Section 5.3

Properties of Fourier Series

Properties of (CT) Fourier Series

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Property	
Parseval's Relation	$\frac{1}{T} \int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Linearity

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where α and β are complex constants.
- That is, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

Time Shifting (Translation)

SKIP SLIDE

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

  where t_0 is a real constant.
- In other words, time shifting a periodic function changes the argument (but not magnitude) of its Fourier series coefficients.

Frequency Shifting (Modulation)

SKIP SLIDE

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_{k-M},$$

  where M is an integer constant.

- In other words, multiplying a periodic function by $e^{jM\omega_0 t}$ shifts the Fourier-series coefficient sequence.

Time Reversal (Reflection)

SKIP SLIDE

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a function results in a time reversal of its Fourier series coefficients.

Conjugation

SKIP SLIDE

- For a T-periodic function x with Fourier series coefficient sequence c, the following property holds:

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a function has the effect of time reversing and conjugating the Fourier series coefficient sequence.

Periodic Convolution

SKIP SLIDE

- Let x and y be two periodic functions with the same period T. If $x(t) \xleftrightarrow{\text{CTFS}} a_k$ and $y(t) \xleftrightarrow{\text{CTFS}} b_k$, then

$$x \circledast y(t) \xleftrightarrow{\text{CTFS}} Ta_k b_k.$$

- In other words, periodic convolution of two functions corresponds to the multiplication (up to a scale factor) of their Fourier-series coefficient sequences.

Multiplication

SKIP SLIDE

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x(t)y(t) \overset{\text{CTFS}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- As we shall see later, the above summation is the DT convolution of a and b.
- In other words, the multiplication of two periodic functions corresponds to the DT convolution of their corresponding Fourier-series coefficient sequences.

Parseval's Relation

SKIP SLIDE

- A function x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 .$$

- The above relationship is simply stating that the amount of energy in x (i.e., $\frac{1}{T}\int_T |x(t)|^2\, dt$) and the amount of energy in the Fourier series coefficient sequence a (i.e., $\sum_{k=-\infty}^{\infty} |a_k|^2$) are equal.
- In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

Even and Odd Symmetry

- For a periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and c always match.

Real Functions

- A function x is ***real*** if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k} \text{ for all } k$$

(i.e., c is ***conjugate symmetric***).

- Thus, for a real-valued function, the negative-indexed Fourier series coefficients are ***redundant***, as they are completely determined by the nonnegative-indexed coefficients.
- From properties of complex numbers, one can show that $c_k = c^*_{-k}$ is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}$$

(i.e., $|c_k|$ is ***even*** and $\arg c_k$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that c is real.

Trigonometric Forms of a Fourier Series

- Consider the periodic function x with the Fourier series coefficients c_k.
- If x is real, then its Fourier series can be rewritten in two other forms, known as the combined trigonometric and trigonometric forms.
- The **combined trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

where $\theta_k = \arg c_k$.
- The **trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t)\right],$$

where $\alpha_k = 2\,\mathrm{Re}\, c_k$ and $\beta_k = -2\,\mathrm{Im}\, c_k$.
- Note that the trigonometric forms contain only ***real*** quantities.

Other Properties of Fourier Series

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:
  1. c_0 is the average value of x over a single period T;
  2. x is real and even $\Leftrightarrow$ c is real and even; and
  3. x is real and odd $\Leftrightarrow$ c is purely imaginary and odd.

Zeroth Coefficient

Section 5.4

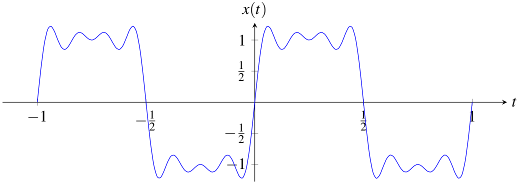
Fourier Series and Frequency Spectra

- The Fourier series provides us with an entirely new way to view functions.
- Instead of viewing a function as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a function as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved ***much more easily*** using the frequency domain than the time domain.
- The Fourier series coefficients of a function x provide a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the ***frequency spectrum*** of the function.

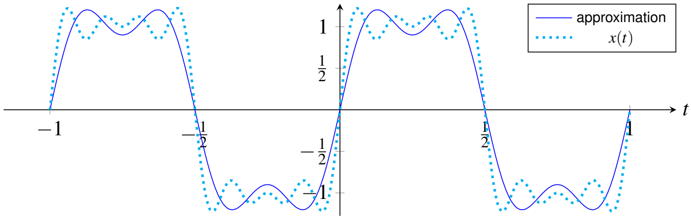
- Consider the real 1-periodic function x having the Fourier series representation

$$x(t) = -\frac{j}{10}e^{-j14\pi t} - \frac{2j}{10}e^{-j10\pi t} - \frac{4j}{10}e^{-j6\pi t} - \frac{13j}{10}e^{-j2\pi t} + \frac{13j}{10}e^{j2\pi t} + \frac{4j}{10}e^{j6\pi t} + \frac{2j}{10}e^{j10\pi t} + \frac{j}{10}e^{j14\pi t}.$$

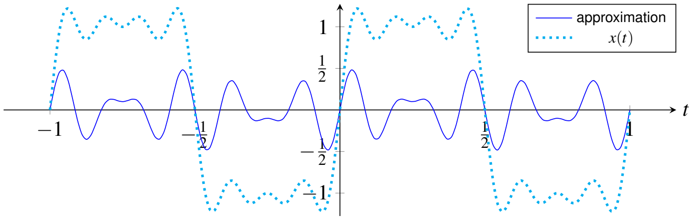
- A plot of x is shown below.



- The terms that make the most dominant contribution to the overall sum are the ones with the largest magnitude coefficients.
- To illustrate this, we consider the problem of determining the best approximation of x that keeps only 4 of the 8 terms in the Fourier series.



Approximation using the 4 terms with the largest magnitude coefficients



Approximation using the 4 terms with the smallest magnitude nonzero coefficients

Fourier Series and Frequency Spectra

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the function x, it is helpful to write the Fourier series with the c_k expressed in ***polar form*** as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| \, e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been ***amplitude scaled*** by a factor of $|c_k|$ and ***time shifted*** by an amount that depends on $\arg c_k$.
- For a given k, the ***larger*** $|c_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the ***larger the contribution*** the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.
- In this way, we can use $|c_k|$ as a ***measure*** of how much information a function x has at the frequency $k\omega_0$.

- The Fourier series coefficients c_k are referred to as the **frequency spectrum** of x.
- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the **magnitude spectrum** of x.
- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the **phase spectrum** of x.
- Normally, the spectrum of a function is plotted against frequency $k\omega_0$ instead of k.
- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is ***discrete*** in the independent variable (i.e., frequency).
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as **line spectra**.

Example 5.7

Frequency Spectra of Real Functions

- Recall that, for a real function x, the Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k}$$

(i.e., c is ***conjugate symmetric***), which is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real function is always ***even***.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real function is always ***odd***.
- Due to the symmetry in the frequency spectra of real functions, we typically ***ignore negative frequencies*** when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

Section 5.5

Fourier Series and LTI Systems

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{e^{st}\}(t) = H_{\mathsf{L}}(s)e^{st}$, where $H_{\mathsf{L}}(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$. (That is, complex exponentials are ***eigenfunctions*** of LTI systems.)
- Since a complex sinusoid is a ***special case*** of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{j\omega t}\}(t) = H(\omega)e^{j\omega t},$$

  where ω is a real constant and

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an ***eigenfunction*** of a LTI system and $H(\omega)$ is the corresponding ***eigenvalue***.
- We refer to H as the **frequency response** of the system $\mathcal{H}$.

- Consider a LTI system with input x, output y, and frequency response H.
- Suppose that the T-periodic input x is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T}.$$

- Using our knowledge about the ***eigenfunctions*** of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}.$$

- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$ then $y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(k\omega_0)c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that ***does not require convolution***.

Example 5.9

Filtering

- In many applications, we want to ***modify the spectrum*** of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.