Example 6.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform X of the function

$$x(t) = A\delta(t - t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

substitute given x into Fourier transform analysis equation

$$X(\omega) = \int_{-\infty}^{\infty} A\delta(t - t_0) e^{-j\omega t} dt$$

pull constant A out of integral

$$= A \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt.$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

sifting property

$= Ae^{-j\omega t}\big|_{t=t_0}$

$$X(\omega) = Ae^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{where} \quad X(\omega) = Ae^{-j\omega t_0}. \qquad \blacksquare$$

Example 6.3 (Fourier transform of the rectangular function)**.** Find the Fourier transform X of the function

$$x(t) = \operatorname{rect} t.$$

$$\left[\operatorname{rect} t = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \right]$$

Solution. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t} dt.$$

substitute given function x into Fourier transform analysis equation

From the definition of the rectangular function, we can simplify this equation to obtain

change limits since rect $t = 0$ for $|t| > \frac{1}{2}$

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

rect $t = 1$ for t in integration interval

Evaluating the integral and simplifying, we have

integrate

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t} \right] \Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \tfrac{1}{j\omega} \left[2j \sin\left(\tfrac{\omega}{2}\right) \right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{\omega}{2}\right) \\ &= \left[\sin\left(\tfrac{\omega}{2}\right) \right] / \left(\tfrac{\omega}{2}\right) \\ &= \operatorname{sinc}\left(\tfrac{\omega}{2}\right). \end{aligned}$$

$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

rewrite in form of sinc

definition of sinc function

Thus, we have shown that

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\tfrac{\omega}{2}\right).$$

Note: This is why the sinc function is of great importance. ■

Example 6.6. Consider the function x shown in Figure 6.5. Let $\hat{x}$ denote the Fourier transform representation of x (i.e., $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, where X denotes the Fourier transform of x). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

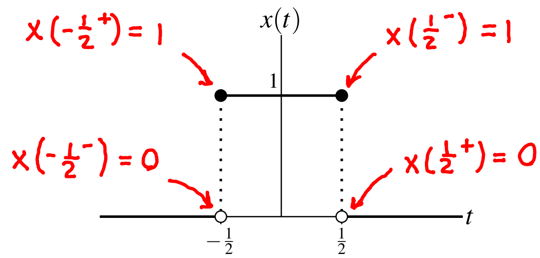

Figure 6.5: Function x.

At a point of discontinuity, the Fourier transform representation converges to the average of the left and right limits.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 6.3 applies. Thus, we have that

$$\begin{aligned}\hat{x}(-\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right] \\ &= \tfrac{1}{2}(0+1) \\ &= \tfrac{1}{2} \quad \text{and}\end{aligned}$$

average of left and right limits

$$\begin{aligned}\hat{x}(\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right] \\ &= \tfrac{1}{2}(1+0) \\ &= \tfrac{1}{2}.\end{aligned}$$

average of left and right limits

■

Example 6.7 (Linearity property of the Fourier transform)**.** Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0), \quad 1$$

find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t),$$

where A and ω_0 are real constants.

Solution. We recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha}+e^{-j\alpha}]$ (2) for any real α. Thus, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{A\cos(\omega_0 t)\})(\omega) \\ &= \left(\mathcal{F}\{\tfrac{A}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})\}\right)(\omega). \end{aligned}$$

from Euler 2

Then, we use the linearity property of the Fourier transform to obtain

linearity

$$X(\omega) = \tfrac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \tfrac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

Using the given Fourier transform pair, we can further simplify the above expression for $X(\omega)$ as follows:

from given FT pair 1

$$\begin{aligned} X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega+\omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega-\omega_0)] \\ &= A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)].$$

■