QUIZ 3

SOLUTIONS

QUESTION 1

(a) The signal is <u>real</u> since $C_k = C_{-k}^{*}$.

(b) The signal is <u>not odd</u> since the condition $C_k = -C_{-k}$ is not satisfied.

(c) The signal is not periodic. Therefore, it <u>cannot</u> be represented by a Fourier series

QUESTION 2

The signal $x(t)$ satisfies the Dirichlet conditions. Therefore, at a point of discontinuity t_0, $\hat{x}(t)$ converges to

$$\hat{x}(t_0) = \frac{1}{2}\left[x(t_0^-) + x(t_0^+)\right]$$

So, we have

$$\hat{x}(1) = \frac{1}{2}[1+2] = \frac{3}{2}$$

$$\hat{x}(3) = \frac{1}{2}[3+1] = 2$$

QUESTION 3

$$T = 2\pi$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\begin{aligned}
C_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jkt}\, dt \\
&= \frac{1}{2\pi}\left[\int_{-\pi/2}^{0} (-1)\, e^{-jkt}\, dt + \int_{0}^{\pi/2} \frac{2}{\pi} t\, e^{-jkt}\, dt\right] \\
&= \frac{1}{2\pi}\left[-A_1 + \frac{2}{\pi} A_2\right]
\end{aligned}$$

$$A_1 = \int_{-\pi/2}^{0} e^{-jkt}\, dt$$

$$A_2 = \int_{0}^{\pi/2} t\, e^{-jkt}\, dt$$

$$\begin{aligned}
A_1 &= \int_{-\pi/2}^{0} e^{-jkt}\, dt \\
&= \left[\frac{1}{-jk} e^{-jkt}\right]\Big|_{-\pi/2}^{0} \qquad \text{assume } k \neq 0 \\
&= \frac{1}{-jk}\left[1 - e^{j\frac{\pi}{2}k}\right] \\
&= \frac{1}{-jk}\left[1 - (e^{j\pi/2})^k\right] \\
&= \frac{1}{-jk}\left[1 - (\cos\tfrac{\pi}{2} + j\sin\tfrac{\pi}{2})^k\right] \\
&= \frac{1}{-jk}\left[1 - j^k\right] \\
&= \frac{j}{k}\left[1 - j^k\right] \checkmark
\end{aligned}$$

$$\begin{aligned}
A_1 &= \int_{-\pi/2}^{0} e^{0}\, dt \qquad \text{for } k = 0 \\
&= [t]\Big|_{-\pi/2}^{0} \\
&= \frac{\pi}{2}
\end{aligned}$$

$$A_1 = \begin{cases} \frac{j}{k}\left[1 - j^k\right] & \text{for } k \neq 0 \\ \frac{\pi}{2} & \text{for } k = 0 \end{cases}$$

assume $k \neq 0$

$$\begin{aligned} A_2 &= \int_0^{\pi/2} t e^{-jkt}\, dt \\ &= \left[\frac{1}{(-jk)^2} e^{-jkt}(-jkt-1)\right]\Big|_0^{\pi/2} \\ &= \frac{1}{k^2}\left[e^{-jkt}(jkt+1)\right]\Big|_0^{\pi/2} \\ &= \frac{1}{k^2}\left[e^{-jk\pi/2}\left(jk\tfrac{\pi}{2}+1\right) - 1\right] \\ &= \frac{1}{k^2}\left[\left(e^{-j\frac{\pi}{2}}\right)^k\left(jk\tfrac{\pi}{2}+1\right) - 1\right] \\ &= \frac{1}{k^2}\left[\left(\cos -\tfrac{\pi}{2} + j\sin -\tfrac{\pi}{2}\right)^k\left(jk\tfrac{\pi}{2}+1\right) - 1\right] \\ &= \frac{1}{k^2}\left[(-j)^k\left(jk\tfrac{\pi}{2}+1\right) - 1\right] \checkmark \end{aligned}$$

$$\begin{aligned} A_2 &= \int_0^{\pi/2} t e^{0}\, dt \qquad \text{for } k=0 \\ &= \left[\frac{t^2}{2}\right]_0^{\pi/2} \\ &= \frac{\pi^2}{8} \end{aligned}$$

$$A_2 = \begin{cases} \frac{1}{k^2}\left[(-j)^k\left(jk\tfrac{\pi}{2}+1\right) - 1\right] & \text{for } k \neq 0 \\ \frac{\pi^2}{8} & \text{for } k = 0 \end{cases}$$

assume $k \neq 0$

$$\begin{aligned} C_k &= \frac{1}{2\pi}\left[-A_1 + \frac{2}{\pi}A_2\right] \\ &= \frac{1}{2\pi}\left[-\frac{j}{k}\left[1-j^k\right] + \left(\frac{2}{\pi}\right)\left(\frac{1}{k^2}\right)\left[(-j)^k\left(jk\tfrac{\pi}{2}+1\right)-1\right]\right] \\ &= \frac{-j}{2\pi k}\left[1-j^k\right] + \frac{1}{\pi^2 k^2}\left[(-j)^k\left(jk\tfrac{\pi}{2}+1\right)-1\right] \qquad \text{for } k \neq 0 \end{aligned}$$

$$C_k = \frac{1}{2\pi}\left[-\frac{\pi}{2} + \frac{2}{\pi}\,\frac{\pi^2}{8}\right] = \frac{1}{2\pi}\left[-\frac{\pi}{2} + \frac{\pi}{4}\right] = \frac{1}{2\pi}\left[-\frac{\pi}{4}\right] = -\frac{1}{8} \qquad \text{for } k=0$$

QUESTION 4

(a)
$$\begin{aligned} x_1(t) &= 3 + 2\cos 2\pi t + \cos 4\pi t \\ &= 3 + 2\left[\tfrac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)\right] + \tfrac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) \\ &= 3 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{2}e^{-j4\pi t} \end{aligned}$$

$$x_1(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

where $C_0 = 3$, $C_1 = C_{-1} = 1$, $C_2 = C_{-2} = \frac{1}{2}$, and $\omega_0 = 2\pi$

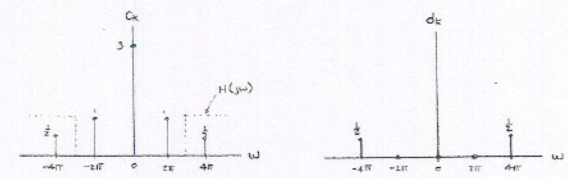
(b)
$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

$$\left[\begin{aligned} y(t) &= \tfrac{1}{2}e^{-j4\pi t} + \tfrac{1}{2}e^{j4\pi t} \\ &= \cos 4\pi t \end{aligned}\right]$$

where $d_k = C_k H(jk\omega_0)$

$$\begin{aligned} d_0 &= C_0 H(j0) = 3(0) = 0 \\ d_1 &= C_1 H(j2\pi) = 1(0) = 0 \\ d_{-1} &= C_{-1} H(-j2\pi) = 1(0) = 0 \\ d_2 &= C_2 H(j4\pi) = \tfrac{1}{2}(1) = \tfrac{1}{2} \\ d_{-2} &= C_{-2} H(-j4\pi) = \tfrac{1}{2}(1) = \tfrac{1}{2} \end{aligned}$$

(c)



(d) The system is an ideal highpass filter