Example 3.33. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right]. \quad (1)$$

(2)

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(t-t_0) &= \tfrac{1}{2}[x(t-t_0) - x(-(t-t_0))] && \text{by substituting } t-t_0 \text{ for } t \text{ in (1)} \\ &= \tfrac{1}{2}[x(t-t_0) - x(-t+t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] && \text{from definition of } \mathcal{H} \text{ in (1)} \\ &= \tfrac{1}{2}[x(t-t_0) - x(-t-t_0)]. && \text{from definition of } x' \text{ in (2)} \end{aligned}$$

not equal if $t_0 \neq 0$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant. ■

only equal if $t_0 = 0$

A system $\mathcal{H}$ is said to be time invariant if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t, \text{ where } x'(t) = x(t-t_0).$$

Example 3.35. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = tx(t). \quad ①$$

②

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

equal for als x_1, x_2, a_1, a_2

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = a_1tx_1(t) + a_2tx_2(t) \quad \text{and}$$

from definition of $\mathcal{H}$ in ①

$$\begin{aligned}\mathcal{H}x'(t) &= tx'(t) \\ &= t\left[a_1x_1(t) + a_2x_2(t)\right] \\ &= a_1tx_1(t) + a_2tx_2(t).\end{aligned}$$

from definition of $\mathcal{H}$ in ①

from definition of x' in ②

Since $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1x_1 + a_2x_2\} = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

Example 3.36. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|. \quad (1)$$

(2)

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = a_1|x_1(t)| + a_2|x_2(t)| \quad \text{and}$$

from definition of $\mathcal{H}$ in (1)

$$\mathcal{H}x'(t) = |x'(t)|$$

from definition of $\mathcal{H}$ in (1)

$$= |a_1x_1(t) + a_2x_2(t)|.$$

from definition of x' in (2)

At this point, we recall the triangle inequality (i.e., for $a, b \in \mathbb{C}$, $|a+b| \leq |a| + |b|$). Thus, $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Counterexample

Therefore, the superposition property does not hold and the system is not linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1x_1 + a_2x_2\} = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2.$$

Example 3.41. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t), \quad (1)$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos 2t$; and
(b) $x(t) = t^3$.

Solution. (a) We have

from definition of $\mathcal{H}$ in (1)

$$\begin{aligned} \mathcal{H}x(t) &= \mathcal{D}^2\{\cos 2t\}(t) \\ &= \mathcal{D}\{-2\sin 2t\}(t) \\ &= -4\cos 2t \\ &= -4x(t). \end{aligned}$$

$\frac{d}{dt} \cos t = -\sin t$

$\frac{d}{dt} \sin t = \cos t$

from definition of x

So, we have $\mathcal{H}x = -4x$.

Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) We have

from definition of $\mathcal{H}$ in (1)

$$\begin{aligned} \mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \tfrac{6}{t^2}x(t). \end{aligned}$$

$\frac{d}{dt} t^3 = 3t^2$

$\frac{d}{dt} 3t^2 = 6t$

from definition of x

$\left(\frac{6t\, x(t)}{x(t)} = \frac{6t\, x(t)}{t^3}\right)$

not a constant

Therefore, x is not an eigenfunction of $\mathcal{H}$. ■

A function x is said to be an eigenfunction of the system $\mathcal{H}$ with eigenvalue λ if

$$\mathcal{H}x = \lambda x.$$