To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{z}{(z-a)^m}$ for some complex constant a and some positive integer m. Fortunately, Table 12.3 does directly contain the entries necessary to handle such inverse z transform calculations (for $m \leq 2$).

Alternatively, we can express X in terms of a partial fraction expansion in the variable z^{-1} (instead of z). Doing this, we obtain a partial fraction expansion for X of the form

$$X(z) = \sum_{k=1}^{P} \sum_{\ell=1}^{q_k} A_{k,\ell} \frac{1}{(1-p_k z^{-1})^{\ell}}. \tag{12.10}$$

To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{1}{(1-az^{-1})^m}$ for some complex constant a and some positive integer m. Fortunately, Table 12.3 does directly contain the entries necessary to handle such inverse z transform calculations. For example, we can use the following pairs from the table:

$$\begin{aligned} \frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(n) &\overset{\text{ZT}}{\longleftrightarrow} \frac{1}{(1-az^{-1})^m} \text{ for } |z| > |a| \quad \text{and} \\ -\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(-n-1) &\overset{\text{ZT}}{\longleftrightarrow} \frac{1}{(1-az^{-1})^m} \text{ for } |z| < |a|. \end{aligned}$$

(A number of other z transform pairs in the table correspond to special cases of the preceding two pairs.)

As seen above, in order to compute an inverse z transform, we can use a partial fraction expansion in the manner shown in (12.9) or in the manner shown in (12.10). The former is likely to be more useful if $X(z)$ is expressed in terms of only positive powers of z, whereas the latter is likely to be more useful if $X(z)$ is expressed in terms of only negative powers of z. Since we tend to write rational z transforms with only positive powers of z herein, we tend to use a partial fraction expansion of the form of (12.9) more often when computing inverse z transforms. In what follows, we will now consider a number of examples of computing inverse z transforms.

Example 12.25. Find the inverse z transform x of the function

$$X(z) = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)} = \frac{z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \text{ for } |z| > \tfrac{1}{2}.$$

Solution. FIRST APPROACH (USING PARTIAL FRACTION EXPANSION IN THE VARIABLE z). As our starting point, we use $X(z)$ expressed in terms of only positive powers of z:

$$X(z) = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

Since we cannot directly obtain the inverse z transform of X from Table 12.3, we employ a partial fraction expansion. We find a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{1}{z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

(Note that $X(z)/z$ is strictly proper.) This expression has an expansion of the form

$$\frac{X(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z+\frac{1}{2}} + \frac{A_3}{z-\frac{1}{2}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left[z\left(\frac{X(z)}{z}\right)\right]\Big|_{z=0} = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\Big|_{z=0} = \frac{1}{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)} = \frac{1}{\left(-\frac{1}{4}\right)} = -4, \\
A_2 &= \left[\left(z+\tfrac{1}{2}\right)\left(\frac{X(z)}{z}\right)\right]\Big|_{z=-1/2} = \frac{1}{z\left(z-\frac{1}{2}\right)}\Big|_{z=-1/2} = \frac{1}{\left(-\frac{1}{2}\right)(-1)} = \frac{1}{\left(\frac{1}{2}\right)} = 2, \quad \text{and} \\
A_3 &= \left[\left(z-\tfrac{1}{2}\right)\left(\frac{X(z)}{z}\right)\right]\Big|_{z=1/2} = \frac{1}{z\left(z+\frac{1}{2}\right)}\Big|_{z=1/2} = \frac{1}{\left(\frac{1}{2}\right)(1)} = \frac{1}{\left(\frac{1}{2}\right)} = 2.
\end{aligned}$$

Thus, we have

$$\frac{X(z)}{z} = -4\left(\frac{1}{z}\right) + 2\left(\frac{1}{z+\frac{1}{2}}\right) + 2\left(\frac{1}{z-\frac{1}{2}}\right).$$

So, we can rewrite X as

$$X(z) = -4 + 2\left(\frac{z}{z+\frac{1}{2}}\right) + 2\left(\frac{z}{z-\frac{1}{2}}\right).$$

Taking the inverse z transform, we have

$$x(n) = -4\mathcal{Z}^{-1}\{1\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{z}{z+\frac{1}{2}}\right\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{2}$, we have

$$\left(-\tfrac{1}{2}\right)^n u(n) \xleftrightarrow{\text{ZT}} \frac{z}{z+\frac{1}{2}} \text{ for } |z| > \tfrac{1}{2} \quad \text{and} \quad \left(\tfrac{1}{2}\right)^n u(n) \xleftrightarrow{\text{ZT}} \frac{z}{z-\frac{1}{2}} \text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned}
x(n) &= -4\delta(n) + 2\left[\left(-\tfrac{1}{2}\right)^n u(n)\right] + 2\left[\left(\tfrac{1}{2}\right)^n u(n)\right] \\
&= -4\delta(n) + 2\left(-\tfrac{1}{2}\right)^n u(n) + 2\left(\tfrac{1}{2}\right)^n u(n).
\end{aligned}$$

SECOND APPROACH (USING PARTIAL FRACTION EXPANSION IN THE VARIABLE z^{-1}). As our starting point, we use $X(z)$ expressed in terms of only negative powers of z:

$$X(z) = \frac{z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.$$

Since we cannot directly obtain the inverse z transform of X from Table 12.3, we employ a partial fraction expansion. First, we observe that the rational expression in the variable z^{-1} for the function X is not strictly proper. So, we first need to express X as the sum of a polynomial in z^{-1} and a strictly proper rational function in z^{-1}. For convenience in what follows, we observe that $\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right) = 1-\frac{1}{4}z^{-2}$. We have

$$\begin{aligned}
X(z) &= \frac{z^{-2} + 4\left(1-\frac{1}{4}z^{-2}\right) - 4\left(1-\frac{1}{4}z^{-2}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\
&= -4 + \frac{z^{-2} + 4 - z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\
&= -4 + \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.
\end{aligned}$$

Let $V(z) = \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$ so that $X(z) = -4+V(z)$. Since V is a strictly proper rational function in z^{-1}, it has a partial fraction expansion. This expansion has the form

$$V(z) = \frac{A_1}{1+\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned} A_1 &= \left(1+\tfrac{1}{2}z^{-1}\right)V(z)\big|_{z=-1/2} = \left.\frac{4}{1-\frac{1}{2}z^{-1}}\right|_{z=-1/2} = \frac{4}{1-\left(\frac{1}{2}\right)(-2)} = 2 \quad \text{and} \\ A_2 &= \left(1-\tfrac{1}{2}z^{-1}\right)V(z)\big|_{z=1/2} = \left.\frac{4}{1+\frac{1}{2}z^{-1}}\right|_{z=1/2} = \frac{4}{1+\left(\frac{1}{2}\right)(2)} = 2. \end{aligned}$$

Thus, we have

$$V(z) = 2\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + 2\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right).$$

So, we can rewrite X as

$$X(z) = -4 + 2\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + 2\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right).$$

Taking the inverse z transform, we have

$$x(n) = -4\mathcal{Z}^{-1}\{1\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{1}{1+\frac{1}{2}z^{-1}}\right\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{2}$, we have

$$\left(-\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1+\frac{1}{2}z^{-1}} \;\text{ for } |z| > \tfrac{1}{2} \quad \text{and} \quad \left(\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \;\text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -4\delta(n) + 2\left[\left(-\tfrac{1}{2}\right)^n u(n)\right] + 2\left[\left(\tfrac{1}{2}\right)^n u(n)\right] \\ &= -4\delta(n) + 2\left(-\tfrac{1}{2}\right)^n u(n) + 2\left(\tfrac{1}{2}\right)^n u(n). \end{aligned}$$

■

Example 12.26. Find the inverse z transform x of the function

$$X(z) = \frac{z(z-1)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)} \quad \text{for } |z| > \tfrac{1}{3}.$$

Solution. Since we cannot directly obtain the answer from Table 12.3, we employ a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{z-1}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)}.$$

This expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z-\frac{1}{3}} + \frac{A_2}{z-\frac{1}{5}}.$$

Calculating the expansion coefficients, we obtain

$$A_1 = \left[\left(z-\tfrac{1}{3}\right)\left(\frac{X(z)}{z}\right)\right]\Big|_{z=1/3} = \left.\frac{z-1}{z-\frac{1}{5}}\right|_{z=1/3} = \frac{\left(-\frac{2}{3}\right)}{\left(\frac{2}{15}\right)} = \left(\frac{-2}{3}\right)\left(\frac{15}{2}\right) = -5 \quad \text{and}$$

$$A_2 = \left[\left(z-\tfrac{1}{5}\right)\left(\frac{X(z)}{z}\right)\right]\Big|_{z=1/5} = \left.\frac{z-1}{z-\frac{1}{3}}\right|_{z=1/5} = \frac{\left(-\frac{4}{5}\right)}{\left(-\frac{2}{15}\right)} = \left(\frac{-4}{5}\right)\left(\frac{-15}{2}\right) = 6.$$

Thus, we have

$$\frac{X(z)}{z} = \frac{-5}{z-\frac{1}{3}} + \frac{6}{z-\frac{1}{5}}.$$

So, we can rewrite X as

$$X(z) = \frac{-5z}{z-\frac{1}{3}} + \frac{6z}{z-\frac{1}{5}}.$$

Taking the inverse z transform, we have

$$x(n) = -5\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{3}}\right\}(n) + 6\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{5}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{3}$, we have

$$\left(\tfrac{1}{3}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{3}} \text{ for } |z| > \tfrac{1}{3} \quad \text{and} \quad \left(\tfrac{1}{5}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{5}} \text{ for } |z| > \tfrac{1}{5}.$$

Thus, we have

$$\begin{aligned} x(n) &= -5\left[\left(\tfrac{1}{3}\right)^n u(n)\right] + 6\left[\left(\tfrac{1}{5}\right)^n u(n)\right] \\ &= \left[-5\left(\tfrac{1}{3}\right)^n + 6\left(\tfrac{1}{5}\right)^n\right] u(n). \end{aligned}$$

■

Example 12.27. Find the inverse z transform x of the function

$$X(z) = \frac{z^2(z-1)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^2} \quad \text{for } z \in R,$$

for each R given below.
(a) $R = \left\{|z| < \frac{1}{4}\right\}$;
(b) $R = \left\{\frac{1}{4} < |z| < \frac{1}{2}\right\}$; and
(c) $R = \left\{|z| > \frac{1}{2}\right\}$.

Solution. Since we cannot directly obtain the answer from Table 12.3, we employ a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{z(z-1)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^2}.$$

This expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z-\frac{1}{4}} + \frac{A_{2,1}}{z-\frac{1}{2}} + \frac{A_{2,2}}{\left(z-\frac{1}{2}\right)^2}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned}
A_1 &= \left(z-\tfrac{1}{4}\right)\left(\frac{X(z)}{z}\right)\bigg|_{z=1/4} = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)^2}\bigg|_{z=1/4} = \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)}{\left(-\frac{1}{4}\right)^2} = \frac{\left(\frac{-3}{16}\right)}{\left(\frac{1}{16}\right)} \\
&= -3, \\
A_{2,1} &= \frac{1}{(2-1)!}\left[\left(\frac{d}{dz}\right)^{2-1}\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\bigg|_{z=1/2} = \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\bigg|_{z=1/2} \\
&= \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\frac{z(z-1)}{z-\frac{1}{4}}\right]\right]\bigg|_{z=1/2} = \left[\left(\frac{d}{dz}\right)\left[(z^2-z)\left(z-\tfrac{1}{4}\right)^{-1}\right]\right]\bigg|_{z=1/2} \\
&= \left[(2z-1)\left(z-\tfrac{1}{4}\right)^{-1}+(-1)\left(z-\tfrac{1}{4}\right)^{-2}(z^2-z)\right]\bigg|_{z=1/2} = (-1)\left(\tfrac{1}{4}\right)^{-2}\left(-\tfrac{1}{4}\right) = \left(\tfrac{1}{4}\right)(16) \\
&= 4, \quad \text{and} \\
A_{2,2} &= \frac{1}{(2-2)!}\left[\left(\frac{d}{dz}\right)^{2-2}\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\bigg|_{z=1/2} = \frac{1}{0!}\left[\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\bigg|_{z=1/2} \\
&= \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\frac{z(z-1)}{z-\frac{1}{4}}\right]\right]\bigg|_{z=1/2} = \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{\left(-\frac{1}{4}\right)} = \frac{\left(\frac{-1}{4}\right)}{\left(\frac{-1}{4}\right)} \\
&= 1.
\end{aligned}$$

Thus, we have

$$\frac{X(z)}{z} = -\frac{3}{z-\frac{1}{4}} + \frac{4}{z-\frac{1}{2}} + \frac{1}{\left(z-\frac{1}{2}\right)^2}.$$

So, we can rewrite X as

$$X(z) = -\frac{3z}{z-\frac{1}{4}} + \frac{4z}{z-\frac{1}{2}} + \frac{z}{\left(z-\frac{1}{2}\right)^2}.$$

Taking the inverse z transform, we have

$$x(n) = -3\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\}(n) + 4\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\}(n) + \mathcal{Z}^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)^2}\right\}(n).$$

To proceed further, we must now consider the ROC R.

(a) Suppose that $R = \left\{|z| < \frac{1}{4}\right\}$. In this case, we have

$$\begin{aligned}
-\left(\tfrac{1}{4}\right)^n u(-n-1) &\overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}} \text{ for } |z| < \tfrac{1}{4}, \\
-n\left(\tfrac{1}{2}\right)^n u(-n-1) &\overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \text{ for } |z| < \tfrac{1}{2}, \quad \text{and} \\
-\left(\tfrac{1}{2}\right)^n u(-n-1) &\overset{\text{ZT}}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)^2} \text{ for } |z| < \tfrac{1}{2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
x(n) &= -3\left[-\left(\tfrac{1}{4}\right)^n u(-n-1)\right] + 4\left[-\left(\tfrac{1}{2}\right)^n u(-n-1)\right] + \left[-n\left(\tfrac{1}{2}\right)^n u(-n-1)\right] \\
&= \left[3\left(\tfrac{1}{4}\right)^n - 4\left(\tfrac{1}{2}\right)^n - n\left(\tfrac{1}{2}\right)^n\right] u(-n-1).
\end{aligned}$$

(b) Suppose that $R = \left\{\frac{1}{4} < |z| < \frac{1}{2}\right\}$. In this case, we have

$$\left(\tfrac{1}{4}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}} \text{ for } |z| > \tfrac{1}{4},$$
$$-n\left(\tfrac{1}{2}\right)^n u(-n-1) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \text{ for } |z| < \tfrac{1}{2}, \quad \text{and}$$
$$-\left(\tfrac{1}{2}\right)^n u(-n-1) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)^2} \text{ for } |z| < \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -3\left[\left(\tfrac{1}{4}\right)^n u(n)\right] + 4\left[-\left(\tfrac{1}{2}\right)^n u(-n-1)\right] + \left[-n\left(\tfrac{1}{2}\right)^n u(-n-1)\right] \\ &= \left[-3\left(\tfrac{1}{4}\right)^n\right] u(n) + \left[-4\left(\tfrac{1}{2}\right)^n - n\left(\tfrac{1}{2}\right)^n\right] u(-n-1). \end{aligned}$$

(c) Suppose that $R = \left\{|z| > \frac{1}{2}\right\}$. In this case, we have

$$\left(\tfrac{1}{4}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}} \text{ for } |z| > \tfrac{1}{4},$$
$$n\left(\tfrac{1}{2}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \text{ for } |z| > \tfrac{1}{2}, \quad \text{and}$$
$$\left(\tfrac{1}{2}\right)^n u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)^2} \text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -3\left[\left(\tfrac{1}{4}\right)^n u(n)\right] + 4\left[\left(\tfrac{1}{2}\right)^n u(n)\right] + \left[n\left(\tfrac{1}{2}\right)^n u(n)\right] \\ &= \left[-3\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^n\right] u(n). \end{aligned}$$ ■

12.10.2 Laurent-Polynomial and Power-Series Expansions

Another approach to finding inverse z transforms is based on Laurent-polynomial or power series expansions. With this approach, in order to find the inverse z transform x of X, we express X as a (Laurent) polynomial or power series. Then, x can be determined by examining the coefficients of this polynomial or power series. A variety of strategies may be used for expressing X as a polynomial or power series. In some cases, X may be a function with a well known power series, such as a trigonometric, exponential, or logarithmic function. In other cases, X may be a rational function, in which case polynomial long division can be used.

Example 12.28. Find the inverse z transform x of the function

$$X(z) = \frac{z^3+2z^2+4z+8}{z^2} \text{ for } |z| > 0.$$

Solution. We can rewrite X as

$$X(z) = z+2+4z^{-1}+8z^{-2}.$$

From the definition of the z transform, it trivially follows that

$$x(n) = \begin{cases} 1 & n=-1 \\ 2 & n=0 \\ 4 & n=1 \\ 8 & n=2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have that

$$x(n) = \delta(n+1) + 2\delta(n) + 4\delta(n-1) + 8\delta(n-2). \qquad \blacksquare$$

Example 12.29. Find the inverse z transform x of the function

$$X(z) = \cos\left(z^{-1}\right) \quad \text{for } |z| > 0.$$

Solution. To begin, we recall the Maclaurin series for the cos function, which is given by

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all complex } z.$$

(Note that a number of useful series, including the preceding one can be found in Section F.6.) Writing X in terms of a Maclaurin series, we have

$$\begin{aligned} X(z) &= \cos\left(z^{-1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(z^{-1}\right)^{2n}. \end{aligned}$$

(Note that $X(z)$ converges for all nonzero complex z.) Since x must be right sided (due to the ROC of X), we have

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-(2n)} \\ &= \tfrac{1}{0!}z^0 - \tfrac{1}{2!}z^{-2} + \tfrac{1}{4!}z^{-4} - \tfrac{1}{6!}z^{-6} + \tfrac{1}{8!}z^{-8} - \ldots \\ &= \sum_{\substack{n\in\mathbb{Z}: \\ n \geq 0 \text{ and } n \text{ even}}} \frac{j^n}{n!} z^{-n}. \end{aligned}$$

Now, we observe that

$$\tfrac{1}{2}\left[1 + (-1)^n\right] u(n) = \begin{cases} 1 & n \geq 0 \text{ and } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Using this observation, we can rewrite the above expression for $X(z)$ as

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left[\tfrac{1}{2}\left[1 + (-1)^n\right] u(n)\right] \frac{j^n}{n!} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \frac{\left[1 + (-1)^n\right] j^n}{2(n!)} u(n) z^{-n}. \end{aligned}$$

Therefore, we conclude that

$$x(n) = \frac{\left[1 + (-1)^n\right] j^n}{2(n!)} u(n). \qquad \blacksquare$$

Example 12.30. Use polynomial long division to find the inverse z transform x of the function

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{for } |z| > |a|.$$

Solution. The ROC of X corresponds to a right-sided sequence. Therefore, we would like for the polynomial long division process to yield decreasing powers of z. This will naturally result from dividing 1 by $1 - az^{-1}$. Performing the first few steps of polynomial long division, we obtain

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \dots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad\qquad a^2z^{-2} \\
 & \qquad\qquad\qquad \dots
\end{array}
$$

At this point, we observe that the long division process is yielding the pattern

$$
\begin{aligned}
X(z) &= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + a^4z^{-4} + \dots \\
&= \sum_{n=0}^{\infty} a^n z^{-n}.
\end{aligned}
$$

Thus, we have that

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}.$$

So, by inspection, we have that

$$x(n) = a^n u(n). \qquad \blacksquare$$

Example 12.31. Use polynomial long division to find the inverse z transform x of the function

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{for } |z| < |a|.$$

Solution. The ROC of X corresponds to a left-sided sequence. Therefore, we would like for the long division process to yield increasing powers of z. So, we rewrite $X(z)$ as

$$X(z) = \frac{z}{z - a}.$$

Performing the first few steps of polynomial long division, we obtain

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \dots \\
\hline
-a + z & z \\
 & \underline{z \qquad - a^{-1}z^2} \\
 & \qquad\quad a^{-1}z^2 \\
 & \qquad\quad \underline{a^{-1}z^2 - a^{-2}z^3} \\
 & \qquad\qquad\qquad a^{-2}z^3 \\
 & \qquad\qquad\qquad\qquad \dots
\end{array}
$$

At this point, we observe that the long division process is yielding the pattern

$$
\begin{aligned}
X(z) &= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 - a^{-5}z^5 - \dots \\
&= \sum_{n=-\infty}^{-1} -a^n z^{-n}.
\end{aligned}
$$

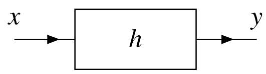

Figure 12.19: Time-domain view of a LTI system with input x, output y, and impulse response h.

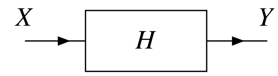

Figure 12.20: z-domain view of a LTI system with input z transform X, output z transform Y, and system function H.

Thus, we have that

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}.$$

So, by inspection, we have that

$$x(n) = -a^n u(-n-1). \qquad \blacksquare$$

12.11 Characterizing LTI Systems Using the z Transform

Consider a LTI system with input x, output y, and impulse response h, as depicted in Figure 12.19. Such a system is characterized by the equation

$$y(n) = x * h(n).$$

Let X, Y, and H denote the z transforms of x, y, and h, respectively. Taking the z transform of both sides of the above equation and using the time-domain convolution property of the z transform, we have

$$Y(z) = H(z)X(z).$$

The quantity H is known as the **system function** or **transfer function** of the system. If the ROC of H contains the unit circle, then $H(e^{j\Omega})$ is the frequency response of the system. The system can be represented with a block diagram labelled in the z domain as shown in Figure 12.20, where the system is labelled by its system function H.

12.12 Interconnection of LTI Systems

From the properties of the z transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 12.21(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1 * h_2(n).$$

Taking the z transform of both sides of this equation yields

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_1 * h_2\}(z) \\ &= \mathcal{Z}h_1(z)\mathcal{Z}h_2(z) \\ &= H_1(z)H_2(z). \end{aligned}$$

Thus, we have the equivalence shown in Figure 12.21.

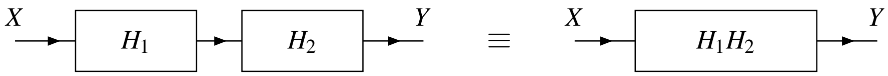

Figure 12.21: Equivalence involving system functions and the series interconnection of LTI systems.

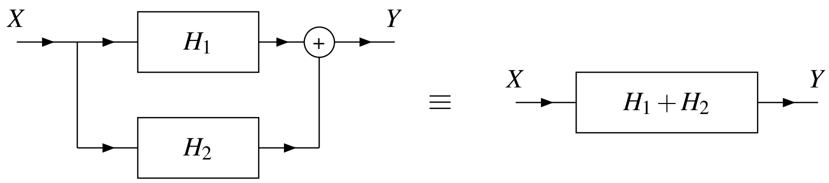

Figure 12.22: Equivalence involving system functions and the parallel interconnection of LTI systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 12.22. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1(n) + h_2(n).$$

Taking the z transform of both sides of the equation yields

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_1 + h_2\}(z) \\ &= \mathcal{Z}h_1(s) + \mathcal{Z}h_2(z) \\ &= H_1(z) + H_2(z). \end{aligned}$$

Thus, we have the equivalence shown in Figure 12.22.

12.13 System Function and System Properties

Many properties of a LTI system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

12.13.1 Causality

From Theorem 9.8, we know that a LTI system is causal if its impulse response is causal. One might wonder, however, how the causality condition manifests itself in the system function of a LTI system. The answer to this question is given by the theorem below.

Theorem 12.14. *A (DT) LTI system with system function H is causal if and only if the ROC of H is:*

1. *the exterior of a circle, including ∞; or*
2. *the entire complex plane, including ∞ and possibly excluding* 0.

Proof. The proof is left as an exercise for the reader. ∎

In the case that the system function is rational, we also have the following result:

Theorem 12.15. *A (DT) LTI system with a rational system function H is causal if and only if:*

1. *the ROC of H is the exterior of a (possibly degenerate) circle outside the outermost (finite) pole of H or, if H has no poles, the entire (finite) complex plane; and*

2. *H is proper (i.e., when H is expressed as a ratio of polynomials in z, the order of the numerator polynomial does not exceed the order of the denominator polynomial).*

Proof. The proof is left as an exercise for the reader. ■

Example 12.32. For the LTI system with each system function H below, determine whether the system is causal.

(a) $H(z) = \dfrac{2z^3}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ for $|z| > \frac{3}{4}$;

(b) $H(z) = \dfrac{10z^2-15z+3}{(z-3)\left(z-\frac{1}{3}\right)}$ for $|z| > 3$;

(c) $H(z) = \dfrac{5z^2-8z+2}{(z-2)\left(z-\frac{1}{2}\right)}$ for $\frac{1}{2} < |z| < 2$; and

(d) $H(z) = \dfrac{1-z^{-9}}{z-1}$ for $|z| > 1$.

Solution. (a) The ROC and poles of H are shown in Figure 12.23(a). The system function H is rational and not proper. Therefore, the ROC cannot contain ∞ and the system is not causal.

In passing, we note that $H(z) = z\left(2 - \frac{1/4}{z-1/4} + \frac{9/4}{z-3/4}\right)$ and $h(n) = 2\delta(n+1) + \frac{1}{4}\left(\frac{1}{4}\right)^n u(n) + \frac{9}{4}\left(\frac{3}{4}\right)^n u(n)$. So, as we would expect, h is not causal.

(b) The ROC and poles of H are shown in Figure 12.23(b). The system function H is rational and proper, and the ROC of H is outside the outermost pole of H (whose magnitude is 3). Therefore, the system is causal.

In passing, we note that $H(z) = z\left(\frac{3}{z} + \frac{1}{z-1/3} + \frac{6}{z-3}\right)$ and $h(n) = 3\delta(n) + \left(\frac{1}{3}\right)^n u(n) + 6(3)^n u(n)$. So, as we would expect, h is causal.

(c) The ROC and poles of H are shown in Figure 12.23(c). The system function H is rational and proper, but the ROC of H is not outside the outermost pole of H (whose magnitude is 2). Therefore, the system is not causal.

In passing, we note that $H(z) = z\left(\frac{2}{z} + \frac{1}{z-1/2} + \frac{2}{z-2}\right)$ and $h(n) = 2\delta(n) + \left(\frac{1}{2}\right)^n u(n) + 2[-2^n u(-n-1)]$. So, as we would expect, h is not causal.

(d) We can rewrite H as

$$H(z) = \frac{z^9-1}{z^9(z-1)}.$$

The ROC and poles of H are shown in Figure 12.23(d). The system function H is rational and proper, and the ROC of H is outside the outermost pole of H (whose magnitude is 1). Therefore, the system is causal.

In passing, we note that $H(z) = z^{-1}\left(\frac{z}{z-1}\right) - z^{-10}\left(\frac{z}{z-1}\right)$ and $h(n) = u(n-1) - u(n-10)$. So, as we would expect, h is causal. ■

12.13.2 BIBO Stability

In this section, we consider the relationship between the system function and BIBO stability. The first important result is given by the theorem below.

Theorem 12.16. *A LTI system is BIBO stable if and only if the ROC of its system function H contains the unit circle (i.e., $|z| = 1$).*

Proof. We present only a partial proof. In particular, we show that the ROC of H containing the unit circle is a necessary condition for BIBO stability. In what follows, let h denote the inverse z transform of H (i.e., h is the impulse response of the system).

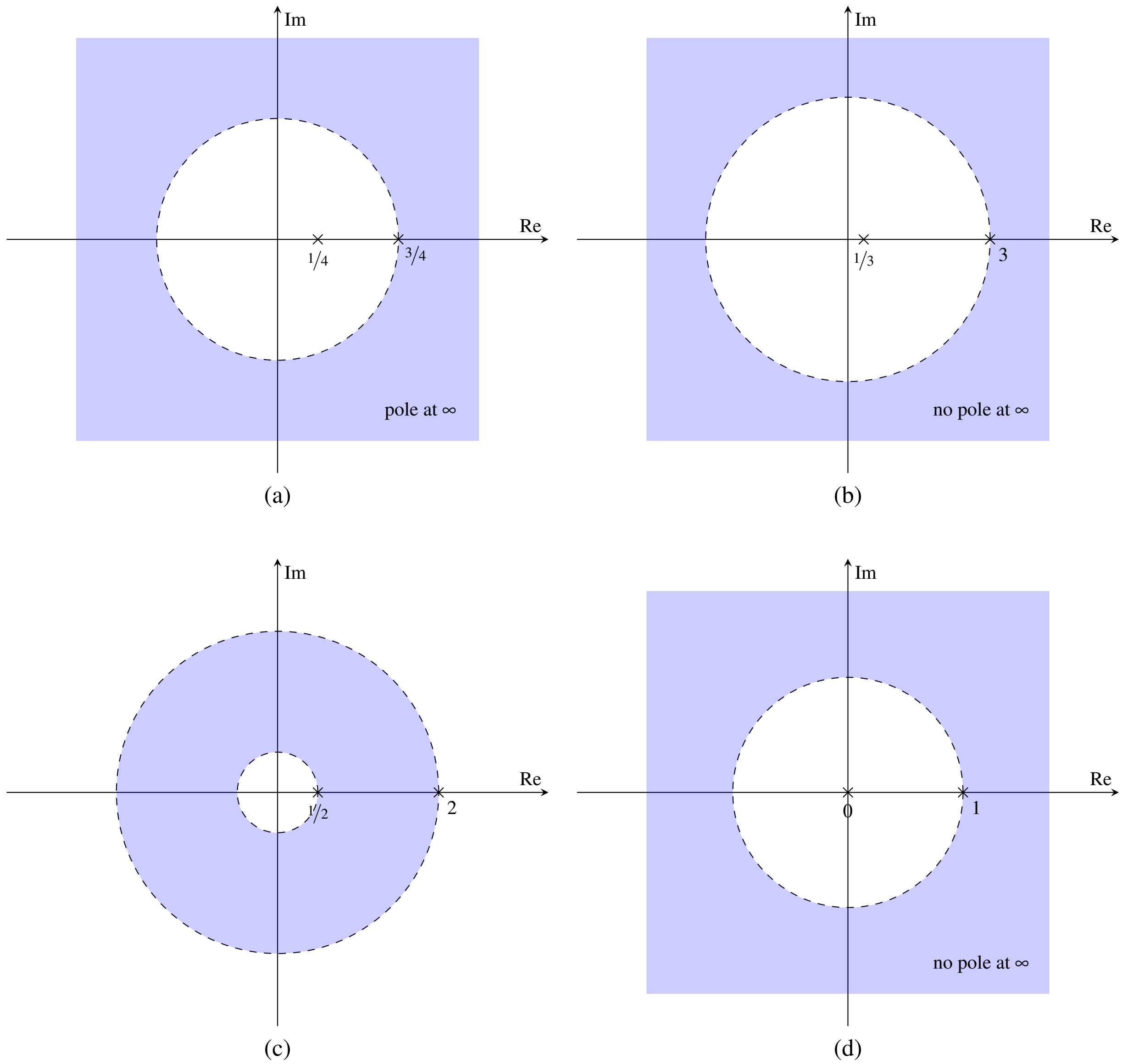

Figure 12.23: Poles and ROCs of the rational system functions in the causality example. The cases of part (a), (b), (c), and (d).

Suppose that the system is BIBO stable. From earlier in Theorem 9.11, we know that this implies

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

(i.e., h is absolutely summable). From the definition of H (evaluated on the unit circle), we have

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}. \tag{12.11}$$

Recall that a sum is convergent if it is absolutely convergent. That is, for any sequence f,

$$\sum_{n=-\infty}^{\infty} f(n) \text{ converges} \quad \text{if} \quad \sum_{n=-\infty}^{\infty} |f(n)| \text{ converges.}$$

From this relationship and (12.11), we can infer that

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n} \text{ converges} \quad \text{if} \quad \sum_{n=-\infty}^{\infty} \left|h(n)e^{-j\Omega n}\right| \text{ converges.} \tag{12.12}$$

We have, however, that

$$\sum_{n=-\infty}^{\infty} \left|h(n)e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} |h(n)| \left|e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} |h(n)|.$$

So, we can rewrite (12.12) as

$$H(e^{j\Omega}) \text{ converges} \quad \text{if} \quad \sum_{n=-\infty}^{\infty} |h(n)| \text{ converges.}$$

The condition for convergence in the preceding statement, however, is always satisfied, since $\sum_{n=-\infty}^{\infty} |h(n)|$ must converge for a BIBO stable system (as noted above). Therefore, we conclude that $H(e^{j\Omega})$ must converge for all Ω (i.e., the ROC of H must contain the unit circle) if the system is BIBO stable. Thus, the ROC of H containing the unit circle is a necessary condition for BIBO stability. ■

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

Theorem 12.17. *A causal LTI system with a (proper) rational system function H is BIBO stable if and only if all of the poles of H are inside the unit circle (i.e., all of the poles have magnitudes less than one).*

Proof. The proof is left as an exercise for the reader. ■

Observe from the preceding two theorems (i.e., Theorems 12.16 and 12.17) that, in the case of a LTI system, the characterization of the BIBO stability property is much simpler in the z domain (via the system function) than the time domain (via the impulse response). For this reason, analyzing the stability of LTI systems is typically performed using the z transform.

Example 12.33. A LTI system has the system function

$$H(z) = \frac{z - \frac{1}{3}}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}.$$

Given that the system is BIBO stable, determine the ROC of H.

Solution. Clearly, the system function H is rational with poles at $\frac{1}{4}$ and $\frac{1}{2}$. Therefore, only three possibilities exist for the ROC:

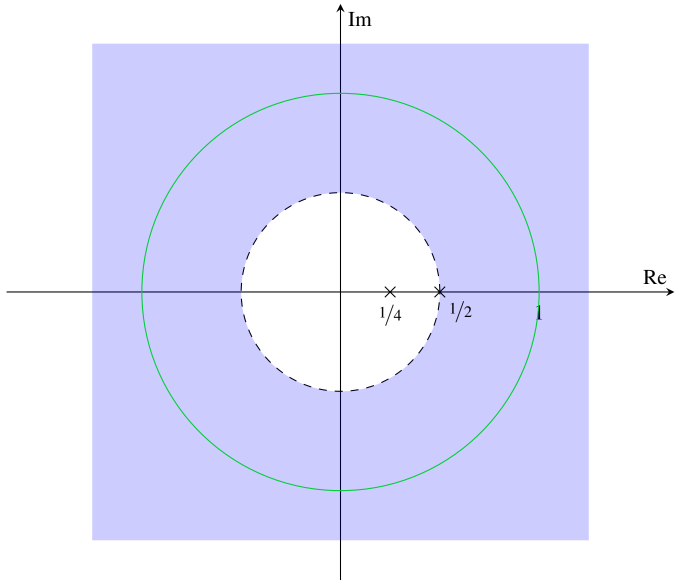

Figure 12.24: ROC for example.

i) $|z| < \frac{1}{4}$,
ii) $\frac{1}{4} < |z| < \frac{1}{2}$, and
iii) $|z| > \frac{1}{2}$.

In order for the system to be BIBO stable, however, the ROC of H must contain the unit circle. Therefore, the ROC must be $|z| > \frac{1}{2}$. This ROC is illustrated in Figure 12.24. ■

Example 12.34. A LTI system is causal and has the system function

$$H(z) = \frac{z^2 - 1}{\left(z^2 - \frac{1}{9}\right)\left(z^2 - \frac{1}{4}\right)}.$$

Determine whether this system is BIBO stable.

Solution. We begin by factoring H to obtain

$$H(z) = \frac{(z+1)(z-1)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

Thus, H has poles at $-\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{2}$. The poles of H are plotted in Figure 12.25. Since the system is causal, the ROC of H must be the exterior of the circle passing through the outermost (finite) pole of H. So, the ROC is $|z| > \frac{1}{2}$. This ROC is shown as the shaded region in Figure 12.25. Since this ROC contains the unit circle, the system is BIBO stable. ■

Example 12.35. For the LTI system with each system function H given below, determine the ROC of H that corresponds to the system being BIBO stable.

(a) $H(z) = \dfrac{z-1}{z^2 - \frac{1}{4}}$;

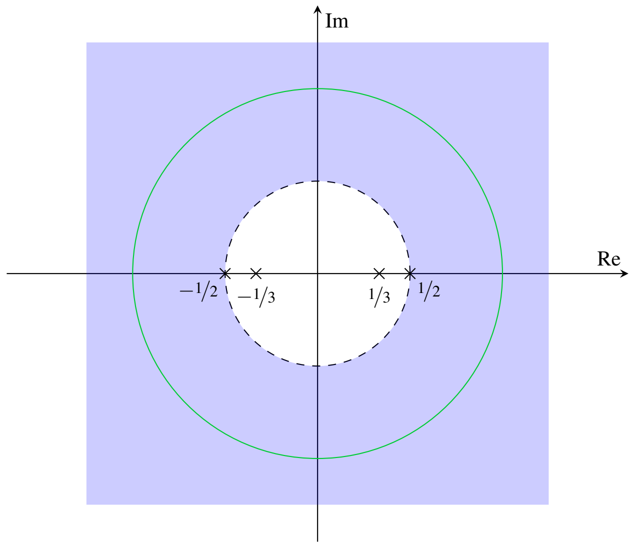

Figure 12.25: The poles and ROC of the system function.

(b) $H(z) = \dfrac{1}{\left(z-\frac{1}{2}e^{j\pi/4}\right)\left(z-\frac{1}{2}e^{-j\pi/4}\right)\left(z-\frac{3}{2}e^{j3\pi/4}\right)\left(z-\frac{3}{2}e^{-j3\pi/4}\right)}$;

(c) $H(z) = \dfrac{z-1}{(z^2+4)(z^2-4)}$; and

(d) $H(z) = \dfrac{z}{z-1}$.

Solution. (a) We can rewrite H as

$$H(z) = \frac{z-1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

So, H has poles at $-\frac{1}{2}$ and $\frac{1}{2}$. The poles are shown in Figure 12.26(a). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles both have the same magnitude (i.e., $\frac{1}{2}$), only two ROCs are possible:

i) $|z| < \frac{1}{2}$ and
ii) $|z| > \frac{1}{2}$.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $|z| > \frac{1}{2}$. This ROC is shown as the shaded region in Figure 12.26(a).

(b) The function H has poles at $\frac{1}{2}e^{-j\pi/4}$, $\frac{1}{2}e^{j\pi/4}$, $\frac{3}{2}e^{-j3\pi/4}$, and $\frac{3}{2}e^{j3\pi/4}$. The poles are shown in Figure 12.26(b). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles have only two distinct magnitudes (i.e., $\frac{1}{2}$ and $\frac{3}{2}$), three distinct ROCs are possible:

i) $|z| < \frac{1}{2}$,
ii) $\frac{1}{2} < |z| < \frac{3}{2}$, and
iii) $|z| > \frac{3}{2}$.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $\frac{1}{2} < |z| < \frac{3}{2}$. This ROC is shown as the shaded region in Figure 12.26(b).

(c) Writing H with its denominator fully factored, we have

$$H(z) = \frac{z-1}{(z+2)(z-2)(z+2j)(z-2j)}.$$

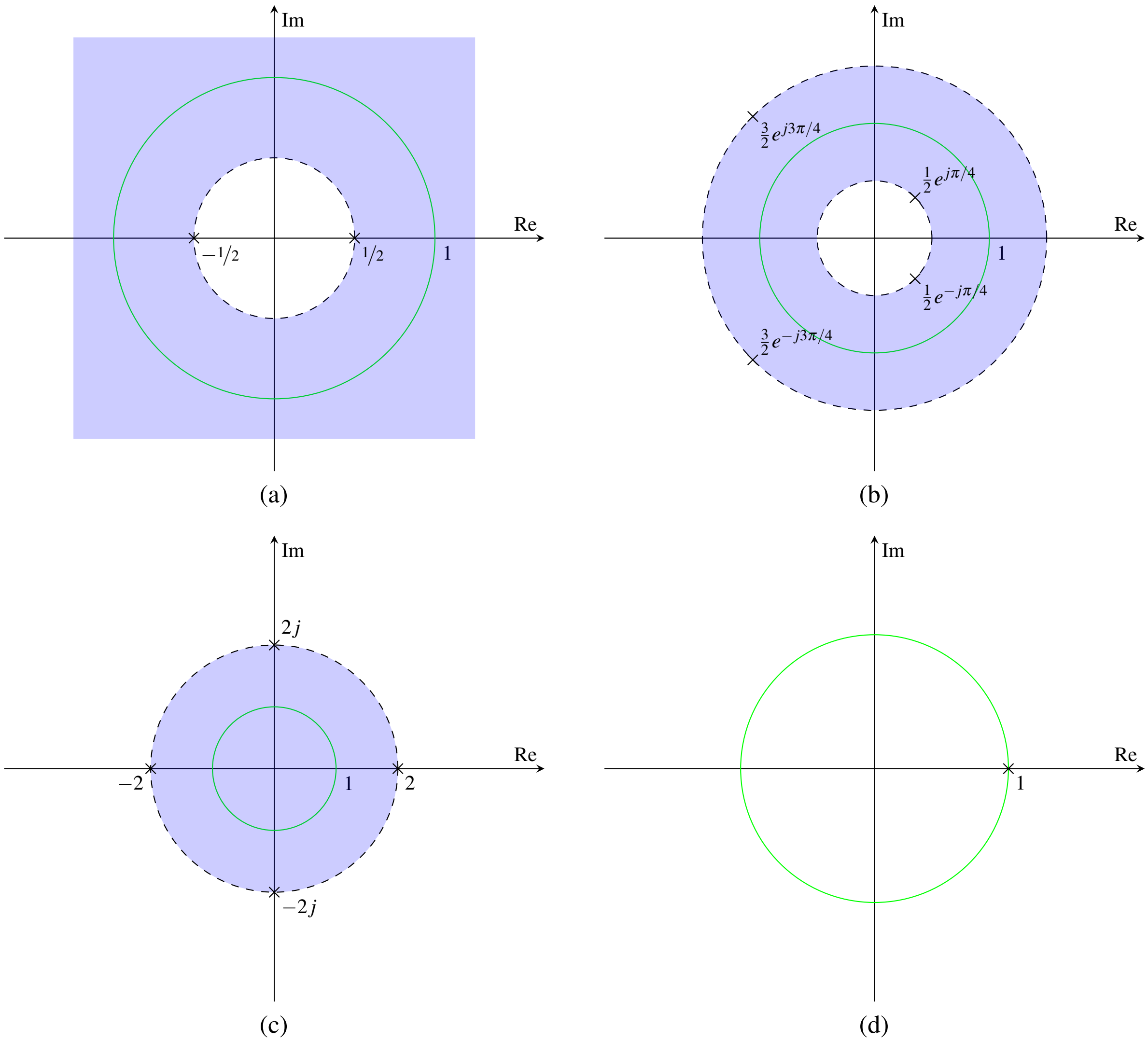

Figure 12.26: Poles and ROCs of the system function H in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

The function H has poles at -2, 2, $-2j$, and $2j$. The poles are shown in Figure 12.26(c). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles of H have the same magnitude (i.e., 2), two distinct ROCs are possible:

i) $|z| < 2$ and
ii) $|z| > 2$.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $|z| < 2$. This ROC is shown as the shaded region in Figure 12.26(c).

(d) The function H has a pole at 1. The pole is shown in Figure 12.26(d). Since H is rational, it cannot converge at 1 (which is a pole of H). Consequently, the ROC can never contain the unit circle. Therefore, the system function H can never be associated with a BIBO stable system. ■

12.13.3 Invertibility

In this section, we consider the relationship between the system function and invertibility. The first important result is given by the theorem below.

Theorem 12.18 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with system function H. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has a system function H_{inv} that satisfies*

$$H(z)H_{\mathsf{inv}}(z) = 1. \tag{12.13}$$

Proof. Let h denote the inverse z transform of H. From Theorem 9.9, we know that the system $\mathcal{H}$ is invertible if and only if there exists another LTI system with impulse response h_{inv} satisfying

$$h * h_{\mathsf{inv}} = \delta.$$

Let H_{inv} denote the z transform of h_{inv}. Taking the z transform of both sides of the above equation, we have

$$\mathcal{Z}\{h * h_{\mathsf{inv}}\} = \mathcal{Z}\delta.$$

From the time-domain convolution property of the z transform and Table 12.3 (i.e., $\mathcal{Z}\delta(z) = 1$), we have

$$H(z)H_{\mathsf{inv}}(z) = 1. \qquad \blacksquare$$

From the preceding theorem, we have the result below.

Theorem 12.19 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists a function H_{inv} satisfying*

$$H(z)H_{\mathsf{inv}}(z) = 1.$$

Proof. The proof follows immediately from the result of Theorem 12.18 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. $\blacksquare$

From the above theorems, we have that a LTI system $\mathcal{H}$ with system function H has an inverse if and only if a solution for H^{inv} exists in (12.13). Furthermore, if an inverse system exists, its system function is given by

$$H_{\mathsf{inv}}(z) = \frac{1}{H(z)}.$$

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a BIBO stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of BIBO stability and/or causality).

Example 12.36. Consider the LTI system with system function

$$H(z) = \frac{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{3}\right)} \quad \text{for } |z| > \tfrac{1}{3}.$$

Determine all possible inverses of this system. Comment on the BIBO stability of each of these inverse systems.

Solution. The system function H_{inv} of the inverse system is given by

$$H_{\mathsf{inv}}(z) = \frac{1}{H(z)} = \frac{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{3}\right)}{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{2}\right)}.$$

Since the poles of H_{inv} have two distinct magnitudes (i.e., $\frac{1}{5}$ and $\frac{1}{2}$), three ROCs are possible for H_{inv}:

i) $|z| < \frac{1}{5}$,
ii) $\frac{1}{5} < |z| < \frac{1}{2}$, and
iii) $|z| > \frac{1}{2}$.

Each ROC is associated with a distinct inverse system. The first ROC is associated with a system that is not BIBO stable, since this ROC does not contain the unit circle. The second ROC is associated with a system that is not BIBO stable, since this ROC does not contain the unit circle. The third ROC is associated with a BIBO stable system, since this ROC contains the unit circle. ■

12.14 LTI Systems and Difference Equations

Many LTI systems of practical interest can be described by Nth-order linear difference equations with constant coefficients. Such a system with input x and output y can be characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k), \tag{12.14}$$

where $M \leq N$. Let X and Y denote the z transforms of x and y, respectively. Let H denote the system function of the system. Taking the z transform of both sides of the above equation, we obtain

$$\mathcal{Z}\left\{\sum_{k=0}^{N} b_k y(n-k)\right\}(z) = \mathcal{Z}\left\{\sum_{k=0}^{M} a_k x(n-k)\right\}(z).$$

Using the linearity property of the z transform, we can rewrite this equation as

$$\sum_{k=0}^{N} b_k \mathcal{Z}\{y(n-k)\}(z) = \sum_{k=0}^{M} a_k \mathcal{Z}\{x(n-k)\}(z).$$

Using the time-shifting property of the z transform, we have

$$\sum_{k=0}^{N} b_k z^{-k} Y(z) = \sum_{k=0}^{M} a_k z^{-k} X(z).$$

Factoring, we have

$$Y(z) \sum_{k=0}^{N} b_k z^{-k} = X(z) \sum_{k=0}^{M} a_k z^{-k}.$$

Dividing both sides of this equation by $\sum_{k=0}^{N} b_k z^{-k}$ and $X(z)$, we obtain

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

Since $H(z) = \frac{Y(z)}{X(z)}$, we have that H is given by

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (12.14)), the system function is always rational. It is for this reason that rational functions are of particular interest.

Example 12.37 (Difference equation to system function)**.** A causal LTI system with input x and output y is characterized by the difference equation

$$y(n) - ay(n-1) = bx(n),$$

where a and b are real constants and $a \neq 0$. Find the system function H of this system.

Solution. Taking the z transform of the given difference equation, we obtain

$$Y(z) - az^{-1}Y(z) = bX(z).$$

Factoring, we have

$$\left(1 - az^{-1}\right)Y(z) = bX(z).$$

Dividing both sides by $1 - az^{-1}$ and $X(z)$, we obtain

$$\frac{Y(z)}{X(z)} = \frac{b}{1 - az^{-1}}.$$

Thus, H is given by

$$H(z) = \frac{b}{1 - az^{-1}} = \frac{bz}{z - a}.$$

Since the system is causal, the ROC of H must be outside the outermost pole at a. Therefore, we conclude

$$H(z) = \frac{bz}{z-a} \text{ for } |z| > |a|.$$

■

Example 12.38 (System function to difference equation)**.** A causal LTI system with input x and output y has the system function

$$H(z) = \frac{b_0 z^2}{z^2 + a_1 z + a_2},$$

where a_1, a_2, and b_0 are real constants and $a_2 \neq 0$. Find the difference equation that characterizes this system.

Solution. Let X and Y denote the z transforms of x and y, respectively. Since $H(z) = \frac{Y(z)}{X(z)}$, we have

$$\frac{Y(z)}{X(z)} = \frac{b_0 z^2}{z^2 + a_1 z + a_2}.$$

Multiplying both sides of this equation by $z^2 + a_1 z + a_2$ and $X(z)$, we obtain

$$z^2 Y(z) + a_1 z Y(z) + a_2 Y(z) = b_0 z^2 X(z).$$

Multiplying both sides of this equation by z^{-2} in order to ensure that the largest power of z is zero, we have

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 X(z).$$

Taking the inverse z transform of both sides of this equation (by using the linearity and translation properties of the z transform), we have

$$\begin{aligned} & \mathcal{Z}^{-1}\{Y(z)\}(n) + a_1 \mathcal{Z}^{-1}\{z^{-1}Y(z)\}(n) + a_2 \mathcal{Z}^{-1}\{z^{-2}Y(z)\}(n) = b_0 \mathcal{Z}^{-1}\{X(z)\}(n) \\ \Rightarrow \quad & y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n). \end{aligned}$$

Therefore, the system is characterized by the difference equation

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n).$$

■

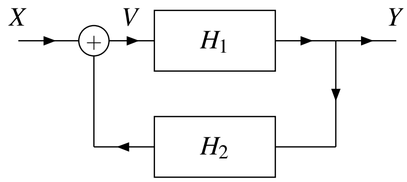

Figure 12.27: Feedback system.

12.15 Stability Analysis

As mentioned earlier, since BIBO stability is more easily characterized for LTI systems in the z domain than the time domain, the z domain is often used to analyze system stability. In what follows, we will consider this application of the z transform in more detail.

Example 12.39 (Feedback system)**.** Consider the system shown in Figure 12.27 that has input z transform X and output z transform Y, and is formed by the interconnection of two causal LTI systems labelled with their system functions H_1 and H_2. The system functions H_1 and H_2 are given by

$$H_1(z) = \frac{10\beta z}{z-1} \quad \text{and} \quad H_2(z) = 1,$$

where β is a real constant. (a) Find the system function H of the (overall) system. (b) Determine the values of the parameter β for which the system is BIBO stable.

Solution. (a) From the system diagram, we can write

$$\begin{aligned} V(z) &= X(z) + H_2(z)Y(z) \quad \text{and} \\ Y(z) &= H_1(z)V(z). \end{aligned}$$

Combining these two equations and simplifying, we obtain

$$\begin{aligned} & Y(z) = H_1(z)[X(z) + H_2(z)Y(z)] \\ \Rightarrow\ & Y(z) = H_1(z)X(z) + H_1(z)H_2(z)Y(z) \\ \Rightarrow\ & Y(z)[1 - H_1(z)H_2(z)] = H_1(z)X(z) \\ \Rightarrow\ & \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 - H_1(z)H_2(z)}. \end{aligned}$$

Since $H(z) = \frac{Y(z)}{X(z)}$, we have

$$H(z) = \frac{H_1(z)}{1 - H_1(z)H_2(z)}.$$

Substituting the given expressions for H_1 and H_2, and simplifying, we can write

$$\begin{aligned}
H(z) &= \frac{\frac{10\beta z}{z-1}}{1-\frac{10\beta z}{z-1}(1)} \\
&= \frac{\left(\frac{10\beta z}{z-1}\right)}{\left(\frac{z-1-10\beta z}{z-1}\right)} \\
&= \frac{10\beta z}{z-1-10\beta z} \\
&= \frac{10\beta z}{(1-10\beta)z-1} \\
&= \frac{10\beta}{1-10\beta}\left(\frac{z}{z-\frac{1}{1-10\beta}}\right).
\end{aligned}$$

(b) In order to assess the BIBO stability of the system, we need to consider the poles of the system function H. From the expression for H above, we can see that H has a single pole at $\frac{1}{1-10\beta}$. Since the system is causal, the system is BIBO stable if and only if all of the poles are strictly inside the unit circle. Thus, we have that

$$\begin{aligned}
& \left|\tfrac{1}{1-10\beta}\right| < 1 \\
\Rightarrow\quad & \tfrac{1}{|1-10\beta|} < 1 \\
\Rightarrow\quad & |1-10\beta| > 1 \\
\Rightarrow\quad & 1-10\beta > 1 \text{ or } 1-10\beta < -1 \\
\Rightarrow\quad & 10\beta < 0 \text{ or } 10\beta > 2 \\
\Rightarrow\quad & \beta < 0 \text{ or } \beta > \tfrac{1}{5}
\end{aligned}$$

Therefore, the system is BIBO stable if and only if

$$\beta < 0 \text{ or } \beta > \tfrac{1}{5}. \qquad \blacksquare$$

12.16 Unilateral z Transform

As mentioned earlier, two different versions of the z transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral z transform. Now, we turn our attention to the unilateral z transform. The **unilateral z transform** of the sequence x is denoted as $\mathcal{Z}_u x$ or X and is defined as

$$\mathcal{Z}_u x(z) = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}. \tag{12.15}$$

The **inverse unilateral z transform** has the same definition as in the case of the bilateral transform, namely (12.3).

Comparing the definitions of the unilateral and bilateral z transforms given by (12.15) and (12.2), respectively, we can see that these definitions only differ in the lower limit of summation. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral z transform of the sequence xu for an arbitrary sequence x. We have

$$\begin{aligned}\mathcal{Z}\{xu\}(z) &= \sum_{n=-\infty}^{\infty} x(n)u(n)z^{-n} \\ &= \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= \mathcal{Z}_{\mathrm{u}}x(z).\end{aligned}$$

In other words, the unilateral z transform of the sequence x is simply the bilateral z transform of the sequence xu. Since $\mathcal{Z}_{\mathrm{u}}x = \mathcal{Z}\{xu\}$ and xu is always right sided, the ROC associated with $\mathcal{Z}_{\mathrm{u}}x$ is always the one corresponding to a right-sided sequence (i.e., the exterior of a circle or the entire complex plane). For this reason, we often do not explicitly indicate the ROC when working with the unilateral z transform.

From earlier in this chapter, we know that the bilateral z transform is invertible. That is, if the sequence x has the bilateral z transform $X = \mathcal{Z}x$, then $\mathcal{Z}^{-1}X = x$. Now, let us consider the invertibility of the unilateral z transform. To do this, we must consider the quantity $\mathcal{Z}_{\mathrm{u}}^{-1}\mathcal{Z}_{\mathrm{u}}x$. Since $\mathcal{Z}_{\mathrm{u}}x = \mathcal{Z}\{xu\}$ and the inverse equations for the unilateral and bilateral z transforms are identical, we can write

$$\begin{aligned}\mathcal{Z}_{\mathrm{u}}^{-1}\mathcal{Z}_{\mathrm{u}}x(n) &= \mathcal{Z}_{\mathrm{u}}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= \mathcal{Z}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\ &= x(n)u(n) \\ &= \begin{cases} x(n) & n \geq 0 \\ 0 & n < 0. \end{cases}\end{aligned}$$

Thus, we have that $\mathcal{Z}_{\mathrm{u}}^{-1}\mathcal{Z}_{\mathrm{u}}x = x$ only if x is causal. In other words, the unilateral z transform is invertible only for causal sequences. For noncausal sequences, we can only recover $x(n)$ for $n \geq 0$. In essence, the unilateral z transform discards all information about the value of the sequence x at n for $n < 0$. Since this information is discarded, it cannot be recovered by an inverse unilateral z transform operation.

Due to the close relationship between the unilateral and bilateral z transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral z transform are summarized in Table 12.4.

By comparing the properties of the unilateral and bilateral z transforms listed in Tables 12.4 and 12.2, respectively, we can see that the unilateral z transform has some of the same properties as its bilateral counterpart, namely, the linearity, modulation, conjugation, upsampling, downsampling, and z-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral z transform.

Since the unilateral and bilateral z transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral z transform properties listed in Table 12.2 with the unilateral z transform properties listed in Table 12.4. In the unilateral case, we can see that:

1. the translation property has been replaced by the time-delay and time-advance properties;

2. the time-reversal property has been dropped;

3. the convolution property has the additional requirement that the sequences being convolved must be causal;

4. the differencing property has an extra term in the expression for $\mathcal{Z}_{\mathrm{u}}\{x(n) - x(n-1)\}$; and

5. the accumulation property has a different lower limit in the summation (namely, 0 instead of $-\infty$).

Since $\mathcal{Z}_{\mathrm{u}}x = \mathcal{Z}\{xu\}$, we can easily generate a table of unilateral z transform pairs from a table of bilateral transform pairs. Using the bilateral z transform pairs from Table 12.3, and the preceding relationship between the unilateral and bilateral z transforms, we can trivially deduce the unilateral z transform pairs in Table 12.5. Since, in the unilateral case, the ROC is always the exterior of a circle (or the entire complex plane), we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be the exterior of a circle (or the entire complex plane).

The inverse unilateral z transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to correspond to a right-sided sequence.

Table 12.4: Properties of the unilateral z transform

Property	Time Domain	z Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$
Time Delay	$x(n-1)$	$z^{-1}X(z)+x(-1)$
Time Advance	$x(n+1)$	$zX(z)-zx(0)$
Complex Modulation	$a^nx(n)$	$X(z/a)$
Conjugation	$x^*(n)$	$X^*(z^*)$
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$
Convolution	$x_1 * x_2(n), x_1$ and x_2 are causal	$X_1(z)X_2(z)$
Z-Domain Differentiation	$nx(n)$	$-z\frac{d}{dz}X(z)$
Differencing	$x(n)-x(n-1)$	$\frac{z-1}{z}X(z)-x(-1)=(1-z^{-1})X(z)-x(-1)$
Accumulation	$\sum_{k=0}^{n} x(k)$	$\frac{z}{z-1}X(z)=\frac{1}{1-z^{-1}}X(z)$

Property	
Initial Value Theorem	$x(0)=\lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n)=\lim_{z\to 1}[(z-1)X(z)]$

Table 12.5: Transform pairs for the unilateral z transform

Pair	$x(n),\ n \geq 0$	$X(z)$
1	$\delta(n)$	1
2	1	$\dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$
3	n	$\dfrac{z}{(z-1)^2} = \dfrac{z^{-1}}{(1-z^{-1})^2}$
4	a^n	$\dfrac{z}{z-a} = \dfrac{1}{1-az^{-1}}$
5	$a^n n$	$\dfrac{az}{(z-a)^2} = \dfrac{az^{-1}}{(1-az^{-1})^2}$
6	$\cos(\Omega_0 n)$	$\dfrac{z(z-\cos\Omega_0)}{z^2-(2\cos\Omega_0)z+1} = \dfrac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$
7	$\sin(\Omega_0 n)$	$\dfrac{z\sin\Omega_0}{z^2-(2\cos\Omega_0)z+1} = \dfrac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$
8	$a^n\cos(\Omega_0 n)$	$\dfrac{z(z-a\cos\Omega_0)}{z^2-(2a\cos\Omega_0)z+a^2} = \dfrac{1-(a\cos\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$
9	$a^n\sin(\Omega_0 n)$	$\dfrac{za\sin\Omega_0}{z^2-(2a\cos\Omega_0)z+a^2} = \dfrac{(a\sin\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$

12.17 Solving Difference Equations Using the Unilateral z Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear difference equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the system will be linear only if the initial conditions for the difference equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral z transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral z transform is in solving constant-coefficient linear difference equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral z transform to this end.

Example 12.40 (Unilateral z transform of two-unit delay)**.** Let x and y be sequences related by

$$y(n) = x(n-2).$$

Find the unilateral z transform Y of y in terms of the unilateral z transform X of x.

Solution. Define the sequence

$$v(n) = x(n-1) \tag{12.16}$$

so that

$$y(n) = v(n-1). \tag{12.17}$$

Let V denote the unilateral z transform of v. Taking the unilateral z transform of (12.16) (using the delay property), we have

$$V(z) = z^{-1}X(z) + x(-1). \tag{12.18}$$

Taking the unilateral z transform of (12.17) (using the delay property), we have

$$Y(z) = z^{-1}V(z) + v(-1). \tag{12.19}$$

Substituting the expression for V in (12.18) into (12.19) and using the fact that $v(-1) = x((-1)-1) = x(-2)$, we have

$$\begin{aligned} Y(z) &= z^{-1}V(z) + v(-1) \\ &= z^{-1}\left[z^{-1}X(z) + x(-1)\right] + x(-2) \\ &= z^{-2}X(z) + z^{-1}x(-1) + x(-2). \end{aligned}$$

Thus, we have that

$$Y(z) = z^{-2}X(z) + z^{-1}x(-1) + x(-2).$$

■

Example 12.41 (Investment earning compound interest)**.** An investment earns compound interest at a fixed rate of r percent annually with the fraction of a year that constitutes the compounding period being α. (For example, $\alpha = \frac{1}{12}$ corresponds to monthly compounding.) Let y be a sequence such that $y(n)$ is the value of the investment at the start of the nth compounding period. Then, y satisfies the difference equation

$$y(n) = \left(1 + \tfrac{\alpha r}{100}\right) y(n-1).$$

Consider an investment with an initial value of \$1000 that earns interest at a fixed rate of 6% annually compounded monthly. Use the unilateral z transform to find the value of the investment after 10 years.

Solution. Since $r = 6$ and $\alpha = \frac{1}{12}$, y is the solution to the difference equation

$$\begin{aligned} y(n) &= \left(1 + \tfrac{6}{12(100)}\right) y(n-1) \\ &= \tfrac{201}{200} y(n-1), \end{aligned}$$

subject to the initial condition $y(-1) = 1000$. Taking the unilateral z transform of this equation (with the help of the delay property) and solving for Y, we obtain

$$\begin{aligned} & Y(z) = \tfrac{201}{200}\left[z^{-1}Y(z) + y(-1)\right] \\ \Rightarrow \quad & Y(z) = \tfrac{201}{200} z^{-1}Y(z) + \tfrac{201}{200} y(-1) \\ \Rightarrow \quad & Y(z) = \tfrac{201}{200} z^{-1}Y(z) + 1005 \\ \Rightarrow \quad & \left[1 - \tfrac{201}{200} z^{-1}\right] Y(z) = 1005 \\ \Rightarrow \quad & Y(z) = \frac{1005}{1 - \frac{201}{200} z^{-1}}. \end{aligned}$$

Taking the inverse z transform of y, we obtain

$$y(n) = 1005 \left(\tfrac{201}{200}\right)^n \quad \text{for } n \geq 0.$$

The value of n corresponding to 10 years is given by

$$n = -1 + 12(10) = 119.$$

Thus, the value of the investment after 10 years is

$$y(119) = 1005 \left(\tfrac{201}{200}\right)^{119} \approx 1819.39.$$

■