System Failure Example: Tacoma Narrows Bridge

- The (original) Tacoma Narrows Bridge was a suspension bridge linking Tacoma and Gig Harbor (WA, USA).
- This mile-long bridge, with a 2,800-foot main span, was the third largest suspension bridge at the time of opening.
- Construction began in Nov. 1938 and took about 19 months to build at a cost of $6,400,000.
- On July 1, 1940, the bridge opened to traffic.
- On Nov. 7, 1940 at approximately 11:00, the bridge collapsed during a moderate (42 miles/hour) wind storm.
- The bridge was supposed to withstand winds of up to 120 miles/hour.
- The collapse was due to wind-induced vibrations and an ***unstable mechanical system***.
- Repair of the bridge was not possible.
- Fortunately, a dog trapped in an abandoned car was the only fatality.

Image of bridge collapse omitted for copyright reasons.

A video of the bridge collapse can be found at
https://youtu.be/j-zczJXSxnw.

Part 2

Preliminaries

Section 2.1

Functions, Sequences, System Operators, and Transforms

Sets

- A **rational number** is a number of the form x/y, where x and y are integers and $y \neq 0$ (i.e., a ratio of integers).
- For example, $-\frac{5}{3}$, $\frac{17}{11}$, and $0 = \frac{0}{1}$ are rational numbers, whereas π and e are irrational numbers (i.e., not rational).
- The symbols employed to denote several commonly-used sets are as follows:

Symbol	Set
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Q}$	rational numbers

Notation for Sets of Consecutive Integers

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- For two integers a and b, we define the following notation for sets of consecutive integers:

$$
\begin{aligned}
[a \,..\, b] &= \{x \in \mathbb{Z} : a \leq x \leq b\}, \\
[a \,..\, b) &= \{x \in \mathbb{Z} : a \leq x < b\}, \\
(a \,..\, b] &= \{x \in \mathbb{Z} : a < x \leq b\}, \quad \text{and} \\
(a \,..\, b) &= \{x \in \mathbb{Z} : a < x < b\}.
\end{aligned}
$$

- In this notation, a and b indicate the endpoints of the range for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicate whether each endpoint is included in the set.
- For example:
  - $[0 \,..\, 4]$ denotes the set of integers $\{0, 1, 2, 3, 4\}$;
  - $[0 \,..\, 4)$ denotes the set of integers $\{0, 1, 2, 3\}$; and
  - $[0 \,..\, N-1]$ and $[0 \,..\, N)$ both denote the set of integers $\{0, 1, 2, \ldots, N-1\}$.

Notation for Intervals on the Real Line

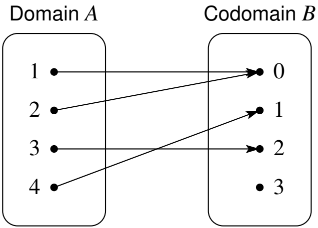
- For two real numbers a and b, we define the following notation for intervals on the real line:

$$\begin{aligned} [a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\ (a,b) &= \{x \in \mathbb{R} : a < x < b\}, \\ [a,b) &= \{x \in \mathbb{R} : a \leq x < b\}, \quad \text{and} \\ (a,b] &= \{x \in \mathbb{R} : a < x \leq b\}. \end{aligned}$$

- In this notation, a and b indicate the endpoints of the interval for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicate whether each endpoint is included in the set.
- For example:
  - $[0, 100]$ denotes the set of all real numbers from 0 to 100, including both 0 and 100;
  - $(-\pi, \pi]$ denotes the set of all real numbers from $-\pi$ to π, excluding $-\pi$ but including π; and
  - $[-\pi, \pi)$ denotes the set of all real numbers from $-\pi$ to π, including $-\pi$ but excluding π.

Mappings

- A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**.
- The notation $f : A \to B$ denotes a mapping f whose domain is the set A and whose codomain is the set B.
- Example:



$$f : A \to B$$
$$A = \{1,2,3,4\}$$
$$B = \{0,1,2,3\}$$
$$f(x) = \begin{cases} 0 & x \in \{1,2\} \\ 1 & x = 4 \\ 2 & x = 3. \end{cases}$$

- Although many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms.

Functions

- A **function** is a mapping where the domain is a set that is ***continuous*** in nature, such as the real numbers or complex numbers.
- In practice, the codomain is typically either the real numbers or complex numbers.
- Functions are also commonly referred to as **continuous-time (CT) signals**.
- Example:
  - Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = t^2$ (i.e., f is the squaring function).
  - The function f maps each real number t to the real number $f(t) = t^2$.
  - The domain and codomain are the real numbers.
  - Note that f is a ***function***, whereas $f(t)$ is a ***number*** (namely, the value of the function f evaluated at t).
- Herein, we will focus almost exclusively on functions of a single independent variable (i.e., one-dimensional functions).

Sequences

- A **sequence** is a mapping where the domain is a set that is ***discrete*** in nature, such as the integers, or a subset thereof.
- In practice, the codomain is typically either the real numbers or complex numbers.
- Sequences are also commonly referred to as **discrete-time (DT) signals**.
- Example:
  - Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n) = n^2$, where $\mathbb{Z}^+$ denotes the set of (strictly) positive integers (i.e., f is the sequence of perfect squares).
  - The sequence f maps each (strictly) positive integer n to the (strictly) positive integer $f(n) = n^2$.
  - The domain and codomain are $\mathbb{Z}^+$ (i.e., the positive integers).
  - Note that f is a ***sequence***, whereas $f(n)$ is a ***number*** (namely, the value of the sequence f evaluated at n).
- As a matter of notation, the nth element of a sequence x is denoted as either $x(n)$ or x_n.
- Herein, we will focus almost exclusively on sequences with a single independent variable (i.e., one-dimensional sequences).

Remarks on Notation for Functions and Sequences

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- For a real-valued function f of a real variable and an arbitrary real number t, the expression f denotes the function f itself and the expression $f(t)$ denotes the value of the function f evaluated at t.
- That is, f is a ***function*** and $f(t)$ is a ***number***.
- Unfortunately, the practice of using $f(t)$ to denote the function f is quite common, although strictly speaking this is an abuse of notation.
- In contexts where imprecise notation may lead to problems, one should be careful to clearly distinguish between a function and its value.
- For the real-valued functions f and g of a real variable and an arbitrary real number t:
  - The expression $f+g$ denotes a ***function***, namely, the function formed by adding the functions f and g.
  - The expression $f(t)+g(t)$ denotes a ***number***, namely, the sum of: 1) the value of the function f evaluated at t; and 2) the value of the function g evaluated at t.
- Similar comments as the ones made above for functions also hold in the case of sequences.

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- To express that two functions f and g are equal, we can write either:
  1. $f = g$; or
  2. $f(t) = g(t)$ for all t.
- Of the preceding two expressions, the first (i.e., $f = g$) is usually preferable, as it is less verbose.
- For the functions f and g and an operation $\circ$ that is defined pointwise for functions (such as addition, subtraction, multiplication, and division), the following relationship holds:

$$(f \circ g)(t) = f(t) \circ g(t).$$

- Some operations $\circ$ involving functions (such as convolution, to be discussed later) cannot be defined in a pointwise manner, in which case $(f \circ g)(t)$ is a valid mathematical expression, while $f(t) \circ g(t)$ is not.
- Again, similar comments as the ones made above for functions also hold in the case of sequences.

System Operators

- A **system operator** is a mapping used to represent a system.
- We will focus exclusively on the case of single-input single-output systems.
- A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system.
- The domain and codomain of a system operator are sets of ***functions or sequences***, not sets of numbers.
- Example:
  - Let $\mathcal{H} : F \to F$ such that $\mathcal{H}x(t) = 2x(t)$ (for all $t \in \mathbb{R}$) and F is the set of functions mapping $\mathbb{R}$ to $\mathbb{R}$.
  - The system $\mathcal{H}$ maps a function to a function.
  - In particular, the domain and codomain are each F, which is a set of functions.
  - The system $\mathcal{H}$ multiplies its input function x by a factor of 2 in order to produce its output function $\mathcal{H}x$.
  - Note that $\mathcal{H}x$ is a function, not a number.

Remarks on Operator Notation for CT Systems

- For a system operator $\mathcal{H}$ and a function x, $\mathcal{H}x$ is the function produced as the output of the system $\mathcal{H}$ when the input is the function x.
- Brackets around the operand of an operator are ***often omitted when not required*** for grouping.
- For example, for an operator $\mathcal{H}$, a function x, and a real number t, we would normally prefer to write:
  1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
  2. $\mathcal{H}x(t)$ instead of the equivalent expression $\mathcal{H}(x)(t)$.
- Also, note that $\mathcal{H}x$ is a ***function*** and $\mathcal{H}x(t)$ is a ***number*** (namely, the value of the function $\mathcal{H}x$ evaluated at t).
- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").
- When multiple operators are applied, they group from ***right to left***.
- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the function x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

Example 2.6 Example 2.7

Remarks on Operator Notation for DT Systems

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- For a system operator $\mathcal{H}$ and a sequence x, $\mathcal{H}x$ is the sequence produced as the output of the system $\mathcal{H}$ when the input is the sequence x.
- Brackets around the operand of an operator are ***often omitted when not required*** for grouping.
- For example, for an operator $\mathcal{H}$, a sequence x, and an integer n, we would normally prefer to write:
  1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
  2. $\mathcal{H}x(n)$ instead of the equivalent expression $\mathcal{H}(x)(n)$.
- Also, note that $\mathcal{H}x$ is a ***sequence*** and $\mathcal{H}x(n)$ is a ***number*** (namely, the value of the sequence $\mathcal{H}x$ evaluated at n).
- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").
- When multiple operators are applied, they group from ***right to left***.
- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the sequence x, the expression $\mathcal{H}_2\mathcal{H}_1 x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

Transforms

- Later, we will be introduced to several types of mappings known as transforms.
- Transforms have a mathematical structure similar to system operators.
- That is, transforms map functions/sequences to functions/sequences.
- Due to this similar structure, many of the earlier comments about system operators also apply to the case of transforms.
- For example, the Fourier transform (introduced later) is denoted as $\mathcal{F}$ and the result of applying the Fourier transform operator to the function/sequence x is denoted as $\mathcal{F}x$.
- Some examples of transforms of interest in the study of signals and systems are listed on the next slide.

Examples of Transforms

Name	Domain	Codomain
CT Fourier Series	T-periodic functions (with domain $\mathbb{R}$)	sequences (with domain $\mathbb{Z}$)
CT Fourier Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{R}$)
Laplace Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{C}$)
DT Fourier Series	N-periodic sequences (with domain $\mathbb{Z}$)	N-periodic sequences (with domain $\mathbb{Z}$)
DT Fourier Transform	sequences (with domain $\mathbb{Z}$)	2π-periodic functions (with domain $\mathbb{R}$)
Z Transform	sequences (with domain $\mathbb{Z}$)	functions (with domain $\mathbb{C}$)

Section 2.2

Properties of Signals

Even Symmetry

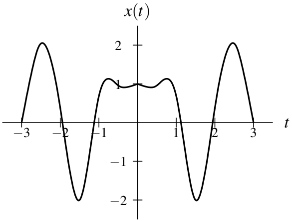
- A function x is said to be **even** if it satisfies

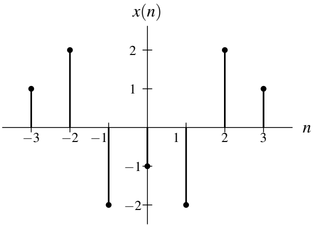
$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Geometrically, the graph of an even signal is ***symmetric*** with respect to the vertical axis.
- Some examples of even signals are shown below.





Odd Symmetry

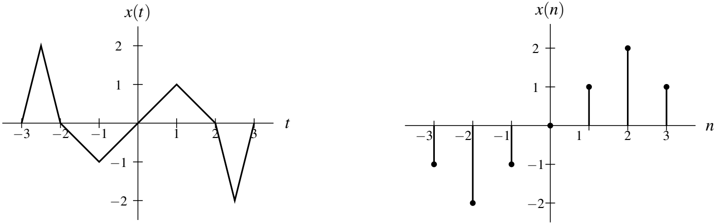
- A function x is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- An odd signal x must be such that $x(0) = 0$.
- Geometrically, the graph of an odd signal is ***symmetric*** with respect to the origin.
- Some examples of odd signals are shown below.



- A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- The real part of a conjugate symmetric function or sequence is even.
- The imaginary part of a conjugate symmetric function or sequence is odd.
- An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos \omega t + j \sin \omega t$, where ω is a real constant.

Periodicity

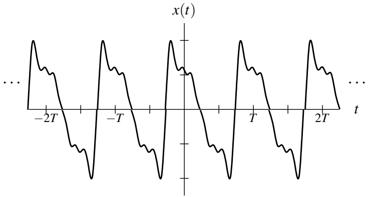
- A function x is said to be **periodic** with **period** T (or T**-periodic**) if, for some strictly-positive real constant T, the following condition holds:

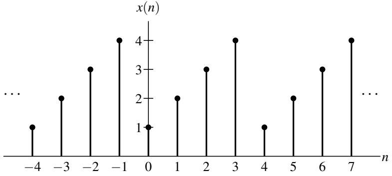
$$x(t) = x(t+T) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **periodic** with **period** N (or N**-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

$$x(n) = x(n+N) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Some examples of periodic signals are shown below.



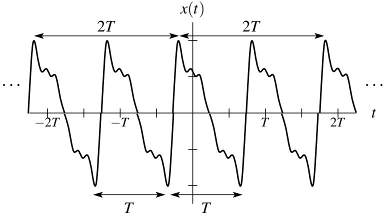


Periodicity (Continued 1)

- A function/sequence that is not periodic is said to be **aperiodic**.
- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.

Periodicity (Continued 2)

- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Part 3

Continuous-Time (CT) Signals and Systems