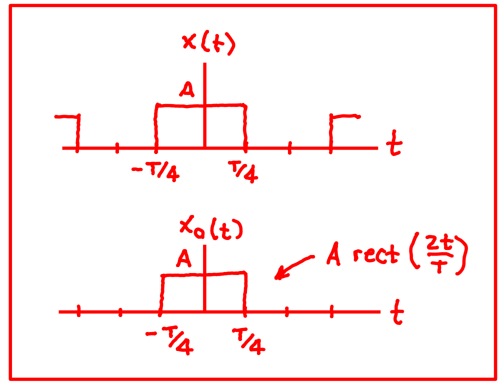


Example 6.24. Consider the periodic function x given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t - kT),$$

where a single period of x is given by

$$x_0(t) = A\,\mathrm{rect}\left(\tfrac{2t}{T}\right)$$

and A is a real constant. Find the Fourier transform X of the function x.

Solution. From (6.16b), we know that

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0)$$

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t - kT)\right\}(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\delta(\omega - k\omega_0). \end{aligned}$$

using (6.16)

table of FT pairs

So, we need to find X_0. Using the linearity property of the Fourier transform and Table 6.2, we have

$$\begin{aligned} X_0(\omega) &= \mathcal{F}\left\{A\,\mathrm{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= A\mathcal{F}\left\{\mathrm{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= \tfrac{AT}{2}\,\mathrm{sinc}\left(\tfrac{\omega T}{4}\right). \end{aligned}$$

from definition of x

linearity

FT table

Thus, we have that

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \left(\tfrac{AT}{2}\right) \mathrm{sinc}\left(\tfrac{k\omega_0 T}{4}\right) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A\,\mathrm{sinc}\left(\tfrac{\pi k}{2}\right) \delta(\omega - k\omega_0). \end{aligned}$$

$\omega_0 = \frac{2\pi}{T}$

■

Example 6.30 (Frequency spectrum of a time-shifted signum function)**.** The function

$$x(t) = \operatorname{sgn}(t-1)$$

has the Fourier transform

$$X(\omega) = \tfrac{2}{j\omega}e^{-j\omega}. \quad 1$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) x has the most information.

Solution. (a) First, we find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

take magnitude of both sides of 1

$$\begin{aligned} |X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right| \\ &= \tfrac{2}{|\omega|}. \end{aligned}$$

$|ab| = |a||b|$

$|e^{j\theta}| = 1$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Next, we find the phase spectrum $\arg X(\omega)$. First, we observe that $\arg X(\omega)$ is not well defined if $\omega = 0$. So, we assume that $\omega \neq 0$. From the expression for $X(\omega)$, we can write (for $\omega \neq 0$)

take argument of both sides of 1

$$\begin{aligned} \arg X(\omega) &= \arg\left\{\tfrac{2}{j\omega}e^{-j\omega}\right\} \\ &= \arg e^{-j\omega} + \arg \tfrac{2}{j\omega} \\ &= -\omega + \arg \tfrac{2}{j\omega} \\ &= -\omega + \arg(-\tfrac{j2}{\omega}) \\ &= \begin{cases} -\frac{\pi}{2} - \omega & \omega > 0 \\ \frac{\pi}{2} - \omega & \omega < 0 \end{cases} \\ &= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega. \end{aligned}$$

$\arg(ab) = \arg a + \arg b$

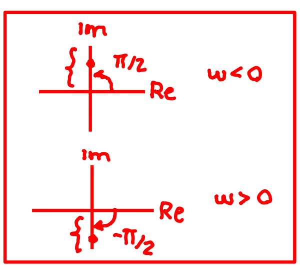
$\arg(e^{j\theta}) = \theta$

$\frac{1}{j} = -j$

$*$

definition of signum function

In the above simplification, we used the fact that



$$* \qquad \arg \tfrac{2}{j\omega} = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0. \end{cases}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown in Figures 6.10(a) and (b).

(b) Since $|X(\omega)|$ is largest for $\omega = 0$, x has the most information at the frequency 0.

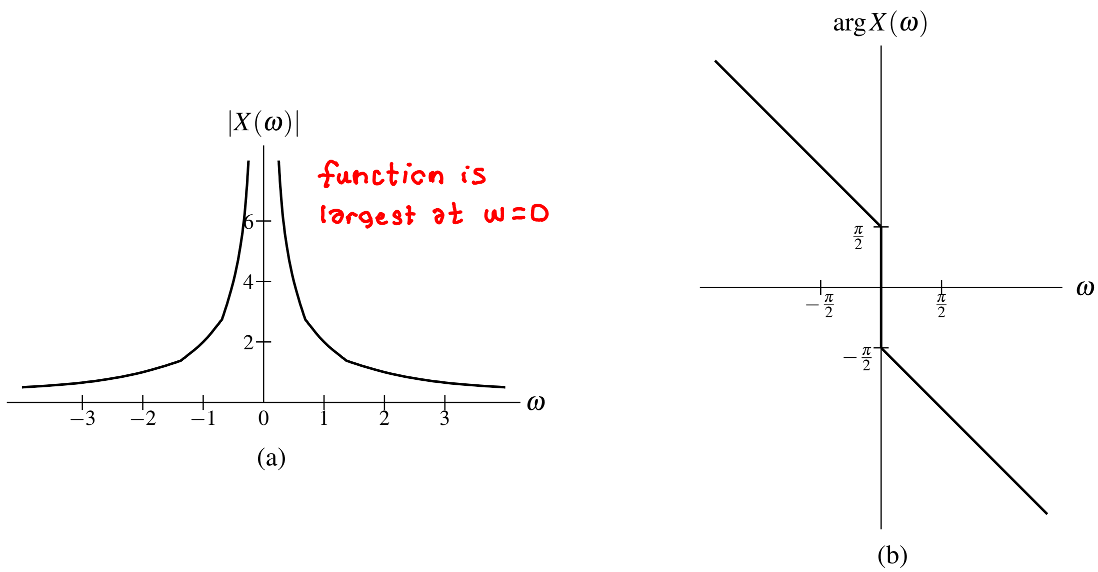


Figure 6.10: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of x.

■

Example 6.34 (Differential equation to frequency response)**.** A LTI system with input x and output y is characterized by the differential equation

$$7y''(t)+11y'(t)+13y(t)=5x'(t)+3x(t),$$

where x', y', and y'' denote the first derivative of x, the first derivative of y, and the second derivative of y, respectively. Find the frequency response H of this system.

Solution. Taking the Fourier transform of the given differential equation, we obtain

$\left(\frac{d}{dt}\right)^n x(t) \overset{FT}{\longleftrightarrow} (j\omega)^n X(\omega)$

$$7(j\omega)^2Y(\omega)+11j\omega Y(\omega)+13Y(\omega)=5j\omega X(\omega)+3X(\omega).$$

move terms containing Y and X to the left- and right-hand sides, respectively and factor

Rearranging the terms and factoring, we have

$$(-7\omega^2+11j\omega+13)Y(\omega)=(5j\omega+3)X(\omega).$$

divide both sides by $(-7\omega^2+11j\omega+13)X(\omega)$

Thus, H is given by

① $\frac{Y(\omega)}{X(\omega)} = \frac{5j\omega+3}{-7\omega^2+11j\omega+13}$

$$H(\omega)=\underbrace{\frac{Y(\omega)}{X(\omega)}=\frac{5j\omega+3}{-7\omega^2+11j\omega+13}}_{①}. \qquad \blacksquare$$

⊛

⊛ Since system is LTI, $Y(\omega) = X(\omega)H(\omega) \implies H(\omega) = \frac{Y(\omega)}{X(\omega)}$