Now, let us employ a change of variable. Let $\lambda = -k$ so that $k = -\lambda$. Applying the change of variable, we obtain

$$\begin{aligned} x * \delta(n) &= \sum_{\lambda=-(-\infty)}^{-(\infty)} x(-\lambda)\delta(n+\lambda) \\ &= \sum_{\lambda=\infty}^{-\infty} x(-\lambda)\delta(n+\lambda) \\ &= \sum_{\lambda=-\infty}^{\infty} x(-\lambda)\delta(\lambda+n). \end{aligned} \tag{9.6}$$

From the equivalence property of δ, we can rewrite the preceding equation as

$$\begin{aligned} x * \delta(n) &= \sum_{\lambda=-\infty}^{\infty} x(-[-n])\delta(\lambda+n) \\ &= \sum_{\lambda=-\infty}^{\infty} x(n)\delta(\lambda+n). \end{aligned}$$

Factoring $x(n)$ out of the summation, we obtain

$$x * \delta(n) = x(n) \sum_{\lambda=-\infty}^{\infty} \delta(\lambda+n).$$

Since $\sum_{\lambda=-\infty}^{\infty} \delta(\lambda) = 1$ implies that $\sum_{\lambda=-\infty}^{\infty} \delta(\lambda+n) = 1$, we have

$$x * \delta(n) = x(n).$$

Thus, δ is the convolutional identity (i.e., $x * \delta = x$). (Alternatively, we could have directly applied the sifting property to (9.6) to show the desired result.) ■

9.4 Periodic Convolution

The convolution of two periodic sequences is usually not well defined. This motivates an alternative notion of convolution for periodic sequences known as periodic convolution. The **periodic convolution** of the N-periodic sequences x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(n) = \sum_{k=\langle N \rangle} x(k)h(n-k),$$

where $\sum_{k=\langle N \rangle}$ denotes summation over an interval of length N. Equivalently, the periodic convolution can be written as

$$x \circledast h(n) = \sum_{k=0}^{N-1} x(k)h(\operatorname{mod}(n-k, N)),$$

where $\operatorname{mod}(a, b)$ is the remainder after division when a is divided by b. The periodic convolution and (linear) convolution of the N-periodic sequences x and h are related as

$$x \circledast h(n) = x_0 * h(n) \quad \text{where} \quad x(n) = \sum_{k=-\infty}^{\infty} x_0(n - Nk)$$

(i.e., $x_0(n)$ equals $x(n)$ over a single period of x and is zero elsewhere).

9.5 Characterizing LTI Systems and Convolution

As a matter of terminology, the **impulse response** h of a system $\mathcal{H}$ is defined as

$$h = \mathcal{H}\delta.$$

In other words, the impulse response of a system is the output that the system produces when presented with δ as an input. As it turns out, a LTI system has a very special relationship between its input, output, and impulse response, as given by the theorem below.

Theorem 9.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. To begin, we assume that $\mathcal{H}$ is LTI (i.e., $\mathcal{H}$ is both linear and time invariant). Using the fact that δ is the convolutional identity, we have

$$\mathcal{H}x = \mathcal{H}\{x * \delta\}.$$

From the definition of convolution, we have

$$\mathcal{H}x = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(\cdot - k)\right\}.$$

Since $\mathcal{H}$ is linear, we can pull the summation and $x(k)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

$$\mathcal{H}x = \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}\{\delta(\cdot - k)\}. \tag{9.7}$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by k. That is, we have

$$\mathcal{H}\{\delta(\cdot - k)\} = h(\cdot - k).$$

Thus, we can rewrite (9.7) as

$$\begin{aligned}\mathcal{H}x &= \sum_{k=-\infty}^{\infty} x(k)h(\cdot - k) \\ &= x * h.\end{aligned}$$

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■

By Theorem 9.5 above, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input. Consequently, the impulse response provides a very powerful tool for the study of LTI systems.

Example 9.6. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(n) = u(n). \tag{9.8}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k) \tag{9.9}$$

(i.e., $\mathcal{H}$ corresponds to an ideal accumulator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(n) = x * h(n).$$

Substituting (9.8) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(n) &= x * h(n) \\
&= x * u(n) \\
&= \sum_{k=-\infty}^{\infty} x(k)u(n-k) \\
&= \sum_{k=-\infty}^{n} x(k)u(n-k) + \sum_{k=n+1}^{\infty} x(k)u(n-k) \\
&= \sum_{k=-\infty}^{n} x(k).
\end{aligned}$$

Therefore, the system with the impulse response h given by (9.8) is, in fact, the ideal accumulator given by (9.9). ■

Example 9.7. Consider a LTI system with input x, output y, and impulse response h, where

$$h(n) = \begin{cases} 1 & 0 \leq n \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response of the system to the particular input x given by

$$x(n) = \begin{cases} n+1 & 0 \leq n \leq 3 \\ 7-n & 4 \leq n \leq 6 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Since the system is LTI, we have

$$y = x * h.$$

So, in this example, we are essentially being asked to compute $x * h$. This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows x and the various time-reversed and shifted versions of h, we can easily compute the elements of $x * h$. The result of this process is shown in Table 9.3. From this table, we have

$$x * h(n) = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 6 & n = 2 \\ 9 & n = 3 \\ 10 & n = 4 \\ 9 & n = 5 \\ 6 & n = 6 \\ 3 & n = 7 \\ 1 & n = 8 \\ 0 & \text{otherwise.} \end{cases}$$

■

Table 9.3: Convolution computation for Example 9.7

$n \backslash k$	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	
	$x(k)$													
				1	2	3	4	3	2	1				
	$h(k)$													
				1	1	1								
	$h(-k)$													
		1	1	1										
	$h(n-k)$													$x*h(n)$
-1	1	1	1											0
0		1	1	1										$0+0+1=1$
1			1	1	1									$0+1+2=3$
2				1	1	1								$1+2+3=6$
3					1	1	1							$2+3+4=9$
4						1	1	1						$3+4+3=10$
5							1	1	1					$4+3+2=9$
6								1	1	1				$3+2+1=6$
7									1	1	1			$2+1+0=3$
8										1	1	1		$1+0+0=1$
9											1	1	1	0

9.6 Unit Step Response of LTI Systems

The **step response** s of a system $\mathcal{H}$ is defined as

$$s = \mathcal{H}u$$

(i.e., the step response of a system is the output it produces for a unit-step sequence input). In the case of a LTI system, it turns out that the step response is closely related to the impulse response, as given by the theorem below.

Theorem 9.6. *The step response s and impulse response h of a LTI system are related as*

$$h(n) = s(n) - s(n-1) \quad \text{and} \quad s(n) = \sum_{k=-\infty}^{n} h(k).$$

That is, the impulse response h is the first difference of the step response s.

Proof. Using the fact that $s = u * h$, we can write

$$\begin{aligned} s(n) &= u * h(n) \\ &= h * u(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} h(k). \end{aligned}$$

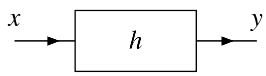


Figure 9.5: Block diagram representation of discrete-time LTI system with input x, output y, and impulse response h.

Thus, s can be obtained by accumulating h. Taking the first difference of s, we obtain

$$\begin{aligned} s(n)-s(n-1) &= \sum_{k=-\infty}^{n} h(k) - \sum_{k=-\infty}^{n-1} h(k) \\ &= h(n) + \sum_{k=-\infty}^{n-1} h(k) - \sum_{k=-\infty}^{n-1} h(k) \\ &= h(n). \end{aligned}$$

Thus, h is the first difference of s. ■

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. In particular, the impulse response can be determined from the step response via differencing.

9.7 Block Diagram Representation of Discrete-Time LTI Systems

Frequently, it is convenient to represent discrete-time LTI systems in block diagram form. Since a LTI system is completely characterized by its impulse response, we often label such a system with its impulse response in a block diagram. That is, we represent a LTI system with input x, output y, and impulse response h, as shown in Figure 9.5.

9.8 Interconnection of Discrete-Time LTI Systems

Suppose that we have a LTI system with input x, output y, and impulse response h. We know that x and y are related as $y = x * h$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 9.6(a). From the block diagram on the left side of Figure 9.6(a), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity of convolution, however, this is equivalent to

$$y = x * (h_1 * h_2).$$

Thus, the series interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 * h_2$. In other words, we have the equivalence shown in Figure 9.6(a).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 9.6(b). From the block diagram on the left side of Figure 9.6(b), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity and commutativity of convolution, this is equivalent to

$$\begin{aligned} y &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1. \end{aligned}$$

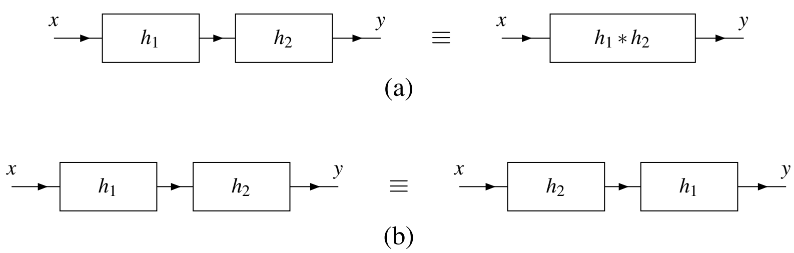


Figure 9.6: Equivalences for the series interconnection of discrete-time LTI systems. The (a) first and (b) second equivalences.

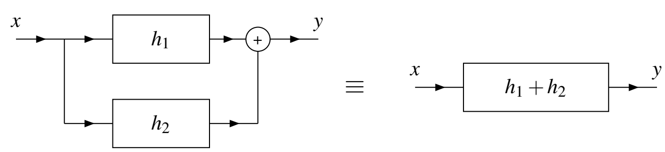


Figure 9.7: Equivalence for the parallel interconnection of discrete-time LTI systems.

Thus, interchanging the two LTI systems does not change the behavior of the overall system with input x and output y. In other words, we have the equivalence shown in Figure 9.6(b).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a parallel configuration, as shown on the left-side of Figure 9.7. From the block diagram on the left side of Figure 9.7, we have

$$y = x*h_1 + x*h_2.$$

Due to convolution being distributive, however, this equation can be rewritten as

$$y = x*(h_1+h_2).$$

Thus, the parallel interconnection of two LTI systems behaves as a single LTI system with impulse response h_1+h_2. In other words, we have the equivalence shown in Figure 9.7.

Example 9.8. Consider the system with input x, output y, and impulse response h as shown in Figure 9.8. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

$$\begin{aligned} v &= x + x*h_1 + x*h_2 \\ &= x*\delta + x*h_1 + x*h_2 \\ &= x*(\delta + h_1 + h_2). \end{aligned}$$

Similarly, from the right half of the block diagram, we can write

$$y = v*h_3.$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y &= v*h_3 \\ &= (x*[\delta + h_1 + h_2])*h_3 \\ &= x*([\delta + h_1 + h_2]*h_3) \\ &= x*(h_3 + h_1*h_3 + h_2*h_3). \end{aligned}$$

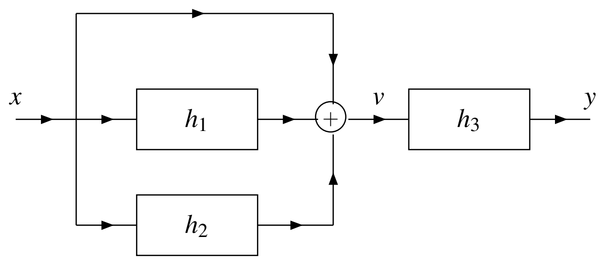


Figure 9.8: System interconnection example.

Thus, the impulse response h of the overall system is

$$h = h_3 + h_1 * h_3 + h_2 * h_3. \quad \blacksquare$$

9.9 Properties of Discrete-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, BIBO stability, and invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

9.9.1 Memory

The first system property to be considered is memory.

Theorem 9.7 (Memorylessness of LTI system)**.** *A LTI system with impulse response h is memoryless if and only if*

$$h(n) = 0 \textit{ for all } n \neq 0.$$

Proof. Recall that a system is memoryless if its output y at any arbitrary time depends only on the value of its input x at that same time. Suppose now that we have a LTI system with input x, output y, and impulse response h. The output y at some arbitrary time n_0 is given by

$$\begin{aligned} y(n_0) &= x * h(n_0) \\ &= h * x(n_0) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k). \end{aligned}$$

Consider the summation in the above equation. In order for the system to be memoryless, the result of the summation is allowed to depend on $x(n)$ only for $n = n_0$. This, however, is only possible if

$$h(n) = 0 \quad \text{for all } n \neq 0. \quad \blacksquare$$

From the preceding theorem, it follows that a memoryless LTI system must have an impulse response h of the form

$$h = K\delta \tag{9.10}$$

where K is a complex constant. As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y &= x * (K\delta) \\ &= K(x*\delta) \\ &= Kx. \end{aligned}$$

In other words, a memoryless LTI system must be an ideal amplifier (i.e., a system that simply performs amplitude scaling).

Example 9.9. Consider the LTI system with the impulse response h given by

$$h(n) = e^{-an}u(n),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(n) \neq 0$ for some $n \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

Example 9.10. Consider the LTI system with the impulse response h given by

$$h(n) = \delta(n).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse sequence δ. Therefore, the system is memoryless (i.e., does not have memory). ■

9.9.2 Causality

The next system property to be considered is causality.

Theorem 9.8 (Causality of LTI system)**.** *A LTI system with impulse response h is causal if and only if*

$$h(n) = 0 \textit{ for all } n < 0.$$

(i.e., h is causal).

Proof. Recall that a system is causal if its output y at any arbitrary time n_0 does not depend on its input x at a time later than n_0. Suppose that we have the LTI system with input x, output y, and impulse response h. The value of the output y at n_0 is given by

$$\begin{aligned} y(n_0) &= x*h(n_0) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n_0-k) \\ &= \sum_{k=-\infty}^{n_0} x(k)h(n_0-k) + \sum_{k=n_0+1}^{\infty} x(k)h(n_0-k). \end{aligned} \tag{9.11}$$

In order for the expression for $y(n_0)$ in (9.11) not to depend on $x(n)$ for $n > n_0$, we must have that

$$h(n) = 0 \quad \text{for } n < 0 \tag{9.12}$$

(i.e., h is causal). In this case, (9.11) simplifies to

$$y(n_0) = \sum_{k=-\infty}^{n_0} x(k)h(n_0 - k).$$

Clearly, the result of this integration does not depend on $x(n)$ for $n > n_0$ (since k varies from $-\infty$ to n_0). Therefore, a LTI system is causal if its impulse response h satisfies (9.12). ∎

Example 9.11. Consider the LTI system with impulse response h given by

$$h(n) = e^{-an}u(n),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(n) = 0$ for $n < 0$ (due to the $u(n)$ factor in the expression for $h(n)$). Therefore, the system is causal. ∎

Example 9.12. Consider the LTI system with impulse response h given by

$$h(n) = \delta(n + n_0),$$

where n_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(n) = 0$ except at $n = -n_0$. Since $-n_0 < 0$, the system is not causal. ∎

9.9.3 Invertibility

The next system property to be considered is invertibility.

Theorem 9.9 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with impulse response h. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has an impulse response h_{inv} that satisfies*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. To begin, we need to show that the inverse of a LTI system, if it exists, must also be LTI. This part of the proof, however, is left as an exercise for the reader in Exercise 9.9. (The general approach to take for this problem is to show that: 1) the inverse of a linear system, if it exists, is linear; and 2) the inverse of a time-invariant system, if it exists, is time invariant.) We assume that this part of the proof has been demonstrated and proceed.

Suppose now that the inverse system $\mathcal{H}^{-1}$ exists. We have that

$$\mathcal{H}x = x * h \quad \text{and} \quad \mathcal{H}^{-1}x = x * h_{\mathsf{inv}}.$$

From the definition of an inverse system, we have that, for every sequence x,

$$\mathcal{H}^{-1}\mathcal{H}x = x.$$

Expanding the left-hand side of the preceding equation, we obtain

$$\begin{aligned} & \mathcal{H}^{-1}[x * h] = x \\ \Leftrightarrow \quad & x * h * h_{\mathsf{inv}} = x. \end{aligned} \tag{9.13}$$

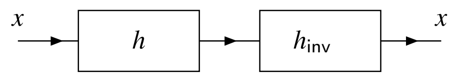


Figure 9.9: System in cascade with its inverse.

This relationship is expressed diagrammatically in Figure 9.9. Since the unit-impulse sequence is the convolutional identity, we can equivalently rewrite (9.13) as

$$x * h * h_{\mathsf{inv}} = x * \delta.$$

This equation, however, must hold for arbitrary x. Thus, by comparing the left- and right-hand sides of this equation, we conclude

$$h * h_{\mathsf{inv}} = \delta. \tag{9.14}$$

Therefore, if $\mathcal{H}^{-1}$ exists, it must have an impulse response h_{inv} that satisfies (9.14). This completes the proof. ■

From the preceding theorem, we have the following result:

Theorem 9.10 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with impulse response h is invertible if and only if there exists a sequence h_{inv} satisfying*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. The proof follows immediately from the result of Theorem 9.9 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. ■

Example 9.13. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(n) = A\delta(n - n_0),$$

where A is a nonzero real constant and n_0 is an integer constant. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathsf{inv}} = \delta. \tag{9.15}$$

So, let us attempt to solve this equation for h_{inv}. Substituting the given sequence h into (9.15) and using straightforward algebraic manipulation, we can write

$$\begin{aligned} & h * h_{\mathsf{inv}}(n) = \delta(n) \\ \Rightarrow \quad & \sum_{k=-\infty}^{\infty} h(k) h_{\mathsf{inv}}(n-k) = \delta(n) \\ \Rightarrow \quad & \sum_{k=-\infty}^{\infty} A\delta(k-n_0) h_{\mathsf{inv}}(n-k) = \delta(n) \\ \Rightarrow \quad & \sum_{k=-\infty}^{\infty} \delta(k-n_0) h_{\mathsf{inv}}(n-k) = \tfrac{1}{A}\delta(n). \end{aligned}$$

Using the sifting property of the unit-impulse sequence, we can simplify the summation on the left-hand side of the preceding equation to obtain

$$h_{\mathsf{inv}}(n - n_0) = \tfrac{1}{A}\delta(n). \tag{9.16}$$

Substituting $n + n_0$ for n in the preceding equation yields

$$\begin{aligned} & h_{\mathsf{inv}}([n+n_0] - n_0) = \tfrac{1}{A}\delta(n+n_0) \\ \Rightarrow \quad & h_{\mathsf{inv}}(n) = \tfrac{1}{A}\delta(n+n_0). \end{aligned}$$

Since $A \neq 0$, the sequence h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

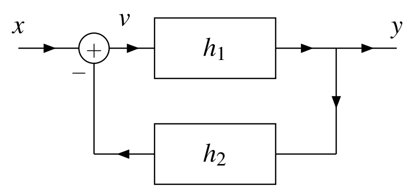


Figure 9.10: Feedback system with input x and output y.

Example 9.14. Consider the system with the input x and output y as shown in Figure 9.10. Each subsystem in the block diagram is LTI and labelled with its impulse response. Use the notion of an inverse system in order to express y in terms of x.

Solution. From Figure 9.10, we can write:

$$v = x - y * h_2 \quad \text{and} \tag{9.17}$$

$$y = v * h_1. \tag{9.18}$$

Substituting (9.17) into (9.18), and simplifying we obtain

$$\begin{aligned} & y = [x - y * h_2] * h_1 \\ \Rightarrow \quad & y = x * h_1 - y * h_2 * h_1 \\ \Rightarrow \quad & y + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * \delta + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * [\delta + h_2 * h_1] = x * h_1. \end{aligned} \tag{9.19}$$

For convenience, we now define the sequence g as

$$g = \delta + h_2 * h_1. \tag{9.20}$$

So, we can rewrite (9.19) as

$$y * g = x * h_1. \tag{9.21}$$

Thus, we have almost solved for y in terms of x. To complete the solution, we need to eliminate g from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response g. This inverse system has an impulse response g_{inv} given by

$$g * g_{\mathsf{inv}} = \delta. \tag{9.22}$$

This relationship follows from the definition of an inverse system. Now, we use g_{inv} in order to simplify (9.21) as follows:

$$\begin{aligned} & y * g = x * h_1 \\ \Rightarrow \quad & y * g * g_{\mathsf{inv}} = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y * \delta = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y = x * h_1 * g_{\mathsf{inv}}. \end{aligned}$$

Thus, we can express the output y in terms of the input x as

$$y = x * h_1 * g_{\mathsf{inv}},$$

where g_{inv} is given by (9.22) and g is given by (9.20). ■

9.9.4 BIBO Stability

The last system property to be considered is BIBO stability.

Theorem 9.11 (BIBO stability of LTI system)**.** *A LTI system with impulse response h is BIBO stable if and only if*

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{9.23}$$

(i.e., h is absolutely summable).

Proof. Recall that a system is BIBO stable if every bounded input to the system produces a bounded output. Suppose that we have a LTI system with input x, output y, and impulse response h.

First, we consider the sufficiency of (9.23) for BIBO stability. Assume that $|x(n)| \leq A < \infty$ for all n (i.e., x is bounded). We can write

$$\begin{aligned} y(n) &= x * h(n) \\ &= h * x(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k). \end{aligned}$$

By taking the magnitude of both sides of the preceding equation, we obtain

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right|. \tag{9.24}$$

One can show, for any two sequences f_1 and f_2, that

$$\left| \sum_{n=-\infty}^{\infty} f_1(n) f_2(n) \right| \leq \sum_{n=-\infty}^{\infty} |f_1(n) f_2(n)|.$$

Using this inequality, we can rewrite (9.24) as

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)x(n-k)| = \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)|.$$

We know (by assumption) that $|x(n)| \leq A$ for all n, so we can replace $|x(n)|$ by its bound A in the above inequality to obtain

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)| \leq \sum_{k=-\infty}^{\infty} A \, |h(k)| = A \sum_{k=-\infty}^{\infty} |h(k)|. \tag{9.25}$$

Thus, we have

$$|y(n)| \leq A \sum_{k=-\infty}^{\infty} |h(k)|. \tag{9.26}$$

Since A is finite, we can deduce from (9.26) that y is bounded if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{9.27}$$

(i.e., h is absolutely summable). Thus, the absolute summability of the impulse response h is a sufficient condition for BIBO stability.

Now, we consider the necessity of (9.23) for BIBO stability. Suppose that h is not absolutely summable. That is, suppose that

$$\sum_{k=-\infty}^{\infty} |h(k)| = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input x given by

$$x(n) = e^{-j\arg[h(-n)]}.$$

Since $\left|e^{j\theta}\right| = 1$ for all real θ, x is bounded (i.e., $|x(n)| \leq 1$ for all n). The output y is given by

$$\begin{aligned} y(n) &= x * h(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} \left[e^{-j\arg[h(-k)]}\right] h(n-k). \end{aligned} \tag{9.28}$$

Now, consider the output value $y(n)$ at $n = 0$. From (9.28), we have

$$y(0) = \sum_{k=-\infty}^{\infty} \left[e^{-j\arg[h(-k)]}\right] h(-k). \tag{9.29}$$

Since $e^{-j\arg z}z = |z|$ for all complex z, we have that $[e^{-j\arg[h(-k)]}]h(-k) = |h(-k)|$ and we can simplify (9.29) to obtain

$$\begin{aligned} y(0) &= \sum_{k=-\infty}^{\infty} |h(-k)| \\ &= \sum_{k=-\infty}^{\infty} |h(k)| \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input x will result in an unbounded output y (where $y(n)$ is unbounded for $n = 0$). Thus, the absolute summability of h is also necessary for BIBO stability. This completes the proof. ■

Example 9.15. Consider the LTI system with impulse response h given by

$$h(n) = e^{an}u(n),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely summable. Suppose that $a \neq 0$. We can write

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} |e^{an}u(n)| \\ &= \sum_{n=-\infty}^{\infty} e^{an}u(n) \\ &= \sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{\infty} e^{an} \\ &= \sum_{n=0}^{\infty} e^{an} \\ &= \sum_{n=0}^{\infty} (e^{a})^{n}. \end{aligned}$$

The right-hand side is an infinite geometric sequence. This converges to a finite value if and only if

$$e^{a} < 1.$$

Taking the logarithm of both sides, we obtain

$$a < \log 1 = 0.$$

Therefore, h is absolutely summable if $a < 0$ and infinite if $a \geq 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 9.16 (Ideal accumulator)**.** Consider the LTI system with input x and output y defined by

$$y(n) = \sum_{k=-\infty}^{n} x(k).$$

Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(n) &= \sum_{k=-\infty}^{n} \delta(k) \\ &= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \\ &= u(n). \end{aligned}$$

Using this expression for h, we now check to see if h is absolutely summable. We have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} |u(n)| \\ &= \sum_{n=0}^{\infty} 1 \\ &= \infty. \end{aligned}$$

Thus, h is not absolutely summable. Therefore, the system is not BIBO stable. ■

9.10 Eigensequences of Discrete-Time LTI Systems

Earlier, in Section 8.7.7, we were introduced to notion of eigensequences of systems. Given that eigensequences have the potential to simplify the mathematics associated with systems, it is natural to wonder what eigensequences LTI systems might have. In this regard, the following theorem is enlightening.

Theorem 9.12 (Eigensequences of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a sequence of the form $x(n) = z^n$, where z is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(n) = H(z)z^n, \tag{9.30a}$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}. \tag{9.30b}$$

That is, x is an eigensequence of $\mathcal{H}$ with the corresponding eigenvalue $H(z)$. Note that (9.30a) only holds for values of z for which $H(z)$ converges (i.e., values of z in the region of convergence of H).

Proof. To begin, we observe that a system $\mathcal{H}$ is LTI if and only if it computes a convolution (i.e., $\mathcal{H}x = x * h$ for some h). We have

$$\begin{aligned}\mathcal{H}x(n) &= x * h(n)\\ &= h * x(n)\\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k)\\ &= \sum_{k=-\infty}^{\infty} h(k)z^{n-k}\\ &= z^n \sum_{k=-\infty}^{\infty} h(k)z^{-k}\\ &= z^n H(z).\end{aligned}$$ ■

As a matter of terminology, the function H that appears in the preceding theorem (i.e., Theorem 9.12) is referred to as the **system function** (or **transfer function**) of the system $\mathcal{H}$. The system function completely characterizes the behavior of a LTI system. Consequently, system functions are often useful when working with LTI systems. As it turns out, a summation of the form appearing in (9.30b) is of great importance, as it defines what is called the z transform. We will study the z transform in great depth later in Chapter 12.

Note that a LTI system can have eigensequences other than complex exponentials. For example, the system in Example 8.38 is LTI and has every sequence as an eigensequence. Also, a system that has every complex exponential as an eigensequence is not necessarily LTI. This is easily demonstrated by the example below (as well as Exercise 9.17).

Example 9.17. Let S denote the set of all complex exponential sequences (i.e., S is the set of all sequences x of the form $x(n) = ba^n$ where $a, b \in \mathbb{C}$). Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x = \begin{cases} x & x \in S \\ 1 & \text{otherwise.} \end{cases}$$

For any sequence $x \in S$, we have $\mathcal{H}x = x$, implying that x is an eigensequence of $\mathcal{H}$ with eigenvalue 1. Therefore, every complex exponential sequence is an eigensequence of $\mathcal{H}$.

Now, we show that $\mathcal{H}$ is not linear. In what follows, let a denote an arbitrary complex constant. Consider the sequence $x(n) = n$. Clearly, $x \notin S$. Since $x \notin S$, we have $\mathcal{H}x = 1$, which implies that

$$a\mathcal{H}x = a.$$

Next, consider the sequence $ax(n) = an$. Since $ax \notin S$, we have

$$\mathcal{H}(ax) = 1.$$

From the above equations, however, we conclude that $\mathcal{H}(ax) = a\mathcal{H}x$ only in the case that $a = 1$. Therefore, $\mathcal{H}$ is not homogeneous and consequently not linear. So, $\mathcal{H}$ is an example of a system that has every complex exponential as an eigensequence, but is not LTI. ■

Let us now consider an application of eigensequences. Since convolution can often be quite painful to handle at the best of times, let us exploit eigensequences in order to devise a means to avoid having to deal with convolution directly in certain circumstances.

Suppose that we have a LTI system with input x, output y, and impulse response h. Suppose now that we can express some arbitrary input x as a sum of complex exponentials as follows:

$$x(n) = \sum_k a_k z_k^n.$$

From the eigensequence properties of LTI systems, the response to the input $a_k z_k^n$ is $a_k H(z_k) z^n$. By using this knowledge and the superposition property, we can write

$$\begin{aligned}
y(n) &= \mathcal{H}x(n) \\
&= \mathcal{H}\left\{\sum_k a_k z_k^n\right\}(n) \\
&= \sum_k a_k \mathcal{H}\{z_k^n\}(n) \\
&= \sum_k a_k H(z_k) z_k^n.
\end{aligned}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials.

Example 9.18. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(n) = \delta(n-1). \tag{9.31}$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(n) = e^n \cos(\pi n).$$

Solution. (a) Substituting (9.31) into (9.30b), we obtain

$$\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \delta(n-1) z^{-n} \\
&= z^{-1}.
\end{aligned}$$

(b) To begin, we can rewrite x as

$$\begin{aligned}
x(n) &= e^n \left[\tfrac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\right] \\
&= \tfrac{1}{2}\left(e^{1+j\pi}\right)^n + \tfrac{1}{2}\left(e^{1-j\pi}\right)^n.
\end{aligned}$$

So, the input x is now expressed in the form

$$x(n) = \sum_{k=0}^{1} a_k z_k^n,$$

where

$$a_0 = a_1 = \tfrac{1}{2}, \quad \text{and} \quad z_k = \begin{cases} e^{1+j\pi} & k=0 \\ e^{1-j\pi} & k=1. \end{cases}$$

In part (a), we found the system function H to be $H(z) = z^{-1}$. So we can calculate y by using the system function as follows:

$$\begin{aligned}
y(n) &= \sum_k a_k H(z_k) z_k^n \\
&= a_0 H(z_0) z_0^n + a_1 H(z_1) z_1^n \\
&= \tfrac{1}{2} H(e^{1+j\pi}) e^{(1+j\pi)n} + \tfrac{1}{2} H(e^{1-j\pi}) e^{(1-j\pi)n} \\
&= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)n} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)n} \\
&= \tfrac{1}{2} e^{n-1+j\pi n-j\pi} + \tfrac{1}{2} e^{n-1-j\pi n+j\pi} \\
&= \tfrac{1}{2} e^{n-1} e^{j\pi(n-1)} + \tfrac{1}{2} e^{n-1} e^{-j\pi(n-1)} \\
&= e^{n-1} \left[\tfrac{1}{2} \left(e^{j\pi(n-1)} + e^{-j\pi(n-1)} \right) \right] \\
&= e^{n-1} \cos[\pi(n-1)].
\end{aligned}$$

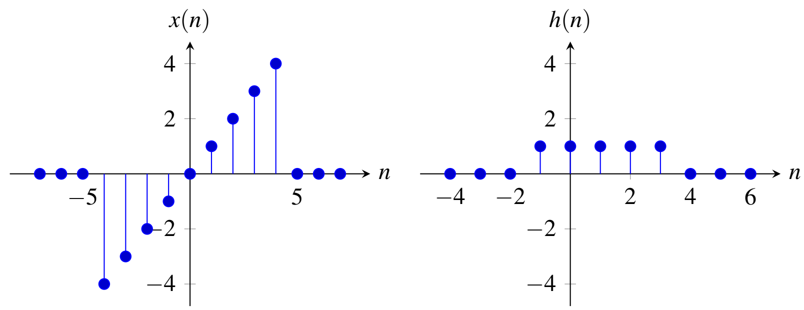
Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(z) = z^{-1}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

9.11 Exercises

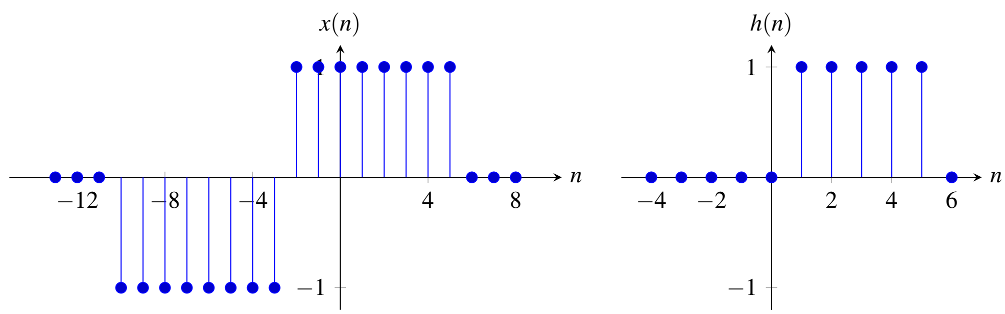
9.11.1 Exercises Without Answer Key

9.1 Compute $x * h$ for each pair of sequences x and h given below.
(a) $x(n) = n[u(n+2) - u(n-5)]$ and $h(n) = u(n)$;
(b) $x(n) = 2^n u(-n)$ and $h(n) = u(n)$;
(c) $x(n) = \begin{cases} n & 0 \leq n \leq 3 \\ 6-n & 4 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$ and $h(n) = u(n) - u(n-6)$; and
(d) $x(n) = u(n)$ and $h(n) = u(n-3)$.

9.2 For each pair of sequences x and h given in the figures below, compute $x * h$.



(a)



(b)

9.3 Show that, for any sequence x, $x * v(n) = x(n - n_0)$, where $v(n) = \delta(n - n_0)$ and n_0 is an arbitrary integer constant.

9.4 Let x, y, h, and v be sequences such that $y = x * h$ and

$$v(n) = \sum_{k=-\infty}^{\infty} x(-k-b)h(k+an),$$

where a and b are integer constants. Express v in terms of y.

9.5 Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

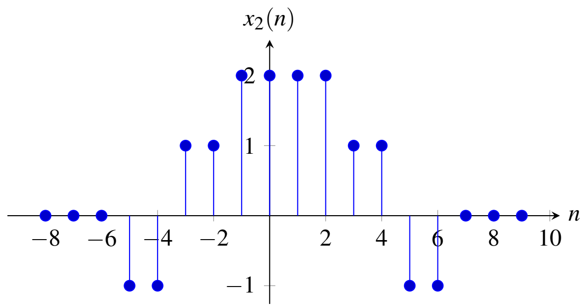
9.6 Let $\mathcal{D}x(n) = x(n) - x(n-1)$ (i.e., $\mathcal{D}$ is the first-difference operator). From the definition of convolution, show that if $y = x * h$, then $\mathcal{D}y = x * \mathcal{D}h$.

9.7 Let x and h be sequences satisfying

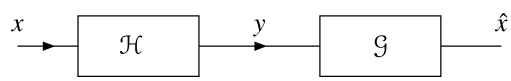
$$x(n) = 0 \quad \text{for } n < A_1 \text{ or } n > A_2, \quad \text{and}$$
$$h(n) = 0 \quad \text{for } n < B_1 \text{ or } n > B_2$$

(i.e., x and h are finite duration). Determine for which values of n the quantity $x * h(n)$ must be zero.

9.8 Consider a LTI system whose response to the sequence $x_1(n) = u(n) - u(n-2)$ is the sequence y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



9.9 Consider the system shown in the figure below, where $\mathcal{H}$ is a LTI system and $\mathcal{G}$ is known to be the inverse system of $\mathcal{H}$. Let $y_1 = \mathcal{H}x_1$ and $y_2 = \mathcal{H}x_2$.



(a) Determine the response of the system $\mathcal{G}$ to the input $y'(n) = a_1y_1(n) + a_2y_2(n)$, where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathcal{G}$ to the input $y_1'(n) = y_1(n - n_0)$, where n_0 is an integer constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathcal{G}$ is linear and/or time invariant.

9.10 Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.

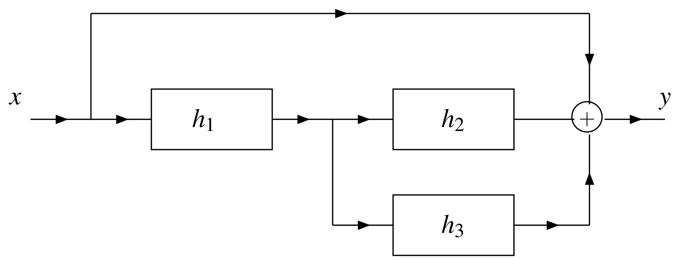
(a) $\mathcal{H}x(n) = \displaystyle\sum_{k=-\infty}^{n+1} x(k)$;

(b) $\mathcal{H}x(n) = \displaystyle\sum_{k=-\infty}^{\infty} x(k+5)e^{k-n+1}u(n-k-2)$;

(c) $\mathcal{H}x(n) = \displaystyle\sum_{k=-\infty}^{n} x(k)v(n-k)$; and

(d) $\mathcal{H}x(n) = \displaystyle\sum_{k=n-1}^{n} x(k)$.

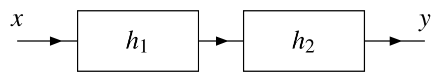
9.11 Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



(a) Find the impulse response h of the overall system in terms of h_1, h_2, and h_3.
(b) Determine the impulse response h in the specific case that

$$h_1(n) = \delta(n+1), \quad h_2(n) = \delta(n), \quad \text{and} \quad h_3(n) = \delta(n).$$

9.12 Consider the system shown in the figure below with input x and output y. This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2.



For each pair of h_1 and h_2 given below, find the output y if the input $x(n) = u(n)$.
(a) $h_1(n) = \delta(n)$ and $h_2(n) = \delta(n)$;
(b) $h_1(n) = \delta(n+1)$ and $h_2(n) = \delta(n+1)$; and
(c) $h_1(n) = 2^{-n}u(n)$ and $h_2(n) = \delta(n)$.

9.13 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(n) = (n+1)u(n-1)$;
(b) $h(n) = 2\delta(n+1)$;
(c) $h(n) = 2^{-n}u(n-1)$;
(d) $h(n) = 2^{n}u(-n-1)$;
(e) $h(n) = 2^{-3|n|}$; and
(f) $h(n) = 3\delta(n)$.

9.14 Suppose that we have two LTI systems with impulse responses

$$h_1(n) = \tfrac{1}{2}\delta(n-1) \quad \text{and} \quad h_2(n) = 2\delta(n+1).$$

Determine whether these systems are inverses of one another.

9.15 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(n) = 2^{an}u(-n)$, where a is a strictly positive real constant;
(b) $h(n) = n^{-1}u(n-1)$;
(c) $h(n) = 2^{n}u(n)$;
(d) $h(n) = \delta(n-10)$;
(e) $h(n) = u(n) - u(n-9)$; and
(f) $h(n) = 2^{-|n|}$.

9.16 Suppose that we have the systems $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, whose responses to a complex exponential input $x(n) = e^{j2n}$ are given by

$$\mathcal{H}_1x(n) = 2e^{j2n}, \quad \mathcal{H}_2x(n) = ne^{j2n}, \quad \mathcal{H}_3x(n) = e^{j2n+\pi/3}, \quad \text{and} \quad \mathcal{H}_4x(n) = \cos(2n).$$

Indicate which of these systems cannot be LTI.

9.17 A system that has every complex exponential sequence as an eigensequence is not necessarily LTI. In this exercise, we prove this fact by way of counterexample. Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x(n) = \begin{cases} \frac{x^2(n)}{x(n-1)} & \text{if } x(n-1) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that every complex exponential sequence is an eigensequence of $\mathcal{H}$.
(b) Show that $\mathcal{H}$ is not linear (and therefore not LTI).

9.11.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.

9.12 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 10

Discrete-Time Fourier Series

10.1 Introduction

One very important tool in the study of signals and systems is the (DT) Fourier series. A very large class of sequences can be represented using Fourier series, namely most practically useful periodic sequences. The Fourier series represents a periodic sequence as a linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy sequences with which to work. This is mainly due to the fact that complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigensequences of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex constant.

10.2 Definition of Discrete-Time Fourier Series

Consider a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all } k \in \mathbb{Z},$$

where N is a (strictly) positive integer constant. Since $e^{j\theta}$ is 2π-periodic in the variable θ, $\phi_k = \phi_{k+mN}$ for all $m \in \mathbb{Z}$. Consequently, the above set of sequences contains only N distinct elements, which can be obtained by choosing k as any set of N consecutive integers (e.g., $k \in [0\,..\,N-1]$). Furthermore, since $\left(\frac{2\pi}{N}k\right)/(2\pi) = \frac{k}{N}$ is a rational number, each ϕ_k is periodic. In particular, ϕ_k is periodic with the fundamental period $\frac{N}{\gcd(k,N)}$. Since it follows from the definition of the GCD that N must by an integer multiple of $\frac{N}{\gcd(k,N)}$, each ϕ_k must be N periodic. Since the sum of periodic sequences with the same period must be periodic with that period, any linear combination of ϕ_k must be N periodic. So, for example, a sum of the following form must be N-periodic:

$$\sum_{k=K_0}^{K_0+N-1} a_k \phi_k(n) = \sum_{k=K_0}^{K_0+N-1} a_k e^{j(2\pi/N)kn},$$

where the a_k are complex constants and K_0 is an integer constant.

Suppose now that we can represent a complex N-periodic sequence x as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N\rangle} c_k e^{jk(2\pi/N)n}, \tag{10.1}$$

where c is a complex N-periodic sequence and $\sum_{k=\langle N\rangle}$ denotes summation over any set of N consecutive integers. Such a representation is known as a (DT) **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. As a matter of terminology, we refer to (10.1) as the **Fourier-series synthesis equation**.

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a sequence is associated with particular Fourier-series coefficients. If a sequence x has the Fourier-series coefficient sequence c, we sometimes indicate this using the notation

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k.$$

10.3 Determining the Fourier-Series Representation of a Sequence

Given an arbitrary periodic sequence x, we need some means for finding its corresponding Fourier-series representation. In other words, we need a method for calculating the Fourier-series coefficient sequence corresponding to x. Such a method is given by the theorem below.

Theorem 10.1 (Fourier-series analysis equation)**.** *The Fourier-series coefficient sequence c of an N-periodic sequence x is given by*

$$c_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} \quad \text{for all } k \in \mathbb{Z}. \tag{10.2}$$

Proof. Recalling the Fourier-series synthesis equation, we have

$$x(n) = \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)\ell n}.$$

Multiplying both sides of this equation by $e^{-j(2\pi/N)kn}$ yields

$$\begin{aligned} x(n)e^{-j(2\pi/N)kn} &= \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)\ell n} e^{-j(2\pi/N)kn} \\ &= \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)(\ell-k)n}. \end{aligned}$$

Summing both sides of this equation over one period N of x, we obtain

$$\sum_{n=\langle N \rangle} x(n)e^{-j(2\pi/N)kn} = \sum_{n=\langle N \rangle} \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)(\ell-k)n}.$$

Interchanging the order of the summations on the right-hand side yields

$$\sum_{n=\langle N \rangle} x(n)e^{-j(2\pi/N)kn} = \sum_{\ell=\langle N \rangle} c_\ell \left(\sum_{n=\langle N \rangle} e^{j(2\pi/N)(\ell-k)n} \right).$$

Rewriting k as $k = Nk_1 + k_0$, where k_1 and k_0 are integers and $k_0 \in [0\,..\,N-1]$ (i.e., $k_1 = N\lfloor k/N \rfloor$ and $k_0 = \operatorname{mod}(k,N)$), and performing the outer summation on the right-hand side over $\ell \in [0\,..\,N-1]$, we obtain

$$\begin{aligned}
\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-[Nk_1+k_0])n} \right) \\
&= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-Nk_1-k_0)n} \right) \\
&= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n-j2\pi k_1 n} \right) \\
&= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{-j2\pi k_1 n} e^{j(2\pi/N)(\ell-k_0)n} \right) \\
&= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} 1^{k_1 n} e^{j(2\pi/N)(\ell-k_0)n} \right) \\
&= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n} \right). \qquad (10.3)
\end{aligned}$$

Consider now the inner summation on the right-hand side of this equation. We observe that

$$\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n} = \begin{cases} N & (\ell-k_0)/N \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \qquad (10.4)$$

(The proof of (10.4) is left as an exercise for the reader in Exercise A.9.) Moreover, since $\ell - k_0 \in [-(N-1)\,..\,N-1]$, the only time that $(\ell-k_0)/N \in \mathbb{Z}$ is when $\ell - k_0 = 0$ (i.e., $\ell = k_0$). Thus, the right-hand side of (10.4) can be simplified to $N\delta(\ell-k_0)$. Substituting (10.4) into (10.3), we have

$$\begin{aligned}
\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} &= \sum_{\ell=0}^{N-1} c_\ell [N\delta(\ell-k_0)] \\
&= c_{k_0} N.
\end{aligned}$$

Rearranging, we obtain

$$c_{k_0} = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn}.$$

Since $k = k_0$ for $k \in [0\,..\,N-1]$, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

Since c is N-periodic and the formula for c_k given by the preceding equation is N-periodic in k, this formula must be valid for all $k \in \mathbb{Z}$. Thus, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \quad \text{for all } k \in \mathbb{Z}. \qquad \blacksquare$$

As a matter of terminology, we refer to (10.2) as the **Fourier-series analysis equation**.

Suppose that we have a complex-valued N-periodic sequence x with Fourier-series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period N. The proof is trivial. Consider the Fourier-series analysis equation given by (10.2). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} \right] \Bigg|_{k=0} \\ &= \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{0} \\ &= \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n). \end{aligned}$$

Thus, c_0 is simply the average value of x over a single period.

Example 10.1. Find the Fourier-series representation of the sequence

$$x(n) = \sin\left(\tfrac{2\pi}{7} n\right).$$

Solution. To begin, we can first confirm that x has a Fourier-series representation. Since $(2\pi)/\left(\frac{2\pi}{7}\right) = (2\pi)\left(\frac{7}{2\pi}\right) = 7$ is rational, the sequence x is periodic. In particular, the period $N = 7$. So, x does have a Fourier-series representation. Using Euler's relation, we can express x as

$$\begin{aligned} \sin\left(\tfrac{2\pi}{7} n\right) &= \tfrac{1}{2j} \left[e^{j(2\pi/7)n} - e^{-j(2\pi/7)n} \right] \\ &= \tfrac{j}{2} e^{-j(2\pi/7)n} - \tfrac{j}{2} e^{j(2\pi/7)n} \\ &= \tfrac{j}{2} e^{j(2\pi/7)(-1)n} - \tfrac{j}{2} e^{j(2\pi/7)(1)n}. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=-3}^{3} c_k e^{j(2\pi/7)kn}$$

where

$$c_k = \begin{cases} \frac{j}{2} & k = -1 \\ -\frac{j}{2} & k = 1 \\ 0 & k \in \{-3, -2, 0, 2, 3\} \end{cases} \quad \text{and} \quad c_k = c_{k+7}.$$

Since c is 7-periodic, we could also change the summation to be taken over any set of 7 consecutive integers and specify c_k for $k \in [0\,..\,6]$ to yield

$$x(n) = \sum_{k=\langle 7 \rangle} c_k e^{j(2\pi/7)kn}$$

where

$$c_k = \begin{cases} -\frac{j}{2} & k = 1 \\ \frac{j}{2} & k = 6 \\ 0 & k \in \{0, 2, 3, 4, 5\} \end{cases} \quad \text{and} \quad c_k = c_{k+7}.$$

■

Example 10.2 (Periodic impulse train)**.** Find the Fourier-series representation of the periodic impulse train

$$x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n - N\ell),$$

where N is a strictly positive integer constant.