ELEC 360 : Control Theory and Systems I

Midterm

February 24th, 2016

Name: ____________________

Student Number: ____________________

Mark: _______ /50

Notes:

- Students are permitted a one page single-side 8.5 by 11 inch handwritten crib sheet.
- Calculators are allowed.
  - No other aids permitted.
  - The use of any other electronic devices, including cell phones, etc., during the exam will result in the confiscation of the exam paper and a zero grade.

1. Determine which transfer functions match which unit step responses: (10 pts)

   Transfer Functions:

   (a) $\frac{C(s)}{R(s)} = \frac{8}{2s^2+8s+4} = \frac{4}{s^2+4s+2}$ $K = 2, \zeta = \sqrt{2} \approx 1.414, \omega_n^2 = 2$

   (b) $\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+2}$ $K = 2, \zeta = \frac{1}{\sqrt{2}} \approx 0.7072, \omega_n^2 = 2$

   (c) $\frac{C(s)}{R(s)} = \frac{16}{4s^2+12s+8} = \frac{4}{s^2+3s+2}$ $K = 2, \zeta = \frac{3}{2\sqrt{2}} \approx 1.0607, \omega_n^2 = 2$

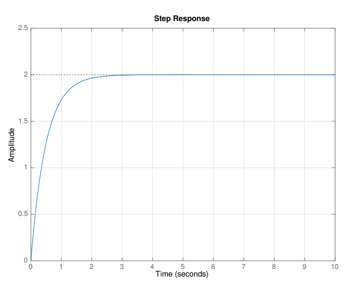
   (d) $\frac{C(s)}{R(s)} = \frac{4}{s+2}$ 1^{st} order system.

   Note: There were errors in the denominator "s^1" terms for (a) and (b) in the copy of the exam you wrote. The correct terms are shown above. You received marks for the question for computing ζ for the 2^{nd} order transfer functions (a)-(c) and then selecting the plots based on this ordering. This exam printing error has been accounted for through the exam scaling which has been increased to "+5" marks for the exam.

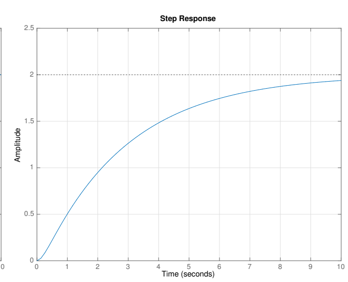
   Plots (b)-(d) all are of a similar shape. Hence, each is generated by one of the (a)-(c) transfer function which are the same functions but with different damping ratios.

   - Oscillations only occur when $0 < \zeta < 1$ so Plot (d) must be generated from transfer function (b) which has $\zeta = \frac{1}{\sqrt{2}}$.
   - As the damping ratio ζ increase past $\zeta = 1$ the transfer function will become more and more overdamped, e.g. produce a slower response that takes longer to reach its steady-state value. Hence, since Plot (b) has the slowest response of the (a)-(c) transfer function it must be generated by transfer function (a) which has the largest damping ratio of $\zeta = \sqrt{2}$.
   - Plot (c) must then be generated by transfer function (c) which has a $\zeta = \frac{3}{2\sqrt{2}}$ which is very close the critically damped case of $\zeta = 1$ which is the fasted response which has no overshoot or oscillations, e.g. the point where the complex conjugate roots that arise for $0 \leq \zeta < 1$ become two real and equal roots at $\zeta = 1$.
   - Plot (a) then must belong to the 1^{st} order transfer function (d).

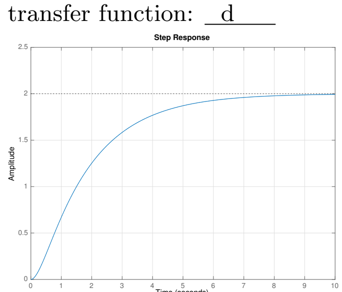
   Unit Step $c(t)$ time domain responses:



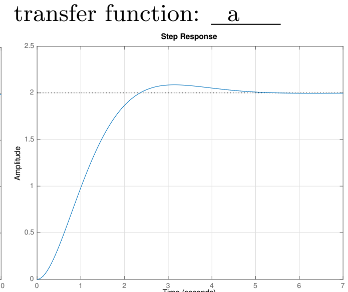
(a) Unit Step Response of transfer function: d



(b) Unit Step Response of transfer function: a

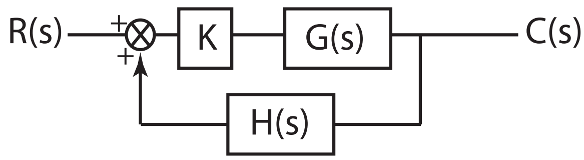


(c) Unit Step Response of transfer function: c



(d) Unit Step Response of transfer function: b

2. For the following closed loop control system determine: (15 pts)



where $G(s) = \frac{s}{s^3+4s^2+2s+1}$ and $H(s) = s^2 + 1$.

(a) The values of K for which the system will be stable.

(b) The steady-state error constants K_p, K_v, and K_a.

(c) The system's steady state error for the unit step, unit ramp, and unit parabola inputs.

[You must show ALL your work]

Answer:

(a) The closed loop transfer function is given by:

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{KG(s)}{1 - KG(s)H(s)} \\ &= \frac{K\left(\frac{s}{s^3+4s^2+2s+1}\right)}{1 - K\left(\frac{s}{s^3+4s^2+2s+1}\right)(s^2+1)} \\ &= \frac{Ks}{s^3 + 4s^2 + 2s + 1 - Ks(s^2+1)} \\ &= \frac{Ks}{s^3 + 4s^2 + 2s + 1 - Ks^3 - Ks} \\ &= \frac{Ks}{(1-K)s^3 + 4s^2 + (2-K)s + 1}\end{aligned}$$

Now construct Routh-Hurwitz table for the denominator polynomial $(1-K)s^3 + 4s^2 + (2-K)s + 1$

s^3:	$(1-K)$	$(2-K)$
s^2:	4	1
s^1:	$\frac{4(2-K)-(1-K)(1)}{4} = \frac{7-3K}{4}$	
s^0:	1	

For stability there can be no sign changes in the first column of the Routh-Hurwitz table.
The s^2 and s^0 first columns have a +4 and +1 entries, therefore all of the other first column rows must be positive.
This gives that:

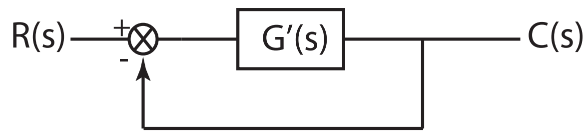
- For s^3:

$$(1-K) \geq 0 \rightarrow K < 1$$

- For s^1:

$$\frac{7-3K}{4} \geq 0 \rightarrow K < \frac{7}{3} = 2.5$$

The conditions must be met concurrently so, $K < 1$ if no sign changes are to occur in the Routh-Hurwitz table's first column.
Therefore the closed loop system will be stable for all $K < 1$ and unstable for any $K \geq 1$.



(b) To compute the steady-state error constants we need to transform the closed loop system into the form: This can be done by applying the following formula:

$$G'(s) = \frac{KG(s)}{1 - KG(s)H(s) - KG(s)}$$

(This can be derived by setting $\frac{C(s)}{R(s)} = \frac{G'(s)}{1+G'(s)} = \frac{KG(s)}{1-KG(s)H(s)}$ and solving for $G'(s)$.)

Substituting the $G(s)$ and $H(s)$ terms from above gives,

$$\begin{aligned}
G'(s) &= \frac{K\left(\frac{s}{s^3+4s^2+2s+1}\right)}{1 - K\left(\frac{s}{s^3+4s^2+2s+1}\right)(s^2+1) - K\left(\frac{s}{s^3+4s^2+2s+1}\right)} \\
&= \frac{Ks}{s^3 + 4s^2 + 2s + 1 - Ks(s^2+1) - Ks} \\
&= \frac{Ks}{s^3 + 4s^2 + 2s + 1 - Ks^3 - 2Ks} \\
&= \frac{Ks}{(1-K)s^3 + 4s^2 + (2-2K)s + 1}
\end{aligned}$$

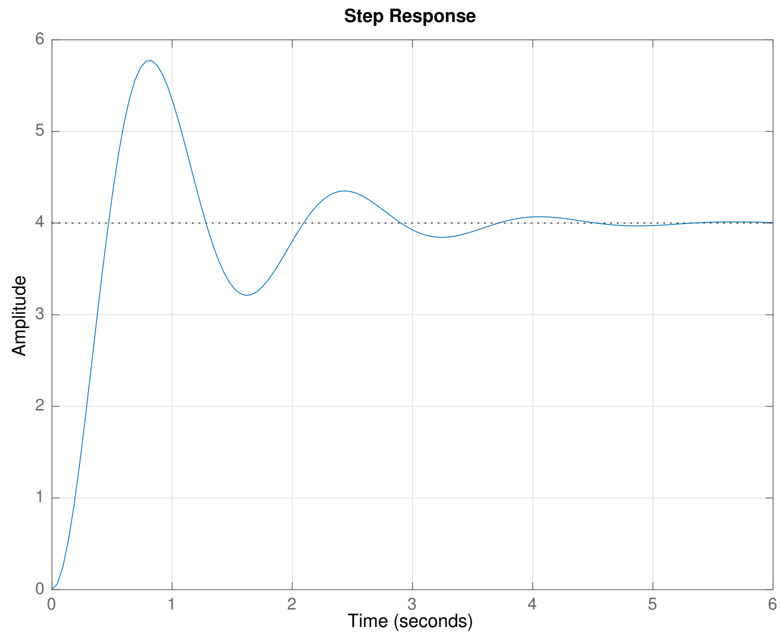
This is a Type 0 system (it has no roots at $s = 0$).
Therefore, $K_v = 0$ and $K_a = 0$ and we only need to solve for

$$\begin{aligned}
K_p &= \lim_{s \to 0} G'(s) \\
&= \frac{K(0)}{(1-K)(0)^3 + (4)(0)^2 + (2-2K)(0) + (1)} \\
&= \frac{0}{(1)} \\
&= 0
\end{aligned}$$

(c) Solving for the steady-state error terms then gives:

- Unit Step
$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1} = 1$$
- Unit Ramp
$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$
- Unit Parabola
$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

3. To have an unknown closed-loop system. The response of this system to a unit set input is shown below.



From the unit step response you have been able to determine that:

- The maximum peak occurs at $t_{peak} = 0.8112$ seconds and has a maximum value of 5.774.
- The rise time $t_{rise} = 0.3172$ seconds.
- The 2% settling time occurs at 4

Estimate the transfer function that best describes this unknown system.

[You must show ALL of your work.]

Answer:

- First, we known that the gain for the system is $K = 4$ as the response to the unit step input goes to the steady-state value of 4.
- Second, the unit step response has the form of that of a response of a underdamped 2^{nd} order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + w_n^2}$$

- For this type of system we have the following formulas:

$$w_d = w_n\sqrt{1-\zeta^2}$$
$$t_{peak} = \frac{\pi}{w_d}$$
$$M_p = \frac{\sigma\pi}{w_d}$$
$$t_{2\%} = \frac{4}{\sigma}$$
$$\sigma = \zeta\omega_n$$

- So we can then compute that,

$$t_{peak} = \frac{\pi}{w_d} = 0.8112 \rightarrow \omega_d = \frac{\pi}{0.8112}$$

$$t_{2\%} = \frac{4}{\sigma} = 4 \rightarrow \sigma = 1$$

$$\beta = tan^{-1}\left(\frac{\omega_d}{\sigma}\right) = tan^{-1}\left(\frac{\frac{\pi}{0.8112}}{1}\right) = 75.5225°$$

$$\zeta = \cos(\beta) = 0.25$$

$$\sigma = \zeta\omega_n \rightarrow \omega_n = \frac{1}{0.25} = 4$$

- Substituting these values of K, ζ and ω_n into the $G(s)$ formula gives,

$$\begin{aligned} G(s) &= \frac{4(4)^2}{s^2 + 2 * 0.25 * 4s + 4^2} \\ &= \frac{64}{s^2 + 2s + 16} \end{aligned}$$

This is identical to the transfer function $G(s) = \frac{64}{s^2+2s+16}$ that was used to generate the unit step response plot.

(Note: Differences can arise in the estimated $G(s)$ do to the accuracy with which the various values are estimated from the time domain unit step function as well as how may significant figures are carried through all of the computations.)

4. Determine the transfer function of the following system via applying Mason's gain formula. (15 pts)

   (a) Clearly label and provide the simplified formulas for each of the forward path gains.

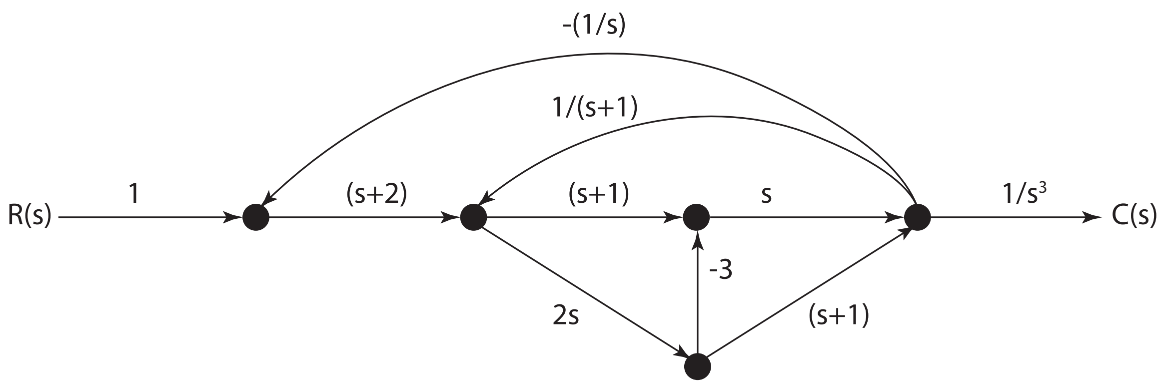
   (b) Clearly separately sketch and label each of the feedback loops and provide the simplified formulas for each of the loop gains.

   (c) Provide the formula for Δ in terms of these forward path and loop gains.

   (d) Provide the formula for each of the Δ_k co-factors associated with each forward path.

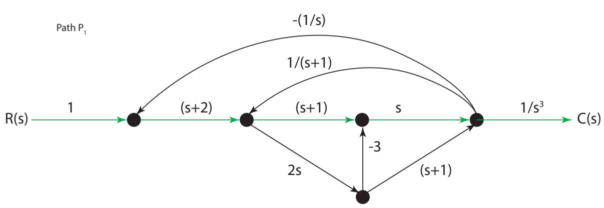
   (e) Provide the formula for the transfer function P in terms of the above forward path gains, loop gains, Δ, and Δ_k co-factor terms.

   You do **NOT** need to provide the final transfer function P in a simplified $P = \frac{B(s)}{A(s)}$ form.

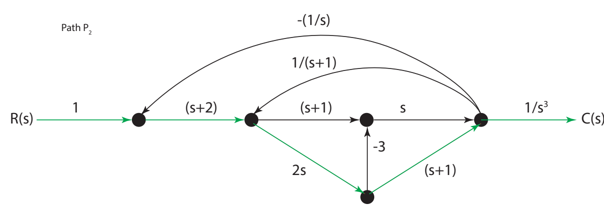


Answer:

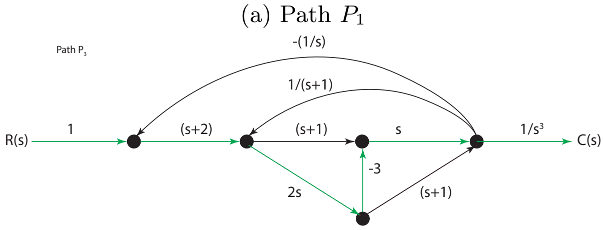
Determine the forward paths between $R(s)$ and $C(s)$:



(a) Path P_1



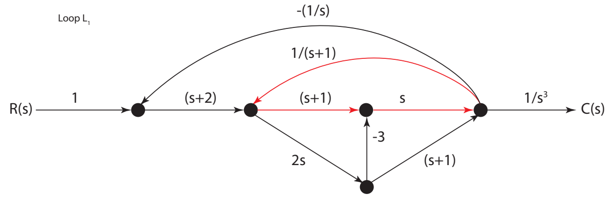
(b) Path P_2



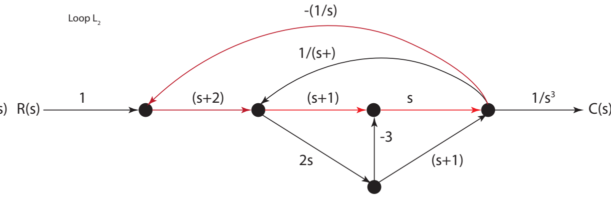
(c) Path P_3

- $P_1 = (1)(s+2)(s+1)(s)\frac{1}{s^3} = \frac{(s+2)(s+1)}{s^2}$
- $P_2 = (1)(s+2)(2s)(s+1)\frac{1}{s^3} = \frac{2(s+2)(s+1)}{s^2}$
- $P_3 = (1)(s+2)(2s)(-3)(s)\frac{1}{s^3} = \frac{-6(s+2)}{s}$

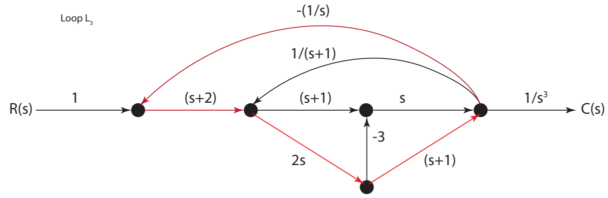
Determine all loops:



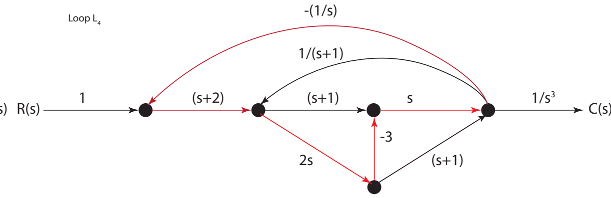
(a) Loop L_1



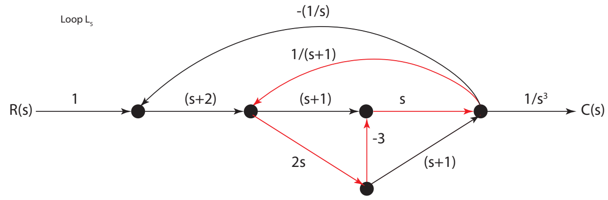
(b) Loop L_2



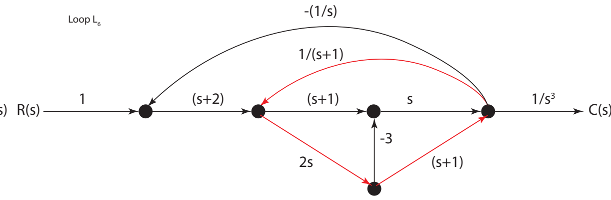
(c) Loop L_3



(d) Loop L_4



(e) Loop L_5



(f) Loop L_6

- $L_1 = (s+1)(s)\frac{1}{s+1} = s$
- $L_2 = (s+2)(s+1)(s)\frac{(-1)}{s} = -(s+2)(s+1)$
- $L_3 = (s+2)(2s)(s+1)\frac{(-1)}{s} = -2(s+2)(s+1)$
- $L_4 = (s+2)(2s)(-3)(s)\frac{(-1)}{s} = 6s(s+2)$
- $L_5 = (2s)(-3)(s)\frac{1}{(s+1)} = \frac{-6s^2}{(s+1)}$
- $L_6 = (2s)(s+1)\frac{1}{(s+1)} = 2s$

Now to solve the Mason's gain formula $P = \frac{1}{\Delta}\sum_{k=1}^{3} P_k\Delta_k$ we need to compute:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + (L_1 L_3)$$

Now remove Path P_1 and compute:

$$\Delta_1 = 1 - L_6$$

Now remove Path P_2 and compute:

$$\Delta_2 = 1 - L_1$$

Now remove Path P_3 and compute:

$$\Delta_3 = 1$$

Substituting these solutions into Mason's gain formula gives:

$$
\begin{aligned}
P &= \frac{1}{1-(L_1+L_2+L_3+L_4+L_5+L_6)-(L_1L_3)}P_1(1-L_6)+P_2(1-L_1)+P_3(1) \\
&= \frac{\frac{(s+2)(s+1)}{s^2}(1-2s)+\frac{2(s+2)(s+1)}{s^2}(1-s)+\frac{-6(s+2)}{s}(1)}{s-(s+2)(s+1)-2(s+2)(s+1)+6s(s+2)-\frac{6s^2}{(s+1)}+2s-2s(s+2)(s+1)} \\
&= \frac{\frac{s^2+3s+2}{s^2}(1-2s)+\frac{2(s^2+3s+2)}{s^2}(1-s)+\frac{-6(s+2)}{s}}{s-(s^2+3s+2)-2(s^2+3s+2)+6s^2+12s-\frac{6s^2}{(s+1)}+2s-2s^3-6s^2-4s} \\
&= \frac{1}{s^2}\left[\frac{(s^2+3s+2)(1-2s)+(2s^2+6s+6)(1-s)+-6s^2+2s}{-s^2-2s-2-2s^2-6s-4+6s^2+12s-\frac{6s^2}{(s+1)}+2s-2s^3-6s^2-4s}\right] \\
&= \frac{1}{s^2}\left[\frac{-4s^3-15s^2+s+8}{-2s^3-3s^2+2s-6-\frac{6s^2}{(s+1)}}\right] \\
&= \frac{(s+1)}{s^2}\left[\frac{-4s^3-15s^2+s+8}{-2s^4-5s^3-7s^2-4s-6}\right] \\
&= -\left[\frac{-4s^4-19s^3-14s^2+9s+8}{2s^6+5s^5+7s^4+4s^3+6s^2}\right]
\end{aligned}
$$

(Full final simplification was not required as part of your solution. You only need to provide the full first line of the $P =$ equation above.)

END OF EXAM