7.13 Consider a LTI system with input x, output y, and system function H, where

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Find the differential equation that characterizes the behavior of the system.

Answer.

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively. The system is characterized by the equation

$$Y(s) = H(s)X(s).$$

So, we have

$$\begin{aligned} & Y(s) = \left(\frac{s+1}{s^2+2s+2}\right)X(s) \\ \Rightarrow \quad & [s^2+2s+2]Y(s) = [s+1]X(s) \\ \Rightarrow \quad & s^2Y(s)+2sY(s)+2Y(s) = sX(s)+X(s). \end{aligned}$$

Taking the inverse Laplace transform of both sides of each of the above equation yields

$$\begin{aligned} & \mathcal{L}^{-1}\{s^2Y(s)\}+2\mathcal{L}^{-1}\{sY(s)\}+2\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{sX(s)\}+\mathcal{L}^{-1}\{X(s)\} \\ \Rightarrow \quad & \tfrac{d^2}{dt^2}y(t)+2\tfrac{d}{dt}y(t)+2y(t) = \tfrac{d}{dt}x(t)+x(t). \end{aligned}$$

7.14 A causal LTI system with input x and output y is characterized by the differential equation

$$y''(t) + 4y'(t) + 3y(t) = 2x'(t) + x(t),$$

where the prime symbol denotes derivative. Find the system function H of the system.

Answer.

We begin by taking the Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
& \mathcal{L}\{\tfrac{d^2}{dt^2}y(t)\} + 4\mathcal{L}\{\tfrac{d}{dt}y(t)\} + 3\mathcal{L}\{y(t)\} = 2\mathcal{L}\{\tfrac{d}{dt}x(t)\} + \mathcal{L}\{x(t)\} \\
\Rightarrow \quad & s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + X(s) \\
\Rightarrow \quad & [s^2 + 4s + 3]Y(s) = [2s + 1]X(s) \\
\Rightarrow \quad & \frac{Y(s)}{X(s)} = \frac{2s+1}{s^2+4s+3}.
\end{aligned}$$

Therefore, the system function $H(s)$ is given by

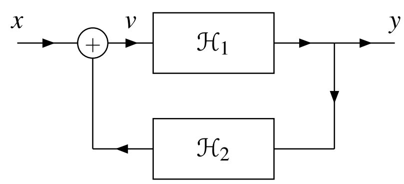
$$\begin{aligned}
H(s) &= \frac{2s+1}{s^2+4s+3} \\
&= \frac{2s+1}{(s+3)(s+1)} \quad \text{for } \operatorname{Re}(s) > -1.
\end{aligned}$$

(The ROC is a right-half plane, since the system is causal. The ROC is to the right of the rightmost pole, since $H(s)$ is also rational.)

7.15 Consider the LTI system with input x, output y, and system function H, as shown in the figure below. Suppose that the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are causal and LTI with the respective system functions

$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where A is a real constant.



(a) Find an expression for H in terms of H_1 and H_2.
(b) Determine for what values of A the system is BIBO stable.

Answer (a).

From the system block diagram, we can write

$$\begin{aligned} &V(s) = X(s) + H_2(s)Y(s) \quad \text{and} \\ &Y(s) = H_1(s)V(s). \end{aligned}$$

Combining these equations yields

$$\begin{aligned} Y(s) &= H_1(s)[X(s) + H_2(s)Y(s)] \\ &= H_1(s)X(s) + H_1(s)H_2(s)Y(s). \end{aligned}$$

So, we know

$$\begin{aligned} & [1 - H_1(s)H_2(s)]Y(s) = H_1(s)X(s) \\ \Rightarrow \quad & \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)}. \end{aligned}$$

Therefore, we have

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given expressions for $H_1(s)$ and $H_2(s)$, we have

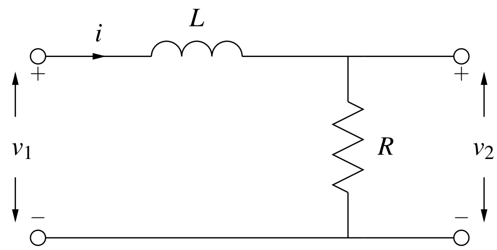
$$\begin{aligned} H(s) &= \frac{\frac{1}{s-1}}{1 - (\frac{1}{s-1})A} \\ &= \frac{1}{s-1-A} \\ &= \frac{1}{s-(A+1)} \quad \text{for } \operatorname{Re}(s) > A+1. \end{aligned}$$

Answer (b).

The system is BIBO stable if and only if the ROC of $H(s)$ includes the entire imaginary axis. We know that $H(s)$ converges for $\operatorname{Re}(s) > A+1$. Therefore, the ROC includes the entire imaginary axis if $A+1 < 0$. So, the system is stable if $A+1 < 0$ which implies $A < -1$. Therefore the system is stable if

$$A < -1.$$

7.16 Consider the LTI resistor-inductor (RL) network with input v_1 and output v_2 shown in the figure below.



(a) Find the system function H of the system.
(b) Determine whether the system is BIBO stable.
(c) Find the step response s of the system.

Answer (a).

From basic circuit analysis, we can write

$$\begin{aligned} v_1(t) &= L\tfrac{d}{dt}i(t) + v_2(t) \\ i(t) &= \tfrac{1}{R}v_2(t) \end{aligned}$$

Combining these two equations yields

$$\begin{aligned} v_1(t) &= L\tfrac{d}{dt}[\tfrac{1}{R}v_2(t)] + v_2(t) \\ &= \tfrac{L}{R}\tfrac{d}{dt}v_2(t) + v_2(t). \end{aligned}$$

Taking the Laplace transform of both sides of this equation, we obtain

$$\begin{aligned} V_1(s) &= \mathcal{L}\{v_1(t)\} \\ &= \mathcal{L}\{\tfrac{L}{R}\tfrac{d}{dt}v_2(t) + v_2(t)\} \\ &= \tfrac{L}{R}\mathcal{L}\{\tfrac{d}{dt}v_2(t)\} + \mathcal{L}\{v_2(t)\} \\ &= \tfrac{L}{R}sV_2(s) + V_2(s). \end{aligned}$$

Rearranging, we have

$$\begin{aligned} & V_1(s) = [\tfrac{L}{R}s+1]V_2(s) \\ \Rightarrow \quad & \frac{V_2(s)}{V_1(s)} = \frac{1}{\frac{L}{R}s+1} = \frac{R/L}{s+R/L}. \end{aligned}$$

Therefore, the system function $H(s)$ is given by

$$H(s) = \frac{R/L}{s+R/L} \quad \text{for } \operatorname{Re}(s) > -\tfrac{R}{L}.$$

(The ROC must be a right-half plane since the system is causal.)

Answer (b).

The rational function $H(s)$ has a single pole at $-\frac{R}{L}$. Since L and R are strictly positive quantities, we have that $-\frac{R}{L} < 0$. In other words, all of the poles of $H(s)$ are in the left-half plane. Since the system is causal, this implies that the system is stable.

Answer (c).

Now, we consider the step response of the system.

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{R/L}{s+R/L}\right)\left(\frac{1}{s}\right) \\ &= \frac{R/L}{s(s+R/L)}. \end{aligned}$$

We must find the inverse Laplace transform of $V_2(s)$. So, we first find a partial fraction expansion of $V_2(s)$. Such an expansion is of the form

$$V_2(s) = \frac{A_1}{s+R/L} + \frac{A_2}{s}.$$

Calculating the expansion coefficients yields

$$\begin{aligned} A_1 &= (s+R/L)V_2(s)|_{s=-R/L} \\ &= \tfrac{R/L}{s}|_{s=-R/L} \\ &= -1 \quad \text{and} \\ A_2 &= sV_2(s)|_{s=0} \\ &= \tfrac{R/L}{s+R/L}|_{s=0} \\ &= 1. \end{aligned}$$

So, we have

$$V_2(s) = \frac{-1}{s+R/L} + \frac{1}{s}.$$

Taking the inverse Laplace transform of $V_2(s)$ yields

$$\begin{aligned} v_2(t) &= \mathcal{L}^{-1}\{\tfrac{-1}{s+R/L} + \tfrac{1}{s}\} \\ &= -\mathcal{L}^{-1}\{\tfrac{1}{s+R/L}\} + \mathcal{L}^{-1}\{\tfrac{1}{s}\} \\ &= -[e^{-(R/L)t}u(t)] + u(t) \\ &= [1-e^{-(R/L)t}]u(t). \end{aligned}$$

Therefore, the step response $s(t)$ of the system is

$$s(t) = [1-e^{-(R/L)t}]u(t).$$

7.19 Consider a LTI system with the system function

$$H(s) = \frac{s^2+7s+12}{s^2+3s+12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

Answer.

All of the inverse systems have the same algebraic expression $H^{\text{inv}}(s)$ for their system function, namely

$$H^{\text{inv}}(s) = \frac{1}{H(s)} = \frac{s^2+3s+12}{s^2+7s+12}.$$

Factoring the denominator of $H^{\text{inv}}(s)$, we have

$$H^{\text{inv}}(s) = \frac{s^2+3s+12}{(s+4)(s+3)}.$$

Obviously, the system function $H^{\text{inv}}(s)$ is rational and has poles at -4 and -3. Consequently, there are three possible ROCs associated with $H^{\text{inv}}(s)$: i) $\operatorname{Re}(s) < -4$; ii) $-4 < \operatorname{Re}(s) < -3$; and iii) $\operatorname{Re}(s) > -3$.

A system is BIBO stable if and only if the ROC of its system function contains the entire imaginary axis. Clearly, only the ROC $\operatorname{Re}(s) > -3$ contains the entire imaginary axis. So, only the inverse system associated with this ROC is stable.

A system with a rational system function is causal if and only if the ROC of the system function is the right-half plane to the right of the rightmost pole. Since $H^{\text{inv}}(s)$ is rational, only the ROC of $\operatorname{Re}(s) > -3$ is associated with a causal system.

7.21 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input x and output y. Suppose that the transmitted signal x propagates along two paths. Along the intended direct path, the channel has a delay of T and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of $T+\tau$ and gain of a. Thus, the received signal y is given by $y(t) = x(t-T) + ax(t-T-\tau)$. Find the transfer function H of a LTI system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal $x(t-T)$ without any distortion. Determine whether this system is physically realizable.

Answer.

Let $G(s)$ denote the system function of the given wireless communication channel. We are given that

$$y(t) = x(t-T) + ax(t-T-\tau).$$

Taking the Laplace transform of the preceding equation, we obtain

$$\begin{aligned} Y(s) &= e^{-sT}X(s) + ae^{-(T+\tau)s}X(s) \\ &= (e^{-sT} + ae^{-(T+\tau)s})X(s). \end{aligned}$$

Using the preceding equation and the fact that $G(s) = \frac{Y(s)}{X(s)}$, we have

$$\begin{aligned} G(s) &= e^{-sT} + ae^{-(T+\tau)s} \\ &= e^{-sT}(1 + ae^{-s\tau}). \end{aligned}$$

Now, we want $G(s)H(s) = e^{-sT}$. Thus, we have

$$H(s) = \frac{e^{-sT}}{G(s)} = \frac{1}{1+ae^{-s\tau}}.$$

Thus, $H(s) = \frac{1}{1+ae^{-s\tau}}$.

A LTI system, consisting of a single negative feedback loop where the system in the feedback path has transfer function $F(s)$, has an overall transfer function $H(s) = \frac{1}{1+F(s)}$. Such a feedback system is physically realizable as long as the system with transfer function $F(s)$ is physically realizable. In the case of this problem, we have the situation where $F(s) = ae^{-s\tau}$. This system (with transfer function $F(s) = ae^{-s\tau}$) simply amplifies the input signal by a and delays the signal by τ, and such a system is physically realizable (since we can build systems that delay and amplify signals).

7.17 Consider the causal (incrementally-linear TI) system with input x and output y that is characterized by the differential equation

$$y''(t)+7y'(t)+12y(t)=x(t),$$

where the prime symbol denotes derivative. If $y(0^-)=-1$, $y'(0^-)=0$, and $x(t)=u(t)$, find y.

Answer.

We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
&\mathcal{L}_\mathsf{u}\{\tfrac{d^2}{dt^2}y(t)\}+7\mathcal{L}_\mathsf{u}\{\tfrac{d}{dt}y(t)\}+12\mathcal{L}_\mathsf{u}\{y(t)\}=\mathcal{L}_\mathsf{u}\{x(t)\}\\
\Rightarrow\quad & s^2Y(s)-sy(0^-)-y'(0^-)+7[sY(s)-y(0^-)]+12Y(s)=X(s)\\
\Rightarrow\quad & [s^2+7s+12]Y(s)=sy(0^-)+y'(0^-)+7y(0^-)+X(s)\\
\Rightarrow\quad & Y(s)=\frac{X(s)+sy(0^-)+y'(0^-)+7y(0^-)}{s^2+7s+12}.
\end{aligned}$$

Since $x(t)=u(t)$, we have

$$X(s)=\mathcal{L}_\mathsf{u}\{u(t)\}=\tfrac{1}{s}.$$

Substituting this expression for $X(s)$ and the given initial conditions into the above equation for $Y(s)$ yields

$$Y(s)=\frac{\frac{1}{s}-s-7}{s^2+7s+12}=\frac{-s^2-7s+1}{s(s^2+7s+12)}=\frac{-s^2-7s+1}{s(s+3)(s+4)}.$$

Now, we need to calculate a partial fraction expansion of $Y(s)$. Such an expansion is of the form

$$Y(s)=\frac{A_1}{s}+\frac{A_2}{s+3}+\frac{A_3}{s+4}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1&=sY(s)|_{s=0}\\
&=\left.\frac{-s^2-7s+1}{(s+3)(s+4)}\right|_{s=0}\\
&=\tfrac{1}{12},\\
A_2&=(s+3)Y(s)|_{s=-3}\\
&=\left.\frac{-s^2-7s+1}{s(s+4)}\right|_{s=-3}\\
&=-\tfrac{13}{3},\quad\text{and}\\
A_3&=(s+4)Y(s)|_{s=-4}\\
&=\left.\frac{-s^2-7s+1}{s(s+3)}\right|_{s=-4}\\
&=\tfrac{13}{4}.
\end{aligned}$$

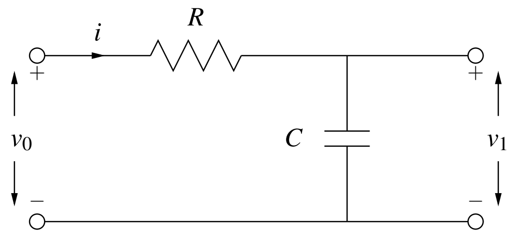
So, we can rewrite $Y(s)$ as

$$Y(s)=\tfrac{1}{12}\left(\frac{1}{s}\right)-\tfrac{13}{3}\left(\frac{1}{s+3}\right)+\tfrac{13}{4}\left(\frac{1}{s+4}\right).$$

Taking the inverse unilateral Laplace transform of $Y(s)$ yields

$$\begin{aligned}
y(t)&=\tfrac{1}{12}\mathcal{L}_\mathsf{u}^{-1}\left\{\frac{1}{s}\right\}-\tfrac{13}{3}\mathcal{L}_\mathsf{u}^{-1}\left\{\frac{1}{s+3}\right\}+\tfrac{13}{4}\mathcal{L}_\mathsf{u}^{-1}\left\{\frac{1}{s+4}\right\}\\
&=\tfrac{1}{12}-\tfrac{13}{3}e^{-3t}+\tfrac{13}{4}e^{-4t}\quad\text{for } t>0^-.
\end{aligned}$$

7.18 Consider the LTI resistor-capacitor (RC) network shown in the figure below, where $R = 1000$ and $C = \frac{1}{1000}$.



(a) Find the differential equation that characterizes the relationship between the input v_0 and output v_1.
(b) If $v_1(0^-) = 2$, and $v_0(t) = 2e^{-3t}$, find v_1.

Answer (a).

From basic circuit analysis, we have

$$\begin{aligned} & v_1(t) = \tfrac{1}{C}\int_{-\infty}^{t} \tfrac{1}{R}[v_0(\tau) - v_1(\tau)]d\tau \\ \Rightarrow\quad & \frac{d}{dt}v_1(t) - \tfrac{1}{RC}v_0(t) + \frac{1}{RC}v_1(t) = 0. \end{aligned}$$

Answer (b).

Taking the unilateral Laplace transform of both sides of the above equation yields

$$\begin{aligned} & \mathcal{L}_{\mathsf{u}}\{\frac{d}{dt}v_1(t)\} - \tfrac{1}{RC}\mathcal{L}_{\mathsf{u}}\{v_0(t)\} + \tfrac{1}{RC}\mathcal{L}_{\mathsf{u}}\{v_1(t)\} = 0 \\ \Rightarrow\quad & sV_1(s) - v_1(0^-) - \tfrac{1}{RC}v_1(0^-) - \tfrac{1}{RC}V_0(s) + \frac{1}{RC}V_1(s) = 0 \\ \Rightarrow\quad & [s + \tfrac{1}{RC}]V_1(s) = v_1(0^-) + \tfrac{1}{RC}V_0(s) \\ \Rightarrow\quad & V_1(s) = \frac{\tfrac{1}{RC}V_0(s) + v_1(0^-)}{s + \tfrac{1}{RC}}. \end{aligned}$$

From the given information, we can compute $V_0(s)$ as

$$V_0(s) = \mathcal{L}_{\mathsf{u}}\{2e^{-3t}\} = \tfrac{2}{s+3}$$

Substituting this expression for $V_0(s)$ into the above equation for $V_1(s)$, we obtain

$$\begin{aligned} V_1(s) &= \frac{\left(\frac{2}{s+3}\right) + 2}{s+1} \\ &= \frac{2s+8}{(s+1)(s+3)} \\ &= \frac{2(s+4)}{(s+1)(s+3)}. \end{aligned}$$

Now, we find a partial fraction expansion of $V_1(s)$. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3}$$

Calculating the expansion coefficients yields

$$\begin{aligned}
A_1 &= (s+1)V_1(s)\big|_{s=-1} \\
&= \left.\frac{2s+8}{s+3}\right|_{s=-1} \\
&= 3 \quad \text{and} \\
A_2 &= (s+3)V_1(s)\big|_{s=-3} \\
&= \left.\frac{2s+8}{s+1}\right|_{s=-3} \\
&= -1.
\end{aligned}$$

Thus, we can rewrite $V_1(s)$ as

$$V_1(s) = \frac{3}{s+1} - \frac{1}{s+3}$$

Taking the inverse unilateral Laplace transform of $V_1(s)$, we obtain

$$\begin{aligned}
v_1(t) &= 3\mathcal{L}_{\mathsf{u}}^{-1}\left\{\tfrac{1}{s+1}\right\} - \mathcal{L}_{\mathsf{u}}\left\{\frac{1}{s+3}\right\} \\
&= 3e^{-t} - e^{-3t} \quad \text{for } t > 0^-.
\end{aligned}$$

7.101 Consider a causal LTI system with the system function

$$H(s) = \frac{1}{-2s^7 - s^6 - 3s^5 + 2s^3 + s - 3}.$$

(a) Use MATLAB to find and plot the poles of H.
(b) Determine whether the system is BIBO stable.

Answer (a).

The poles of the rational function $H(s)$ are simply the roots of the denominator polynomial. The roots can be obtained by the following code fragment:

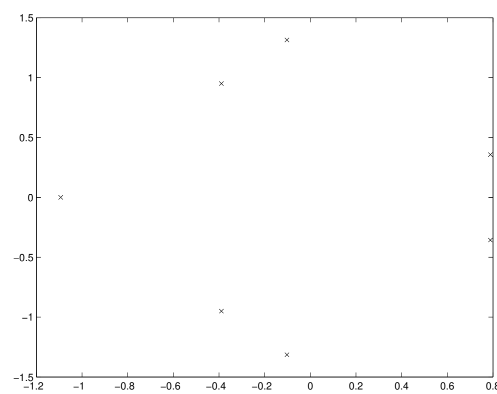
```
p = roots([-2 -1 -3 0 2 0 1 -3])
```

This yields the following poles:

```
 -1.09309528728402
 -0.10265208819816 + 1.31411369725196i
 -0.10265208819816 - 1.31411369725196i
 -0.38945961543237 + 0.94998954538106i
 -0.38945961543237 - 0.94998954538106i
  0.78865934727254 + 0.35672610951156i
  0.78865934727254 - 0.35672610951156i
```

We can then plot the poles (as a set of points) with the following code fragment:

```
plot(real(p), imag(p), 'x');
print -dps poles.ps
```



Answer (b).

Since the system is causal, in order for it to be BIBO stable, all of the poles must be in the left half of the complex plane. We observe that two of the poles have nonnegative real parts. Therefore, the system is not BIBO stable.

7.102 Consider a LTI system with the system function

$$H(s) = \frac{1}{1.0000s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1.0000}.$$

(This system corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the response y of the system to each input x given below. In each case, plot y over the interval $[0, 20]$.
(a) $x = \delta$; and
(b) $x = u$.
(Hint: The `tf`, `impulse`, and `step` functions may be helpful.)

Answer (a,b).

We can find the desired responses with the code given below.

```
tfnum = [0 0 0 0 1];
tfdenom = [1.0000 2.6131 3.4142 2.6131 1.0000];
finaltime = 20;
sys = tf(tfnum, tfdenom);
subplot(2, 1, 1);
step(sys, finaltime);
subplot(2, 1, 2);
impulse(sys, finaltime);
```