Example 7.15 (Time-domain differentiation property)**.** Find the Laplace transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

LT table

Solution. From Table 7.2, we have that

$$\delta(t) \stackrel{\text{LT}}{\longleftrightarrow} 1 \ \text{ for all } s.$$

Using the time-domain differentiation property, we can deduce

multiply by s

ROC contains original ROC

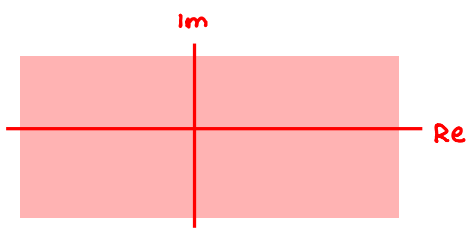
differentiate

$$x(t) = \tfrac{d}{dt}\delta(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \underbrace{s(1)}_{s} \ \text{ for all } s.$$

Therefore, we have

$$X(s) = s \quad \text{for all } s. \qquad \blacksquare$$

obviously, ROC cannot be larger



Sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, since no poles, ROC fills entire plane

Example 7.16 (Laplace-domain differentiation property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform of the function

$$x(t) = te^{-2t}u(t).$$

Solution. We are given

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

multiply by t   -d/ds   ROC unchanged

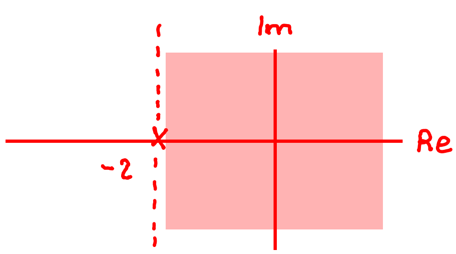
$$x(t) = te^{-2t}u(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) \quad \text{for } \operatorname{Re}(s) > -2.$$

Simplifying the algebraic expression for X, we have

$$X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) = -\tfrac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$



Sanity check:
are the stated algebraic expression and stated ROC self consistent?

yes, the ROC is bounded by poles or extends to $\pm\infty$

Example 7.17 (Time-domain integration property)**.** Find the Laplace transform of the function

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau.$$

LT table

Solution. From Table 7.2, we have that

$$e^{-2t}\sin(t)u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{(s+2)^2+1} \quad \text{for } \operatorname{Re}(s) > -2.$$

ROC is intersected with Re(s) > 0 (cannot be larger since no poles cancelled)

Using the time-domain integration property, we can deduce

integrate

multiply by 1/s

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau \overset{\text{LT}}{\longleftrightarrow} X(s) = \underbrace{\frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right)}_{\text{simplify}} \quad \text{for } \underbrace{\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}}_{\operatorname{Re}(s)>0}.$$

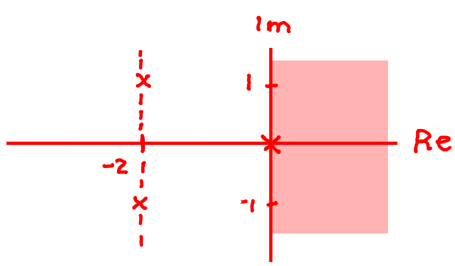
The ROC of X is $\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for X, we have

$$X(s) = \frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+4+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+5}\right).$$

Therefore, we have

$$X(s) = \frac{1}{s\underbrace{(s^2+4s+5)}} \quad \text{for } \operatorname{Re}(s) > 0.$$

[Note: $s^2+4s+5 = (s+2-j)(s+2+j)$.] $(s+2-j)(s+2+j)$ ■



sanity check:

are the stated algebraic expression and stated ROC self consistent?

yes, the ROC is bounded by poles or extends to $\pm\infty$

Example 7.18 (Initial and final value theorems)**.** A bounded causal function x with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s} \text{ for } \operatorname{Re}(s) > 0.$$

Determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Solution. Since x is causal (i.e., $x(t) = 0$ for all $t < 0$) and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) \\
&= \lim_{s\to\infty} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] \\
&= \lim_{s\to\infty} \frac{2s^2+3s+2}{s^2+2s+2} \\
&= 2.
\end{aligned}$$

substitute given X

multiply

take limit (highest power terms dominate)

Since x is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\
&= \lim_{s\to 0} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] \\
&= \left.\frac{2s^2+3s+2}{s^2+2s+2}\right|_{s=0} \\
&= 1.
\end{aligned}$$

substitute given X

multiply

evaluate at s=0

In passing, we note that the inverse Laplace transform x of X can be shown to be

$$x(t) = [1+e^{-t}\cos t]u(t).$$

As we would expect, the values calculated above for $x(0^+)$ and $\lim_{t\to\infty} x(t)$ are consistent with this formula for x. ■