ECE 260

EXAM 4

SOLUTIONS

(FALL 2024)

QUESTION 1(A)

$$|H(\omega)| = \left|\frac{\omega^2 - 2j\omega}{(j\omega-1)^2}\right| = \frac{|\omega^2-2j\omega|}{|(j\omega-1)^2|} = \frac{\sqrt{\omega^4+4\omega^2}}{|j\omega-1|^2} = \frac{\sqrt{\omega^2(\omega^2+4)}}{\left(\sqrt{1+\omega^2}\right)^2}$$

$$= \frac{|\omega|\sqrt{\omega^2+4}}{\omega^2+1}$$

ALTERNATIVE SOLUTION

$$|H(\omega)| = \left|\frac{\omega^2 - 2j\omega}{(j\omega-1)^2}\right| = \frac{|\omega^2-2j\omega|}{|(j\omega-1)^2|} = \frac{|\omega(\omega-2j)|}{|j\omega-1|^2} = \frac{|\omega||\omega-2j|}{\left(\sqrt{\omega^2+1}\right)^2}$$

$$= \frac{|\omega|\sqrt{\omega^2+4}}{\omega^2+1}$$

QUESTION 1(B)

$$\boxed{|H(\omega)| = \frac{|\omega|\sqrt{\omega^2+4}}{\omega^2+1}}$$

$$|H(0)| = \frac{0(2)}{1} = 0$$

$$\lim_{|\omega|\to\infty} |H(\omega)| = \lim_{|\omega|\to\infty} \frac{|\omega|\sqrt{\omega^2+4}}{\omega^2+1} = \lim_{|\omega|\to\infty} \frac{|\omega|\sqrt{\omega^2}}{\omega^2}$$

$$= \lim_{|\omega|\to\infty} \frac{|\omega||\omega|}{\omega^2} = \lim_{|\omega|\to\infty} \frac{\omega^2}{\omega^2} = 1$$

since $|H(0)| = 0$, the filter eliminates low frequencies

Since $\lim_{|\omega|\to\infty} |H(\omega)| = 1$, the filter passes high frequencies

therefore, the filter best approximates an ideal highpass filter

QUESTION 2(A)

The sampling theorem requires that the condition $\omega_s > 2\omega_m$ be satisfied in order to avoid aliasing, where ω_s is the sampling rate and ω_m is the highest magnitude frequency in the signal being sampled.

For the given function x, we have

$$\omega_m = 4000\pi$$

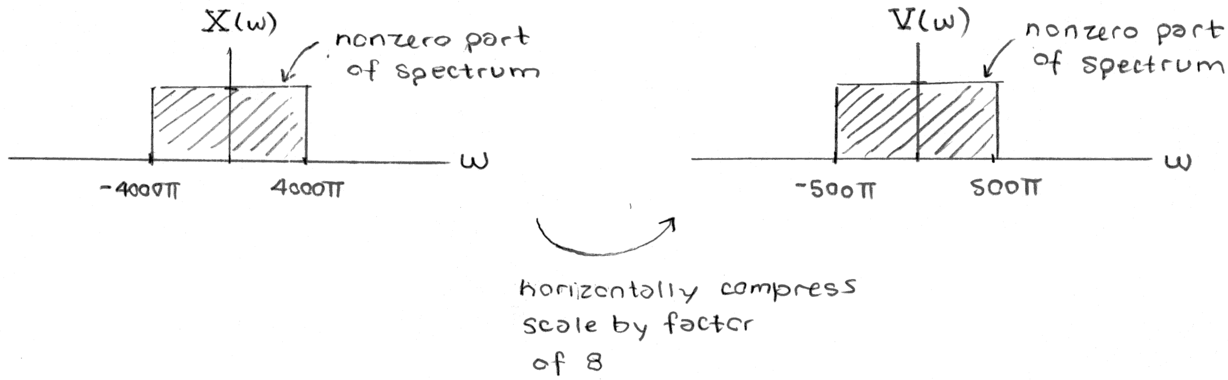
$$\omega_s > 2\omega_m = 2(4000\pi) = 8000\pi$$

Therefore, we must sample x at a rate of $\omega_s > 8000\pi$.

QUESTION 2(B)

$$\boxed{v(t) = x(t/8)}$$

$$V(\omega) = 8\,X(8\omega)$$



For the signal v, we have

$$\omega_m = 500\pi$$

$$\omega_s > 2\omega_m = 2(500\pi) = 1000\pi$$

Therefore, we must sample v at a rate of

$\omega_s > 1000\pi$.

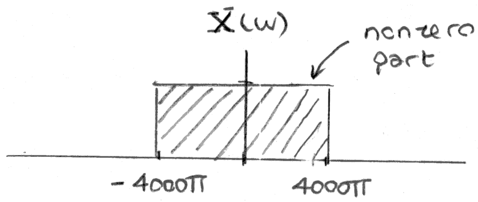
QUESTION 2(c)

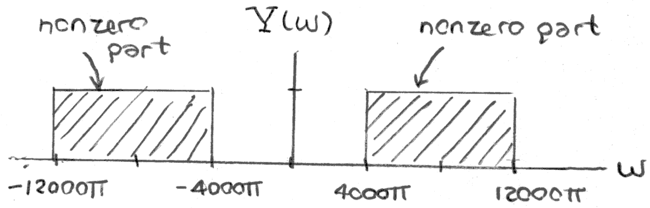
$$\boxed{y(t) = x(t)\cos(8000\pi t)}$$

$$y(t) = \frac{1}{2}\left(e^{j8000\pi t} + e^{-j8000\pi t}\right)x(t)$$

$$= \frac{1}{2}\left[e^{j8000\pi t}x(t) + e^{-j8000\pi t}x(t)\right]$$

$$Y(\omega) = \frac{1}{2}\left[X(\omega - 8000\pi) + X(\omega + 8000\pi)\right]$$





For the signal y, we have

$\omega_m = 12000\pi$

$\omega_s > 2\omega_m = 2(12000\pi) = 24000\pi$

Therefore, we must sample y at a rate of

$\omega_s > 24000\pi$.

QUESTION 3

$$\boxed{H(w) = \frac{w+7}{3w^3 - 2w^2 + 1}}$$

$$H(w) = \frac{Y(w)}{X(w)}$$

$$\frac{Y(w)}{X(w)} = \frac{w+7}{3w^3 - 2w^2 + 1}$$

$$(3w^3 - 2w^2 + 1)\, Y(w) = (w+7)\, X(w)$$

$$3w^3\, Y(w) - 2w^2\, Y(w) + Y(w) = w\, X(w) + 7\, X(w)$$

$$j3\, (jw)^3\, Y(w) - 2(-1)(jw)^2\, Y(w) + Y(w) = -j\, (jw)\, X(w) + 7X(w)$$

$$j3\, D^3 y(t) + 2\, D^2 y(t) + y(t) = -j\, D\, x(t) + 7\, x(t)$$

QUESTION 4

$$x(t) = e^{-|2t-3|}$$

$$v_1(t) = e^{-|t|} \quad ①$$

$$v_2(t) = v_1(t-3) \quad ②$$

$$x(t) = v_2(2t) \quad ③$$

$$V_1(w) = \frac{2}{w^2+1} \quad ④ \quad \left[\text{from FT of } ①\right]$$

$$V_2(w) = e^{-j3w} V_1(w) \quad ⑤ \quad \left[\text{from FT of } ②\right]$$

$$X(w) = \tfrac{1}{2} V_2(w/2) \quad ⑥ \quad \left[\text{from FT of } ③\right]$$

$$\begin{aligned}
X(w) &= \tfrac{1}{2} V_2(w/2) \qquad \text{from } ⑥ \\
&\overset{⑤}{=} \tfrac{1}{2} e^{-j3w/2} V_1(w/2) \\
&\overset{④}{=} \tfrac{1}{2} e^{-j3w/2} \frac{2}{(w/2)^2+1} \\
&= \frac{e^{-j3w/2}}{w^2/4+1} \\
&= \frac{e^{-j3w/2}}{\left(\frac{w^2+4}{4}\right)} \\
&= \frac{4e^{-j3w/2}}{w^2+4} \\
&= \frac{4}{(w^2+4)\, e^{j3w/2}}
\end{aligned}$$

QUESTION 5

$$\boxed{i(t) = 4v_1(t) \quad \text{and} \quad Dv_o(t) = 4D^2 i(t) + 2i(t) + Dv_1(t)}$$

$$I(w) = 4V_1(w) \quad ①$$

$$jw\,V_o(w) = 4(jw)^2 I(w) + 2I(w) + jw\,V_1(w)$$

$$jw\,V_o(w) = -4w^2 I(w) + 2I(w) + jw\,V_1(w) \quad ②$$

substituting ① into ②, we have

$$jw\,V_o(w) = -4w^2[4V_1(w)] + 2[4V_1(w)] + jw\,V_1(w)$$

$$jw\,V_o(w) = -16w^2 V_1(w) + 8V_1(w) + jw\,V_1(w)$$

$$jw\,V_o(w) = [-16w^2 + jw + 8]\,V_1(w)$$

$$\frac{V_1(w)}{V_o(w)} = \frac{jw}{-16w^2 + jw + 8}$$

Therefore, we have

$$H(w) = \frac{jw}{-16w^2 + jw + 8}$$