Lecture 5: NFA Equivalence

CSC 320: Foundations of Computer Science

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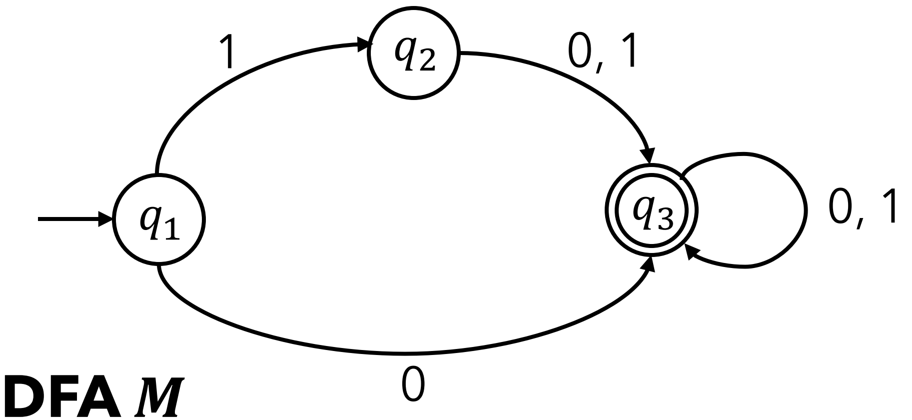
NFAs and DFAs

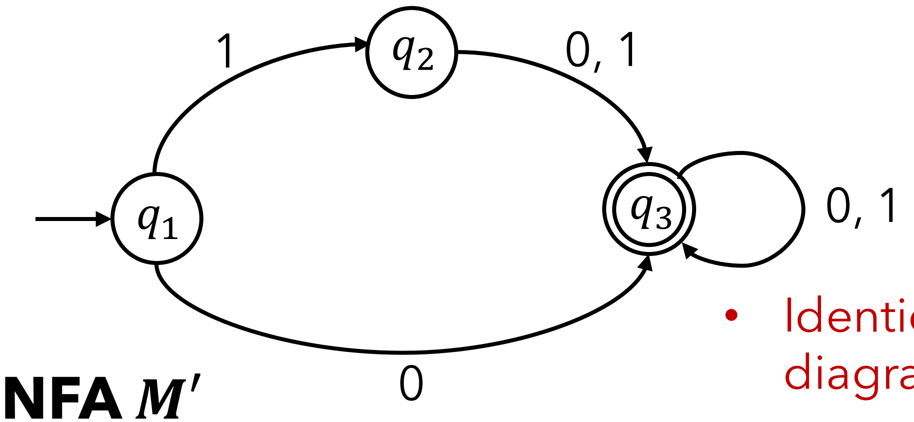
- A DFA can be considered a **special case** of an NFA
- We just have to change the **transition function** definition slightly



DFA M

$(\{q_1, q_2, q_3\}, \{0, 1\}, \delta, q_1, \{q_3\})$ with
$\delta: Q \times \Sigma \rightarrow Q$ defined by

δ	0	1
q_1	q_3	q_2
q_2	q_3	q_3
q_3	q_3	q_3



NFA M'

- Identical state diagram

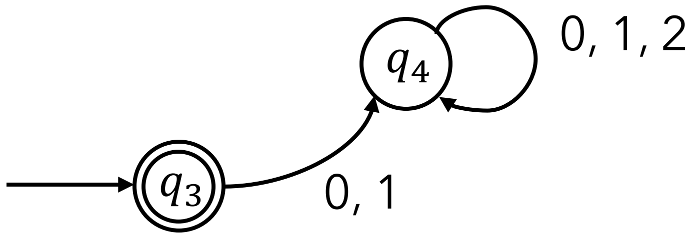
$(\{q_1, q_2, q_3\}, \{0, 1\}, \delta, q_1, \{q_3\})$ with
$\delta: Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$ defined by

δ	0	1	ε
q_1	$\{q_3\}$	$\{q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\{q_3\}$	$\emptyset$
q_3	$\{q_3\}$	$\{q_3\}$	$\emptyset$

- Next states are singleton sets
- No ε transitions

NFAs and DFAs

- Recall that this state diagram was **not a valid DFA**
- $\boldsymbol{q_3}$ does not have an outgoing transition for symbol **2**



- However, this is a **valid state diagram for an NFA**
- In NFAs, we do not require outgoing transitions for each alphabet symbol from every state

NFAs and DFAs

- **Definition**: Let $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ each be a DFA or an NFA. We call $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ **equivalent** if $\boldsymbol{L(M_1) = L(M_2)}$.
- Clearly, for **every DFA** there exists an **equivalent NFA**
- Recall, we claimed that the **computational power** of DFAs and NFAs are **equal**
- So, we will also prove that for **every NFA** there also exists an **equivalent DFA**

Equivalence of NFAs and DFAs

First: We show that for **every DFA**, there exists an **equivalent NFA**

Proof: **(easy)**

Let $\boldsymbol{D} = (\boldsymbol{Q_D}, \boldsymbol{\Sigma}, \boldsymbol{\delta_D}, \boldsymbol{q_D}, \boldsymbol{F_D})$ be a DFA.

We can build NFA $\boldsymbol{N} = (\boldsymbol{Q_N}, \boldsymbol{\Sigma}, \boldsymbol{\delta_N}, \boldsymbol{q_N}, \boldsymbol{F_N})$ with $\boldsymbol{L(D) = L(N)}$ as follows:

- $\boldsymbol{Q_N} \coloneqq \boldsymbol{Q_D}$
- $\boldsymbol{q_N} \coloneqq \boldsymbol{q_D}$
- $\boldsymbol{F_N} \coloneqq \boldsymbol{F_D}$

States, start state, and accept states are the same

- $\boldsymbol{\delta_N} \colon \boldsymbol{Q_N} \times (\boldsymbol{\Sigma} \cup \{\boldsymbol{\varepsilon}\}) \to \boldsymbol{\mathcal{P}}(\boldsymbol{Q_N})$ with
  - $\boldsymbol{\delta_N(q, a)} \coloneqq \{\boldsymbol{\delta_D(q, a)}\}$ for all $\boldsymbol{a} \in \boldsymbol{\Sigma}$
  - $\boldsymbol{\delta_N(q, \varepsilon)} \coloneqq \emptyset$

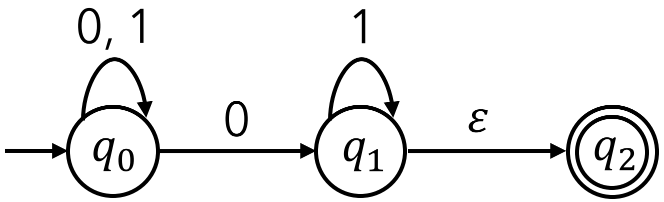
- Exact same transitions, but written as set containing the outgoing state in DFA
- No $\boldsymbol{\varepsilon}$-transitions

Equivalence of NFAs and DFAs

Next: We show that for **every NFA**, there exists an **equivalent DFA**

Proof: Let $\boldsymbol{N} = (\boldsymbol{Q_N}, \boldsymbol{\Sigma}, \boldsymbol{\delta_N}, \boldsymbol{q_N}, \boldsymbol{F_N})$ be an NFA. Construct DFA $\boldsymbol{D} = (\boldsymbol{Q_D}, \boldsymbol{\Sigma}, \boldsymbol{\delta_D}, \boldsymbol{q_D}, \boldsymbol{F_D})$ such that $\boldsymbol{L}(\boldsymbol{D}) = \boldsymbol{L}(\boldsymbol{N})$.

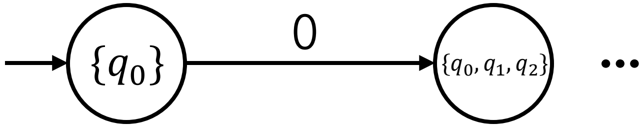
$\boldsymbol{N}$



Idea: Build $\boldsymbol{D}$ such that is simulates the computation of $\boldsymbol{N}$.

- Since states can go to a set of states in NFAs, **create a state in $\boldsymbol{D}$ for every possible subset of states of $\boldsymbol{Q_N}$**

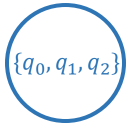
$\boldsymbol{D}$



NFA to DFA Construction

Proof continued: Given NFA $N = (Q_N, \Sigma, \delta_N, q_N, F_N)$, build DFA $D = (Q_D, \Sigma, \delta_D, q_D, F_D)$:

For now, let's ignore the ε-transitions of N.



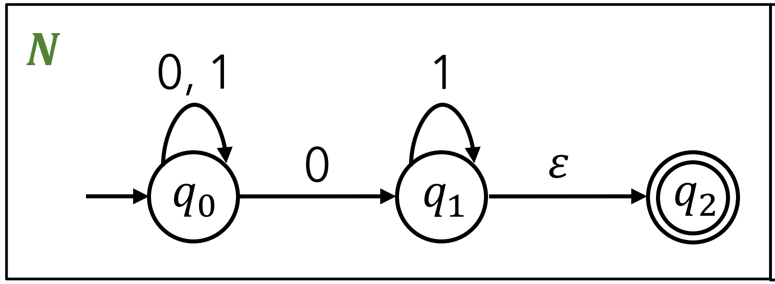
- $Q_D \coloneqq \mathcal{P}(Q_N)$ New states are **all possible subsets** of states in NFA
- $q_D \coloneqq \{q_N\}$ Start state is **the singleton set** containing NFA start state
- δ_D is defined as follows: Next state is the state corresponding to **set of all reachable states in NFA** from the states in current DFA state
  - Let $S \in Q_D$ be a state of the DFA D and let $a \in \Sigma$. Recall, $S \subseteq Q_N$.
  - $\delta_D(S, a) \coloneqq \{q \in Q_N \mid q \in \delta_N(s, a) \text{ for some } s \in S\}$
- $F_D \coloneqq \{S \in Q_D \mid \text{there exists a } q \in S \text{ with } q \in F_N\}$ Accept states are the states which **contain an accept state** in NFA

NFA to DFA Construction (ε-transitions)

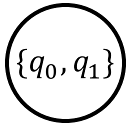
Proof continued: Next, modify the $\boldsymbol{D}$ to simulate the $\boldsymbol{\varepsilon}$-transitions in NFA

All states in $\boldsymbol{S}$ plus states reachable by $\boldsymbol{\varepsilon}$-transitions (**0 or more**) from states in $\boldsymbol{S}$

- For any state $\boldsymbol{S} \in \boldsymbol{Q_D}$, let $\boldsymbol{E}(\boldsymbol{S}) = \{\boldsymbol{q} \mid \boldsymbol{q}$ can be reached from some state in $\boldsymbol{S}$ by **traveling 0 or more $\boldsymbol{\varepsilon}$-transitions**$\}$

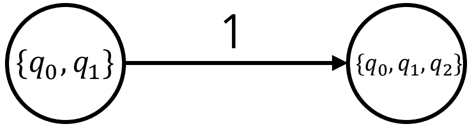


State $\boldsymbol{S}$ in $\boldsymbol{D}$

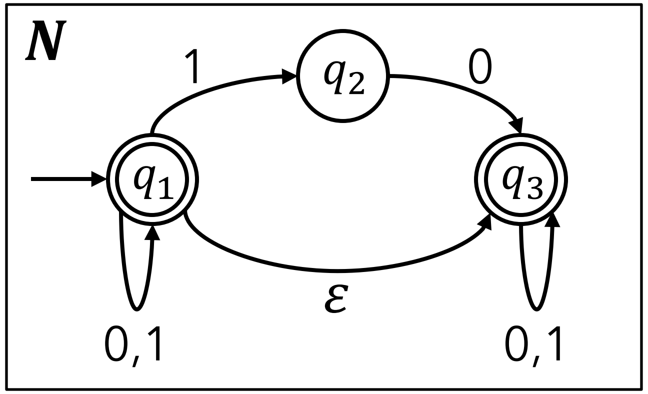


$\boldsymbol{E}(\boldsymbol{S}) = \{q_0, q_1, \boldsymbol{q_2}\}$

- Modify $\boldsymbol{D}$ as follows:
  - $\boldsymbol{q_D} \coloneqq \boldsymbol{E}(\{\boldsymbol{q_N}\})$ Start state is $E()$ of start state in NFA
  - $\boldsymbol{\delta_D}(\boldsymbol{S}, \boldsymbol{a}) \coloneqq \{\boldsymbol{q} \in \boldsymbol{Q_N} \mid \boldsymbol{q} \in \boldsymbol{E}\big(\boldsymbol{\delta_N}(\boldsymbol{s}, \boldsymbol{a})\big)$ for some $\boldsymbol{s} \in \boldsymbol{S}\}$



NFA to DFA Example



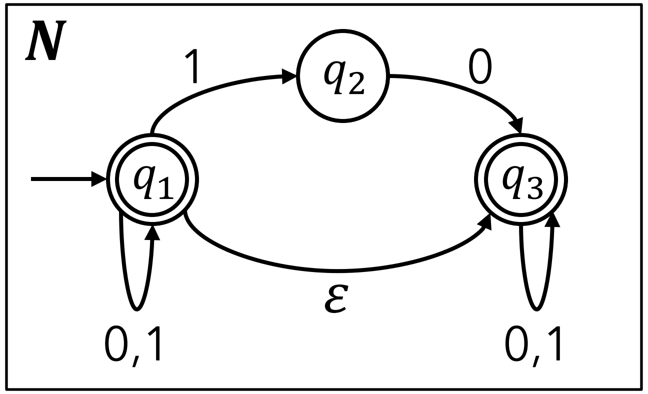
Transition table for DFA $\boldsymbol{D}$

$\boldsymbol{\delta_D}$	**0**	**1**
$\emptyset$		
$\{q_1\}$		
$\{q_2\}$		
$\{q_3\}$		
$\{q_1, q_2\}$		
$\{q_1, q_3\}$		
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$		

$\boldsymbol{D}$

$$Q_D := \mathcal{P}(Q_N)$$

NFA to DFA Example



Transition table for DFA ***D***

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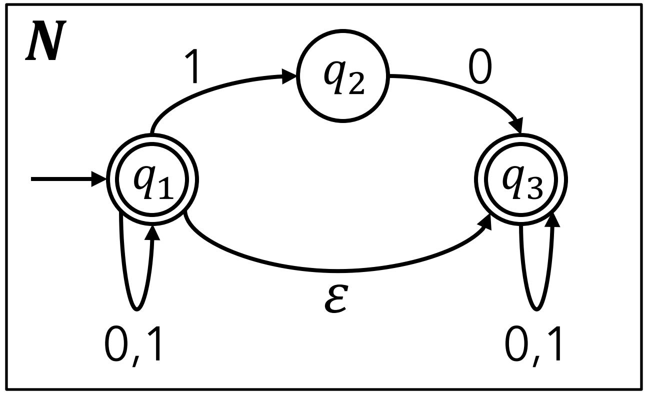
$\emptyset$		
$\{q_1\}$		
$\{q_2\}$		
$\{q_3\}$		
$\{q_1, q_2\}$		
→ $\{q_1, q_3\}$		
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$		

D

$$q_D := E(\{q_N\})$$

NFA to DFA Example



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→ $\{q_1, q_3\}$		
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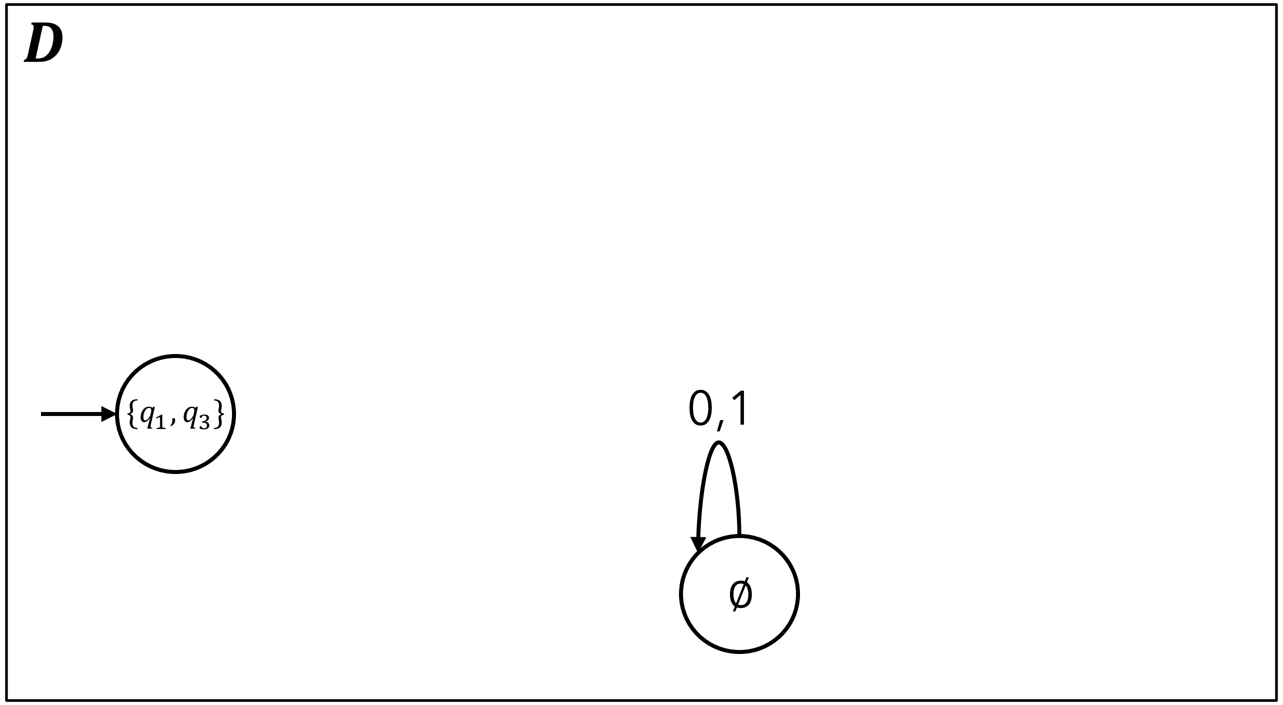
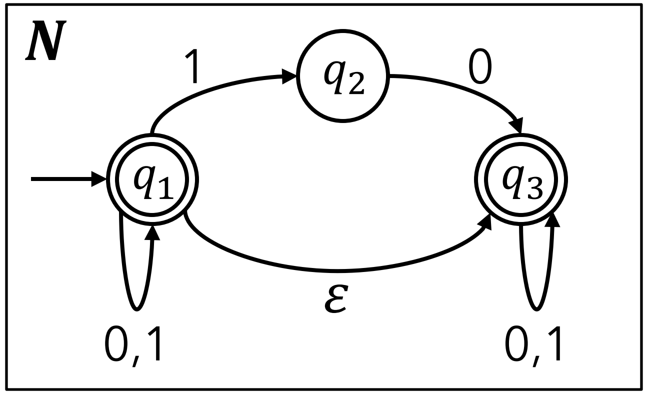


DFA state $\emptyset$ represents when an NFA state has **no outgoing transitions** for a symbol

NFA to DFA Example



Transition table for DFA ***D***

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$\{q_3\}$		
$\{q_1, q_2\}$		
$\rightarrow \{q_1, q_3\}$		
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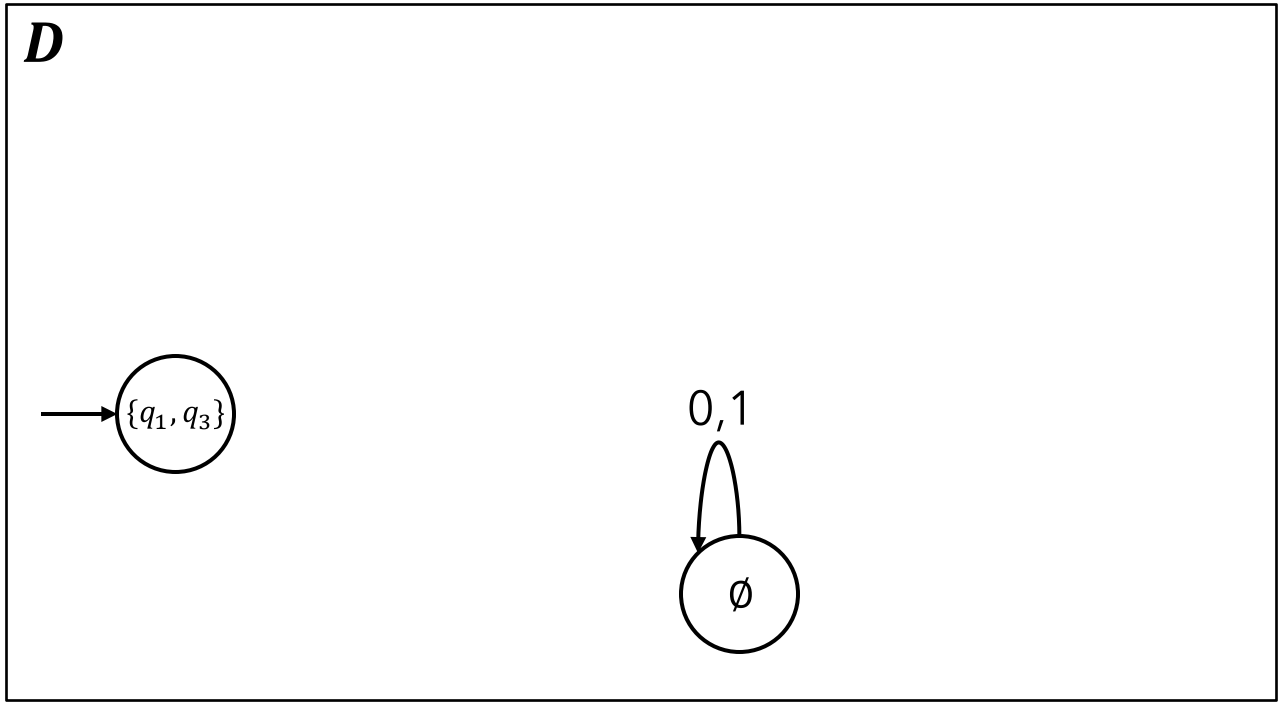
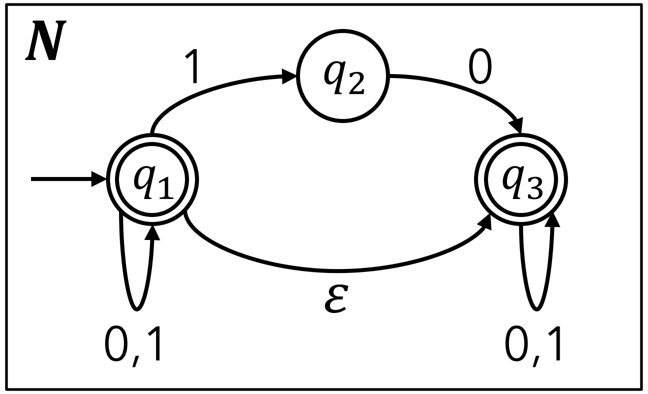


$$\delta_D(S, a) := \{q \in Q_N \mid q \in E(\delta_N(s, a)) \text{ for some } s \in S\}$$

NFA to DFA Example



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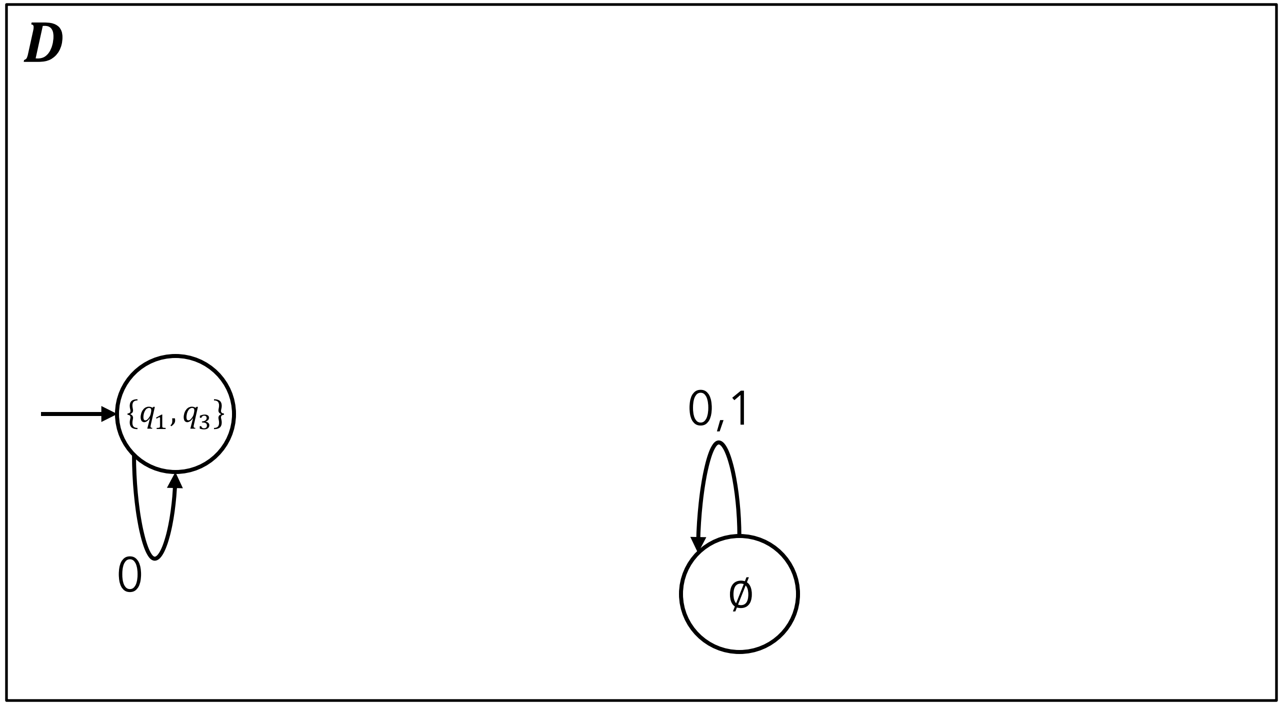
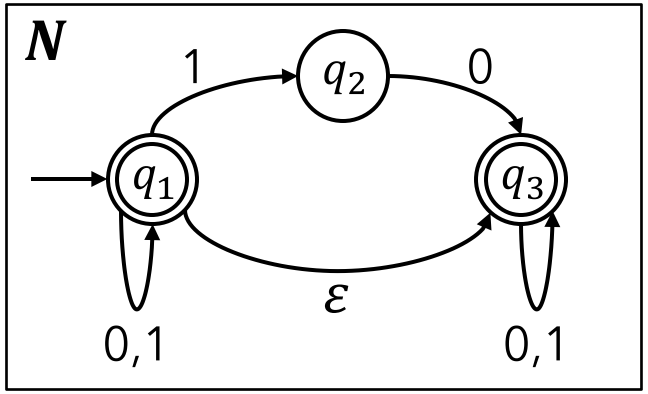
→ $\{q_1, q_3\}$	$\{q_1, q_3\}$	
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$		



$$\delta_D(S, a) \coloneqq \{q \in Q_N \mid q \in E\big(\delta_N(s, a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



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$\{q_1, q_2\}$		
$\rightarrow \{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2, q_3\}$		
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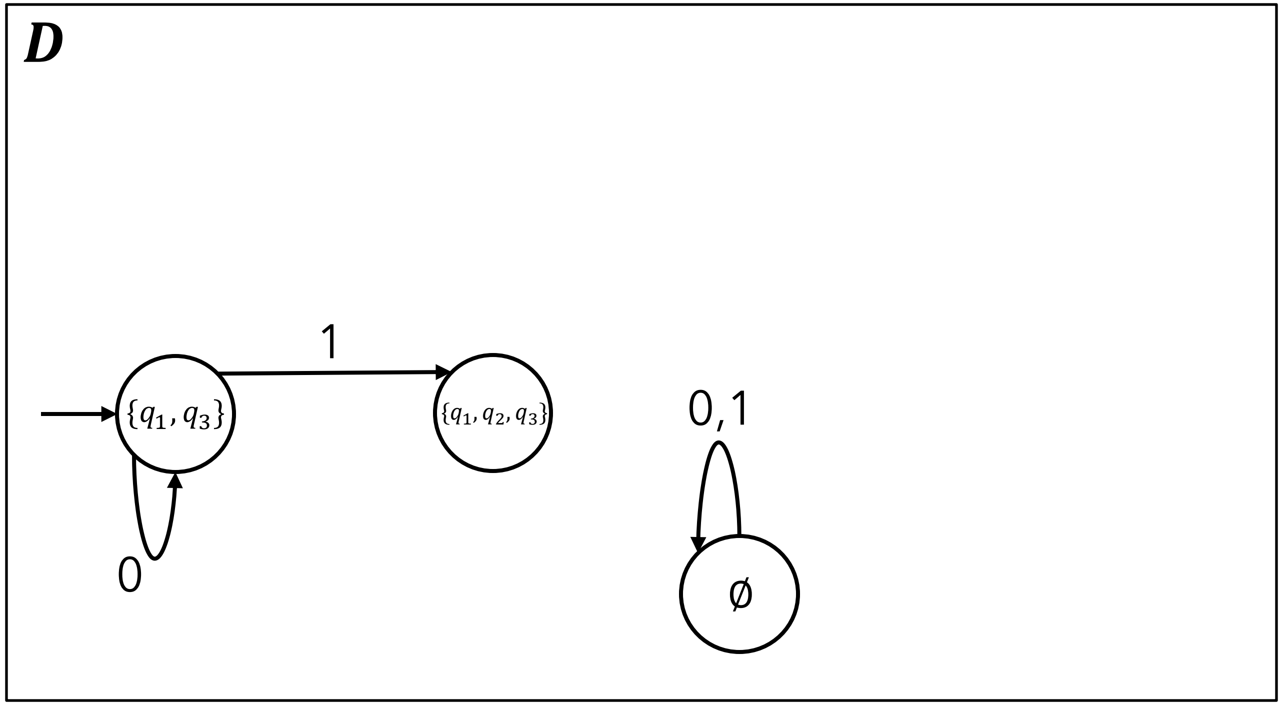

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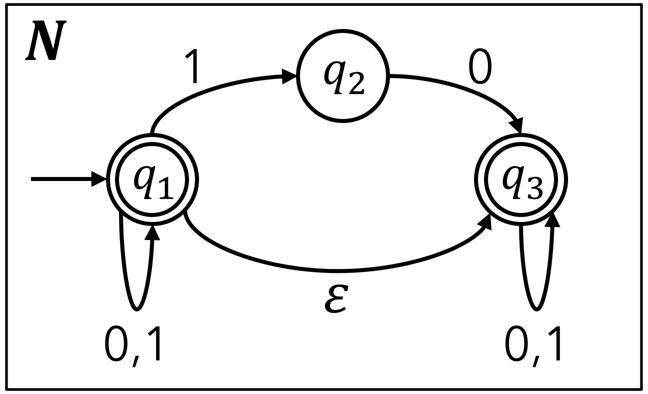
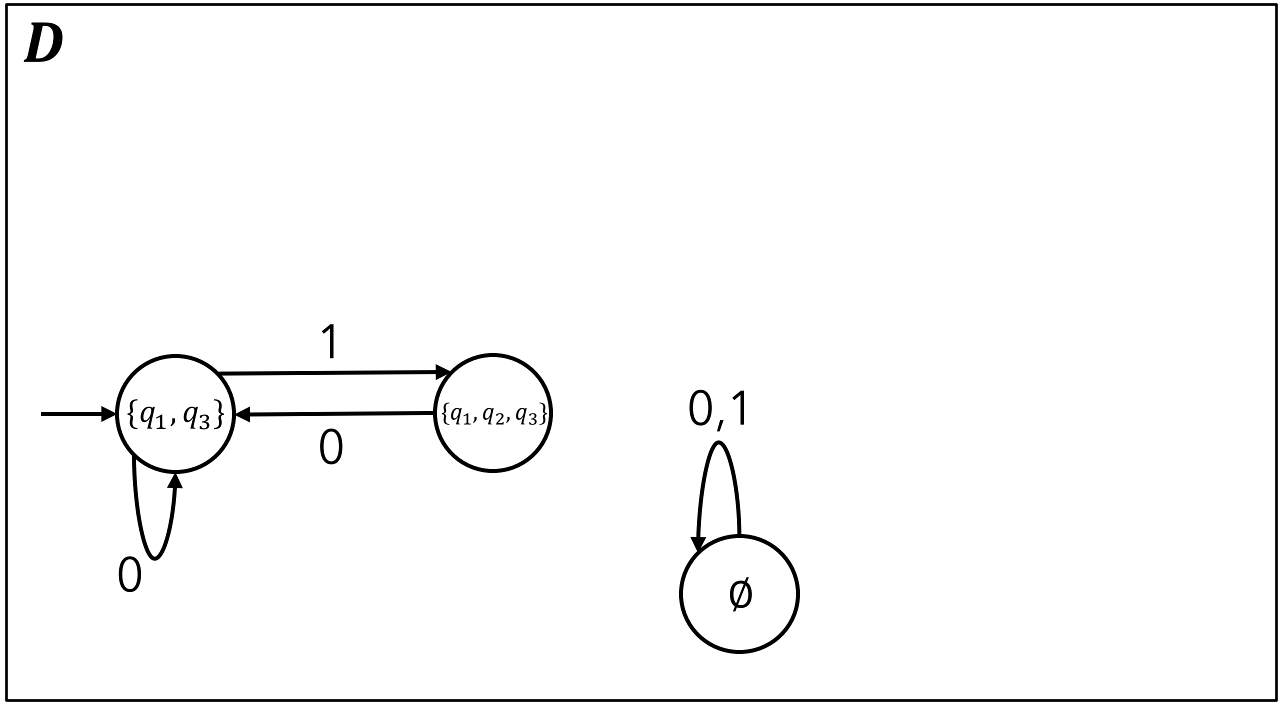
NFA to DFA Example



Transition table for DFA D

δ_D	**0**	**1**
$\emptyset$	$\emptyset$	$\emptyset$
$\{q_1\}$		
$\{q_2\}$		
$\{q_3\}$		
$\{q_1, q_2\}$		
→ $\{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$	$\{q_1, q_3\}$	



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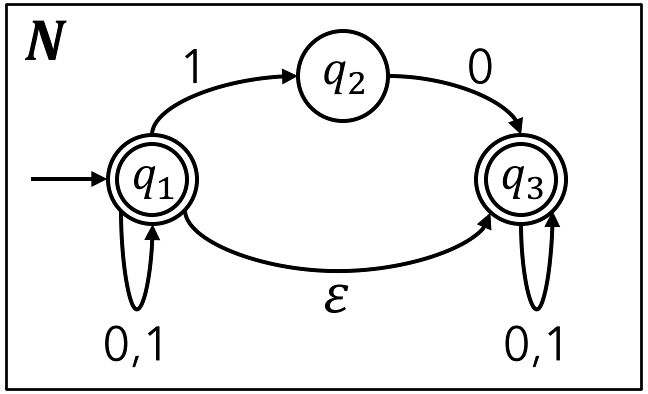
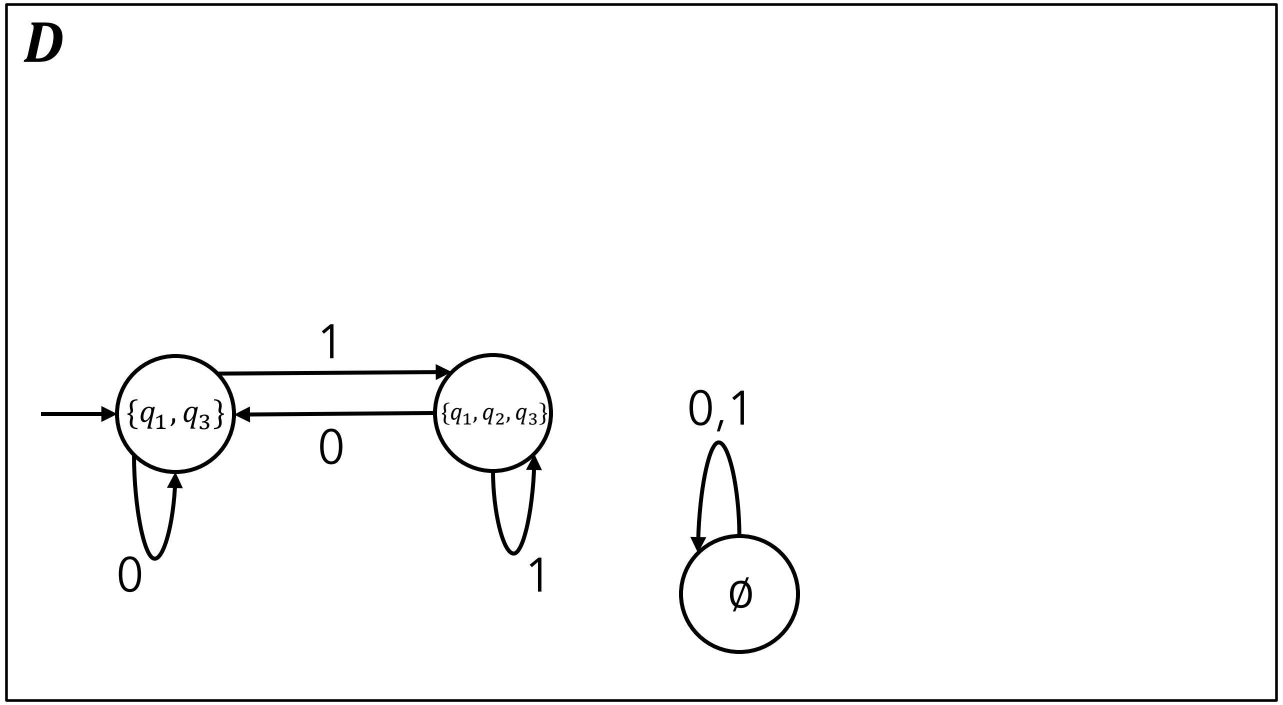
NFA to DFA Example



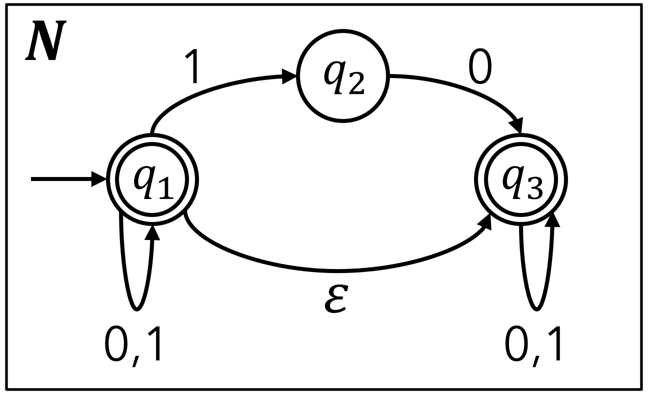
Transition table for DFA D

δ_D	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\{q_1\}$		
$\{q_2\}$		
$\{q_3\}$		
$\{q_1, q_2\}$		
→ $\{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$



$$\delta_D(S, a) := \{q \in Q_N \mid q \in E\big(\delta_N(s, a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



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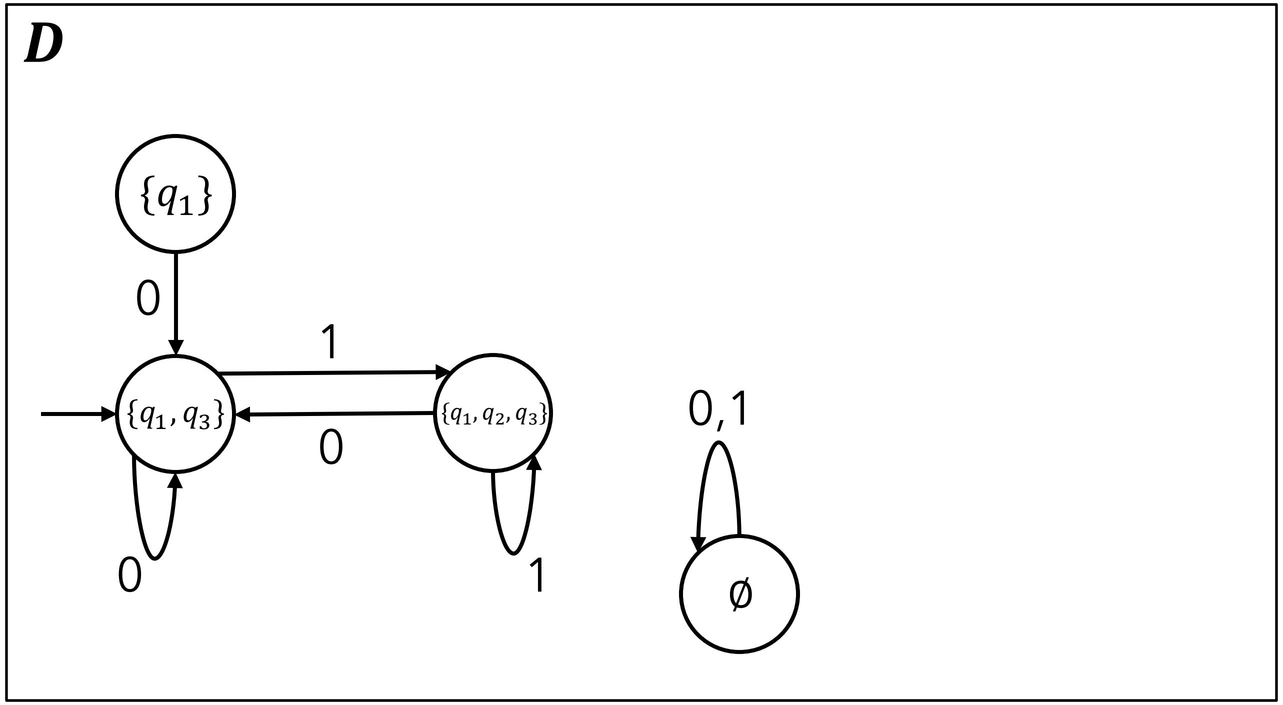
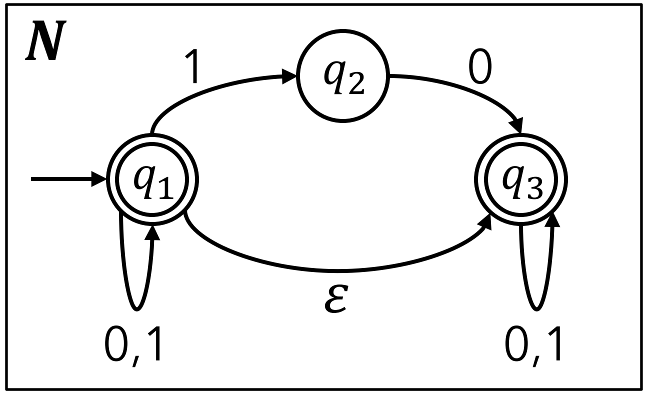
→ $\{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2, q_3\}$		
$\{q_1, q_2, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$



$$\delta_D(S, a) := \{q \in Q_N \mid q \in E\big(\delta_N(s, a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



Transition table for DFA D

δ_D	0	1
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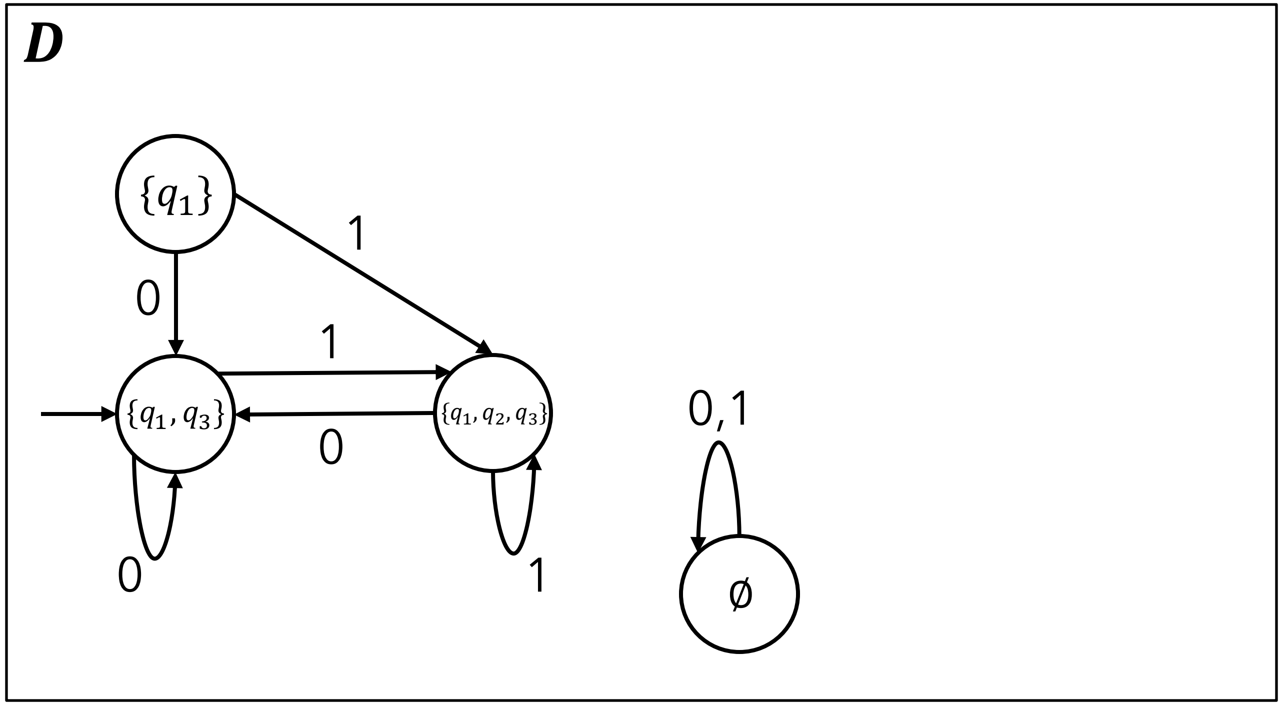
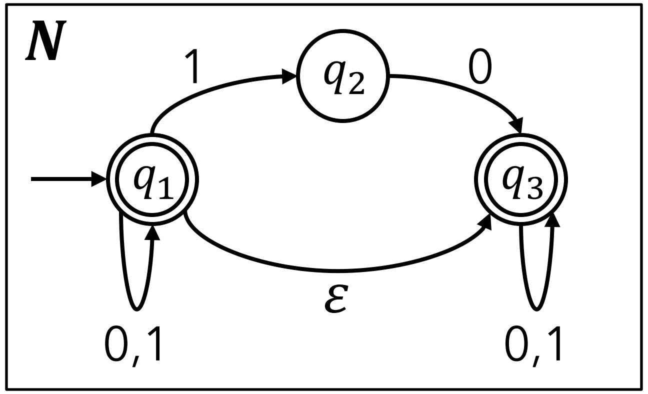


$$\delta_D(S, a) := \{q \in Q_N \mid q \in E\big(\delta_N(s, a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



Transition table for DFA ***D***

δ_D	**0**	**1**
$\emptyset$	$\emptyset$	$\emptyset$
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→ $\{q_1,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$
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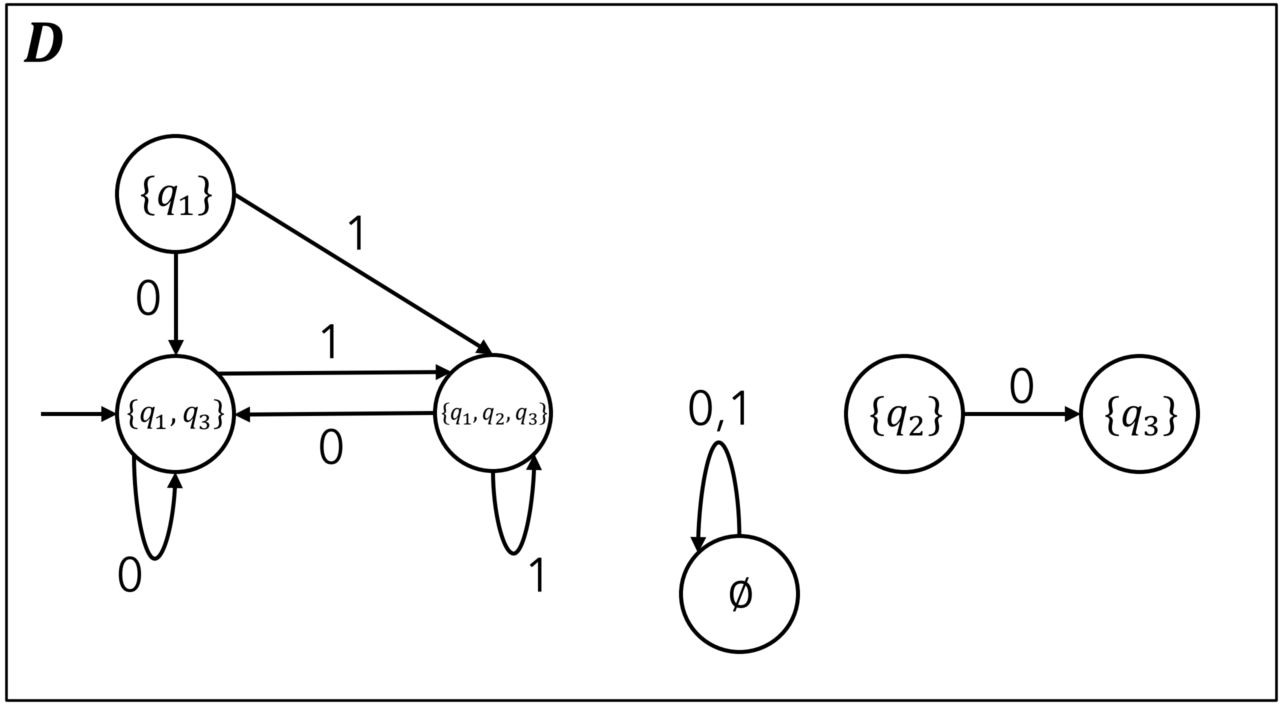
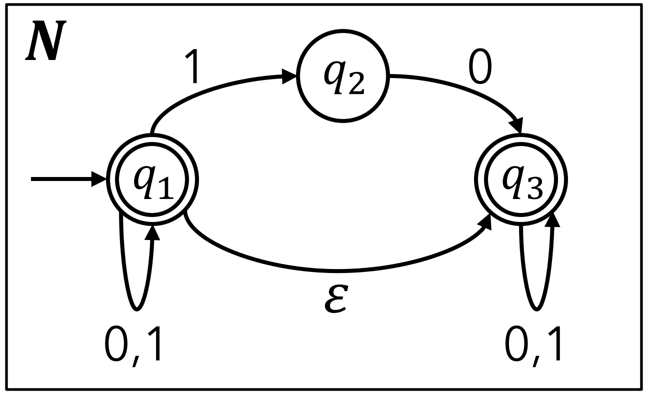


$$\delta_D(S,a) \coloneqq \{q \in Q_N \mid q \in E\big(\delta_N(s,a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



Transition table for DFA $\boldsymbol{D}$

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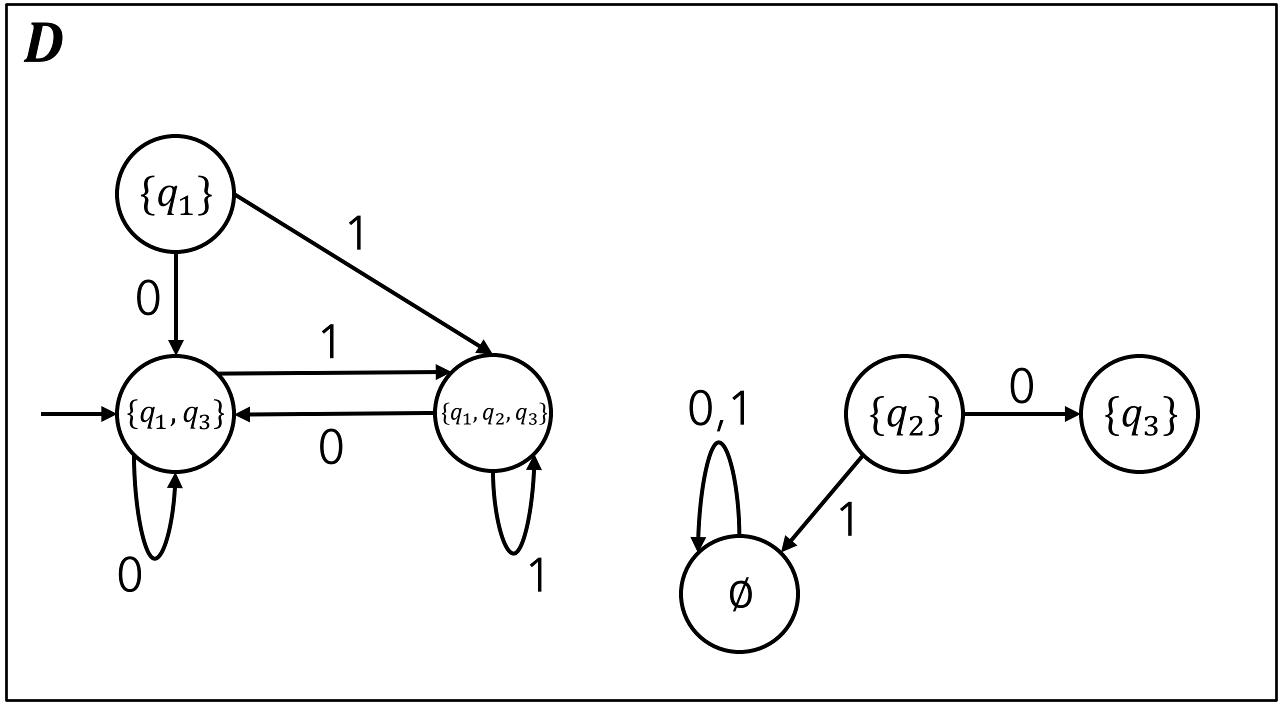
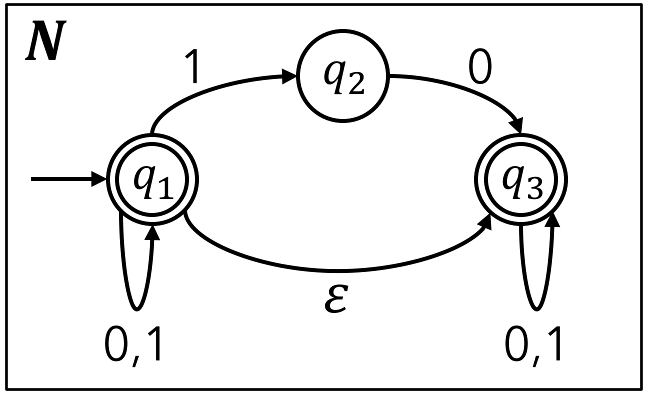
$\{q_2\}$	$\{q_3\}$	$\emptyset$
$\{q_3\}$		
$\{q_1,q_2\}$		
→ $\{q_1,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$
$\{q_2,q_3\}$		
$\{q_1,q_2,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$



$$\delta_D(S,a) \coloneqq \{q \in Q_N \mid q \in E\big(\delta_N(s,a)\big) \text{ for some } s \in S\}$$

NFA to DFA Example



Transition table for DFA ***D***

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$\emptyset$	$\emptyset$	$\emptyset$
$\{q_1\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2\}$	$\{q_3\}$	$\emptyset$
$\{q_3\}$	$\{q_3\}$	$\{q_3\}$
$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
→ $\{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
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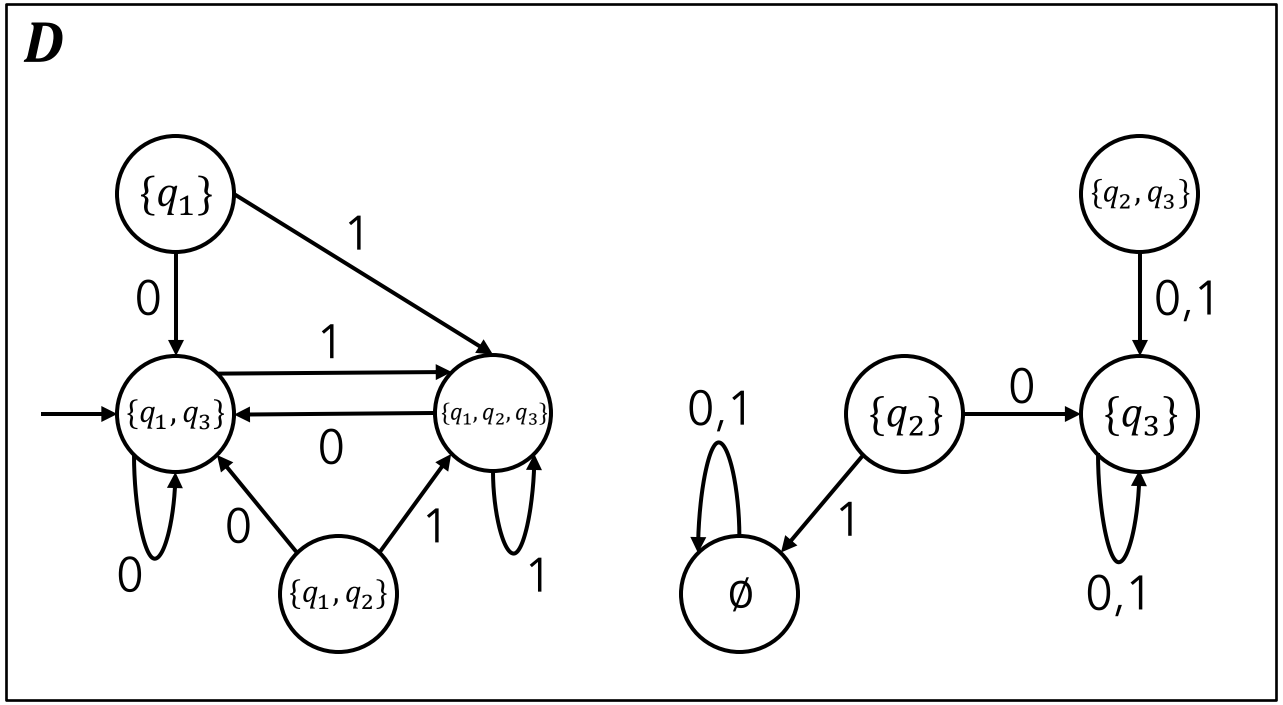
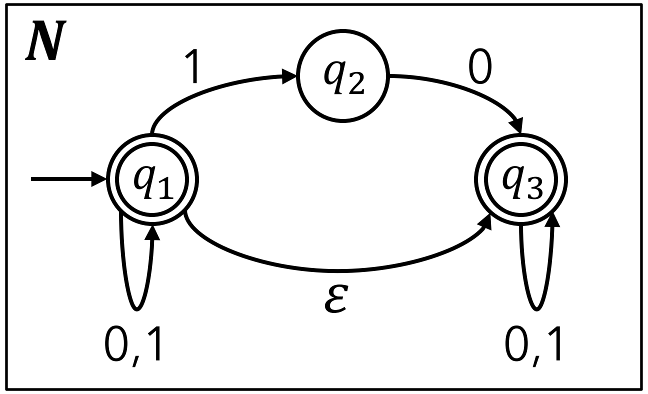


NFA to DFA Example



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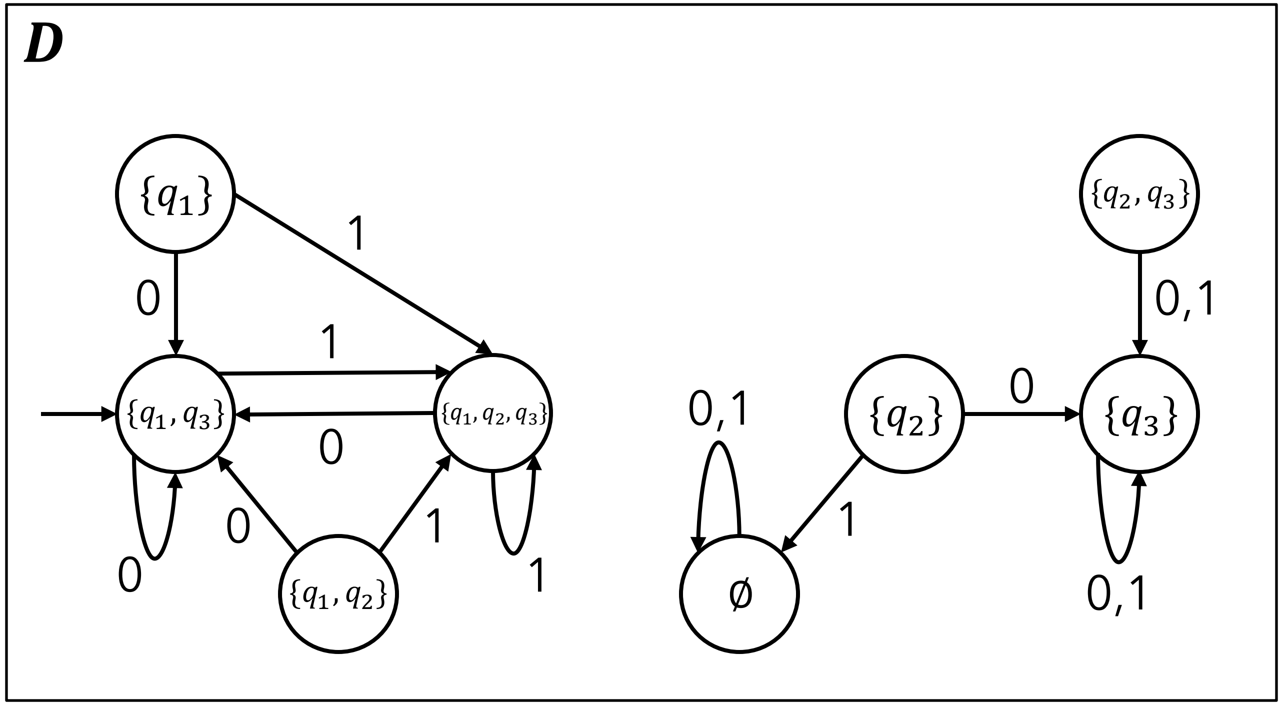
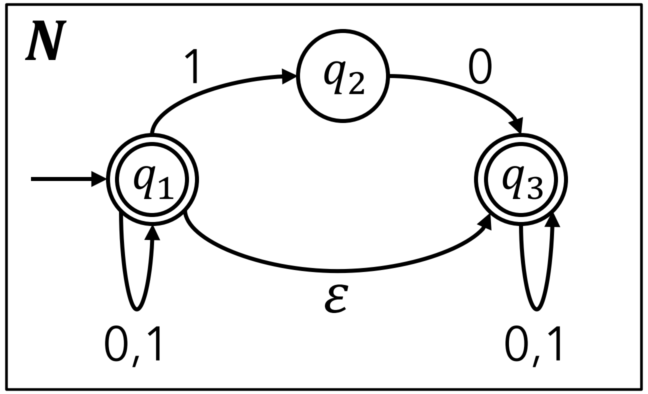
$\boldsymbol{\delta_D}$	**0**	**1**
$\emptyset$	$\emptyset$	$\emptyset$
$\{q_1\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$
$\{q_2\}$	$\{q_3\}$	$\emptyset$
$\{q_3\}$	$\{q_3\}$	$\{q_3\}$
$\{q_1,q_2\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$
→ $\{q_1,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$
$\{q_2,q_3\}$	$\{q_3\}$	$\{q_3\}$
$\{q_1,q_2,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3\}$



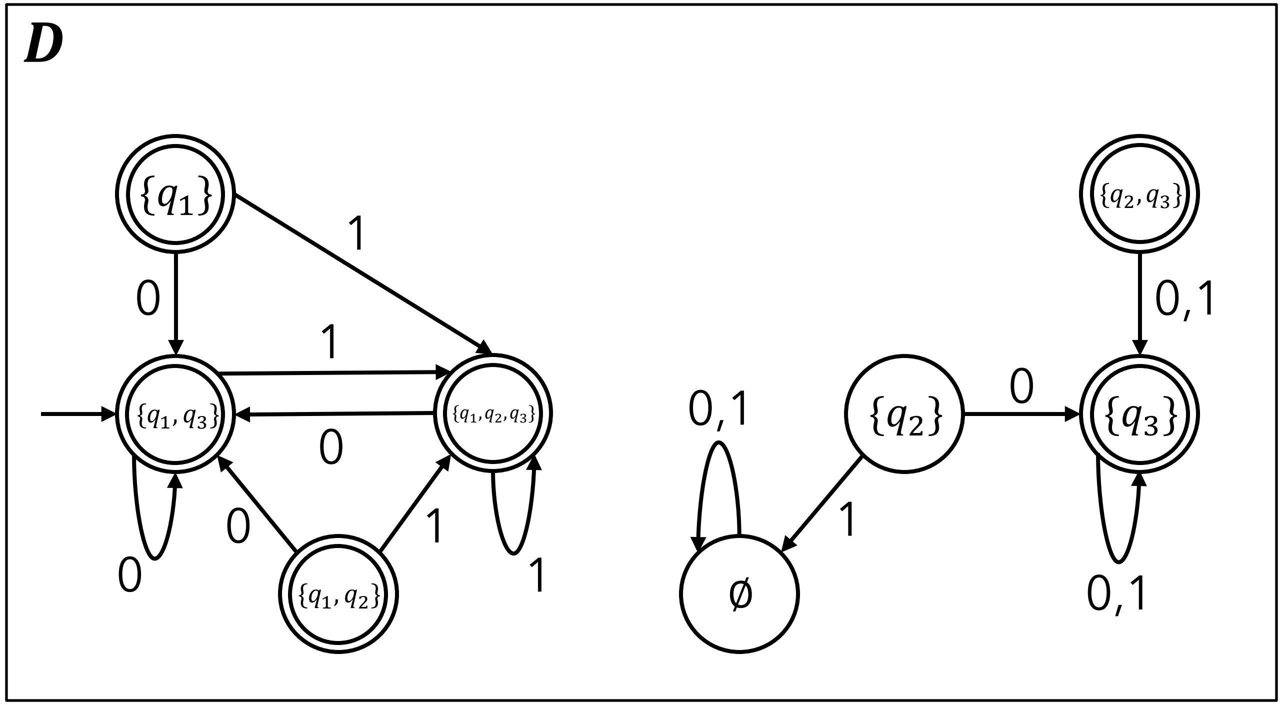
$$F_D \coloneqq \{S \in Q_D \mid \text{there exists a } q \in S \text{ with } q \in F_N\}$$

NFA to DFA Example



Transition table for DFA $\boldsymbol{D}$

$\boldsymbol{\delta_D}$	**0**	**1**
$\emptyset$	$\emptyset$	$\emptyset$
$\boldsymbol{\{q_1\}}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2\}$	$\{q_3\}$	$\emptyset$
$\boldsymbol{\{q_3\}}$	$\{q_3\}$	$\{q_3\}$
$\boldsymbol{\{q_1, q_2\}}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
→ $\boldsymbol{\{q_1, q_3\}}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\boldsymbol{\{q_2, q_3\}}$	$\{q_3\}$	$\{q_3\}$
$\boldsymbol{\{q_1, q_2, q_3\}}$	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$



Equivalence of NFAs and DFAs

- We have shown that:
  - For **every DFA** there exists **an equivalent NFA**
  - For **every NFA** there exists **an equivalent DFA**

- So, we know that DFAs and NFAs produce the **same set of languages**

- Therefore, the **languages recognized by NFAs** is exactly the set of **regular languages**

Regular Languages are Closed under Concatenation

Recall: Given languages $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ over alphabet $\boldsymbol{\Sigma}$, their **concatenation** denoted $\boldsymbol{L_1L_2}$ is defined as $\boldsymbol{L_1L_2} = \{\boldsymbol{w} \in \boldsymbol{\Sigma}^* \mid \boldsymbol{w} = \boldsymbol{xy} \text{ for some } \boldsymbol{x} \in \boldsymbol{L_1} \text{ and } \boldsymbol{y} \in \boldsymbol{L_2}\}$

- E.g. Let $\boldsymbol{L_1} = \{0, 10\}$ and $\boldsymbol{L_2} = \{0, 11\}$. Then $\boldsymbol{L_1L_2} = \{00, 011, 100, 1011\}$

Theorem: If $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are **regular languages** over alphabet Σ, then the language $\boldsymbol{L_1L_2}$ is a **regular language**

Proof: Since $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are regular languages, then there exist DFAs $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ where $\boldsymbol{L_1} = \boldsymbol{L(M_1)}$ and $\boldsymbol{L_2} = \boldsymbol{L(M_2)}$

We can create an NFA $\boldsymbol{M}$ that accepts the strings where the **first part** is accepted by $\boldsymbol{M_1}$ and the **second part** is accepted by $\boldsymbol{M_2}$.

Regular Languages are Closed under Concatenation

$\boldsymbol{M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)}$

$\boldsymbol{M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)}$

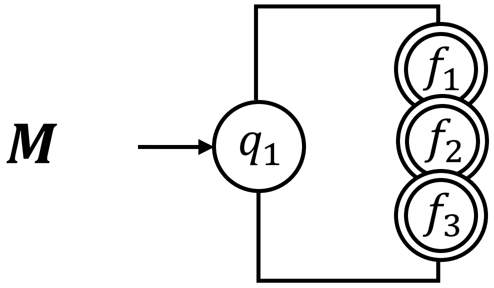
Create **NFA** $\boldsymbol{M}$ as follows:

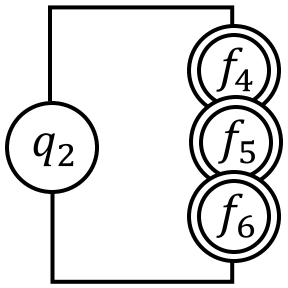
- $\boldsymbol{M}$ inherits **all states** from DFAs $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$

Regular Languages are Closed under Concatenation

$\boldsymbol{M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)}$

$\boldsymbol{M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)}$





Create **NFA** $\boldsymbol{M}$ as follows:

- $\boldsymbol{M}$ inherits **all states** from DFAs $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$
- The **start state** of $\boldsymbol{M}$ is the start state of $\boldsymbol{M_1}$

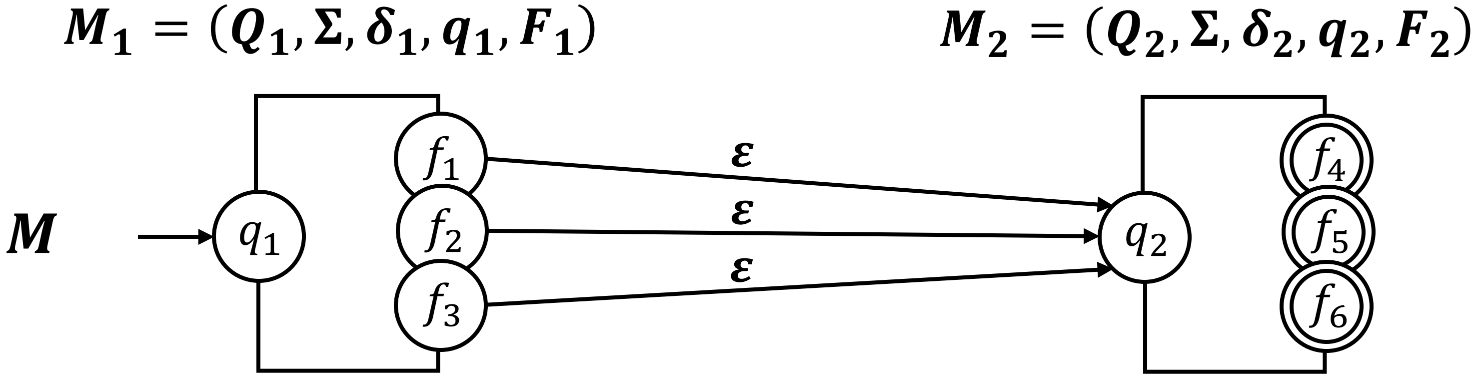
Regular Languages are Closed under Concatenation

$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$

Create **NFA** M as follows:

- M inherits **all states** from DFAs M_1 and M_2
- The **start state** of M is the start state of M_1
- The **accept states** of M are the accept states of M_2 (accept states of M_1 are no longer accept states)

Regular Languages are Closed under Concatenation



$M = (Q_1 \cup Q_2, \Sigma, \delta, q_1, F_2)$

Create **NFA** M as follows:

- M inherits **all states** from DFAs M_1 and M_2
- The **start state** of M is the start state of M_1
- The **accept states** of M are the accept states of M_2 (accept states of M_1 are no longer accept states)
- The **transitions** of M consist of:
    - All transitions of M_1 and all transitions of M_2 (remain the same)
    - **Add ε-transitions** between the previous **accept states in M_1** to the **start state of M_2**

Regular Language Closure

We have now shown that given **regular languages** $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$:

- $\boldsymbol{L_1} \cup \boldsymbol{L_2}$ is a regular language
- $\boldsymbol{L_1} \cap \boldsymbol{L_2}$ is a regular language
- $\boldsymbol{L_1 L_2}$ is a regular language

Regular languages are also closed under **complement** and **Kleene star**:

- $\overline{\boldsymbol{L_1}}$ is a regular language
- $\boldsymbol{L_1}^*$ is a regular language