

**5A 6.1** Using the Fourier transform analysis equation, find the Fourier transform  $X$  of each function  $x$  below.

(a)  $x(t) = \text{rect}(t - t_0)$ , where  $t_0$  is a real constant;

(b)  $x(t) = e^{-4t}u(t - 1)$ ;

(c)  $x(t) = 3[u(t) - u(t - 2)]$ ; and

(d)  $x(t) = e^{-|t|}$ .

**5A Answer (c).**

We are asked to find the Fourier transform of the function

$$x(t) = 3[u(t) - u(t - 2)].$$

Let  $X$  denote the Fourier transform of  $x$ . From the Fourier transform analysis equation, we have

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} 3[u(t) - u(t - 2)]e^{-j\omega t} dt \\ &= 3 \int_{-\infty}^{\infty} [u(t) - u(t - 2)]e^{-j\omega t} dt \\ &= 3 \int_0^2 e^{-j\omega t} dt \\ &= 3 \left[ \frac{1}{-j\omega} e^{-j\omega t} \right]_0^2 \\ &= \frac{3}{-j\omega} [e^{-j2\omega} - 1] \\ &= \frac{j3}{\omega} [e^{-j2\omega} - 1] \\ &= \frac{j3}{\omega} [e^{-j\omega}][e^{-j\omega} - e^{j\omega}] \\ &= \frac{j3}{\omega} e^{-j\omega} [-2j \sin \omega] \\ &= \frac{6}{\omega} e^{-j\omega} \sin \omega \\ &= 6e^{-j\omega} \text{sinc } \omega. \end{aligned}$$

**5A Answer (d).**

We are asked to find the Fourier transform of the function

$$x(t) = e^{-|t|}.$$

Let  $X$  denote the Fourier transform of  $x$ . From the Fourier transform analysis equation, we have

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
 &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt \\
 &= \int_{-\infty}^0 e^{-|t|} e^{-j\omega t} dt + \int_0^{\infty} e^{-|t|} e^{-j\omega t} dt \\
 &= \int_{-\infty}^0 e^t e^{-j\omega t} dt + \int_0^{\infty} e^{-t} e^{-j\omega t} dt \\
 &= \int_{-\infty}^0 e^{(1-j\omega)t} dt + \int_0^{\infty} e^{(-1-j\omega)t} dt \\
 &= \frac{1}{1-j\omega} \left[ e^{(1-j\omega)t} \right]_{-\infty}^0 - \frac{1}{1+j\omega} \left[ e^{(-1-j\omega)t} \right]_0^{\infty} \\
 &= \frac{1}{1-j\omega} [1 - 0] - \frac{1}{1+j\omega} [0 - 1] \\
 &= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \\
 &= \frac{1+j\omega + 1-j\omega}{(1+j\omega)(1-j\omega)} \\
 &= \frac{2}{1+\omega^2}.
 \end{aligned}$$

**5A 6.3** Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform  $X$  of each function  $x$  below.

- (a)  $x(t) = \cos(t - 5)$ ;
- (b)  $x(t) = e^{-j5t}u(t + 2)$ ;
- (c)  $x(t) = \cos(t)u(t)$ ;
- (d)  $x(t) = 6[u(t) - u(t - 3)]$ ;
- (e)  $x(t) = 1/t$ ;
- (f)  $x(t) = t \text{rect}(2t)$ ;
- (g)  $x(t) = e^{-j3t} \sin(5t - 2)$ ;
- (h)  $x(t) = \cos(5t - 2)$ ;
- (i)  $x(t) = x_1 * x_2(t)$ , where  $x_1(t) = e^{-2t}u(t)$  and  $x_2(t) = te^{-3t}u(t)$ ; and
- (j)  $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT)$ , where  $a$  is a complex constant satisfying  $|a| < 1$  and  $T$  is a strictly-positive real constant; (Hint: Recall the formula for the sum of an infinite geometric sequence (i.e., (F.9)).)

**5A Answer (c).**

We are asked to find the Fourier transform  $X$  of

$$x(t) = \cos(t)u(t).$$

We begin by rewriting  $x(t)$  as

$$x(t) = v_1(t)v_2(t),$$

where

$$\begin{aligned} v_1(t) &= \cos(t) \quad \text{and} \\ v_2(t) &= u(t). \end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} X(\omega) &= \frac{1}{2\pi} V_1 * V_2(\omega), \\ V_1(\omega) &= \pi[\delta(\omega - 1) + \delta(\omega + 1)], \quad \text{and} \\ V_2(\omega) &= \pi\delta(\omega) + \frac{1}{j\omega}. \end{aligned}$$

Combining the above results, we obtain

$$\begin{aligned} X(\omega) &= \frac{1}{2\pi} V_1 * V_2(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V_1(\lambda) V_2(\omega - \lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi [\delta(\lambda - 1) + \delta(\lambda + 1)] \left[ \pi\delta(\omega - \lambda) + \frac{1}{j(\omega - \lambda)} \right] d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left[ \pi\delta(\lambda - 1)\delta(\omega - \lambda) + \delta(\lambda - 1)\frac{1}{j(\omega - \lambda)} + \pi\delta(\lambda + 1)\delta(\omega - \lambda) + \delta(\lambda + 1)\frac{1}{j(\omega - \lambda)} \right] d\lambda \\ &= \frac{1}{2} \left[ \pi\delta(\omega - 1) + \frac{1}{j(\omega - 1)} + \pi\delta(\omega + 1) + \frac{1}{j(\omega + 1)} \right] \\ &= \frac{1}{2} \left[ \pi\delta(\omega - 1) + \pi\delta(\omega + 1) - \frac{j}{\omega - 1} - \frac{j}{\omega + 1} \right] \\ &= \frac{1}{2} \left[ \pi\delta(\omega - 1) + \pi\delta(\omega + 1) + \frac{-j(\omega - 1) - j(\omega + 1)}{\omega^2 - 1} \right] \\ &= \frac{1}{2} \left[ \pi\delta(\omega - 1) + \pi\delta(\omega + 1) - \frac{j2\omega}{\omega^2 - 1} \right] \\ &= \frac{\pi}{2} [\delta(\omega - 1) + \delta(\omega + 1)] - \frac{j\omega}{\omega^2 - 1}. \end{aligned}$$

**5A Answer (d).**

We are asked to find the Fourier transform  $X$  of

$$x(t) = 6[u(t) - u(t-3)].$$

We begin by rewriting  $x(t)$  as

$$x(t) = 6v_3(t),$$

where

$$\begin{aligned} v_3(t) &= v_2(t/3), \\ v_2(t) &= v_1\left(t - \frac{1}{2}\right), \quad \text{and} \\ v_1(t) &= \text{rect}(t). \end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} X(\omega) &= 6V_3(\omega), \\ V_3(\omega) &= 3V_2(3\omega), \\ V_2(\omega) &= e^{-j\omega/2}V_1(\omega), \quad \text{and} \\ V_1(\omega) &= \text{sinc}(\omega/2). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(\omega) &= 6V_3(\omega) \\ &= 6(3)V_2(3\omega) \\ &= 18V_2(3\omega) \\ &= 18e^{-j3\omega/2}V_1(3\omega) \\ &= 18e^{-j3\omega/2}\text{sinc}\left(\frac{3\omega}{2}\right). \end{aligned}$$

Alternatively, we can restate this result in a slightly different form (i.e., in terms of complex exponentials) as follows:

$$\begin{aligned} X(\omega) &= 18e^{-j3\omega/2}\text{sinc}\left(\frac{3\omega}{2}\right) \\ &= 18e^{-j3\omega/2}\frac{2}{3\omega}\left[\frac{1}{2j}\left[e^{j3\omega/2} - e^{-j3\omega/2}\right]\right] \\ &= \frac{6}{j\omega}[1 - e^{-j3\omega}]. \end{aligned}$$

ALTERNATIVE SOLUTION. We have

$$\begin{aligned} X(\omega) &= 6[\mathcal{F}\{u(t)\}(\omega) - \mathcal{F}\{u(t-3)\}(\omega)] \\ &= 6\left(\pi\delta(\omega) + \frac{1}{j\omega} - e^{-j3\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right) \\ &= 6\left(\pi\delta(\omega) + \frac{1}{j\omega} - \pi\delta(\omega)e^{-j3\omega} - \frac{1}{j\omega}e^{-j3\omega}\right) \\ &= 6\left(\pi\delta(\omega) + \frac{1}{j\omega} - \pi\delta(\omega) - \frac{1}{j\omega}e^{-j3\omega}\right) \\ &= \frac{6}{j\omega}(1 - e^{-j3\omega}) \\ &= \frac{6}{j\omega}e^{-j3\omega/2}(e^{j3\omega/2} - e^{-j3\omega/2}) \\ &= \frac{6}{j\omega}e^{-j3\omega/2}(2j)\sin(3\omega/2) \\ &= \frac{12}{\omega}e^{-j3\omega/2}\sin(3\omega/2) \\ &= \left(\frac{3\omega}{2}\right)\left(\frac{12}{\omega}\right)e^{-j3\omega/2}\left(\frac{3\omega}{2}\right)^{-1}\sin(3\omega/2) \\ &= 18e^{-j3\omega/2}\text{sinc}(3\omega/2). \end{aligned}$$

**5A Answer (e).**

We are asked to find the Fourier transform  $X$  of

$$x(t) = 1/t.$$

From a table of Fourier transforms, we have

$$\text{sgn } t \xleftrightarrow{\text{CTFT}} \frac{2}{j\omega}.$$

From this transform pair, we can use the duality property of the Fourier transform to deduce

$$\begin{aligned}\mathcal{F}\left\{\frac{2}{j\omega}\right\}(\omega) &= 2\pi \text{sgn}(-\omega) \\ &= -2\pi \text{sgn } \omega.\end{aligned}$$

Using this result and the linearity property of the Fourier transform, we can write

$$\begin{aligned}X(\omega) &= \mathcal{F}\{1/t\}(\omega) \\ &= \frac{j}{2} \mathcal{F}\left\{\frac{2}{j\omega}\right\}(\omega) \\ &= \frac{j}{2} [-2\pi \text{sgn } \omega] \\ &= -j\pi \text{sgn } \omega.\end{aligned}$$

**5A Answer (f).**

We are asked to find the Fourier transform  $X$  of

$$x(t) = t \text{rect}(2t).$$

We begin by rewriting  $x(t)$  as

$$x(t) = tv_2(t),$$

where

$$\begin{aligned}v_2(t) &= v_1(2t) \quad \text{and} \\ v_1(t) &= \text{rect}(t).\end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations, we obtain

$$\begin{aligned}V_1(\omega) &= \text{sinc}(\omega/2), \\ V_2(\omega) &= \frac{1}{2}V_1\left(\frac{\omega}{2}\right), \quad \text{and} \\ X(\omega) &= j\frac{d}{d\omega}V_2(\omega).\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}X(\omega) &= j\frac{d}{d\omega}V_2(\omega) \\ &= j\frac{d}{d\omega}\left[\frac{1}{2}V_1\left(\frac{\omega}{2}\right)\right] \\ &= \frac{j}{2}\frac{d}{d\omega}V_1\left(\frac{\omega}{2}\right) \\ &= \frac{j}{2}\frac{d}{d\omega}\text{sinc}\left(\frac{\omega}{4}\right) \\ &= \frac{j}{2}\left[\frac{\frac{\omega}{4}\left(\frac{1}{4}\cos\left(\frac{\omega}{4}\right)\right) - \frac{1}{4}\sin\left(\frac{\omega}{4}\right)}{\omega^2/16}\right] \\ &= \frac{j}{2}\left[\frac{16\left(\frac{\omega}{16}\cos\left(\frac{\omega}{4}\right) - \frac{1}{4}\sin\left(\frac{\omega}{4}\right)\right)}{\omega^2}\right] \\ &= \frac{j}{2}\left[\frac{1}{\omega}\cos\left(\frac{\omega}{4}\right) - \frac{4}{\omega^2}\sin\left(\frac{\omega}{4}\right)\right] \\ &= \frac{j}{2\omega}\cos\left(\frac{\omega}{4}\right) - \frac{j^2}{\omega^2}\sin\left(\frac{\omega}{4}\right).\end{aligned}$$

**5A Answer (g).**

We are asked to find the Fourier transform  $X$  of

$$x(t) = e^{-j3t} \sin(5t - 2).$$

We begin by rewriting  $x(t)$  as

$$x(t) = e^{-j3t} v_3(t),$$

where

$$\begin{aligned} v_3(t) &= v_2(5t), \\ v_2(t) &= v_1(t - 2), \quad \text{and} \\ v_1(t) &= \sin t. \end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} V_1(\omega) &= \frac{\pi}{j} [\delta(\omega - 1) - \delta(\omega + 1)], \\ V_2(\omega) &= e^{-j2\omega} V_1(\omega), \\ V_3(\omega) &= \frac{1}{5} V_2\left(\frac{\omega}{5}\right), \quad \text{and} \\ X(\omega) &= V_3(\omega + 3). \end{aligned}$$

Combining the above results, we obtain

$$\begin{aligned} X(\omega) &= V_3(\omega + 3) \\ &= \frac{1}{5} V_2\left(\frac{\omega + 3}{5}\right) \\ &= \frac{1}{5} e^{-j2(\omega + 3)/5} V_1\left(\frac{\omega + 3}{5}\right) \\ &= \frac{\pi}{j5} e^{-j2(\omega + 3)/5} \left[ \delta\left(\frac{\omega + 3}{5} - 1\right) - \delta\left(\frac{\omega + 3}{5} + 1\right) \right] \\ &= -\frac{j\pi}{5} e^{-j2(\omega + 3)/5} \left[ \delta\left(\frac{\omega - 2}{5}\right) - \delta\left(\frac{\omega + 8}{5}\right) \right] \\ &= -\frac{j\pi}{5} e^{-j2(\omega + 3)/5} [5\delta(\omega - 2) - 5\delta(\omega + 8)] \\ &= j\pi e^{-j2(\omega + 3)/5} [\delta(\omega + 8) - \delta(\omega - 2)] \\ &= j\pi \left[ e^{-j2(\omega + 3)/5} \delta(\omega + 8) - e^{-j2(\omega + 3)/5} \delta(\omega - 2) \right] \\ &= j\pi \left( \left[ e^{-j2(\omega + 3)/5} \right] \Big|_{\omega=-8} \delta(\omega + 8) - \left[ e^{-j2(\omega + 3)/5} \right] \Big|_{\omega=2} \delta(\omega - 2) \right) \\ &= j\pi [e^{j2} \delta(\omega + 8) - e^{-j2} \delta(\omega - 2)]. \end{aligned}$$

(In the above simplification, we used the fact that  $\delta(at) = \frac{1}{|a|} \delta(t)$ .)

**5A 6.4** For each function  $y$  given below, find the Fourier transform  $Y$  of  $y$  in terms of the Fourier transform  $X$  of  $x$ .

- (a)  $y(t) = x(at - b)$ , where  $a$  and  $b$  are constants and  $a \neq 0$ ;
- (b)  $y(t) = \int_{-\infty}^{2t} x(\tau) d\tau$ ;
- (c)  $y(t) = \int_{-\infty}^t x^2(\tau) d\tau$ ;
- (d)  $y(t) = \mathcal{D}(x * x)(t)$ , where  $\mathcal{D}$  denotes the derivative operator;
- (e)  $y(t) = tx(2t - 1)$ ;
- (f)  $y(t) = e^{j2t}x(t - 1)$ ;
- (g)  $y(t) = (te^{-j5t}x(t))^*$ ; and
- (h)  $y(t) = (\mathcal{D}x) * x_1(t)$ , where  $x_1(t) = e^{-jt}x(t)$  and  $\mathcal{D}$  denotes the derivative operator.

**5A Answer (a).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = x(at - b), \quad \text{where } a, b \in \mathbb{R} \text{ and } a \neq 0.$$

We rewrite  $y(t)$  as

$$y(t) = v_1(at)$$

where

$$v_1(t) = x(t - b).$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \frac{1}{|a|} V_1\left(\frac{\omega}{a}\right) \quad \text{and} \\ V_1(\omega) &= e^{-j\omega b} X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{|a|} V_1\left(\frac{\omega}{a}\right) \\ &= \frac{1}{|a|} e^{-j(\omega/a)b} X(\omega/a) \\ &= \frac{1}{|a|} e^{-jb\omega/a} X(\omega/a). \end{aligned}$$

**5A Answer (b).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = \int_{-\infty}^{2t} x(\tau) d\tau.$$

We rewrite  $y(t)$  as

$$y(t) = v_1(2t)$$

where

$$v_1(t) = \int_{-\infty}^t x(\tau) d\tau.$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{v_1(2t)\}(\omega) \\ &= \frac{1}{2}V_1\left(\frac{\omega}{2}\right) \quad \text{and} \\ V_1(\omega) &= \mathcal{F}\left\{\int_{-\infty}^t x(\tau)d\tau\right\}(\omega) \\ &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned} Y(\omega) &= \frac{1}{2}V_1\left(\frac{\omega}{2}\right) \\ &= \frac{1}{2}\left(\frac{1}{j(\omega/2)}X\left(\frac{\omega}{2}\right) + \pi X(0)\delta\left(\frac{\omega}{2}\right)\right) \\ &= \frac{1}{j\omega}X\left(\frac{\omega}{2}\right) + \frac{\pi}{2}X(0)\delta\left(\frac{\omega}{2}\right). \end{aligned}$$

**5A Answer (c).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = \int_{-\infty}^t x^2(\tau)d\tau.$$

We rewrite  $y(t)$  as

$$y(t) = \int_{-\infty}^t v_1(\tau)d\tau$$

where

$$v_1(t) = x^2(t).$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} V_1(\omega) &= \frac{1}{2\pi}X * X(\omega), \quad \text{and} \\ Y(\omega) &= \frac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} Y(\omega) &= \frac{1}{j\omega}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\lambda)X(\omega-\lambda)d\lambda\right] + \pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\lambda)X(-\lambda)d\lambda\right]\delta(\omega) \\ &= \frac{1}{j2\pi\omega}\int_{-\infty}^{\infty}X(\lambda)X(\omega-\lambda)d\lambda + \frac{1}{2}\delta(\omega)\int_{-\infty}^{\infty}X(\lambda)X(-\lambda)d\lambda. \end{aligned}$$

**5A Answer (d).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = \mathcal{D}(x * x)(t), \quad \text{where } \mathcal{D} \text{ denotes the derivative operator.}$$

We rewrite  $y(t)$  as

$$y(t) = \frac{d}{dt}v_1(t)$$

where

$$v_1(t) = x * x(t).$$



Taking the Fourier transform of both sides of these equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\left\{\frac{d}{dt}v_1(t)\right\}(\omega) \\ &= j\omega V_1(\omega) \quad \text{and} \\ V_1(\omega) &= \mathcal{F}\{x * x\}(\omega) \\ &= X^2(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= j\omega V_1(\omega) \\ &= j\omega X^2(\omega). \end{aligned}$$

**5A Answer (e).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = tx(2t - 1).$$

We rewrite  $y(t)$  as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t) \quad \text{and} \\ v_2(t) &= x(t - 1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{tv_1(t)\}(\omega) \\ &= j\frac{d}{d\omega}V_1(\omega), \\ V_1(\omega) &= \mathcal{F}\{v_2(2t)\}(\omega) \\ &= \frac{1}{2}V_2\left(\frac{\omega}{2}\right), \quad \text{and} \\ V_2(\omega) &= \mathcal{F}\{x(t - 1)\}(\omega) \\ &= e^{-j\omega}X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= j\frac{d}{d\omega}V_1(\omega) \\ &= j\frac{d}{d\omega}\left[\left(\frac{1}{2}\right)V_2\left(\frac{\omega}{2}\right)\right] \\ &= \frac{j}{2}\left[\frac{d}{d\omega}e^{-j\omega/2}X\left(\frac{\omega}{2}\right)\right]. \end{aligned}$$

ALTERNATE SOLUTION. In what follows, we use the prime symbol to denote derivative (i.e.,  $f'$  denotes the derivative of  $f$ ). We can rewrite  $y(t)$  as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t), \quad \text{and} \\ v_2(t) &= x(t - 1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations, we obtain

$$\begin{aligned} Y(\omega) &= jV_1'(\omega), \\ V_1(\omega) &= \frac{1}{2}V_2(\omega/2), \text{ and} \\ V_2(\omega) &= e^{-j\omega}X(\omega). \end{aligned}$$

In anticipation of what is to come, we compute the quantities:

$$\begin{aligned} V_1'(\omega) &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)V_2'(\omega/2) = \frac{1}{4}V_2'(\omega/2) \text{ and} \\ V_2'(\omega) &= -je^{-j\omega}X(\omega) + X'(\omega)e^{-j\omega}. \end{aligned}$$

Combining the above equations, we have

$$\begin{aligned} Y(\omega) &= jV_1'(\omega) \\ &= j\frac{1}{4}V_2'(\omega/2) \\ &= \frac{j}{4}\left[-je^{-j\omega/2}X(\omega/2) + e^{-j\omega/2}X'(\omega/2)\right]. \end{aligned}$$

**5A Answer (f).**

We are asked to find the Fourier transform  $Y$  of

$$y(t) = e^{j2t}x(t-1).$$

We begin by rewriting  $y(t)$  as

$$y(t) = e^{j2t}v_1(t)$$

where

$$v_1(t) = x(t-1).$$

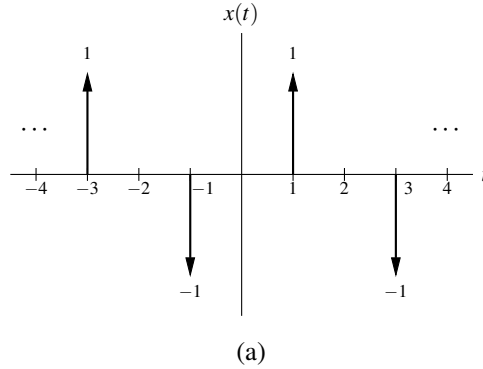
Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} V_1(\omega) &= e^{-j\omega}X(\omega) \text{ and} \\ Y(\omega) &= V_1(\omega-2). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} Y(\omega) &= V_1(\omega-2) \\ &= e^{-j(\omega-2)}X(\omega-2). \end{aligned}$$

**5A 6.5** Find the Fourier transform  $X$  of each periodic function  $x$  shown below.



**5A Answer (a).**

The frequency  $\omega_0$  is given by  $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$ . Consider the period of  $x(t)$  for  $-2 \leq t < 2$ . Let us denote this single period as  $x_T(t)$ . We have

$$x_T(t) = -\delta(t+1) + \delta(t-1).$$

Taking the Fourier transform of  $x_T(t)$ , we obtain

$$\begin{aligned} X_T(\omega) &= \mathcal{F}\{\delta(t-1) - \delta(t+1)\}(\omega) \\ &= \mathcal{F}\{\delta(t-1)\}(\omega) - \mathcal{F}\{\delta(t+1)\}(\omega) \\ &= e^{-j\omega} - e^{j\omega} \\ &= -[e^{j\omega} - e^{-j\omega}] \\ &= -2j \sin \omega. \end{aligned}$$

Using the formula for the Fourier transform of a periodic signal, we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}x(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \frac{\pi}{2} [-2j \sin(k\frac{\pi}{2})] \delta(\omega - k\frac{\pi}{2}) \\ &= \sum_{k=-\infty}^{\infty} -j\pi [\sin(\frac{k\pi}{2})] \delta(\omega - \frac{k\pi}{2}). \end{aligned}$$

**5A 6.10** For each function  $x$  given below, compute the frequency spectrum of  $x$ , and find and plot the corresponding magnitude and phase spectra.

(a)  $x(t) = e^{-at}u(t)$ , where  $a$  is a positive real constant; and

(b)  $x(t) = \text{sinc}\left(\frac{1}{200}t - \frac{1}{200}\right)$ .

**5A Answer (a).**

Taking the Fourier transform of  $x$ , we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}\{e^{-at}u(t)\}(\omega) \\ &= \frac{1}{a + j\omega}. \end{aligned}$$

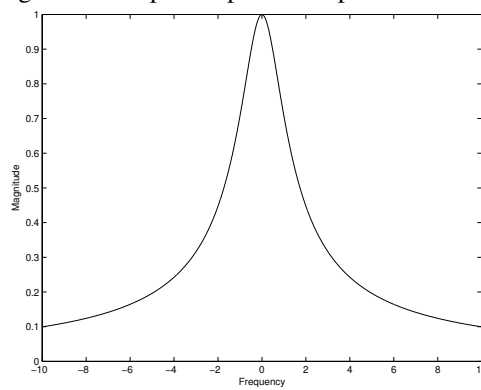
Computing the magnitude spectrum, we obtain

$$\begin{aligned} |X(\omega)| &= \left| \frac{1}{a + j\omega} \right| \\ &= \frac{|1|}{|a + j\omega|} \\ &= \frac{1}{\sqrt{a^2 + \omega^2}}. \end{aligned}$$

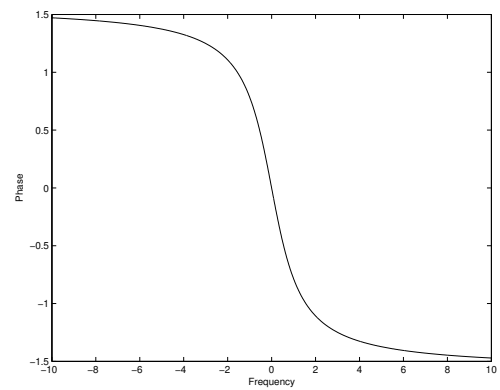
Computing the phase spectrum, we obtain

$$\begin{aligned} \arg X(\omega) &= \arg \left[ \frac{1}{a + j\omega} \right] \\ &= \arg 1 - \arg(a + j\omega) \\ &= 0 - \arg(a + j\omega) \\ &= -\arg(a + j\omega) \\ &= -\arctan\left(\frac{\omega}{a}\right). \end{aligned}$$

The magnitude and phase spectra are plotted below for  $a = 1$ .



(a)



(b)