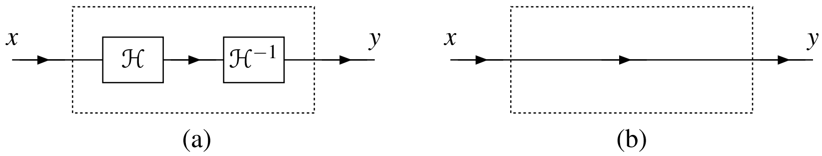



Figure 8.19: Systems that are equivalent (assuming $\mathcal{H}^{-1}$ exists). (a) First and (b) second system.

**Example 8.19.** Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = x(n-n_0)$$

and $n_0$ is an integer constant.

*Solution.* Let $y = \mathcal{H}x$. By substituting $n+n_0$ for $n$ in $y(n) = x(n-n_0)$, we obtain

$$\begin{aligned} y(n+n_0) &= x(n+n_0-n_0) \\ &= x(n). \end{aligned}$$

Thus, we have shown that

$$x(n) = y(n+n_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(n) = \mathcal{H}^{-1}y(n)$$

where

$$\mathcal{H}^{-1}y(n) = y(n+n_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system is invertible. ■

**Example 8.20** (Ideal squarer)**.** Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = x^2(n).$$

*Solution.* Consider the sequences $x_1(n) = -1$ and $x_2(n) = 1$. We have

$$\mathcal{H}x_1(n) = 1 \quad \text{and} \quad \mathcal{H}x_2(n) = 1.$$

Thus, we have found two distinct inputs that result in the same output. Therefore, the system is not invertible. ■

**Example 8.21.** Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = \frac{e^{x(n)}}{n^2+1}.$$

*Solution.* Let $y = \mathcal{H}x$. Attempting to solve for $x$ in terms of $y$, we have

$$\begin{aligned} & y(n) = \frac{e^{x(n)}}{n^2+1} \\ \Rightarrow \quad & \left(n^2+1\right) y(n) = e^{x(n)} \\ \Rightarrow \quad & \ln\left[(n^2+1)y(n)\right] = x(n). \end{aligned}$$

Thus, we have shown that

$$x(n) = \ln[(n^2+1)y(n)].$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(n) = \mathcal{H}^{-1}y(n)$$

where

$$\mathcal{H}^{-1}y(n) = \ln\left[\left(n^2+1\right)y(n)\right].$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system is invertible. ■

**Example 8.22.** Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

*Solution.* Consider the response $\mathcal{H}x$ of the system to an input $x$ of the form

$$x(n) = \alpha,$$

where $\alpha$ is a real constant. We have that

$$\begin{aligned}\mathcal{H}x(n) &= \tfrac{1}{2}[x(n) - x(-n)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0.\end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible. ■

### 8.7.4 BIBO Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system $\mathcal{H}$ is **BIBO stable** if, for every bounded sequence $x$, $\mathcal{H}x$ is also bounded (i.e., $|x(n)| < \infty$ for all $n$ implies that $|\mathcal{H}x(n)| < \infty$ for all $n$). In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output). Often, the triangle inequality (F.16) can be quite helpful when trying to show that a system is BIBO stable.

In practical terms, a BIBO stable system is well behaved in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time. Usually, a system that is not BIBO stable will have serious safety issues. For example, a portable music player with a battery input of 3.7 volts and headset output of $\infty$ volts would result in one vaporized human (and likely one big lawsuit as well).

**Example 8.23** (Ideal squarer)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = x^2(n).$$

*Solution.* Suppose that the input $x$ is bounded such that (for all $n$)

$$|x(n)| \leq A,$$

where $A$ is a (finite) real constant. Squaring both sides of the inequality, we obtain

$$|x(n)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations, we have

$$\left|x^2(n)\right| \leq A^2.$$

Using the fact that $\mathcal{H}x(n) = x^2(n)$, we can write

$$|\mathcal{H}x(n)| \leq A^2.$$

Since $A$ is finite, $A^2$ is also finite. Thus, we have that $\mathcal{H}x$ is bounded (i.e., $|\mathcal{H}x(n)| \leq A^2 < \infty$ for all $n$). Therefore, the system is BIBO stable. ■

**Example 8.24** (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

*Solution.* Suppose that we choose the input $x(n) = 1$. Clearly, $x$ is bounded (i.e., $|x(n)| \leq 1$ for all $n$). We can calculate the response $\mathcal{H}x$ to this input as follows:

$$\begin{aligned}
\mathcal{H}x(n) &= \sum_{k=-\infty}^{n} x(k) \\
&= \sum_{k=-\infty}^{n} 1 \\
&= \infty.
\end{aligned}$$

Thus, the output $\mathcal{H}x$ is unbounded for the bounded input $x$. Therefore, the system is not BIBO stable. ■

**Example 8.25.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

*Solution.* Suppose that $x$ is bounded. Then, $x(-n)$ is also bounded. Since the difference of two bounded sequences is bounded, $x(n) - x(-n)$ is bounded. Multiplication of a bounded sequence by a finite constant yields a bounded result. So, the sequence $\frac{1}{2}[x(n) - x(-n)]$ is bounded. Thus, $\mathcal{H}x$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable. ■

**Example 8.26.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \frac{x(n)}{x^2(n) - 4}.$$

*Solution.* Consider the input $x(n) = 2$. Clearly, $x$ is bounded (i.e., $|x(n)| \leq 2$ for all $n$). We can calculate the response $\mathcal{H}x$ to this input as follows:

$$\begin{aligned}
\mathcal{H}x(n) &= \frac{2}{2^2 - 4} = \frac{2}{0} \\
&= \infty.
\end{aligned}$$

Thus, the output $\mathcal{H}x$ is unbounded for the bounded input $x$. Therefore, the system is not BIBO stable. ■

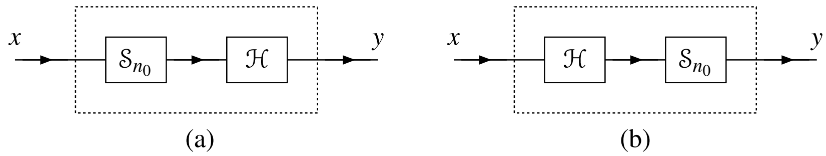



Figure 8.20: Systems that are equivalent if $\mathcal{H}$ is time invariant (i.e., $\mathcal{H}$ commutes with $\mathcal{S}_{n_0}$). (a) A system that first time shifts by $n_0$ and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}\mathcal{S}_{n_0}x$); and (b) a system that first applies $\mathcal{H}$ and then time shifts by $n_0$ (i.e., $y = \mathcal{S}_{n_0}\mathcal{H}x$).

### 8.7.5 Time Invariance

A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every sequence $x$ and every integer $n_0$, the following condition holds:

$$\mathcal{H}x(n-n_0) = \mathcal{H}x'(n) \text{ for all } n \quad \text{where} \quad x'(n) = x(n-n_0)$$

(i.e., $\mathcal{H}$ *commutes with time shifts*). In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input sequence results in an identical time shift in the output sequence. A system that is not time invariant is said to be **time varying** (or **shift varying**). In effect, time invariance means that the two systems shown in Figure 8.20 are equivalent, where $\mathcal{S}_{n_0}$ denotes an operator that applies a time shift of $n_0$ to a sequence (i.e., $\mathcal{S}_{n_0}x(n) = x(n-n_0)$).

In simple terms, a time invariant system is a system whose behavior does not change with respect to time. Practically speaking, compared to time-varying systems, time-invariant systems are much easier to design and analyze, since their behavior does not change over time.

**Example 8.27.** Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \sin[x(n)].$$

*Solution.* Let $x'(n) = x(n-n_0)$, where $n_0$ is an arbitrary integer constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(n-n_0) &= \sin[x(n-n_0)] \quad \text{and} \\ \mathcal{H}x'(n) &= \sin[x'(n)] \\ &= \sin[x(n-n_0)]. \end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ for all $x$ and $n_0$, the system is time invariant. ■

**Example 8.28.** Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = nx(n).$$

*Solution.* Let $x'(n) = x(n-n_0)$, where $n_0$ is an arbitrary integer constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(n-n_0) &= (n-n_0)x(n-n_0) \quad \text{and} \\ \mathcal{H}x'(n) &= nx'(n) \\ &= nx(n-n_0). \end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ does not hold for all $x$ and $n_0$, the system is not time invariant (i.e., the system is time varying). ■

**Example 8.29.** Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \sum_{k=-10}^{10} kx(n-k).$$

*Solution.* Let $x'(n) = x(n-n_0)$, where $n_0$ is an arbitrary integer constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned}\mathcal{H}x(n-n_0) &= \sum_{k=-10}^{10} kx(n-n_0-k) \quad \text{and} \\ \mathcal{H}x'(n) &= \sum_{k=-10}^{10} kx'(n-k) \\ &= \sum_{k=-10}^{10} kx(n-k-n_0) \\ &= \sum_{k=-10}^{10} kx(n-n_0-k).\end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ for all $x$ and $n_0$, the system is time invariant. ■

**Example 8.30.** Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

*Solution.* Let $x'(n) = x(n-n_0)$, where $n_0$ is an arbitrary integer constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}\mathcal{H}x(n-n_0) &= \tfrac{1}{2}[x(n-n_0) - x(-(n-n_0))] \\ &= \tfrac{1}{2}[x(n-n_0) - x(-n+n_0)] \quad \text{and} \\ \mathcal{H}x'(n) &= \tfrac{1}{2}[x'(n) - x'(-n)] \\ &= \tfrac{1}{2}[x(n-n_0) - x(-n-n_0)].\end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ does not hold for all $x$ and $n_0$, the system is not time invariant. ■

### 8.7.6 Linearity

Two of the most and frequently-occurring mathematical operations are addition and scalar multiplication. For this reason, it is often extremely helpful to know if these operations commute with the operation performed by a given system. The system properties to be introduced next relate to this particular issue.

A system $\mathcal{H}$ is said to be **additive** if, for all sequences $x_1$ and $x_2$, the following condition holds:

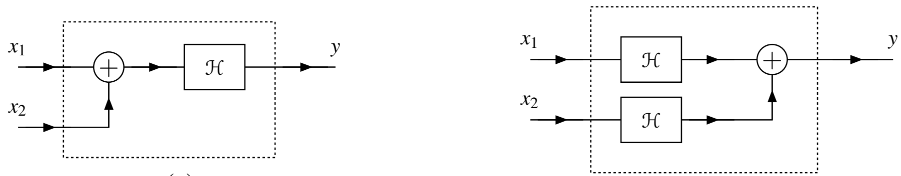
$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with addition*). Essentially, a system $\mathcal{H}$ being additive means that the two systems shown in Figure 8.21 are equivalent.

A system $\mathcal{H}$ is said to be **homogeneous** if, for every sequence $x$ and every complex constant $a$, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

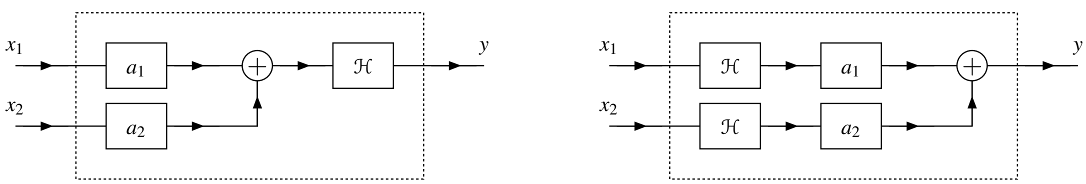



(a) (b)

Figure 8.21: Systems that are equivalent if $\mathcal{H}$ is additive (i.e., $\mathcal{H}$ commutes with addition). (a) A system that first performs addition and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(x_1 + x_2)$); and (b) a system that first applies $\mathcal{H}$ and then performs addition (i.e., $y = \mathcal{H}x_1 + \mathcal{H}x_2$).


(a) (b)

Figure 8.22: Systems that are equivalent if $\mathcal{H}$ is homogeneous (i.e., $\mathcal{H}$ commutes with scalar multiplication). (a) A system that first performs scalar multiplication and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(ax)$); and (b) a system that first applies $\mathcal{H}$ and then performs scalar multiplication (i.e., $y = a\mathcal{H}x$).




(a) (b)

Figure 8.23: Systems that are equivalent if $\mathcal{H}$ is linear (i.e., $\mathcal{H}$ commutes with linear combinations). (a) A system that first computes a linear combination and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(a_1x_1 + a_2x_2)$); and (b) a system that first applies $\mathcal{H}$ and then computes a linear combination (i.e., $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$).

(i.e., $\mathcal{H}$ *commutes with scalar multiplication*). Essentially, a system $\mathcal{H}$ being homogeneous means that the two systems shown in Figure 8.22 are equivalent.

The additivity and homogeneity properties can be combined into a single property known as superposition. In particular, a system $\mathcal{H}$ is said to have the **superposition** property, if for all sequences $x_1$ and $x_2$ and all complex constants $a_1$ and $a_2$, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with linear combinations*). A system that is both additive and homogeneous (or equivalently satisfies superposition) is said to be **linear**. Essentially, a system $\mathcal{H}$ being linear means that the two systems shown in Figure 8.23 are equivalent. To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds. Practically speaking, linear systems are much easier to design and analyze than nonlinear systems.

**Example 8.31.** Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = nx(n).$$

*Solution.* Let $x'(n) = a_1x_1(n) + a_2x_2(n)$, where $x_1$ and $x_2$ are arbitrary sequences and $a_1$ and $a_2$ are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= a_1nx_1(n)+a_2nx_2(n) \quad \text{and} \\ \mathcal{H}x'(n) &= nx'(n) \\ &= n\left[a_1x_1(n)+a_2x_2(n)\right] \\ &= a_1nx_1(n)+a_2nx_2(n). \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all $x_1$, $x_2$, $a_1$, and $a_2$, the superposition property holds and the system is linear. ■

**Example 8.32.** Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = |x(n)|.$$

*Solution.* Let $x'(n) = a_1x_1(n)+a_2x_2(n)$, where $x_1$ and $x_2$ are arbitrary sequences and $a_1$ and $a_2$ are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= a_1|x_1(n)|+a_2|x_2(n)| \quad \text{and} \\ \mathcal{H}x'(n) &= \left|x'(n)\right| \\ &= |a_1x_1(n)+a_2x_2(n)|. \end{aligned}$$

At this point, we recall the triangle inequality (F.16). Thus, $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ cannot hold for all $x_1$, $x_2$, $a_1$, and $a_2$ due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(n) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(n) = 0,$$

in which case

$$a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) = -1 \quad \text{and} \quad \mathcal{H}x'(n) = 1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

**Example 8.33.** Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n)-x(-n)\right].$$

*Solution.* Let $x'(n) = a_1x_1(n)+a_2x_2(n)$, where $x_1$ and $x_2$ are arbitrary sequences and $a_1$ and $a_2$ are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= \tfrac{1}{2}a_1[x_1(n)-x_1(-n)]+\tfrac{1}{2}a_2[x_2(n)-x_2(-n)] \quad \text{and} \\ \mathcal{H}x'(n) &= \tfrac{1}{2}[x'(n)-x'(-n)] \\ &= \tfrac{1}{2}[a_1x_1(n)+a_2x_2(n)-[a_1x_1(-n)+a_2x_2(-n)]] \\ &= \tfrac{1}{2}[a_1x_1(n)-a_1x_1(-n)+a_2x_2(n)-a_2x_2(-n)] \\ &= \tfrac{1}{2}a_1[x_1(n)-x_1(-n)]+\tfrac{1}{2}a_2[x_2(n)-x_2(-n)]. \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all $x_1$, $x_2$, $a_1$, and $a_2$, the system is linear. ■

**Example 8.34.** Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = x(n)x(n-1).$$

*Solution.* Let $x'(n) = a_1x_1(n) + a_2x_2(n)$, where $x_1$ and $x_2$ are arbitrary sequences and $a_1$ and $a_2$ are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(n) + a_2\mathcal{H}x_2(n) &= a_1x_1(n)x_1(n-1) + a_2x_2(n)x_2(n-1) \quad \text{and} \\ \mathcal{H}x'(n) &= x'(n)x'(n-1) \\ &= [a_1x_1(n) + a_2x_2(n)][a_1x_1(n-1) + a_2x_2(n-1)] \\ &= a_1^2x_1(n)x_1(n-1) + a_1a_2x_1(n)x_2(n-1) + a_1a_2x_1(n-1)x_2(n) + a_2^2x_2(n)x_2(n-1). \end{aligned}$$

Clearly, the expressions for $\mathcal{H}(a_1x_1 + a_2x_2)$ and $a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ are quite different. Consequently, these expressions are not equal for many choices of $a_1$, $a_2$, $x_1$, and $x_2$ (e.g., $a_1 = 2$, $a_2 = 0$, $x_1(n) = 1$, and $x_2(n) = 0$). Therefore, the superposition property does not hold and the system is not linear. ■

**Example 8.35** (Ideal accumulator)**.** A system $\mathcal{H}$ is defined by the equation

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

Determine whether this system is additive and/or homogeneous. Also, determine whether this system is linear.

*Solution.* First, we consider the additivity property. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x_1(n) + \mathcal{H}x_2(n) &= \sum_{k=-\infty}^{n} x_1(k) + \sum_{k=-\infty}^{n} x_2(k) \quad \text{and} \\ \mathcal{H}\{x_1 + x_2\}(n) &= \sum_{k=-\infty}^{n} [x_1(k) + x_2(k)] \\ &= \sum_{k=-\infty}^{n} x_1(k) + \sum_{k=-\infty}^{n} x_2(k). \end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all $x_1$ and $x_2$, the system is additive.

Second, we consider the homogeneity property. Let $a$ denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a\mathcal{H}x(n) &= a\sum_{k=-\infty}^{n} x(k) \quad \text{and} \\ \mathcal{H}\{ax\}(n) &= \sum_{k=-\infty}^{n} ax(k) \\ &= a\sum_{k=-\infty}^{n} x(k). \end{aligned}$$

Since $\mathcal{H}(ax) = a\mathcal{H}x$ for all $x$ and $a$, the system is homogeneous.

Lastly, we consider the linearity property. The system is linear since it has both the additivity and homogeneity properties. ■

**Example 8.36.** A system $\mathcal{H}$ is given by

$$\mathcal{H}x(n) = \operatorname{Re}[x(n)].$$

Determine whether this system is additive and/or homogeneous. Also, determine whether this system is linear.

*Solution.* First, we check if the additivity property is satisfied. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}\mathcal{H}x_1(n) + \mathcal{H}x_2(n) &= \operatorname{Re}[x_1(n)] + \operatorname{Re}[x_2(n)] \quad \text{and} \\ \mathcal{H}\{x_1 + x_2\}(n) &= \operatorname{Re}[x_1(n) + x_2(n)] \\ &= \operatorname{Re}[x_1(n)] + \operatorname{Re}[x_2(n)].\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all $x_1$ and $x_2$, the system is additive.

Second, we check if the homogeneity property is satisfied. Let $a$ denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}a\mathcal{H}x(n) &= a\operatorname{Re}x(n) \quad \text{and} \\ \mathcal{H}\{ax\}(n) &= \operatorname{Re}[ax(n)].\end{aligned}$$

In order for $\mathcal{H}$ to be homogeneous, $a\mathcal{H}x = \mathcal{H}(ax)$ must hold for all $x$ and all complex $a$. Suppose that $a = j$ and $x$ is not identically zero (i.e., $x$ is not the sequence $x(n) = 0$). In this case, we have

$$\begin{aligned}a\mathcal{H}x(n) &= j\operatorname{Re}[x(n)] \quad \text{and} \\ \mathcal{H}\{ax\}(n) &= \operatorname{Re}[jx(n)] \\ &= \operatorname{Re}\left[j\left(\operatorname{Re}[x(n)] + j\operatorname{Im}[x(n)]\right)\right] \\ &= \operatorname{Re}(-\operatorname{Im}[x(n)] + j\operatorname{Re}[x(n)]) \\ &= -\operatorname{Im}[x(n)].\end{aligned}$$

Thus, the quantities $\mathcal{H}(ax)$ and $a\mathcal{H}x$ are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

### 8.7.7 Eigensequences

An **eigensequence** of a system $\mathcal{H}$ is a sequence $x$ that satisfies

$$\mathcal{H}x = \lambda x,$$

for some complex constant $\lambda$, which is called an **eigenvalue**. Essentially, a system behaves as an ideal amplifier (i.e., performs amplitude scaling) when presented with one of its eigensequences as input. The significance of the eigensequence property cannot be overstated. No matter how complicated a system might be, it exhibits extremely simple behavior for its eigensequences. We can often exploit this simplicity to reduce the complexity of solving many types of problems involving systems. In fact, as we will see later, eigensequences essentially form the basis for many of the mathematical tools that we use for studying systems.

**Example 8.37.** Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(n) = x(n) - x(n-1).$$

For each sequence $x$ given below, determine if $x$ is an eigensequence of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(n) = 2$; and
(b) $x(n) = n$.

*Solution.* (a) We have

$$\begin{aligned}\mathcal{H}x(n) &= 2 - 2 \\ &= 0 \\ &= 0x(n).\end{aligned}$$

Therefore, $x$ is an eigensequence of $\mathcal{H}$ with the eigenvalue 0.

(b) We have

$$\begin{aligned}\mathcal{H}x(n) &= n-(n-1) \\ &= 1 \\ &= \left(\tfrac{1}{n}\right)n \\ &= \tfrac{1}{n}x(n).\end{aligned}$$

Therefore, $x$ is not an eigensequence of $\mathcal{H}$. ■

**Example 8.38** (Ideal amplifier)**.** Consider the system $\mathcal{H}$ given by

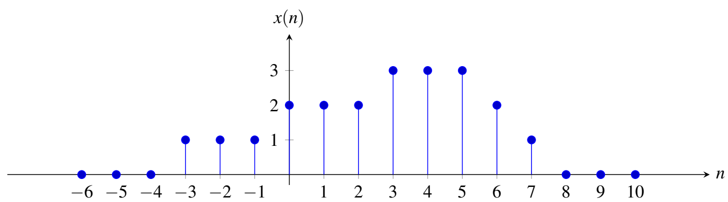
$$\mathcal{H}x(n) = ax(n),$$

where $a$ is a complex constant. Clearly, every sequence is an eigensequence of $\mathcal{H}$ with eigenvalue $a$. ■

## 8.8 Exercises

### 8.8.1 Exercises Without Answer Key

**8.1** Given the sequence $x$ shown in the figure below, sketch a graph of each sequence $y$ given below.
(a) $y(n) = x(n-2)$;
(b) $y(n) = x(n+2)$;
(c) $y(n) = x(2n)$;
(d) $y(n) = x(2n-1)$;
(e) $y(n) = x(3n+1)$; and
(f) $y(n) = x(1-2n)$.




**8.2** Let $x$ and $y$ be sequences that are related by $y(n) = x(an-b)$, where $a$ and $b$ are integers and $a \geq 1$.
(a) Show that $y$ can be obtained by time shifting $x$ by $b$ and then downsampling the resulting sequence by a factor of $a$.
(b) Show that, if $\frac{b}{a}$ is an integer, $y$ can be obtained by downsampling $x$ by a factor of $a$ and then time shifting the resulting sequence by $\frac{b}{a}$.

**8.3** Determine if each sequence $x$ given below is periodic, and if it is, find its fundamental period $N$.
(a) $x(n) = 2e^{j(3\pi/10)n}$;
(b) $x(n) = 2e^{j(3/10)n}$;
(c) $x(n) = 5e^{j3n/2} + 3e^{j5n/2}$;
(d) $x(n) = e^{j7(2n+1)/3} + e^{j7(3n+2)/5}$;
(e) $x(n) = \cos(2n)$;
(f) $x(n) = \cos\left(\frac{8\pi}{31}n\right)$;
(g) $x(n) = \sin\left(\frac{6\pi}{7}n+1\right)$;
(h) $x(n) = \sin\left(\frac{1}{10}n-\pi\right)$; and
(i) $x(n) = \cos\left(\frac{\pi}{9}n\right) + \sin\left(\frac{\pi}{10}n\right) + \cos\left(\frac{\pi}{4}n\right)$.

**8.4** Let $x$ be an $N$-periodic sequence. Determine whether each sequence $y$ given below is periodic.
(a) $y(n) = \mathrm{Odd}\{x\}(n)$;
(b) $y(n) = x(Nn)$; and
(c) $y(n) = x(2n)$.

**8.5** Let $x$ denote an arbitrary $N$-periodic sequence $x$. Show that:
(a) if $x$ is even, then $x(n) = x(N-n)$;
(b) if $x$ is odd, then $x(n) = -x(N-n)$; and
(c) if $x$ is odd, then $x(0) = 0$ for both even and odd $N$, and $x\left(\frac{N}{2}\right) = 0$ for even $N$.

**8.6** Show that a complex sinusoid of the form $x(n) = e^{j(2\pi\ell/m)n}$ has the fundamental period $N$ given by $N = \frac{m}{\gcd(\ell,m)}$, where $\gcd(\ell,m)$ denotes the greatest common divisor (GCD) of $\ell$ and $m$.

**8.7** Let $y$ be the sequence given by $y(n) = \sum_{k=-\infty}^{\infty} x(n-Nk)$, where $x$ is any arbitrary sequence and $N$ is a strictly positive integer constant. Show that $y$ is $N$ periodic.

**8.8** Determine whether each sequence $x$ given below is even, odd, or neither even nor odd.
(a) $x(n) = n^3$;
(b) $x(n) = n^3 |n|$;
(c) $x(n) = \left|n^3\right|$;
(d) $x(n) = \frac{1}{2}(e^n + e^{-n})$; and
(e) $x(n) = n+1$.

**8.9** Prove each of the assertions given below.
(a) The sum of two even sequences is even.
(b) The sum of two odd sequences is odd.
(c) The sum of an even sequence and an odd sequence, where neither of the sequences is identically zero, is neither even nor odd.
(d) The product of two even sequences is even.
(e) The product of two odd sequences is even.
(f) The product of an even sequence and an odd sequence is odd.

**8.10** Show that, if a sequence $x$ is odd, then $\sum_{k=-n}^{n} x(k) = 0$, where $n$ is a nonnegative integer constant.

**8.11** Show that the only sequence that is both even and odd is the zero sequence (i.e., the sequence $x$ satisfying $x(n) = 0$ for all $n$).

**8.12** Show that, for any sequence $x$,

$$\sum_{k=-\infty}^{\infty} x^2(k) = \sum_{k=-\infty}^{\infty} x_{\mathsf{e}}^2(k) + \sum_{k=-\infty}^{\infty} x_{\mathsf{o}}^2(k),$$

where $x_{\mathsf{e}}$ and $x_{\mathsf{o}}$ denote the even and odd parts of $x$, respectively.

**8.13** Consider the sequence

$$x(n) = \begin{cases} n^2 & 0 \leq n \leq 2 \\ 5 & 3 \leq n \leq 5 \\ 9-n & 6 \leq n \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Use the unit-step sequence to find a single expression for $x(n)$ that is valid for all $n$.

**8.14** Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(n) = 3x(n)$;
(b) $\mathcal{H}x(n) = x(n+1) - x(n)$;
(c) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$; and
(d) $\mathcal{H}x(n) = 42$.

**8.15** Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(n) = 3x(n)$;
(b) $\mathcal{H}x(n) = x(n-1)+1$;
(c) $\mathcal{H}x(n) = x(n+1)-x(n)$;
(d) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$;
(e) $\mathcal{H}x(n) = \sum_{k=n-4}^{n} x(k)$; and
(f) $\mathcal{H}x(n) = 3x(3n+3)$.

**8.16** Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(n) = x(n-3)$;
(b) $\mathcal{H}x(n) = x(2n-1)$;
(c) $\mathcal{H}x(n) = e^{x(n)}$;
(d) $\mathcal{H}x(n) = x(n)-x(n-1)$; and
(e) $\mathcal{H}x(n) = \operatorname{Even}\{x\}(n)$.

**8.17** Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(n) = 2x(n)+1$;
(b) $\mathcal{H}x(n) = \sum_{k=n}^{n+4} x(k)$;
(c) $\mathcal{H}x(n) = \frac{1}{x(n)}$;
(d) $\mathcal{H}x(n) = \frac{1}{1+x^2(n)}$;
(e) $\mathcal{H}x(n) = e^{-|x(n)|}$;
(f) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$; and
(g) $\mathcal{H}x(n) = \frac{1}{1+|x(n)|}$.

**8.18** Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(n) = x(n)-x(n-1)$;
(b) $\mathcal{H}x(n) = n^2x(n)$;
(c) $\mathcal{H}x(n) = \operatorname{Even}\{x\}(n)$;
(d) $\mathcal{H}x(n) = \sum_{k=-\infty}^{\infty} x(k)x(n-k)$; and
(e) $\mathcal{H}x(n) = \sum_{k=n-n_0}^{n} x(k)$, where $n_0$ is a strictly positive integer constant.

**8.19** Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(n) = x(n)+1$;
(b) $\mathcal{H}x(n) = \sum_{k=n-1}^{n+1} x(k)$;
(c) $\mathcal{H}x(n) = e^{x(n)}$;
(d) $\mathcal{H}x(n) = \operatorname{Even}\{x\}(n)$;
(e) $\mathcal{H}x(n) = x^2(n)$; and
(f) $\mathcal{H}x(n) = n^2x(n)$.

**8.20** Show that, for any system $\mathcal{H}$ that is either additive or homogeneous, if a sequence $x$ is identically zero (i.e., $x(n) = 0$ for all $n$), then $\mathcal{H}x$ is identically zero (i.e., $\mathcal{H}x(n) = 0$ for all $n$).

**8.21** For each system $\mathcal{H}$ and the sequences $\{x_k\}$ given below, determine if each of the $x_k$ is an eigensequence of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(n) = x^2(n)$, $x_1(n) = a$, $x_2(n) = e^{-an}$, and $x_3(n) = \cos n$, where $a$ is a complex constant;
(b) $\mathcal{H}x(n) = x(n+1)-x(n)$, $x_1(n) = e^{an}$, $x_2(n) = e^{an^2}$, and $x_3(n) = 42$, where $a$ is a real constant; and
(c) $\mathcal{H}x(n) = |x(n)|$, $x_1(n) = a$, $x_2(n) = n$, $x_3(n) = n^2$, where $a$ is a strictly positive real constant.

### 8.8.2 Exercises With Answer Key

**8.101** Determine if each sequence $x$ given below is periodic and, if it is, find its fundamental period.
(a) $x(n) = e^{j2\pi n/3}$;
(b) $x(n) = e^{j3\pi n/4}$;
(c) $x(n) = \cos(2\pi n)$;
(d) $x(n) = e^{j(3\pi/7)n} + e^{j(\pi/2)n} + \sin\left(\frac{5\pi}{9}n\right)$;
(e) $x(n) = \cos(\pi n) + \sin\left(\frac{\pi}{2}n\right) + e^{j(\pi/3)n}$;
(f) $x(n) = \sin\left(\frac{16\pi}{3}n + \frac{\pi}{11}\right)$; and
(g) $x(n) = \sin\left(-\frac{7\pi}{13}n + \frac{\pi}{4}\right)$.

**Short Answer.** (a) 3-periodic; (b) 8-periodic; (c) 1-periodic (d) 252-periodic; (e) 12-periodic; (f) 3-periodic; (g) 26-periodic

**8.102** Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(n) = x(-n)$; and
(b) $\mathcal{H}x(n) = 3x(3n+3)$.

**Short Answer.** (a) has memory; (b) has memory

**8.103** Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(n) = x(3n)$.

**Short Answer.** (a) not invertible

**8.104** Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

**Short Answer.** (a) BIBO stable

**8.105** Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

**Short Answer.** (a) not time invariant

**8.106** Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

**Short Answer.** (a) linear

**8.107** For each system $\mathcal{H}$ and the sequences $\{x_k\}$ given below, determine if each of the $x_k$ is an eigensequence of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k)$, $x_1(n) = 2^n$, $x_2(n) = \delta(n)$, and $x_3(n) = u(n)$.

**Short Answer.** (a) $x_1$ is an eigensequence with eigenvalue 2; $x_2$ is not an eigensequence; $x_3$ is not an eigensequence

# Chapter 9

# Discrete-Time Linear Time-Invariant Systems

## 9.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were linearity and time invariance. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant (LTI)** systems.

## 9.2 Discrete-Time Convolution

In the context of LTI systems, a mathematical operation known as convolution turns out to be particularly important. The **convolution** of the sequences $x$ and $h$, denoted $x * h$, is defined as the sequence

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \tag{9.1}$$

Herein, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

Notationally, $x * h$ denotes a sequence, namely the sequence that results from convolving $x$ and $h$. In contrast, $x * h(n)$ denotes the sequence $x * h$ evaluated at $n$. Although we could equivalently write $x * h(n)$ with an extra pair of brackets as $(x * h)(n)$, we usually omit this extra pair of brackets, since doing so does not introduce any ambiguity and leads to more compact notation. That is, there is only one sensible way to group operations in the expression $x * h(n)$. The grouping $x * [h(n)]$ would not make sense since a convolution requires two sequences as operands and $h(n)$ is not a sequence, but rather the value of $h$ evaluated at $n$. Thus, the only sensible way to interpret the expression $x * h(n)$ is as $(x * h)(n)$.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolutional sum. Suppose that, for the given sequences $x$ and $h$, we wish to compute

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

Of course, we could naively attempt to compute $x * h$ by evaluating $x * h(n)$ as a separate summation for each possible value of $n$. This approach, however, is not feasible, as $n$ can assume an infinite number of values, and therefore, an infinite number of summations would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the summation in terms of the intermediate sequence $w_n(k)$ where

$$w_n(k) = x(k)h(n-k).$$

(Note that $w_n(k)$ is implicitly a function of $n$.) This means that we need to compute

$$x * h(n) = \sum_{k=-\infty}^{\infty} w_n(k).$$

Now, we observe that, for most sequences $x$ and $h$ of practical interest, the form of $w_n(k)$ typically remains fixed over particular ranges of $n$. Thus, we can compute the convolution result $x * h$ by first identifying each of the distinct expressions for $w_n(k)$ and the range over which each expression is valid. Then, for each range, we evaluate a summation. In this way, we typically only need to compute a small number of summations instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for computing a convolution:

1. Plot $x(k)$ and $h(n-k)$ with respect to $k$.

2. Initially, consider an arbitrarily large negative value for $n$. This will result in $h(n-k)$ being shifted very far to the left on the time axis.

3. Write the mathematical expression for $w_n(k)$.

4. Increase $n$ gradually until the expression for $w_n(k)$ changes form. Record the interval over which the expression for $w_n(k)$ was valid.

5. Repeat steps 3 and 4 until $n$ is an arbitrarily large positive value. This corresponds to $h(n-k)$ being shifted very far to the right on the time axis.

6. For each of the intervals identified above, sum $w_n(k)$ in order to find an expression for $x * h(n)$. This will yield an expression for $x * h(n)$ for each interval.

7. The results for the various intervals can be combined in order to obtain the convolution result $x * h(n)$ for all $n$.

Depending on the form of the sequences being convolved, we may employ a number of variations on the above strategy. In most cases, however, drawing graphs of the various sequences involved in the convolution computation is extremely helpful, as this typically makes the general form of the answer much easier to visualize. Sometimes, the information from the graphs can be more conveniently represented in table form. So, the use of tables can also be quite helpful. In what follows, we consider several examples of computing convolutions.

**Example 9.1.** Compute $x * h$, where

$$x(n) = 2^{-|n|} \quad \text{and} \quad h(n) = u(n-2).$$

*Solution.* From the definition of convolution, we have

$$\begin{aligned} x * h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} 2^{-|k|} u(n-k-2). \end{aligned}$$

Since $u(n-k-2)=0$ for $k>n-2$, we can write

$$
\begin{aligned}
x*h(n) &= \sum_{k=-\infty}^{n-2} 2^{-|k|} \\
&= \begin{cases} \sum_{k=-\infty}^{n-2} 2^{k} & n-2 \leq 0 \\ \sum_{k=-\infty}^{0} 2^{k} + \sum_{k=1}^{n-2} 2^{-k} & n-2 > 0 \end{cases} \\
&= \begin{cases} \sum_{k=-\infty}^{n-2} 2^{k} & n \leq 2 \\ \sum_{k=-\infty}^{0} 2^{k} + \sum_{k=1}^{n-2} 2^{-k} & n > 2. \end{cases}
\end{aligned}
$$

Often, it can be somewhat tricky to identify the various cases that arise in the convolution computation. In this example, we have two cases: $n \leq 2$ and $n > 2$. The reason for these cases is more easily seen by examining the plots of $x(k)$ and $h(n-k)$ versus $k$, as shown in Figure 9.1. Now, we simplify each of the summations appearing in the above expression for $x*h$. We have

$$
\begin{aligned}
\sum_{k=-\infty}^{n-2} 2^{k} &= \sum_{k=2-n}^{\infty} \left(\tfrac{1}{2}\right)^{k} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k+2-n} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^{k} \\
&= \frac{\left(\frac{1}{2}\right)^{2-n}}{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{1}{2}\right)^{1-n} \\
&= 2^{n-1}, \\
\sum_{k=-\infty}^{0} 2^{k} &= \sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k} \\
&= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = (1)(2) \\
&= 2, \quad \text{and} \\
\sum_{k=1}^{n-2} 2^{-k} &= \sum_{k=0}^{n-3} 2^{-(k+1)} = \sum_{k=0}^{n-3} \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{k} \\
&= \left(\tfrac{1}{2}\right)\left(\frac{\left(\frac{1}{2}\right)^{n-2}-1}{\frac{1}{2}-1}\right) = \left(\tfrac{1}{2}\right)\left(\frac{\left(\frac{1}{2}\right)^{n-2}-1}{-\frac{1}{2}}\right) \\
&= 1-\left(\tfrac{1}{2}\right)^{n-2}.
\end{aligned}
$$

Substituting these simplified expressions into the earlier formula for $x*h$ yields

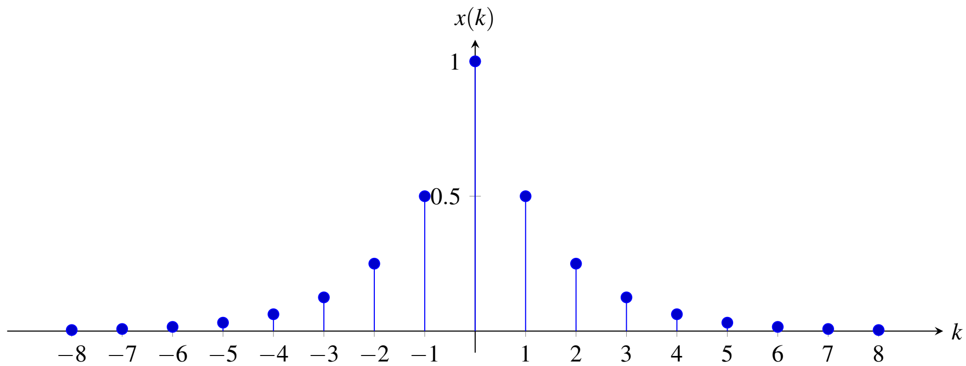
$$
x*h(n) = \begin{cases} 2^{n-1} & n \leq 2 \\ 3-\left(\frac{1}{2}\right)^{n-2} & n > 2. \end{cases}
$$

■

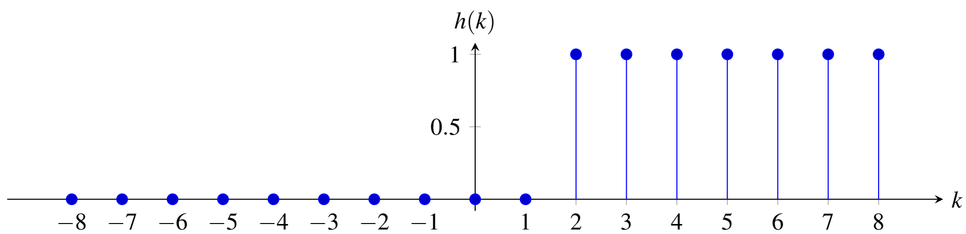
**Example 9.2.** Compute $x*h$, where

$$
x(n) = \begin{cases} \left(\frac{3}{4}\right)^{-n-4} & n \leq -5 \\ 1 & -4 \leq n \leq 4 \\ \left(\frac{3}{4}\right)^{n-4} & n \geq 5. \end{cases} \quad \text{and} \quad h(n) = u(n+2)-u(n-5).
$$

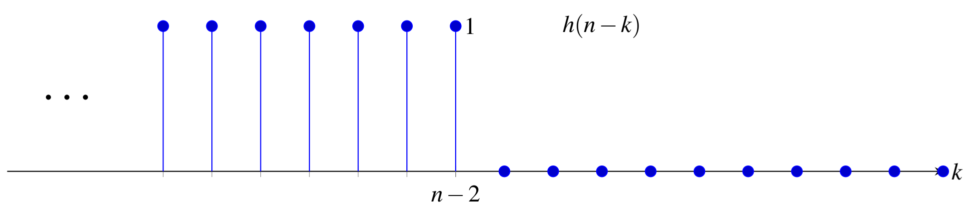
 




(a)




(b)




(c)

Figure 9.1: Plots for Example 9.1. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus $k$.

*Solution.* To assist in visualizing the various cases involved in the convolution computation, we plot $x(k)$, $h(k)$, and $h(n-k)$ versus $k$ in Figure 9.2. From these figures, we can deduce that that there are five cases in the convolution computation, namely:

1. $n \leq -7$, which comes from $n+2 \leq -5$;
2. $-6 \leq n \leq -1$, which comes from $n-4 \leq -5$ and $n+2 \geq -4$;
3. $0 \leq n \leq 2$, which comes from $n-4 \geq -4$ and $n+2 \leq 4$;
4. $3 \leq n \leq 8$, which comes from $n-4 \leq 4$ and $n+2 \geq 5$; and
5. $n \geq 9$, which comes from $n-4 \geq 5$.

We consider each of the preceding cases in turn.

In the first case, we have

$$\begin{aligned} x*h(n) &= \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{-k-4}(1) = \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{-k-4} \\ &= \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{-[k+n-4]-4} = \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{-k-n} = \sum_{k=0}^{6} \left(\tfrac{4}{3}\right)^{n} \left(\tfrac{4}{3}\right)^{k} = \left(\tfrac{4}{3}\right)^{n} \frac{\left(\tfrac{4}{3}\right)^{7}-1}{\tfrac{4}{3}-1} \\ &= \tfrac{14197}{729} \left(\tfrac{4}{3}\right)^{n}. \end{aligned}$$

In the second case, we have

$$\begin{aligned} x*h(n) &= \sum_{k=n-4}^{-5} \left(\tfrac{3}{4}\right)^{-k-4}(1) + \sum_{k=-4}^{n+2}(1)(1) = \left[\sum_{k=n-4}^{-5} \left(\tfrac{4}{3}\right)^{k+4}\right] + n + 7 \\ &= n+7+\sum_{k=0}^{-n-1} \left(\tfrac{4}{3}\right)^{-(k+n-4)-4} = n+7+\sum_{k=0}^{-n-1} \left(\tfrac{4}{3}\right)^{n} \left(\tfrac{4}{3}\right)^{k} = n+7+\left(\tfrac{4}{3}\right)^{n} \frac{\left(\tfrac{4}{3}\right)^{-n}-1}{\tfrac{4}{3}-1} \\ &= n+7+3\left(\tfrac{4}{3}\right)^{n}\left[\left(\tfrac{4}{3}\right)^{-n}-1\right] = n+7+3\left[1-\left(\tfrac{4}{3}\right)^{n}\right] \\ &= n-3\left(\tfrac{4}{3}\right)^{n}+10. \end{aligned}$$

In the third case, we have

$$\begin{aligned} x*h(n) &= \sum_{k=n-4}^{n+2}(1)(1) \\ &= 7. \end{aligned}$$

In the fourth case, we have

$$\begin{aligned} x*h(n) &= \sum_{k=n-4}^{4}(1)(1) + \sum_{k=5}^{n+2} \left(\tfrac{3}{4}\right)^{k-4}(1) = 8-n+\sum_{k=5}^{n+2} \left(\tfrac{3}{4}\right)^{k-4} \\ &= 8-n+\sum_{k=0}^{n-3} \left(\tfrac{3}{4}\right)^{k+5-4} = 8-n+\sum_{k=0}^{n-3} \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)^{k} = 8-n+\tfrac{3}{4}\frac{\left(\tfrac{3}{4}\right)^{n-2}-1}{\tfrac{3}{4}-1} \\ &= 8-n-4\left(\tfrac{3}{4}\right)\left[\left(\tfrac{3}{4}\right)^{n-2}-1\right] = 8-n-\left[3\left(\tfrac{3}{4}\right)^{n-2}-4\right] = 12-n-3\left(\tfrac{3}{4}\right)^{n-2} \\ &= 12-\tfrac{16}{3}\left(\tfrac{3}{4}\right)^{n}-n. \end{aligned}$$

In the fifth case, we have

$$\begin{aligned}
x*h(n) &= \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{k-4}(1) = \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{k-4} \\
&= \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{k+n-4-4} = \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{n-8} \left(\tfrac{3}{4}\right)^{k} = \left(\tfrac{3}{4}\right)^{n-8} \frac{\left(\frac{3}{4}\right)^7 - 1}{\frac{3}{4} - 1} \\
&= -4\left(\tfrac{3}{4}\right)^{n-8} \left[\left(\tfrac{3}{4}\right)^7 - 1\right] = \tfrac{14197}{4096} \left(\tfrac{3}{4}\right)^{n-8} \\
&= \tfrac{227152}{6561} \left(\tfrac{3}{4}\right)^{n}.
\end{aligned}$$

Combining the above results, we conclude

$$x*h(n) = \begin{cases} \frac{14197}{729}\left(\frac{4}{3}\right)^n & n \leq 7 \\ n - 3\left(\frac{4}{3}\right)^n + 10 & -6 \leq n \leq -1 \\ 7 & 0 \leq n \leq 2 \\ 12 - \frac{16}{3}\left(\frac{3}{4}\right)^n - n & 3 \leq n \leq 8 \\ \frac{227152}{6561}\left(\frac{3}{4}\right)^n & n \geq 9. \end{cases}$$

A plot of $x*h$ is given in Figure 9.3. ■

**Example 9.3.** Compute $x*h$, where

$$x(n) = \delta(n) + 2\delta(n-1) + \delta(n-2) \quad \text{and} \quad h(n) = \tfrac{1}{2}\delta(n+1) + \tfrac{1}{2}\delta(n).$$

*Solution.* This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows $x$ and the various time-reversed and shifted versions of $h$, we can easily compute the elements of $x*h$. The result of this process is shown in Table 9.1. From this table, we have

$$x*h(n) = \tfrac{1}{2}\delta(n+1) + \tfrac{3}{2}\delta(n) + \tfrac{3}{2}\delta(n-1) + \tfrac{1}{2}\delta(n-2).$$ ■

**Example 9.4.** Compute $x*h$, where

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) \quad \text{and} \quad h(n) = u(n).$$

*Solution.* From the definition of convolution, we have

$$\begin{aligned}
x*h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
&= \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^k u(k)u(n-k).
\end{aligned}$$

Since $u(k) = 0$ for $k < 0$ and $u(n-k) = 0$ for $k > n$, we can write

$$\begin{aligned}
x*h(n) &= \begin{cases} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k & n \geq 0 \\ 0 & \text{otherwise.} \end{cases} \\
&= \left(\sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k\right) u(n).
\end{aligned}$$

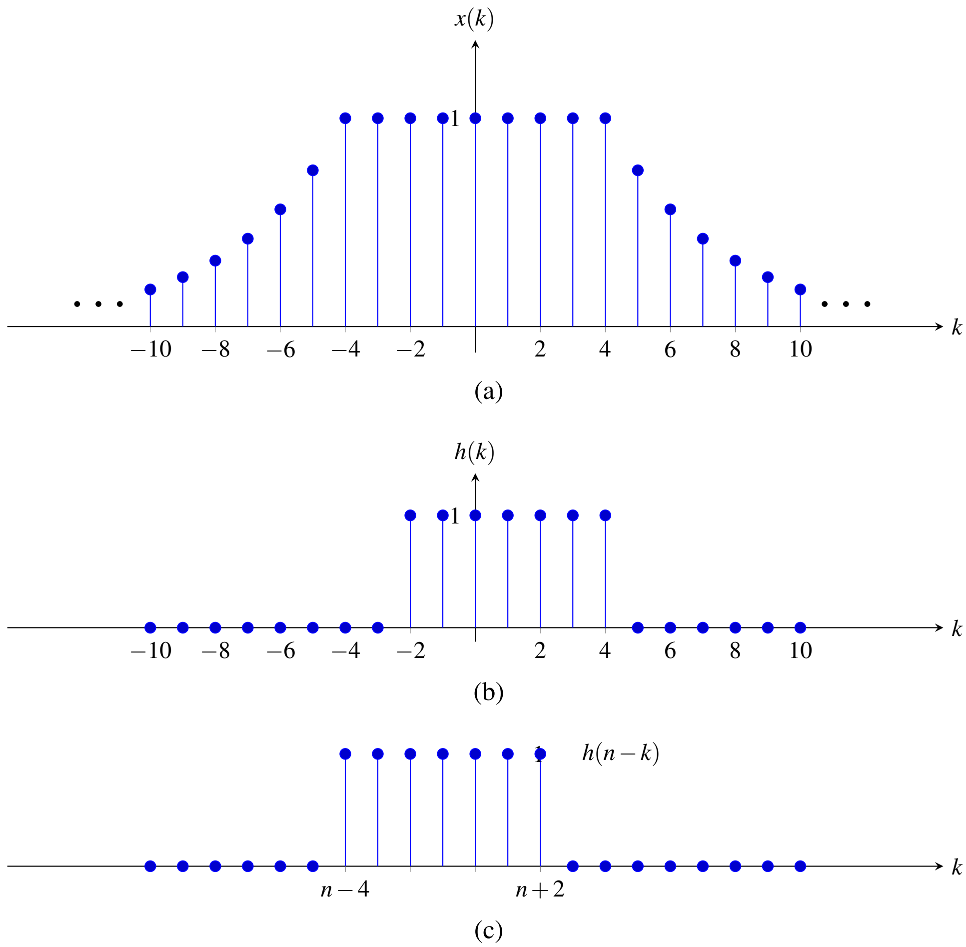



Figure 9.2: Plots for Example 9.2. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus $k$.

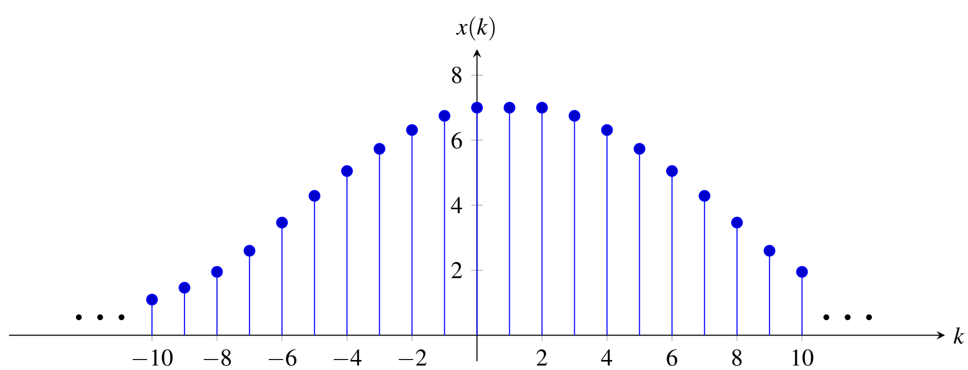



Figure 9.3: The sequence $x*h$ for Example 9.2.

Table 9.1: Convolution computation for Example 9.3

| $n \backslash k$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | |
|---|---|---|---|---|---|---|---|---|
| | $x(k)$ | | | | | | | |
| | | | $1$ | $2$ | $1$ | | | |
| | $h(k)$ | | | | | | | |
| | | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | | |
| | $h(-k)$ | | | | | | | |
| | | | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | |
| | $h(n-k)$ | | | | | | | $x*h(n)$ |
| $-2$ | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | | | $0$ |
| $-1$ | | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | | $\frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$ |
| $0$ | | | $\frac{1}{2}$ | $\frac{1}{2}$ | | | | $\frac{1}{2}(1)+\frac{1}{2}(2)=\frac{3}{2}$ |
| $1$ | | | | $\frac{1}{2}$ | $\frac{1}{2}$ | | | $\frac{1}{2}(2)+\frac{1}{2}(1)=\frac{3}{2}$ |
| $2$ | | | | | $\frac{1}{2}$ | $\frac{1}{2}$ | | $\frac{1}{2}(1)+\frac{1}{2}(0)=\frac{1}{2}$ |
| $3$ | | | | | | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ |

Often, it can be somewhat tricky to identify the various cases that arise in the convolution computation. In this example, we have two cases: $n \geq 0$ and $n < 0$. The reason for these cases is more easily seen by examining the plots of $x(k)$ and $h(n-k)$ versus $k$, as shown in Figure 9.4. Using the formula for the sum of a geometric sequence, we can write

$$\begin{aligned}
x*h(n) &= \left[\frac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1}\right]u(n) \\
&= -2\left[\left(\tfrac{1}{2}\right)^{n+1}-1\right]u(n) \\
&= \left[2-\left(\tfrac{1}{2}\right)^{n}\right]u(n).
\end{aligned}$$

■

**Example 9.5.** Compute $x*h$, where

$$x(n) = \delta(n)+3\delta(n-1)+9\delta(n-2)+9\delta(n-3)+3\delta(n-4)+\delta(n-5) \quad \text{and}$$
$$h(n) = \delta(n)-\delta(n-1).$$

*Solution.* This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows $x$ and the various time-reversed and shifted versions of $h$, we can easily compute the elements of $x*h$. The result of this process is shown in Table 9.2. From this table, we have

$$x*h(n) = \delta(n)+2\delta(n-1)+6\delta(n-2)-6\delta(n-4)-2\delta(n-5)-\delta(n-6).$$

■

## 9.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties.

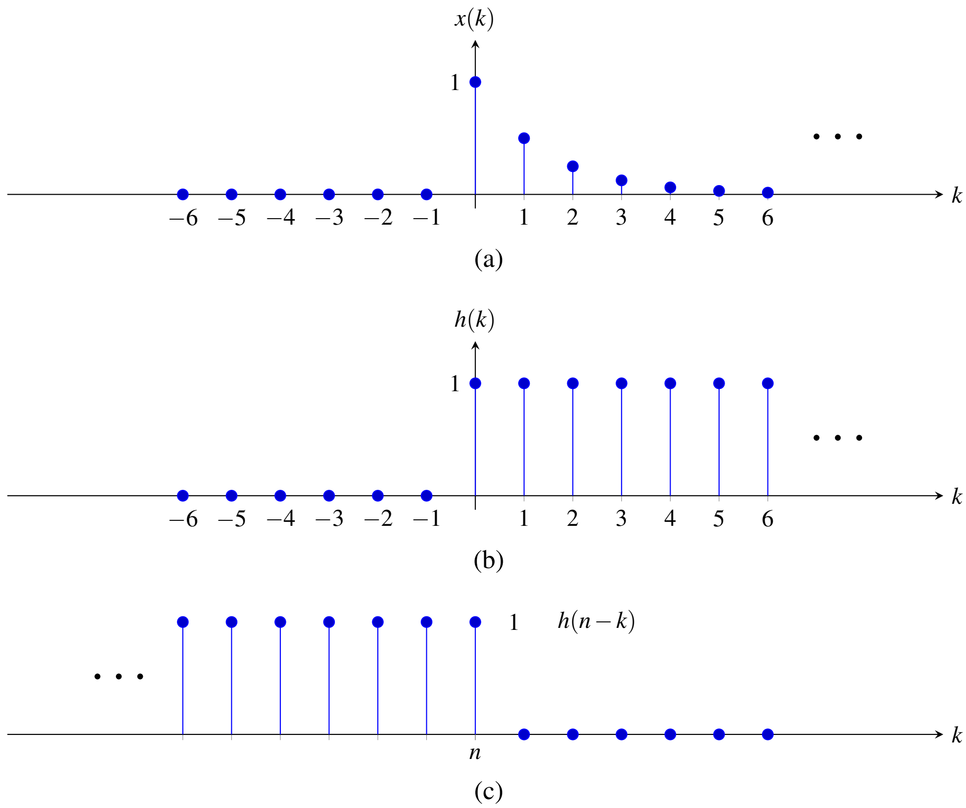



Figure 9.4: Plots for Example 9.4. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus $k$.

Table 9.2: Convolution computation for Example 9.5

| $n$ \ $k$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x(k)$ | | | | | | | | | | |
| | | | 1 | 3 | 9 | 9 | 3 | 1 | | | |
| | $h(k)$ | | | | | | | | | | |
| | | | 1 | $-1$ | | | | | | | |
| | $h(-k)$ | | | | | | | | | | |
| | | $-1$ | 1 | | | | | | | | |
| | $h(n-k)$ | | | | | | | | | | $x*h(n)$ |
| $-1$ | $-1$ | 1 | | | | | | | | | 0 |
| 0 | | $-1$ | 1 | | | | | | | | $1-0=1$ |
| 1 | | | $-1$ | 1 | | | | | | | $3-1=2$ |
| 2 | | | | $-1$ | 1 | | | | | | $9-3=6$ |
| 3 | | | | | $-1$ | 1 | | | | | $9-9=0$ |
| 4 | | | | | | $-1$ | 1 | | | | $3-9=-6$ |
| 5 | | | | | | | $-1$ | 1 | | | $1-3=-2$ |
| 6 | | | | | | | | $-1$ | 1 | | $0-1=-1$ |
| 7 | | | | | | | | | $-1$ | 1 | 0 |

**Theorem 9.1** (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two sequences $x$ and $h$,*

$$x * h = h * x. \tag{9.2}$$

*In other words, the result of a convolution is not affected by the order of its operands.*

*Proof.* We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (9.2) as follows:

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

Next, we perform a change of variable. Let $v = n-k$ which implies that $k = n-v$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned}
x * h(n) &= \sum_{v=n+\infty}^{n-\infty} x(n-v)h(v) \\
&= \sum_{v=\infty}^{-\infty} x(n-v)h(v) \\
&= \sum_{v=-\infty}^{\infty} x(n-v)h(v) \\
&= \sum_{v=-\infty}^{\infty} h(v)x(n-v) \\
&= h * x(n).
\end{aligned}$$

Thus, we have proven that convolution is commutative. ■

**Theorem 9.2** (Associativity of convolution)**.** *Convolution is associative. That is, for any three sequences $x$, $h_1$, and $h_2$,*

$$(x * h_1) * h_2 = x * (h_1 * h_2). \tag{9.3}$$

*In other words, the final result of multiple convolutions does not depend on how the convolution operations are grouped.*

*Proof.* To begin, we use the definition of the convolution operation to expand the left-hand side of (9.3) as follows:

$$\begin{aligned}
([x * h_1] * h_2)(n) &= \sum_{\ell=-\infty}^{\infty} [x * h_1(\ell)]h_2(n-\ell) \\
&= \sum_{\ell=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x(k)h_1(\ell-k) \right) h_2(n-\ell).
\end{aligned}$$

Now, we change the order of summation to obtain

$$([x * h_1] * h_2)(n) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} x(k)h_1(\ell-k)h_2(n-\ell).$$

Pulling the factor of $x(k)$ out of the inner summation yields

$$([x * h_1] * h_2)(n) = \sum_{k=-\infty}^{\infty} x(k) \sum_{\ell=-\infty}^{\infty} h_1(\ell-k)h_2(n-\ell).$$

Next, we perform a change of variable. Let $\lambda = \ell - k$ which implies that $\ell = \lambda + k$. Using this change of variable, we can write

$$\begin{aligned}
([x*h_1]*h_2)(n) &= \sum_{k=-\infty}^{\infty} x(k) \sum_{\lambda=-\infty-k}^{\infty-k} h_1(\lambda)h_2(n-\lambda-k) \\
&= \sum_{k=-\infty}^{\infty} x(k) \sum_{\lambda=-\infty}^{\infty} h_1(\lambda)h_2(n-\lambda-k) \\
&= \sum_{k=-\infty}^{\infty} x(k) \left( \sum_{\lambda=-\infty}^{\infty} h_1(\lambda)h_2([n-k]-\lambda) \right) \\
&= \sum_{k=-\infty}^{\infty} x(k)\,[h_1*h_2(n-k)] \\
&= (x*[h_1*h_2])(n).
\end{aligned}$$

Thus, we have proven that convolution is associative. ∎

**Theorem 9.3** (Distributivity of convolution)**.** *Convolution is distributive. That is, for any three sequences $x$, $h_1$, and $h_2$,*

$$x*(h_1+h_2) = x*h_1 + x*h_2. \tag{9.4}$$

*In other words, convolution can be distributed across addition.*

*Proof.* The proof of this property is relatively simple. Expanding the left-hand side of (9.4), we have:

$$\begin{aligned}
(x*[h_1+h_2])(n) &= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k)+h_2(n-k)] \\
&= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)h_2(n-k) \\
&= x*h_1(n) + x*h_2(n).
\end{aligned}$$

Thus, we have shown that convolution is distributive. ∎

The identity for an operation defined on elements of a set is often extremely helpful to know. Consider the operations of addition and multiplication as defined for real numbers. For any real number $a$, $a+0=a$. Since adding zero to $a$ has no effect (i.e., the result is $a$), we call 0 the **additive identity**. For any real number $a$, $1\cdot a = a$. Since multiplying $a$ by 1 has no effect (i.e., the result is $a$), we call 1 the **multiplicative identity**. Imagine for a moment how difficult arithmetic would be if we did not know that $a+0=a$ or $1\cdot a = a$. For this reason, identity values are clearly of fundamental importance.

Earlier, we were introduced to a new operation known as convolution. So, in light of the above, it is natural to wonder if there is a convolutional identity. In fact, there is, as given by the theorem below.

**Theorem 9.4** (Convolutional identity)**.** *For any sequence $x$,*

$$x*\delta = x. \tag{9.5}$$

*In other words, $\delta$ is the convolutional identity (i.e., convolving any sequence $x$ with $\delta$ simply yields $x$).*

*Proof.* Suppose that we have an arbitrary sequence $x$. From the definition of convolution, we can write

$$x*\delta(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$