STAT 260 Spring 2023: Assignment 5

Due: Friday March 3rd BEFORE 11:59pm PT to Crowdmark

Please read the instructions below and in the Written Assignment 5 assignment on Crowdmark.

For full marks, your work must be neatly written, and contain enough detail that it is clear how you arrived at your solutions. Messy, poorly formatted work will receive deductions, or may not be graded at all.

Talking to your classmates about assigned work is a healthy practice that is encouraged. However, in the end, each person is expected to write their own solutions, in their own words, and in a way that reflects their own understanding. Assignment questions are not to be posted to homework "help" websites.

Late policy: Late assignments will be accepted until the final cutoff of 11:59pm on Sunday March 5th. Solutions submitted within 1 hour of the Friday deadline will have a 5% late penalty automatically applied within Crowdmark. Solutions submitted after 1 hour of the Friday deadline but before the final Sunday cutoff will have a 20% late penalty applied. Solutions submitted after the final Sunday cutoff will be graded for feedback, but marks will not be awarded.

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1. At a particular intersection, vehicles arrive in the left turn lane at an average rate of 5 cars every 30 seconds.

   (a) **[0.5 marks]** What is the probabilty that exactly 32 cars arrive in the left turn lane within a random 3 minute interval?

3 minutes = 6 × 30 seconds

$\lambda = 5 \cdot 6 = 30$

$$P(X=32) = \frac{30^{32} e^{-30}}{32!} = 0.0659$$

   (b) **[1 mark]** Suppose that in a random 90 second interval at least 11 cars will arrive in the left turn lane. What is the probability that at most 19 cars will arrive in the left turn lane?

90 seconds = 3 × 30 seconds

$\lambda = 5 \cdot 3 = 15$

$$P(X \le 19 \mid X \ge 11) = \frac{P(11 \le X \le 19)}{P(X \ge 11)}$$

$$= \frac{P(X \le 19) - P(X \le 10)}{1 - P(X \le 10)} = \frac{0.8752 - 0.1185}{1 - 0.1185} = \frac{0.7567}{0.8815} = 0.8584$$

   (c) **[1 mark]** Suppose the left turn lane has a traffic light sensor cycle that is 15 seconds long. If at least one car arrives in the left turn lane in the sensor cycle, then the left turn traffic light will be triggered to turn green. What is the probability that in 8 independent random sensor cycles, exactly 6 of sensor cycles will have the left turn traffic light turn green?

15 seconds = $\frac{1}{2}$ × 30 seconds

$\lambda = \frac{1}{2} \cdot 5 = 2.5$

sucess = at least 1 car

$$p = P(\text{sucess}) = P(X \ge 1) = 1 - P(X=0)$$

$$= 1 - 0.0821 = 0.9179$$

binomial $n = 8$ $p = 0.9179$

$$P(X=6) = \binom{8}{6} (0.9179)^6 (0.0821)^2 = 0.1129$$

2. In traffic flow, "time headway" is the amount of time between when one car finishes passing a fixed point and the instant the next car begins to pass the point. Let the continuous random variable X equal the time headway for two consecutive cars on a busy highway, measured in seconds.

Suppose the random variable X has the pdf

$$f(x) = \begin{cases} 0.3e^{-0.3(x-0.25)} & x \geq 0.25 \\ 0 & \text{otherwise} \end{cases}$$

If integration is used below, all steps must be shown. If ∞ is used as an integration endpoint, it must be handled correctly with a limit. (Remember, ∞ is not a numerical value.)

(a) **[0.5 marks]** For two randomly chosen consecutive cars, what is the probability that the time headway is greater than 4 seconds?

$$P(X \geq 4) = \int_4^{\infty} f(x)\,dx = \int_4^{\infty} 0.3e^{-0.3(x-0.25)}\,dx = \lim_{c\to\infty} \int_4^{c} 0.3e^{-0.3(x-0.25)}\,dx$$

$$= \lim_{c\to\infty} \left[-e^{-0.3(x-0.25)}\right]\Big|_4^c = \lim_{c\to\infty} e^{-0.3(4-0.25)} - e^{-0.3(c-0.25)}$$

$$= e^{-0.3(3.75)} - 0 = e^{-1.125} = 0.3247$$

$\underline{\underline{\text{or}}}$ $$P(X \geq 4) = 1 - P(X \leq 4) = 1 - \int_{0.25}^{4} 0.3e^{-0.3(x-0.25)}\,dx = e^{-1.125} = 0.3247$$

(b) **[1 mark]** Find the cdf for X.

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{0.25}^{x} 0.3e^{-0.3(y-0.25)}\,dy$$

$$= -e^{-0.3(y-0.25)}\Big|_{0.25}^{x} = -e^{-0.3(x-0.25)} - (-e^0)$$

$$= 1 - e^{-0.3(x-0.25)}$$

$$F(x) = \begin{cases} 0 & x < 0.25 \\ 1 - e^{-0.3(x-0.25)} & x \geq 0.25 \end{cases}$$

3. Suppose the continuous random variable X has the pdf of

$$f(x) = \begin{cases} \frac{6}{5}\left(1 - \frac{1}{x^2}\right) & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

If integration is used below, all steps must be shown.

(a) **[1 mark]** Find the 50^{th} percentile. That is, find $\widetilde{\mu}$.

$$0.5 = \int_2^{x_0} \frac{6}{5}\left(1-\frac{1}{x^2}\right) dx = \int_2^{x_0} \frac{6}{5}\left(1-x^{-2}\right) dx = \frac{6}{5}\left(x + x^{-1}\right) \Big|_2^{x_0}$$

$$0.5 = \frac{6}{5}\left(x_0 + \frac{1}{x_0}\right) - \frac{6}{5}\left(2 + \frac{1}{2}\right)$$

$$0.5 = \frac{6}{5}\left(\frac{x_0^2+1}{x_0}\right) - 3$$

$$0 = \frac{6}{5}\left(\frac{x_0^2+1}{x_0}\right) - \frac{7}{2}$$

$$0 = 12x_0^2 + 12 - 35x_0$$

$$x_0 = \frac{35 \pm \sqrt{35^2 - 4(12)(12)}}{2(12)}$$

$$x_0 = \frac{35 \pm \sqrt{649}}{24}$$

$$x_0 = 0.3969, \quad x_0 = 2.5198$$

$\uparrow$ not in $2 \leq x \leq 3$

so $x_0 = \widetilde{\mu} = 2.5198$

(b) **[1 mark]** Find the $V(X)$, the variance of X.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_2^3 x \cdot \frac{6}{5}\left(1-\frac{1}{x^2}\right) dx = \int_2^3 \frac{6}{5}\left(x - x^{-1}\right) dx$$

$$= \frac{6}{5}\left(\frac{x^2}{2} - \ln x\right)\Big|_2^3 = \frac{6}{5}\left(\frac{9}{2} - \ln 3\right) - \frac{6}{5}\left(2 - \ln 2\right) = 3 + \frac{6}{5}\left(\ln 2 - \ln 3\right)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_2^3 x^2 \cdot \frac{6}{5}\left(1-\frac{1}{x^2}\right) dx = \int_2^3 \frac{6}{5}\left(x^2 - 1\right) dx$$

$$= \frac{6}{5}\left(\frac{x^3}{3} - x\right)\Big|_2^3 = \frac{6}{5}\left(9-3\right) - \frac{6}{5}\left(\frac{8}{3} - 2\right) = \frac{32}{5}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{32}{5} - \left(3 + \frac{6}{5}\left(\ln 2 - \ln 3\right)\right)^2 = 0.0826$$