ASSIGNMENT 01

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ECE 260, A01

A.1 Express each complex number given below in Cartesian form.
(a) $2e^{j2\pi/3}$;
(b) $\sqrt{2}e^{j\pi/4}$;
(c) $2e^{j7\pi/6}$; and
(d) $3e^{j\pi/2}$.

ANSWER:

Given $\boxed{z = 2e^{\left(j\frac{\pi}{6}\right)}}$

This is given in polar form $\boxed{z = re^{i\theta}}$

where, $r = 2$, $\theta = \frac{7\pi}{6}$

The cartesian form is $\boxed{z = x + jy}$

where $z = r(\cos\theta + j\sin\theta)$

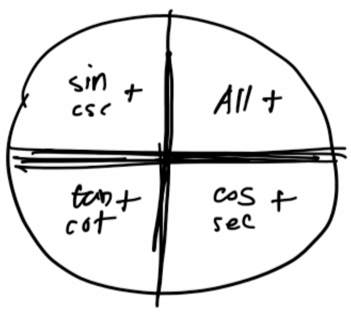
$x = r\cos\theta = 2\cos\left(\frac{7\pi}{6}\right)$

$y = r\sin\theta = 2\sin\left(7\pi/6\right)$

$\downarrow$

$z = 2\left[\cos\left(\frac{7\pi}{6}\right) + j\sin\left(\frac{7\pi}{6}\right)\right]$

$= 2\left[\cos\left(\frac{7 \cdot 180}{6}\right) + j\sin\left(\frac{7 \cdot 180}{6}\right)\right]$



$= 2\left[\cos 210^\circ + j\sin 210^\circ\right]$

$= 2\left[\cos(180^\circ + 30^\circ) + j(\sin(180^\circ + 30^\circ))\right]$

$= 2\left[-\cos 30^\circ - j\sin 30^\circ\right]$

$= 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}j\right]$

$= -\sqrt{3} - 1j$

$= -\sqrt{3} - j$

(Ans)

A.2 Express each complex number given below in polar form. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument.

(a) $-\sqrt{3}+j$;

(b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$;

(c) $\sqrt{2}-j\sqrt{2}$;

(d) $1+j\sqrt{3}$;

(e) $-1-j\sqrt{3}$; and

(f) $-3+4j$.

(b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$;

$$z = re^{j\theta} = x + jy = r(\cos\theta + j\sin\theta)$$
$$= r\cos\theta + rj\sin\theta$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore x = -\frac{1}{2} \qquad y = -\sqrt{3}/2$$

$$r = |z| = \sqrt{x^2+y^2} = \sqrt{(-1/2)^2 + (-\sqrt{3}/2)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{4/4} = \sqrt{1}$$
$$= 1$$

$$\tan^{-1}(y/x) = \tan^{-1}\left(-\sqrt{3}/2 \times \frac{1}{-\frac{1}{2}}\right)$$

$$= \tan^{-1}\left(\sqrt{3}/2 \times 2\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \pi/3$$

since $x<0$ and $y<0$, $\operatorname{Arg} z = \tan^{-1}(y/x) - \pi$

$$= \pi/3 - \pi$$

$$= \frac{\pi - 3\pi}{3}$$

$$= -2\pi/3 = \theta$$

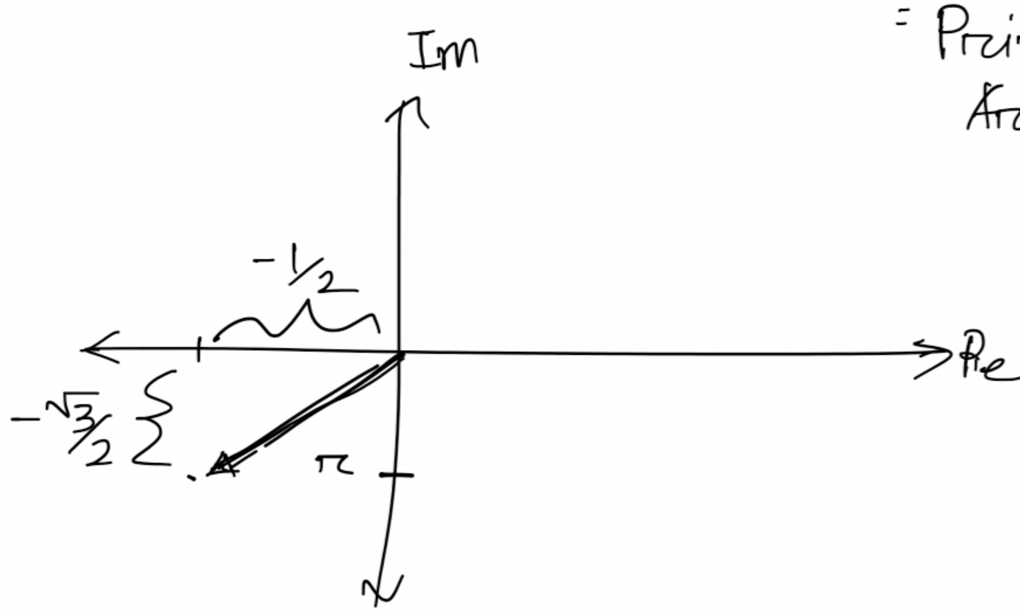
= Principal Argument

$x = -1/2$

$y = -\sqrt{3}/2$

$r = 1$

$\theta =$



$$z = r(\cos\theta + j\sin\theta)$$

$$= 1\left(\cos\left(-\frac{2\pi}{3}\right) + j\sin\left(-\frac{2\pi}{3}\right)\right)$$

(Ans)

(d) $1 + j\sqrt{3}$

$x = 1 \quad y = \sqrt{3}$

$|z| = r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \pm 2 = 2$

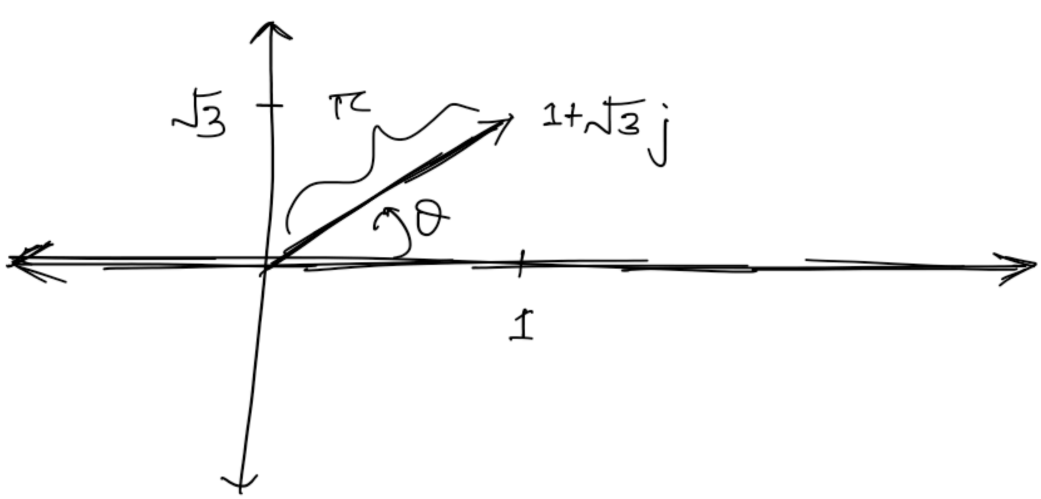
$\arctan(y/x) = \tan^{-1}\left(\sqrt{3}/1\right) = \pi/3$

since $x > 0$ and $y > 0$, $\mathrm{Arg}\, z = \theta = \dfrac{\pi}{3}$ = Principal Argument

$\arg z = \pi/3 + 2\pi n;\ n \in \mathbb{Z}$.

$z = r(\cos\theta + j\sin\theta)$

$= r(\cos\pi/3 + j\sin\pi/3)$



A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.

(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in Cartesian form;

(b) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in polar form;

(c) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1+j)$ in polar form;

(d) $e^{1+j\pi/4}$ in Cartesian form;

(e) $\left[\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right]^8$ in polar form;

(f) $(1+j)^{10}$ in Cartesian form;

(g) $\frac{1+j}{1-j}$ in polar form;

(h) $\frac{1}{1+re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$; and

(i) $\frac{1}{1-re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$.

(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\frac{\pi}{4})}\right) \longrightarrow \boxed{x + yj = z}$

$= (\sqrt{3} - j) + j\left[\frac{1}{\sqrt{2}}\left\{\cos\left(-3\frac{\pi}{4}\right) + \sin\left(-3\frac{\pi}{4}\right)\right\}\right]$

$= (\sqrt{3} - j) + j\left(-\frac{1}{2} - \frac{1}{2}j\right)$

$= \sqrt{3} - j - \frac{1}{2}j - \frac{1}{2}(j^2)$

$= \sqrt{3} + \frac{1}{2} - \frac{3}{2}j$

$= \left(\sqrt{3} + \frac{1}{2}\right) - j\left(3/2\right)$

(5) $\underbrace{\left(\frac{\sqrt{3}}{2} - j\,\frac{1}{2}\right)}\left(\frac{1}{\sqrt{2}}\,e^{j\left(-3\frac{\pi}{4}\right)}\right) \longrightarrow \boxed{z = re^{j\theta}}$

$\boxed{x = \sqrt{3}/2;\ y = \frac{1}{2};\ r = \sqrt{x^2+y^2} = 1}$

$\tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right)$

$= \tan^{-1}\left(1/\sqrt{3}\right)$

$= \pi/6$

$= \left[1 \cdot e^{-j\left(\frac{\pi}{6}\right)}\right]$

$\theta = -\pi/6$

as position is in 4th Quadrant

$\left[\frac{1}{\sqrt{2}} \cdot e^{-j\left(3\pi/4\right)}\right]$

$= \frac{1}{\sqrt{2}}\,e^{-j\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)}$

$= \frac{1}{\sqrt{2}}\,e^{-j\left(\frac{11\pi}{12}\right)}$.

(Ans)

A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.

(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in Cartesian form;

(b) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in polar form;

(c) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1+j)$ in polar form;

(d) $e^{1+j\pi/4}$ in Cartesian form;

(e) $\left[\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right]^8$ in polar form;

(f) $(1+j)^{10}$ in Cartesian form;

(g) $\dfrac{1+j}{1-j}$ in polar form;

(h) $\dfrac{1}{1+re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$; and

(i) $\dfrac{1}{1-re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$.

(f) $(1+j)^{10} \longrightarrow x + jy$

$$= \left[(1+j)^2\right]^5 = \left[1^2 + j^2 + 2\cdot 1\cdot j\right]^5$$

$$= \left[1 - 1 + 2j\right]^5 = \left[2j\right]^5 = 2^5 j^5$$

$$= 32\cdot(-1)(-1)\cdot j$$

$$= 32j$$

$$= 0 + 32j$$

Ans

(g) $\left[\dfrac{1+j}{1-j}\right] \longrightarrow$ polar $\longrightarrow z = e^{-j\theta}$

Simplifying the expression,

$$\frac{1+j}{1-j} = \frac{(1+j)(1-j)}{(1-j)(1-j)} = \frac{1+j^2+2j}{1-j^2}$$

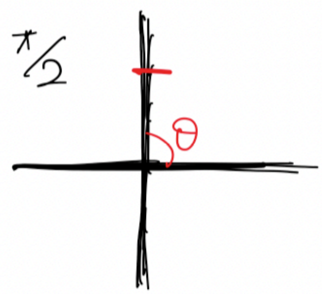
$$= \frac{1-1+2j}{1-(-1)} = \frac{2j}{2} = j = (0+1j)$$

$x=0$

$y=1$

$$\boxed{r = \sqrt{x^2+y^2} = 1}$$

$\theta = \pi/2$



$z = (0+1j) = 1 \cdot e^{\left(\frac{\pi}{2}\right)j}$

(Ans)

A.4 Show that each of the identities below holds, where z, z_1, and z_2 are arbitrary complex numbers.

(a) $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$;

(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \neq 0$;

(c) $z + z^* = 2\,\mathrm{Re}\{z\}$;

(d) $zz^* = |z|^2$; and

(e) $(z_1 z_2)^* = z_1^* z_2^*$.

(b) Let $z_1 = a+bi$ $\quad |z_1| = \sqrt{a^2+b^2}$

$z_2 = c+di$ $\quad |z_2| = \sqrt{c^2+d^2}$

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

$$= \frac{ac-adi+bci+bd}{c^2+d^2}$$

$$= \frac{ac+bd+i(bc-ad)}{c^2+d^2}$$

$$\left|z_1/z_2\right| = \left|\frac{ac+bd+i(bc-ad)}{c^2+d^2}\right|$$

$$= \sqrt{\frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\left[\frac{(ac)^2+(bd)^2+(bc)^2+(ad)^2}{(c^2+d^2)^2}\right]}$$

$$= \sqrt{\left[\frac{a^2(c^2+d^2)+b^2(c^2+d^2)}{(c^2+d^2)^2}\right]}$$

$$= \sqrt{\frac{a^2}{(c^2+d^2)}+\frac{b^2}{(c^2+d^2)}}$$

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{|z_1|}{|z_2|}$$

(Ans)

(e)

Let $z_1 = x_1 + iy$, $z_2 = x_2 + iy$ $\Big\}$ LHS $= z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2)$

$+ i(x_1 y_2 + y_1 x_2)$

Now, $(z_1 z_2)^* = z_1^* z_2^*$

$z_1^* z_2^* = ((x_1 x_2 - y_1 y_2) + i(x_1 y_1 + y_1 x_2))^*$

$= ((x_1 x_2 - y_1 y_2) - i(x_1 y_1 + y_1 x_2))$

RHS; $z_1^* z_2^* = (x_1 + iy_1)^* (x_2 + iy_2)^*$

$= (x_1 - iy_1)(x_2 - iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(-x_1 y_2 - y_1 x_2)$

$= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$

$=$ LHS

$\therefore (z_1 z_2)^* = z_1^* z_2^*$ (proved)

A.5 Use Euler's relation to show that each of the identities below holds, where θ is an arbitrary real constant.

(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$;

(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$; and

(c) $\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$.

From Euler's relation

we get

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

When argument is negative,

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we have to prove $\sin\theta = \left(\frac{1}{2j}\right)\left[e^{j\theta} - e^{-j\theta}\right]$

$$\text{RHS} = \frac{1}{2j} \cdot \left(e^{j\theta} - e^{-j\theta}\right)$$

$$= \frac{1}{2j} \cdot \left((\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)\right)$$

$$= \frac{1}{2j} \cdot \left(\cos\theta + j\sin\theta - \cos\theta + j\sin\theta\right)$$

$$= \frac{1}{2j} \cdot \left(0 + 2j\sin\theta\right)$$

$$= \frac{1}{2j} \cdot 2j\sin\theta = \sin\theta$$

$$= \text{RHS}.$$

Ans

A.6 For each rational function f of a complex variable given below, find the (finite) poles and zeros of f and their orders. Also, plot these poles and zeros in the complex plane.

(a) $f(z) = z^2 + jz + 3$;

(b) $f(z) = z + 3 + 2z^{-1}$;

(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$;

(d) $f(z) = \dfrac{z^3 - z}{z^2 - 4}$;

(e) $f(z) = \dfrac{z + \frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and

(f) $f(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

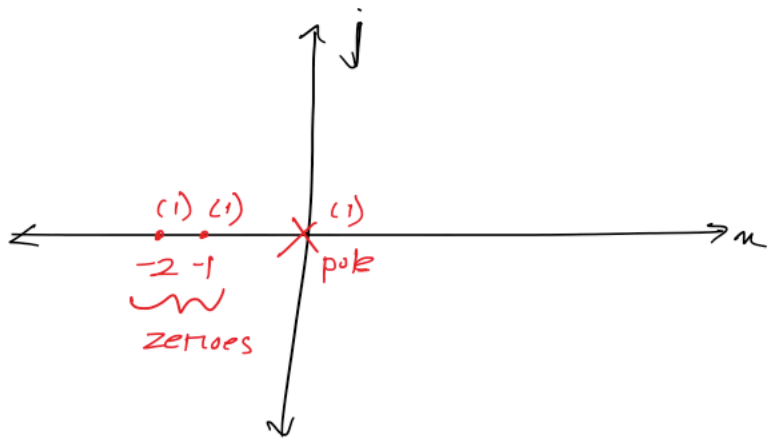
(b) $f(z) = z+3+(2z)^{-1}$

$$= z+3+\frac{2}{z}$$

$$= \frac{z^2+3z+2}{z}$$

$$= \frac{(z+1)(z+2)}{z}$$

zeroes	ones
$z=-1$	$z=0$
$z=-2$	



(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$

PTO

(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$

(1) For $z^2+2z+5=0$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4\times1\times5}}{2\times1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4j}{2}$$

$$\Rightarrow \boxed{z = -1 \pm 2j}$$

(2) $z^2+1=0$

$\Rightarrow z^2 = -1$

$\Rightarrow z = \sqrt{-1}$

$\Rightarrow \boxed{z = \pm j}$

(3) $z^2+2z+2=0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2-4\times2\times1}}{2\times1}$$

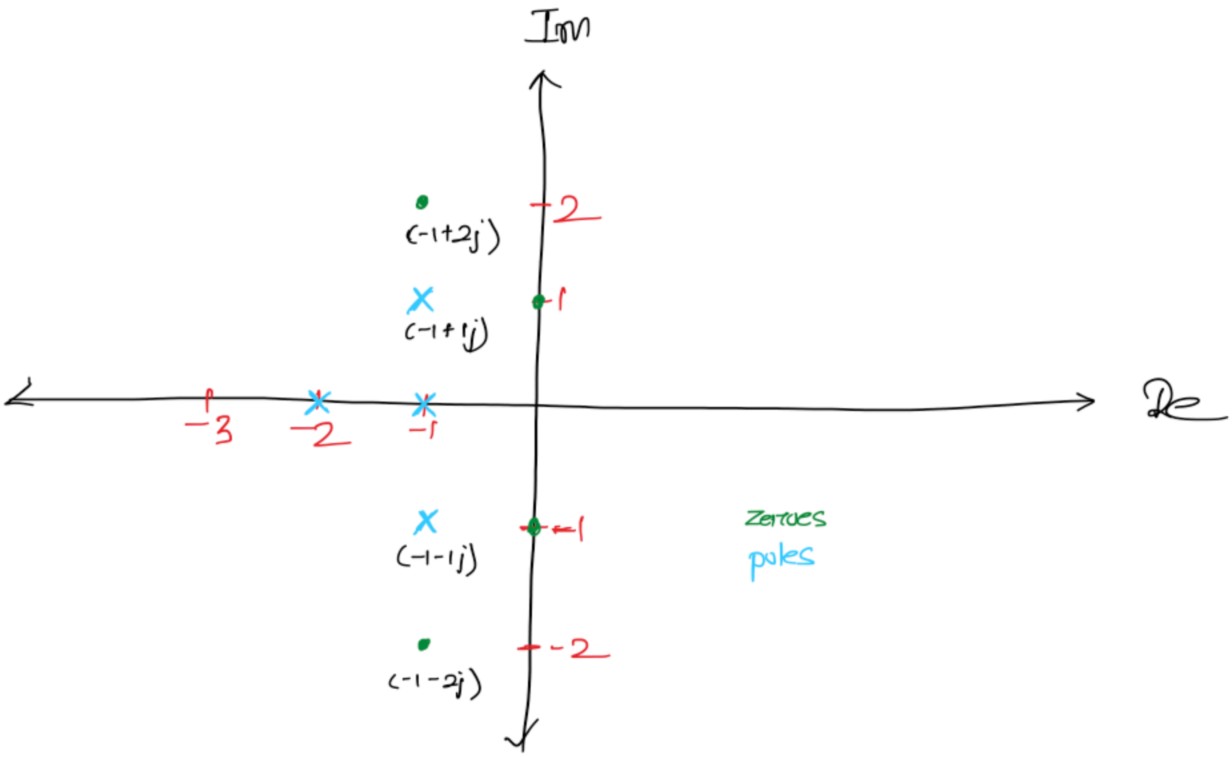
$$\Rightarrow z = \frac{-2 \pm \sqrt{-4}}{2} = \boxed{-1 \pm j = z}$$

(4) $z^2+3z+2=0$

$\Rightarrow (z+1)(z+2)=0$

$\Rightarrow \boxed{\begin{matrix} z=-1 \\ z=-2 \end{matrix}}$

$$f(z) = \frac{(z+1+j^2)(z+1-j^2)(z+j)(z-j)}{(z+1-j)(z+1+j)(z+1)(z-2)}$$



A.7 Determine the points at which each function f given below is: i) continuous, ii) differentiable, and iii) analytic. To deduce the answer, use your knowledge about polynomial and rational functions. Simply state the final answer along with a short justification (i.e., two or three sentences). (In other words, it is not necessary to use the Cauchy-Riemann equations for this problem.)

(a) $f(z) = 3z^3 - jz^2 + z - \pi$;

(b) $f(z) = \dfrac{z-1}{(z^2+3)(z^2+z+1)}$;

(c) $f(z) = \dfrac{z}{z^4-16}$; and

(d) $f(z) = z + 2 + z^{-1}$.

ⓒ $f(z) = \dfrac{z}{z^4 - 16}$ Let $f(z) = \dfrac{A(z)}{B(z)}$ where $A(z) = z$

$B(z) = z^4 - 16$

Individually, $A(z)$ and $B(z)$ are all differentiable continuous and analytical functions.

Since, $f(z) = A(z)/B(z)$; $f(z)$ is a rational function, and will be a continuous, differentiable & analytical given $B(z)$/ denominator is not 0.

$\therefore z^4 - 16 \neq 0$

$\Rightarrow z^4 \neq 16$

$\Rightarrow \sqrt{z^4} \neq \pm\sqrt{16}$

$\Rightarrow z \neq \pm 4 \neq \pm 2, \pm 2i$

$\therefore$ $f(z)$ is differentiable, continuous and analytical everywhere except $\pm 2, \pm 2i$ in the complex plane

$\downarrow$

$$\boxed{\mathbb{C} - \{\pm 2, \pm 2i\}}$$

(d) $f(z) = z+2+z^{-1}$

$$= \underbrace{(z+2)}_{A(z)} + \underbrace{\frac{1}{z}}_{B(z)}$$

Let,

$A(z) = (z+2)$ = polynomial function
= rational function with denominator 1
= continuous, Analytical and differentiable everywhere in the complex plane $\mathbb{C}$.

$B(z)$ = rational function $= 1/z$
= continuous, differentiable and Analytical everywhere except 0 in the complex plane $\mathbb{C}$.

$\therefore f(z) = A(z) + B(z)$ = continuous, Analytical and differentiable everywhere except 0 in the complex plane.

$\therefore \boxed{\mathbb{C} - \{0\}}$.

Answer

A.9 For each function f of a real variable given below, find an expression for $|f(\omega)|$ and $\arg f(\omega)$.

(a) $f(\omega) = \dfrac{1}{(1+j\omega)^{10}}$;

(b) $f(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$;

(c) $f(\omega) = \dfrac{2e^{j11\omega}}{(3+j5\omega)^7}$;

(d) $f(\omega) = \dfrac{-5}{(-1-j\omega)^4}$;

(e) $f(\omega) = \dfrac{j\omega^2}{(j\omega-1)^{10}}$; and

(f) $f(\omega) = \dfrac{j\omega-1}{j\omega+1}$.

(f) $F(\omega) = \dfrac{\omega j - 1}{\omega j + 1}$. Given ω is a real variable of the function f.

$$|F(\omega)| = \left|\frac{\omega j - 1}{\omega j + 1}\right| = \frac{\sqrt{(\omega j)^2 + (-1)^2}}{\sqrt{(\omega j)^2 + (1)^2}} = \frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+1}} = 1$$

$$\operatorname{Arg}(F(\omega)) = \arg\left(\frac{j\omega - 1}{j\omega + 1}\right)$$

$$= \arg(j\omega - 1) - \arg(j\omega + 1)$$

$$= \arctan\left(\frac{\omega}{-1}\right) - \arctan\left(\frac{\omega}{1}\right)$$

$$= -2\arctan(\omega)$$

(Answer)

c. $f(\omega) = \dfrac{2e^{11j\omega}}{(3+j5\omega)^7}$

$$|f(\omega)| = \left| \frac{2e^{(11j\omega)}}{(3+j5w)^7} \right| = \frac{|2 \cdot e^{(11jw)}|}{|3+5jw|^7} \longrightarrow |re^{\theta j}| = r$$

$$= \frac{2}{\left[\sqrt{3^2+(5w)^2}\right]^7}$$

(Ans)

$$\arg(f(w)) = \arg\left(\frac{2e^{j11w}}{(3+5jw)^7}\right)$$

$$= \arg(2e^{j11w}) - \arg((3+5jw)^7)$$

$$= 11w - 7\arctan(5w/3).$$

(Answer)