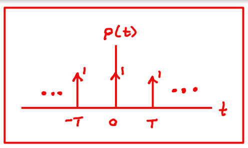
Sampling: Fourier Series for a Periodic Impulse Train



✱

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

1 $$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$ p has Fourier series representation, since p is periodic

2 $$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t}\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t}\,dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t}\,dt \\ &= \frac{1}{T} \\ &= \frac{\omega_s}{2\pi} \end{aligned}$$

Fourier series analysis equation

see plot of p in figure ✱

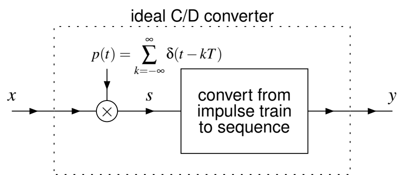
integrand is zero everywhere outside integration interval

sifting property

$T = \frac{2\pi}{\omega_s}$ by definition

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$ substitute 2 into 1

Sampling: Multiplication by a Periodic Impulse Train



use modulation property, not multiplication property !!!

$$\underset{①}{s(t) = p(t)x(t)}, \quad p(t) = \underset{②}{\sum_{k=-\infty}^{\infty} \delta(t-kT)}, \quad \omega_s = \frac{2\pi}{T}$$

③ $$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

result of finding Fourier series representation of p in ②

$$s(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} x(t)$$

substitute Fourier series representation of p in ③ into ①

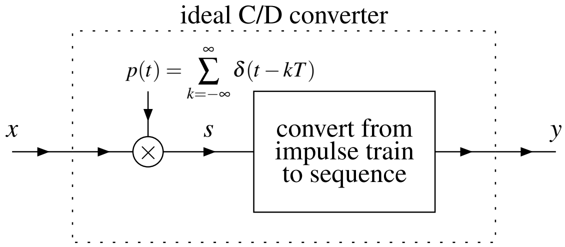
$$X = \mathcal{F}x, \quad S = \mathcal{F}s$$

take FT using modulation property

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Analysis of Sampling System

s(t) = x(t) p(t)   (6.51)



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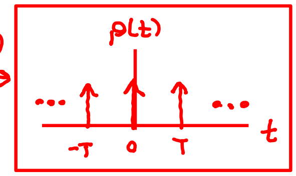

Figure 6.36: Model of ideal C/D converter with input function x and output sequence y.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original function x and its impulse-train sampled version s. In what follows, let X, Y, P, and S denote the Fourier transforms of x, y, p, and s, respectively. Since p is T-periodic, it can be represented in terms of a Fourier series as

from definition of Fourier series

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{6.52}$$

Using the Fourier series analysis equation, we calculate the coefficients c_k to be

Fourier series analysis equation

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}\int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \\ &= \frac{\omega_s}{2\pi}. \end{aligned} \tag{6.53}$$

⊛

sifting property

$T = \frac{2\pi}{\omega_s}$

Substituting (6.52) and (6.53) into (6.51), we obtain

$s(t) = x(t)\,p(t)$

replace p(t) by its Fourier series representation

$$\begin{aligned} s(t) &= x(t) \sum_{k=-\infty}^{\infty} \frac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}. \end{aligned}$$

rearrange

Taking the Fourier transform of s yields

take FT using frequency-domain shifting property

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{6.54}$$

Thus, the spectrum of the impulse-train sampled function s is a scaled sum of an infinite number of shifted copies of the spectrum of the original function x.

Example 6.41. Let x denote a continuous-time audio signal with Fourier transform X. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period T with which x can be sampled that will allow x to be exactly recovered from its samples.

44100π rad/s = 22.05 kHz

Solution. The function x is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

from sampling theorem

$$\begin{aligned} \omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi. \end{aligned}$$

$\omega_m = 44100\pi$

88200π rad/s = 44.1 kHz

Thus, the largest permissible sampling period is given by

take reciprocal for corresponding sampling period

$$\begin{aligned} T &= \frac{2\pi}{\omega_s} \\ &= \frac{2\pi}{88200\pi} \\ &= \frac{1}{44100}. \end{aligned}$$

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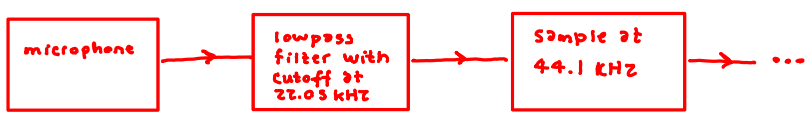
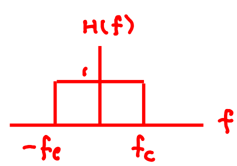
Why does CD-quality audio use a sampling rate of 44.1 kHz?

In practice, how do we ensure the audio signal to be sampled is sufficiently bandlimited?

The human auditory system (assuming pristine hearing) can sense frequencies up to about 22.05 kHz.



- filter prevents aliasing
- removed frequencies cannot be detected by humans



$f_c = 22.05$ kHz