**ELEC 360 : Control Theory and Systems I**

**Midterm**

**February 20th, 2015**

Name: ______________________________

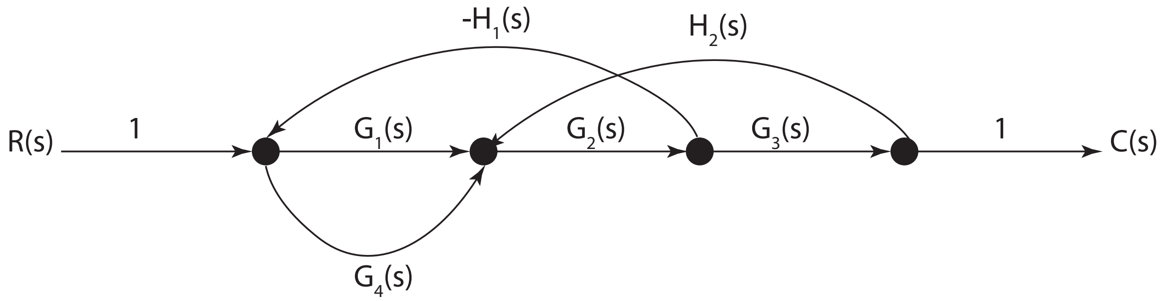
Student Number: ______________________________

Mark: _______ /40

*Notes*:

- Students are permitted a one page single-side 8.5 by 11 inch handwritten crib sheet.
- Calculators are allowed.
  - No other aids permitted.
  - The use of any other electronic devices, including cell phones, etc., during the exam will result in the confiscation of the exam paper and a zero grade.

1. Determine the transfer function of the following system via applying Mason's gain formula. (10 pts)




There are two forward paths:

$$P_1 = G_1G_2G_3$$
$$P_2 = G_4G_2G_3$$

There are three loops:

$$L_1 = -G_1G_2H_1$$
$$L_2 = G_2G_3H_2$$
$$L_3 = -G_4G_2H_1$$

.

Note: All of these loops touch as they all share the common edge $G_2$.

The determinant $\Delta$ of the graph is:

$$\begin{aligned}\Delta &= 1 - [L_1 + L_2 + L_3] \\ &= 1 + G_1G_2H_1 - G_2G_3H_2 + G_4G_2H_1\end{aligned}$$

Both of the $P_1$ and $P_2$ forward paths touch all three loops so,

$$\Delta_1 = \Delta_2 = 1$$

Hence, the transfer function for this system is given by:

$$\begin{aligned}P &= \frac{1}{\Delta}\sum_{k=1}^{2}\Delta_k P_k \\ &= \left[\frac{1}{1 + G_1G_2H_1 - G_2G_3H_2 + G_4G_2H_1}\right][G_1G_2G_3(1) + G_4G_2G_3(1)] \\ &= \frac{G_1G_2G_3 + G_4G_2G_3}{1 + G_1G_2H_1 - G_2G_3H_2 + G_4G_2H_1}\end{aligned}$$

2. Determine which transfer functions match which unit step responses: (10 pts)

   Transfer Functions:

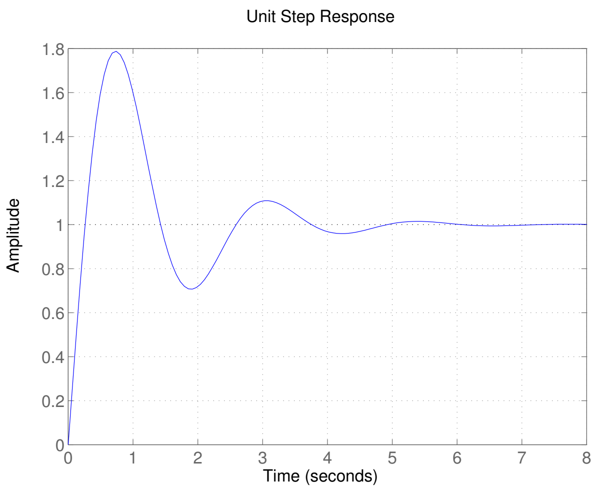
   (a) $\frac{C(s)}{R(s)} = \frac{8}{s^2+1.2\sqrt{2}s+8}$

   (b) $\frac{C(s)}{R(s)} = \frac{8}{s^2+5.2\sqrt{2}s+8}$

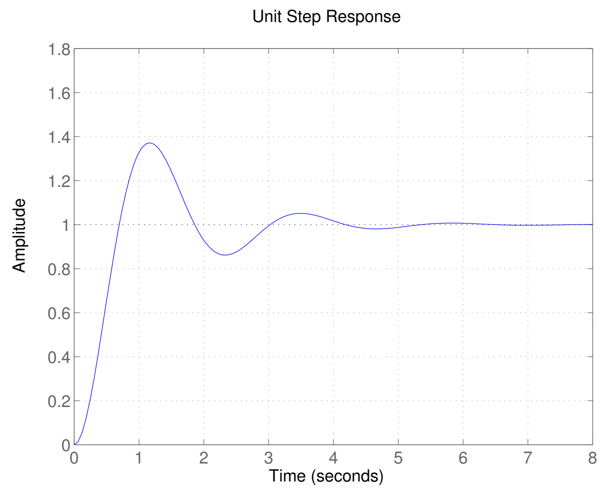
   (c) $\frac{C(s)}{R(s)} = \frac{4(s+2)}{s^2+1.2\sqrt{2}s+8}$

   (d) $\frac{C(s)}{R(s)} = \frac{24}{s^2+0.8\sqrt{6}s+24}$

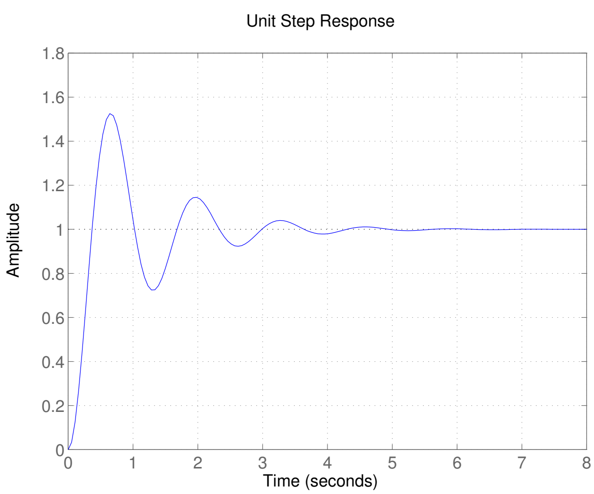
   Unit Step $c(t)$ time domain responses:



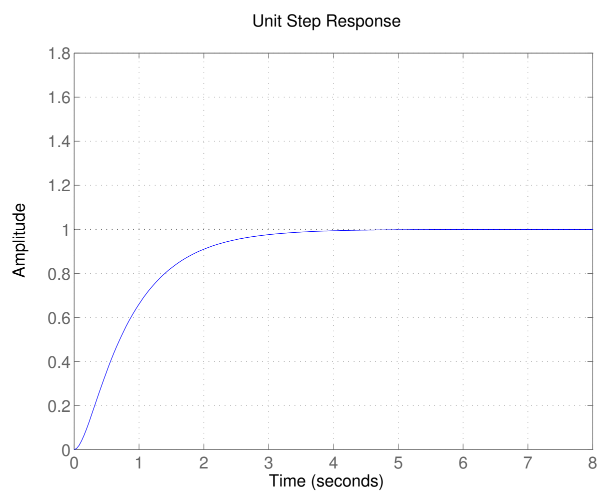

(a) Unit Step Response of transfer function: c




(b) Unit Step Response of transfer function: a




(c) Unit Step Response of transfer function: d




(d) Unit Step Response of transfer function: b

To produce the correct solutions you need to find the damping ratio $\eta$ for each of the transfer functions, the value of the natural frequency $\omega_n$, and know the impact that adding a zero has on the transfer function. These can be found by applying the standard form of a second order transfer function $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\eta\omega_n s+\omega_n^2}$.

Transfer function (d) has the highest $\omega_n^2$ so it corresponds to subplot (c). All of the remaining transfer functions have the same $\omega_n^2$. But, subplot (a) has a faster response and higher overshoot which is the effect of adding a zero to a transfer function - so transfer function (c) is subplot (a). Transfer functions (a) and (c) have identical denominators - so subplot (b) must be the plot of transfer function (a). This leaves transfer function (b) as subplot (d) - the only transfer function with a $\eta > 1$.

3. For what values of $K$ will the following differential equation be stable? (10 pts)

[You must show ALL your work]

$$r(t) = 12\frac{d^5c(t)}{dt^5} + 6\frac{d^4c(t)}{dt^4} + 3\frac{d^3c(t)}{dt^3} + K\frac{dr^c(t)}{dt^2} + \frac{dc(t)}{dt}$$

*Solution:*

Transform to Laplace domain:

$$\begin{aligned}\frac{C(s)}{R(s)} &= 12s^5 + 6s^4 + 3s^3 + Ks^2 + s \\ &= 12s^5 + 6s^4 + 3s^3 + Ks^2 + s + 0\end{aligned}$$

All coefficients are of the same sign - hence, the system can be stable or unstable.

So, now construct the Routh-Hurwitz table to check which case occurs for which values of $K$,

| | | | |
|---|---|---|---|
| $s^5$: | 12 | 3 | 1 |
| $s^4$: | 6 | K | 0 |
| $s^3$: | $\frac{(6)(3)-(12)(K)}{6} = 3 - 2K$ | $\frac{(6)(1)-(12)(0)}{6} = 1$ | |
| $s^2$: | $\frac{(3-2K)(K)-(6)(1)}{(3-2K)} = K - \frac{6}{3-2K}$ | $\frac{(3-2K)(0)-(-6)(0)}{3-2K} = 0$ | |
| $s^1$ : | $\frac{\left(K-\frac{6}{3-2K}\right)(1)-(3-2K)(0)}{\left(K-\frac{6}{3-2K}\right)} = 1$ | $\frac{\left(K-\frac{6}{3-2K}\right)(0)-(3-2K)(0)}{\left(K-\frac{6}{3-2K}\right)} = 0$ | |
| $s^0$ : | $\frac{(1)(0)-\left(K-\frac{6}{3-2K}\right)(0)}{1} = 0 \to \epsilon^+$ | | |

For the system to be stable we require that there are no sign changes in the 1st column, therefore:

It must be that case that both $3 - 2K \geq 0$ and $K - \frac{6}{3-2K} \geq 0$ hold.

Solving, for $K$ in $3 - 2K \geq 0$ gives that $K \leq \frac{3}{2}$

Solving for $K$ in $K - \frac{6}{3-2K} \geq 0$ clearly at a minimum requires that $K \geq 0$ as even for $K = 0$ we get a $-6 < 0$.

Hence, we need the $-\frac{6}{3-2K}$ to give a positive quantity if $K - \frac{6}{3-2K} \geq 0$ (clearly, it can only give a zero in the limit as $K \to \infty$). [<u>Note</u>: There is a minus sign in $-\frac{6}{3-2K}$].

Therefore having $K - \frac{6}{3-2K} \geq 0$ requires that $3 - 2K < 0$.

This only occurs when $K < \frac{3}{2}$.

Hence, when $K > 3/2$ then $-\frac{6}{3-2K}$ will be $> 0$ and, therefore, $K - \frac{6}{3-2K} \geq 0$ will be true.

But, we now have two conditions that must *both* be meet in that,

- The first condition requires that $K \leq \frac{3}{2}$ and,
- The second condition requires that $K > \frac{3}{2}$.

Clearly, there can be **no** $K$ that can meet both conditions *simultaneously.*

Hence, there is **no** value of $K$ that can make the above system stable, as any value we choose for $K$ will lead to sign changes occurring in the 1st column of Routh-Hurwitz tabls.

Therefore, the above systems will always be an unstable system for all values of $K$ for $K \in (-\infty, \infty)$.

4. You are given a standard close-loop transfer function for which $G(s) = s+1$ and $H(s) = s^2(s^3+9s^2+28s+40)$.

   (a) Determine the values for the steady-state static error constants $K_p$, $K_v$, and $K_a$. (5 pts)

   (b) Determine the steady-state errors for the unit step, unit ramp, and unit parabolic inputs for this closed-loop control system. (5 pts)

   [You must show ALL of your work.]

   *Solution:*

   The closed-loop transfer function for this system is given by,

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G(s)}{1+G(s)H(s)} \\ &= \frac{s+1}{1+s^2(s+1)(s^3+9s^2+28s+40)}\end{aligned}$$

   We now need to put this in the form of a transfer function that has $H(s) = 1$ as it is this case that is the case covered by our formulas for $K_p$, $K_v$, and $K_a$.

   We know (or can easily deduce) that a general closed-loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$ can be converted into an equivalent transfer function involving $G'(s)$ and $H'(s) = 1$ as follows,

$$\begin{aligned}\frac{G(s)}{1+G(s)H(s)} &= \frac{G'(s)}{1+G'(s)} \\ G(s)\left[1+G'(s)\right] &= G'(s)\left[1+G(s)H(s)\right] \\ G(s) &= G'(s)\left[1+G(s)H(s)-G(s)\right] \\ G'(s) &= \frac{G(s)}{1+G(s)H(s)-G(s)}\end{aligned}$$

   Hence, we want to evaluate the behavior of the open-loop transfer function that is given by,

$$\begin{aligned}G'(s) &= \frac{s+1}{1+s^2(s+1)(s^3+9s^2+28s+40)-(s+1)} \\ &= \frac{s+1}{s^2(s+1)(s^3+9s^2+28s+40)-s} \\ &= \frac{s+1}{(s^3+s^2)(s^3+9s^2+28s+40)-s} \\ &= \frac{s+1}{s^6+9s^5+28s^4+40s^3+s^5+9s^4+28s^3+40s^2-s} \\ &= \frac{s+1}{s(s^5+10s^4+37s^3+68s^2+40s-1)} \qquad \text{(Type 1 system)}\end{aligned},$$

   to determine the values of $K_p$, $K_v$, and $K_a$.

$$\begin{aligned}K_p &= \lim_{s\to 0} G'(s) = \infty \text{ and } e_{ss_{step}} = 0 \\ K_v &= \lim_{s\to 0} sG'(s) = -1 \text{ and } e_{ss_{ramp}} = -1 \\ K_a &= \lim_{s\to 0} s^2G'(s) = 0 \text{ and } e_{ss_{parabolic}} = \infty\end{aligned}$$

**END OF EXAM**