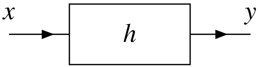
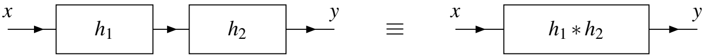
Block Diagram Representation of LTI Systems

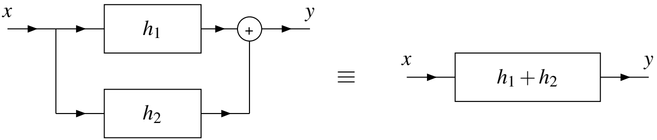
- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.



- The ***series*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 * h_2$. That is, we have the equivalence shown below.



- The ***parallel*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 + h_2$. That is, we have the equivalence shown below.



Section 4.3

Properties of LTI Systems

Memory

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t),$$

  where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

  (i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

Example 4.8 Example 4.9

Causality

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

  (i.e., h is a causal function).
- It is due to the above relationship that we call a function x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

  a causal function.

Example 4.10 Example 4.11

Invertibility

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

Example 4.12

BIBO Stability

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., h is ***absolutely integrable***).

Example 4.14 Example 4.15

Eigenfunctions of LTI Systems

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st},$$

  where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

Representations of Functions Using Eigenfunctions

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials
$$x(t) = \sum_k a_k e^{s_k t},$$
where the a_k and s_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude
$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$
- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Corollary of Theorem 4.12 Example 4.16

Part 5

Continuous-Time Fourier Series (CTFS)

Introduction

- The (CT) Fourier series is a representation for ***periodic*** functions.
- With a Fourier series, a function is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are ***eigenfunctions*** of LTI systems.

Section 5.1

Fourier Series

Harmonically-Related Complex Sinusoids

- A set of complex sinusoids is said to be **harmonically related** if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{for all integer } k.$$

- The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0.
- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.
- More specifically, a linear combination of these complex sinusoids is periodic with period $T = \frac{2\pi}{\omega_0}$.

CT Fourier Series

- A periodic (complex-valued) function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a (CT) **Fourier series**, and the c_k are called **Fourier series coefficients**.
- The above formula for x is often referred to as the **Fourier series synthesis equation**.
- The terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$.
- To denote that a function x has the Fourier series coefficient sequence c_k, we write

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

- The periodic function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

where $\int_T$ denotes integration over an arbitrary interval of length T (i.e., one period of x).

- The above equation for c_k is often referred to as the **Fourier series analysis equation**.

Example 5.1 Example 5.3

Section 5.2

Convergence Properties of Fourier Series

Remarks on Equality of Functions

- The equality of functions can be defined in more than one way.
- Two functions x and y are said to be **equal in the pointwise sense** if $x(t) = y(t)$ for all t (i.e., x and y are equal at every point).
- Two functions x and y are said to be **equal in the mean-squared error (MSE) sense** if $\int |x(t) - y(t)|^2 dt = 0$ (i.e., the energy in $x - y$ is zero).
- Pointwise equality is a stronger condition than MSE equality (i.e., pointwise equality implies MSE equality but the converse is not true).
- Consider the functions

$$x_1(t) = 1 \text{ for all } t, \quad x_2(t) = 1 \text{ for all } t, \quad \text{and}$$

$$x_3(t) = \begin{cases} 2 & t = 0 \\ 1 & \text{otherwise.} \end{cases}$$

- The functions x_1 and x_2 are equal in both the pointwise sense and MSE sense.
- The functions x_1 and x_3 are equal in the MSE sense, but not in the pointwise sense.

Convergence of Fourier Series

- Since a Fourier series can have an infinite number of (nonzero) terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.
- That is, when we claim that a periodic function x is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?
- Consider a periodic function x that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let x_N denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N$ is equal (in some sense) to x.

- Again, let x_N denote the Fourier series for the periodic function x truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- If $\lim_{N\to\infty} x_N(t) = x(t)$ for all t (i.e., $\lim_{N\to\infty} x_N$ is equal to x in the pointwise sense), the Fourier series is said to converge **pointwise** to x.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be **uniform**.
- If $\lim_{N\to\infty} \frac{1}{T}\int_T |x_N(t) - x(t)|^2 \, dt = 0$ (i.e., $\lim_{N\to\infty} x_N$ is equal to x in the MSE sense), the Fourier series is said to converge to x in the **MSE** sense.
- Pointwise convergence is a stronger condition than MSE convergence (i.e., pointwise convergence implies MSE convergence, but the converse is not true).

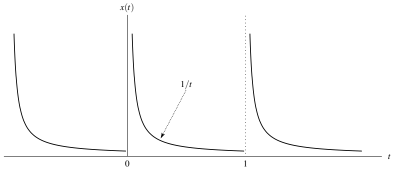
Convergence of Fourier Series: Continuous Case

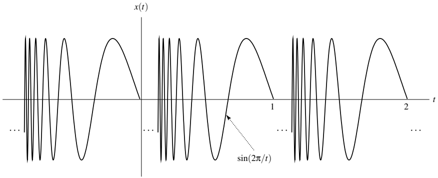
- If a periodic function x is ***continuous*** and its Fourier series coefficients c_k are ***absolutely summable*** (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges ***uniformly*** (i.e., pointwise at the same rate everywhere).
- Since, in practice, we often encounter functions with discontinuities (e.g., a square wave), the above result is of somewhat limited value.

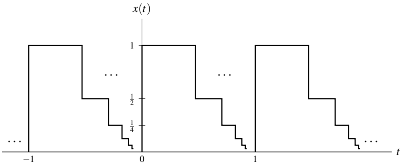
Convergence of Fourier Series: Finite-Energy Case

- If a periodic function x has ***finite energy*** in a single period (i.e., $\int_T |x(t)|^2\,dt < \infty$), the Fourier series converges in the ***MSE*** sense.
- Since, in situations of practice interest, the finite-energy condition in the above theorem is typically satisfied, the theorem is usually applicable.
- It is important to note, however, that MSE convergence (i.e., $E = 0$) does not necessarily imply pointwise convergence (i.e., $\tilde{x}(t) = x(t)$ for all t).
- Thus, the above convergence theorem does not provide much useful information regarding the value of $\tilde{x}(t)$ for specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier series converges.

- The **Dirichlet conditions** for the periodic function x are as follows:
    1. over a single period, x is ***absolutely integrable*** (i.e., $\int_T |x(t)|\, dt < \infty$);
    2. over a single period, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***); and
    3. over any finite interval, x has a ***finite number of discontinuities***, each of which is ***finite***.
- Examples of functions violating the Dirichlet conditions are shown below.







Convergence of Fourier Series: Dirichlet Case

- If a periodic function x satisfies the ***Dirichlet conditions***, then:
  1. the Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x; and
  2. at each point t_a of discontinuity of x, the Fourier series $\tilde{x}$ converges to

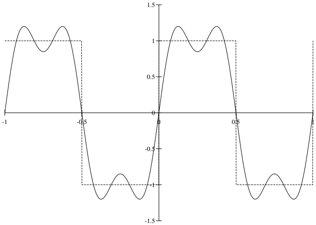
  $$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^-) + x(t_a^+)\right],$$

  where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.
- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.

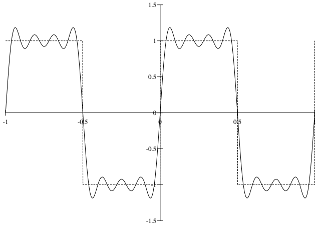
Gibbs Phenomenon

- In practice, we frequently encounter functions with discontinuities.
- When a function x has discontinuities, the Fourier series representation of x does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains approximately constant.
- This behavior is known as **Gibbs phenomenon**.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).

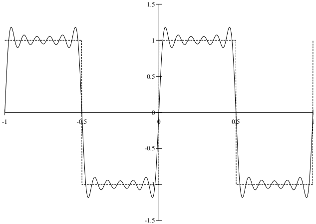
Gibbs Phenomenon: Periodic Square Wave Example



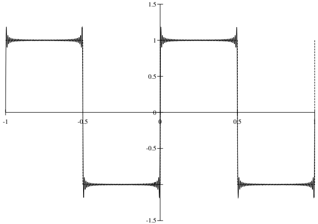
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101st harmonic components

Annotated example