**Example 6.26.** Let $X$ and $Y$ denote the Fourier transforms of $x$ and $y$, respectively. Suppose that $y(t) = x(t)\cos(at)$, where $a$ is a nonzero real constant. Find an expression for $Y$ in terms of $X$.

*Solution.* Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

You really don't want to do this!

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. To allow for simpler notation in what follows, we define

$$v(t) = \cos(at)$$

table of FT pairs

and let $V$ denote the Fourier transform of $v$. From Table 6.2, we have that

from table of FT pairs

$$V(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)].$$

Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t)v(t)\})(\omega) && \text{from definition of } y \\ &= \tfrac{1}{2\pi} X * V(\omega) && \text{time-domain multiplication property} \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. && \text{definition of convolution} \end{aligned}$$

Substituting the above expression for $V$, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)])d\lambda && \text{cancel } \pi\text{'s} \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)]d\lambda && \text{split into two integrals} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda-a})d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda+a})d\lambda\right] && \delta \text{ is even} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-a)d\lambda\right] && \text{make shifts explicit} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-a)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+a)]d\lambda\right] && \text{sifting property} \\ &= \tfrac{1}{2}[X(\omega-a)+X(\omega+a)] \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

Note that the above solution is essentially identical to the one appearing earlier in Example 6.15 on page 1.

This approach is much simpler

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t)\cos(at)\})(\omega) && \text{from definition of } y \\ &= \left(\mathcal{F}\{\tfrac{1}{2}(e^{jat}+e^{-jat})x(t)\}\right)(\omega) && \text{Euler} \\ &= \tfrac{1}{2}\left(\mathcal{F}\{e^{jat}x(t)\}\right)(\omega) + \tfrac{1}{2}\left(\mathcal{F}\{e^{-jat}x(t)\}\right)(\omega) && \text{linearity property} \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). && \text{frequency-domain shifting property} \end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. ∎

THE TAKEAWAY: Only use the time-domain multiplication property when absolutely necessary, since its use will result in the appearance of a convolution operation.

**Answer (j).**

We are asked to find the Fourier transform $X$ of

$$x(t) = \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1) d\tau.$$

$$\begin{aligned}
x(t) &= \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1)\, d\tau \\
&= \int_{-\infty}^{5t} e^{-2} \underbrace{e^{-(\tau-1)} u(\tau-1)}_{v_1(\tau-1)}\, d\tau \quad \text{where } v_1(t) = e^{-t}u(t) \quad 1 \\
&= \int_{-\infty}^{5t} e^{-2} \underbrace{v_1(\tau-1)}_{v_2(\tau)}\, d\tau \quad \text{where } v_2(t) = v_1(t-1) \quad 2 \\
&= \underbrace{e^{-2} \int_{-\infty}^{5t} v_2(\tau)\, d\tau}_{v_3(5t)} \quad \text{where } v_3(t) = e^{-2}\int_{-\infty}^{t} v_2(\tau)\, d\tau \quad 3 \\
&= v_3(5t) \quad 4
\end{aligned}$$

We begin by rewriting $x(t)$ as

$$4 \longrightarrow \quad x(t) = v_3(5t),$$

where

$$1 \longrightarrow \quad v_1(t) = e^{-t}u(t),$$

$$2 \longrightarrow \quad v_2(t) = v_1(t-1), \quad \text{and}$$

$$3 \longrightarrow \quad v_3(t) = \int_{-\infty}^{t} e^{-2} v_2(\tau) d\tau. = e^{-2}\int_{-\infty}^{t} v_2(\tau)\, d\tau$$

Taking the Fourier transform of both sides of each of the above equations yields

$$5 \quad V_1(\omega) = \frac{1}{1+j\omega}, \longleftarrow \text{FT of 1 using FT table}$$

$$6 \quad V_2(\omega) = e^{-j\omega}V_1(\omega), \longleftarrow \text{FT of 2 using time shifting property}$$

$$7 \quad V_3(\omega) = e^{-2}\left[\frac{1}{j\omega}V_2(\omega) + \pi V_2(0)\delta(\omega)\right], \longleftarrow \text{FT of 3 using integration property} \quad \text{and}$$

$$8 \quad X(\omega) = \tfrac{1}{5}V_3(\omega/5). \longleftarrow \text{FT of 4 using time scaling property}$$

Combining the above results, we have

$$\begin{aligned}
8 \longrightarrow \quad X(\omega) &= \tfrac{1}{5}V_3(\omega/5) && \text{substitute 7} \\
&= \tfrac{1}{5}e^{-2}\left[\left(\frac{1}{j(\omega/5)}\right)V_2(\omega/5) + \pi V_2(0)\delta(\omega/5)\right] \\
&= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right)V_2(\omega/5) + \pi V_2(0)\delta(\omega/5)\right] && \text{substitute 6} \\
&= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right)e^{-j\omega/5}V_1(\omega/5) + \pi V_1(0)\delta(\omega/5)\right] && \text{substitute 5} \\
&= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right)e^{-j\omega/5}\left(\frac{1}{1+j(\omega/5)}\right) + \pi\delta(\omega/5)\right] \\
&= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right)\left(\frac{5}{5+j\omega}\right)e^{-j\omega/5} + \pi\delta(\omega/5)\right] && \text{simplify} \\
&= \tfrac{1}{5e^2}\left[\left(\frac{25}{j5\omega-\omega^2}\right)e^{-j\omega/5} + \pi\delta(\omega/5)\right].
\end{aligned}$$

**Example 6.20.** Let $X_1$ and $X_2$ denote the Fourier transforms of $x_1$ and $x_2$, respectively. Suppose that $X_1$ and $X_2$ are as shown in Figures 6.6(a) and (b). Determine whether $x_1$ and $x_2$ are periodic.

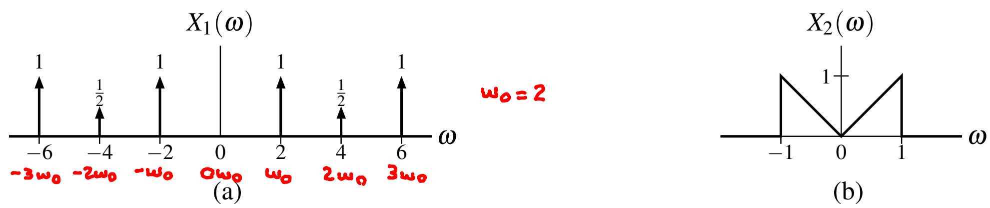


Figure 6.6: Frequency spectra. The frequency spectra (a) $X_1$ and (b) $X_2$.

*Solution.* We know that the Fourier transform $X$ of a $T$-periodic function $x$ must be of the form

$$X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0),$$

where $\omega_0 = \frac{2\pi}{T}$ and the $\{\alpha_k\}$ are complex constants. The spectrum $X_1$ does have this form, with $\omega_0 = 2$ and $T = \frac{2\pi}{2} = \pi$. Therefore, $x_1$ must be $\pi$-periodic. The spectrum $X_2$ does not have this form. Therefore, $x_2$ must not be periodic. ■

**Example 6.21.** Consider the periodic function $x$ with fundamental period $T = 2$ as shown in Figure 6.7. Using the Fourier transform, find the Fourier series representation of $x$.

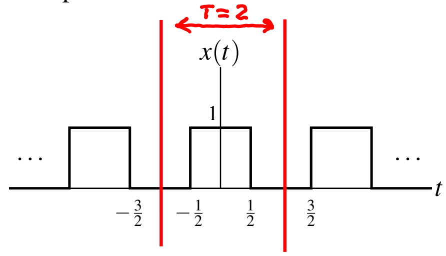


Figure 6.7: Periodic function $x$.

since T=2

*Solution.* Let $\omega_0$ denote the fundamental frequency of $x$. We have that $\omega_0 = \frac{2\pi}{T} = \pi$. Let $y(t) = \operatorname{rect} t$ (i.e., $y$ corresponds to a single period of the periodic function $x$). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t-2k).$$

Let $Y$ denote the Fourier transform of $y$. Taking the Fourier transform of $y$, we obtain

FT table

$$Y(\omega) = (\mathcal{F}\{\operatorname{rect} t\})(\omega) = \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \quad (1)$$

Now, we seek to find the Fourier series representation of $x$, which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} Y(k\omega_0) && \text{sample FT of } y \text{ at } k\omega_0 \text{ for } k^{th} \text{ FS coefficient} \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\omega_0}{2}k\right) && \text{substitute (1)} \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\pi}{2}k\right). && \omega_0 = \pi \end{aligned}$$

■