1 **A.1** Express each complex number given below in Cartesian form.
(a) $2e^{j2\pi/3}$;
(b) $\sqrt{2}e^{j\pi/4}$;
(c) $2e^{j7\pi/6}$; and
(d) $3e^{j\pi/2}$.

1 **Answer (c).**

We have

$$\begin{aligned}2e^{j7\pi/6} &= 2\left(\cos\tfrac{7\pi}{6} + j\sin\tfrac{7\pi}{6}\right)\\ &= 2\left(-\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right)\\ &= -\sqrt{3} - j.\end{aligned}$$

1 **A.2** Express each complex number given below in polar form. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument.
(a) $-\sqrt{3}+j$;
(b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$;
(c) $\sqrt{2}-j\sqrt{2}$;
(d) $1+j\sqrt{3}$;
(e) $-1-j\sqrt{3}$; and
(f) $-3+4j$.

1 **Answer (b).**

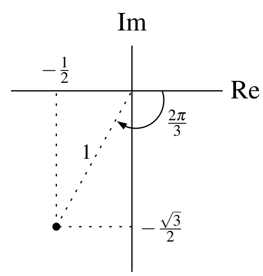
Let z denote the given complex value. Taking the magnitude and argument of z, we have

$$|z| = \sqrt{\left(-\tfrac{1}{2}\right)^2+\left(-\tfrac{\sqrt{3}}{2}\right)^2} = 1 \quad \text{and}$$
$$\arg z = \arctan\left(\left[-\tfrac{\sqrt{3}}{2}\right] / \left[-\tfrac{1}{2}\right]\right) - \pi = \arctan(\sqrt{3}) - \pi = \tfrac{\pi}{3} - \pi = -\tfrac{2\pi}{3}.$$

Thus, z has the polar-form representation

$$z = e^{j(-2\pi/3)}.$$

A plot of z in the complex plane is shown below.



1 **Answer (d).**

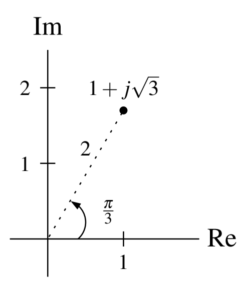
Let z denote the given complex value. Taking the magnitude and argument of z, we have

$$|z| = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{4} = 2 \quad \text{and}$$
$$\arg z = \arctan(\sqrt{3}/1) = \arctan(\sqrt{3}) = \tfrac{\pi}{3}.$$

Thus, z has the polar-form representation

$$z = 2e^{j\pi/3}.$$

A plot of z in the complex plane is shown below.



1 **A.3** Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.

(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in Cartesian form;

(b) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in polar form;

(c) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1+j)$ in polar form;

(d) $e^{1+j\pi/4}$ in Cartesian form;

(e) $\left[\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right]^8$ in polar form;

(f) $(1+j)^{10}$ in Cartesian form;

(g) $\dfrac{1+j}{1-j}$ in polar form;

(h) $\dfrac{1}{1+re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$; and

(i) $\dfrac{1}{1-re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$.

1 **Answer (a).**

We have

$$\begin{aligned}
2\left(\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right) + j\left(\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right) &= \sqrt{3} - j + \left(-j\tfrac{1}{2} + \tfrac{1}{2}\right) \\
&= \sqrt{3} + \tfrac{1}{2} - j\left(1 + \tfrac{1}{2}\right) \\
&= \tfrac{2\sqrt{3}+1}{2} - j\tfrac{3}{2}.
\end{aligned}$$

1 **Answer (b).**

We have

$$\begin{aligned}
\left(\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right) &= e^{j(-\pi/6)}\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)} \\
&= \tfrac{1}{\sqrt{2}}e^{j(-11\pi/12)}.
\end{aligned}$$

1 **Answer (f).**

We have

$$\begin{aligned}
(1+j)^{10} &= \left(\sqrt{2}e^{j\arctan(1)}\right)^{10} \\
&= (\sqrt{2})^{10}e^{j10\arctan(1)} \\
&= 2^5 e^{j10(\pi/4)} \\
&= 32e^{j5\pi/2} \\
&= 32e^{j(5\pi/2-4\pi/2)} \\
&= 32e^{j\pi/2} \\
&= 32(\cos\tfrac{\pi}{2} + j\sin\tfrac{\pi}{2}) \\
&= j32.
\end{aligned}$$

1 **Answer (g).**

We have

$$\begin{aligned}\frac{1+j}{1-j} &= \frac{\sqrt{2}e^{j\arctan(1/1)}}{\sqrt{2}e^{j\arctan(-1/1)}} \\ &= \frac{\sqrt{2}e^{j\pi/4}}{\sqrt{2}e^{-j\pi/4}} \\ &= e^{j\pi/2}.\end{aligned}$$

1 **A.4** Show that each of the identities below holds, where z, z_1, and z_2 are arbitrary complex numbers and n is an arbitrary integer.
(a) $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$;
(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \neq 0$;
(c) $z + z^* = 2\operatorname{Re}\{z\}$;
(d) $zz^* = |z|^2$;
(e) $(z_1 z_2)^* = z_1^* z_2^*$;
(f) $|z^n| = |z|^n$; and
(g) $\arg(z^n) = n \arg z$.

1 **Answer (b).**

We are asked to show that, for all complex numbers z_1 and z_2 such that $z_2 \neq 0$, the following identity holds:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$$

Let z_1 and z_2 be arbitrary complex numbers (where $z_2 \neq 0$) with the polar representations

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2},$$

where r_1, r_2, θ_1, and θ_2 are real constants, and $r_1 \geq 0$ and $r_2 > 0$. Consider the left-hand side of the given equation, which we can manipulate as follows:

$$\begin{aligned}
\arg\left[\frac{z_1}{z_2}\right] &= \arg\left[\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right] \\
&= \arg\left[\left(\frac{r_1}{r_2}\right)\left(\frac{e^{j\theta_1}}{e^{j\theta_2}}\right)\right] \\
&= \arg\left[\left(\frac{r_1}{r_2}\right) e^{j(\theta_1-\theta_2)}\right] \\
&= \theta_1 - \theta_2 \\
&= \arg z_1 - \arg z_2.
\end{aligned}$$

(In the preceding steps, we used the fact that $r_1/r_2 \geq 0$, which must be true, since $r_1 \geq 0$ and $r_2 > 0$.) Thus, the given identity holds.

1 **Answer (e).**

We are asked to show that, for all complex z_1 and z_2, the following identity holds:

$$(z_1 z_2)^* = z_1^* z_2^*.$$

Let z_1 and z_2 be arbitrary complex numbers with the Cartesian representations

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2,$$

where x_1, y_1, x_2, and y_2 are real constants. Consider the left-hand side of the given equation, which we can manipulate as follows:

$$\begin{aligned}
(z_1 z_2)^* &= [(x_1 + jy_1)(x_2 + jy_2)]^* \\
&= [(x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)]^* \\
&= x_1 x_2 - y_1 y_2 - j(x_1 y_2 + x_2 y_1) \\
&= x_1 x_2 - y_1 y_2 - jx_1 y_2 - jx_2 y_1.
\end{aligned}$$

Consider the right-hand side of the given equation, which we can manipulate as follows:

$$\begin{aligned} z_1^* z_2^* &= (x_1 - jy_1)(x_2 - jy_2) \\ &= x_1 x_2 - y_1 y_2 - jx_1 y_2 - jx_2 y_1. \end{aligned}$$

Comparing the preceding expressions for the left-hand and right-hand sides, we conclude that $(z_1 z_2)^* = z_1^* z_2^*$. Thus, the given identity holds.

1 **A.5** For each function f of a real variable given below, find an expression for $|f(\omega)|$ and $\arg f(\omega)$.

(a) $f(\omega) = \dfrac{1}{(1+j\omega)^{10}}$;

(b) $f(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$;

(c) $f(\omega) = \dfrac{2e^{j11\omega}}{(3+j5\omega)^7}$;

(d) $f(\omega) = \dfrac{-5}{(-1-j\omega)^4}$;

(e) $f(\omega) = \dfrac{j\omega^2}{(j\omega-1)^{10}}$; and

(f) $f(\omega) = \dfrac{j\omega-1}{j\omega+1}$.

1 **Answer (c).**

We are given the function

$$f(\omega) = \frac{2e^{j11\omega}}{(3+j5\omega)^7}.$$

First, we compute the magnitude of $f(\omega)$ to obtain

$$\begin{aligned}|f(\omega)| &= \frac{\left|2e^{j11\omega}\right|}{|(3+j5\omega)^7|} \\ &= \frac{2}{|3+j5\omega|^7} \\ &= \frac{2}{(\sqrt{9+25\omega^2})^7} \\ &= \frac{2}{(9+25\omega^2)^{7/2}}.\end{aligned}$$

Next, we calculate the argument of $f(\omega)$ as

$$\begin{aligned}\arg f(\omega) &= \arg(2e^{j11\omega}) - \arg((3+j5\omega)^7) \\ &= \arg(2e^{j11\omega}) - \arg\left[\left(\sqrt{9+25\omega^2}e^{j\arctan(5\omega/3)}\right)^7\right] \\ &= \arg(2e^{j11\omega}) - \arg\left(\left(\sqrt{9+25\omega^2}\right)^7\left(e^{j\arctan(5\omega/3)}\right)^7\right) \\ &= \arg(2e^{j11\omega}) - \arg\left(\left(\sqrt{9+25\omega^2}\right)^7 e^{j7\arctan(5\omega/3)}\right) \\ &= 11\omega - 7\arctan(5\omega/3).\end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg f(\omega) = 11\omega - 7\arctan(5\omega/3) + 2\pi k$$

for all integer k.

1 **Answer (f).**

We are given the function

$$f(\omega) = \frac{j\omega-1}{j\omega+1}.$$

First, we compute the magnitude of $f(\omega)$ to obtain

$$\begin{aligned} |f(\omega)| &= \frac{|j\omega - 1|}{|j\omega + 1|} \\ &= \frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+1}} \\ &= 1. \end{aligned}$$

Next, we calculate the argument of $f(\omega)$ as

$$\begin{aligned} \arg f(\omega) &= \arg(j\omega - 1) - \arg(j\omega + 1) \\ &= \pi + \arctan(-\omega) - \arctan(\omega) \\ &= \pi - \arctan(\omega) - \arctan(\omega) \\ &= \pi - 2\arctan(\omega). \end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg f(\omega) = -2\arctan(\omega) + (2k+1)\pi$$

for all integer k.

1 **A.6** Use Euler's relation to show that each of the identities below holds, where θ is an arbitrary real constant.
(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$;
(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$; and
(c) $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

1 **Answer (b).**

From Euler's relation, we know

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

Thus, we can write

$$\begin{aligned}\tfrac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right] &= \tfrac{1}{2j}[\cos\theta + j\sin\theta - [\cos(-\theta) + j\sin(-\theta)]] \\ &= \tfrac{1}{2j}[\cos\theta + j\sin\theta - \cos(-\theta) - j\sin(-\theta)]].\end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can further simplify the above equation to obtain

$$\begin{aligned}\tfrac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right] &= \tfrac{1}{2j}[\cos\theta + j\sin\theta - \cos\theta + j\sin\theta] \\ &= \tfrac{1}{2j}[2j\sin\theta] \\ &= \sin\theta.\end{aligned}$$

Thus, we have that $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$.

1 **A.11** Determine the points at which each function f given below is: i) continuous, ii) differentiable, and iii) analytic. To deduce the answer, use your knowledge about polynomial and rational functions. Simply state the final answer along with a short justification (i.e., two or three sentences). (In other words, it is not necessary to use the Cauchy-Riemann equations for this problem.)

(a) $f(z) = 3z^3 - jz^2 + z - \pi$;

(b) $f(z) = \dfrac{z-1}{(z^2+3)(z^2+z+1)}$;

(c) $f(z) = \dfrac{z}{z^4-16}$; and

(d) $f(z) = z+2+z^{-1}$.

1 **Answer (c).**

The function f is a rational function. Rational functions are continuous, differentiable, and analytic everywhere, except at points where the denominator polynomial becomes zero. So, we find these points. We factor f as

$$f(z) = \frac{z}{(z^2)^2-4^2} = \frac{z}{(z^2-4)(z^2+4)} = \frac{z}{(z+2)(z-2)(z+j2)(z-j2)}.$$

Therefore, the denominator becomes zero for

$$z \in \{-2, 2, -2j, 2j\}.$$

Therefore, f is continuous, differentiable, and analytic everywhere, except at the points: -2, 2, $-2j$, and $2j$.

1 **Answer (d).**

The function f is a rational function. Rational functions are continuous, differentiable, and analytic everywhere, except at points where the denominator polynomial becomes zero. So, we find these points. We factor f as

$$f(z) = z^{-1}(z^2+2z+1) = \frac{z^2+2z+1}{z}.$$

Therefore, the denominator becomes zero for

$$z = 0.$$

Therefore, f is continuous, differentiable, and analytic everywhere, except at 0.

1 **A.13** For each rational function f of a complex variable given below, find the (finite) poles and zeros of f and their orders. Also, plot these poles and zeros in the complex plane.
(a) $f(z) = z^2 + jz + 3$;
(b) $f(z) = z + 3 + 2z^{-1}$;
(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$;
(d) $f(z) = \dfrac{z^3 - z}{z^2 - 4}$;
(e) $f(z) = \dfrac{z+\frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and
(f) $f(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

1 **Answer (b).**

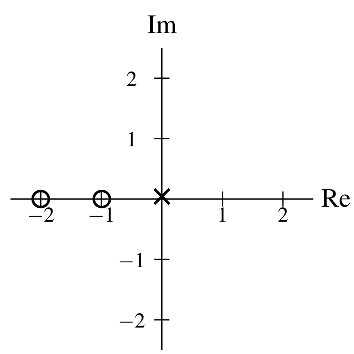
We are given the rational function

$$f(z) = z + 3 + 2z^{-1}.$$

By inspection, we can factor the given rational function as follows:

$$\begin{aligned} z + 3 + 2z^{-1} &= z^{-1}(z^2+3z+2) \\ &= z^{-1}(z+2)(z+1) \\ &= \frac{(z+2)(z+1)}{z}. \end{aligned}$$

Thus, f has first order zeros at -2 and -1, and a first order pole at 0.



1 **Answer (c).**

We are given the rational function

$$f(z) = \frac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}.$$

To begin, we fully factor each of the nonlinear factors in the expression given for f. First, let us consider the factor z^2+2z+5. Solving for the roots of $z^2+2z+5=0$ using the quadratic formula, we obtain

$$\begin{aligned} \tfrac{-2\pm\sqrt{2^2-4(1)(5)}}{2} &= \tfrac{-2\pm\sqrt{-16}}{2} \\ &= -1 \pm j2. \end{aligned}$$

Thus, we have the factorization

$$z^2+2z+5 = (z+1-j2)(z+1+j2).$$

Next, we consider the factor z^2+1, which can be factored (using a sum of squares rule) as

$$z^2+1=(z+j)(z-j).$$

Next, let us consider the factor z^2+2z+2. Solving for the roots of $z^2+2z+2=0$ using the quadratic formula, we obtain

$$\begin{aligned}\tfrac{-2\pm\sqrt{2^2-4(1)(2)}}{2} &= \tfrac{-2\pm\sqrt{-4}}{2}\\ &= -1\pm j.\end{aligned}$$

Thus, we have the factorization

$$z^2+2z+2=(z+1-j)(z+1+j).$$

Lastly, we consider the factor z^2+3z+2, which can be factored, by inspection, to yield

$$z^2+3z+2=(z+2)(z+1).$$

Combining all of the above results, we have that f has the factorization

$$f(z)=\frac{(z+1-j2)(z+1+j2)(z+j)(z-j)}{(z+1-j)(z+1+j)(z+2)(z+1)}.$$

Thus, f has:

- first order zeros at $-1-j2$, $-1+j2$, $-j$, and j; and
- first order poles at -2, -1, $-1+j$, and $-1-j$.