approximate u. The step response s is given by

$$\begin{aligned}
s(t) &= \mathcal{L}^{-1}\left\{U(s)H(s)\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{s(s-[1-10\beta])}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{10\beta-1}\left(\frac{1}{s}-\frac{1}{s-(1-10\beta)}\right)\right\}(t) \\
&= \frac{10\beta}{10\beta-1}\left(1-e^{-(10\beta-1)t}\right)u(t) \\
&\approx u(t) \quad \text{for large } \beta.
\end{aligned}$$

Clearly, as β increases, s better approximates the desired response u.

(c) The scheme in part (a) for stabilizing the unstable plant relies on pole-zero cancellation. Unfortunately, in practice, it is not possible to achieve pole-zero cancellation. In short, the issue is one of approximation. Our analysis of systems is based on theoretical models specified in terms of equations. These theoretical models, however, are only approximations of real-world systems. This approximate nature is due to many factors, including (but not limited to) the following:

1. We cannot determine the system function of a system exactly, since this involves measurement, which always has some error.

2. We cannot build a system with such precision that it will have exactly some prescribed system function. The system function will only be close to the desired one.

3. The system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, and so on).

4. Although a system may be represented by a LTI model, the likely reality is that the system is not exactly LTI, which introduces error.

For reasons such as these, the effective poles and zeros of the system function will only be approximately where we expect them to be. Pole-zero cancellation, however, requires a pole and zero to be placed at exactly the same location. So, any error will prevent the pole-zero cancellation from occurring. Since at least some small error is unavoidable in practice, the desired pole-zero cancellation will not be achieved.

The scheme in part (b) for stabilizing the unstable plant is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros. Instead, the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to approximation error), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does. ■

7.17 Unilateral Laplace Transform

As mentioned earlier, two different versions of the Laplace transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral Laplace transform. Now, we turn our attention to the unilateral Laplace transform. The **unilateral Laplace transform** of the function x is denoted as $\mathcal{L}_{\mathsf{u}}x$ or X and is defined as

$$\mathcal{L}_{\mathsf{u}}x(s) = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt. \tag{7.16}$$

The **inverse unilateral Laplace transform** has the same definition as in the case of the bilateral transform, namely (7.3).

Comparing the definitions of the unilateral and bilateral Laplace transforms given by (7.16) and (7.2), respectively, we can see that these definitions only differ in the lower limit of integration. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral Laplace transform of the function xu for an arbitrary function x. We have

$$\begin{aligned}\mathcal{L}\{xu\}(s) &= \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} x(t)e^{-st}dt \\ &= \mathcal{L}_{\mathrm{u}}x(s).\end{aligned}$$

In other words, the unilateral Laplace transform of the function x is simply the bilateral Laplace transform of the function xu. Since $\mathcal{L}_{\mathrm{u}}x = \mathcal{L}\{xu\}$ and xu is always right sided, the ROC associated with $\mathcal{L}_{\mathrm{u}}x$ is always a right-half plane (or the entire complex plane). For this reason, we often do not explicitly indicate the ROC when working with the unilateral Laplace transform.

From earlier in this chapter, we know that the bilateral Laplace transform is invertible. That is, if the function x has the bilateral Laplace transform $X = \mathcal{L}x$, then $\mathcal{L}^{-1}X = x$. Now, let us consider the invertibility of the unilateral Laplace transform. To do this, we must consider the quantity $\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x$. Since $\mathcal{L}_{\mathrm{u}}x = \mathcal{L}\{xu\}$ and the inverse equations for the unilateral and bilateral Laplace transforms are identical, we can write

$$\begin{aligned}\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x(t) &= \mathcal{L}_{\mathrm{u}}^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= x(t)u(t) \\ &= \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0. \end{cases}\end{aligned}$$

Thus, we have that $\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x = x$ only if x is causal. In other words, the unilateral Laplace transform is invertible only for causal functions. For noncausal functions, we can only recover $x(t)$ for $t \geq 0$. In essence, the unilateral Laplace transform discards all information about the value of the function x at t for $t < 0$. Since this information is discarded, it cannot be recovered by an inverse unilateral Laplace transform operation.

Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral Laplace transform are summarized in Table 7.3.

By comparing the properties of the unilateral and bilateral Laplace transform listed in Tables 7.3 and 7.1, respectively, we can see that the unilateral Laplace transform has some of the same properties as its bilateral counterpart, namely, the linearity, Laplace-domain shifting, conjugation, and Laplace-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral Laplace transform.

Since the unilateral and bilateral Laplace transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral Laplace transform properties listed in Table 7.1 with the unilateral Laplace transform properties listed in Table 7.3. In the unilateral case, we can see that:

1. the time-domain convolution property has the additional requirement that the functions being convolved must be causal;

2. the time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive;

3. the time-domain differentiation property has an extra term in the expression for $\mathcal{L}_{\mathrm{u}}\{\mathcal{D}x\}(t)$, where $\mathcal{D}$ denotes the derivative operator (namely, $-x(0^-)$); and

4. the time-domain integration property has a different lower limit in the time-domain integral (namely, 0^- instead of $-\infty$).

Table 7.3: Properties of the unilateral Laplace transform

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Laplace-Domain Scaling	$x(at), a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1 * x_2(t), x_1$ and x_2 are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Table 7.4: Transform pairs for the unilateral Laplace transform

Pair	$x(t)$, $t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos(\omega_0 t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin(\omega_0 t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Also, in the unilateral case, the time-domain shifting property does not hold (except in special circumstances).

Since $\mathcal{L}_u x = \mathcal{L}\{xu\}$, we can easily generate a table of unilateral Laplace transform pairs from a table of bilateral transform pairs. Using the bilateral Laplace transform pairs from Table 7.2, and the preceding relationship between the unilateral and bilateral Laplace transforms, we can trivially deduce the unilateral Laplace transform pairs in Table 7.4. Since, in the unilateral case, the ROC always corresponds to a right-sided function, we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be the one that corresponds to a right-sided function (i.e., a right-half plane or the entire complex plane).

The inverse unilateral Laplace transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to correspond to a right-sided function.

7.18 Solving Differential Equations Using the Unilateral Laplace Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the system will be linear only if the initial conditions for the differential equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral Laplace transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral Laplace transform to this end.

Example 7.42. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y'(t) + 3y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = e^{-t}u(t)$ and $y(0^-) = 1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
& \mathcal{L}_{\mathsf{u}}\left\{y'\right\}(s) + 3\mathcal{L}_{\mathsf{u}}y(s) = \mathcal{L}_{\mathsf{u}}x(s) \\
\Rightarrow\quad & sY(s) - y(0^-) + 3Y(s) = X(s) \\
\Rightarrow\quad & (s+3)Y(s) = X(s) + y(0^-) \\
\Rightarrow\quad & Y(s) = \frac{X(s)+y(0^-)}{s+3}.
\end{aligned}$$

Since $x(t) = e^{-t}u(t)$, we have

$$X(s) = \mathcal{L}_{\mathsf{u}}\{e^{-t}\}(s) = \frac{1}{s+1}.$$

Substituting this expression for X and the given initial conditions (i.e., $y(0^-) = 1$) into the above equation for Y, we obtain

$$Y(s) = \frac{\left(\frac{1}{s+1}\right)+1}{s+3} = \frac{\left(\frac{s+2}{s+1}\right)}{s+3} = \frac{s+2}{(s+1)(s+3)}.$$

Now, we find a partial fraction expansion for Y. Such an expansion has the form

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= (s+1)Y(s)\big|_{s=-1} \\
&= \left.\frac{s+2}{s+3}\right|_{s=-1} \\
&= \tfrac{1}{2} \quad \text{and} \\
A_2 &= (s+3)Y(s)\big|_{s=-3} \\
&= \left.\frac{s+2}{s+1}\right|_{s=-3} \\
&= \tfrac{1}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \tfrac{1}{2}\left(\frac{1}{s+1}\right) + \tfrac{1}{2}\left(\frac{1}{s+3}\right).$$

Taking the inverse unilateral Laplace transform of Y, we obtain

$$\begin{aligned}
y(t) &= \mathcal{L}_{\mathsf{u}}^{-1}Y(t) \\
&= \tfrac{1}{2}\mathcal{L}_{\mathsf{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{1}{2}\mathcal{L}_{\mathsf{u}}^{-1}\left\{\frac{1}{s+3}\right\}(t) \\
&= \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t} \quad \text{for } t \geq 0.
\end{aligned}$$

■

Example 7.43 (Unilateral Laplace transform of second-order derivative)**.** Find the unilateral Laplace transform Y of y in terms of the unilateral Laplace transform X of x, where

$$y(t) = x''(t)$$

and the prime symbol denotes derivative (e.g., x'' is the second derivative of x)

Solution. Define the function

$$v(t) = x'(t) \tag{7.17}$$

so that

$$y(t) = v'(t). \tag{7.18}$$

Let V denote the unilateral Laplace transform of v. Taking the unilateral Laplace transform of (7.17) (using the time-domain differentiation property), we have

$$\begin{aligned} V(s) &= \mathcal{L}_{\mathsf{u}}\left\{x'\right\}(s) \\ &= sX(s) - x(0^-). \end{aligned} \tag{7.19}$$

Taking the unilateral Laplace transform of (7.18) (using the time-domain differentiation property), we have

$$\begin{aligned} Y(s) &= \mathcal{L}_{\mathsf{u}}\left\{v'\right\}(s) \\ &= sV(s) - v(0^-). \end{aligned} \tag{7.20}$$

Substituting (7.19) into (7.20), we have

$$\begin{aligned} Y(s) &= s\left[sX(s) - x(0^-)\right] - v(0^-) \\ &= s^2X(s) - sx(0^-) - x'(0^-). \end{aligned}$$

Thus, we have that

$$Y(s) = s^2X(s) - sx(0^-) - x'(0^-). \qquad \blacksquare$$

Example 7.44. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = 5u(t)$, $y(0^-) = 1$, and $y'(0^-) = -1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned} & \mathcal{L}_{\mathsf{u}}\left\{y'' + 3y' + 2y\right\}(s) = \mathcal{L}_{\mathsf{u}}x(s) \\ \Rightarrow\quad & \mathcal{L}_{\mathsf{u}}\left\{y''\right\}(s) + 3\mathcal{L}_{\mathsf{u}}\left\{y'\right\}(s) + 2\mathcal{L}_{\mathsf{u}}y(s) = \mathcal{L}_{\mathsf{u}}x(s) \\ \Rightarrow\quad & \left[s^2Y(s) - sy(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = X(s) \\ \Rightarrow\quad & s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = X(s) \\ \Rightarrow\quad & \left[s^2 + 3s + 2\right]Y(s) = X(s) + sy(0^-) + y'(0^-) + 3y(0^-) \\ \Rightarrow\quad & Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 3y(0^-)}{s^2 + 3s + 2}. \end{aligned}$$

Since $x(t) = 5u(t)$, we have

$$X(s) = \mathcal{L}_{\mathrm{u}}\{5u(t)\}(s) = \tfrac{5}{s}.$$

Substituting this expression for X and the given initial conditions into the above equation yields

$$Y(s) = \frac{\left(\frac{5}{s}\right) + s - 1 + 3}{s^2 + 3s + 2} = \frac{s^2 + 2s + 5}{s(s+1)(s+2)}.$$

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sY(s)|_{s=0} \\
&= \left.\frac{s^2+2s+5}{(s+1)(s+2)}\right|_{s=0} \\
&= \tfrac{5}{2}, \\
A_2 &= (s+1)Y(s)|_{s=-1} \\
&= \left.\frac{s^2+2s+5}{s(s+2)}\right|_{s=-1} \\
&= -4, \quad \text{and} \\
A_3 &= (s+2)Y(s)|_{s=-2} \\
&= \left.\frac{s^2+2s+5}{s(s+1)}\right|_{s=-2} \\
&= \tfrac{5}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \frac{5/2}{s} - \frac{4}{s+1} + \frac{5/2}{s+2}.$$

Taking the inverse unilateral Laplace transform of Y yields

$$\begin{aligned}
y(t) &= \mathcal{L}_{\mathrm{u}}^{-1}Y(t) \\
&= \tfrac{5}{2}\mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{5}{2}\mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&= \tfrac{5}{2} - 4e^{-t} + \tfrac{5}{2}e^{-2t} \quad \text{for } t \geq 0.
\end{aligned}$$

■

Example 7.45 (RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.41 with input v_0 and output v_1. If $R = 100$, $C = \frac{1}{100}$, $v_0(t) = 3e^{-2t}u(t)$, and $v_1(0^-) = 1$, find v_1.

Solution. From basic circuit analysis, we have

$$\begin{aligned}
v_0(t) &= Ri(t) + v_1(t) \quad \text{and} \\
i(t) &= C\tfrac{d}{dt}v_1(t).
\end{aligned}$$

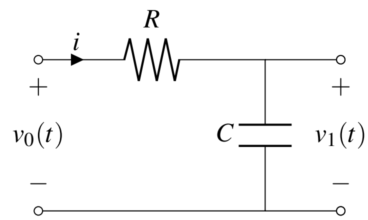


Figure 7.41: RC network.

Combining the preceding two equations, we obtain

$$\begin{aligned} v_0(t) &= R\left[C\tfrac{d}{dt}v_1(t)\right] + v_1(t) \\ &= RC\tfrac{d}{dt}v_1(t) + v_1(t). \end{aligned}$$

Taking the unilateral Laplace transform of both sides of this equation yields

$$\begin{aligned} & \mathcal{L}_{\mathsf{u}}v_0(s) = \mathcal{L}_{\mathsf{u}}\left\{RC\tfrac{d}{dt}v_1(t) + v_1(t)\right\}(s) \\ \Rightarrow\quad & \mathcal{L}_{\mathsf{u}}v_0(s) = RC\mathcal{L}_{\mathsf{u}}\left\{\tfrac{d}{dt}v_1(t)\right\}(s) + \mathcal{L}_{\mathsf{u}}v_1(s) \\ \Rightarrow\quad & V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + V_1(s) \\ \Rightarrow\quad & V_0(s) = RCsV_1(s) - RCv_1(0^-) + V_1(s) \\ \Rightarrow\quad & V_0(s) + RCv_1(0^-) = RCsV_1(s) + V_1(s) \\ \Rightarrow\quad & V_1(s) = \frac{V_0(s) + RCv_1(0^-)}{RCs+1}. \end{aligned}$$

Since $v_0(t) = 3e^{-2t}u(t)$, we have

$$V_0(s) = \frac{3}{s+2}.$$

Substituting this expression for V_0 into the above equation for V_1, we obtain

$$\begin{aligned} V_1(s) &= \frac{\left(\frac{3}{s+2}\right)+1}{s+1} \\ &= \frac{3}{(s+1)(s+2)} + \frac{1}{s+1} \\ &= \frac{s+5}{(s+1)(s+2)}. \end{aligned}$$

Now, we find a partial fraction expansion of V_1. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients yields

$$\begin{aligned} A_1 &= (s+1)V_1(s)\big|_{s=-1} \\ &= \left.\frac{s+5}{s+2}\right|_{s=-1} \\ &= 4 \quad \text{and} \\ A_2 &= (s+2)V_1(s)\big|_{s=-2} \\ &= \left.\frac{s+5}{s+1}\right|_{s=-2} \\ &= -3. \end{aligned}$$

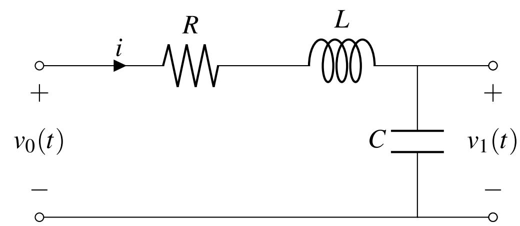


Figure 7.42: RLC network.

Thus, we can rewrite V_1 as

$$V_1(s) = \frac{4}{s+1} - \frac{3}{s+2}.$$

Taking the inverse unilateral Laplace transform of V_1, we obtain

$$\begin{aligned} v_1(t) &= \mathcal{L}_u^{-1} V_1(t) \\ &= \mathcal{L}_u^{-1}\left\{\frac{4}{s+1} - \frac{3}{s+2}\right\}(t) \\ &= 4\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathcal{L}_u^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &= 4e^{-t} - 3e^{-2t} \quad \text{for } t \geq 0. \end{aligned}$$

■

Example 7.46 (RLC network)**.** Consider the resistor-inductor-capacitor (RLC) network shown in Figure 7.42 with input v_0 and output v_1. If $R = 2$, $L = 1$, $C = 1$, $v_0(t) = u(t)$, $v_1(0^-) = 0$, and $v_1'(0^-) = 1$, find v_1.

Solution. From basic circuit analysis, we can write

$$v_0(t) = Ri(t) + L\frac{d}{dt}i(t) + v_1(t) \quad \text{and}$$
$$v_1(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \Rightarrow \quad i(t) = C\frac{d}{dt}v_1(t).$$

Combining the above equations, we obtain

$$\begin{aligned} v_0(t) &= R\left[C\tfrac{d}{dt}v_1(t)\right] + L\tfrac{d}{dt}\left[C\tfrac{d}{dt}v_1(t)\right] + v_1(t) \\ &= RC\tfrac{d}{dt}v_1(t) + LC\left(\tfrac{d}{dt}\right)^2 v_1(t) + v_1(t). \end{aligned}$$

Taking the unilateral Laplace transform of both sides of the preceding equation yields

$$\begin{aligned} &\mathcal{L}_u v_0(s) = RC\mathcal{L}_u\left\{\tfrac{d}{dt}v_1(t)\right\}(s) + LC\mathcal{L}_u\left\{\left(\tfrac{d}{dt}\right)^2 v_1(t)\right\}(s) + \mathcal{L}_u v_1(s) \\ \Rightarrow\quad & V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + LC\left[s^2V_1(s) - sv_1(0^-) - v_1'(0^-)\right] + V_1(s) \\ \Rightarrow\quad & V_0(s) = RCsV_1(s) - RCv(0^-) + LCs^2V_1(s) - LCsv_1(0^-) - LCv_1'(0^-) + V_1(s) \\ \Rightarrow\quad & \left[LCs^2 + RCs + 1\right]V_1(s) = V_0(s) + RCv(0^-) + LCsv_1(0^-) + LCv_1'(0^-) \\ \Rightarrow\quad & V_1(s) = \frac{V_0(s) + [RC + LCs]\,v_1(0^-) + LCv_1'(0^-)}{LCs^2 + RCs + 1}. \end{aligned}$$

Since $v_0 = u$, we have $V_0(s) = \frac{1}{s}$. Substituting this expression for V_0 into the preceding equation for V_1, we obtain

$$\begin{aligned}
V_1(s) &= \frac{\left(\frac{1}{s}\right)+1}{s^2+2s+1} \\
&= \frac{\left(\frac{s+1}{s}\right)}{(s+1)^2} \\
&= \frac{1}{s(s+1)}.
\end{aligned}$$

Now, we find a partial fraction expansion of V_1. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s} + \frac{A_2}{s+1}.$$

Solving for the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sV_1(s)|_{s=0} \\
&= \left.\frac{1}{s+1}\right|_{s=0} \\
&= 1 \quad \text{and} \\
A_2 &= (s+1)V_1(s)|_{s=-1} \\
&= \left.\frac{1}{s}\right|_{s=-1} \\
&= -1.
\end{aligned}$$

Thus, we can rewrite V_1 as

$$V_1(s) = \frac{1}{s} - \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform of V_1, we obtain

$$\begin{aligned}
v_1(t) &= \mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) \\
&= 1 - e^{-t} \quad \text{for } t \geq 0.
\end{aligned}$$

■

7.19 Exercises

7.19.1 Exercises Without Answer Key

7.1 Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.
(a) $x(t) = e^{-at}u(t)$;
(b) $x(t) = e^{-a|t|}$; and
(c) $x(t) = \cos(\omega_0 t)u(t)$. [Note: Use (F.3).]

7.2 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x below.
(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$, where a and b are real constants and $a > 0$;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

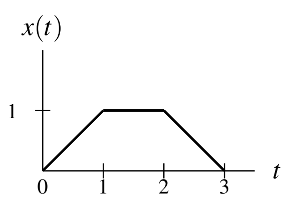
7.3 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x below.

(a) $x(t) = \begin{cases} t & 0 \le t < 1 \\ t-2 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

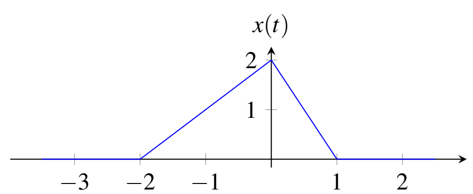
(b) $x(t) = \begin{cases} 1+t & -1 \le t < 0 \\ 1-t & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$ and

(c) $x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ t-1 & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$

7.4 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x shown in the figure below.



(a)



(b)

7.5 For each case below, using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x, where the ROCs of X and Y are R_X and R_Y, respectively.

(a) $y(t) = x(at-b)$, where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}\left[x * x(t-1)\right]$;
(c) $y(t) = tx(3t-2)$;
(d) $y(t) = \mathcal{D}x_1(t)$, where $x_1(t) = x^*(t-3)$ and $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = e^{-5t}x(3t+7)$; and
(f) $y(t) = e^{-j5t}x(t+3)$.

7.6 A causal function x has the Laplace transform

$$X(s) = \frac{-2s}{s^2+3s+2}.$$

(a) Assuming that x has no singularities at 0, find $x(0^+)$.
(b) Assuming that $\lim_{t\to\infty} x(t)$ exists, find this limit.

7.7 The function x has the Laplace transform

$$X(s) = \frac{(s+\frac{1}{2})(s-\frac{1}{2})}{s(s+1)(s-1)}.$$

Determine whether x is left sided but not right sided, right sided but not left sided, two sided, or finite duration for each ROC of X below.
(a) $\operatorname{Re}(s) < -1$;
(b) $-1 < \operatorname{Re}(s) < 0$;
(c) $0 < \operatorname{Re}(s) < 1$; and
(d) $\operatorname{Re}(s) > 1$.

7.8 A function x has the Laplace transform

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)}.$$

Plot the ROC of X if x is (a) left sided (but not right sided); (b) right sided (but not left sided); (c) two sided; (d) causal.

7.9 A function x has the Laplace transform

$$X(s) = \frac{s+\frac{1}{2}}{(s-1)(s+1-j)(s+1+j)(s+2)}.$$

Determine how many distinct possibilities exist for x. (It is not necessary to explicitly find all of them.)

7.10 Find the inverse Laplace transform x of each function X below.
(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;
(b) $X(s) = \dfrac{2s^2+4s+5}{(s+1)(s+2)}$ for $\operatorname{Re}(s) > -1$;
(c) $X(s) = \dfrac{3s+1}{s^2+3s+2}$ for $-2 < \operatorname{Re}(s) < -1$;
(d) $X(s) = \dfrac{s^2-s+1}{(s+3)^2(s+2)}$ for $\operatorname{Re}(s) > -2$; and
(e) $X(s) = \dfrac{s+2}{(s+1)^2}$ for $\operatorname{Re}(s) < -1$.

7.11 Find the causal inverse Laplace transform x of each function X below.
(a) $X(s) = \dfrac{s^2+4s+5}{s^2+2s+1}$; and
(b) $X(s) = \dfrac{-3s^2-6s-2}{(s+1)^2(s+2)}$.

7.12 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

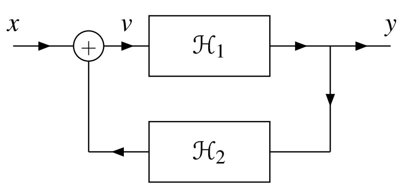
7.13 For the LTI system with input x and output y and each system function H given below, find the differential equation that characterizes the system.
(a) $H(s) = \dfrac{s+1}{s^2+2s+2}$.

7.14 For the causal LTI system with input x and output y that is characterized by each differential equation given below, find the system function H of the system.
(a) $\mathcal{D}^2y(t) + 4\mathcal{D}y(t) + 3y(t) = 2\mathcal{D}x(t) + x(t)$.

7.15 Consider the LTI system with input x, output y, and system function H, as shown in the figure below. Suppose that the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are causal and LTI with the respective system functions

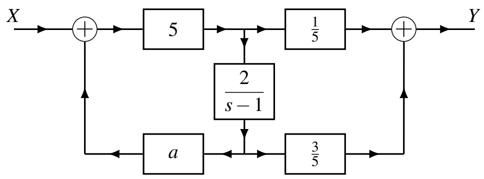
$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where A is a real constant.

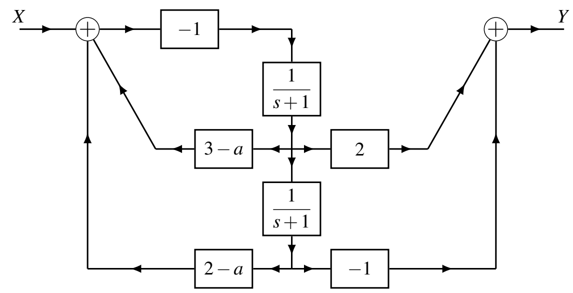


(a) Find an expression for H in terms of H_1 and H_2.
(b) Determine for what values of A the system is BIBO stable.

7.16 Each figure below shows a system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y. Each subsystem in the figure is LTI and causal and labelled with its system function, and a is a real constant. In each case where a subsystem has a system function that depends on a, it may be assumed that a is constrained so that the system function cannot be identically zero. (i) Find the system function H of the system $\mathcal{H}$. (ii) Determine whether the system $\mathcal{H}$ is BIBO stable.

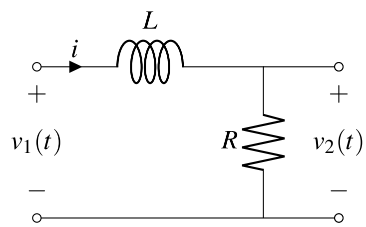


(a)



(b)

7.17 Consider the LTI resistor-inductor (RL) network with input v_1 and output v_2 shown in the figure below.



(a) Find the system function H of the system.
(b) Determine whether the system is BIBO stable.
(c) Determine the type of ideal frequency-selective filter that the system best approximates.
(d) Find the step response g of the system.

7.18 Consider a LTI system with the system function

$$H(s) = \frac{s^2+7s+12}{s^2+3s+12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

7.19 Consider the causal LTI system with input x and output y that is characterized by the differential equation

$$\mathcal{D}^2y(t) - y(t) = \mathcal{D}x(t) + ax(t),$$

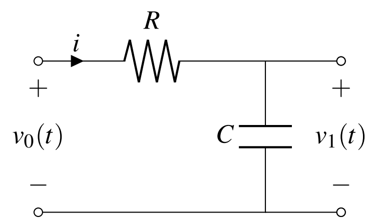
where a is a real constant and $\mathcal{D}$ denotes the derivative operator. Determine for what values of a the system is BIBO stable.

7.20 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input x and output y. Suppose that the transmitted signal x propagates along two paths. Along the intended direct path, the channel has a delay of T (where $T > 0$) and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of $T+\tau$ and gain of a (where $\tau > 0$). Thus, the received signal y is given by $y(t) = x(t-T) + ax(t-T-\tau)$. Find the system function H of a LTI system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal $x(t-T)$ without any distortion. Determine whether this system is physically realizable.

7.21 For each differential equation given below that characterizes a causal (incrementally-linear TI) system with input x and output y, solve for y subject to the given initial conditions.
(a) $\mathcal{D}^2 y(t) + 7\mathcal{D}y(t) + 12y(t) = x(t)$, where $y(0^-) = -1$, $\mathcal{D}y(0^-) = 0$, and $x(t) = u(t)$.

7.22 Consider the resistor-capacitor (RC) network shown in the figure below, where $R = 1000$ and $C = \frac{1}{1000}$.



(a) Find the differential equation that characterizes the relationship between the input v_0 and output v_1.
(b) If $v_1(0^-) = 2$, and $v_0(t) = 2e^{-3t}$, find v_1.

7.19.2 Exercises With Answer Key

7.101 Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.
(a) $x(t) = \sin(at)u(t)$, where a is a real constant; and
(b) $x(t) = \sinh(at)u(t)$, where $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and a is a real constant.
(c) $x(t) = \cosh(at)u(t)$, where $\cosh t = \frac{1}{2}(e^t + e^{-t})$ and a is a real constant.
(d) $x(t) = u(t-a) - u(t-b)$, where a and b are real constants and $a < b$.

Short Answer. (a) $X(s) = \dfrac{a}{s^2+a^2}$ for $\operatorname{Re}(s) > 0$; (b) $X(s) = \dfrac{a}{s^2-a^2}$ for $\operatorname{Re}(s) > |a|$; (c) $X(s) = \dfrac{s}{s^2-a^2}$ for $\operatorname{Re}(s) > |a|$; (d) $X(s) = \dfrac{e^{-as}-e^{-bs}}{s}$ for all $s \in \mathbb{C}$

7.102 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x given below.
(a) $x(t) = t^2 e^{-t} u(t-1)$;
(b) $x(t) = t^2 u(t-1)$;
(c) $x(t) = (t+1)u(t-1)$; and
(d) $x(t) = u(t-1) - u(t-2)$.

Short Answer. (a) $X(s) = e^{-s-1}\left[\dfrac{s^2+4s+5}{(s+1)^3}\right]$ for $\operatorname{Re}(s) > -1$; (b) $X(s) = e^{-s}\left(\dfrac{s^2+2s+2}{s^3}\right)$ for $\operatorname{Re}(s) > 0$; (c) $X(s) = e^{-s}\left(\dfrac{2s+1}{s^2}\right)$ for $\operatorname{Re}(s) > 0$; (d) $X(s) = \dfrac{e^{-s}-e^{-2s}}{s}$ for all s.

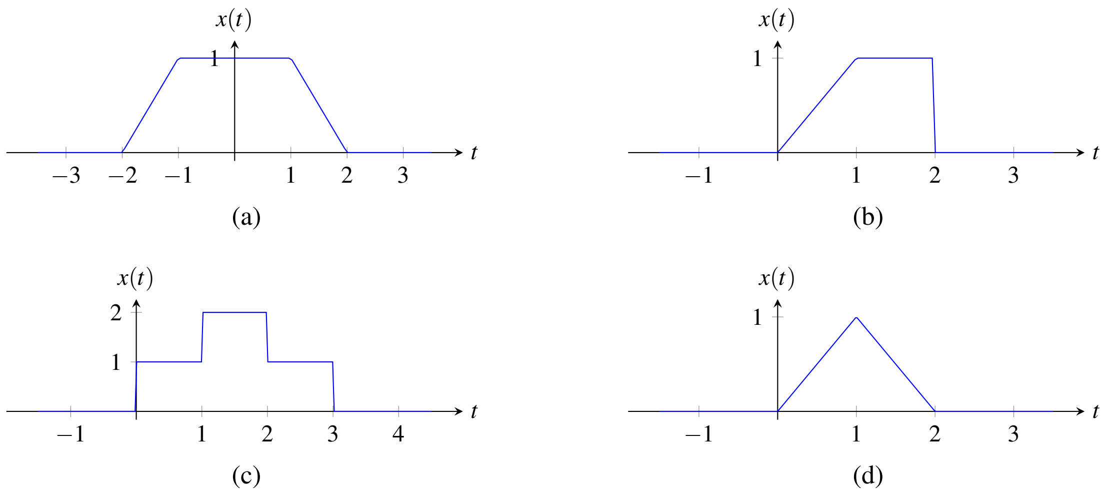
7.103 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x below.

(a) $x(t) = \begin{cases} t+1 & -1 \le t < 0 \\ (t-1)^2 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$ and

(b) $x(t) = \begin{cases} t & -1 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$

Short Answer. (a) $X(s) = \dfrac{se^s - 3s - 2e^{-s} + 2}{s^3}$ for all $s \in \mathbb{C}$; (b) $X(s) = \dfrac{e^s - e^{-s} - se^s - se^{-s}}{s^2}$ for all $s \in \mathbb{C}$

7.104 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x shown in the figure below.



Short Answer. (a) $X(s) = \dfrac{e^{2s} - e^s - e^{-s} + e^{-2s}}{s^2}$ for all $s \in \mathbb{C}$; (b) $X(s) = \dfrac{1 - e^{-s} - se^{-2s}}{s^2}$ for all $s \in \mathbb{C}$; (c) $X(s) = \dfrac{e^{3s} + e^{2s} - e^s - 1}{e^{3s}s}$ for all $s \in \mathbb{C}$; (d) $X(s) = \dfrac{1 - 2e^{-s} + e^{-2s}}{s^2}$ for all $s \in \mathbb{C}$

7.105 For each case below, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x, where the ROCs of X and Y are R_X and R_Y, respectively.
(a) $y(t) = t(x*x)(t)$;
(b) $y(t) = x*h\left(\frac{1}{3}t - 1\right)$, where h is an arbitrary function whose Laplace transform is H with ROC R_H;
(c) $y(t) = (t+1)^{100}x(t+1)$.
(d) $y(t) = t^{100}x(t+1)$; and
(e) $y(t) = (t+1)^{100}x(t)$.

Short Answer.
(a) $Y(s) = -2X(s)\frac{d}{ds}[X(s)]$ for $s \in R_X$;
(b) $Y(s) = 3e^{-3s}X(3s)H(3s)$ for $s \in \frac{1}{3}(R_X \cap R_H)$;
(c) $Y(s) = e^s\left(\frac{d}{ds}\right)^{100}X(s)$ for $s \in R_X$;
(d) $Y(s) = \left(\frac{d}{ds}\right)^{100}[e^sX(s)]$ for $s \in R_X$;
(e) $Y(s) = e^s\left(\frac{d}{ds}\right)^{100}[e^{-s}X(s)]$ for $s \in R_X$

7.106 Using properties of the Laplace transform and the given Laplace transform pair, find the Laplace transform X of each function x.
(a) $x(t) = te^{-3t}e^{-|t-2|}$ and $e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{-2}{s^2-1}$ for $-1 < \operatorname{Re}\{s\} < 1$; and
(b) $x(t) = te^{-5t}u(t+3)$, where $u(t) \stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s}$ for $\operatorname{Re}\{s\} > 0$.

Short Answer.
(a) $X(s) = \frac{d}{ds}\left(\frac{2e^{-2(s+3)}}{s^2+6s+8}\right)$ for $-4 < \operatorname{Re}(s) < -2$; and
(b) $X(s) = -e^{15}e^{3s}\frac{3s+14}{(s+5)^2}$ for $\operatorname{Re}(s) > -5$.

7.107 Find the inverse Laplace transform x of each function X given below.
(a) $X(s) = e^{-7s}\frac{6s+13}{(s+2)(s+3)}$ for $\operatorname{Re}(s) > -2$;
(b) $X(s) = \frac{-3s+2}{(s+1)^2}$ for $\operatorname{Re}(s) > -1$;
(c) $X(s) = \frac{7s^2+19s+17}{(s+1)^2(s+2)}$ for $\operatorname{Re}(s) > -1$;
(d) $X(s) = \frac{s^2+s+2}{(s+1)^2}$ for $\operatorname{Re}(s) > -1$;
(e) $X(s) = \frac{3s-5}{s^2-2s-3}$ for $\operatorname{Re}(s) < -1$; and
(f) $X(s) = \frac{s-7}{s^2-1}$ for $-1 < \operatorname{Re}(s) < -1$.

Short Answer. (a) $x(t) = e^{-2(t-7)}u(t-7) + 5e^{-3(t-7)}u(t-7)$; (b) $x(t) = 5te^{-t}u(t) - 3e^{-t}u(t)$; (c) $x(t) = 7e^{-2t}u(t) + 5te^{-t}u(t)$; (d) $x(t) = \delta(t) - e^{-t}u(t) + 2te^{-t}u(t)$; (e) $x(t) = -(2e^{-t} + e^{3t})u(-t)$; (f) $x(t) = 4e^{-t}u(t) + 3e^{t}u(-t)$

7.108 Find the inverse Laplace transform x of the function $X(s) = \frac{-3}{(s+2)(s-1)}$ if the ROC of X is:
(a) $-2 < \operatorname{Re}(s) < 1$;
(b) $\operatorname{Re}(s) > 1$; and
(c) $\operatorname{Re}(s) < -2$.

Short Answer. (a) $x(t) = e^{-2t}u(t) + e^{t}u(-t)$; (b) $x(t) = \left(e^{-2t} - e^{t}\right)u(t)$; (c) $x(t) = \left(-e^{-2t} + e^{t}\right)u(-t)$

7.109 Use the Laplace transform to compute the convolution $y(t) = x_1 * x_2(t)$ for each pair of functions x_1 and x_2 given below.
(a) $x_1(t) = e^{-at}u(t)$ and $x_2(t) = e^{-bt}u(t)$, where a and b are strictly positive real constants. [Note that there are two cases to consider, namely $a = b$ and $a \neq b$.]

Short Answer. (a) if $a \neq b$, $y(t) = \frac{1}{b-a}(e^{-at} - e^{-bt})u(t)$; if $a = b$, $y(t) = te^{-at}u(t)$

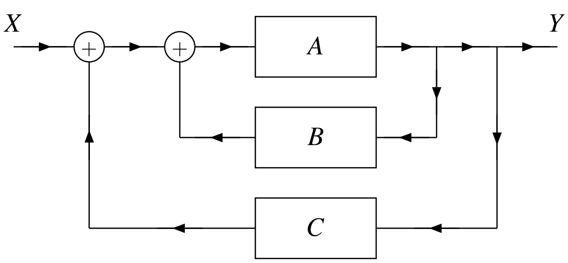
7.110 For the causal LTI system with input x and output y that is characterized by each differential equation given below, find the system function H of the system.
(a) $\mathcal{D}^2y(t) + 3\mathcal{D}y(t) + 2y(t) = 5\mathcal{D}x(t) + 7x(t)$; and
(b) $\mathcal{D}^2y(t) - 5\mathcal{D}y(t) + 6y(t) = \mathcal{D}x(t) + 7x(t)$.

Short Answer. (a) $H(s) = \dfrac{5s+7}{(s+1)(s+2)}$ for $\operatorname{Re}(s) > -1$; (b) $H(s) = \dfrac{s+7}{(s-2)(s-3)}$ for $\operatorname{Re}(s) > 3$

7.111 For the LTI system with input x and output y and each system function H given below, find the differential equation that characterizes the system.
(a) $H(s) = \dfrac{7s+3}{15s^2+4s+1}$.
(b) $H(s) = \dfrac{s^2+4}{s^3-s}$; and
(c) $H(s) = \dfrac{s^2+1}{s^3+6s^2+11s+6}$.

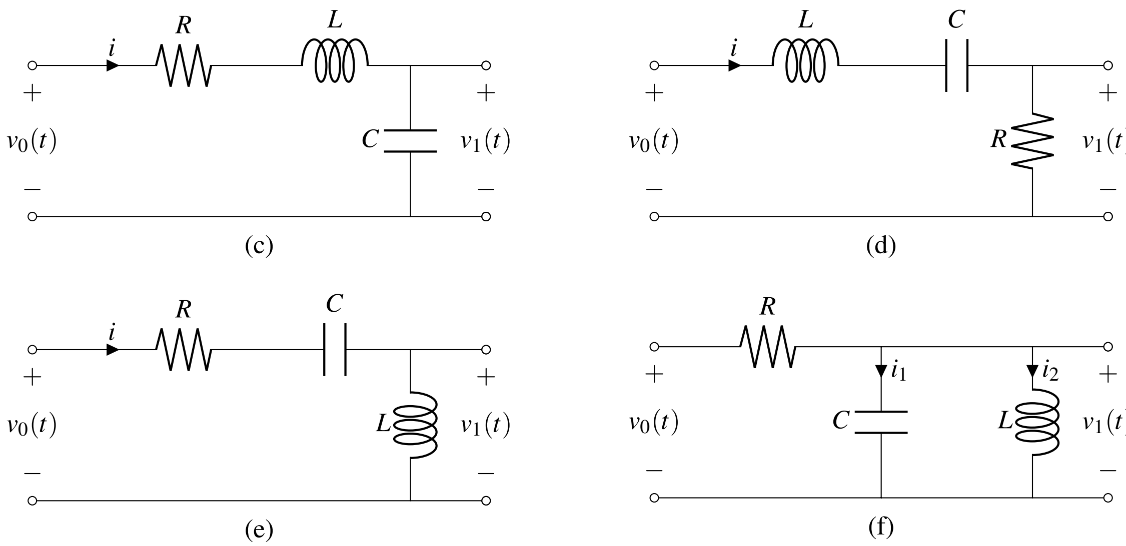
Short Answer.
(a) $15\mathcal{D}^2y(t)+4\mathcal{D}y(t)+y(t) = 7\mathcal{D}x(t)+3x(t)$;
(b) $\mathcal{D}^3y(t)-\mathcal{D}y(t) = \mathcal{D}^2x(t)+4x(t)$;
(c) $\mathcal{D}^3y(t)+6\mathcal{D}^2y(t)+11\mathcal{D}y(t)+6y(t) = \mathcal{D}^2x(t)+x(t)$

7.112 Consider the LTI system $\mathcal{H}$ shown in the figure below, where the Laplace transforms of the input and output are denoted as X and Y, respectively. Each of the subsystems in the diagram are LTI and labelled with their system functions. Find the system function H of the system $\mathcal{H}$ in terms of A, B, and C.



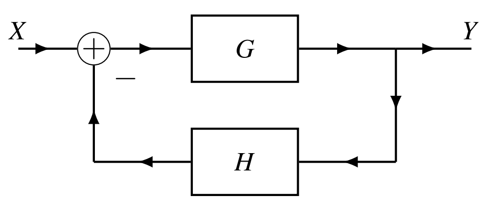
Short Answer. $H(s) = \dfrac{A(s)}{1-A(s)[B(s)+C(s)]}$.

7.113 For each of the LTI circuits with input v_0 and output v_1 shown in the figures below:
(i) Find the differential equation that characterizes the circuit.
(ii) Find the system function H of the circuit.
(iii) Determine whether the circuit is BIBO stable.
(iv) Determine the type of ideal frequency-selective filter that the circuit best approximates.



Short Answer.
(a) $\mathcal{D}v_0(t) = \mathcal{D}v_1(t) + \frac{1}{RC}v_1(t)$; $H(s) = \frac{RCs}{RCs+1}$ for $\operatorname{Re}(s) > -\frac{1}{RC}$; BIBO stable; highpass filter;
(b) $v_1(t) = \frac{L}{R}\mathcal{D}v_0(t) - \frac{L}{R}\mathcal{D}v_1(t)$; $H(s) = \frac{Ls}{Ls+R}$ for $\operatorname{Re}(s) > -\frac{R}{L}$; BIBO stable; highpass filter;
(c) $v_0(t) = LC\mathcal{D}^2v_1(t) + RC\mathcal{D}v_1(t) + v_1(t)$; $H(s) = \frac{1}{LCs^2+RCs+1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}]\right)$; BIBO stable; lowpass filter;
(d) $\mathcal{D}v_0(t) = \frac{L}{R}\mathcal{D}^2v_1(t) + \mathcal{D}v_1(t) + \frac{1}{RC}v_1(t)$; $H(s) = \frac{RCs}{LCs^2+RCs+1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}\right)$; BIBO stable; bandpass filter with passband center at $\pm\frac{1}{\sqrt{LC}}$;
(e) $\mathcal{D}^2v_0(t) = \mathcal{D}^2v_1(t) + \frac{R}{L}\mathcal{D}v_1(t) + \frac{1}{LC}v_1(t)$; $H(s) = \frac{LCs}{LCs-RCs-1}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-RC+\sqrt{(RC)^2-4LC}}{2LC}\right)$; BIBO stable; highpass filter;
(f) $\mathcal{D}v_0(t) = RC\mathcal{D}^2v_1(t) + \mathcal{D}v_1(t) + \frac{R}{L}v_1(t)$; $H(s) = \frac{Ls}{RLCs^2+Ls+R}$ for $\operatorname{Re}(s) > \operatorname{Re}\left(\frac{-L+\sqrt{L^2-4R^2LC}}{2RLC}\right)$; BIBO stable; bandpass filter with passband center $\pm\frac{1}{\sqrt{LC}}$

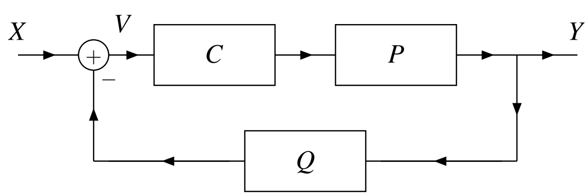
7.114 Consider the system $\mathcal{A}$ with input Laplace transform X and output Laplace transform Y. Each of the subsystems in the diagram is causal and LTI and labelled with its system function.



For each of the cases below, (i) find the system function A of $\mathcal{A}$; and (ii) determine if $\mathcal{A}$ is BIBO stable.
(a) $G(s) = \dfrac{1}{s}$ and $H(s) = a$, where a is a real constant;
(b) $G(s) = \dfrac{a}{s}$ and $H(s) = 1$, where a is a real constant;
(c) $G(s) = \dfrac{2s}{s-1}$ and $H(s) = -3a$, where a is a real constant; and
(d) $G(s) = \dfrac{a}{s(s+2)}$ and $H(s) = 1$, where a is a nonzero real constant.

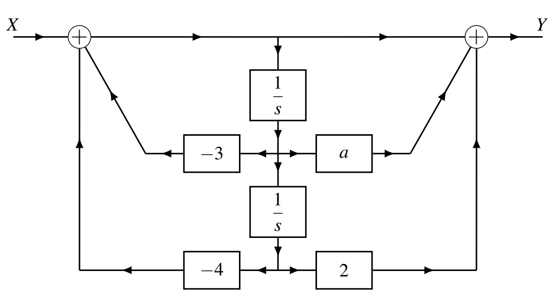
Short Answer.
(a.i) $A(s) = \dfrac{1}{s+a}$ for $\operatorname{Re}(s) > -a$; (a.ii) BIBO stable if and only if $a > 0$;
(b.i) $A(s) = \begin{cases} \frac{a}{s+a} & \text{for } s \in \{\operatorname{Re}(s) > -a\} & \text{if } a \neq 0 \\ 0 & \text{for } s \in \mathbb{C} & \text{if } a = 0; \end{cases}$ (b.ii) BIBO stable if and only if $a \geq 0$;
(c.i) $A(s) = \dfrac{2s}{(1-6a)s-1}$ for $\operatorname{Re}(s) > \frac{1}{1-6a}$; (c.ii) BIBO stable if and only if $a > \frac{1}{6}$;
(d.i) $A(s) = \dfrac{a}{s^2+2s+a}$ for $\operatorname{Re}(s) > -1+\operatorname{Re}\left\{\sqrt{1-a}\right\}$ (assuming $a \neq 0$); (d.ii) BIBO stable if and only if $a > 0$ (assuming $a \neq 0$);

7.115 Consider the system with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function; and $P(s) = 1/s$, $C(s) = as+3$, $Q(s) = 1$, and a is a real constant. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine for what values of a the system $\mathcal{H}$ is BIBO stable.

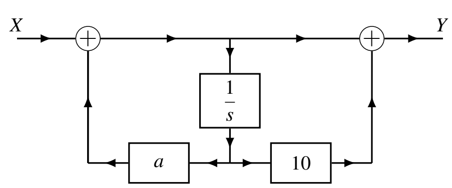


Short Answer. (a) $H(s) = \dfrac{as+3}{s+as+3}$ for $\operatorname{Re}(s) > \dfrac{-3}{a+1}$; (b) $a > -1$

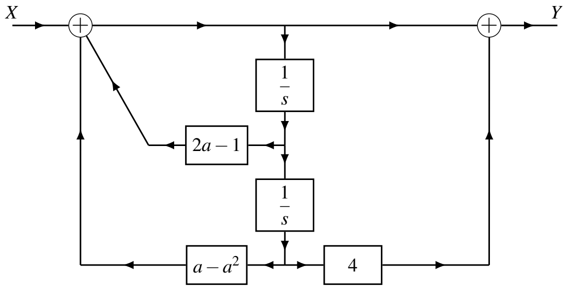
7.116 Each figure below shows a system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y. Each subsystem in the figure is LTI and causal and labelled with its system function, and a is a real constant. In each case where a subsystem has a system function that depends on a, it may be assumed that a is constrained so that the system function cannot be identically zero. (i) Find the system function H of the system $\mathcal{H}$. (ii) Determine whether the system $\mathcal{H}$ is BIBO stable.



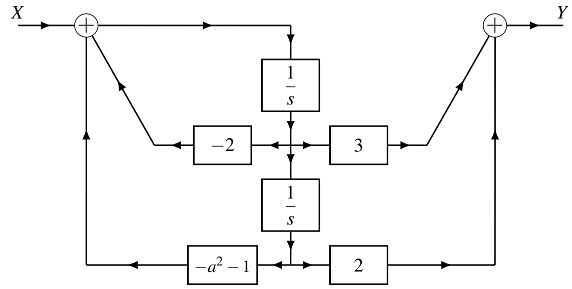
(a)



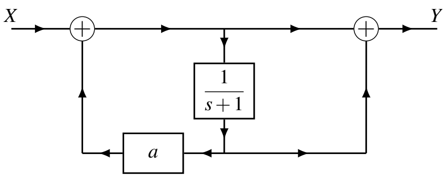
(b)



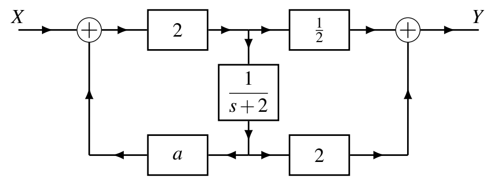
(c)



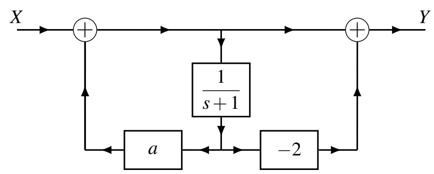
(d)



(e)



(f)



(g)

Short Answer.

(a.i) $H(s) = \dfrac{s^2+as+2}{s^2+3s+4}$ for $\operatorname{Re}(s) > -\frac{3}{2}$; (a.ii) BIBO stable for all a;

(b.i) $H(s) = \dfrac{s+10}{s-a}$ for $\operatorname{Re}(s) > a$ (except when $a = -10$); (b.ii) BIBO stable if and only if $a < 0$;

(c.i) $H(s) = \dfrac{s^2+4}{s^2+(1-2a)s+a^2-a} = \dfrac{(s+2j)(s-2j)}{(s-a+1)(s-a)}$ for $\operatorname{Re}(s) > a$; (c.ii) BIBO stable if and only if $a < 0$;

(d.i) $H(s) = \dfrac{3s+2}{s^2+2s+a^2+1} = \dfrac{3(s+\frac{2}{3})}{(s+1+ja)(s+1-ja)}$ for $\operatorname{Re}(s) > -1$; (d.ii) BIBO stable for all a;

(e.i) $H(s) = \dfrac{s+2}{s-a+1}$ for $\operatorname{Re}(s) > a-1$ (except when $a = -1$); (e.ii) BIBO stable if and only if $a < 1$;

(f.i) $H(s) = \dfrac{s+6}{s-2a+2}$ for $\operatorname{Re}(s) > 2a-2$ (except when $a = -2$); (f.ii) BIBO stable if and only if $a < 1$;

(g.i) $H(s) = \dfrac{s-1}{s-a+1} = \dfrac{s-1}{s-(a-1)}$ for $\operatorname{Re}(s) > a-1$ (except when $a = 2$); (g.ii) BIBO stable if and only if $a < 1$ or $a = 2$

7.117 Consider the design of a thermostat system for a room in a building, where the input x is the desired room temperature and the output y is the actual room temperature. For the purposes of this design, the system will be modelled as a LTI system with system function H. Determine which of the functions H_1 and H_2 given below would be a more desirable choice for H. Explain the rationale for your decision. [Hint: Consider the unit-step response of the system.]

(a) $H_1(s) = \dfrac{s^2}{s^2+9}$; and

(b) $H_2(s) = 1 + \dfrac{1}{20}\left[\dfrac{s(s+10)}{(s+10)^2+1}\right]$.

Short Answer. H_2

7.118 For each differential equation given below that characterizes a causal (incrementally-linear TI) system with input x and output y, find y for the case of the given x and initial conditions.
(a) $\mathcal{D}^2y(t)+4\mathcal{D}y(t)+3y(t)=x(t)$, where $x(t)=u(t)$ and $y(0^-)=0$ and $\mathcal{D}y(0^-)=1$; and
(b) $\mathcal{D}^2y(t)+5\mathcal{D}y(t)+6y(t)=x(t)$, where $x(t)=\delta(t)$ and $y(0^-)=1$ and $\mathcal{D}y(0^-)=-1$.

Short Answer. (a) $y(t)=\frac{1}{3}-\frac{1}{3}e^{-3t}$ for $t\geq 0$; (b) $y(t)=3e^{-2t}-2e^{-3t}$ for $t\geq 0$

7.119 Let X denote the Laplace transform of x. Suppose that x is absolutely integrable and X is rational with a pole at 1 and the other poles unknown. Determine if x can be each of the following:
(a) left sided (but not right sided);
(b) right sided (but not left sided);
(c) two sided;
(d) finite duration.

Short Answer. (a) yes; (b) no; (c) yes; (d) no

7.20 MATLAB Exercises

7.201 Consider a causal LTI system with the system function

$$H(s)=\frac{1}{-2s^7-s^6-3s^5+2s^3+s-3}.$$

(a) Use MATLAB to find and plot the poles of H.
(b) Determine whether the system is BIBO stable.

7.202 Consider a LTI system with the system function

$$H(s)=\frac{1}{1.0000s^4+2.6131s^3+3.4142s^2+2.6131s+1.0000}.$$

(This system corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the response y of the system to each input x given below. In each case, plot y over the interval $[0,20]$.
(a) $x=\delta$; and
(b) $x=u$.
(Hint: The `tf`, `impulse`, and `step` functions may be helpful.)

Part II

Discrete-Time Signals and Systems

Chapter 8

Discrete-Time Signals and Systems

8.1 Overview

In this chapter, we will examine discrete-time signals and systems in more detail.

8.2 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

8.2.1 Time Shifting (Translation)

The first type of signal transformation that we shall consider is known as time shifting. **Time shifting** (also known as **translation**) maps a sequence x to the sequence y given by

$$y(n) = x(n-b), \tag{8.1}$$

where b is an integer constant. In other words, the sequence y is formed by replacing n by $n-b$ in the expression for the $x(n)$. Geometrically, the transformation (8.1) shifts the sequence x along the time axis to yield y. If $b > 0$, y is shifted to the right relative to x (i.e., delayed in time). If $b < 0$, y is shifted to the left relative to x (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 8.1. By applying a time-shifting transformation to the sequence x shown in Figure 8.1(a), each of the sequences in Figures 8.1(b) and (c) can be obtained.

8.2.2 Time Reversal (Reflection)

The next type of signal transformation that we consider is called **time reversal**. **Time reversal** (also known as **reflection**) maps a sequence x to the sequence y given by

$$y(n) = x(-n). \tag{8.2}$$

In other words, the sequence y is formed by replacing n with $-n$ in the expression for $x(n)$. Geometrically, the transformation 8.2 reflects the sequence x about the origin to yield y.

To illustrate the effects of time reversal, an example is provided in Figure 8.2. Applying a time-reversal transformation to the sequence x in Figure 8.2(a) yields the sequence in Figure 8.2(b).