

3B 4.11 Find the impulse response of the LTI system \mathcal{H} characterized by each of the equations below.

(a) $\mathcal{H}x(t) = \int_{-\infty}^{t+1} x(\tau) d\tau;$

(b) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau;$

(c) $\mathcal{H}x(t) = \int_{-\infty}^t x(\tau)v(t-\tau)d\tau;$ and

(d) $\mathcal{H}x(t) = \int_{t-1}^t x(\tau)d\tau.$

3B Answer (a).

Let h denote the impulse response of the system.

$$\begin{aligned} h(t) &= \int_{-\infty}^{t+1} \delta(\tau) d\tau \\ &= \begin{cases} 1 & t > -1 \\ 0 & t < -1 \end{cases} \\ &= u(t+1). \end{aligned}$$

3B Answer (b).

Let h denote the impulse response of the system.

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} \delta(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau \\ &= e^{-5-t+1}u(t-[-5]-2) \\ &= e^{-t-4}u(t+3). \end{aligned}$$

3B Answer (c).

Let h denote the impulse response of the system. We have

$$h(t) = \int_{-\infty}^t \delta(\tau)v(t-\tau)d\tau.$$

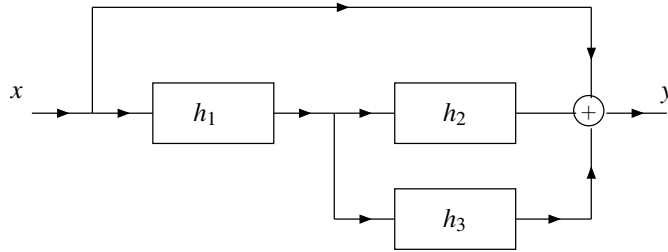
Now, we employ a change of variable. Let $\lambda = \tau - t$ so that $\tau = \lambda + t$ and $d\lambda = d\tau$. Applying this change of variable, we obtain

$$\begin{aligned} h(t) &= \int_{-\infty}^0 \delta(\lambda+t)v(t-[\lambda+t])d\lambda \\ &= \int_{-\infty}^0 \delta(\lambda+t)v(-\lambda)d\lambda. \end{aligned}$$

From the equivalence property of the unit-impulse function, we can write

$$\begin{aligned} h(t) &= \int_{-\infty}^0 \delta(\lambda+t)v(t)d\lambda \\ &= v(t) \int_{-\infty}^0 \delta(\lambda+t)d\lambda \\ &= v(t)u(t). \end{aligned}$$

3B 4.12 Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



- (a) Find the impulse response h of the overall system in terms of h_1 , h_2 , and h_3 .
 (b) Determine the impulse response h in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

3B Answer (a).

Let v denote the output of the system with impulse response h_1 . From the block diagram, we have

$$\begin{aligned} v(t) &= x * h_1(t) \\ y(t) &= v * [h_2 + h_3](t) + x(t). \end{aligned}$$

Combining these equations yields

$$\begin{aligned} y(t) &= v * [h_2 + h_3](t) + x(t) \\ &= [x * h_1] * [h_2 + h_3](t) + x(t) \\ &= x * [h_1 * [h_2 + h_3]](t) + x(t) \\ &= x * [h_1 * h_2 + h_1 * h_3](t) + x * \delta(t) \\ &= x * [h_1 * h_2 + h_1 * h_3 + \delta](t). \end{aligned}$$

Therefore, we have

$$h(t) = h_1 * h_2(t) + h_1 * h_3(t) + \delta(t).$$

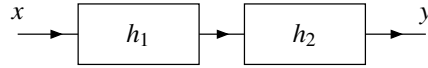
3B Answer (b).

Substituting the given expressions for h_1 , h_2 , and h_3 into the expression for h , we obtain

$$\begin{aligned} h(t) &= \delta(\cdot+1) * \delta(t) + \delta(\cdot+1) * \delta(t) + \delta(t) \\ &= \delta(t+1) + \delta(t+1) + \delta(t) \\ &= 2\delta(t+1) + \delta(t). \end{aligned}$$

(Note that $\delta(\cdot+1)$ simply means the function v defined by the equation $v(t) = \delta(t+1)$.)

- 3B 4.13** Consider the system shown in the figure below with input x and output y . This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2 .



For each pair of h_1 and h_2 given below, find the output y if the input $x(t) = u(t)$.

- (a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
 (b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$; and
 (c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

3B Answer (b).

First, we calculate the impulse response h of the overall system. We have

$$\begin{aligned} h(t) &= h_1 * h_2(t) \\ &= (\delta(\cdot + 1) * \delta(\cdot + 1))(t) \\ &= \int_{-\infty}^{\infty} \delta(\tau + 1) \delta(t - \tau + 1) d\tau \\ &= \delta(t + 2). \end{aligned}$$

(Note that $\delta(\cdot + 1)$ is simply an abbreviated notation for the function $v(t) = \delta(t + 1)$.) The output y is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= u * \delta(t + 2) \\ &= u(t + 2). \end{aligned}$$

3B Answer (c).

First, we calculate the impulse response h of the overall system. We have that

$$\begin{aligned} h(t) &= h_1 * h_2(t) \\ &= [e^{-3t}u(t)] * \delta(t) \\ &= e^{-3t}u(t). \end{aligned}$$

Now, we compute the convolution $y(t) = x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$. By drawing graphs of $x(\tau)$ and $h(t - \tau)$ and applying the appropriate logic, we conclude that there are two cases to consider for the computation of $y(t)$: $t < 0$ and $t \geq 0$. In the case that $t < 0$, we have that $y(t)$ is trivially zero. So, we now consider the case that $t \geq 0$. For $t \geq 0$, we have that $y(t)$ is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)u(t - \tau)d\tau \\ &= \int_0^t e^{-3\tau}d\tau \quad \text{for } t > 0 \\ &= \left[-\frac{1}{3}e^{-3\tau}\right]_0^t \\ &= -\frac{1}{3}[e^{-3t} - 1] \\ &= -\frac{1}{3}e^{-3t} + \frac{1}{3}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} y(t) &= \begin{cases} -\frac{1}{3}e^{-3t} + \frac{1}{3} & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= \left[-\frac{1}{3}e^{-3t} + \frac{1}{3}\right]u(t). \end{aligned}$$

3B 4.14 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.

(a) $h(t) = (t + 1)u(t - 1)$;

(b) $h(t) = 2\delta(t + 1)$;

(c) $h(t) = \frac{\omega_c}{\pi} \text{sinc}(\omega_c t)$;

(d) $h(t) = e^{-4t}u(t - 1)$;

(e) $h(t) = e^t u(-1 - t)$;

(f) $h(t) = e^{-3|t|}$; and

(g) $h(t) = 3\delta(t)$.

3B Answer (a–g).

A LTI system with impulse response h is memoryless if $h(t) = 0$ for all $t \neq 0$. Therefore, the systems in (a), (b), (c), (d), (e), and (f) all have memory, while the system in (g) is memoryless.

A LTI system with impulse response h is causal if $h(t) = 0$ for all $t < 0$. Therefore, the systems in (a), (d), and (g) are causal, while the systems in (b), (c), (e), and (f) are not causal.

3B 4.15 Determine whether the LTI system with each impulse response h given below is BIBO stable.

- (a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
- (b) $h(t) = t^{-1}u(t-1)$;
- (c) $h(t) = e^t u(t)$;
- (d) $h(t) = \delta(t-10)$;
- (e) $h(t) = \text{rect}(t)$; and
- (f) $h(t) = e^{-|t|}$.

3B Answer (a).

A LTI system with impulse response h is BIBO stable if and only if h is absolutely integrable. From the given h , we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} |e^{at}u(-t)| dt \\
 &= \int_{-\infty}^{\infty} e^{at}u(-t) dt \\
 &= \int_{-\infty}^0 e^{at} dt \\
 &= \left[\frac{1}{a} e^{at} \right]_{-\infty}^0 \quad \text{for } a \neq 0 \\
 &= \frac{1}{a} [e^{at}]_{-\infty}^0 \\
 &= \frac{1}{a} [1 - 0] \\
 &= \frac{1}{a} \\
 &< \infty.
 \end{aligned}$$

Therefore, the system is BIBO stable.

3B Answer (b).

A LTI system with impulse response h is BIBO stable if and only if h is absolutely integrable. From the given h , can write

$$\begin{aligned}
 \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} |t^{-1}u(t-1)| dt \\
 &= \int_{-\infty}^{\infty} t^{-1}u(t-1) dt \\
 &= \int_1^{\infty} t^{-1} dt \\
 &= [\ln(t)]_1^{\infty} \\
 &= \ln(\infty) - \ln(1) \\
 &= \infty - 0 \\
 &= \infty.
 \end{aligned}$$

Therefore, the system is not BIBO stable.

3B 4.16 Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \frac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

3B Answer.

These systems are inverses if $h_1 * h_2(t) = \delta(t)$. We have

$$\begin{aligned} h_1 * h_2(t) &= \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\delta(\tau-1)2\delta(t-\tau+1)d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau-1)\delta(t-\tau+1)d\tau \\ &= \delta(t). \end{aligned}$$

Therefore, the systems are inverses of one another.

3B 4.17 For each case below, find the response y of the LTI system with system function H to the input x .

(a) $H(s) = \frac{1}{s+1}$ for $\text{Re}(s) > -1$; and $x(t) = 10 + 4\cos(3t) + 2\sin(5t)$; and

(b) $H(s) = \frac{1}{e^s(s+1)}$ for $\text{Re}(s) > -1$; and $x(t) = 10 + 2e^{3t} - e^t$.

3B Answer (a).

Rewriting x in terms of eigenfunctions of LTI systems, we obtain

$$\begin{aligned} x(t) &= 10 + 4 \left[\frac{1}{2} (e^{j3t} + e^{-j3t}) \right] + 2 \left[\frac{1}{2j} (e^{j5t} - e^{-j5t}) \right] \\ &= 10 + 2e^{j3t} + 2e^{-j3t} - je^{j5t} + je^{-j5t}. \end{aligned}$$

Since the system is LTI, we have

$$\begin{aligned} y(t) &= H(0)(10) + H(j3)(2e^{j3t}) + H(-j3)(2e^{-j3t}) + H(j5)(-je^{j5t}) + H(-j5)(je^{-j5t}) \\ &= 10H(0) + 2H(j3)e^{j3t} + 2H(-j3)e^{-j3t} - jH(j5)e^{j5t} + jH(-j5)e^{-j5t}. \end{aligned}$$

Now, we compute the eigenvalues $H(0)$, $H(j3)$, $H(-j3)$, $H(j5)$, and $H(-j5)$. We have

$$\begin{aligned} H(0) &= \frac{1}{0+1} \\ &= 1, \\ H(j3) &= \frac{1}{1+j3} = \frac{1-j3}{1+9} = \frac{1-j3}{10} = \frac{1}{10} - j\frac{3}{10} = \sqrt{\frac{1}{10}}e^{j\arctan(-3)} \\ &= \frac{1}{\sqrt{10}}e^{-j\arctan(3)}, \\ H(-j3) &= \frac{1}{1-j3} = \frac{1+j3}{1+9} = \frac{1+j3}{10} = H(j3)^* \\ &= \frac{1}{\sqrt{10}}e^{j\arctan(3)}, \\ H(j5) &= \frac{1}{1+j5} = \frac{1-j5}{1+25} = \frac{1-j5}{26} = \frac{1}{26} - j\frac{5}{26} = \sqrt{\frac{26}{676}}e^{j\arctan(-5)} = \sqrt{\frac{1}{26}}e^{-j\arctan(5)} \\ &= \frac{1}{\sqrt{26}}e^{-j\arctan(5)}, \quad \text{and} \\ H(-j5) &= \frac{1}{1-j5} = \frac{1+j5}{1+25} = \frac{1+j5}{26} = H(j5)^* \\ &= \frac{1}{\sqrt{26}}e^{j\arctan(5)}. \end{aligned}$$

Substituting the computed eigenvalues into the above formula for $y(t)$, we have

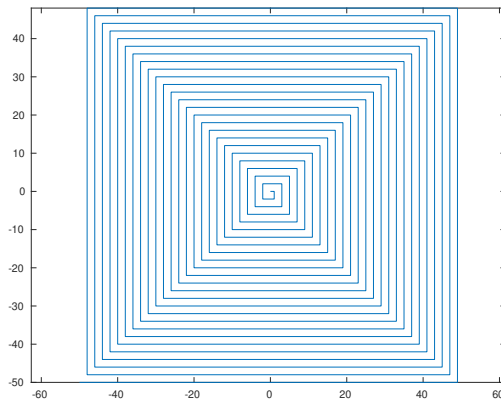
$$\begin{aligned} y(t) &= 10(1) + 2 \left(\frac{1}{\sqrt{10}}e^{-j\arctan(3)}e^{j3t} \right) + 2 \left(\frac{1}{\sqrt{10}}e^{j\arctan(3)}e^{-j3t} \right) - j \left(\frac{1}{\sqrt{26}}e^{-j\arctan(5)}e^{j5t} \right) + j \left(\frac{1}{\sqrt{26}}e^{j\arctan(5)}e^{-j5t} \right) \\ &= 10 + \frac{2}{\sqrt{10}}e^{-j\arctan(3)}e^{j3t} + \frac{2}{\sqrt{10}}e^{j\arctan(3)}e^{-j3t} - \frac{j}{\sqrt{26}}e^{-j\arctan(5)}e^{j5t} + \frac{j}{\sqrt{26}}e^{j\arctan(5)}e^{-j5t} \\ &= 10 + \frac{2}{\sqrt{10}} \left(e^{j[3t-\arctan(3)]} + e^{-j[3t-\arctan(3)]} \right) - \frac{j}{\sqrt{26}} \left(e^{j[5t-\arctan(5)]} - e^{-j[5t-\arctan(5)]} \right) \\ &= 10 + \frac{2}{\sqrt{10}}[2\cos(3t - \arctan(3))] - \frac{j}{\sqrt{26}}[2j\sin(5t - \arctan(5))] \\ &= 10 + \frac{4}{\sqrt{10}}\cos(3t - \arctan(3)) + \frac{2}{\sqrt{26}}\sin(5t - \arctan(5)). \end{aligned}$$

3B D.8 In this exercise, we consider an algorithm for generating a sequence p of n points in the plane (i.e., p_0, p_1, \dots, p_{n-1}). The first point p_0 is chosen as the origin (i.e., $p_0 = [0 \ 0]^T$), with the remaining points being given by the formula

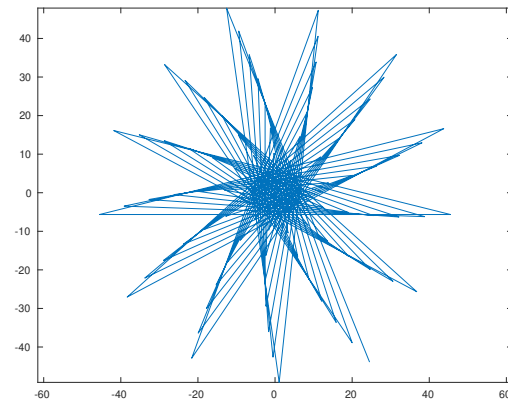
$$p_i = p_{i-1} + \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{i-1} \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

(a) Using MATLAB, write a function called `drawpattern` that takes n and θ as input arguments (in that order) with θ being specified in degrees, and then computes and plots the points p_0, p_1, \dots, p_{n-1} connected by straight lines (i.e., draw a line from p_0 to p_1 , p_1 to p_2 , p_2 to p_3 , and so on). When performing the plotting, be sure to use `axis('equal')` in order to maintain the correct aspect ratio for the plot. For illustrative purposes, the plots produced for two sets of θ and n values are shown in Figures (a) and (b) below.

(b) Generate the plots obtained by invoking `drawpattern` with $n = 100$ and θ set to each of the following values: 89° , 144° , and 154° . [Note: In MATLAB, the `sin` and `cos` functions take values in radians, not degrees.]



(a) $\theta = 90^\circ$ and $n = 100$



(b) $\theta = 166^\circ$ and $n = 100$

3B Answer (a).

The `drawpattern` function can be implemented using the code below.

Listing D.6: `drawpattern.m`

```
function drawpattern(n, theta)
    % drawpattern - Draw a pattern
    % This function takes two arguments:
    %     the number of iterations
    %     the rotation angle

    % Convert from degrees to radians.
    t = theta * pi / 180;

    % Generate the list of points.
    p = [0 0]';
    x = p';
    for i = 1 : (n - 1)
        p = p + [cos(t) sin(t); -sin(t) cos(t)] ^ (i - 1) * [i 0]';
        x = [x; p'];
    end

    % Plot the list of points.
    plot(x(:, 1), x(:, 2));
```



```

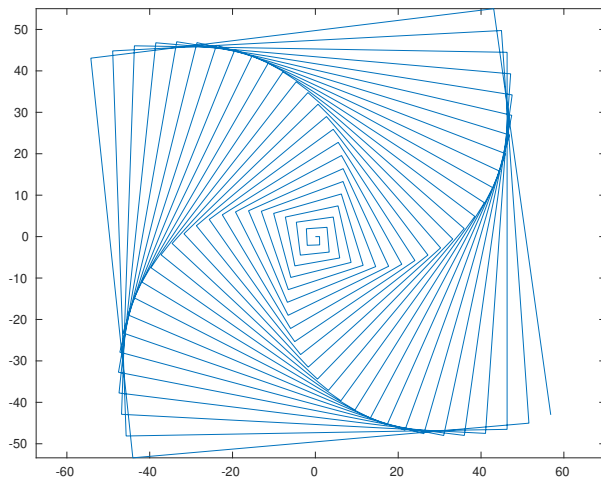
axis('equal');

% Print the plot to a file.
% eval(sprintf('print -dps data/drawpattern_%d_%d.ps', theta, n))
end

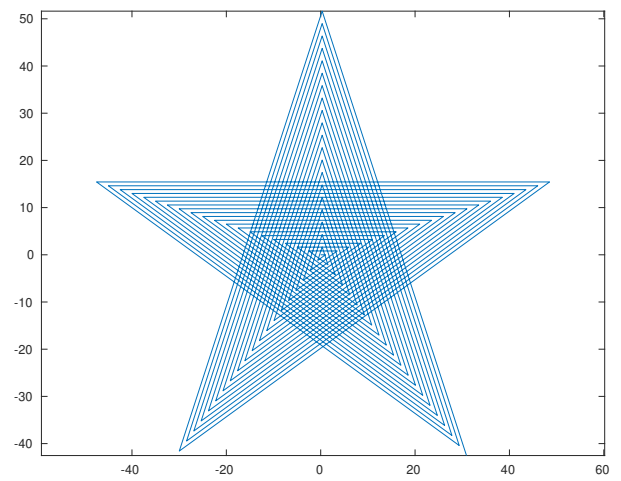
```

3B Answer (b).

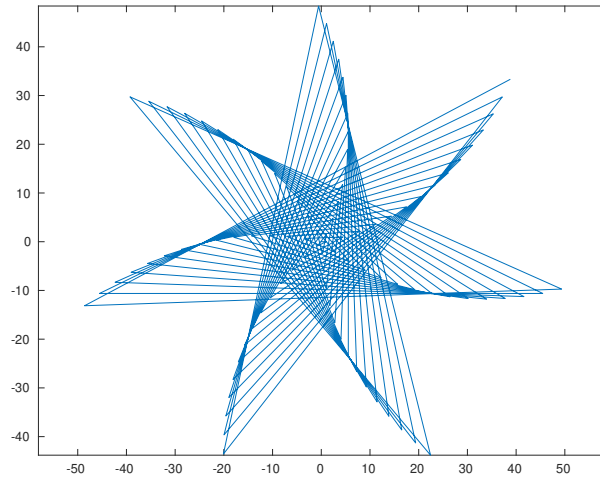
The plots obtained with the drawpattern function are shown below.



$\theta = 89^\circ, n = 100$



$\theta = 144^\circ, n = 100$



$\theta = 154^\circ, n = 100$