Example 6.35 (Frequency response to differential equation). A LTI system with input x and output y has the frequency response

$$H(\omega) = \frac{-7\omega^2 + 11 j\omega + 3}{-5\omega^2 + 2}.$$

Find the differential equation that characterizes this system.

Solution. From the given frequency response H, we have

$$\frac{Y(\omega)}{X(\omega)} = \frac{-7\omega^2 + 11 j\omega + 3}{-5\omega^2 + 2}.$$

since system is LTI,
$Y(\omega) = X(\omega) H(\omega) \Rightarrow$
$H(\omega) = \frac{Y(\omega)}{X(\omega)}$

Multiplying both sides by $(-5\omega^2 + 2)X(\omega)$, we have

$$-5\omega^2 Y(\omega) + 2Y(\omega) = -7\omega^2 X(\omega) + 11 j\omega X(\omega) + 3X(\omega).$$

write with powers of jw

Applying some simple algebraic manipulation yields

$$5(j\omega)^2 Y(\omega) + 2Y(\omega) = 7(j\omega)^2 X(\omega) + 11(j\omega)X(\omega) + 3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$5y''(t) + 2y(t) = 7x''(t) + 11x'(t) + 3x(t). \qquad \blacksquare$$

$\left(\frac{d}{dt}\right)^n x(t) \overset{FT}{\longleftrightarrow} (j\omega)^n X(\omega)$