From the eigensequence properties of LTI systems, we have

$$y(n) = \sum_{k=-1}^{2} b_k e^{j(2\pi/4)kn} \quad \text{where} \quad b_k = a_k H\left(\tfrac{2\pi}{4}k\right).$$

Computing the b_k, we have

$$\begin{aligned}
b_{-1} &= a_{-1}H\left(\tfrac{2\pi}{4}[-1]\right) = a_{-1}H\left(-\tfrac{\pi}{2}\right) \\
&= \left(\tfrac{1}{4}\right)\left(\frac{1}{2-e^{-j(-\pi/2)}}\right) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2-j}\right) = \tfrac{1}{8-4j} = \left(4\sqrt{5}e^{-j\pi/4}\right)^{-1} \\
&= \tfrac{1}{4\sqrt{5}}e^{j\pi/4}, \\
b_0 &= a_0 H\left(\tfrac{2\pi}{4}[0]\right) = a_0 H(0) \\
&= (1)\left(\frac{1}{2-e^{-j0}}\right) \\
&= 1, \\
b_1 &= a_1 H\left(\tfrac{2\pi}{4}[1]\right) = a_1 H\left(\tfrac{\pi}{2}\right) \\
&= \left(\tfrac{1}{4}\right)\left(\frac{1}{2-e^{-j\pi/2}}\right) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2+j}\right) = \tfrac{1}{8+4j} = \left(4\sqrt{5}e^{j\pi/4}\right)^{-1} \\
&= \tfrac{1}{4\sqrt{5}}e^{-j\pi/4}, \quad \text{and} \\
b_2 &= a_2 H\left(\tfrac{2\pi}{4}[2]\right) = a_2 H(\pi) \\
&= \left(\tfrac{1}{8}\right)\left(\frac{1}{2-e^{-j\pi}}\right) = \left(\tfrac{1}{8}\right)\left(\tfrac{1}{2+1}\right) \\
&= \tfrac{1}{24}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
y(n) &= \sum_{k=-1}^{2} b_k e^{j(2\pi/4)kn} \\
&= b_{-1}e^{j(2\pi/4)(-1)n} + b_0 e^{j(2\pi/4)(0)n} + b_1 e^{j(2\pi/4)(1)n} + b_2 e^{j(2\pi/4)(2)n} \\
&= \tfrac{1}{4\sqrt{5}}e^{j\pi/4}e^{-j(\pi/2)n} + 1 + \tfrac{1}{4\sqrt{5}}e^{-j\pi/4}e^{j(\pi/2)n} + \tfrac{1}{24}e^{j\pi n} \\
&= 1 + \tfrac{1}{4\sqrt{5}}\left[e^{-j\pi/4}e^{j(\pi/2)n} + e^{j\pi/4}e^{j(-\pi/2)n}\right] + \tfrac{1}{24}\cos(\pi n) \\
&= 1 + \tfrac{1}{4\sqrt{5}}\left[e^{j[(\pi/2)n-\pi/4]} + e^{-j[(\pi/2)n-\pi/4]}\right] + \tfrac{1}{24}\cos(\pi n) \\
&= 1 + \tfrac{1}{4\sqrt{5}}\cos\left(\tfrac{\pi}{2}n - \tfrac{\pi}{4}\right) + \tfrac{1}{24}\cos(\pi n).
\end{aligned}$$

The magnitude spectrum of the output y is shown in Figure 10.4.

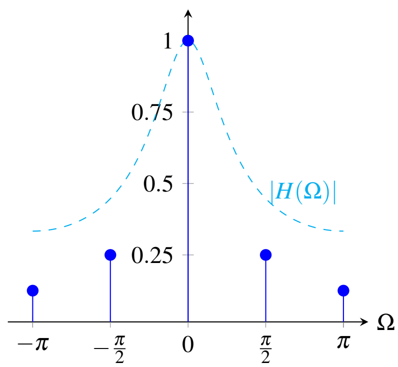


Figure 10.3: Magnitude spectrum of input sequence x.

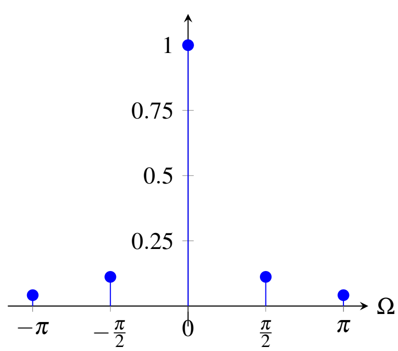


Figure 10.4: Magnitude spectrum of output sequence y.

In passing, we note that H can be written in Cartesian form as

$$\begin{aligned}
H(\Omega) &= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}[\cos(-\Omega)+j\sin(-\Omega)]}\right) \\
&= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}\cos\Omega+\frac{j}{2}\sin\Omega}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{(1-\frac{1}{2}\cos\Omega)^2+(-\frac{1}{2}\sin\Omega)^2}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{1-\cos\Omega+\frac{1}{4}\cos^2\Omega+\frac{1}{4}\sin^2\Omega}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{\frac{5}{4}-\cos\Omega}\right) \\
&= \frac{2-\cos\Omega-j\sin\Omega}{5-4\cos\Omega} \\
&= \frac{\cos\Omega-2}{4\cos\Omega-5}+j\frac{\sin\Omega}{4\cos\Omega-5}.
\end{aligned}$$

This expression was used to assist in the generation the plot of $|H(\cdot)|$ in Figure 10.3. ■

Example 10.11. Consider a LTI system with the frequency response

$$H(\Omega) = e^{-j\Omega}.$$

Find the response y of the system to the input x, where

$$x(n) = \tfrac{1}{2}\cos\left(\tfrac{2\pi}{5}n\right).$$

Solution. To begin, we observe that x is 5-periodic. We rewrite x as

$$x(n) = \tfrac{1}{4}\left(e^{j(2\pi/5)n}+e^{-j(2\pi/5)n}\right).$$

Thus, the Fourier series for x is given by

$$x(n) = \sum_{k=-2}^{2} c_k e^{j(2\pi/5)kn},$$

where

$$c_k = \begin{cases} \frac{1}{4} & k\in\{-1,1\} \\ 0 & k\in\{-2,0,2\}. \end{cases}$$

Thus, we can write

$$\begin{aligned}
y(n) &= \sum_{k=-2}^{2} c_k H(\tfrac{2\pi}{5}k) e^{j(2\pi/5)kn} \\
&= c_{-1} H(-\tfrac{2\pi}{5}) e^{-j(2\pi/5)n} + c_1 H(\tfrac{2\pi}{5}) e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} H(-\tfrac{2\pi}{5}) e^{-j(2\pi/5)n} + \tfrac{1}{4} H(\tfrac{2\pi}{5}) e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} e^{j(2\pi/5)} e^{-j(2\pi/5)n} + \tfrac{1}{4} e^{-j(2\pi/5)} e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} [e^{-j(2\pi/5)(n-1)} + e^{j(2\pi/5)(n-1)}] \\
&= \tfrac{1}{4} \left(2\cos\left[\tfrac{2\pi}{5}(n-1)\right]\right) \\
&= \tfrac{1}{2} \cos\left[\tfrac{2\pi}{5}(n-1)\right].
\end{aligned}$$

Observe that $y(n) = x(n-1)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\Omega) = e^{-j\Omega}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

10.9 Filtering

In some applications, we want to change the magnitude or phase of the frequency components of a sequence or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a sequence is referred to as **filtering** and the system that performs such processing is called a **filter**.

For the sake of simplicity, we consider only LTI filters here. If a filter is LTI, then it is completely characterized by its frequency response. Since the frequency spectra of sequences are 2π-periodic, we only need to consider frequencies over an interval of length 2π. Normally, we choose this interval to be centered about the origin, namely, $(-\pi, \pi]$. When we consider this interval, low frequencies are those closer to the origin, while high frequencies are those closer to $\pm\pi$.

Various types of filters exist. Frequency-selective filters pass some frequency components with little or no distortion, while significantly attenuating others. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency greater in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \le \Omega_c \\ 0 & \Omega_c < |\Omega| \le \pi, \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 10.5(a).

The **ideal highpass filter** eliminates all frequency components with a frequency lesser in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & \Omega_c < |\Omega| \le \pi \\ 0 & |\Omega| \le \Omega_c, \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 10.5(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0 & |\Omega| < \Omega_{c1} \text{ or } \Omega_{c2} < |\Omega| < \pi, \end{cases}$$

where the limits of the passband are Ω_{c1} and Ω_{c2}. A plot of this frequency response is shown in Figure 10.5(c).

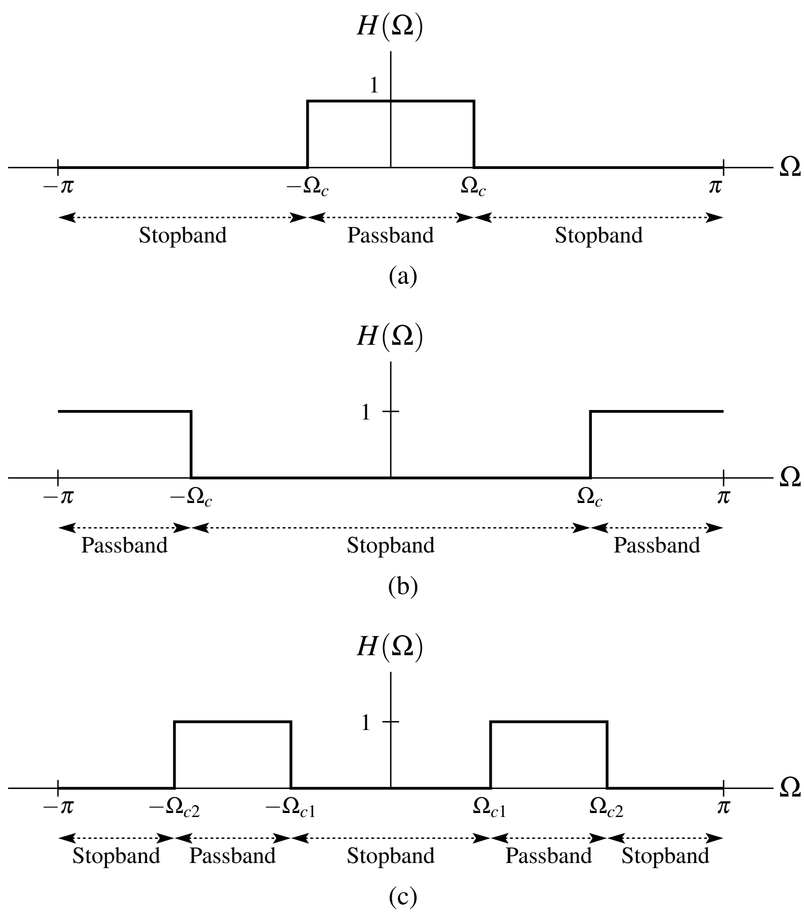


Figure 10.5: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 10.12 (Lowpass filtering)**.** Consider a LTI system with input x, output y, and frequency response H, where

$$H(\Omega) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\Omega| \leq \pi. \end{cases}$$

Suppose that the input x is the periodic sequence

$$x(n) = 1 + \cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{5}\cos(\pi n).$$

(a) Find the Fourier-series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

Solution. (a) We begin by finding the Fourier-series representation of x. The fundamental period N of x is given by

$$N = \operatorname{lcm}\{8, 8, 2\} = \operatorname{lcm}\{2^3, 2^3, 2^1\} = 2^3 = 8.$$

Using Euler's formula, we can re-express x as

$$\begin{aligned} x(n) &= 1 + \cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{5}\cos(\pi n) \\ &= 1 + \left[\tfrac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right)\right] + \tfrac{1}{2}\left[\tfrac{1}{2}\left(e^{j(3\pi/4)n} + e^{-j(3\pi/4)n}\right)\right] + \tfrac{1}{5}e^{j\pi n} \\ &= 1 + \tfrac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right) + \tfrac{1}{4}\left(e^{j(3\pi/4)n} + e^{-j(3\pi/4)n}\right) + \tfrac{1}{5}e^{j\pi n} \\ &= \tfrac{1}{4}e^{-j(3\pi/4)n} + \tfrac{1}{2}e^{-j(\pi/4)n} + 1 + \tfrac{1}{2}e^{j(\pi/4)n} + \tfrac{1}{4}e^{j(3\pi/4)n} + \tfrac{1}{5}e^{j\pi n} \\ &= \tfrac{1}{4}e^{j(2\pi/8)(-3)n} + \tfrac{1}{2}e^{j(2\pi/8)(-1)n} + 1 + \tfrac{1}{2}e^{j(2\pi/8)(1)n} + \tfrac{1}{4}e^{j(2\pi/8)(3)n} + \tfrac{1}{5}e^{j(2\pi/8)(4)n}. \end{aligned}$$

Thus, we have that the Fourier series of x is given by

$$x(n) = \sum_{k=-3}^{4} a_k e^{j(2\pi/8)kn},$$

where

$$a_k = \begin{cases} \frac{1}{4} & k \in \{-3, 3\} \\ \frac{1}{2} & k \in \{-1, 1\} \\ 1 & k = 0 \\ \frac{1}{5} & k = 4 \\ 0 & k \in \{-2, 2\}. \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

$$y(n) = \sum_{k=-\infty}^{\infty} b_k e^{j(2\pi/8)kn}$$

where

$$b_k = a_k H\left(\tfrac{2\pi}{8}k\right).$$

For each nonzero a_k, we compute the corresponding b_k to obtain:

$$\begin{aligned}
b_0 &= a_0 H\left[\left(\tfrac{\pi}{4}\right)(0)\right] = 1(1) \\
&= 1, \\
b_1 &= a_1 H\left[\left(\tfrac{\pi}{4}\right)(1)\right] = \left(\tfrac{1}{2}\right)(1) \\
&= \tfrac{1}{2}, \\
b_{-1} &= a_{-1} H\left[\left(\tfrac{\pi}{4}\right)(-1)\right] = \left(\tfrac{1}{2}\right)(1) \\
&= \tfrac{1}{2}, \\
b_3 &= a_3 H\left[\left(\tfrac{\pi}{4}\right)(3)\right] = \left(\tfrac{1}{4}\right)(0) \\
&= 0, \\
b_{-3} &= a_{-3} H\left[\left(\tfrac{\pi}{4}\right)(-3)\right] = \left(\tfrac{1}{4}\right)(0) \\
&= 0, \\
b_4 &= a_4 H\left[\left(\tfrac{\pi}{4}\right)(4)\right] = \left(\tfrac{1}{5}\right)(0) \\
&= 0.
\end{aligned}$$

Thus, we have

$$b_k = \begin{cases} 1 & k = 0 \\ \frac{1}{2} & k \in \{-1, 1\} \\ 0 & k \in \{-3, -2, 2, 3, 4\}. \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 10.6(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes. ■

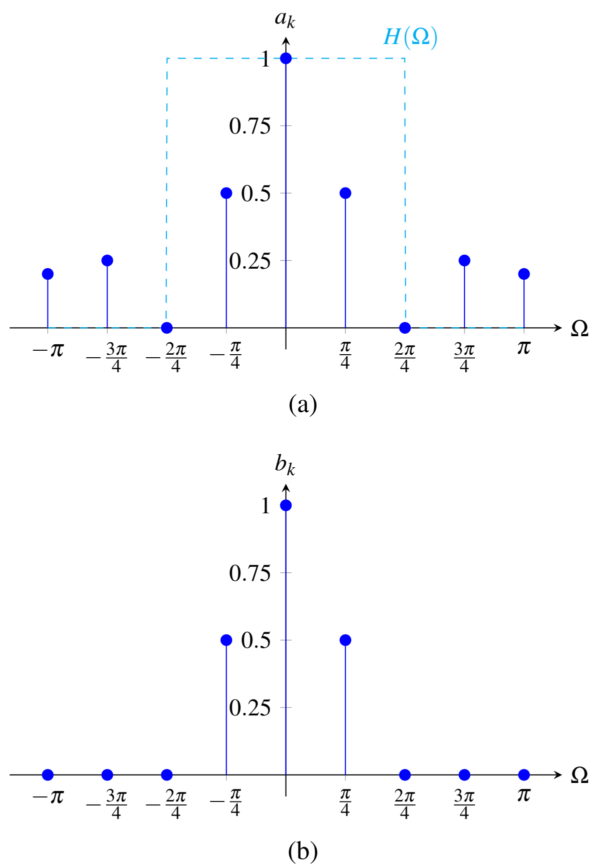


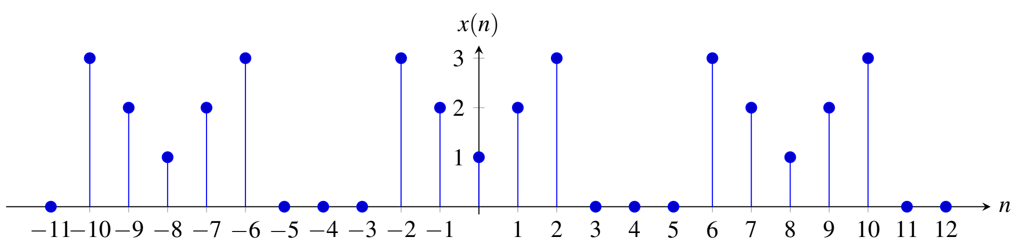
Figure 10.6: Frequency spectra of the (a) input sequence x and (b) output sequence y.

10.10 Exercises

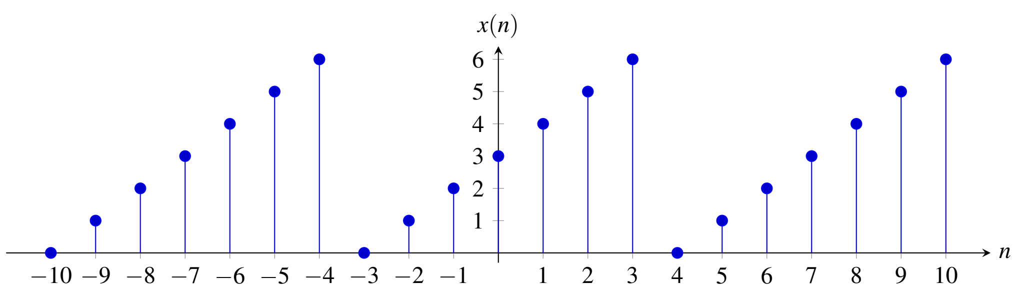
10.10.1 Exercises Without Answer Key

10.1 Find the Fourier series representation of each sequence x given below.
(a) $x(n) = \left(\frac{1}{2}\right)^n$ for $n \in [0\,..\,7]$ and $x(n) = x(n+8)$;
(b) $x(n) = n$ for $n \in [-3\,..\,3]$ and $x(n) = x(n+7)$;
(c) $x(n) = 1 + \cos\left(\frac{2\pi}{7}n\right) + \sin\left(\frac{4\pi}{7}n - \frac{\pi}{6}\right)$; and
(d) $x(n) = \begin{cases} 1 & n \in [-4\,..\,4] \\ 0 & n \in [-15\,..\,-5] \text{ or } [5\,..\,16] \end{cases}$ and $x(n) = x(n) + 32$.

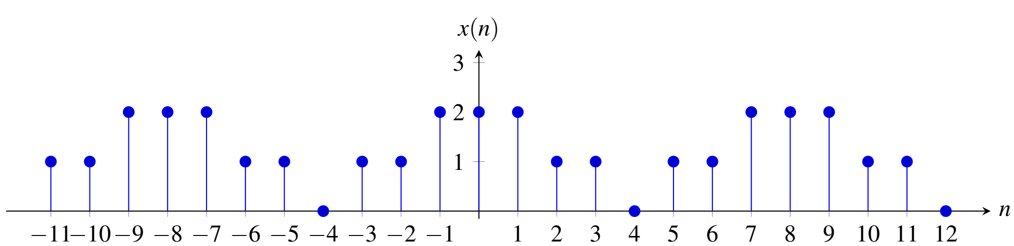
10.2 For each periodic sequence x shown in the figures below, find the corresponding Fourier series coefficient sequence a. The number of samples appearing in each plot is an integer multiple of the period. Properties of Fourier series may be used if helpful. [Hint: For part (c) use the result of Exercise 10.101.]



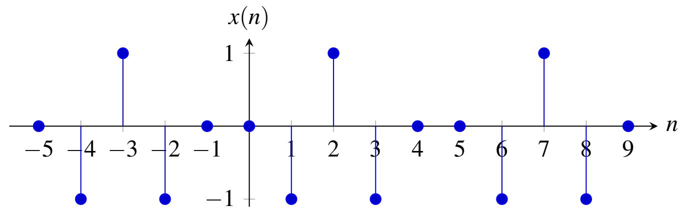
(a)



(b)



(c)



(d)

10.3 Show that, for a complex periodic sequence x with the Fourier series coefficient sequence a:
(a) x is even if and only if a is even; and
(b) x is odd if and only if a is odd.

10.4 Let x and y denote N-periodic sequences with the Fourier-series coefficient sequences a and b, respectively. For each sequence y given below, find an expression for b in terms of a.
(a) $y(n) = x(n) - x(n-1)$; and
(b) $y(n) = x(n+1) - x(n-1)$.

10.5 Let x be a 31-periodic sequence with the Fourier-series coefficient sequence a given by

$$a_k = k^2 \left(2^k + 2^{-k}\right) e^{j(2\pi/31)k} \quad \text{for } k \in [-15\,..\,15].$$

Using Fourier series properties as appropriate, determine if each of the following assertions is true:
(a) x is real;
(b) x is even; and
(c) the sequence $y(n) = x(n-1)$ is even.

10.6 An N-periodic sequence x, where N is even, has the Fourier series coefficient sequence c.
(a) Show that if, $c_k = 0$ for all even $k \in \mathbb{Z}$, then $x(n) = -x(n - \frac{N}{2})$ for all $n \in \mathbb{Z}$.
(b) Show that if $x(n) = -x(n - \frac{N}{2})$ for all $n \in \mathbb{Z}$, then $c_k = 0$ for all even $k \in \mathbb{Z}$.

10.7 Let x and y denote N-periodic sequences with the Fourier series coefficient sequences a and b, respectively. For each sequence y given below, find b in terms of a.
(a) $y(n) = \operatorname{Even}\{x\}(n)$; and
(b) $y(n) = \operatorname{Re}\{x\}(n)$.

10.8 Find and plot the magnitude and phase spectrum of each sequence x given below for frequencies in $[-\pi, \pi]$.
(a) $x(n) = 1 + \sin\left(\frac{\pi}{5}n + \frac{\pi}{3}\right)$;
(b) $x(n) = \frac{3}{4} + \frac{1}{2}\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}n\right) + \frac{1}{8}\cos(\pi n)$; and
(c) $x(n) = \begin{cases} -1 & n \in [-3\,..\,0] \\ 1 & n \in [1\,..\,4] \end{cases}$ and $x(n) = x(n+8)$.

10.9 Consider a LTI system with frequency response

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & |\Omega| \in \left(\frac{\pi}{2}, \pi\right]. \end{cases}$$

Find the response y of the system to the input x, where

$$x(n) = 1 + \frac{1}{3}\cos\left(\frac{4\pi}{9}n\right) + \frac{1}{6}\cos\left(\frac{6\pi}{9}n\right).$$

10.10 A LTI system has the impulse response

$$h(n) = \left(\tfrac{1}{2}\right)^{n+1} u(n).$$

Find the response y of this system to the input x, where

$$x(n) = 1 + \cos\left(\tfrac{\pi}{4}n\right) + \sin\left(\tfrac{\pi}{2}n\right).$$

10.10.2 Exercises With Answer Key

10.101 Find the Fourier-series coefficient sequence a of an N-periodic square-wave sequence x of the form

$$x(n) = \begin{cases} 1 & n \in [N_1 \mathinner{..} N_2] \\ 0 & n \in [N_0 \mathinner{..} N_1 - 1] \cup [N_2 + 1 \mathinner{..} N_0 + N - 1], \end{cases}$$

where N_0 is an integer, $N_1, N_2 \in [N_0 \mathinner{..} N_0 + N - 1]$ and $N_1 \leq N_2$. (Note that this exercise is closely related to Exercise A.10.)

Short Answer.

$$a_k = \begin{cases} \frac{1}{N} e^{-j\pi(N_1+N_2)k/N} \left[\frac{\sin[\pi(N_2-N_1+1)k/N]}{\sin(\pi k/N)}\right] & \frac{k}{N} \notin \mathbb{Z} \\ \frac{N_2-N_1+1}{N} & \frac{k}{N} \in \mathbb{Z}. \end{cases}$$

10.11 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 11

Discrete-Time Fourier Transform

11.1 Introduction

The (DT) Fourier series provides an extremely useful representation for periodic sequences. Often, however, we need to deal with sequences that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) sequences, known as the (DT) Fourier transform.

11.2 Development of the Discrete-Time Fourier Transform for Aperiodic Sequences

As demonstrated earlier, the Fourier series is an extremely useful representation for sequences. Unfortunately, this representation can only be used for sequences that are periodic, since a Fourier series is inherently periodic. Many sequences, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic sequences. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic sequence can be viewed as a periodic sequence with a period of infinity. By viewing an aperiodic sequence as this limiting case of a periodic sequence where the period is infinite, we can use the Fourier series to develop a more general representation for sequences that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the sequence being considered exists.)

Suppose that we have an aperiodic sequence x. From x, let us define the sequence x_M as

$$x_M(n) = \begin{cases} x(n) & n \in [-M \mathinner{..} M-1] \\ 0 & \text{otherwise.} \end{cases} \tag{11.1}$$

Essentially, x_M is identical in value to x for the $2M$ elements (approximately) centered about the origin. Let us now repeat the portion of $x_M(n)$ for $n \in [-M \mathinner{..} M-1]$ to form a $2M$-periodic sequence $\tilde{x}$. That is, we define $\tilde{x}$ as

$$\tilde{x}(n) = x_M(n) \;\text{ for } n \in [-M \mathinner{..} M-1] \quad \text{and} \quad \tilde{x}(n) = \tilde{x}(n+2M).$$

Before proceeding further, we make two important observations that we will use later. First, from the definition of x_M, we have

$$\lim_{M\to\infty} x_M(n) = x(n). \tag{11.2}$$

Second, from the definition of x_M and $\tilde{x}$, we have

$$\lim_{M\to\infty} \tilde{x}(n) = x(n). \tag{11.3}$$

Now, let us consider the sequence $\tilde{x}$. Since $\tilde{x}$ is $2M$-periodic, we can represent it using a Fourier series as

$$\begin{aligned}\tilde{x}(n) &= \sum_{k=\langle 2M\rangle} a_k e^{j[2\pi/(2M)]kn} \\ &= \sum_{k=\langle 2M\rangle} a_k e^{j(\pi/M)kn} \\ &= \sum_{k=-M}^{M-1} a_k e^{j(\pi/M)kn}. \end{aligned} \tag{11.4}$$

The coefficient sequence a is then given by

$$\begin{aligned}a_k &= \tfrac{1}{2M} \sum_{\ell=\langle 2M\rangle} \tilde{x}(\ell) e^{-j(2\pi/(2M))k\ell} \\ &= \tfrac{1}{2M} \sum_{\ell=\langle 2M\rangle} \tilde{x}(\ell) e^{-j(\pi/M)k\ell} \\ &= \tfrac{1}{2M} \sum_{\ell=-M}^{M-1} \tilde{x}(\ell) e^{-j(\pi/M)k\ell}.\end{aligned}$$

Moreover, since $x_M(\ell) = \tilde{x}(\ell)$ for $\ell \in [-M\,..\,M-1]$, we can rewrite the preceding equation for a_k as

$$a_k = \tfrac{1}{2M} \sum_{\ell=-M}^{M-1} x_M(\ell) e^{-j(\pi/M)k\ell}.$$

Substituting this expression for a_k into (11.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}$:

$$\tilde{x}(n) = \sum_{k=-M}^{M-1} \left(\tfrac{1}{2M} \sum_{\ell=-M}^{M-1} x_M(\ell) e^{-j(\pi/M)k\ell} \right) e^{j(\pi/M)kn}.$$

Now, we define $\Delta\Omega = \frac{2\pi}{2M} = \frac{\pi}{M}$. Rewriting the previous equation for $\tilde{x}$ in terms of $\Delta\Omega$, we obtain

$$\begin{aligned}\tilde{x}(n) &= \sum_{k=-M}^{M-1} \left(\tfrac{1}{2\pi}\Delta\Omega \sum_{\ell=-M}^{M-1} x_M(\ell) e^{-j\Delta\Omega k\ell} \right) e^{j\Delta\Omega kn} \\ &= \tfrac{1}{2\pi} \sum_{k=-M}^{M-1} \left(\Delta\Omega \sum_{\ell=-M}^{M-1} x_M(\ell) e^{-j\Delta\Omega k\ell} \right) e^{j\Delta\Omega kn}.\end{aligned}$$

Substituting the above expression for $\tilde{x}$ into (11.3), we obtain

$$x(n) = \lim_{M\to\infty} \tfrac{1}{2\pi} \sum_{k=-M}^{M-1} \left(\Delta\Omega \sum_{\ell=-M}^{M-1} x_M(\ell) e^{-j\Delta\Omega k\ell} \right) e^{j\Delta\Omega kn} \tag{11.5}$$

Now, we must evaluate the above limit. As $M \to \infty$, we have that $\Delta\Omega \to 0$. Thus, in the limit above, $k\Delta\Omega$ becomes a continuous variable which we denote as Ω, $\Delta\Omega$ becomes the infinitesimal $d\Omega$, and the summation becomes an integral whose lower and upper limits are respectively given by

$$\lim_{M\to\infty} \left(k\Delta\Omega|_{k=-M} \right) = \lim_{M\to\infty} -M\Delta\Omega = \lim_{M\to\infty} -M\left(\tfrac{\pi}{M}\right) = -\pi \quad \text{and}$$
$$\lim_{M\to\infty} \left(k\Delta\Omega|_{k=M-1} \right) = \lim_{M\to\infty} (M-1)\Delta\Omega = \lim_{M\to\infty} (M-1)\left(\tfrac{\pi}{M}\right) = \pi.$$

From (11.2), as $M \to \infty$, we have that $x_M \to x$. Combining these results, we can rewrite (11.5) to obtain

$$x(n) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x(\ell) e^{-j\Omega\ell} \right) e^{j\Omega n} d\Omega,$$

Since the integrand is 2π-periodic, we can rewrite this equation as

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi}\left(\sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}\right)e^{j\Omega n}d\Omega,$$

Thus, we have that

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega,$$

where

$$X(\Omega) = \sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}.$$

Thus, we have found a representation of the aperiodic sequence x in terms of complex sinusoids at all frequencies. We call this the (DT) Fourier transform representation of the sequence x.

11.3 Generalized Fourier Transform

In the previous section, we used a limiting process involving the analysis and synthesis equations for Fourier series in order to develop a new mathematical tool known as the (DT) Fourier transform. As it turns out, many sequences of practical interest do not have a Fourier transform in the sense of the definition developed previously. That is, for a given sequence x, the Fourier transform summation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

may fail to converge, in which case the Fourier transform X of x does not exist. For example, the preceding summation does not converge if x is any of the following (as well as many other possibilities):

- a nonzero constant sequence;
- a periodic sequence (e.g., a real or complex sinusoid); or
- the unit-step sequence (i.e., u).

Sequences such as these are of great practical interest, however. Therefore, it is highly desirable to have a mathematical tool that can handle such sequences. This motivates the development of what is called the **generalized Fourier transform**. The generalized Fourier transform exists for periodic sequences, nonzero constant sequences, and many other types of sequences as well. The underlying math associated with the generalized Fourier transform is quite complicated, however. So, we will not attempt to formally develop the generalized Fourier transform here. Although not entirely correct, one can think of the generalized Fourier transform as being defined by the same formulas as the classical Fourier transform. So, for this and other reasons, we can mostly ignore the distinction between the generalized Fourier transform and classical Fourier transform, and think of them as being one and the same. In what follows, we will avoid making a distinction between the classical Fourier transform and generalized Fourier transform, except in a very small number of places where it is beneficial to do so. The main disadvantage of not formally introducing the generalized Fourier transform is that some results presented later (which actually rely on the use of the generalized Fourier transform) must be accepted on faith since their proof would require formal knowledge of the generalized Fourier transform, which is not introduced herein. As long as the generalized Fourier transform is used, both periodic and aperiodic sequences can be handled, and in this sense we have a more general tool than Fourier series (which require periodic sequences). Later, when we discuss the Fourier transform of periodic sequences, we will implicitly be using the generalized Fourier transform in that context. In fact, in much of what follows, when we speak of the Fourier transform, we are often referring to the generalized Fourier transform.

11.4 Definition of the Discrete-Time Fourier Transform

Earlier, we developed the Fourier transform representation of a sequence. This representation expresses a sequence in terms of complex sinusoids at all frequencies. More formally, the **Fourier transform** of the sequence x, denoted as $\mathcal{F}x$ or X, is defined as

$$\mathcal{F}x(\Omega) = X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}. \tag{11.6}$$

Similarly, the inverse Fourier transform of X, denoted as $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(n) = x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega. \tag{11.7}$$

We refer to (11.6) as the **Fourier transform analysis equation** and (11.7) as the **Fourier transform synthesis equation**.

To denote that a sequence x has the Fourier transform X, we can write

$$x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega).$$

As a matter of terminology, x and X are said to constitute a **Fourier transform pair**.

Example 11.1 (Fourier transform of a shifted and scaled delta sequence)**.** Find the Fourier transform X of the sequence

$$x(n) = A\delta(n-n_0),$$

where A is a real constant and n_0 is an integer constant. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} A\delta(n-n_0)e^{-j\Omega n} \\ &= A\sum_{n=-\infty}^{\infty} \delta(n-n_0)e^{-j\Omega n}. \end{aligned}$$

Using the sifting property of the delta sequence, we can simplify the above result to obtain

$$X(\Omega) = Ae^{-j\Omega n_0}.$$

Thus, we have shown that

$$A\delta(n-n_0) \overset{\text{DTFT}}{\longleftrightarrow} Ae^{-j\Omega n_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega, \quad \text{where} \quad X(\Omega) = Ae^{-j\Omega n_0}. \qquad \blacksquare$$

Example 11.2 (Fourier transform of a rectangular pulse)**.** Find the Fourier transform X of the sequence

$$x(n) = u(n-a) - u(n-b),$$

where a and b are integer constants such that $a < b$.

Solution. To begin, we observe that

$$x(n) = \begin{cases} 1 & n \in [a\,..\,b) \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} \left(e^{-j\Omega}\right)^n \\ &= e^{-ja\Omega} \sum_{n=0}^{b-a-1} \left(e^{-j\Omega}\right)^n. \end{aligned}$$

The summation on the right-hand side corresponds to the sum of a geometric sequence. Using the formula for the sum of a geometric sequence (F.8), we can write

$$\begin{aligned} X(\Omega) &= e^{-j\Omega a} \frac{\left(e^{-j\Omega}\right)^{b-a} - 1}{e^{-j\Omega} - 1} \\ &= \frac{e^{-jb\Omega} - e^{-ja\Omega}}{e^{-j\Omega} - 1} \\ &= \frac{e^{-ja\Omega} - e^{-jb\Omega}}{1 - e^{-j\Omega}}. \end{aligned}$$

From the relationship between exponentials and sinusoids, we can rewrite this as

$$\begin{aligned} X(\Omega) &= \frac{e^{-j(a+b)\Omega/2}\left(e^{-j(a-b)\Omega/2} - e^{j(a-b)\Omega/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)} \\ &= \frac{e^{-j(a+b)\Omega/2}\left(2j\sin\left[\frac{(b-a)\Omega}{2}\right]\right)}{e^{-j\Omega/2}\left(2j\sin\left[\frac{\Omega}{2}\right]\right)} \\ &= \frac{e^{-j(a+b)\Omega/2}\sin\left[\frac{(b-a)\Omega}{2}\right]}{e^{-j\Omega/2}\sin\left[\frac{\Omega}{2}\right]} \\ &= e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{(b-a)\Omega}{2}\right]}{\sin\left[\frac{\Omega}{2}\right]}\right). \end{aligned}$$

Thus, we have shown that

$$u(n-a) - u(n-b) \xleftrightarrow{\text{DTFT}} e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{1}{2}(b-a)\Omega\right]}{\sin\left[\frac{1}{2}\Omega\right]}\right).$$ ■

Example 11.3. Find the Fourier transform X of the sequence

$$x(n) = a^n u(n),$$

where a is a real constant satisfying $|a| < 1$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n. \end{aligned}$$

The summation on the right-hand side of the preceding equation is a geometric series. Using the formula for the sum of a geometric series (F.9), we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned} X(\Omega) &= \frac{1}{1 - ae^{-j\Omega}} \\ &= \frac{e^{j\Omega}}{e^{j\Omega} - a}. \end{aligned}$$

Thus, we have shown that

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{for } |a| < 1.$$

■

Example 11.4. Find the Fourier transform X of the sequence

$$x(n) = a^{|n|},$$

where a is a real constant satisfying $|a| < 1$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=1}^{\infty} a^n e^{j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=1}^{\infty} \left(ae^{j\Omega}\right)^n + \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n. \end{aligned}$$

Each of the summations on the right-hand side of this equation is a geometric series. Using the formula for the sum of a geometric series (F.9), we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned} X(\Omega) &= \frac{ae^{j\Omega}}{1 - ae^{j\Omega}} + \frac{1}{1 - ae^{-j\Omega}} \\ &= \frac{1 - ae^{j\Omega} + ae^{j\Omega}(1 - ae^{-j\Omega})}{(1 - ae^{j\Omega})(1 - ae^{-j\Omega})} \\ &= \frac{1 - ae^{j\Omega} + ae^{j\Omega} - a^2}{1 - ae^{-j\Omega} - ae^{j\Omega} + a^2} \\ &= \frac{1 - a^2}{1 - a(e^{j\Omega} + e^{-j\Omega}) + a^2} \\ &= \frac{1 - a^2}{1 - 2a\cos\Omega + a^2}. \end{aligned}$$

Thus, we have shown that

$$a^{|n|} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2} \quad \text{for } |a|<1. \qquad \blacksquare$$

Example 11.5. Find the inverse Fourier transform x of the (2π-periodic) sequence

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

(Note that, in the preceding formula, δ denotes the delta function, not the delta sequence.)

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(n) &= \tfrac{1}{2\pi} \int_{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \right] e^{j\Omega n} d\Omega \\ &= \int_{2\pi} \delta(\Omega) e^{j\Omega n} d\Omega \\ &= \int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega. \end{aligned}$$

Using the sifting property of the delta function, we can simplify the preceding equation to obtain

$$x(n) = e^0 = 1.$$

Thus, we have that

$$1 \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \qquad \blacksquare$$

11.5 Remarks on Notation Involving the Fourier Transform

The Fourier transform operator $\mathcal{F}$ maps a sequence to a (2π-periodic) function, and the inverse Fourier transform operator $\mathcal{F}^{-1}$ maps a (2π-periodic) function to a sequence. Consequently, the operand for each of these operators must be a function/sequence (not a number). Consider the unnamed sequence that maps n to $e^{-|n/10|}$ as shown in Figure 11.1. Suppose that we would like to write an expression that denotes the Fourier transform of this sequence. At first, we might be inclined to write "$\mathcal{F}\{e^{-|n/10|}\}$". Strictly speaking, however, this notation is not correct, since the Fourier transform operator requires a sequence as an operand and "$e^{-|n/10|}$" (strictly speaking) denotes a number (i.e., the value of the sequence in the figure evaluated at n). Essentially, the cause of our problems here is that the sequence in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a sequence x using the equation $x(n) = e^{-|n/10|}$ and then write the Fourier transform as "$\mathcal{F}x$". Unfortunately, introducing a new sequence name just for the sake of strictly correct notation is often undesirable as it frequently leads to overly verbose writing.

One way to avoid overly verbose writing when referring to functions or sequences without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the sequence from Figure 11.1 that maps n to $e^{-|n/10|}$. Using strictly correct notation, we could write the Fourier transform of this sequence as "$\mathcal{F}\left\{e^{-|\cdot/10|}\right\}$". In other words, we can indicate that an expression refers to a sequence (as opposed to the value of sequence) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 11.6. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

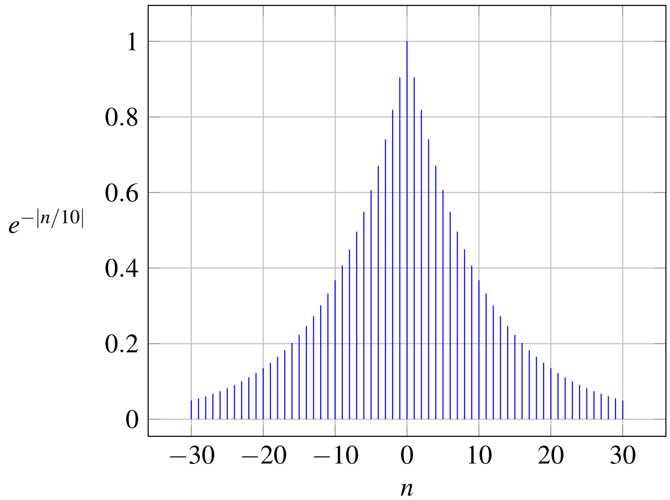


Figure 11.1: A plot of $e^{-|n/10|}$ versus n.

Example 11.6 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|\cdot|+3}\right\}$.

2. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ evaluated at $2\Omega - 3$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|\cdot|+3}\right\}(2\Omega - 3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos(\cdot)+a^2}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ evaluated at $n-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos(\cdot)+a^2}\right\}(n-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Fourier transform operator $\mathcal{F}$, the variable "n" is assumed to be the independent variable for the sequence to which the Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "n" were a "$\cdot$");

- unless indicated otherwise, in an expression for the operand of the inverse Fourier transform operator $\mathcal{F}^{-1}$, the variable "Ω" is assumed to be the independent variable for the function to which the inverse Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "Ω" were a "$\cdot$")

Some examples of using these book-sanctioned notational conventions can be found below in Example 11.7. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 11.7 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|n|+3}\right\}$.

2. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ evaluated at $2\Omega - 3$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|n|+3}\right\}(2\Omega-3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos\Omega+a^2}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ evaluated at $n-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos\Omega+a^2}\right\}(n-3)$. ■

Since applying the Fourier transform operator or inverse Fourier transform operator to a sequence/function yields another function/sequence, we can evaluate this other function/sequence at some value. Again, consider the sequence from Figure 11.1 that maps n to $e^{-|n/10|}$. To denote the value of the Fourier transform of this sequence evaluated at $\Omega - 1$, we would write "$\mathcal{F}\{e^{-|\cdot/10|}\}(\Omega-1)$" using dot notation or "$\mathcal{F}\{e^{-|n/10|}\}(\Omega-1)$" using the book-sanctioned notational conventions described above.

11.6 Convergence Issues Associated with the Discrete-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the summations/integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary sequence x. This sequence has the Fourier transform representation $\hat{x}$ given by

$$\hat{x}(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega, \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}$ is a valid representation of x.

The first important result concerning convergence is given by the theorem below.

Theorem 11.1. *If a sequence x is absolutely summable (i.e., $\sum_{n=-\infty}^{\infty}|x(n)| < \infty$), then the Fourier transform X of x converges uniformly.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

Since, in practice, we often encounter sequences that are not absolutely summable, the above result is sometimes not helpful. This motivates us to consider additional results concerning convergence.

The next important result concerning convergence relates to finite-energy sequences as stated by the theorem below.

Theorem 11.2 (Convergence of Fourier transform (finite-energy case))**.** *If a sequence x is of finite energy (i.e., $\sum_{n=-\infty}^{\infty}|x(n)|^2 < \infty$), then its Fourier transform representation $\hat{x}$ converges in the MSE sense.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

11.7 Properties of the Discrete-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 11.1 (on page 469).

11.7.1 Periodicity

A particularly important property of the (DT) Fourier transform is that is it always periodic, as elucidated by the theorem below.

Theorem 11.3 (Periodicity)**.** *If* $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$X(\Omega) = X(\Omega + 2\pi)$$

(i.e., X is 2π-periodic).

Proof. To prove this property, we proceed as follows. From the definition of the Fourier transform, we have

$$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi n} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) \left(e^{-j2\pi}\right)^n e^{-j\Omega n}. \end{aligned}$$

Since $e^{-j2\pi} = 1$, we have

$$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x(n)(1)^n e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= X(\Omega). \end{aligned}$$

Thus, $X(\Omega + 2\pi) = X(\Omega)$ (i.e., X is 2π-periodic). ∎

11.7.2 Linearity

Arguably, one of the most important properties of the Fourier transform is linearity, as introduced below.

Theorem 11.4 (Linearity)**.** *If* $x_1(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ *and* $x_2(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$*, then*

$$a_1 x_1(n) + a_2 x_2(n) \stackrel{\text{DTFT}}{\longleftrightarrow} a_1 X_1(\Omega) + a_2 X_2(\Omega),$$

where a_1 and a_2 are arbitrary complex constants. This is known as the ***linearity property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = a_1 x_1(n) + a_2 x_2(n)$ and let $Y = \mathcal{F} y$. We have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} a_1 x_1(n) e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) e^{-j\Omega n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\Omega n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\Omega n} \\ &= a_1 X_1(\Omega) + a_2 X_2(\Omega). \end{aligned}$$

Thus, we have shown that the linearity property holds. ∎

Example 11.8. Using the Fourier transform pairs

$$\delta(n) \overset{\text{DTFT}}{\longleftrightarrow} 1 \quad \text{and} \quad u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = 2\delta(n) - u(n).$$

Solution. Taking the Fourier transform of x, we trivially have

$$X(\Omega) = \mathcal{F}\{2\delta(n) - u(n)\}(\Omega).$$

Using the linearity property of the Fourier transform, we can write

$$X(\Omega) = 2\mathcal{F}\delta(\Omega) - \mathcal{F}u(\Omega).$$

Using the given Fourier transform pairs, we obtain

$$\begin{aligned} X(\Omega) &= 2(1) - \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)\right] \\ &= \frac{2e^{j\Omega} - 2 - e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega} - 2}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

■

11.7.3 Translation (Time Shifting)

The next property of the Fourier transform to be introduced is the translation (or time-domain shifting) property, as given below.

Theorem 11.5 (Translation (i.e., time shifting))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$x(n - n_0) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega),$$

where n_0 is an arbitrary integer. This is known as the ***translation property*** *(or* ***time-domain shifting property****) of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(n - n_0)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we can write

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $\lambda = n - n_0$ so that $n = \lambda + n_0$. Applying the change of variable, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{\lambda=-\infty}^{\infty} x(\lambda) e^{-j\Omega(\lambda + n_0)} \\ &= e^{-j\Omega n_0} \sum_{\lambda=-\infty}^{\infty} x(\lambda) e^{-j\Omega\lambda} \\ &= e^{-j\Omega n_0} X(\Omega). \end{aligned}$$

Thus, we have proven that the translation property holds. ■

Example 11.9. Using the Fourier transform pair

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a} \quad \text{for } |a|<1,$$

find the Fourier transform X of the sequence

$$x(n) = a^n u(n-3),$$

where a is a complex constant satisfying $|a| < 1$.

Solution. To begin, we observe that

$$x(n) = a^3 a^{n-3} u(n-3)$$

Define the sequence

$$v_1(n) = a^n u(n)$$

Using the definition of v, we can rewrite x as

$$x(n) = a^3 v_1(n-3).$$

Taking the Fourier transforms of v_1 and x, we obtain

$$\begin{aligned} V_1(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-a} \quad \text{and} \\ X(\Omega) &= a^3 e^{-j3\Omega} V_1(\Omega). \end{aligned}$$

Substituting the formula for V_1 into the formula for X, we obtain

$$\begin{aligned} X(\Omega) &= a^3 e^{-j3\Omega} V_1(\Omega) \\ &= a^3 e^{-j3\Omega} \left(\frac{e^{j\Omega}}{e^{j\Omega}-a} \right) \\ &= \frac{a^3 e^{-j2\Omega}}{e^{j\Omega}-a}. \end{aligned}$$

■

11.7.4 Modulation (Frequency-Domain Shifting)

The next property of the Fourier transform to be introduced is the modulation (i.e., frequency-domain shifting) property, as given below.

Theorem 11.6 (Modulation (i.e., frequency-domain shifting))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$e^{j\Omega_0 n} x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega - \Omega_0),$$

where Ω_0 is an arbitrary real constant. This is known as the ***modulation property*** *(or* ***frequency-domain shifting property****) of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = e^{j\Omega_0 n} x(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x(n) e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega-\Omega_0)n} \\ &= X(\Omega - \Omega_0). \end{aligned}$$

Thus, the modulation property holds. ■

Example 11.10. Using the Fourier transform pair,

$$1 \stackrel{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = \cos(\Omega_0 n),$$

where Ω_0 is a nonzero real constant.

Solution. To begin, we rewrite x as

$$x(n) = \tfrac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right).$$

Taking the Fourier transform of x, we trivially have

$$X(\Omega) = \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)\right\}(\Omega).$$

From the linearity property of the Fourier transform, we have

$$X(\Omega) = \tfrac{1}{2}\left[\mathcal{F}\{1e^{j\Omega_0 n}\}(\Omega) + \mathcal{F}\{1e^{-j\Omega_0 n}\}(\Omega)\right].$$

Using the modulation property of the Fourier transform, we can write

$$X(\Omega) = \tfrac{1}{2}\left[\mathcal{F}\{1\}(\Omega - \Omega_0) + \mathcal{F}\{1\}(\Omega + \Omega_0)\right].$$

Substituting the result from the given Fourier transform pair, we obtain

$$\begin{aligned} X(\Omega) &= \tfrac{1}{2}\left(2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k)\right) \\ &= \pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\right]. \end{aligned}$$

■

11.7.5 Conjugation

The next property of the Fourier transform to be introduced is the conjugation property, as given below.

Theorem 11.7 (Conjugation)**.** *If* $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x^*(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X^*(-\Omega).$$

This is known as the ***conjugation property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x^*(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\Omega n}.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned}
Y(\Omega) &= \left(\sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n}\right)^{**} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n}\right)^{*} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n}\right)^{*} \\
&= X^*(-\Omega).
\end{aligned}$$

Thus, we have shown that the conjugation property holds. ■

Example 11.11 (Fourier transform of a real sequence)**.** Show that the Fourier transform X of a real sequence x is conjugate symmetric (i.e., $X(\Omega) = X^*(-\Omega)$ for all Ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(n)\}(\Omega) = X^*(-\Omega).$$

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\Omega) = X^*(-\Omega).$$

So, we have

$$X(\Omega) = X^*(-\Omega).$$

Thus, the Fourier transform of a real sequence is conjugate symmetric. ■

11.7.6 Time Reversal

The next property of the Fourier transform to be introduced is the time-reversal (i.e., reflection) property, as given below.

Theorem 11.8 (Time reversal (i.e., reflection))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\Omega).$$

*This is known as the **time-reversal property** of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(-n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $n' = -n$ so that $n = -n'$. Applying the change of variable and dropping the primes, we have

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n} \\
&= X(-\Omega).
\end{aligned}$$

Thus, we have shown that the time-reversal property holds. ■

