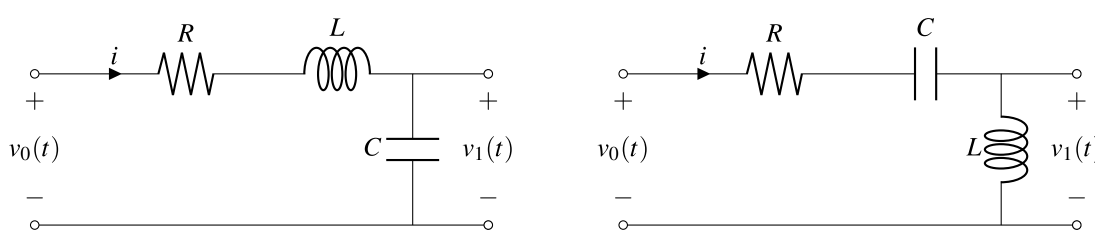

(a) Find the frequency response $H$ of the system.
(b) Determine the magnitude and phase responses of the system.
(c) Determine the type of frequency-selective filter that this system best approximates.
(d) Find $v_1$ in the case that $v_0(t) = \mathrm{sgn}\,t$.
(e) Find the impulse response $h$ of the system.

**6.18** Consider the LTI system with input $v_0$ and output $v_1$ as shown in the figure below, where $R = 1$, $C = \frac{1}{1000}$, and $L = \frac{1}{1000}$.




(a) Find the frequency response $H$ of the system.
(b) Use a computer to plot the magnitude and phase responses of the system.
(c) From the plots in part (b), identify the type of ideal filter that this system approximates.

**6.19** For each LTI circuit with input $v_0$ and output $v_1$, find the differential equation that characterizes the circuit, find the frequency response $H$ of the circuit, and determine which type of frequency-selective filter this circuit best approximates.




(a) (b)