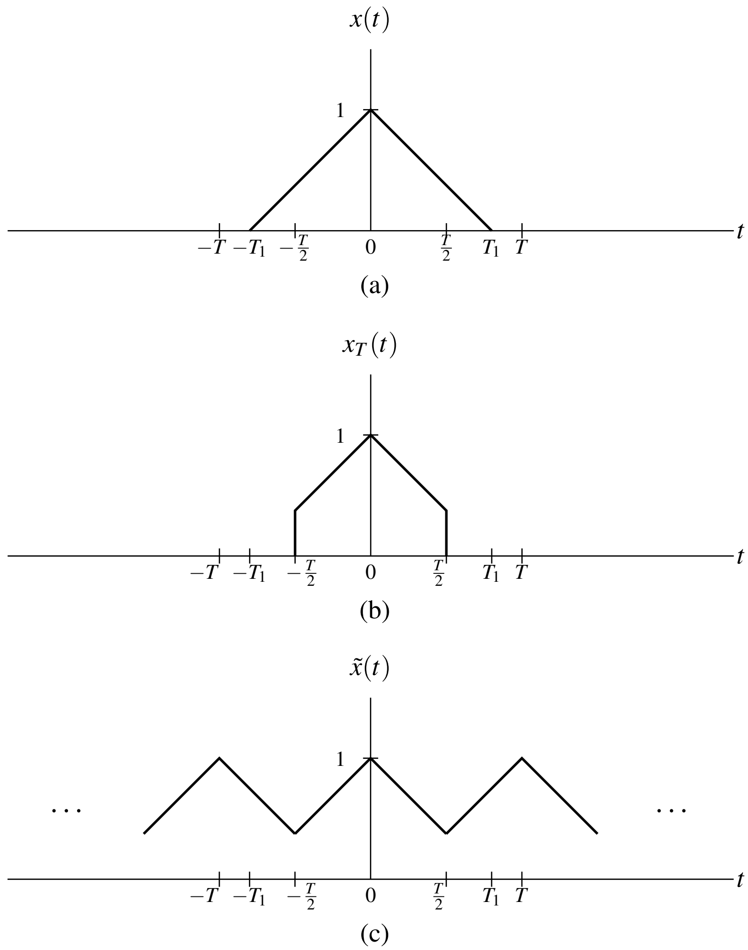

Figure 6.1: An example of the functions used in the derivation of the Fourier transform representation, where $T_1 > \frac{T}{2}$. (a) An aperiodic function x; (b) the function x_T; and (c) the T-periodic function $\tilde{x}$.

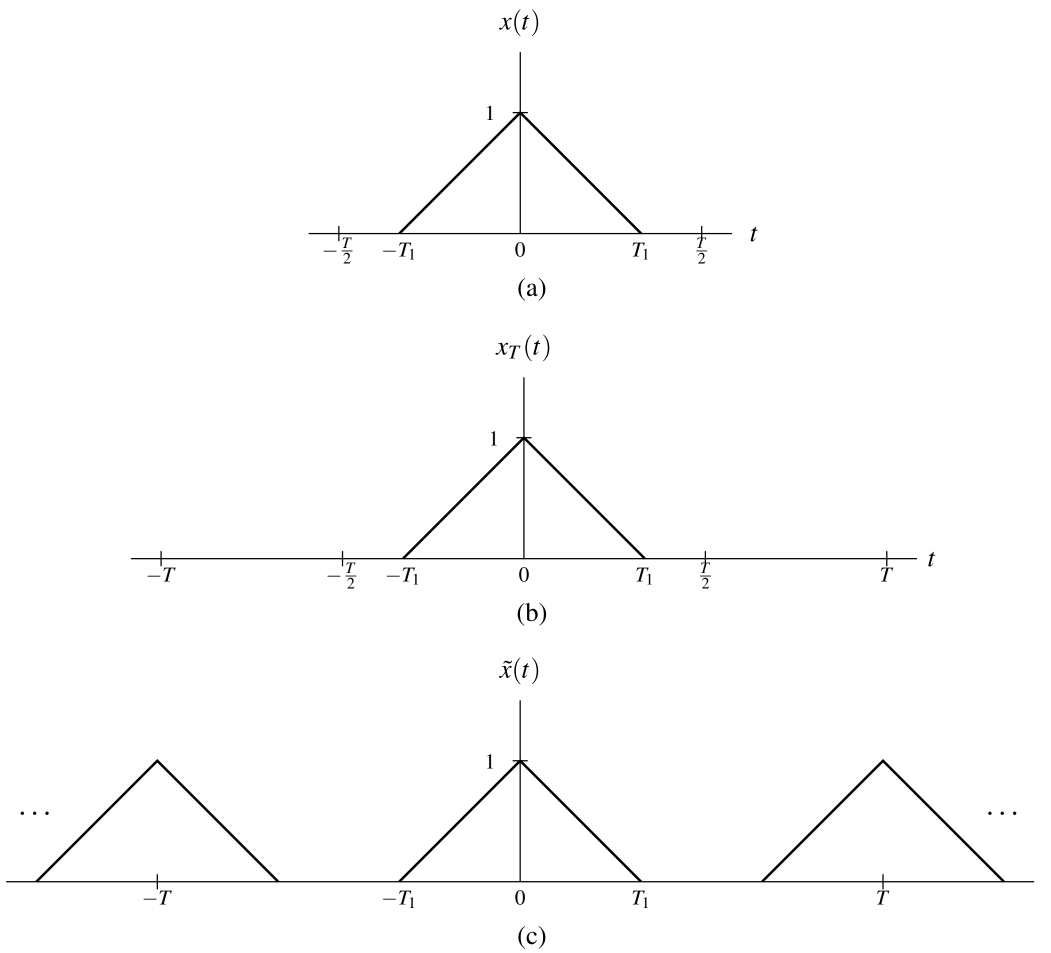

Figure 6.2: An example of the functions used in the derivation of the Fourier transform representation, where $T_1 < \frac{T}{2}$. (a) An aperiodic function x; (b) the function x_T; and (c) the T-periodic function $\tilde{x}$.

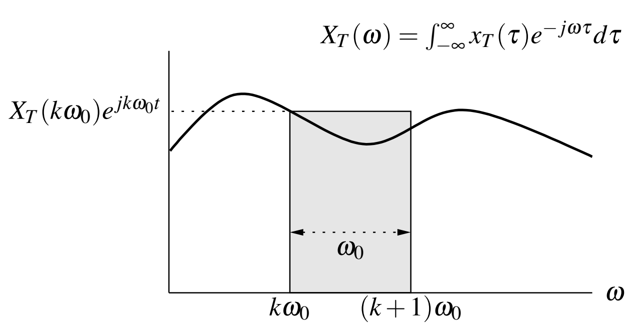

Figure 6.3: Integral obtained in the derivation of the Fourier transform representation.

practical interest do not have a Fourier transform in the sense of the definition developed previously. That is, for a given function x, the Fourier transform integral

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

may fail to converge, in which case the Fourier transform X of x does not exist. For example, the preceding integral does not converge if x is any of the following (as well as many other possibilities):

- a nonzero constant function;
- a periodic function (e.g., a real or complex sinusoid);
- the unit-step function (i.e., u); or
- the signum function (i.e., sgn).

Functions such as these are of great practical interest, however. Therefore, it is highly desirable to have a mathematical tool that can handle such functions. This motivates the development of what is called the **generalized Fourier transform**. The generalized Fourier transform exists for periodic functions, nonzero constant functions, and many other types of functions as well. The underlying math associated with the generalized Fourier transform is quite complicated. So, we will not attempt to formally develop the generalized Fourier transform here. Although not entirely correct, one can think of the generalized Fourier transform as being defined by the same formulas as the classical Fourier transform. So, for this and other reasons, we can mostly ignore the distinction between the generalized Fourier transform and classical Fourier transform, and think of them as being one and the same. In what follows, we will avoid making a distinction between the classical Fourier transform and generalized Fourier transform, except in a very small number of places where it is beneficial to do so. The main disadvantage of not formally introducing the generalized Fourier transform is that some results presented later (which actually rely on the use of the generalized Fourier transform) must be accepted on faith since their proof would require formal knowledge of the generalized Fourier transform, which is not introduced herein. As long as the generalized Fourier transform is used, both periodic and aperiodic functions can be handled, and in this sense we have a more general tool than Fourier series (which require periodic functions). Later, when we discuss the Fourier transform of periodic functions, we will implicitly be using the generalized Fourier transform in that context. In fact, in much of what follows, when we speak of the Fourier transform, we are often referring to the generalized Fourier transform.

6.4 Definition of the Continuous-Time Fourier Transform

Earlier, we developed the Fourier transform representation of a function. This representation expresses a function in terms of complex sinusoids at all frequencies. More formally, the **Fourier transform** of the function x, denoted as $\mathcal{F}x$ or X, is defined as

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{6.7}$$

Similarly, the inverse Fourier transform of X, denoted as $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega. \tag{6.8}$$

We refer to (6.7) as the **Fourier transform analysis equation** and (6.8) as the **Fourier transform synthesis equation**. To denote that a function x has the Fourier transform X, we can write

$$x(t) \xleftrightarrow{\text{CTFT}} X(\omega).$$

As a matter of terminology, x and X are said to constitute a **Fourier transform pair**.

Example 6.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform X of the function

$$x(t) = A\delta(t-t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j\omega t}dt \\ &= A\int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$\begin{aligned} X(\omega) &= A\left[e^{-j\omega t}\right]\Big|_{t=t_0} \\ &= Ae^{-j\omega t_0}. \end{aligned}$$

Thus, we have shown that

$$A\delta(t-t_0) \overset{\text{CTFT}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = Ae^{-j\omega t_0}.$$ ■

Example 6.2 (Inverse Fourier transform of the unit-impulse function)**.** Find the inverse Fourier transform x of the function

$$X(\omega) = 2\pi A\delta(\omega-\omega_0),$$

where A and ω_0 are real constants.

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\omega-\omega_0)e^{j\omega t}d\omega \\ &= A\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$x(t) = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi A\delta(\omega-\omega_0).$$ ■

Example 6.3 (Fourier transform of the rectangular function)**.** Find the Fourier transform X of the function

$$x(t) = \operatorname{rect} t.$$

Solution. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t} dt.$$

From the definition of the rectangular function, we can simplify this equation to obtain

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

Evaluating the integral and simplifying, we have

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t} \right] \Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \tfrac{1}{j\omega} \left[2j \sin\left(\tfrac{1}{2}\omega\right) \right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{1}{2}\omega\right) \\ &= \left[\sin\left(\tfrac{1}{2}\omega\right) \right] / \left(\tfrac{1}{2}\omega\right) \\ &= \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \end{aligned}$$

Thus, we have shown that

$$\operatorname{rect} t \stackrel{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \qquad \blacksquare$$

6.5 Remarks on Notation Involving the Fourier Transform

Each of the Fourier transform operator $\mathcal{F}$ and inverse Fourier transform operator $\mathcal{F}^{-1}$ map a function to a function. Consequently, the operand for each of these operators must be a function (and not a number). Consider the unnamed function that maps t to $e^{-|t|}$ as shown in Figure 6.4. Suppose that we would like to write an expression that denotes the Fourier transform of this function. At first, we might be inclined to write "$\mathcal{F}\{e^{-|t|}\}$". Strictly speaking, however, this notation is not correct, since the Fourier transform operator requires a function as an operand and "$e^{-|t|}$" (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at t). Essentially, the cause of our problems here is that the function in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a function x using the equation $x(t) = e^{-|t|}$ and then write the Fourier transform as "$\mathcal{F}x$". Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to overly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 6.4 that maps t to $e^{-|t|}$. Using strictly correct notation, we could write the Fourier transform of this function as "$\mathcal{F}\left\{e^{-|\cdot|}\right\}$". In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 6.4. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 6.4 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ (without the need to introduce the named function x), we can write: $\mathcal{F}\left\{e^{(\cdot)^2}\right\}$.

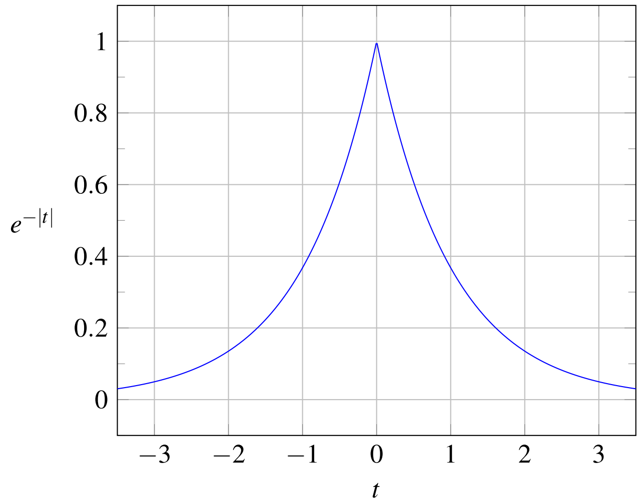

Figure 6.4: A plot of $e^{-|t|}$ versus t.

2. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ evaluated at $2\omega - 3$ (without the need to introduce the named function x), we can write: $\mathcal{F}\left\{e^{(\cdot)^2}\right\}(2\omega - 3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ evaluated at $t - 3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}(t-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Fourier transform operator $\mathcal{F}$, the variable "t" is assumed to be the independent variable for the function to which the Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "t" were a "$\cdot$");

- unless indicated otherwise, in an expression for the operand of the inverse Fourier transform operator $\mathcal{F}^{-1}$, the variable "ω" is assumed to be the independent variable for the function to which the inverse Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "ω" were a "$\cdot$")

Some examples of using these book-sanctioned notational conventions can be found below in Example 6.5. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 6.5 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ (without the need to introduce the named function x), we can write: $\mathcal{F}\{e^{t^2}\}$.

2. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ evaluated at $2\omega - 3$ (without the need to introduce the named function x), we can write: $\mathcal{F}\{e^{t^2}\}(2\omega - 3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}(t-3)$. ■

Since applying the Fourier transform operator or inverse Fourier transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 6.4 that maps t to $e^{-|t|}$. To denote the value of the Fourier transform of this function evaluated at $\omega-1$, we would write "$\mathcal{F}\{e^{-|\cdot|}\}(\omega-1)$" using dot notation or "$\mathcal{F}\{e^{-|t|}\}(\omega-1)$" using the book-sanctioned notational conventions described above.

6.6 Convergence of the Continuous-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary function x. This function has the Fourier transform representation $\hat{x}$ given by

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}$ is a valid representation of x. In our earlier derivation of the Fourier transform, we relied heavily on the Fourier series. Therefore, one might expect that the convergence of the Fourier transform representation is closely related to the convergence properties of Fourier series. This is, in fact, the case. The convergence properties of the Fourier transform are very similar to the convergence properties of the Fourier series (as studied in Section 5.4).

The first important result concerning convergence relates to continuous functions as stated by the theorem below.

Theorem 6.1 (Convergence of the Fourier transform (continuous case))**.** *If a function x is continuous and absolutely integrable (i.e., $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty}|X(\omega)|\,d\omega < \infty$), then the Fourier transform representation of x converges pointwise (i.e., $x(t) = \hat{x}(t)$ for all t).*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value. This motivates us to consider additional results concerning convergence.

The next important result concerning convergence relates to finite-energy functions as stated by the theorem below.

Theorem 6.2 (Convergence of Fourier transform (finite-energy case))**.** *If a function x is of finite energy (i.e., $\int_{-\infty}^{\infty}|x(t)|^2\,dt < \infty$), then its Fourier transform representation $\hat{x}$ converges in the MSE sense.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

In other words, the preceding theorem states that, if x is of finite energy, then

$$E = \int_{-\infty}^{\infty}|\hat{x}(t) - x(t)|^2\,dt = 0.$$

Although x and $\hat{x}$ may differ at individual points, the energy E in the difference is zero.

The last result concerning convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet conditions for the function x are as follows:

1. The function x is absolutely integrable (i.e., $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$).

2. The function x has a finite number of maxima and minima on any finite interval (i.e., is of bounded variation).

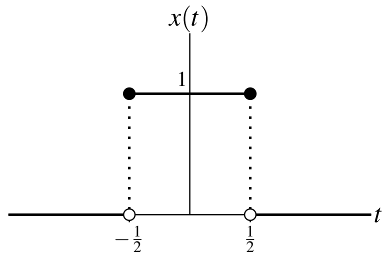

Figure 6.5: Function x.

3. The function x has a finite number of discontinuities on any finite interval, and each discontinuity is itself finite.

For a function satisfying the Dirichlet conditions, we have the important convergence result stated below.

Theorem 6.3 (Convergence of Fourier transform (Dirichlet case))**.** *If a function x satisfies the Dirichlet conditions, then its Fourier transform representation $\hat{x}$ converges pointwise everywhere except at points of discontinuity. Furthermore, at each discontinuity point t_a, we have that*

$$\hat{x}(t_a) = \tfrac{1}{2}\left[x\left(t_a^-\right) + x\left(t_a^+\right)\right],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

In other words, the preceding theorem states that, if a function x satisfies the Dirichlet conditions, then the Fourier transform representation $\hat{x}$ is such that $\hat{x}(t) = x(t)$ for all t, except at points of discontinuity where $\hat{x}(t)$ equals the average value of x on the two sides of the discontinuity.

The finite-energy and Dirichlet conditions mentioned above are only sufficient conditions for the convergence of the Fourier transform representation. They are not necessary. In other words, a function may violate these conditions and still have a valid Fourier transform representation.

Example 6.6. Consider the function x shown in Figure 6.5. Let $\hat{x}$ denote the Fourier transform representation of x (i.e., $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, where X denotes the Fourier transform of x). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 6.3 applies. Thus, we have that

$$\begin{aligned}\hat{x}(-\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right]\\ &= \tfrac{1}{2}(0+1)\\ &= \tfrac{1}{2} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\hat{x}(\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right]\\ &= \tfrac{1}{2}(1+0)\\ &= \tfrac{1}{2}.\end{aligned}$$

■

6.7 Properties of the Continuous-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 6.1 (on page 179). Also, for convenience, several Fourier-transform pairs are given later in Table 6.2 (on page 183). In what follows, we will sometimes refer to transform pairs in this table.

6.7.1 Linearity

Arguably, the most important property of the Fourier transform is linearity, as introduced below.

Theorem 6.4 (Linearity)**.** *If* $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ *and* $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$*, then*

$$a_1x_1(t) + a_2x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega),$$

where a_1 *and* a_2 *are arbitrary complex constants.*

Proof. To prove the above property, we proceed as follows. Let $y(t) = a_1x_1(t) + a_2x_2(t)$ and let $Y = \mathcal{F}y$. We have

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} a_1x_1(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-j\omega t}dt \\
&= a_1\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt \\
&= a_1X_1(\omega) + a_2X_2(\omega).
\end{aligned}$$

Thus, we have shown that the linearity property holds. ■

Example 6.7 (Linearity property of the Fourier transform)**.** Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0),$$

find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t),$$

where A and ω_0 are real constants.

Solution. We recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned}
X(\omega) &= (\mathcal{F}\{A\cos(\omega_0 t)\})(\omega) \\
&= \left(\mathcal{F}\{\tfrac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\}\right)(\omega).
\end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

$$X(\omega) = \tfrac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \tfrac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

Using the given Fourier transform pair, we can further simplify the above expression for $X(\omega)$ as follows:

$$\begin{aligned}
X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega + \omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega - \omega_0)] \\
&= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].
\end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$ ■

Example 6.8 (Fourier transform of the unit-step function)**.** Using properties of the Fourier transform and the transform pairs

$$1 \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega) \quad \text{and} \quad \operatorname{sgn} t \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{2}{j\omega},$$

find the Fourier transform X of the function $x = u$.

Solution. First, we observe that x can be expressed in terms of the signum function as

$$x(t) = u(t) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t.$$

Taking the Fourier transform of both sides of this equation yields

$$X(\omega) = \left(\mathcal{F}\{\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t\}\right)(\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$X(\omega) = \tfrac{1}{2}\mathcal{F}\{1\}(\omega) + \tfrac{1}{2}\mathcal{F}\{\operatorname{sgn} t\}(\omega).$$

Evaluating the two Fourier transforms using the given transform pairs, we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{1}{2}(\tfrac{2}{j\omega}) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

Thus, we have shown that

$$u(t) \overset{\text{CTFT}}{\longleftrightarrow} \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

■

6.7.2 Time-Domain Shifting (Translation)

The next property of the Fourier transform to be introduced is the time-domain shifting (i.e., translation) property, as given below.

Theorem 6.5 (Time-domain shifting (i.e., translation))**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

where t_0 is an arbitrary real constant.

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(t - t_0)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt.$$

Now, we use a change of variable. Let $\lambda = t - t_0$ so that $t = \lambda + t_0$ and $dt = d\lambda$. Performing the change of variable and simplifying, we obtain

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + t_0)} d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} e^{-j\omega t_0} d\lambda \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\ &= e^{-j\omega t_0} X(\omega). \end{aligned}$$

Thus, we have proven that the time-shifting property holds. ■

Example 6.9 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

Solution. Let $v(t) = A\cos(\omega_0 t)$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. Also, let $V = \mathcal{F}v$. From Table 6.2, we have that

$$\cos(\omega_0 t) \xleftrightarrow{\text{CTFT}} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

Using this transform pair and the linearity property of the Fourier transform, we have that

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) \\ &= A\mathcal{F}\{\cos(\omega_0 t)\}(\omega) \\ &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \xleftrightarrow{\text{CTFT}} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \qquad \blacksquare$$

6.7.3 Frequency-Domain Shifting (Modulation)

The next property of the Fourier transform to be introduced is the frequency-domain shifting (i.e., modulation) property, as given below.

Theorem 6.6 (Frequency-domain shifting (i.e., modulation))**.** *If* $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, *then*

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\text{CTFT}} X(\omega - \omega_0),$$

where ω_0 *is an arbitrary real constant.*

Proof. To prove the above property, we proceed as follows. Let $y(t) = e^{j\omega_0 t}$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform and straightforward algebraic manipulation, we can write

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t}\, dt \\ &= X(\omega - \omega_0). \end{aligned}$$

Thus, we have shown that the frequency-domain shifting property holds. $\blacksquare$

Example 6.10 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = \cos(\omega_0 t)\cos(20\pi t),$$

where ω_0 is a real constant.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{\cos(\omega_0 t)(\tfrac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})\}(\omega) \\ &= \mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t) + \tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}(\omega) \\ &= \tfrac{1}{2}\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}(\omega) + \tfrac{1}{2}\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}(\omega). \end{aligned}$$

From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega - 20\pi) + \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega + 20\pi) \\ &= \tfrac{1}{2}\left[\pi\left[\delta(v - \omega_0) + \delta(v + \omega_0)\right]\right]\big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi\left[\delta(v - \omega_0) + \delta(v + \omega_0)\right]\right]\big|_{v=\omega+20\pi} \\ &= \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]\right) \\ &= \tfrac{\pi}{2}\left[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)\right]. \end{aligned}$$

■

6.7.4 Time- and Frequency-Domain Scaling (Dilation)

The next property of the Fourier transform to be introduced is the time/frequency-scaling (i.e., dilation) property, as given below.

Theorem 6.7 (Time/frequency-domain scaling (i.e., dilation))**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(at)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we can write

$$Y(\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt.$$

Now, we use a change of variable. Let $\lambda = at$ so that $t = \lambda/a$ and $dt = d\lambda/a$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$\begin{aligned} Y(\omega) &= \begin{cases} \displaystyle\int_{-a(\infty)}^{a(\infty)} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a > 0 \\ \displaystyle\int_{-a(\infty)}^{a(\infty)} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a < 0 \end{cases} \\ &= \begin{cases} \displaystyle\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a > 0 \\ \displaystyle\int_{\infty}^{-\infty} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a < 0 \end{cases} \\ &= \begin{cases} \displaystyle\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & a > 0 \\ \displaystyle-\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & a < 0. \end{cases} \end{aligned}$$

Combining the two cases (i.e., for $a < 0$ and $a > 0$), we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda \\ &= \tfrac{1}{|a|}X\left(\tfrac{\omega}{a}\right). \end{aligned}$$

Thus, we have shown that the time/frequency-scaling property holds. ■

Example 6.11 (Time scaling property of the Fourier transform)**.** Using the Fourier transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right),$$

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let $v(t) = \operatorname{rect} t$ so that $x(t) = v(at)$. Also, let $V = \mathcal{F}v$. From the given transform pair, we know that

$$V(\omega) = \mathcal{F}\{\operatorname{rect} t\}(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right). \tag{6.9}$$

From the definition of v and the time-scaling property of the Fourier transform, we have

$$X(\omega) = \frac{1}{|a|}V\left(\frac{\omega}{a}\right).$$

Substituting the expression for V in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|}\operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

Thus, we have shown that

$$\operatorname{rect}(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|}\operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

■

6.7.5 Conjugation

The next property of the Fourier transform to be introduced is the conjugation property, as given below.

Theorem 6.8 (Conjugation)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = x^*(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned} Y(\omega) &= \left[\left(\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt\right)^*\right]^* \\ &= \left[\int_{-\infty}^{\infty} [x(t)^*]^* \left(e^{-j\omega t}\right)^* dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt\right]^* \\ &= X^*(-\omega). \end{aligned}$$

Thus, we have shown that the conjugation property holds. ■

Example 6.12 (Fourier transform of a real function)**.** Let X denote the Fourier transform of the function x. Show that, if x is real, then X is conjugate symmetric (i.e., $X(\omega) = X^*(-\omega)$ for all ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(t)\}(\omega) = X^*(-\omega).$$

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\omega) = X^*(-\omega),$$

or equivalently

$$X(\omega) = X^*(-\omega). \qquad \blacksquare$$

6.7.6 Duality

The next property of the Fourier transform to be introduced is the duality property, as given below.

Theorem 6.9 (Duality)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega).$$

Proof. To prove the above property, we proceed as follows. From the Fourier transform synthesis equation, we have

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda.$$

Substituting $-\omega$ for t, we obtain

$$x(-\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda\omega}d\lambda.$$

Now, we multiply both sides of the equation by 2π to yield

$$\begin{aligned} 2\pi x(-\omega) &= \int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda\omega}d\lambda \\ &= \mathcal{F}X(\omega). \end{aligned}$$

Thus, we have shown that the duality property holds. $\blacksquare$

The duality property stated in the preceding theorem follows from the high degree of similarity in the equations for the forward and inverse Fourier transforms, given by (6.7) and (6.8), respectively. To make this similarity more obvious, we can rewrite the forward and inverse Fourier transform equations, respectively, as

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta)e^{-j\theta\lambda}d\theta \quad \text{and} \quad x(\lambda) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)e^{j\theta\lambda}d\theta.$$

Observe that these two equations are identical except for: 1) a factor of 2π; and 2) a different sign in the parameter for the exponential function. Consequently, if we were to accidentally use one equation in place of the other, we would obtain an almost correct result. In fact, this almost correct result could be made to be correct by compensating for the above two differences (i.e., the factor of 2π and the sign difference in the exponential function). This is, in effect, what the duality property states.

Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform X of x, the duality property allows us to indirectly infer the Fourier transform of X. Consequently, the duality property can be used to effectively double the number of Fourier transform pairs that we know.

Example 6.13 (Fourier transform of the sinc function)**.** Using the transform pair

$$\operatorname{rect} t \stackrel{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right),$$

find the Fourier transform X of the function

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right).$$

Solution. From the given Fourier transform pair, we have

$$v(t) = \operatorname{rect} t \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad V(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right).$$

By duality, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi v(-\omega) = 2\pi \operatorname{rect}(-\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi \operatorname{rect} \omega.$$

Observing that $V = x$ and $\mathcal{F}V = X$, we can rewrite the preceding relationship as

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad X(\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have shown that

$$X(\omega) = 2\pi \operatorname{rect} \omega. \qquad \blacksquare$$

6.7.7 Time-Domain Convolution

The next property of the Fourier transform to be introduced is the time-domain convolution property, as given below.

Theorem 6.10 (Time-domain convolution)**.** *If* $x_1(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ *and* $x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$*, then*

$$x_1 * x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

Proof. The proof of this property is as follows. Let $y(t) = x_1 * x_2(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform and convolution, we have

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} [x_1 * x_2(t)]\, e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-j\omega t} d\tau dt. \end{aligned}$$

Changing the order of integration, we obtain

$$Y(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-j\omega t} dt d\tau.$$

Now, we use a change of variable. Let $\lambda = t-\tau$ so that $t = \lambda+\tau$ and $d\lambda = dt$. Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega(\lambda+\tau)}d\lambda d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega\lambda}e^{-j\omega\tau}d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda\right]d\tau \\
&= \left[\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau\right]\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda\right] \\
&= X_1(\omega)X_2(\omega).
\end{aligned}$$

Thus, we have shown that the time-domain convolution property holds. ■

The time-domain convolution property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts a convolution into a multiplication, the Fourier transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when solving problems involving LTI systems, for example, since such problems almost inevitably involve convolution (due to the fact that a LTI system computes a convolution).

Example 6.14 (Time-domain convolution property of the Fourier transform)**.** With the aid of Table 6.2, find the Fourier transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

Solution. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$\begin{aligned}
X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\
&= X_1(\omega)X_2(\omega).
\end{aligned} \tag{6.10}$$

From Table 6.2, we know that

$$\begin{aligned}
X_1(\omega) &= \left(\mathcal{F}\{e^{-2t}u(t)\}\right)(\omega) \\
&= \tfrac{1}{2+j\omega} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
X_2(\omega) &= \mathcal{F}u(\omega) \\
&= \pi\delta(\omega) + \tfrac{1}{j\omega}.
\end{aligned}$$

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (6.10), we obtain

$$\begin{aligned}
X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}.
\end{aligned}$$

From the equivalence property of the delta function, we have

$$X(\omega) = \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}.$$ ■

6.7.8 Time-Domain Multiplication

The next property of the Fourier transform to be introduced is the time-domain multiplication (or frequency-domain convolution) property, as given below.

Theorem 6.11 (Time-domain multiplication)**.** *If* $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ *and* $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$*, then*

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{2\pi} X_1 * X_2(\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

Proof. To prove the above property, we proceed as follows. Let $Y(\omega) = \frac{1}{2\pi}X_1 * X_2(\omega)$ and let $y = \mathcal{F}^{-1}Y$. From the definition of the inverse Fourier transform, we have

$$\begin{aligned}
y(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\tfrac{1}{2\pi}X_1 * X_2(\omega)\right]e^{j\omega t}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)d\lambda\right]e^{j\omega t}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\lambda d\omega.
\end{aligned}$$

Reversing the order of integration, we obtain

$$y(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\omega d\lambda.$$

Now, we employ a change of variable. Let $v = \omega - \lambda$ so that $\omega = v + \lambda$ and $dv = d\omega$. Applying the change of variable and simplifying yields

$$\begin{aligned}
y(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}X_1(\lambda)X_2(v)e^{j(v+\lambda)t}dv d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}X_1(\lambda)X_2(v)e^{jvt}e^{j\lambda t}dv d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right]d\lambda \\
&= \left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}d\lambda\right]\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right] \\
&= \left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\omega)e^{j\omega t}d\omega\right]\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)e^{j\omega t}d\omega\right] \\
&= x_1(t)x_2(t).
\end{aligned}$$

Thus, we have shown that the frequency-domain convolution property holds. ■

From the time-domain multiplication property in the preceding theorem, we can see that the Fourier transform effectively converts a multiplication operation into a convolution operation (with a scale factor of $\frac{1}{2\pi}$). Since convolution is significantly more complicated than multiplication, we normally prefer to avoid using this property in a manner that would result in the introduction of additional convolution operations into our work.

Example 6.15 (Frequency-domain convolution property)**.** Let x and y be functions related as

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a nonzero real constant. Let $Y = \mathcal{F}y$ and $X = \mathcal{F}x$. Find an expression for Y in terms of X.

Solution. To allow for simpler notation in what follows, we define

$$v(t) = \cos(\omega_c t)$$

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)].$$

From the definition of v, we have

$$y(t) = x(t)v(t).$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega).$$

Using the frequency-domain convolution property of the Fourier transform, we obtain

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi} X * V(\omega) \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda.
\end{aligned}$$

Substituting the above expression for V, we obtain

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)])d\lambda \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)]d\lambda \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda-\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda+\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-\omega_c)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+\omega_c)]d\lambda\right] \\
&= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\
&= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c).
\end{aligned}$$

■

6.7.9 Time-Domain Differentiation

The next property of the Fourier transform to be introduced is the time-domain differentiation property, as given below.

Theorem 6.12 (Time-domain differentiation). *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, *then*

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

Proof. To prove the above property, we proceed as follows. Let $Y(\omega) = j\omega X(\omega)$ and let $y = \mathcal{F}^{-1}Y$. We begin by using the definition of the inverse Fourier transform to write

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Now, we differentiate both sides of the preceding equation with respect to t and simplify to obtain

$$\begin{aligned}
\tfrac{dx(t)}{dt} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega \\
&= y(t).
\end{aligned}$$

Thus, we have shown that the time-differentiation property holds. ■

By repeated application of the preceding theorem, we can obtain the more general result that

$$\left(\tfrac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega).$$

The time-domain differentiation property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts differentiation into multiplication (by $j\omega$), the Fourier transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations.

Example 6.16 (Time-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. Taking the Fourier transform of both sides of the given equation for x yields

$$X(\omega) = \mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}(\omega).$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}(\omega) \\ &= j\omega\mathcal{F}\delta(\omega). \end{aligned}$$

Evaluating the Fourier transform of δ using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j\omega(1) \\ &= j\omega. \end{aligned}$$

■

6.7.10 Frequency-Domain Differentiation

The next property of the Fourier transform to be introduced is the frequency-domain differentiation property, as given below.

Theorem 6.13 (Frequency-domain differentiation)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\tfrac{d}{d\omega}X(\omega).$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = tx(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we differentiate both sides of this equation with respect to ω and simplify to obtain

$$\begin{aligned} \tfrac{d}{d\omega}X(\omega) &= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt \\ &= -j\int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt \\ &= -jY(\omega). \end{aligned}$$

Multiplying both sides of the preceding equation by j yields

$$j\tfrac{d}{d\omega}X(\omega) = Y(\omega).$$

Thus, we have shown that the frequency-domain differentiation property holds. ■

Example 6.17 (Frequency-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = t\cos(\omega_0 t),$$

where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for x yields

$$X(\omega) = \mathcal{F}\{t\cos(\omega_0 t)\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{t\cos(\omega_0 t)\}(\omega) \\ &= j\left(\mathcal{D}\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega), \end{aligned}$$

where $\mathcal{D}$ denotes the derivative operator. Evaluating the Fourier transform on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). \end{aligned}$$ ■

6.7.11 Time-Domain Integration

The next property of the Fourier transform to be introduced is the time-domain integration property, as given below.

Theorem 6.14 (Time-domain integration)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

Proof. The above property can be proven as follows. Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, $Y = \mathcal{F}y$, and $U = \mathcal{F}u$. First, we observe that

$$y(t) = x * u(t).$$

Taking the Fourier transform of both sides of the preceding equation and using the time-domain convolution property of the Fourier transform, we have

$$Y(\omega) = X(\omega)U(\omega). \tag{6.11}$$

From Example 6.8, we know that $u(t) \overset{\text{CTFT}}{\longleftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}$. Using this fact, we can rewrite (6.11) as

$$\begin{aligned} Y(\omega) &= X(\omega)\left[\pi\delta(\omega)+\tfrac{1}{j\omega}\right] \\ &= \tfrac{1}{j\omega}X(\omega)+\pi X(\omega)\delta(\omega). \end{aligned}$$

From the equivalence property of the delta function, we have

$$Y(\omega) = \tfrac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

Thus, we have shown that the time-domain integration property holds. ■

The time-domain integration property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts integration into an operation involving division (by $j\omega$), the Fourier transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations.

Example 6.18 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform X of the function $x = u$.

Solution. We begin by observing that x can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

Now, we consider the Fourier transform of x. We have

$$X(\omega) = \mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)d\tau\right\}(\omega).$$

From the time-domain integration property, we can write

$$X(\omega) = \tfrac{1}{j\omega}\mathcal{F}\delta(\omega) + \pi\mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \tfrac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

Thus, we have shown that $u(t) \stackrel{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. ■

6.7.12 Parseval's Relation

The next property of the Fourier transform to be introduced, given below, relates to signal energy and is known as Parseval's relation.

Theorem 6.15 (Parseval's relation)**.** *If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then*

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \tag{6.12}$$

That is, the energy of x and energy of X are equal within a scaling factor of 2π. (Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 dt$.)

Proof. To prove the above relationship, we proceed as follows. Consider the left-hand side of (6.12) which we can write as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\ &= \int_{-\infty}^{\infty} x(t)\mathcal{F}^{-1}\{\mathcal{F}(x^*)\}(t)dt. \end{aligned}$$

From the conjugation property of the Fourier transform, we have that $x^*(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X^*(-\omega)$. So, we can rewrite the above equation as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\left(\mathcal{F}^{-1}\{X^*(-\omega)\}\right)(t)dt \\ &= \int_{-\infty}^{\infty} x(t)\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega)e^{j\omega t}d\omega\right]dt. \end{aligned}$$

Now, we employ a change of variable (i.e., replace ω by $-\omega$) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|x(t)|^2\,dt &= \int_{-\infty}^{\infty}x(t)\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)e^{-j\omega t}d\omega\right]dt\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)X^*(\omega)e^{-j\omega t}d\omega dt.\end{aligned}$$

Reversing the order of integration and simplifying, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|x(t)|^2\,dt &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(t)X^*(\omega)e^{-j\omega t}dtd\omega\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)\left(\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right)d\omega\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}X^*(\omega)X(\omega)d\omega\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega.\end{aligned}$$

Thus, Parseval's relation holds. ■

Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform preserves energy (up to a scale factor). For example, if we are solving a problem in the Fourier domain, we do not have to return to the time domain to compute energy, since we can do this directly in the Fourier domain by using Parseval's relation.

Example 6.19 (Energy of the sinc function)**.** Consider the function $x(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right)$, which has the Fourier transform X given by $X(\omega) = 2\pi\operatorname{rect}\omega$. Compute the energy of x.

Solution. We could directly compute the energy of x as

$$\begin{aligned}E &= \int_{-\infty}^{\infty}|x(t)|^2\,dt\\ &= \int_{-\infty}^{\infty}\left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2dt.\end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned}E &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|2\pi\operatorname{rect}\omega|^2\,d\omega\\ &= \frac{1}{2\pi}\int_{-1/2}^{1/2}(2\pi)^2d\omega\\ &= 2\pi\int_{-1/2}^{1/2}d\omega\\ &= 2\pi[\omega]|_{-1/2}^{1/2}\\ &= 2\pi[\tfrac{1}{2}+\tfrac{1}{2}]\\ &= 2\pi.\end{aligned}$$

Thus, we have

$$E = \int_{-\infty}^{\infty}\left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2dt = 2\pi.$$ ■

6.7.13 Even/Odd Symmetry

The Fourier transform preserves symmetry. In other words, we have the result below.

Theorem 6.16 (Even/odd symmetry)**.** *For a function x with Fourier transform X, the following assertions hold:*

- *x is even if and only if X is even; and*
- *x is odd if and only if X is odd.*

Proof. First, we show that, if a function x is even/odd, then its Fourier transform X is even/odd. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Since x is even/odd, we have that $x(t) = \pm x(-t)$, where the plus case and minus case in the "$\pm$" correspond to x being even and odd, respectively. Using this, we can rewrite the above expression for $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} \pm x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda = -t$ so that $d\lambda = -dt$. Applying this change of variable, we obtain

$$\begin{aligned}
X(\omega) &= \int_{\infty}^{-\infty} \pm x(\lambda)e^{-j\omega(-\lambda)}(-1)d\lambda \\
&= \mp \int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda}d\lambda \\
&= \pm \int_{-\infty}^{\infty} x(\lambda)e^{j\omega\lambda}d\lambda \\
&= \pm \int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda}d\lambda \\
&= \pm X(-\omega).
\end{aligned}$$

Therefore, X is even/odd.

Next, we show that if X is even/odd, then x is even/odd. From the definition of the inverse Fourier transform, we have

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Since X is even/odd, we have that $X(\omega) = \pm X(-\omega)$, where the plus case and minus case in the "$\pm$" correspond to X being even and odd, respectively. Using this, we can rewrite the above expression for $x(t)$ as

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \pm X(-\omega)e^{j\omega t}d\omega.$$

Now, we employ a change of variable. Let $\lambda = -\omega$ so that $d\lambda = -d\omega$. Applying this change of variable, we obtain

$$\begin{aligned}
x(t) &= \tfrac{1}{2\pi}\int_{\infty}^{-\infty} \pm X(\lambda)e^{-j\lambda t}(-1)d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda t}d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda(-t)}d\lambda \\
&= \pm x(-t).
\end{aligned}$$

Therefore, x is even/odd. This completes the proof. ■

In other words, the preceding theorem simply states that the forward and inverse Fourier transforms preserve even/odd symmetry.

6.7.14 Real Functions

As it turns out, the Fourier transform of a real-valued function has a special structure, as given by the theorem below.

Theorem 6.17 (Real-valued functions)**.** *A function x is real-valued if and only if its Fourier transform X satisfies*

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is conjugate symmetric).

Proof. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt. \tag{6.13}$$

Substituting $-\omega$ for ω in the preceding equation, we have

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt.$$

Conjugating both sides of this equation, we obtain

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt. \tag{6.14}$$

First, we show that x being real-valued implies that X is conjugate symmetric. Suppose that x is real-valued. Since x is real-valued, we can replace x^* with x in (6.14) to yield

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

Observing that the right-hand side is simply $X(\omega)$, we have

$$X^*(-\omega) = X(\omega).$$

Thus, x being real-valued implies that X is conjugate symmetric.

Next, we show that X being conjugate symmetric implies that x is real-valued. Suppose that X is conjugate symmetric. Since X is conjugate symmetric, the expressions for $X(\omega)$ in (6.13) and $X^*(-\omega)$ in (6.14) must be equal. Thus, we can write

$$\begin{aligned} X(\omega) - X^*(-\omega) &= 0 \\ \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt - \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt &= 0 \\ \int_{-\infty}^{\infty} [x(t) - x^*(t)]e^{-j\omega t}\,dt &= 0. \end{aligned}$$

This implies that $x^* = x$. Therefore, x is real-valued. Thus, X being conjugate symmetric implies that x is real-valued. This completes the proof. ∎

Suppose that X is the Fourier transform of a real-valued function x so that X is conjugate symmetric. From properties of complex numbers, we can show that that X being conjugate symmetric is equivalent to

$$|X(\omega)| = |X(-\omega)| \text{ for all } \omega \quad \text{and} \tag{6.15a}$$

$$\arg X(\omega) = -\arg X(-\omega) \text{ for all } \omega \tag{6.15b}$$

(i.e., the magnitude and argument of X are even and odd, respectively).

Since the Fourier transform X of a real-valued function x is conjugate symmetric, the graph of X for negative values is completely redundant and can be determined from the graph of X for nonnegative values. Lastly, note that x being real-valued does not necessarily imply that X is real-valued, since a conjugate-symmetric function need not be real-valued.

Table 6.1: Properties of the CT Fourier transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Even	x is even $\Leftrightarrow$ X is even
Odd	x is odd $\Leftrightarrow$ X is odd
Real	x is real $\Leftrightarrow$ X is conjugate symmetric