6.8 Continuous-Time Fourier Transform of Periodic Functions

By making use of the generalized Fourier transform briefly discussed in Section 6.3, the Fourier transform can also be applied to periodic functions. In particular, the Fourier transform of a periodic function can be computed using the result below.

Theorem 6.18 (Fourier transform of a periodic function)**.** *Let x be a T-periodic function with frequency $\omega_0 = \frac{2\pi}{T}$ and Fourier series coefficient sequence a. Let x_T denote the function*

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x_T is a truncated/windowed version of the function x). (Note that x_T is a function equal to x over a single period and zero elsewhere.) Let X_T denote the Fourier transform of x_T. The Fourier transform X of x is given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0), \tag{6.16a}$$

or equivalently,

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0). \tag{6.16b}$$

Furthermore, a and X_T are related by

$$a_k = \tfrac{1}{T} X_T(k\omega_0). \tag{6.17}$$

Proof. Since x is T-periodic, we can express it using a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{6.18a}$$

where

$$a_k = \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt. \tag{6.18b}$$

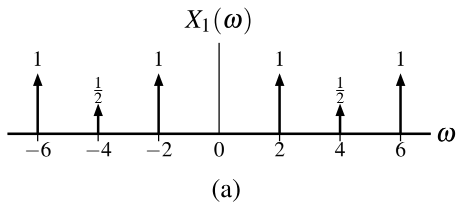
Consider the expression for a_k in (6.18b). Since $x_T(t) = x(t)$ for a single period of x and is zero otherwise, we can rewrite (6.18b) as

$$\begin{aligned} a_k &= \tfrac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} X_T(k\omega_0). \end{aligned} \tag{6.19}$$

Thus, we have shown (6.17) to be correct.

Now, let us consider the Fourier transform X of x. By taking the Fourier transform of both sides of (6.18a), we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\}(\omega) \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-j\omega t} dt. \end{aligned}$$



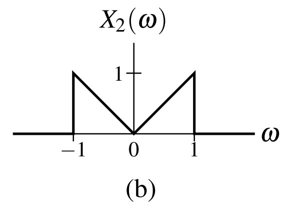


Figure 6.6: Frequency spectra. The frequency spectra (a) X_1 and (b) X_2.

Reversing the order of summation and integration, we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\}(\omega). \end{aligned} \tag{6.20}$$

From Table 6.2, we know that $e^{j\lambda t} \stackrel{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega - \lambda)$. So, we can simplify (6.20) to obtain

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k [2\pi\delta(\omega - k\omega_0)] \\ &= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0). \end{aligned} \tag{6.21}$$

Thus, we have shown (6.16a) to be correct. Furthermore, by substituting (6.19) into (6.21), we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} 2\pi[\tfrac{1}{T} X_T(k\omega_0)]\delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0). \end{aligned}$$

Thus, we have shown (6.16b) to be correct. This completes the proof. ■

Theorem 6.18 above provides two formulas for computing the Fourier transform X of a periodic function x. One formula is in written terms of the Fourier series coefficient sequence a of x, while the other formula is in written in terms of the Fourier transform X_T of a function consisting of a single period of x. The choice of which formula to use would be driven by what information is available or most easily determined. For example, if the Fourier series coefficients of x were known, the use of (6.16b) would likely be preferred.

From Theorem 6.18, we can also make a few important observations. First, the Fourier transform of a periodic function is a series of impulse functions located at integer multiples of the fundamental frequency ω_0. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Second, the Fourier series coefficient sequence a of the periodic function x is produced by sampling the Fourier transform of x_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.

Example 6.20. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. Suppose that X_1 and X_2 are as shown in Figures 6.6(a) and (b). Determine whether x_1 and x_2 are periodic.

Solution. We know that the Fourier transform X of a T-periodic function x must be of the form

$$X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0),$$

where $\omega_0 = \frac{2\pi}{T}$ and the $\{\alpha_k\}$ are complex constants. The spectrum X_1 does have this form, with $\omega_0 = 2$ and $T = \frac{2\pi}{2} = \pi$. Therefore, x_1 must be π-periodic. The spectrum X_2 does not have this form. Therefore, x_2 must not be periodic. ■

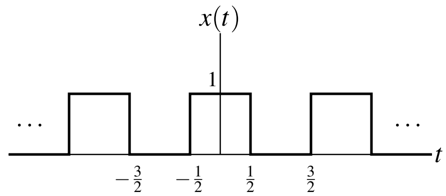


Figure 6.7: Periodic function x.

Example 6.21. Consider the periodic function x with fundamental period $T = 2$ as shown in Figure 6.7. Using the Fourier transform, find the Fourier series representation of x.

Solution. Let ω_0 denote the fundamental frequency of x. We have that $\omega_0 = \frac{2\pi}{T} = \pi$. Let $y(t) = \operatorname{rect} t$ (i.e., y corresponds to a single period of the periodic function x). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t-2k).$$

Let Y denote the Fourier transform of y. Taking the Fourier transform of y, we obtain

$$Y(\omega) = \mathcal{F}\{\operatorname{rect} t\}(\omega) = \operatorname{sinc}\left(\tfrac{1}{2}\omega\right).$$

Now, we seek to find the Fourier series representation of x, which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} Y(k\omega_0) \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\omega_0}{2}k\right) \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\pi}{2}k\right). \end{aligned}$$

■

6.9 More Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 6.2. Using the various Fourier transform properties listed in Table 6.1 and the Fourier transform pairs listed in Table 6.2, we can determine (more easily) the Fourier transform of more complicated functions.

Example 6.22. Suppose that $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, $y(t) \overset{\text{CTFT}}{\longleftrightarrow} Y(\omega)$, and

$$y(t) = \mathcal{D}^2 x(t-2),$$

where $\mathcal{D}$ denotes the derivative operator. Express Y in terms of X.

Solution. Let $v_1(t) = \mathcal{D}^2 x(t)$ so that $y(t) = v_1(t-2)$. Now, we take the Fourier transforms of each of these two equations. Taking the Fourier transform of v_1 using the time-domain differentiation property of the Fourier transform, we obtain

$$\begin{aligned} V_1(\omega) &= (j\omega)^2 X(\omega) \\ &= -\omega^2 X(\omega). \end{aligned} \tag{6.22}$$

Table 6.2: Transform pairs for the CT Fourier transform

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn} t$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}\left(\frac{t}{T}\right)$	$\lvert T\rvert \operatorname{sinc}\left(\frac{T\omega}{2}\right)$
9	$\operatorname{sinc}(Bt)$	$\frac{\pi}{\lvert B\rvert}\operatorname{rect}\left(\frac{\omega}{2B}\right)$
10	$e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}\left(\frac{t}{T}\right)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2\left(\frac{T\omega}{4}\right)$

Taking the Fourier transform of y using the time-shifting property of the Fourier transform, we have

$$Y(\omega) = e^{-j2\omega}V_1(\omega). \tag{6.23}$$

Substituting (6.22) into (6.23), we obtain

$$\begin{aligned} Y(\omega) &= e^{-j2\omega}[-\omega^2 X(\omega)] \\ &= -e^{-j2\omega}\omega^2 X(\omega). \end{aligned}$$ ■

Example 6.23. Suppose that $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, $y(t) \xleftrightarrow{\text{CTFT}} Y(\omega)$, and

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$. Express Y in terms of X.

Solution. We rewrite y as

$$y(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

We now take the Fourier transform of both sides of each of the preceding equations. Using the time-shifting property of the Fourier transform, we can write

$$V_1(\omega) = e^{-jb\omega}X(\omega). \tag{6.24}$$

Using the time-scaling property of the Fourier transform, we can write

$$Y(\omega) = \tfrac{1}{|a|}V_1\left(\tfrac{\omega}{a}\right). \tag{6.25}$$

Substituting the expression for $V_1(\omega)$ in (6.24) into (6.25), we obtain

$$Y(\omega) = \tfrac{1}{|a|}e^{-j(b/a)\omega}X\left(\tfrac{\omega}{a}\right).$$ ■

Example 6.24. Consider the periodic function x given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT),$$

where

$$x_0(t) = A\operatorname{rect}\left(\tfrac{2t}{T}\right)$$

and A and T are real constants with $T > 0$. Find the Fourier transform X of the function x.

Solution. From (6.16b), we know that

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t-kT)\right\}(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\delta(\omega-k\omega_0). \end{aligned}$$

So, we need to find X_0. Using the linearity property of the Fourier transform and Table 6.2, we have

$$\begin{aligned} X_0(\omega) &= \mathcal{F}\left\{A \operatorname{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= A\mathcal{F}\left\{\operatorname{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= \tfrac{AT}{2} \operatorname{sinc}\left(\tfrac{\omega T}{4}\right). \end{aligned}$$

Thus, we have that

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \left(\tfrac{AT}{2}\right) \operatorname{sinc}\left(\tfrac{k\omega_0 T}{4}\right) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A \operatorname{sinc}\left(\tfrac{\pi k}{2}\right) \delta(\omega - k\omega_0). \end{aligned}$$ ■

Example 6.25. Find the Fourier transform X of the function x given by

$$x(t) = \int_{-\infty}^{t} e^{-(3+j2)\tau} u(\tau) d\tau.$$

Solution. We can rewrite x as

$$x(t) = \int_{-\infty}^{t} v_1(\tau) d\tau, \tag{6.26}$$

where

$$v_1(t) = e^{-j2t} v_2(t) \quad \text{and} \tag{6.27}$$
$$v_2(t) = e^{-3t} u(t). \tag{6.28}$$

Taking the Fourier transform of (6.26) (using the time-domain integration property of the Fourier transform), we have

$$X(\omega) = \tfrac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega). \tag{6.29}$$

Taking the Fourier transform of (6.27) (using the frequency-domain shifting property of the Fourier transform), we have

$$V_1(\omega) = V_2(\omega + 2). \tag{6.30}$$

Taking the Fourier transform of (6.28) by using Table 6.2 (i.e., the entry for $\mathcal{F}\{e^{-at}u(t)\}$), we have

$$V_2(\omega) = \frac{1}{3 + j\omega}. \tag{6.31}$$

Combining (6.29), (6.30), and (6.31), we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega) \\ &= \tfrac{1}{j\omega} V_2(\omega + 2) + \pi V_2(2) \delta(\omega) \\ &= \frac{1}{j\omega} \left(\frac{1}{3 + j(\omega + 2)}\right) + \pi \left(\frac{1}{3 + j2}\right) \delta(\omega) \\ &= \frac{-1}{\omega(\omega + 2 - j3)} + \pi \left(\frac{1}{3 + j2}\right) \delta(\omega). \end{aligned}$$ ■

Example 6.26. Let X and Y denote the Fourier transforms of x and y, respectively. Suppose that $y(t) = x(t)\cos(at)$, where a is a nonzero real constant. Find an expression for Y in terms of X.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. To allow for simpler notation in what follows, we define

$$v(t) = \cos(at)$$

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)].$$

Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{x(t)v(t)\}(\omega) \\
&= \tfrac{1}{2\pi} X * V(\omega) \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)V(\omega - \lambda)d\lambda.
\end{aligned}$$

Substituting the above expression for V, we obtain

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega - \lambda - a) + \delta(\omega - \lambda + a)])d\lambda \\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} X(\lambda)[\delta(\omega - \lambda - a) + \delta(\omega - \lambda + a)]d\lambda \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\omega - \lambda - a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\omega - \lambda + a)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda - \omega + a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda - \omega - a)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda - (\omega - a)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda - (\omega + a)]d\lambda\right] \\
&= \tfrac{1}{2}[X(\omega - a) + X(\omega + a)] \\
&= \tfrac{1}{2}X(\omega - a) + \tfrac{1}{2}X(\omega + a).
\end{aligned}$$

Note that the above solution is essentially identical to the one appearing earlier in Example 6.15 on page 171.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{x(t)\cos(at)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}(e^{jat} + e^{-jat})x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\mathcal{F}\left\{e^{jat}x(t)\right\}(\omega) + \tfrac{1}{2}\mathcal{F}\left\{e^{-jat}x(t)\right\}(\omega) \\
&= \tfrac{1}{2}X(\omega - a) + \tfrac{1}{2}X(\omega + a).
\end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. ∎

6.10 Frequency Spectra of Functions

The Fourier transform representation expresses a function in terms of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a function. For example, suppose that we have a function x with Fourier transform X. If X is nonzero at some frequency ω_0, then the function x contains some information at the frequency ω_0. On the other hand, if X is zero at the frequency ω_0, then the function x has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a function. This distribution of information in a function over different frequencies is referred to as the **frequency spectrum** of the function. That is, X is the frequency spectrum of x.

To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with X expressed in polar form as follows:

$$\begin{aligned}x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j\arg X(\omega)}e^{j\omega t}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t+\arg X(\omega)]}d\omega.\end{aligned}$$

In effect, the quantity $|X(\omega)|$ is a weight that determines how much the complex sinusoid at frequency ω contributes to the integration result $x(t)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles (i.e., $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=-\infty}^{\infty}\Delta x f(k\Delta x)$). Expressing x in this way, we obtain

$$\begin{aligned}x(t) &= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\,|X(k\Delta\omega)|\, e^{j[k\Delta\omega t+\arg X(k\Delta\omega)]} \\ &= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\,\left|X(\omega')\right| e^{j[\omega' t+\arg X(\omega')]},\end{aligned}$$

where $\omega' = k\Delta\omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\omega' = k\Delta\omega$) corresponds to a complex sinusoid with fundamental frequency ω' that has had its amplitude scaled by a factor of $|X(\omega')|$ and has been time-shifted by an amount that depends on $\arg X(\omega')$. For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\omega')|$ as a measure of how much information a function x has at the frequency ω'.

Note that, since the Fourier transform X is a function of a real variable, a function can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic functions), where a function can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 6.8, in the case of periodic functions the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Since the frequency spectrum is complex (in the general case), it is usually represented using two plots, one showing the magnitude of X and one showing the argument. We refer to $|X(\omega)|$ as the **magnitude spectrum** of the function x. Similarly, we refer to $\arg X(\omega)$ as the **phase spectrum** of the function x. In the special case that X is a real (or purely imaginary) function, we usually plot the frequency spectrum directly on a single graph.

Recall, from Theorem 6.17 earlier, that the Fourier transform X of a real-valued function x must be conjugate symmetric. So, if x is real-valued, then

$$\begin{aligned}&|X(\omega)| = |X(-\omega)| \text{ for all } \omega \quad \text{and} \\ &\arg X(\omega) = -\arg X(-\omega) \text{ for all } \omega\end{aligned}$$

(i.e., the magnitude and argument of X are even and odd, respectively). (See (6.15a) and (6.15b).) Due to the symmetry in the frequency spectra of real-valued functions, we typically ignore negative frequencies when dealing with such

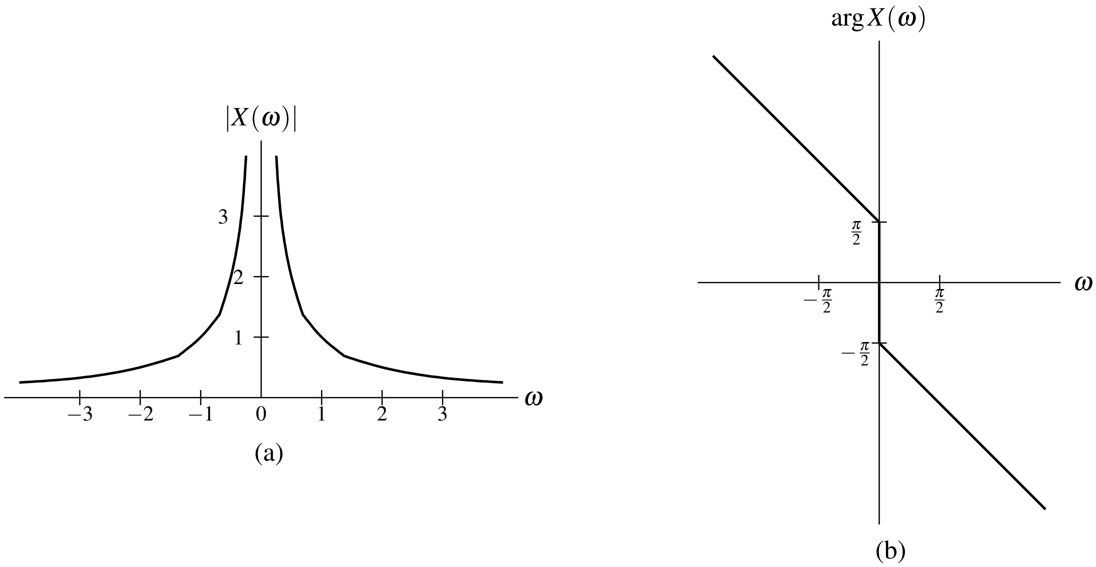


Figure 6.8: Frequency spectrum of the amplitude-scaled time-shifted signum function x. (a) Magnitude spectrum and (b) phase spectrum of x.

functions. In the case of functions that are complex-valued but not real-valued, frequency spectra do not possess the above symmetry, and negative frequencies become important.

Example 6.27 (Frequency spectrum of an amplitude-scaled time-shifted signum function)**.** Consider the function

$$x(t) = \tfrac{1}{2}\operatorname{sgn}(t-1).$$

We can show that this function has the Fourier transform

$$X(\omega) = \tfrac{1}{j\omega}e^{-j\omega}.$$

In this case, X is neither purely real nor purely imaginary, so we use two separate graphs to present the frequency spectrum X. We plot the magnitude spectrum and phase spectrum as shown in Figures 6.8(a) and (b), respectively. ■

Example 6.28 (Frequency spectrum of a time-scaled sinc function)**.** Consider the function

$$x(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right).$$

We can show that this function has the Fourier transform

$$X(\omega) = 2\pi \operatorname{rect}\omega.$$

Since, in this case, X is real, we can plot the frequency spectrum X on a single graph, as shown in Figure 6.9. ■

Example 6.29 (Frequency spectrum of a time-shifted signum function)**.** The function

$$x(t) = \operatorname{sgn}(t-1)$$

has the Fourier transform

$$X(\omega) = \tfrac{2}{j\omega}e^{-j\omega}.$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) x has the most information.

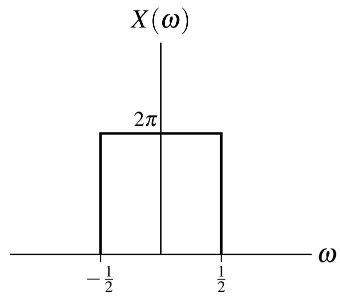


Figure 6.9: Frequency spectrum of the time-scaled sinc function x.

Solution. (a) First, we find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

$$\begin{aligned}|X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right| \\ &= \tfrac{2}{|\omega|}.\end{aligned}$$

Next, we find the phase spectrum $\arg X(\omega)$. First, we observe that $\arg X(\omega)$ is not well defined if $\omega = 0$. So, we assume that $\omega \neq 0$. From the expression for $X(\omega)$, we can write (for $\omega \neq 0$)

$$\begin{aligned}\arg X(\omega) &= \arg\left(\tfrac{2}{j\omega}e^{-j\omega}\right) \\ &= \arg e^{-j\omega} + \arg\tfrac{2}{j\omega} \\ &= -\omega + \arg\tfrac{2}{j\omega} \\ &= -\omega + \arg(-\tfrac{j2}{\omega}) \\ &= \begin{cases} -\frac{\pi}{2} - \omega & \omega > 0 \\ \frac{\pi}{2} - \omega & \omega < 0 \end{cases} \\ &= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega.\end{aligned}$$

In the above simplification, we used the fact that

$$\arg\left(\tfrac{2}{j\omega}\right) = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0. \end{cases}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown in Figures 6.10(a) and (b).

(b) Since $|X(\omega)|$ is largest for $\omega = 0$, x has the most information at the frequency 0. ■

6.11 Bandwidth of Functions

A function x with Fourier transform X is said to be **bandlimited** if, for some nonnegative real constant B,

$$X(\omega) = 0 \text{ for all } |\omega| > B.$$

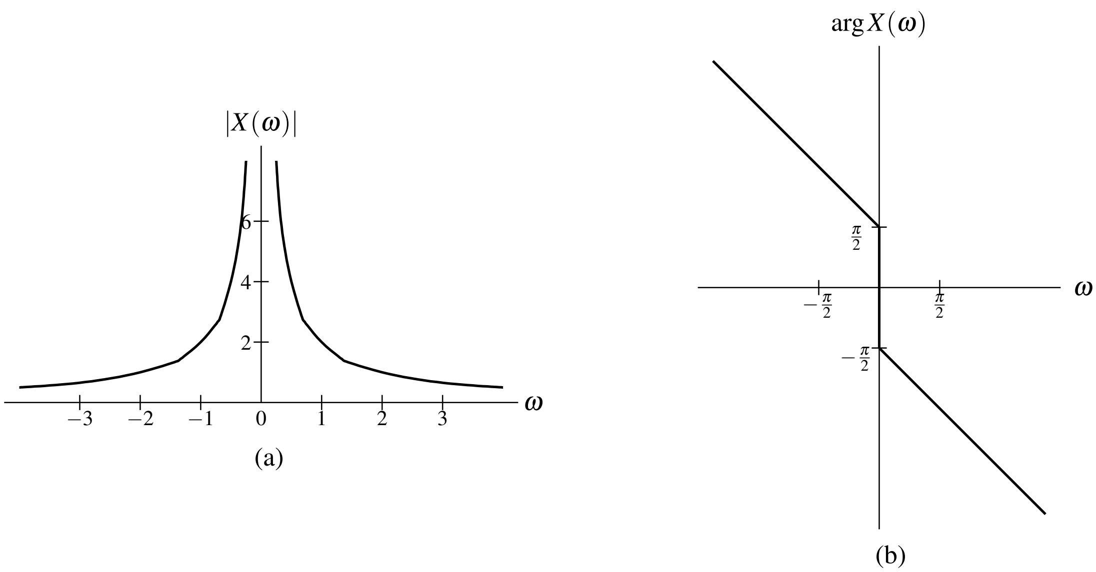


Figure 6.10: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of x.

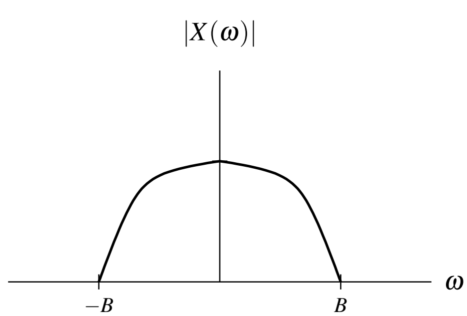


Figure 6.11: Bandwidth of a function x with the Fourier transform X.

We sometimes refer to B as the **bandwidth** of x. An illustrative example is provided in Figure 6.11.

One can show that a function cannot be both time limited and bandlimited. (A proof of this fact is considered in Exercise 6.12.) To help understand why this is so we recall the time/frequency scaling property of the Fourier transform. From this property, we know that as we compress a function x (by time scaling), its Fourier transform X will expand (by time scaling). Similarly, as we compress the Fourier transform X (by time scaling), x will expand (by time scaling). So, clearly, there is an inverse relationship between the time-extent and bandwidth of a function.

6.12 Energy-Density Spectra

Suppose that we have a function x with finite energy E and Fourier transform X. By definition, the energy contained in x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

We can use Parseval's relation (6.12) to express E in terms of X as

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

Thus, the energy E is given by

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} E_x(\omega) d\omega,$$

where

$$E_x(\omega) = |X(\omega)|^2.$$

We refer to E_x as the **energy-density spectrum** of the function x. The function E_x indicates how the energy in x is distributed with respect to frequency. For example, the energy contributed by frequencies in the range $[\omega_1, \omega_2]$ is simply given by

$$\tfrac{1}{2\pi} \int_{\omega_1}^{\omega_2} E_x(\omega) d\omega.$$

Example 6.30. Compute the energy-density spectrum E_x of the function

$$x(t) = \operatorname{sinc}\left(\tfrac{1}{2} t\right).$$

Determine the amount of energy contained in the frequency components in the range $\omega \in [-\frac{1}{4}, \frac{1}{4}]$. Also, determine the total amount of energy in the function.

Solution. First, we compute the Fourier transform X of x. We obtain

$$X(\omega) = 2\pi \operatorname{rect} \omega.$$

Computing the energy spectral density E_x, we have

$$\begin{aligned} E_x(\omega) &= |X(\omega)|^2 \\ &= |2\pi \operatorname{rect} \omega|^2 \\ &= 4\pi^2 \operatorname{rect}^2 \omega \\ &= 4\pi^2 \operatorname{rect} \omega. \end{aligned}$$

Let E_1 denote the energy contained in x for frequencies $|\omega| \in [-\frac{1}{4}, \frac{1}{4}]$. Then, we have

$$\begin{aligned} E_1 &= \tfrac{1}{2\pi}\int_{-1/4}^{1/4} E_x(\omega)d\omega \\ &= \tfrac{1}{2\pi}\int_{-1/4}^{1/4} 4\pi^2 \operatorname{rect}(\omega)d\omega \\ &= \int_{-1/4}^{1/4} 2\pi d\omega \\ &= \pi. \end{aligned}$$

Let E denote the total amount of energy in x. Then, we have

$$\begin{aligned} E &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} E_x(\omega)d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 4\pi^2 \operatorname{rect}(\omega)d\omega \\ &= \int_{-1/2}^{1/2} 2\pi d\omega \\ &= 2\pi. \end{aligned}$$

■

6.13 Characterizing LTI Systems Using the Fourier Transform

Consider a LTI system with input x, output y, and impulse response h. Such a system is depicted in Figure 6.12. The behavior of such a system is governed by the equation

$$y(t) = x * h(t). \tag{6.32}$$

Let X, Y, and H denote the Fourier transforms of x, y, and h, respectively. Taking the Fourier transform of both sides of (6.32) and using the the time-domain convolution property of the Fourier transform, we obtain

$$Y(\omega) = X(\omega)H(\omega). \tag{6.33}$$

This result provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions. In other words, we have a system resembling that in Figure 6.13. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to H as the **frequency response** of the system. The system behavior is completely characterized by the frequency response H. If we know the input, we can compute its Fourier transform X, and then determine the Fourier transform Y of the output. Using the inverse Fourier transform, we can then determine the output y.

In the most general case, the frequency response H is a complex-valued function. Thus, we can represent H in terms of its magnitude and argument. We refer to the magnitude of H as the **magnitude response** of the system. Similarly, we refer to the argument of H as the **phase response** of the system.

From (6.33), we can write

$$\begin{aligned} |Y(\omega)| &= |X(\omega)H(\omega)| \\ &= |X(\omega)|\,|H(\omega)| \quad \text{and} \end{aligned} \tag{6.34a}$$

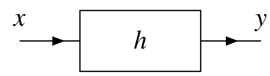


Figure 6.12: Time-domain view of a LTI system with input x, output y, and impulse response h.

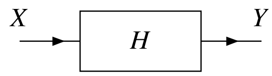


Figure 6.13: Frequency-domain view of a LTI system with input spectrum X, output spectrum Y, and frequency response H.

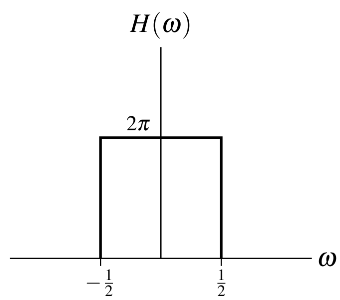


Figure 6.14: Frequency response of example system.

$$\begin{aligned}\arg Y(\omega) &= \arg[X(\omega)H(\omega)] \\ &= \arg X(\omega) + \arg H(\omega).\end{aligned} \tag{6.34b}$$

From (6.34a), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response. From (6.34b), we have that the phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is real, then (for the reasons explained in Section 6.10)

$$|H(\omega)| = |H(-\omega)| \text{ for all } \omega \quad \text{and}$$
$$\arg H(\omega) = -\arg H(-\omega) \text{ for all } \omega$$

(i.e., the magnitude and phase responses are even and odd, respectively).

Example 6.31. Consider a LTI system with impulse response

$$h(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right).$$

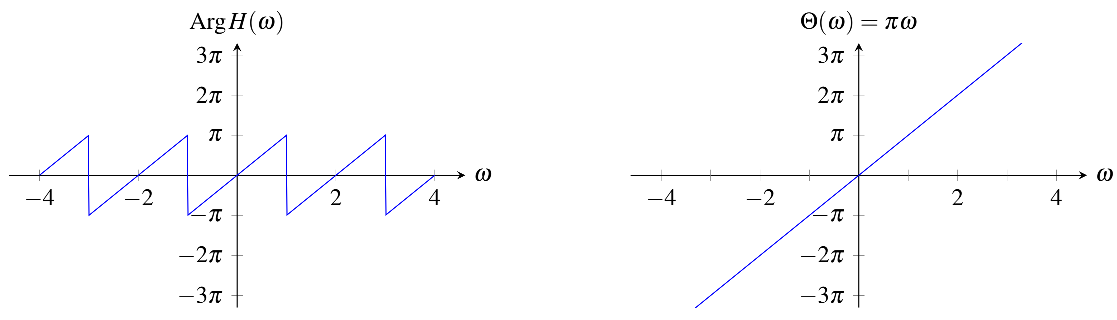
This system has the frequency response

$$H(\omega) = 2\pi \operatorname{rect} \omega.$$

In this particular case, H is real. So, we can plot the frequency response H on a single graph, as shown in Figure 6.14. ■

6.13.1 Unwrapped Phase

Since the argument of a complex number is not uniquely determined, the argument of a complex-valued function is also not uniquely determined. Consequently, we have some freedom in how we define a function that corresponds to the phase (i.e., argument) of a complex-valued function. Often, for convenience, we restrict the argument to lie in an



(a) (b)

Figure 6.15: Unwrapped phase example. (a) The phase function restricted such that its range is in $(-\pi, \pi]$ and (b) the corresponding unwrapped phase.

interval of length 2π, such as the interval $(-\pi, \pi]$ which corresponds to the principal argument. Defining the phase of a complex-valued function in this way, however, can often result in a phase function with unnecessary discontinuities. This motivates the notion of unwrapped phase. The **unwrapped phase** is simply the phase defined in such a way so as not to restrict the phase to an interval of length 2π and to keep the phase function continuous to the greatest extent possible. An example illustrating the notion of unwrapped phase is given below.

Example 6.32 (Unwrapped phase)**.** Consider the phase response of a LTI system with the frequency response

$$H(\omega) = e^{j\pi\omega}.$$

We can choose to define the phase (i.e., argument) of H by simply using the principal argument (i.e., $\operatorname{Arg} H(\omega)$). This yields the phase function shown in Figure 6.15(a). Using the principal argument in this way, however, unnecessarily introduces discontinuities into the phase function. For this reason, we sometimes prefer to define the phase function in such a way as to eliminate such unnecessary discontinuities. This motivates the use of the unwrapped phase. The function H has the unwrapped phase Θ given by

$$\Theta(\omega) = \pi\omega.$$

A plot of Θ is shown in Figure 6.15(b). Unlike the function in Figure 6.15(a) (which has numerous discontinuities), the function in Figure 6.15(b) is continuous. Although the functions in these two figures are distinct, these functions are equivalent in the sense that they correspond to the same physical angular displacement (i.e., $e^{j\operatorname{Arg} H(\omega)} = e^{j\Theta(\omega)}$ for all $\omega \in \mathbb{R}$). ■

6.13.2 Magnitude and Phase Distortion

Recall, from Corollary 5.1, that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\left\{e^{j\omega t}\right\}(t) = H(\omega)e^{j\omega t}$$

(i.e., $e^{j\omega t}$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $H(\omega)$). Expressing $H(\omega)$ in polar form, we have

$$\begin{aligned}
\mathcal{H}\{e^{j\omega t}\}(t) &= |H(\omega)|\, e^{j\arg H(\omega)} e^{j\omega t} \\
&= |H(\omega)|\, e^{j[\omega t + \arg H(\omega)]} \\
&= |H(\omega)|\, e^{j\omega(t + \arg[H(\omega)]/\omega)}.
\end{aligned}$$

This equation can be rewritten as

$$\mathcal{H}\{e^{j\omega t}\}(t) = |H(\omega)|\, e^{j\omega[t - \tau_p(\omega)]}, \tag{6.35a}$$

where

$$\tau_p(\omega) = -\frac{\arg H(\omega)}{\omega}. \tag{6.35b}$$

Thus, the response of the system to the function $e^{j\omega t}$ is produced by applying two transformations to this function:

- (amplitude) scaling by $|H(\omega)|$; and
- translating by $\tau_p(\omega)$.

Therefore, the magnitude response determines how different complex sinusoids are (amplitude) scaled by the system. Similarly, the phase response determines how different complex sinusoids are translated (i.e., delayed/advanced) by the system.

A system for which $|H(\omega)| = 1$ for all ω is said to be **allpass**[1]. In the case of an allpass system, the magnitude spectra of the system's input and output are identical. If a system is not allpass, it modifies the magnitude spectrum in some way. In situations where the magnitude spectrum is changed in an undesirable manner, **magnitude distortion** (i.e., distortion of the magnitude spectrum) is said to occur. If $|H(\omega)| = a$ for all ω, where a is a constant, every complex sinusoid is scaled by the same amount a when passing through the system. In practice, this type of change to the magnitude spectrum may sometimes be undesirable if $a \neq 1$. If $|H(\omega)|$ is not a constant, different complex sinusoids are scaled by different amounts. In practice, this type of change to the magnitude spectrum is usually undesirable and deemed to constitute magnitude distortion.

The function τ_p appearing in (6.35b) is known as the **phase delay** of the system. A system for which $\tau_p(\omega) = 0$ for all ω is said to have **zero phase**. In the case of a system having zero phase, the phase spectra of the system's input and output are identical. In the case that the system does not have zero phase, the phase spectra of the system's input and output differ. In situations where the phase spectrum is changed in an undesirable manner, **phase distortion** (i.e., distortion of the phase spectrum) is said to occur. If $\tau_p(\omega) = t_d$ for all ω, where t_d is a constant, the system shifts all complex sinusoids by the same amount t_d. Note that $\tau_p(\omega) = t_d$ is equivalent to the (unwrapped) phase response being of the form

$$\arg H(\omega) = -t_d\omega,$$

which is a linear function with a zero constant term. For this reason, a system with a constant phase delay is said to have **linear phase**. If $\tau_p(\omega)$ is not a constant, different complex sinusoids are shifted by different amounts. In many practical applications, shifting different complex sinusoids by different amounts is undesirable. Therefore, systems that are not linear phase are typically deemed to introduce phase distortion. For this reason, in contexts where phase spectra are important, systems with either zero phase or linear phase are typically used.

Example 6.33 (Distortionless transmission)**.** Consider a LTI system with input x and output y given by

$$y(t) = x(t - t_0),$$

where t_0 is a real constant. That is, the output of the system is simply the input delayed by t_0. This type of system behavior is referred to as distortionless transmission, since the system allows the input to pass through to the output unmodified, except for a delay being introduced. This type of behavior is the ideal for which we strive in real-world communication systems (i.e., the received signal y equals a delayed version of the transmitted signal x). Taking the Fourier transform of the above equation, we have

$$Y(\omega) = e^{-j\omega t_0}X(\omega).$$

Thus, the system has the frequency response H given by

$$H(\omega) = e^{-j\omega t_0}.$$

[1] Some authors (e.g., [9, 12]) define an allpass system as one for which $|H(\omega)| = c$ for all ω, where c is a constant (and c is not necessarily 1).

Since $|H(\omega)| = 1$ for all ω, the system is allpass and does not introduce any magnitude distortion. The phase delay τ_p of the system is given by

$$\begin{aligned}\tau_p(\omega) &= -\frac{\arg H(\omega)}{\omega} \\ &= -\left(\frac{-\omega t_0}{\omega}\right) \\ &= t_0.\end{aligned}$$

Since the phase delay is a constant, the system has linear phase and does not introduce any phase distortion (except for a trivial time shift of t_0). ■

Example 6.34 (Frequency spectra of images)**.** The human visual system is more sensitive to the phase spectrum of an image than its magnitude spectrum. This can be aptly demonstrated by separately modifying the magnitude and phase spectra of an image, and observing the effect. Below, we consider two variations on this theme.

Consider the `potatohead` and `hongkong` images shown in Figures 6.16(a) and (b), respectively. Replacing the magnitude spectrum of the `potatohead` image with the magnitude spectrum of the `hongkong` image (and leaving the phase spectrum unmodified), we obtain the new image shown in Figure 6.16(c). Although changing the magnitude spectrum has led to distortion, the basic essence of the original image has not been lost. On the other hand, replacing the phase spectrum of the `potatohead` image with the phase spectrum of the `hongkong` image (and leaving the magnitude spectrum unmodified), we obtain the image shown in Figure 6.16(d). Clearly, by changing the phase spectrum of the image, the fundamental nature of the image has been altered, with the new image more closely resembling the `hongkong` image than the original `potatohead` image.

A more extreme scenario is considered in Figure 6.17. In this case, we replace each of the magnitude and phase spectra of the `potatohead` image with random data, with this data being taken from the image consisting of random noise shown in Figure 6.17(b). When we completely replace the magnitude spectrum of the `potatohead` image with random values, we can still recognize the resulting image in Figure 6.17(c) as a very grainy version of the original `potatohead` image. On the other hand, when the phase spectrum of the `potatohead` image is replaced with random values, all visible traces of the original `potatohead` image are lost in the resulting image in Figure 6.17(d). ■

6.14 Interconnection of LTI Systems

From the properties of the Fourier transform and the definition of the frequency response, we can derive a number of equivalences involving the frequency response and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 6.18(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned}H(\omega) &= \mathcal{F}\{h_1 * h_2\}(\omega) \\ &= \mathcal{F}h_1(\omega)\mathcal{F}h_2(\omega) \\ &= H_1(\omega)H_2(\omega).\end{aligned}$$

Thus, we have the equivalence shown in Figure 6.18(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 6.18(b).

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 6.19. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

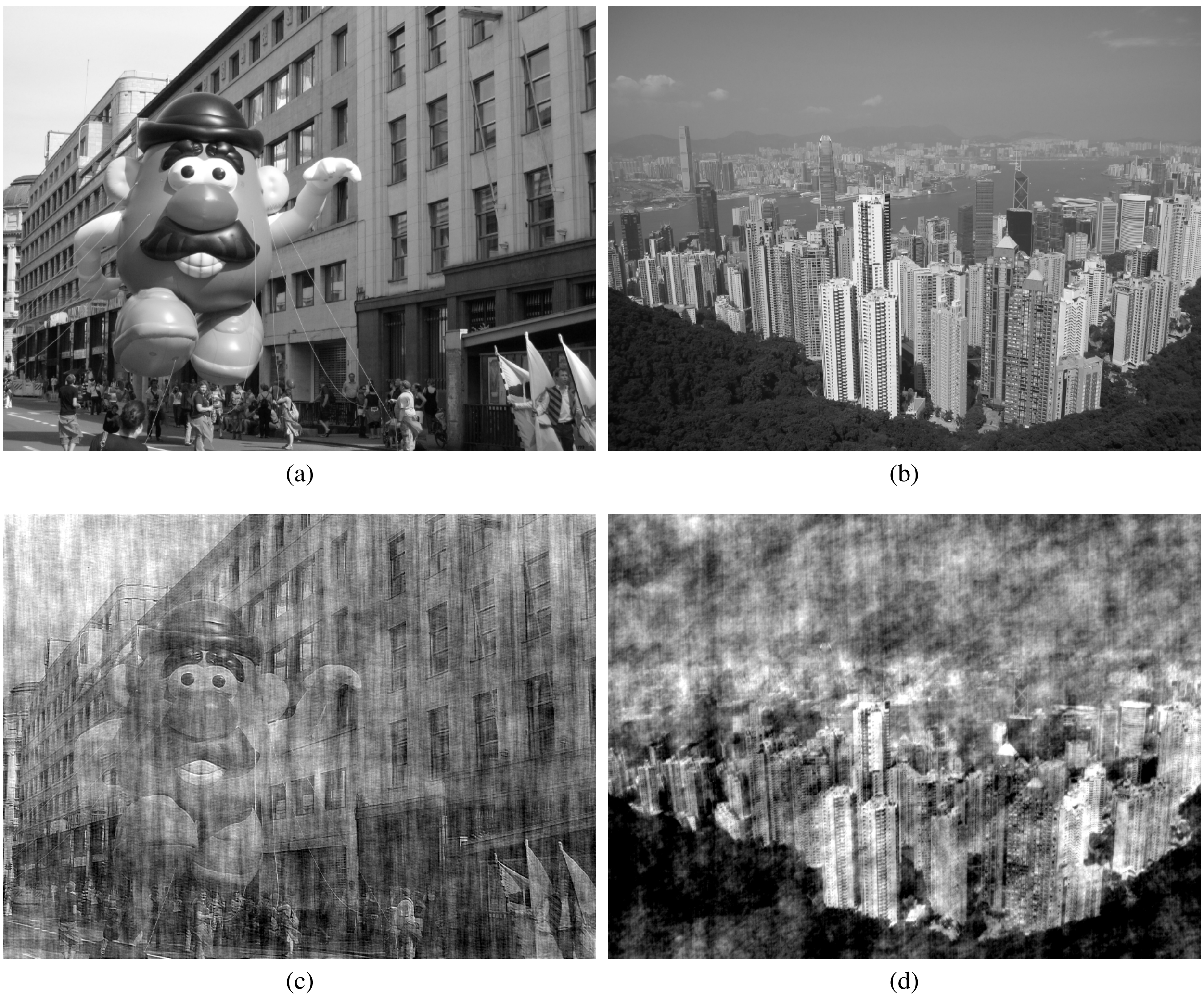


Figure 6.16: Importance of phase information in images. The (a) `potatohead` and (b) `hongkong` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `hongkong` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `hongkong` image.

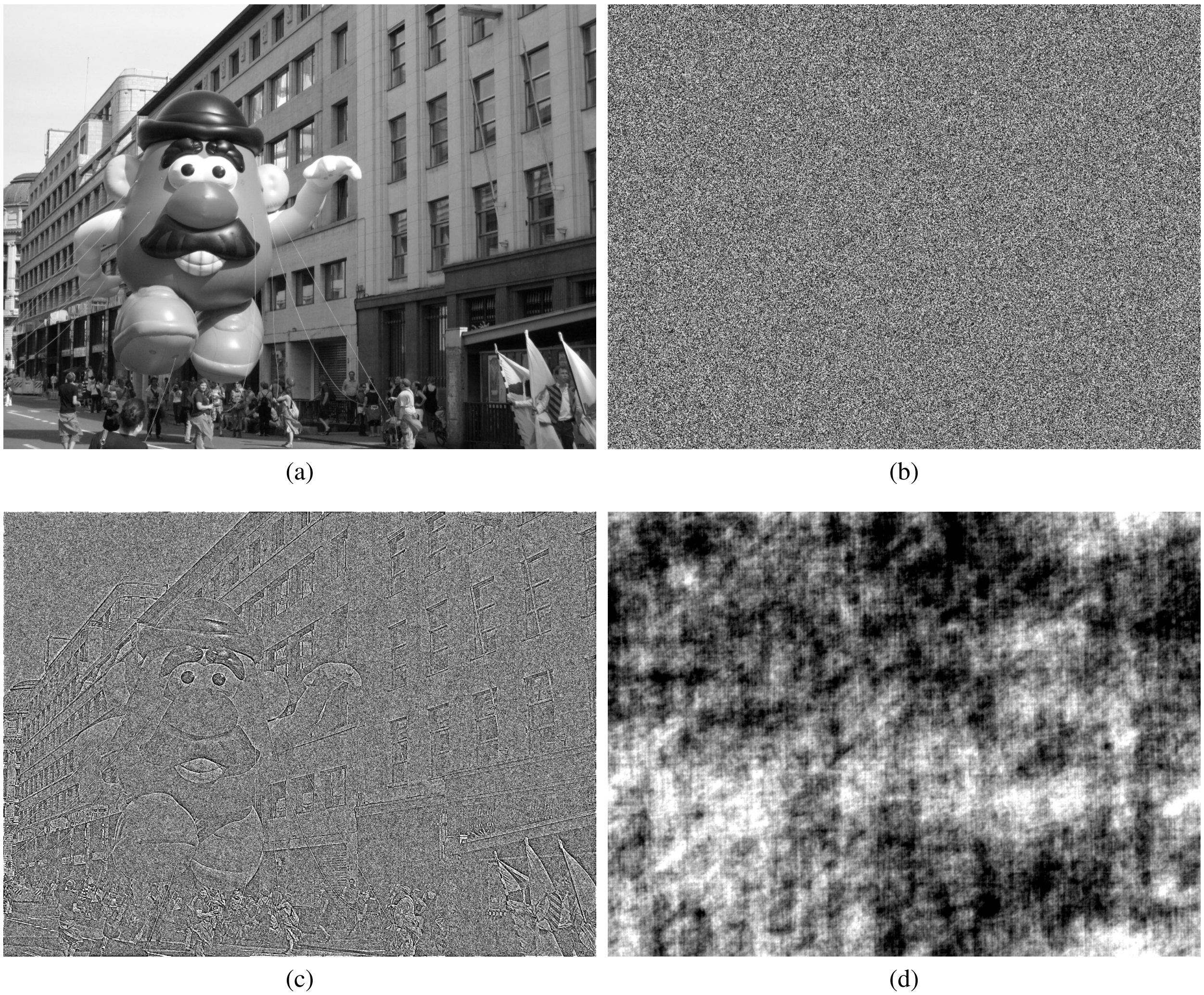


Figure 6.17: Importance of phase information in images. The (a) `potatohead` and (b) `random` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `random` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `random` image.

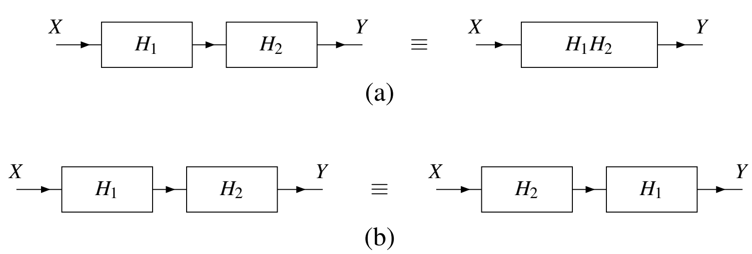


Figure 6.18: Equivalences involving frequency responses and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

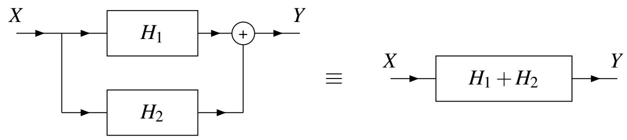


Figure 6.19: Equivalence involving frequency responses and the parallel interconnection of LTI systems.

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned} H(\omega) &= \mathcal{F}\{h_1 + h_2\}(\omega) \\ &= \mathcal{F}h_1(\omega) + \mathcal{F}h_2(\omega) \\ &= H_1(\omega) + H_2(\omega). \end{aligned}$$

Thus, we have the equivalence shown in Figure 6.19.

6.15 LTI Systems and Differential Equations

Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients. Suppose that we have such a system with input x and output y. Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)$$

(where $M \leq N$). Let X and Y denote the Fourier transforms of x and y, respectively. Taking the Fourier transform of both sides of the above equation yields

$$\mathcal{F}\left\{\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t)\right\}(\omega) = \mathcal{F}\left\{\sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)\right\}(\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathcal{F}\left\{\left(\tfrac{d}{dt}\right)^k y(t)\right\}(\omega) = \sum_{k=0}^{M} a_k \mathcal{F}\left\{\left(\tfrac{d}{dt}\right)^k x(t)\right\}(\omega).$$

Using the time-differentiation property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} a_k (j\omega)^k X(\omega).$$

Then, factoring we have

$$Y(\omega) \sum_{k=0}^{N} b_k (j\omega)^k = X(\omega) \sum_{k=0}^{M} a_k (j\omega)^k.$$

Rearranging this equation, we obtain

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k (j\omega)^k}{\sum_{k=0}^{N} b_k (j\omega)^k} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Since the system is LTI, $Y(\omega) = X(\omega)H(\omega)$ (or equivalently, $H(\omega) = \frac{Y(\omega)}{X(\omega)}$), and we can conclude from the above equation that the frequency response H is given by

$$H(\omega) = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Observe that, for a system of the form considered above, the frequency response is a rational function—hence, our interest in rational functions.

Example 6.35 (Differential equation to frequency response)**.** A LTI system with input x and output y is characterized by the differential equation

$$7y''(t)+11y'(t)+13y(t)=5x'(t)+3x(t),$$

where x', y', and y'' denote the first derivative of x, the first derivative of y, and the second derivative of y, respectively. Find the frequency response H of this system.

Solution. Taking the Fourier transform of the given differential equation, we obtain

$$7(j\omega)^2Y(\omega)+11j\omega Y(\omega)+13Y(\omega)=5j\omega X(\omega)+3X(\omega).$$

Rearranging the terms and factoring, we have

$$(-7\omega^2+11j\omega+13)Y(\omega)=(5j\omega+3)X(\omega).$$

Thus, H is given by

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{5j\omega+3}{-7\omega^2+11j\omega+13}. \quad \blacksquare$$

Example 6.36 (Frequency response to differential equation)**.** A LTI system with input x and output y has the frequency response

$$H(\omega)=\frac{-7\omega^2+11j\omega+3}{-5\omega^2+2}.$$

Find the differential equation that characterizes this system.

Solution. From the given frequency response H, we have

$$\frac{Y(\omega)}{X(\omega)}=\frac{-7\omega^2+11j\omega+3}{-5\omega^2+2}.$$

Multiplying both sides by $(-5\omega^2+2)X(\omega)$, we have

$$-5\omega^2Y(\omega)+2Y(\omega)=-7\omega^2X(\omega)+11j\omega X(\omega)+3X(\omega).$$

Applying some simple algebraic manipulation yields

$$5(j\omega)^2Y(\omega)+2Y(\omega)=7(j\omega)^2X(\omega)+11(j\omega)X(\omega)+3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$5y''(t)+2y(t)=7x''(t)+11x'(t)+3x(t). \quad \blacksquare$$

6.16 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. One type is frequency-selective filters. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \lvert\omega\rvert \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(a).

An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \lvert\omega\rvert \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie between two cutoff frequencies, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq \lvert\omega\rvert \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where ω_{c1} and ω_{c2} are the cutoff frequencies. A plot of this frequency response is given in Figure 6.20(c).

Example 6.37 (Ideal filters)**.** For each LTI system whose impulse response h is given below, find and plot the frequency response H of the system, and identify the type of frequency-selective filter to which the system corresponds.
(a) $h(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$, where ω_c is a positive real constant;
(b) $h(t) = \delta(t) - \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$, where ω_c is a positive real constant; and
(c) $h(t) = \frac{2\omega_b}{\pi}[\operatorname{sinc}(\omega_b t)]\cos(\omega_a t)$, where ω_a and ω_b are positive real constants.

Solution. In what follows, let us denote the input and output of the system as x and y, respectively. Also, let X and Y denote the Fourier transforms of x and y, respectively.

(a) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)\}(\omega) \\ &= \tfrac{\omega_c}{\pi}\mathcal{F}\{\operatorname{sinc}(\omega_c t)\}(\omega) \\ &= \tfrac{\omega_c}{\pi}\left[\tfrac{\pi}{\omega_c}\operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right)\right] \\ &= \operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right) \\ &= \begin{cases} 1 & \lvert\omega\rvert \leq \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response H is plotted in Figure 6.21(a). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $\lvert\omega\rvert > \omega_c$, Y will contain only those frequency components in X that lie in the frequency range $\lvert\omega\rvert \leq \omega_c$. In other words, only the lower frequency components from X are kept. Thus, the system represents a lowpass filter.

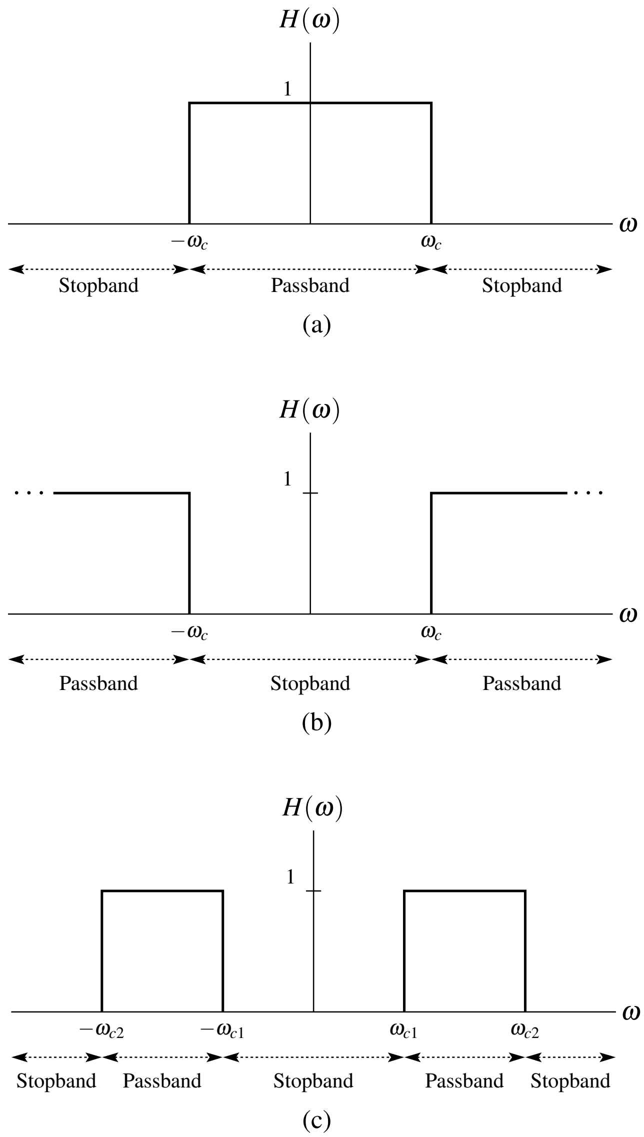


Figure 6.20: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

(b) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\{\delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)\}(\omega) \\
&= \mathcal{F}\delta(\omega) - \tfrac{\omega_c}{\pi}\mathcal{F}\{\operatorname{sinc}(\omega_c t)\}(\omega) \\
&= 1 - \tfrac{\omega_c}{\pi}\left[\tfrac{\pi}{\omega_c}\operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right)\right] \\
&= 1 - \operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right) \\
&= \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 6.21(b). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $|\omega| < \omega_c$, Y will contain only those frequency components in X that lie in the frequency range $|\omega| \geq \omega_c$. In other words, only the higher frequency components from X are kept. Thus, the system represents a highpass filter.

(c) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\{\tfrac{2\omega_b}{\pi}[\operatorname{sinc}(\omega_b t)]\cos(\omega_a t)\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\mathcal{F}\{[\operatorname{sinc}(\omega_b t)](2\cos[\omega_a t])\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\mathcal{F}\{[\operatorname{sinc}(\omega_b t)][e^{j\omega_a t} + e^{-j\omega_a t}]\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\left[\mathcal{F}\{e^{j\omega_a t}\operatorname{sinc}(\omega_b t)\}(\omega) + \mathcal{F}\{e^{-j\omega_a t}\operatorname{sinc}(\omega_b t)\}(\omega)\right] \\
&= \tfrac{\omega_b}{\pi}\left[\tfrac{\pi}{\omega_b}\operatorname{rect}\left(\tfrac{\omega-\omega_a}{2\omega_b}\right) + \tfrac{\pi}{\omega_b}\operatorname{rect}\left(\tfrac{\omega+\omega_a}{2\omega_b}\right)\right] \\
&= \operatorname{rect}\left(\tfrac{\omega-\omega_a}{2\omega_b}\right) + \operatorname{rect}\left(\tfrac{\omega+\omega_a}{2\omega_b}\right) \\
&= \begin{cases} 1 & \omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 6.21(c). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $|\omega| < \omega_a - \omega_b$ or $|\omega| > \omega_a + \omega_b$, Y will contain only those frequency components in X that lie in the frequency range $\omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b$. In other words, only the middle frequency components of X are kept. Thus, the system represents a bandpass filter. ■

Example 6.38 (Lowpass filtering)**.** Consider a LTI system with impulse response

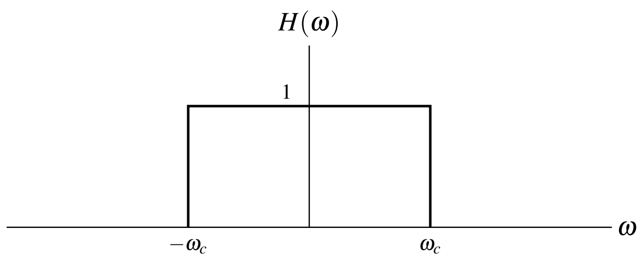
$$h(t) = 300\operatorname{sinc}(300\pi t).$$

Using frequency-domain methods, find the response y of the system to the input

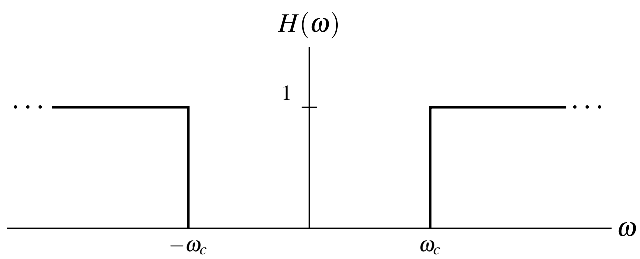
$$x(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t) + \tfrac{1}{2}\cos(400\pi t) - \tfrac{1}{4}\cos(600\pi t).$$

Solution. To begin, we find the Fourier transform X of x. Computing X, we have

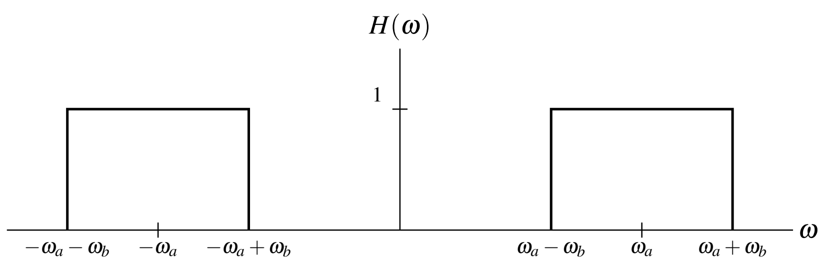
$$\begin{aligned}
X(\omega) &= \mathcal{F}\left\{\tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t) + \tfrac{1}{2}\cos(400\pi t) - \tfrac{1}{4}\cos(600\pi t)\right\}(\omega) \\
&= \tfrac{1}{2}\mathcal{F}\{1\}(\omega) + \tfrac{3}{4}\mathcal{F}\{\cos(200\pi t)\}(\omega) + \tfrac{1}{2}\mathcal{F}\{\cos(400\pi t)\}(\omega) - \tfrac{1}{4}\mathcal{F}\{\cos(600\pi t)\}(\omega) \\
&= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{3\pi}{4}[\delta(\omega+200\pi) + \delta(\omega-200\pi)] + \tfrac{\pi}{2}[\delta(\omega+400\pi) + \delta(\omega-400\pi)] \\
&\qquad - \tfrac{\pi}{4}[\delta(\omega+600\pi) + \delta(\omega-600\pi)] \\
&= -\tfrac{\pi}{4}\delta(\omega+600\pi) + \tfrac{\pi}{2}\delta(\omega+400\pi) + \tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi) \\
&\qquad + \tfrac{\pi}{2}\delta(\omega-400\pi) - \tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}$$



(a)



(b)



(c)

Figure 6.21: Frequency responses of each of the (a) first, (b) second, and (c) third systems from the example.