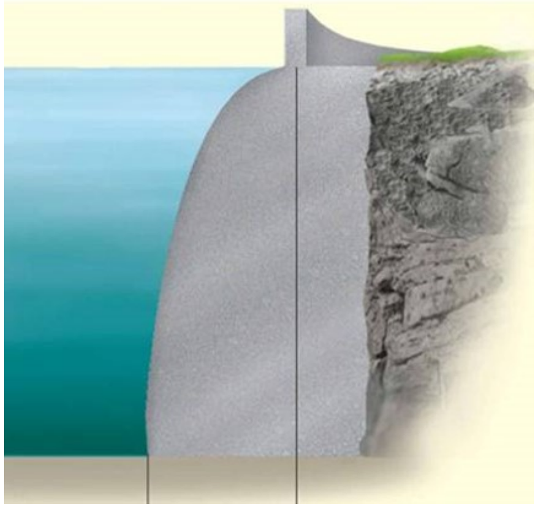
Chapter 9 – Centre of Gravity & Centroid

Contents



Centre of Gravity, Centre of Mass, & Centroid

The *centroid* is the geometric centre of the body, the mean position of all the points in all the coordinate directions. It is required for determining the resultant of a *distributed load*

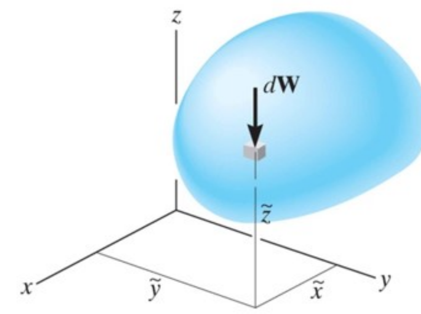
The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single *equivalent force* equal to the weight of the body and applied at the *centre of gravity* for the body.

The *centre of mass* is the mean position of all the elements of mass, of great importance in *spatial dynamics* and *composite bodies*.

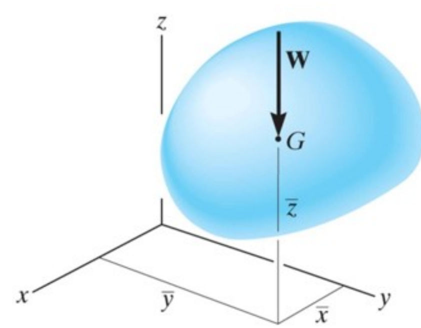
In a homogeneous body with a uniform gravitational field all these centres correspond to the same point.

Centre of Gravity

A body is composed of an infinite number of particles, and so if the body is located within a gravitational field, then each of these particles will have a weight $d\boldsymbol{W}$.



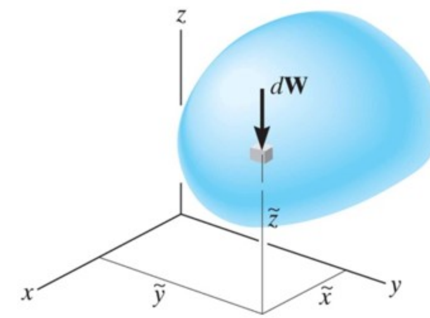
The **center of gravity (G)** is the point that locates the resultant weight of a system of particles or a solid body.



From the definition of a resultant force, the sum of moments due to all particle weights about any point is the same as the moment due to the resultant weight located at G. The sum of moments due to all individual particle's weights about point G is equal to zero.

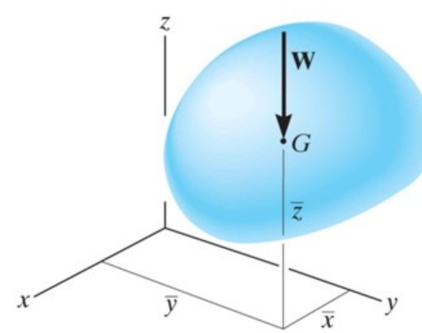
Centre of Gravity

The location of the centre of gravity, measured from the *x* axis, is determined by equating the moment of **W** about the *y*-axis to the sum of the moments of the weights of the particles about this same axis.



If dW is located at point $(\tilde{x}, \tilde{y}, \tilde{z})$, then

$$\bar{x}W = \int \tilde{x}\, dW \qquad \bar{y}W = \int \tilde{y}\, dW \qquad \bar{z}W = \int \tilde{z}\, dW$$



Therefore, the location of the centre of gravity G with respect to the x, y, z-axes becomes

1st moments of weight

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$

Total weight of the body

Centre of Mass & Centroids

By replacing the *W* with an *m* in these equations, the coordinates of the center of mass can be found.

1st moments of mass

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\, dm}{\int dm}$$

Total mass of the body

Similarly, the coordinates of the centroid of volume, area, or length can be obtained by replacing *W* by *V*, *A*, or *L*, respectively. For example, recalling Chapter 7 (distributed loads):

1st moments of area.

$$\bar{x} = \frac{\int_{x=0}^{x=L} x\, dA}{\int_{x=0}^{x=L} dA} \quad \text{and by analogy:} \quad \bar{y} = \frac{\int_{y=0}^{y=h} y\, dA}{\int_{y=0}^{y=h} dA}$$

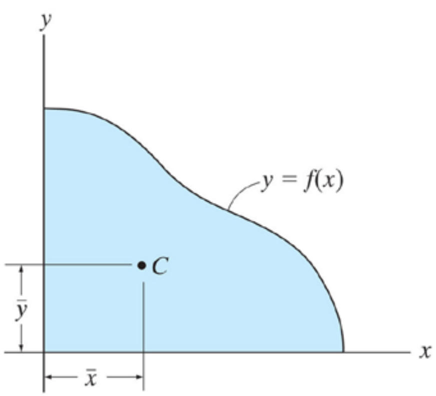
Centroids

Centroid of an Area

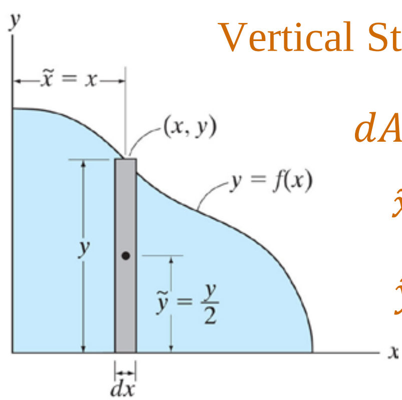
If an area lies in the x-y plane and is bounded by a curve *y = f(x)*, then its centroid is defined as

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

This integrals can be evaluated by performing a single integration using a rectangular strip of the differential area element.



Vertical Strip

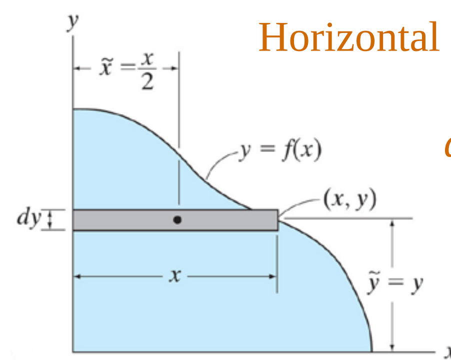


$$dA = y\,dx$$

$$\tilde{x} = x$$

$$\tilde{y} = {}^{y}/_{2}$$

Horizontal Strip



$$dA = x\,dy$$

$$\tilde{y} = y$$

$$\tilde{x} = {}^{x}/_{2}$$

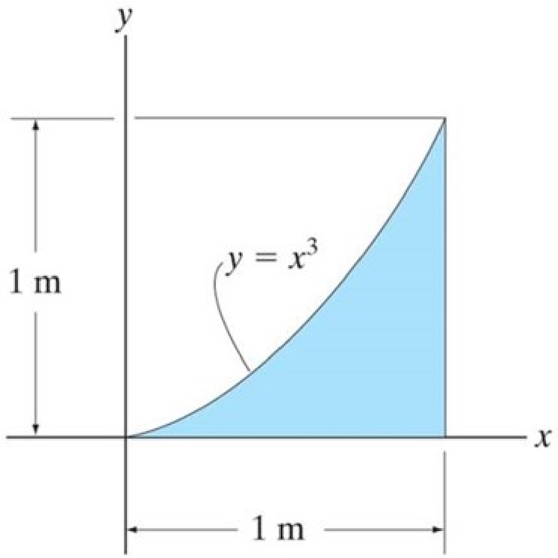
You must be careful when the differential area does not start from the axis.

Centroids

Steps to determine the centroid of an Area

1. Choose an appropriate differential element dA at a general point (x, y). Use vertical strip when $y = f(x)$ and horizontal strips when $x = f(y)$.
2. Express dA in terms of the differentiating element dx (or dy).
3. Determine $(\tilde{x}, \tilde{y})$ of the centroid rectangular strip in terms of the general point (x, y).
4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

Sample Problem (§ 9.1)



Given: The area shown

Find: The centroid of the area

Plan:

Since $y = f(x)$, use vertical strips.

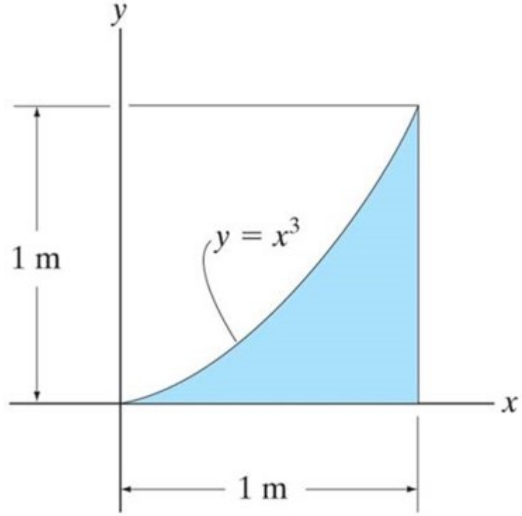
Find dA and $(\tilde{x}, \tilde{y})$.

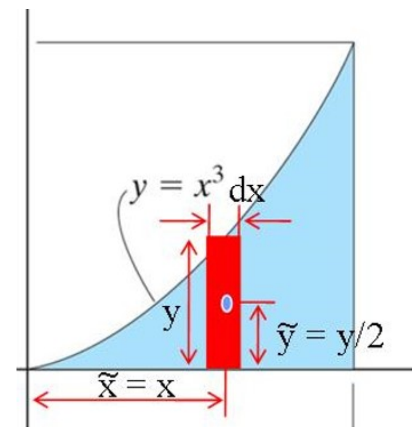
Find centroid $(\bar{x}, \bar{y})$

Find dA and $(\tilde{x}, \tilde{y})$

$$dA = ydx = x^3dx$$

$$\tilde{x} = x \quad \text{and} \quad \tilde{y} = \frac{y}{2} = \frac{x^3}{2}$$



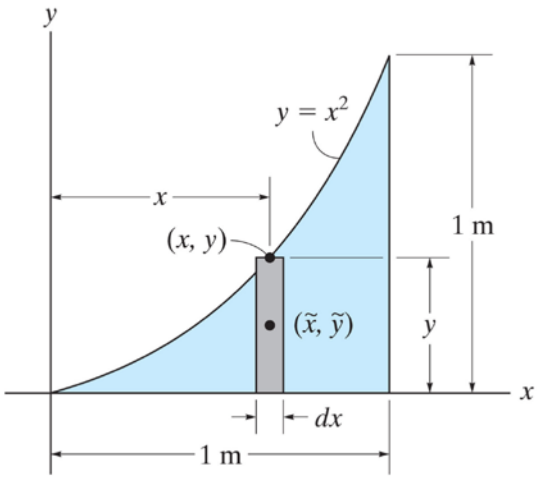


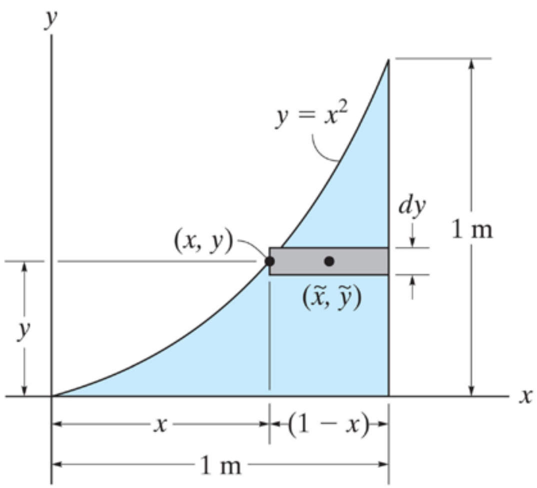
Find $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} = \frac{\int_0^1 x(x^3)dx}{\int_0^1 (x^3)dx} = \frac{\frac{1}{5}[x^5]_0^1}{\frac{1}{4}[x^4]_0^1} = \frac{\left(\frac{1}{5}\right)}{\left(\frac{1}{4}\right)} = 0.8 \text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^1 \left(\frac{x^3}{2}\right)(x^3)dx}{\int_0^1 x^3dx} = \frac{\frac{1}{14}[x^7]_0^1}{\frac{1}{4}} = \frac{\left(\frac{1}{14}\right)}{\left(\frac{1}{4}\right)} = 0.2857\text{m}$$

Example





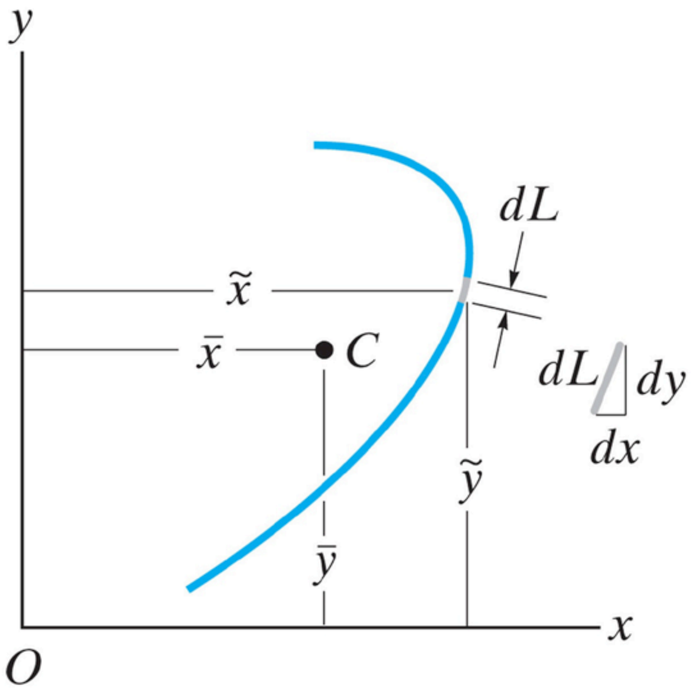
Given: The area shown

Find: The centroid of the area using both solutions (vertical and horizontal strips)

Centroids

Centroid of a Line

If a line lies in the x-y plane and is bounded by a curve $y = f(x)$, then its centroid is defined as



$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL}$$

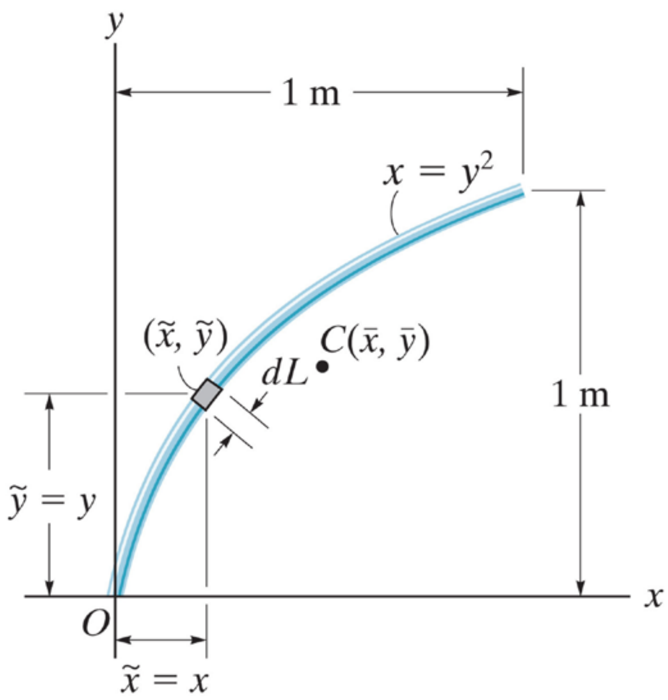
Pythagorean triangle $\quad dL = \sqrt{dx^2 + dy^2}$

$$dy = \frac{dy}{dx}dx \Rightarrow dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$dx = \frac{dx}{dy}dy \Rightarrow dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Use either of these expressions, i.e., the one that results in an easier integration

Example



Given: The bent rod shown below

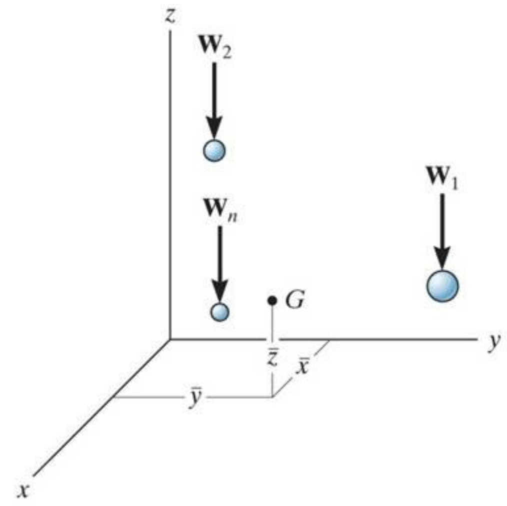
Find: The centroid

Composite Bodies

Consider a composite body which consists of a series of particles (or bodies) as shown in the figure. The net or resultant weight is given as

$$W_R = \sum W_i$$

Summing the moments about the y-axis



$$\bar{x} W_R = \tilde{x}_1 W_1 + \tilde{x}_2 W_2 + \ldots\ldots\ldots + \tilde{x}_n W_n$$

where $\tilde{x}_1$ represents the coordinate of W_1.

Similarly, we can sum moments about the x- and z-axes to find the other coordinates of the centre of gravity (or centre of mass if *m* replaces W).

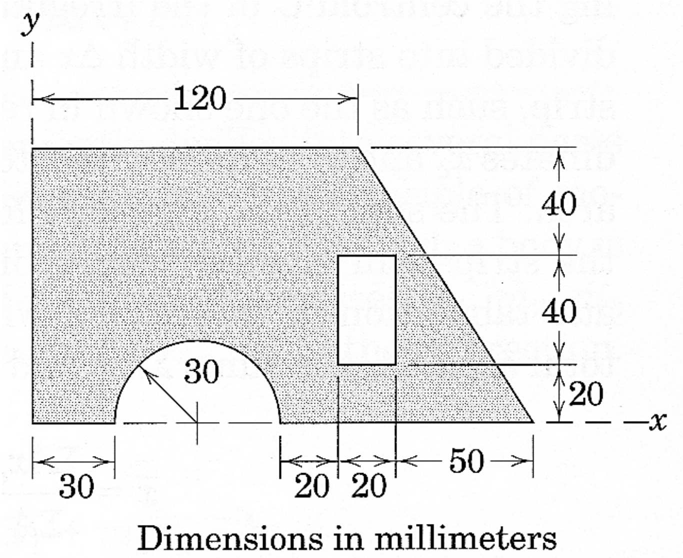
$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}$$

Composite Bodies

Steps to find the CG, CM, or Centroid

- Divide the body into pieces that are known shapes. *Holes are considered as pieces with negative weight or size.*
- Make a table that includes the segment number, the parameter depending on the problem (weight, mass, or size), the moment arms, and required calculations.
- Define the coordinate axes, determine the coordinates of the centre of gravity of centroid of each piece, and then fill in the table.
- Sum the columns to get $\bar{x}$, $\bar{y}$, and $\bar{z}$.

Sample Problem (§ 9.2)



Given: The area below.

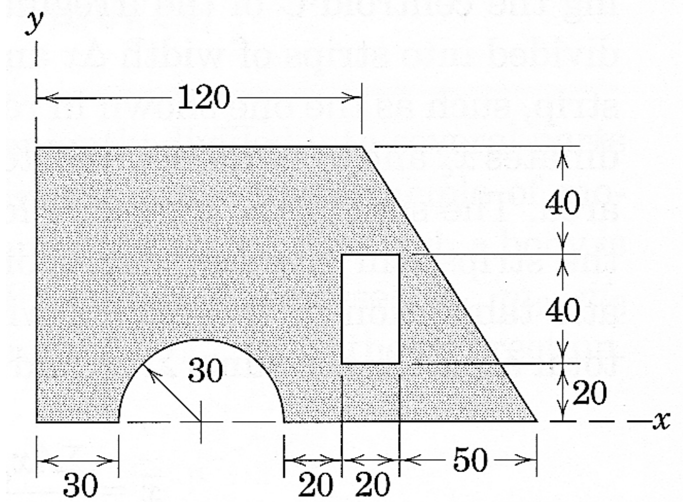
Find: The centroid

Plan:

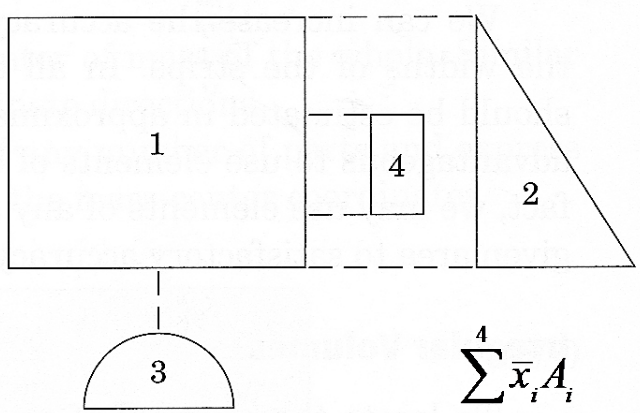
Identify common area shapes

Create a table, including shape areas, moment arms relative to the reference frame, and the products of area × moment arms.

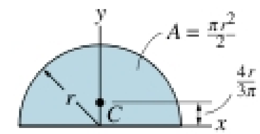
Calculate the centroid



Dimensions in millimeters



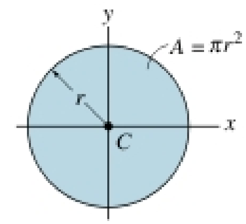
$$\bar{x} = \frac{\sum_{i=1}^{4} \bar{x}_i A_i}{\sum_{i=1}^{4} A_i} \qquad \bar{y} = \frac{\sum_{i=1}^{4} \bar{y}_i A_i}{\sum_{i=1}^{4} A_i}$$



$I_x = \frac{1}{8}\pi r^4$

$I_y = \frac{1}{8}\pi r^4$

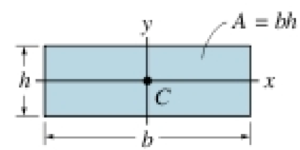
Semicircular area



$I_x = \frac{1}{4}\pi r^4$

$I_y = \frac{1}{4}\pi r^4$

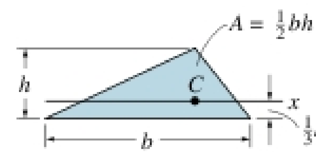
Circular area



$I_x = \frac{1}{12}bh^3$

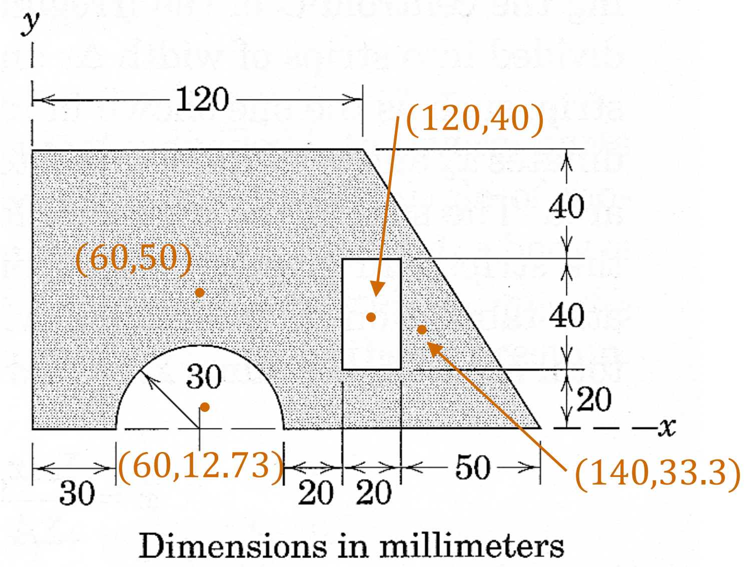
$I_y = \frac{1}{12}hb^3$

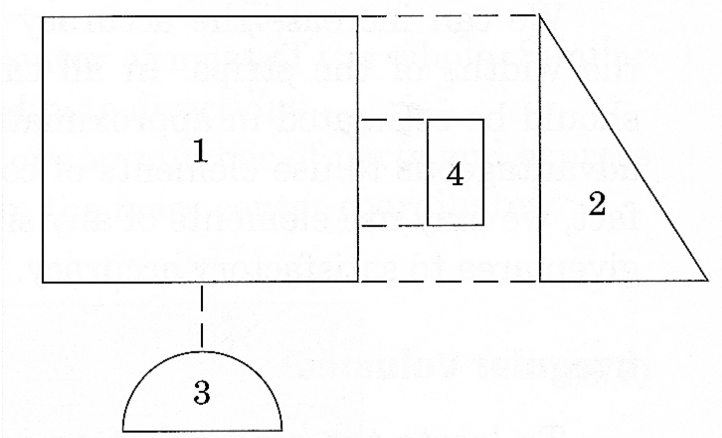
Rectangular area



$I_x = \frac{1}{36}bh^3$

Triangular area





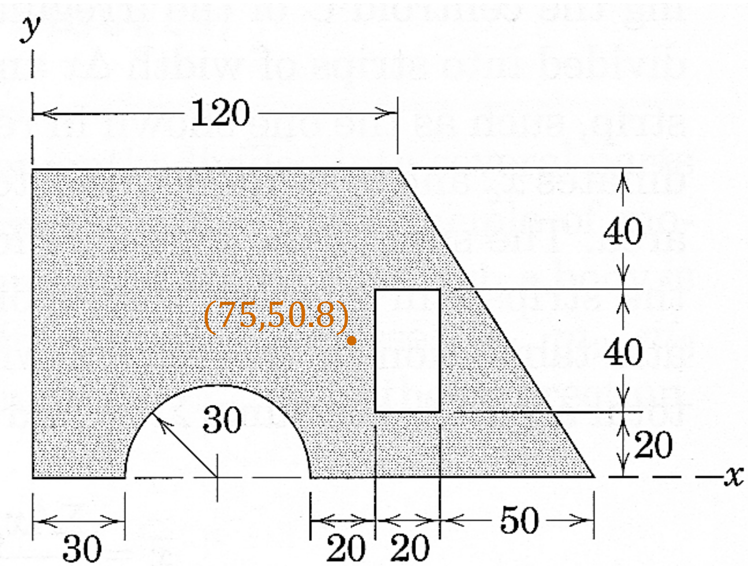
PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	−1414	60	12.73	−84 800	−18 000
4	−800	120	40	−96 000	−32 000
TOTALS	12 790			959 000	650 000

The location of the centroid is

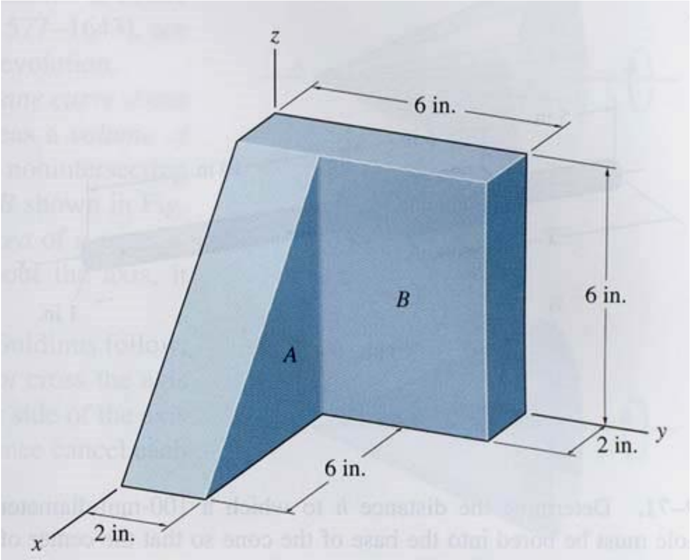
$$\bar{x} = \frac{\sum_{i=1}^{4} \bar{x}_i A_i}{\sum_{i=1}^{4} A_i} = \frac{959000\text{mm}^3}{12790\text{mm}^2} = 75\text{mm}$$

$$\bar{y} = \frac{\sum_{i=1}^{4} \bar{y}_i A_i}{\sum_{i=1}^{4} A_i} = \frac{650000\text{mm}^3}{12790\text{mm}^2} = 50.8\text{mm}$$



Dimensions in millimeters

Example



Given: Two blocks of different materials are assembled as shown.

The densities of the materials are:

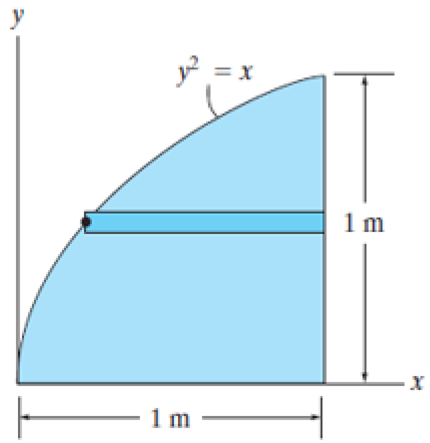
$$\rho_A = 150 \text{ lb / ft}^3 \quad \text{and}$$
$$\rho_B = 400 \text{ lb / ft}^3.$$

Find: The center of gravity of this assembly.

Sample Problems for Students to Review

Chapter 9

Sample Problem (§ 9.1)



Given: The area shown

Find: The centroid of the area

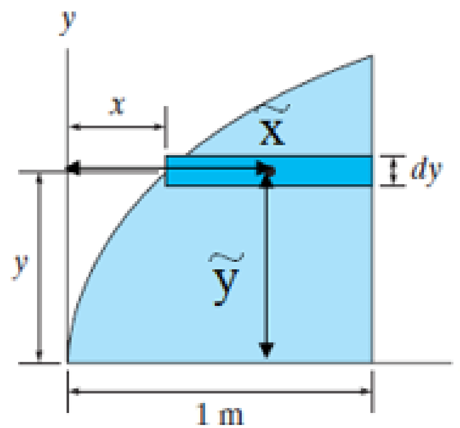
Plan:

Since $x = f(y)$, use horizontal strips.

Find dA and $(\tilde{x}, \tilde{y})$

Find centroid $(\bar{x}, \bar{y})$

Find dA and $(\tilde{x}, \tilde{y})$



$$dA = (1-x)dy \ = (1-y^2)dy$$

$$\tilde{y} = y$$

$$\tilde{x} = x + \frac{(1\ -x)}{2} = \frac{(1\ +x)}{2} = \frac{(1+\mathrm{y}^2)}{2}$$

Find $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int_A \ \tilde{x}dA}{\int_A \ dA} = \frac{\int_0^1 \left(\frac{1+y^2}{2}\right)(1-\mathrm{y}^2)\mathrm{d}y}{\int_0^1 (1-\mathrm{y}^2)\mathrm{d}y} = \frac{\frac{1}{2}[y]_0^1 - \frac{1}{10}[\mathrm{y}^5]_0^1}{[y]_0^1 - \frac{1}{3}[\mathrm{y}^3]_0^1} = \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{3}\right)} = 0.8 \text{ m}$$

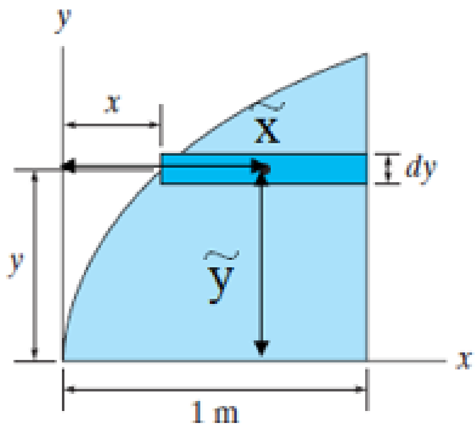
$$\bar{y} = \frac{\int_A \ \tilde{y}dA}{\int_A \ dA} = \frac{\int_0^1 y\,(1-\mathrm{y}^2)\mathrm{d}y}{\int_0^1 (1-\mathrm{y}^2)\mathrm{d}y} = \frac{\frac{1}{2}[\mathrm{y}^2]_0^1 - \frac{1}{4}[\mathrm{y}^4]_0^1}{[y]_0^1 - \frac{1}{3}[\mathrm{y}^3]_0^1} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{2}{3}\right)} = 0.375\text{m}$$

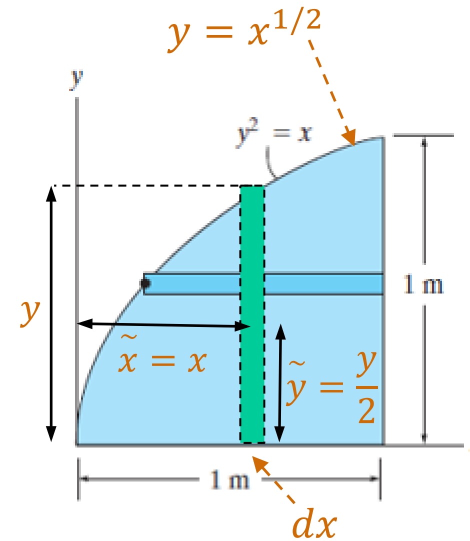
Another solution is $y = x^{1/2}$

Find dA and $(\tilde{x}, \tilde{y})$

$$dA = ydx = x^{1/2}dx$$

$$\tilde{x} = x \qquad \tilde{y} = \frac{y}{2} = \frac{x^{1/2}}{2}$$



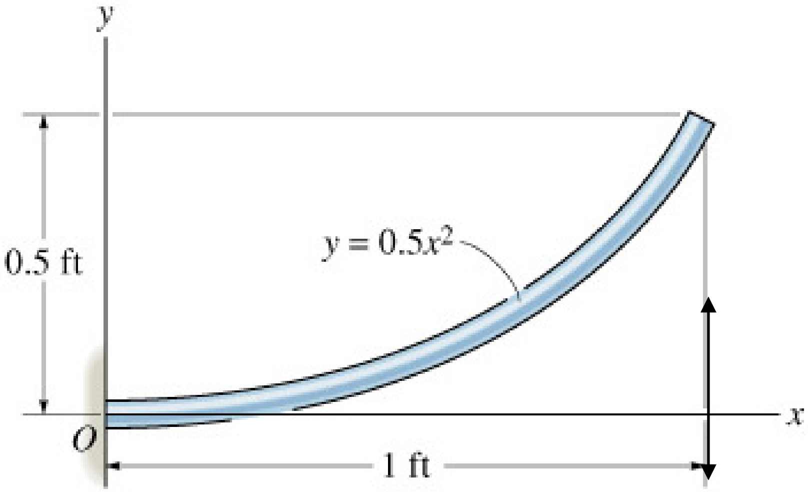


Find $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\int_A \tilde{x}dA}{\int_A dA} = \frac{\int_0^1 x(x^{1/2})\mathrm{d}x}{\int_0^1 (x^{1/2})\mathrm{d}x} = \frac{\frac{2}{5}[\mathrm{x}^{5/2}]_0^1}{\frac{2}{3}[\mathrm{x}^{3/2}]_0^1} = \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{3}\right)} = 0.8\text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y}dA}{\int_A dA} = \frac{\int_0^1 \frac{x^{1/2}}{2}(x^{1/2})\mathrm{d}x}{\int_0^1 (x^{1/2})\mathrm{d}x} = \frac{\frac{1}{4}[\mathrm{x}^2]_0^1}{\frac{2}{3}[\mathrm{x}^{3/2}]_0^1} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{2}{3}\right)} = 0.375\text{m}$$

Sample Problem (§ 9.1)



Given: The curved rod shown

Find: The centroid of the curved rod

Plan:

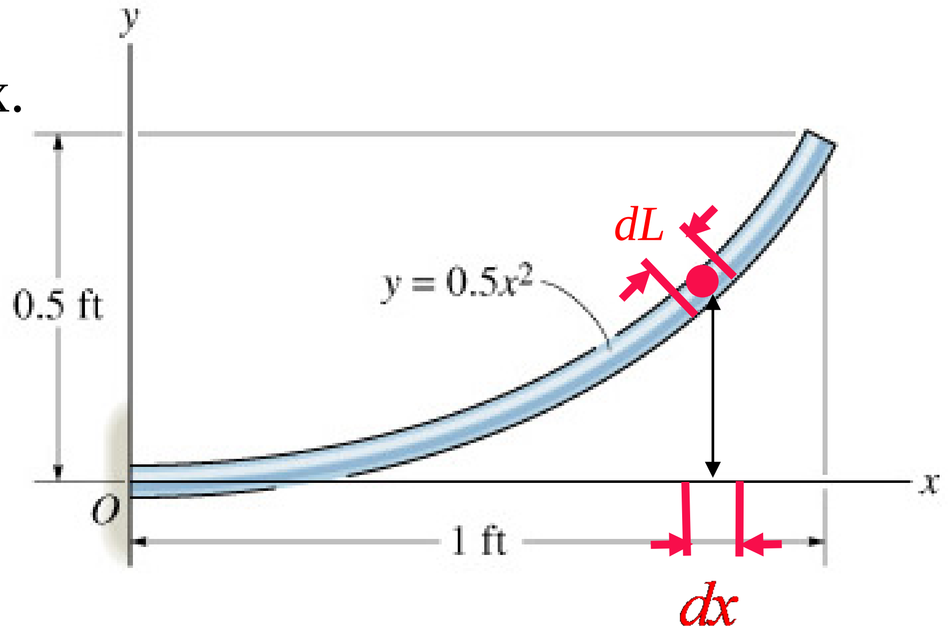
Define dL.

Use equations of integration

Given: a definition of y in terms of x.

We need to evaluate:

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL}$$



Define *dL* using the Pythagorean expression.

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

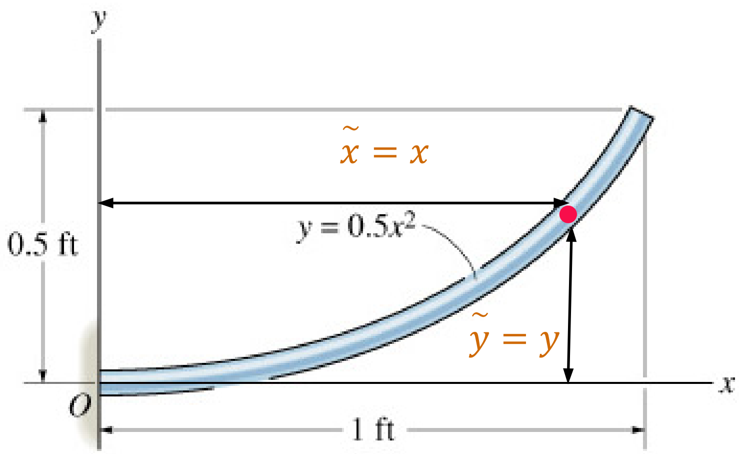
By expressing the arc length dL in terms of x we are setting x up as the variable of integration.

Substitute *dL* in the equation of the centroid

$$\bar{y}=\frac{\int_L y\,dL}{\int_L dL}=\frac{\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$y=0.5x^2 \Rightarrow \frac{dy}{dx}=\frac{d}{dx}0.5x^2=x$$

$$\bar{y}=\frac{\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}=\frac{\int_0^1 0.5x^2\sqrt{1+x^2}\,dx}{\int_0^1 \sqrt{1+x^2}\,dx}$$



Solving these integrals is not easy. Appendix A of the “Statics” textbook has a short integral table that will help in solving problems.

The solution of these integrals are,

$$\bar{y} = \frac{\int_{x=0}^{x=1} 0.5x^2\sqrt{1+x^2}\,dx}{\int_{x=0}^{x=1} \sqrt{1+x^2}\,dx}$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{\left(x^2 \pm a^2\right)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2 \pm a^2}\right) + C$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x+\sqrt{x^2 \pm a^2}\right)\right] + C$$

Therefore,

$$\int_{x=0}^{x=1} 0.5x^2\sqrt{1+x^2}\,dx = 0.5\left[\frac{x}{4}\sqrt{\left(x^2+1\right)^3} - \frac{1}{8}x\sqrt{x^2+1} - \frac{1}{8}\ln\left(x+\sqrt{x^2+1}\right)\right]_0^1 = 0.5\left[\frac{\sqrt{8}}{4} - \frac{\sqrt{2}}{8} - \frac{\ln\left(1+\sqrt{2}\right)}{8}\right] = 0.21$$

$$\int_{x=0}^{x=1} \sqrt{1+x^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+1} + \ln\left(x+\sqrt{x^2+1}\right)\right]_0^1 = \frac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] = 1.148$$

$$\bar{y} = \frac{0.21}{1.148} = 0.183 \text{ ft}$$

A similar procedure is conducted to find $\bar{x}$.