R **6.103** Using properties of the Fourier transform and a table of Fourier transform pairs, find the Fourier transform X of each function x given below.

(a) $x(t) = \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right]$;
(b) $x(t) = e^{-j2t}\operatorname{sgn}(-t-1)$.
(c) $x(t) = e^{-j2t}\frac{1}{3t+1}$;
(d) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1}u(\tau-1)d\tau$;
(e) $x(t) = (t+1)\sin(5t-3)$;
(f) $x(t) = \sin(2\pi t)\delta(t-\frac{\pi}{2})$;
(g) $x(t) = e^{-jt}\frac{1}{3t-2}$;
(h) $x(t) = e^{j5t}\cos(2t)u(t)$;
(i) $x(t) = \operatorname{sinc}^2(at)$, where a is a nonzero real constant; and
(j) $x(t) = x_1 * x_2(t)$, where $x_1(t) = t^2e^{-t}u(t)$ and $x_2(t) = (t-1)e^{-(t-1)}u(t-1)$.

Short Answer.
(a) $X(\omega) = u(\omega)$;
(b) $X(\omega) = e^{j(\omega+2)}\frac{j2}{\omega+2}$;
(c) $X(\omega) = -\frac{j\pi}{3}e^{j(\omega+2)/3}\operatorname{sgn}(\omega+2)$;
(d) $X(\omega) = \frac{1}{e^2}\left[\left(\frac{5}{j5\omega-\omega^2}\right)e^{-j\omega/5} + \pi\delta(\omega)\right]$;
(e) $X(\omega) = j\pi\left[e^{j3}\delta(\omega+5) - e^{-j3}\delta(\omega-5)\right] + \pi\frac{d}{d\omega}\left[e^{-j3}\delta(\omega-5) - e^{j3}\delta(\omega+5)\right]$;
(f) $X(\omega) = \sin(\pi^2)e^{-j\pi\omega/2}$;
(g) $X(\omega) = -\frac{j\pi}{3}e^{-j2(\omega+1)/3}\operatorname{sgn}(\omega+1)$;
(h) $X(\omega) = \frac{1}{2}\left(\pi\delta(\omega-7) + \frac{1}{j(\omega-7)} + \pi\delta(\omega-3) + \frac{1}{j(\omega-3)}\right)$;
(i) $X(\omega) = \frac{\pi}{|a|}\operatorname{tri}\left(\frac{1}{4a}\omega\right)$;
(j) $X(\omega) = e^{-j\omega}\frac{2}{(1+j\omega)^5}$

Exercise 6.103

R Answer (c).

We are asked to find the Fourier transform X of

$$x(t) = e^{-j2t}\frac{1}{3t+1}.$$

We begin by rewriting $x(t)$ as

$$x(t) = e^{-j2t}v_3(t),$$

where

$$\begin{aligned} v_1(t) &= 1/t, \\ v_2(t) &= v_1(t+1), \quad \text{and} \\ v_3(t) &= v_2(3t). \end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} V_1(\omega) &= -j\pi\operatorname{sgn}(\omega), \\ V_2(\omega) &= e^{j\omega}V_1(\omega), \\ V_3(\omega) &= \tfrac{1}{3}V_2(\omega/3), \quad \text{and} \\ X(\omega) &= V_3(\omega+2). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(\omega) &= V_3(\omega+2) \\ &= \tfrac{1}{3}V_2\left(\tfrac{\omega+2}{3}\right) \\ &= \tfrac{1}{3}e^{j(\omega+2)/3}V_1\left(\tfrac{\omega+2}{3}\right) \\ &= \tfrac{1}{3}e^{j(\omega+2)/3}\left[-j\pi\operatorname{sgn}\left(\tfrac{\omega+2}{3}\right)\right] \\ &= -\tfrac{j\pi}{3}e^{j(\omega+2)/3}\operatorname{sgn}\left(\tfrac{\omega+2}{3}\right) \\ &= -\tfrac{j\pi}{3}e^{j(\omega+2)/3}\operatorname{sgn}(\omega+2). \end{aligned}$$

R **6.109** Using properties of the Fourier transform and a table of Fourier transform pairs, find the Fourier transform Y of each function y given below in terms of the Fourier transform X of the function x.
(a) $y(t) = r(t)x(t)$, where $r(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}(50t - 5k)$.

Short Answer. (a) $Y(\omega) = \frac{1}{5}\sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{\pi}{5}k\right) X(\omega - 20\pi k)$

Exercise 6.109

R Answer (a).

From the form of the summation appearing in the definition of r, we can immediately conclude that r is periodic. We have

$$r(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[50(t - \tfrac{1}{10}k)\right].$$

Letting $r_T(t) = \operatorname{rect}(50t)$, we have

$$r(t) = \sum_{k=-\infty}^{\infty} r_T\left(t - \tfrac{1}{10}k\right).$$

So, r is periodic with period $T = \frac{1}{10}$ and the corresponding frequency $\omega_0 = \frac{2\pi}{1/10} = 20\pi$. Thus, r has the Fourier series representation

$$r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j20\pi kt}.$$

Now, we observe that

$$c_k = \tfrac{1}{T} R_T(k\omega_0).$$

From a table of Fourier transforms, we have

$$R_T(\omega) = \tfrac{1}{50} \operatorname{sinc}\left(\tfrac{1}{100}\omega\right).$$

Substituting the expression for $R_T(\omega)$ into the formula for c_k, we have

$$\begin{aligned} c_k &= \tfrac{1}{1/10} R_T(20\pi k) \\ &= 10\left(\tfrac{1}{50}\right) \operatorname{sinc}\left(\tfrac{1}{100}[20\pi k]\right) \\ &= \tfrac{1}{5} \operatorname{sinc}\left(\tfrac{\pi}{5}k\right). \end{aligned}$$

Replacing r by its Fourier series representation in $y(t) = r(t)x(t)$, we obtain

$$y(t) = \sum_{k=-\infty}^{\infty} \tfrac{1}{5} \operatorname{sinc}\left(\tfrac{\pi}{5}k\right) e^{j20\pi kt} x(t).$$

Taking the Fourier transform of y, we have

$$\begin{aligned} Y(\omega) &= \sum_{k=-\infty}^{\infty} \tfrac{1}{5} \operatorname{sinc}\left(\tfrac{\pi}{5}k\right) X(\omega - 20\pi k) \\ &= \tfrac{1}{5} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\tfrac{\pi}{5}k\right) X(\omega - 20\pi k). \end{aligned}$$

R **6.111** For each case below, where the function x has the Fourier transform X and the Fourier transform representation $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, find $y(t)$ at the specified values of t.

(a) $x(t) = \begin{cases} 8t^2+1 & 0 \le t < \frac{1}{2} \\ t-\frac{3}{2} & \frac{1}{2} \le t < \frac{3}{2} \\ \pi & \frac{3}{2} \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $t \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$; and

(b) $x(t) = \begin{cases} e^{-t} & -1 \le t < 0 \\ t+\frac{1}{2} & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$ and $t \in \{-1, 0\}$.

Short Answer. (a) $y\left(\frac{1}{2}\right) = 1$ and $y\left(\frac{3}{2}\right) = \frac{\pi}{2}$; (b) $y(-1) = \frac{e}{2}$ and $y(0) = \frac{3}{4}$

Exercise 6.111

R Answer (a).

Since x satisfies the Dirichlet conditions, $y(t_a) = \frac{1}{2}[x(t_a^-) + x(t_a^+)]$. So, we have

$$\begin{aligned}
y\left(\tfrac{1}{2}\right) &= \tfrac{1}{2}\left[x\left(\tfrac{1}{2}^-\right) + x\left(\tfrac{1}{2}^+\right)\right] \\
&= \tfrac{1}{2}[3 + (-1)] \\
&= 1 \quad \text{and} \\
y\left(\tfrac{3}{2}\right) &= \tfrac{1}{2}\left[x\left(\tfrac{3}{2}^-\right) + x\left(\tfrac{3}{2}^+\right)\right] \\
&= \tfrac{1}{2}(0 + \pi) \\
&= \tfrac{\pi}{2}.
\end{aligned}$$

R **6.116** For each differential/integral equation below that characterizes a LTI system with input x and output y, find the frequency response H of the system. Note that $\mathcal{I}$ denotes the integration operator $\mathcal{I}x(t) = \int_{-\infty}^{t} x(\tau)d\tau$ and $\mathcal{D}$ denotes the derivative operator.
(a) $\mathcal{D}y(t)+3y(t)=x(t)$;
(b) $\mathcal{D}^2y(t)+4\mathcal{D}y(t)+3y(t)=\mathcal{D}x(t)+2x(t)$;
(c) $\mathcal{D}y(t)+3y(t)+2\mathcal{I}y(t)=\mathcal{D}x(t)+5x(t)$; and
(d) $5\mathcal{D}y(t)-2y(t)+7\mathcal{I}y(t)=3\mathcal{I}x(t)-x(t)$.

Short Answer. (a) $H(\omega)=\dfrac{1}{j\omega+3}$; (b) $H(\omega)=\dfrac{j\omega+2}{-\omega^2+4j\omega+3}$; (c) $H(\omega)=\dfrac{\omega^2-5j\omega}{\omega^2-3j\omega-2}$; (d) $H(\omega)=\dfrac{3-j\omega}{-5\omega^2-2j\omega+7}$

Exercise 6.116

R Answer (c).

We are given that the system is characterized by the equation

$$\mathcal{D}y(t) + 3y(t) + 2\mathcal{I}y(t) = \mathcal{D}x(t) + 5x(t).$$

To eliminate the integration operator $\mathcal{I}$ (which would cause difficulties later), we differentiate the preceding equation. (Note that $\mathcal{D}\mathcal{I}x(t) = \frac{d}{dt}\int_{-\infty}^{t} x(\tau)d\tau = x(t)$.) Taking the derivative, we obtain

$$\mathcal{D}^2 y(t) + 3\mathcal{D}y(t) + 2y(t) = \mathcal{D}^2 x(t) + 5\mathcal{D}x(t).$$

Taking the Fourier transform of the preceding equation, we obtain

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = (j\omega)^2 X(\omega) + 5j\omega X(\omega).$$

Rearranging, we have

$$\begin{aligned}
-\omega^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = -\omega^2 X(\omega) + 5j\omega X(\omega) \quad &\Rightarrow \\
(-\omega^2 + 3j\omega + 2)Y(\omega) = (-\omega^2 + 5j\omega)X(\omega) \quad &\Rightarrow \\
\frac{Y(\omega)}{X(\omega)} = \frac{-\omega^2 + 5j\omega}{-\omega^2 + 3j\omega + 2} \quad &\Rightarrow \\
\frac{Y(\omega)}{X(\omega)} = \frac{\omega^2 - 5j\omega}{\omega^2 - 3j\omega - 2}&.
\end{aligned}$$

Since the system is LTI, $H(\omega) = \frac{Y(\omega)}{X(\omega)}$. Thus, we have

$$H(\omega) = \frac{\omega^2 - 5j\omega}{\omega^2 - 3j\omega - 2}.$$

R **6.118** For each case below, use frequency-domain methods to find the response y of the LTI system with impulse response h and frequency response H to the input x.
(a) $H(\omega) = j\omega$ and $x(t) = 1 + \frac{1}{4}\cos(2t) + \frac{1}{9}\sin(3t)$;
(b) $H(\omega) = -\mathcal{D}\delta(t)$ where $\mathcal{D}$ denotes the derivative operator and $x(t) = 10 + \cos(2t) + \sin(6t)$;
(c) $h(t) = (\pi t)^{-1}$ and $x(t) = 1 - \frac{1}{2}\cos(2t) + \frac{1}{3}\sin(3t)$; and
(d) $h(t) = e^{-3(t-1)}u(t-1)$ and $x(t) = t^2 e^{-3t}u(t)$.

Short Answer. (a) $y(t) = -\frac{1}{2}\sin(2t) + \frac{1}{3}\cos(3t)$; (b) $y(t) = 2\sin(2t) - 6\cos(6t)$; (c) $y(t) = -\frac{1}{2}\sin(2t) - \frac{1}{3}\cos(3t)$; (d) $y(t) = \frac{1}{3}(t-1)^3 e^{-3(t-1)}u(t-1)$

Exercise 6.118

R Answer (d).

We are given $h(t) = e^{-3(t-1)}u(t-1)$ and $x(t) = t^2 e^{-3t}u(t)$. First, we find the Fourier transform H of h. We rewrite h as

$$h(t) = v_1(t-1), \quad \text{where} \quad v_1(t) = e^{-3t}u(t).$$

Taking the Fourier transform of the preceding two equations, we obtain

$$H(\omega) = e^{-j\omega}V_1(\omega) \quad \text{and} \quad V_1(\omega) = \frac{1}{3+j\omega}.$$

Combining these equations, we obtain

$$\begin{aligned} H(\omega) &= e^{-j\omega}V_1(\omega) \\ &= \frac{e^{-j\omega}}{3+j\omega}. \end{aligned}$$

Next, we find the Fourier transform X of x. From a table of Fourier transforms, we have

$$X(\omega) = \frac{2!}{(3+j\omega)^3} = \frac{2}{(3+j\omega)^3}.$$

Since the system is LTI, $Y(\omega) = X(\omega)H(\omega)$. So, we have

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) \\ &= \left[\frac{e^{-j\omega}}{3+j\omega}\right]\left[\frac{2}{(3+j\omega)^3}\right] \\ &= \frac{2e^{-j\omega}}{(3+j\omega)^4}. \end{aligned}$$

Taking the inverse Fourier transform of Y, we obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{\frac{2e^{-j\omega}}{(3+j\omega)^4}\right\}(t) \\ &= \tfrac{1}{3}\mathcal{F}^{-1}\left\{e^{-j\omega}\frac{3!}{(3+j\omega)^4}\right\}(t) \\ &= \tfrac{1}{3}\mathcal{F}^{-1}\left\{\frac{3!}{(3+j\omega)^4}\right\}(t-1) \\ &= \tfrac{1}{3}(t-1)^3 e^{-3(t-1)}u(t-1). \end{aligned}$$

R **6.121** For each function x below, by direct application of the Nyquist sampling theorem, determine the lowest sampling rate ω_s at which x can be sampled such that it can be exactly reconstructed from its samples.
(a) $x(t) = \sin(15t)$;
(b) $x(t) = 10 + 4\sin(15t) + 2\cos(20t)$;
(c) $x(t) = \operatorname{sinc}(5t - 3)$;
(d) $x(t) = \operatorname{sinc}^2(20t)$;
(e) $x(t) = \cos(10t)\operatorname{sinc}(30t)$; and
(f) $x(t) = x_1 * x_2(t)$, where $x_1(t) = e^{-t}u(t)$ and $x_2(t) = \operatorname{sinc}(10t)$.

Short Answer. (a) 30; (b) 40; (c) 10; (d) 80; (e) 80; (f) 20

Exercise 6.121

R Answer (c).

We are given $x(t) = \operatorname{sinc}(5t-3)$. Let X denote the Fourier transform of x. First, we find X. We have that

$$\begin{aligned} v_1(t) = \operatorname{sinc}(t) &\overset{\text{CTFT}}{\longleftrightarrow} V_1(\omega) = \pi \operatorname{rect}\left(\tfrac{1}{2}\omega\right) \quad \Rightarrow \\ v_2(t) = v_1(t-3) = \operatorname{sinc}(t-3) &\overset{\text{CTFT}}{\longleftrightarrow} V_2(\omega) = e^{-j3\omega} V_1(\omega) = \pi e^{-j3\omega} \operatorname{rect}\left(\tfrac{1}{2}\omega\right) \quad \Rightarrow \\ x(t) = v_2(5t) = \operatorname{sinc}(5t-3) &\overset{\text{CTFT}}{\longleftrightarrow} X(\omega) = \tfrac{1}{5} V_2(\omega/5) = \tfrac{\pi}{5} e^{-j3(\omega/5)} \operatorname{rect}\left(\tfrac{1}{10}\omega\right). \end{aligned}$$

So, $|X(\omega)| = \frac{\pi}{5} \operatorname{rect}\left(\frac{1}{10}\omega\right)$. Thus, $X(\omega)$ is only nonzero if $\omega \in [-5, 5]$. Therefore, by the sampling theorem, we have that

$$\omega_s > 2(5) = 10.$$

R **6.122** A real sinusoidal function x having frequency ω_0 is ideally sampled with a sampling rate ω_s, yielding the sequence v. Bandlimited interpolation is then applied to v to produce the function y. For each case given below, determine the frequencies present in the spectrum of y.
(a) $\omega_0 = 50$, $\omega_s = 90$;
(b) $\omega_0 = 50$, $\omega_s = 110$;
(c) $\omega_0 = 100$, $\omega_s = 50$; and
(d) $\omega_0 = 179$, $\omega_s = 60$.

Short Answer. (a) ± 40; (b) ± 50; (c) 0; (d) ± 1

Exercise 6.122

R Answer (a).

Since x is a real sinusoidal function with frequency $\omega_0 = 50$, it has spectral information at the frequencies ± 50. Since x is sampled with frequency $\omega_s = 90$, frequencies separated by integer multiples of 90 will alias. Thus, the sequence v has spectral information at the frequencies

$$-50 + 90k \text{ and } 50 + 90k \text{ for all integer } k.$$

That is, v has spectral information at the frequencies

$$\ldots, -320, -230, -140, -50, 40, 130, 220, \ldots \quad \text{and} \quad \ldots, -220, -130, -40, 50, 140, 230, 320, \ldots.$$

Bandlimited interpolation will discard all frequencies outside $\left[-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right] = [-45, 45]$. So, y has spectral information at the frequencies ± 40.

