Stat 260 Lecture Notes
Set 18 - Joint Probability Distributions

Suppose X and Y are two discrete random variables on a sample space $\mathcal{S}$. The **joint probability mass function**, (**joint pmf**) $p(x, y)$ is defined as

$$p(x, y) = P(X = x \cap Y = y) = P(X = x, Y = y)$$

Rules for joint pmfs:

- $p(x, y) \geq 0$ for all x and y
- $\sum_x \sum_y p(x, y) = 1$

The **marginal pmf** of the random variable X, denoted by $p_X(x)$ is $p_X(x) = P(X = x) = \sum_y p(x, y)$. In other words, it's the probability where we focus on a specific x value and add up over all cases of y. The marginal pmf of X is the same as just the pmf of X.

Example 1: Suppose we have random variables X and Y with joint pmf:

	$p(x,y)$	Y = 5	10	15	
X	4	0.20	0.16	0.03	0.39
	14	0.15	0.14	0.02	0.31
	26	0.12	0.14	0.04	0.30

P(14,10)

(a) Find $P(X = 14 \cap Y = 10)$. $= 0.14$

(b) Find $P(Y = 10)$. $= 0.16 + 0.14 + 0.14 = 0.44$

(c) Find $P(Y \geq 10)$. $= P(X=10) + P(X=15) = 0.16 + 0.14 + 0.14 + 0.03 + 0.02 + 0.04 = 0.53$

(d) Find $E(X)$. $= \sum x \cdot f(x) = 13.7$

marginal pmf for X

x	4	14	26
$f(x)$	0.39	0.31	0.30

Recall that we calculate conditional probabilities as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. We can calculate conditional probabilities with joint probability distributions in the same way.

The **conditional pmf** of Y given $X = x$ is

$P(Y=y|X=x)$

$$p_{Y|X=x}(y) = \frac{p(x,y)}{p_X(x)} = \frac{P(X=x \cap Y=y)}{P(X=x)}$$

Example 2: Use the joint pmf from Example 1 to find $P(Y = 10|X = 14)$.

	$p(x,y)$	Y 5	10	15	
	4	0.20	0.16	0.03	0.39
X	14	0.15	0.14	0.02	0.31
	26	0.12	0.14	0.04	0.30
		0.47		0.09	

$$P(Y=10|X=14) = \frac{P(Y=10 \cap X=14)}{P(X=14)} = \frac{0.14}{0.31} = 0.4516$$

$$P(Y=10|X \geq 14) = \frac{P(Y=10 \cap X \geq 14)}{P(X \geq 14)} = \frac{0.14+0.14}{0.31+0.30} =$$

Two random variables X and Y are **independent** if for all pairs of values for x and y we have that $P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y)$. If there is any pair of values for x and y that does not satisfy this, then the random variables X and Y are not independent.

Example 3: Use the joint pmf from Examples 1 and 2. Look at the values $x = 4$ and $y = 5$.

$\therefore$ not independent

$P(X=4 \cap Y=5) = 0.20$

$P(X=4) \cdot P(Y=5) = (0.39)(0.47) = 0.1833$

This definition of independence says that if random variables X_1, X_2, $\ldots$, X_n are all independent, then the way to find the joint pmf is to multiply the marginal probabilities. In other words,

$$p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \cdots \cdot P(X_n = x_n)$$

$P(X_1=x_1 \cap X_2=x_2 \cap \ldots \cap X_n=x_n)$