Exercise 6.23

L **Answer (a).**

From the system block diagram, we have

$$
\begin{aligned}
v_2(t) &= x * h(t), && \text{1}\\
v_1(t) &= \cos(\omega_c t)x(t), && \text{2}\\
v_3(t) &= \sin(\omega_c t)v_2(t), \quad \text{and} && \text{3}\\
y(t) &= v_1(t) + v_3(t). && \text{4}
\end{aligned}
$$

Taking the Fourier transform of these equations, we have

$$
\begin{aligned}
V_2(\omega) &= X(\omega)H(\omega) && \text{FT of 1}\\
&= -j\operatorname{sgn}(\omega)X(\omega), && \text{substitute given } H \qquad (6.2)\\
V_1(\omega) &= \mathcal{F}\{x(t)\cos(\omega_c t)\}(\omega) && \text{FT of 2}\\
&= \mathcal{F}\{\tfrac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t)\}(\omega) && \text{Euler}\\
&= \tfrac{1}{2}\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \tfrac{1}{2}\mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega) && \text{linearity}\\
&= \tfrac{1}{2}X(\omega-\omega_c) + \tfrac{1}{2}X(\omega+\omega_c), && \text{modulation} \qquad (6.3)\\
V_3(\omega) &= \mathcal{F}\{v_2(t)\sin(\omega_c t)\}(\omega) && \text{FT of 3}\\
&= \mathcal{F}\left\{\tfrac{1}{2j}[e^{j\omega_c t} - e^{-j\omega_c t}]v_2(t)\right\}(\omega) && \text{Euler}\\
&= \tfrac{1}{2j}\mathcal{F}\{e^{j\omega_c t}v_2(t)\}(\omega) - \tfrac{1}{2j}\mathcal{F}\{e^{-j\omega_c t}v_2(t)\}(\omega) && \text{linearity}\\
&= \tfrac{1}{2j}V_2(\omega-\omega_c) - \tfrac{1}{2j}V_2(\omega+\omega_c), \quad \text{and} && \text{modulation} \qquad (6.4)\\
Y(\omega) &= V_1(\omega) + V_3(\omega). && \text{FT of 4} \qquad (6.5)
\end{aligned}
$$

Substituting the expression for $V_2(\omega)$ in (6.2) into (6.4), we have

$$
\begin{aligned}
V_3(\omega) &= \tfrac{1}{2j}V_2(\omega-\omega_c) - \tfrac{1}{2j}V_2(\omega+\omega_c) && \text{(6.4)}\\
&= \tfrac{1}{2j}[-j\operatorname{sgn}(\omega-\omega_c)X(\omega-\omega_c)] - \tfrac{1}{2j}[-j\operatorname{sgn}(\omega+\omega_c)X(\omega+\omega_c)] && \text{substitute (6.2)}\\
&= -\tfrac{1}{2}\operatorname{sgn}(\omega-\omega_c)X(\omega-\omega_c) + \tfrac{1}{2}\operatorname{sgn}(\omega+\omega_c)X(\omega+\omega_c). && \text{simplify} \qquad (6.6)
\end{aligned}
$$

Substituting the expressions for $V_1(\omega)$ and $V_3(\omega)$ from (6.3) and (6.6), respectively, into (6.5), we have

$$
\begin{aligned}
Y(\omega) &= V_1(\omega) + V_3(\omega) && \text{(6.5)}\\
&= \tfrac{1}{2}X(\omega-\omega_c) + \tfrac{1}{2}X(\omega+\omega_c) - \tfrac{1}{2}\operatorname{sgn}(\omega-\omega_c)X(\omega-\omega_c) + \tfrac{1}{2}\operatorname{sgn}(\omega+\omega_c)X(\omega+\omega_c) && \text{substitute (6.3) and (6.6)}\\
&= \left[\tfrac{1}{2} - \tfrac{1}{2}\operatorname{sgn}(\omega-\omega_c)\right]X(\omega-\omega_c) + \left[\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn}(\omega+\omega_c)\right]X(\omega+\omega_c) && \text{factor}\\
&= u(-\omega+\omega_c)X(\omega-\omega_c) + u(\omega+\omega_c)X(\omega+\omega_c). && \text{rewrite in terms of sgn}
\end{aligned}
$$