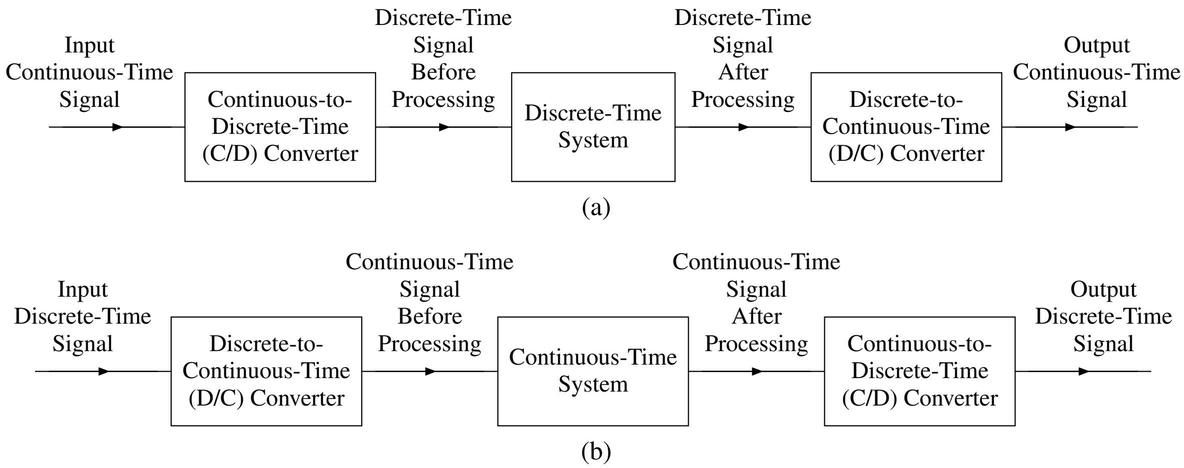

Figure 1.6: Signal processing systems. (a) Processing a continuous-time signal with a discrete-time system. (b) Processing a discrete-time signal with a continuous-time system.

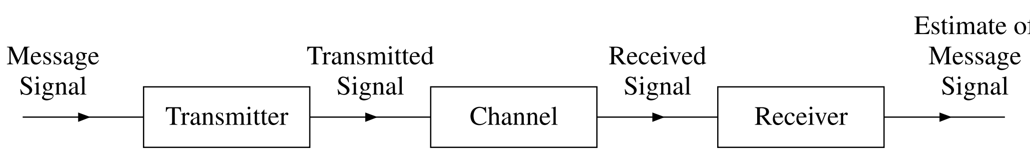

Figure 1.7: Communication system.

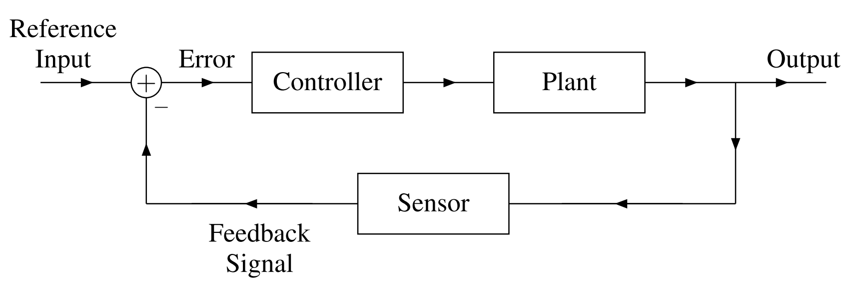

Figure 1.8: Feedback control system.

systems is absolutely critical. Without such a framework, there is little hope that any system that we design would operate as desired, meeting all of the required specifications.

1.5 Overview of This Book

This book presents the mathematical tools used for studying both signals and systems. Although most systems considered herein are SISO, the mathematics extends in a very straightforward manner to systems that have multiple inputs and/or multiple outputs. Only the one-dimensional case is considered herein, however, as the multi-dimensional case is considerably more complicated and beyond the scope of this book.

The remainder of this book is organized as follows. Chapter 2 presents some mathematical preliminaries that are essential for both the continuous-time and discrete-time cases. Then, Chapters 3, 4, 5, 6, and 7 cover material that relates primarily to the continuous-time case. Then, Chapters 8, 9, 10, 11, and 12 cover material that relates primarily to the discrete-time case.

The material on continuous-time signals and systems consists of the following. Chapter 3 examines signals and systems in more depth than covered in earlier chapters. A particular class of systems known as (continuous-time) linear time-invariant (LTI) systems is introduced. Chapter 4 then studies continuous-time LTI systems in depth. Finally, Chapters 5, 6, and 7 introduce continuous-time Fourier series, the continuous-time Fourier transform, and the Laplace transform, respectively, which are important mathematical tools for studying continuous-time signals and systems.

The material on discrete-time signals and systems consists of the following. Chapter 8 examines discrete-time signals and systems in more depth than earlier chapters. A particular class of systems known as (discrete-time) linear time-invariant (LTI) systems is introduced. Chapter 9 then studies discrete-time LTI systems in depth. Finally, Chapters 10, 11, and 12 introduce discrete-time Fourier series, the discrete-time Fourier transform, and the z transform, respectively, which are important mathematical tools for studying discrete-time signals and systems.

This book also includes several appendices, which contain supplemental material related to the topics covered in the main chapters of the book. Appendix A provides a review of complex analysis. Appendix B introduces partial fraction expansions. Appendix C presents time-domain methods for solving for differential equations. Appendix D presents a detailed introduction to MATLAB. Appendix E provides some additional exercises. Appendix F offers a list of some useful mathematical formulas. Finally, Appendix G presents some information about video lectures available for this book.

Chapter 2

Preliminaries

2.1 Overview

Before we can proceed to study signals and systems in more depth, we must first establish some basic mathematical preliminaries. In what follows, we introduce some background related to sets, mappings, functions, sequences, operators, and transforms. Also, we present some basic properties of functions and sequences.

2.2 Sets

As a matter of terminology, a **rational number** is a number of the form x/y, where x and y are integers and $y \neq 0$. That is, a rational number is a ratio of integers (that does not result in division by zero). For example, $\frac{1}{2}$, $5 = \frac{5}{1}$, and $-\frac{2}{3}$ are rational numbers. In contrast, π is not a rational number, as it cannot be represented exactly as a ratio of integers.

The sets of integers, rational numbers, real numbers, and complex numbers are denoted as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. It is important to note that the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, do not include infinity. The omission of infinity from these sets is necessary in order to allow for a consistent system with respect to all of the usual arithmetic operations (i.e., addition, subtraction, multiplication, and division). For readers not familiar with complex numbers, detailed coverage of complex numbers can be found in Appendix A.

Often, it is convenient to be able to concisely denote a set consisting of a consecutive range of integers. Consequently, we introduce some notation for this purpose. For two integers a and b, we define the following notation for sets of consecutive integers:

$$\begin{aligned}
[a\,..\,b] &= \{x \in \mathbb{Z} : a \leq x \leq b\}, \\
[a\,..\,b) &= \{x \in \mathbb{Z} : a \leq x < b\}, \\
(a\,..\,b] &= \{x \in \mathbb{Z} : a < x \leq b\}, \quad \text{and} \\
(a\,..\,b) &= \{x \in \mathbb{Z} : a < x < b\}.
\end{aligned}$$

In this notation, a and b indicate the endpoints of the range for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicates whether each endpoint is included in the set. A square bracket indicates that the corresponding endpoint is included in the set, while a parenthesis (i.e., round bracket) indicates that the corresponding endpoint is not included. For example, we have that:

- $[0\,..\,4]$ denotes the set of integers $\{0,1,2,3,4\}$,
- $[0\,..\,4)$ denotes the set of integers $\{0,1,2,3\}$,
- $(0\,..\,4)$ denotes the set of integers $\{1,2,3\}$, and
- $(0\,..\,4]$ denotes the set of integers $\{1,2,3,4\}$.

The variants of this notation that exclude one or both of the endpoints of the range are typically useful when excluding an endpoint would result in a simpler expression. For example, if one wanted to denote the set $\{0,1,2,\ldots,N-1\}$, we could write this as either $[0\,..\,N)$ or $[0\,..\,N-1]$, but the first of these would likely be preferred since it is more compact.

Often, it is convenient to be able to concisely denote an interval (i.e., range) on the real line. Consequently, we introduce some notation for this purpose. For two real numbers a and b, we define the following notation for intervals:

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\
(a,b) &= \{x \in \mathbb{R} : a < x < b\}, \\
[a,b) &= \{x \in \mathbb{R} : a \leq x < b\}, \quad \text{and} \\
(a,b] &= \{x \in \mathbb{R} : a < x \leq b\}.
\end{aligned}$$

In this notation, a and b indicate the endpoints of the interval for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicates whether each endpoint is included in the set. A square bracket indicates that the corresponding endpoint is included in the set, while a parenthesis (i.e., round bracket) indicates that the corresponding endpoint is not included. For example, we have that:

- $[0,100]$ denotes the set of all real numbers from 0 to 100, including both 0 and 100;
- $(-\pi,\pi]$ denotes the set of all real numbers from $-\pi$ to π, excluding $-\pi$ but including π;
- $[-\pi,\pi)$ denotes the set of all real numbers from $-\pi$ to π, including $-\pi$ but excluding π; and
- $(0,1)$ denotes the set of all real numbers from 0 to 1, excluding both 0 and 1.

2.3 Mappings

A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**. The notation $f : A \to B$ denotes a mapping f whose domain is the set A and whose codomain is the set B. An example of a very simple mapping is given below.

Example 2.1. Let A and B be the sets given by

$$A = \{1,2,3,4\} \quad \text{and} \quad B = \{0,1,2,3\}.$$

Let $f : A \to B$ such that

$$f(x) = \begin{cases} 0 & x \in \{1,2\} \\ 1 & x = 4 \\ 2 & x = 3. \end{cases}$$

The mapping f maps each element x in the domain A to the element $f(x)$ in the codomain B. The mapping f is expressed in pictorial form in Figure 2.1. ∎

The mapping in the above example is quite simple. So, we now consider a few more interesting examples of mappings. The sine function (i.e., sin) is an example of a mapping. For each real number t, the sine function maps t to the sine of t (i.e., $\sin t$). In the case of the sine function, both the domain and codomain are the real numbers. As another example, consider the function trunc that rounds a real number t to an integer by discarding the fractional part of t (i.e., trunc rounds towards zero). The trunc function is a mapping where the domain and codomain are the real numbers and integers, respectively.

Although very many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms. In the sections that follow, we will introduce (or further elaborate upon) each of these types of mappings.

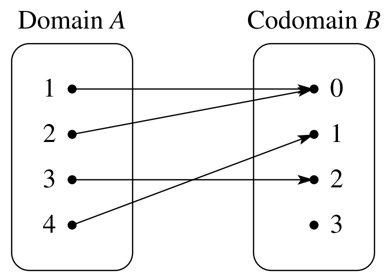

Figure 2.1: The mapping f.

2.4 Functions

A **function** is a mapping where the domain is a set that is continuous in nature, such as the real numbers, complex numbers, or a subset of either of these. In practice, the codomain is typically either the real numbers or complex numbers. Functions are also commonly referred to as **continuous-time (CT) signals**. Herein, we focus mostly on functions of a single independent variable (i.e., one-dimensional functions). The trigonometric sine (i.e., sin) function is an example of a function. It associates each real number t with the real number $\sin t$. In this case, the domain and codomain are the real numbers.

For a function f, the expression "$f(t)$" denotes the value of the function f evaluated at t. Note that "$f(t)$" does not denote the function f itself. This distinction is an extremely important one to make—a point that we shall revisit shortly. Normally, the parameters to a function are placed in (round, square, or curly) brackets. For standard mathematical functions with only a single parameter (e.g., sin and cos), brackets generally tend to be omitted when not needed for grouping. That is, we typically write "$\sin t$" instead of "$\sin(t)$". In the case of a non-standard function like f, we would always write "$f(t)$", not "ft".

Many notational schemes can be used to specify a function mathematically. In engineering, however, it is arguably most common to specify a function in terms of a defining equation. For example, we can specify a function f that maps the real number t to t^2 (i.e., a squaring function) using the equation $f(t) = t^2$. Here, f is the function, and t is a dummy variable that is used only for the purposes of writing the equation $f(t) = t^2$ that defines the function f. Since t is only a dummy variable, the equations $f(t) = t^2$ and $f(x) = x^2$ define exactly the same function. Note that "f" and "$f(t) = t^2$" are, strictly speaking, different things. That is, "f" is the function itself whereas "$f(t) = t^2$" is not the function f but rather an equation that defines the function f. So, strictly speaking, to define a function f, we should use wording like: "let f be the function defined by the equation $f(t) = \ldots$". Often, in practice, we abuse notation somewhat and simply write "the function $f(t) = \ldots$" or simply "$f(t) = \ldots$". Fortunately, this type notational abuse does not usually lead to problems in most cases.

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with functions.

Example 2.2. Let f and g each denote a real-valued function of a real variable. Let t denote an arbitrary real number. The expression $f+g$ denotes a *function*, namely, the function formed by adding the functions f and g. The expression $f(t)+g(t)$ denotes a *number*, namely, the sum of: 1) the value of the function f evaluated at t; and 2) the value of the function g evaluated at t. The expression $(f+g)(t)$ denotes the result obtained by: 1) first computing a new function h that is the sum of the functions f and g; and 2) then evaluating h at t. Note that the meanings of the expressions $(f+g)(t)$ and $f(t)+g(t)$ are subtly different. In the first case, the addition operation is being applied to two functions, while in the second case, the addition operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(f+g)(t) = f(t)+g(t) \quad \text{for all } t.$$

This is due to the fact that the preceding equation is precisely how the addition of functions is defined. In other words, we add functions by adding their values at corresponding points (i.e., the addition of functions is defined in a pointwise manner). ■

Example 2.3. For two functions x_1 and x_2, the expression $x_1 + x_2$ denotes the function that is the sum of the functions x_1 and x_2. The expression $(x_1 + x_2)(t)$ denotes the function $x_1 + x_2$ evaluated at t. Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1 + x_2)(t) = x_1(t) + x_2(t) \quad \text{for all } t.$$

Similarly, since subtraction, multiplication, and division can also be defined pointwise, the following relationships also hold:

$$\begin{gathered}(x_1 - x_2)(t) = x_1(t) - x_2(t) \quad \text{for all } t, \\ (x_1 x_2)(t) = x_1(t) x_2(t) \quad \text{for all } t, \quad \text{and} \\ (x_1/x_2)(t) = x_1(t)/x_2(t) \quad \text{for all } t.\end{gathered}$$

It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to functions, then the expression $(x_1 \diamond x_2)(t)$ is mathematically valid, but the expression $x_1(t) \diamond x_2(t)$ is not. The latter expression is not valid since the $\diamond$ operator requires two functions as operands, but the provided operands $x_1(t)$ and $x_2(t)$ are numbers (namely, the values of the functions x_1 and x_2 each evaluated at t). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

2.5 Sequences

A **sequence** is a mapping where the domain is a set that is discrete in nature, such as the integers, or a subset thereof. In practice, the codomain is typically either the real numbers or complex numbers. Sequences are also commonly referred to as **discrete-time (DT) signals**. Herein, we focus mostly on sequences with a single independent variable (i.e., one-dimensional sequences). The sequence $1^2, 2^2, 3^2, 4^2, \ldots$ of perfect squares is an example of a sequence. It associates each (strictly) positive integer n with the integer n^2. In this case, the domain and codomain are the positive integers.

Similar comments as above also apply to expressions involving sequences. For a sequence f, the expression "$f(n)$" denotes the value of the sequence f evaluated at n, whereas the expression "f" denotes the sequence f itself. It is often critically important to make a clear distinction between a sequence and its value. A sequence is typically specified by a defining equation. For example, a sequence f that maps the integer n to n^2 (i.e., a sequence of squares) can be specified using the equation $f(n) = n^2$.

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with sequences.

Example 2.4. Let f and g each denote a real-valued sequence with an integer index. Let n denote an arbitrary integer. The expression fg denotes a *sequence*, namely, the sequence formed by multiplying the sequences f and g. The expression $f(n)g(n)$ denotes a *number*, namely, the product of: 1) the value of the sequence f evaluated at n; and 2) the value of the sequence g evaluated at n. The expression $(fg)(n)$ denotes the result obtained by: 1) first computing a new sequence h that is the product of the sequences f and g; and 2) then evaluating h at n. Note that the meanings of the expressions $(fg)(n)$ and $f(n)g(n)$ are subtly different. In the first case, the multiplication operation is being applied to two sequences, while in the second case, the multiplication operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(fg)(n) = f(n)g(n) \quad \text{for all } n.$$

This is due to the fact that the preceding equation is precisely how the multiplication of sequences is defined. In other words, we multiply sequences by multiplying their values at corresponding points (i.e., the multiplication of sequences is defined in a pointwise manner). ■

Example 2.5. For two sequences x_1 and x_2, we have two ways in which we can express the equality of these sequences. First, we can simply write that $x_1 = x_2$. Second, we can write that $x_1(n) = x_2(n)$ for all n. The first approach is probably preferable since it is less verbose. ■

Example 2.6. For two sequences x_1 and x_2, the expression $x_1 + x_2$ denotes the sequence that is the sum of the sequences x_1 and x_2. The expression $(x_1 + x_2)(n)$ denotes the sequence $x_1 + x_2$ evaluated at n. Since the addition of sequences can be defined pointwise (i.e., we can add two sequences by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1 + x_2)(n) = x_1(n) + x_2(n) \quad \text{for all } n.$$

Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

$$\begin{aligned}
(x_1 - x_2)(n) &= x_1(n) - x_2(n) \quad \text{for all } n, \\
(x_1 x_2)(n) &= x_1(n)x_2(n) \quad \text{for all } n, \quad \text{and} \\
(x_1/x_2)(n) &= x_1(n)/x_2(n) \quad \text{for all } n.
\end{aligned}$$

It is important to note, however, that not all mathematical operations involving sequences can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be sequences. The convolution operation (for sequences), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to sequences, then the expression $(x_1 \diamond x_2)(n)$ is mathematically valid, but the expression $x_1(n) \diamond x_2(n)$ is not. The latter expression is not valid since the $\diamond$ operator requires two sequences as operands, but the provided operands $x_1(n)$ and $x_2(n)$ are numbers (namely, the values of the sequences x_1 and x_2 each evaluated at n). Due to issues like this, one must be careful in the use of mathematical notation related to sequences. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

2.6 Remarks on Abuse of Notation

Unfortunately, for a function f, it is common practice for engineers to abuse mathematical notation and write "$f(t)$" to refer to the function f itself. A similar issue also exists for the case of sequences. The abuse of mathematical notation, however, can often lead to trouble. In some contexts, it is critically important to make a clear distinction between a function (or sequence) and its value, and failing to do so can lead to many problems, such as writing mathematical expressions that are ambiguous, contradictory, or nonsensical. For this reason, it is strongly recommended that one try to maintain a clear distinction between a function (or sequence) and its value.

With the above said, notational abuse in trivial cases is not likely to cause problems. For example, if we simply write "the function $f(t)$" instead of more correctly writing the "the function f", this is unlikely to cause confusion. Where notational abuse is much more likely to become problematic is when the expressions that are being referred to as functions contain mathematical operations or more than one variable, such as expressions like: $f(t-1)$, $f(7t)$, $f(at-b)$, $tf(t)$, and $f(t-\tau)$. In cases like these, abuse of notation makes the intended meaning much less clear, opening the possibility of misinterpretation. For example, in the case of "$f(t-\tau)$", more than one reasonable interpretation exists if one allows notation to be abused. In particular, "$f(t-\tau)$" might mean:

1. a number that is equal to the function f evaluated at $t-\tau$;
2. an anonymous (i.e., unnamed) function that is equal to the function g, where $g(x) = f(x-\tau)$ (i.e., t and τ are interpreted as a variable and constant, respectively);
3. an anonymous function that is equal to the function g, where $g(x) = f(t-x)$ (i.e., t and τ are interpreted as a constant and variable, respectively);

Table 2.1: Examples of dot notation for functions and sequences. Examples for (a) functions and (b) sequences.

(a)

Named Function f	Corresponding Unnamed Function
$f(t) = g(t)$	$g(\cdot)$
$f(t) = t^2$	$(\cdot)^2$
$f(t) = \sqrt[3]{t}$	$\sqrt[3]{\cdot}$
$f(t) = e^t$	$e^{(\cdot)}$
$f(t) = \lvert t \rvert$	$\lvert \cdot \rvert$
$f(t) = t^2 + 3t + 1$	$(\cdot)^2 + 3\cdot + 1$
$f(t) = g(at - b)$	$g(a \cdot - b)$
$f(t) = g(t - 1)$	$g(\cdot - 1)$
$f(t) = g(3t)$	$g(3\cdot)$

(b)

Named Sequence f	Corresponding Unnamed Sequence
$f(n) = g(n)$	$g(\cdot)$
$f(n) = n^2$	$(\cdot)^2$
$f(n) = \sin(\frac{2\pi}{3}n)$	$\sin[\frac{2\pi}{3}(\cdot)]$
$f(n) = g(3n)$	$g(3\cdot)$
$f(n) = g(n - 1)$	$g(\cdot - 1)$
$f(n) = ng(n)$	$(\cdot)g(\cdot)$
$f(n) = 2n^2 + n + 5$	$2(\cdot)^2 + (\cdot) + 5$

4. an anonymous function that is equal to the constant function g, where $g(x) = f(t-\tau)$ (i.e., t and τ are both interpreted as constants).

In circumstances like this one, notational problems can be easily avoided by simply specifying the desired function in terms of an equation. In other words, we can give a name to the function being specified and then define the function in terms of an equation using the given name. For example, instead of saying "the function $f(t-\tau)$", we can say "the function $g(t) = f(t-\tau)$". This latter notation makes clear that τ is a constant, for example. As another example, instead of saying "the function $f(t-1)$", we can say "the function $g(t) = f(t-1)$".

Due to problems like those above, great care must be exercised when using anonymous functions in order to avoid ambiguous notation. Since ambiguous notation is a frequent source of problems, the author would suggest that anonymous functions are best avoided in most circumstances.

2.7 Dot Notation for Functions and Sequences

Sometimes a situation may arise where one would like to distinguish a function from the value of a function, but without resorting to giving the function a name or other more verbose notational approaches. A similar comment also applies for the case of sequences. In this regard, the dot notation for functions and sequences is quite useful. If we wish to indicate that an expression corresponds to a function (as opposed to the value of a function), we can denote this using the interpunct symbol (i.e., "·"). In each place where the variable for the function would normally appear, we simply replace it with an interpunct symbol (i.e., "·"). For example, $\sqrt{\cdot}$ denotes the square root function, whereas $\sqrt{t}$ denotes the value of the square root function evaluated at t. Some additional examples of the dot notation for functions can be found in Table 2.1(a). A similar convention can also be applied to sequences. Some examples of the dot notation for sequences can be found in Table 2.1(b). Since some readers may find this dot notation to be somewhat strange, this book minimizes the use of this notation. It is, however, used in a few limited number of places in order to achieve clarity without the need for being overly verbose. Although the dot notation may appear strange at first, it is a very commonly used notation by mathematicians. Sadly, it is not used as much by engineers, in spite of its great utility.

2.8 System Operators

A system operator is a mapping used to represent a system. In what follows, we will focus exclusively on the case of single-input single-output systems, since this case is our primary focus herein. A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system. For example, the system $\mathcal{H}$ that maps a function to a function and is given by

$$\mathcal{H}x(t) = 2x(t)$$

multiplies its input function x by a factor of 2 in order to produce its output function. The system $\mathcal{H}$ that maps a sequence to a sequence and is given by

$$\mathcal{H}x(n) = x(n) + 1$$

adds one to its input sequence x in order to produce its output sequence.

For a system operator $\mathcal{H}$ and function x, the expression $\mathcal{H}(x)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function x. Since only a single symbol x follows the operator $\mathcal{H}$, there is only one way to group the operations in this expression. Therefore, the parentheses can be omitted without any risk of changing the meaning of the expression. In other words, we can equivalently write $\mathcal{H}(x)$ as $\mathcal{H}x$. Since $\mathcal{H}x$ is a function, we can evaluate this function at t, which corresponds to the expression $(\mathcal{H}x)(t)$. We can omit the first set of parentheses in this expression without changing its meaning. In other words, the expressions $(\mathcal{H}x)(t)$ and $\mathcal{H}x(t)$ have identical meanings. This is due to the fact that there is only one possible way to group the operations in $\mathcal{H}x(t)$ that is mathematically valid. For example, the grouping $\mathcal{H}[x(t)]$ is not mathematically valid since $\mathcal{H}$ must be provided a function as an operand, but the provided operand $x(t)$ is a number (namely, the value of the function x evaluated at t).

Again, since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with system operators.

Example 2.7. For a system operator $\mathcal{H}$, a function x, a real variable t, and a real constant t_0, the expression $\mathcal{H}x(t-t_0)$ denotes the result obtained by taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x and then evaluating y at $t-t_0$. ■

Example 2.8. For a system operator $\mathcal{H}$, function x', and real number t, the expression $\mathcal{H}x'(t)$ denotes result of taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x' and then evaluating y at t. ■

Example 2.9. For a system operator $\mathcal{H}$, function x, and a complex constant a, the expression $\mathcal{H}(ax)$ denotes the output from the system $\mathcal{H}$ when the input is the function ax (i.e., a times x). ■

Example 2.10. For a system operator $\mathcal{H}$ and the functions x_1 and x_2, the expression $\mathcal{H}(x_1+x_2)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function x_1+x_2. Note that, in this case, we cannot omit the parentheses without changing the meaning of the expression. That is, $\mathcal{H}x_1+x_2$ means $(\mathcal{H}x_1)+x_2$, which denotes the sum of the function x_2 and the output produced by the system $\mathcal{H}$ when the input is the function x_1. ■

Example 2.11. For a system operator $\mathcal{H}$ and the functions x_1 and x_2, and the complex constants a_1 and a_2, the expression $\mathcal{H}(a_1x_1+a_2x_2)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function $a_1x_1+a_2x_2$. ■

Example 2.12. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the operators representing two systems and let x denote a function. Consider the expression $\mathcal{H}_2\mathcal{H}_1x$. The implied grouping of operations in this expression is $\mathcal{H}_2(\mathcal{H}_1x)$. So, this expression denotes the output produced by the system $\mathcal{H}_2$ when its input is the function y, where y is the output produced by the system $\mathcal{H}_1$ when its input is the function x. ■

2.9 Transforms

Later, we will be introduced to several types of mappings known as transforms. Transforms have a mathematical structure similar to system operators. That is, transforms map functions/sequences to functions/sequences. Due to

Table 2.2: Examples of transforms

Name	Domain	Codomain
CT Fourier Series	T-periodic functions (with domain $\mathbb{R}$)	sequences (with domain $\mathbb{Z}$)
CT Fourier Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{R}$)
Laplace Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{C}$)
DT Fourier Series	N-periodic sequences (with domain $\mathbb{Z}$)	N-periodic sequences (with domain $\mathbb{Z}$)
DT Fourier Transform	sequences (with domain $\mathbb{Z}$)	2π-periodic functions (with domain $\mathbb{R}$)
Z Transform	sequences (with domain $\mathbb{Z}$)	functions (with domain $\mathbb{C}$)

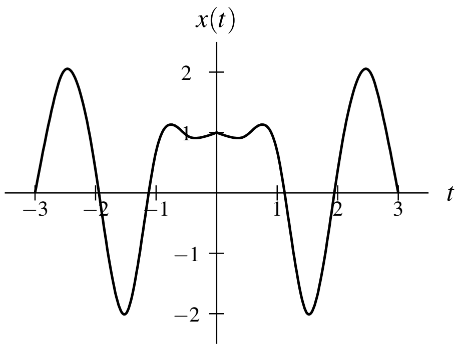

Figure 2.2: Example of an even function.

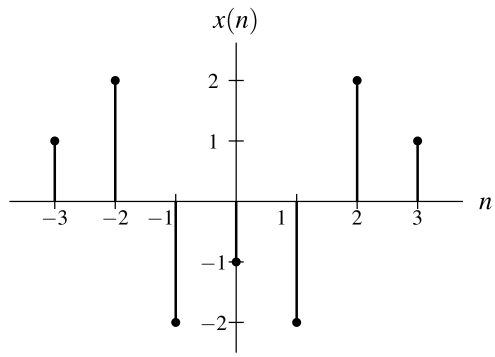

Figure 2.3: Example of an even sequence.

this similar structure, many of the earlier comments about system operators also apply to the case of transforms. Some examples of transforms of interest in the study of signals and systems are listed in Table 2.2. For example, the Fourier transform (introduced later) is denoted as $\mathcal{F}$ and the result of applying the Fourier transform operator to the function/sequence x is denoted as $\mathcal{F}x$.

2.10 Basic Signal Properties

Signals can possess a number of interesting properties. In what follows, we introduce a few very basic properties that are frequently useful.

2.10.1 Symmetry of Functions and Sequences

A function x is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.1}$$

Similarly, a sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.2}$$

Geometrically, an even function or sequence is symmetric with respect to the vertical axis. Examples of an even function and sequence are given in Figures 2.2 and 2.3. Some other examples of even functions include the cosine, absolute value, and square functions. Some other examples of even sequences include the unit-impulse and rectangular sequences (to be introduced later).

A function x is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.3}$$

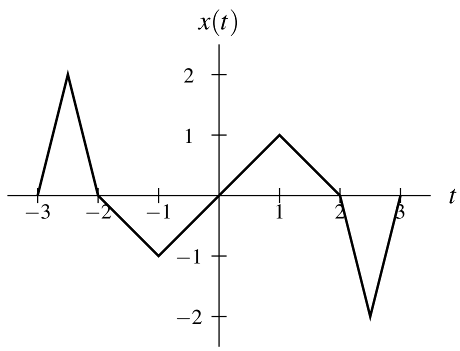

Figure 2.4: Example of an odd function.

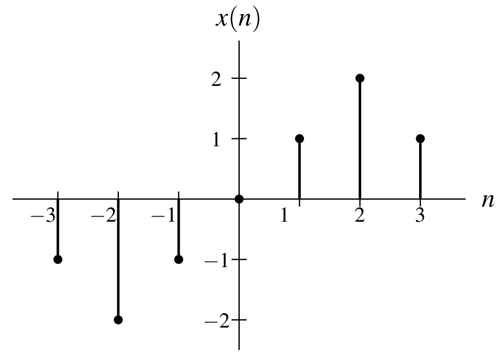

Figure 2.5: Example of an odd sequence.

Similarly, a sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.4}$$

Geometrically, an odd function or sequence is symmetric with respect to the origin. Examples of an odd function and odd sequence are shown in Figures 2.4 and 2.5. One can easily show that an odd function or sequence x must be such that $x(0) = 0$, assuming that the domain of x includes 0 (i.e., $x(0)$ is defined). Some other examples of odd functions include the sine, signum, and cube (i.e., $x(t) = t^3$) functions.

A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.5}$$

Similarly, a sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.6}$$

An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos(\omega t) + j\sin(\omega t)$, where ω is a real constant. The real part of a conjugate symmetric function or sequence is even. The imaginary part of a conjugate symmetric function or sequence is odd.

2.10.2 Periodicity of Functions and Sequences

A function x is said to be **periodic** with **period** T (or simply T**-periodic**) if, for some strictly positive real constant T,

$$x(t) = x(t+T) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.7}$$

In other words, the graph of a T-periodic function repeats in value every T units along the horizontal axis. A T-periodic function x is said to have the **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$. Similarly, a sequence x is said to be **periodic** with **period** N (or simply N**-periodic**) if, for some strictly positive integer N,

$$x(n) = x(n+N) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.8}$$

An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$. A function or sequence that is not periodic is said to be **aperiodic**. Examples of a periodic function and sequence are shown in Figures 2.6 and 2.7. Some other examples of periodic functions include the cosine and sine functions.

The period of a periodic function or sequence is not uniquely determined. In particular, a function or sequence that is periodic with period T is also periodic with period kT, for every strictly positive integer k. In most cases, we are interested in the smallest (positive) value of T or N for which (2.7) or (2.8) is satisfied, respectively. We refer to this value as the **fundamental period**. Moreover, the frequency corresponding to the fundamental period is called the **fundamental frequency**.

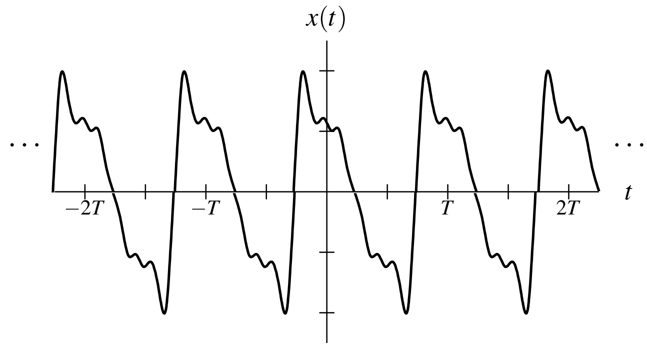

Figure 2.6: Example of a periodic function (with a fundamental period of T).

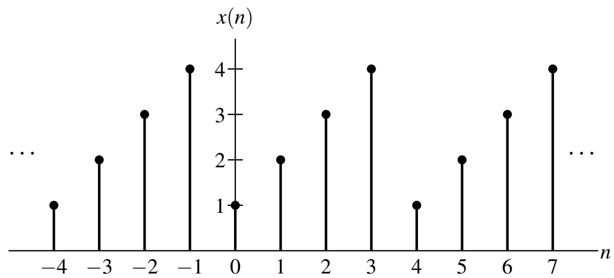

Figure 2.7: Example of a periodic sequence (with a fundamental period of 4).

Consider, for example, the periodic function x shown in Figure 2.6. The function x is periodic with periods T, $2T, 3T$, and so on. The fundamental period of x, however, is uniquely determined and has the value T.

When a function or sequence is periodic, it cycles through a particular set of values in each period. Normally, we are interested simply in the rate of oscillation (i.e., the number of cycles completed per unit time). In some contexts, however, it can be meaningful to associate the oscillations with a direction (or order). For example, a function or sequence might cycle through the set of values in each period in either the forward or reverse direction. In contexts where oscillations can be associated with a direction and this direction is of key importance, it is often convenient to attach a sign (i.e., positive or negative) to the frequency to indicate the direction in which periodic cycles occur. This leads to the notion of signed frequency. That is, a **signed frequency** is simply a quantity whose magnitude corresponds to the rate of oscillation (i.e., number of cycles per unit time) and whose sign (i.e., positive or negative) indicates the direction of oscillation.

To give a more concrete example, consider a wheel rotating at some fixed rate. Not only might we be interested in how many times the wheel rotates per second (i.e., what we would normally refer to as frequency), we might also be interested in whether the wheel is turning in the counterclockwise or clockwise direction. In this case, we could employ the notion of signed frequency. That is, we could define the frequency as a signed quantity, where the magnitude of the signed frequency indicates the oscillation rate (i.e., the number of rotations per second), while the sign (i.e., positive or negative) of the signed frequency indicates the direction of rotation. Suppose that we adopt the convention that the counterclockwise and clockwise directions correspond to positive and negative signed frequencies, respectively. Then, a signed frequency of 10 would correspond to 10 rotations of the wheel per second in the counterclockwise direction, while a signed frequency of -10 would correspond to 10 rotations per second in the clockwise direction.

Often, in the interest of brevity, we simply refer to "signed frequency" as "frequency". This does not normally cause confusion, since it is usually clear from the context when signed frequency is being employed.

2.11 Exercises

2.11.1 Exercises Without Answer Key

2.1 Let each of $\mathcal{G}$ and $\mathcal{H}$ denote a system operator that maps a function to a function; let x and y denote functions; and assume that all other variables denote numbers. Fully parenthesize each of the expressions below in order to show the implied grouping of all operations.
(a) $\mathcal{H}x(t) = t^2 + 1$;
(b) $\mathcal{G}\mathcal{H}y(t)$;
(c) $\mathcal{H}x + y$; and
(d) $x\mathcal{H}\mathcal{G}y$.

2.2 Let $\mathcal{H}$ denote a system operator that maps a function to a function; let x and y denote functions; and let all other variables denote numbers. Using strictly-correct mathematical notation, write an expression for each quantity specified below. Only use brackets for grouping when strictly required. Use $\mathcal{D}$ to denote the derivative operator.
(a) the output of the system $\mathcal{H}$ when its input is y;
(b) the output of the system $\mathcal{H}$ evaluated at $2t-1$ when the input to the system is x;
(c) the output of the system $\mathcal{H}$ evaluated at t when the input to the system is ax;
(d) the output of the system $\mathcal{H}$ evaluated at $5t$ when the input to the system is $x+y$;
(e) the derivative of the output of the system $\mathcal{H}$ when its input is ax;
(f) the output of the system $\mathcal{H}$ when its input is the derivative of ax;
(g) the sum of: 1) the output of the system $\mathcal{H}$ when its input is x; and 2) the output of the system $\mathcal{H}$ when its input is y;
(h) the output of the system $\mathcal{H}$ when its input is $x+y$; and
(i) the derivative of x evaluated at $5t-3$.

2.3 Let $\mathcal{D}$ denote the derivative operator; let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ denote the system operators $\mathcal{H}_1x(t) = tx(t)$, $\mathcal{H}_2x(t) = x(t-1)$, and $\mathcal{H}_3x(t) = x(3t)$; and let x denote the function $x(t) = t-1$. Evaluate each of the expressions given below.
(a) $\mathcal{H}_2\mathcal{H}_1x(t)$;
(b) $\mathcal{H}_1\mathcal{H}_2x(t)$;
(c) $\mathcal{H}_1x(t-1)$;
(d) $\mathcal{D}\mathcal{H}_3x(t)$;
(e) $\mathcal{H}_3\mathcal{D}x(t)$; and
(f) $\mathcal{D}\{3x\}(t)$.

2.11.2 Exercises With Answer Key

2.101 In each case below, find a fully-simplified expression for the function y.
(a) $y(t) = \mathcal{D}x(3t)$, where $x(t) = t^2 + 2t + 1$ and $\mathcal{D}$ denotes the derivative operator;
(b) $y(t) = \mathcal{D}\{x(3\cdot)\}(t)$, where $x(t) = t^2 + 2t + 1$ and $\mathcal{D}$ denotes the derivative operator;
(c) $y(t) = \mathcal{H}x(t-3)$, where $\mathcal{H}x(t) = tx(t)$ and $x(t) = 2t+1$; and
(d) $y(t) = \mathcal{H}\{x(\cdot - 3)\}(t)$, where $\mathcal{H}x(t) = tx(t)$ and $x(t) = 2t+1$.

Short Answer. (a) $y(t) = 6t+2$; (b) $y(t) = 18t+6$; (c) $y(t) = 2t^2 - 11t + 15$; (d) $y(t) = 2t^2 - 5t$

Part I

Continuous-Time Signals and Systems

Chapter 3

Continuous-Time Signals and Systems

3.1 Overview

In this chapter, we will examine continuous-time signals and systems in more detail.

3.2 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

3.2.1 Time Shifting (Translation)

The first type of signal transformation that we shall consider is known as time shifting. **Time shifting** (also known as **translation**) maps a function x to the function y given by

$$y(t) = x(t-b), \tag{3.1}$$

where b is a real constant. In other words, the function y is formed by replacing t by $t-b$ in the expression for $x(t)$. Geometrically, the transformation (3.1) shifts the function x (to the left or right) along the time axis to yield y. If $b > 0$, y is shifted to the right relative to x (i.e., delayed in time). If $b < 0$, y is shifted to the left relative to x (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 3.1. By applying a time-shifting transformation to the function x shown in Figure 3.1(a), each of the functions in Figures 3.1(b) and (c) can be obtained.

3.2.2 Time Reversal (Reflection)

The next type of signal transformation that we consider is called time reversal. **Time reversal** (also known as **reflection**) maps a function x to the function y given by

$$y(t) = x(-t). \tag{3.2}$$

In other words, the function y is formed by replacing t with $-t$ in the expression for $x(t)$. Geometrically, the transformation (3.2) reflects the function x about the origin to yield y.

To illustrate the effects of time reversal, an example is provided in Figure 3.2. Applying a time-reversal transformation to the function x in Figure 3.2(a) yields the function in Figure 3.2(b).

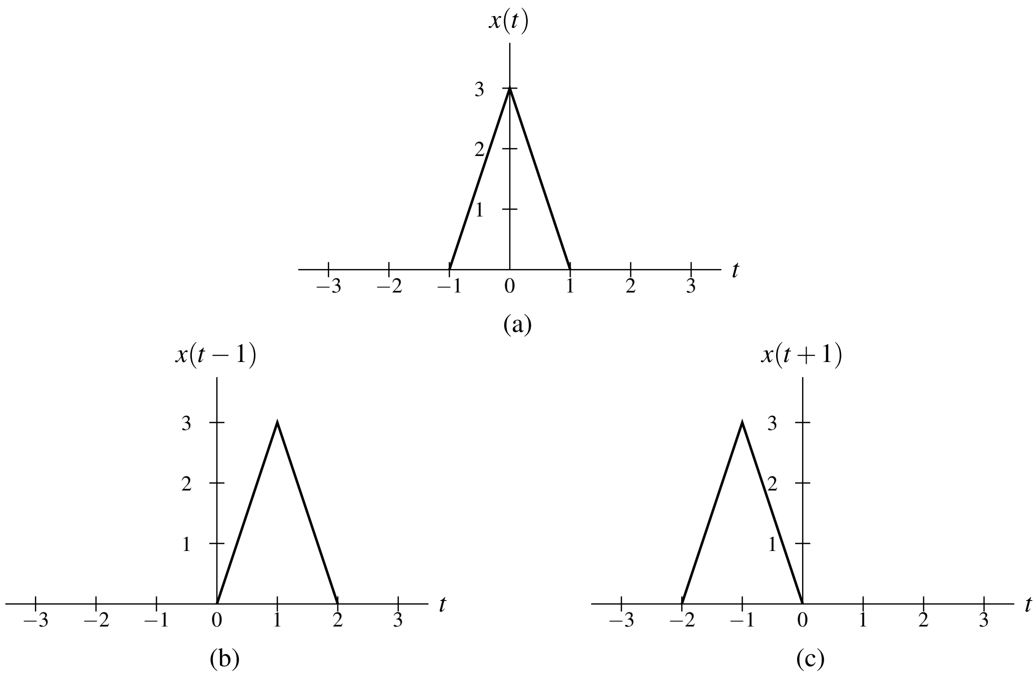

Figure 3.1: Example of time shifting. (a) The function x; and the result of applying a time-shifting transformation to x with a shift of (b) 1 and (c) -1.

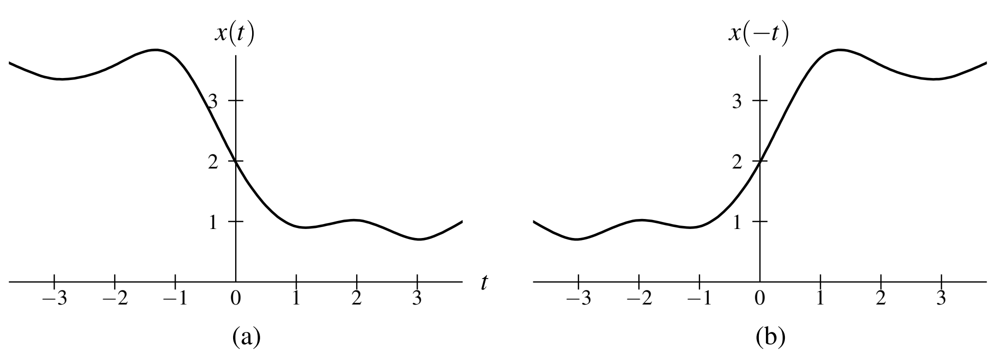

Figure 3.2: Example of time reversal. (a) The function x; and (b) the result of applying a time-reversal transformation to x.

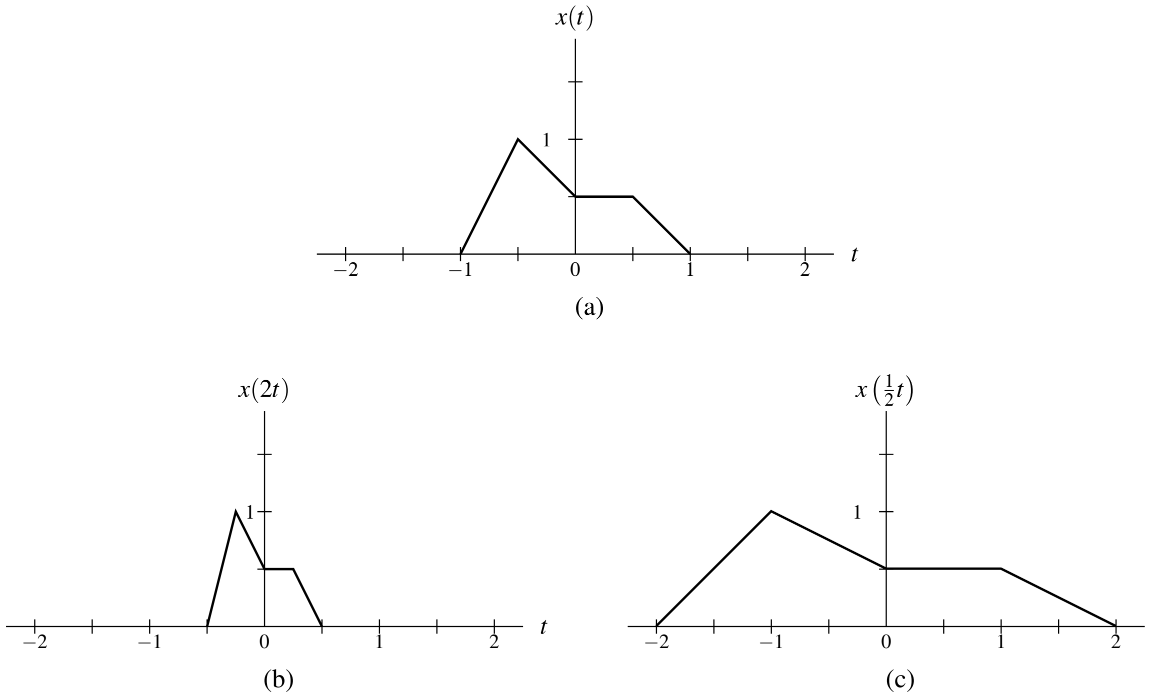

Figure 3.3: Example of time compression/expansion. (a) The function x; and the result of applying a time compression/expansion transformation to x with a scaling factor of (b) 2 and (c) $\frac{1}{2}$.

3.2.3 Time Compression/Expansion (Dilation)

The next transformation to be considered is called time compression/expansion. **Time compression/expansion** (also known as **dilation**) maps a function x to the function y given by

$$y(t) = x(at), \tag{3.3}$$

where a is a strictly positive real constant. In other words, the function y is formed by replacing t by at in the expression for $x(t)$. The constant a is referred to as the scaling factor. The transformation in (3.3) is associated with a compression/expansion along the time axis. If $a > 1$, y is compressed along the horizontal axis by a factor of a, relative to x. If $a < 1$, y is expanded (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

To illustrate the effects of time compression/expansion, an example is provided in Figure 3.3. By applying a time compression/expansion transformation to the function x in Figure 3.3(a), each of the functions shown in Figures 3.3(b) and (c) can be obtained.

3.2.4 Time Scaling (Dilation/Reflection)

Another type of signal transformation is called time scaling. **Time scaling** maps a function x to the function y given by

$$y(t) = x(at), \tag{3.4}$$

where a is a nonzero real constant. In other words, the function y is formed by replacing t with at in the expression for $x(t)$. The quantity a is referred to as the scaling factor. Geometrically, the transformation (3.4) is associated with a compression/expansion along the time axis and/or reflection about the origin. If $|a| < 1$, the function is expanded (i.e., stretched) along the time axis. If $|a| > 1$, the function is instead compressed. If $|a| = 1$, the function is neither expanded nor compressed. Lastly, if $a < 0$, the function is reflected about the origin. Observe that time scaling includes both time compression/expansion and time reversal as special cases.

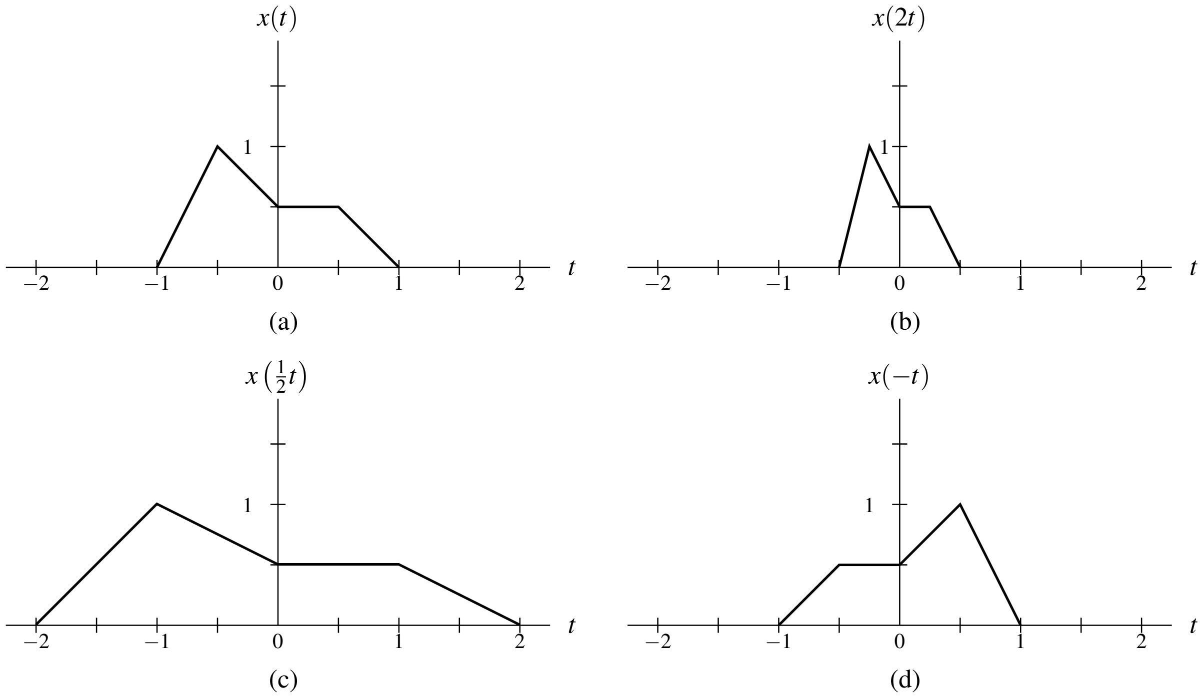

Figure 3.4: Example of time scaling. (a) The function x; and the result of applying a time-scaling transformation to x with a scaling factor of (b) 2, (c) $\frac{1}{2}$, and (d) -1.

To illustrate the behavior of time scaling, we provide an example in Figure 3.4. Each of the functions shown in Figures 3.4(b), (c), and (d) can be obtained by applying a time-scaling transformation to the function x given in Figure 3.4(a).

3.2.5 Combining Time Shifting and Time Scaling

In the preceding sections, we introduced the time shifting and time scaling transformations. Moreover, we observed that time scaling includes time compression/expansion and time reversal as special cases. Some independent-variable transformations commute, while others do not. The issue of commutativity is important, for example, when trying to simplify or manipulate expressions involving combined transformations. Time-scaling, time-reversal, and time-compression/expansion operations commute. Time-shifting (with a nonzero shift) and each of time-scaling, time-reversal, and time-compression/expansion operations do not commute.

Now, we introduce a more general transformation that combines the effects of time shifting and time scaling. This new transformation maps a function x to the function y given by

$$y(t) = x(at - b), \tag{3.5}$$

where a and b are real constants and $a \neq 0$. In other words, the function y is formed by replacing t with $at - b$ in the expression for $x(t)$. One can show that the transformation (3.5) is equivalent to first time shifting x by b, and then time scaling the resulting function by a. Geometrically, this transformation preserves the shape of x except for a possible expansion/compression along the time axis and/or a reflection about the origin. If $|a| < 1$, the function is stretched along the time axis. If $|a| > 1$, the function is instead compressed. If $a < 0$, the function is reflected about the origin.

The above transformation has two distinct but equivalent interpretations. That is, it is equivalent to each of the following:

1. first, time shifting x by b, and then time scaling the result by a.
2. first, time scaling x by a, and then time shifting the result by $\frac{b}{a}$.

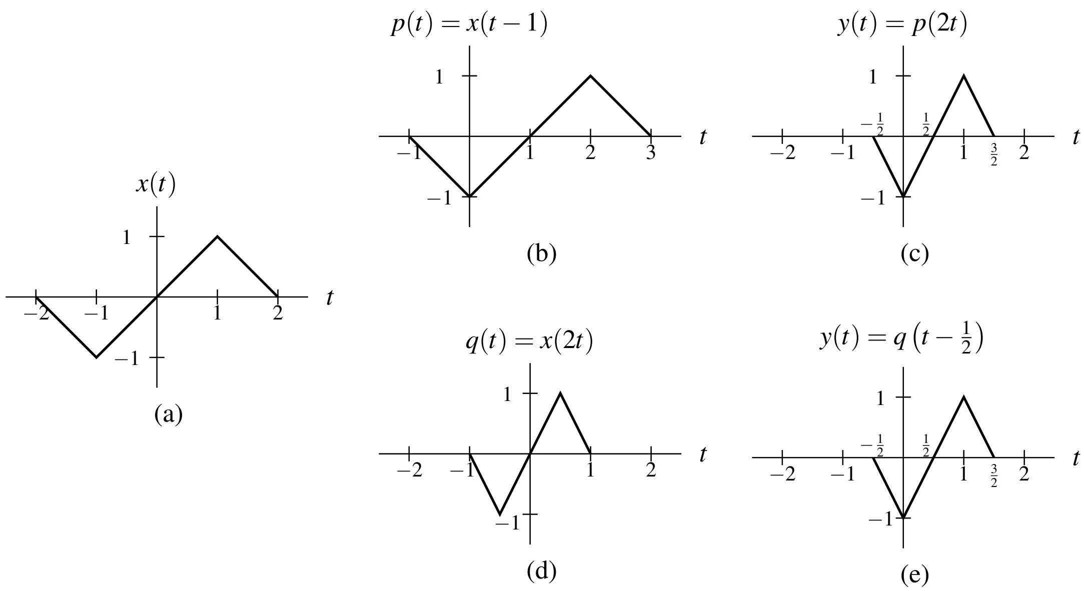

Figure 3.5: Two different interpretations of a combined time-shifting and time-scaling transformation. (a) Original function. Results obtained by shifting followed by scaling: (b) intermediate result and (c) final result. Results obtained by scaling followed by shifting: (d) intermediate result and (e) final result.

Observe that the shifting amount differs in these two interpretations (i.e., b versus $\frac{b}{a}$). This is due to the fact that time shifting and time scaling do not commute. The proof that the above two interpretations are valid is left as an exercise for the reader in Exercise 3.3.

Example 3.1. To illustrate the two equivalent interpretations of this combined transformation, we consider a simple example. Consider the function x shown in Figure 3.5(a). Let us now determine the transformed function $y(t) = x(at - b)$, where $a = 2$ and $b = 1$.

Solution. First, we consider the shift-then-scale method. In this case, we first shift the function x by b (i.e., 1). This yields the function in Figure 3.5(b). Then, we scale this new function by a (i.e., 2) in order to obtain y as shown in Figure 3.5(c). Second, we consider the scale-then-shift method. In this case, we first scale the function x by a (i.e., 2). This yields the function in Figure 3.5(d). Then, we shift this new function by $\frac{b}{a}$ (i.e., $\frac{1}{2}$) in order to obtain y as shown in Figure 3.5(e). ■

3.2.6 Two Perspectives on Independent-Variable Transformations

A transformation of the independent variable can be viewed in terms of:

1. the effect that the transformation has on the *function*; or

2. the effect that the transformation has on the *horizontal axis*.

This distinction is important because such a transformation has *opposite* effects on the function and horizontal axis. For example, the (time-shifting) transformation that replaces t by $t - b$ (where b is a real number) in the expression for $x(t)$ can be viewed as a transformation that

1. shifts the function x *right* by b units; or

2. shifts the horizontal axis *left* by b units.

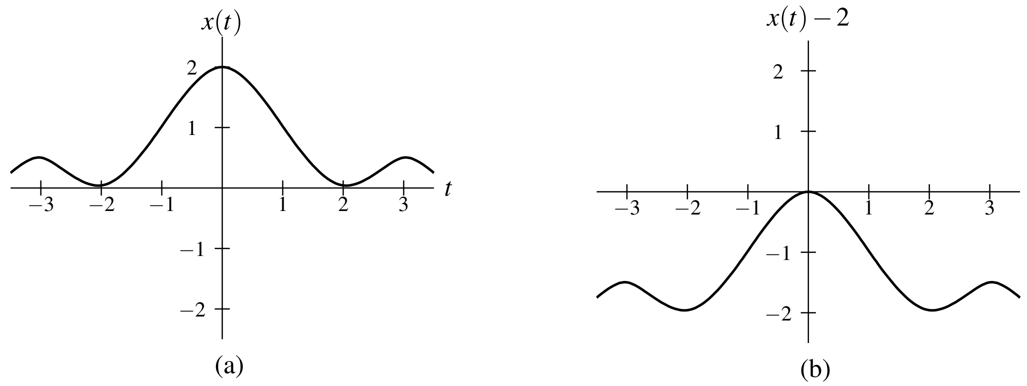

Figure 3.6: Example of amplitude shifting. (a) The function x; and the result obtained by applying an amplitude-shifting transformation to x with a shifting value of -2.

In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *function*. If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

3.3 Transformations of the Dependent Variable

In the preceding sections, we examined several transformations of the independent variable. Now, we consider some transformations of the dependent variable.

3.3.1 Amplitude Shifting

The first transformation that we consider is referred to as amplitude shifting. **Amplitude shifting** maps a function x to the function y given by

$$y(t) = x(t) + b,$$

where b is a scalar constant. Geometrically, the function y is displaced vertically relative to x. If $b > 0$, y is shifted upwards by $|b|$ relative to x. If $b < 0$, y is shifted downwards by $|b|$ relative to x.

The effects of amplitude shifting are illustrated by Figure 3.6. The function shown in Figure 3.6(b) can be obtained by applying an amplitude-shifting transformation to the function x given in Figure 3.6(a).

3.3.2 Amplitude Scaling

The next transformation that we consider is referred to as amplitude scaling. **Amplitude scaling** maps a function x to the function y given by

$$y(t) = ax(t),$$

where a is a scalar constant. Geometrically, the function y is expanded/compressed in amplitude and/or reflected about the horizontal axis, relative to x.

To illustrate the effects of amplitude scaling, an example is given in Figure 3.7. Each of the functions shown in Figures 3.7(b), (c), and (d) can be obtained by applying an amplitude-scaling transformation to the function x given in Figure 3.7(a).

3.3.3 Combining Amplitude Shifting and Scaling

In the previous sections, we considered the amplitude-shifting and amplitude-scaling transformations. We can define a new transformation that combines the effects of amplitude shifting and amplitude scaling. This transformation maps

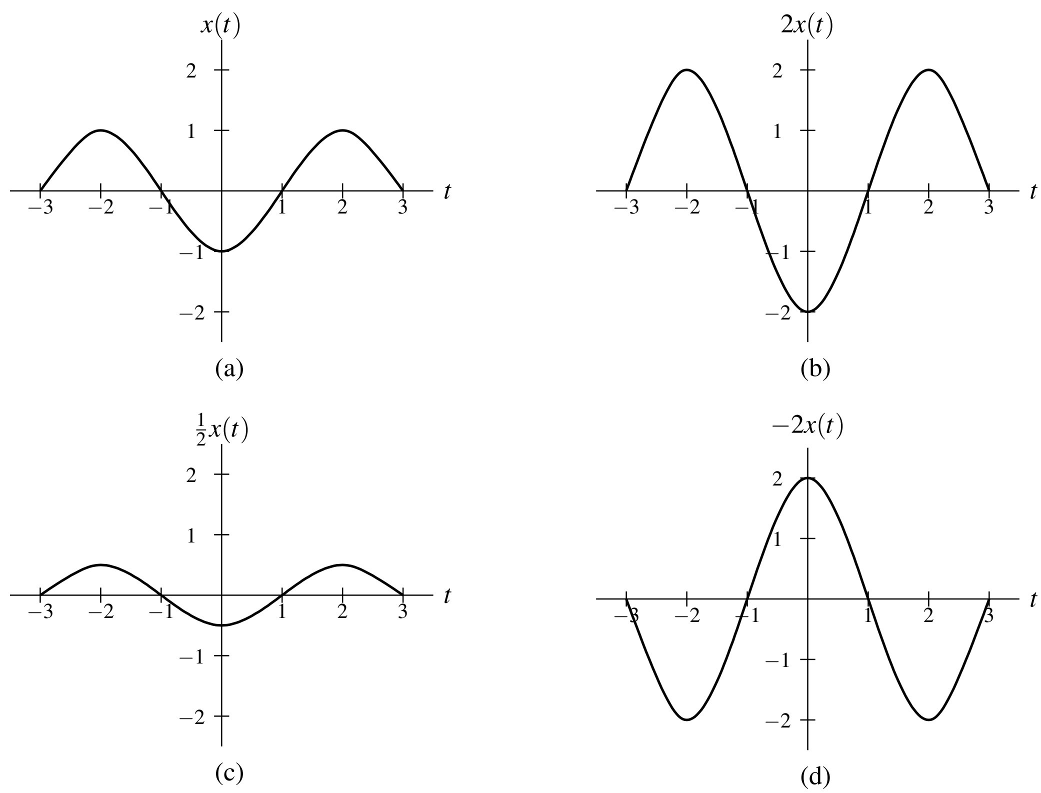

Figure 3.7: Example of amplitude scaling. (a) The function x; and the result of applying an amplitude-scaling transformation to x with a scaling factor of (b) 2, (c) $\frac{1}{2}$, and (d) -2.

the function x to the function y given by

$$y(t) = ax(t) + b, \tag{3.6}$$

where a and b are scalar constants. One can show that this transformation is equivalent to first scaling x by a and then shifting the resulting function by b. Moreover, since (3.6) can be rewritten as $y(t) = a[x(t) + \frac{b}{a}]$, this transformation is also equivalent to first amplitude shifting x by $\frac{b}{a}$ and then amplitude scaling the resulting function by a.

3.4 Properties of Functions

Functions can possess a number of interesting properties. In what follows, we consider the properties of symmetry and periodicity (introduced earlier) in more detail. Also, we present several other function properties. The properties considered are frequently useful in the analysis of signals and systems.

3.4.1 Remarks on Symmetry

At this point, we make some additional comments about even and odd functions (introduced earlier). Since functions are often summed or multiplied, one might wonder what happens to the even/odd symmetry properties of functions under these operations. In what follows, we introduce a few results in this regard.

Sums involving even and odd functions have the following properties:

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of an even function and odd function is neither even nor odd, provided that the functions are not identically zero.

Products involving even and odd functions have the following properties:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.

(The proofs of the above properties involving sums and products of even and odd functions is left as an exercise for the reader in Exercise 3.10.)

As it turns out, any arbitrary function can be expressed as the sum of an even and odd function, as elaborated upon by the theorem below.

Theorem 3.1 (Decomposition of function into even and odd parts)**.** *Any arbitrary function x can be uniquely represented as the sum of the form*

$$x(t) = x_\mathrm{e}(t) + x_\mathrm{o}(t), \tag{3.7}$$

where x_e and x_o are even and odd, respectively, and given by

$$x_\mathrm{e}(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \textit{and} \tag{3.8}$$

$$x_\mathrm{o}(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right]. \tag{3.9}$$

As a matter of terminology, x_e is called the ***even part*** *of x and is denoted* $\operatorname{Even}\{x\}$*, and x_o is called the* ***odd part*** *of x and is denoted* $\operatorname{Odd}\{x\}$.

Proof. From (3.8) and (3.9), we can easily confirm that $x_\mathrm{e} + x_\mathrm{o} = x$ as follows:

$$\begin{aligned} x_\mathrm{e}(t) + x_\mathrm{o}(t) &= \tfrac{1}{2}[x(t) + x(-t)] + \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\ &= x(t). \end{aligned}$$

Furthermore, we can easily verify that x_e is even and x_o is odd. From the definition of x_e in (3.8), we have

$$\begin{aligned} x_e(-t) &= \tfrac{1}{2}[x(-t) + x(-[-t])] \\ &= \tfrac{1}{2}[x(t) + x(-t)] \\ &= x_e(t). \end{aligned}$$

Thus, x_e is even. From the definition of x_o in (3.9), we have

$$\begin{aligned} x_o(-t) &= \tfrac{1}{2}[x(-t) - x(-[-t])] \\ &= \tfrac{1}{2}[-x(t) + x(-t)] \\ &= -x_o(t). \end{aligned}$$

Thus, x_o is odd.

Lastly, we show that the decomposition of x into the sum of an even function and odd function is unique. Suppose that x can be written as the sum of an even function and odd function in two ways as

$$x(t) = f_e(t) + f_o(t) \quad \text{and} \tag{3.10a}$$

$$x(t) = g_e(t) + g_o(t), \tag{3.10b}$$

where f_e and g_e are even and f_o and g_o are odd. Equating these two expressions for x, we have

$$f_e(t) + f_o(t) = g_e(t) + g_o(t).$$

Rearranging this equation, we have

$$f_e(t) - g_e(t) = g_o(t) - f_o(t).$$

Now, we consider the preceding equation more carefully. Since the sum of even functions is even and the sum of odd functions is odd, we have that the left- and right-hand sides of the preceding equation correspond to even and odd functions, respectively. Thus, we have that the even function $f_e(t) - g_e(t)$ is equal to the odd function $g_o(t) - f_o(t)$. The only function, however, that is both even and odd is the zero function. (A proof of this fact is left as an exercise for the reader in Exercise 3.16.) Therefore, we have that

$$f_e(t) - g_e(t) = g_o(t) - f_o(t) = 0.$$

In other words, we have that

$$f_e(t) = g_e(t) \quad \text{and} \quad f_o(t) = g_o(t).$$

This implies that the two decompositions of x given by (3.10a) and (3.10b) must be the same decomposition (i.e., they cannot be distinct). Thus, the decomposition of x into the sum of an even function and odd function is unique. ■

3.4.2 Remarks on Periodicity

Since we often add functions, it is helpful to know if the sum of periodic functions is also periodic. We will consider this issue next, but before doing so we first must introduce the notion of a least common multiple.

The **least common multiple (LCM)** of two strictly positive real numbers a_1 and a_2, denoted $\operatorname{lcm}(a_1, a_2)$, is the smallest positive real number that is an integer multiple of each a_1 and a_2. For example, $\operatorname{lcm}(6\pi, 10\pi) = 30\pi$, since 30π is the smallest positive real number that can be evenly divided by both 6π and 10π. More generally, the **least common multiple** of a set of positive real numbers $\{a_1, a_2, \ldots, a_N\}$, denoted $\operatorname{lcm}\{a_1, a_2, \ldots, a_N\}$, is the smallest positive real number that is an integer multiple of each a_k for $k = 1, 2, \ldots, N$.

Having introduced the notation of a least common multiple, we can now consider whether the sum of two periodic functions is periodic. In this regard, the theorem below is enlightening.

