Chapter 5 – Equilibrium Rigid Bodies

Contents

The submarine is a rigid body subjected to multiple forces (own weight, tension of cables, etc.) All these forces are not concurrent. If the submarine does not translate or rotate, then the sum of forces and moments is zero.

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Equilibrium of rigid bodies is the core of statics. In this Chapter we will apply the concepts that we learned in the previous Chapters.

- Add and subtract vector quantities.
- Take dot products of vectors using the vectors' Cartesian coordinates.
- Take cross products of vectors using the vectors' Cartesian coordinates.
- Find the moments of a force about a reference point and axis.
- Calculate the magnitude and direction of the couple moment exerted by a force couple.
- Reduce a given set of loads down to a simple equivalent loading.

Chapter 5 – Equilibrium Rigid Bodies

The significant step that we will add in this Chapter is the implementation of Free Body Diagrams that will expose all of the loads acting on a rigid body (including reactions exerted by other bodies).

We will classify the type of reactions that will be exerted on a rigid body.

Once we have defined all of the loads that are acting on the rigid body, we will impose the equilibrium equations to solve for any unknowns.

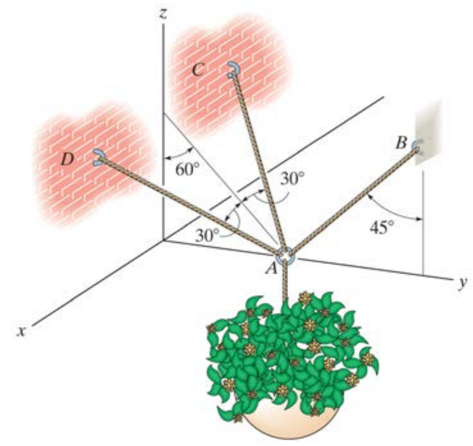
Conditions for Rigid Body Equilibrium

Conditions of Rigid Body Equilibrium

A particle is in equilibrium, when the net force acting on it is zero, $\sum \mathbf{F} = 0$.

$$(\Sigma F_x)\,\boldsymbol{i} + (\Sigma F_y)\,\boldsymbol{j} + (\Sigma F_z)\,\boldsymbol{k} = 0$$

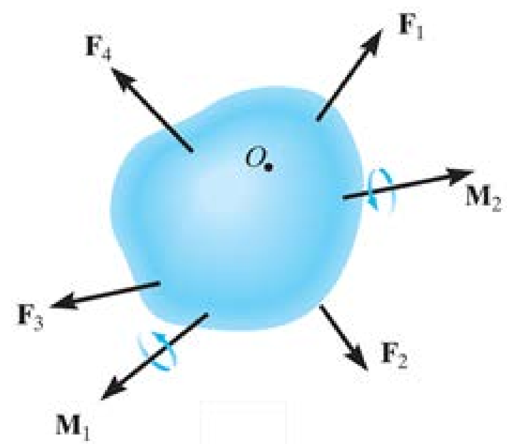
or $\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$



A rigid body is in equilibrium when the net force and the net moment about any arbitrary point *O* are equal to zero.

$\sum \mathbf{F} = 0$ (no translation)

and $\sum \mathbf{M}_O = 0$ (no rotation)



Free Body Diagram - Coplanar

Free Body Diagram (FBD) – Coplanar Systems.

Free Body Diagram is the tool that we use to identify all the forces acting on a particular collection of particles (the rigid body).

To draw the FBD of a rigid body we remove the body from its surroundings (the body becomes free).

When the body is removed from its surroundings, the effects of any joints/interconnections/supports are exposed.

The loads exerted by joints/interconnections are referred to as reaction loads or support loads.

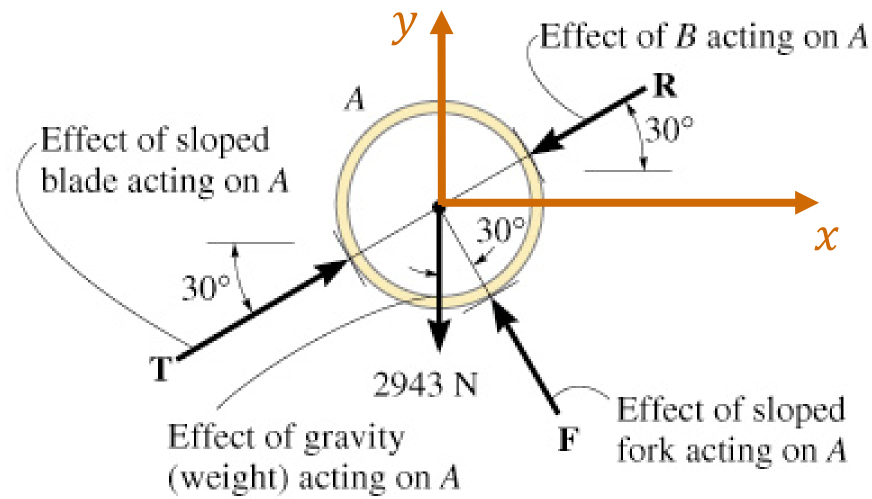
Support loads represent the ability of the joint to prevent translation and/or rotation.

Free Body Diagram - Coplanar

Examples of FBDs

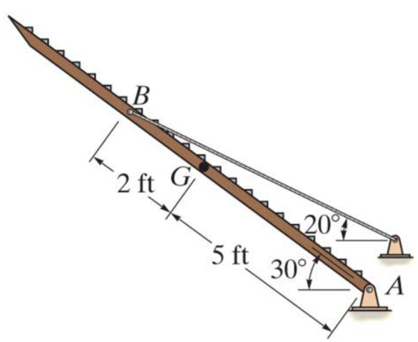
Physical System

FDB of Rigid Body under study

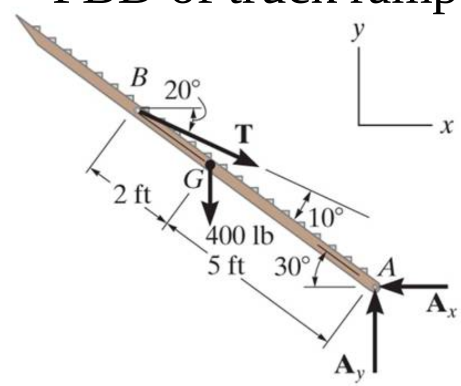


Physical System

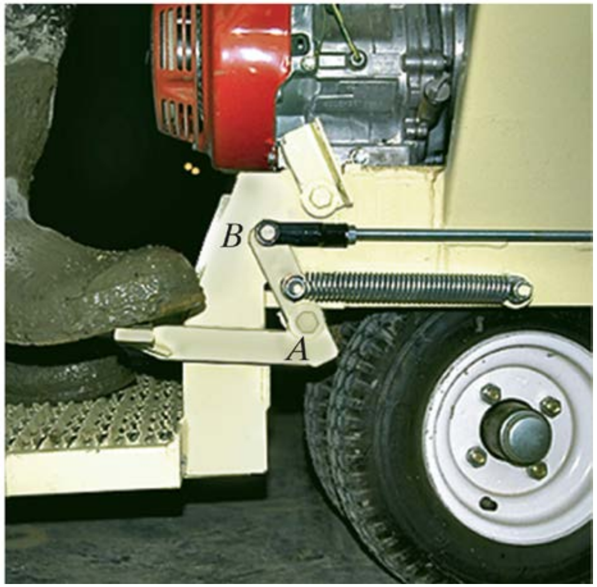
Idealized Model



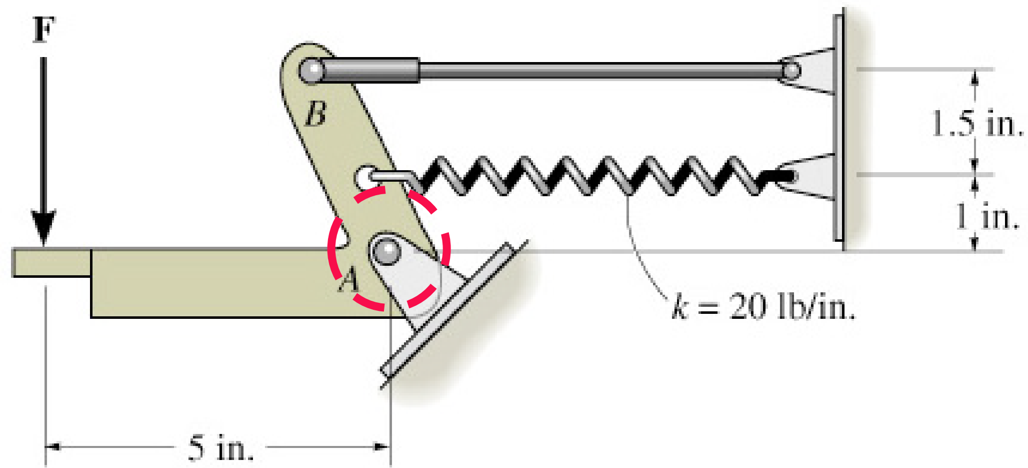
FBD of truck ramp



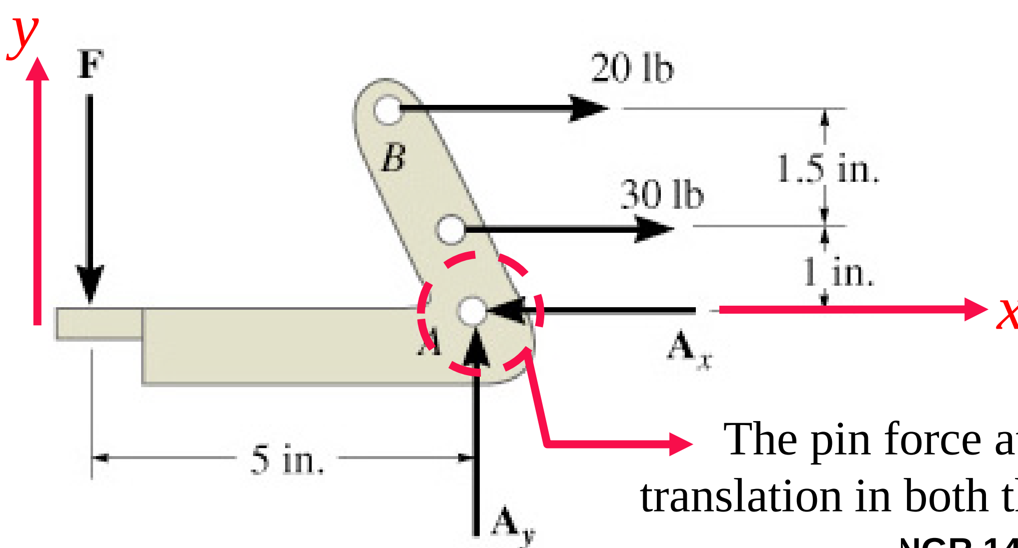
Free Body Diagram - Coplanar



Operator applies a vertical force to the pedal



FBD

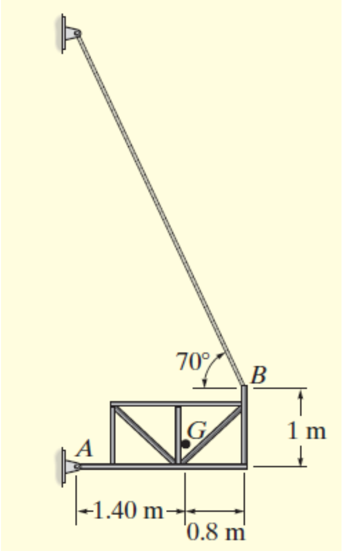


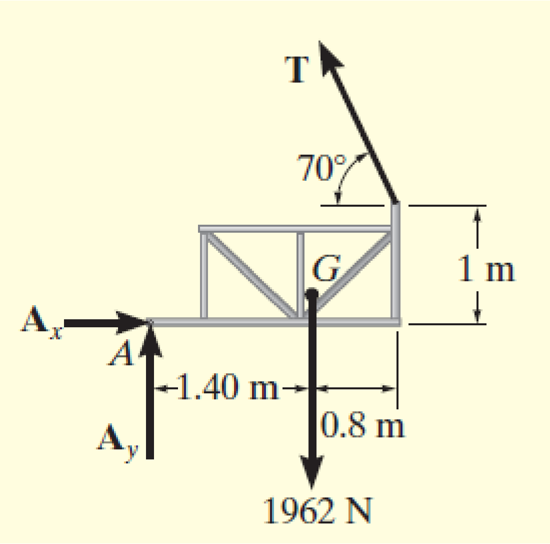
The pin force at 'A' constraints translation in both the x and y directions.

Free Body Diagram - Coplanar

Unloaded platform is suspended off the edge of the oil rig.

The idealized model of the platform is considered in two dimensions because the loading and the dimensions are all symmetrical about a vertical plane passing through its center.





Equations of Equilibrium - Coplanar

Equations of equilibrium – Coplanar systems

For coplanar systems the two vector equations for rigid body equilibrium reduce to 3 scalar equations:

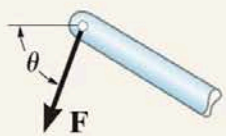
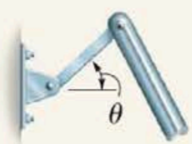
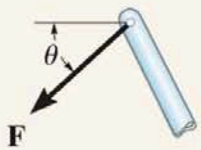
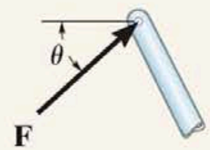
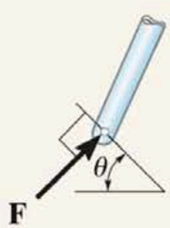
$$\boxed{\begin{array}{l} \mathbf{F}_R = \sum \mathbf{F} = \mathbf{0} \\ \mathbf{M}_O = \sum \mathbf{M}_C + \sum \mathbf{r} \times \mathbf{F} = \mathbf{0} \end{array}}$$

$\Downarrow$ Coplanar systems

$$\boxed{\sum F_{Rx} = 0 \qquad \sum F_{Ry} = 0 \qquad \sum M_{O_z} = 0}$$

Support Reactions - Coplanar

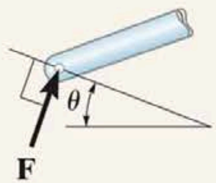
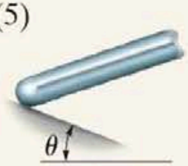
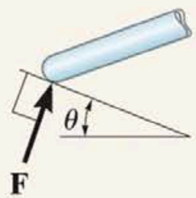
TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

If a **support prevents translation** of a body in a given direction, then **a force is developed** on the body in the opposite direction. Similarly, if **rotation is prevented, a couple moment** is applied on the body.

Support Reactions - Coplanar

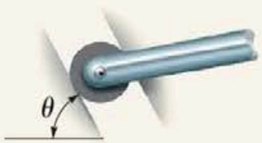
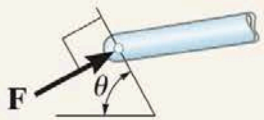
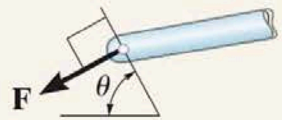
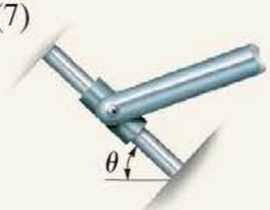
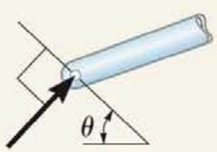
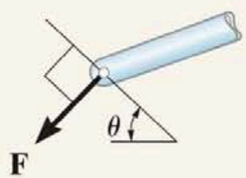
TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

If a **support prevents translation** of a body in a given direction, then **a force is developed** on the body in the opposite direction. Similarly, if **rotation is prevented, a couple moment** is applied on the body.

Support Reactions - Coplanar

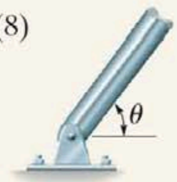
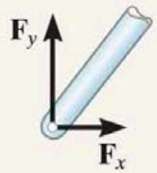
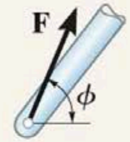
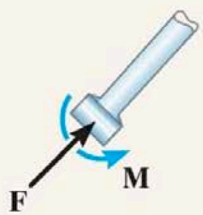
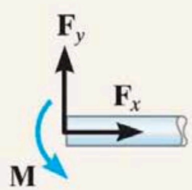
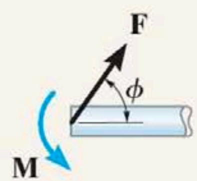
TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

If a **support prevents translation** of a body in a given direction, then **a force is developed** on the body in the opposite direction. Similarly, if **rotation is prevented, a couple moment** is applied on the body.

Support Reactions - Coplanar

TABLE 5–1 Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge	 or 	Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support	 or 	Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Support Reactions - Coplanar

Examples

When the link of the awning window mechanism is extended, it exerts a force on the slider, which results on a normal force being applied to the rod. This causes the window to open.

(One Unknown)

The abutment-mounted rocker bearing is used to support the roadway of a bridge. It allows horizontal movement so the bridge is free to expand and contract due to temperature.

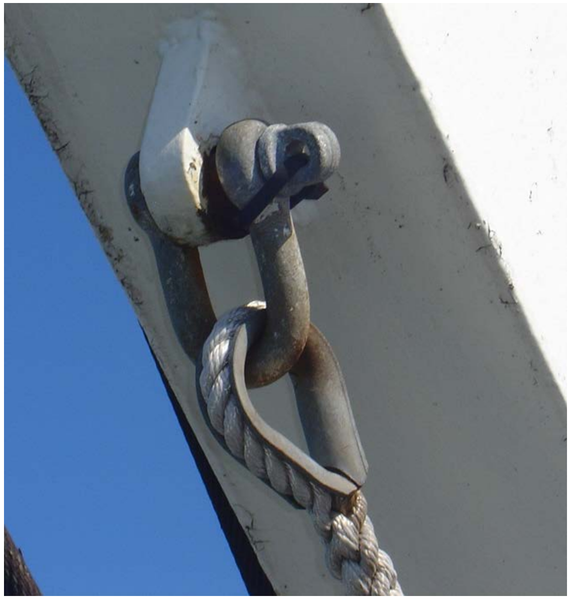
(One Unknown)

Support Reactions - Coplanar

Examples

Wooden columns are supported by pins.

(Frictionless pin or hinge: two unknowns)



The cable exerts a force on the bracket in the direction of the cable.

(Two Unknowns)

(One Unknown)

Support Reactions - Coplanar

Examples

The pole is bolted to the ground (bolts are part of the free body diagram).

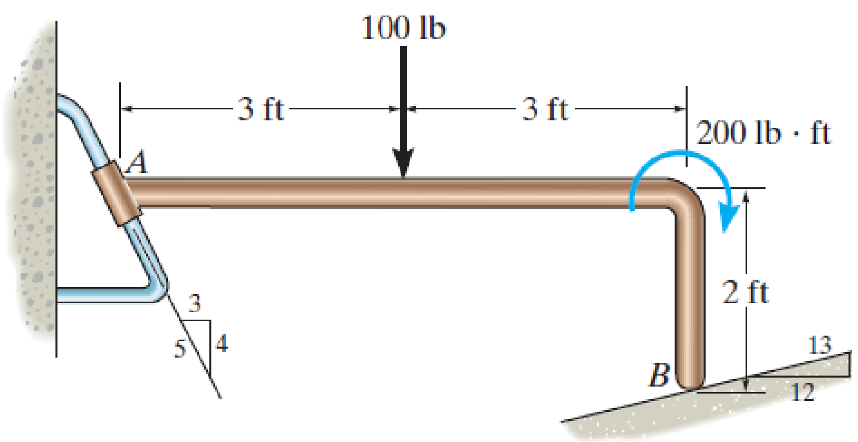
Concrete girder rests on the ledge, which is assumed to act as a smooth contacting surface.

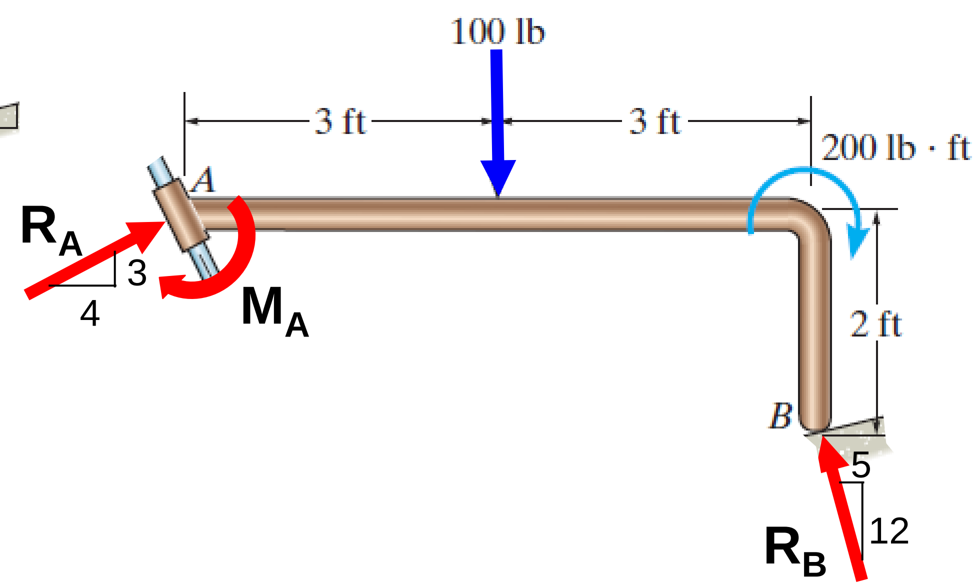
(Three Unknowns)

(One Unknown)

Example

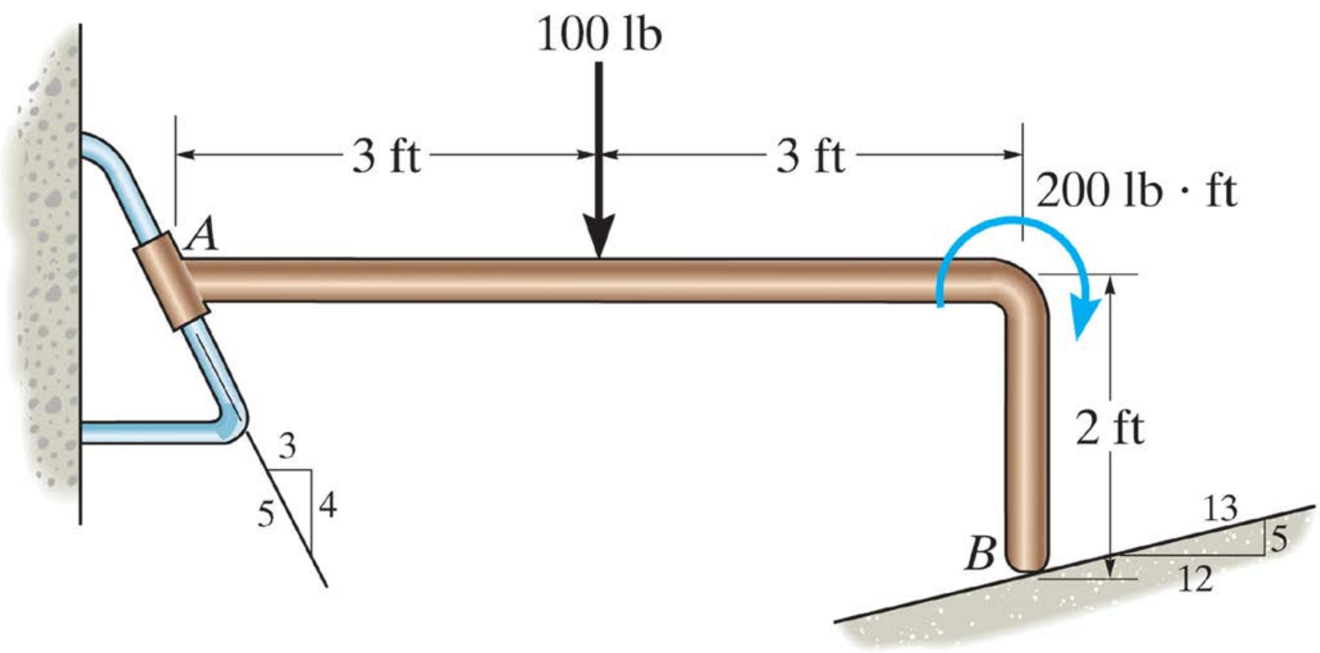
Draw a FBD of the bent rod supported by a smooth surface at *B* and by a collar at *A*, which is fixed to the rod and is free to slide over the fixed inclined rod.





Example

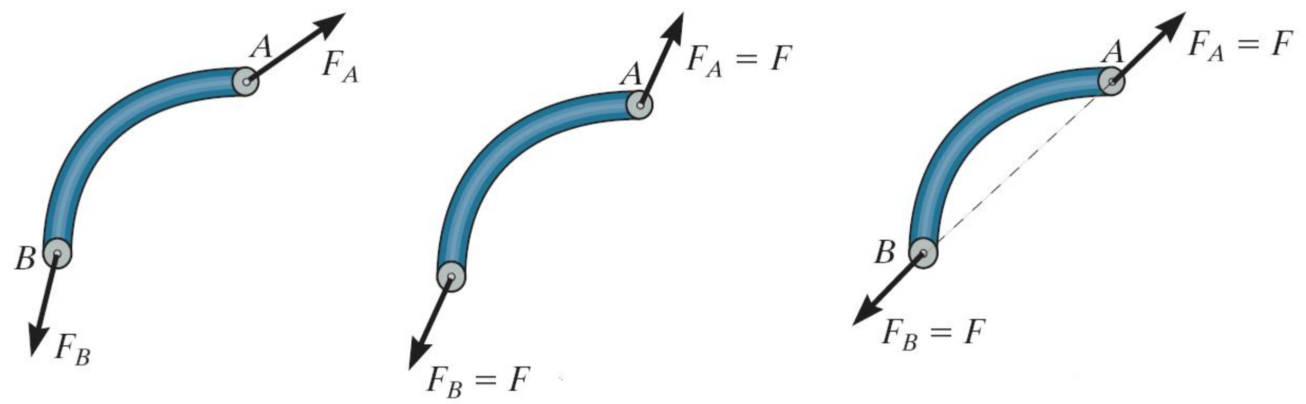
Determine the reactions on the bent rod which is supported by a smooth surface at B and by a collar at A, which is fixed to the rod and is free to slide over the fixed inclined rod.



Two- and Three-Force Members

Two-Force Members

Assume a *weightless* link. If the member is subjected to forces at only two points.

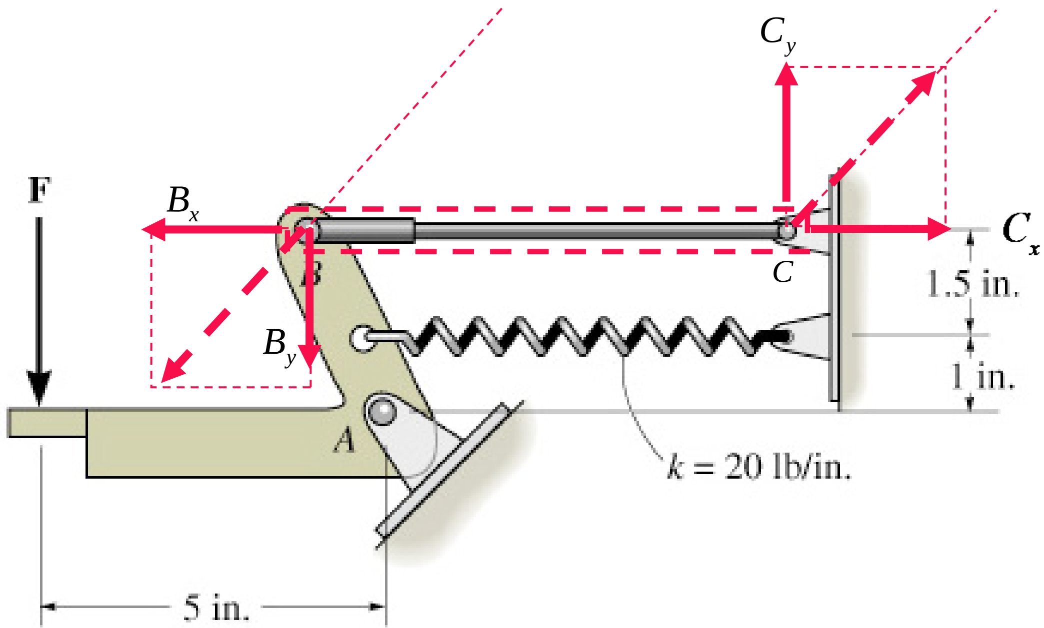


Then, we can recognize that for this object to be in static equilibrium the resultant forces at A and B must be equal in magnitude and act in the opposite directions along the line joining points A and B.

Two-force member must be equal opposite and collinear.

Two- and Three-Force Members

Example

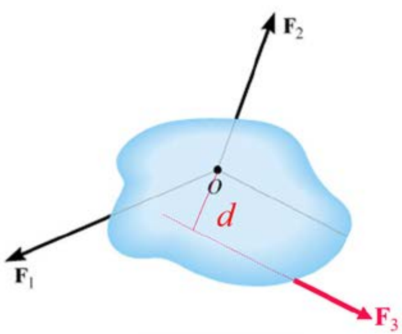


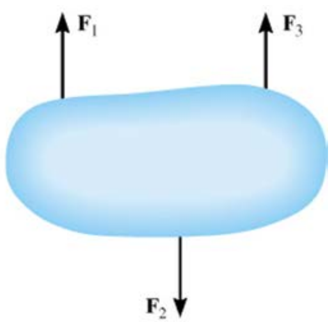
Two- and Three-Force Members

Three-Force Members

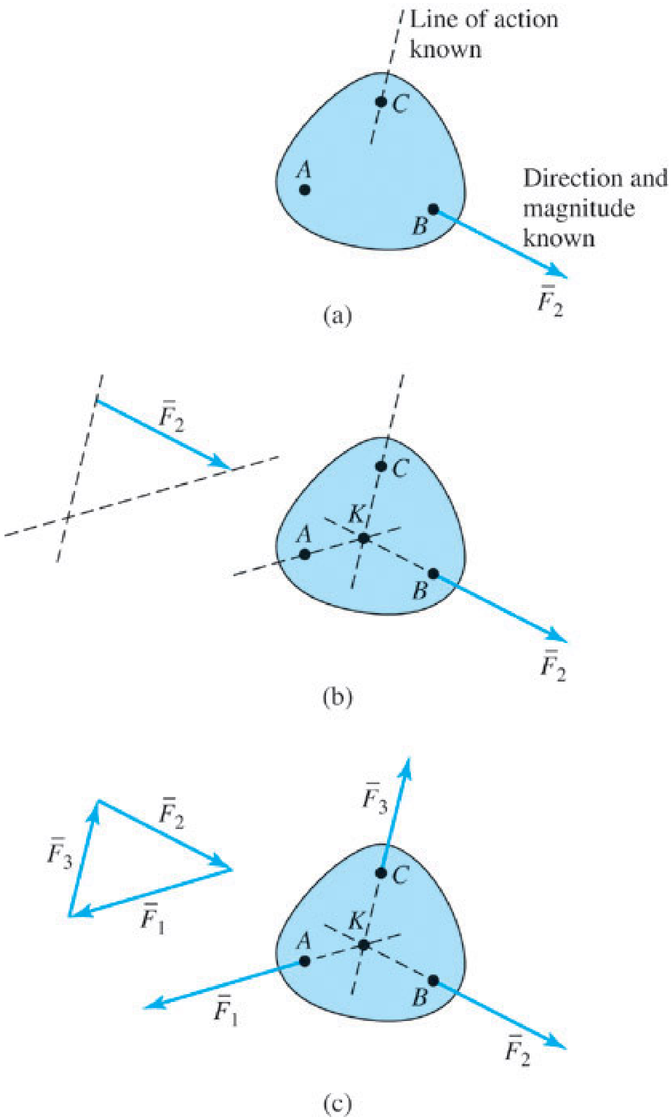
Assume a *weightless* link subjected to forces at three points A, B, and C.

For the member to be in static equilibrium the three forces must intersect at a point (or at infinity), otherwise $\sum M_A \neq 0$



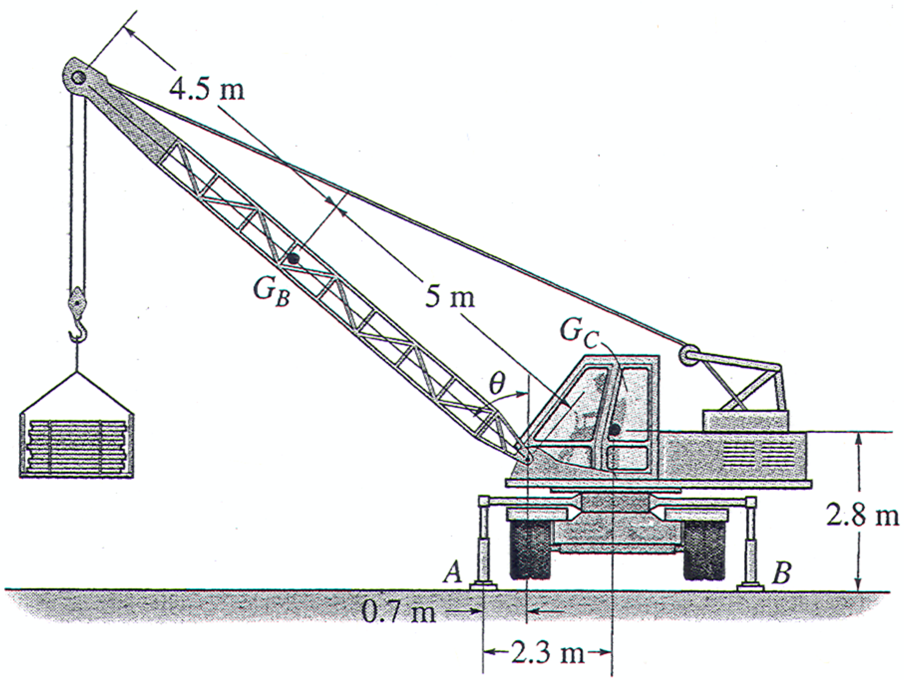


A graphical solution can be accomplished by forming a closed force polygon.



Example

Outriggers A and B are used to stabilize the crane from overturning when lifting large loads. If the load to be lifted is 3 Mg, determine the maximum boom angle θ so that the crane does not overturn. The crane has a mass of 5 Mg and center of mass at G_C , whereas the boom has a mass of 0.6 Mg and center of mass at G_B.



Chapter 5 – Equilibrium Rigid Bodies

Contents

The submarine is a rigid body subjected to multiple forces (own weight, tension of cables, etc.) All these forces are not concurrent. If the submarine does not translate or rotate, then the sum of forces and moments is zero.

Equations of Equilibrium

Equations of equilibrium – Spatial systems

For a spatial (3D) system, the equations of equilibrium lead to two vector equations, or six scalar equations

$$\sum \mathbf{F} = \mathbf{0}$$

$$\sum \mathbf{M}_O = \underbrace{\sum \mathbf{M}_C}_{\text{Couple moments}} + \underbrace{\sum \mathbf{r} \times \mathbf{F}}_{\text{Moments of all the forces}} = \mathbf{0}$$

If a problem is statically determinant (discussed later) then these 6 scalar equations provide all the necessary information. We must:

- Isolate, or free, the rigid body in question, include all the reactions forces using a free body diagram.
- Apply the equilibrium equations to the isolated rigid body.

Free Body Diagram - Spatial

Free Body Diagram of a Spatial Rigid Body

The first step in resolving problems related to the equilibrium of spatial (3D) rigid bodies is to draw a Free-Body Diagram.

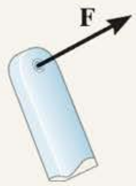
One must be careful to incorporate all the reactive forces and couple moments acting at supports and connections.

If a **support prevents translation** of a body in a given direction (x, y, z), then **a force is developed** on the body in the opposite direction.

If a support **support prevents rotation** of a body about any axis (x, y, z), then **a couple moment** is applied on the body.

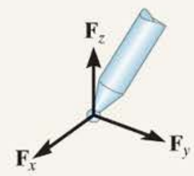
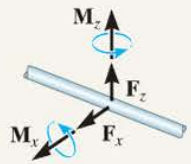
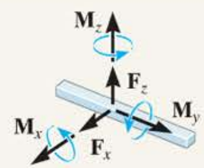
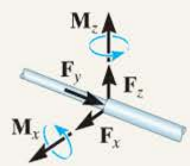
Support Reactions - Spatial

TABLE 5–2 Supports for Rigid Bodies Subjected to Three-Dimensional Force Systems

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a force which acts away from the member in the known direction of the cable.
(2)  smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

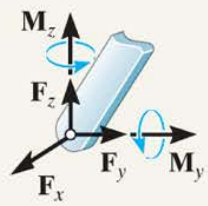
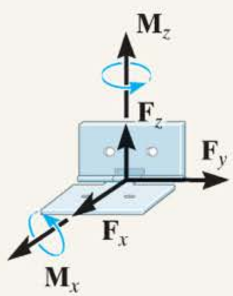
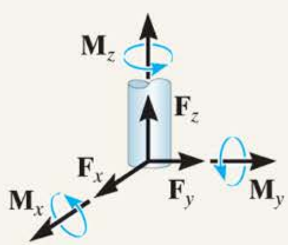
Support Reactions - Spatial

TABLE 5–2 Continued

Types of Connection	Reaction	Number of Unknowns
(4)  ball and socket		Three unknowns. The reactions are three rectangular force components.
(5)  single journal bearing		Four unknowns. The reactions are two force and two couple-moment components which act perpendicular to the shaft.
(6)  single journal bearing with square shaft		Five unknowns. The reactions are two force and three couple-moment components.
(7)  single thrust bearing		Five unknowns. The reactions are three force and two couple-moment components.

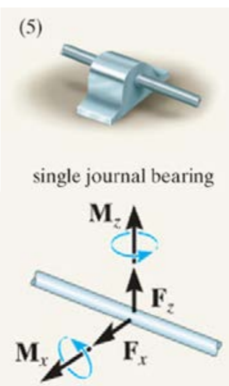
Support Reactions - Spatial

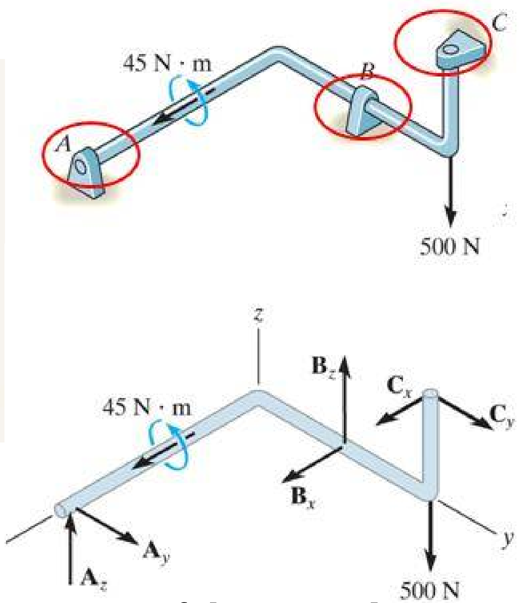
TABLE 5–2 Continued

Types of Connection	Reaction	Number of Unknowns
(8)  single smooth pin		Five unknowns. The reactions are three force and two couple-moment components.
(9)  single hinge		Five unknowns. The reactions are three force and two couple-moment components.
(10)  fixed support		Six unknowns. The reactions are three force and three couple-moment components.

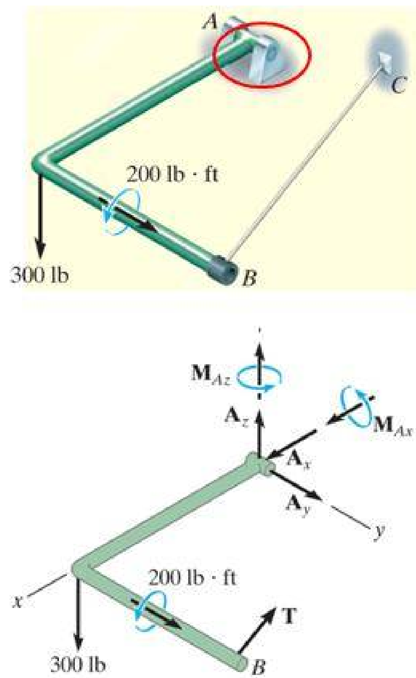
Support Reactions - Observation

A single bearing, a hinge, or a pin can prevent rotation by providing a resistive couple moment. However, if two or more properly aligned bearings or hinges are used, it is safe to assume that only force reactions are generated and no moment reactions are created.

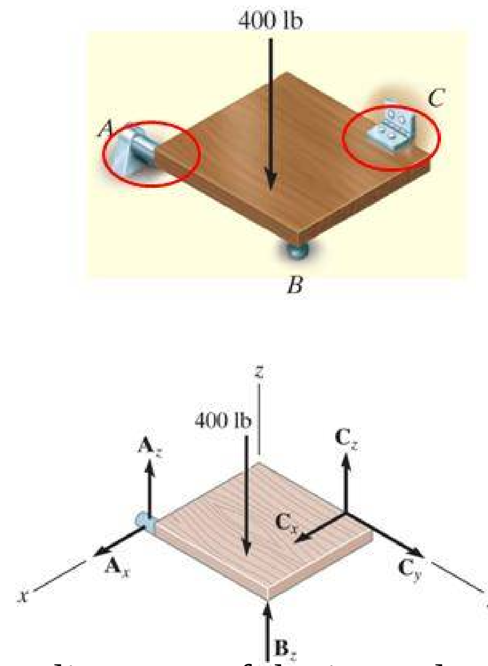




The alignment of the journal bearings prevent the shaft rotation. No couple moments are developed.



No simplification



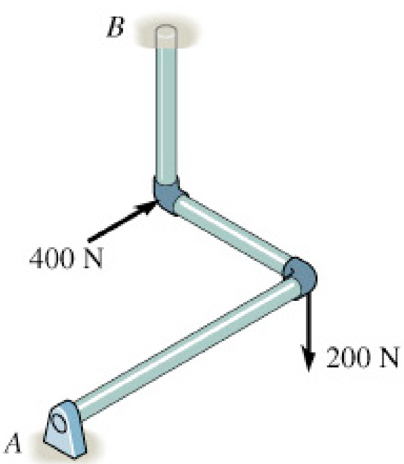
The alignment of the journal bearings prevent the shaft rotation. No couple moments are developed.

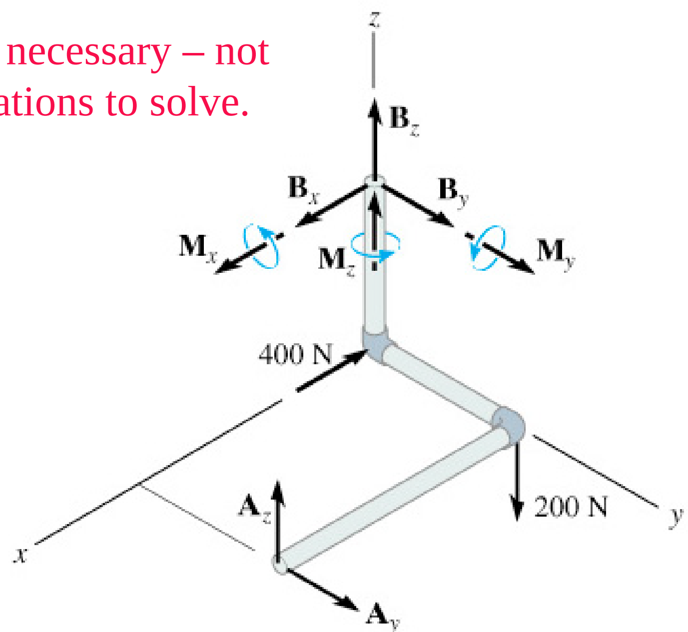
Constraints and Statical Determinacy

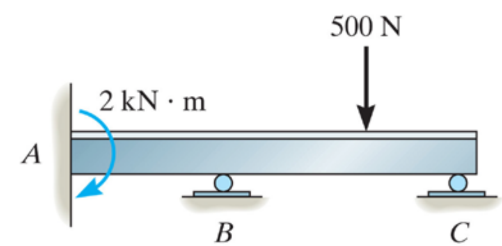
Redundant Constraints

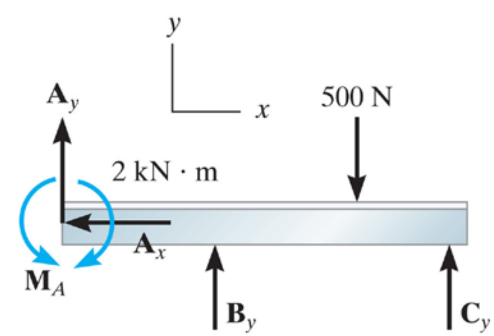
When a body has more supports than necessary to hold it in equilibrium, it becomes statically indeterminate. A problem that is statically indeterminate has more unknowns than equations of equilibrium.

More support forces than necessary – not enough equilibrium equations to solve.









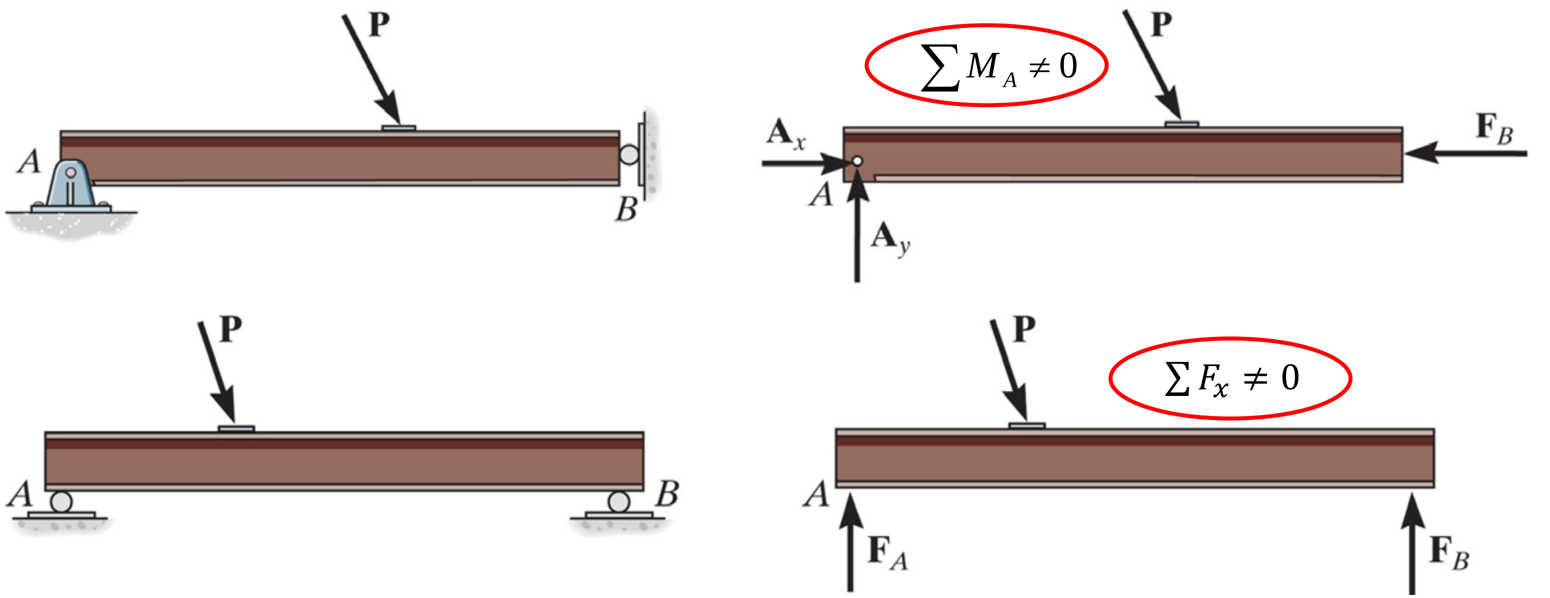
These problems must be solved by analyzing the deformation of the bodies.

Constraints and Statical Determinacy

Improper Constraints

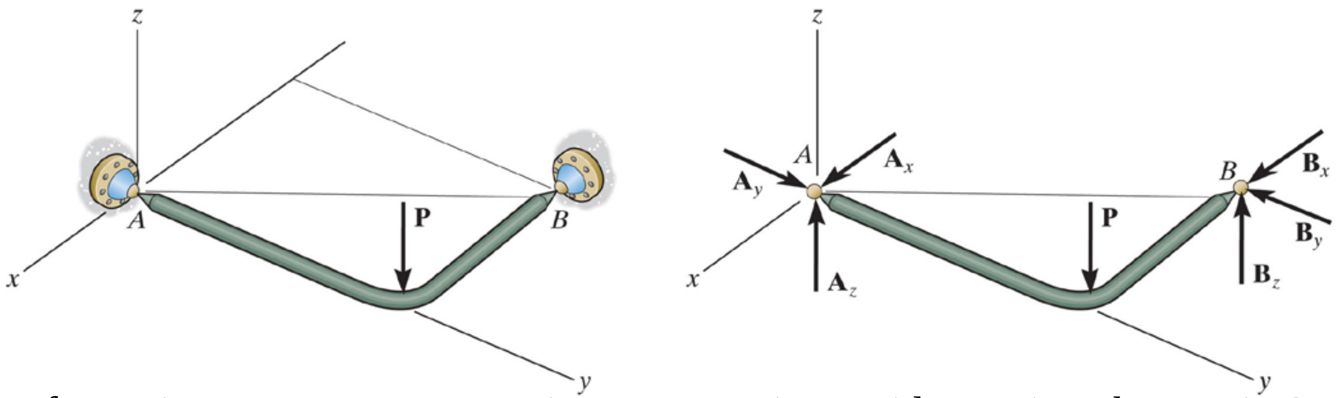
Having the same number of unknown reactive forces as number of equations does not guarantee a that the body is in equilibrium.

However, if the supports are not properly constrained, the body may become unstable for some loading cases.

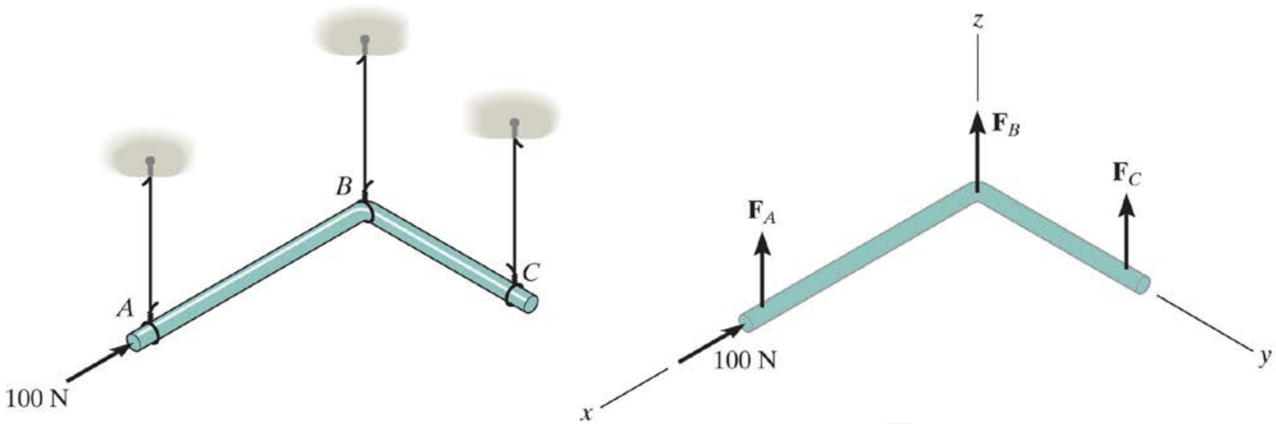


Constraints and Statical Determinacy

A body is considered *improperly constrained* if all the reactive forces intersect at a common point, pass through a common axis, or if all the forces are parallel. In engineering, these designs must be avoided.



All reactive forces intersect a common axis, no constraint avoids rotation about axis AB, $\sum M_{AB} \neq 0$.



All the reactive forces are parallel, $\sum F_x \neq 0$.

Equations of Equilibrium

Therefore, the equations of equilibrium lead to two vector equations, or six scalar equations

$$\boxed{\begin{aligned}&\sum \mathbf{F} = \mathbf{0}\\ &\sum \mathbf{M}_O = \sum \mathbf{M}_C + \sum \mathbf{r} \times \mathbf{F} = \mathbf{0}\end{aligned}}$$

$$\sum F_x = 0; \quad \sum F_y = 0; \quad \sum F_z = 0;$$

$$\sum M_{O_x} = 0; \quad \sum M_{O_y} = 0; \quad \sum M_{O_z} = 0;$$

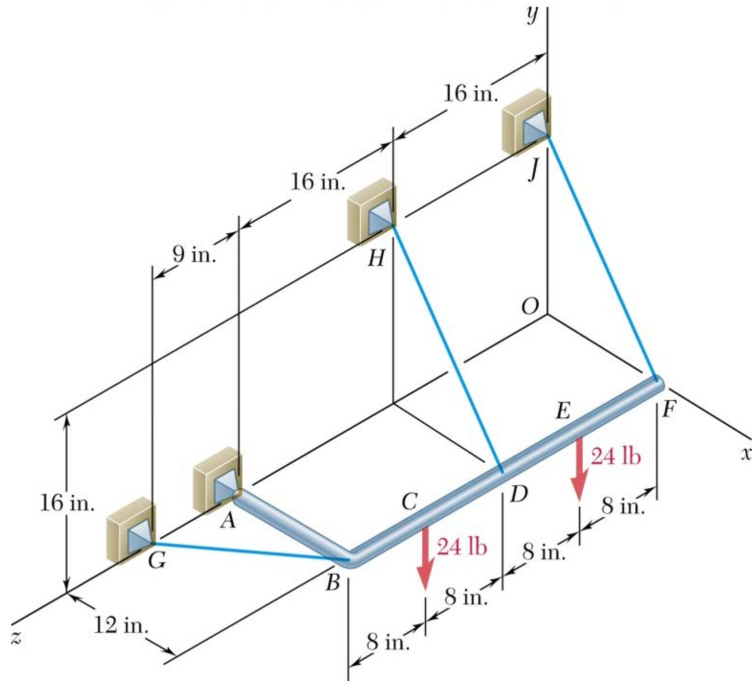
The moment equations can be determined about any point.

It is recommended to choose a point where the largest number of unknown forces are present, as any forces passing through that point where moments are taken do not appear in the moment equation.

Note, moments about any axis must also be equal to zero.

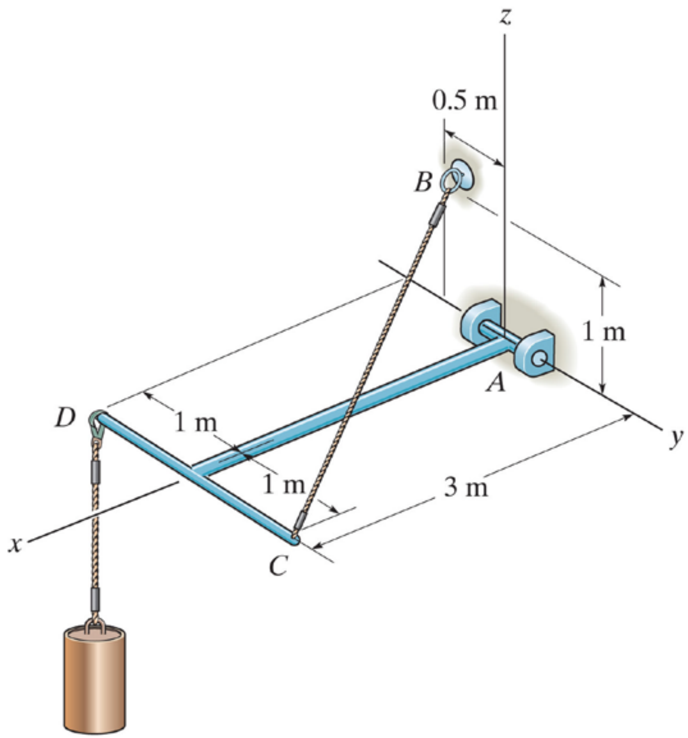
Example

The bent rod is supported by a ball-and-socket joint at A and by three cables. Determine the tension in each cable and the reaction at A.



Example

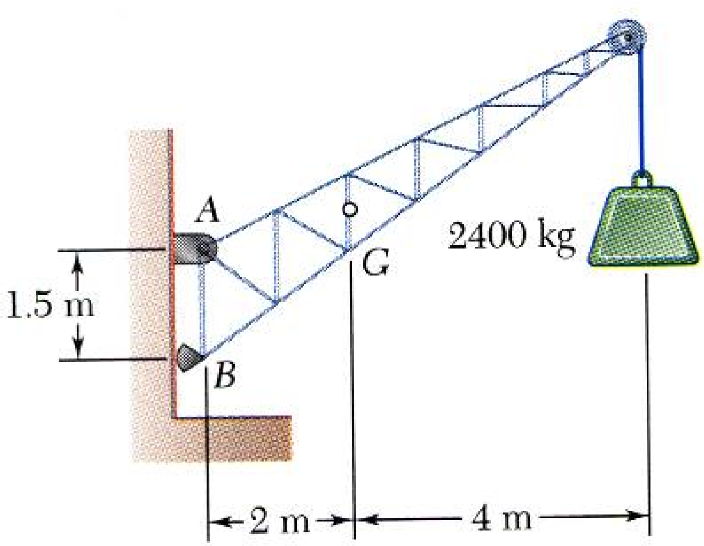
The member is supported by a pin at *A* and cable *BC*. Determine the components of reaction at these supports if the cylinder has a mass of 40 kg.



Sample Problems for Students to Review

Chapter 5

Sample Problem (§ 5.3)



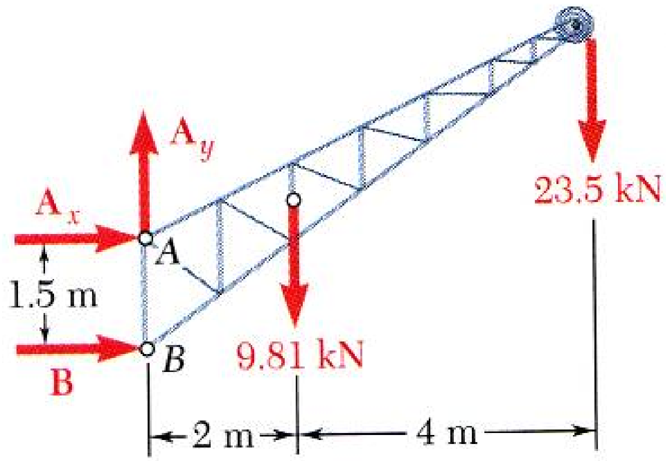
Given: A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The centre of gravity of the crane is located at G.

Find: Determine the components of the reactions at *A* and *B*.

Plan:

1) Create a free-body diagram for the crane and identify reaction forces.

2) Apply equations of equilibrium.



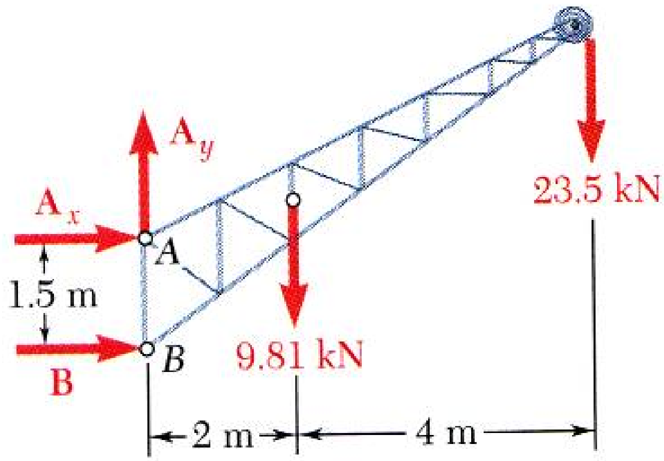
We place the forces and support reactions at A (two unknows - pin) and B (one unknown - rocker)

We can start this problem by taking moments about A, so the reaction force at B is the only unknown

Determine *B* by solving the equation for the sum of the moments of all forces about *A*.

$$\sum M_A = 0: \quad +B(1.5\text{m}) - 9.81\text{kN}(2\text{m}) - 23.5\text{kN}(6\text{m}) = 0$$

$$B = +107.1\text{kN}$$



Determine the reactions at A by solving the force equations related to the net force along x and y.

Sum of forces along x, with $B = +107.1\text{kN}$

$$\sum F_x = 0: \quad A_x + B = 0$$

The negative sign indicates that direction of Ax that we assumed in the FBD was incorrect.

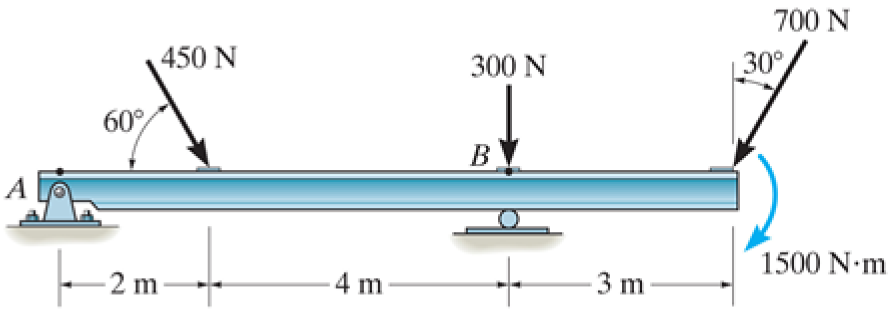
$$A_x = -107.1\text{kN}$$

Sum of forces along y,

$$\sum F_y = 0: \quad A_y - 9.81\text{kN} - 23.5\text{kN} = 0$$

$$A_y = +33.3\text{ kN}$$

Sample Problem (§ 5.3)



Given: A 2-D force and couple system as shown.

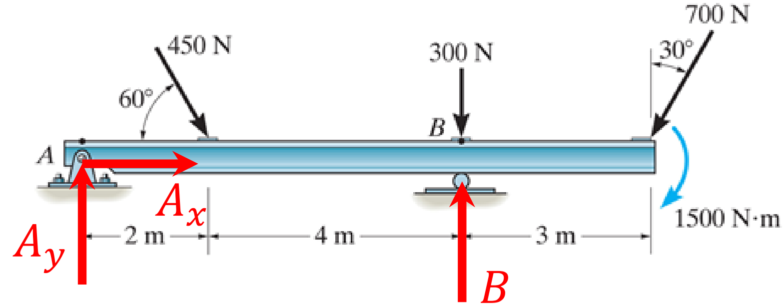
Find: The support reactions forces at *A* and *B* that maintains this beam in static equilibrium.

Plan:

1) Create a free-body diagram for the crane and identify reaction forces.

2) Apply equations of equilibrium.

The two support reactions are a pin at A (two unknowns) and a roller support at B (one unknown)



We can find A_x with $\sum F_x = 0$, B with $\sum M_A = 0$, or A with $\sum M_B = 0$.

$$\Sigma F_x = 450\ (\cos 60) - 700\ (\sin 30) + A_x = 0$$

$$= -125 + A_x = 0 \qquad A_x = 125\ \text{N}$$

$$M_A = -450\ (\sin 60)\ (2) - 300\ (6) + B(6) - 700\ (\cos 30)\ (9) - 1500 = 0$$

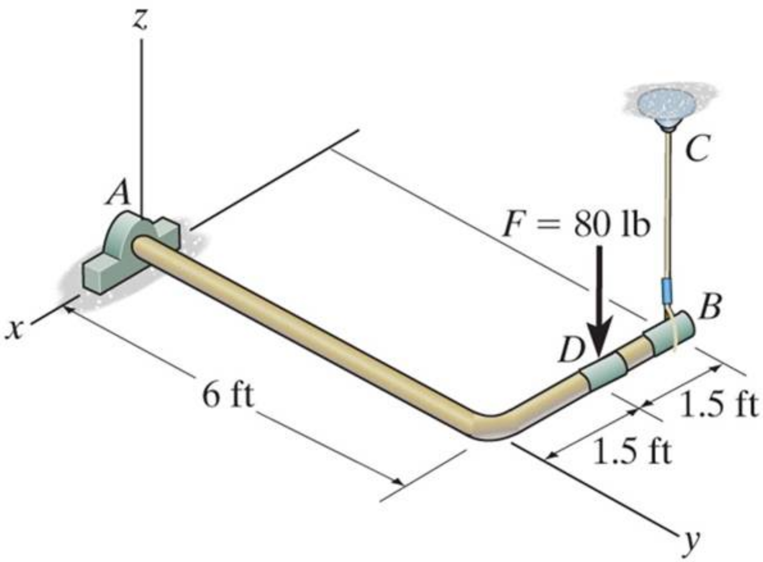
$$= B(6) - 9535 = 0 \qquad B = 1589\ \text{N}$$

$$M_B = -A_y(6) + 450\ (\sin 60)\ (4) - 700\ (\cos 30)\ (3) - 1500 = 0$$

$$= -A_y\ (6) - 1759.8 = 0 \qquad A_y = -293.3\ \text{N}$$

Verify $\Sigma F_y = -293.3 - 450\ (\sin 60) - 300 + 1589 - 700\ (\cos 30) = 0$

Sample Problem (§ 5.6)



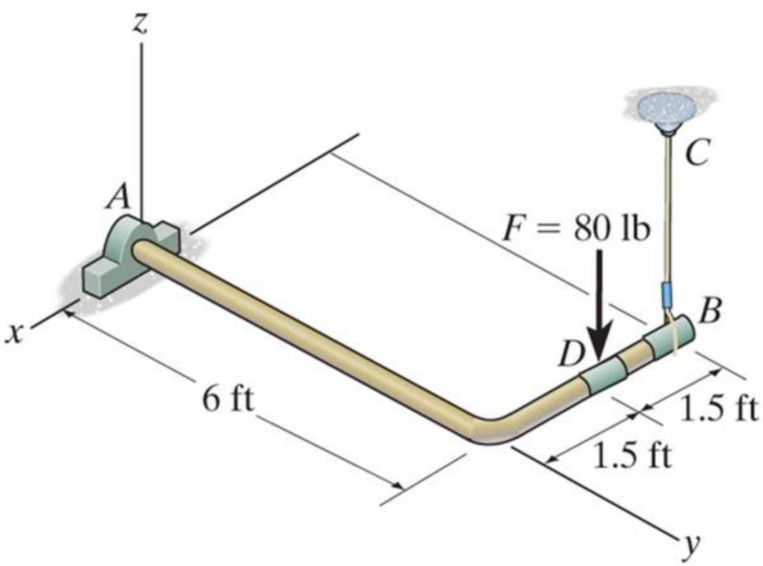
Given: The rod, supported by thrust bearing at A and cable BC, is subjected to an 80 lb force.

Find: Reactions at the thrust bearing A and cable BC.

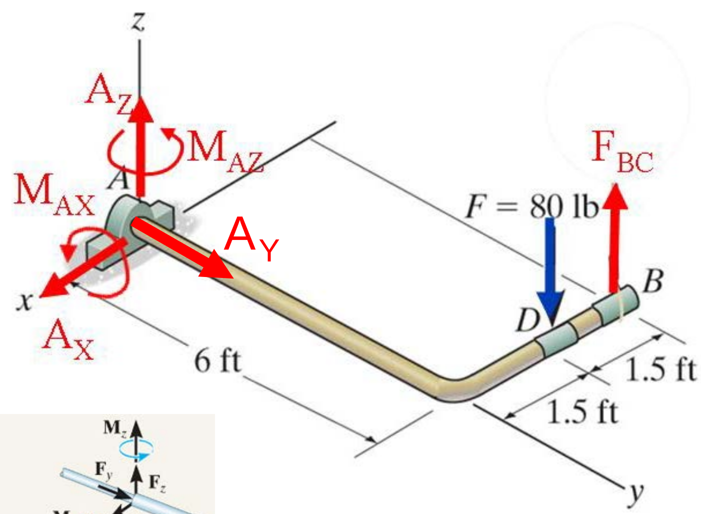
Plan:

a) Draw a FBD of the rod including all the reaction forces.

b) Apply scalar equations of equilibrium to solve for the unknown forces



FBD of the rod:



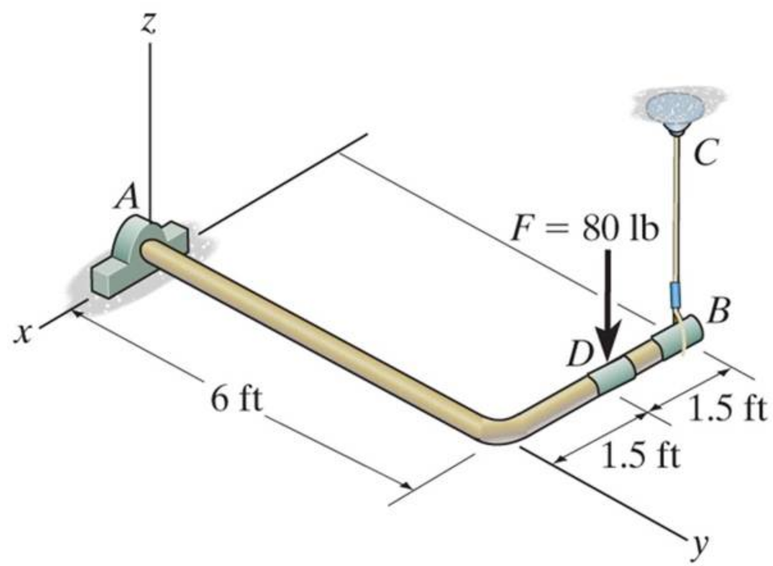
single thrust bearing

Apply scalar equations of equilibrium. Let us start with the forces

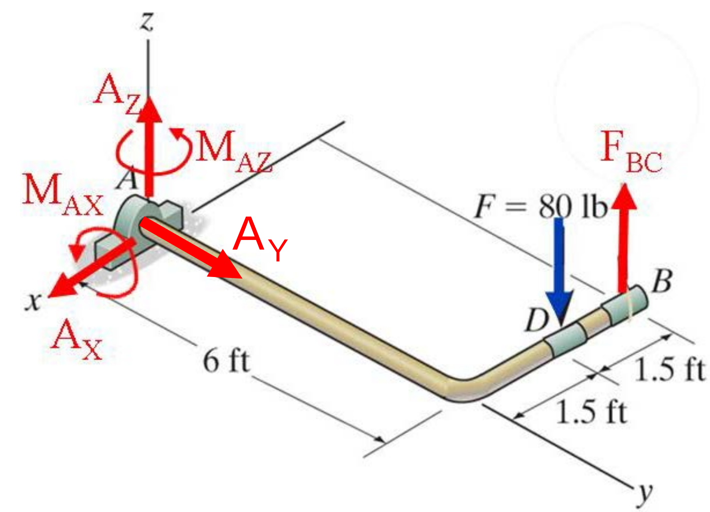
$\sum F_X = A_X = 0;$ $\quad A_X = 0 \quad (1)$

$\sum F_Y = A_Y = 0;$ $\quad A_Y = 0 \quad (2)$

$\sum F_Z = A_Z + F_{BC} - 80 = 0;$ $\quad A_Z + F_{BC} - 80 = 0 \quad (3)$



FBD of the rod:



Scalar moment equations about point A

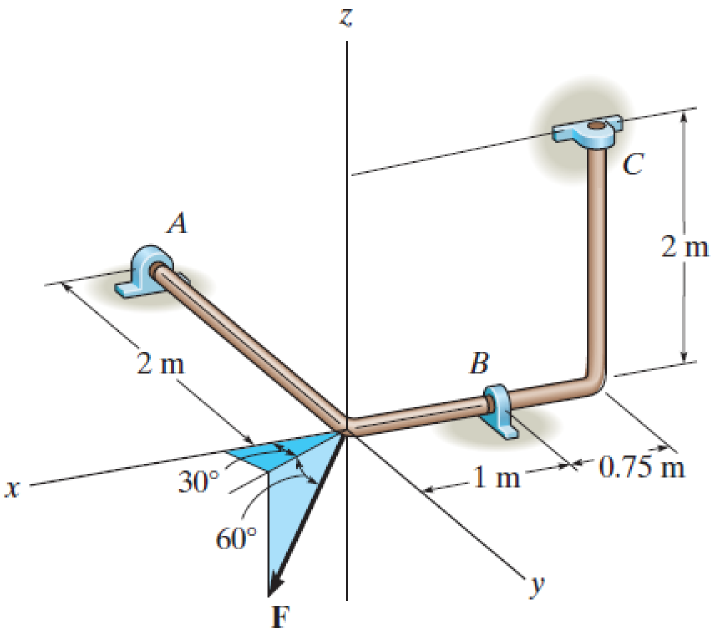
$\sum M_Y = -80\,(1.5) + F_{BC}\,(3.0) = 0;$ $F_{BC} = 40$ lb (4)

Solve equation (3): $A_Z + F_{BC} - 80 = 0$ $A_Z = 40$ lb

$\sum M_X = M_{AX} + 40\,(6) - 80\,(6) = 0\,;$ $M_{AX} = 240$ lb ft CCW (5)

$\sum M_Z = M_{AZ} = 0\,;$ $M_{AZ} = 0$ (6)

Sample Problem (§ 5.6)



Given: A bent rod is supported by smooth journal bearings at A, B, and C. $F = 800$ N. Assume the rod is properly aligned.

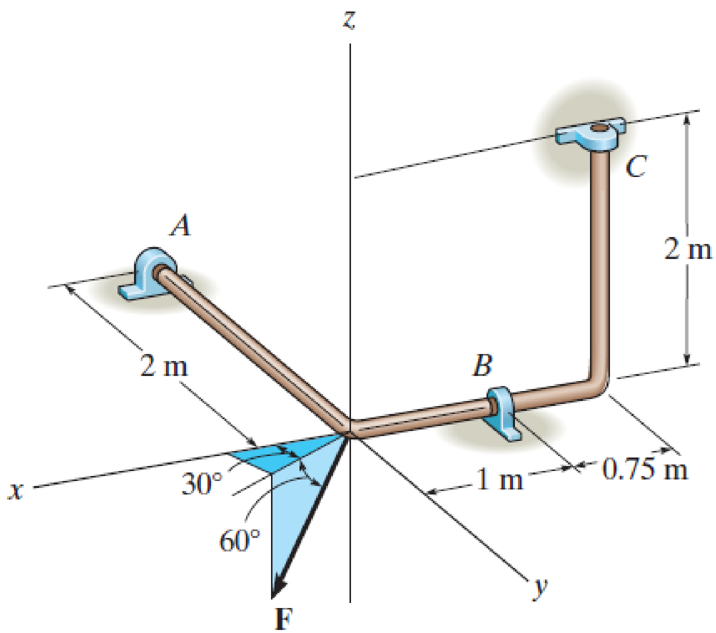
Find: The reactions at all the supports.

Plan:

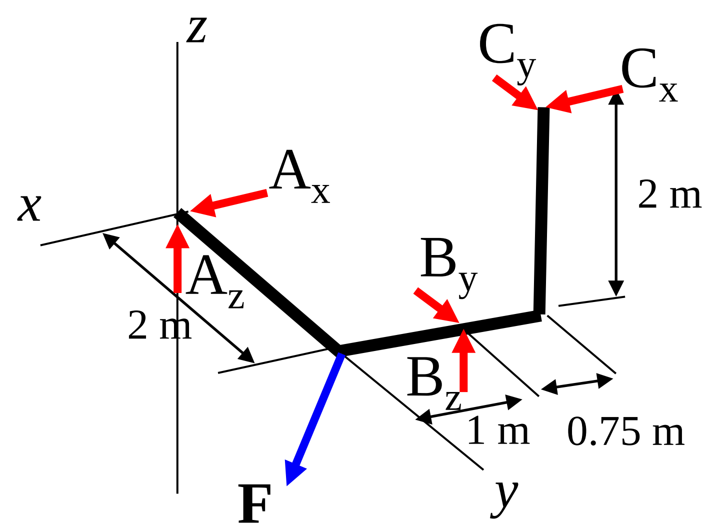
a) Resolve the force **F** in the Cartesian form vector.

b) Draw a FBD of the bend rod including all the reaction forces.

c) Apply scalar equations of equilibrium to solve for the unknown forces



FBD of the bent rod



The x, y and z components of force F are

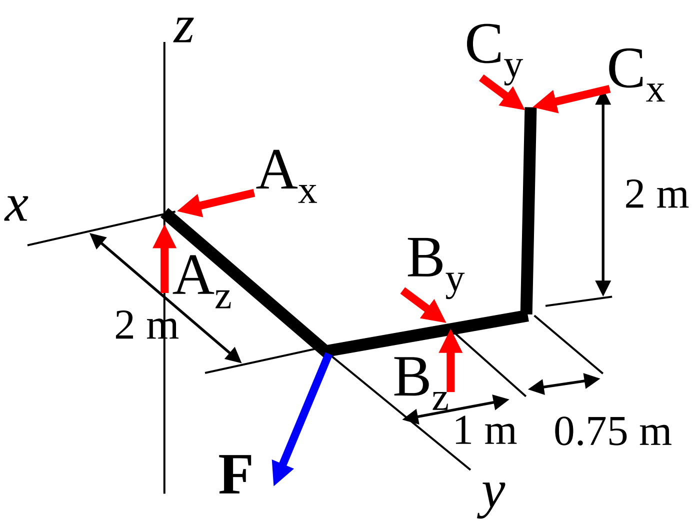
$$F_x = (800 \cos 60°) \cos 30° = 346.4 \text{ N}$$

$$F_y = (800 \cos 60°) \sin 30° = 200 \text{ N}$$

$$F_z = -800 \sin 60° = -692.8 \text{ N}$$

$$\mathbf{F} = 346.4\, \boldsymbol{i} + 200\, \boldsymbol{j} - 692.8\, \boldsymbol{k}$$

FBD of the bent rod



$\mathbf{F} = 346.4\,\boldsymbol{i} + 200\,\boldsymbol{j} - 692.8\,\boldsymbol{k}$

Apply scalar equations of equilibrium related to forces

$$\Sigma F_x = A_x + C_x + 346.4 = 0 \qquad (1)$$

$$\Sigma F_y = 200 + B_y + C_y = 0 \qquad (2)$$

$$\Sigma F_z = A_z + B_z - 692.8 = 0 \qquad (3)$$

related to moments about point A

$$\Sigma M_x = -C_y(2) + B_z(2) - 692.8(2) = 0 \qquad (4)$$

$$\Sigma M_y = B_z(1) + C_x(2) = 0 \qquad (5)$$

$$\Sigma M_z = -C_y(1.75) - C_x(2) - B_y(1) - 346.4(2) = 0 \qquad (6)$$

This leads to a linear system of six equations in six unknowns

$A_x = -400$ N, $B_y = 600$ N, $C_x = 53.6$ N

$A_z = 800$ N, $B_z = -107$ N, $C_y = -800$ N