# ECE 260

# EXAM 4

# SOLUTIONS

(FALL 2023)

QUESTION 1

PART (A)

x is real if and only if $X$ is conjugate symmetric (i.e., $X(\omega) = X^*(-\omega)$ for all $\omega$).

$X$ is not conjugate symmetric since $X(\omega) = X^*(-\omega)$ does not hold for $\omega \in [1, 2]$; that is $1 \neq -1^*$

therefore, x is not real

PART (B)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$
$$= \frac{1}{2\pi} [1 + 1]$$
$$= \frac{1}{\pi}$$

therefore, x has finite energy (since $\frac{1}{\pi}$ is clearly finite)

PART (C)

x and $X$ cannot both be finite duration functions since $X$ is finite duration, x cannot be finite duration

QUESTION 2 (WITHOUT NEEDING CHAIN RULE)

$$y(t) = e^{-j3t}(t/2)\,x(t/2)$$

Let $v_1(t) = t\,x(t)$ ①

$$= e^{-j3t}\,v_1(t/2)$$

Let $v_2(t) = v_1(t/2)$ ②

$$= e^{-j3t}\,v_2(t) \quad ③$$

taking the Fourier transforms of ①, ②, and ③, we have

$$V_1(\omega) = j\,X'(\omega) \quad ④$$

$$V_2(\omega) = 2\,V_1(2\omega) \quad ⑤$$

$$Y(\omega) = V_2(\omega+3) \quad ⑥$$

combining ④, ⑤, and ⑥, we have

$$\begin{aligned} Y(\omega) &= V_2(\omega+3) \\ &= 2\,V_1(2[\omega+3]) \\ &= 2\,[j\,X'(2[\omega+3]) \\ &= 2j\,X'(2[\omega+3]) \\ &= 2j\,X'(2\omega+6) \end{aligned}$$

(Note: The prime symbol is used to denote the derivative.)

QUESTION 2 (REQUIRING CHAIN RULE)

$$y(t) = e^{-j3t}(t/2)\,x(t/2)$$
$$= \tfrac{1}{2}e^{-j3t}\,t\,x(t/2)$$
$$= \tfrac{1}{2}e^{-j3t}\,t\,v_1(t)$$
$$= \tfrac{1}{2}e^{-j3t}\,v_2(t) \quad ③$$

Let $v_1(t) = x(t/2)$ ①

Let $v_2(t) = t\,v_1(t)$ ②

taking the Fourier transforms of ①, ②, and ③, we have

$$V_1(\omega) = 2X(2\omega) \quad ④$$

$$V_2(\omega) = j\,V_1'(\omega) \quad ⑤$$

$$Y(\omega) = \tfrac{1}{2}V_2(\omega+3) \quad ⑥$$

computing $V_1'$, we have (via the chain rule)

$$V_1'(\omega) = 2\left[2X'(2\omega)\right] = 4X'(2\omega) \quad ⑦$$

combining ⑦, ⑤, and ⑥, we have

$$Y(\omega) = \tfrac{1}{2}V_2(\omega+3)$$
$$= \tfrac{1}{2}\left[j\,V_1'(\omega+3)\right]$$
$$= \tfrac{j}{2}V_1'(\omega+3)$$
$$= \tfrac{j}{2}\left[4X'(2[\omega+3])\right]$$
$$= 2j\,X'(2[\omega+3])$$
$$= 2j\,X'(2\omega+6)$$

(Note: The prime symbol is used to denote derivative.)

QUESTION 3

$$H(\omega) = \frac{\omega^2}{\omega^2 - 2j\omega - 1} \qquad \left[ \text{Note: } H(\omega) = \frac{\omega^2}{(\omega - j)^2} \right]$$

PART (A)

$$|H(\omega)| = \left| \frac{\omega^2}{\omega^2 - 2j\omega - 1} \right| = \frac{|\omega^2|}{|\omega^2 - 2j\omega - 1|} = \frac{\omega^2}{|(\omega^2 - 1) + j(-2\omega)|}$$

$$= \frac{\omega^2}{\sqrt{(\omega^2-1)^2 + (-2\omega)^2}} = \frac{\omega^2}{\sqrt{(\omega^2-1)^2 + 4\omega^2}} = \frac{\omega^2}{\sqrt{\omega^4 - 2\omega^2 - 1 + 4\omega^2}}$$

$$= \frac{\omega^2}{\sqrt{\omega^4 + 2\omega^2 - 1}} = \frac{\omega^2}{\sqrt{(\omega^2+1)^2}} = \frac{\omega^2}{\omega^2 + 1}$$

PART (B)

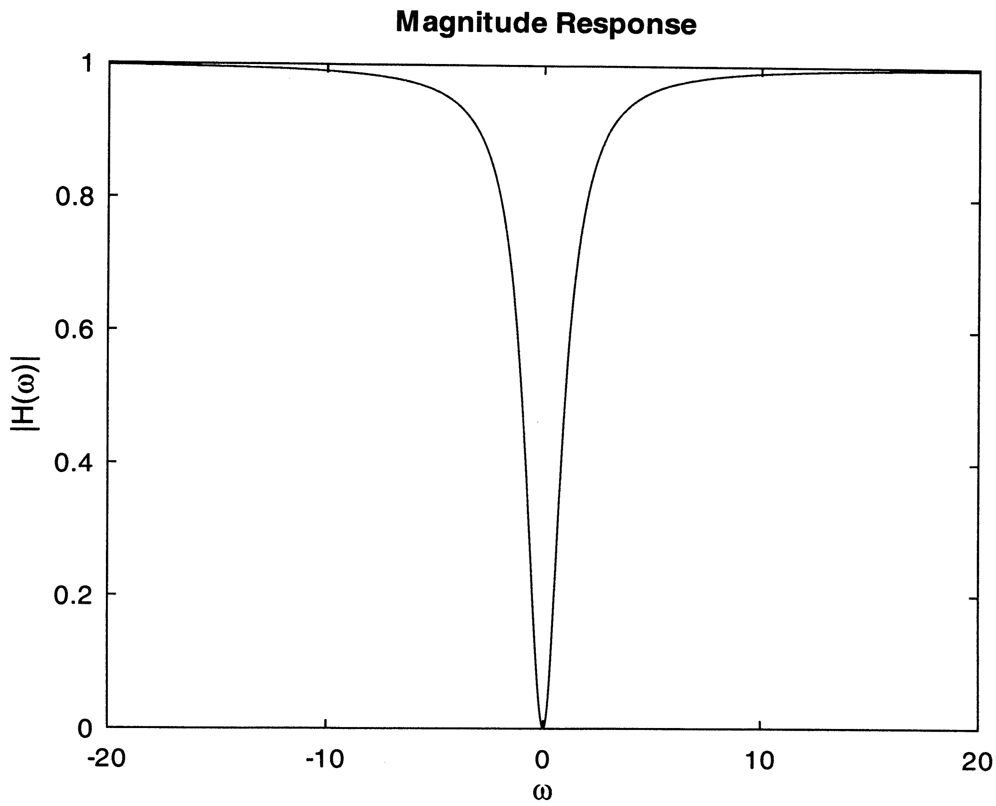
$$|H(0)| = \frac{0}{0+1} = 0$$

$$\lim_{|\omega| \to \infty} |H(\omega)| = \lim_{|\omega| \to \infty} \frac{\omega^2}{\omega^2 + 1} = 1$$

Since the circuit attenuates lower frequencies and allows higher frequencies to pass through without much change, the circuit most closely approximates an ideal highpass filter.

QUESTION 3

This graph is not formally part of the solution.
It is included simply for illustrative purposes.



QUESTION 4

PART (A)

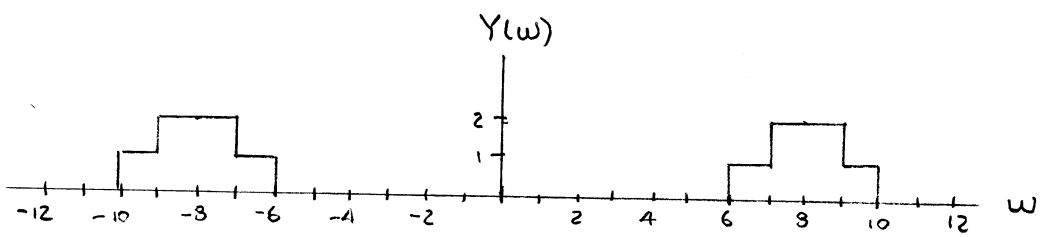
From the sampling theorem, a signal bandlimited to frequencies in the range $[-\omega_b, \omega_b]$ must be sampled at a frequency $\omega_s$ satisfying $\omega_s > 2\omega_b$ to avoid aliasing.

For the given function $x$, $\omega_b = 2$.

So, $\omega_s > 2\omega_b = 2(2) = 4$.

PART (B)

$$y(t) = x(t)\cos(8t)$$
$$= \frac{1}{2}\left(e^{j8t} + e^{-j8t}\right) x(t)$$
$$= \frac{1}{2} e^{j8t} x(t) + \frac{1}{2} e^{-j8t} x(t)$$
$$= \frac{1}{2} X(\omega - 8) + \frac{1}{2} X(\omega + 8)$$




$y$ is bandlimited to frequencies in $[-\omega_b, \omega_b]$ where $\omega_b = 10$

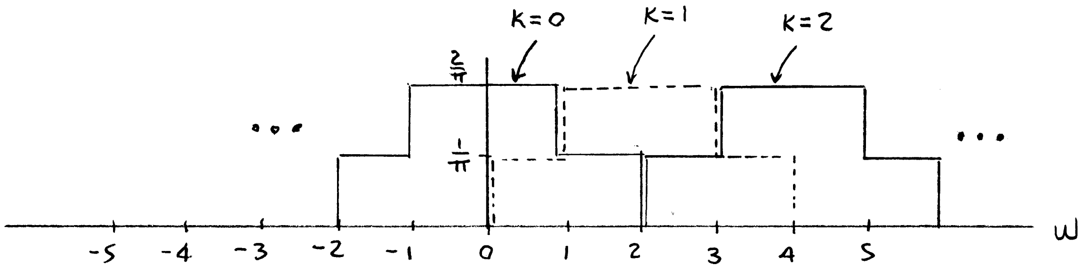
let $\omega_s$ denote the sampling frequency for $y$

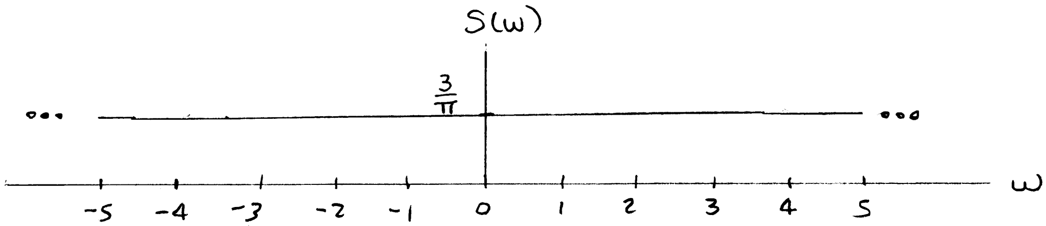
$\omega_s > 2\omega_b = 2(10) = 20$ (from sampling theorem)

QUESTION 4

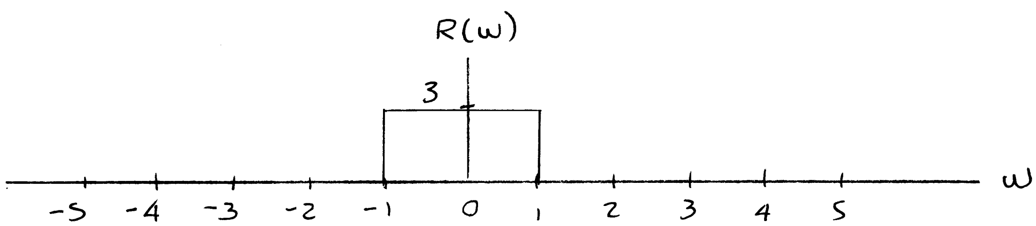
PART (c)

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$
$$= \frac{2}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - 2k)$$
$$= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} X(\omega - 2k)$$




$$R(\omega) = \frac{2\pi}{\omega_s} \operatorname{rect}\left(\frac{\omega}{\omega_s}\right) S(\omega)$$
$$= \frac{2\pi}{2} \operatorname{rect}(\omega/2)\, S(\omega)$$
$$= \pi \operatorname{rect}(\omega/2)\, S(\omega)$$



QUESTION 5

$$h(t) = e^{-3t} u(t)$$

$$x(t) = e^{-3t} \cos(2t)\, u(t)$$

$$y(t) = x * h(t)$$

$$Y(w) = X(w)\, H(w)$$

$$X(w) = \frac{3+jw}{(3+jw)^2 + 2^2} = \frac{3+jw}{(3+jw)^2+4}$$

$$H(w) = \frac{1}{3+jw}$$

$$Y(w) = X(w)\, H(w)$$

$$= \left[\frac{3+jw}{(3+jw)^2+4}\right]\left[\frac{1}{3+jw}\right]$$

$$= \frac{1}{(3+jw)^2+4}$$

$$= \frac{1}{2}\left[\frac{2}{(3+jw)^2+2^2}\right]$$

$$y(t) = \frac{1}{2}\left[e^{-3t} \sin(2t)\, u(t)\right]$$

$$= \frac{1}{2} e^{-3t} \sin(2t)\, u(t)$$