Example 7.40 (Stabilization of unstable plant)**.** Consider the causal LTI system with input Laplace transform X, output Laplace transform Y, and system function

$$P(s) = \frac{10}{s-1},$$

as depicted in Figure 7.27. One can easily confirm that this system is not BIBO stable, due to the pole of P at 1. (Since the system is causal, the ROC of P is the RHP given by $\mathrm{Re}(s) > 1$. Clearly, this ROC does not include the imaginary axis. Therefore, the system is not stable.) In what follows, we consider two different strategies for stabilizing this unstable system as well as their suitability for use in practice.

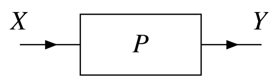


Figure 7.27: Plant.

(a) STABILIZATION OF UNSTABLE PLANT VIA POLE-ZERO CANCELLATION. Suppose that the system in Figure 7.27 is connected in series with another causal LTI system with system function

$$W(s) = \frac{s-1}{10(s+1)},$$

in order to yield a new system with input Laplace transform X and output Laplace transform Y, as shown in Figure 7.28(a). Show that this new system is BIBO stable.

(b) STABILIZATION OF UNSTABLE PLANT VIA FEEDBACK. Suppose now that the system in Figure 7.27 is interconnected with two other causal LTI systems with system functions C and Q, as shown in Figure 7.28(b), in order to yield a new system with input Laplace transform X, output Laplace transform Y, and system function H. Moreover, suppose that

$$C(s) = \beta \quad \text{and} \quad Q(s) = 1,$$

where β is a real constant. Show that, with an appropriate choice of β, the resulting system is BIBO stable.

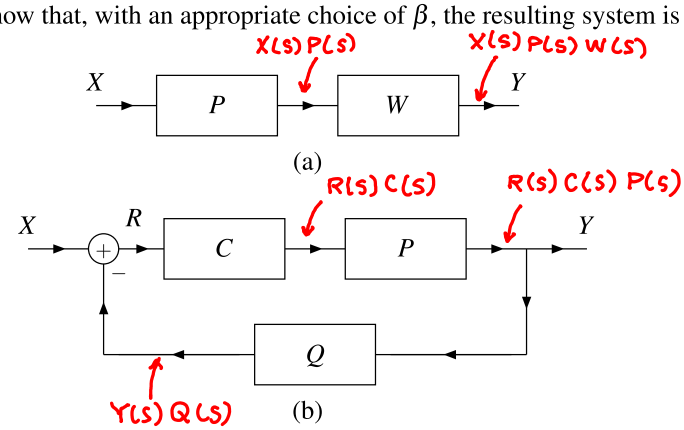


Figure 7.28: Two configurations for stabilizing the unstable plant. (a) Simple cascade system and (b) feedback control system.

(c) PRACTICAL ISSUES. Parts (a) and (b) of this example consider two different schemes for stabilizing the unstable system in Figure 7.27. As it turns out, a scheme like the one in part (a) is not useful in practice. Identify the practical problems associated with this approach. Indicate whether the scheme in part (b) suffers from the same shortcomings.

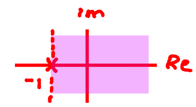
Solution. (a) From the block diagram in Figure 7.28(a), the system function H of the overall system is

$\underbrace{P(s)W(s)}_{H(s)}$ $Y(s) = \underbrace{P(s)W(s)}_{H(s)}X(s)$

$$
\begin{aligned}
H(s) &= P(s)W(s) \\
&= \left(\frac{10}{s-1}\right)\left(\frac{s-1}{10(s+1)}\right) \\
&= \frac{1}{s+1}.
\end{aligned}
$$

substitute given P and W

Simplify



Since the system is causal and H is rational, the ROC of H is $\operatorname{Re}(s) > -1$. Since the ROC includes the imaginary axis, the system is BIBO stable.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a somewhat reasonable step response. Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

system has reasonable step response

$$
\begin{aligned}
s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t) \\
&= (1-e^{-t})u(t).
\end{aligned}
$$

Evidently, s is a somewhat crude approximation of the desired response u.

(b) From the block diagram in Figure 7.28(b), we can write

$$
R(s) = X(s) - Q(s)Y(s) \quad \text{and} \quad (B1)
$$
$$
Y(s) = C(s)P(s)R(s). \quad (B2)
$$

Combining these equations (by substituting the expression for R in the first equation into the second equation), we obtain

Substituting B1 into B2

$$
\begin{aligned}
& Y(s) = C(s)P(s)[X(s) - Q(s)Y(s)] \\
\Rightarrow\quad & Y(s) = C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \\
\Rightarrow\quad & [1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s) \\
\Rightarrow\quad & \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.
\end{aligned}
$$

rearrange and factor

divide both sides by $[1 + C(s)P(s)Q(s)]X(s)$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

Since $H(s) = \frac{Y(s)}{X(s)}$

$$
H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.
$$

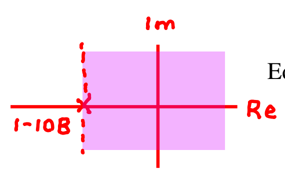
Substituting the given expressions for P, C, and Q, we have

substituting given P, C, and Q

$$
\begin{aligned}
H(s) &= \frac{\beta(\frac{10}{s-1})}{1 + \beta(\frac{10}{s-1})(1)} \\
&= \frac{10\beta}{s - 1 + 10\beta} \\
&= \frac{10\beta}{s - \underbrace{(1-10\beta)}_{\text{pole}}}.
\end{aligned}
$$

Simplify

explicitly show poles



The system function H is rational and has a single pole at $1-10\beta$. Since the system is causal, the ROC must be the RHP given by $\operatorname{Re}(s) > 1-10\beta$. For the system to be stable, we require that the ROC includes the imaginary axis. Thus, the system is stable if $1-10\beta < 0$ which implies $10\beta > 1$, or equivalently $\beta > \frac{1}{10}$.

system has reasonable step response

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a reasonable step response. (This is not by chance, however. Some care had to be exercised in the choice of the form of the compensator system function C. The process involved in making this choice requires knowledge of control systems beyond the scope of this book, however.) Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

$$
\begin{aligned}
s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{s(s-[1-10\beta])}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{10\beta-1}\left(\frac{1}{s}-\frac{1}{s-(1-10\beta)}\right)\right\}(t) \\
&= \frac{10\beta}{10\beta-1}\left(1-e^{-(10\beta-1)t}\right)u(t) \\
&\approx u(t) \quad \text{for large } \beta.
\end{aligned}
$$

Clearly, as β increases, s better approximates the desired response u.

(c) The scheme in part (a) for stabilizing the unstable plant relies on pole-zero cancellation. Unfortunately, in practice, it is not possible to achieve pole-zero cancellation. In short, the issue is one of approximation. Our analysis of systems is based on theoretical models specified in terms of equations. These theoretical models, however, are only approximations of real-world systems. This approximate nature is due to many factors, including (but not limited to) the following:

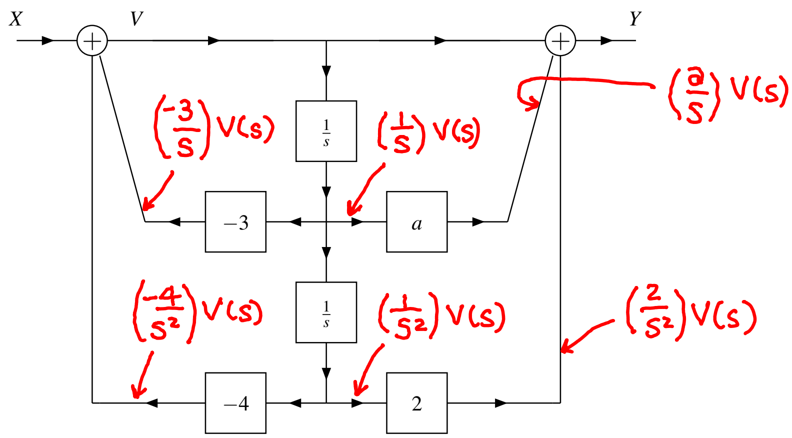
1. We cannot determine the system function of a system exactly, since this involves measurement, which always has some error.
2. We cannot build a system with such precision that it will have exactly some prescribed system function. The system function will only be close to the desired one.
3. The system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, and so on).
4. Although a system may be represented by a LTI model, the likely reality is that the system is not exactly LTI, which introduces error.

For reasons such as these, the effective poles and zeros of the system function will only be approximately where we expect them to be. Pole-zero cancellation, however, requires a pole and zero to be placed at exactly the same location. So, any error will prevent the pole-zero cancellation from occurring. Since at least some small error is unavoidable in practice, the desired pole-zero cancellation will not be achieved.

The scheme in part (b) for stabilizing the unstable plant is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros. Instead, the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to approximation error), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does. ■

7.30 Consider the system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and a is a real constant. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine whether the system $\mathcal{H}$ is BIBO stable.

systematic approach to obtaining system function:
1) label system input and system output
2) label each adder output
3) write equation for each adder output and system output
4) combine equations to obtain system function



Short Answer. (a) $H(s) = \frac{s^2+as+2}{s^2+3s+4}$ for $\operatorname{Re}(s) > -\frac{3}{2}$; (b) system is BIBO stable.

Answer (a,b).
From the system block diagram, we have:

$$Y(s) = V(s) + \left(\frac{a}{s}\right)V(s) + \left(\frac{2}{s^2}\right)V(s) \quad \text{and} \quad ①$$

$$V(s) = X(s) + \left(-\frac{3}{s}\right)V(s) + \left(-\frac{4}{s^2}\right)V(s). \quad ②$$

The preceding two equations can be rearranged to yield

$$③ \quad Y(s) = \left(1 + \frac{a}{s} + \frac{2}{s^2}\right)V(s) \quad \text{and}$$ rearrange ①

$$④ \quad X(s) = \left(1 + \frac{3}{s} + \frac{4}{s^2}\right)V(s).$$ rearrange ②

divide ③ by ④

Thus, $H(s)$ is given by

* $Y(s) = X(s)H(s) \Rightarrow$
$H(s) = \frac{Y(s)}{X(s)}$

$$H(s) = \underset{*}{\boxed{\frac{Y(s)}{X(s)} = \frac{1+a/s+2/s^2}{1+3/s+4/s^2}}} = \frac{s^2+as+2}{s^2+3s+4}$$

rearrange

Solving for the poles of $H(s)$, we obtain

$$\frac{-3 \pm \sqrt{9-4(1)(4)}}{2(1)} = -\tfrac{3}{2} \pm \tfrac{j\sqrt{7}}{2}.$$

Since the poles have negative real parts, the system is BIBO stable.