Exercise A.4

L Answer (a).

We are asked to show that, for all complex numbers z_1 and z_2 such that $z_2 \neq 0$, the following identity holds:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Let z_1 and z_2 be arbitrary complex numbers (where $z_2 \neq 0$) with the polar representations

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2},$$

where r_1, r_2, θ_1, and θ_2 are real constants, and $r_1 \geq 0$ and $r_2 > 0$. Consider the left-hand side of the given equation, which we can manipulate as follows:

replace z_1 and z_2 with their polar representations

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right| \\
&= \left|\left(\frac{r_1}{r_2}\right)\left(\frac{e^{j\theta_1}}{e^{j\theta_2}}\right)\right| && \text{rearrange factors} \\
&= \left|\left(\frac{r_1}{r_2}\right) e^{j(\theta_1-\theta_2)}\right| && \text{exponent laws} \\
&= \frac{r_1}{r_2} && \text{definition of polar form} \\
&= \frac{|z_1|}{|z_2|}. && \text{definition of } z_1 \text{ and } z_2
\end{aligned}$$

(In the preceding steps, we used the fact that $r_1/r_2 \geq 0$, which must be true, since $r_1 \geq 0$ and $r_2 > 0$.) Thus, the given identity holds.