Experiment 3: Position Control Using a DC Motor

1 Objective

The objective of this laboratory project is to develop an understanding of Proportional and Derivative (PD) Control as applied to a position control application. In particular, you will explore:

- Qualitative properties of proportional and derivative action.
- Design of controllers for specifications on the set-point response.
- Tracking of triangular signals.

2 Introduction

The following nomenclature, as described in Table 3.1, is used.

Symbol	**Description**	**Units**
θ_m	Motor angle	rad
ω_m	Motor angular velocity	rad/s
u_m	Voltage from amplifier which drives the motor	V
u_e	Back-emf voltage	V
T_m	Torque generated by motor	Nm
T_d	Disturbance torque externally applied to the inertial load	Nm
V_d	Disturbance voltage corresponding to T_d	V
V_{sd}	Simulated disturbance voltage	V
i_m	Motor armature current	A
k_m	Motor torque constant	Nm/A
R_m	Motor armature resistance	Ω
J_{eq}	Total moment of inertia of motor rotor and the load	kgm^2
K	Open-loop steady-state gain	rad/(V.s)
τ	Open-loop time constant	s
ω_n	Undamped Natural Frequency	rad
ζ	Damping Ratio	-
k_p	Proportional gain	V.s/rad
k_d	Derivative gain	V/rad
b_{sp}	Set-Point Weight on proportional Control	-
b_{sd}	Set-Point weight on derivative Control	-
u	Control signal	V
r	Reference signal	rad/s
y	Measured process output	rad/s

Table 1: Nomenclature used for position control

Consider a DC motor whose angular position $\theta_m(t)$ is supposed to follow a reference signal $r(t)$. Such a system can be represented by the block diagram shown in Figure 3.1. This block diagram illustrates the parts of the system that are relevant for position control:

The process is represented by a block which has input voltage u_m and torque T_d as inputs and motor angle θ_m as the output. The torque is typically a disturbance torque similar to the disturbance torque used in experiment 2. The controller to be used in this experiment will be proportional and derivative (PD) control.

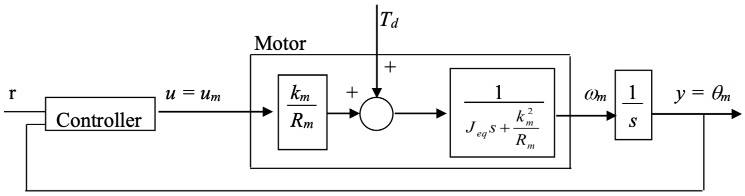

Figure 1: Block diagram of a position control system

2.1 PD Control Law

The controller function in Fig. 3.1 can be described using the following equation:

$$u_m(t) = k_p \left(b_{sp} r(t) - \theta_m(t)\right) + k_d \left[b_{sd} \frac{\partial}{\partial t} r(t) - \frac{\partial}{\partial t} \theta_m(t) \right] \tag{1}$$

where

- k_p: Proportional Gain
- k_d: Derivative Gain
- b_{sp}: Reference Signal Weight for Proportional Part
- b_{sd}: Reference Signal Weight for Derivative Part

The derivative term can be seen as a predictor of future measurements and improves the possibility of introducing damping. One possible disadvantage of derivative control is that it may be noise sensitive due to differentiation.

3 Pre-Laboratory Assignments

You may find the short videos discussing the theory behind the preparatory questions useful:
`https://www.youtube.com/watch?v=QX4b-9BcIY4&list=PLJ-LoEuClIty7ewNzrOhHwqw20VoEXBHBmlz`

3.1 Proportional Control

3.1.1. The open-loop transfer function between the control input $u_m(t)$ and the position signal $\theta_m(t)$ of the DC motor in Fig. 3.1 can be given by:

$$G(s) = \frac{\theta_m(s)}{U_m(s)} = \frac{K}{\tau s + 1} \times \frac{1}{s} \tag{2}$$

Using the values of K and τ obtained in experiment 1 and proportional control,

$$u_m(t) = k_p(r(t) - \theta_m(t)) \tag{3}$$

Solution: The open-loop transfer function for the system is:

$$G(s) = \frac{K}{(\tau s + 1)s}$$

Using proportional control, the closed-loop transfer function is given by:

$$\frac{\theta_m(s)}{R(s)} = \frac{k_p K}{s(\tau s + 1) + k_p K}$$

3.1.2. Determine the location of the poles of the closed-loop system when the proportional gain k_p is changed. That is, derive the poles of the closed-loop system as a function of k_p. How does the unit step response of the system change when k_p is changed?

Solution: The characteristic equation of the closed-loop system is:

$$\tau s^2 + s + k_p K = 0$$

The poles are given by:

$$s = \frac{-1 \pm \sqrt{1 - 4\tau k_p K}}{2\tau}$$

As k_p increases, the poles move left, leading to faster responses and reduced settling time. For large k_p, the system may become underdamped, resulting in oscillations. For small k_p, the system remains overdamped with slower responses.

3.1.3. Consider a step of amplitude r_0 for $r(t)$ and use the Final Value Theorem to find θ_{ss}, the value of the output signal of the closed-loop system at steady state. How does it compare to the input signal $r(t)$?

Solution: Using the Final Value Theorem:

$$\theta_{ss} = \lim_{s \to 0} s \cdot \frac{k_p K}{s(\tau s + 1) + k_p K} \cdot \frac{r_0}{s}$$

Simplifying:

$$\theta_{ss} = r_0$$

Therefore, with proportional control, the steady-state output equals the reference input, meaning there is no steady-state error for step inputs.

3.2 Design of Proportional and Derivative Control Parameters

3.2.1. Using the open-loop transfer function, eq. (2) and PD control, eq. (1) with $b_{sp} = 1$, obtain the closed-loop transfer function $G_{PD}(s)$ between the reference signal $r(t)$ as input and the motor position $\theta_m(t)$ as output (Assume disturbance torque $T_d = 0$).

Solution: The open-loop transfer function is:

$$G(s) = \frac{K}{(\tau s + 1)s}$$

Using PD control with $b_{sp} = 1$ and $T_d = 0$, the control law is:

$$u_m(t) = k_p(r(t) - \theta_m(t)) + k_d \left(\dot{r}(t) - \dot{\theta}_m(t) \right)$$

In Laplace domain:

$$U_m(s) = k_p(R(s) - \Theta_m(s)) + k_d s(R(s) - \Theta_m(s))$$

The closed-loop transfer function $G_{PD}(s)$ is:

$$G_{PD}(s) = \frac{\theta_m(s)}{r(s)} = \frac{K(k_p + k_d s)}{s(\tau s + 1) + K(k_p + k_d s)}$$

3.2.2. One possible way to design a controller is to choose controller parameters that give a specified transfer function. The controller parameters can be determined by using the mathematical model of the process and applying pole placement design. Determine the PD controller parameters k_p, k_d, and b_{sd} so that the closed-loop system, i.e., $G_{PD}(s)$, becomes the following quadratic transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4}$$

Solution: Comparing coefficients:

$$\tau s^2 + (1 + K k_d)s + K k_p = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Therefore:

$$k_d = \frac{2\zeta\omega_n\tau - 1}{K}, \quad k_p = \frac{\omega_n^2\tau}{K}, \quad b_{sd} = 1$$

3.2.3. For a second order system, eq. (4), with two complex conjugate poles (underdamped), the maximum Percentage Overshoot, PO, over the steady-state response is given by:

$$PO = 100e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \tag{5}$$

The time to first peak t_p is given by:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{6}$$

and the settling time is given by:

$$T_s = \frac{4}{\zeta\omega_n} \tag{7}$$

Using the following values as design specifications:

$$PO \leq 18\% \quad \text{and} \quad T_s = 0.43\,s \tag{8}$$

choose a damping ratio, ζ, and a natural frequency ω_n, that satisfies the above requirements. Is the value $\zeta = 0.5$ acceptable? Explain.

Solution: First, for $PO \leq 18\%$, we use the overshoot formula:

$$PO = 100e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \leq 18$$

Solving this inequality gives $\zeta \geq 0.48$. Now, using the settling time formula $T_s = \frac{4}{\zeta\omega_n}$, and substituting $T_s = 0.43$ s, we get:

$$0.43 = \frac{4}{\zeta\omega_n}$$

Choosing $\zeta = 0.5$, we find:

$$\omega_n = \frac{4}{0.43 \times 0.5} \approx 18.6\,\text{rad/s}$$

Hence, $\zeta = 0.5$ is acceptable as it satisfies both the overshoot and settling time requirements.

3.2.4. Using the numerical results from 3.2.3 and the expressions derived for k_p, k_d, and b_{sd} in 3.2.2 compute k_p, k_d, and b_{sd}.

Solution: Using the values $\zeta = 0.5$ and $\omega_n = 18.6$ rad/s from 3.2.3, we substitute them into the formulas for k_p and k_d derived in 3.2.2:

$$k_p = \frac{(18.6)^2 \cdot \tau}{K}, \quad k_d = \frac{2(0.5)(18.6) - 1}{K}, \quad b_{sd} = 1$$

The exact values of k_p and k_d depend on the values of τ and K obtained in experiment 1.

3.2.5. Consider a step with amplitude r_0 for $r(t)$ and use the Final Value Theorem to find θ_{ss_PD}, the value of the output signal θ_m of the closed-loop system at steady state. How does it compare to the input signal $r(t)$?

Solution: Using the Final Value Theorem:

$$\theta_{ss\_PD} = \lim_{s \to 0} s \cdot \Theta_m(s)$$

The transfer function is:

$$\Theta_m(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{r_0}{s}$$

Simplifying for $s \to 0$, we find that:

$$\theta_{ss\_PD} = r_0$$

This shows that with PD control, there is no steady-state error, and the output tracks the reference signal exactly.

3.3 Tracking Triangular Signals

3.3.1. Consider the closed-loop system with PD control, i.e., $G_{PD}(s)$, and assume that you have as input a ramp signal given by:

$$r(t) = r_0\, t \quad \text{for} \quad t \geq 0 \tag{9}$$

Apply the Final Value Theorem to calculate the steady-state error, e_{ss_PD}.

Solution: For a ramp input $r(t) = r_0 t$, with Laplace transform $R(s) = \frac{r_0}{s^2}$:

$$e_{ss\_PD} = \lim_{s \to 0} s \cdot \frac{sR(s)}{1 + G_{PD}(s)}$$

Using the designed quadratic transfer function:

$$e_{ss\_PD} = \frac{r_0}{\omega_n^2}$$

3.3.2. Compute e_{ss_PD} using $r_0 = 32$ [rad/s] and the values of k_p, k_d, and b_{sd} obtained in 3.2.4.

Solution: Using $r_0 = 32$ rad/s and $\omega_n = 18.6$ rad/s:

$$e_{ss\_PD} = \frac{32}{(18.6)^2} \approx 0.09 \text{ rad}$$

3.4 Pre-Laboratory Results Summary Table

Description	**Symbol**	**Value**	**Units**
Open-loop steady-state gain	K	19.92	rad/(V.s)
Open-loop time constant	τ	0.093	s
PD Controller Design			
Proportional Gain	k_p	32.15	V/rad
Derivative Gain	k_d	0.93	V.s/rad
Set-Point Weight on Derivative Part	b_{sd}	1	
Desired Damping Ratio	ζ	0.5	
Desired Undamped Natural Frequency	ω_n	18.6	rad/s
Given Maximum Percentage Overshoot	PO	≤ 18	%
Given 2% Settling Time	T_s	0.43	s
Desired Peak Time	t_p	0.19	s
Steady-State Value of Position using PD	θ_{ss_PD}	32	rad
Tracking Triangular Signals			
Steady-State Error using PD control	e_{ss_PD}	0.09	rad

Table 2: Position Control pre-laboratory assignment results

4 In-Laboratory Session

4.1 Position Control Module

The *Position Control* module of the QICii software package is the main tool for this laboratory.
Table 3.3. Lists and describes the main elements of the *Position Control* module interface.

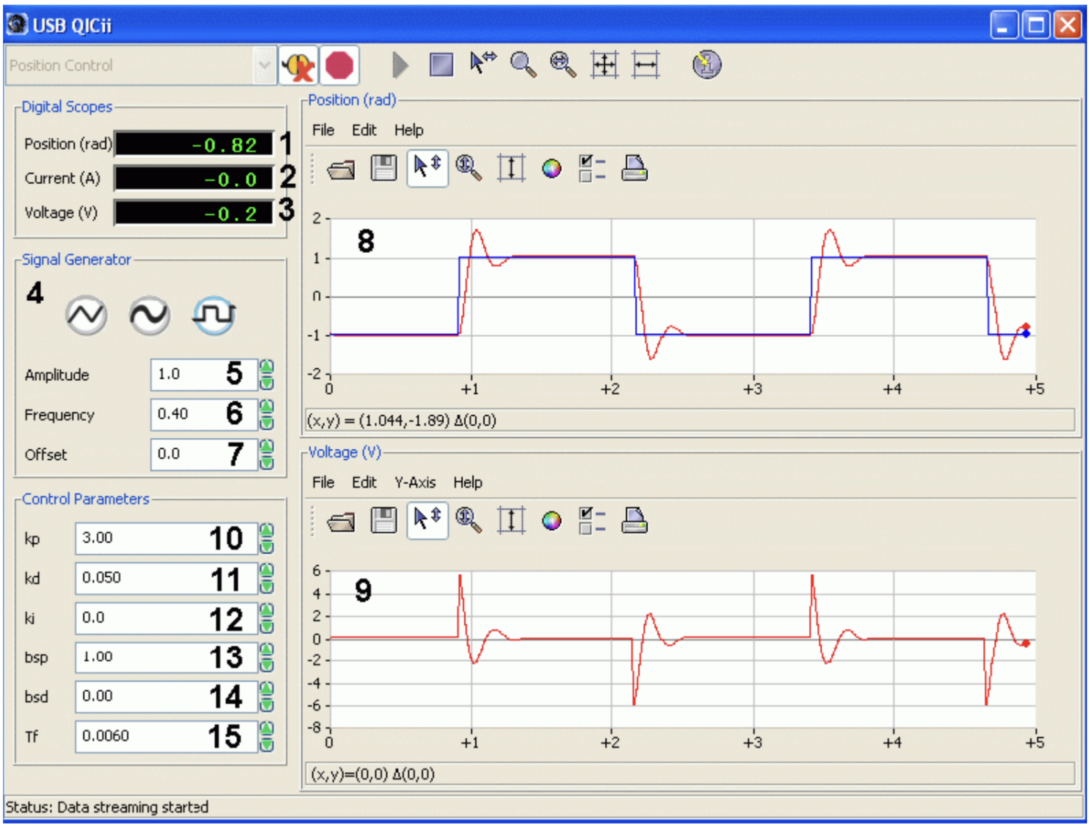

Figure 2: Interface of the Position Control module.

The *Position Control* module program runs the process in closed-loop with the motor reference position angle given by the signal generator. There are two windows that show the time histories of motor position (control output) and motor voltage (control input). To compute the control signal, eq. (1), it is required that the term:

$$e_d(t) = \frac{\partial}{\partial t}\left(b_{sd}r(t) - \theta_m(t)\right) \tag{10}$$

is computed. The signal $e_d(t)$ is obtained by filtering the error signal using the following filter:

$$E_d(s) = \frac{s}{T_f s + 1}\left[b_{sp}R(s) - \Theta_m(s)\right] \tag{11}$$

where $E_d(s)$, $R(s)$, and $\Theta_m(s)$ are the Laplace transforms of the signals $e_d(t)$, $r(t)$, and the position $\theta_m(t)$. A good value for T_f is 0.006 and should be left unchanged during this experiment.

5 Quantitative Properties of Proportional and Derivative Control

The goal of the following procedures is to develop an intuitive feel for the properties of proportional and derivative control actions.

5.1 Proportional Control

Step 1. Set reference signal to a square wave. A reasonable amplitude is 3 rad. When you change the reference signal level ensure that the control signal does not saturate. Set both integral and derivative gains to zero ($k_i = k_d = 0$). Set the proportional gain to 0.2 V/rad to start with. Ensure that the following parameters of the Position Control window are set properly.

Signal Type	Amplitude [rad]	Frequency [Hz]	Offset [rad]	k_p [V/rad]	b_{sp}
Square Wave	3	0.4	0	0.2	1

ID #	Label	Parameter	Description	Unit
1	Position	θ_m	Motor Output position Numeric Display	rad
2	Current	i_m	Motor Armature Current Numeric Display	A
3	Voltage	u_m	Motor Input Voltage Numeric Display	V
4	Signal Generator		Type of Generator for the Angle Reference Signal: Sawtooth Wave or Square Wave	
5	Amplitude		Generated Signal Amplitude Input Box	rad
6	Frequency		Generated Signal Frequency Input Box	Hz
7	Offset		Generated Signal Offset Input Box	rad
8	Speed	ω_m	Scope with Actual (in red) and Reference (in blue) Angles	rad
9	Voltage	u_m	Scope with Applied Motor Voltage (in red)	V
10	kp	k_p	Controller Proportional Gain Input Box	V/rad
11	kd	k_d	Controller Derivative Gain Input Box	V.s/rad
12	ki	k_i	Controller Integral Gain Input Box	V/(rad.s)
13	bsp	b_{sp}	Controller Proportional Reference Signal Weight Input Box	
14	bsd	b_{sd}	Controller Derivative Reference Signal Weight Input Box	
15	Tf	T_f	Time Constant of Filter for obtaining the Derivative of the Error Signal	s

Table 3: QICii Position Control Module Nomenclature

Step 2. To investigate the closed-loop system for proportional controllers with different gains change the proportional gain to the following values: $k_p = 1$, 2, and $4\,V/\text{rad}$. What are your observations?

Step 3. Describe the steady-state error to a step input.

Step 4. Repeat the previous observations. Change the Amplitude of the reference signal and observe under what conditions the control signal saturates.

6 Proportional and Derivative (PD) Control

The combination of proportional and derivative control will now be explored. Follow the steps below:

Step 1. Fix the proportional gain to 2.0 V/rad and set the derivative gain to 0.0 V.s/rad to start with ($b_{sp} = b_{sd} = 1$). Set the parameters of the QICii module window as listed in Table 3.5. Set the integral gain to zero ($k_i = 0$).

Signal Type	Amplitude [rad]	Frequency [Hz]	Offset [rad]	$k_p\,[V/\text{rad}]$	$k_d\,[V \cdot s/\text{rad}]$	b_{sp}	b_{sd}
Square Wave	2	0.4	0	2.0	0	1	1

Table 3.5. Module Parameters for the Proportional and Derivative Control Test

Step 2. Change the derivative gain by incremental steps of 0.05 V.s/rad to investigate the closed-loop system for PD controllers with different derivative gains. Try the following gains: $k_d = 0$, 0.05, 0.1, and 0.15 V.s/rad. What are your observations?

Step 3. Determine the lowest value of derivative gain k_d which gives a step response without overshoot (k_p still 2 V/rad). Determine the settling time for the closed loop system.

7 PD Controller Design to Given Specifications

This section provides the experimental verification of the PD controller design to given specifications, as carried out in the pre-lab assignment in Section 3.2. The performance of the closed-loop system with the control parameters obtained using the basic pole assignment method in Section 3.2. will be evaluated and compared to the performance specifications given in eq. (8). Please follow the steps below:

Step 1. Set the parameters of the QICii module window as described in Table 3.6.

Signal Type	Frequency [Hz]	Amplitude [rad]	Offset [rad]	b_{sp}	T_f [s]
Square Wave	0.4	4.5	0	1	0.006

Table 3.6. Module Parameters for PD Controller Design to Given Specifications

Ensure that the integral gain is zero ($k_i = 0$) and set b_{sd} and both proportional and derivative gains, k_p and k_d, to the values you calculated in Section 3.2, Question 3.2.4.

Step 2. Make sure that the motor input voltage is below its saturation limit. If not, adjust the square wave reference signal Amplitude. Measure the resulting Percent Overshoot (PO), settling time T_s, and peak time t_p. Does the system's actual response meet the desired requirements? How close are the measurements to the values you calculated in Question 3.2.3?

Step 3. Summarize your observations and your calculations in your report. Select some representative results, screen-captures, and plots.

8 Tracking Triangular Signals

Step 1. Select a triangular reference signal and set the parameters of the QICii module window as listed in Table 3.7. Set the controller gains to the PD controller parameters (i.e k_p, k_d, b_{sd}) to the values calculated in Question 3.2.4.

Signal Type	Amplitude [rad]	Frequency [Hz]	Offset [rad]	b_{sp}	T_f [s]
Triangular Wave	20	0.4	0	1	0.006

Table 3.7. QICii Module Parameters for Triangular Wave Tracking Test

Step 2. From the triangular wave specifications given in Table 3.7, calculate the slope r_0 of the ramp signal (see eq. (9)).

Step 3. Observe how well the output signal tracks the triangular signal (in particular in terms of the tracking error). Measure the actual asymptotic tracking error and compare it with the analytic estimate obtained in Question 3.3.2.

Step 4. Change the controller proportional gain k_p by steps of $\pm 0.5\, V$/rad and explore its effect on the tracking error. Select some representative results and plots and include in your report.

9 References

1. K. Ogata, Modern Control Engineering, 5^{th} Edition, Prentice Hall, Englewood Cliffs, N.J., 2010.
2. K. J. Astrom, J. Apkarian, H. Lacharery, USB QICii Laboratory Workbook, Quanser Engineering Trainer Series.