Taking the Laplace transform of both sides of the preceding equation, we obtain

$$\begin{aligned}
X(s) &= -\mathcal{L}\{-tu(t+1)\}(s) + 2\mathcal{L}\{u(t+1)\}(s) + 2\mathcal{L}\{-tu(t)\}(s) - \mathcal{L}\{-tu(t-1)\}(s) - 2\mathcal{L}\{u(t-1)\}(s) \\
&= -\tfrac{d}{ds}\left(\frac{e^s}{s}\right) + 2\left(\frac{e^s}{s}\right) + 2\tfrac{d}{ds}\left(\frac{1}{s}\right) - \tfrac{d}{ds}\left(\frac{e^{-s}}{s}\right) - 2\left(\frac{e^{-s}}{s}\right) \\
&= -\left(\frac{se^s - e^s}{s^2}\right) + 2\left(\frac{e^s}{s}\right) + 2\left(-\frac{1}{s^2}\right) - \left(\frac{s(-e^{-s}) - e^{-s}}{s^2}\right) - 2\left(\frac{e^{-s}}{s}\right) \\
&= \frac{-se^s + e^s + se^{-s} + e^{-s} - 2 + 2se^s - 2e^{-s}}{s^2} \\
&= \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2}.
\end{aligned}$$

Since x is (bounded and) finite duration, the ROC of X must be the entire complex plane. Thus, we have that

$$X(s) = \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2} \quad \text{for all } s.$$

Second solution (which incurs less work by avoiding differentiation). Alternatively, we can rearrange our above expression for x to obtain

$$\begin{aligned}
x(t) &= [tu(t+1) + u(t+1) - u(t+1)] + 2u(t+1) - 2tu(t) + [tu(t-1) - u(t-1) + u(t-1)] - 2u(t-1) \\
&= (t+1)u(t+1) + u(t+1) - 2tu(t) + (t-1)u(t-1) - u(t-1).
\end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we have

$$\begin{aligned}
X(s) &= e^s\mathcal{L}\{tu(t)\}(s) + \mathcal{L}\{u(t+1)\}(s) - 2\mathcal{L}\{tu(t)\}(s) + e^{-s}\mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{u(t-1)\}(s) \\
&= e^s\tfrac{1}{s^2} + e^s\tfrac{1}{s} - 2\tfrac{1}{s^2} + e^{-s}\tfrac{1}{s^2} - e^{-s}\tfrac{1}{s} \\
&= \frac{e^s + se^s - 2 + e^{-s} - se^{-s}}{s^2}.
\end{aligned}$$

In the case of this alternate solution, the expression for X is considerably easier to simplify. ■

7.10 Determination of the Inverse Laplace Transform

As suggested earlier, in practice, we rarely use (7.3) directly in order to compute the inverse Laplace transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions for which we can usually find the inverse Laplace transform in a table (e.g., such as Table 7.2). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions is provided in Appendix B for those who might not be acquainted with such expansions.

Example 7.27. Find the inverse Laplace transform x of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \operatorname{Re}(s) < 2.$$

Solution. We begin by rewriting X in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

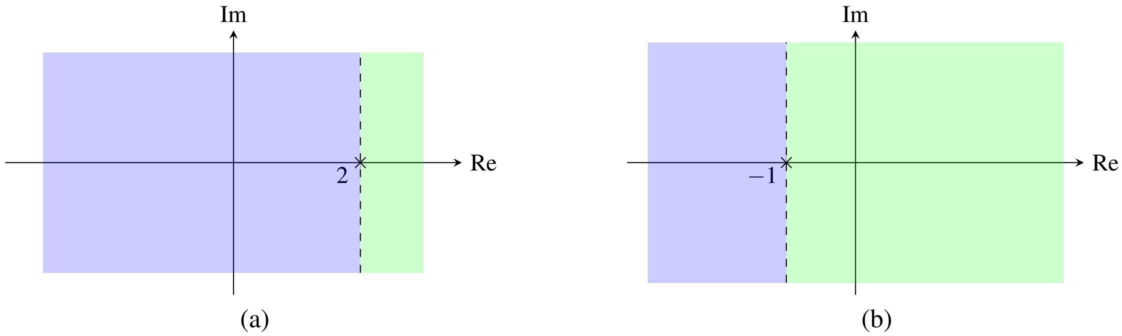

Figure 7.24: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s-2}$; and (b) $\frac{1}{s+1}$.

Then, we find a partial fraction expansion of X. We know that X has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} = \left.\frac{2}{s-2}\right|_{s=-1} = -\tfrac{2}{3} \quad \text{and} \\ A_2 &= (s-2)X(s)|_{s=2} = \left.\frac{2}{s+1}\right|_{s=2} = \tfrac{2}{3}. \end{aligned}$$

So, X has the expansion

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

$$x(t) = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t). \tag{7.6}$$

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the two inverse Laplace transforms appearing in (7.6), we must associate a ROC with each of the two expressions $\frac{1}{s-2}$ and $\frac{1}{s+1}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.24. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $-1 < \operatorname{Re}(s) < 2$). Using Table 7.2, we have

$$\begin{aligned} -e^{2t}u(-t) &\stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s-2} \quad \text{for } \operatorname{Re}(s) < 2 \quad \text{and} \\ e^{-t}u(t) &\stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

Substituting these results into (7.6), we obtain

$$\begin{aligned} x(t) &= \tfrac{2}{3}[-e^{2t}u(-t)] - \tfrac{2}{3}[e^{-t}u(t)] \\ &= -\tfrac{2}{3}e^{2t}u(-t) - \tfrac{2}{3}e^{-t}u(t). \end{aligned}$$

■

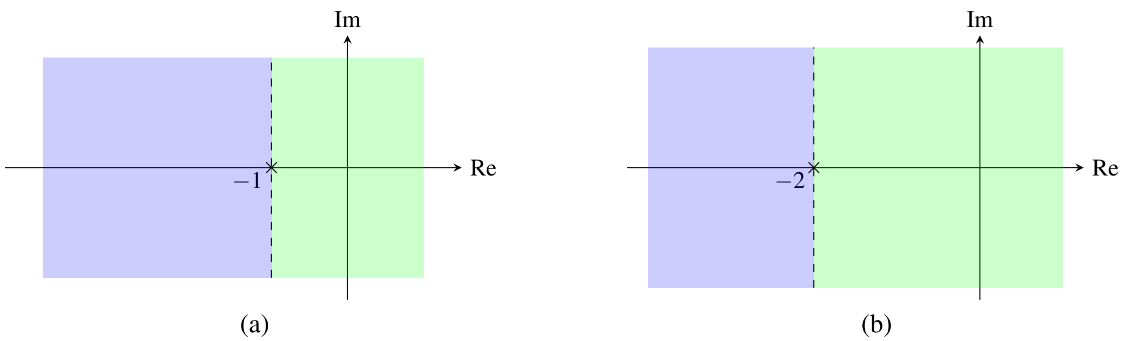

Figure 7.25: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s+1}$ and $\frac{1}{(s+1)^2}$; and (b) $\frac{1}{s+2}$.

Example 7.28 (Rational function with a repeated pole)**.** Find the inverse Laplace transform x of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

Solution. To begin, we find a partial fraction expansion of X. We know that X has an expansion of the form

$$X(s) = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned}
A_{1,1} &= \tfrac{1}{(2-1)!}\left[\left(\tfrac{d}{ds}\right)^{2-1}[(s+1)^2X(s)]\right]\Big|_{s=-1} = \tfrac{1}{1!}\left[\tfrac{d}{ds}[(s+1)^2X(s)]\right]\Big|_{s=-1} = \left[\tfrac{d}{ds}\left(\frac{2s+1}{s+2}\right)\right]\Big|_{s=-1} \\
&= \left[\frac{(s+2)(2)-(2s+1)(1)}{(s+2)^2}\right]\Big|_{s=-1} = \left[\frac{2s+4-2s-1}{(s+2)^2}\right]\Big|_{s=-1} = \left[\frac{3}{(s+2)^2}\right]\Big|_{s=-1} = 3, \\
A_{1,2} &= \tfrac{1}{(2-2)!}\left[\left(\tfrac{d}{ds}\right)^{2-2}\left[(s+1)^2X(s)\right]\right]\Big|_{s=-1} = \tfrac{1}{0!}\left[(s+1)^2X(s)\right]\Big|_{s=-1} = \frac{2s+1}{s+2}\Big|_{s=-1} = \tfrac{-1}{1} = -1, \quad \text{and} \\
A_{2,1} &= (s+2)X(s)|_{s=-2} = \frac{2s+1}{(s+1)^2}\Big|_{s=-2} = \tfrac{-3}{1} = -3.
\end{aligned}$$

Thus, X has the expansion

$$X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.7}$$

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the three inverse Laplace transforms appearing in (7.7), we must associate a ROC with each of the three expressions $\frac{1}{s+1}$, $\frac{1}{(s+1)^2}$, and $\frac{1}{s+2}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.25. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $\operatorname{Re}(s) > -1$). From Table 7.2, we have

$$e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1,$$
$$te^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{(s+1)^2} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and}$$
$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Substituting these results into (7.7), we obtain

$$\begin{aligned} x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t) \\ &= \left(3e^{-t} - te^{-t} - 3e^{-2t}\right)u(t). \end{aligned}$$

■

Example 7.29 (Inverse Laplace transform of improper rational function)**.** Find the inverse Laplace transform x of

$$X(s) = \frac{2s^2+4s+5}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

Solution. We begin by observing that, although X is rational, it is not strictly proper. So, we need to perform polynomial long division in order to express X as the sum of a polynomial and a strictly-proper rational function. By long division, we have

$$\begin{array}{r|l} & \phantom{2s^2 + 4s +{}} 2 \\ \cline{2-2} s^2+3s+2 & 2s^2 + 4s + 5 \\ & \underline{2s^2 + 6s + 4} \\ & - 2s + 1. \end{array}$$

In other words, we have

$$X(s) = 2 + \frac{-2s+1}{s^2+3s+2}.$$

For convenience, we define

$$V(s) = \frac{-2s+1}{(s+1)(s+2)}$$

so that

$$X(s) = 2 + V(s).$$

Observe that V is a strictly-proper rational function. So, we can find a partial fraction expansion of V. Now, we find this expansion. We know that such an expansion is of the form

$$V(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned} A_1 &= (s+1)V(s)|_{s=-1} \\ &= \left.\frac{-2s+1}{s+2}\right|_{s=-1} \\ &= 3 \quad \text{and} \\ A_2 &= (s+2)V(s)|_{s=-2} \\ &= \left.\frac{-2s+1}{s+1}\right|_{s=-2} \\ &= -5 \end{aligned}$$

So, we have

$$\begin{aligned} X(s) &= 2 + V(s) \\ &= 2 + \frac{3}{s+1} - \frac{5}{s+2}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}X(t) \\ &= 2\mathcal{L}^{-1}\{1\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 5\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \end{aligned}$$

Considering the ROC of X, we can obtain the following from Table 7.2:

$$\begin{aligned} \delta(t) &\stackrel{\text{LT}}{\longleftrightarrow} 1, \\ e^{-t}u(t) &\stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and} \\ e^{-2t}u(t) &\stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2. \end{aligned}$$

Finally, we can write

$$\begin{aligned} x(t) &= 2\delta(t) + 3e^{-t}u(t) - 5e^{-2t}u(t) \\ &= 2\delta(t) + (3e^{-t} - 5e^{-2t})u(t). \end{aligned}$$

■

Example 7.30. Find all possible inverse Laplace transforms of

$$X(s) = \frac{1}{s^2+3s+2}. \tag{7.8}$$

Solution. We begin by rewriting X in factored form as

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

Then, we find the partial fraction expansion of X. We know that such an expansion has the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the coefficients of the expansion, we have

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = 1 \quad \text{and} \\ A_2 &= (s+2)X(s)|_{s=-2} = \left.\frac{1}{s+1}\right|_{s=-2} = -1. \end{aligned}$$

So, we have

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.9}$$

For the Laplace transform X, three possible ROCs exist:

i) $\mathrm{Re}(s) < -2$,
ii) $-2 < \mathrm{Re}(s) < -1$, and
iii) $\mathrm{Re}(s) > -1$.

Thus, three possible inverse Laplace transforms exist for X, depending on the choice of ROC.

First, let us consider the case of the ROC $\mathrm{Re}(s) < -2$. From Table 7.2, we have

$$-e^{-t}u(-t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \mathrm{Re}(s) < -1 \quad \text{and}$$
$$-e^{-2t}u(-t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \mathrm{Re}(s) < -2.$$

Substituting these results into (7.9), we have

$$\begin{aligned} x(t) &= -e^{-t}u(-t) + e^{-2t}u(-t) \\ &= \left(-e^{-t} + e^{-2t}\right)u(-t). \end{aligned}$$

Second, let us consider the case of the ROC $-2 < \mathrm{Re}(s) < -1$. From Table 7.2, we have

$$-e^{-t}u(-t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \mathrm{Re}(s) < -1 \quad \text{and}$$
$$e^{-2t}u(t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \mathrm{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t).$$

Third, let us consider the case of the ROC $\mathrm{Re}(s) > -1$. From Table 7.2, we have

$$e^{-t}u(t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \mathrm{Re}(s) > -1 \quad \text{and}$$
$$e^{-2t}u(t) \stackrel{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \mathrm{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$\begin{aligned} x(t) &= e^{-t}u(t) - e^{-2t}u(t) \\ &= \left(e^{-t} - e^{-2t}\right)u(t). \end{aligned}$$

■

7.11 Characterizing LTI Systems Using the Laplace Transform

Consider a LTI system with input x, output y, and impulse response h, as depicted in Figure 7.26. Such a system is characterized by the equation

$$y(t) = x * h(t).$$

Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively. Taking the Laplace transform of both sides of the above equation and using the time-domain convolution property of the Laplace transform, we have

$$Y(s) = H(s)X(s).$$

The quantity H is known as the **system function** or **transfer function** of the system. If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the frequency response of the system. The system can be represented with a block diagram labelled in the Laplace domain as shown in Figure 7.27, where the system is labelled by its system function H.

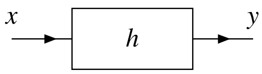

Figure 7.26: Time-domain view of a LTI system with input x, output y, and impulse response h.

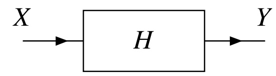

Figure 7.27: Laplace-domain view of a LTI system with input Laplace transform X, output Laplace transform Y, and system function H.

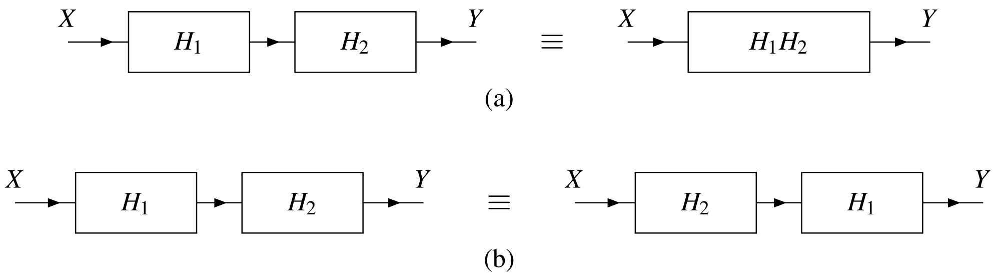

Figure 7.28: Equivalences involving system functions and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

7.12 Interconnection of LTI Systems

From the properties of the Laplace transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 7.28(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Laplace transform of both sides of this equation yields

$$\begin{aligned} H(s) &= \mathcal{L}\{h_1 * h_2\}(s) \\ &= \mathcal{L}h_1(s)\mathcal{L}h_2(s) \\ &= H_1(s)H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 7.28(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 7.28(b).

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 7.29. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

Taking the Laplace transform of both sides of the equation yields

$$\begin{aligned} H(s) &= \mathcal{L}\{h_1 + h_2\}(s) \\ &= \mathcal{L}h_1(s) + \mathcal{L}h_2(s) \\ &= H_1(s) + H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 7.29.

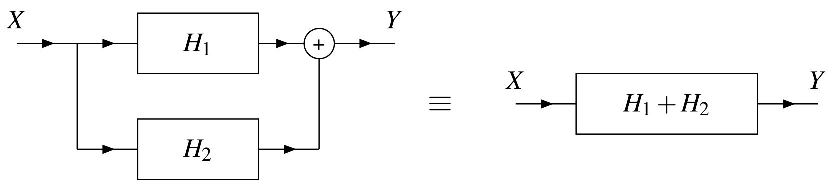

Figure 7.29: Equivalence involving system functions and the parallel interconnection of LTI systems.

7.13 System Function and System Properties

Many properties of a system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

7.13.1 Causality

From Theorem 4.8, we know that, a LTI system is causal if its impulse response is causal. A causal function, however, is inherently right sided. Consequently, from the properties of the ROC of the Laplace transform discussed in Section 7.7, we have the theorem below.

Theorem 7.12. *The ROC associated with the system function of a causal LTI system is a right-half plane or the entire complex plane.*

Proof. Let h and H denote the impulse response and system function of a causal LTI system $\mathcal{H}$. The system $\mathcal{H}$ is causal if and only if h is causal. So, $\mathcal{H}$ being causal implies that h is causal, which in turn implies (by definition) that h is right sided. From the properties of the ROC, h being right sided implies that the ROC of H is either a right-half plane (in the case that h is not left sided) or the entire complex plane (in the case that h is left sided). Thus, the ROC of H has the stated form. ■

In general, the converse of the above theorem is not necessarily true. This is, it is not always true that a system function H with a ROC that is either a right-half plane or the entire complex plane is associated with a causal system. If H is rational, however, we have that the converse does hold, as indicated by the theorem below.

Theorem 7.13. *For a LTI system with a rational system function H, causality of the system is equivalent to the ROC of H being the right-half plane to the right of the rightmost pole or, if H has no poles, the entire complex plane.*

Proof. The proof is left as an exercise for the reader. ■

Example 7.31. For the LTI system with each system function H below, determine whether the system is causal.

(a) $H(s) = \dfrac{1}{s+1}$ for $\operatorname{Re}(s) > -1$;

(b) $H(s) = \dfrac{1}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;

(c) $H(s) = \dfrac{e^s}{s+1}$ for $\operatorname{Re}(s) < -1$; and

(d) $H(s) = \dfrac{e^s}{s+1}$ for $\operatorname{Re}(s) > -1$.

Solution. (a) The poles of H are plotted in Figure 7.30(a) and the ROC is indicated by the shaded area. The system function H is rational and the ROC is the right-half plane to the right of the rightmost pole. Therefore, the system is causal.

(b) The poles of H are plotted in Figure 7.30(b) and the ROC is indicated by the shaded area. The system function is rational but the ROC is not a right-half plane or the entire complex plane. Therefore, the system is not causal.

(c) The system function H has a left-half plane ROC. Therefore, h is left sided but not right sided. Thus, the system is not causal.

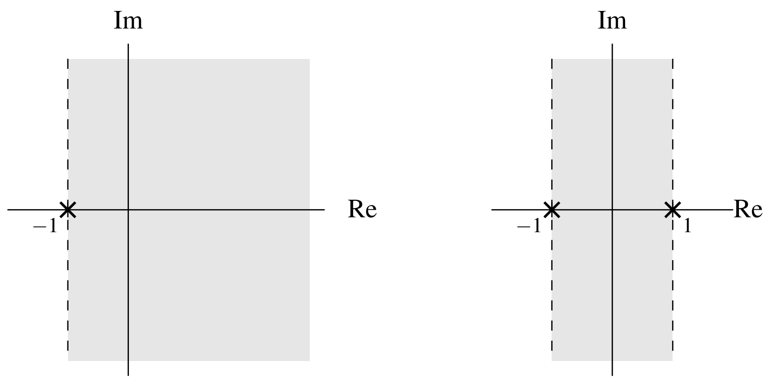

Figure 7.30: Pole and ROCs of the rational system functions in the causality example. The cases of the (a) first (b) second system functions.

(d) The system function H has a right-half plane ROC but is not rational. Thus, we cannot make any conclusion directly from (the ROC of) the system function. Instead, we draw our conclusion from the impulse response h. Taking the inverse Laplace transform of H, we obtain

$$h(t) = e^{-(t+1)}u(t+1).$$

Thus, the impulse response h is not causal. Therefore, the system is not causal. ■

7.13.2 BIBO Stability

In this section, we consider the relationship between the system function and BIBO stability. The first important result is given by the theorem below.

Theorem 7.14. *A LTI system is BIBO stable if and only if the ROC of its system function H contains the imaginary axis (i.e.,* $\operatorname{Re}(s) = 0$*).*

Proof. We present only a partial proof. In particular, we show that the ROC of H containing the imaginary axis is a necessary condition for BIBO stability. In what follows, let h denote the inverse Laplace transform of H (i.e., h is the impulse response of the system).

Suppose that the system is BIBO stable. From earlier in Theorem 4.11, we know that this implies

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

(i.e., h is absolutely integrable). From the definition of H (evaluated on the imaginary axis), we have

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt. \tag{7.10}$$

Recall that a function is integrable if it is absolutely integrable. That is, for any function f,

$$\int_{-\infty}^{\infty} f(t)dt \text{ converges} \quad \text{if } \int_{-\infty}^{\infty} |f(t)|\,dt \text{ converges.}$$

From this relationship and (7.10), we can infer that

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \text{ converges} \quad \text{if } \int_{-\infty}^{\infty} \left|h(t)e^{-j\omega t}\right| dt \text{ converges.} \tag{7.11}$$

We have, however, that

$$\int_{-\infty}^{\infty} \left|h(t)e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)| \left|e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)|\, dt.$$

So, we can rewrite (7.11) as

$$H(j\omega) \text{ converges} \quad \text{if} \int_{-\infty}^{\infty} |h(t)|\, dt \text{ converges.}$$

The condition for convergence in the preceding statement, however, is always satisfied, since $\int_{-\infty}^{\infty} |h(t)|\, dt$ must converge for a BIBO stable system (as noted above). Therefore, we conclude that $H(j\omega)$ must converge for all ω (i.e., the ROC of H must contain the imaginary axis) if the system is BIBO stable. Thus, the ROC of H containing the imaginary axis is a necessary condition for BIBO stability. ■

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

Theorem 7.15. *A causal LTI system with a (proper) rational system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have negative real parts).*

Proof. The proof is left as an exercise for the reader. ■

Observe from the preceding two theorems (i.e., Theorems 7.14 and 7.15) that, in the case of a LTI system, the characterization of the BIBO stability property is much simpler in the Laplace domain (via the system function) than the time domain (via the impulse response). For this reason, analyzing the stability of LTI systems is typically performed using the Laplace transform.

Example 7.32. A LTI system has the system function

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

Given that the system is BIBO stable, determine the ROC of H.

Solution. Clearly, the system function H is rational with poles at -1 and -2. Therefore, only three possibilities exist for the ROC:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$, and
iii) $\operatorname{Re}(s) > -1$.

In order for the system to be BIBO stable, however, the ROC of H must include the entire imaginary axis. Therefore, the ROC must be $\operatorname{Re}(s) > -1$. This ROC is illustrated in Figure 7.31. ■

Example 7.33. A LTI system is causal and has the system function

$$H(s) = \frac{1}{(s+2)(s^2+2s+2)}.$$

Determine whether this system is BIBO stable.

Solution. We begin by factoring H to obtain

$$H(s) = \frac{1}{(s+2)(s+1-j)(s+1+j)}.$$

(Using the quadratic formula, one can confirm that $s^2+2s+2=0$ has roots at $s=-1\pm j$.) Thus, H has poles at -2, $-1+j$, and $-1-j$. The poles are plotted in Figure 7.32. Since the system is causal and all of the poles of H are in the left half of the plane, the system is BIBO stable. ■

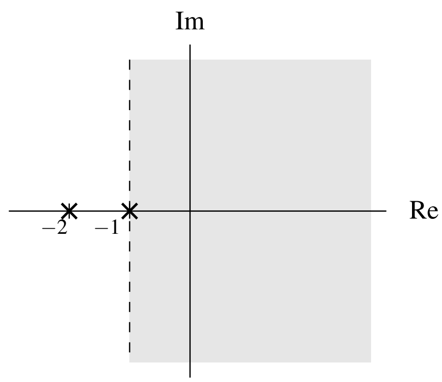

Figure 7.31: ROC for example.

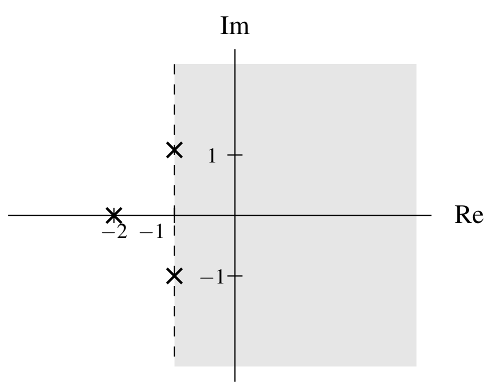

Figure 7.32: Poles of the system function.

Example 7.34. For each LTI system with system function H given below, determine the ROC of H that corresponds to a BIBO stable system.
(a) $H(s) = \dfrac{s(s-1)}{(s+2)(s+1+j)(s+1-j)}$;
(b) $H(s) = \dfrac{s}{(s+1)(s-1)(s-1-j)(s-1+j)}$;
(c) $H(s) = \dfrac{(s+j)(s-j)}{(s+2-j)(s+2+j)}$; and
(d) $H(s) = \dfrac{s-1}{s}$.

Solution. (a) The function H has poles at -2, $-1+j$, and $-1-j$. The poles are shown in Figure 7.33(a). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:
i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$, and
iii) $\operatorname{Re}(s) > -1$.
Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\operatorname{Re}(s) > -1$. This is the shaded region in the Figure 7.33(a).

(b) The function H has poles at -1, 1, $1+j$, and $1-j$. The poles are shown in Figure 7.33(b). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:
i) $\operatorname{Re}(s) < -1$,
ii) $-1 < \operatorname{Re}(s) < 1$, and
iii) $\operatorname{Re}(s) > 1$.
Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $-1 < \operatorname{Re}(s) < 1$. This is the shaded region in Figure 7.33(b).

(c) The function H has poles at $-2+j$ and $-2-j$. The poles are shown in Figure 7.33(c). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only two distinct ROCs are possible:
i) $\operatorname{Re}(s) < -2$ and
ii) $\operatorname{Re}(s) > -2$.
Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\operatorname{Re}(s) > -2$. This is the shaded region in Figure 7.33(c).

(d) The function H has a pole at 0. The pole is shown in Figure 7.33(d). Since H is rational, it cannot converge at 0 (which is a pole of H). Consequently, the ROC can never include the entire imaginary axis. Therefore, the system function H can never be associated with a BIBO stable system. ■

7.13.3 Invertibility

In this section, we consider the relationship between the system function and invertibility. The first important result is given by the theorem below.

Theorem 7.16 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with system function H. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has a system function H_{inv} that satisfies*

$$H(s)H_{\mathsf{inv}}(s) = 1. \tag{7.12}$$

Proof. Let h denote the inverse Laplace transform of H. From Theorem 4.9, we know that the system $\mathcal{H}$ is invertible if and only if there exists another LTI system with impulse response h_{inv} satisfying

$$h * h_{\mathsf{inv}} = \delta.$$

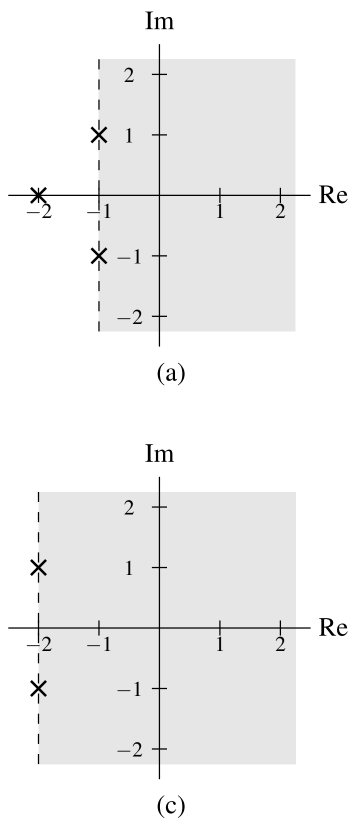

Figure 7.33: Poles and ROCs of the system function H in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

Let H_{inv} denote the Laplace transform of h_{inv}. Taking the Laplace transform of both sides of the above equation, we have

$$\mathcal{L}\{h * h_{\mathsf{inv}}\} = \mathcal{L}\delta.$$

From the time-domain convolution property of the Laplace transform and Table 7.2 (i.e., $\mathcal{L}\delta(s) = 1$), we obtain

$$H(s)H_{\mathsf{inv}}(s) = 1. \qquad \blacksquare$$

From the preceding theorem, we have the result below.

Theorem 7.17 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists a function H_{inv} satisfying*

$$H(s)H_{\mathsf{inv}}(s) = 1.$$

Proof. The proof follows immediately from the result of Theorem 7.16 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. $\blacksquare$

From the above theorems, we have that a LTI system $\mathcal{H}$ with system function H has an inverse if and only if a solution for H^{inv} exists in (7.12). Furthermore, if an inverse system exists, its system function is given by

$$H_{\mathsf{inv}}(s) = \frac{1}{H(s)}.$$

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of stability and/or causality).

Example 7.35. Consider the LTI system with system function

$$H(s) = \frac{s+1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Determine all possible inverses of this system. Comment on the BIBO stability of each of these inverse systems.

Solution. The system function H_{inv} of the inverse system is given by

$$H_{\mathsf{inv}}(s) = \frac{1}{H(s)} = \frac{s+2}{s+1}.$$

Two ROCs are possible for H_{inv}:

i) $\operatorname{Re}(s) < -1$ and
ii) $\operatorname{Re}(s) > -1$.

Each ROC is associated with a distinct inverse system. The first ROC is associated with a system that is not BIBO stable, since this ROC does not contain the imaginary axis. The second ROC is associated with a BIBO stable system, since this ROC contains the imaginary axis. $\blacksquare$

7.14 LTI Systems and Differential Equations

Many LTI systems of practical interest can be described by Nth-order linear differential equations with constant coefficients. Such a system with input x and output y can be characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t), \tag{7.13}$$

where $M \leq N$. Let X and Y denote the Laplace transforms of x and y, respectively. Let H denote the system function of the system. Taking the Laplace transform of both sides of the above equation, we obtain

$$\mathcal{L}\left\{\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t)\right\}(s) = \mathcal{L}\left\{\sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)\right\}(s).$$

Using the linearity property of the Laplace transform, we can rewrite this equation as

$$\sum_{k=0}^{N} b_k \mathcal{L}\left\{\left(\tfrac{d}{dt}\right)^k y(t)\right\}(s) = \sum_{k=0}^{M} a_k \mathcal{L}\left\{\left(\tfrac{d}{dt}\right)^k x(t)\right\}(s).$$

Using the time differentiation property of the Laplace transform, we have

$$\sum_{k=0}^{N} b_k s^k Y(s) = \sum_{k=0}^{M} a_k s^k X(s).$$

Factoring, we have

$$Y(s) \sum_{k=0}^{N} b_k s^k = X(s) \sum_{k=0}^{M} a_k s^k.$$

Dividing both sides of this equation by $X(s)\sum_{k=0}^{N} b_k s^k$, we obtain

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have that H is given by

$$H(s) = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (7.13)), the system function is always rational. It is for this reason that rational functions are of particular interest.

Example 7.36 (Differential equation to system function)**.** A LTI system with input x and output y is characterized by the differential equation

$$y''(t) + \tfrac{D}{M} y'(t) + \tfrac{K}{M} y(t) = x(t),$$

where D, K, and M are positive real constants, and the prime symbol is used to denote derivative. Find the system function H of this system.

Solution. Taking the Laplace transform of the given differential equation, we obtain

$$s^2 Y(s) + \tfrac{D}{M} s Y(s) + \tfrac{K}{M} Y(s) = X(s).$$

Rearranging the terms and factoring, we have

$$\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)Y(s) = X(s).$$

Dividing both sides by $\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)X(s)$, we obtain

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}.$$

Thus, H is given by

$$H(s) = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}. \qquad \blacksquare$$

Example 7.37 (System function to differential equation)**.** A LTI system with input x and output y has the system function

$$H(s) = \frac{s}{s + R/L},$$

where L and R are positive real constants. Find the differential equation that characterizes this system.

Solution. Let X and Y denote the Laplace transforms of x and y, respectively. To begin, we have

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{s}{s + R/L}\right)X(s). \end{aligned}$$

Rearranging this equation, we obtain

$$\begin{aligned} & (s + \tfrac{R}{L})Y(s) = sX(s) \\ \Rightarrow \quad & sY(s) + \tfrac{R}{L}Y(s) = sX(s). \end{aligned}$$

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\begin{aligned} & \mathcal{L}^{-1}\{sY(s)\}(t) + \tfrac{R}{L}\mathcal{L}^{-1}Y(t) = \mathcal{L}^{-1}\{sX(s)\}(t) \\ \Rightarrow \quad & \tfrac{d}{dt}y(t) + \tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \end{aligned}$$

Therefore, the system is characterized by the differential equation

$$\tfrac{d}{dt}y(t) + \tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \qquad \blacksquare$$

7.15 Circuit Analysis

One application of the Laplace transform is circuit analysis. In this section, we consider this particular application. The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 7.34(a), is governed by the relationship

$$v(t) = Ri(t) \qquad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

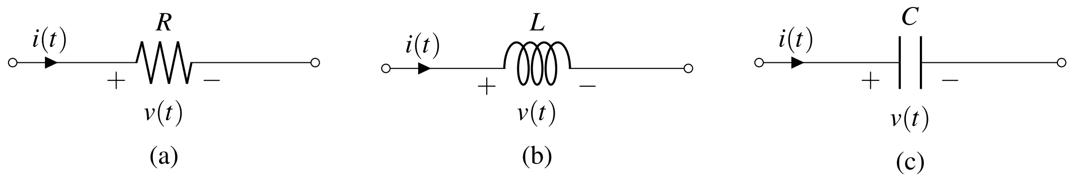

Figure 7.34: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

where R, v and i denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance R is a nonnegative quantity (i.e., $R \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \quad \left(\text{or equivalently, } I(s) = \tfrac{1}{R}V(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 7.34(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

where L, v, and i denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance L is a nonnegative quantity (i.e., $L \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \quad \left(\text{or equivalently, } I(s) = \tfrac{1}{sL}V(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 7.34(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\right),$$

where C, v, and i denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance C is a nonnegative quantity (i.e., $C \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = \tfrac{1}{sC}I(s) \quad \left(\text{or equivalently, } I(s) = sCV(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

Observe that the equations that characterize inductors and capacitors are arguably much simpler to express in the Laplace domain than in the time domain. Consequently, the use of the Laplace transform can greatly simplify the process of analyzing circuits containing inductors and capacitors.

Example 7.38 (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.35 with input v_1 and output v_2. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function H of this system. (b) Determine whether the system is BIBO stable. (c) Determine the type of ideal frequency-selective filter that this system best approximates. (d) Determine the step response of the system.

Solution. (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \tag{7.14a}$$

$$i(t) = C\tfrac{d}{dt}v_2(t). \tag{7.14b}$$

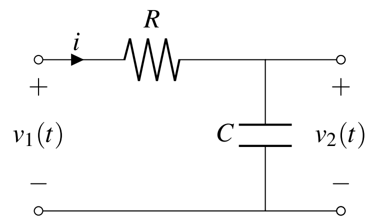

Figure 7.35: Simple RC network.

Taking the Laplace transform of (7.14) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \tag{7.15a}$$
$$I(s) = CsV_2(s). \tag{7.15b}$$

Substituting (7.15b) into (7.15a) and rearranging, we obtain

$$\begin{aligned}
& V_1(s) = R[CsV_2(s)] + V_2(s) \\
\Rightarrow \quad & V_1(s) = RCsV_2(s) + V_2(s) \\
\Rightarrow \quad & V_1(s) = [1 + RCs]V_2(s) \\
\Rightarrow \quad & \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + RCs}.
\end{aligned}$$

Thus, we have that the system function H is given by

$$\begin{aligned}
H(s) &= \frac{1}{1 + RCs} \\
&= \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \\
&= \frac{\frac{1}{RC}}{s - (-\frac{1}{RC})}.
\end{aligned}$$

Since the system can be physically realized, it must be causal. Therefore, the ROC of H must be a right-half plane. Thus, we may infer that the ROC of H is $\operatorname{Re}(s) > -\frac{1}{RC}$. So, we have

$$H(s) = \frac{1}{1 + RCs} \quad \text{for } \operatorname{Re}(s) > -\tfrac{1}{RC}.$$

(b) Since resistance and capacitance are (strictly) positive quantities, $R > 0$ and $C > 0$. Thus, $-\frac{1}{RC} < 0$. Consequently, the ROC contains the imaginary axis and the system is BIBO stable.

(c) Let H_F denote the frequency response of the system. Since the system is BIBO stable, $H_F(\omega) = H(j\omega)$. Evaluating $|H_F(0)|$ and $\lim_{|\omega|\to\infty} |H_F(\omega)|$, we obtain

$$|H_F(0)| = \left| \frac{1}{1 + RC(0)} \right| = 1 \quad \text{and} \quad \lim_{|\omega|\to\infty} |H_F(\omega)| = \lim_{|\omega|\to\infty} \left| \frac{1}{1 + RCj\omega} \right| = 0.$$

Thus, the system best approximates an ideal lowpass filter.

(d) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$\begin{aligned}
V_2(s) &= H(s)V_1(s) \\
&= \left(\frac{1}{1 + RCs} \right) V_1(s).
\end{aligned}$$

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1 = u$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for V_1 into the above expression for V_2, we have

$$\begin{aligned} V_2(s) &= \left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right) \\ &= \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}. \end{aligned}$$

Now, we need to compute the inverse Laplace transform of V_2 in order to determine v_2. To simplify this task, we find the partial fraction expansion for V_2. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+\tfrac{1}{RC})V_2(s)\big|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that V_2 has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}(t).$$

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= \left(1 - e^{-t/(RC)}\right)u(t). \end{aligned}$$

■

7.16 Stability Analysis

As mentioned earlier, since BIBO stability is more easily characterized for LTI systems in the Laplace domain than the time domain, the Laplace domain is often used to analyze system stability. In what follows, we will consider this application of the Laplace transform in more detail.

Example 7.39. Consider the system shown in Figure 7.36 that has input Laplace transform X and output Laplace transform Y, and is formed by the interconnection of two causal LTI systems labelled with their system functions H_1 and H_2. The system functions H_1 and H_2 are given by

$$H_1(s) = \frac{1}{s^2+as+(a-2)} \quad \text{and} \quad H_2(s) = -1,$$

where a is a real constant. (a) Find the system function H of the (overall) system (including the ROC). (b) Determine the values of the parameter a for which the system is BIBO stable.

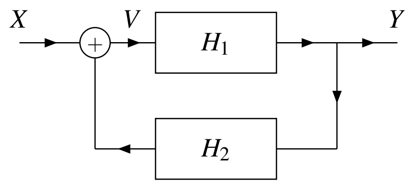

Figure 7.36: Feedback system.

Solution. (a) From the system diagram, we can write

$$\begin{aligned} V(s) &= X(s) + H_2(s)Y(s) \quad \text{and} \\ Y(s) &= H_1(s)V(s). \end{aligned}$$

Combining these two equations and simplifying, we obtain

$$\begin{aligned} & Y(s) = H_1(s)[X(s) + H_2(s)Y(s)] \\ \Rightarrow \quad & Y(s) = H_1(s)X(s) + H_1(s)H_2(s)Y(s) \\ \Rightarrow \quad & Y(s)[1 - H_1(s)H_2(s)] = H_1(s)X(s) \\ \Rightarrow \quad & \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)}. \end{aligned}$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given expressions for H_1 and H_2, and simplifying, we can write

$$\begin{aligned} H(s) &= \frac{\left(\frac{1}{s^2+as+(a-2)}\right)}{1 + \left(\frac{1}{s^2+as+(a-2)}\right)} \\ &= \frac{1}{s^2 + as + (a-2) + 1} \\ &= \frac{1}{s^2 + as + (a-1)}. \end{aligned}$$

(b) In order to assess the BIBO stability of the system, we need to know the poles of the system function H. So, we use the quadratic formula in order to factor the denominator of H. Solving for the roots s of the denominator of H, we obtain

$$\begin{aligned} s &= \frac{-a \pm \sqrt{a^2 - 4(a-1)}}{2} \\ &= \frac{-a \pm \sqrt{a^2 - 4a + 4}}{2} \\ &= \frac{-a \pm \sqrt{(a-2)^2}}{2} \\ &= \frac{-a \pm (a-2)}{2} \\ &= \{-1, 1-a\}. \end{aligned}$$

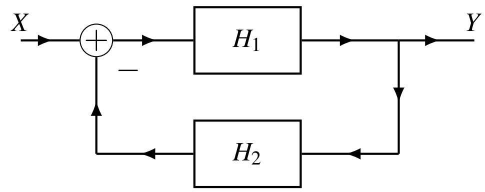

Figure 7.37: Feedback system.

So, $s^2 + as + (a-1) = (s+1)(s+a-1)$. Thus, we have

$$H(s) = \frac{1}{(s+1)(s+a-1)}.$$

Since the system is causal, the system is BIBO stable if and only if all of the poles are strictly to the left of the imaginary axis. The system has two poles, one at -1 and one at $1-a$. Thus, we know that

$$1 - a < 0 \Rightarrow a > 1.$$

Therefore, the system is BIBO stable if and only if $a > 1$. ■

Example 7.40. Consider the system shown in Figure 7.37 that has input Laplace transform X and output Laplace transform Y, and is formed by the interconnection of two causal LTI systems labelled with their system functions H_1 and H_2. The system functions H_1 and H_2 are given by

$$H_1(s) = \frac{1}{(s+1)(s+2)} \quad \text{and} \quad H_2(s) = a,$$

where a is a real constant. (a) Find the system function H of the (overall) system. (b) Determine the values of the parameter a for which the system is BIBO stable.

Solution. (a) In what follows, let V denote the output of the adder in the block diagram. From the system diagram, we can write

$$V(s) = X(s) - V(s)H_1(s)H_2(s) \quad \Rightarrow \quad X(s) = [1 + H_1(s)H_2(s)]V(s) \quad \text{and}$$
$$Y(s) = V(s)H_1(s).$$

Combining these two equations, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)V(s)}{[1 + H_1(s)H_2(s)]V(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

Substituting the given expressions for H_1 and H_2 and simplifying, we have

$$H(s) = \frac{\frac{1}{(s+1)(s+2)}}{1 + \left(\frac{1}{(s+1)(s+2)}\right)(a)} = \frac{1}{(s+1)(s+2)+a} = \frac{1}{s^2 + 3s + a + 2}.$$

Rewriting H in factored form, we have

$$H(s) = \frac{1}{(s-p_1)(s-p_2)},$$

where the poles p_1 and p_2 are to be determined. From the quadratic formula, we have

$$p_k = \frac{-3 \pm \sqrt{3^2 - 4(a+2)}}{2} = -\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{9-4a-8} = -\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{1-4a}.$$

So, we have

$$p_1 = -\tfrac{3}{2} - \tfrac{1}{2}\sqrt{1-4a} \quad \text{and} \quad p_2 = -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{1-4a}.$$

Since the system is causal, the ROC R_H of H will be the RHP to the right of the rightmost pole of H. That is,

$$R_H = \{\mathrm{Re}(s) > \max\{\mathrm{Re}(p_1), \mathrm{Re}(p_2)\}.$$

Now, we compute $\mathrm{Re}(p_k)$. There are two cases to consider:

1. $1-4a \leq 0$ (i.e., $\mathrm{Re}\left(\sqrt{1-4a}\right) = 0$), or equivalently, $4a \geq 1 \Rightarrow a \geq \frac{1}{4}$; and
2. $1-4a > 0$ (i.e., $\mathrm{Re}\left(\sqrt{1-4a}\right) = \sqrt{1-4a}$), or equivalently, $4a < 1 \Rightarrow a < \frac{1}{4}$.

First, we consider the case that $1-4a \leq 0$ (i.e., $a \geq \frac{1}{4}$). In this case, $\mathrm{Re}\left(\sqrt{1-4a}\right) = 0$. So, we have

$$\mathrm{Re}(p_1) = \mathrm{Re}(p_2) = -\tfrac{3}{2}.$$

Next, we consider the case that $1-4a > 0$ (i.e., $a < \frac{1}{4}$). In this case, $\mathrm{Re}\left(\sqrt{1-4a}\right) = \sqrt{1-4a}$. So, we have

$$\mathrm{Re}(p_1) = -\tfrac{3}{2} - \tfrac{1}{2}\sqrt{1-4a} \quad \text{and} \quad \mathrm{Re}(p_2) = -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{1-4a}.$$

Clearly, p_2 is the rightmost of p_1 and p_2. So, the real part of the rightmost pole is

$$\mathrm{Re}(p_2) = -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{1-4a}.$$

Thus, the ROC R_H of H is

$$R_H = \begin{cases} \mathrm{Re}(s) > -\frac{3}{2} & a \geq \frac{1}{4} \\ \mathrm{Re}(s) > -\frac{3}{2} + \frac{1}{2}\sqrt{1-4a} & a < \frac{1}{4}. \end{cases}$$

(b) For the system to be BIBO stable, R_H must contain the imaginary axis. First, consider the case that $a \geq \frac{1}{4}$. In this case, R_H contains the imaginary axis, and the system is therefore BIBO stable. Next, consider the case that $a < \frac{1}{4}$. In this case, R_H contains the imaginary axis if

$$\begin{gathered} -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{1-4a} < 0 \quad \Rightarrow \quad \tfrac{1}{2}\sqrt{1-4a} < \tfrac{3}{2} \quad \Rightarrow \quad \sqrt{1-4a} < 3 \quad \Rightarrow \\ 1-4a < 9 \quad \Rightarrow \quad 4a > -8 \quad \Rightarrow \quad a > -2. \end{gathered}$$

Combining the results of the two cases, the system is BIBO stable if and only if

$$\begin{gathered} a \geq \tfrac{1}{4} \text{ or } (a < \tfrac{1}{4} \text{ and } a > -2) \quad \Rightarrow \\ a \geq \tfrac{1}{4} \text{ or } (-2 < a < \tfrac{1}{4}) \quad \Rightarrow \\ a > -2. \end{gathered}$$

Thus, the system is BIBO stable if and only if $a > -2$. ■

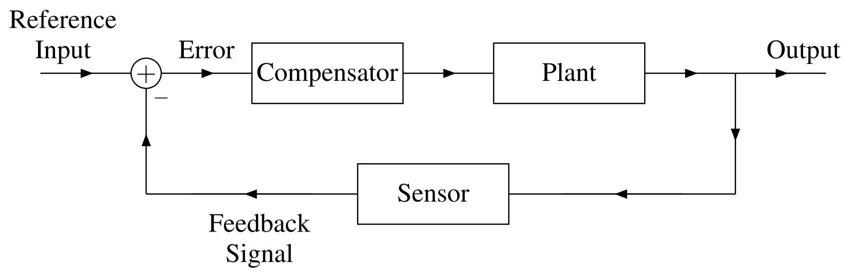

Figure 7.38: Feedback control system.

7.16.1 Feedback Control Systems

In control systems applications, we wish to have some physical quantity (such as a position or force) track a desired value over time. The input to the control system is the desired (i.e., reference) value for the quantity being controlled, and the output is the actual value of the quantity. The difference between the actual value and reference value constitutes an error. The goal of the control system is to force this error to be as close to zero as possible. When the error is zero, this corresponds to the actual value being equal to the desired value. For example, in a simple heating/cooling thermostat system, the reference input would be the temperature setting on the thermostat wall control and the output would be the actual room temperature. In a flight control system, the reference input would be the desired values for the position and orientation of the plane specified by the pilot's flight controls and the output would be the plane's actual position and orientation.

A very commonly occurring configuration used for control system applications is known as a feedback control system. A feedback control system consists of three interconnected subsystems:

1. a plant, which is the system whose output corresponds to the quantity being controlled;
2. a sensor, which is used to measure the actual value of the quantity being controlled; and
3. a compensator (also called a controller), which is used to ensure that the overall system behaves in a manner that the output closely tracks the reference input in addition to possibly satisfying other criteria such as being stable.

The general structure of a feedback control system is shown in Figure 7.38. The reference input corresponds to the desired value for the quantity being controlled. The output corresponds to the actual value of the quantity being controlled (which is measured by a sensor). The adder output corresponds to error (i.e., the difference between the desired and actual values of the controlled quantity). Again, in control applications, the objective is to have the output track the reference input as closely as possible over time. In other words, we wish for the error to be as close to zero as possible. It is the responsibility of the compensator to ensure that this happens. If the compensator is well designed, the error will remain close to zero over time.

Consider a simple control system application corresponding to a robot arm. In this system, the reference input corresponds to the desired position and orientation of the robot-arm end effector (i.e., the device at the end of the robot arm), while the output corresponds to the actual position and orientation of the end effector. If the compensator is well designed, then the actual position and orientation of the end effector should track the desired value over time.

Having introduced the notion of feedback control systems, we now consider the application of stabilizing an unstable plant with a feedback control system.

Example 7.41 (Stabilization of unstable plant)**.** Consider the causal LTI system with input Laplace transform X, output Laplace transform Y, and system function

$$P(s) = \frac{10}{s-1},$$

as depicted in Figure 7.39. One can easily confirm that this system is not BIBO stable, due to the pole of P at 1. (Since the system is causal, the ROC of P is the RHP given by $\operatorname{Re}(s) > 1$. Clearly, this ROC does not include the imaginary

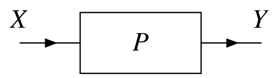

Figure 7.39: Plant.

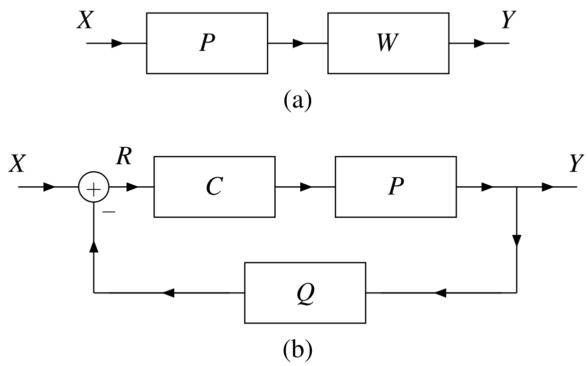

Figure 7.40: Two configurations for stabilizing the unstable plant. (a) Simple cascade system and (b) feedback control system.

axis. Therefore, the system is not BIBO stable.) In what follows, we consider two different strategies for stabilizing this unstable system as well as their suitability for use in practice.

(a) STABILIZATION OF UNSTABLE PLANT VIA POLE-ZERO CANCELLATION. Suppose that the system in Figure 7.39 is connected in series with another causal LTI system with system function

$$W(s) = \frac{s-1}{10(s+1)},$$

in order to yield a new system with input Laplace transform X and output Laplace transform Y, as shown in Figure 7.40(a). Show that this new system is BIBO stable.

(b) STABILIZATION OF UNSTABLE PLANT VIA FEEDBACK. Suppose now that the system in Figure 7.39 is interconnected with two other causal LTI systems with system functions C and Q, as shown in Figure 7.40(b), in order to yield a new system with input Laplace transform X, output Laplace transform Y, and system function H. Moreover, suppose that

$$C(s) = \beta \quad \text{and} \quad Q(s) = 1,$$

where β is a real constant. Show that, with an appropriate choice of β, the resulting system is BIBO stable.

(c) PRACTICAL ISSUES. Parts (a) and (b) of this example consider two different schemes for stabilizing the unstable system in Figure 7.39. As it turns out, a scheme like the one in part (a) is not useful in practice. Identify the practical problems associated with this approach. Indicate whether the scheme in part (b) suffers from the same shortcomings.

Solution. (a) From the block diagram in Figure 7.40(a), the system function H of the overall system is

$$\begin{aligned} H(s) &= P(s)W(s) \\ &= \left(\frac{10}{s-1}\right)\left(\frac{s-1}{10(s+1)}\right) \\ &= \frac{1}{s+1}. \end{aligned}$$

Since the system is causal and H is rational, the ROC of H is $\text{Re}(s) > -1$. Since the ROC includes the imaginary axis, the system is BIBO stable.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a somewhat reasonable step response. Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

$$\begin{aligned} s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\}(t) \\ &= (1-e^{-t})u(t). \end{aligned}$$

Evidently, s is a somewhat crude approximation of the desired response u.

(b) From the block diagram in Figure 7.40(b), we can write

$$R(s) = X(s) - Q(s)Y(s) \quad \text{and}$$
$$Y(s) = C(s)P(s)R(s).$$

Combining these equations (by substituting the expression for R in the first equation into the second equation), we obtain

$$\begin{aligned} & Y(s) = C(s)P(s)[X(s)-Q(s)Y(s)] \\ \Rightarrow\quad & Y(s) = C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \\ \Rightarrow\quad & [1+C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s) \\ \Rightarrow\quad & \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1+C(s)P(s)Q(s)}. \end{aligned}$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

$$H(s) = \frac{C(s)P(s)}{1+C(s)P(s)Q(s)}.$$

Substituting the given expressions for P, C, and Q, we have

$$\begin{aligned} H(s) &= \frac{\beta(\frac{10}{s-1})}{1+\beta(\frac{10}{s-1})(1)} \\ &= \frac{10\beta}{s-1+10\beta} \\ &= \frac{10\beta}{s-(1-10\beta)}. \end{aligned}$$

The system function H is rational and has a single pole at $1-10\beta$. Since the system is causal, the ROC must be the RHP given by $\operatorname{Re}(s) > 1-10\beta$. For the system to be BIBO stable, we require that the ROC includes the imaginary axis. Thus, the system is BIBO stable if $1-10\beta < 0$ which implies $10\beta > 1$, or equivalently $\beta > \frac{1}{10}$.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a reasonable step response. (This is not by chance, however. Some care had to be exercised in the choice of the form of the compensator system function C. The process involved in making this choice requires knowledge of control systems beyond the scope of this book, however.) Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least

