Complex Exponential Functions

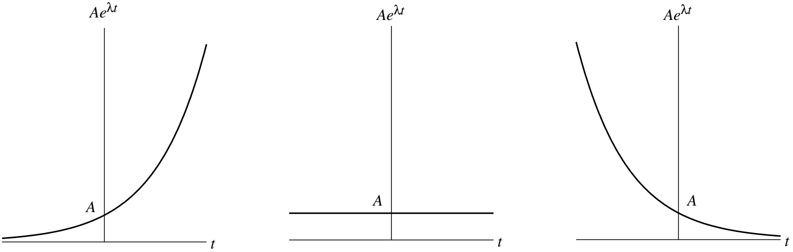
- A **complex exponential function** is a function of the form

$$x(t) = Ae^{\lambda t},$$

  where A and λ are ***complex*** constants.
- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

Real Exponential Functions

- A **real exponential function** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



$\lambda > 0$ $\lambda = 0$ $\lambda < 0$

Complex Sinusoidal Functions

- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a **complex sinusoidal function** is a function of the form

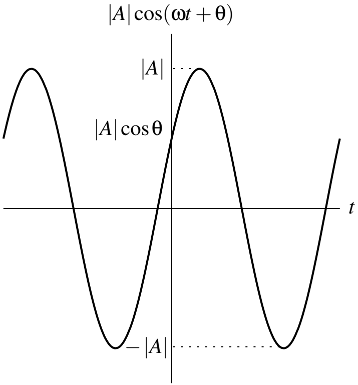
$$x(t) = Ae^{j\omega t},$$

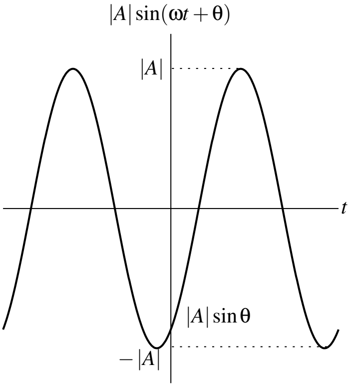
  where A is ***complex*** and ω is ***real***.
- By expressing A in polar form as $A = |A|\,e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

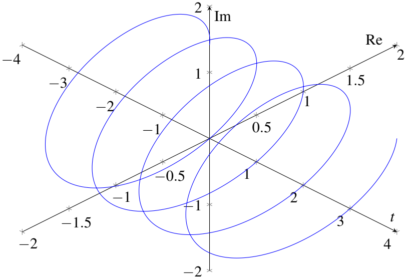
- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.
- Also, x is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.

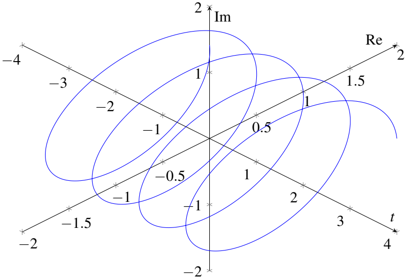




Plot of $x(t) = e^{j\omega t}$ for $\omega \in \{2\pi, -2\pi\}$



$\omega = 2\pi$



$\omega = -2\pi$

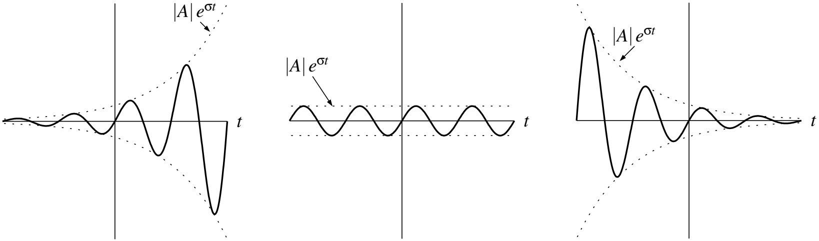
General Complex Exponential Functions

- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both ***complex***.
- Letting $A = |A| e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| e^{\sigma t} \cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j \underbrace{|A| e^{\sigma t} \sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of ***three distinct modes*** of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are ***real sinusoids***.
- If $\sigma > 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $\sigma < 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

- The ***three modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



$\sigma > 0$ $\quad$ $\sigma = 0$ $\quad$ $\sigma < 0$

Relationship Between Complex Exponentials and Real Sinusoids

SKIP SLIDE

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t).$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

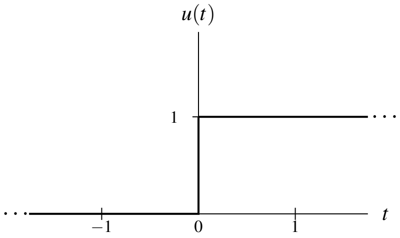
$$A\cos(\omega t + \theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$
$$A\sin(\omega t + \theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

- Note that, above, we are simply ***restating results*** from the (appendix) material on complex analysis.

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

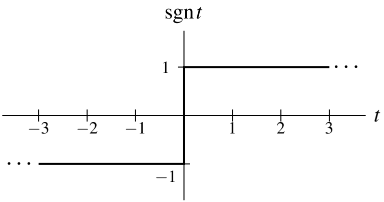
- Due to the manner in which u is used in practice, the actual ***value of*** $u(0)$ is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.



- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the ***sign*** of a number.
- A plot of this function is shown below.

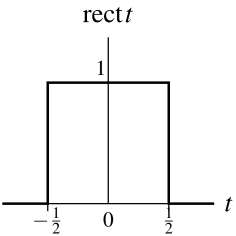


Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\operatorname{rect} t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\operatorname{rect} t$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.



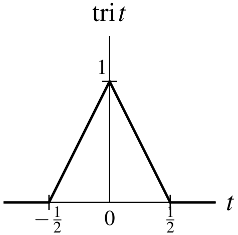
Triangular Function

SKIP SLIDE

- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2\left|t\right| & \left|t\right| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

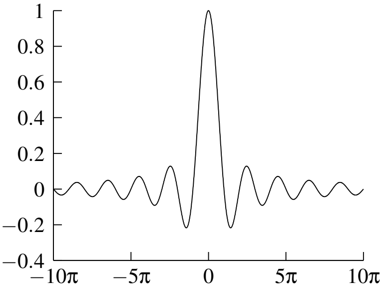
- A plot of this function is shown below.



- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc} t = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc} t$ do not die out for finite t.]



Floor and Ceiling Functions

SKIP SLIDE

- The **floor function**, denoted $\lfloor\cdot\rfloor$, is a function that maps a real number x to the largest integer not more than x.
- In other words, the floor function rounds a real number to the nearest integer in the direction of negative infinity.
- For example,

$$\left\lfloor -\tfrac{1}{2}\right\rfloor = -1, \quad \left\lfloor \tfrac{1}{2}\right\rfloor = 0, \quad \text{and} \quad \lfloor 1\rfloor = 1.$$

- The **ceiling function**, denoted $\lceil\cdot\rceil$, is a function that maps a real number x to the smallest integer not less than x.
- In other words, the ceiling function rounds a real number to the nearest integer in the direction of positive infinity.
- For example,

$$\left\lceil -\tfrac{1}{2}\right\rceil = 0, \quad \left\lceil \tfrac{1}{2}\right\rceil = 1, \quad \text{and} \quad \lceil 1\rceil = 1.$$

Some Properties of the Floor and Ceiling Functions

SKIP SLIDE

- Several useful properties of the floor and ceiling functions include:

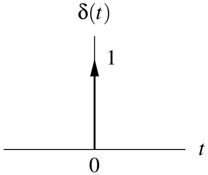
$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$
$$\lceil x+n \rceil = \lceil x \rceil + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$
$$\lceil x \rceil = -\lfloor -x \rfloor \quad \text{for } x \in \mathbb{R};$$
$$\lfloor x \rfloor = -\lceil -x \rceil \quad \text{for } x \in \mathbb{R};$$
$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0; \quad \text{and}$$
$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil = \left\lceil \frac{m+1}{n} \right\rceil - 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0.$$

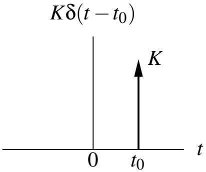
Unit-Impulse Function

- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

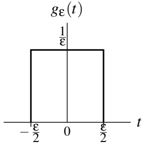
$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.





- Define
$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$
- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:
$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$
- That is, δ can be viewed as a ***limiting case of a rectangular pulse*** where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Properties of the Unit-Impulse Function

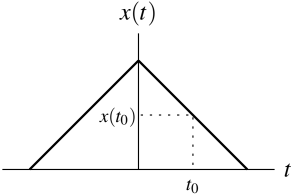
- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

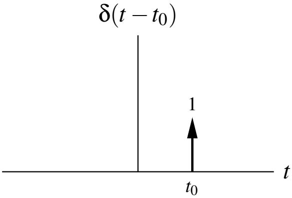
- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

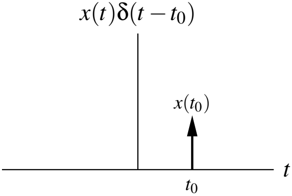
Example 3.8 Example 3.9 Example 3.10 NEXT SLIDE



Function x



Time-Shifted Unit-Impulse Function



Product

Representing a Rectangular Pulse (Using Unit-Step Functions)

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., x is a ***rectangular pulse*** of height one, with a ***rising edge at a*** and ***falling edge at b***).
- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).
- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

Example 3.11

Representing Functions Using Unit-Step Functions

- The idea from the previous slide can be extended to handle any function that is defined in a ***piecewise manner*** (i.e., via an expression involving multiple cases).
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Example 3.12

Section 3.4

Continuous-Time (CT) Systems

CT Systems

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}x,$$

  where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a function to a function*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

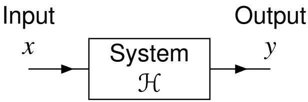
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \rightarrow y.$$

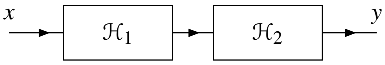
- Note that the symbols "$\rightarrow$" and "=" have ***very different*** meanings.
- The symbol "$\rightarrow$" should be read as "***produces***" (not as "equals").

Block Diagram Representations

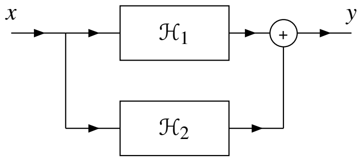
- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.



- ***Two basic ways*** in which systems can be interconnected are shown below.



Series



Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2\mathcal{H}_1 x.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x.$$