Solution. We begin with the given equation for y, namely,

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} x(-k).$$

Letting $v_1(k) = x(-k)$, we can rewrite the preceding equation for y as

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} v_1(k).$$

Letting $v_2(k) = e^{-j3k} v_1(k)$, we can rewrite the preceding equation as

$$y(n) = \sum_{k=-\infty}^{n} v_2(k).$$

Taking the Fourier transform of the equation for each of v_1, v_2, and y, we have

$$\begin{aligned}
v_1(n) = x(-n) \quad &\Leftrightarrow \quad V_1(\Omega) = X(-\Omega), \\
v_2(n) = e^{-j3n} v_1(n) \quad &\Leftrightarrow \quad V_2(\Omega) = V_1(\Omega+3), \quad \text{and} \\
y(n) = \sum_{k=-\infty}^{n} v_2(k) \quad &\Leftrightarrow \quad Y(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-1} V_2(\Omega) + \pi V_2(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).
\end{aligned}$$

Substituting the formulas for V_2 and V_1 into the formula for $Y(\Omega)$ on the preceding line, we obtain

$$\begin{aligned}
Y(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-1} V_2(\Omega) + \pi V_2(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1} V_1(\Omega+3) + \pi V_1(3) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1} X(-\Omega-3) + \pi X(-3) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).
\end{aligned}$$

■

Example 11.27. Let x and y be two sequences related by

$$y(n) = x(n)\cos(\Omega_0 n),$$

where Ω_0 is a nonzero real constant. Let X and Y denote the Fourier transforms of x and y, respectively. Find an expression for Y in terms of X.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two different ways in which to do this. One is to use the multiplication property of the Fourier transform, and another is to use the modulation property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort. The moral of this example is that, when multiple solution techniques are possible for a problem, one should always select the simpler technique, as this saves time and reduces the likelihood of errors.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use an approach based on the multiplication property of the Fourier transform. Taking the Fourier transform of both sides of the given equation, we trivially have

$$Y(\Omega) = \mathcal{F}\{x(n)\cos(\Omega_0 n)\}(\Omega).$$

From the multiplication property of the Fourier transform, we have

$$Y(\Omega) = \tfrac{1}{2\pi}\left(X * \mathcal{F}\{\cos(\Omega_0 n)\}\right)(\Omega).$$

Using Table 11.2 to determine $\mathcal{F}\{\cos(\Omega_0 n)\}$, we have

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2\pi}\left\{X * \left(\pi \sum_{k=-\infty}^{\infty}[\delta(\cdot-\Omega_0-2\pi k)+\delta(\cdot+\Omega_0-2\pi k)]\right)\right\}(\Omega) \\
&= \tfrac{1}{2\pi}\int_{2\pi} X(\theta)\left[\pi \sum_{k=-\infty}^{\infty}[\delta(\Omega-\theta-\Omega_0-2\pi k)+\delta(\Omega-\theta+\Omega_0-2\pi k)]\right]d\theta \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi^-} X(\theta)\left[\pi \sum_{k=-\infty}^{\infty}[\delta(\Omega-\theta-\Omega_0-2\pi k)+\delta(\Omega-\theta+\Omega_0-2\pi k)]\right]d\theta.
\end{aligned}$$

Since the delta function is even, we can rewrite the preceding equation as

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2}\int_{-\pi}^{\pi^-} X(\theta)\sum_{k=-\infty}^{\infty}[\delta(-\Omega+\theta+\Omega_0+2\pi k)+\delta(-\Omega+\theta-\Omega_0+2\pi k)]\,d\theta \\
&= \tfrac{1}{2}\int_{-\pi}^{\pi^-} X(\theta)\sum_{k=-\infty}^{\infty}[\delta(\theta-[\Omega-\Omega_0-2\pi k])+\delta(\theta-[\Omega+\Omega_0-2\pi k])]\,d\theta \\
&= \tfrac{1}{2}\left[\int_{-\pi}^{\pi^-} X(\theta)\sum_{k=-\infty}^{\infty}\delta[\theta-(\Omega-\Omega_0-2\pi k)]d\theta + \int_{-\pi}^{\pi^-} X(\theta)\sum_{\ell=-\infty}^{\infty}\delta[\theta-(\Omega+\Omega_0-2\pi\ell)]d\theta\right].
\end{aligned}$$

Consider the two integrals on the right-hand side of the preceding equation. The summation in the leftmost integral only contains one term that is nonzero in the integration interval $[-\pi,\pi)$. Let this term correspond to $k=k'$. Similarly, the summation in the rightmost integral only contains one term that is nonzero in the integration interval $[-\pi,\pi)$. Let this term correspond to $\ell=\ell'$. Now, we can rewrite the above equation for Y as

$$Y(\Omega) = \tfrac{1}{2}\left[\int_{-\pi}^{\pi^-} X(\theta)\delta[\theta-(\Omega-\Omega_0-2\pi k')]d\theta + \int_{-\pi}^{\pi^-} X(\theta)\delta[\theta-(\Omega+\Omega_0-2\pi\ell')]d\theta\right].$$

Using the sifting property of the delta function, we have

$$Y(\Omega) = \tfrac{1}{2}\left[X(\Omega-\Omega_0-2\pi k')+X(\Omega+\Omega_0-2\pi\ell')\right].$$

Since X is 2π-periodic, we can rewrite the preceding equation as

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2}[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)] \\
&= \tfrac{1}{2}X(\Omega-\Omega_0)+\tfrac{1}{2}X(\Omega+\Omega_0).
\end{aligned}$$

Although we have managed to solve the problem at hand, the above solution is quite tedious. Fortunately, there is a much better way to approach the problem, which we consider next.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use an approach based on the modulation property of the Fourier transform. Taking the Fourier transform of both sides of the given equation, we trivially have

$$Y(\Omega) = \mathcal{F}\{x(n)\cos(\Omega_0 n)\}(\Omega).$$

Expressing $\cos(\Omega_0 n)$ in terms of complex sinusoids, we have

$$\begin{aligned}
Y(\Omega) &= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\Omega_0 n}+e^{-j\Omega_0 n}\right)x(n)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}e^{j\Omega_0 n}x(n)+\tfrac{1}{2}e^{-j\Omega_0 n}x(n)\right\}(\Omega).
\end{aligned}$$

Using the linearity property of the Fourier transform, we have

$$Y(\Omega) = \tfrac{1}{2}\mathcal{F}\left\{e^{j\Omega_0 n}x(n)\right\}(\Omega)+\tfrac{1}{2}\mathcal{F}\left\{e^{-j\Omega_0 n}x(n)\right\}(\Omega).$$

Using the modulation property of the Fourier transform, we obtain

$$Y(\Omega) = \tfrac{1}{2}X(\Omega - \Omega_0) + \tfrac{1}{2}X(\Omega + \Omega_0).$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the multiplication property of the Fourier transform is usually best avoided whenever possible, since its use introduces convolution into the solution. ■

11.10 Frequency Spectra of Sequences

The Fourier transform representation expresses a sequence in terms of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a sequence. For example, suppose that we have a sequence x with Fourier transform X. If X is nonzero at some frequency Ω_0, then the sequence x contains some information at the frequency Ω_0. On the other hand, if X is zero at the frequency Ω_0, then the sequence x has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a sequence. This distribution of information in a sequence over different frequencies is referred to as the **frequency spectrum** of the sequence. That is, X is the frequency spectrum of x.

To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with X expressed in polar form as follows:

$$\begin{aligned}
x(n) &= \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega \\
&= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j\arg X(\Omega)}e^{j\Omega n}d\Omega \\
&= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j[\Omega n+\arg X(\Omega)]}d\Omega.
\end{aligned}$$

In effect, the quantity $|X(\Omega)|$ is a weight that determines how much the complex sinusoid at frequency Ω contributes to the integration result $x(n)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles (i.e., $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=-\infty}^{\infty}\Delta x f(k\Delta x)$). Expressing x in this way, we obtain

$$\begin{aligned}
x(n) &= \lim_{\Delta\Omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\Omega\,|X(k\Delta\Omega)|\, e^{j[k\Delta\Omega n+\arg X(k\Delta\Omega)]} \\
&= \lim_{\Delta\Omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\Omega\,\left|X(\Omega')\right| e^{j[\Omega' n+\arg X(\Omega')]},
\end{aligned}$$

where $\Omega' = k\Delta\Omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\Omega' = k\Delta\Omega$) corresponds to a complex sinusoid with frequency Ω' that has had its amplitude scaled by a factor of $|X(\Omega')|$ and has been time-shifted by an amount that depends on $\arg X(\Omega')$. For a given $\Omega' = k\Delta\Omega$ (which is associated with the kth term in the summation), the larger $|X(\Omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\Omega' n}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\Omega')|$ as a measure of how much information a sequence x has at the frequency Ω'.

To formalize the notion of frequency spectrum, the **frequency spectrum** of a sequence x is given by its Fourier transform X. Due to the above interpretation of the Fourier transform in terms of the polar form, we are often interested in $|X(\cdot)|$ and $\arg X(\cdot)$. As a matter of terminology, we refer to $|X(\cdot)|$ as the **magnitude spectrum** of x and $\arg X(\cdot)$ as the **phase spectrum** of x.

Since the graphical presentation of information is often helpful for visualization purposes, we often want to plot frequency spectra of sequences. Since three-dimensional plots are usually more difficult to generate (especially by hand) than two-dimensional ones and can often be more difficult to interpret accurately, we usually present frequency spectra in graphical form using only two-dimensional plots. In the case that the frequency spectrum is either purely real or purely imaginary, we typically plot the frequency spectrum directly on a single pair of axes. Most often, however, the frequency spectrum will be complex (but neither purely real nor purely imaginary), in which case we plot the frequency spectrum in polar form by using two plots, one showing the magnitude spectrum and one showing the phase spectrum.

Note that, since the Fourier transform X is a function of a real variable, a sequence x can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic sequences), where a sequence can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 11.8, in the case of periodic sequences the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Recall (from Theorem 11.18) that, for a real-valued sequence x, the Fourier transform X is conjugate symmetric (i.e., $X(\Omega) = X^*(-\Omega)$ for all $\Omega \in \mathbb{R}$). This, however, implies that

$$\begin{aligned} |X(\Omega)| &= |X(-\Omega)| \quad \text{for all } \Omega \in \mathbb{R} \quad \text{and} \\ \arg X(\Omega) &= -\arg X(-\Omega) \quad \text{for all } \Omega \in \mathbb{R} \end{aligned}$$

(i.e., the magnitude and argument of X are even and odd, respectively). (See (11.11a) and (11.11b).) Due to the symmetry in the frequency spectra of real-valued sequences, we typically ignore negative frequencies when dealing with such sequences. In the case of sequences that are complex-valued but not real-valued, frequency spectra do not possess the above symmetry, and negative frequencies become important.

Example 11.28. Find and plot the frequency spectrum of the sequence

$$x(n) = u(n) - u(n-16).$$

Solution. From Table 11.2, the Fourier transform X of x is given by

$$X(\Omega) = e^{-j(15/2)\Omega}\left[\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right] \quad \text{for } \Omega \in (-\pi, \pi].$$

In this case, X is neither purely real nor purely imaginary, so we will plot the frequency spectrum X in polar form using two graphs, one for the magnitude spectrum and one for the phase spectrum. Taking the magnitude of X, we have

$$|X(\Omega)| = \left|\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right|.$$

Taking the argument of X, we have

$$\arg[X(\Omega)] = -\tfrac{15}{2}\Omega + \arg\left[\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right].$$

The magnitude spectrum and phase spectrum are shown plotted in Figures 11.6(a) and (b), respectively. ■

Example 11.29. Find and plot the frequency spectrum of the sequence

$$x(n) = \left(\tfrac{1}{2}\right)^{|n|}.$$

Solution. From Table 11.2, the Fourier transform X of x is given by

$$X(\Omega) = \frac{3}{5 - 4\cos\Omega} \quad \text{for } \Omega \in (-\pi, \pi].$$

Since, in this case, X is real, we can plot the frequency spectrum X on a single graph, as shown in Figure 11.7. ■

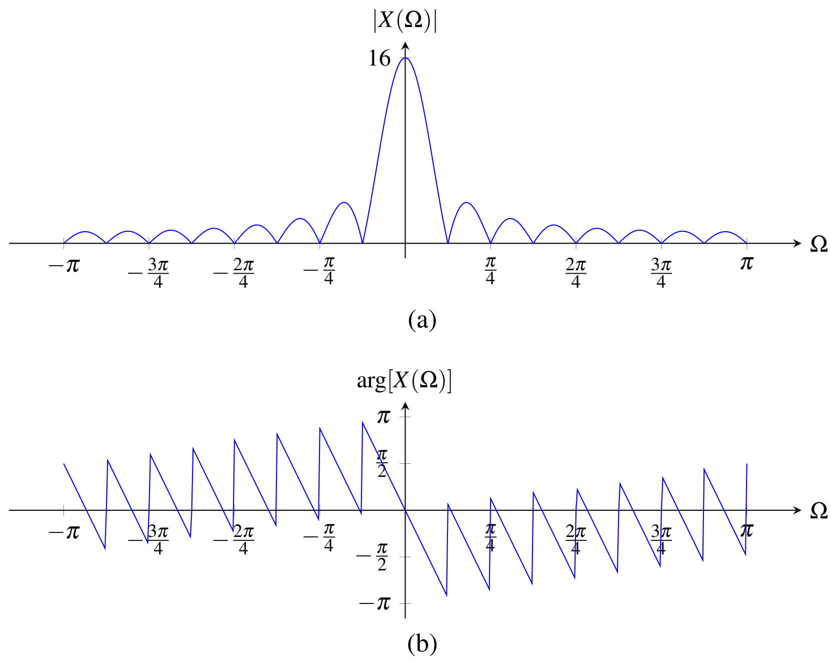

Figure 11.6: Frequency spectrum X of the sequence x. (a) Magnitude spectrum and (b) phase spectrum.

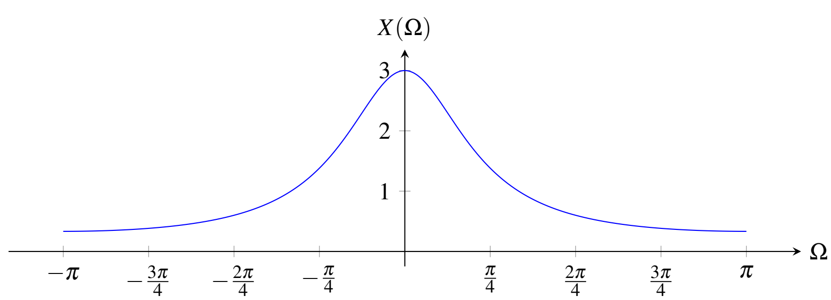

Figure 11.7: Frequency spectrum X of the sequence x.

Example 11.30. The sequence

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

has the Fourier transform

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}.$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) in the range $(-\pi, \pi]$ the sequence x has the most information.

Solution. (a) First, we find the magnitude spectrum $|X(\Omega)|$. From the expression for $X(\Omega)$, we can write

$$\begin{aligned}
|X(\Omega)| &= \left| \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}} \right| \\
&= \frac{1}{\left|e^{j\Omega} - \frac{1}{2}\right|} \\
&= \frac{1}{\left|\cos(\Omega) + j\sin(\Omega) - \frac{1}{2}\right|} \\
&= \frac{1}{\sqrt{\left(\cos(\Omega) - \frac{1}{2}\right)^2 + j\,(\sin\Omega)^2}} \\
&= \frac{1}{\sqrt{\cos^2(\Omega) - \cos\Omega + \frac{1}{4} + \sin^2\Omega}} \\
&= \frac{1}{\sqrt{\frac{5}{4} - \cos\Omega}}.
\end{aligned}$$

Next, we find the phase spectrum $\arg[X(\Omega)]$. From the expression for $X(\Omega)$, we can write

$$\begin{aligned}
\arg[X(\Omega)] &= \arg\left[\frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}\right] \\
&= \arg\left[\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right] \\
&= \arg 1 - \arg\left(1 - \tfrac{1}{2}e^{-j\Omega}\right) \\
&= -\arg\left(1 - \tfrac{1}{2}e^{-j\Omega}\right) \\
&= -\arg\left[1 - \tfrac{1}{2}\left[\cos(-\Omega) + j\sin(-\Omega)\right]\right] \\
&= -\arg\left(\left[1 - \tfrac{1}{2}\cos\Omega\right] + j\left[\tfrac{1}{2}\sin\Omega\right]\right) \\
&= -\arg\left([2 - \cos(\Omega)] + j\,[\sin(\Omega)]\right) \\
&= -\arctan\left[\frac{\sin(\Omega)}{2 - \cos(\Omega)}\right].
\end{aligned}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\Omega)|$ and $\arg X(\Omega)$ to obtain the results shown in Figures 11.8(a) and (b).

(b) For $\Omega \in (-\pi, \pi]$, $|X(\Omega)|$ is greatest at $\Omega = 0$. Therefore, x has the most information at the frequency 0. ■

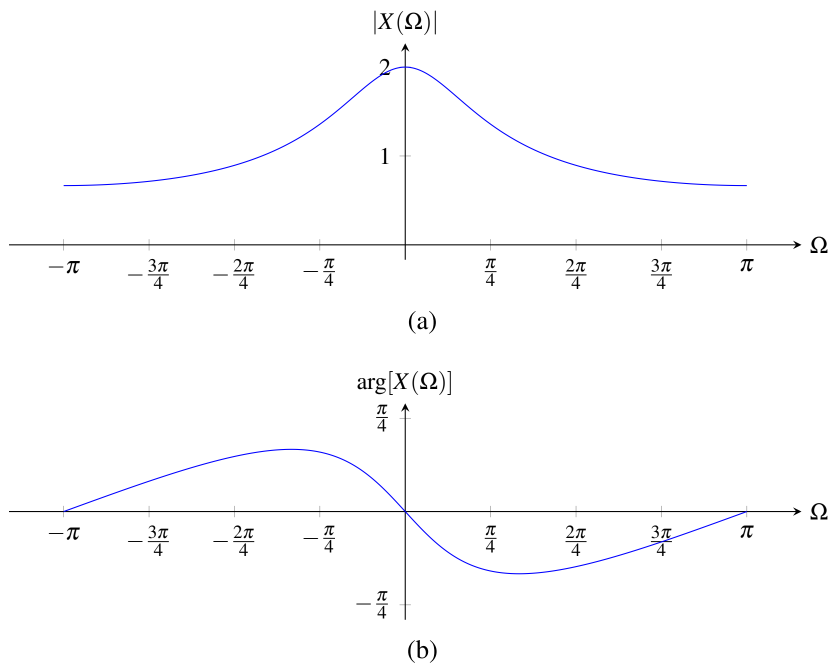

Figure 11.8: Frequency spectrum X of the sequence x. (a) Magnitude spectrum and (b) phase spectrum.

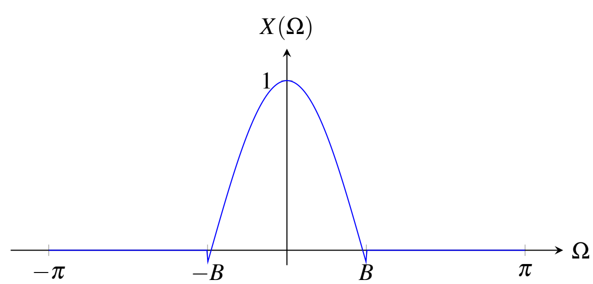

Figure 11.9: Example of the Fourier transform X of a sequence x that is bandlimited to frequencies in $[-B, B]$.

11.11 Bandwidth of Sequences

The range of frequencies over which the spectrum of a sequence is nonzero is often of interest. This is analogous to the notion of bandwidth that was introduced in the context of continuous-time signals (i.e., functions) in Section 6.11. Since, the spectrum of a discrete-time signal (i.e., sequence) is always 2π-periodic, when we speak of the bandwidth of a sequence, we only consider the spectrum over a single interval of length 2π. In particular, we normally consider an interval (of length 2π) centered at the origin (e.g., $(-\pi, \pi]$). When a sequence x has a Fourier transform X satisfying $X(\Omega) = 0$ for all Ω in $(-\pi, \pi]$ except for some interval I, we say that x is **bandlimited** to frequencies in I. Moreover, we define the **bandwidth** of a sequence x with Fourier transform X as the length of the interval in $(-\pi, \pi]$ over which X is nonzero. For example, the sequence x whose Fourier transform X is shown in Figure 11.9 is bandlimited to frequencies in $[-B, B]$ and has bandwidth $B - (-B) = 2B$. Sometimes, when dealing with real sequences, negative frequencies are ignored. Since x is real in this example (as X is conjugate symmetric), we might choose to ignore negative frequencies, in which case x would be deemed to be bandlimited to frequencies in $[0, B]$ and have bandwidth $B - 0 = B$.

11.12 Energy-Density Spectra

Suppose that we have a sequence x with finite energy E and Fourier transform X. By definition, the energy contained in x is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

We can use Parseval's relation (6.12) to express E in terms of X as

$$E = \tfrac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega.$$

Thus, the energy E is given by

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} E_x(\Omega) d\Omega,$$

where

$$E_x(\Omega) = |X(\Omega)|^2.$$

We refer to E_x as the **energy-density spectrum** of the sequence x. The function E_x indicates how the energy in x is distributed with respect to frequency. For example, the energy contributed by frequencies in the range $[\Omega_1, \Omega_2]$ is given by

$$\tfrac{1}{2\pi} \int_{\Omega_1}^{\Omega_2} E_x(\Omega) d\Omega.$$

Example 11.31. Consider the sequence

$$x(n) = \operatorname{sinc}\left(\tfrac{\pi}{4} n\right).$$

Compute the energy-density spectrum E_x of x. Determine the amount of energy contained in x in the frequency range $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Also, determine the total amount of energy in x.

Solution. First, we compute the Fourier transform X of x. We obtain

$$X(\Omega) = 4 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right).$$

Computing the energy-density spectrum E_x, we have

$$\begin{aligned} E_x(\Omega) &= |X(\Omega)|^2 \\ &= \left|4 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)\right|^2 \\ &= 16 \operatorname{rect}^2\left(\tfrac{2}{\pi}\Omega\right) \\ &= 16 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right). \end{aligned}$$

Let E_1 denote the energy contained in x for frequencies $\Omega \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Then, we have

$$\begin{aligned}
E_1 &= \tfrac{1}{2\pi}\int_{-\pi/8}^{\pi/8} E_x(\Omega)d\Omega \\
&= \tfrac{1}{2\pi}\int_{-\pi/8}^{\pi/8} 16\operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/8}^{\pi/8} \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/8}^{\pi/8} 1 d\Omega \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{8}+\tfrac{\pi}{8}\right) \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{4}\right) \\
&= 2.
\end{aligned}$$

Let E denote the total amount of energy in x. Then, we have

$$\begin{aligned}
E &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} E_x(\Omega)d\Omega \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} 16\operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi}^{\pi} \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/4}^{\pi/4} 1 d\Omega \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{2}\right) \\
&= 4.
\end{aligned}$$

■

11.13 Characterizing LTI Systems Using the Fourier Transform

Consider a LTI system with input x, output y, and impulse response h. Such a system is depicted in Figure 11.10. The behavior of such a system is governed by the equation

$$y(n) = x * h(n). \tag{11.16}$$

Let X, Y, and H denote the Fourier transforms of x, y, and h, respectively. Taking the Fourier transform of both sides of (11.16) and using the convolution property of the Fourier transform, we obtain

$$Y(\Omega) = X(\Omega)H(\Omega). \tag{11.17}$$

This result provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output sequences. In other words, we have a system resembling that in Figure 11.11. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to H as the **frequency response** of the system. The system behavior is completely characterized by the frequency response H. If we know the input, we can compute its Fourier transform X, and then determine the Fourier transform Y of the output. Using the inverse Fourier transform, we can then determine the output y.

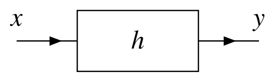

Figure 11.10: Time-domain view of a LTI system with input x, output y, and impulse response h.

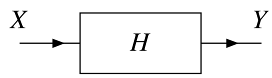

Figure 11.11: Frequency-domain view of a LTI system with input spectrum X, output spectrum Y, and frequency response H.

In the most general case, the frequency response H is a complex-valued function. Thus, we can represent H in terms of its magnitude and argument. We refer to the magnitude of H as the **magnitude response** of the system. Similarly, we refer to the argument of H as the **phase response** of the system.

From (11.17), we can write

$$\begin{aligned} |Y(\Omega)| &= |X(\Omega)H(\Omega)| \\ &= |X(\Omega)|\,|H(\Omega)| \quad \text{and} \end{aligned} \tag{11.18a}$$

$$\begin{aligned} \arg Y(\Omega) &= \arg[X(\Omega)H(\Omega)] \\ &= \arg X(\Omega) + \arg H(\Omega). \end{aligned} \tag{11.18b}$$

From (11.18a), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system (i.e., the magnitude spectrum of the impulse response). From (11.18b), we can see that the phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system (i.e., the phase spectrum of the impulse response).

Since the frequency response H is simply the frequency spectrum of the impulse response h, for the reasons explained in Section 11.10, if h is real, then

$$\begin{aligned} |H(\Omega)| &= |H(-\Omega)| \text{ for all } \Omega \quad \text{and} \\ \arg H(\Omega) &= -\arg H(-\Omega) \text{ for all } \Omega \end{aligned}$$

(i.e., the magnitude and phase responses are even and odd, respectively).

Example 11.32. A LTI system has the impulse response

$$h(n) = u(n+5) - u(n-6).$$

Find the frequency response H of the system.

Solution. The frequency response is simply the Fourier transform of h. Using Table 11.2 (or the result of Example 11.2), we can easily determine H to be

$$H(\Omega) = \frac{\sin\left(\frac{11}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}.$$

Observe that H is real. So, we can plot the frequency response H on a single graph, as shown in Figure 11.12. ■

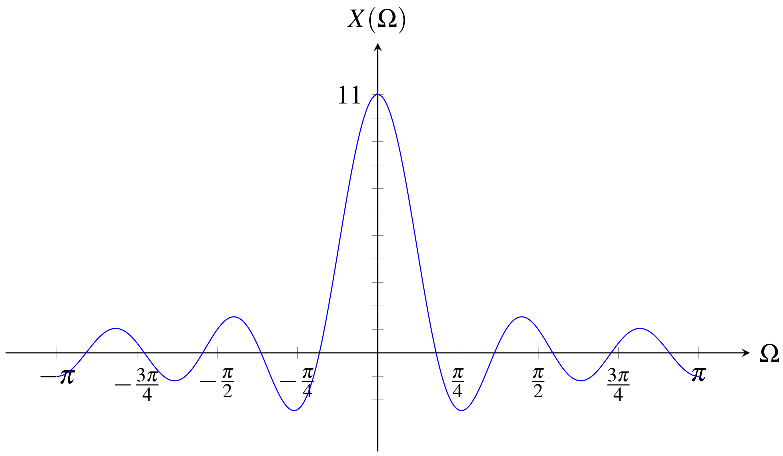

Figure 11.12: Frequency response of example system.

11.13.1 Unwrapped Phase

Since the argument of a complex number is not uniquely determined, the argument of a complex-valued function is also not uniquely determined. Consequently, we have some freedom in how we define a function that corresponds to the phase (i.e., argument) of a complex-valued function. Often, for convenience, we restrict the argument to lie in an interval of length 2π, such as the interval $(-\pi, \pi]$ which corresponds to the principal argument. Defining the phase of a complex-valued function in this way, however, can often result in a phase function with unnecessary discontinuities. This motivates the notion of unwrapped phase. The **unwrapped phase** is simply the phase defined in such a way so as not to restrict the phase to an interval of length 2π and to keep the phase function continuous to the greatest extent possible. An example illustrating the notion of unwrapped phase is given below.

Example 11.33 (Unwrapped phase)**.** Consider the phase response of a LTI system with the frequency response

$$H(\Omega) = e^{j3\Omega}.$$

We can choose to define the phase (i.e., argument) of H by simply using the principal argument (i.e., $\operatorname{Arg} H(\Omega)$). This yields the phase function shown in Figure 11.13(a). Using the principal argument in this way, however, unnecessarily introduces discontinuities into the phase function. For this reason, we sometimes prefer to define the phase function in such a way as to eliminate such unnecessary discontinuities. This motivates the use of the unwrapped phase. The function H has the unwrapped phase Θ given by

$$\Theta(\Omega) = 3\Omega.$$

A plot of Θ is shown in Figure 11.13(b). Unlike the function in Figure 11.13(a) (which has numerous discontinuities), the function in Figure 11.13(b) is continuous. Although the functions in these two figures are distinct, these functions are equivalent in the sense that they correspond to the same physical angular displacement (i.e., $e^{j\operatorname{Arg} H(\Omega)} = e^{j\Theta(\Omega)}$ for all $\Omega \in \mathbb{R}$). ■

11.13.2 Magnitude and Phase Distortion

Recall, from Corollary 10.1, that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\left\{e^{j\Omega n}\right\}(n) = H(\Omega)e^{j\Omega n}$$

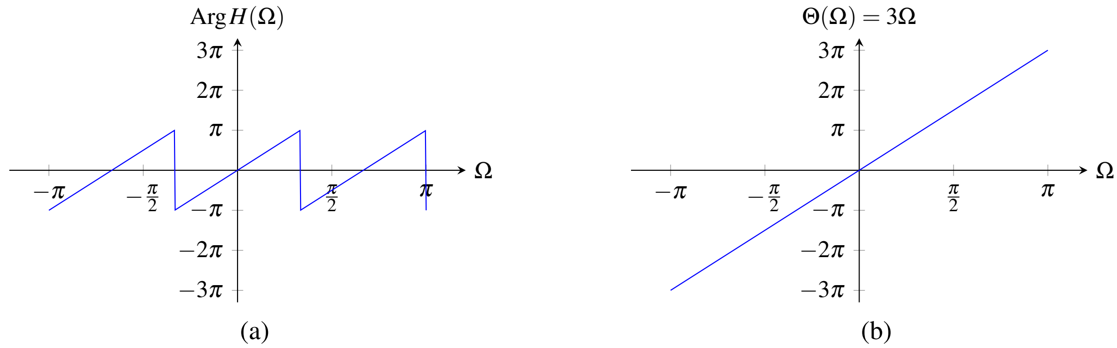

Figure 11.13: Unwrapped phase example. (a) The phase function restricted such that its range is in $(-\pi, \pi]$ and (b) the corresponding unwrapped phase.

(i.e., $e^{j\Omega n}$ is an eigensequence of $\mathcal{H}$ with eigenvalue $H(\Omega)$). Expressing $H(\Omega)$ in polar form, we have

$$\begin{aligned}\mathcal{H}\{e^{j\Omega n}\}(n) &= |H(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} \\ &= |H(\Omega)|\, e^{j[\Omega n + \arg H(\Omega)]} \\ &= |H(\Omega)|\, e^{j\Omega(n + \arg[H(\Omega)]/\Omega)}.\end{aligned}$$

This equation can be rewritten as

$$\mathcal{H}\{e^{j\Omega n}\}(n) = |H(\Omega)|\, e^{j\Omega[n - \tau_{\mathsf{p}}(\Omega)]}, \tag{11.19a}$$

where

$$\tau_{\mathsf{p}}(\Omega) = -\frac{\arg H(\Omega)}{\Omega}. \tag{11.19b}$$

Thus, the response of the system to the sequence $e^{j\Omega n}$ is produced by applying two transformations to this sequence:

- (amplitude) scaling by $|H(\Omega)|$; and
- if $\tau_{\mathsf{p}}(\Omega) \in \mathbb{Z}$, translating by $\tau_{\mathsf{p}}(\Omega)$; otherwise (i.e., $\tau_{\mathsf{p}}(\Omega) \notin \mathbb{Z}$), the corresponding continuous-time complex sinusoid is translated by $\tau_{\mathsf{p}}(\Omega)$ and then sampled (i.e., bandlimited interpolation is performed).

Therefore, the magnitude response determines how different complex sinusoids are (amplitude) scaled by the system. Similarly, the phase response determines how different complex sinusoids are translated (i.e., delayed/advanced) by the system (possibly using interpolation).

A system for which $|H(\Omega)| = 1$ for all Ω is said to be **allpass**[1]. In the case of an allpass system, the magnitude spectra of the system's input and output are identical. If a system is not allpass, it modifies the magnitude spectrum in some way. In situations where the magnitude spectrum is changed in an undesirable manner, **magnitude distortion** (i.e., distortion of the magnitude spectrum) is said to occur. If $|H(\Omega)| = a$ for all Ω, where a is a constant, every complex sinusoid is scaled by the same amount a when passing through the system. In practice, this type of change to the magnitude spectrum may sometimes be undesirable if $a \neq 1$. If $|H(\Omega)|$ is not a constant, different complex sinusoids are scaled by different amounts. In practice, this type of change to the magnitude spectrum is usually undesirable and deemed to constitute magnitude distortion.

The function τ_{p} appearing in (11.19b) is known as the **phase delay** of the system. A system for which $\tau_{\mathsf{p}}(\Omega) = 0$ for all Ω is said to have **zero phase**. In the case of a system having zero phase, the phase spectra of the system's input and output are identical. In the case that the system does not have zero phase, the phase spectra of the system's input

[1] Some authors (e.g., [9, 12]) define an allpass system as one for which $|H(\Omega)| = c$ for all Ω, where c is a constant (and c is not necessarily 1).

and output differ. In situations where the phase spectrum is changed in an undesirable manner, **phase distortion** (i.e., distortion of the phase spectrum) is said to occur. If $\tau_\mathsf{p}(\Omega) = n_\mathsf{d}$ for all Ω, where n_d is a constant, the system shifts all complex sinusoids by the same amount n_d. Note that $\tau_\mathsf{p}(\Omega) = n_\mathsf{d}$ is equivalent to the (unwrapped) phase response being of the form

$$\arg H(\Omega) = -n_\mathsf{d}\Omega,$$

which is a linear function with a zero constant term. For this reason, a system with a constant phase delay is said to have **linear phase**. If $\tau_\mathsf{p}(\Omega)$ is not a constant, different complex sinusoids are shifted by different amounts. In many practical applications, shifting different complex sinusoids by different amounts is undesirable. Therefore, systems that are not linear phase are typically deemed to introduce phase distortion. For this reason, in contexts where phase spectra are important, systems with either zero phase or linear phase are typically used.

Example 11.34 (Distortionless transmission)**.** Consider a LTI system with input x and output y given by

$$y(n) = x(n - n_0),$$

where n_0 is an integer constant. That is, the output of the system is simply the input delayed by n_0. This type of system behavior is referred to as distortionless transmission, since the system allows the input to pass through to the output unmodified, except for a delay being introduced. This type of behavior is the ideal for which we strive in real-world communication systems (i.e., the received signal y equals a delayed version of the transmitted signal x). Taking the Fourier transform of the above equation, we have

$$Y(\Omega) = e^{-j\Omega n_0} X(\Omega).$$

Thus, the system has the frequency response H given by

$$H(\Omega) = e^{-j\Omega n_0}.$$

Since $|H(\Omega)| = 1$ for all Ω, the system is allpass and does not introduce any magnitude distortion. The phase delay τ_p of the system is given by

$$\begin{aligned}
\tau_\mathsf{p}(\Omega) &= -\frac{\arg H(\Omega)}{\Omega} \\
&= -\left(\frac{-\Omega n_0}{\Omega}\right) \\
&= n_0.
\end{aligned}$$

Since the phase delay is a constant, the system has linear phase and does not introduce any phase distortion (except for a trivial time shift of n_0). ■

11.14 Interconnection of LTI Systems

From the properties of the Fourier transform and the definition of the frequency response, we can derive a number of equivalences involving the frequency response and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 11.14(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1 * h_2(n).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\{h_1 * h_2\}(\Omega) \\
&= \mathcal{F}h_1(\Omega)\mathcal{F}h_2(\Omega) \\
&= H_1(\Omega)H_2(\Omega).
\end{aligned}$$

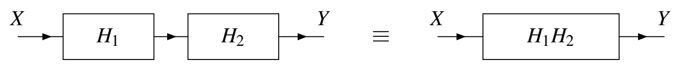

Figure 11.14: Equivalence involving frequency responses and the series interconnection of LTI systems.

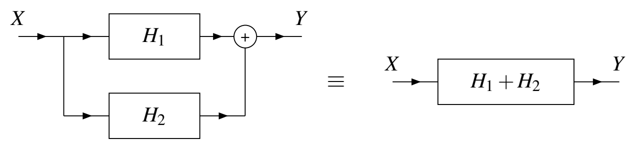

Figure 11.15: Equivalence involving frequency responses and the parallel interconnection of LTI systems.

Thus, we have the equivalence shown in Figure 11.14.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 11.15. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1(n) + h_2(n).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned} H(\Omega) &= \mathcal{F}\{h_1 + h_2\}(\Omega) \\ &= \mathcal{F}h_1(\Omega) + \mathcal{F}h_2(\Omega) \\ &= H_1(\Omega) + H_2(\Omega). \end{aligned}$$

Thus, we have the equivalence shown in Figure 11.15.

11.15 LTI Systems and Difference Equations

Many LTI systems of practical interest can be represented using an Nth-order linear difference equation with constant coefficients. Suppose that we have such a system with input x and output y. Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k)$$

(where $M \leq N$). Let X and Y denote the Fourier transforms of x and y, respectively. Taking the Fourier transform of both sides of the above equation yields

$$\mathcal{F}\left\{\sum_{k=0}^{N} b_k y(n-k)\right\}(\Omega) = \mathcal{F}\left\{\sum_{k=0}^{M} a_k x(n-k)\right\}(\Omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathcal{F}\{y(n-k)\}(\Omega) = \sum_{k=0}^{M} a_k \mathcal{F}\{x(n-k)\}(\Omega).$$

Using the differencing property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k e^{-j\Omega k} Y(\Omega) = \sum_{k=0}^{M} a_k e^{-j\Omega k} X(\Omega).$$

Then, factoring we have

$$Y(\Omega)\sum_{k=0}^{N} b_k e^{-j\Omega k} = X(\Omega)\sum_{k=0}^{M} a_k e^{-j\Omega k}.$$

Rearranging this equation, we obtain

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} a_k e^{-j\Omega k}}{\sum_{k=0}^{N} b_k e^{-j\Omega k}}.$$

Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, the frequency response H is given by

$$H(\Omega) = \frac{\sum_{k=0}^{M} a_k e^{-j\Omega k}}{\sum_{k=0}^{N} b_k e^{-j\Omega k}} = \frac{\sum_{k=0}^{M} a_k (e^{-j\Omega})^k}{\sum_{k=0}^{N} b_k (e^{-j\Omega})^k}.$$

Observe that each of the numerator and denominator of H is a polynomial in the variable $e^{-j\Omega}$. Thus, H is a (proper) rational function in the variable $e^{-j\Omega}$. As it turns out, this is one reason why rational functions are of particular interest in the study of signals and systems.

Example 11.35 (Difference equation to frequency response)**.** A LTI system with input x and output y is characterized by the difference equation

$$5y(n) + 2y(n-1) + 3y(n-2) = x(n) - 2x(n-1).$$

Find the frequency response H of this system.

Solution. Let $X = \mathcal{F}x$ and $Y = \mathcal{F}y$. Taking the Fourier transform of the given difference equation, we obtain

$$5Y(\Omega) + 2e^{-j\Omega}Y(\Omega) + 3e^{-j2\Omega}Y(\Omega) = X(\Omega) - 2e^{-j\Omega}X(\Omega).$$

Rearranging the terms and factoring, we have

$$(5 + 2e^{-j\Omega} + 3e^{-j2\Omega})Y(\Omega) = (1 - 2e^{-j\Omega})X(\Omega).$$

Dividing both sides of the equation by $5 + 2e^{-j\Omega} + 3e^{-j2\Omega}$ and $X(\Omega)$, we have

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{1 - 2e^{-j\Omega}}{5 + 2e^{-j\Omega} + 3e^{-j2\Omega}}.$$

Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, we have

$$H(\Omega) = \frac{1 - 2e^{-j\Omega}}{5 + 2e^{-j\Omega} + 3e^{-j2\Omega}}.$$ ■

Example 11.36 (Frequency response to difference equation)**.** A LTI system with input x and output y has the frequency response

$$H(\Omega) = \frac{e^{j2\Omega} - e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}}.$$

Find the differential equation that characterizes this system.

Solution. Let $X = \mathcal{F}x$ and $Y = \mathcal{F}y$. Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, we have (from the given frequency response H)

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{e^{j2\Omega} - e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}}.$$

Multiplying both sides of the equation by $e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}$ and $X(\Omega)$, we have

$$e^{j2\Omega}Y(\Omega) - e^{j\Omega}Y(\Omega) + \tfrac{1}{4}Y(\Omega) = e^{j2\Omega}X(\Omega) - e^{j\Omega}X(\Omega).$$

Multiplying both sides by $e^{-j2\Omega}$ (so that the largest power of $e^{j\Omega}$ is zero), we obtain

$$Y(\Omega) - e^{-j\Omega}Y(\Omega) + \tfrac{1}{4}e^{-j2\Omega}Y(\Omega) = X(\Omega) - e^{-j\Omega}X(\Omega).$$

Taking the inverse Fourier transform of this equation, we obtain

$$y(n) - y(n-1) + \tfrac{1}{4}y(n-2) = x(n) - x(n-1).$$ ■

11.16 Filtering

In some applications, we want to change the magnitude or phase of the frequency components of a sequence or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a sequence is referred to as **filtering** and the system that performs such processing is called a **filter**.

For the sake of simplicity, we consider only LTI filters here. If a filter is LTI, then it is completely characterized by its frequency response. Since the frequency spectra of sequences are 2π-periodic, we only need to consider frequencies over an interval of length 2π. Normally, we choose this interval to be centered about the origin, namely, $(-\pi, \pi]$. When we consider this interval, low frequencies are those closer to the origin, while high frequencies are those closer to $\pm\pi$.

Many types of filters exist. One important class of filters are those that are frequency selective. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [0, \Omega_c] \\ 0 & |\Omega| \in (\Omega_c, \pi], \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 11.16(a).

An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [\Omega_c, \pi] \\ 0 & |\Omega| \in [0, \Omega_c), \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 11.16(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie between two cutoff frequencies, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [\Omega_{c1}, \Omega_{c2}] \\ 0 & |\Omega| \in [0, \Omega_{c1}) \cup (\Omega_{c2}, \pi], \end{cases}$$

where Ω_{c1} and Ω_{c2} are the cutoff frequencies. A plot of this frequency response is given in Figure 11.16(c).

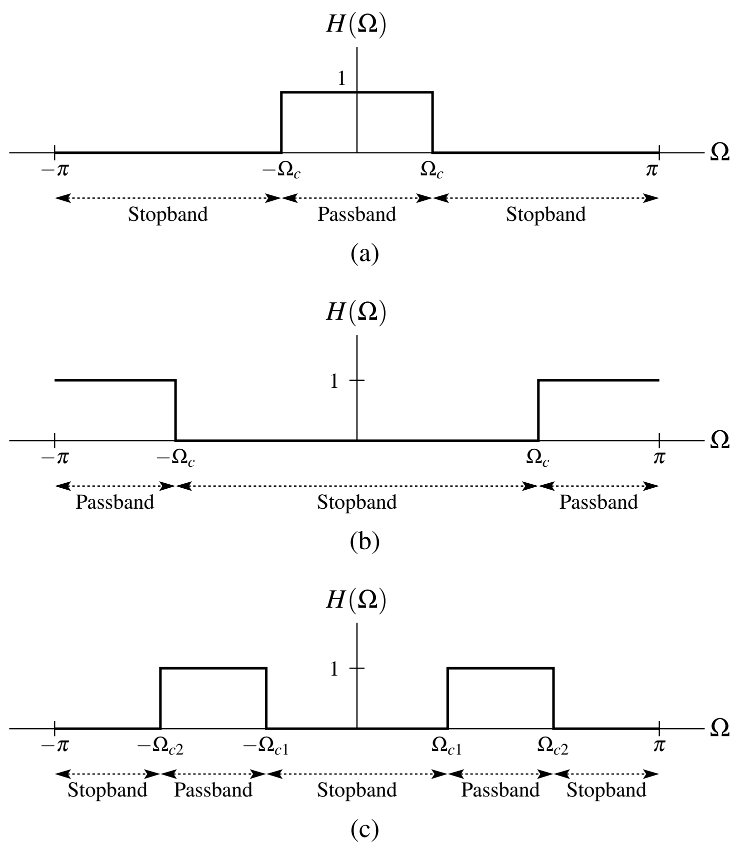

Figure 11.16: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 11.37 (Ideal filters)**.** For each LTI system whose impulse response h is given below, find and plot the frequency response H of the system, and identify the type of frequency-selective filter to which the system corresponds.
(a) $h(n) = \frac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n)$, where Ω_c is a real constant in the interval $(0, \pi)$;
(b) $h(n) = \delta(n) - \frac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n)$, where Ω_c is a real constant in the interval $(0, \pi)$; and
(c) $h(n) = \frac{2\Omega_b}{\pi} \operatorname{sinc}(\Omega_b n) \cos(\Omega_a n)$, where Ω_a and Ω_b are real constants in the interval $(0, \pi)$.

Solution. In what follows, let us denote the input and output of the system as x and y, respectively. Also, let X and Y denote the Fourier transforms of x and y, respectively.

(a) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{ \tfrac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n) \right\}(\Omega) \\
&= \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2\Omega_c}(\Omega - 2\pi k) \right] \\
&= \operatorname{rect}\left(\tfrac{1}{2\Omega_c}\Omega \right) \quad \text{for } |\Omega| \in (-\pi, \pi] \\
&= \begin{cases} 1 & |\Omega| \in [0, \Omega_c] \\ 0 & |\Omega| \in (\Omega_c, \pi]. \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 11.17(a). Since $Y(\Omega) = H(\Omega)X(\Omega)$ and $H(\Omega) = 0$ for $|\Omega| \in (\Omega_c, \pi]$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega| \in [0, \Omega_c]$. In other words, only the lower frequency components from X are kept. Thus, the system corresponds to a lowpass filter.

(b) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{ \delta(n) - \tfrac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n) \right\}(\Omega) \\
&= \mathcal{F}\delta(\Omega) - \mathcal{F}\left\{ \tfrac{\Omega_c}{\pi} \operatorname{sinc}(\Omega_c n) \right\}(\Omega) \\
&= 1 - \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2\Omega_c}(\Omega - 2\pi k) \right] \\
&= 1 - \operatorname{rect}\left(\tfrac{1}{2\Omega_c}\Omega \right) \quad \text{for } \Omega \in (-\pi, \pi] \\
&= \begin{cases} 1 & |\Omega| \in [\Omega_c, \pi] \\ 0 & |\Omega| \in [0, \Omega_c). \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 11.17(b). Since $Y(\Omega) = H(\Omega)X(\Omega)$ and $H(\Omega) = 0$ for $|\Omega| \in [0, \Omega_c)$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega| \in [\Omega_c, \pi]$. In other words, only the higher frequency components from X are kept. Thus, the system corresponds to a highpass filter.

(c) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we

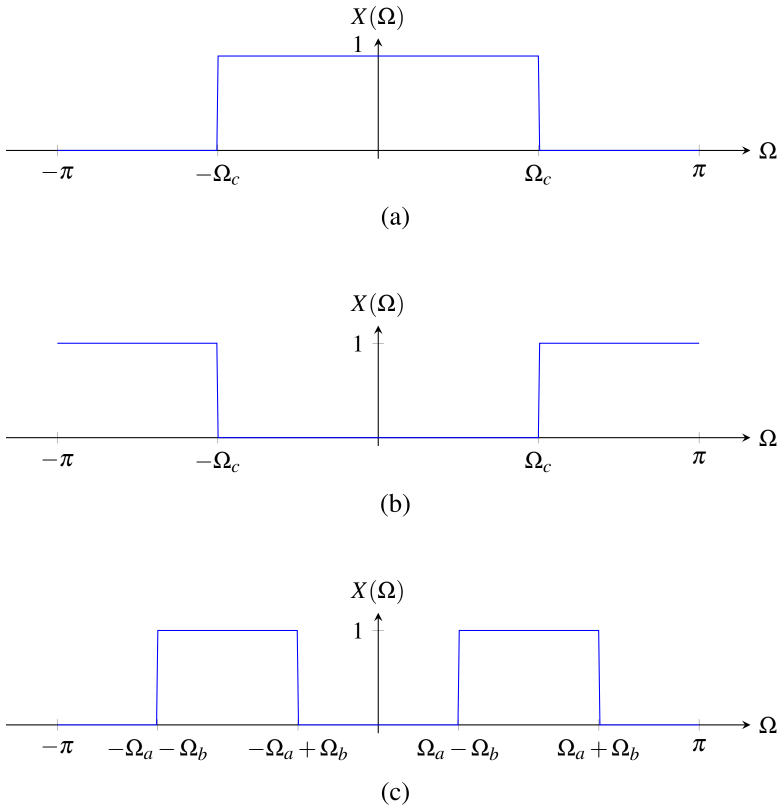

Figure 11.17: Frequency responses of each of the (a) first, (b) second, and (c) third systems from the example.

have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{2\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)\cos(\Omega_a n)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)[2\cos(\Omega_a n)]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)\left[e^{j\Omega_a n}+e^{-j\Omega_a n}\right]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}e^{j\Omega_a n}\operatorname{sinc}(\Omega_b n)\right\}(\Omega) + \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}e^{-j\Omega_a n}\operatorname{sinc}(\Omega_b n)\right\}(\Omega) \\
&= \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_b}([\Omega-\Omega_a]-2\pi k)\right] + \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_b}([\Omega+\Omega_a]-2\pi k)\right] \\
&= \operatorname{rect}\left[\tfrac{1}{2\Omega_b}(\Omega-\Omega_a)\right] + \operatorname{rect}\left[\tfrac{1}{2\Omega_b}(\Omega+\Omega_a)\right] \quad \text{for } \Omega \in (-\pi,\pi] \\
&= \begin{cases} 1 & |\Omega| \in [\Omega_a-\Omega_b, \Omega_a+\Omega_b] \\ 0 & |\Omega| \in [0,\Omega_a-\Omega_b) \cup (\Omega_a+\Omega_b,\pi]. \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 11.17(c). Since $Y(\Omega) = H(\Omega)X(\Omega)$ and $H(\Omega) = 0$ for $|\Omega| \in [0,\Omega_a-\Omega_b) \cup (\Omega_a+\Omega_b,\pi]$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega| \in [\Omega_a-\Omega_b, \Omega_a+\Omega_b]$. In other words, only the middle frequency components of X are kept. Thus, the system corresponds to a bandpass filter. ■

Example 11.38 (Lowpass filtering)**.** Consider a LTI system with impulse response

$$h(n) = \tfrac{1}{3}\operatorname{sinc}\left(\tfrac{\pi}{3}n\right).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(n) = \tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{6}\cos\left(\pi n\right).$$

Solution. To begin, we find the Fourier transform X of x. Computing $X(\Omega)$ for $\Omega \in (-\pi,\pi]$, we have

$$\begin{aligned}
X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{6}\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{2}\mathcal{F}\{1\}(\Omega) + \tfrac{2}{3}\mathcal{F}\left\{\cos(\tfrac{\pi}{4}n)\right\}(\Omega) + \tfrac{1}{2}\mathcal{F}\left\{\cos(\tfrac{3\pi}{4}n)\right\}(\Omega) + \tfrac{1}{6}\mathcal{F}\left\{\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{2}[2\pi\delta(\Omega)] + \tfrac{2}{3}\left(\pi\left[\delta\left(\Omega+\tfrac{\pi}{4}\right)+\delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right) + \tfrac{1}{2}\left(\pi\left[\delta\left(\Omega+\tfrac{3\pi}{4}\right)+\delta\left(\Omega-\tfrac{3\pi}{4}\right)\right]\right) + \tfrac{1}{6}[2\pi\delta(\Omega-\pi)] \\
&= \pi\delta(\Omega) + \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega+\tfrac{3\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega-\tfrac{3\pi}{4}\right) + \tfrac{\pi}{3}\delta\left(\Omega-\pi\right) \\
&= \tfrac{\pi}{2}\delta\left(\Omega+\tfrac{3\pi}{4}\right) + \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega-\tfrac{3\pi}{4}\right) + \tfrac{\pi}{3}\delta\left(\Omega-\pi\right).
\end{aligned}$$

A plot of the frequency spectrum X is shown in Figure 11.18(a). Now, we compute the Fourier transform H of h. For $\Omega \in (-\pi,\pi]$, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{1}{3}\operatorname{sinc}\left(\tfrac{\pi}{3}n\right)\right\} \\
&= \operatorname{rect}\left(\tfrac{3}{2\pi}\Omega\right) \\
&= \begin{cases} 1 & |\Omega| \in [0,\tfrac{\pi}{3}] \\ 0 & |\Omega| \in (\tfrac{\pi}{3},\pi]. \end{cases}
\end{aligned}$$

The frequency response H is shown in Figure 11.18(b). The frequency spectrum Y of the output can be computed as

$$\begin{aligned}
Y(\Omega) &= H(\Omega)X(\Omega) \\
&= \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right).
\end{aligned}$$

The frequency spectrum Y is shown in Figure 11.18(c). Taking the inverse Fourier transform of Y yields

$$\begin{aligned}
y(n) &= \mathcal{F}^{-1}\left\{\pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\left[\delta\left(\Omega+\tfrac{\pi}{4}\right)+\delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right\}(n) \\
&= \tfrac{1}{2}\mathcal{F}^{-1}\left\{2\pi\delta\left(\Omega\right)\right\}(n) + \tfrac{2}{3}\mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega+\tfrac{\pi}{4}\right)+\delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right\}(n) \\
&= \tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right).
\end{aligned}$$

■

Example 11.39 (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(n) = \tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\cos(\tfrac{6}{5}n).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(n) = \tfrac{1}{4} + \cos\left(\tfrac{4}{5}n\right) + \tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right) + \tfrac{1}{4}\cos\left(\tfrac{12}{5}n\right) + \tfrac{1}{6}\cos(\pi n).$$

In passing, we note that x is not periodic.

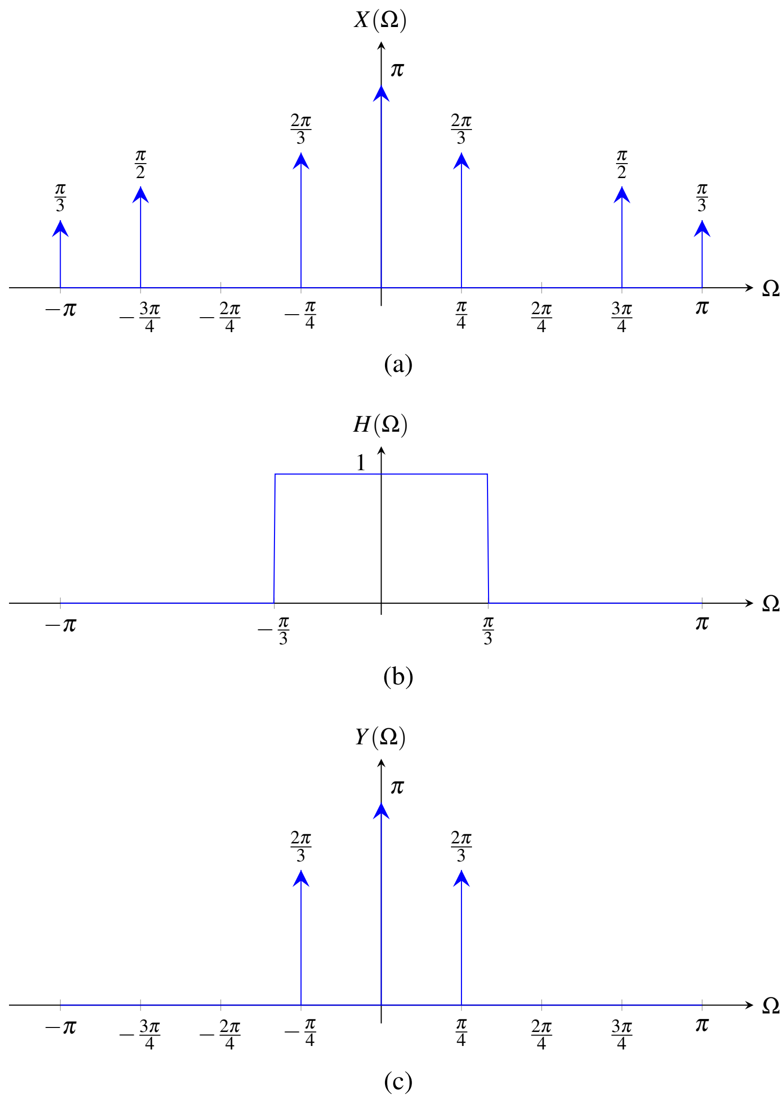

Figure 11.18: Frequency spectra for the lowpass filtering example. (a) Frequency spectrum X of the input x. (b) Frequency response H of the system. (c) Frequency spectrum Y of the output y.

Solution. Taking the Fourier transform of x, we have

$$\begin{aligned}
X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{4} + \cos\left(\tfrac{4}{5}n\right) + \tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right) + \tfrac{1}{4}\cos\left(\tfrac{12}{5}n\right) + \tfrac{1}{6}\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{4}\mathcal{F}\{1\}(\Omega) + \mathcal{F}\left\{\cos\left(\tfrac{4}{5}n\right)\right\}(\Omega) + \tfrac{2}{3}\mathcal{F}\left\{\cos\left(\tfrac{8}{5}n\right)\right\}(\Omega) + \tfrac{1}{4}\mathcal{F}\left\{\cos\left(\tfrac{12}{5}n\right)\right\}(\Omega) \\
&\quad + \tfrac{1}{6}\mathcal{F}\{\cos(\pi n)\}(\Omega) \\
&= \tfrac{1}{4}[2\pi\delta(\Omega)] + \left(\pi\left[\delta\left(\Omega - \tfrac{4}{5}\right) + \delta\left(\Omega + \tfrac{4}{5}\right)\right]\right) + \tfrac{2}{3}\left(\pi\left[\delta\left(\Omega - \tfrac{8}{5}\right) + \delta\left(\Omega + \tfrac{8}{5}\right)\right]\right) \\
&\quad + \tfrac{1}{4}\left(\pi\left[\delta\left(\Omega - \tfrac{12}{5}\right) + \delta\left(\Omega + \tfrac{12}{5}\right)\right]\right) + \tfrac{1}{6}[2\pi\delta(\Omega - \pi)] \\
&= \tfrac{\pi}{2}\delta(\Omega) + \pi\left[\delta\left(\Omega - \tfrac{4}{5}\right) + \delta\left(\Omega + \tfrac{4}{5}\right)\right] + \tfrac{2\pi}{3}\left[\delta\left(\Omega - \tfrac{8}{5}\right) + \delta\left(\Omega + \tfrac{8}{5}\right)\right] \\
&\quad + \tfrac{\pi}{4}\left(\left[\delta\left(\Omega - \tfrac{12}{5}\right) + \delta\left(\Omega + \tfrac{12}{5}\right)\right]\right) + \tfrac{\pi}{3}[\delta(\Omega - \pi)] \\
&= \tfrac{\pi}{2}\delta(\Omega) + \pi\delta\left(\Omega + \tfrac{4}{5}\right) + \pi\delta\left(\Omega - \tfrac{4}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega + \tfrac{8}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega - \tfrac{8}{5}\right) + \tfrac{\pi}{4}\delta\left(\Omega + \tfrac{12}{5}\right) \\
&\quad + \tfrac{\pi}{4}\delta\left(\Omega - \tfrac{12}{5}\right) + \tfrac{\pi}{3}\delta(\Omega - \pi) \\
&= \tfrac{\pi}{4}\delta\left(\Omega + \tfrac{12}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega + \tfrac{8}{5}\right) + \pi\delta\left(\Omega + \tfrac{4}{5}\right) + \tfrac{\pi}{2}\delta(\Omega) + \pi\delta\left(\Omega - \tfrac{4}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega - \tfrac{8}{5}\right) \\
&\quad + \tfrac{\pi}{4}\delta\left(\Omega - \tfrac{12}{5}\right) + \tfrac{\pi}{3}\delta(\Omega - \pi).
\end{aligned}$$

The frequency spectrum X is shown in Figure 11.19(a). Now, we compute the frequency response H of the system. We have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\cos(\tfrac{6}{5}n)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\left[\tfrac{1}{2}\left(e^{j(6/5)n} + e^{-j(6/5)n}\right)\right]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}e^{j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right) + \tfrac{4}{5\pi}e^{-j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}e^{j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega) + \mathcal{F}\left\{\tfrac{4}{5\pi}e^{-j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}\left(\Omega - \tfrac{6}{5}\right) + \mathcal{F}\left\{\tfrac{4}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}\left(\Omega + \tfrac{6}{5}\right) \\
&= \operatorname{rect}\left[\tfrac{5}{8}(\Omega - \tfrac{6}{5})\right] + \operatorname{rect}\left[\tfrac{5}{8}(\Omega + \tfrac{6}{5})\right] \\
&= \begin{cases} 1 & |\Omega| \in \left[\tfrac{2}{5}, \tfrac{10}{5}\right] \\ 0 & |\Omega| \in \left[0, \tfrac{2}{5}\right) \cup \left(\tfrac{10}{5}, \pi\right]. \end{cases}
\end{aligned}$$

The frequency response H is shown in Figure 11.19(b). The frequency spectrum Y of the output is given by

$$\begin{aligned}
Y(\Omega) &= H(\Omega)X(\Omega) \\
&= \tfrac{2\pi}{3}\delta\left(\Omega + \tfrac{8}{5}\right) + \pi\delta\left(\Omega + \tfrac{4}{5}\right) + \pi\delta\left(\Omega - \tfrac{4}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega - \tfrac{8}{5}\right).
\end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
y(n) &= \mathcal{F}^{-1}\left\{\tfrac{2\pi}{3}\delta\left(\Omega + \tfrac{8}{5}\right) + \pi\delta\left(\Omega + \tfrac{4}{5}\right) + \pi\delta\left(\Omega - \tfrac{4}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega - \tfrac{8}{5}\right)\right\}(n) \\
&= \mathcal{F}^{-1}\left\{\tfrac{2\pi}{3}\delta\left(\Omega + \tfrac{8}{5}\right) + \tfrac{2\pi}{3}\delta\left(\Omega - \tfrac{8}{5}\right)\right\}(n) + \mathcal{F}^{-1}\left\{\pi\delta\left(\Omega + \tfrac{4}{5}\right) + \pi\delta\left(\Omega - \tfrac{4}{5}\right)\right\}(n) \\
&= \tfrac{2}{3}\mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega + \tfrac{8}{5}\right) + \delta\left(\Omega - \tfrac{8}{5}\right)\right]\right\}(n) + \mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega + \tfrac{4}{5}\right) + \delta\left(\Omega - \tfrac{4}{5}\right)\right]\right\}(n) \\
&= \cos\left(\tfrac{4}{5}n\right) + \tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right).
\end{aligned}$$

■

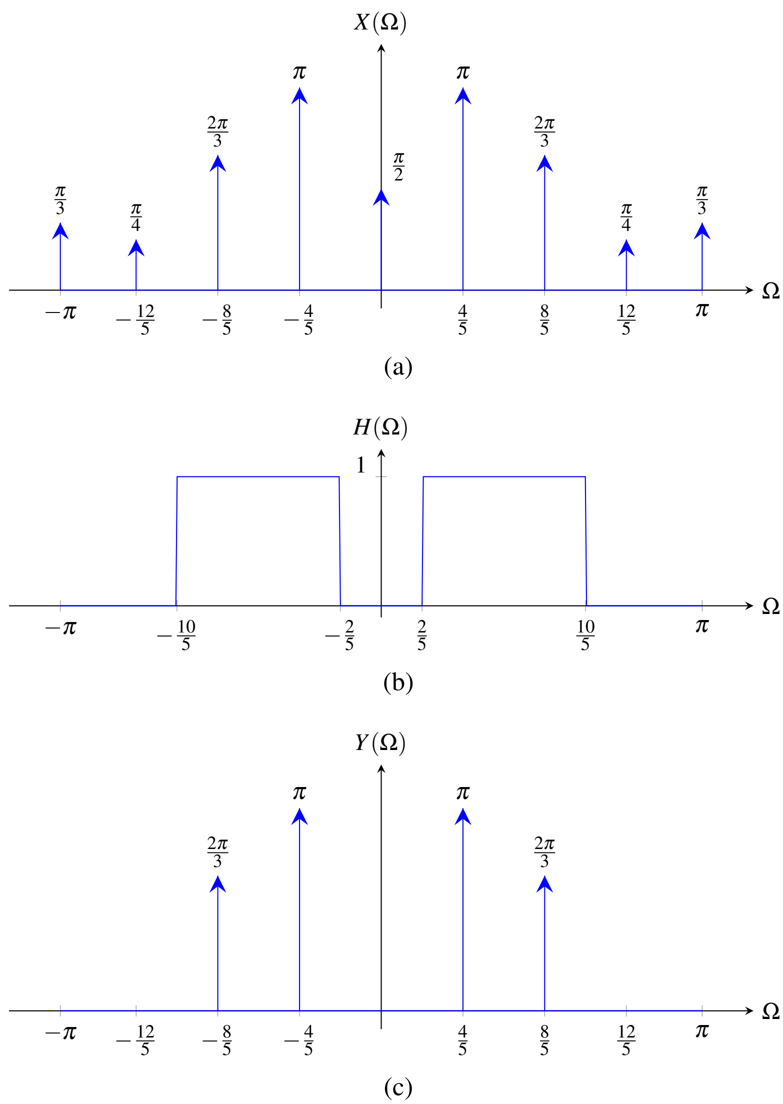

Figure 11.19: Frequency spectra for the bandpass filtering example. (a) Frequency spectrum X of the input x. (b) Frequency response H of the system. (c) Frequency spectrum Y of the output y.

11.17 Relationship Between DT Fourier Transform and CT Fourier Series

A duality relationship exists between the DT Fourier transform and CT Fourier series. Recall that the DT Fourier transform analysis and synthesis equations are, respectively, given by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x(k)e^{-jk\Omega} \quad \text{and} \tag{11.20a}$$

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{jn\Omega}d\Omega. \tag{11.20b}$$

Also, recall that the CT Fourier series synthesis and analysis equations are, respectively, given by

$$y(t) = \sum_{k=-\infty}^{\infty} Y(k)e^{jk(2\pi/T)t} \quad \text{and}$$

$$Y(n) = \tfrac{1}{T}\int_{T} y(t)e^{-jn(2\pi/T)t}dt,$$

which can be rewritten, respectively, as

$$y(t) = \sum_{k=-\infty}^{\infty} Y(-k)e^{-jk(2\pi/T)t} \quad \text{and} \tag{11.21a}$$

$$Y(-n) = \tfrac{1}{T}\int_{T} y(t)e^{jn(2\pi/T)t}dt. \tag{11.21b}$$

Observe that, if $T = 2\pi$, the pair of equations in (11.20) is essentially the same as the pair of equations in (11.21). In particular, (11.21a) with $T = 2\pi$ is identical to (11.20a) with $X = y$, $\Omega = t$, and $x(n) = Y(-n)$. Furthermore, (11.21b) with $T = 2\pi$ is identical to (11.20b) with $X = y$, $\Omega = t$, and $x(n) = Y(-n)$. Consequently, the DT Fourier transform X of the sequence x can be viewed as a CT Fourier-series representation of the 2π-periodic spectrum X. We can formalize this result in the following theorem.

Theorem 11.20. *Let X be a 2π-periodic function and let x be a sequence. Then, the following are equivalent:*

1. $X \overset{\text{CTFS}}{\longleftrightarrow} x$
2. $\mathcal{R}x \overset{\text{DTFT}}{\longleftrightarrow} X$*, where $\mathcal{R}$ denotes time reversal (i.e., $\mathcal{R}x(n) = x(-n)$).*

Proof. The result of this theorem follows immediately from the definition of the analysis and synthesis equations for the DT Fourier transform and CT Fourier series (as explained above). ■

The duality relationship in the preceding theorem can be quite helpful in some situations. We consider one such situation below. In particular, we use this relationship to assist in the computation of a DT Fourier transform.

Example 11.40 (Fourier transform of sinc sequence)**.** Let B denote a real constant in the interval $(0, \pi)$.
(a) Show that the 2π-periodic function

$$y(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(t - 2\pi k)\right],$$

has the Fourier series coefficient sequence

$$Y(k) = \tfrac{B}{\pi}\operatorname{sinc}(Bk).$$

(b) Use the result of part (a) to find the DT Fourier transform X of the sequence

$$x(n) = \tfrac{B}{\pi}\operatorname{sinc}(Bn).$$

Solution. (a) From the Fourier series analysis equation, we have

$$\begin{aligned}
Y(k) &= \tfrac{1}{T}\int_T y(t)e^{-j(2\pi/T)kt}\,dt \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} y(t)e^{-jkt}\,dt \\
&= \tfrac{1}{2\pi}\int_{-B}^{B} e^{-jkt}\,dt \\
&= \begin{cases} \tfrac{1}{2\pi}\left[\tfrac{1}{-jk}e^{-jkt}\right]\Big|_{-B}^{B} & k \neq 0 \\ \tfrac{1}{2\pi}(2B) & k = 0. \end{cases}
\end{aligned}$$

Now, we simplify the preceding formula for Y for each of the cases $k \neq 0$ and $k = 0$. First, we consider $k \neq 0$. We have

$$\begin{aligned}
Y(k) &= \tfrac{1}{2\pi}\left[\frac{1}{-jk}e^{-jkt}\right]\Big|_{-B}^{B} \\
&= \tfrac{j}{2\pi k}e^{-jkt}\Big|_{-B}^{B} \\
&= \tfrac{j}{2\pi k}\left[e^{-jBk} - e^{jBk}\right] \\
&= \tfrac{j}{2\pi k}\left[2j\sin(-Bk)\right] \\
&= -\tfrac{1}{\pi k}\sin(-Bk) \\
&= \tfrac{1}{\pi k}\sin(Bk) \\
&= \tfrac{B}{\pi}\left[\frac{\sin(Bk)}{Bk}\right] \\
&= \tfrac{B}{\pi}\operatorname{sinc}(Bk).
\end{aligned}$$

Next, we consider $k = 0$. We have

$$\begin{aligned}
Y(k) &= \tfrac{1}{2\pi}(2B) \\
&= \tfrac{B}{\pi}.
\end{aligned}$$

Since, by chance, the formula for $Y(k)$ for $k \neq 0$ yields the correct result for $k = 0$, we can use this formula for both cases. Thus, we conclude

$$y(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(t - 2\pi k)\right] \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad Y(k) = \tfrac{B}{\pi}\operatorname{sinc}(Bk).$$

(b) Since $x = Y$, X is the Fourier transform of Y. So, in effect, we are being asked to find the Fourier transform of Y. Since $y \overset{\text{CTFS}}{\longleftrightarrow} Y$, we have by duality (i.e., Theorem 11.20) that $\mathcal{R}Y \overset{\text{DTFT}}{\longleftrightarrow} y$, where $\mathcal{R}$ denotes time reversal (i.e., $\mathcal{R}x(n) = x(-n)$). Moreover, since Y is even, $\mathcal{R}Y = Y$, which implies $Y \overset{\text{DTFT}}{\longleftrightarrow} y$, or equivalently, $x \overset{\text{DTFT}}{\longleftrightarrow} y$. Thus, $X = y$. Therefore, we conclude

$$\begin{aligned}
X(\Omega) &= y(\Omega) \\
&= \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(\Omega - 2\pi k)\right].
\end{aligned}$$

■

