Chapter 5

Laplace Transform (Chapter 6)

6.1 Using the definition of the Laplace transform, find the Laplace transform of each of the following signals (including the corresponding ROC):
(a) $x(t) = e^{-at}u(t)$;
(b) $x(t) = e^{-a|t|}$; and
(c) $x(t) = [\cos\omega_0 t]u(t)$. [Note: $\int e^{ax}\cos bx dx = \frac{1}{a^2+b^2}(e^{ax}[a\cos bx + b\sin bx])$.]

Solution.

(c) Let $s = \sigma + j\omega$. We have

$$\begin{aligned}\mathscr{L}\{[\cos\omega_0 t]u(t)\} &= \int_{-\infty}^{\infty}[\cos\omega_0 t]u(t)e^{-st}dt \\ &= \int_0^{\infty}[\cos\omega_0 t]e^{-st}dt.\end{aligned}$$

Since this integral does not converge if $s = 0$, we assume that $s \neq 0$. From this assumption, we have

$$\begin{aligned}\mathscr{L}\{[\cos\omega_0 t]u(t)\} &= \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(-s)^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0\sin\omega_0 t]}{s^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-(\sigma+j\omega)t}[-(\sigma+j\omega)\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(\sigma+j\omega)^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-\sigma t}e^{-j\omega t}[-(\sigma+j\omega)\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(\sigma+j\omega)^2+\omega_0^2}\right]\Bigg|_0^{\infty}.\end{aligned}$$

The preceding expression only converges to a finite limit if $\sigma > 0$ (i.e., $\operatorname{Re}\{s\} > 0$). We proceed to compute this limit as follows:

$$\begin{aligned}\mathscr{L}\{[\cos\omega_0 t]u(t)\} &= 0 - \left[\frac{-(\sigma+j\omega)}{(\sigma+j\omega)^2+\omega_0^2}\right] \\ &= \frac{\sigma+j\omega}{(\sigma+j\omega)^2+\omega_0^2} \\ &= \frac{s}{s^2+\omega_0^2} \quad \text{for } \operatorname{Re}\{s\} > 0.\end{aligned}$$

6.2 Find the Laplace transform (including the ROC) of the following signals:
(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$ where a is a positive real constant and b is a real constant;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

Solution.

(a) From a Laplace transform table, we can write

$$X(s) = \tfrac{1}{s+2} \text{for } \operatorname{Re}\{s\} > -2.$$

(b)

$$\begin{aligned}
X(s) &= \mathscr{L}\{3e^{-2t}u(t) + 2e^{5t}u(-t)\} \\
&= 3\mathscr{L}\{e^{-2t}u(t)\} + 2\mathscr{L}\{e^{5t}u(-t)\} \\
&= 3\left(\frac{1}{s+2}\right) - 2\left(\tfrac{1}{s-5}\right) \quad \text{for } \operatorname{Re}\{s\} > -2 \cap \operatorname{Re}\{s\} < 5 \\
&= \frac{3(s-5) - 2(s+2)}{(s+2)(s-5)} \\
&= \frac{3s - 15 - 2s - 4}{(s+2)(s-5)} \\
&= \frac{s-19}{(s+2)(s-5)} \quad \text{for } \operatorname{Re}\{s\} > -2 \cap \operatorname{Re}\{s\} < 5 \\
&= \frac{s-19}{(s+2)(s-5)} \quad \text{for } -2 < \operatorname{Re}\{s\} < 5.
\end{aligned}$$

(c) To begin, let $v_1(t) = x(t-4)$ so that

$$\begin{aligned}
x(t) &= v_1(t+4) \quad \text{and} \\
v_1(t) &= e^{-2(t-4)}u(t-4+4) \\
&= e^8 e^{-2t}u(t).
\end{aligned}$$

Taking the Laplace transform of these equations yields

$$\begin{aligned}
X(s) &= \mathscr{L}\{x(t)\} \\
&= \mathscr{L}\{v_1(t+4)\} \\
&= e^{4s}V_1(s) \quad \text{for ROC of } V_1(s), \quad \text{and} \\
V_1(s) &= \mathscr{L}\{v_1(t)\} \\
&= \mathscr{L}\{e^8 e^{-2t}u(t)\} \\
&= e^8\mathscr{L}\{e^{-2t}u(t)\} \\
&= e^8\tfrac{1}{s+2} \quad \text{for } \operatorname{Re}\{s\} > -2.
\end{aligned}$$

Substituting the above expression for $V_1(s)$ into the expression for $X(s)$, we obtain

$$\begin{aligned}X(s) &= e^{4s}V_1(s)\\ &= e^{4s}\left[e^{8}\tfrac{1}{s+2}\right]\\ &= \frac{e^{4s+8}}{s+2} \quad \text{for } \operatorname{Re}\{s\} > -2.\end{aligned}$$

(d) We rewrite $x(t)$ as

$$x(t) = \int_{-\infty}^{t} v_1(\tau)d\tau$$

where

$$\begin{aligned}v_1(t) &= e^{-2t}v_2(t)\\ v_2(t) &= u(t).\end{aligned}$$

Let R_X, R_{V_1}, and R_{V_2} denote the ROCs of $X(s)$, $V_1(s)$, and $V_2(s)$, respectively. Taking the Laplace transform of both sides of each of the above equations, we obtain

$$\begin{aligned}X(s) &= \tfrac{1}{s}V_1(s) \quad \text{for } R_X \supset R_{V_1} \cap [\operatorname{Re}\{s\} > 0]\\ V_1(s) &= V_2(s+2) \quad \text{for } R_{V_1} = R_{V_2} + 2\\ V_2(s) &= \tfrac{1}{s} \quad \text{for } \operatorname{Re}\{s\} > 0.\end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned}X(s) &= \tfrac{1}{s}\left[V_2(s+2)\right]\\ &= \tfrac{1}{s}\left[\tfrac{1}{s+2}\right]\\ &= \frac{1}{s(s+2)} \quad \text{for } \operatorname{Re}\{s\} > 0.\end{aligned}$$

(e) Let us rewrite $x(t)$ as

$$x(t) = v_1(-t)$$

where

$$\begin{aligned}v_1(t) &= v_2(t+b)\\ v_2(t) &= -e^{ab}e^{-at}u(t).\end{aligned}$$

Let R_X, R_{V_1}, and R_{V_2} denote the ROCs of $X(s)$, $V_1(s)$, and $V_2(s)$, respectively. Taking the Laplace transform of both sides of each of the above equations, we obtain

$$\begin{aligned}X(s) &= \mathscr{L}\{v_1(-t)\}\\ &= V_1(-s) \quad \text{for } R_X = -R_{V_1}\\ V_1(s) &= \mathscr{L}\{v_2(t+b)\}\\ &= e^{bs}V_2(s) \quad \text{for } R_{V_2}\\ V_2(s) &= \mathscr{L}\{-e^{ab}e^{-at}u(t)\}\\ &= -e^{ab}\mathscr{L}\{e^{-at}u(t)\}\\ &= -e^{ab}\tfrac{1}{s+a} \quad \text{for } \operatorname{Re}\{s\} > -a.\end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(s) &= V_1(-s) \\ &= e^{-bs}V_2(-s) \\ &= e^{-bs}[-e^{ab}\tfrac{1}{-s+a}] \quad \text{for } \operatorname{Re}\{s\} < a \\ &= e^{-b(s-a)}\tfrac{1}{s-a} \quad \text{for } \operatorname{Re}\{s\} < a \\ &= e^{b(a-s)}\tfrac{1}{s-a} \quad \text{for } \operatorname{Re}\{s\} < a. \end{aligned}$$

6.3 Suppose that $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$ with ROC R_X and $y(t) \overset{\mathscr{L}}{\longleftrightarrow} Y(s)$ with ROC R_Y. Then, express $Y(s)$ in terms of $X(s)$, and express R_Y in terms of R_X, in each of the cases below.
(a) $y(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}\ [x(\lambda) * x(\lambda)]|_{\lambda=t-1}$;
(c) $y(t) = tx(3t-2)$;
(d) $y(t) = \frac{d}{dt}[x^*(t-3)]$;
(e) $y(t) = e^{-5t}x(3t+7)$; and
(f) $y(t) = e^{-j5t}x(t+3)$.

Solution.

(e) Define

$$\begin{aligned} v_1(t) &= x(t+7), \\ v_2(t) &= v_1(3t), \end{aligned}$$

so that we can express $y(t)$ as

$$y(t) = e^{-5t}v_2(t).$$

Taking the Laplace transforms of both sides of the above equations, we obtain

$$\begin{aligned} V_1(s) &= e^{7s}X(s), \quad R_{V_1} = R_X, \\ V_2(s) &= \tfrac{1}{3}V_1(s/3), \quad R_{V_2} = 3R_{V_1}, \\ Y(s) &= V_2(s+5), \quad R_Y = R_{V_2} - 5, \end{aligned}$$

where R_{V_1} and R_{V_2} denote the ROCs of $V_1(s)$ and $V_2(s)$, respectively. Combining the above equations, we have

$$\begin{aligned} Y(s) &= V_2(s+5) \\ &= \tfrac{1}{3}V_1(\tfrac{s+5}{3}) \\ &= \tfrac{1}{3}e^{7(s+5)/3}X(\tfrac{s+5}{3}). \end{aligned}$$

Also, we have a ROC of

$$\begin{aligned} R_Y &= R_{V_2} - 5 \\ &= 3R_{V_1} - 5 \\ &= 3R_X - 5. \end{aligned}$$

6.8 The Laplace transform $X(s)$ of the causal signal $x(t)$ is given by

$$X(s) = \frac{-2s}{s^2+3s+2}.$$

(a) Assuming that $x(t)$ has no singularities at $t = 0$, find $x(0^+)$.
(b) Assuming that $x(t)$ has a finite limit as $t \to \infty$, find this limit.

Solution.

(a) Since $x(t)$ is causal and has no singularities at the origin, we can compute $x(0^+)$ using the initial value theorem as follows:

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty} sX(s) \\ &= \lim_{s\to\infty} \frac{s(-2s)}{s^2+3s+2} \\ &= -2. \end{aligned}$$

(b) Since $x(t)$ is causal and we are told that the limit of $x(t)$ as $t \to \infty$ exists, we can compute this limit using the final value theorem as follows:

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\ &= \left.\frac{s(-2s)}{s^2+3s+2}\right|_{s=0} \\ &= 0. \end{aligned}$$

6.9 Find the inverse Laplace transform of each of the following:
(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \operatorname{Re}\{s\} < 1$;
(b) $X(s) = \dfrac{2s^2+4s+5}{(s+1)(s+2)}$ for $\operatorname{Re}\{s\} > -1$;
(c) $X(s) = \dfrac{3s+1}{s^2+3s+2}$ for $-2 < \operatorname{Re}\{s\} < -1$.

Solution.

(a) First, we rewrite $X(s)$ as

$$X(s) = \frac{s-5}{(s+1)(s-1)}.$$

From this, we can see that $X(s)$ has a partial fraction expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-1}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} \\ &= \left.\frac{s-5}{s-1}\right|_{s=-1} \\ &= \tfrac{-6}{-2} \\ &= 3 \quad \text{and} \\ A_2 &= (s-1)X(s)|_{s=1} \\ &= \frac{s-5}{s+1}|_{s=1} \\ &= \tfrac{-4}{2} \\ &= -2. \end{aligned}$$

Thus, we have

$$X(s) = \frac{3}{s+1} - \frac{2}{s-1}.$$

Taking the inverse Laplace transform, we have

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\{X(s)\} \\ &= 3\mathscr{L}^{-1}\left\{\tfrac{1}{s+1}\right\} - 2\mathscr{L}^{-1}\left\{\tfrac{1}{s-1}\right\}. \end{aligned}$$

From Laplace transform tables, we know

$$\begin{aligned} e^{-t}u(t) &\stackrel{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}\{s\} > -1 \quad \text{and} \\ -e^{t}u(-t) &\stackrel{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s-1} \quad \text{for } \operatorname{Re}\{s\} < 1. \end{aligned}$$

(The ROCs must include the ROC of $X(s)$ which is $-1 < \operatorname{Re}\{s\} < 1$.) Substituting these results, we obtain

$$\begin{aligned} x(t) &= 3e^{-t}u(t) - 2[-e^{t}u(-t)] \\ &= 3e^{-t}u(t) + 2e^{t}u(-t). \end{aligned}$$

6.11 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

Solution.

Each distinct ROC for $H(s)$ will yield a distinct inverse Laplace transform. Since $H(s)$ is a rational function with poles at -1 and 1, three distinct ROCs are possible: i) $\operatorname{Re}\{s\} < -1$; ii) $-1 < \operatorname{Re}\{s\} < 1$; and iii) $\operatorname{Re}\{s\} > 1$. From the expression for $H(s)$, we have

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\{H(s)\} \\ &= \mathscr{L}^{-1}\left\{\tfrac{4}{s+1} + \tfrac{3}{s-1}\right\} \\ &= 4\mathscr{L}^{-1}\left\{\tfrac{1}{s+1}\right\} + 3\mathscr{L}^{-1}\left\{\tfrac{1}{s-1}\right\}. \end{aligned}$$

For $\operatorname{Re}\{s\} < -1$, we have

$$\begin{aligned} h(t) &= 4[-e^{-t}u(-t)] + 3[-e^{t}u(-t)] \\ &= [-4e^{-t} - 3e^{t}]u(-t). \end{aligned}$$

For $-1 < \operatorname{Re}\{s\} < 1$, we have

$$\begin{aligned} h(t) &= 4[e^{-t}u(t)] + 3[-e^{t}u(-t)] \\ &= 4e^{-t}u(t) - 3e^{t}u(-t). \end{aligned}$$

For $\operatorname{Re}\{s\} > 1$, we have

$$\begin{aligned} h(t) &= 4[e^{-t}u(t)] + 3[e^{t}u(t)] \\ &= [4e^{-t} + 3e^{t}]u(t). \end{aligned}$$

6.12 Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$, where

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Find the differential equation that characterizes the behavior of the system.

Solution.

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively. The system is characterized by the equation

$$Y(s) = H(s)X(s).$$

So, we have

$$\begin{aligned} & Y(s) = \left(\frac{s+1}{s^2+2s+2}\right)X(s) \\ \Rightarrow \quad & [s^2+2s+2]Y(s) = [s+1]X(s) \\ \Rightarrow \quad & s^2Y(s)+2sY(s)+2Y(s) = sX(s)+X(s). \end{aligned}$$

Taking the inverse Laplace transform of both sides of each of the above equation yields

$$\begin{aligned} & \mathscr{L}^{-1}\{s^2Y(s)\} + 2\mathscr{L}^{-1}\{sY(s)\} + 2\mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\{sX(s)\} + \mathscr{L}^{-1}\{X(s)\} \\ \Rightarrow \quad & \tfrac{d^2}{dt^2}y(t) + 2\tfrac{d}{dt}y(t) + 2y(t) = \tfrac{d}{dt}x(t) + x(t). \end{aligned}$$

6.13 Suppose that we have a causal LTI system characterized by the differential equation

$$\tfrac{d^2}{dt^2}y(t) + 4\tfrac{d}{dt}y(t) + 3y(t) = 2\tfrac{d}{dt}x(t) + x(t),$$

where $x(t)$ and $y(t)$ denote the input and output of the system, respectively. Find the system function $H(s)$.

Solution.

We begin by taking the Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned} & \mathscr{L}\{\tfrac{d^2}{dt^2}y(t)\} + 4\mathscr{L}\{\tfrac{d}{dt}y(t)\} + 3\mathscr{L}\{y(t)\} = 2\mathscr{L}\{\tfrac{d}{dt}x(t)\} + \mathscr{L}\{x(t)\} \\ \Rightarrow \quad & s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + X(s) \\ \Rightarrow \quad & [s^2+4s+3]Y(s) = [2s+1]X(s) \\ \Rightarrow \quad & \frac{Y(s)}{X(s)} = \frac{2s+1}{s^2+4s+3}. \end{aligned}$$

Therefore, the system function $H(s)$ is given by

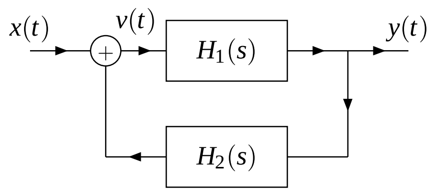
$$\begin{aligned} H(s) &= \frac{2s+1}{s^2+4s+3} \\ &= \frac{2s+1}{(s+3)(s+1)} \quad \text{for } \operatorname{Re}\{s\} > -1. \end{aligned}$$

(The ROC is a right-half plane, since the system is causal. The ROC is to the right of the rightmost pole, since $H(s)$ is also rational.)

6.14 Consider the LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$, as shown in the figure below. Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are causal LTI systems with system functions $H_1(s)$ and $H_2(s)$, respectively, given by

$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where A is a real constant.



(a) Find an expression for $H(s)$ in terms of $H_1(s)$ and $H_2(s)$.
(b) Determine for what values of A the system is BIBO stable.

Solution.

(a) From the system block diagram, we can write

$$\begin{aligned} V(s) &= X(s) + H_2(s)Y(s) \quad \text{and} \\ Y(s) &= H_1(s)V(s). \end{aligned}$$

Combining these equations yields

$$\begin{aligned} Y(s) &= H_1(s)[X(s) + H_2(s)Y(s)] \\ &= H_1(s)X(s) + H_1(s)H_2(s)Y(s). \end{aligned}$$

So, we know

$$\begin{aligned} & [1 - H_1(s)H_2(s)]Y(s) = H_1(s)X(s) \\ \Rightarrow \quad & \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)}. \end{aligned}$$

Therefore, we have

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

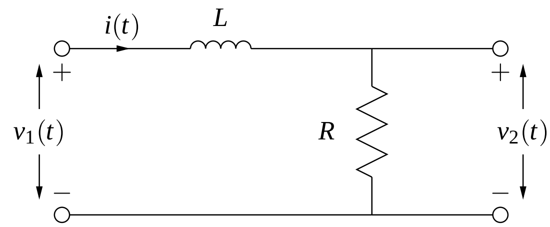
Substituting the given expressions for $H_1(s)$ and $H_2(s)$, we have

$$\begin{aligned} H(s) &= \frac{\frac{1}{s-1}}{1 - (\frac{1}{s-1})A} \\ &= \frac{1}{s-1-A} \\ &= \frac{1}{s-(A+1)} \quad \text{for } \operatorname{Re}\{s\} > A+1. \end{aligned}$$

(b) The system is BIBO stable if and only if the ROC of $H(s)$ includes the entire imaginary axis. We know that $H(s)$ converges for $\operatorname{Re}\{s\} > A+1$. Therefore, the ROC includes the entire imaginary axis if $A+1 < 0$. So, the system is stable if $A+1 < 0$ which implies $A < -1$. Therefore the system is stable if

$$A < -1.$$

6.15 Suppose that we have the RL network with input $v_1(t)$ and output $v_2(t)$ as shown in the figure below. This system is LTI and can be characterized by a linear differential equation with constant coefficients.



(a) Find the system function $H(s)$ of the system.
(b) Determine whether the system is stable.
(c) Find the step response $s(t)$ of the system.

Solution.

(a) From basic circuit analysis, we can write

$$\begin{aligned} v_1(t) &= L\tfrac{d}{dt}i(t) + v_2(t) \\ i(t) &= \tfrac{1}{R}v_2(t) \end{aligned}$$

Combining these two equations yields

$$\begin{aligned} v_1(t) &= L\tfrac{d}{dt}[\tfrac{1}{R}v_2(t)] + v_2(t) \\ &= \tfrac{L}{R}\tfrac{d}{dt}v_2(t) + v_2(t). \end{aligned}$$

Taking the Laplace transform of both sides of this equation, we obtain

$$\begin{aligned} V_1(s) &= \mathscr{L}\{v_1(t)\} \\ &= \mathscr{L}\{\tfrac{L}{R}\tfrac{d}{dt}v_2(t) + v_2(t)\} \\ &= \tfrac{L}{R}\mathscr{L}\{\tfrac{d}{dt}v_2(t)\} + \mathscr{L}\{v_2(t)\} \\ &= \tfrac{L}{R}sV_2(s) + V_2(s). \end{aligned}$$

Rearranging, we have

$$\begin{aligned} & V_1(s) = [\tfrac{L}{R}s + 1]V_2(s) \\ \Rightarrow \quad & \frac{V_2(s)}{V_1(s)} = \frac{1}{\frac{L}{R}s + 1} = \frac{R/L}{s + R/L}. \end{aligned}$$

Therefore, the system function $H(s)$ is given by

$$H(s) = \frac{R/L}{s + R/L} \quad \text{for } \operatorname{Re}\{s\} > -\tfrac{R}{L}.$$

(The ROC must be a right-half plane since the system is causal.)

(b) The rational function $H(s)$ has a single pole at $-\frac{R}{L}$. Since L and R are strictly positive quantities, we have that $-\frac{R}{L} < 0$. In other words, all of the poles of $H(s)$ are in the left-half plane. Since the system is causal, this implies that the system is stable.

(c) Now, we consider the step response of the system.

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{R/L}{s + R/L}\right)\left(\frac{1}{s}\right) \\ &= \frac{R/L}{s(s + R/L)}. \end{aligned}$$

We must find the inverse Laplace transform of $V_2(s)$. So, we first find a partial fraction expansion of $V_2(s)$. Such an expansion is of the form

$$V_2(s) = \frac{A_1}{s+R/L} + \frac{A_2}{s}.$$

Calculating the expansion coefficients yields

$$\begin{aligned} A_1 &= (s+R/L)V_2(s)|_{s=-R/L} \\ &= \tfrac{R/L}{s}|_{s=-R/L} \\ &= -1 \quad \text{and} \\ A_2 &= sV_2(s)|_{s=0} \\ &= \tfrac{R/L}{s+R/L}|_{s=0} \\ &= 1. \end{aligned}$$

So, we have

$$V_2(s) = \frac{-1}{s+R/L} + \frac{1}{s}.$$

Taking the inverse Laplace transform of $V_2(s)$ yields

$$\begin{aligned} v_2(t) &= \mathscr{L}^{-1}\{\tfrac{-1}{s+R/L} + \tfrac{1}{s}\} \\ &= -\mathscr{L}^{-1}\{\tfrac{1}{s+R/L}\} + \mathscr{L}^{-1}\{\tfrac{1}{s}\} \\ &= -[e^{-(R/L)t}u(t)] + u(t) \\ &= [1-e^{-(R/L)t}]u(t). \end{aligned}$$

Therefore, the step response $s(t)$ of the system is

$$s(t) = [1-e^{-(R/L)t}]u(t).$$

6.16 Suppose that we have a causal (incrementally-linear TI) system with input $x(t)$ and output $y(t)$ characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 12y(t) = x(t).$$

If $y(0^-) = -1$, $y'(0^-) = 0$, and $x(t) = u(t)$, find $y(t)$.

Solution.

We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned} & \mathscr{UL}\{\tfrac{d^2}{dt^2}y(t)\} + 7\mathscr{UL}\{\tfrac{d}{dt}y(t)\} + 12\mathscr{UL}\{y(t)\} = \mathscr{UL}\{x(t)\} \\ \Rightarrow \quad & s^2Y(s) - sy(0^-) - y'(0^-) + 7[sY(s) - y(0^-)] + 12Y(s) = X(s) \\ \Rightarrow \quad & [s^2+7s+12]Y(s) = sy(0^-) + y'(0^-) + 7y(0^-) + X(s) \\ \Rightarrow \quad & Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 7y(0^-)}{s^2+7s+12}. \end{aligned}$$

Since $x(t) = u(t)$, we have

$$X(s) = \mathscr{UL}\{u(t)\} = \tfrac{1}{s}.$$

Substituting this expression for $X(s)$ and the given initial conditions into the above equation for $Y(s)$ yields

$$Y(s) = \frac{\frac{1}{s} - s - 7}{s^2 + 7s + 12} = \frac{-s^2 - 7s + 1}{s(s^2 + 7s + 12)} = \frac{-s^2 - 7s + 1}{s(s+3)(s+4)}.$$

Now, we need to calculate a partial fraction expansion of $Y(s)$. Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+3} + \frac{A_3}{s+4}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sY(s)|_{s=0} \\
&= \left.\frac{-s^2 - 7s + 1}{(s+3)(s+4)}\right|_{s=0} \\
&= \tfrac{1}{12}, \\
A_2 &= (s+3)Y(s)|_{s=-3} \\
&= \left.\frac{-s^2 - 7s + 1}{s(s+4)}\right|_{s=-3} \\
&= -\tfrac{13}{3}, \quad \text{and} \\
A_3 &= (s+4)Y(s)|_{s=-4} \\
&= \left.\frac{-s^2 - 7s + 1}{s(s+3)}\right|_{s=-4} \\
&= \tfrac{13}{4}.
\end{aligned}$$

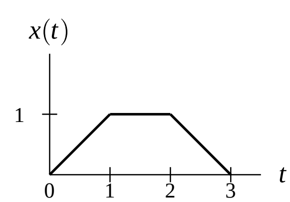
So, we can rewrite $Y(s)$ as

$$Y(s) = \tfrac{1}{12}\left(\frac{1}{s}\right) - \tfrac{13}{3}\left(\frac{1}{s+3}\right) + \tfrac{13}{4}\left(\frac{1}{s+4}\right).$$

Taking the inverse unilateral Laplace transform of $Y(s)$ yields

$$\begin{aligned}
y(t) &= \tfrac{1}{12}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{13}{3}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} + \tfrac{13}{4}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\} \\
&= \tfrac{1}{12} - \tfrac{13}{3}e^{-3t} + \tfrac{13}{4}e^{-4t} \quad \text{for } t > 0^-.
\end{aligned}$$

6.18 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform $X(s)$ of the signal $x(t)$ shown in the figure below.



Solution.

We have

$$x(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < 2 \\ -t+3 & \text{for } 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$

We rewrite $x(t)$ using unit-step functions to obtain

$$\begin{aligned} x(t) &= t\,[u(t)-u(t-1)] + [u(t-1)-u(t-2)] + [-t+3]\,[u(t-2)-u(t-3)] \\ &= tu(t) + (-t+1)u(t-1) + (-t+2)u(t-2) + (t-3)u(t-3) \\ &= tu(t) - (t-1)u(t-1) - (t-2)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Taking the Laplace transform of both sides of this equation, we have

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} \\ &= \frac{1-e^{-s}-e^{-2s}+e^{-3s}}{s^2}. \end{aligned}$$

Since $x(t)$ is of finite duration, the ROC of $X(s)$ is the entire complex plane.

6.19 Suppose that we have a LTI system with system function $H(s)$ given by

$$H(s) = \frac{s^2+7s+12}{s^2+3s+12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

Solution.

All of the inverse systems have the same algebraic expression $H^{\text{inv}}(s)$ for their system function, namely

$$H^{\text{inv}}(s) = \frac{1}{H(s)} = \frac{s^2+3s+12}{s^2+7s+12}.$$

Factoring the denominator of $H^{\text{inv}}(s)$, we have

$$H^{\text{inv}}(s) = \frac{s^2+3s+12}{(s+4)(s+3)}.$$

Obviously, the system function $H^{\text{inv}}(s)$ is rational and has poles at -4 and -3. Consequently, there are three possible ROCs associated with $H^{\text{inv}}(s)$: i) $\operatorname{Re}\{s\} < -4$; ii) $-4 < \operatorname{Re}\{s\} < -3$; and iii) $\operatorname{Re}\{s\} > -3$.

A system is BIBO stable if and only if the ROC of its system function contains the entire imaginary axis. Clearly, only the ROC $\operatorname{Re}\{s\} > -3$ contains the entire imaginary axis. So, only the inverse system associated with this ROC is stable.

A system with a rational system function is causal if and only if the ROC of the system function is the right-half plane to the right of the rightmost pole. Since $H^{\text{inv}}(s)$ is rational, only the ROC of $\operatorname{Re}\{s\} > -3$ is associated with a causal system.

6.101 Suppose that we have a causal LTI system with the system function

$$H(s) = \frac{1}{-2s^7 - s^6 - 3s^5 + 2s^3 + s - 3}.$$

(a) Use MATLAB to find and plot the poles of $H(s)$.
(b) Is this system stable?

Solution.

(a) The poles of the rational function $H(s)$ are simply the roots of the denominator polynomial. The roots can be obtained by the following code fragment:

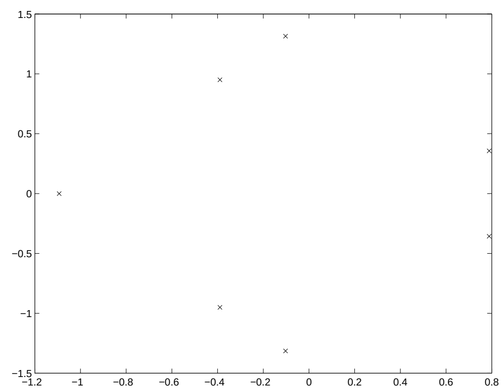
```
p = roots([-2 -1 -3 0 2 0 1 -3])
```

This yields the following poles:

```
-1.09309528728402
-0.10265208819816 + 1.31411369725196i
-0.10265208819816 - 1.31411369725196i
-0.38945961543237 + 0.94998954538106i
-0.38945961543237 - 0.94998954538106i
 0.78865934727254 + 0.35672610951156i
 0.78865934727254 - 0.35672610951156i
```

We can then plot the poles (as a set of points) with the following code fragment:

```
plot(real(p), imag(p), 'x');
print -dps poles.ps
```



(b) Since the system is causal, in order for it to be BIBO stable, all of the poles must be in the left half of the complex plane. We observe that two of the poles have nonnegative real parts. Therefore, the system is not BIBO stable.

6.102 Suppose that we have a LTI system with the system function

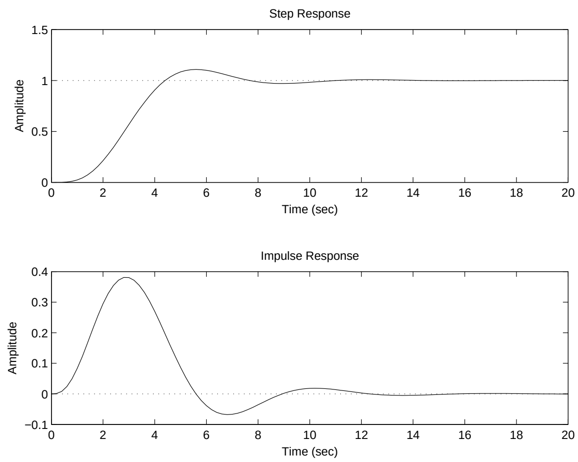
$$H(s) = \frac{1}{1.0000s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1.0000}.$$

(This corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the responses of the system to each of the inputs given below. In each case, plot the response over the interval $[0, 20]$.
(a) Dirac delta function;
(b) unit-step function.
(Hint: The `tf`, `impulse`, and `step` functions may be helpful.)

Solution.

(a) and (b) We can find the desired responses with the code given below.


```
tfnum = [0 0 0 0 1];
tfdenom = [1.0000 2.6131 3.4142 2.6131 1.0000];
finaltime = 20;
sys = tf(tfnum, tfdenom);
subplot(2, 1, 1);
step(sys, finaltime);
subplot(2, 1, 2);
impulse(sys, finaltime);
```



Chapter 13

Review (Appendix F)

F.2 Consider a system consisting of a communication channel with input $x(t)$ and output $y(t)$. Since the channel is not ideal, $y(t)$ is typically a distorted version of $x(t)$. Suppose that the channel can be modelled as a causal LTI system with impulse response $h(t) = e^{-t}u(t) + \delta(t)$. Determine whether we can devise a physically-realizable stable system that recovers $x(t)$ from $y(t)$. If such a system exists, find its impulse response $g(t)$.

Answer:

$g(t) = \delta(t) - e^{-2t}u(t)$

Solution. Essentially, we must determine whether the given system has a causal stable inverse. Let $g(t)$ denote the impulse response of the inverse of the given system. Let $H(s)$ and $G(s)$ denote the Laplace transforms of $h(t)$ and $g(t)$, respectively. Taking the Laplace transform of $h(t)$, we have

$$H(s) = 1 + \tfrac{1}{s+1} = \tfrac{s+1+1}{s+1} = \tfrac{s+2}{s+1}.$$

Since the given system is causal, $H(s)$ has a RHP ROC, namely, $\mathrm{Re}\{s\} > -1$. Next, we determine $G(s)$ to be

$$G(s) = \tfrac{1}{H(s)} = \tfrac{s+1}{s+2}.$$

Since the inverse system is required to be causal, the ROC of $G(s)$ is a RHP, namely, $\mathrm{Re}\{s\} > -2$. Since this ROC includes the imaginary axis, the inverse system is stable. Taking the inverse Laplace transform of $G(s)$, we obtain

$$\begin{aligned} g(t) &= \mathscr{L}^{-1}\{\tfrac{s+1}{s+2}\} = \mathscr{L}^{-1}\{\tfrac{s+2-1}{s+2}\} = \mathscr{L}^{-1}\{1 - \tfrac{1}{s+2}\} = \mathscr{L}^{-1}\{1\} - \mathscr{L}^{-1}\{\tfrac{1}{s+2}\} \\ &= \delta(t) - e^{-2t}u(t). \end{aligned}$$

Thus, a causal stable inverse of the given system does exist and has the impulse response $g(t) = \delta(t) - e^{-2t}u(t)$.

F.36 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input $x(t)$ and output $y(t)$. Suppose that the transmitted signal $x(t)$ propagates along two paths. Along the intended direct path, the channel has a delay of T and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of $T+\tau$ and gain of a. Thus, the received signal $y(t)$ is given by $y(t) = x(t-T) + ax(t-T-\tau)$. Find the transfer function $H(s)$ of a system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal $x(t-T)$ without any distortion.

Answer:

$H(s) = \frac{1}{1+ae^{-s\tau}}$

Solution. Let $G(s)$ denote the system function of the given wireless communication channel. We are given that

$$y(t) = x(t-T) + ax(t-T-\tau).$$

Taking the Laplace transform of the preceding equation, we obtain

$$\begin{aligned} Y(s) &= e^{-sT}X(s) + ae^{-(T+\tau)s}X(s) \\ &= (e^{-sT} + ae^{-(T+\tau)s})X(s). \end{aligned}$$

Using the preceding equation and the fact that $G(s) = \frac{Y(s)}{X(s)}$, we have

$$\begin{aligned} G(s) &= e^{-sT} + ae^{-(T+\tau)s} \\ &= e^{-sT}(1+ae^{-s\tau}). \end{aligned}$$

Now, we want $G(s)H(s) = e^{-sT}$. Thus, we have

$$H(s) = \frac{e^{-sT}}{G(s)} = \frac{1}{1+ae^{-s\tau}}.$$

Thus, $H(s) = \frac{1}{1+ae^{-s\tau}}$.

