Example 11.12. Using properties of the Fourier transform and the fact that

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a},$$

find the Fourier transform X of the sequence

$$x(n) = a^{|n|}.$$

Solution. We begin by rewriting x as

$$x(n) = a^n u(n) + a^{-n} u(-n) - \delta(n).$$

Taking the Fourier transform of both sides of this equation, we obtain

$$X(\Omega) = \mathcal{F}\{a^n u(n) + a^{-n} u(-n) - \delta(n)\}(\Omega).$$

Using the linearity property of the Fourier transform, we can rewrite the preceding equation as

$$X(\Omega) = \mathcal{F}\{a^n u(n)\}(\Omega) + \mathcal{F}\{a^{-n} u(-n)\}(\Omega) - \mathcal{F}\delta(\Omega).$$

Using the time-reversal property of the Fourier transform, we have

$$X(\Omega) = \mathcal{F}\{a^n u(n)\}(\Omega) + \mathcal{F}\{a^n u(n)\}(-\Omega) - \mathcal{F}\delta(\Omega).$$

Using the given Fourier transform pair and the fact that $\mathcal{F}\delta(\Omega) = 1$, we have

$$\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-a} + \frac{e^{-j\Omega}}{e^{-j\Omega}-a} - 1 \\
&= \frac{e^{j\Omega}(e^{-j\Omega}-a) + e^{-j\Omega}(e^{j\Omega}-a) + (e^{j\Omega}-a)(e^{-j\Omega}-a)}{(e^{j\Omega}-a)(e^{-j\Omega}-a)} \\
&= \frac{1 - ae^{j\Omega} + 1 - ae^{-j\Omega} - (1 - ae^{j\Omega} - ae^{-j\Omega} + a^2)}{1 - ae^{j\Omega} - ae^{-j\Omega} + a^2} \\
&= \frac{1-a^2}{1 - a(e^{j\Omega} + e^{-j\Omega}) + a^2} \\
&= \frac{1-a^2}{1 - 2a\cos\Omega + a^2}.
\end{aligned}$$

■

11.7.7 Upsampling

The next property of the Fourier transform to be introduced is the upsampling property, as given below.

Theorem 11.9 (Upsampling)**.** *If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then*

$$(\uparrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(M\Omega),$$

where M is a (strictly) positive integer. This is known as the ***upsampling property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = (\uparrow M)x(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} y(n)e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} [(\uparrow M)x](n)e^{-j\Omega n}.
\end{aligned}$$

From the definition of upsampling, we have

$$Y(\Omega) = \sum_{\substack{n\in\mathbb{Z}:\\ M \text{ divides } n}} x(n/M)e^{-j\Omega n}.$$

Rewriting the summation so that only the terms where M divides n are considered, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-jM\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(M\Omega)n} \\ &= X(M\Omega). \end{aligned}$$

Thus, we have shown that the upsampling property holds. ■

Example 11.13 (Upsampling property)**.** Using the Fourier transform pair

$$u(n) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = \begin{cases} 1 & n \geq 0 \text{ and } 3 \text{ divides } n \text{ (i.e., } n/3 \text{ is an integer)} \\ 0 & \text{otherwise.} \end{cases}$$

Solution. To begin, we observe that

$$x(n) = (\uparrow 3)u(n).$$

So, we have

$$X(\Omega) = \mathcal{F}\{(\uparrow 3)u(n)\}(\Omega).$$

From the upsampling property of the Fourier transform, we can write

$$X(\Omega) = \mathcal{F}u(3\Omega).$$

Using the given Fourier transform pair, we have

$$\begin{aligned} X(\Omega) &= \left.\left[\frac{e^{j\lambda}}{e^{j\lambda}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\lambda - 2\pi k)\right]\right|_{\lambda=3\Omega} \\ &= \frac{e^{j3\Omega}}{e^{j3\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(3\Omega - 2\pi k). \end{aligned}$$ ■

11.7.8 Downsampling

The next property of the Fourier transform to be introduced is the downsampling property, as given below.

Theorem 11.10 (Downsampling)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x(Mn) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{1}{M} \sum_{k=0}^{M-1} X\left[\tfrac{1}{M}(\Omega - 2\pi k)\right],$$

where M is a (strictly) positive integer. This is known as the ***downsampling property*** *of the Fourier transform.*

Proof. The proof immediately follows from Theorem 12.7 in the special case that $z = e^{j\Omega}$. ■

Example 11.14. Find the Fourier transform Y of the sequence

$$y(n) = (\downarrow 2)x(n),$$

where

$$x(n) = \tfrac{1}{4}\operatorname{sinc}\left(\tfrac{\pi}{4}n\right).$$

Solution. To begin, we compute the Fourier transform X of x. From Table 11.2, we have

$$\begin{aligned} X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{4}\operatorname{sinc}\left(\tfrac{\pi}{4}n\right)\right\}(\Omega) \\ &= \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{2}{\pi}(\Omega-2\pi k)\right]. \end{aligned}$$

Now, we consider Y. From the downsampling property of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \tfrac{1}{2}\sum_{\ell=0}^{1} X\left[\tfrac{1}{2}(\Omega-2\pi\ell)\right] \\ &= \tfrac{1}{2}X\left(\tfrac{1}{2}\Omega\right) + \tfrac{1}{2}X\left[\tfrac{1}{2}(\Omega-2\pi)\right]. \end{aligned}$$

Substituting the expression for X computed above into the preceding equation for Y, we have

$$\begin{aligned} Y(\Omega) &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{2}{\pi}(\tfrac{1}{2}\Omega-2\pi k)\right] + \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{2}{\pi}\left(\left[\tfrac{1}{2}(\Omega-2\pi)\right]-2\pi k\right)\right] \\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{2}{\pi}\left(\tfrac{1}{2}\right)(\Omega-4\pi k)\right] + \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{2}{\pi}\left(\tfrac{1}{2}\right)([(\Omega-2\pi)]-4\pi k)\right] \\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-4\pi k)\right] + \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{\pi}([(\Omega-2\pi)]-4\pi k)\right] \\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-4\pi k)\right] + \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-2\pi(2k+1))\right]. \end{aligned}$$

Note that each of the two summations on the right-hand side of the preceding equation are 4π-periodic functions. Consider $\Omega \in (-\pi,\pi]$. Over this interval, only the $k=0$ term from the first summation is nonzero and none of the terms in the second summation are nonzero. Therefore, we can simplify the preceding expression for Y to obtain

$$Y(\Omega) = \tfrac{1}{2}\operatorname{rect}\left(\tfrac{1}{\pi}\Omega\right) \quad \text{for } \Omega \in (-\pi,\pi].$$

To visualize how the two infinite summations collapse into a single term, some graphs are helpful. The spectrum X is shown in Figure 11.2(a). The two 4π-periodic functions from above are denoted as Y_0 and Y_1 so that $Y = Y_0 + Y_1$. One full period of each of Y_0 and Y_1 is shown in Figures 11.2(b) and (c). Observe that of all of the rectangular pulses in the 4π-periodic functions Y_0 and Y_1, only one pulse (namely, from Y_0) falls in the interval $(-\pi,\pi]$. The spectrum $Y = Y_0 + Y_1$ is shown in Figure 11.2(d). ■

11.7.9 Convolution

The next property of the Fourier transform to be introduced is the convolution property, as given below.

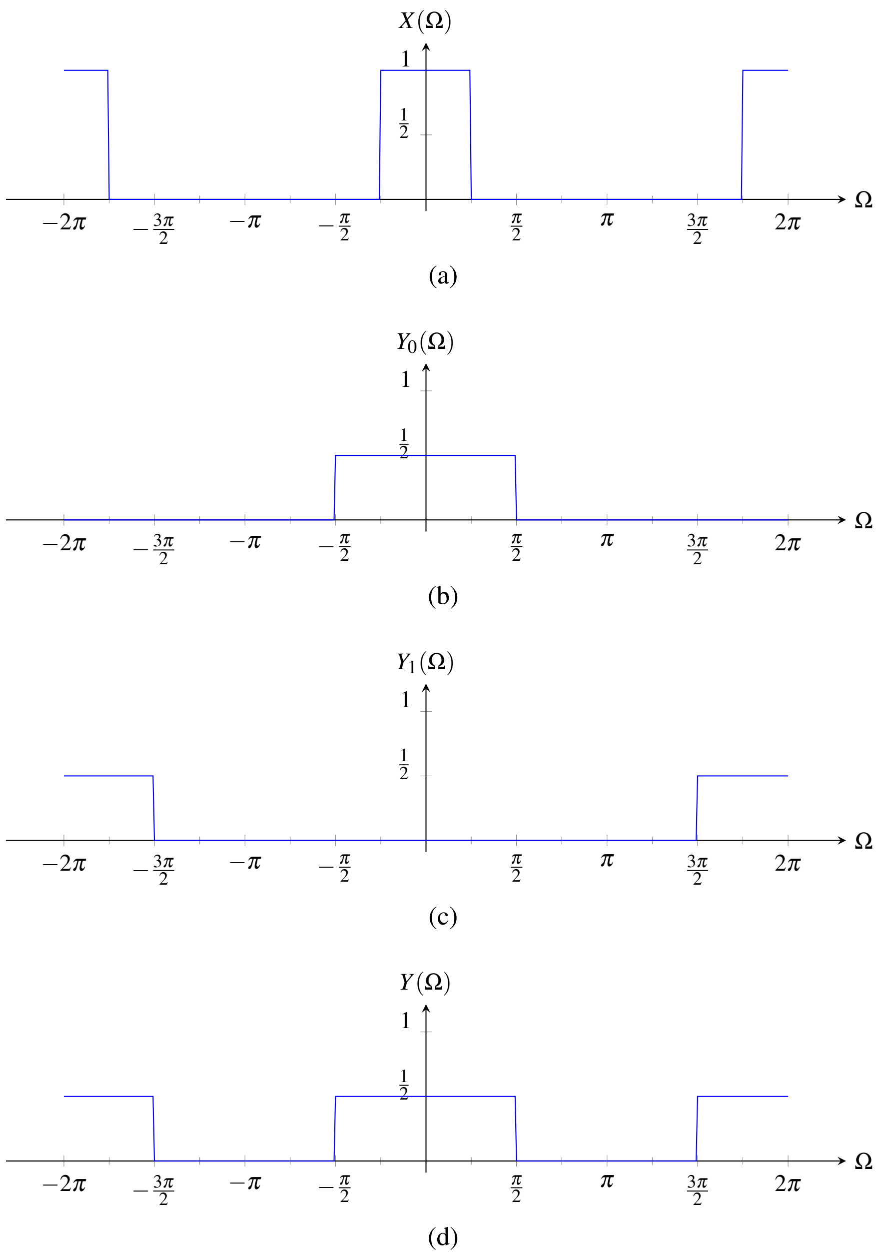

Figure 11.2: Spectra for downsampling example. (a) Spectrum X of x. (b) First summation Y_0 in expression for Y. (c) Second summation Y_1 in expression for Y. (d) Spectrum Y of y.

Theorem 11.11 (Convolution)**.** *If* $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ *and* $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$*, then*

$$x_1 * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)X_2(\Omega).$$

This is known as the ***convolution property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x_1 * x_2(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} [x_1 * x_2(n)]\, e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)e^{-j\Omega n}. \end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying the change of variable, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega(\lambda+k)} \\ &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega\lambda}e^{-j\Omega k} \\ &= \sum_{k=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega\lambda}e^{-j\Omega k} \\ &= \left[\sum_{k=-\infty}^{\infty} x_1(k)e^{-j\Omega k}\right]\left[\sum_{\lambda=-\infty}^{\infty} x_2(\lambda)e^{-j\Omega\lambda}\right] \\ &= X_1(\Omega)X_2(\Omega). \end{aligned}$$

Thus, we have shown that the time-domain convolution property holds. ■

The convolution property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts a convolution into a multiplication, the Fourier transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when solving problems involving LTI systems, for example, since such problems almost inevitably involve convolution (due to the fact that a LTI system computes a convolution).

Example 11.15 (Convolution property)**.** Find the Fourier transform X of the sequence

$$x(n) = x_1 * x_2(n)$$

where

$$x_1(n) = a^n u(n), \quad x_2(n) = a^{|n|},$$

and a is a complex constant satisfying $|a| < 1$.

Solution. We have

$$\begin{aligned} X(\Omega) &= \mathcal{F}\{x_1 * x_2\}(\Omega) \\ &= \mathcal{F}x_1(\Omega)\mathcal{F}x_2(\Omega). \end{aligned}$$

Using the Fourier transform pairs in Table 11.2, we have

$$\begin{aligned} X(\Omega) &= \left(\frac{e^{j\Omega}}{e^{j\Omega}-a}\right)\left(\frac{1-a^2}{1-2a\cos\Omega+a^2}\right) \\ &= \frac{(e^{j\Omega})(1-a^2)}{(e^{j\Omega}-a)(1-2a\cos\Omega+a^2)}. \end{aligned}$$ ■

11.7.10 Multiplication

The next property of the Fourier transform to be introduced is the multiplication property, as given below.

Theorem 11.12 (Multiplication)**.** *If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then*

$$x_1(n)x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta.$$

*This is known as the **multiplication property** of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x_1(n)x_2(n)$, and let Y denote the Fourier transform of y. We have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)e^{-j\Omega n}.$$

Rewriting $x_1(n)$ in terms of the formula for the inverse Fourier transform of X_1, we obtain

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x_2(n)\left[\tfrac{1}{2\pi}\int_{2\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right]e^{-j\Omega n}.$$

Interchanging the order of the integration and summation, we have

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2\pi}\int_{2\pi} X_1(\lambda)\sum_{n=-\infty}^{\infty} x_2(n)e^{j\lambda n}d\lambda e^{-j\Omega n} \\
&= \tfrac{1}{2\pi}\int_{2\pi} X_1(\lambda)\left[\sum_{n=-\infty}^{\infty} x_2(n)e^{j\lambda n}e^{-j\Omega n}\right]d\lambda \\
&= \tfrac{1}{2\pi}\int_{2\pi} X_1(\lambda)\left[\sum_{n=-\infty}^{\infty} x_2(n)e^{-j(\Omega-\lambda)n}\right]d\lambda \\
&= \tfrac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\Omega-\lambda)d\lambda.
\end{aligned}$$

Thus, we have shown that the multiplication property holds. ■

From the multiplication property in the preceding theorem, we can see that the Fourier transform effectively converts a multiplication operation into a convolution operation. Since convolution is significantly more complicated than multiplication, we normally prefer to avoid using this property in a manner that would result in the introduction of additional convolution operations into our work.

Example 11.16. Using the Fourier transform pair

$$\tfrac{B}{\pi}\operatorname{sinc}(Bn) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right) \quad \text{for } 0 < B < \pi,$$

find the Fourier transform X of the sequence

$$x(n) = \tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right),$$

where B is a real constant satisfying $0 < B < \frac{\pi}{2}$.

Solution. We were given the Fourier transform pair

$$\tfrac{B}{\pi}\operatorname{sinc}(Bn) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right).$$

Substituting $B/2$ for B in this pair yields

$$\tfrac{B}{2\pi}\operatorname{sinc}\left(\tfrac{B}{2}n\right) \stackrel{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{B}[\Omega-2\pi k]\right).$$

So, from the linearity property of the Fourier transform, we can infer

$$\operatorname{sinc}\left(\tfrac{B}{2}n\right) \stackrel{\text{DTFT}}{\longleftrightarrow} \tfrac{2\pi}{B}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{B}[\Omega-2\pi k]\right).$$

Taking the Fourier transform of x, we have

$$\begin{aligned} X(\Omega) &= \mathcal{F}\left\{\tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right)\right\}(\Omega) \\ &= \tfrac{B}{2\pi}\mathcal{F}\left\{\operatorname{sinc}^2\left(\tfrac{B}{2}n\right)\right\}(\Omega). \end{aligned}$$

Using the multiplication property of the Fourier transform and choosing the interval $[-\pi,\pi]$ for the resulting integration, we obtain

$$\begin{aligned} X(\Omega) &= \tfrac{B}{2\pi}\left(\tfrac{1}{2\pi}\left[\mathcal{F}\left\{\operatorname{sinc}\left(\tfrac{B}{2}n\right)\right\} \circledast \mathcal{F}\left\{\operatorname{sinc}\left(\tfrac{B}{2}n\right)\right\}\right](\Omega)\right) \\ &= \tfrac{B}{4\pi^2}\int_{-\pi}^{\pi}\left(\tfrac{2\pi}{B}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\theta-2\pi k]\right)\right)\left(\tfrac{2\pi}{B}\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta \\ &= \tfrac{1}{B}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\theta-2\pi k]\right)\right)\left(\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta. \end{aligned}$$

Since, for $\theta\in[-\pi,\pi]$, the left summation is zero for $k\neq 0$, we can rewrite the preceding equation as

$$X(\Omega) = \tfrac{1}{B}\int_{-\pi}^{\pi}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\left(\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta.$$

Since, for $\theta\in[-\pi,\pi]$, $\operatorname{rect}\left(\tfrac{1}{B}\theta\right)$ and the right summation are never both simultaneously nonzero for $\ell\neq 0$, we can rewrite the preceding equation as

$$X(\Omega) = \tfrac{1}{B}\int_{-\pi}^{\pi}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta]\right)d\theta.$$

Using the fact that the $\operatorname{rect}\left(\tfrac{1}{B}\theta\right)$ factor in the integrand is zero everywhere outside of the integration interval, we can rewrite the preceding equation as

$$\begin{aligned} X(\Omega) &= \tfrac{1}{B}\int_{-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta]\right)d\theta \\ &= \tfrac{1}{B}\left\{\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right]*\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right]\right\}(\Omega). \end{aligned}$$

One can show that $\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right]*\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right] = B\operatorname{tri}\left[\tfrac{1}{2B}(\cdot)\right]$. (The proof of this is left as an exercise for the reader in Exercise 4.102(b).) Using this fact, we can write

$$\begin{aligned} X(\Omega) &= \tfrac{1}{B}\left[B\operatorname{tri}\left(\tfrac{1}{2B}\Omega\right)\right] \\ &= \operatorname{tri}\left(\tfrac{1}{2B}\Omega\right). \end{aligned}$$

Thus, for all Ω, we have that

$$X(\Omega) = \sum_{k=-\infty}^{\infty}\operatorname{tri}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right).$$

Therefore, we have that

$$\tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right) \stackrel{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty}\operatorname{tri}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right) \quad \text{for } 0<B<\tfrac{\pi}{2}. \qquad \blacksquare$$

11.7.11 Frequency-Domain Differentiation

The next property of the Fourier transform to be introduced is the frequency-domain differentiation property, as given below.

Theorem 11.13 (Frequency-domain differentiation)**.** *If* $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$nx(n) \stackrel{\text{DTFT}}{\longleftrightarrow} j\tfrac{d}{d\Omega}X(\Omega).$$

This is known as the ***frequency-domain differentiation property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = nx(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} nx(n)e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n)\left(ne^{-j\Omega n}\right). \end{aligned}$$

Using that fact that $ne^{-j\Omega n} = \tfrac{d}{d\Omega}je^{-j\Omega n}$, we can rewrite the preceding equation as

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)\tfrac{d}{d\Omega}je^{-j\Omega n} \\ &= j\tfrac{d}{d\Omega}\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\ &= j\tfrac{d}{d\Omega}X(\Omega). \end{aligned}$$

Thus, we have shown that the frequency-domain differentiation property holds. (Alternatively, we could have simply differentiated both sides of (11.6) with respect to Ω to show the desired result.) ■

Example 11.17 (Frequency-domain differentiation property)**.** Using the Fourier transform pair

$$a^n u(n) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a} \quad \text{for } |a| < 1$$

and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = na^n u(n).$$

Solution. Let $v_1(n) = a^n u(n)$ so that

$$x(n) = nv_1(n).$$

From the given Fourier transform pair, we have

$$V_1(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-a}.$$

For convenience, let the prime symbol denote derivative in what follows. Taking the Fourier transform of x with the help of the frequency-domain differentiation property, we obtain

$$\begin{aligned}
X(\Omega) &= jV_1'(\Omega) \\
&= j\tfrac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-a}\right] \\
&= j\tfrac{d}{d\Omega}\left[(e^{j\Omega})(e^{j\Omega}-a)^{-1}\right] \\
&= j\left[(je^{j\Omega})(e^{j\Omega}-a)^{-1} + [(-1)(e^{j\Omega}-a)^{-2}(je^{j\Omega})](e^{j\Omega})\right] \\
&= -\frac{e^{j\Omega}}{e^{j\Omega}-a} + \frac{e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-e^{j\Omega}(e^{j\Omega}-a)+e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-e^{j2\Omega}+ae^{j\Omega}+e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}.
\end{aligned}$$

Thus, we conclude that

$$na^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2} \text{ for } |a|<1.$$ ■

11.7.12 Differencing

The next property of the Fourier transform to be introduced is the differencing property, as given below.

Theorem 11.14 (Differencing)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$x(n)-x(n-1) \overset{\text{DTFT}}{\longleftrightarrow} \left(1-e^{-j\Omega}\right)X(\Omega)$$

This is known as the ***differencing property*** *of the Fourier transform.*

Proof. The result of this theorem follows immediately from the linearity and translation properties of the Fourier transform. ■

Example 11.18 (Differencing property)**.** Given the Fourier transform pair

$$a^{|n|}, |a|<1 \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2},$$

find the Fourier transform X of the sequence

$$x(n) = a^{|n|} - a^{|n-1|},$$

where a is a complex constant satisfying $|a|<1$.

Solution. From the differencing property, we have

$$X(\Omega) = \frac{(1-e^{-j\Omega})(1-a^2)}{1-2a\cos\Omega+a^2}.$$ ■

11.7.13 Accumulation

The next property of the Fourier transform to be introduced is the accumulation property, as given below.

Theorem 11.15 (Accumulation)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

This is known as the ***accumulation property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = \sum_{k=-\infty}^{n} x(k)$, and let Y denote the Fourier transform of y. To begin, we observe that

$$y(n) = x * u(n).$$

That is, we have

$$\begin{aligned} y(n) &= x * u(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} x(k)u(n-k) + \sum_{k=n+1}^{\infty} x(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} x(k). \end{aligned}$$

Thus, from the convolution property of the Fourier transform, we have

$$Y(\Omega) = X(\Omega)U(\Omega).$$

From Table 11.2, we have the Fourier transform pair

$$u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k).$$

Using this pair, we can simplify the above expression for Y as follows:

$$\begin{aligned} Y(\Omega) &= X(\Omega)\left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)\right] \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \sum_{k=-\infty}^{\infty} \pi X(\Omega)\delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \sum_{k=-\infty}^{\infty} \pi X(2\pi k)\delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

Thus, we have shown that the accumulation property holds. ■

Example 11.19 (Accumulation property)**.** Find the Fourier transform X of the sequence

$$x(n) = u(n).$$

Solution. To begin, we observe that

$$x(n) = \sum_{k=-\infty}^{n} \delta(k).$$

So, using the accumulation property of the Fourier transform and the fact that $\mathcal{F}\delta(\Omega) = 1$ (from Table 11.2), we can write

$$\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-1}\mathcal{F}\delta(\Omega) + \pi\mathcal{F}\delta(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1}(1) + \pi(1)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k).
\end{aligned}$$

■

11.7.14 Parseval's Relation

The next property of the Fourier transform to be introduced, given below, relates to signal energy and is known as Parseval's relation.

Theorem 11.16 (Parseval's relation)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega. \tag{11.8}$$

*This is known as **Parseval's relation**.*

Proof. To prove the above property, we proceed as follows. We start from the expression on the left-hand side of (11.8). From properties of complex numbers, we have

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*(n)x(n).$$

Rewriting $x(n)$ in terms of the inverse Fourier transform of X, we have

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*(n)\tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega.$$

Interchanging the order of integration and summation, we obtain

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x(n)|^2 &= \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)\sum_{n=-\infty}^{\infty} x^*(n)e^{j\Omega n}d\Omega \\
&= \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)\left(\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}\right)^* d\Omega \\
&= \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)X^*(\Omega)d\Omega \\
&= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega.
\end{aligned}$$

Thus, we have shown that Parseval's relation holds. ■

Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform preserves energy (up to a scale factor). For example, if we are solving a problem in the Fourier domain, we do not have to return to the time domain to compute energy, since we can do this directly in the Fourier domain by using Parseval's relation.

Example 11.20. Find the energy E of the sequence

$$x(n) = \operatorname{sinc}\left(\tfrac{1}{2}n\right).$$

Solution. We could try to find E directly from its definition in terms of x using

$$\begin{aligned}
E &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}n\right)\right|^2 \\
&= \sum_{n=-\infty}^{\infty} \left|\frac{2\sin\left(\tfrac{1}{2}n\right)}{n}\right|^2.
\end{aligned}$$

This sum, however, would be rather tedious to compute. So, we instead use Parseval's relation to compute E in terms of X. From Table 11.2, we know that

$$2\pi\left[\frac{1}{2\pi}\operatorname{sinc}\left(\frac{n}{2}\right)\right] \stackrel{\text{DTFT}}{\longleftrightarrow} 2\pi\sum_{k=-\infty}^{\infty}\operatorname{rect}(\Omega-2\pi k).$$

Thus, we have

$$X(\Omega) = 2\pi\operatorname{rect}\Omega \quad \text{for } \Omega\in[-\pi,\pi).$$

Using this fact, we can find E as follows:

$$\begin{aligned}
E &= \tfrac{1}{2\pi}\int_{2\pi}|X(\Omega)|^2\,d\Omega \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi}|2\pi\operatorname{rect}\Omega|^2\,d\Omega \\
&= \tfrac{1}{2\pi}\int_{-1/2}^{1/2}(2\pi)^2 d\Omega \\
&= (\tfrac{1}{2\pi})(4\pi^2) \\
&= 2\pi.
\end{aligned}$$

■

11.7.15 Even/Odd Symmetry

The Fourier transform preserves symmetry. In other words, we have the result below.

Theorem 11.17 (Even/odd symmetry)**.** *For a sequence x with Fourier transform X, the following assertions hold:*

- *x is even if and only if X is even; and*
- *x is odd if and only if X is odd.*

Proof. First, we show that, if a sequence x is even/odd, then its Fourier transform X is even/odd. From the definition of the Fourier transform, we have

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Since x is even/odd, we have that $x(n) = \pm x(-n)$, where the plus case and minus case in the "$\pm$" correspond to x being even and odd, respectively. Using this, we can rewrite the above expression for $X(\Omega)$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \pm x(-n)e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $n' = -n$ so that $n = -n'$. Applying this change of variable and dropping the primes, we obtain

$$\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} \pm x(n)e^{-j\Omega(-n)} \\
&= \pm \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n} \\
&= \pm \sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n} \\
&= \pm X(-\Omega).
\end{aligned}$$

Therefore, X is even/odd.

Next, we show that if X is even/odd, then x is even/odd. From the definition of the inverse Fourier transform, we have

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega.$$

Since X is even/odd, we have that $X(\Omega) = \pm X(-\Omega)$, where the plus case and minus case in the "$\pm$" correspond to X being even and odd, respectively. Using this, we can rewrite the above expression for $x(n)$ as

$$\begin{aligned}
x(n) &= \tfrac{1}{2\pi}\int_{2\pi} \pm X(-\Omega)e^{j\Omega n}d\Omega \\
&= \tfrac{1}{2\pi}\int_{a}^{a+2\pi} \pm X(-\Omega)e^{j\Omega n}d\Omega.
\end{aligned}$$

Now, we employ a change of variable. Let $\lambda = -\Omega$ so that $\Omega = -\lambda$ and $d\Omega = -d\lambda$. Applying this change of variable, we obtain

$$\begin{aligned}
x(n) &= \tfrac{1}{2\pi}\int_{-a}^{-a-2\pi} \pm X(\lambda)e^{-j\lambda n}(-1)d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{-a-2\pi}^{-a} X(\lambda)e^{-j\lambda n}d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{2\pi} X(\lambda)e^{j\lambda(-n)}d\lambda \\
&= \pm x(-n).
\end{aligned}$$

Therefore, x is even/odd. This completes the proof. ■

In other words, the preceding theorem simply states that the forward and inverse Fourier transforms preserve even/odd symmetry.

11.7.16 Real Sequences

As it turns out, the Fourier transform of a real-valued sequence has a special structure, as given by the theorem below.

Theorem 11.18 (Real-valued sequences)**.** *A sequence x is real-valued if and only if its Fourier transform X satisfies*

$$X(\Omega) = X^*(-\Omega) \text{ for all } \Omega$$

(i.e., X is conjugate symmetric).

Proof. From the definition of the Fourier transform, we have

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}. \tag{11.9}$$

Substituting $-\Omega$ for Ω in the preceding equation, we have

$$X(-\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n}.$$

Conjugating both sides of this equation, we obtain

$$X^*(-\Omega) = \sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n}. \tag{11.10}$$

First, we show that x being real-valued implies that X is conjugate symmetric. Suppose that x is real-valued. Since x is real-valued, we can replace x^* with x in (11.10) to yield

$$X^*(-\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Observing that the right-hand side is simply $X(\Omega)$, we have

$$X^*(-\Omega) = X(\Omega).$$

Thus, x being real-valued implies that X is conjugate symmetric.

Next, we show that X being conjugate symmetric implies that x is real-valued. Suppose that X is conjugate symmetric. Since X is conjugate symmetric, the expressions for $X(\Omega)$ in (11.9) and $X^*(-\Omega)$ in (11.10) must be equal. Thus, we can write

$$\begin{aligned} & X(\Omega) - X^*(-\Omega) = 0 \\ \Rightarrow \quad & \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n} = 0 \\ \Rightarrow \quad & \sum_{n=-\infty}^{\infty} [x(n) - x^*(n)]e^{-j\Omega n} = 0. \end{aligned}$$

This implies that $x^* = x$. Therefore, x is real-valued. Thus, X being conjugate symmetric implies that x is real-valued. This completes the proof. ∎

Suppose that X is the Fourier transform of a real-valued sequence x so that X is conjugate symmetric. From properties of complex numbers, we can show that that X being conjugate symmetric is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{for all } \Omega \in \mathbb{R} \quad \text{and} \tag{11.11a}$$

$$\arg X(\Omega) = -\arg X(-\Omega) \quad \text{for all } \Omega \in \mathbb{R} \tag{11.11b}$$

(i.e., the magnitude and argument of X are even and odd, respectively).

Since the Fourier transform X of a real-valued sequence x is conjugate symmetric, the graph of X for negative values is completely redundant and can be determined from the graph of X for nonnegative values. Lastly, note that x being real-valued does not necessarily imply that X is real-valued, since a conjugate-symmetric function need not be real-valued.

Table 11.1: Properties of the DT Fourier transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Translation	$x(n-n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Modulation	$e^{j\Omega_0 n}x(n)$	$X(\Omega-\Omega_0)$
Conjugation	$x^*(n)$	$X^*(-\Omega)$
Time Reversal	$x(-n)$	$X(-\Omega)$
Upsampling	$(\uparrow M)x(n)$	$X(M\Omega)$
Downsampling	$x(Mn)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\Omega-2\pi k}{M}\right)$
Convolution	$x_1 * x_2(n)$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta$
Frequency-Domain Differentiation	$nx(n)$	$j\frac{d}{d\Omega}X(\Omega)$
Differencing	$x(n)-x(n-1)$	$\left(1-e^{-j\Omega}\right)X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega)+\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$

Property	
Periodicity	$X(\Omega)=X(\Omega+2\pi)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty}\lvert x(n)\rvert^2=\frac{1}{2\pi}\int_{2\pi}\lvert X(\Omega)\rvert^2 d\Omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

11.8 Discrete-Time Fourier Transform of Periodic Sequences

By making use of the generalized Fourier transform briefly discussed in Section 11.3, the Fourier transform can also be applied to periodic sequences. In particular, the Fourier transform of a periodic sequence can be computed using the result below.

Theorem 11.19 (Fourier transform of a periodic sequence)**.** *Let x be an N-periodic sequence with the corresponding Fourier-series coefficient sequence a. Let x_N denote the sequence*

$$x_N(n) = \begin{cases} x(n) & n \in [0 \,..\, N-1] \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x_N is a truncated/windowed version of the sequence x). (Note that x_N is a sequence equal to x over a single period and zero elsewhere.) Let X_N denote the Fourier transform of x_N. The Fourier transform X of x is given by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right), \tag{11.12a}$$

or equivalently,

$$X(\Omega) = \tfrac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\tfrac{2\pi}{N}k\right) \delta\left(\Omega - \tfrac{2\pi}{N}k\right). \tag{11.12b}$$

Furthermore, a and X_N are related by

$$a_k = \tfrac{1}{N} X_N\left(\tfrac{2\pi}{N}k\right). \tag{11.13}$$

Proof. Since x is N-periodic, we can express it using a Fourier series as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn} \tag{11.14a}$$

where

$$a_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}. \tag{11.14b}$$

Consider the expression for a_k in (11.14b). Since $x_N(n) = x(n)$ for a single period of x and is zero otherwise, we can rewrite (11.14b) as

$$\begin{aligned} a_k &= \tfrac{1}{N} \sum_{n=-\infty}^{\infty} x_N(n) e^{-j[(2\pi/N)k]n} \\ &= \tfrac{1}{N} X_N\left(\tfrac{2\pi}{N}k\right). \end{aligned} \tag{11.15}$$

Thus, we have shown (11.13) to be correct.

Now, let us consider the Fourier transform X of x. By taking the Fourier transform of both sides of (11.14a), we obtain

$$\begin{aligned} X(\Omega) &= \left(\mathcal{F}\left\{\sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn}\right\}\right)(\Omega) \\ &= \sum_{k=\langle N \rangle} a_k \mathcal{F}\left\{e^{j(2\pi/N)kn}\right\}(\Omega) \end{aligned}$$

Now, consider the Fourier transform on the right-hand side of the preceding equation. From Table 11.2 and the modulation property of the Fourier transform, we can deduce

$$\begin{aligned}
\mathcal{F}\{e^{j(2\pi/N)kn}\}(\Omega) &= \mathcal{F}\{e^{j(2\pi/N)kn} \cdot 1\}(\Omega) \\
&= 2\pi \sum_{\ell=-\infty}^{\infty} \delta\left([\Omega - \tfrac{2\pi}{N}k] - 2\pi\ell\right) \\
&= 2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\Omega - \tfrac{2\pi}{N}k - 2\pi\ell\right).
\end{aligned}$$

Substituting this Fourier transform into the above expression for $X(\Omega)$, we obtain

$$\begin{aligned}
X(\Omega) &= \sum_{k=\langle N\rangle} a_k \mathcal{F}\left\{e^{j(2\pi/N)kn}\right\}(\Omega) \\
&= \sum_{k=\langle N\rangle} a_k \left[2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\Omega - \tfrac{2\pi}{N}k - 2\pi\ell\right)\right].
\end{aligned}$$

Reversing the order of the two summations and then choosing the take the summation over k using the range $[0\,..\,N-1]$, we have

$$\begin{aligned}
X(\Omega) &= 2\pi \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k \delta\left(\Omega - \tfrac{2\pi}{N}k - 2\pi\ell\right) \\
&= 2\pi \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k \delta\left(\Omega - \tfrac{2\pi}{N}[N\ell + k]\right).
\end{aligned}$$

Now, we observe that the combined effect of the two summations is that $N\ell + k$ takes on every integer value exactly once. Consequently, we can combine the two summations into a single summation to yield

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right).$$

Thus, (11.12a) holds. Substituting (11.15) into the preceding equation for $X(\Omega)$, we have

$$\begin{aligned}
X(\Omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right) \\
&= 2\pi \sum_{k=-\infty}^{\infty} \tfrac{1}{N} X_N\left(\tfrac{2\pi}{N}k\right) \delta\left(\Omega - \tfrac{2\pi}{N}k\right) \\
&= \tfrac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\tfrac{2\pi}{N}k\right) \delta\left(\Omega - \tfrac{2\pi}{N}k\right).
\end{aligned}$$

Thus, (11.12b) holds. This completes the proof. ■

Theorem 11.19 above provides two formulas for computing the Fourier transform X of a periodic sequence x. One formula is in written terms of the Fourier series coefficient sequence a of x, while the other formula is in written in terms of the Fourier transform X_N of a sequence consisting of a single period of x. The choice of which formula to use would be driven by what information is available or most easily determined. For example, if the Fourier series coefficients of x were known, the use of (11.12b) would likely be preferred.

From Theorem 11.19, we can also make a few important observations. First, the Fourier transform of a periodic sequence is a series of impulse functions located at integer multiples of the fundamental frequency $\frac{2\pi}{N}$. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Second, the Fourier series coefficient sequence a of the periodic sequence x is produced by sampling the Fourier transform of x_N at integer multiples of the fundamental frequency $\frac{2\pi}{N}$ and scaling the resulting sequence by $\frac{1}{N}$.

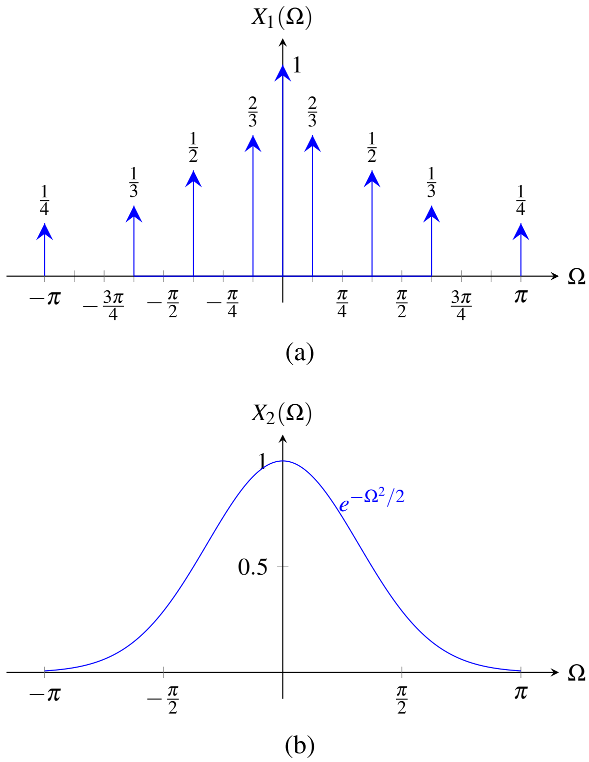

Figure 11.3: Frequency spectra. The frequency spectra (a) X_1 and (b) X_2.

Example 11.21. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. Suppose that X_1 and X_2 are as shown in Figures 11.3(a) and (b). Determine whether x_1 and x_2 are periodic.

Solution. We know that the Fourier transform X of an N-periodic sequence x must be of the form

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right),$$

where the $\{\alpha_k\}$ are complex constants. The spectrum X_1 does have this form, with $N = 16$. That is, impulses reside at integer multiples of the frequency $\frac{2\pi}{16} = \frac{\pi}{8}$. Therefore, x_1 must be 16-periodic. The spectrum X_2 does not have the required form. Therefore, x_2 must not be periodic. ■

Example 11.22. Consider the N-periodic sequence x shown in Figure 11.4, where $N = 16$. Using the Fourier transform, find the Fourier series representation of x.

Solution. Define the sequence y as the truncated/windowed version of x given by

$$\begin{aligned} y(n) &= \begin{cases} x(n) & n \in [0\,..\,15] \\ 0 & \text{otherwise} \end{cases} \\ &= u(n) - u(n-8). \end{aligned}$$

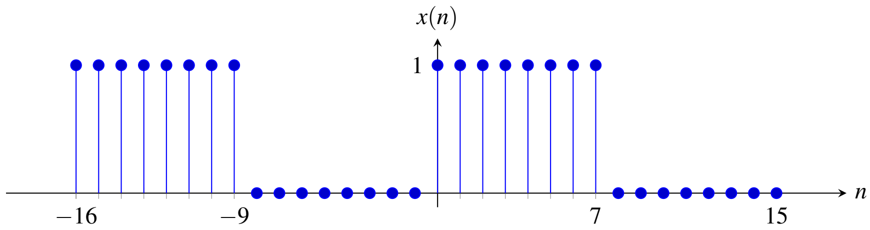

Figure 11.4: The 16-periodic sequence x.

Thus, we have that

$$\begin{aligned} x(n) &= \sum_{k=-\infty}^{\infty} y(n-Nk) \\ &= \sum_{k=-\infty}^{\infty} y(n-16k). \end{aligned}$$

From Table 11.2, the Fourier transform Y of y is given by

$$\begin{aligned} Y(\Omega) &= \left(\mathcal{F}\{u(n)-u(n-8)\}\right)(\Omega) \\ &= e^{-j\Omega(8-1)/2}\left[\frac{\sin\left(\frac{8}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right] \\ &= e^{-j(7/2)\Omega}\left[\frac{\sin(4\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right]. \end{aligned}$$

Now, we seek to find the Fourier series representation of x, which has the form

$$\begin{aligned} x(n) &= \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle 16\rangle} c_k e^{j(\pi/8)kn}. \end{aligned}$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{N}Y\left(\tfrac{2\pi}{N}k\right) \\ &= \tfrac{1}{16}Y\left(\tfrac{\pi}{8}k\right) \\ &= \tfrac{1}{16}e^{-j(7/2)(\pi/8)k}\left(\frac{\sin\left[4\left(\frac{\pi}{8}k\right)\right]}{\sin\left[\frac{1}{2}\left(\frac{\pi}{8}k\right)\right]}\right) \\ &= \tfrac{1}{16}e^{-j(7\pi/16)k}\left[\frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{16}k\right)}\right]. \end{aligned}$$

■

11.9 More Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 11.2. Using the various Fourier transform properties listed in Table 11.1 and the Fourier transform pairs listed in Table 11.2, we can determine (more easily) the Fourier transforms of more complicated sequences.

Table 11.2: Transform pairs for the DT Fourier transform

Pair	$x(n)$	$X(\Omega)$
1	$\delta(n)$	1
2	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega-2\pi k)$
3	$u(n)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega-2\pi k)$
4	$a^n u(n), \lvert a\rvert<1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
5	$-a^n u(-n-1), \lvert a\rvert>1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
6	$a^{\lvert n\rvert}, \lvert a\rvert<1$	$\frac{1-a^2}{1-2a\cos\Omega+a^2}$
7	$\cos(\Omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)]$
8	$\sin(\Omega_0 n)$	$j\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)]$
9	$\cos(\Omega_0 n)u(n)$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1} + \frac{\pi}{2}\sum_{k=-\infty}^{\infty} [\delta(\Omega-2\pi k-\Omega_0)+\delta(\Omega-2\pi k+\Omega_0)]$
10	$\sin(\Omega_0 n)u(n)$	$\frac{e^{j\Omega}\sin\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1} + \frac{\pi}{2j}\sum_{k=-\infty}^{\infty} [\delta(\Omega-2\pi k-\Omega_0)-\delta(\Omega-2\pi k+\Omega_0)]$
11	$\frac{B}{\pi}\operatorname{sinc}(Bn), 0<B<\pi$	$\sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{\Omega-2\pi k}{2B}\right)$
12	$u(n)-u(n-M)$	$e^{-j\Omega(M-1)/2}\left(\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\right)$
13	$na^n u(n), \lvert a\rvert<1$	$\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}$

Example 11.23. Using a table of Fourier transform pairs and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = n\left(\tfrac{1}{2}\right)^{|n+3|}.$$

Solution. We reexpress x as

$$x(n) = nv_2(n),$$

where

$$v_2(n) = v_1(n+3) \quad \text{and}$$
$$v_1(n) = \left(\tfrac{1}{2}\right)^{|n|}.$$

Let V_1 and V_2 denote the Fourier transforms of v_1 and v_2, respectively. Taking the Fourier transform of v_1 using Table 11.2, we have

$$\begin{aligned}
V_1(\Omega) &= \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\Omega+\left(\frac{1}{2}\right)^2} \\
&= \frac{\frac{3}{4}}{1-\cos\Omega+\frac{1}{4}} \\
&= \frac{3}{4-4\cos\Omega+1} \\
&= \frac{3}{5-4\cos\Omega} \\
&= 3(5-4\cos\Omega)^{-1}.
\end{aligned}$$

Taking the Fourier transform of v_2, we have

$$V_2(\Omega) = e^{j3\Omega}V_1(\Omega).$$

Taking the Fourier transform of x, we have

$$X(\Omega) = jV_2'(\Omega).$$

Taking the derivative of V_1 yields

$$\begin{aligned}
V_1'(\Omega) &= \tfrac{d}{d\Omega}\left[3(5-4\cos\Omega)^{-1}\right] \\
&= -3(5-4\cos\Omega)^{-2}(4\sin\Omega) \\
&= \frac{-12\sin\Omega}{(5-4\cos\Omega)^2}.
\end{aligned}$$

Taking the derivative of V_2 yields

$$V_2'(\Omega) = 3je^{j3\Omega}V_1(\Omega) + e^{j3\Omega}V_1'(\Omega).$$

Substituting the preceding formula for V_2' into the above expression for $X(\Omega)$, we have

$$
\begin{aligned}
X(\Omega) &= jV_2'(\Omega) \\
&= j\left[3je^{j3\Omega}V_1(\Omega) + e^{j3\Omega}V_1'(\Omega)\right] \\
&= -3e^{j3\Omega}V_1(\Omega) + je^{j3\Omega}V_1'(\Omega) \\
&= -3e^{j3\Omega}\frac{3}{5-4\cos\Omega} + je^{j3\Omega}\frac{-12\sin\Omega}{(5-4\cos\Omega)^2} \\
&= e^{j3\Omega}\left[\frac{-9}{5-4\cos\Omega} + \frac{-12j\sin\Omega}{(5-4\cos\Omega)^2}\right] \\
&= e^{j3\Omega}\left[\frac{-9(5-4\cos\Omega)-12j\sin\Omega}{(5-4\cos\Omega)^2}\right] \\
&= e^{j3\Omega}\left[\frac{36\cos\Omega-12j\sin\Omega-45}{(5-4\cos\Omega)^2}\right] \\
&= 3e^{j3\Omega}\left[\frac{12\cos\Omega-4j\sin\Omega-15}{(5-4\cos\Omega)^2}\right].
\end{aligned}
$$

■

Example 11.24. Using a table of Fourier transform pairs and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = (n+1)a^n u(n),$$

where a is complex constant satisfying $|a| < 1$.

Solution. Let $v(n) = a^n u(n)$. We rewrite x as

$$
\begin{aligned}
x(n) &= (n+1)a^n u(n) \\
&= na^n u(n) + a^n u(n) \\
&= nv(n) + v(n).
\end{aligned}
$$

Let V denote the Fourier transform of v. Taking the Fourier transform of v, we have

$$
\begin{aligned}
V(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-a} \\
&= e^{j\Omega}\left(e^{j\Omega}-a\right)^{-1}.
\end{aligned}
$$

Taking the Fourier transform of x, we have

$$X(\Omega) = jV'(\Omega) + V(\Omega).$$

Taking the derivative of V, we have

$$\begin{aligned}
V'(\Omega) &= je^{j\Omega}\left(e^{j\Omega}-a\right)^{-1}+(-1)\left(e^{j\Omega}-a\right)^{-2}\left(je^{j\Omega}\right)e^{j\Omega} \\
&= \frac{je^{j\Omega}}{e^{j\Omega}-a}-\frac{je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{je^{j\Omega}(e^{j\Omega}-a)-je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{je^{j2\Omega}-aje^{j\Omega}-je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-jae^{j\Omega}}{(e^{j\Omega}-a)^2}.
\end{aligned}$$

Substituting the above formulas for V' and V into the expression for $X(\Omega)$ from above, we obtain

$$\begin{aligned}
X(\Omega) &= j\left[\frac{-jae^{j\Omega}}{(e^{j\Omega}-a)^2}\right]+\frac{e^{j\Omega}}{e^{j\Omega}-a} \\
&= \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}+\frac{e^{j\Omega}}{e^{j\Omega}-a} \\
&= \frac{ae^{j\Omega}+e^{j\Omega}\left(e^{j\Omega}-a\right)}{(e^{j\Omega}-a)^2} \\
&= \frac{ae^{j\Omega}+e^{j2\Omega}-ae^{j\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{e^{j2\Omega}}{[e^{j\Omega}(1-ae^{-j\Omega})]^2} \\
&= \frac{e^{j2\Omega}}{e^{j2\Omega}(1-ae^{-j\Omega})^2} \\
&= \frac{1}{(1-ae^{-j\Omega})^2}.
\end{aligned}$$

■

Example 11.25. Consider the N-periodic sequence x shown in Figure 11.5, where $N=7$. Find the Fourier transform X of x.

Solution. In what follows, we will use the prime symbol to denote the derivative. To begin, we observe that x is given by

$$x(n)=\sum_{k=-\infty}^{\infty} y(n-7k),$$

where

$$y(n)=n[u(n+3)-u(n-4)].$$

Now, we re-express y as

$$y(n)=nv_2(n),$$

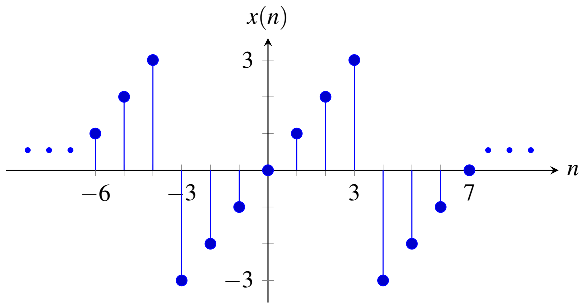

Figure 11.5: The 7-periodic sequence x.

where

$$\begin{aligned} v_2(n) &= v_1(n+3) \quad \text{and} \\ v_1(n) &= u(n) - u(n-7). \end{aligned}$$

Let Y, V_1, and V_2 denote the Fourier transforms of y, v_1, and v_2, respectively. Taking the Fourier transform of the equation for v_1 with the assistance of Table 11.2, we have

$$\begin{aligned} V_1(\Omega) &= e^{-j\Omega(7-1)/2}\left(\frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right) \\ &= e^{-j3\Omega}\left(\frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right). \end{aligned}$$

Taking the Fourier transform of the equation for v_2 using the translation property of the Fourier transform, we have

$$\begin{aligned} V_2(\Omega) &= e^{j3\Omega}V_1(\Omega) \\ &= \frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

Now, we compute the derivative of V_2. We have

$$\begin{aligned} V_2'(\Omega) &= \frac{\left[\sin\left(\frac{1}{2}\Omega\right)\right]\left[\frac{7}{2}\cos\left(\frac{7}{2}\Omega\right)\right] - \left[\sin\left(\frac{7}{2}\Omega\right)\right]\left[\frac{1}{2}\cos\left(\frac{1}{2}\Omega\right)\right]}{\sin^2\left(\frac{1}{2}\Omega\right)} \\ &= \frac{\frac{7}{2}\cos\left(\frac{7}{2}\Omega\right)\sin\left(\frac{1}{2}\Omega\right) - \frac{1}{2}\cos\left(\frac{1}{2}\Omega\right)\sin\left(\frac{7}{2}\Omega\right)}{\sin^2\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

Taking the Fourier transform of the equation for y using the frequency-domain differentiation property of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= jV_2'(\Omega) \\ &= \frac{\frac{7j}{2}\cos\left(\frac{7}{2}\Omega\right)\sin\left(\frac{1}{2}\Omega\right) - \frac{j}{2}\cos\left(\frac{1}{2}\Omega\right)\sin\left(\frac{7}{2}\Omega\right)}{\sin^2\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

From Theorem 11.19, we know that

$$X(\Omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\Omega - \tfrac{2\pi}{7}k\right),$$

where

$$a_k = \tfrac{1}{7} Y\left(\tfrac{2\pi}{7}k\right).$$

Evaluating a_k, we have

$$\begin{aligned}
a_k &= \tfrac{1}{7} Y\left(\tfrac{2\pi}{7}k\right) \\
&= \tfrac{1}{7}\left[\frac{\frac{7j}{2}\cos\left[\frac{7}{2}\left(\frac{2\pi}{7}\right)k\right]\sin\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right] - \frac{j}{2}\cos\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right]\sin\left[\frac{7}{2}\left(\frac{2\pi}{7}\right)k\right]}{\sin^2\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right]}\right] \\
&= \tfrac{1}{7}\left[\frac{\frac{7j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right) - \frac{j}{2}\cos\left(\frac{\pi}{7}k\right)\sin(\pi k)}{\sin^2\left(\frac{\pi}{7}k\right)}\right] \\
&= \frac{\frac{j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right) - \frac{j}{14}\cos\left(\frac{\pi}{7}k\right)\sin(\pi k)}{\sin^2\left(\frac{\pi}{7}k\right)}.
\end{aligned}$$

If $\frac{k}{7} \notin \mathbb{Z}$, we have

$$\begin{aligned}
a_k &= \frac{\frac{j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right)}{\sin^2\left(\frac{\pi}{7}k\right)} \\
&= \frac{j(-1)^k\sin\left(\frac{\pi}{7}k\right)}{2\sin^2\left(\frac{\pi}{7}k\right)} \\
&= \frac{j(-1)^k}{2\sin\left(\frac{\pi}{7}k\right)}.
\end{aligned}$$

(In the above simplification, we used the fact $\sin(\pi k) = 0$ and $\cos(\pi k) = (-1)^k$ for all $k \in \mathbb{Z}$.) Otherwise (i.e., if $\frac{k}{7} \in \mathbb{Z}$), we have

$$\begin{aligned}
a_k &= a_0 \\
&= \tfrac{1}{7} Y(0) \\
&= \tfrac{1}{7} \sum_{n=-\infty}^{\infty} y(n) \\
&= 0.
\end{aligned}$$

Thus, we conclude

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{7}k\right),$$

where

$$a_k = \begin{cases} \dfrac{j(-1)^k}{2\sin\left(\frac{\pi}{7}k\right)} & \frac{k}{7} \notin \mathbb{Z} \\ 0 & \frac{k}{7} \in \mathbb{Z}. \end{cases}$$

■

Example 11.26. Let x and y denote two sequences related by

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} x(-k).$$

Find the Fourier transform Y of y in terms of the Fourier transform X of x.

