ECE 260

EXAM 1

SOLUTIONS

(FALL 2022)

QUESTION 1

(a) $$f(z) = \frac{z^2+1}{z^5 - z^3} = \frac{(z+j)(z-j)}{z^3(z^2-1)} = \frac{(z+j)(z-j)}{z^3(z+1)(z-1)}$$

poles of f:

3rd order pole at 0

1st order pole at -1

1st order pole at 1

zeros of f:

1st order zero at j

1st order zero at $-j$

(b) Since f is rational, it is analytic everywhere except at its poles 0, 1, and -1.

QUESTION 2

```
function y = sum_positive(x)
    y = 0;
    for r = 1 : height(x)
        for c = 1 : width(x)
            if x(r, c) >= 0
                y = y + x(r, c);
            end
        end
    end
end
```

QUESTION 3

(a) A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following relationship holds:

$$\mathcal{H}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{H} x_1 + a_2 \mathcal{H} x_2$$

(b) Let $x' = a_1 x_1 + a_2 x_2$.

$$\mathcal{H}x'(t) = x'(t) + 3$$
$$= a_1 x_1(t) + a_2 x_2(t) + 3$$

$$a_1 \mathcal{H} x_1(t) + a_2 \mathcal{H} x_2(t) = a_1[x_1(t) + 3] + a_2[x_2(t) + 3]$$
$$= a_1 x_1(t) + a_2 x_2(t) + 3(a_1 + a_2)$$

Clearly, $\mathcal{H}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{H} x_1 + a_2 \mathcal{H} x_2$ does not hold for all x_1, x_2, a_1, a_2. In particular, this condition does not hold if $a_1 + a_2 \neq 1$.

Therefore, $\mathcal{H}$ is not linear.

QUESTION 4

$$X(t) = (t+2)[u(t+2)-u(t+1)] + t^2[u(t+1)-u(t-1)] + (1)[u(t-1)-u(t-\infty)]$$

$$= (t+2)[u(t+2)-u(t+1)] + t^2[u(t+1)-u(t-1)] + u(t-1)$$

$$= (t+2)u(t+2) + [-(t+2)+t^2]u(t+1) + [-t^2+1]u(t-1)$$

$$= (t+2)u(t+2) + [t^2-t-2]u(t+1) + [1-t^2]u(t-1)$$

QUESTION 5 (with typo on exam paper)

① $x(t) = 2 - t$ for $0 \leq t < 1$

② the function $v(t) = x(t) - 2$ is causal

③ the function $w(t) = x(t-1)$ is odd

From ②, we have

$$v(t) = 0 \text{ for all } t < 0 \Rightarrow$$

$$x(t) - 2 = 0 \text{ for all } t < 0 \Rightarrow$$

$$x(t) = 2 \text{ for all } t < 0$$

From ③, $x(t) = w(t+1)$

So, x is obtained by shifting w to the left by 1

Therefore, x has odd symmetry about -1.

This odd symmetry implies $x(-1) = 0$.

Since $x(-1)$ cannot be both 0 and 2 at the same time, no function x exists that satisfies the stated properties.

The function x is such that:
1) x(t) = 2 - t for 0 <= t < 1;
2) v(t) = x(t) - 2 is causal; and
3) w(t) = x(t+1) is odd.

This was the intended exam question (i.e., the question without the typo on the exam paper).

Note the "+1"

Answer

First, we consider the consequences of v being causal. From the fact that $v(t) = x(t) - 2$ is causal, we have

$$\begin{aligned} v(t) &= 0 \text{ for all } t < 0 \Rightarrow \\ x(t) - 2 &= 0 \text{ for all } t < 0 \Rightarrow \\ x(t) &= 2 \text{ for all } t < 0. \end{aligned}$$

Now, we consider the consequences of w being odd. Since $w(t) = x(t+1)$ we have

$$x(t) = w(t-1).$$

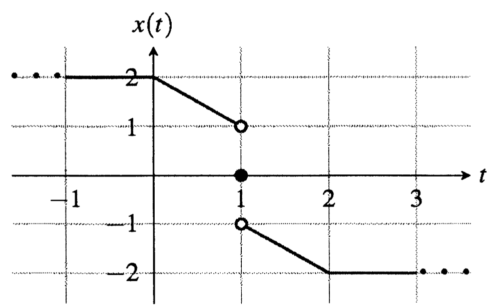
(i.e., x is w shifted to the right by 1). Thus, since w is odd and x is w shifted right by 1, x has odd symmetry about 1. Therefore, $x(1) = 0$. Next, we determine $x(t)$ for $1 < t \leq 2$. This can be deduced either graphically or algebraically. Since a graphical approach is easier, we will use this approach here. (An algebraic approach is presented at the end of this solution.) With a graphical approach, we can simply visualize the consequences of the symmetry in x from a graph of $x(t)$ for $0 \leq t < 1$. (See the part of the plot of $x(t)$ below for $0 \leq t < 1$, which is known from the information given in the problem statement.) This allows us to deduce that

$$x(t) = -t \text{ for } 1 < t \leq 2.$$

Combining the results from above, we conclude

$$x(t) = \begin{cases} 2 & t < 0 \\ 2 - t & 0 \leq t < 1 \\ 0 & t = 1 \\ -t & 1 < t \leq 2 \\ -2 & t > 2. \end{cases}$$

A plot of x is shown in the figure below.



REMARKS ON ALGEBRAIC APPROACH. As mentioned above, the formula for $x(t)$ for $1 < t \leq 2$ can also be deduced algebraically (instead of graphically). Now, we will perform this deduction using an algebraic approach. Since x has odd symmetry about 1, we know that

$$\begin{aligned} x(1+t) &= -x(1-t) \text{ for all } t \quad \Rightarrow \\ x(t) &= -x(1-(t-1)) \text{ for all } t \quad \Rightarrow \\ x(t) &= -x(2-t) \text{ for all } t. \end{aligned}$$

Substituting into the preceding equation for the case that $0 \leq t < 1$, we have

$$\begin{aligned} x(t) &= -x(2-t) \text{ for } 0 \leq t < 1 \quad \Rightarrow \\ x(t) &= -[2-(2-t)] \text{ for } 0 \leq 2-t < 1 \quad \Rightarrow \\ x(t) &= -2+2-t \text{ for } 0 \leq 2-t < 1 \quad \Rightarrow \\ x(t) &= -t \text{ for } 1 < t \leq 2. \end{aligned}$$

(Above, we used that fact that $0 \leq 2-t < 1 \Leftrightarrow 0 \leq 2-t$ and $2-t < 1 \Leftrightarrow t \leq 2$ and $1 < t \Leftrightarrow 1 < t \leq 2$.)

QUESTION 6

$$\boxed{\mathcal{H}x(t) = tDx(t)}$$

(a) $\mathcal{H}x_1(t) = tDx_1(t)$

$= t(6t)$

$= 6t^2$

$= 2(3t^2)$

$= 2x_1(t)$

$\therefore$ x_1 is an eigenfunction of $\mathcal{H}$ with an eigenvalue of 2

(b) $\mathcal{H}x_2(t) = tDx_2(t)$

$= t(0)$

$= 0$

$= 0\,x_2(t)$

$\therefore$ x_2 is an eigenfunction of $\mathcal{H}$ with an eigenvalue of 0.