4.2 Integral Control (Answers)

4.2.1 Closed-Loop Transfer Function with Integral Control

The control signal for integral control is: $u_m(t) = k_i \int_0^t (r(\tau) - \omega_m(\tau))\, d\tau$

In the Laplace domain: $U_m(s) = \frac{k_i}{s} (R(s) - \Omega_m(s))$

Substitute $U_m(s)$ into the open-loop transfer function: $\Omega_m(s) = \frac{K}{\tau s+1} U_m(s)$

This leads to: $\Omega_m(s) = \frac{K}{\tau s+1} \frac{k_i}{s} (R(s) - \Omega_m(s))$

Rearranging for $\Omega_m(s)$: $\Omega_m(s)(\tau s^2 + s + Kk_i) = Kk_i R(s)$

Thus, the closed-loop transfer function $G_I(s)$ is: $G_I(s) = \frac{\Omega_m(s)}{R(s)} = \frac{Kk_i}{\tau s^2+s+Kk_i}$

4.2.2 Location of Poles as a Function of k_i

The characteristic equation is: $\tau s^2 + s + Kk_i = 0$

The poles are the roots of this quadratic equation:

$$s = \frac{-1 \pm \sqrt{1 - 4\tau K k_i}}{2\tau}$$

As k_i increases, the poles shift. For small k_i, the system has a slower response. As k_i increases, the system responds faster, but increasing k_i too much may lead to oscillations or instability if the discriminant becomes negative (complex poles).

4.2.3 Steady-State Value Using Final Value Theorem

For a step input $r(t) = r_0$, the Laplace transform is: $R(s) = \frac{r_0}{s}$

Using the closed-loop transfer function: $\Omega_m(s) = \frac{Kk_i}{\tau s^2+s+Kk_i} \cdot \frac{r_0}{s}$

Applying the **Final Value Theorem**:

$$\omega_m(\infty) = \lim_{s \to 0} s \cdot \Omega_m(s)$$

At $s = 0$: $\omega_m(\infty) = r_0$. Thus, with integral control, the steady-state output matches the input exactly, eliminating steady-state error.
