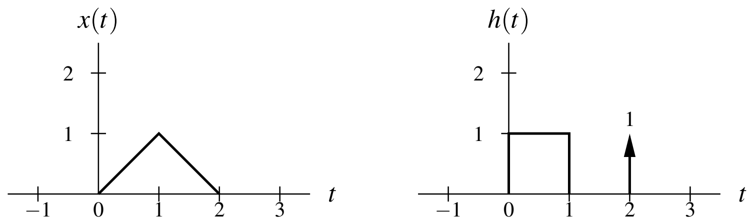
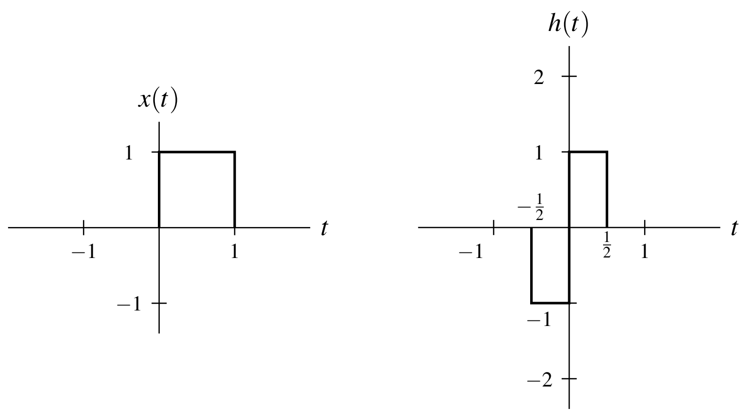
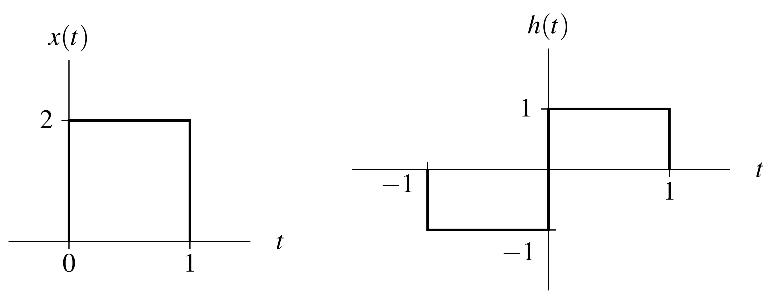
3A **4.1** Using the graphical method, for each pair of functions x and h given in the figures below, compute $x * h$.



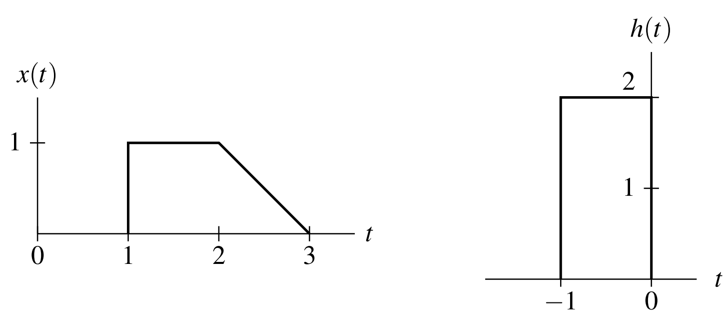
(a)



(b)



(c)



(d)

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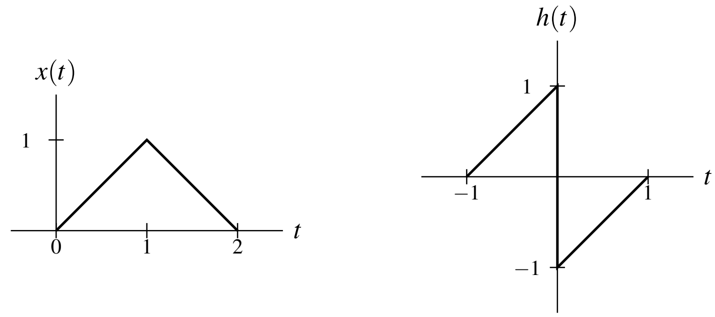
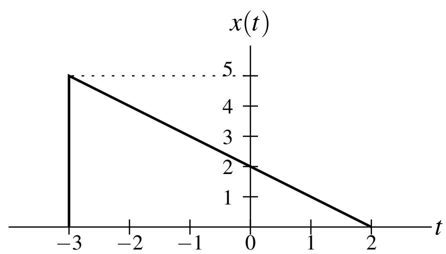


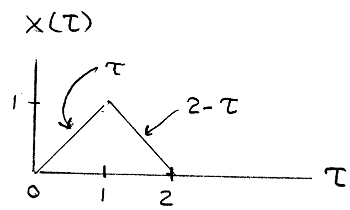
(e)



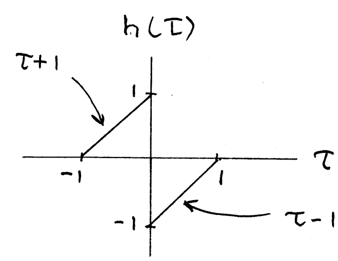
(f)

3A Answer (e).

To assist in the convolution computation, we first plot $x(\tau)$ and $h(t-\tau)$ versus τ as shown below.



(a)



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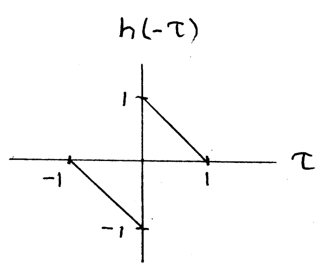
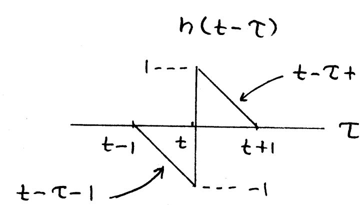


(c)



(d)

From the above plots, we can deduce that there are six cases (i.e., intervals of t) to be considered, which correspond to the scenarios shown in the graphs below.

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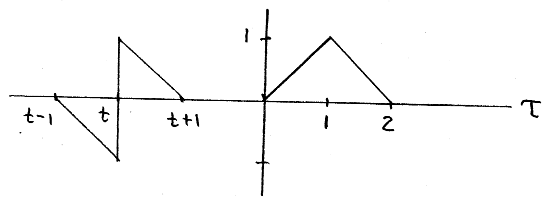
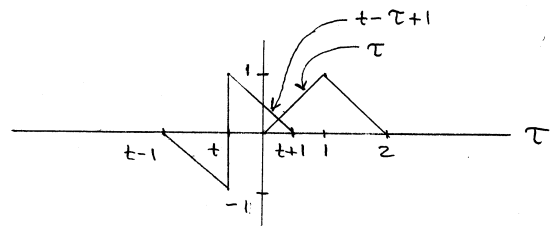
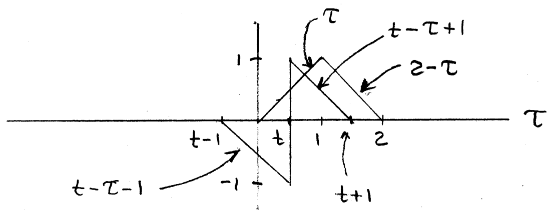


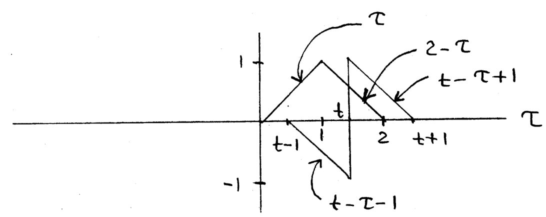
(a)



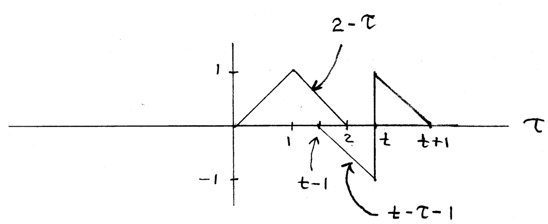
(b)



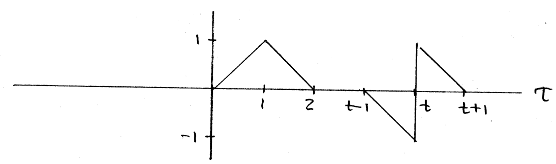
(c)



(d)



(e)



(f)

In the case that $t < -1$, which corresponds to Figure (a), we trivially have

$$x * h(t) = 0.$$

In the case that $-1 \leq t < 0$, which corresponds to Figure (b), we have

$$x * h(t) = \int_{0}^{t+1} (\tau)(t - \tau + 1) d\tau.$$

In the case that $0 \leq t < 1$, which corresponds to Figure (c), we have

$$x * h(t) = \int_{0}^{t} (\tau)(t - \tau - 1) d\tau + \int_{t}^{1} (\tau)(t - \tau + 1) d\tau + \int_{1}^{t+1} (2 - \tau)(t - \tau + 1) d\tau.$$

In the case that $1 \leq t < 2$, which corresponds to Figure (d), we have

$$x * h(t) = \int_{t-1}^{1} (\tau)(t - \tau - 1) d\tau + \int_{1}^{t} (2 - \tau)(t - \tau - 1) d\tau + \int_{t}^{2} (2 - \tau)(t - \tau + 1) d\tau.$$

In the case that $2 \leq t < 3$, which corresponds to Figure (e), we have

$$x * h(t) = \int_{t-1}^{2} (2 - \tau)(t - \tau - 1) d\tau.$$

In the case that $t \geq 3$, which corresponds to Figure (f), we trivially have

$$x * h(t) = 0.$$

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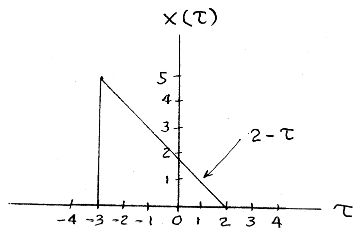
Combining the above results, we have that

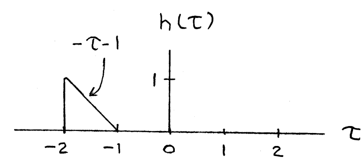
$$x*h(t)=\begin{cases}\int_0^{t+1}(\tau)(t-\tau+1)d\tau & -1\leq t<0\\ \int_0^t(\tau)(t-\tau-1)d\tau+\int_t^1(\tau)(t-\tau+1)d\tau+\int_1^{t+1}(2-\tau)(t-\tau+1)d\tau & 0\leq t<1\\ \int_{t-1}^1(\tau)(t-\tau-1)d\tau+\int_1^t(2-\tau)(t-\tau-1)d\tau+\int_t^2(2-\tau)(t-\tau+1)d\tau & 1\leq t<2\\ \int_{t-1}^2(2-\tau)(t-\tau-1)d\tau & 2\leq t<3\\ 0 & \text{otherwise.}\end{cases}$$

3A Answer (f).

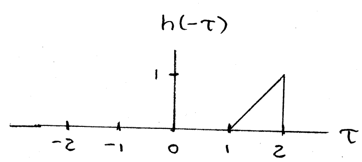
To assist in the convolution computation, we first plot $x(\tau)$ and $h(t-\tau)$ versus τ as shown below.



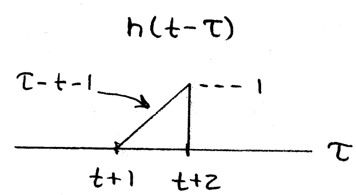
(a)



(b)

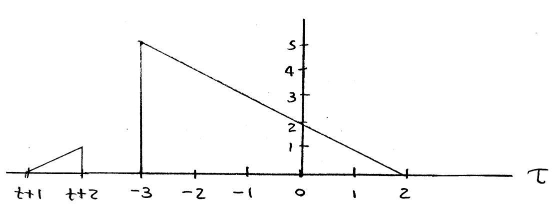


(c)

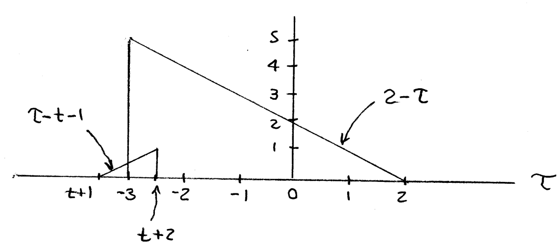


(d)

From the above plots, we can deduce that there are five cases (i.e., intervals of t) to be considered, which correspond to the scenarios shown in the graphs below.



(a)



(b)

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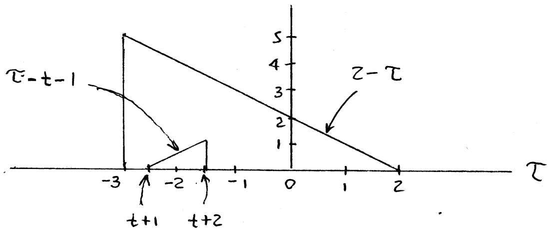
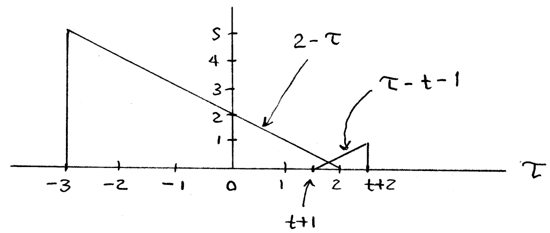
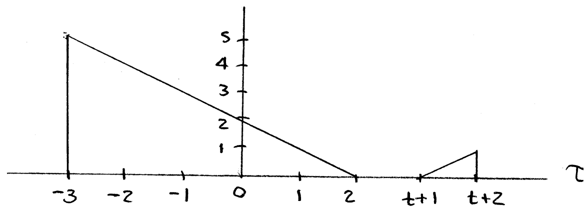


(c)



(d)



(e)

In the case that $t < -5$, which corresponds to Figure (a), we trivially have

$$x * h(t) = 0.$$

In the case that $-5 \le t < -4$, which corresponds to Figure (b), we have

$$x * h(t) = \int_{-3}^{t+2} (2-\tau)(\tau - t - 1)d\tau.$$

In the case that $-4 \le t < 0$, which corresponds to Figure (c), we have

$$x * h(t) = \int_{t+1}^{t+2} (2-\tau)(\tau - t - 1)d\tau.$$

In the case that $0 \le t < 1$, which corresponds to Figure (d), we have

$$x * h(t) = \int_{t+1}^{2} (2-\tau)(\tau - t - 1)d\tau.$$

In the case that $t \ge 1$, which corresponds to Figure (e), we trivially have

$$x * h(t) = 0.$$

Combining all of the above results, we have

$$x * h(t) = \begin{cases} \int_{-3}^{t+2} (2-\tau)(\tau - t - 1)d\tau & -5 \le t < -4 \\ \int_{t+1}^{t+2} (2-\tau)(\tau - t - 1)d\tau & -4 \le t < 0 \\ \int_{t+1}^{2} (2-\tau)(\tau - t - 1)d\tau & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

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3A **4.3** Using the graphical method, compute $x * h$ for each pair of functions x and h given below.

(a) $x(t) = e^t u(-t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}(\frac{1}{3}[t - \frac{1}{2}])$;

(c) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(d) $x(t) = \operatorname{rect}\left(\frac{1}{2}t\right)$ and $h(t) = e^{2-t}u(t-2)$;

(e) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t+2 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$

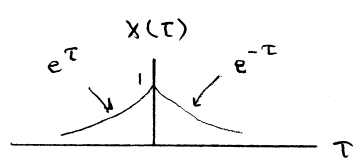
(f) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x(t) = \begin{cases} 1 - \frac{1}{4}t & 0 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

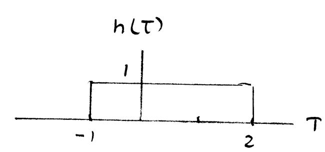
(h) $x(t) = \operatorname{rect}\left(\frac{1}{4}t\right)$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$ and

(i) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-2 & 2 \le t < 4 \\ 0 & \text{otherwise.} \end{cases}$

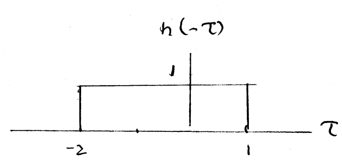
3A **Answer (b).**



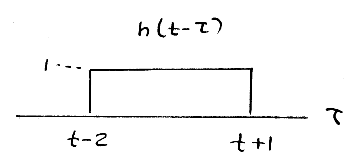
(a)



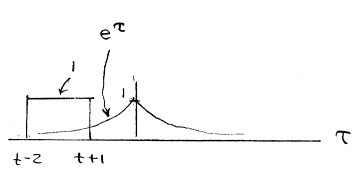
(b)



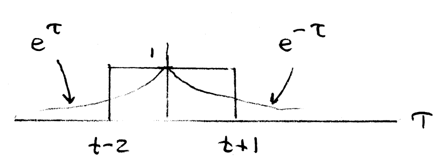
(c)



(d)



(e)



(f)

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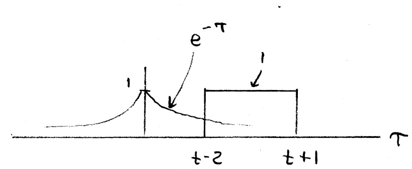
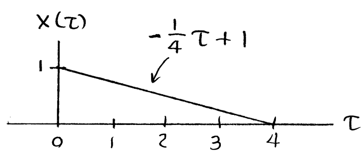


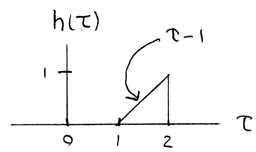
(g)

$$x*h(t)=\begin{cases}\int_{t-2}^{t+1} e^{\tau}d\tau & t<-1\\ \int_{t-2}^{0} e^{\tau}d\tau+\int_{0}^{t+1} e^{-\tau}d\tau & -1\le t<2\\ \int_{t-2}^{t+1} e^{-\tau}d\tau & t\ge 2\end{cases}$$

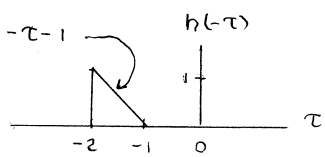
3A Answer (g).



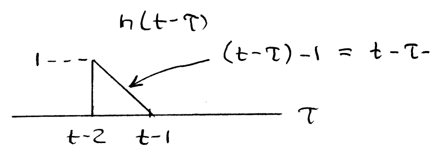
(a)



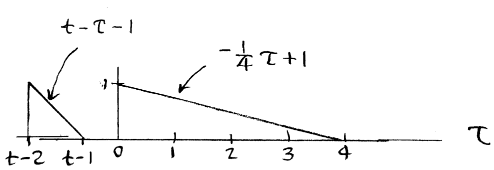
(b)



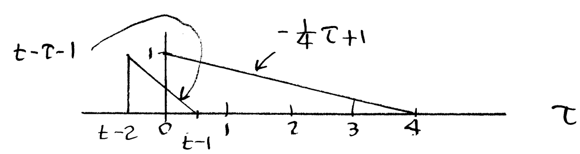
(c)



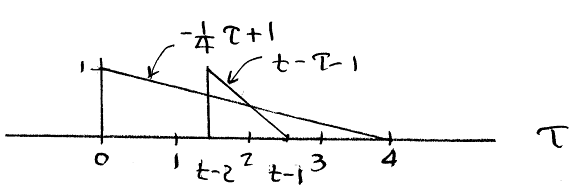
(d)



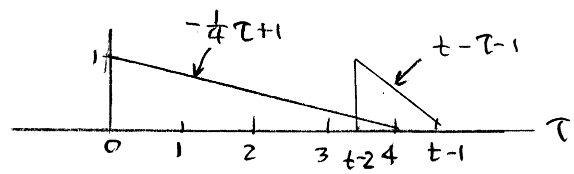
(e)



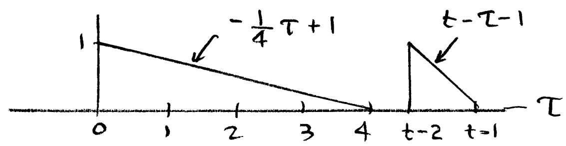
(f)



(g)



(h)



(i)

$$x*h(t) = \begin{cases} \int_0^{t-1}\left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1}\left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 2 \le t < 5 \\ \int_{t-2}^{4}\left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 5 \le t < 6 \\ 0 & \text{otherwise} \end{cases}$$

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3A **4.5** Let x, y, h, and v be functions such that $y = x * h$ and

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where a and b are real constants. Express v in terms of y.

3A **Answer.**

From the definition of v, we have

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau.$$

Now, we employ a change of variable. Let $\lambda = -\tau - b$ so that $\tau = -\lambda - b$ and $d\tau = -d\lambda$. Applying this change of variable and simplifying, we obtain

$$\begin{aligned}
v(t) &= \int_{\infty}^{-\infty} x(\lambda)h([-\lambda - b] + at)(-1)d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)h(at - b - \lambda)d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)h([at - b] - \lambda)d\lambda \\
&= x * h(at - b) \\
&= y(at - b).
\end{aligned}$$

Therefore, we have that $v(t) = y(at - b)$.

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3A **4.6** Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

3A **Answer (a).**

From the definition of convolution, we have

$$\begin{aligned} y(t) &= x * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \end{aligned}$$

Suppose that x is periodic with period T. Then, we have $x(t) = x(t+T)$ and we can rewrite the above integral as

$$y(t) = \int_{-\infty}^{\infty} x(\tau+T)h(t-\tau)d\tau.$$

Now, we employ a change of variable. Let $\lambda = \tau + T$ so that $\tau = \lambda - T$ and $d\lambda = d\tau$. This yields

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\lambda)h(t-[\lambda-T])d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h(t+T-\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h([t+T]-\lambda)d\lambda \\ &= x * h(t+T) \\ &= y(t+T). \end{aligned}$$

Therefore, y is periodic with period T.

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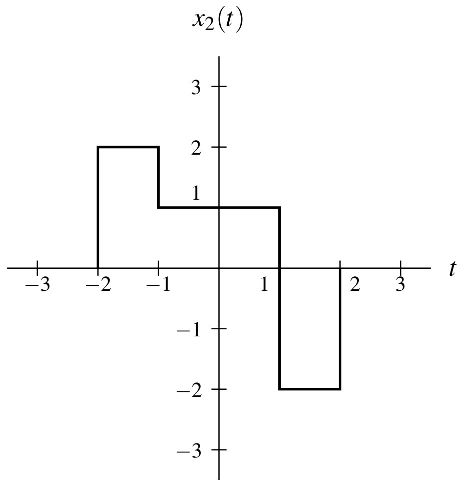
3A **4.9** Consider a LTI system whose response to the function $x_1(t) = u(t) - u(t-1)$ is the function y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



3A **Answer.**

First, we express x_2 in terms of x_1. This yields

$$x_2(t) = 2x_1(t+2) + x_1(t+1) + x_1(t) - 2x_1(t-1).$$

Then, we observe that the system is LTI. This implies that

$$2x_1(t+2) + x_1(t+1) + x_1(t) - 2x_1(t-1) \to 2y_1(t+2) + y_1(t+1) + y_1(t) - 2y_1(t-1).$$

Therefore, we have

$$y_2(t) = 2y_1(t+2) + y_1(t+1) + y_1(t) - 2y_1(t-1).$$

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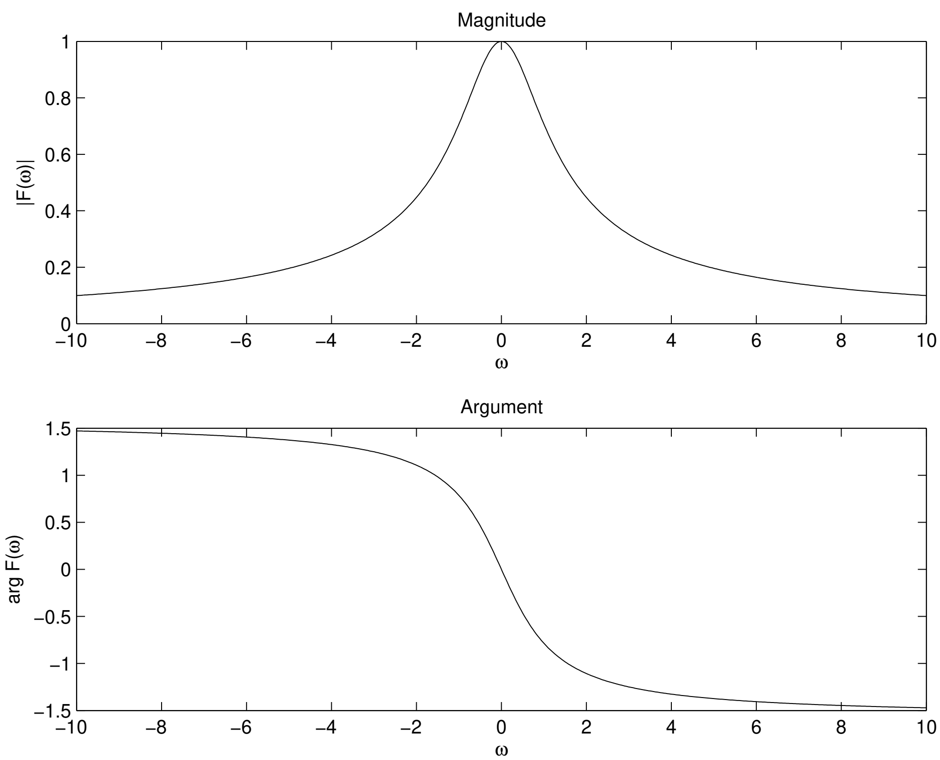
3A **D.103** Let F denote the complex-valued function of a real variable given by

$$F(\omega) = \frac{1}{j\omega + 1}.$$

Write a program to plot $|F(\omega)|$ and $\arg F(\omega)$ for ω in the interval $[-10, 10]$. Use `subplot` to place both plots on the same figure.

3A **Answer.**

```
w = linspace(-10, 10, 500);
f = (j * w + 1) .^ (-1);
subplot(2, 1, 1);
plot(w, abs(f));
title('Magnitude');
xlabel('\omega');
ylabel('|F(\omega)|');
subplot(2, 1, 2);
plot(w, unwrap(angle(f)));
title('Argument');
xlabel('\omega');
ylabel('arg F(\omega)');
```



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