A.1 Express each complex number given below in Cartesian form.

(a) $2e^{j2\pi/3}$;

(b) $\sqrt{2}e^{j\pi/4}$;

(c) $2e^{j7\pi/6}$; and

(d) $3e^{j\pi/2}$.

$$2e^{j(7\pi/6)} \qquad \begin{array}{|l} r = 2 \\ \theta = 7\frac{\pi}{6} \end{array}$$

$$= 2\cos\left(\frac{7\pi}{6}\right) + j \cdot 2\sin\left(\frac{7\pi}{6}\right)$$

$$= 2\cos\left(\frac{2\pi}{2} + \frac{\pi}{6}\right) + j \cdot 2 \cdot \sin\left(2\frac{\pi}{2} + \frac{\pi}{6}\right)$$

$$= -2\cos\left(\frac{\pi}{6}\right) - 2j\sin\left(\frac{\pi}{6}\right)$$

$$= -\cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} - \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot j$$

$$= -\sqrt{3} - j$$

A.2 Express each complex number given below in polar form. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument.

(a) $-\sqrt{3}+j$;
(b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$;
(c) $\sqrt{2}-j\sqrt{2}$;
(d) $1+j\sqrt{3}$;
(e) $-1-j\sqrt{3}$; and
(f) $-3+4j$.

(b) $r = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

$x = -\frac{1}{2} \quad y = -\frac{\sqrt{3}}{2}$

$r = \sqrt{x^2+y^2} = \sqrt{(-\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2}$

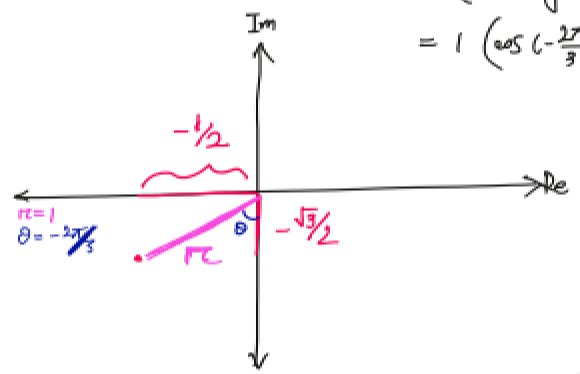
$= 1$

$\operatorname{arctan2}(y/x) \triangleq \arctan(y/x) - \pi$

$= \tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right) - \pi = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) - \pi$

$= \pi/3 - \pi = -2\pi/3 = -120^\circ = \theta$

$z = r(\cos\theta + j\sin\theta)$
$= 1\left(\cos\left(-\frac{2\pi}{3}\right) + j\sin\left(-\frac{2\pi}{3}\right)\right)$ Answer A.2.b



(d) $1 + j\sqrt{3}$ $\quad x = 1$

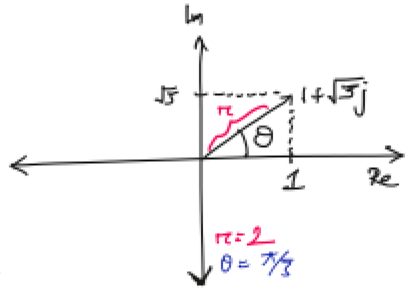
$y = \sqrt{3}$

$r = \sqrt{1^2+3} = \sqrt{4} = 2$

$\theta = \arctan(y/x) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi/3$

$\arg z = (\pi/3 + 2\pi n)$

$z = r(\cos\theta + j\sin\theta) = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$



A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.

(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in Cartesian form;

(b) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in polar form;

(c) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1+j)$ in polar form;

(d) $e^{1+j\pi/4}$ in Cartesian form;

(e) $\left[\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right]^8$ in polar form;

(f) $(1+j)^{10}$ in Cartesian form;

(g) $\frac{1+j}{1-j}$ in polar form;

(h) $\frac{1}{1+re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$; and

(i) $\frac{1}{1-re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$.

ⓐ $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j\left(\frac{-3\pi}{4}\right)}\right)$

$= \sqrt{3} - j + j\left(\frac{1}{\sqrt{2}}\cos\left(-\frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}\sin\left(-\frac{3\pi}{4}\right)j\right)$

$= \sqrt{3} - j + j\left(-\frac{1}{2} + j\left(-\frac{1}{2}\right)\right)$

$= \sqrt{3} - j - \frac{1}{2}j - \frac{1}{2}j^2$

$= \sqrt{3} - \frac{3}{2}j + \frac{1}{2} = \left[\frac{1+2\sqrt{3}}{2}\right] - j\left(\frac{3}{2}\right)$ (Ans)

ⓑ $\left[\frac{\sqrt{3}}{2} - j\frac{1}{2}\right]\left[\frac{1}{\sqrt{2}}e^{j\left(-\frac{3\pi}{4}\right)}\right]$

$= \left[\left(\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right) \cdot e^{-j\cdot\tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right)}\right]\left[\frac{1}{\sqrt{2}}e^{j\left(-\frac{3\pi}{4}\right)}\right]$

$= 1 \cdot e^{-j\left(\frac{\pi}{6}\right)} \cdot \frac{1}{\sqrt{2}} \cdot e^{j\left(-\frac{3\pi}{4}\right)}$

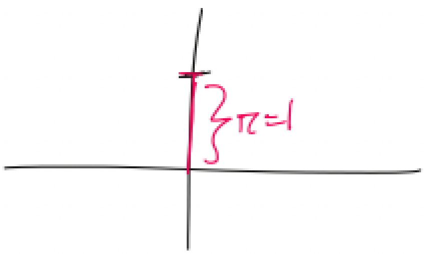
$= \frac{1}{\sqrt{2}}e^{-j\left(\frac{11}{12}\right)\pi}$ (Ans)

f. $(1+j)^{10} = \left((1+j)^2\right)^5 = (1+j^2+2j)^5 = (2j)^5 = 32j^5$

$= 32j$ Ans

g. $\dfrac{1+j}{1-j} = \dfrac{(1+j)(1+j)}{(1-j)(1+j)} = \dfrac{(1+j)^2}{1-j^2} = \dfrac{1+j^2+2j}{2} = j$

$= 0+1j$



$0+1j$

$r=1$

$\theta = \frac{\pi}{2}$

$$\boxed{\begin{aligned} z &= 1j+0 \\ &= 1e^{(\pi/2)j} \end{aligned}}$$

A.4 Show that each of the identities below holds, where z, z_1, and z_2 are arbitrary complex numbers and n is an arbitrary integer.

(a) $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$;
(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \neq 0$;
(c) $z + z^* = 2\operatorname{Re}\{z\}$;
(d) $zz^* = |z|^2$;
(e) $(z_1 z_2)^* = z_1^* z_2^*$;
(f) $|z^n| = |z|^n$; and
(g) $\arg(z^n) = n \arg z$.

(b) $\arg\left(\frac{z_1}{z_2}\right) = \arg\left[\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right]$ (assuming $z_2 \neq 0$)

$= \arg\left(\frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}\right)$

$= \theta_1 - \theta_2$

$= \arg(z_1) - \arg(z_2)$

(e) $(z_1 z_2)^*$

$= \left[(x_1 + jy_1)(x_2 + jy_2)\right]^*$

$= \left[x_1x_2 + jx_1y_2 + jx_2y_1 + j^2 y_1 y_2\right]^*$

$= \left[x_1x_2 - y_1y_2 + j(x_1y_2 + x_2y_1)\right]^* = x_1x_2 - y_1y_2 - j(x_1y_2 + x_2y_1)$

$(z_1^* z_2^*) = (x_1 - jy_1)(x_2 - jy_2) = x_1(x_2 - jy_2) - jy_1(x_2 - jy_2)$

$= x_1x_2 - jx_1y_2 - jx_2y_1 + j^2 y_1y_2$

$= x_1x_2 - y_1y_2 - j(x_1y_2 + x_2y_1)$

$= (z_1z_2)^*$

A.5 For each function f of a real variable given below, find an expression for $|f(\omega)|$ and $\arg f(\omega)$.

(a) $f(\omega) = \dfrac{1}{(1+j\omega)^{10}}$;

(b) $f(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$;

(c) $f(\omega) = \dfrac{2e^{j11\omega}}{(3+j5\omega)^7}$;

(d) $f(\omega) = \dfrac{-5}{(-1-j\omega)^4}$;

(e) $f(\omega) = \dfrac{j\omega^2}{(j\omega-1)^{10}}$; and

(f) $f(\omega) = \dfrac{j\omega-1}{j\omega+1}$.

(c) $$f(\omega) = \frac{2e^{j11\omega}}{(3+j5\omega)^7}$$

$$r = \text{magnitude} = \left| f(\omega) \right| = \frac{|2e^{j(11\omega)}|}{|(3+j5\omega)^7|}$$

$$= \frac{2}{\left(\sqrt{3^2+(5\omega)^2}\right)^7} = \boxed{\frac{2}{(9+25\omega^2)^{7/2}}}$$

$$\arg(f(\omega)) = \arg\left(\frac{2e^{j(11\omega)}}{(3+j(5\omega))^7}\right) = \arg\left[2e^{j(11\omega)}\right] - \arg\left[(3+j5\omega)^7\right]$$

$$= 11\omega - \arg\left[\left(\sqrt{3^2+(5\omega)^2}\cdot e^{\tan^{-1}\left(\frac{5\omega}{3}\right)j}\right)^7\right]$$

$$= 11\omega - \arg\left[\left(\sqrt{9+25\omega^2}\right)^7\cdot\left(e^{\tan^{-1}(5\omega/3)j}\right)^7\right]$$

$$= 11\omega - \left[\tan^{-1}\left(\frac{5\omega}{3}\right)j\right]^7 = \boxed{11\omega - 7\tan^{-1}\left(5\omega/3\right)j + 2\pi k} \text{ for all } k\in\mathbb{Z}$$

(f) $$f(\omega) = \frac{j\omega - 1}{j\omega + 1}$$

$$|f(\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|} = \frac{\sqrt{\omega^2 + 1}}{\sqrt{\omega^2 + 1}} = 1$$

$$\arg(f(\omega)) = \arg(j\omega - 1) - \arg(j\omega + 1)$$

$$= \arg\left(\sqrt{\omega^2 + 1}\, e^{j\left(\pi + \tan^{-1}\left(\frac{\omega}{-1}\right)\right)}\right) -$$

$$\arg\left(\sqrt{\omega^2 + 1}\, e^{j\left(\tan^{-1}\left(\frac{\omega}{1}\right)\right)}\right)$$

$$= \pi + \tan^{-1}(-\omega) - \tan^{-1}\omega$$

$$= \pi - \tan^{-1}\omega - \tan^{-1}\omega$$

$$= \pi - 2\tan^{-1}\omega$$

A.6 Use Euler's relation to show that each of the identities below holds, where θ is an arbitrary real constant.

(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$;

(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$; and

(c) $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

Left Hand Side $= \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$

$= \frac{1}{2j}\left[\cos\theta + j\sin\theta - \left(\cos(-\theta) + j\sin(-\theta)\right)\right]$

$= \frac{1}{2j}\left[\cos\theta + j\sin\theta - \cos(-\theta) - j\sin(-\theta)\right]$ $\quad \because \cos(-\theta) = \cos\theta$

$= \frac{1}{2j}\left[\cos\theta + j\sin\theta - \cos\theta + j\sin\theta\right]$

$= \frac{1}{\cancel{2j}}\left[\cancel{2j}\sin\theta\right]$

$= \sin\theta$

A.11 Determine the points at which each function f given below is: i) continuous, ii) differentiable, and iii) analytic. To deduce the answer, use your knowledge about polynomial and rational functions. Simply state the final answer along with a short justification (i.e., two or three sentences). (In other words, it is not necessary to use the Cauchy-Riemann equations for this problem.)

(a) $f(z) = 3z^3 - jz^2 + z - \pi$;

(b) $f(z) = \dfrac{z-1}{(z^2+3)(z^2+z+1)}$;

(c) $f(z) = \dfrac{z}{z^4-16}$; and

(d) $f(z) = z + 2 + z^{-1}$.

$$\cdots = \frac{z}{(z+2j)(z-2j)(z+2)(z-2)}$$

ⓒ $f(z) = \dfrac{z}{z^4-16} = \dfrac{z}{(z^2)^2-(4)^2} = \dfrac{z}{(z^2+4)(z^2-4)} = \dfrac{z}{(z^2+4)(z+2)(z-2)}$

for the function, the denominator becomes 0 when $z = +2, -2, +2j, -2j$

∴ continuous, differentiable, analytical when $z \neq 2, -2, 2j, -2j$

ⓓ $f(z) = z+2+z^{-1} = z+2+\dfrac{1}{z} = \dfrac{z^2+2z+1}{z}$

for the function, the denominator becomes 0 when $z=0$

∴ continuous, differentiable, analytical when $z \neq 0$

A.13 For each rational function f of a complex variable given below, find the (finite) poles and zeros of f and their orders. Also, plot these poles and zeros in the complex plane.

(a) $f(z) = z^2 + jz + 3$;

(b) $f(z) = z + 3 + 2z^{-1}$;

(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$;

(d) $f(z) = \dfrac{z^3 - z}{z^2 - 4}$;

(e) $f(z) = \dfrac{z + \frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and

(f) $f(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

(b) $f(z) = z + 3 + 2z^{-1}$

$$= z + 3 + 2 \cdot \frac{1}{z} = \frac{z^2+3z+2}{z}$$

$$= \frac{z^2+2z+z+2}{z}$$

$$= \frac{z(z+2)+1(z+2)}{z}$$

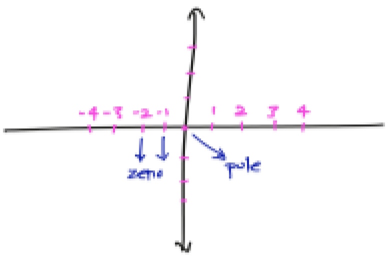
$$= \frac{(z+1)(z+2)}{z}$$

First Order Pole

$z = 0$

First Order Zero

$z = -1, -2$



(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$

$z^2+2z+5 = (z+1-2j)(z+1+2j)$

$z^2+3z+2 = (z+2)(z+1)$

$z^2+2z+2 = (z+1-j)(z+1+j)$

$z^2+1 = (z+j)(z-j)$

$$\therefore f(z) = \frac{(z+j)(z-j)(z+1+2j)(z+1-2j)}{(z+2)(z+1)(z+1+j)(z+1-j)}$$