Example 4.16. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.49). Substituting the given function h into (4.49), we obtain

$$\begin{aligned}
H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt && \text{(4.49)} \\
&= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt && \text{substitute given } h \\
&= \left[e^{-st}\right]\Big|_{t=1} && \text{sifting property} \\
&= e^{-s}.
\end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned}
x(t) &= e^t \cos(\pi t) && \text{Euler} \\
&= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] && \text{exponent rules} \\
&= \tfrac{1}{2}e^{(1+j\pi)t} + \tfrac{1}{2}e^{(1-j\pi)t}.
\end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \quad \text{and} \quad s_k = \begin{cases} 1+j\pi & k=0 \\ 1-j\pi & k=1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

$$\begin{aligned}
y(t) &= \sum_{k=0}^{1} a_k H(s_k)e^{s_k t} && \mathcal{H}\{a_k e^{s_k t}\}(t) = a_k H(s_k)e^{s_k t} \\
&&& \text{expand summation} \\
&= a_0 H(s_0)e^{s_0 t} + a_1 H(s_1)e^{s_1 t} \\
&= \tfrac{1}{2}H(1+j\pi)e^{(1+j\pi)t} + \tfrac{1}{2}H(1-j\pi)e^{(1-j\pi)t} && \text{substitute for } a_k, s_k \\
&= \tfrac{1}{2}e^{-(1+j\pi)}e^{(1+j\pi)t} + \tfrac{1}{2}e^{-(1-j\pi)}e^{(1-j\pi)t} && \text{evaluate } H(\ldots) \\
&= \tfrac{1}{2}e^{t-1+j\pi t-j\pi} + \tfrac{1}{2}e^{t-1-j\pi t+j\pi} \\
&= \tfrac{1}{2}e^{t-1}e^{j\pi(t-1)} + \tfrac{1}{2}e^{t-1}e^{-j\pi(t-1)} && \text{rearrange} \\
&= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] && \text{Euler} \\
&= e^{t-1}\cos\left[\pi(t-1)\right].
\end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

NOTE: THIS SOLUTION DID NOT REQUIRE THE COMPUTATION OF A CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

Interlude

Interlude

1) LTI systems are relatively simple mathematically and are extremely useful in practice (e.g., for modelling real-world systems).

2) LTI systems, while relatively simpler, involve convolution.

3) Are we doomed to directly face convolution in every problem we solve that involves LTI systems?

4) Often, there is a better way. Employ transform-based solution techniques that utilize mathematical tools such as:

- CT Fourier series
- CT Fourier transform
- Laplace transform

Unit:
CT Fourier Series