Theorem 4.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. Using the fact that δ is the convolutional identity, we can write

convolutional identity

$$\mathcal{H}x(t) = \mathcal{H}\{x * \delta\}(t).$$

Rewriting the convolution in terms of an integral, we have

rewrite convolution as integral

$$\mathcal{H}x(t) = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot - \tau)d\tau\right\}(t).$$

Since $\mathcal{H}$ is a linear operator, we can pull the integral and $x(\tau)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

interchange $\mathcal{H}$ with both $x(\tau)$ and integral (linearity)

$$\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{\mathcal{H}\{\delta(\cdot - \tau)\}(t)}d\tau.$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by τ (i.e., $\underbrace{\mathcal{H}\{\delta(\cdot - \tau)\}}_{\text{shift by } \tau \text{ then } \mathcal{H}} = \underbrace{H\delta(\cdot - \tau)}_{\mathcal{H} \text{ then shift by } \tau}$) and then use the fact that $h = \mathcal{H}\delta$ to obtain

interchange $\mathcal{H}$ and time shift (time invariance)

$$\begin{aligned}\mathcal{H}x(t) &= \int_{-\infty}^{\infty} x(\tau)\underbrace{\mathcal{H}\delta(t - \tau)}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\ &= x * h(t).\end{aligned}$$

$h = \mathcal{H}\delta$ (by definition)

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■

Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.23}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.24}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t). \quad 1$$

Substituting (4.23) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{from 1} \\
&= x * u(t) && \text{substitute given function } h \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau && \text{definition of convolution} \\
&= \int_{-\infty}^{t} x(\tau)\underbrace{u(t-\tau)}_{1}d\tau + \int_{t^+}^{\infty} x(\tau)\underbrace{u(t-\tau)}_{0}d\tau && \text{split into two integrals} \\
&= \int_{-\infty}^{t} x(\tau)d\tau. && \text{second integral is 0}
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.23) is, in fact, the ideal integrator given by (4.24). ■

Example 4.7. Consider the system with input x, output y, and impulse response h as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

To begin, we label all signals in Figure 4.9.

①

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

δ is convolutional identity

distributive property

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t). \quad ②$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [\underbrace{h_3 + h_1 * h_3 + h_2 * h_3}_{h}](t). \end{aligned}$$

from ②

substituting ① for v

distributive and associative properties and convolutional identity

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t). \qquad \blacksquare$$

Recall that, for any LTI system with input x, output y, and impulse response h, y = x * h.

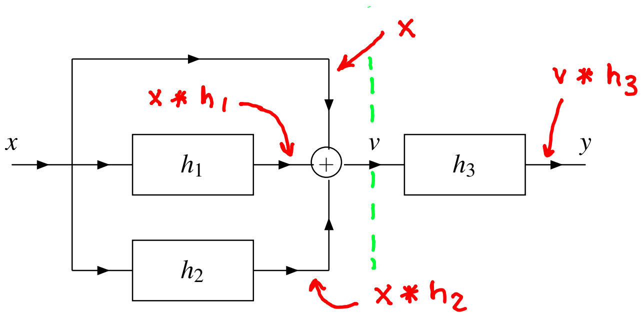


Figure 4.9: System interconnection example.

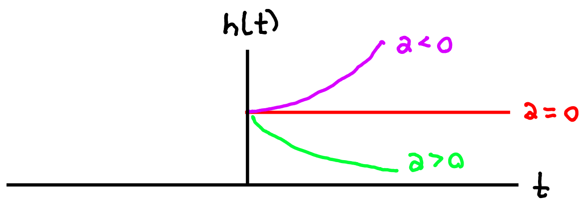
Example 4.8. Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

condition for memorylessness violated



memoryless $\Leftrightarrow$ $h(t) = 0$ for all $t \neq 0$