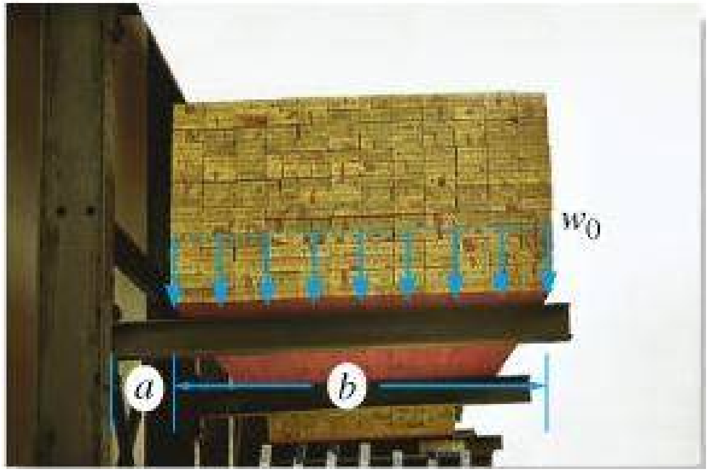
Chapter 7 – Internal Forces

Contents

Reduction of a Simple Distributed Loading

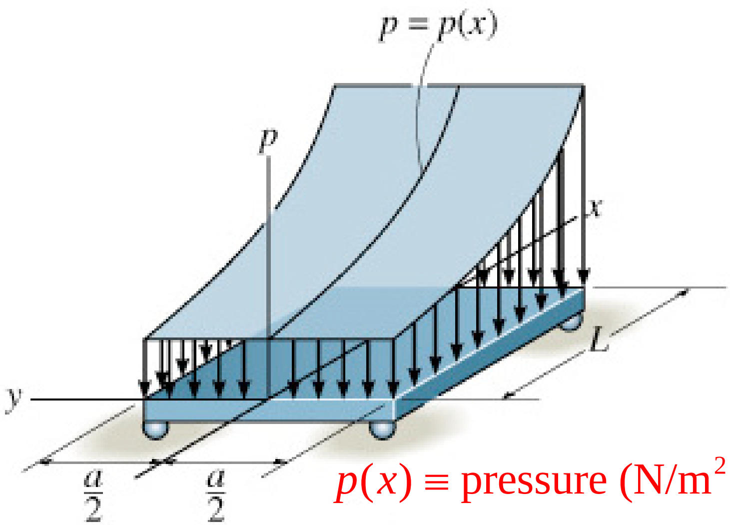
Sometimes a body is subjected to a loading that is distributed over its surface



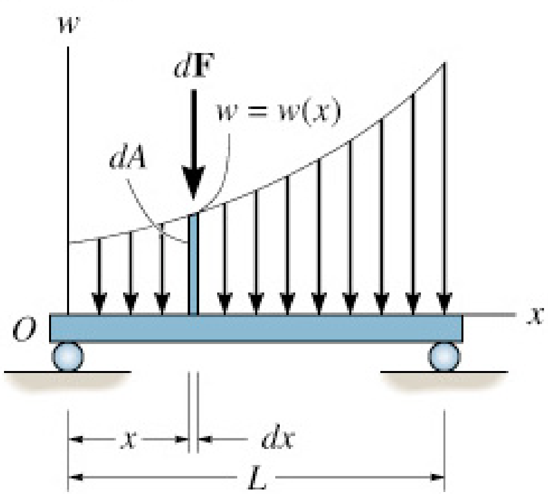
A bundle (bunk) of 2” x 4” boards is stored. The lumber places a distributed load (due to the weight of the wood) on the beams.

The roof of these houses are supporting a distributed load of snow.

Reduction of a Simple Distributed Loading



$p(x) \equiv$ pressure (N/m^2)



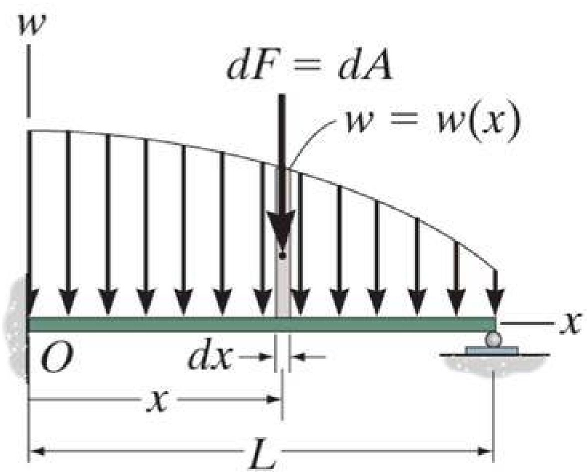
$w(x) = ap(x) \equiv$ distributed load (N/m)

Sometimes the surface of a body is subjected to a distributed load. Such forces are caused by weight, wind, fluid pressure, etc.

We will analyze the most common case of a distributed pressure loading: a *uniform load* along one axis of a flat rectangular body.

In such cases, w is a function of x and has units of force per length.

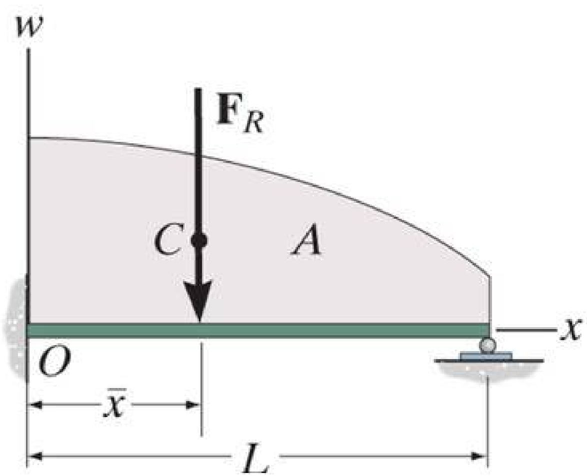
Reduction of a Simple Distributed Loading



Consider an element of length *dx*.

The force magnitude *dF* acting on it is given as

$$dF = w(x)\,dx$$

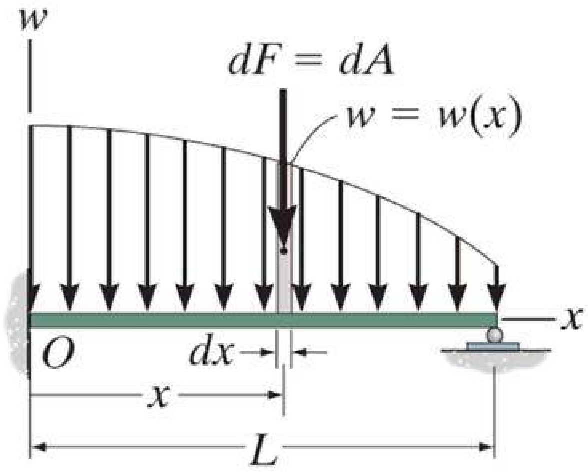


The *net force* on the beam is given by

$$F_R = \int_{x=0}^{x=L} dF = \int_{x=0}^{x=L} w(x)\,dx = A$$

Here *A* is the area under the loading curve *w(x)*.

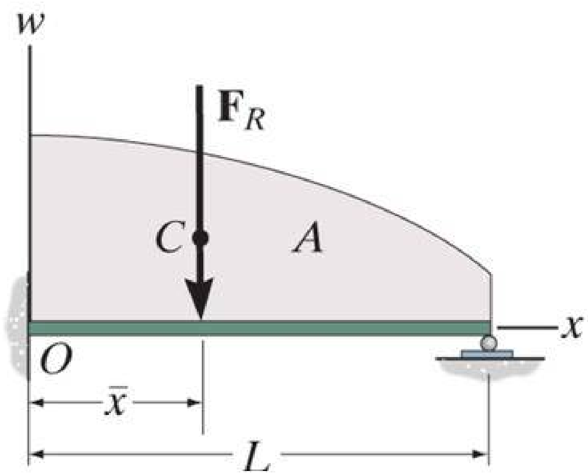
Reduction of a Simple Distributed Loading



The force dF will produce a moment of $(x)(dF)$ about point O.

The total moment about point O is given as

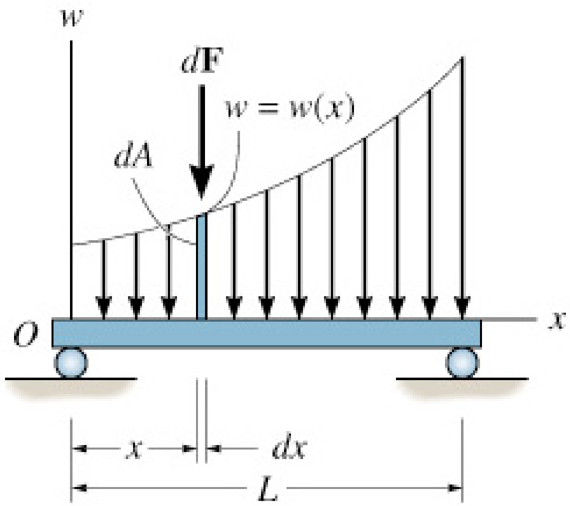
$$M_O = \int_{x=0}^{x=L} x\,dF = \int_{x=0}^{x=L} x w(x)\,dx$$



Assuming that F_R acts at $\bar{x}$, it will produce the moment about point O as

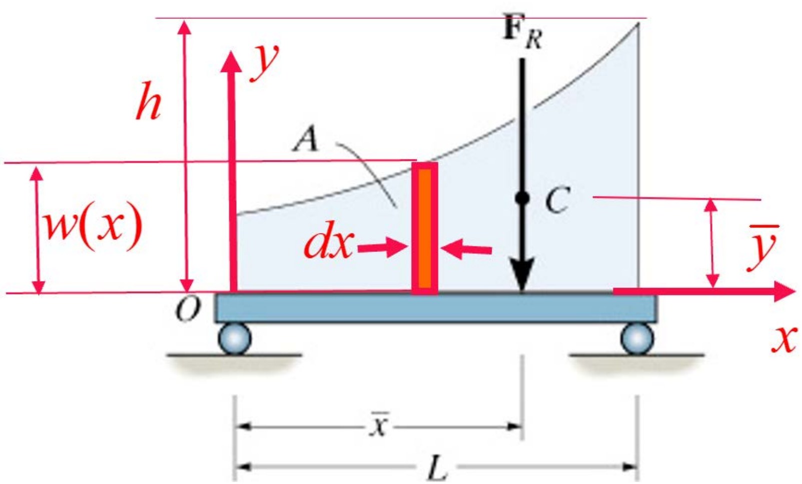
$$M_O = \bar{x}\, F_R$$

Centroids



Thus, $\bar{x}$ can be obtained from $M_O = \bar{x}\, F_R$ as follows

$$\bar{x} = \frac{\int_L x w(x)\, dx}{\int_L w(x)\, dx} = \frac{\int_A x\, dA}{\int_A dA}$$



The *centroid*, *C*, of the area is defined by a ratio of the *1st moments of area* to the total area.

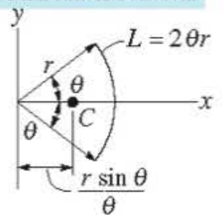
1st moments of area.

$$\bar{x} = \frac{\int_{x=0}^{x=L} x\, dA}{\int_{x=0}^{x=L} dA} \quad \text{and by analogy:} \quad \bar{y} = \frac{\int_{y=0}^{y=h} y\, dA}{\int_{y=0}^{y=h} dA}$$

Centroids

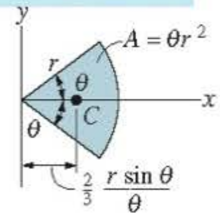
Geometric Properties of Line and Area Elements

Centroid Location



Circular arc segment

Centroid Location

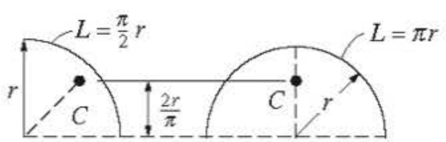


Circular sector area

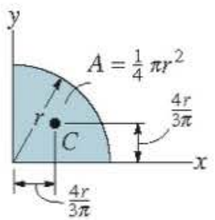
Area Moment of Inertia

$$I_x = \tfrac{1}{4} r^4 (\theta - \tfrac{1}{2}\sin 2\theta)$$

$$I_y = \tfrac{1}{4} r^4 (\theta + \tfrac{1}{2}\sin 2\theta)$$



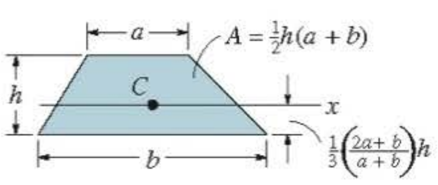
Quarter and semicircle arcs



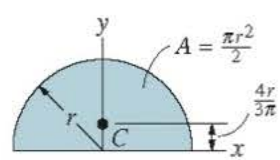
Quarter circle area

$$I_x = \tfrac{1}{16}\pi r^4$$

$$I_y = \tfrac{1}{16}\pi r^4$$



Trapezoidal area



Semicircular area

$$I_x = \tfrac{1}{8}\pi r^4$$

$$I_y = \tfrac{1}{8}\pi r^4$$

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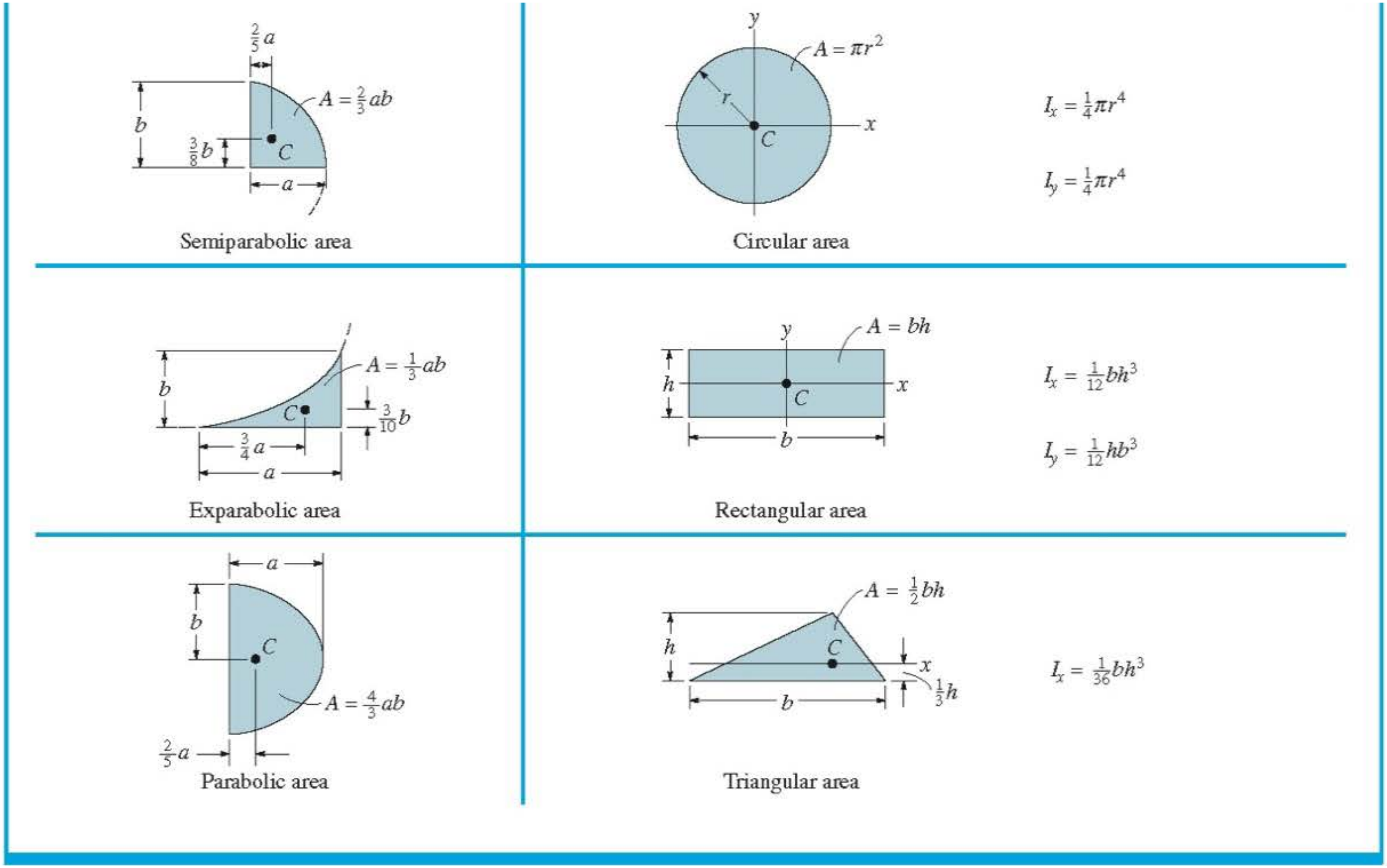
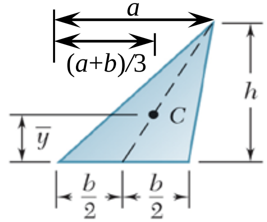
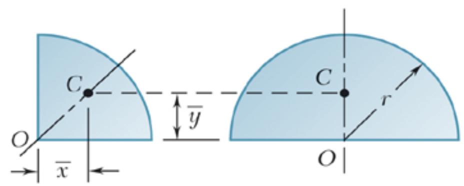
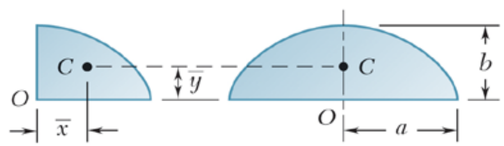
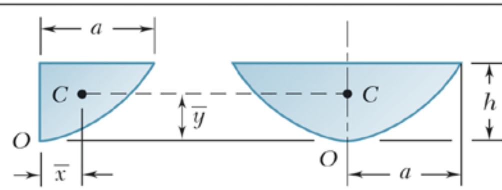
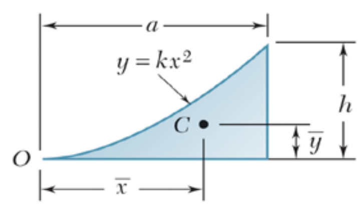
Centroids



Centroids

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area		$(a+b)/3$	$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

Centroids

The *centroid* is the geometric centre of the body, the mean position of all the points in all the coordinate directions.

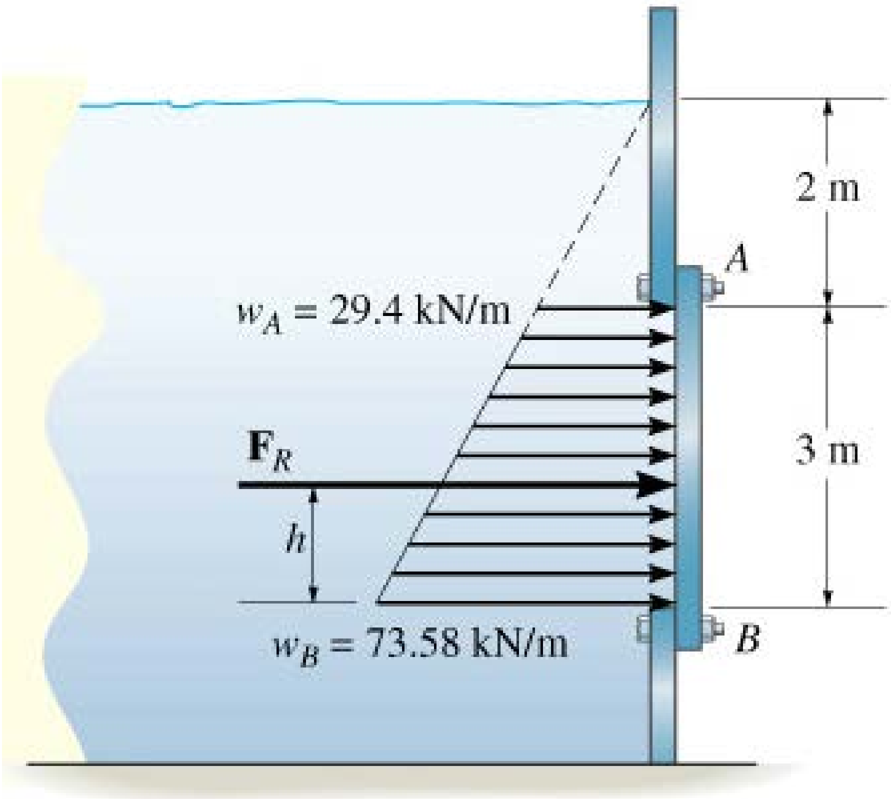
The *centre of mass* is the mean position of all the elements of mass.

The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *centre of gravity* for the body.

In a homogeneous body with a uniform gravitational field all these centres correspond to the same point.

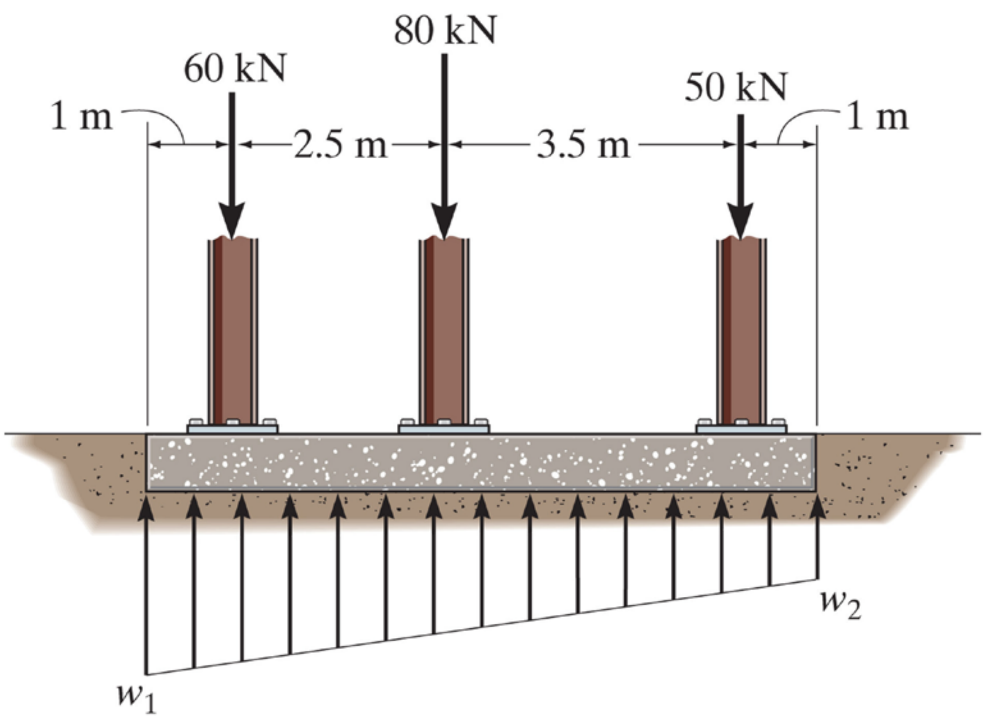
Example

Replace the loading with an equivalent system, i.e., find F_R and h.



Example

If the soil exerts a trapezoidal distribution of load on the bottom of the footing, determine the intensities w_1 and w_2 of this distribution needed to support the column loadings.



Internal Forces

In truss analysis we explored the concept of internal forces. We "cut open" rigid bodies to expose forces between particles in our FBD's.

Truss members are two force members and thus the internal forces were purely ***axial*** or ***normal loads***.

- Tensile.
- Compressive.

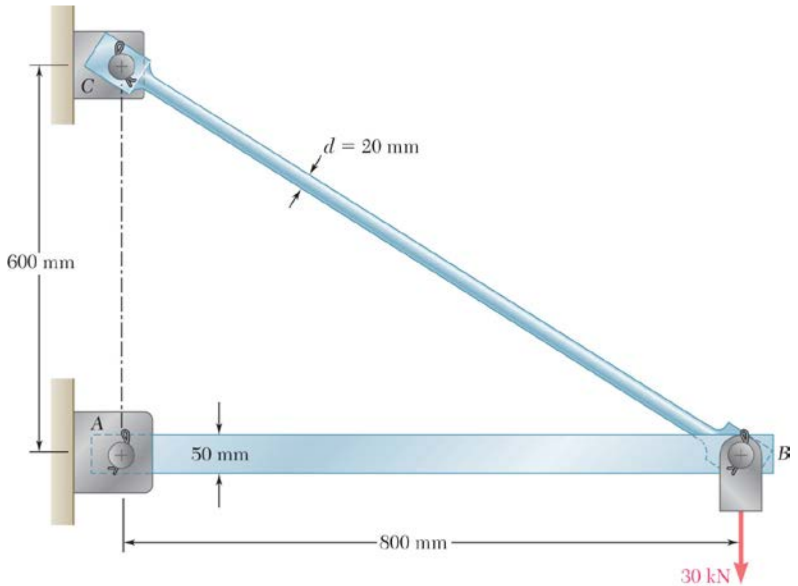
For other types of bodies the state of internal loading is more complex.

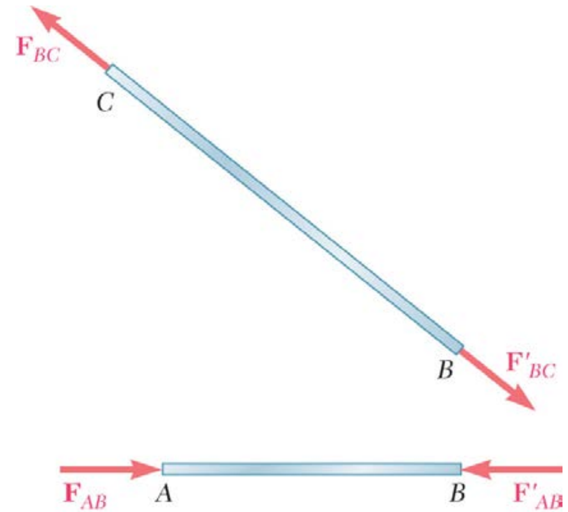
Both the analysis and design of a given structure involve the determination of *stresses* and *deformations*. More advanced courses (MECH 220 & MECH 320) will deal with this topics.

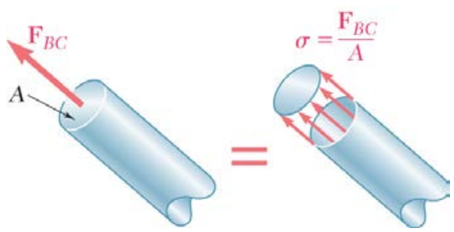
Internal Forces

Axial/Normal Stress

The developed normal stress is proportional to the cross-section area



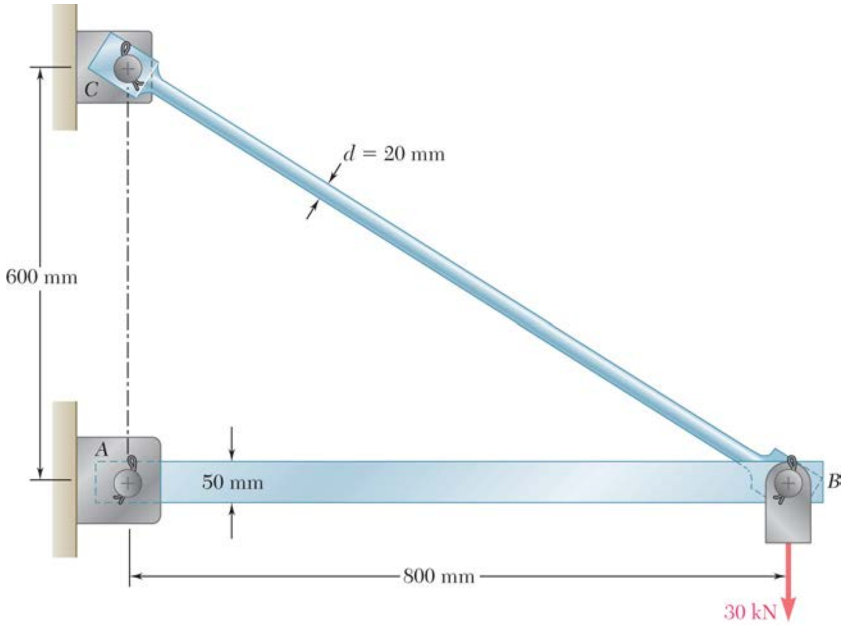


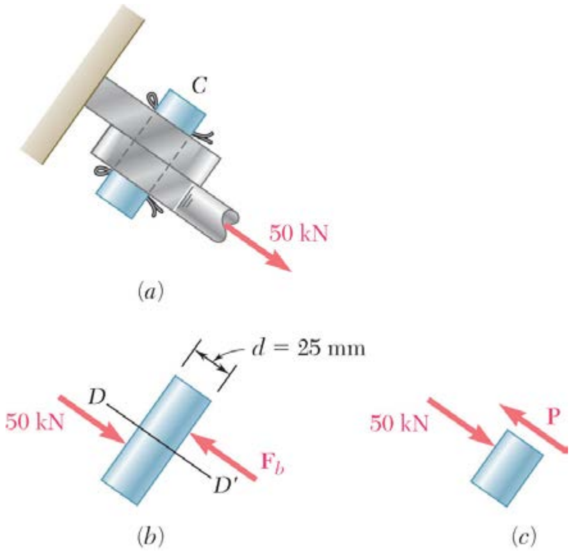


Internal Forces

Shearing Stress

Pins and bolts are commonly subjected to shearing stress





Chapter 7 – Internal Forces

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~~Reduction of a Simple Distributed Loading (§ 4.9)~~

~~Internal Loadings Developed in Structural Members (§ 7.1)~~

Shear and Moment Equations and Diagrams (§ 7.2)

Relations between Distributed Load, Shear, and Moment (§ 7.3)

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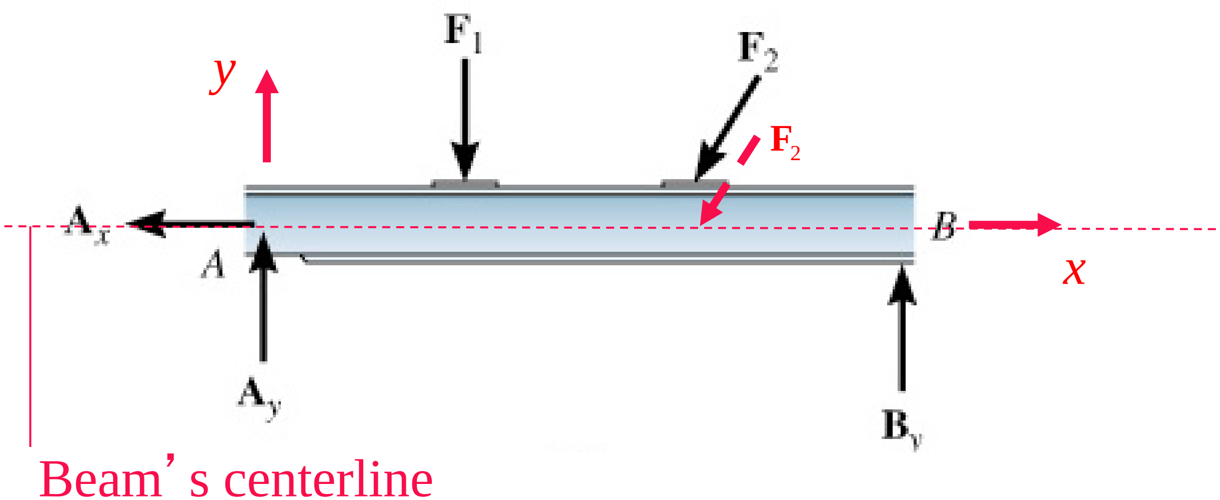
Beams

A **beam** is a structural member designed to withstand transverse loads.

Beam analysis is normally confined to a plane.

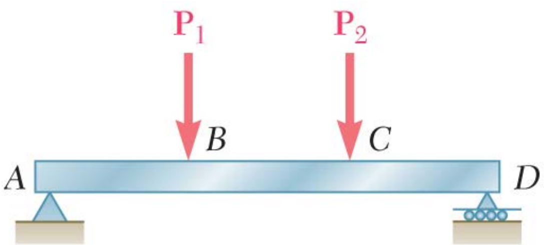
Beams are referred to as line elements or filament elements.

Applied loads are considered to be applied at the centerline.

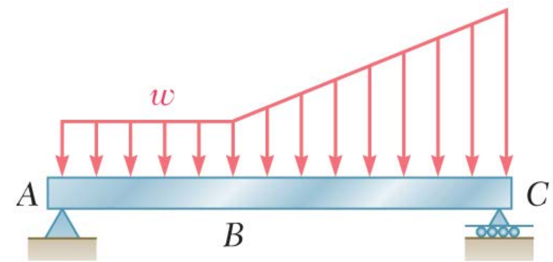


Beams

Classification of Loads



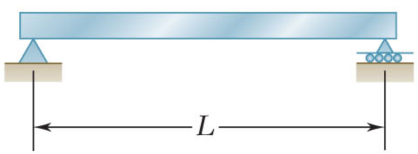
(a) Concentrated loads



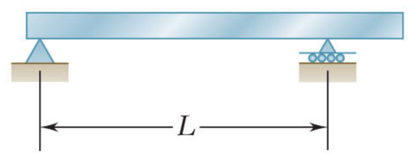
(b) Distributed loads

Classification of Beam Supports

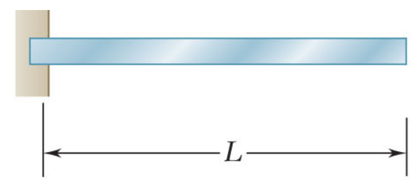
Statically Determinate Beams



(a) Simply supported beam

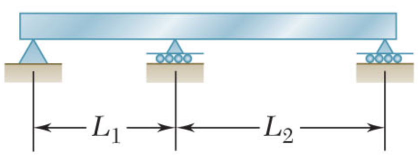


(b) Overhanging beam

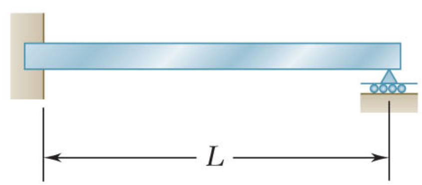


(c) Cantilever beam

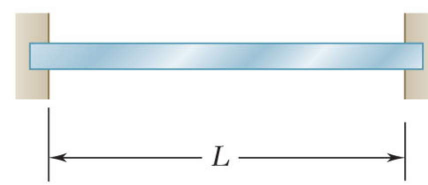
Statically Indeterminate Beams



(d) Continuous beam



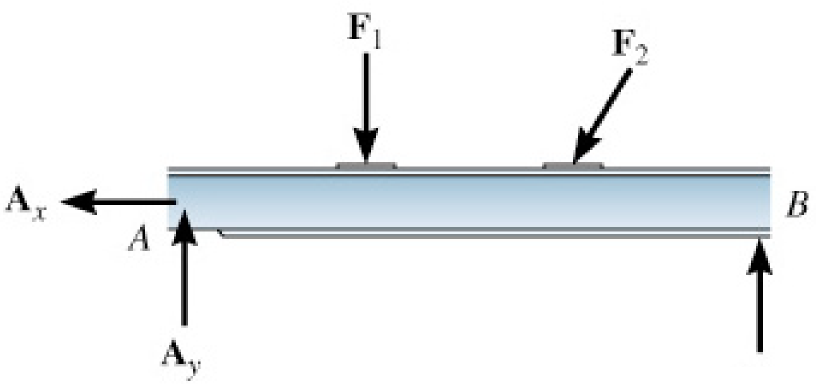
(e) Beam fixed at one end and simply supported at the other end

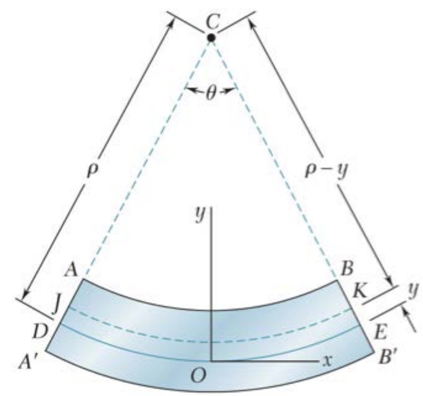


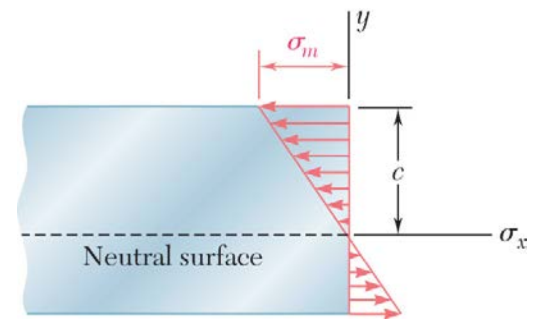
(f) Fixed beam

Beams

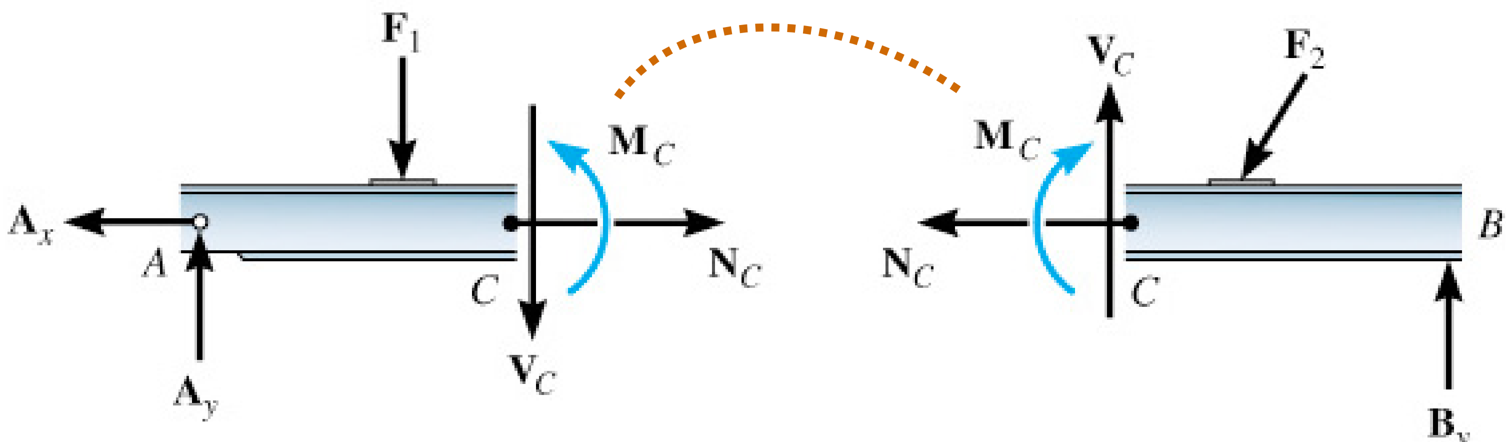
Applied loads result in internal forces consisting of a ***shear force*** (from the shear stress distribution) and a ***bending couple*** (from the normal stress distribution)



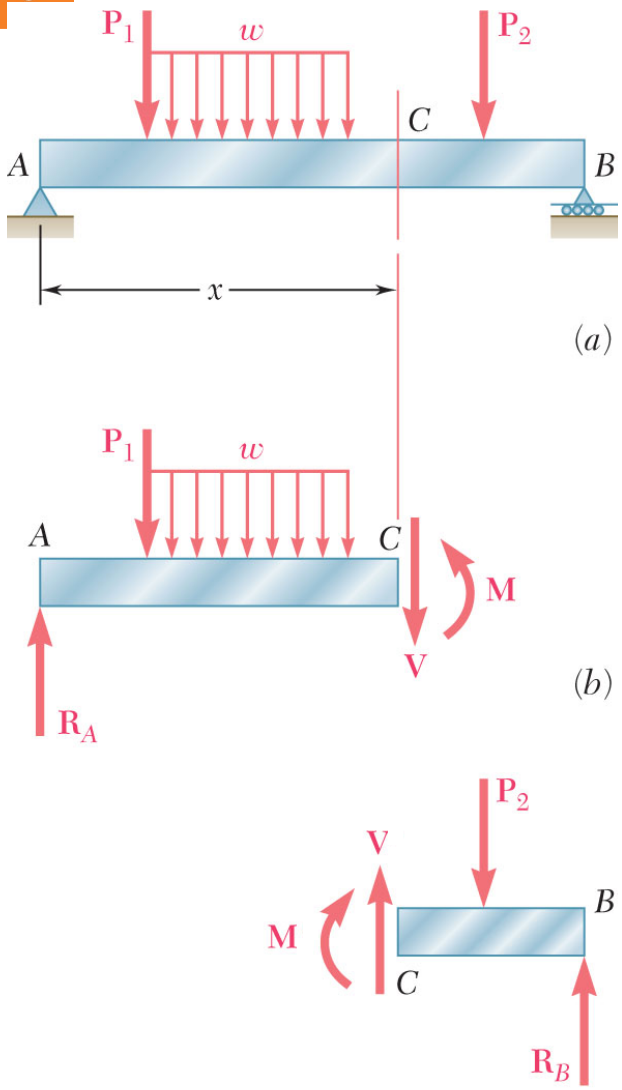




3rd Law



Beams

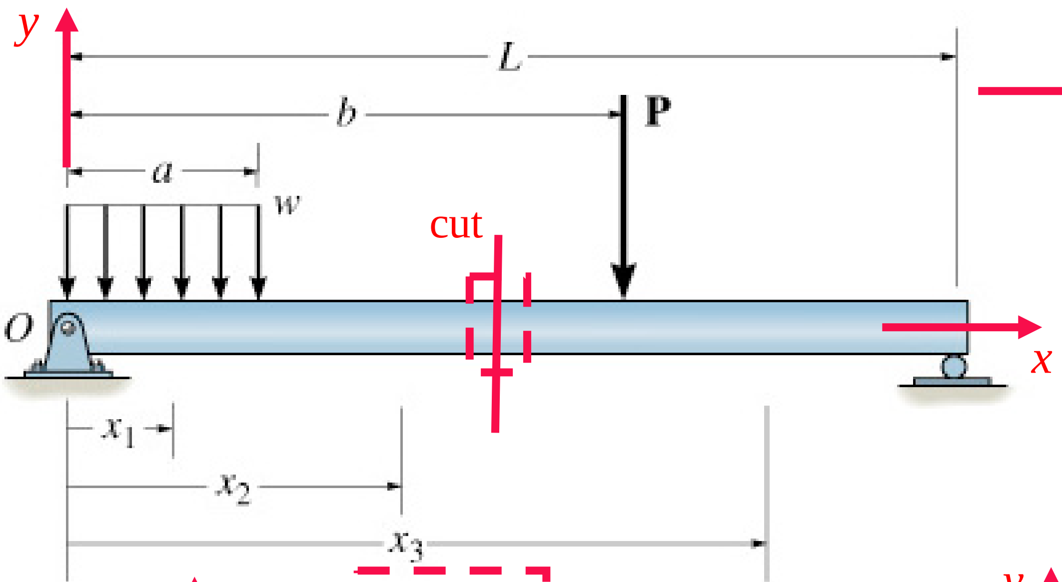


Determination of maximum normal and shearing stresses requires identification of maximum internal shear force and bending couple.

Shear force and bending couple at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions on either side of the section.

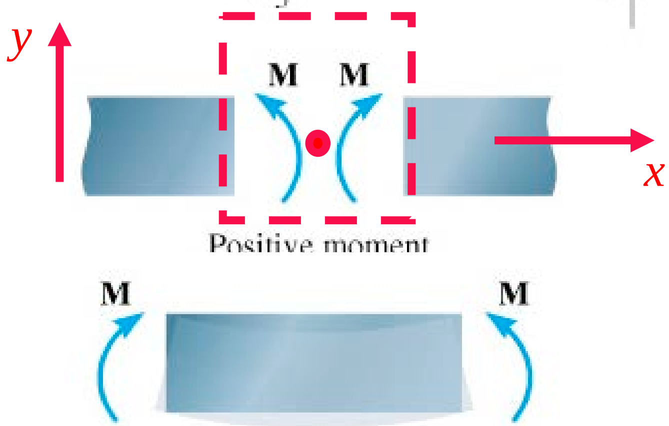
Beams

Sign Convention

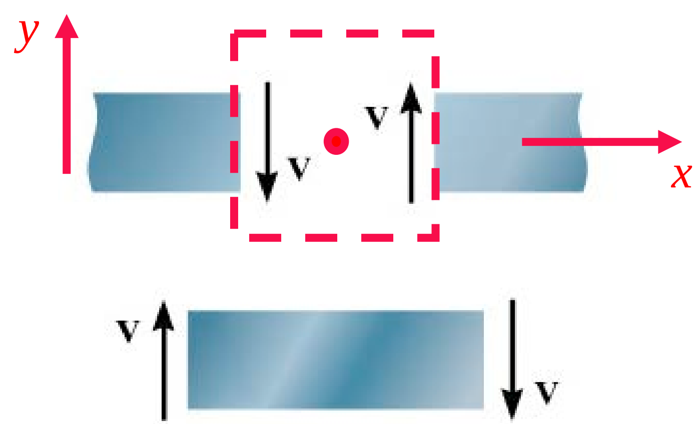


Typical beam analysis: transverse loads are only significant loads.

We are primarily concerned with the transverse shear force and the bending moment inside the beam.



Positive moment

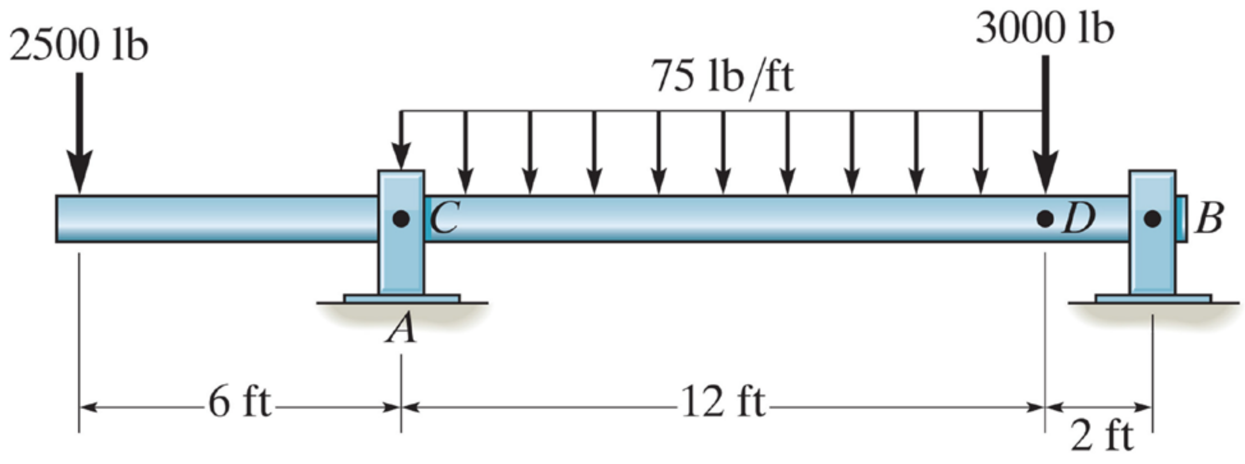


Positive shear

Example

The shaft is supported by a journal bearing at *A* and a thrust bearing at *B*. Determine the normal force, shear force, and moment at a section passing through

a) point *C*, which is just to the right of the bearing at *A*
b) point *D*, which is just to the left of the 3000-lb force.



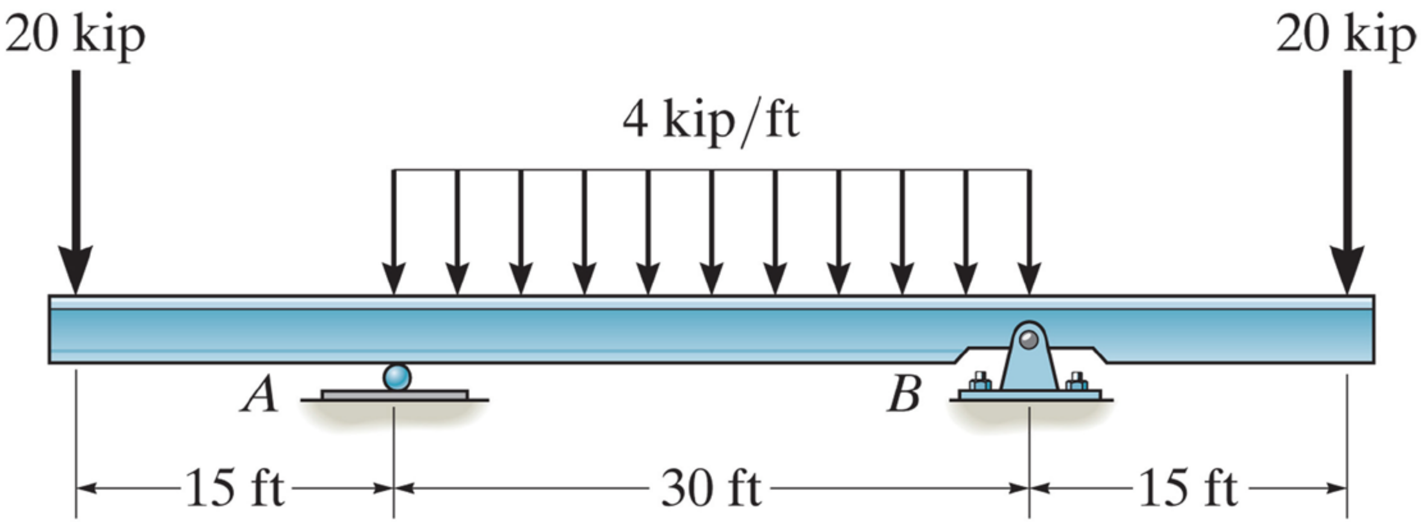
Shear and Bending-Moment Diagrams

The *shear and bending-moment diagrams* for a beam show how the shear and moment vary throughout the beam.

- Determine all the support reactions acting on the beam
- Section the beam at each distance *x* (from the origin of the reference frame to each location where there is a concentrated load, a couple moment, or a distributed load).
- Draw free body diagram including **V** and **M** (use positive sign convention)
- Solve for **V** ($\sum F_y = 0$) and **M** ($\sum M = 0$).
- Plot shear vs *x* and moment vs *x* diagrams. Positive values of **V** and **M** are plotted above the *x* axis.
- Maximum bending moment occurs at x^*, where $V(x^*) = 0$.

Example

Draw the shear and moment diagrams for the beam.



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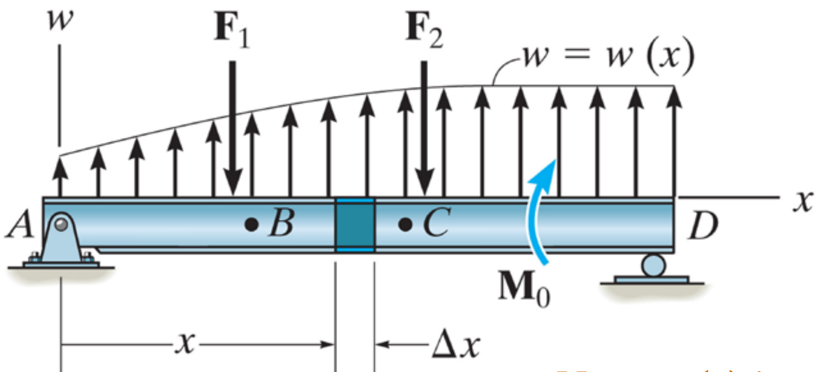
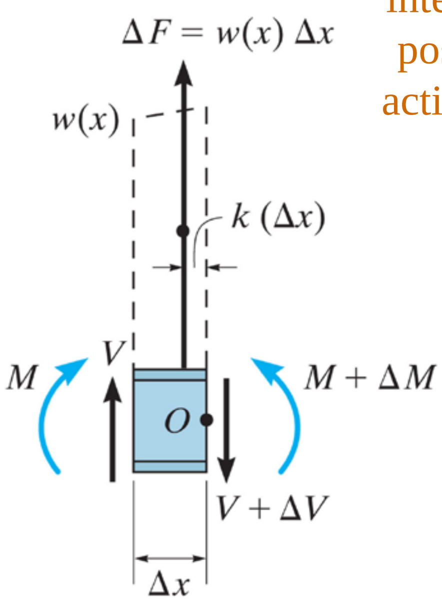
Relations between distributed load, shear, and bending moment



Note: $w(x)$ is interpreted as a positive when acting upwards.



Relationship between load and shear:

$$\sum F_y = 0$$

$$V + w(x)\Delta x - (V + \Delta V) = 0$$

$$\Delta V = w(x)\Delta x$$

Dividing by Δx, and letting $\Delta x \to 0$.

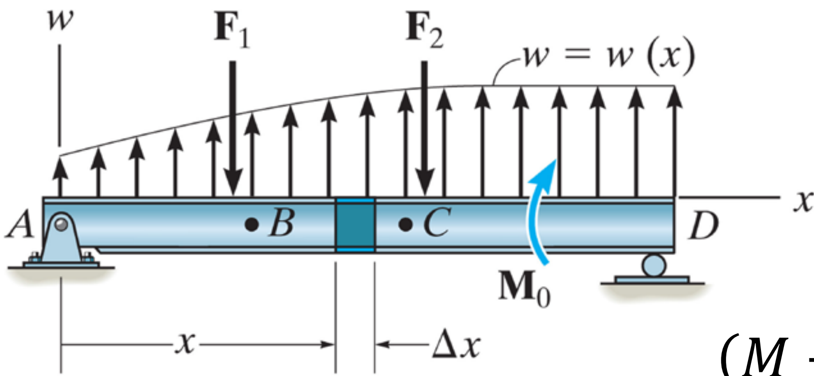
$$\frac{dV}{dx} = w(x)$$

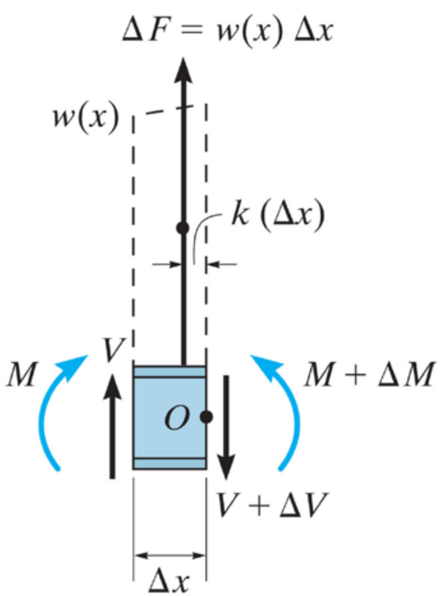
Slope of shear diagram is the distributed load intensity

$$V_C - V_B = \int_{x_B}^{x_C} w(x)dx$$

Change in shear is the area under the load curve

Relations between distributed load, shear, and bending moment





Relationship between shear and moment:

$$\sum M_O = 0$$

$$(M + \Delta M) - ((w(x)\Delta x)k\Delta x - V\Delta x - M = 0$$

$$\Delta M = V\Delta x - kw(x)\Delta x^2$$

Dividing by Δx, and letting $\Delta x \rightarrow 0$.

$$\frac{dM}{dx} = V$$

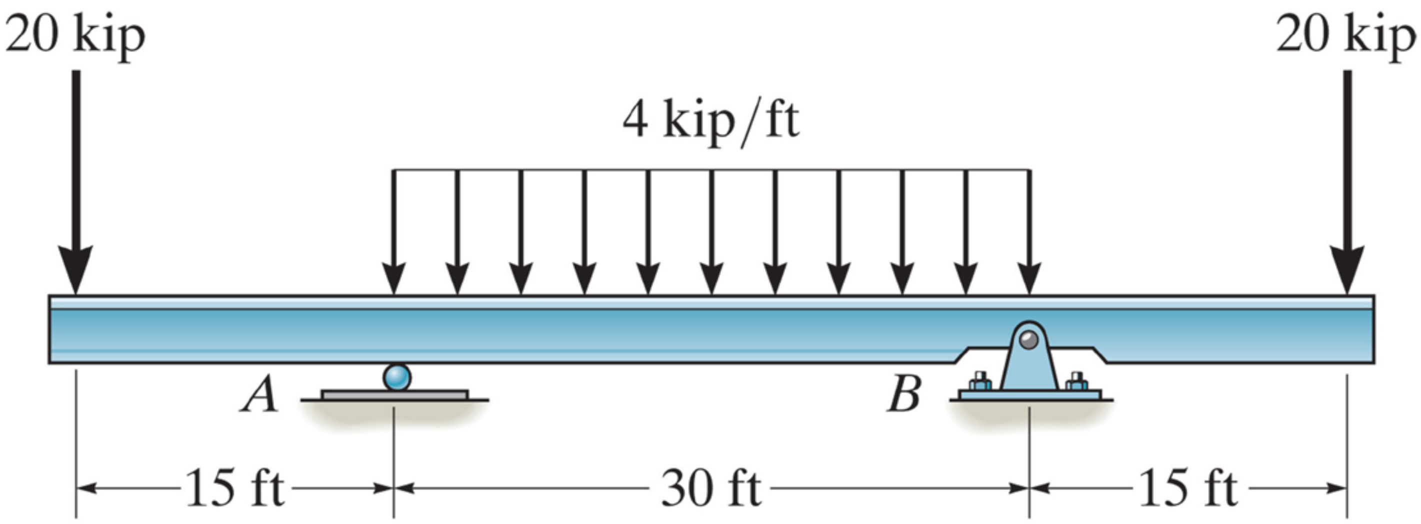
Slope of moment diagram is the shear

$$M_C - M_B = \int_{x_B}^{x_C} V dx$$

Change in moment is the area under the shear curve

Example

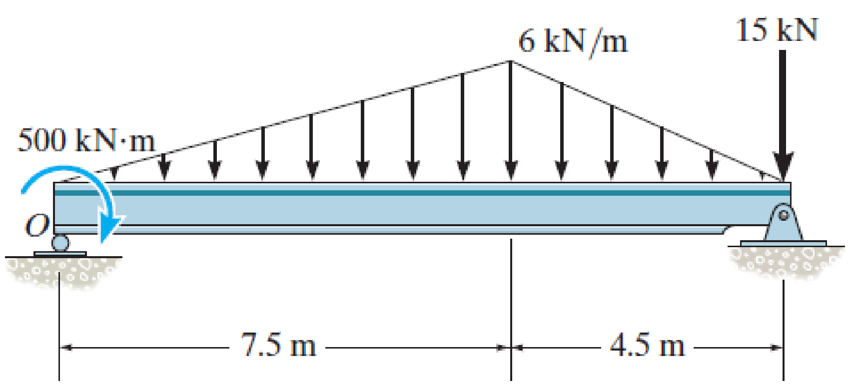
Draw the shear and moment diagrams for the beam, using the relationship between distributed load, shear and bending moment.



Sample Problems for Students to Review

Chapter 7

Sample Problem (§ 4.9)



Given: The distributed loading on the beam as shown.

Find: The equivalent force of the distributed loading and the support reactions.

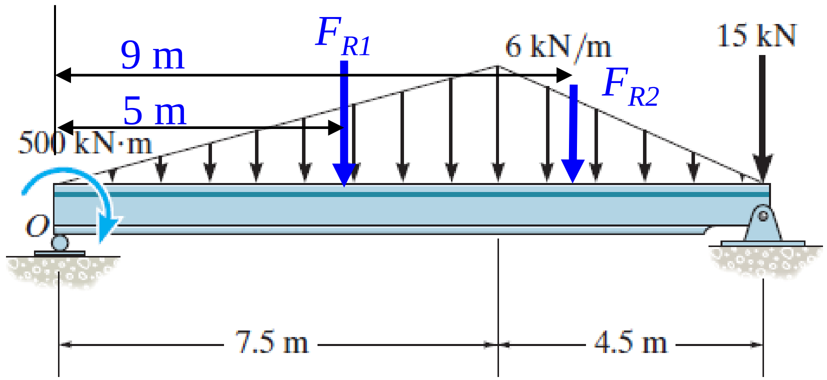
Plan:

The distributed loading can be divided into two triangular loads.

Find F_R and its location for each of these distributed loads.

Determine the overall F_R of the point loadings and find $\bar{x}$

Find the support reactions by applying equations of equilibrium.



For the left triangular loading of height 6 kN/m and width 7.5 m,

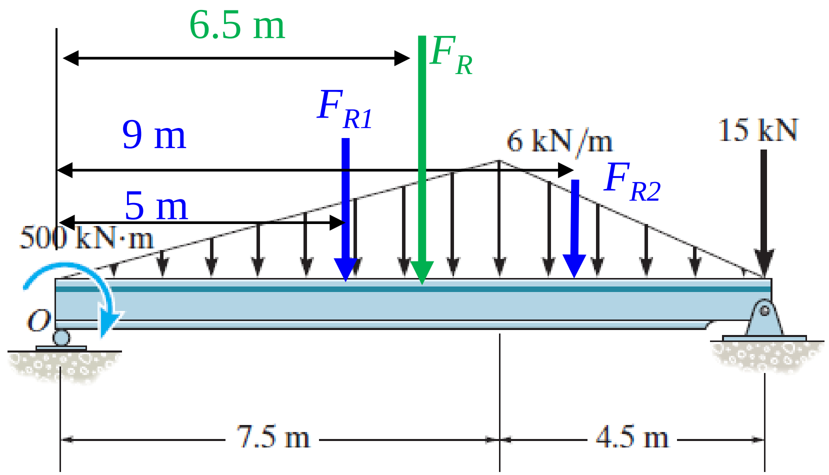
$$F_{R1} \;=\; (0.5)\,(6)\,(7.5) \;=\; 22.5\ \text{kN}$$

and its line of action is at $\bar{x}_1 = (2/3)(7.5) = 5$ m from O

For the right triangular loading of height 6 kN/m and width 4.5 m,

$$F_{R2} \;=\; (0.5)\,(6)\,(4.5) \;=\; 13.5\ \text{kN}$$

and its line of action is at $\bar{x}_2 = 7.5 + (1/3)(4.5) = 9$ m from O



For the combined loading of the two resultant forces:

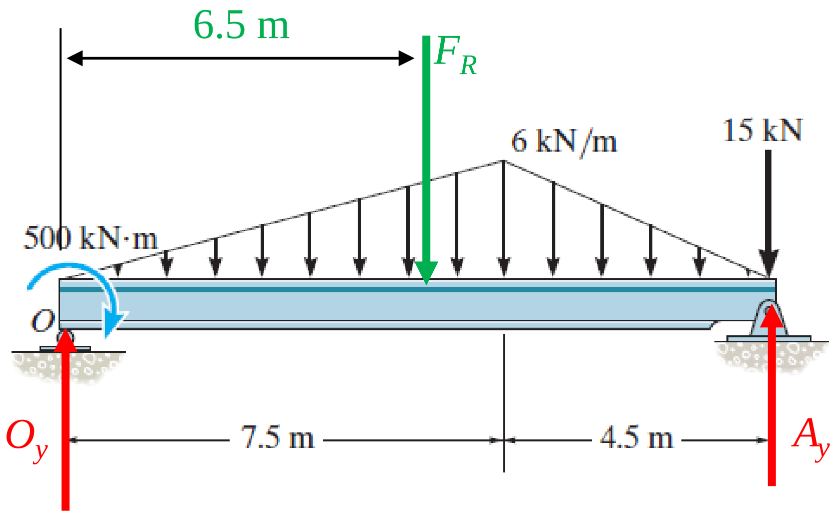
$$F_R = 22.5 + 13.5 = 36 \text{ kN}$$

The couple moment at point *O* relative to the

$$M_{RO} = -5\,(22.5) - 9\,(13.5) = -234 \text{ kN·m}$$

The line of action of the net force passes through the centroid of the area under the curve.

$$\bar{x} = \frac{234}{36} = 6.5 \text{ m}$$



Find support reactions by taking moments about *O*:

$$\sum M_O = 0 \qquad -500 - 36\,(6.5) - 15\,(12) + A_y\,(12) = 0$$

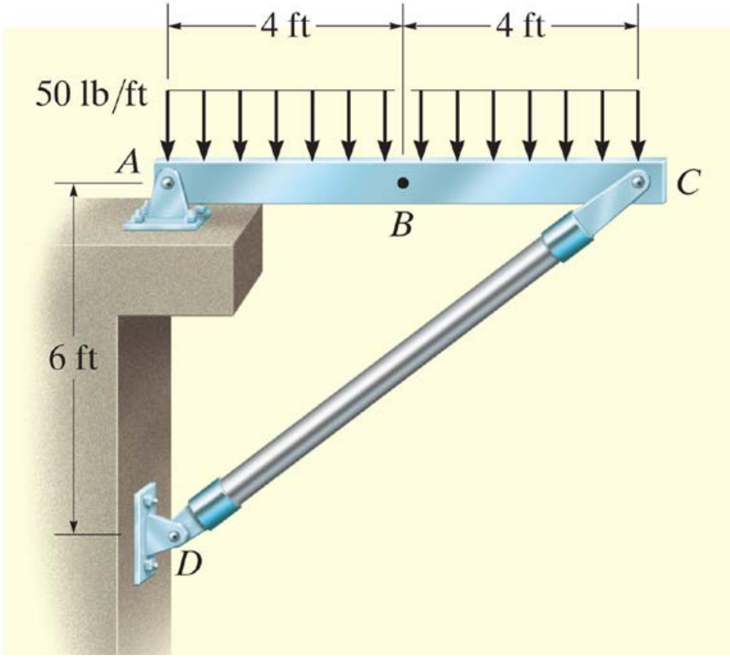
$$A_y = 76.2 \text{ kN}$$

and sum of forces along *y*:

$$\sum F_y = 0 \qquad O_y - 36 - 15 + 76.2 = 0$$

$$O_y = -25.17 \text{ kN}$$

Sample Problem (§ 7.1)



Given: The beam as shown.

Find: The normal force, shear force, and bending moment at point *B*.

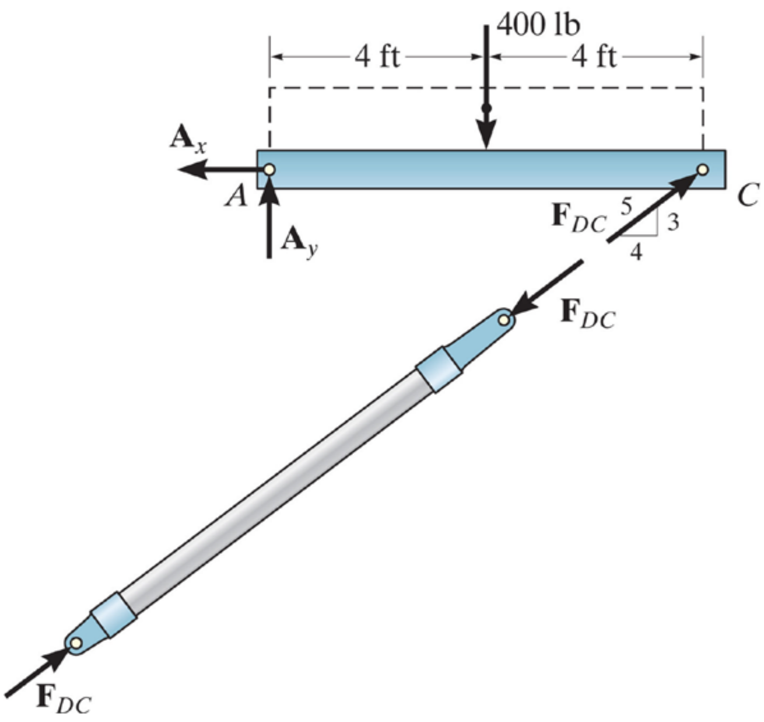
Plan:

Find the support reactions and A and C.

Draw FBC of section AB, make sure to use the right sign convention

Find normal force, shear force, and bending moment.

Support reactions



DC is a two-force member,

$$\sum M_A = 0$$

$$-400(4) + \frac{3}{5}F_{DC}(8) = 0$$

$$F_{DC} = 333.3 \text{ lb}$$

$$\sum F_x = 0$$

$$-A_x + \frac{4}{5}F_{DC} = 0$$

$$A_x = 266.7 \text{ lb}$$

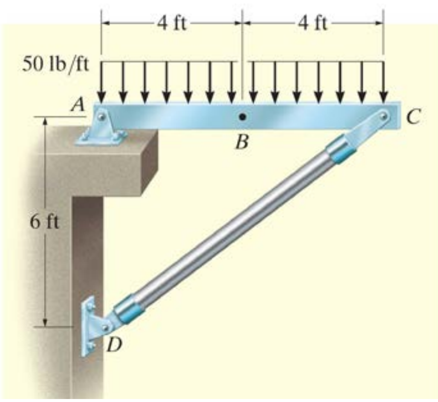
$$\sum F_y = 0$$

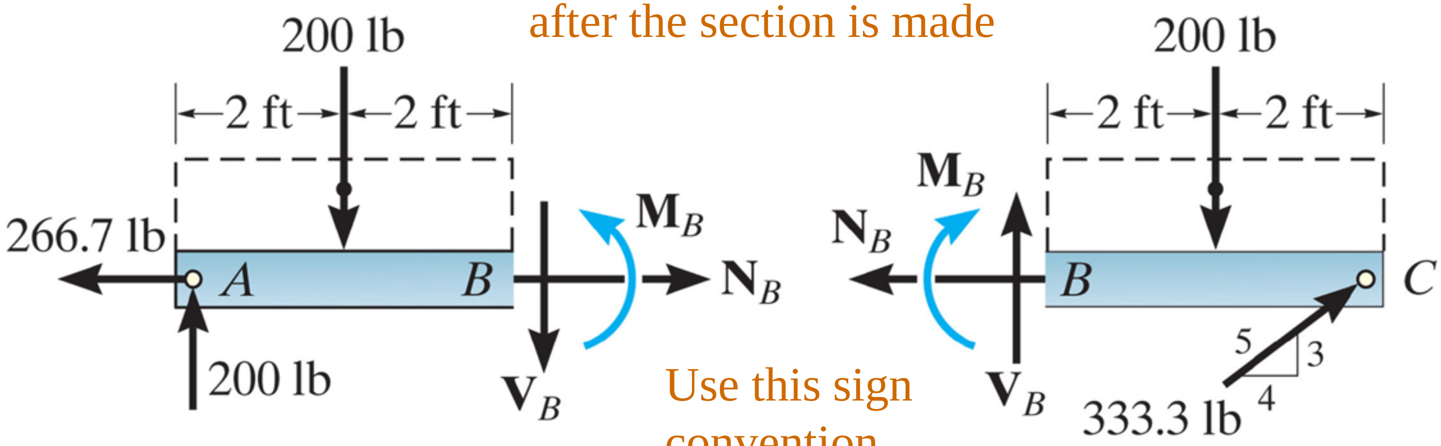
$$A_y - 400 + \frac{3}{5}F_{DC} = 0$$

$$A_y = 200 \text{ lb}$$

Internal Forces

Keep distributed load until after the section is made





Use this sign convention

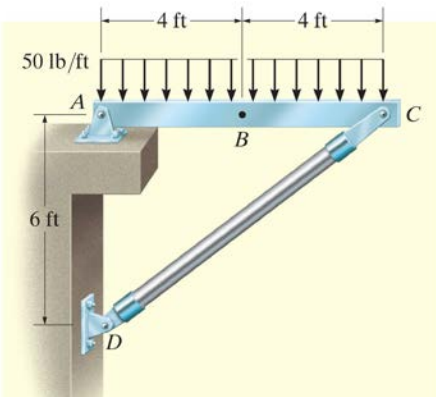
Using left section *AB*,

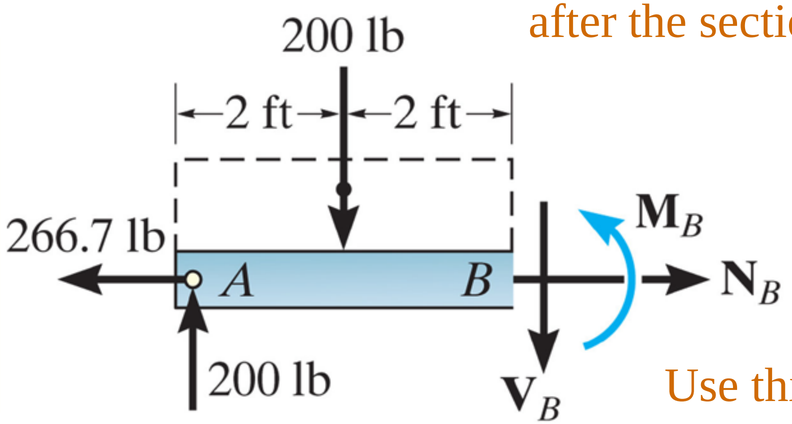
$$\sum F_x = 0 \qquad N_B - 266.7 = 0 \qquad N_B = 266.7 \text{ lb}$$

$$\sum F_y = 0 \qquad 200 - 200 - V_B = 0 \qquad V_B = 0 \text{ lb}$$

$$\sum M_B = 0 \qquad M_B - 200(4) + 200(2) = 0 \qquad M_B = 400 \text{ lb·ft}$$

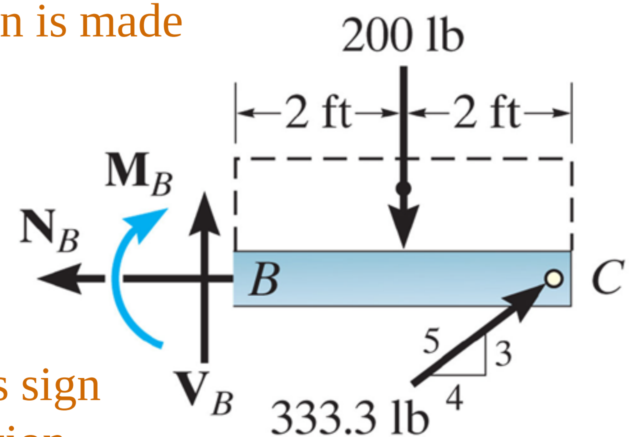
Internal Forces





Keep distributed load until after the section is made

Use this sign convention



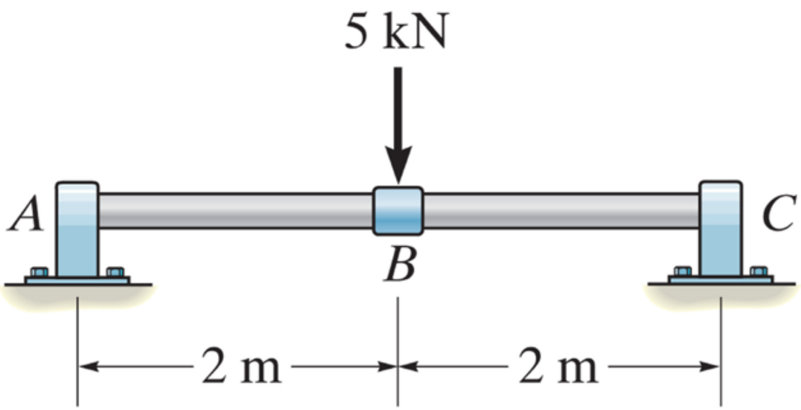
Using left section BC,

$$\sum F_x = 0 \qquad -N_B + 333.3\frac{4}{5} = 0 \qquad N_B = 266.7\ \text{lb}$$

$$\sum F_y = 0 \qquad V_B - 200 + 333.3\frac{3}{5} = 0 \qquad V_B = 0\ \text{lb}$$

$$\sum M_B = 0 \qquad -M_B - 200(2) + 333.3\frac{3}{5}(4) = 0 \qquad M_B = 400\ \text{lb·ft}$$

Sample Problem (§ 7.2)



Given: The shaft as shown.

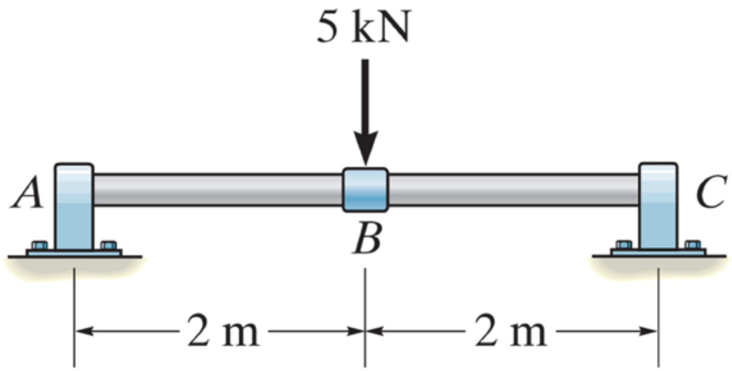
Find: The shear and moment diagram.

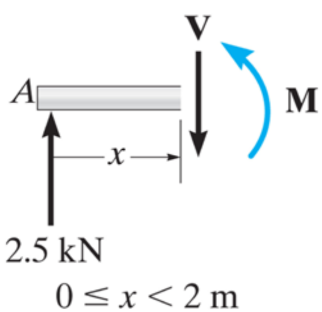
Plan:

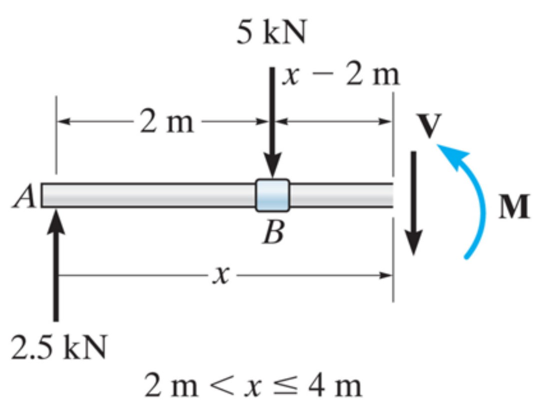
Find the support reactions and *A* and *C*.

Calculate shear force and bending moment at each interval

Plot the shear and moment diagram







Support reactions,

$A_y = 2.5$ kN and $C_y = 2.5$ kN

Shear and moment functions

between segment *AB*

$\sum F_y = 0 \quad V = 2.5 \text{ kN}$

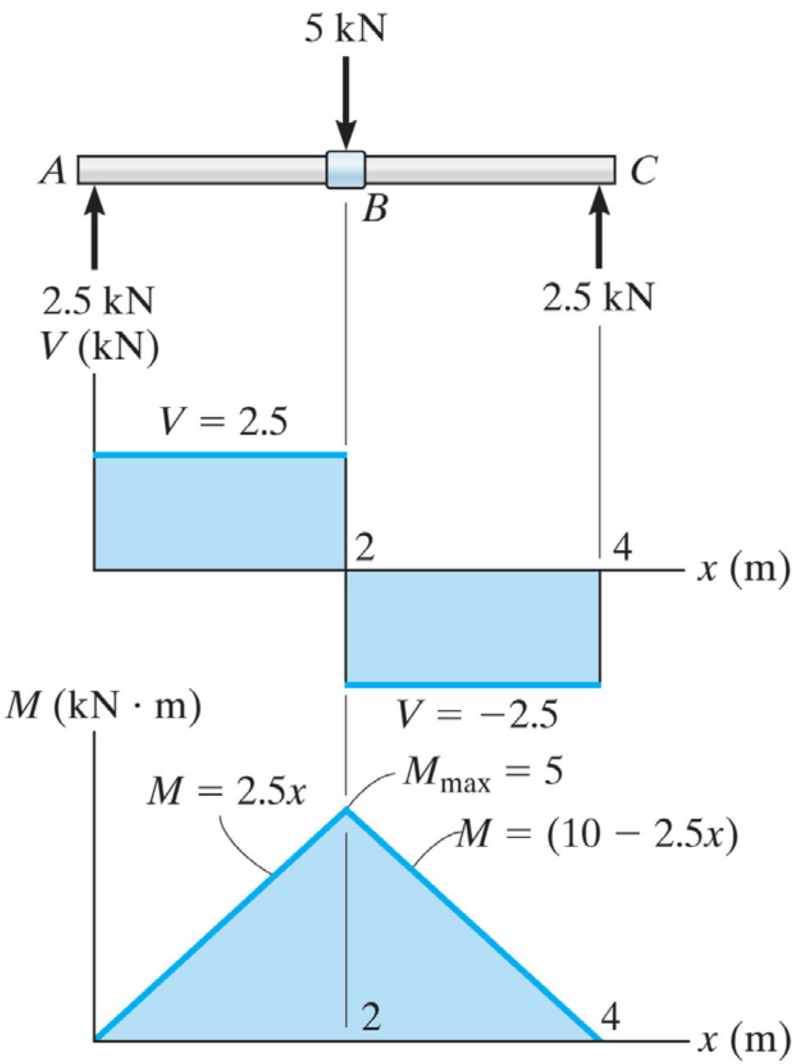
$\sum M = 0 \quad M = 2.5x \text{ kN·m}$

between segment *BC* (from *A*)

$\sum F_y = 0 \quad V = -2.5 \text{ kN}$

$\sum M = 0 \quad M + 5(x - 2) - 2.5x = 0$

$M = (10 - 2.5x) \text{ kN·m}$



Shear and moment diagrams

between segment *AB*

$\sum F_y = 0$

$V = 2.5$ kN (constant)

$\sum M = 0$

$M = 2.5x$ kN·m (linear)

between segment *BC* (from *A*)

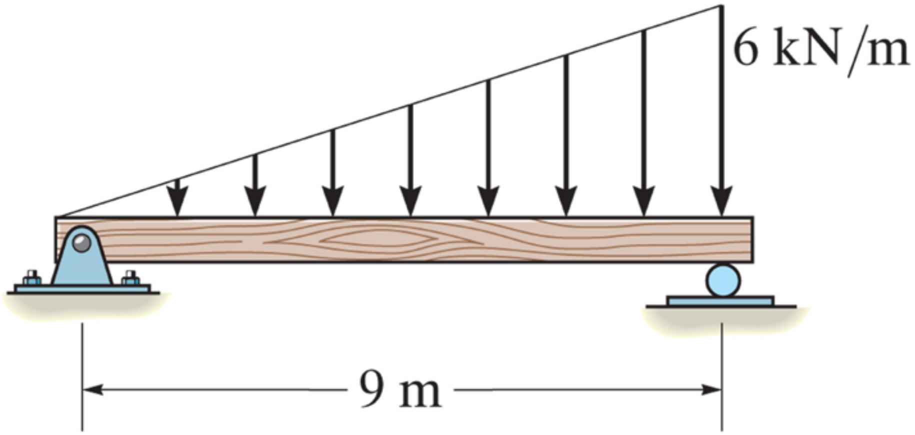
$\sum F_y = 0$

$V = -2.5$ kN (constant)

$\sum M = 0 \quad M + 5(x - 2) - 2.5x = 0$

$M = (10 - 2.5x)$ kN·m (linear)

Sample Problem (§ 7.2)



Given: The beam as shown.

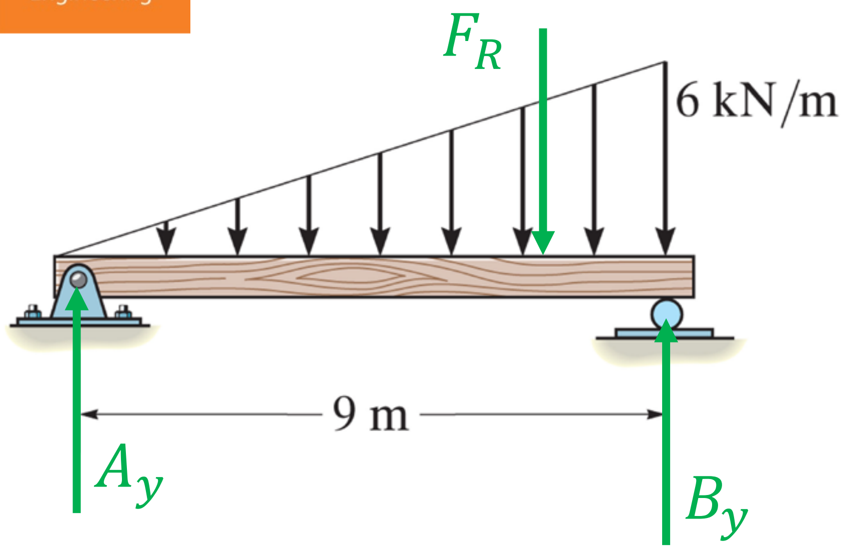
Find: The shear and moment diagram.

Plan:

Find the support reactions and *A* and *B*.

Calculate shear force and bending moment through the distributed load

Plot the shear and moment diagram



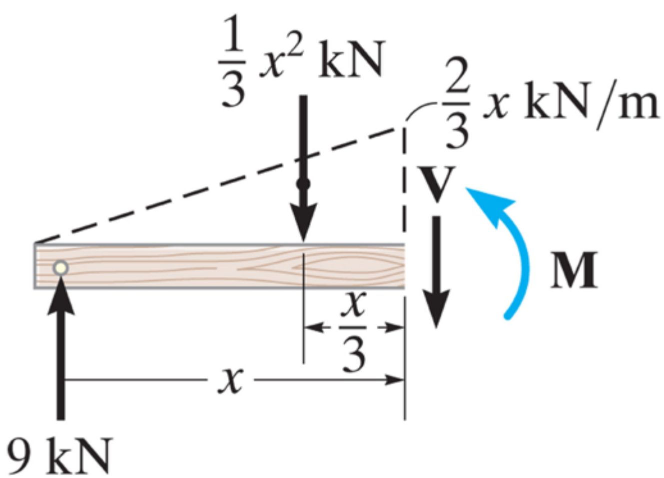
Support reactions,

$$F_R = {}^1/_2\, 6(9) = 27 \text{ kN}$$

$$\sum M_B = 0$$

$$-A_y(9) + F_R\,(3) = 0 \qquad A_y = 9 \text{ kN}$$

$$\sum F_y = 0 \quad B_y = 27 - 9 = 18 \text{ kN}$$



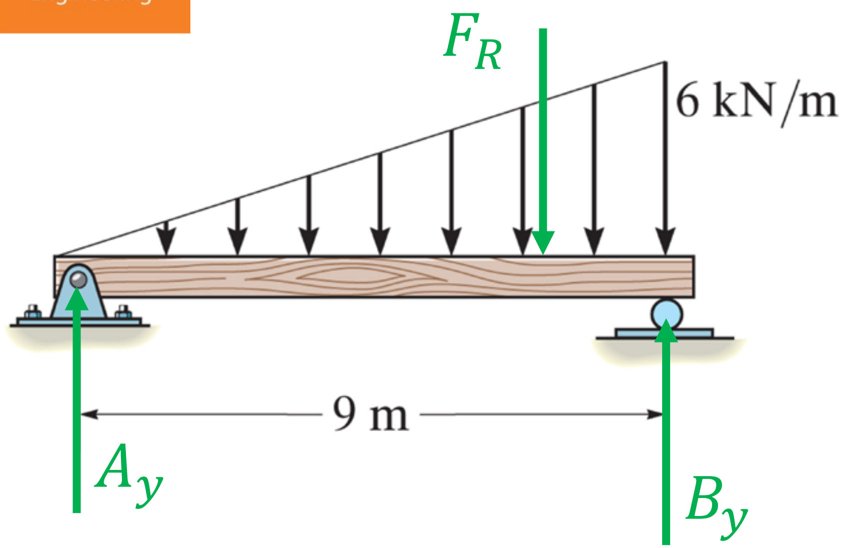
Shear and moment functions,

For each segment, the intensity of the distributed loading is

$$w/x = {}^6/_9 \qquad \therefore \quad w = {}^2/_3\, x$$

and the resultant force

$$F_R = {}^1/_2\, x\left({}^2/_3\, x\right) = {}^1/_3\, x^2$$



Shear and moment functions (cont'd)

$$\sum F_y = 0$$

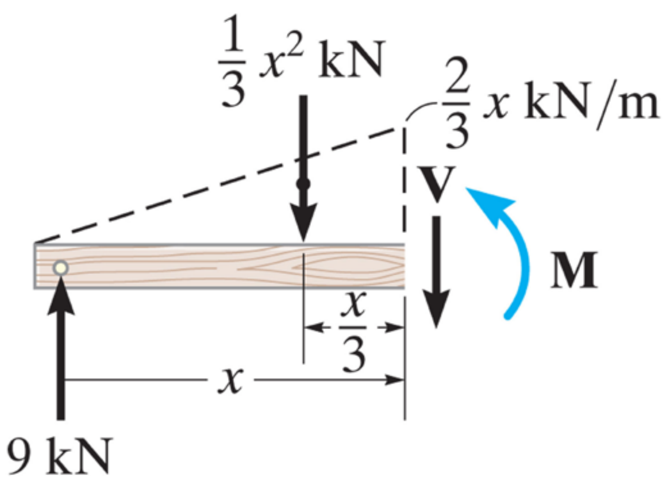
$$9 - {}^1\!/_3\, x^2 - V = 0$$

$$V = 9 - {}^1\!/_3\, x^2$$

$$\sum M = 0$$

$$M + {}^1\!/_3\, x^2 \left({}^x\!/_3\right) - 9x = 0$$

$$M = 9x - {}^1\!/_9\, x^3$$

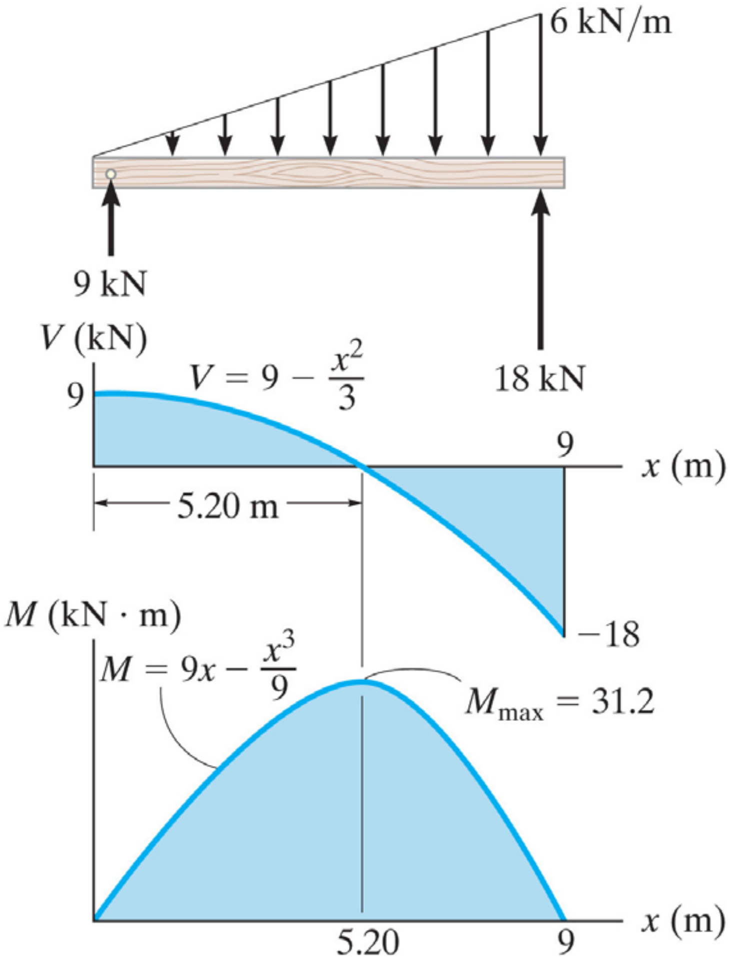


Note:

A concentrated force leads to constant shear and linear moment

A constant distribution leads to linear shear and quadratic moment

A linear distribution leads to quadratic shear and cubic moment, and so on.



Shear and moment functions

$$V = 9 - {}^{1}\!/_{3}\,x^2$$

$$M = 9x - {}^{1}\!/_{9}\,x^3$$

Shear and moment diagrams

plot the diagrams from $0 \le x \le 9\text{m}$

The maximum moment occurs at the point of zero shear

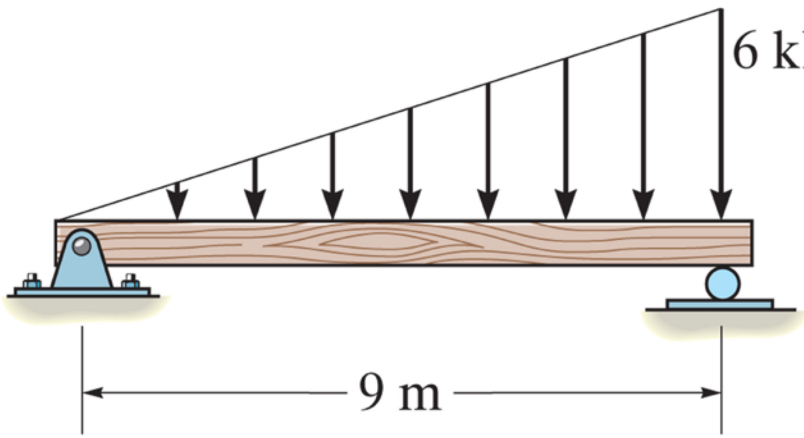
$$V = 9 - {}^{1}\!/_{3}\,x^2 = 0 \qquad x = 5.2\text{m}$$

$$M = 9(5.2) - {}^{1}\!/_{9}\,(5.2)^3$$

$$M = 31.2\ \text{kN} \cdot \text{m}$$

Note: The maximum bending moment is of interest as the strength of the beam is related to this value.

Sample Problem (§ 7.3)



Given: The beam as shown.

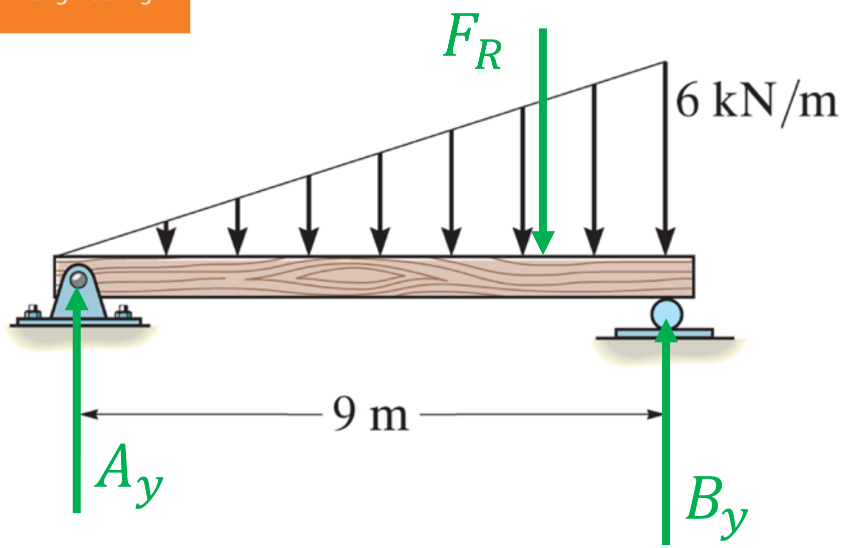
Find: The shear and moment diagram using the relationships between load, shear and bending moment.

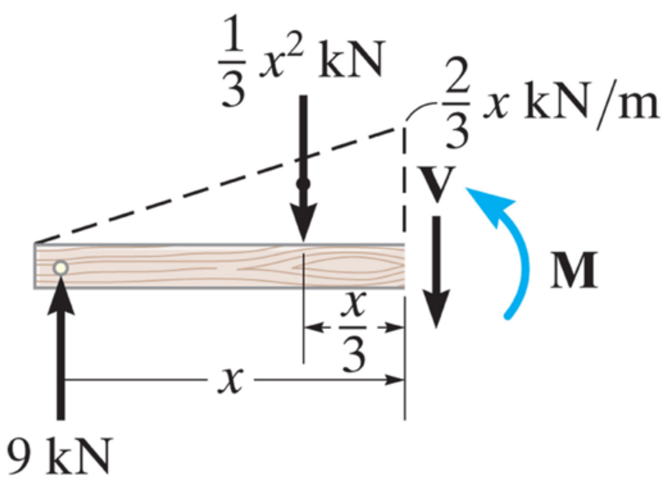
Plan:

Use the results obtained previously for the support reactions and the shear and moment functions.

Apply the relationships between load, shear and bending moment and plot the shear and bending moment diagram.

Sample Problem





From previous example.

Support reactions,

$A_y = 9$ kN and $B_y = 18$ kN

Load, shear and moment functions,

$$w = -{}^{2}/_{3}\,x$$

w(x) points down, negative increase

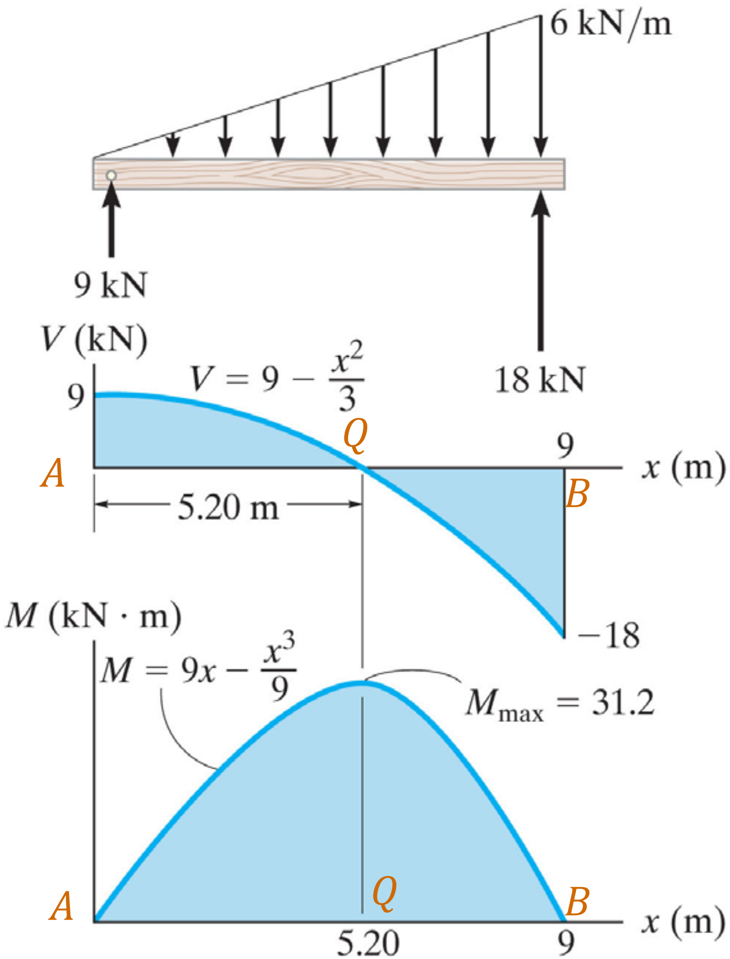
$$V = 9 - {}^{1}/_{3}\,x^2$$

$$M = 9x - {}^{1}/_{9}\,x^3$$

Zero shear occurs at,

$$V = 9 - {}^{1}/_{3}\,x^2 = 0$$

$$x = 5.2\text{m}$$



Shear diagram

$$V_A = 9 \text{ kN}$$

$$V_B - V_A = \int_{x_A}^{x_B} w(x)dx = \int_{0}^{9} -2/3(x)dx$$

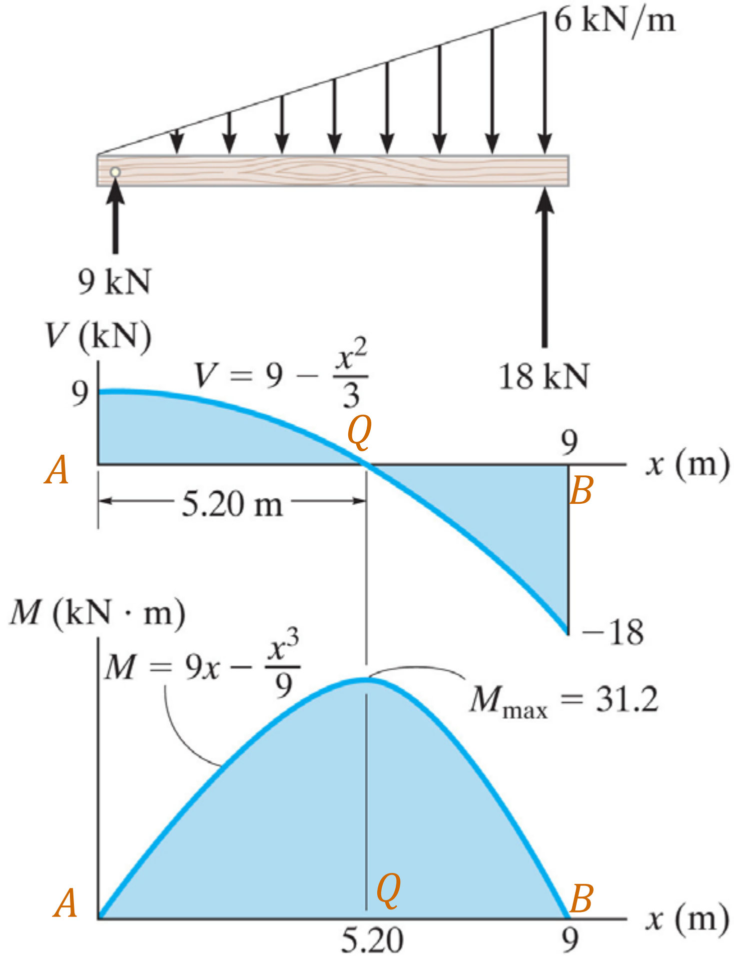
$$V_B = 9 - \left({}^{1}/_{3}\,(9)^2\right) = -18\,kN$$

Using the area of distributed load

w(x) points down, negative shear

$$V_B - V_A = \int_{x_A}^{x_B} w(x)dx = -A_{AB}$$

$$V_B = V_A - A_{AB} = 9 - \frac{1}{2}(6)(9) = -18\,kN$$



Bending moment diagram

$$M_A = 0\ \mathrm{kN}\cdot\mathrm{m}$$

$$M_Q - M_A = \int_{x_A}^{x_Q} V\,dx = \int_{0}^{5.2} (9 - {}^{1}/_{3}\,x^2)\,dx$$

$$M_Q = 0 + \int_{0}^{5.2} (9 - {}^{1}/_{3}\,x^2)\,dx = 31.2\ \mathrm{kN}\cdot\mathrm{m}$$

$$M_B - M_Q = \int_{x_Q}^{x_B} V\,dx = \int_{5.2}^{9} (9 - {}^{1}/_{3}\,x^2)\,dx$$

$$M_B = 31.2 + (0 - 31.2) = 0\ \mathrm{kN}\cdot\mathrm{m}$$