Inverse Laplace Transform via Partial Fractions

Math Solutions

Introduction

This document provides a step-by-step guide to solving the inverse Laplace transform using the method of partial fractions. We'll cover various cases, including distinct linear factors, repeated factors, and irreducible quadratic factors. Each example is accompanied by a detailed explanation of each step.

Problem 1: Distinct Linear Factors

Solve the inverse Laplace transform of

$$F(s) = \frac{6}{(s+1)(s+3)}$$

Solution:

Step 1: Express $F(s)$ as partial fractions.

$$\frac{6}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

Multiply both sides by $(s+1)(s+3)$:

$$6 = A(s+3) + B(s+1)$$

Expand and collect terms:

$$6 = As + 3A + Bs + B$$

$$6 = (A+B)s + (3A+B)$$

Equating coefficients:

$$A + B = 0, \quad 3A + B = 6$$

From $A + B = 0$, we have $B = -A$. Substituting into the second equation:

$$3A - A = 6 \quad \Rightarrow \quad 2A = 6 \quad \Rightarrow \quad A = 3, \quad B = -3$$

Thus,

$$F(s) = \frac{3}{s+1} - \frac{3}{s+3}$$

Step 2: Apply the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} = 3e^{-t}, \quad \mathcal{L}^{-1}\left\{\frac{3}{s+3}\right\} = 3e^{-3t}$$

$$f(t) = 3e^{-t} - 3e^{-3t}$$

Problem 2: Repeated Linear Factor

Solve the inverse Laplace transform of

$$F(s) = \frac{5}{(s+2)^2}$$

Solution:

Step 1: Express $F(s)$ as partial fractions.

For repeated factors, we write:

$$\frac{5}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

Multiply both sides by $(s+2)^2$:

$$5 = A(s+2) + B$$

Substitute $s = -2$ to solve for B:

$$5 = B \quad \Rightarrow \quad B = 5$$

Now substitute $B = 5$ into the equation:

$$5 = A(s+2) + 5$$

Simplify:

$$A(s+2) = 0 \quad \Rightarrow \quad A = 0$$

Thus,

$$F(s) = \frac{5}{(s+2)^2}$$

Step 2: Apply the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2}\right\} = 5te^{-2t}$$

$$f(t) = 5te^{-2t}$$

Problem 3: Irreducible Quadratic Factor

Solve the inverse Laplace transform of

$$F(s) = \frac{2s+1}{s^2+4s+5}$$

Solution:

Step 1: Complete the square in the denominator.

$$s^2 + 4s + 5 = (s+2)^2 + 1$$

Thus, we rewrite $F(s)$ as:

$$F(s) = \frac{2(s+2) - 3}{(s+2)^2 + 1}$$

Split into two fractions:

$$F(s) = 2\frac{s+2}{(s+2)^2+1} - 3\frac{1}{(s+2)^2+1}$$

Step 2: Apply the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} = e^{-2t}\cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$$

Thus,

$$f(t) = 2e^{-2t}\cos t - 3e^{-2t}\sin t$$

Problem 4: Combination of Repeated and Distinct Linear Factors

Solve the inverse Laplace transform of

$$F(s) = \frac{7s+4}{s(s+1)^2}$$

Solution:

Step 1: Express $F(s)$ as partial fractions.

We write:

$$\frac{7s+4}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiply both sides by $s(s+1)^2$:

$$7s+4 = A(s+1)^2 + Bs(s+1) + Cs$$

Expand and collect terms:

$$7s+4 = A(s^2+2s+1) + B(s^2+s) + Cs$$

$$7s+4 = (A+B)s^2 + (2A+B+C)s + A$$

Equating coefficients:

$$A+B=0, \quad 2A+B+C=7, \quad A=4$$

From $A=4$, we substitute into the other equations:

$$4+B=0 \quad \Rightarrow \quad B=-4$$

$$2(4)-4+C=7 \quad \Rightarrow \quad 8-4+C=7 \quad \Rightarrow \quad C=3$$

Thus,

$$F(s) = \frac{4}{s} - \frac{4}{s+1} + \frac{3}{(s+1)^2}$$

Step 2: Apply the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{4}{s}\right\} = 4, \quad \mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} = 4e^{-t}, \quad \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2}\right\} = 3te^{-t}$$

Thus,

$$f(t) = 4 - 4e^{-t} + 3te^{-t}$$

Problem 5: Distinct Linear Factors with Complex Roots

Solve the inverse Laplace transform of

$$F(s) = \frac{4s}{(s^2+4)(s+3)}$$

Solution: We decompose:

$$\frac{4s}{(s^2+4)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

Multiply both sides by $(s^2+4)(s+3)$:

$$4s = A(s^2+4) + (Bs+C)(s+3)$$

Expand and collect terms:

$$4s = A(s^2+4) + Bs^2 + 3Bs + Cs + 3C$$

$$4s = (A+B)s^2 + (3B+C)s + (4A+3C)$$

Equating coefficients:

$$A+B=0, \quad 3B+C=4, \quad 4A+3C=0$$

Solving, we get:

$$A=0, \quad B=0, \quad C=4$$

Thus,

$$F(s) = \frac{4}{(s^2+4)}$$

Now apply the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\} = 2\sin 2t$$

Thus,

$$f(t) = 2\sin 2t$$

Conclusion

The method of partial fractions is an essential tool for finding the inverse Laplace transform of rational functions. By breaking down complicated fractions into simpler components, we can easily apply standard inverse transforms and find the solution in the time domain.