Example 7.12 (Time-domain scaling property)**.** Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \mathrm{Re}(s) < 1,$$

find the Laplace transform of the function

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \text{ for } -1 < \mathrm{Re}(s) < 1.$$

time and amplitude scale

ROC scales by 3

Using the time-domain scaling property, we can deduce

time scale by 3

$$x(t) = e^{-|3t|} \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{|3|}\frac{2}{1-(\frac{s}{3})^2} \quad \text{for } \underbrace{3(-1)}_{-3} < \mathrm{Re}(s) < \underbrace{3(1)}_{3}.$$

Thus, we have

$$X(s) = \frac{2}{3\left[1-(\frac{s}{3})^2\right]} \quad \text{for } -3 < \mathrm{Re}(s) < 3.$$

not strictly necessary except as sanity check

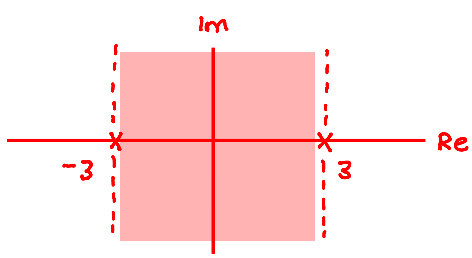
Simplifying, we have

$$X(s) = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \mathrm{Re}(s) < 3.$$

■



sanity check: are stated algebraic expression and stated ROC self consistent?

yes, ROC is bounded by poles

Example 7.19. Using properties of the Laplace transform and the Laplace transform pair

$$e^{-a|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{-2a}{(s+a)(s-a)} \quad \text{for } -a < \operatorname{Re}(s) < a,$$

find the Laplace transform X of the function

$$x(t) = e^{-5|3t-7|}.$$

Solution. We begin by re-expressing x in terms of the following equations:

$$\begin{aligned} &(1) & v_1(t) &= e^{-5|t|}, \\ &(2) & v_2(t) &= v_1(t-7), \quad \text{and} \\ &(3) & x(t) &= v_2(3t). \end{aligned}$$

$$\boxed{\begin{aligned} &\text{sanity check:} \\ x(t) &= v_2(3t) \\ &= v_1(3t-7) \\ &= e^{-5|3t-7|} \end{aligned}}$$

In what follows, let R_{V_1}, R_{V_2}, and R_X denote the ROCs of V_1, V_2, and X, respectively. Taking the Laplace transform of the above three equations, we obtain

algebraic expression | ROC

$$\begin{aligned} &(4) & V_1(s) &= \frac{-10}{(s+5)(s-5)}, & R_{V_1} &= (-5 < \operatorname{Re}(s) < 5), && \leftarrow \text{from LT of (1) using given LT pair} \\ &(5) & V_2(s) &= e^{-7s}V_1(s), & R_{V_2} &= R_{V_1}, && \leftarrow \text{from LT of (2) using time-domain shifting property} \\ &(6) & X(s) &= \tfrac{1}{3}V_2(s/3), \quad \text{and} & R_X &= 3R_{V_2}. && \leftarrow \text{from LT of (3) using time-scaling property} \end{aligned}$$

Combining the above equations, we have

$$\begin{aligned} (6) \longrightarrow \quad X(s) &= \tfrac{1}{3}V_2(s/3) && \text{substituting (5) for } V_2 \\ &= \tfrac{1}{3}e^{-7(s/3)}V_1(s/3) && \text{multiply} \\ &= \tfrac{1}{3}e^{-7s/3}V_1(s/3) && \text{substituting (4) for } V_1 \\ &= \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} && \text{and} \end{aligned}$$

$$\begin{aligned} (6) \longrightarrow \quad R_X &= 3R_{V_2} && \text{substituting (5) for } R_{V2} \\ &= 3R_{V_1} && \text{substituting (4) for } R_{V1} \\ &= 3(-5 < \operatorname{Re}(s) < 5) && \text{multiply} \\ &= -15 < \operatorname{Re}(s) < 15. \end{aligned}$$

Thus, we have shown that

$$X(s) = \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{for } -15 < \operatorname{Re}(s) < 15. \qquad \blacksquare$$

Example 7.13 (Conjugation property)**.** Using only properties of the Laplace transform and the transform pair

$$\underbrace{e^{(-1-j)t}u(t)}_{v(t)} \stackrel{\text{LT}}{\longleftrightarrow} \underbrace{\frac{1}{s+1+j}}_{V(s)} \quad \text{for } \operatorname{Re}(s) > -1,$$

find the Laplace transform of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. To begin, let $v(t) = e^{(-1-j)t}u(t)$ (i.e., v is the function whose Laplace transform is given in the Laplace-transform pair above) and let V denote the Laplace transform of v. First, we determine the relationship between x and v. We have

$$\begin{aligned} x(t) &= \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^* \\ &= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^* \\ &= \left[e^{(-1-j)t}u(t)\right]^* \\ &= v^*(t). \end{aligned}$$

$z^{**} = z$

$(z_1 z_2)^* = z_1^* z_2^*$

u is real

from definition of v

Thus, $x = v^*$. Next, we find the Laplace transform of x. We are given

$$v(t) = e^{(-1-j)t}u(t) \stackrel{\text{LT}}{\longleftrightarrow} V(s) = \frac{1}{s+1+j} \quad \text{for } \operatorname{Re}(s) > -1.$$

Using the conjugation property, we can deduce

conjugate

conjugate s and overall

ROC unchanged

$$x(t) = e^{(-1+j)t}u(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{1}{s^*+1+j}\right)^* \quad \text{for } \operatorname{Re}(s) > -1.$$

Simplifying the algebraic expression for X, we have

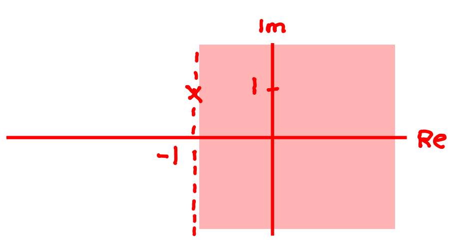
$$X(s) = \left(\frac{1}{s^*+1+j}\right)^* = \frac{1^*}{[s^*+1+j]^*} = \frac{1}{s+1-j}.$$

$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$ $\qquad$ $(z_1+z_2)^* = z_1^* + z_2^*$

Therefore, we can conclude

$$X(s) = \frac{1}{s+1-j} \quad \text{for } \operatorname{Re}(s) > -1.$$

■



sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, the ROC is bounded by poles or extends to $\pm\infty$

Example 7.14 (Time-domain convolution property)**.** Find the Laplace transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

LT table

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

Solution. From Table 7.2, we have that

$$\left.\begin{aligned} x_1(t) = \sin(3t)u(t) &\overset{\text{LT}}{\longleftrightarrow} X_1(s) = \frac{3}{s^2+9} \text{ for } \operatorname{Re}(s) > 0 \quad \text{and} \\ x_2(t) = tu(t) &\overset{\text{LT}}{\longleftrightarrow} X_2(s) = \frac{1}{s^2} \text{ for } \operatorname{Re}(s) > 0. \end{aligned}\right\} \text{from LT table}$$

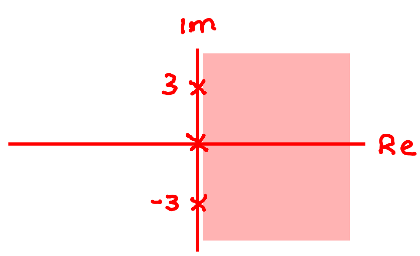
Using the time-domain convolution property, we have

ROC equals intersection since no pole-zero cancellation

$$\underbrace{x_1 * x_2(t)}_{\text{convolve}} = x(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \underbrace{\left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right)}_{\text{multiply}} \text{ for } \{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of X is $\{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation occurs. Simplifying the expression for X, we conclude

$$X(s) = \frac{3}{s^2\underbrace{(s^2+9)}_{(s+3j)(s-3j)}} \text{ for } \operatorname{Re}(s) > 0. \qquad A \cap A = A \qquad \blacksquare$$



sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, the ROC is bounded by poles or extends to $\pm\infty$