R **5.102** Find the Fourier series coefficient sequence c of each periodic function x given below with fundamental period T.
(a) $x(t) = e^{-t}$ for $-1 \le t < 1$ and $T = 2$;
(b) $x(t) = \operatorname{rect}(t - \frac{3}{2}) - \operatorname{rect}(t + \frac{3}{2})$ for $-\frac{5}{2} \le t < \frac{5}{2}$ and $T = 5$;
(c) $x(t) = e^{-2|t|}$ for $-2 \le t < 2$ and $T = 4$;
(d) $x(t) = -\delta(t+1) + \delta(t) + \delta(t-1)$ for $-2 \le t < 2$ and $T = 4$;
(e) $x(t) = 5e^{3t}$ for $0 \le t < 5$ and $T = 5$;
(f) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$ for $-2 \le t < 2$ and $T = 4$;
(g) $x(t) = t^2$ for $-1 \le t < 1$ and $T = 2$ [Hint: See (F.2).];
(h) $x(t) = \sin\left(\frac{\pi}{2}t\right)[u(t-1) - u(t-2)]$ for $0 \le t < 2$ and $T = 2$ [Hint: See (F.4).];
(i) $x(t) = t[u(t) - u(t-1)]$ for $0 \le t < 2$ and $T = 2$ [Hint: See (F.1).];
(j) $x(t) = |t|\,[u(t+1) - u(t-1)]$ for $-2 \le t < 2$ and $T = 4$ [Hint: See (F.5).];
(k) $x(t) = 4\delta(t-1) - 4\delta(t-2) + 6\delta(t-3) + 6\delta(t-5)$ for $0 \le t < 8$ and $T = 8$;
(l) $x(t) = 2\delta(t) + \delta(t-1) + \delta(t-2)$ for $0 \le t < 4$ and $T = 4$; and
(m) $x(t) = \sum_{k=-\infty}^{\infty} 3\delta(t - 4k)$ (where T is implicit in the definition of x).

Short Answer.
(a) $c_k = \dfrac{(-1)^k(e - e^{-1})}{j2\pi k + 2}$;
(b) $c_k = \begin{cases} \frac{1}{j\pi k}\left(\cos\left(\frac{2\pi k}{5}\right) - \cos\left(\frac{4\pi k}{5}\right)\right) & k \ne 0 \\ 0 & k = 0; \end{cases}$
(c) $c_k = \dfrac{4\left[1 - e^{-4}(-1)^k\right]}{16 + \pi^2 k^2}$;
(d) $c_k = -\frac{j}{2}\sin(\frac{\pi}{2}k) + \frac{1}{4}$;
(e) $c_k = \dfrac{5(e^{15} - 1)}{15 - j2\pi k}$;
(f) $c_k = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}k\right)$;
(g) $c_k = \begin{cases} \dfrac{(-1)^k 2}{\pi^2 k^2} & k \ne 0 \\ \frac{1}{3} & k = 0; \end{cases}$
(h) $c_k = \dfrac{1 + (-1)^k j2k}{4\pi\left(\frac{1}{4} - k^2\right)}$;
(i) $c_k = \begin{cases} \dfrac{(-1)^k(j\pi k + 1) - 1}{2\pi^2 k^2} & k \ne 0 \\ \frac{1}{4} & k = 0; \end{cases}$
(j) $c_k = \begin{cases} \dfrac{\pi k \sin\left(\frac{\pi}{2}k\right) + 2\cos\left(\frac{\pi}{2}k\right) - 2}{\pi^2 k^2} & k \ne 0 \\ \frac{1}{4} & k = 0; \end{cases}$
(k) $c_k = je^{-j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right) + \frac{3}{2}(-1)^k\cos\left(\frac{\pi}{4}k\right)$;
(l) $c_k = \frac{1}{2}\left[1 + e^{-j(3\pi/4)k}\cos\left(\frac{\pi}{4}k\right)\right]$;
(m) $c_k = \frac{3}{4}$

Exercise 5.102

R Answer (c).

We are given

$$x(t) = e^{-2|t|} \text{ for } -2 \le t < 2 \text{ and } T = 4.$$

Let ω_0 denote the fundamental frequency of x. So, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$. We have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \tfrac{1}{4}\int_{-2}^{2} e^{-2|t|}e^{-jk(\pi/2)t}\,dt \\
&= \tfrac{1}{4}\left[\int_{-2}^{0} e^{2t}e^{-jk(\pi/2)t}\,dt + \int_{0}^{2} e^{-2t}e^{-jk(\pi/2)t}\,dt\right] = \tfrac{1}{4}\left[\int_{-2}^{0} e^{(2-jk\pi/2)t}\,dt + \int_{0}^{2} e^{(-2-jk\pi/2)t}\,dt\right] \\
&= \tfrac{1}{4}\left(\left[\frac{e^{(2-jk\pi/2)t}}{2-\frac{j\pi}{2}k}\right]\Bigg|_{-2}^{0} + \left[\frac{e^{(-2-jk\pi/2)t}}{-2-\frac{j\pi}{2}k}\right]\Bigg|_{0}^{2}\right) \\
&= \tfrac{1}{4}\left[\left(\frac{1}{2-\frac{j\pi}{2}k}\right)\left(1-e^{(2-jk\pi/2)(-2)}\right) + \left(\frac{1}{-2-\frac{j\pi}{2}k}\right)\left(e^{(-2-jk\pi/2)(2)}-1\right)\right] \\
&= \tfrac{1}{4}\left[\left(\frac{1}{2-\frac{j\pi}{2}k}\right)\left(1-e^{-4+jk\pi}\right) + \left(\frac{1}{-2-\frac{j\pi}{2}k}\right)\left(e^{-4-jk\pi}-1\right)\right] \\
&= \tfrac{1}{4}\left[\frac{1-e^{-4}(-1)^k}{2-\frac{j\pi}{2}k} + \frac{1-e^{-4}(-1)^k}{2+\frac{j\pi}{2}k}\right] = \tfrac{1}{4}\left[1-e^{-4}(-1)^k\right]\left[\frac{1}{2-\frac{j\pi}{2}k} + \frac{1}{2+\frac{j\pi}{2}k}\right] \\
&= \tfrac{1}{4}\left[1-e^{-4}(-1)^k\right]\left[\frac{2+\frac{j\pi}{2}k+2-\frac{j\pi}{2}k}{4+\frac{\pi^2}{4}k^2}\right] = \frac{1-e^{-4}(-1)^k}{4+\frac{\pi^2}{4}k^2} \\
&= \frac{4\left[1-e^{-4}(-1)^k\right]}{16+\pi^2k^2}.
\end{aligned}$$

R **5.104** For each case below, where the function x has the Fourier series $y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$ (where T denotes the fundamental period of x), find $y(t)$ for the specified values of t.

(a) $x(t) = \begin{cases} e^{t+2} & -2 \le t < -1 \\ 1 & -1 \le t < 1 \\ e^{-t+3} - e & 1 \le t < 2 \end{cases}$ and $x(t) = x(t+4)$, $t \in \{-1, 2\}$;

(b) $x(t) = \begin{cases} e^{t} & 0 \le t < 2 \\ -t^2 & 2 \le t < 5 \end{cases}$ and $x(t) = x(t+5)$, $t \in \{0, 2\}$;

(c) $x(t) = \begin{cases} 1 + e^{t} & -1 < t < 0 \\ e^{-2t} & 0 \le t \le 1 \end{cases}$ and $x(t) = x(t+2)$, $t \in \{0, 1\}$;

(d) $x(t) = \begin{cases} t^2 + 2t + 1 & -2 \le t < 0 \\ -t^2 + 2t - \pi & 0 \le t < 2 \end{cases}$ and $x(t) = x(t+4)$, $t \in \{0, 1\}$; and

(e) $x(t) = \begin{cases} e^{-t} & 0 \le t < 1 \\ t - 1 & 1 \le t < 2 \\ e^{t-3} & 2 \le t < 3 \end{cases}$ and $x(t) = x(t+3)$, $t \in \{1, 2\}$.

Short Answer. (a) $y(-1) = \frac{e+1}{2}$ and $y(2) = \frac{1}{2}$; (b) $y(0) = -12$ and $y(2) = \frac{e^2-4}{2}$; (c) $y(0) = \frac{3}{2}$ and $y(1) = \frac{e^2+e+1}{2e^2}$; (d) $y(0) = \frac{1-\pi}{2}$ and $y(1) = 1 - \pi$; (e) $y(1) = \frac{1}{2e}$ and $y(2) = \frac{e+1}{2e}$

Exercise 5.104

R Answer (b).

We are given the function x, where

$$x(t) = \begin{cases} e^t & 0 \le t < 2 \\ -t^2 & 2 \le t < 5 \end{cases} \quad \text{and} \quad x(t) = x(t+5).$$

To begin, we observe that the function x satisfies the Dirichlet conditions. Therefore, at each point t_a of discontinuity of x, we have $y(t_a) = \frac{1}{2}\left[x(t_a^-) + x(t_a^+)\right]$. Thus, we have

$$\begin{aligned} y(0) &= \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] \\ &= \tfrac{1}{2}[-25+1] \\ &= \tfrac{1}{2}[-24] \\ &= -12 \quad \text{and} \\ y(2) &= \tfrac{1}{2}\left[x(2^-) + x(2^+)\right] \\ &= \tfrac{1}{2}\left[e^2 + (-2^2)\right] \\ &= \tfrac{1}{2}\left[e^2 - 4\right] \\ &= \tfrac{e^2-4}{2}. \end{aligned}$$

R **5.105** For each case below, where the T-periodic function x has the Fourier series coefficient sequence c, find the magnitude and phase spectra of x.

(a) $c_k = \dfrac{jk-1}{jk+1}$ and $T = 2\pi$;

(b) $c_k = \dfrac{4jk+4}{(jk-1)^2}$ and $T = 4$;

(c) $c_k = \dfrac{-1}{(2+j\pi k)^2}$ and $T = 2$;

(d) $c_k = \left(\dfrac{e^{j3k}}{j2k-1}\right)^2$ and $T = 2$; and

(e) $c_k = \dfrac{j4\pi^2k^2}{(j2\pi k-1)^{10}}$ and $T = 1$.

Short Answer. (a) $|c_k| = 1$ and $\arg c_k = -2\arctan(k) + (2\ell+1)\pi$ (where $\ell \in \mathbb{Z}$); (b) $|c_k| = \frac{4}{\sqrt{k^2+1}}$ and $\arg c_k = 3\arctan(k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$); (c) $|c_k| = \frac{1}{4+\pi^2k^2}$ and $\arg c_k = -2\arctan\left(\frac{\pi}{2}k\right) + (2\ell+1)\pi$ (where $\ell \in \mathbb{Z}$); (d) $|c_k| = \frac{1}{4k^2+1}$ and $\arg c_k = 6k + 2\arctan(2k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$); (e) $|c_k| = \frac{4\pi^2k^2}{(4\pi^2k^2+1)^5}$ and $\arg(c_k) = \frac{\pi}{2} + 10\arctan(2\pi k) + 2\pi\ell$ (where $\ell \in \mathbb{Z}$)

Exercise 5.105

R Answer (b).

We are given

$$c_k = \frac{4jk+4}{(jk-1)^2} \quad \text{and} \quad T = 4.$$

First, we compute the magnitude spectrum of x. We have

$$\begin{aligned}|c_k| &= \left|\frac{4jk+4}{(jk-1)^2}\right| = \frac{|4jk+4|}{|(jk-1)^2|} = \frac{4\,|jk+1|}{\left(\sqrt{k^2+1}\right)^2} = \frac{4\sqrt{k^2+1}}{\left(\sqrt{k^2+1}\right)^2} \\ &= \frac{4}{\sqrt{k^2+1}}.\end{aligned}$$

Next, we compute the phase spectrum of x. We have

$$\begin{aligned}\arg c_k &= \arg\left[\frac{4jk+4}{(jk-1)^2}\right] = \arg(4jk+4) - \arg\left[(jk-1)^2\right] \\ &= \arctan\left(\frac{4k}{4}\right) - 2\left[\arctan\left(\frac{k}{-1}\right) + \pi\right] \\ &= \arctan(k) - 2\arctan(-k) - 2\pi = \arctan(k) + 2\arctan(k) - 2\pi \\ &= 3\arctan(k) - 2\pi.\end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg c_k = 3\arctan(k) + 2\pi\ell$$

for all integer ℓ.

R **5.106** For each case below, where the periodic function x has the Fourier series coefficient sequence c, determine whether x is each of the following: real, even, odd.

(a) $c_k = e^{-|k|}$;

(b) $c_k = \dfrac{e^{-j3k}}{k^2}$ if $k \neq 0$ and $c_0 = 0$;

(c) $c_k = \operatorname{sgn}(k)e^{-|k|}$;

(d) $c_k = j\operatorname{sgn}(k)e^{-|3k|}$;

(e) $c_k = j\,|k|\,e^{-k^2}$;

(f) $c_k = \dfrac{1}{k+j}$;

(g) $c_k = \begin{cases} j\sin\left(\frac{\pi}{2}k\right) & k \in [-32\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$

(h) $c_k = \begin{cases} \cos(\pi k) & k \in [-32\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$

(i) $c_k = \begin{cases} 2^{-k} & k \in [0\,..\,32] \\ 0 & \text{otherwise;} \end{cases}$ and

(j) $c_k = \begin{cases} k^3 & k \in [-8\,..\,8] \\ 0 & \text{otherwise.} \end{cases}$

Short Answer. (a) real and even; (b) real but not even/odd; (c) odd but not real; (d) real and odd; (e) even but not real; (f) not real and not even/odd; (g) real and odd; (h) real and even; (i) not real and not even/odd; (j) odd but not real

Exercise 5.106

R **Answer (d).**

We are given the Fourier series coefficient sequence c, where

$$c_k = j\,\mathrm{sgn}(k)e^{-|3k|}.$$

To begin, we observe that sgn is an odd function. So, we have

$$\begin{aligned}
c_{-k} &= j\,\mathrm{sgn}(-k)e^{-|-3k|} \\
&= j[-\mathrm{sgn}(k)]e^{-|3k|} \\
&= -j\,\mathrm{sgn}(k)e^{-|3k|} \\
&= -c_k.
\end{aligned}$$

Therefore, c is odd. (Or, alternatively, the sequence c is the product of the odd sequence $v_1(k) = \mathrm{sgn}(k)$ and the even sequence $v_2(k) = je^{-|3k|}$. Since the product of an odd sequence and an even sequence is an odd sequence, c is odd.) Since c is purely imaginary and the conjugate of a purely imaginary number is its negative, c being odd implies that c is conjugate symmetric. Since c is conjugate symmetric, x is real. Therefore, we conclude that x is real and odd.

R **5.107** For each case below, find the response y of the LTI system with frequency response H to the input x.

(a) $H(\omega) = \begin{cases} -3 & \omega < -1 \\ 0 & -1 \le \omega \le 0 \\ 3 & \omega > 0 \end{cases}$ and $x(t) = 4\cos(t) + 2\cos(2t)$;

(b) $H(\omega) = \operatorname{rect}\left(\frac{1}{10\pi}\omega\right)$ and $x(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{1}{2}k\right)$;

(c) $H(\omega) = \operatorname{sgn}(\omega)$ and $x(t) = 4 + 3\cos(t) + 2\cos(3t)$;

(d) $H(\omega) = \begin{cases} 2 & 4 \le |\omega| \le 11 \\ 0 & \text{otherwise} \end{cases}$ and $x(t) = 1 + \frac{1}{2}\sin(5t) + \frac{1}{4}\cos(10t) + \frac{1}{8}\sin(15t)$;

(e) $H(\omega) = \frac{5}{j\omega}$ and $x(t) = 8\sin(2t) + 6\cos(3t)$; and

(f) $H(\omega) = \frac{1}{4+j\omega}$ and $x(t) = 8 + \cos(3t)$.

Short Answer. (a) $y(t) = 6e^{jt} + 6j\sin(2t)$; (b) $y(t) = 4\cos(4\pi t) + 2$; (c) $y(t) = 3j\sin(t) + 2j\sin(3t)$; (d) $y(t) = \sin(5t) + \frac{1}{2}\cos(10t)$; (e) $y(t) = 10\sin(3t) - 20\cos(2t)$; (f) $y(t) = 2 + \frac{1}{5}\cos\left[3t - \arctan\left(\frac{3}{4}\right)\right]$

Exercise 5.107

R Answer (b).

We are given the LTI system with input x, output y, and frequency response H, where

$$H(\omega) = \operatorname{rect}\left(\tfrac{1}{10\pi}\omega\right) \quad \text{and} \quad x(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - \tfrac{1}{2}k\right).$$

Clearly, x is periodic with period $T = \frac{1}{2}$. So, x has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}$$

From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk(2\pi/T)t}dt \\
&= \tfrac{1}{1/2}\int_{-1/4}^{1/4} \delta(t)e^{-jk4\pi t}dt \\
&= 2\int_{-\infty}^{\infty} \delta(t)e^{-jk4\pi t}dt \\
&= 2\left[e^{-jk4\pi t}\right]\Big|_{t=0} \\
&= 2.
\end{aligned}$$

Since the system is LTI, we have

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} H(\tfrac{2\pi}{T}k)c_k e^{jk(2\pi/T)t} \\
&= \sum_{k=-\infty}^{\infty} H(4\pi k)c_k e^{j4\pi kt} \\
&= H(-4\pi)c_{-1}e^{j4\pi(-1)t} + H(0)c_0 + H(4\pi)c_1 e^{j4\pi(1)t} \\
&= (1)(2)e^{-j4\pi t} + (1)(2) + (1)(2)e^{j4\pi t} \\
&= 2e^{-j4\pi t} + 2e^{j4\pi t} + 2 \\
&= 2\left(e^{j4\pi t} + e^{-j4\pi t}\right) + 2 \\
&= 4\cos(4\pi t) + 2.
\end{aligned}$$