CHAPTER 4

B-4-1. For this system

$$C\,dH = -Q\,dt, \quad H = 3r, \quad C = r^2\pi = \left(\frac{H}{3}\right)^2 \pi$$

Hence

$$\left(\frac{H}{3}\right)^2 \pi\, dH = -0.005\sqrt{H}\, dt$$

or

$$H^{\frac{3}{2}}\, dH = -0.005\,\frac{9}{\pi}\, dt$$

Assume that the head moves down from H = 2m to x for the 60 sec period. Then

$$\int_2^x H^{\frac{3}{2}}\, dH = -0.005\,\frac{9}{\pi}\int_0^{60} dt$$

or

$$\frac{2}{5}\left(x^{\frac{5}{2}} - 2^{\frac{5}{2}}\right) = -0.014324(60-0)$$

which can be rewritten as

$$x^{\frac{5}{2}} - (1.41413)^5 = -2.1486$$

or

$$x^{\frac{5}{2}} = 5.65684 - 2.1486 = 3.50824$$

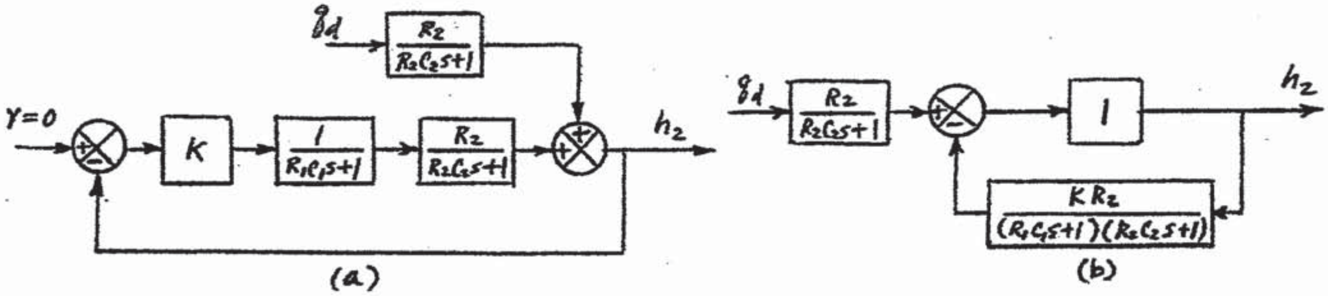
Taking logarithm of both sides of this last equation, we obtain

$$\frac{5}{2}\log_{10} x = \log_{10} 3.50824$$

or

$$x = 1.6521\,m$$

B-4-2. Figure (a) shown is a block diagram of the given system when changes in the variables are small. Since the set point of the controller is fixed, r = 0. (Note that r is the change in the set point.) To investigate the response of the level of the second tank subjected to a unit-step disturbance input q_d, we find it convenient to modify the block diagram of Figure (a) to the one shown in Figure (b).



The transfer function between $H_2(s)$ and $Q_d(s)$ can be obtained as

$$\frac{H_2(s)}{Q_d(s)} = \frac{R_2(R_1C_1s+1)}{(R_1C_1s+1)(R_2C_2s+1)+KR_2}$$

From this equation, the response $H_2(s)$ to the disturbance input $Q_d(s)$ can be obtained. For the unit-step disturbance input $Q_d(s)$, we obtain

$$h_2(\infty) = \lim_{s\to 0} s\,H_2(s) = \frac{R_2}{1+KR_2}$$

or

$$\text{steady-state error} = -\frac{R_2}{1+KR_2}$$

The system exhibits offset in the response to a unit-step disturbance input.

B-4-3. Note that

$$C\,dp_o = q\,dt$$

where q is the flow rate through the valve and is given by

$$q = \frac{p_i - p_o}{R}$$

Hence

$$C\frac{dp_o}{dt} = \frac{p_i - p_o}{R}$$

from which we obtain

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs+1}$$

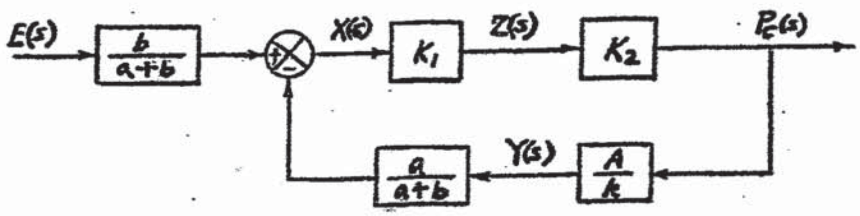
For the bellows and spring, we have the following equation:

$$Ap_o = kx$$

The transfer function $X(s)/P_i(s)$ is then given by

$$\frac{X(s)}{P_i(s)} = \frac{X(s)}{P_o(s)}\,\frac{P_o(s)}{P_i(s)} = \frac{A}{k}\,\frac{1}{RCs+1}$$

B-4-4.



In this block diagram, Z(s) is the Laplace transform of the small displacement of the diaphragm of the pneumatic relay. The transfer function $P_c(s)/E(s)$ is given by

$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b} \frac{K_1 K_2}{1 + K_1 K_2 \frac{a}{a+b} \frac{A}{k}} = K_p$$

The control action of this controller is proportional. Thus, the controller is a proportional controller.

B-4-5. Define the pressure of air in the bellows as $\bar{P}_c + p_o$. Then

$$C\,dp_o = q\,dt \,, \qquad q = \frac{p_c - p_o}{R}$$

Hence

$$C \frac{dp_o}{dt} = \frac{p_c - p_o}{R}$$

or

$$RC \frac{dp_o}{dt} + p_o = p_c \tag{1}$$

Define the area of bellows as A and the displacement of the bellows as $\bar{Y} + y$. Then, noting that $p_o A = ky$, Equation (1) becomes as

$$RC \frac{k}{A} \frac{dy}{dt} + \frac{k}{A} y = p_c$$

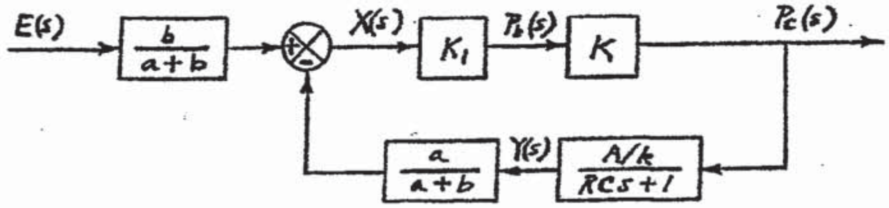
or

$$RC \frac{dy}{dt} + y = \frac{A}{k} p_c$$

Thus

$$\frac{Y(s)}{P_c(s)} = \frac{\frac{A}{k}}{RCs + 1}$$

A block diagram for this system is shown below.



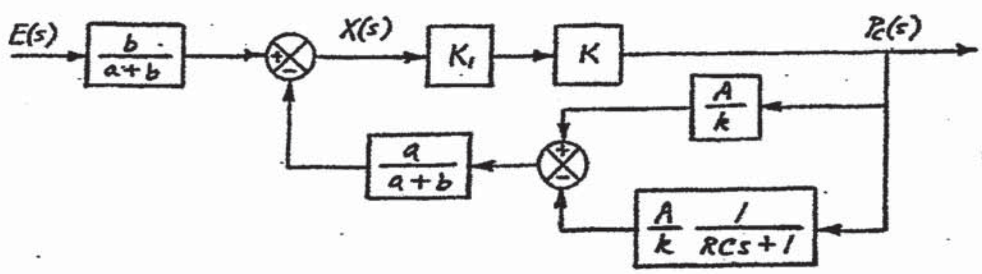
$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{K_1 K}{1 + K_1 K \dfrac{a}{a+b}\,\dfrac{A/k}{RCs+1}}$$

Assume that $K_1K \gg 1$. Then

$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{a+b}{a}\,\frac{RCs+1}{\dfrac{A}{k}} = \left(\frac{bk}{aA}\right)(RCs+1)$$

Thus, the control action is proportional-plus-derivative. The controller is a proportional-plus-derivative controller.

B-4-6.



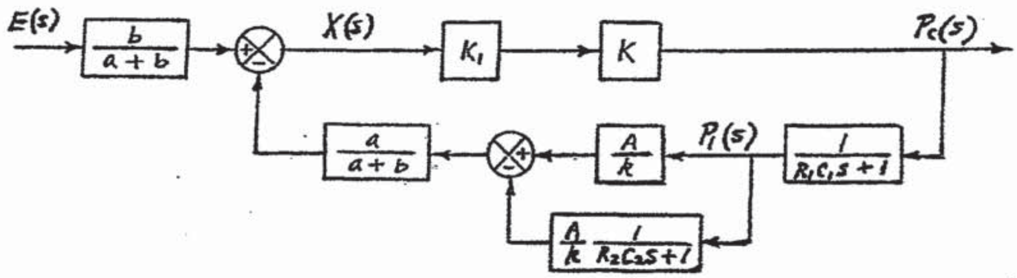
$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{K_1 k}{1 + \dfrac{K_1 K a}{a+b}\left(\dfrac{A}{k} - \dfrac{A}{k}\,\dfrac{1}{RCs+1}\right)}$$

If $K_1K \gg 1$, then

$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{K_1 K}{\dfrac{K_1 K a}{a+b}\,\dfrac{A}{k}\,\dfrac{RCs}{RCs+1}} = \left(\frac{bk}{aA}\right)\left(1 + \frac{1}{RCs}\right)$$

The controller is a proportional-plus-integral controller.

B-4-7.



$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{K_1K}{1 + K_1K\,\dfrac{a}{a+b}\,\dfrac{A}{k}\,\dfrac{R_2C_2s}{R_2C_2s+1}\,\dfrac{1}{R_1C_1s+1}}$$

If $K_1K \gg 1$, then

$$\frac{P_c(s)}{E(s)} = \frac{b}{a+b}\,\frac{1}{\dfrac{a}{a+b}\,\dfrac{A}{k}\,\dfrac{R_2C_2s}{R_2C_2s+1}\,\dfrac{1}{R_1C_1s+1}}$$

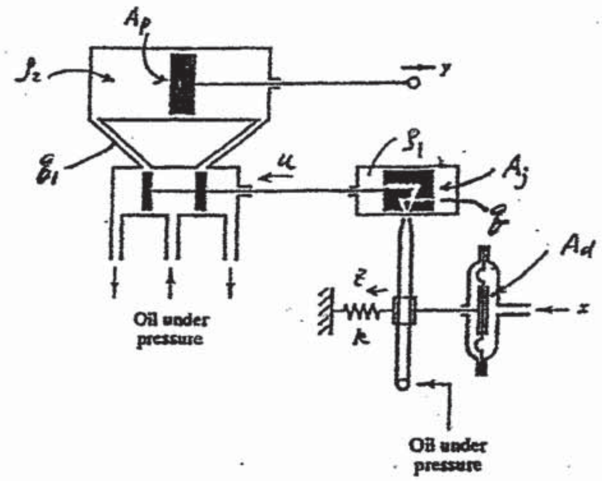
$$= \left(\frac{bk}{aA}\right)\left(\frac{R_2C_2s+1}{R_2C_2s}\right)(R_1C_1s+1)$$

$$= \frac{bk}{aA}\left(1 + \frac{1}{R_2C_2s}\right)(R_1C_1s+1)$$

$$= \frac{bk}{aA}\left(1 + \frac{R_1C_1}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1s\right)$$

Thus, the control action is proportional-plus-integral-plus-derivative. The controller is a PID controller.

B-4-8. **Referring to the figure shown on the next page, we can obtain the equations for the system.**



For the diaphragm and spring assembly,

$$A_d x = k z$$

or

$$\frac{Z(s)}{X(s)} = \frac{A_d}{k}$$

For the jet pipe,

$$q = K_1 z \ , \qquad A_j \rho_1 du = q\, dt$$

$$\frac{du}{dt} = \frac{q}{A_j \rho_1} = \frac{K_1}{A_j \rho_1} z$$

or

$$\frac{U(s)}{Z(s)} = \frac{K_1}{A_j \rho_1 s}$$

For the pilot valve,

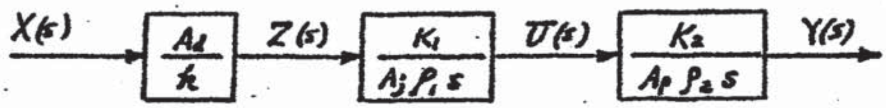
$$A_p \rho_2 dy = q_1 dt \ , \qquad q_1 = K_2 u$$

$$\frac{dy}{dt} = \frac{q_1}{A_p \rho_2} = \frac{K_2 u}{A_p \rho_2}$$

or

$$\frac{Y(s)}{U(s)} = \frac{K_2}{A_p \rho_2 s}$$

A simplified block diagram for the system is shown below.



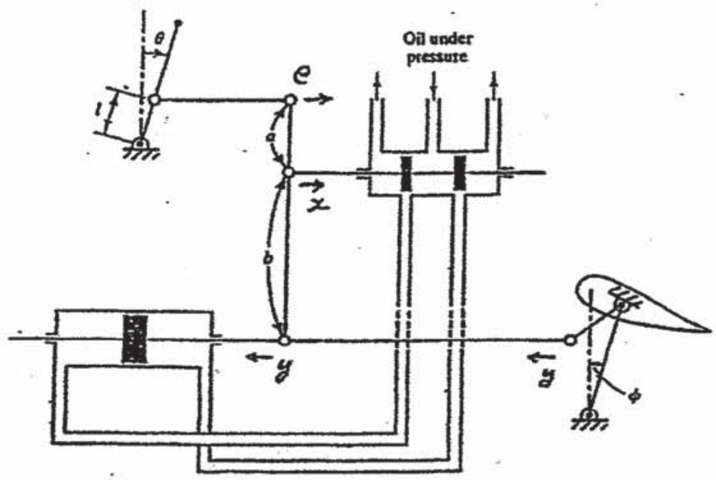
From this block diagram we obtain

$$\frac{Y(s)}{X(s)} = \frac{Y(s)}{U(s)} \frac{U(s)}{Z(s)} \frac{Z(s)}{X(s)} = \frac{K_2}{A_p \rho_2 s} \frac{K_1}{A_j \rho_1 s} \frac{A_d}{k} = \frac{K}{s^2}$$

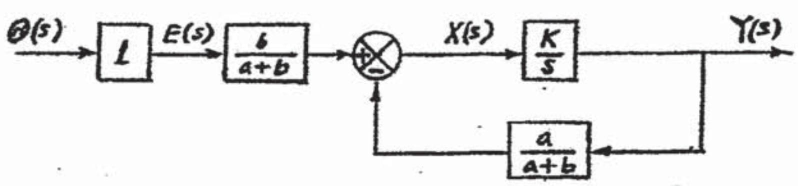
where

$$K = \frac{K_2 K_1 A_d}{A_p \rho_2 A_j \rho_1 k}$$

B-4-9. Define displacements e, x, and y as shown in the figure below.



From this figure we can construct a block diagram as shown below.



From the block diagram we obtain the transfer function Y(s)/Θ(s) as follows:

$$\frac{Y(s)}{\Theta(s)} = \frac{l \dfrac{b}{a+b} \dfrac{K}{s}}{1 + \dfrac{K}{s} \dfrac{a}{a+b}} \doteqdot \frac{lb}{a+b} \frac{a+b}{a} = l \frac{b}{a}$$

We see that the piston displacement y is proportional to the deflection angle θ of the control lever. Also, from the system diagram we see that for each value of y, there is a corresponding value of angle ø. Therefore, for each angle θ of the control lever, there is a corresponding steady-state elevator angle ø.

B-4-10. Since the increase of water in the tank during dt seconds is equal to the net inflow to the tank during the same dt seconds, we have

$$C\,dh = (q_i + q_d - q_o)\,dt \tag{1}$$

where

$$q_o = \frac{h}{R}$$

For the feedback lever mechanism, we have

$$x = \frac{a}{a+b} h$$

Equation (1) can now be written as follows:

$$C\frac{dh}{dt} = q_i + q_d - q_o = -K_v y + q_d - \frac{h}{R} \tag{2}$$

Note that

$$\frac{dy}{dt} = K_1 x = K_1 \frac{a}{a+b} h \tag{3}$$

By substituting the given numerical values into Equations (2) and (3), we obtain

$$2\frac{dh}{dt} = -y + q_d - 2h$$

$$\frac{dy}{dt} = h$$

Taking the Laplace transforms of the preceding two equations, assuming zero initial conditions, we obtain

$$2s\,H(s) = -Y(s) + Q_d(s) - 2H(s)$$

$$s\,Y(s) = H(s)$$

By eliminating Y(s) from the last two equations, we get

$$2s^2 H(s) = -H(s) + s\,Q_d(s) - 2s\,H(s)$$

Hence

$$(2s^2 + 2s + 1)\,H(s) = s\,Q_d(s)$$

from which we get

$$\frac{H(s)}{Q_d(s)} = \frac{s}{2s^2 + 2s + 1}$$

B-4-11. For the system

$$p_i A = k(x - z)$$

where A is the area of the bellows and z is the displacement of the lower end of the spring. Also,

$$y = K\int x\,dt\ , \qquad y = -z$$

Thus

$$Y(s) = \frac{K}{s} X(s)\ , \qquad Y(s) = -Z(s)$$

Hence

$$AP_i(s) = k\left[X(s) - Z(s)\right] = k\left[X(s) + Y(s)\right] = k\left(1 + \frac{K}{s}\right)X(s)$$

Therefore,

$$\frac{Y(s)}{P_i(s)} = \frac{K}{s}\frac{X(s)}{P_i(s)} = \frac{KA}{sk\left(1+\frac{K}{s}\right)} = \frac{KA}{k(s+K)}$$

B-4-12. Define

θ_0 = ambient temperature
θ_1 = temperature of thermocouple
θ_2 = temperature of thermal well
R_1 = thermal resistance of thermocouple
R_2 = thermal resistance of thermal well
C_1 = thermal capacitance of thermocouple
C_2 = thermal capacitance of thermal well
h_1 = heat input rate to thermocouple
h_2 = heat input rate to thermal well

Then, the equations for the system can be written as

$$C_1 d\theta_1 = h_1 dt$$

$$C_2 d\theta_2 = (h_2 - h_1)dt$$

where $h_1 = (\theta_2 - \theta_1)/R_1$ and $h_2 = (\theta_0 - \theta_2)/R_2$. Thus we have

$$R_1 C_1 \frac{d\theta_1}{dt} + \theta_1 = \theta_2$$

$$C_2 \frac{d\theta_2}{dt} = \frac{\theta_0 - \theta_2}{R_2} - \frac{\theta_2 - \theta_1}{R_1}$$

By eliminating θ_2 from the last two equations, we obtain

$$\frac{\Theta_1(s)}{\Theta_0(s)} = \frac{1}{R_1C_1R_2C_2 s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1}$$

Noting that

R_1C_1 = time constant of thermocouple = 2 sec
R_2C_2 = time constant of thermal well = 30 sec

we have

$$R_2C_1 = R_2C_2\frac{C_1}{C_2} = 30\frac{8}{40} = 6 \text{ sec}$$

Hence the denominator of $\Theta_1(s)/\Theta_2(s)$ becomes as

$$R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1$$
$$= 60s^2 + 38s + 1 = (1.651s + 1)(36.35s + 1)$$

Thus, the time constants of the system are

$$T_1 = 1.651 \text{ sec}, \qquad T_2 = 36.35 \text{ sec}$$
