Example 7.4. Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we can write

$$\begin{aligned}
X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) && \text{definition of LT}\\
&= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt && \text{use } u \text{ to change limits}\\
&= \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt && \text{combine exponentials}\\
&= \int_{-\infty}^{0} -e^{-(s+a)t}\,dt && \text{integrate}\\
&= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}.
\end{aligned}$$

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

real exponential
$\begin{cases} 0 & \sigma+a<0 \\ \infty & \sigma+a>0 \end{cases}$
complex sinusoid
finite but limit not well defined

$$\begin{aligned}
X(s) &= \left[\left(\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} && \text{split exponential}\\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0} && \text{take difference}\\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[1-\underbrace{e^{(\sigma+a)\infty}e^{j\omega\infty}}\right].
\end{aligned}$$

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\text{Re}(s) < -a$). In this case, we have

if $\text{Re}(s) < -a$

$$\begin{aligned}
X(s) &= \left(\tfrac{1}{\sigma+a+j\omega}\right)[1-0] && \text{rewrite in terms of } s \;\; (s=\sigma+j\omega)\\
&= \frac{1}{s+a}.
\end{aligned}$$

Thus, we have that

$$-e^{-at}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s+a} \quad \text{for } \text{Re}(s) < -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.3(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.

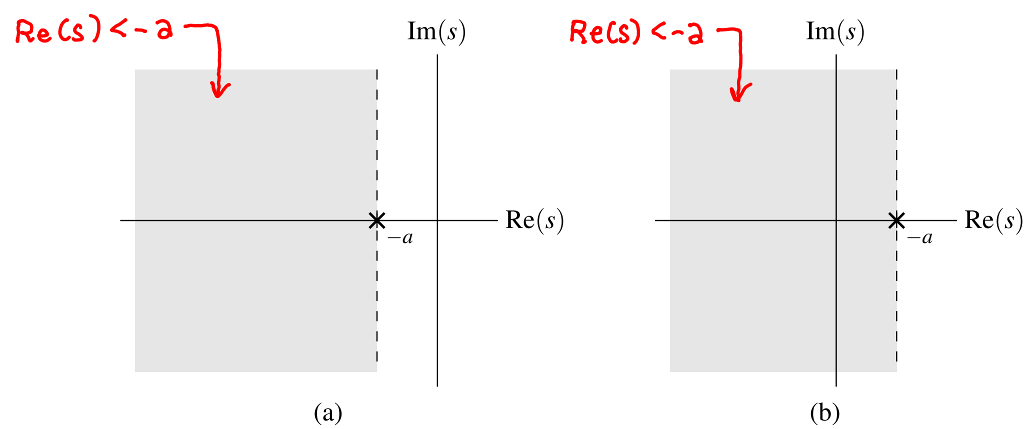

Figure 7.3: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

NOTE:

	different		same		different (and this is critical for invertibility of LT)
Example 7.3	$e^{-at}u(t)$	$\xleftrightarrow{\text{LT}}$	$\frac{1}{s+a}$	for	$\text{Re}(s) > -a$
Example 7.4	$-e^{-at}u(-t)$	$\xleftrightarrow{\text{LT}}$	$\frac{1}{s+a}$	for	$\text{Re}(s) < -a$

Example 7.7. The Laplace transform X of the function x has the algebraic expression

$$X(s) = \frac{s+\frac{1}{2}}{(s^2+2s+2)(s^2+s-2)}.$$

← rational function

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding function x is left sided, right sided, two sided, or finite duration.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)(s-1)}.$$

these factors obtained by using quadratic formula

Thus, X has poles at -2, $-1-j$, $-1+j$, and 1. Since these poles only have three distinct real parts (namely, -2, -1, and 1), there are four possible ROCs:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$,
iii) $-1 < \operatorname{Re}(s) < 1$, and
iv) $\operatorname{Re}(s) > 1$.

These ROCs are plotted in Figures 7.8(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding x must be left sided. The second ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding x must be two sided. The third ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding x must be two sided. The fourth ROC is a right-half plane, so the corresponding x must be right sided.

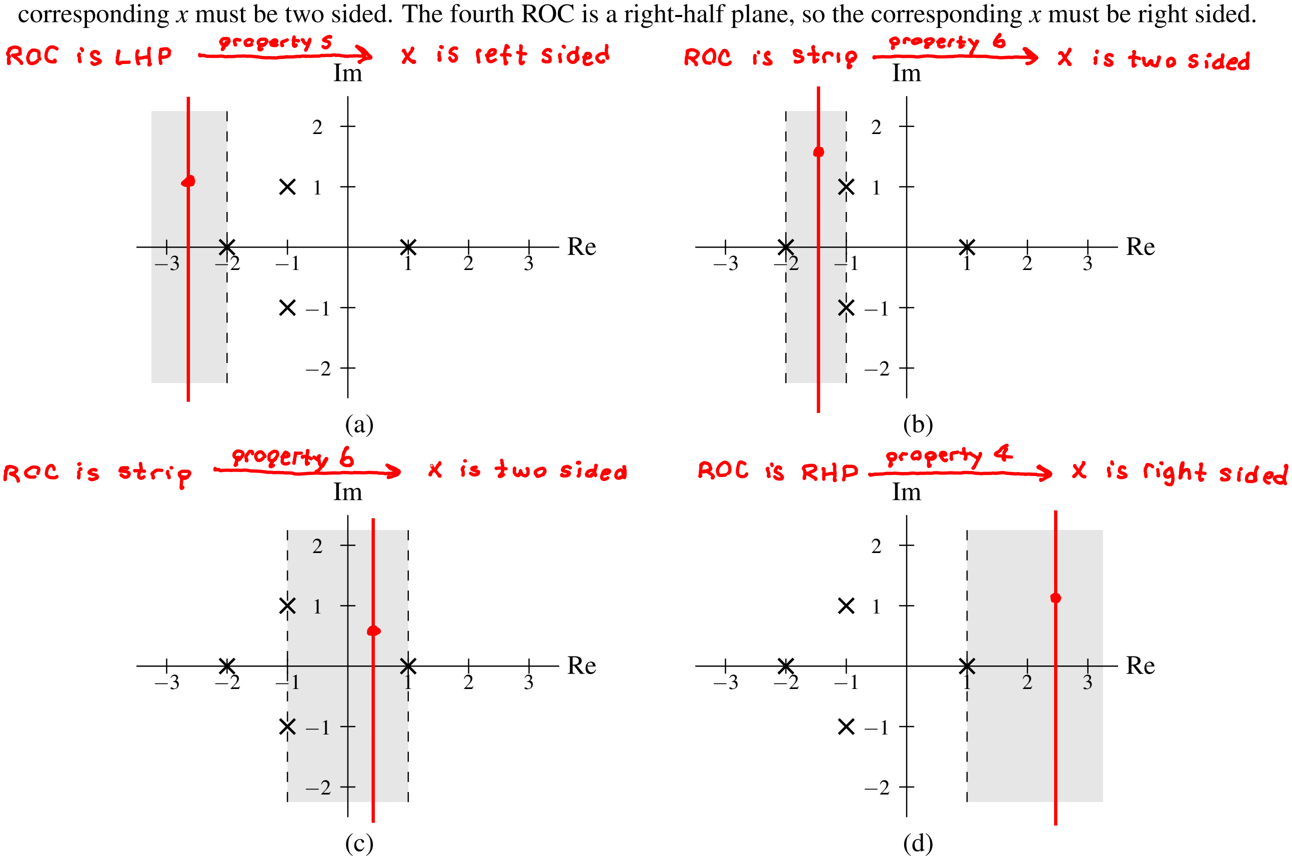

Figure 7.8: ROCs for example. The (a) first, (b) second, (c) third, and (d) fourth possible ROCs for X.

■

Example 7.8 (Linearity property of the Laplace transform)**.** Find the Laplace transform of the function

$$x = x_1 + x_2,$$

where

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

Solution. Using Laplace transform pairs from Table 7.2, we have

$$\begin{aligned} ①\quad X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) \\ &= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\ ②\quad X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) \\ &= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) \\ &= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \\ &= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

from LT table

linearity

from LT table and ⊛

common denominator

So, from the definition of X, we can write

⊛ $[\operatorname{Re}(s) > -2] \cap [\operatorname{Re}(s) > -1]$
$= \operatorname{Re}(s) > -1$

$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 + x_2\}(s) \\ &= X_1(s) + X_2(s) \\ &= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2+1}{(s+1)(s+2)} \\ &= \frac{s+3}{(s+1)(s+2)}. \end{aligned}$$

linearity

substitute expressions for X_1 and X_2 in ① and ②

common denominator

simplify

but is it larger than the intersection?

Now, we must determine the ROC of X. We know that the ROC of X must contain the intersection of the ROCs of X_1 and X_2. So, the ROC must contain $\operatorname{Re}(s) > -1$. Furthermore, the ROC cannot be larger than this intersection, since X has a pole at -1. Therefore, the ROC of X is $\operatorname{Re}(s) > -1$. The various ROCs are illustrated in Figure 7.9. So, in conclusion, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \qquad \blacksquare$$

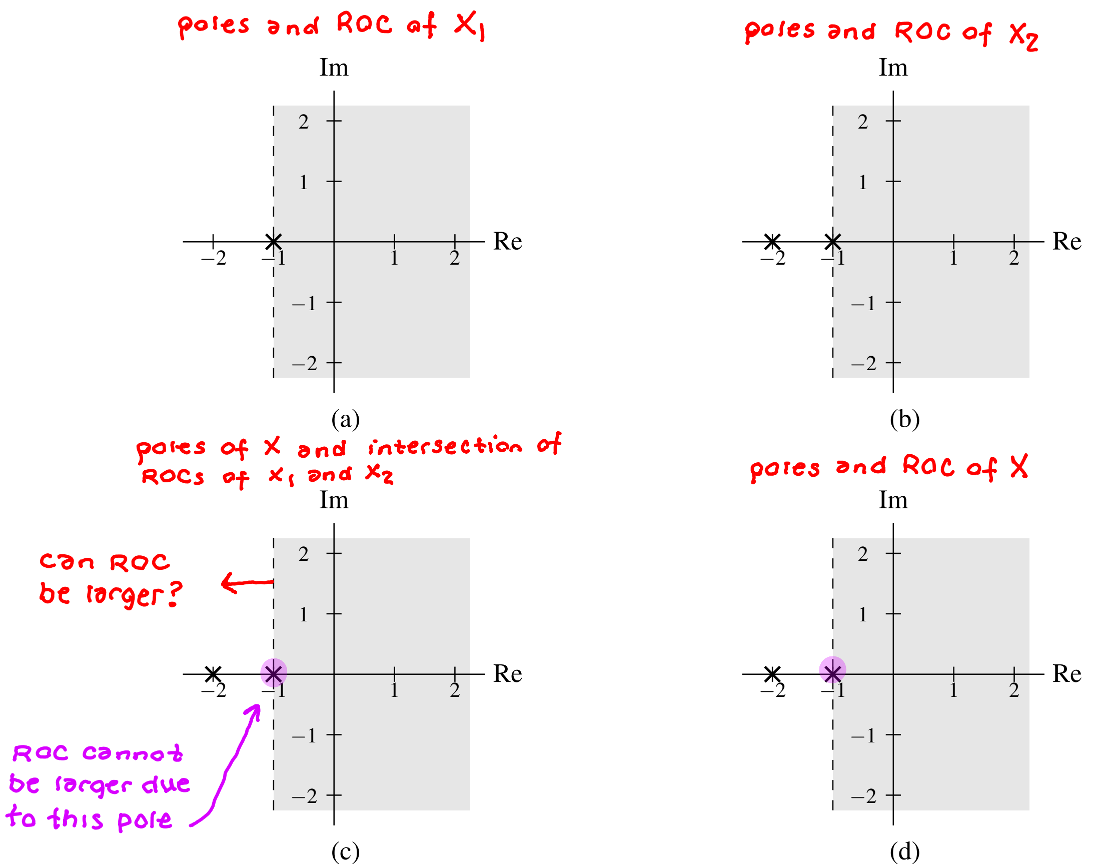

Figure 7.9: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

