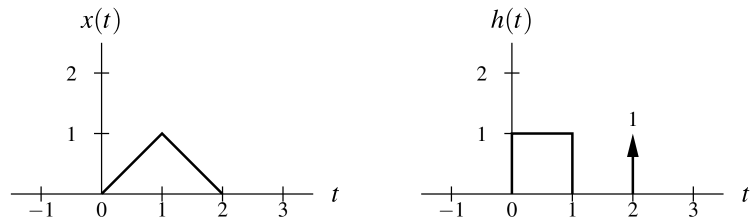
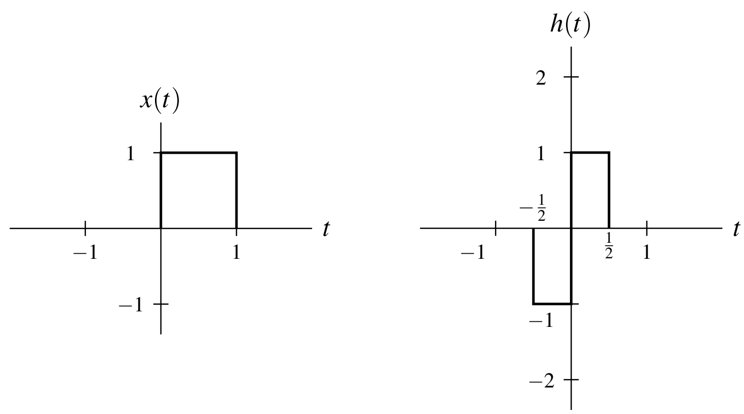
4.11 Exercises

4.11.1 Exercises Without Answer Key

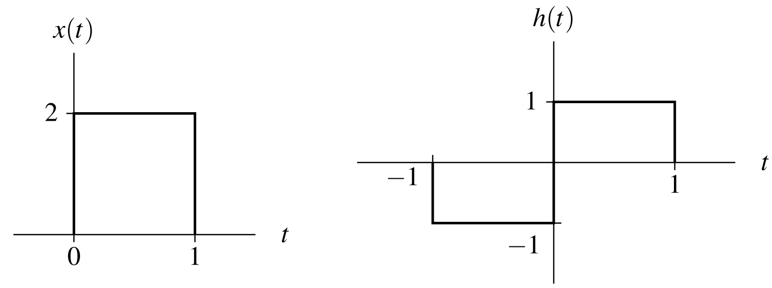
4.1 Using the graphical method, for each pair of functions x and h given in the figures below, directly compute $x*h$. (Do not compute $x*h$ indirectly by instead computing $h*x$ and using the commutative property of convolution.)



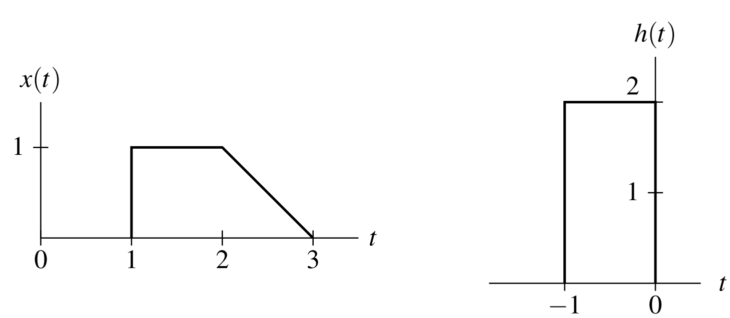
(a)



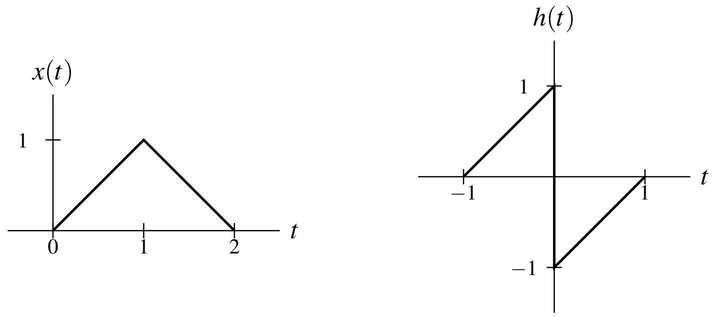
(b)



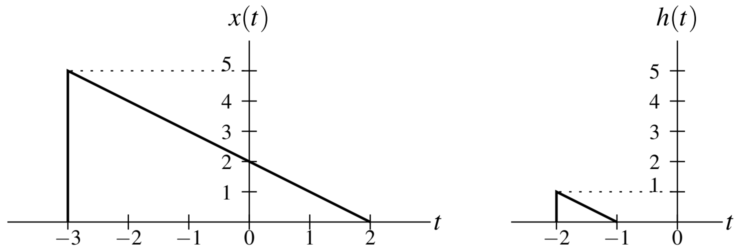
(c)



(d)



(e)



(f)

4.2 For each pair of functions x and h given below, compute $x * h$.
(a) $x(t) = e^{at}u(-t)$ and $h(t) = e^{-at}u(t)$, where a is a (strictly) positive real constant;
(b) $x(t) = e^{-j\omega_0 t}u(t)$ and $h(t) = e^{j\omega_0 t}u(t)$ where ω_0 is a (strictly) positive real constant;
(c) $x(t) = u(t-2)$ and $h(t) = u(t+3)$;
(d) $x(t) = u(t)$ and $h(t) = e^{-2t}u(t-1)$; and
(e) $x(t) = u(t-1) - u(t-2)$ and $h(t) = e^{t}u(-t)$.

4.3 Using the graphical method, compute $x * h$ for each pair of functions x and h given below.
(a) $x(t) = e^{t}u(-t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(b) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}(\frac{1}{3}[t - \frac{1}{2}])$;
(c) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(d) $x(t) = \operatorname{rect}\left(\frac{1}{2}t\right)$ and $h(t) = e^{2-t}u(t-2)$;
(e) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t+2 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$
(f) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(g) $x(t) = \begin{cases} 1 - \frac{1}{4}t & 0 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(h) $x(t) = \operatorname{rect}\left(\frac{1}{4}t\right)$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$ and
(i) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-2 & 2 \le t < 4 \\ 0 & \text{otherwise.} \end{cases}$

4.4 Show that, for any function x, $x * v(t) = x(t - t_0)$, where $v(t) = \delta(t - t_0)$ and t_0 is an arbitrary real constant.

4.5 Let x, y, h, and v be functions such that $y = x * h$ and

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b) h(\tau + at) d\tau,$$

where a and b are real constants. Express v in terms of y.

4.6 Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

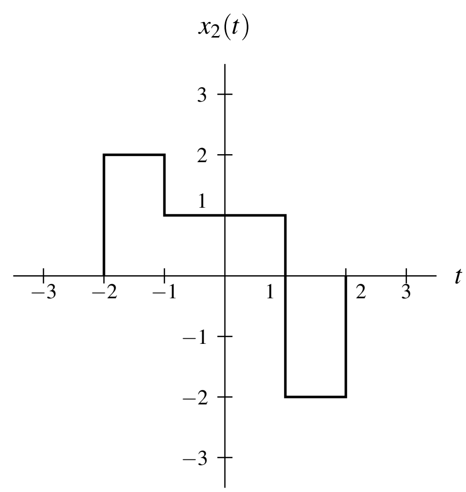
4.7 From the definition of convolution, show that if $y = x * h$, then $\mathcal{D}y(t) = [x * (\mathcal{D}h)](t)$, where $\mathcal{D}$ denotes the derivative operator.

4.8 Let x and h be functions satisfying

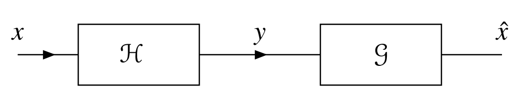
$$\begin{aligned} x(t) &= 0 \quad \text{for } t < A_1 \text{ or } t > A_2, \quad \text{and} \\ h(t) &= 0 \quad \text{for } t < B_1 \text{ or } t > B_2 \end{aligned}$$

(i.e., x and h are finite duration). Determine for which values of t the quantity $x * h(t)$ must be zero.

4.9 Consider a LTI system whose response to the function $x_1(t) = u(t) - u(t-1)$ is the function y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



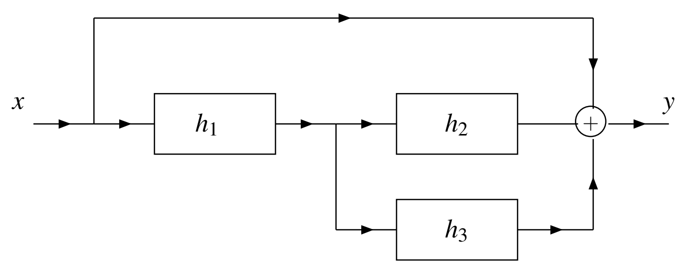
4.10 Consider the system shown in the figure below, where $\mathcal{H}$ is a LTI system and $\mathcal{G}$ is known to be the inverse system of $\mathcal{H}$. Let $y_1 = \mathcal{H}x_1$ and $y_2 = \mathcal{H}x_2$.



(a) Determine the response of the system $\mathcal{G}$ to the input $y'(t) = a_1 y_1(t) + a_2 y_2(t)$, where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathcal{G}$ to the input $y_1'(t) = y_1(t - t_0)$, where t_0 is a real constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathcal{G}$ is linear and/or time invariant.

4.11 Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau$;
(c) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)v(t-\tau)d\tau$; and
(d) $\mathcal{H}x(t) = \int_{t-1}^{t} x(\tau)d\tau$.

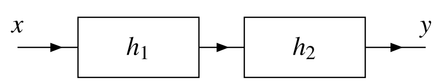
4.12 Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



(a) Find the impulse response h of the overall system in terms of h_1, h_2, and h_3.
(b) Determine the impulse response h in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

4.13 Consider the system shown in the figure below with input x and output y. This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2.



For each pair of h_1 and h_2 given below, find the output y if the input $x(t) = u(t)$.
(a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
(b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$; and
(c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

4.14 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(t) = (t+1)u(t-1)$;
(b) $h(t) = 2\delta(t+1)$;
(c) $h(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$;
(d) $h(t) = e^{-4t}u(t-1)$;
(e) $h(t) = e^{t}u(-1-t)$;
(f) $h(t) = e^{-3|t|}$; and
(g) $h(t) = 3\delta(t)$.

4.15 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
(b) $h(t) = t^{-1}u(t-1)$;
(c) $h(t) = e^{t}u(t)$;
(d) $h(t) = \delta(t-10)$;
(e) $h(t) = \operatorname{rect}(t)$; and
(f) $h(t) = e^{-|t|}$.

4.16 Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \tfrac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

4.17 For each case below, find the response y of the LTI system with system function H to the input x.
(a) $H(s) = \dfrac{1}{s+1}$ for $\operatorname{Re}(s) > -1$; and $x(t) = 10 + 4\cos(3t) + 2\sin(5t)$; and
(b) $H(s) = \dfrac{1}{e^{s}(s+1)}$ for $\operatorname{Re}(s) > -1$; and $x(t) = 10 + 2e^{3t} - e^{t}$.

4.18 Suppose that we have the systems $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, whose responses to a complex exponential input $x(t) = e^{j2t}$ are given by

$$\mathcal{H}_1 x(t) = 2e^{j2t}, \quad \mathcal{H}_2 x(t) = te^{j2t}, \quad \mathcal{H}_3 x(t) = e^{j2t+\pi/3}, \quad \text{and} \quad \mathcal{H}_4 x(t) = \cos(2t).$$

Indicate which of these systems cannot be LTI.

4.11.2 Exercises With Answer Key

4.101 Using the graphical method, compute $x*h$ for each pair of functions x and h given below. (Do not compute $x*h$ indirectly by instead computing $h*x$ and using the commutative property of convolution.)

(a) $x(t) = 2\operatorname{rect}(t-\tfrac{1}{2})$ and $h(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = u(t-1)$ and $h(t) = \begin{cases} t+1 & -1 \le t < 0 \\ t-1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x(t) = \begin{cases} t-2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \operatorname{rect}\left[\tfrac{1}{2}(t+2)\right]$;

(d) $x(t) = \operatorname{rect}\left[\tfrac{1}{3}\left(t-\tfrac{3}{2}\right)\right]$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x(t) = \begin{cases} \tfrac{1}{4}(t-1)^2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x(t) = \begin{cases} 2\cos\left(\tfrac{\pi}{4}t\right) & 0 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left[\tfrac{1}{2}(t-2)\right]$;

(h) $x(t) = \begin{cases} \frac{1}{2}t - \frac{1}{2} & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-1 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{tri}\left[\frac{1}{2}(t-3)\right]$;

(j) $x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{4} & 1 \le t < 5 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} \frac{3}{2} - \frac{1}{2}t & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x(t) = \begin{cases} 1 - \frac{1}{100}t & 0 \le t < 100 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = e^{-t}u(t-1)$;

(m) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} 1-(t-2)^2 & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x(t) = e^{-t}u(t)$ and $h(t) = e^{-3t}u(t-2)$;

(o) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left(t - \frac{3}{2}\right)$;

(p) $x(t) = e^{-2t}u(t)$ and $h(t) = \operatorname{rect}\left(t - \frac{5}{2}\right)$;

(q) $x(t) = u(t-1)$ and $h(t) = \begin{cases} \sin[\pi(t-1)] & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(r) $x(t) = u(t)$ and $h(t) = \operatorname{rect}\left(\frac{1}{4}[t-4]\right)$;

(s) $x(t) = e^{-t}u(t)$ and $h(t) = e^{2-2t}u(t-1)$;

(t) $x(t) = e^{-3t}u(t)$ and $h(t) = u(t+1)$; and

(u) $x(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-2 & -3 \le t < -2 \\ 0 & \text{otherwise.} \end{cases}$

Short Answer.

(a) $x * h(t) = \begin{cases} \int_0^{t+1} -2d\tau & -1 \le t < 0 \\ \int_0^t 2d\tau + \int_t^1 -2d\tau & 0 \le t < 1 \\ \int_{t-1}^1 2d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x * h(t) = \begin{cases} \int_1^{t+1} (-\tau+t+1)d\tau & 0 \le t < 1 \\ \int_1^t (-\tau+t-1)d\tau + \int_t^{t+1} (-\tau+t+1)d\tau & 1 \le t < 2 \\ \int_{t-1}^t (-\tau+t-1)d\tau + \int_t^{t+1} (-\tau+t+1)d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x * h(t) = \begin{cases} \int_1^{t+3} (\tau-2)d\tau & -2 \le t < 0 \\ \int_{t+1}^3 (\tau-2)d\tau & 0 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(d) $x * h(t) = \begin{cases} \int_0^{t-1} (t-\tau-1)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} (t-\tau-1)d\tau & 2 \le t < 4 \\ \int_{t-2}^3 (t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x * h(t) = \begin{cases} \int_1^{t-1} \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1} \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 3 \le t < 4 \\ \int_{t-2}^3 \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x*h(t) = \begin{cases} \int_0^{t-1} 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 2 \le t < 3 \\ \int_{t-2}^{2} 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 3 \le t < 4 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x*h(t) = \begin{cases} \int_{t-3}^{t-1} e^{\tau}d\tau & t < 1 \\ \int_{t-3}^{0} e^{\tau}d\tau + \int_0^{t-1} e^{-\tau}d\tau & 1 \le t < 3 \\ \int_{t-3}^{t-1} e^{-\tau}d\tau & t \ge 3; \end{cases}$

(h) $x*h(t) = \begin{cases} \int_1^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau-t-1)d\tau & -1 \le t < 0 \\ \int_{t+1}^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau-t-1)d\tau & 0 \le t < 1 \\ \int_{t+1}^{3} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau-t-1)d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x*h(t) = \begin{cases} \int_{t-4}^{t-3} e^{\tau}(\tau-t+4)d\tau + \int_{t-3}^{t-2} e^{\tau}(t-\tau-2)d\tau & t < 2 \\ \int_{t-4}^{t-3} e^{\tau}(\tau-t+4)d\tau + \int_{t-3}^{0} e^{\tau}(t-\tau-2)d\tau + \int_0^{t-2} e^{-\tau}(t-\tau-2)d\tau & 2 \le t < 3 \\ \int_{t-4}^{0} e^{\tau}(\tau-t+4)d\tau + \int_0^{t-3} e^{-\tau}(t-\tau+4)d\tau + \int_{t-3}^{t-2} e^{-\tau}(t-\tau-2)d\tau & 3 \le t < 4 \\ \int_{t-4}^{t-3} e^{-\tau}(\tau-t+4)d\tau + \int_{t-3}^{t-2} e^{-\tau}(t-\tau-2)d\tau & t \ge 4; \end{cases}$

(j) $x*h(t) = \begin{cases} \int_1^{t-1} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 2 \le t < 4 \\ \int_{t-3}^{t-1} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 4 \le t < 6 \\ \int_{t-3}^{5} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 6 \le t < 8 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x*h(t) = \begin{cases} \int_{-10}^{t-1}(t-\tau-1)d\tau & -9 \le t < -8 \\ \int_{t-2}^{t-1}(t-\tau-1)d\tau & -8 \le t < 11 \\ \int_{t-2}^{10}(t-\tau-1)d\tau & 11 \le t < 12 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x*h(t) = \begin{cases} 0 & t < 1 \\ \int_0^{t-1} \left(1 - \frac{1}{100}\tau\right) e^{\tau-t}d\tau & 1 \le t < 101 \\ \int_0^{100} \left(1 - \frac{1}{100}\tau\right) e^{\tau-t}d\tau & t \ge 101; \end{cases}$

(m) $x*h(t) = \begin{cases} \int_{-10}^{t-1}[1-(t-\tau-2)^2]d\tau & -9 \le t < -7 \\ \int_{t-3}^{t-1}[1-(t-\tau-2)^2]d\tau & -7 \le t < 11 \\ \int_{t-3}^{10}[1-(t-\tau-2)^2]d\tau & 11 \le t < 13 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x*h(t) = \begin{cases} \int_0^{t-2} e^{-\tau}e^{3\tau-3t}d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(o) $x*h(t) = \begin{cases} \int_{t-2}^{t-1} e^{\tau}d\tau & t < 1 \\ \int_{t-2}^{0} e^{\tau}d\tau + \int_0^{t-1} e^{-\tau}d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} e^{-\tau}d\tau & t \ge 2; \end{cases}$

(p) $x*h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} e^{-2\tau}d\tau & 2 \le t < 3 \\ \int_{t-3}^{t-2} e^{-2\tau}d\tau & t \ge 3; \end{cases}$

(q) $x*h(t) = \begin{cases} 0 & t < 2 \\ \int_1^{t-1} \sin(\pi t - \pi\tau - \pi)d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1} \sin(\pi t - \pi\tau - \pi)d\tau & t \ge 3; \end{cases}$

(r) $x*h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} 1 d\tau & 2 \leq t < 6 \\ \int_{t-6}^{t-2} 1 d\tau & t \geq 6; \end{cases}$

(s) $x*h(t) = \begin{cases} \int_0^{t-1} e^{\tau-2t+2} d\tau & t \geq 1 \\ 0 & \text{otherwise;} \end{cases}$

(t) $x*h(t) = \begin{cases} \int_0^{t+1} e^{-3\tau} d\tau & t \geq -1 \\ 0 & \text{otherwise;} \end{cases}$

(u) $x*h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \leq t < -1 \\ -\frac{1}{6}t^3 & -1 \leq t < 0 \\ 0 & \text{otherwise} \end{cases}$

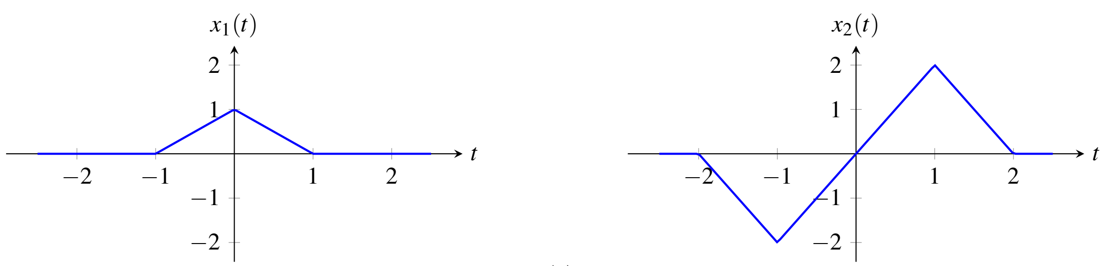
4.102 Using the graphical method, compute $x*h$ for each pair of functions x and h given below.
(a) $x(t) = \operatorname{rect}\left(\frac{1}{2a}t\right)$ and $h(t) = \operatorname{rect}\left(\frac{1}{2a}t\right)$; and
(b) $x(t) = \operatorname{rect}\left(\frac{1}{a}t\right)$ and $h(t) = \operatorname{rect}\left(\frac{1}{a}t\right)$.

Short Answer. (a) $x*h(t) = 2a\operatorname{tri}\left(\frac{1}{4a}t\right)$; (b) $x*h(t) = a\operatorname{tri}\left(\frac{1}{2a}t\right)$.

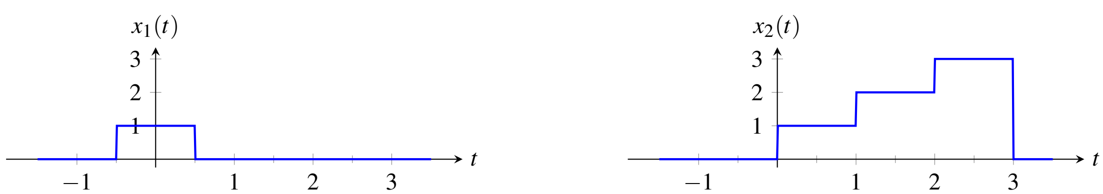
4.103 Let $\mathcal{H}$ denote an operator corresponding to a LTI system; and let $\mathcal{S}_{t_0}$ denote an operator that shifts a function by t_0 (i.e., $S_{t_0}x(t) = x(t-t_0)$ for all t). Further, let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, x, and y denote functions such that $y_k = \mathcal{H}x_k$ and $y = \mathcal{H}x$. For each case below, find y in terms of $y_1, y_2, \ldots, y_n$.
(a) $x = \pi x_1$;
(b) $x = x_1 + x_2$;
(c) $x = jx_1 + \pi x_2$;
(d) $x = \mathcal{S}_{42}x_1$;
(e) $x = 3\mathcal{S}_{-2}x_1$;
(f) $x = 2\mathcal{S}_3 x_1 + j\pi S_{-5}x_2$; and
(g) $x = \sum_{k=1}^{10} k\mathcal{S}_k x_k$.

Short Answer. (a) $y = \pi y_1$; (b) $y = y_1 + y_2$; (c) $y = jy_1 + \pi y_2$; (d) $y = \mathcal{S}_{42}y_1$; (e) $y = 3\mathcal{S}_{-2}y_1$; (f) $y = 2\mathcal{S}_3 y_1 + j\pi\mathcal{S}_{-5}y_2$; (g) $y = \sum_{k=1}^{10} k\mathcal{S}_k y_k$

4.104 Let $\mathcal{H}$ denote an operator corresponding to a LTI system; and let x_1, x_2, y_1, y_2 denote functions such that $y_1 = \mathcal{H}x_1$ and $y_2 = \mathcal{H}x_2$. For the case of each pair of functions x_1 and x_2 given below, find y_2 in terms of y_1.



(a)



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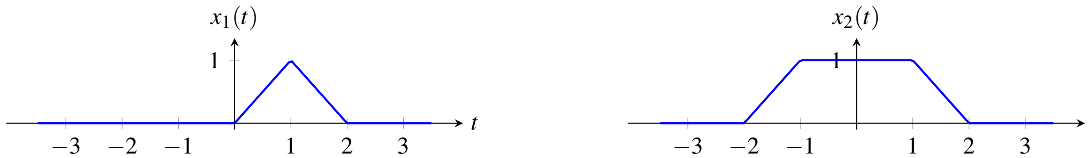

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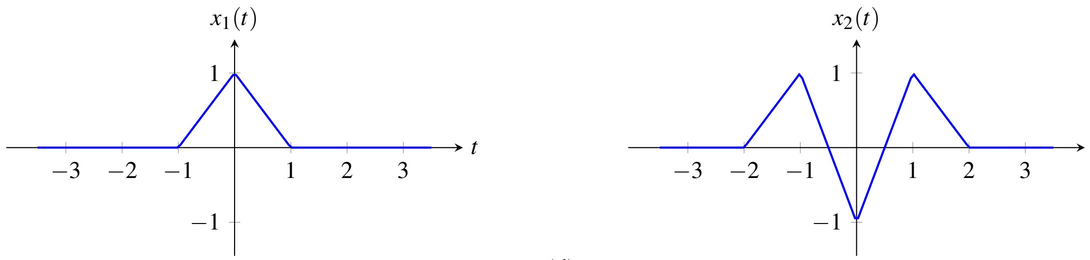

(d)

Short Answer. (a) $y_2(t) = -2y_1(t+1) + 2y_1(t-1)$; (b) $y_2(t) = y_1\left(t - \frac{1}{2}\right) + 2y_1\left(t - \frac{3}{2}\right) + 3y_1\left(t - \frac{5}{2}\right)$; (c) $y_2(t) = y_1(t+2) + y_1(t+1) + y_1(t)$; (d) $y_2(t) = y_1(t+1) - y_1(t) + y_1(t-1)$

4.105 Let $\mathcal{H}_k$ (where k is an integer) denote an operator representing a LTI system with impulse response h_k. Express the impulse response h of the system $\mathcal{H}$ characterized by each equation below in terms of h_k.
(a) $\mathcal{H}x = \frac{1}{2}\mathcal{H}_1(4x) - \frac{1}{2}\mathcal{H}_2(4x)$;
(b) $\mathcal{H}x = 2\mathcal{H}_2\mathcal{H}_1\left(\frac{1}{3}x\right)$;
(c) $\mathcal{H}x = 5x - \mathcal{H}_1(3x) + 2\mathcal{H}_2\left(\frac{1}{5}x\right)$;
(d) $\mathcal{H}x = \mathcal{H}_1\left(\mathcal{H}_2\mathcal{H}_3x + \mathcal{H}_3x + x\right)$; and
(e) $\mathcal{H}x = \mathcal{G}\mathcal{G}(2x)$, where $\mathcal{G}x = \mathcal{H}_1x + \mathcal{H}_2x$.

Short Answer. (a) $h = 2(h_1 - h_2)$; (b) $h = \frac{2}{3}h_1 * h_2$; (c) $h = 5\delta - 3h_1 + \frac{2}{5}h_2$; (d) $h = h_1 * (h_2 * h_3 + h_3 + \delta)$; (e) $h = 2h_1 * h_1 + 4h_1 * h_2 + 2h_2 * h_2$

4.106 Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.
(a) $\mathcal{H}x(t) = \displaystyle\int_t^\infty x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \displaystyle\int_{-\infty}^\infty e^{-|\tau|}x(t-\tau)d\tau$;
(c) $\mathcal{H}x(t) = \displaystyle\int_{t-5}^{t-4} x(\tau)d\tau$; and
(d) $\mathcal{H}x(t) = x(t) + x(t-1)$.

Short Answer. (a) $h(t) = u(-t)$; (b) $h(t) = e^{-|t|}$; (c) $h(t) = u(t-4) - u(t-5)$; (d) $h(t) = \delta(t) + \delta(t-1)$

4.107 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(t) = u(t+1) - u(t-1)$;
(b) $h(t) = e^{-5t}u(t-1)$;
(c) $h(t) = (t^2-1)\sin(t)\delta(t+1)$; and
(d) $h(t) = \pi\delta(t+42)$.

Short Answer. (a) has memory, not causal; (b) has memory, causal; (c) memoryless, causal; (d) has memory, not causal

4.108 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = u(t-1) - u(t-2)$;
(b) $h(t) = e^{-2t^2}$ [Hint: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$];
(c) $h(t) = t^{-2}u(-t-1)$;
(d) $h(t) = e^{-t}\sin(t)u(t)$;
(e) $h(t) = e^{-t}u(-t)$;
(f) $h(t) = te^{-3t}u(t-1)$ [Hint: See (F.1).]; and
(g) $h(t) = te^{-2t}u(1-t)$ [Hint: See (F.1).].

Short Answer. (a) BIBO stable ($\int_{-\infty}^{\infty}|h(t)|\,dt = 1$); (b) BIBO stable ($\int_{-\infty}^{\infty}|h(t)|\,dt = \sqrt{\frac{\pi}{2}}$); (c) BIBO stable ($\int_{-\infty}^{\infty}|h(t)|\,dt = 1$); (d) BIBO stable ($\int_{-\infty}^{\infty}|h(t)|\,dt = 1$); (e) not BIBO stable; (f) BIBO stable ($\int_{-\infty}^{\infty}|h(t)|\,dt = \frac{4}{9e^3}$); (g) not BIBO stable

4.109 For each case below, find the response y of the LTI system with system function H to the input x.
(a) $H(s) = \dfrac{1}{(s+1)(s+2)}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -1$; and $x(t) = 1 + \frac{1}{2}e^{-t/2} + \frac{1}{3}e^{-t/3}$;
(b) $H(s) = s$ for all $s \in \mathbb{C}$; and $x(t) = 1 + 2e^{-t/2} + 3e^{-t/3}$;
(c) $H(s) = \dfrac{1}{s+1}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -1$; and $x(t) = 2\cos(t)$;
(d) $H(s) = se^{-s}$ for all $s \in \mathbb{C}$; and $x(t) = 4\cos(t) + 2\sin(3t)$;
(e) $H(s) = \dfrac{1}{e^s(s+4)}$ for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > -4$; and $x(t) = 11 + 7e^{-2t} + 5e^{-3t}$; and
(f) $H(s) = s^2$ for all $s \in \mathbb{C}$; and $x(t) = 7 + e^{-5t} + 4\cos(3t)$.

Short Answer. (a) $y(t) = \frac{1}{2} + \frac{2}{3}e^{-t/2} + \frac{3}{10}e^{-t/3}$; (b) $y(t) = -e^{-t/2} - e^{-t/3}$; (c) $y(t) = \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$; (d) $y(t) = 6\cos(3t-3) - 4\sin(t-1)$; (e) $y(t) = \frac{11}{4} + \frac{7}{2}e^{2(1-t)} + 5e^{3(1-t)}$; (f) $y(t) = 25e^{-5t} - 36\cos(3t)$

Chapter 5

Continuous-Time Fourier Series

5.1 Introduction

One very important tool in the study of signals and systems is the Fourier series. A very large class of functions can be represented using Fourier series, namely most practically useful periodic functions. The Fourier series represents a periodic function as a (possibly infinite) linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy functions with which to work. For example, complex sinusoids are easy to integrate and differentiate. Also, complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigenfunctions of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex constant.

5.2 Definition of Continuous-Time Fourier Series

Suppose that we have a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm 1, \pm 2, \ldots.$$

The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0. Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic. More specifically, a linear combination of these complex sinusoids is periodic with period $T = \frac{2\pi}{\omega_0}$.

Suppose that we can represent a periodic complex-valued function x as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{5.1}$$

Such a representation is known as a **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. As a matter of terminology, we refer to (5.1) as the **Fourier series synthesis equation**. The terms in the summation for $k = 1$ and $k = -1$ are known as the fundamental frequency components or **first harmonic components**, and have the fundamental frequency ω_0. More generally, the terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$. Since the complex sinusoids are harmonically related, the function x is periodic with period $T = \frac{2\pi}{\omega_0}$ (and frequency ω_0).

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a function is associated with particular Fourier series coefficients. If a function x has the Fourier series coefficient sequence c, we sometimes indicate this using the notation

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

5.3 Determining the Fourier Series Representation of a Continuous-Time Periodic Function

Given an arbitrary periodic function x, we need some means for finding its corresponding Fourier series representation. In other words, we need a method for calculating the Fourier series coefficient sequence c. Such a method is given by the theorem below.

Theorem 5.1 (Fourier series analysis equation)**.** *The Fourier series coefficient sequence c of a periodic function x with fundamental period T is given by*

$$c_k = \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt, \tag{5.2}$$

where $\int_T$ denotes integration over an interval of length T.

Proof. We begin with the definition of the Fourier series in (5.1). Multiplying both sides of this equation by $e^{-jn\omega_0 t}$ yields

$$\begin{aligned} x(t)e^{-jn\omega_0 t} &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}e^{-jn\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}. \end{aligned}$$

As a matter of notation, we use $\int_T$ to denote the integral over an arbitrary interval of length T (i.e., the interval (t_0, t_0+T) for arbitrary t_0). Integrating both sides of this equation over one period T of x, we obtain

$$\int_T x(t)e^{-jn\omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}dt.$$

Reversing the order of integration and summation yields

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} c_k \left(\int_T e^{j(k-n)\omega_0 t}dt\right). \tag{5.3}$$

Now, we note that the following identity holds:

$$\int_T e^{j(k-n)\omega_0 t}dt = \begin{cases} T & k=n \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

(The proof of this fact is left as an exercise for the reader in Exercise A.9.) Substituting (5.4) into (5.3), we obtain

$$\int_T x(t)e^{-jn\omega_0 t}dt = c_n T. \tag{5.5}$$

Rearranging, we obtain

$$c_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt. \qquad \blacksquare$$

As a matter of terminology, we refer to (5.2) as the **Fourier series analysis equation**.

Suppose that we have a complex-valued periodic function x with period T and Fourier series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (5.2). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\right]\bigg|_{k=0} \\ &= \tfrac{1}{T}\int_T x(t)e^{0}dt \\ &= \tfrac{1}{T}\int_T x(t)dt. \end{aligned}$$

Thus, c_0 is simply the average value of x over a single period.

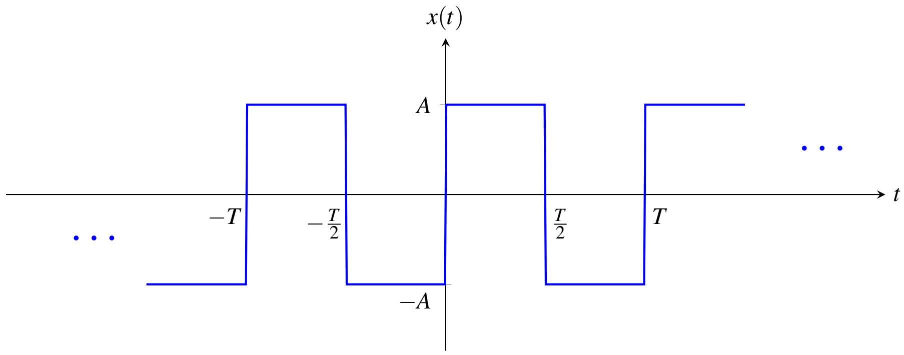

Figure 5.1: Periodic square wave.

Example 5.1 (Fourier series of a periodic square wave)**.** Find the Fourier series representation of the periodic square wave x shown in Figure 5.1.

Solution. Let us consider the single period of $x(t)$ for $0 \le t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & 0 \le t < \frac{T}{2} \\ -A & \frac{T}{2} \le t < T. \end{cases}$$

Let $\omega_0 = \frac{2\pi}{T}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} \left(\int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^{T} (-A) e^{-jk\omega_0 t} dt \right) \\ &= \begin{cases} \dfrac{1}{T} \left(\left[\dfrac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\dfrac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) & k \ne 0 \\ \dfrac{1}{T} \left([At]|_0^{T/2} + [-At]|_{T/2}^{T} \right) & k = 0. \end{cases} \end{aligned}$$

Now, we simplify the expression for c_k for each of the cases $k \ne 0$ and $k = 0$ in turn. First, suppose that $k \ne 0$. We have

$$\begin{aligned} c_k &= \frac{1}{T} \left(\left[\frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{-A}{j2\pi k} \left(\left[e^{-jk\omega_0 t} \right] \Big|_0^{T/2} - \left[e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{jA}{2\pi k} \left(\left[e^{-j\pi k} - 1 \right] - \left[e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\ &= \frac{jA}{2\pi k} \left[2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\ &= \frac{jA}{2\pi k} \left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right]. \end{aligned}$$

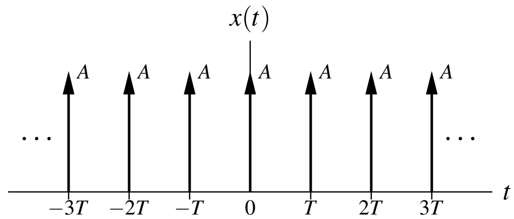

Figure 5.2: Periodic impulse train.

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

$$\begin{aligned} c_k &= \frac{jA}{2\pi k}[2(-1)^k - 1^k - 1] \\ &= \frac{jA}{2\pi k}[2(-1)^k - 2] \\ &= \frac{jA}{\pi k}[(-1)^k - 1] \\ &= \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0. \end{cases} \end{aligned}$$

Now, suppose that $k = 0$. We have

$$\begin{aligned} c_0 &= \tfrac{1}{T}\left([At]|_0^{T/2} + [-At]|_{T/2}^{T} \right) \\ &= \tfrac{1}{T}\left[\tfrac{AT}{2} - \tfrac{AT}{2} \right] \\ &= 0. \end{aligned}$$

Thus, the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt},$$

where

$$c_k = \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases}$$

■

Example 5.2 (Fourier series of a periodic impulse train)**.** Consider the periodic impulse train x shown in Figure 5.2. Find the Fourier series representation of x.

Solution. Let $\omega_0 = \frac{2\pi}{T}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T}\int_{-T/2}^{T/2} A\delta(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{A}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt. \end{aligned}$$

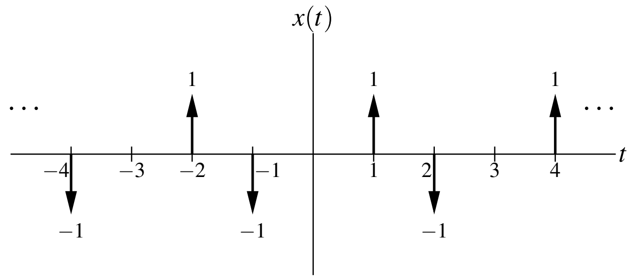

Figure 5.3: Periodic impulse train.

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$c_k = \tfrac{A}{T}.$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \tfrac{A}{T} e^{j(2\pi/T)kt}. \qquad \blacksquare$$

Example 5.3. Consider the periodic function x with fundamental period $T = 3$ as shown in Figure 5.3. Find the Fourier series representation of x.

Solution. The function x has the fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \leq t < \frac{3}{2}$). From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{3} \int_{-3/2}^{3/2} x(t) e^{-j(2\pi/3)kt} dt \\
&= \tfrac{1}{3} \int_{-3/2}^{3/2} [-\delta(t+1) + \delta(t-1)] e^{-j(2\pi/3)kt} dt \\
&= \tfrac{1}{3} \left[\int_{-3/2}^{3/2} -\delta(t+1) e^{-j(2\pi/3)kt} dt + \int_{-3/2}^{3/2} \delta(t-1) e^{-j(2\pi/3)kt} dt \right] \\
&= \tfrac{1}{3} \left[\int_{-\infty}^{\infty} -\delta(t+1) e^{-j(2\pi/3)kt} dt + \int_{-\infty}^{\infty} \delta(t-1) e^{-j(2\pi/3)kt} dt \right] \\
&= \tfrac{1}{3} \left(\left[-e^{-j(2\pi/3)kt} \right] \Big|_{t=-1} + \left[e^{-j(2\pi/3)kt} \right] \Big|_{t=1} \right) \\
&= \tfrac{1}{3} \left[-e^{-jk(2\pi/3)(-1)} + e^{-jk(2\pi/3)(1)} \right] \\
&= \tfrac{1}{3} \left[e^{-j(2\pi/3)k} - e^{j(2\pi/3)k} \right] \\
&= \tfrac{1}{3} \left[2j \sin\left(-\tfrac{2\pi}{3}k\right) \right] \\
&= \tfrac{2j}{3} \sin\left(-\tfrac{2\pi}{3}k\right) \\
&= -\tfrac{2j}{3} \sin\left(\tfrac{2\pi}{3}k\right).
\end{aligned}$$

Thus, x has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$
$$= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3} \sin\left(\tfrac{2\pi}{3}k\right) e^{j(2\pi/3)kt}. \quad \blacksquare$$

Example 5.4 (Fourier series of an even real function)**.** Let x be an arbitrary periodic real function that is even. Let c denote the Fourier series coefficient sequence for x. Show that

- c is real (i.e., $\operatorname{Im}\{c_k\} = 0$ for all k);
- c is even (i.e., $c_k = c_{-k}$ for all k); and
- $c_0 = \frac{1}{T}\int_0^T x(t)dt$.

Solution. From the Fourier series analysis equation (5.2) and using Euler's relation, we can write

$$c_k = \tfrac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$
$$= \tfrac{1}{T}\int_T \left(x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]\right) dt.$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$c_k = \tfrac{1}{T}\int_T \left(x(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right]\right) dt$$
$$= \tfrac{1}{T}\left[\int_T x(t)\cos(k\omega_0 t)dt - j\int_T x(t)\sin(k\omega_0 t)dt\right].$$

Consider the first integral above (i.e., the one involving the cos function). Since x is even and $\cos(k\omega_0 t)$ is even, we have that $x(t)\cos(k\omega_0 t)$ is even. Thus, $\int_T x(t)\cos(k\omega_0 t)dt = 2\int_0^{T/2} x(t)\cos(k\omega_0 t)dt$. Consider the second integral above (i.e., the one involving the sin function). Since x is even and $\sin(k\omega_0 t)$ is odd, we have that $x(t)\sin(k\omega_0 t)$ is odd. If we integrate an odd periodic function over one period (or an integer multiple thereof), the result is zero. Therefore, the second integral is zero. Combining these results, we can write

$$c_k = \tfrac{1}{T}\left[2\int_0^{T/2} x(t)\cos(k\omega_0 t)dt\right]$$
$$= \tfrac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)dt. \tag{5.6}$$

Since x is real, the quantity c_k must also be real. Thus, we have that $\operatorname{Im}(c_k) = 0$.

Consider now the expression for c_{-k}. We substitute $-k$ for k in (5.6) to obtain

$$c_{-k} = \tfrac{2}{T}\int_0^{T/2} x(t)\cos(-k\omega_0 t)dt.$$

Since cos is an even function, we can simplify this expression to obtain

$$c_{-k} = \tfrac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)dt$$
$$= c_k.$$

Thus, $c_k = c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ into (5.6), we can write

$$\begin{aligned}
c_0 &= \tfrac{2}{T}\int_0^{T/2} x(t)\cos(0)dt \\
&= \tfrac{2}{T}\int_0^{T/2} x(t)dt \\
&= \tfrac{2}{T}\left[\tfrac{1}{2}\int_0^{T} x(t)dt\right] \\
&= \tfrac{1}{T}\int_0^{T} x(t)dt.
\end{aligned}$$

Thus, $c_0 = \frac{1}{T}\int_0^T x(t)dt$. ■

Example 5.5 (Fourier series of an odd real function)**.** Let x be a periodic real function that is odd. Let c denote the Fourier series coefficient sequence for x. Show that

- c is purely imaginary (i.e., $\operatorname{Re}\{c_k\} = 0$ for all k);
- c is odd (i.e., $c_k = -c_{-k}$ for all k); and
- $c_0 = 0$.

Solution. From the Fourier series analysis equation (5.2) and Euler's formula, we can write

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{T}\left[\int_T x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]dt\right].
\end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned}
c_k &= \tfrac{1}{T}\left[\int_T x(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right]dt\right] \\
&= \tfrac{1}{T}\left[\int_T x(t)\cos(k\omega_0 t)dt - j\int_T x(t)\sin(k\omega_0 t)dt\right].
\end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since x is odd and $\cos(k\omega_0 t)$ is even, we have that $x(t)\cos(k\omega_0 t)$ is odd. If we integrate an odd periodic function over a single period (or an integer multiple thereof), the result is zero. Therefore, the first integral is zero. Consider the second integral above (i.e., the one involving the sin function). Since x is odd and $\sin(k\omega_0 t)$ is odd, we have that $x(t)\sin(k\omega_0 t)$ is even. Thus, $\int_T x(t)\sin(k\omega_0 t)dt = 2\int_0^{T/2} x(t)\sin(k\omega_0 t)dt$. Combining these results, we can write

$$\begin{aligned}
c_k &= \tfrac{-j}{T}\int_T x(t)\sin(k\omega_0 t)dt \\
&= \tfrac{-j2}{T}\int_0^{T/2} x(t)\sin(k\omega_0 t)dt. \qquad (5.7)
\end{aligned}$$

Since x is real, the result of the integration is real, and consequently c_k is purely imaginary. Thus, $\operatorname{Re}(c_k) = 0$.

Consider the quantity c_{-k}. Substituting $-k$ for k in (5.7), we obtain

$$\begin{aligned}
c_{-k} &= \tfrac{-j2}{T}\int_0^{T/2} x(t)\sin(-k\omega_0 t)dt \\
&= \tfrac{-j2}{T}\int_0^{T/2} x(t)[-\sin(k\omega_0 t)]dt \\
&= \tfrac{j2}{T}\int_0^{T/2} x(t)\sin(k\omega_0 t)dt \\
&= -c_k.
\end{aligned}$$

Thus, $c_k = -c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ in the expression (5.7), we have

$$\begin{aligned} c_0 &= \tfrac{-j2}{T}\int_0^{T/2} x(t)\sin(0)dt \\ &= 0. \end{aligned}$$

Thus, $c_0 = 0$. ■

5.4 Convergence of Continuous-Time Fourier Series

So far we have assumed that a given periodic function x can be represented by a Fourier series. Since a Fourier series consists of an infinite number of terms (infinitely many of which may be nonzero), we need to more carefully consider the issue of convergence. That is, we want to know under what circumstances the Fourier series of x converges (in some sense) to x.

Suppose that we have an arbitrary periodic function x. This function has the Fourier series representation given by (5.1) and (5.2). Let x_N denote the finite series

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(i.e., x_N is a Fourier series truncated after the Nth harmonic components). The approximation error is given by

$$e_N(t) = x(t) - x_N(t).$$

Let us also define the mean-squared error (MSE) as

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2\, dt.$$

Before we can proceed further, we need to more precisely specify what we mean by convergence. This is necessary because convergence can be defined in more than one way. For example, two common types of convergence are: pointwise and MSE. In the case of pointwise convergence, the error goes to zero at every point. If convergence is pointwise and the rate of convergence is the same everywhere, we call this uniform convergence. In the case of MSE convergence, the MSE goes to zero, which does not necessarily imply that the error goes to zero at every point.

Now, we introduce a few important results regarding the convergence of Fourier series for various types of periodic functions. The first result that we consider is for the case of continuous functions as given below.

Theorem 5.2 (Convergence of Fourier series (continuous case))**.** *If a periodic function x is continuous and its Fourier series coefficient sequence c is absolutely summable (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges uniformly (i.e., converges pointwise and at the same rate everywhere).*

Proof. A rigorous proof of this theorem is somewhat involved and omitted here. ■

In other words, in the above theorem, we have that if x is continuous, then as $N \to \infty$, $e_N(t) \to 0$ for all t. Often, however, we must work with functions that are not continuous. For example, many useful periodic functions are not continuous (e.g., a square wave). Consequently, we must consider the matter of convergence for functions with discontinuities.

Another important result regarding convergence applies to functions that have finite energy over a single period. Mathematically, a function x has finite energy over a single period if it satisfies

$$\int_T |x(t)|^2\, dt < \infty.$$

In the case of such a function, we have the following important result.

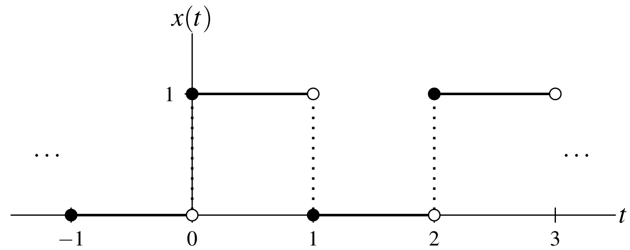

Figure 5.4: Periodic function x.

Theorem 5.3 (Convergence of Fourier series (finite-energy case))**.** *If the periodic function x has finite energy in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the MSE sense.*

Proof. A rigorous proof of this theorem is somewhat involved and omitted here. ∎

In other words, in the above theorem, we have that if x is of finite energy, then as $N \to \infty$, $E_N \to 0$.

The last important result regarding convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet[1] conditions for the periodic function x are as follows:

1. Over a single period, x is absolutely integrable (i.e., $\int_T |x(t)| dt < \infty$).
2. Over a single period, x has a finite number of maxima and minima (i.e., x is of bounded variation).
3. Over any finite interval, x has a finite number of discontinuities, each of which is finite.

Theorem 5.4 (Convergence of Fourier series (Dirichlet case))**.** *If x is a periodic function satisfying the Dirichlet conditions, then:*

1. *The Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x.*
2. *At each point t_a of discontinuity of x, the Fourier series converges to $\frac{1}{2}(x(t_a^-) + x(t_a^+))$, where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function on the left- and right-hand sides of the discontinuity, respectively.*

In other words, if the Dirichlet conditions are satisfied, then as $N \to \infty$, $e_N(t) \to 0$ for all t except at discontinuities. Furthermore, at each discontinuity, the Fourier series converges to the average of the function values on the left- and right-hand side of the discontinuity.

Example 5.6. Consider the periodic function x with period $T = 2$ as shown in Figure 5.4. Let $\hat{x}$ denote the Fourier series representation of x (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 5.4 applies. Thus, we have that

$$\begin{aligned}
\hat{x}(0) &= \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] \\
&= \tfrac{1}{2}(0+1) \\
&= \tfrac{1}{2} \quad \text{and} \\
\hat{x}(1) &= \tfrac{1}{2}\left[x(1^-) + x(1^+)\right] \\
&= \tfrac{1}{2}(1+0) \\
&= \tfrac{1}{2}.
\end{aligned}$$
∎

[1] Pronounced Dee-ree-klay.

Although many functions of practical interest satisfy the Dirichlet conditions, not all functions satisfy these conditions. In what follows, some examples of functions that violate these conditions are given.

Consider the 1-periodic function x defined by

$$x(t) = t^{-1} \text{ for } 0 \le t < 1 \quad \text{and} \quad x(t) = x(t+1).$$

A plot of x is shown in Figure 5.5(a). The function x violates the first Dirichlet condition, since x is not absolutely integrable over a single period.

Consider the 1-periodic function x defined by

$$x(t) = \sin\left(2\pi t^{-1}\right) \text{ for } 0 < t \le 1 \quad \text{and} \quad x(t) = x(t+1).$$

A plot of x is shown in Figure 5.5(b). Since x has an infinite number of minima and maxima over a single period, x violates the second Dirichlet condition.

Consider the 1-periodic function x shown in Figure 5.5(c). As t goes from 0 to 1, $x(t)$ traces out a sequence of steps in a staircase, where each step is half the size of the previous step. As it turns out, the number of steps in this staircase between 0 and 1 is infinite. Thus, x violates the third Dirichlet condition, as x has an infinite number of discontinuities over a single period.

One might wonder how the Fourier series converges for periodic functions with discontinuities. Let us consider the periodic square wave from Example 5.1. In Figure 5.6, we have plotted the truncated Fourier series x_N for the square wave (with period $T = 1$ and amplitude $A = 1$) for several values of N. At the discontinuities of x, we can see that the series appears to converge to the average of the function values on either side of the discontinuity. In the vicinity of a discontinuity, however, the truncated series x_N exhibits ripples and the peak amplitude of the ripples does not seem to decrease with increasing N. As it turns out, as N increases, the ripples get compressed towards the discontinuity, but, for any finite N, the peak amplitude of the ripples remains constant. This behavior is known as **Gibbs phenomenon**.

5.5 Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties. In the sections that follow, we introduce a number of these properties. For convenience, these properties are also summarized later in Table 5.1 (on page 135).

5.5.1 Linearity

Arguably, the most important property of Fourier series is linearity, as introduced below.

Theorem 5.5 (Linearity)**.** *Let x and y denote two periodic functions with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \textit{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k,$$

then

$$Ax(t) + By(t) \overset{\text{CTFS}}{\longleftrightarrow} Aa_k + Bb_k,$$

where A and B are complex constants. In other words, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

Proof. To prove the above property, we proceed as follows. First, we express x and y in terms of their corresponding Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

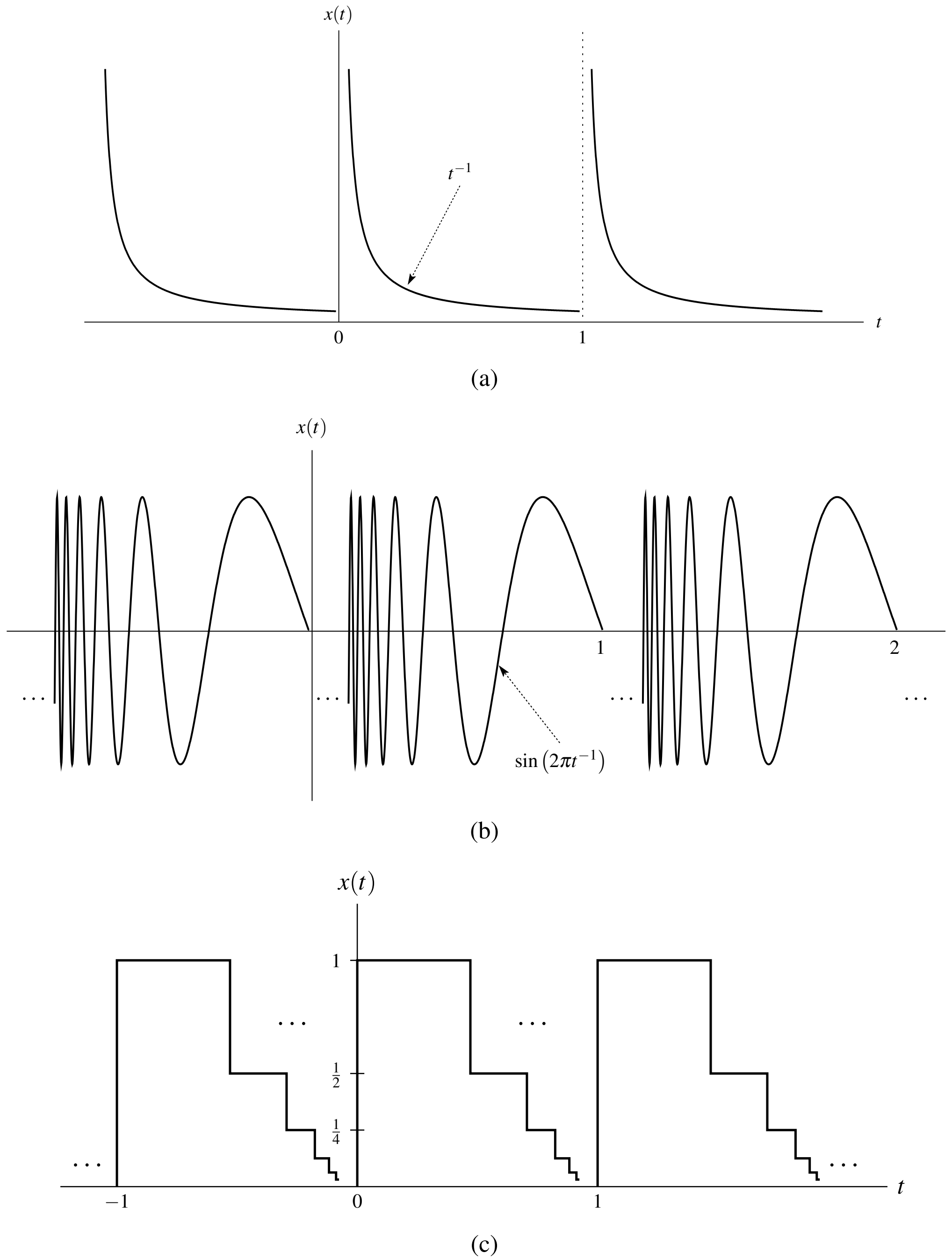

Figure 5.5: Examples of functions that violate the Dirichlet conditions. (a) A function that is not absolutely integrable over a single period. (b) A function that has an infinite number of maxima and minima over a single period. (c) A function that has an infinite number of discontinuities over a single period.

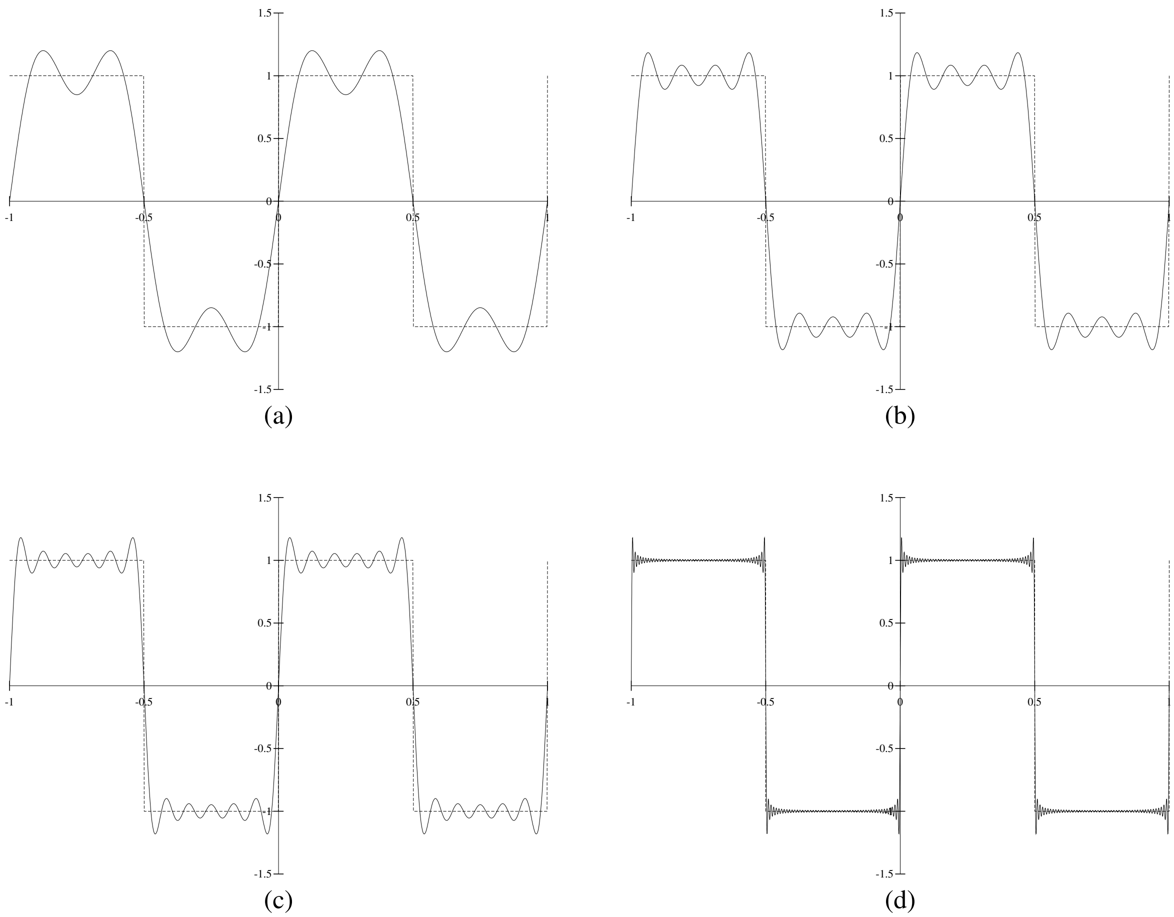

Figure 5.6: Gibbs phenomenon. The Fourier series for the periodic square wave truncated after the Nth harmonic components for (a) $N = 3$, (b) $N = 7$, (c) $N = 11$, and (d) $N = 101$.

Now, we determine the Fourier series of $Ax+By$. We have

$$\begin{aligned} Ax(t)+By(t) &= A\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} Aa_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} Bb_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} (Aa_k+Bb_k) e^{jk\omega_0 t}. \end{aligned}$$

Therefore, we have that $Ax(t)+By(t) \xleftrightarrow{\text{CTFS}} Aa_k+Bb_k$. ■

5.5.2 Time Shifting (Translation)

The next property of Fourier series to be introduced is the time-shifting (i.e., translation) property, as given below.

Theorem 5.6 (Time shifting (i.e., translation))**.** *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \xleftrightarrow{\mathit{CTFS}} a_k,$$

then

$$x(t-t_0) \xleftrightarrow{\mathit{CTFS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

where t_0 is a real constant.

Proof. To prove the time-shifting property, we proceed as follows. The Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{5.8}$$

We express $x(t-t_0)$ in terms of its Fourier series, and then use algebraic manipulation to obtain

$$\begin{aligned} x(t-t_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} \\ &= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}. \end{aligned} \tag{5.9}$$

Comparing (5.8) and (5.9), we have that $x(t-t_0) \xleftrightarrow{\text{CTFS}} e^{-jk\omega_0 t_0} a_k$. ■

From the above theorem, we can see that time shifting a periodic function does not change the magnitude of its Fourier series coefficients (since $\left|e^{j\theta}\right| = 1$ for all real θ).

5.5.3 Frequency Shifting (Modulation)

The next property of Fourier series to be introduced is the frequency-shifting (i.e., modulation) property, as given below.

Theorem 5.7 (Frequency shifting (i.e., modulation))**.** *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \xleftrightarrow{\mathit{CTFS}} a_k,$$

then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_{k-M},$$

where M is an integer constant.

Proof. To prove the frequency-shifting property, we proceed as follows. We have

$$\begin{aligned}
e^{j(2\pi/T)Mt}x(t) &= e^{j(2\pi/T)Mt}\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt} \\
&= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)Mt} e^{j(2\pi/T)kt} \\
&= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)(k+M)t}.
\end{aligned}$$

Now, we employ a change of variable. Let $k' = k+M$ so that $k = k'-M$. Applying the change of variable and dropping the primes, we obtain

$$e^{j(2\pi/T)Mt}x(t) = \sum_{k=-\infty}^{\infty} a_{k-M} e^{j(2\pi/T)kt}.$$

Now, we observe that the right-hand side of this equation is a Fourier series. Therefore, the Fourier series coefficient sequence for $e^{j(2\pi/T)Mt}x(t)$ (i.e., the left-hand side of the equation) is $a'(k) = a_{k-M}$. ■

5.5.4 Time Reversal (Reflection)

The next property of Fourier series to be introduced is the time-reversal (i.e., reflection) property, as given below.

Theorem 5.8 (Time reversal (i.e., reflection)). *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} a_{-k}.$$

Proof. To prove the time-reversal property, we proceed in the following manner. The Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{5.10}$$

Now, we consider the Fourier series expansion of $x(-t)$. The Fourier series in this case is given by

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)}. \tag{5.11}$$

Now, we employ a change of variable. Let $l = -k$ so that $k = -l$. Performing this change of variable, we can rewrite (5.11) to obtain

$$\begin{aligned}
x(-t) &= \sum_{l=-\infty}^{\infty} a_{-l} e^{j(-l)\omega_0(-t)} \\
&= \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\omega_0 t}.
\end{aligned} \tag{5.12}$$

Comparing (5.10) and (5.12), we have that $x(-t) \overset{\text{CTFS}}{\longleftrightarrow} a_{-k}$. ■

In other words, the above theorem states that time reversing a function time reverses the corresponding sequence of Fourier series coefficients.

5.5.5 Conjugation

The next property of Fourier series to be introduced is the conjugation property, as given below.

Theorem 5.9 (Conjugation)**.** *For a T-periodic function x with Fourier series coefficient sequence c,*

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}.$$

Proof. From the definition of a Fourier series, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing k by $-k$ in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c^*_{-k} e^{jk\omega_0 t}.$$

Thus, the Fourier series coefficient sequence c' of x^* is $c'_k = c^*_{-k}$. ■

In other words, the above theorem states that conjugating a function has the effect of time reversing and conjugating the corresponding Fourier series coefficient sequence.

5.5.6 Periodic Convolution

The next property of Fourier series to be introduced is the periodic-convolution property, as given below.

Theorem 5.10 (Periodic convolution)**.** *Let x and y be T-periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$. Let $z(t) = x \circledast y(t)$, where

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences a, b, and c are related as

$$c_k = T a_k b_k.$$

