Example A.12 (Poles and zeros of a rational function)**.** Find and plot the poles and (finite) zeros of the function

$$f(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

Solution. We observe that f is a rational function, so we can easily determine the poles and zeros of f from its factored form. We now proceed to factor f. First, we factor z^2+3z+2. To do this, we solve for the roots of $z^2+3z+2=0$ to obtain

or factor by hand

$$z = \frac{-3 \pm \sqrt{3^2-4(1)(2)}}{2(1)} = -\tfrac{3}{2} \pm \tfrac{1}{2} = \{-1,-2\}. \qquad z^2+3z+2=(z+2)(z+1)$$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have

$$z^2+3z+2 = (z+1)(z+2). \quad 1$$

Second, we factor z^2+2z+2. To do this, we solve for the roots of $z^2+2z+2=0$ to obtain

$$z = \frac{-2 \pm \sqrt{2^2-4(1)(2)}}{2(1)} = -1 \pm j = \{-1+j, -1-j\}.$$

So, we have

$$z^2+2z+2 = (z+1-j)(z+1+j). \quad 2$$

Lastly, we factor z^2+1. Using the well-known factorization for a sum of squares, we obtain

$$z^2+1 = (z+j)(z-j). \quad 3 \qquad a^2+b^2 = (a+jb)(a-jb)$$

Combining the above results, we can rewrite f as

1, 2, 3

$$f(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}. = \frac{z^2(z+j)^1(z-j)^1(z-1)^1}{(z+1)^2(z+2)^1(z+1-j)^1(z+1+j)^1}$$

From this expression, we can trivially deduce that f has:

- first order zeros at 1, j, and $-j$,
- a second order zero at 0,

} from numerator

- first order poles at $-1+j$, $-1-j$, -2, and
- a second order pole at -1.

} from denominator

The zeros and poles of this function are plotted in Figure A.9. In such plots, the poles and zeros are typically denoted by the symbols "x" and "o", respectively.

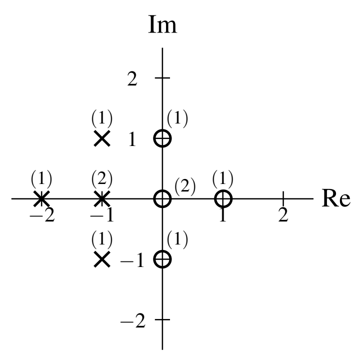


Figure A.9: Plot of the poles and zeros of f (with their orders indicated in parentheses).

■

Unit:
Preliminaries

Example 2.2. For two functions x_1 and x_2, the expression $x_1 + x_2$ denotes the function that is the sum of the functions x_1 and x_2. The expression $(x_1 + x_2)(t)$ denotes the function $x_1 + x_2$ evaluated at t. Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

$$\overset{\text{adding functions}}{(x_1 + x_2)(t)} = \overset{\text{adding numbers}}{x_1(t) + x_2(t)} \quad \text{for all } t.$$

Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

$$\overset{\text{subtracting functions}}{(x_1 - x_2)(t)} = \overset{\text{subtracting numbers}}{x_1(t) - x_2(t)} \quad \text{for all } t,$$

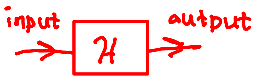
$$\text{multiplying functions} \rightarrow (x_1 x_2)(t) = x_1(t) x_2(t) \leftarrow \text{multiplying numbers} \quad \text{for all } t, \quad \text{and}$$

$$\text{dividing functions} \rightarrow (x_1 / x_2)(t) = x_1(t) / x_2(t) \leftarrow \text{dividing numbers} \quad \text{for all } t.$$

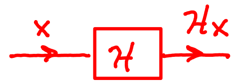
It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to functions, then the expression $(x_1 \diamond x_2)(t)$ is mathematically valid, but the expression $x_1(t) \diamond x_2(t)$ is not. The latter expression is not valid since the $\diamond$ operator requires two functions as operands, but the provided operands $x_1(t)$ and $x_2(t)$ are numbers (namely, the values of the functions x_1 and x_2 each evaluated at t). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

Example 2.6. For a system operator $\mathcal{H}$, a function x, a real variable t, and a real constant t_0, the expression $\mathcal{H}x(t-t_0)$ denotes the result obtained by taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x and then evaluating y at $t-t_0$. ■

$\mathcal{H}$ is a system.



$\underbrace{\mathcal{H}x}_{\text{function}}$ is the output of the system $\mathcal{H}$ when the input is $\underbrace{x}_{\text{function}}$.



Since $\mathcal{H}x$ is a function, we can evaluate it at some point such as $t-t_0$.

$$\overbrace{\underbrace{\mathcal{H}x}_{\text{function}}\;\underbrace{(t-t_0)}_{\substack{\text{point at}\\ \text{which}\\ \text{function is}\\ \text{evaluated}}}}^{\text{number}}$$