Example 3.10. Evaluate the integral $\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau$.

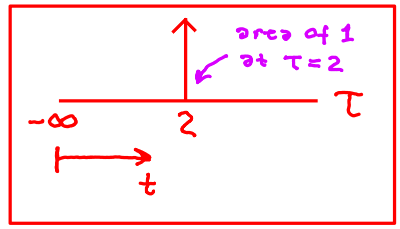
Solution. Using the equivalence property of the delta function given by (3.23), we can write

equivalence property

$$\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau = \int_{-\infty}^{t}(2^2+1)\delta(\tau-2)d\tau$$
$$= 5\underbrace{\int_{-\infty}^{t}\delta(\tau-2)d\tau}.$$

consider simplification of the underlined integral

Using the defining properties of the delta function given by (3.22), we have that

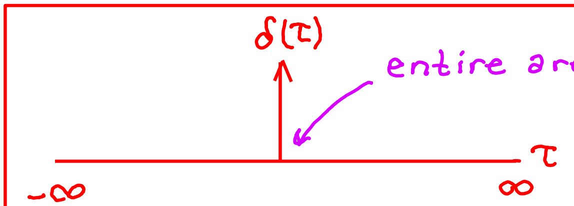


$$\int_{-\infty}^{t}\delta(\tau-2)d\tau = \begin{cases}1 & t \geq 2\\ 0 & t < 2\end{cases}$$
$$= u(t-2).$$

Therefore, we conclude that

$$\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau = \begin{cases}5 & t \geq 2\\ 0 & t < 2\end{cases} = 5\int_{-\infty}^{t}\delta(\tau-2)\,d\tau$$
$$= 5u(t-2).$$

■



$$\int_a^b \delta(\tau)\,d\tau = \begin{cases}1 & 0 \in [a,b]\\ 0 & \text{otherwise}\end{cases}$$

(where $a \leq b$)

Example 3.11 (Rectangular function). Show that the rect function can be expressed in terms of u as

$$\operatorname{rect} t = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right).$$

Solution. Using the definition of u and time-shift transformations, we have

$$u\left(t+\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq -\tfrac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u\left(t-\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq \tfrac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\begin{aligned} u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right) &= \begin{cases} 0 & t < -\tfrac{1}{2} \\ 1 & -\tfrac{1}{2} \leq t < \tfrac{1}{2} \\ 0 & t \geq \tfrac{1}{2} \end{cases} \\ &= \begin{cases} 1 & -\tfrac{1}{2} \leq t < \tfrac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \operatorname{rect} t. \end{aligned}$$

$$= \begin{cases} 0-0 & t < -\tfrac{1}{2} \\ 1-0 & -\tfrac{1}{2} \leq t < \tfrac{1}{2} \\ 1-1 & t \geq \tfrac{1}{2} \end{cases}$$

Graphically, we have the scenario depicted in Figure 3.24.

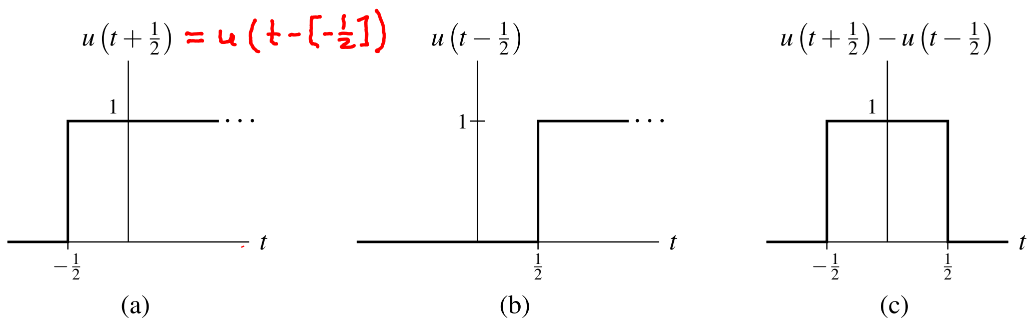
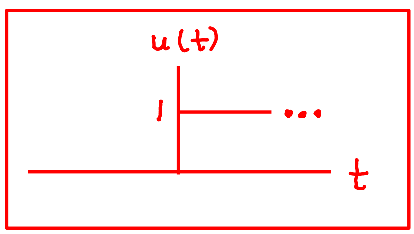


Figure 3.24: Representing the rectangular function using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their difference (which is the rectangular function).

■

recall:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Example 3.12 (Piecewise-linear function)**.** Consider the piecewise-linear function x given by

$$x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. A plot of x is shown in Figure 3.25(a). We consider each segment of the piecewise-linear function separately. The first segment (i.e., for $0 \le t < 1$) can be expressed as

$$v_1(t) = t[u(t) - u(t-1)].$$

This function is plotted in Figure 3.25(b). The second segment (i.e., for $1 \le t < 2$) can be expressed as

$$v_2(t) = [u(t-1) - u(t-2)]. \quad (1)$$

This function is plotted in Figure 3.25(c). The third segment (i.e., for $2 \le t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

This function is plotted in Figure 3.25(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t.

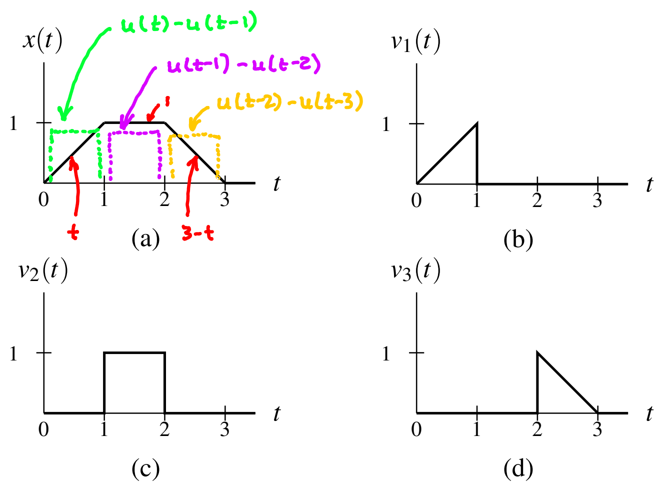


Figure 3.25: Representing a piecewise-linear function using unit-step functions. (a) The function x. (b), (c), and (d) Three functions whose sum is x.

■

Example 3.15 (Ideal amplifier)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

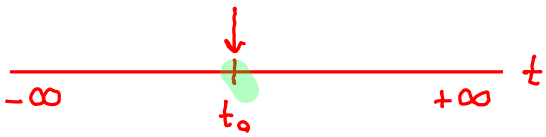
and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ∎

compute output at this point



at what points must input be known?