Example 6.13 (Fourier transform of the sinc function)**.** Using the transform pair

$$\underbrace{\operatorname{rect} t}_{v(t)} \overset{\text{CTFT}}{\longleftrightarrow} \underbrace{\operatorname{sinc}\left(\frac{\omega}{2}\right)}_{V(\omega)}, \quad 1$$

find the Fourier transform X of the function

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right).$$

Solution. From the given Fourier transform pair, we have

$$v(t) = \operatorname{rect} t \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad V(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right).$$

simply restating given FT pair 1

By duality, we have $\mathcal{F}V(\omega) = 2\pi v(-\omega)$

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) \underset{\text{duality}}{=} 2\pi v(-\omega) \underset{\text{given FT pair 1}}{=} 2\pi \operatorname{rect}(-\omega) \underset{\text{rect is even}}{=} 2\pi \operatorname{rect} \omega.$$

Thus, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi \operatorname{rect} \omega.$$

Observing that $V = x$ and $\mathcal{F}V = X$, we can rewrite the preceding relationship as

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X(\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have shown that

$$X(\omega) = 2\pi \operatorname{rect} \omega.$$

■

table of FT pairs

Example 6.14 (Time-domain convolution property of the Fourier transform)**.** With the aid of Table 6.2, find the Fourier transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

Solution. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\ &= X_1(\omega)X_2(\omega). \end{aligned} \tag{6.10}$$

time-domain convolution property

table of FT pairs

From Table 6.2, we know that

①
$$\begin{aligned} X_1(\omega) &= \left(\mathcal{F}\{e^{-2t}u(t)\}\right)(\omega) \\ &= \tfrac{1}{2+j\omega} \quad \text{and} \end{aligned}$$

table of FT pairs

②
$$\begin{aligned} X_2(\omega) &= \mathcal{F}u(\omega) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

table of FT pairs

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (6.10), we obtain

$X(\omega) = X_1(\omega)\,X_2(\omega)$ (6.10)

substituting ① and ② into (6.10)

$$\begin{aligned} X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2} \\ &= \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}. \end{aligned}$$

$\left.\dfrac{\pi}{2+j\omega}\right|_{\omega=0}$

equivalence property of δ function

■

Example 6.15 (Frequency-domain convolution property)**.** Let x and y be functions related as

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a nonzero real constant. Let $Y = \mathcal{F}y$ and $X = \mathcal{F}x$. Find an expression for Y in terms of X.

Solution. To allow for simpler notation in what follows, we define

$$v(t) = \cos(\omega_c t) \qquad ①$$

table of FT pairs

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$② \qquad V(\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]. \qquad \leftarrow \text{from table of FT pairs}$$

From the definition of v, we have

$$③ \qquad y(t) = x(t)v(t). \qquad \leftarrow \text{since } y(t) = x(t)\underbrace{\cos(\omega_c t)}_{v(t)}$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega).$$

taking FT of both sides of ③

Using the frequency-domain convolution property of the Fourier transform, we obtain

frequency-domain convolution property

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi} X * V(\omega) \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)V(\omega - \lambda)\,d\lambda.
\end{aligned}$$

definition of convolution

Substituting the above expression for V, we obtain

substitute V from ②

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega - \lambda - \omega_c) + \delta(\omega - \lambda + \omega_c)])\,d\lambda \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} X(\lambda)[\delta(\omega - \lambda - \omega_c) + \delta(\omega - \lambda + \omega_c)]\,d\lambda \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\omega - \lambda - \omega_c)\,d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\omega - \lambda + \omega_c)\,d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda - \omega + \omega_c)\,d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda - \omega - \omega_c)\,d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda - (\omega - \omega_c)]\,d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda - (\omega + \omega_c)]\,d\lambda\right] \\
&= \tfrac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)] \\
&= \tfrac{1}{2}X(\omega - \omega_c) + \tfrac{1}{2}X(\omega + \omega_c).
\end{aligned}$$

cancel π's

split into two integrals

δ is even

regroup

sifting property

expand

■

Example 6.16 (Time-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. Taking the Fourier transform of both sides of the given equation for x yields

$$X(\omega) = \left(\mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}\right)(\omega).$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \left(\mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}\right)(\omega) \\ &= j\omega\mathcal{F}\delta(\omega). \end{aligned}$$

from definition of X

time-domain differentiation property

Evaluating the Fourier transform of δ using Table 6.2, we obtain

$F\delta(\omega) = 1$

$$\begin{aligned} X(\omega) &= j\omega(1) \\ &= j\omega. \end{aligned}$$

■