ECE 260

EXAM 5

SOLUTIONS

(FALL 2024)

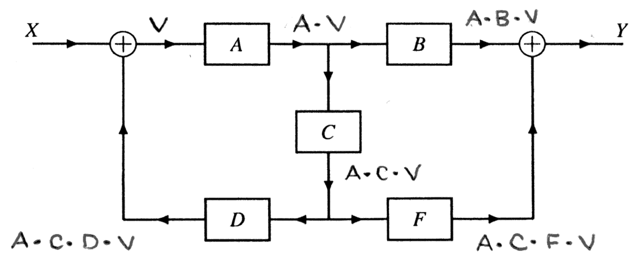
QUESTION 1

$$
\begin{aligned}
x(t) &= (t+1)^2[u(t+1)-u(t)] + (t-1)^2[u(t)-u(t-1)] \\
&= (t+1)^2u(t+1) - (t+1)^2u(t) + (t-1)^2u(t) - (t-1)^2u(t-1) \\
&= (t+1)^2u(t+1) + [(t-1)^2-(t+1)^2]u(t) - (t-1)^2u(t-1) \\
&= (t+1)^2u(t+1) + [t^2-2t+1-(t^2+2t+1)]u(t) - (t-1)^2u(t-1) \\
&= (t+1)^2u(t+1) + [-4t]u(t) - (t-1)^2u(t-1) \\
&= (t+1)^2u(t+1) - 4tu(t) - (t-1)^2u(t-1)
\end{aligned}
$$

$$
\begin{aligned}
X(s) &= e^s L\{(\cdot)^2u(\cdot)\}(s) - 4L\{(\cdot)u(\cdot)\}(s) - e^{-s}L\{(\cdot)^2u(\cdot)\}(s) \\
&= e^s\left(\frac{2!}{s^3}\right) - 4\left(\frac{1!}{s^2}\right) - e^{-s}\left(\frac{2!}{s^3}\right) \\
&= \frac{2e^s}{s^3} - \frac{4}{s^2} - \frac{2e^{-s}}{s^3} \\
&= \frac{2e^s-2e^{-s}-4s}{s^3} \quad \text{for all } s \in \mathbb{C}
\end{aligned}
$$

(Since x is finite duration, the ROC of X is the entire complex plane.)

QUESTION 2(A)



from the block diagram (as labelled)

$$V(s) = X(s) + A(s)\,C(s)\,D(s)\,V(s) \Rightarrow$$

$$[1 - A(s)\,C(s)\,D(s)]\,V(s) = X(s) \Rightarrow$$

$$V(s) = \frac{X(s)}{1 - A(s)\,C(s)\,D(s)} \qquad (1)$$

$$Y(s) = A(s)\,B(s)\,V(s) + A(s)\,C(s)\,F(s)\,V(s)$$

$$= A(s)\,[B(s) + C(s)\,F(s)]\,V(s) \qquad (2)$$

substituting (1) into (2)

$$Y(s) = \frac{A(s)\,[B(s) + C(s)\,F(s)]}{1 - A(s)\,C(s)\,D(s)}\,X(s) \Rightarrow$$

$$\frac{Y(s)}{X(s)} = \frac{A(s)\,[B(s) + C(s)\,F(s)]}{1 - A(s)\,C(s)\,D(s)}$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$H(s) = \frac{A(s)\,[B(s) + C(s)\,F(s)]}{1 - A(s)\,C(s)\,D(s)}$$

The ROC of H is a RHP since the system is causal.

QUESTION 2(B)

A LTI system with system function G is BIBO stable if and only if the ROC of G contains the (entire) imaginary axis.

QUESTION 2(C)

$$\boxed{A(s) = 1, \quad B(s) = 0, \quad C(s) = 1/s, \quad D(s) = 2\alpha + 1, \quad F(s) = 1, \quad \alpha \in \mathbb{R}}$$

$$H(s) = \frac{A(s)\,[B(s) + C(s)F(s)]}{1 - A(s)\,C(s)\,D(s)}$$

$$= \frac{(1)\,[\frac{1}{s}(1)]}{1 - (1)(\frac{1}{s})(2\alpha+1)} = \frac{(\frac{1}{s})}{(1 - \frac{2\alpha+1}{s})} = \frac{(\frac{1}{s})}{(\frac{s-2\alpha-1}{s})} = \left(\frac{1}{s}\right)\left(\frac{s}{s-2\alpha-1}\right)$$

$$= \frac{1}{s-2\alpha-1} = \frac{1}{s-(2\alpha+1)}$$

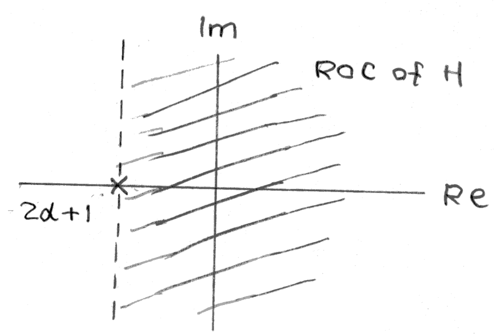
So H has a pole at $2\alpha+1$.

Since the system is causal, the ROC of H is the RHP to the right of $2\alpha+1$.

For BIBO stability, we require

$$2\alpha + 1 < 0 \Rightarrow 2\alpha < -1 \Rightarrow \alpha < -\tfrac{1}{2}$$

Therefore, the system is BIBO stable if and only if $\alpha < -\frac{1}{2}$.



QUESTION 3(A)

$$\boxed{D^2y(t) = D^2x(t) + 3Dx(t) + 2x(t)}$$

$$D^2y(t) = D^2x(t) + 3Dx(t) + 2x(t)$$

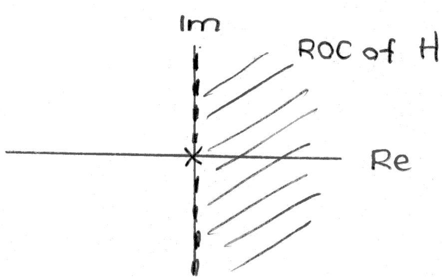
$$s^2 Y(s) = s^2 X(s) + 3s X(s) + 2X(s)$$

$$s^2 Y(s) = [s^2 + 3s + 2] X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{s^2 + 3s + 2}{s^2}$$

$$H(s) = \frac{s^2 + 3s + 2}{s^2} \quad \text{for } Re(s) > 0$$

Since the system is causal, the ROC of H is the RHP to the right of the rightmost pole (which is at 0).



QUESTION 3(B)

$$H(s) = \frac{s^2 + 3s + 2}{s^2} \quad \text{for } \operatorname{Re}(s) > 0$$

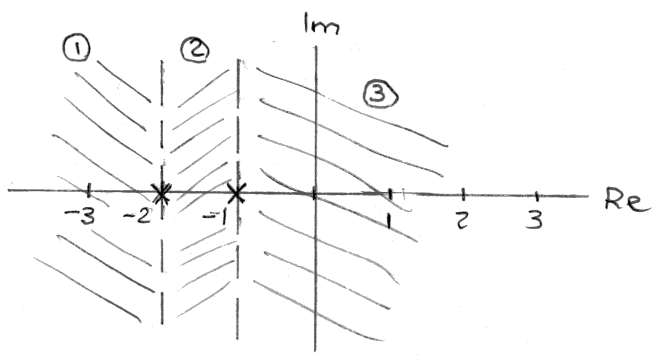
$$G(s) = \frac{1}{H(s)} = \frac{s^2}{s^2 + 3s + 2} = \frac{s^2}{(s+2)(s+1)}$$

G has three possible ROCs:

① $\operatorname{Re}(s) < -2$

② $-2 < \operatorname{Re}(s) < -1$

③ $\operatorname{Re}(s) > -1$



ROC ③ contains the imaginary axis and therefore corresponds to a BIBO stable system.

So, a BIBO stable inverse exists and has the system function G, where

$$G(s) = \frac{s^2}{(s+2)(s+1)} \quad \text{for } \operatorname{Re}(s) > -1.$$

QUESTION 4

$$\boxed{X(s) = \frac{s-8}{s^2-s-6} \quad \text{for } \mathrm{Re}(s) < -2}$$

$$X(s) = \frac{s-8}{s^2-s-6} = \frac{s-8}{(s+2)(s-3)}$$

$$X(s) = \frac{A_1}{s+2} + \frac{A_2}{s-3}$$

$$A_1 = [(s+2)X(s)]\Big|_{s=-2} = \frac{s-8}{s-3}\Big|_{s=-2} = \frac{-10}{-5} = 2$$

$$A_2 = [(s-3)X(s)]\Big|_{s=3} = \frac{s-8}{s+2}\Big|_{s=3} = \frac{-5}{5} = -1$$

$$X(s) = \frac{2}{s+2} - \frac{1}{s-3} \quad \leftarrow \text{[Note: Both terms have LHP ROCs.]}$$

$$x(t) = 2\,L^{-1}\left\{\frac{1}{(\cdot)+2}\right\}(t) - L^{-1}\left\{\frac{1}{(\cdot)-3}\right\}(t)$$

$$= 2\left[-e^{-2t}u(-t)\right] - \left[-e^{3t}u(-t)\right]$$

$$= -2e^{-2t}u(-t) + e^{3t}u(-t)$$

$$= \left[e^{3t} - 2e^{-2t}\right]u(-t)$$