Exercise 6.19

L **Answer (b).**

From circuit analysis, we have

$$v_0(t) = Ri(t) + \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + v_1(t) \quad \text{and} \quad 1$$
$$i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v_1(\tau)d\tau. \quad 2$$

Taking the derivative of the preceding two equations (in order to eliminate the integral operators which would cause difficulties later), we have

derivative of 1

$$3 \quad \mathcal{D}v_0(t) = R\mathcal{D}i(t) + \tfrac{1}{C}i(t) + \mathcal{D}v_1(t) \quad \text{and}$$
$$4 \quad \mathcal{D}i(t) = \tfrac{1}{L}v_1(t).$$

derivative of 2

Taking the Fourier transform of these two equations, we obtain

take FT of 3

$$5 \quad j\omega V_0(\omega) = jR\omega I(\omega) + \tfrac{1}{C}I(\omega) + j\omega V_1(\omega) \quad \text{and}$$
$$j\omega I(\omega) = \tfrac{1}{L}V_1(\omega) \quad \Rightarrow \quad I(\omega) = \tfrac{1}{jL\omega}V_1(\omega). \quad 6$$

take FT of 4 and solve for I

Combining the preceding two equations, we obtain

substitute 6 into 5

$$j\omega V_0(\omega) = jR\omega\left[\frac{1}{jL\omega}V_1(\omega)\right] + \tfrac{1}{C}\left[\frac{1}{jL\omega}V_1(\omega)\right] + j\omega V_1(\omega) \quad \Rightarrow$$
$$j\omega V_0(\omega) = \frac{R}{L}V_1(\omega) + \frac{1}{jLC\omega}V_1(\omega) + j\omega V_1(\omega).$$

simplify

Rearranging the preceding equation, we have

factor out V_0 and V_1

$$[j\omega]V_0(\omega) = \left[\frac{R}{L} + \frac{1}{jLC\omega} + j\omega\right]V_1(\omega) \quad \Rightarrow$$
$$[j\omega]V_0(\omega) = \left[\frac{jRC\omega + 1 - LC\omega^2}{jLC\omega}\right]V_1(\omega).$$

common denominator

Solving for $\frac{V_1(\omega)}{V_0(\omega)}$, we obtain

multiply both sides by $\left(\frac{jLC\omega}{jRC\omega+1-LC\omega^2}\right)\left(\frac{1}{V_0(\omega)}\right)$

$$\begin{aligned}\frac{V_1(\omega)}{V_0(\omega)} &= j\omega\left[\frac{jLC\omega}{jRC\omega + 1 - LC\omega^2}\right] \\ &= \frac{-LC\omega^2}{-LC\omega^2 + jRC\omega + 1} \\ &= \frac{LC\omega^2}{LC\omega^2 - jRC\omega - 1}.\end{aligned}$$

multiply

flip sign of numerator and denominator

Since the system is LTI, $H(\omega) = \frac{V_1(\omega)}{V_0(\omega)}$. Thus, we have

$H(\omega) = \frac{V_1(\omega)}{V_0(\omega)}$

$$H(\omega) = \frac{LC\omega^2}{LC\omega^2 - jRC\omega - 1}. \quad 7$$

Computing $|H(0)|$ and $\lim_{|\omega|\to\infty}|H(\omega)|$, we obtain

take magnitude of 7 and let $\omega = 0$

take magnitude of 7 and $|\omega| \to \infty$

$$|H(0)| = \left|\frac{0}{-1}\right| = 0 \quad \text{and} \quad \lim_{|\omega|\to\infty}|H(\omega)| = \lim_{|\omega|\to\infty}\left|\frac{LC\omega^2}{LC\omega^2}\right| = 1.$$

Since $|H(0)| = 0$ and $\lim_{|\omega|\to\infty}|H(\omega)| = 1$, the system best approximates a highpass filter.