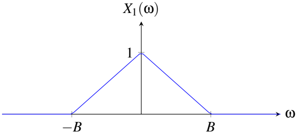
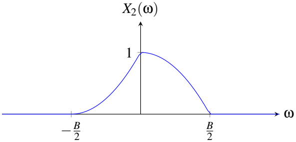
Bandwidth

- A function with the Fourier transform X is said to be **bandlimited** if, for some (finite) nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- The **bandwidth** B of a function with the Fourier transform X is defined as $B = \omega_1 - \omega_0$, where $X(\omega) = 0$ for all $\omega \notin [\omega_0, \omega_1]$.
- In the case of ***real-valued*** functions, however, this definition of bandwidth is usually amended to consider ***only nonnegative*** frequencies.
- The real-valued function x_1 and complex-valued function x_2 with the respective Fourier transforms X_1 and X_2 shown below each have bandwidth B (where only nonnegative frequencies are considered in the case of x_1).





- One can show that a function ***cannot be both time limited and bandlimited***.

Energy-Density Spectra

SKIP SLIDE

- By Parseval's relation, the energy E in a function x with Fourier transform X is given by

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_x(\omega)d\omega,$$

where

$$E_x(\omega) = |X(\omega)|^2.$$

- We refer to E_x as the **energy-density spectrum** of the function x.
- The function E_x indicates how the energy in x is distributed with respect to frequency.
- For example, the energy contributed by frequencies in the range $[\omega_1, \omega_2]$ is given by

$$\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} E_x(\omega)d\omega.$$

Section 6.6

Fourier Transform and LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is ***completely characterized*** by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

Frequency Response of LTI Systems (Continued 1)

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\omega)$ is called the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

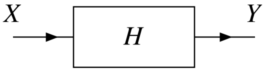
- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is ***real***, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is ***even*** and the phase response $\arg H(\omega)$ is ***odd***).

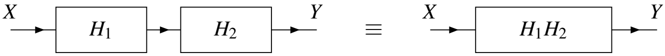
Block Diagram Representations of LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.

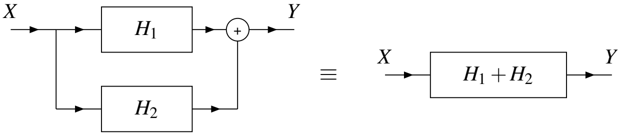


- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

- The ***series*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with frequency response $H_1 H_2$. That is, we have the equivalence shown below.



- The ***parallel*** interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with the frequency response $H_1 + H_2$. That is, we have the equivalence shown below.



LTI Systems and Differential Equations

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

  where the a_k and b_k are complex constants and $M \leq N$.
- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a *rational function*.

Example 6.34 Example 6.35

Section 6.7

Application: Filtering

Filtering

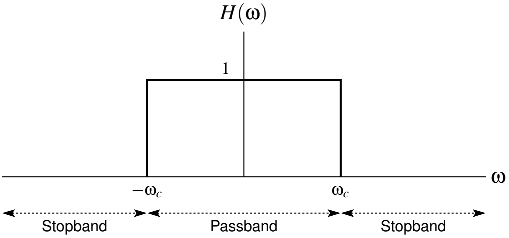
- In many applications, we want to ***modify the spectrum*** of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

Ideal Lowpass Filter

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

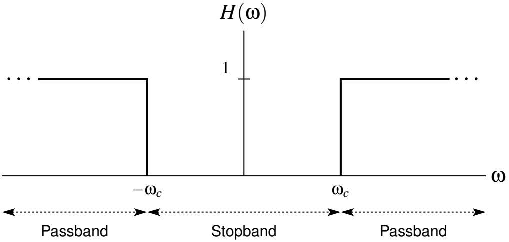
  where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.



- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

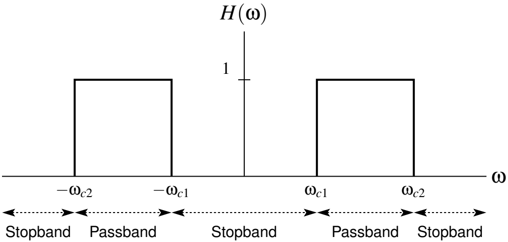


Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



Example 6.38

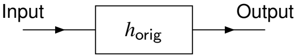
Section 6.8

Application: Equalization

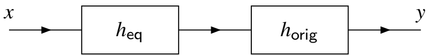
Equalization

- Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand.
- As a result, we would like to change the frequency response of the system to be something more desirable.
- This process of modifying the frequency response in this way is referred to as **equalization**. [Essentially, equalization is just a filtering operation.]
- Equalization is used in many applications.
- In real-world ***communication systems***, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel.
- In ***audio applications***, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, equalization can be used to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Equalization (Continued)



Original System



New System with Equalization

- Let H_{orig} denote the frequency response of ***original*** system (i.e., without equalization).
- Let H_{d} denote the ***desired*** frequency response.
- Let H_{eq} denote the frequency response of the ***equalizer***.
- The new system with equalization has frequency response

$$H_{\text{new}}(\omega) = H_{\text{eq}}(\omega)H_{\text{orig}}(\omega).$$

- By choosing $H_{\text{eq}}(\omega) = H_{\text{d}}(\omega)/H_{\text{orig}}(\omega)$, the new system with equalization will have the frequency response

$$H_{\text{new}}(\omega) = [H_{\text{d}}(\omega)/H_{\text{orig}}(\omega)]\, H_{\text{orig}}(\omega) = H_{\text{d}}(\omega).$$

- In effect, by using an equalizer, we can obtain a new system with the frequency response that we desire.

Example 6.39

Section 6.9

Application: Circuit Analysis

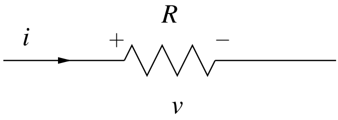
- An **electronic circuit** is a network of one or more interconnected circuit elements.
- The three most basic types of circuit elements are:
  1. resistors;
  2. inductors; and
  3. capacitors.
- Two fundamental quantities of interest in electronic circuits are current and voltage.
- **Current** is the rate at which electric charge flows through some part of a circuit, such as a circuit element, and is measured in units of amperes (A).
- **Voltage** is the difference in electric potential between two points in a circuit, such as across a circuit element, and is measured in units of volts (V).
- Voltage is essentially a force that makes electric charge (or current) flow.

Resistors

- A **resistor** is a circuit element that opposes the flow of current.
- A resistor is characterized by an equation of the form

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

  where R is a nonnegative real constant, and v and i respectively denote the voltage across and current through the resistor as a function of time.
- As a matter of terminology, the quantity R is known as the **resistance** of the resistor.
- Resistance is measured in units of ohms (Ω).
- In circuit diagrams, a resistor is denoted by the symbol shown below.

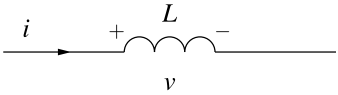


Inductors

- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor uses the energy stored in a magnetic field in order to ***oppose changes in current*** (through the inductor).
- An inductor is characterized by an equation of the form

$$v(t) = L\tfrac{d}{dt}i(t) \quad \text{(or equivalently, } i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\text{)},$$

  where L is a nonnegative real constant, and v and i respectively denote the voltage across and current through the inductor as a function of time.
- As a matter of terminology, the quantity L is known as the **inductance** of the inductor.
- Inductance is measured in units of henrys (H).
- In circuit diagrams, an inductor is denoted by the symbol shown below.

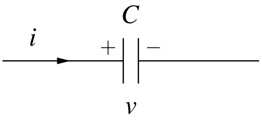


Capacitors

- A **capacitor** is a circuit element that stores electric charge.
- A capacitor uses the energy stored in an electric field in order to ***oppose changes in voltage*** (across the capacitor).
- A capacitor is characterized by an equation of the form

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \text{(or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\text{)},$$

  where C is a nonnegative real constant, and v and i respectively denote the voltage across and current through the capacitor as a function of time.
- As a matter of terminology, the quantity C is known as the **capacitance** of the capacitor.
- Capacitance is measured in units of farads (F).
- In circuit diagrams, a capacitor is denoted by the symbol shown below.



Circuit Analysis with the Fourier Transform

- The Fourier transform is a very useful tool for circuit analysis.
- The utility of the Fourier transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.
- Let v and i denote the voltage across and current through a circuit element, and let V and I denote the Fourier transforms of v and i, respectively.
- In the frequency domain, the equations characterizing a resistor, an inductor, and a capacitor respectively become:

$$V(\omega) = RI(\omega) \quad \text{(or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega)\text{);}$$
$$V(\omega) = j\omega LI(\omega) \quad \text{(or equivalently, } I(\omega) = \tfrac{1}{j\omega L}V(\omega)\text{);} \quad \text{and}$$
$$V(\omega) = \tfrac{1}{j\omega C}I(\omega) \quad \text{(or equivalently, } I(\omega) = j\omega CV(\omega)\text{).}$$

- Note the absence of differentiation and integration in the above equations for an inductor and a capacitor.

Section 6.10

Application: Amplitude Modulation (AM)

Motivation for Amplitude Modulation (AM)

- In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal.
- For example, voice/audio signals typically have information in the range of 0 to 22 kHz.
- Often, it is not practical to transmit such a signal using its original frequency range.
- Two potential problems with such an approach are:
  1. interference; and
  2. constraints on antenna length.
- Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference.
- Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals.
- For the preceding reasons, we often need to change the frequency range associated with a signal before transmission.