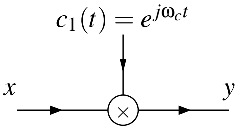
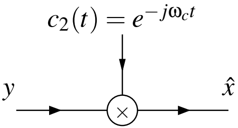
Trivial Amplitude Modulation (AM) System



Transmitter



Receiver

- The transmitter is characterized by

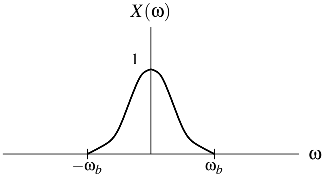
$$y(t) = e^{j\omega_c t}x(t) \quad \Longleftrightarrow \quad Y(\omega) = X(\omega - \omega_c).$$

- The receiver is characterized by

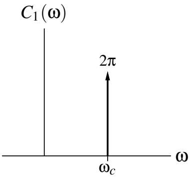
$$\hat{x}(t) = e^{-j\omega_c t}y(t) \quad \Longleftrightarrow \quad \hat{X}(\omega) = Y(\omega + \omega_c).$$

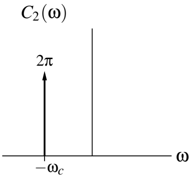
- Clearly, $\hat{x}(t) = e^{j\omega_c t}e^{-j\omega_c t}x(t) = x(t)$.

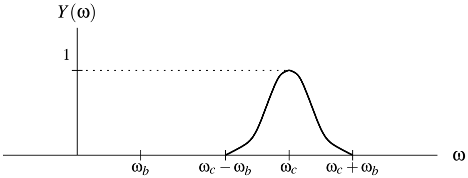
Trivial Amplitude Modulation (AM) System: Example



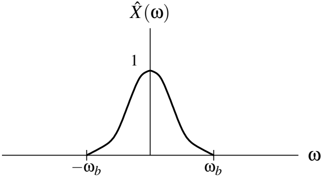
Transmitter Input





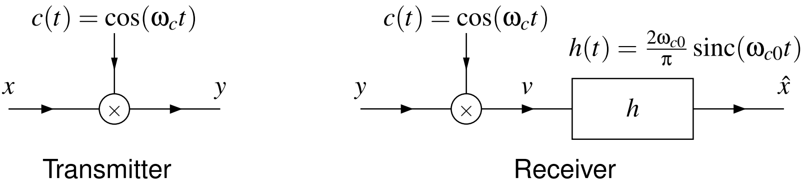


Transmitter Output



Receiver Output

Double-Sideband Suppressed-Carrier (DSB-SC) AM



- Let $X = \mathcal{F}x$, $Y = \mathcal{F}y$, and $\hat{X} = \mathcal{F}\hat{x}$.
- Suppose that $X(\omega) = 0$ for all $\omega \notin [-\omega_b, \omega_b]$.
- The transmitter is characterized by

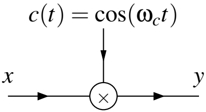
$$Y(\omega) = \tfrac{1}{2}\left[X(\omega+\omega_c) + X(\omega-\omega_c)\right].$$

- The receiver is characterized by

$$\hat{X}(\omega) = \left[Y(\omega+\omega_c) + Y(\omega-\omega_c)\right]\operatorname{rect}\left(\frac{\omega}{2\omega_{c_0}}\right).$$

- If $\omega_b < \omega_{c_0} < 2\omega_c - \omega_b$, we have $\hat{X}(\omega) = X(\omega)$ (implying $\hat{x}(t) = x(t)$).

DSB-SC AM: Transmitter

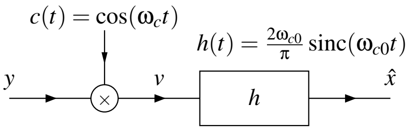


$$y(t) = \cos(\omega_c t) x(t)$$

$$X = \mathcal{F}x, \quad Y = \mathcal{F}y$$

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{\cos(\omega_c t) x(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right) x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t} x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t} x(t)\}(\omega)\right] \\
&= \tfrac{1}{2}\left[X(\omega - \omega_c) + X(\omega + \omega_c)\right]
\end{aligned}$$

DSB-SC AM: Receiver



$$v(t) = \cos(\omega_c t) y(t), \quad h(t) = \frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0} t), \quad \hat{x}(t) = v * h(t)$$

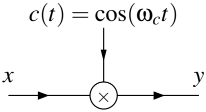
$$Y = \mathcal{F}y, \quad V = \mathcal{F}v, \quad H = \mathcal{F}h, \quad \hat{X} = \mathcal{F}\hat{x}$$

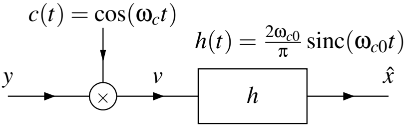
$$\begin{aligned}
V(\omega) &= \mathcal{F}\{\cos(\omega_c t) y(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right) y(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t} y(t)\right\}(\omega) + \mathcal{F}\left\{e^{-j\omega_c t} y(t)\right\}(\omega)\right] \\
&= \tfrac{1}{2}\left[Y(\omega - \omega_c) + Y(\omega + \omega_c)\right]
\end{aligned}$$

$$\begin{aligned}
H(\omega) &= \mathcal{F}\left\{\frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0} t)\right\}(\omega) \\
&= 2\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right)
\end{aligned}$$

$$\hat{X}(\omega) = H(\omega) V(\omega)$$

DSB-SC AM: Complete System



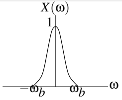


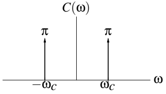
$$Y(\omega) = \tfrac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

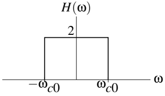
$$\begin{aligned}
V(\omega) &= \tfrac{1}{2}[Y(\omega-\omega_c) + Y(\omega+\omega_c)] \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega-\omega_c]-\omega_c) + X([\omega-\omega_c]+\omega_c)\right] + \right. \\
&\qquad \left. \tfrac{1}{2}\left[X([\omega+\omega_c]-\omega_c) + X([\omega+\omega_c]+\omega_c)\right]\right] \\
&= \tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega-2\omega_c) + \tfrac{1}{4}X(\omega+2\omega_c)
\end{aligned}$$

$$\begin{aligned}
\hat{X}(\omega) &= H(\omega)V(\omega) \\
&= H(\omega)\left[\tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega-2\omega_c) + \tfrac{1}{4}X(\omega+2\omega_c)\right] \\
&= \tfrac{1}{2}H(\omega)X(\omega) + \tfrac{1}{4}H(\omega)X(\omega-2\omega_c) + \tfrac{1}{4}H(\omega)X(\omega+2\omega_c) \\
&= \tfrac{1}{2}[2X(\omega)] + \tfrac{1}{4}(0) + \tfrac{1}{4}(0) \\
&= X(\omega)
\end{aligned}$$

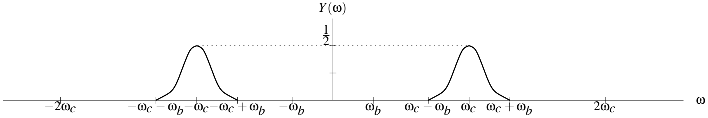
DSB-SC AM: Example



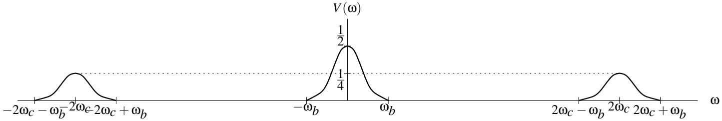


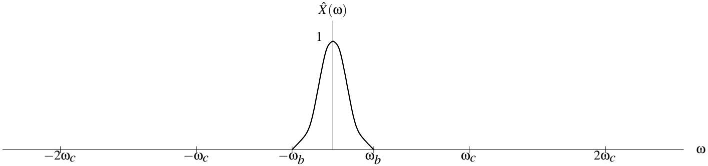


Transmitter Input

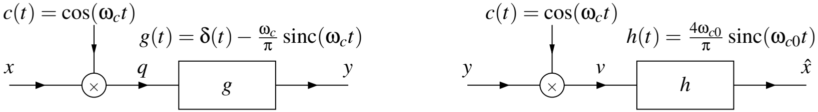


Transmitter Output





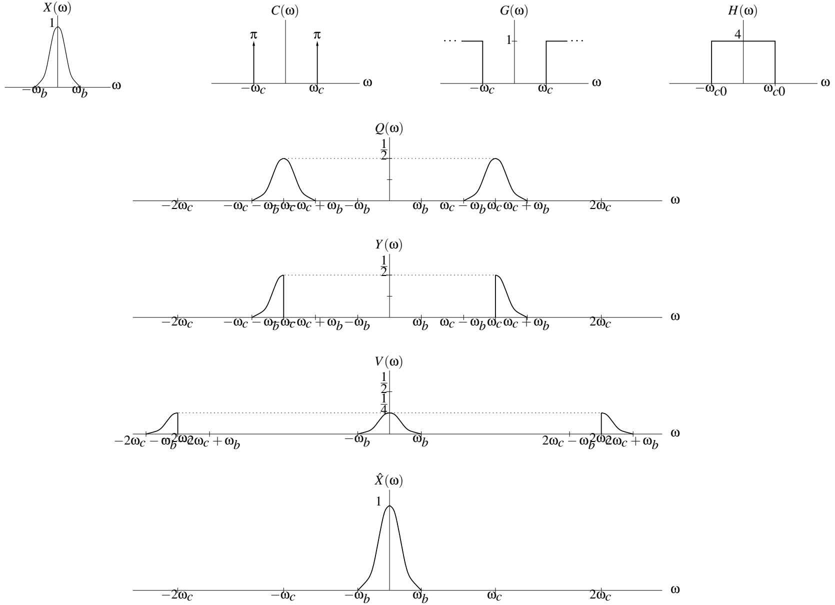
Receiver Output



Transmitter

Receiver

- The basic analysis of the SSB-SC AM system is similar to the DSB-SC AM system.
- SSB-SC AM requires half as much bandwidth for the transmitted signal as DSB-SC AM.

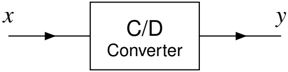

$X(\omega)$
1
$-\omega_b$ ω_b ω
$C(\omega)$
π π
$-\omega_c$ ω_c ω
$G(\omega)$
$\cdots$ 1 $\cdots$
$-\omega_c$ ω_c ω
$H(\omega)$
4
$-\omega_{c0}$ ω_{c0} ω
$Q(\omega)$
$\frac{1}{2}$
$-2\omega_c$ $-\omega_c-\omega_b$ $-\omega_c$ $-\omega_c+\omega_b$ $-\omega_b$ ω_b $\omega_c-\omega_b$ ω_c $\omega_c+\omega_b$ $2\omega_c$ ω
$Y(\omega)$
$\frac{1}{2}$
$-2\omega_c$ $-\omega_c-\omega_b$ $-\omega_c$ $-\omega_c+\omega_b$ $-\omega_b$ ω_b $\omega_c-\omega_b$ ω_c $\omega_c+\omega_b$ $2\omega_c$ ω
$V(\omega)$
$\frac{1}{2}$
$\frac{1}{4}$
$-2\omega_c-\omega_b$ $-2\omega_c$ $-2\omega_c+\omega_b$ $-\omega_b$ ω_b $2\omega_c-\omega_b$ $2\omega_c$ $2\omega_c+\omega_b$ ω
$\hat{X}(\omega)$
1
$-2\omega_c$ $-\omega_c$ $-\omega_b$ ω_b ω_c $2\omega_c$ ω

Section 6.11

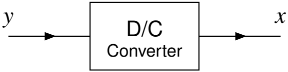
Application: Sampling and Interpolation

Sampling and Interpolation

- Often, we want to be able to transform a continuous-time signal (i.e., a function) into a discrete-time signal (i.e., a sequence) and vice versa.
- This is accomplished through processes known as ***sampling*** and ***interpolation***.
- **Sampling**, which is performed by a **continuous-time to discrete-time (C/D) converter** shown below, transforms a function x to a sequence y.



- **Interpolation**, which is performed by a **discrete-time to continuous-time (D/C) converter** shown below, transforms a sequence y to a function x.



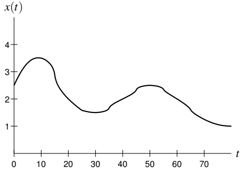
- Note that, unless very special conditions are met, the sampling process loses information (i.e., is ***not invertible***).

- Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**.
- With this scheme, a sequence y of samples is obtained from a function x according to the relation

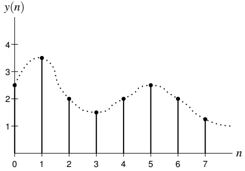
$$y(n) = x(Tn) \quad \text{for all integer } n,$$

where T is a (strictly) positive real constant.

- As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = \frac{2\pi}{T}$ as the (angular) **sampling frequency**.
- An example of periodic sampling is shown below, where the function x has been sampled with ***sampling period*** $T = 10$, yielding the sequence y.



Function to Be Sampled



Sequence Produced by Sampling

Invertibility of Sampling

- Unless constraints are placed on the functions being sampled, the sampling process is ***not invertible***.
- In other words, in the absence of any constraints, a function cannot be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the functions x_1 and x_2 given by

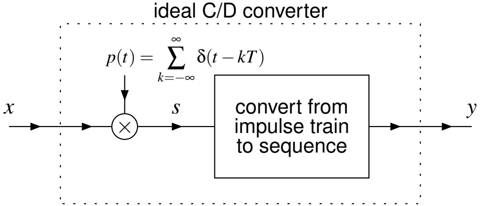
$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

- Sampling x_1 and x_2 with the sampling period $T = 1$ yields the respective sequences

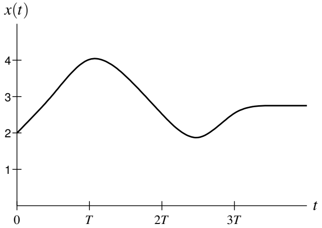
$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

- So, although x_1 and x_2 are ***distinct***, y_1 and y_2 are ***identical***.
- Given the sequence y where $y = y_1 = y_2$, it is impossible to determine which function was sampled to produce y.
- Only by imposing a carefully chosen set of constraints on the functions being sampled can we ensure that a function can be exactly recovered from only its samples.

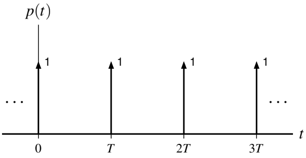
- An **impulse train** is a function of the form $v(t) = \sum_{k=-\infty}^{\infty} c_k \delta(t-kT)$, where c_k and T are real constants.
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.



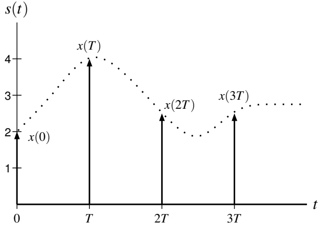
- The sampling of a function x to produce a sequence y consists of the following two steps (in order):
  1. Multiply the function x to be sampled by a periodic impulse train p, yielding the impulse train $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$.
  2. Convert the impulse train s to a sequence y by forming y from the weights of successive impulses in s so that $y(n) = x(nT)$.



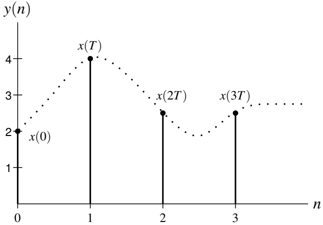
Input Function



Periodic Impulse Train

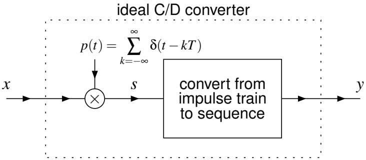


Impulse-Sampled Function *(Continuous-Time)*

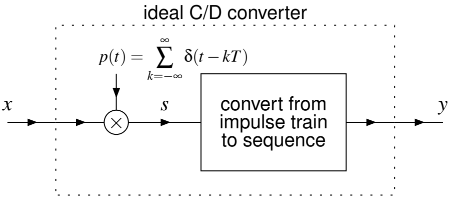


Output Sequence *(Discrete-Time)*

Model of Sampling: Invertibility of Sampling Revisited



- Since sampling is not invertible and our model of sampling consists of only two steps, at least one of these two steps must not be invertible.
- Recall the two steps in our model of sampling are as follows (in order):
  1. $x \longrightarrow s(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$; and
  2. $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \longrightarrow y(n) = x(nT)$.
- Step 1 cannot be undone (unless we somehow restrict which functions x can be sampled).
- Step 2 is always invertible.
- Therefore, the fact that sampling is not invertible is entirely due to step 1.



- In the time domain, the impulse-sampled function s is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad (\text{where } \omega_s = \tfrac{2\pi}{T}).$$

- Thus, the spectrum of the impulse-sampled function s is a scaled sum of an infinite number of ***shifted copies*** of the spectrum of the original function x.

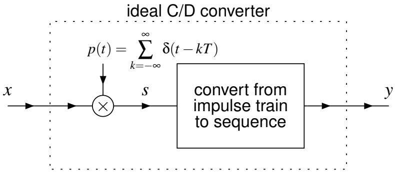
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$\begin{aligned}
c_k &= \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt \\
&= \frac{1}{T} \int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t} dt \\
&= \frac{1}{T} \\
&= \frac{\omega_s}{2\pi}
\end{aligned}$$

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

Sampling: Multiplication by a Periodic Impulse Train



$$s(t) = p(t)x(t), \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

$$s(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} x(t)$$

$$X = \mathcal{F}x, \quad S = \mathcal{F}s$$

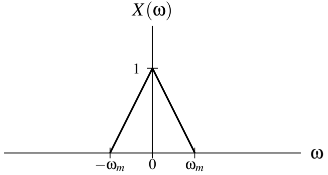
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Model of Sampling: Aliasing

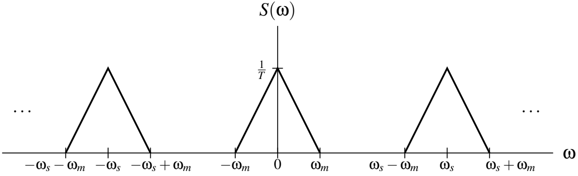
- Consider frequency spectrum S of the impulse-sampled function s given by

$$S(\omega) = \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

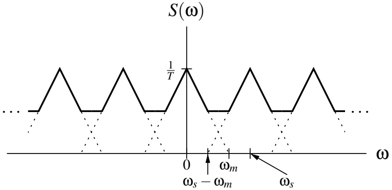
- The function S is a scaled sum of an infinite number of ***shifted copies*** of X.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.
- In particular, the nonzero portions of the different shifted copies of X can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as **aliasing**.
- When aliasing occurs, the original function x cannot be recovered from its samples in y.



Spectrum of Input Function (Bandwidth ω_m)

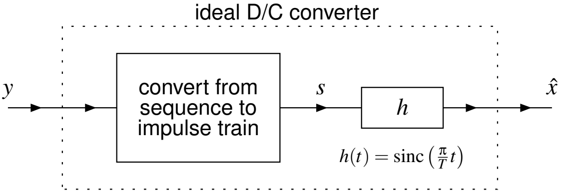


Spectrum of Impulse-Sampled Function: No Aliasing Case ($\omega_s > 2\omega_m$)



Spectrum of Impulse-Sampled Function: Aliasing Case ($\omega_s \leq 2\omega_m$)

- For the purposes of analysis, interpolation can be modelled as shown below.



- The reconstruction of a function x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s by using the samples in y as the weights of successive impulses in s so that $s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn)$.
  2. Apply the lowpass filter with impulse response h to s to produce $\hat{x}$ so that $\hat{x}(t) = s * h(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\frac{\pi}{T}(t - Tn)\right]$.
- The lowpass filter is used to eliminate the extra copies of the originally-sampled function's spectrum present in the spectrum of s.

- **Sampling Theorem.** Let x be a function with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(Tn)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = \frac{2\pi}{T}$. The preceding inequality is known as the **Nyquist condition**. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right],$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left(\tfrac{\omega_s}{2}t - \pi n\right).$$

- We call $\frac{\omega_s}{2}$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**.

Example 6.41 NEXT SLIDE: PFEs

Part 7

Laplace Transform (LT)

Motivation Behind the Laplace Transform

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a ***generalization of the Fourier transform***.
- Due to its more general nature, the Laplace transform has a number of ***advantages*** over the Fourier transform.
- First, the Laplace transform representation ***exists for some functions that do not have a Fourier transform representation***. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform.
- Second, since the Laplace transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.