5A **6.1** Using the Fourier transform analysis equation, find the Fourier transform X of each function x below.
(a) $x(t) = \operatorname{rect}(t - t_0)$, where t_0 is a constant;
(b) $x(t) = e^{-4t}u(t-1)$;
(c) $x(t) = 3[u(t) - u(t-2)]$; and
(d) $x(t) = e^{-|t|}$.

5A **Answer (c).**

Let X denote the Fourier transform of x. From the Fourier transform analysis equation, we can write

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} 3[u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{-\infty}^{\infty} [u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{0}^{2} e^{-j\omega t}dt \\
&= 3\left[\tfrac{1}{-j\omega}e^{-j\omega t}\right]\Big|_0^2 \\
&= \tfrac{3}{-j\omega}\left[e^{-j\omega t}\right]\Big|_0^2 \\
&= \tfrac{j3}{\omega}\left[e^{-j2\omega} - 1\right] \\
&= \tfrac{j3}{\omega}[e^{-j\omega}][e^{-j\omega} - e^{j\omega}] \\
&= \tfrac{j3}{\omega}e^{-j\omega}[-2j\sin\omega] \\
&= \tfrac{6}{\omega}e^{-j\omega}\sin\omega \\
&= 6e^{-j\omega}\operatorname{sinc}\omega.
\end{aligned}$$

5A **Answer (d).**

Let X denote the Fourier transform of x. From the Fourier transform analysis equation, we have

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{-|t|}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-|t|}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t}dt + \int_{0}^{\infty} e^{(-1-j\omega)t}dt \\
&= \tfrac{1}{1-j\omega}\left[e^{(1-j\omega)t}\right]\Big|_{-\infty}^{0} - \tfrac{1}{1+j\omega}\left[e^{(-1-j\omega)t}\right]\Big|_{0}^{\infty} \\
&= \tfrac{1}{1-j\omega}[1-0] - \tfrac{1}{1+j\omega}[0-1] \\
&= \tfrac{1}{1-j\omega} + \tfrac{1}{1+j\omega} \\
&= \frac{1+j\omega+1-j\omega}{(1+j\omega)(1-j\omega)} \\
&= \frac{2}{1+\omega^2}.
\end{aligned}$$

5A **6.3** Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform X of each function x below.
(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t}u(t+2)$;
(c) $x(t) = \cos(t)u(t)$;
(d) $x(t) = 6[u(t) - u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\,\mathrm{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = x_1 * x_2(t)$, where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = te^{-3t}u(t)$; and
(j) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT)$, where a is a complex constant satisfying $|a| < 1$ and T is a strictly-positive real constant; (Hint: Recall the formula for the sum of an infinite geometric sequence (i.e., (F.6)).)

5A **Answer (c).**

We are asked to find the Fourier transform X of

$$x(t) = \cos(t)u(t).$$

We begin by rewriting $x(t)$ as

$$x(t) = v_1(t)v_2(t),$$

where

$$v_1(t) = \cos t \quad \text{and}$$
$$v_2(t) = u(t).$$

Taking the Fourier transform of both sides of each of the above equations yields

$$X(\omega) = \tfrac{1}{2\pi} V_1 * V_2(\omega),$$
$$V_1(\omega) = \pi[\delta(\omega-1) + \delta(\omega+1)], \quad \text{and}$$
$$V_2(\omega) = \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

Combining the above results, we obtain

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2\pi} V_1 * V_2(\omega) \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} V_1(\lambda)V_2(\omega-\lambda)d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\lambda-1)+\delta(\lambda+1)\right]\left[\pi\delta(\omega-\lambda)+\tfrac{1}{j(\omega-\lambda)}\right]d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\pi\delta(\lambda-1)\delta(\omega-\lambda)+\delta(\lambda-1)\tfrac{1}{j(\omega-\lambda)}+\pi\delta(\lambda+1)\delta(\omega-\lambda)+\delta(\lambda+1)\tfrac{1}{j(\omega-\lambda)}\right]d\lambda \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1)+\tfrac{1}{j(\omega-1)}+\pi\delta(\omega+1)+\tfrac{1}{j(\omega+1)}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1)+\pi\delta(\omega+1)-\tfrac{j}{\omega-1}-\tfrac{j}{\omega+1}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1)+\pi\delta(\omega+1)+\tfrac{-j(\omega-1)-j(\omega+1)}{\omega^2-1}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1)+\pi\delta(\omega+1)-\tfrac{j2\omega}{\omega^2-1}\right] \\
&= \tfrac{\pi}{2}\left[\delta(\omega-1)+\delta(\omega+1)\right]-\frac{j\omega}{\omega^2-1}.
\end{aligned}$$

5A **Answer (d).**

We are asked to find the Fourier transform X of

$$x(t) = 6[u(t) - u(t-3)].$$

We begin by rewriting $x(t)$ as

$$x(t) = 6v_3(t),$$

where

$$\begin{aligned} v_3(t) &= v_2(t/3), \\ v_2(t) &= v_1(t - \tfrac{1}{2}), \quad \text{and} \\ v_1(t) &= \operatorname{rect}(t). \end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} X(\omega) &= 6V_3(\omega), \\ V_3(\omega) &= 3V_2(3\omega), \\ V_2(\omega) &= e^{-j\omega/2}V_1(\omega), \quad \text{and} \\ V_1(\omega) &= \operatorname{sinc}(\omega/2). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(\omega) &= 6V_3(\omega) \\ &= 6(3)V_2(3\omega) \\ &= 18V_2(3\omega) \\ &= 18e^{-j3\omega/2}V_1(3\omega) \\ &= 18e^{-j3\omega/2}\operatorname{sinc}\left(\tfrac{3\omega}{2}\right). \end{aligned}$$

Alternatively, we can restate this result in a slightly different form (i.e., in terms of complex exponentials) as follows:

$$\begin{aligned} X(\omega) &= 18e^{-j3\omega/2}\operatorname{sinc}\left(\tfrac{3\omega}{2}\right) \\ &= 18e^{-j3\omega/2}\tfrac{2}{3\omega}\left[\tfrac{1}{2j}\left[e^{j3\omega/2} - e^{-j3\omega/2}\right]\right] \\ &= \tfrac{6}{j\omega}[1 - e^{-j3\omega}]. \end{aligned}$$

ALTERNATIVE SOLUTION. We have

$$\begin{aligned} X(\omega) &= 6\left[\mathcal{F}\{u(t)\}(\omega) - \mathcal{F}\{u(t-3)\}(\omega)\right] \\ &= 6\left(\pi\delta(\omega) + \tfrac{1}{j\omega} - e^{-j3\omega}(\pi\delta(\omega) + \tfrac{1}{j\omega})\right) \\ &= 6\left(\pi\delta(\omega) + \tfrac{1}{j\omega} - \pi\delta(\omega)e^{-j3\omega} - \tfrac{1}{j\omega}e^{-j3\omega}\right) \\ &= 6\left(\pi\delta(\omega) + \tfrac{1}{j\omega} - \pi\delta(\omega) - \tfrac{1}{j\omega}e^{-j3\omega}\right) \\ &= \tfrac{6}{j\omega}(1 - e^{-j3\omega}) \\ &= \tfrac{6}{j\omega}e^{-j3\omega/2}(e^{j3\omega/2} - e^{-j3\omega/2}) \\ &= \tfrac{6}{j\omega}e^{-j3\omega/2}(2j)\sin(3\omega/2) \\ &= \tfrac{12}{\omega}e^{-j3\omega/2}\sin(3\omega/2) \\ &= \tfrac{3\omega}{2}\tfrac{12}{\omega}e^{-j3\omega/2}(\tfrac{3\omega}{2})^{-1}\sin(3\omega/2) \\ &= 18e^{-j3\omega/2}\operatorname{sinc}(3\omega/2). \end{aligned}$$

5A **Answer (e).**

We are asked to find the Fourier transform X of

$$x(t) = 1/t.$$

From a table of Fourier transforms, we have

$$\operatorname{sgn} t \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{2}{j\omega}.$$

From this transform pair, we can use the duality property of the Fourier transform to deduce

$$\begin{aligned}\mathcal{F}\{\tfrac{2}{jt}\}(\omega) &= 2\pi \operatorname{sgn}(-\omega) \\ &= -2\pi \operatorname{sgn} \omega.\end{aligned}$$

Using this result and the linearity property of the Fourier transform, we can write

$$\begin{aligned}X(\omega) &= \mathcal{F}\{1/t\}(\omega) \\ &= \tfrac{j}{2}\mathcal{F}\{\tfrac{2}{jt}\}(\omega) \\ &= \tfrac{j}{2}[-2\pi \operatorname{sgn} \omega] \\ &= -j\pi \operatorname{sgn} \omega.\end{aligned}$$

5A **Answer (f).**

We are asked to find the Fourier transform X of

$$x(t) = t \operatorname{rect}(2t).$$

We begin by rewriting $x(t)$ as

$$x(t) = t v_2(t),$$

where

$$\begin{aligned}v_2(t) &= v_1(2t) \quad \text{and} \\ v_1(t) &= \operatorname{rect}(t).\end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations, we obtain

$$\begin{aligned}V_1(\omega) &= \operatorname{sinc}(\omega/2), \\ V_2(\omega) &= \tfrac{1}{2}V_1(\tfrac{\omega}{2}), \quad \text{and} \\ X(\omega) &= j\tfrac{d}{d\omega}V_2(\omega).\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}X(\omega) &= j\tfrac{d}{d\omega}V_2(\omega) \\ &= j\tfrac{d}{d\omega}[\tfrac{1}{2}V_1(\tfrac{\omega}{2})] \\ &= \tfrac{j}{2}\tfrac{d}{d\omega}V_1(\tfrac{\omega}{2}) \\ &= \tfrac{j}{2}\tfrac{d}{d\omega}\operatorname{sinc}\left(\tfrac{\omega}{4}\right) \\ &= \tfrac{j}{2}\left[\frac{\frac{\omega}{4}(\frac{1}{4}\cos\left(\frac{\omega}{4}\right)) - \frac{1}{4}\sin\left(\frac{\omega}{4}\right)}{\omega^2/16}\right] \\ &= \tfrac{j}{2}\left[\frac{16(\frac{\omega}{16}\cos\left(\frac{\omega}{4}\right) - \frac{1}{4}\sin\left(\frac{\omega}{4}\right))}{\omega^2}\right] \\ &= \tfrac{j}{2}\left[\tfrac{1}{\omega}\cos\left(\tfrac{\omega}{4}\right) - \tfrac{4}{\omega^2}\sin\left(\tfrac{\omega}{4}\right)\right] \\ &= \tfrac{j}{2\omega}\cos\left(\tfrac{\omega}{4}\right) - \tfrac{j2}{\omega^2}\sin\left(\tfrac{\omega}{4}\right).\end{aligned}$$

5A Answer (g).

We are asked to find the Fourier transform X of

$$x(t) = e^{-j3t}\sin(5t-2).$$

We begin by rewriting $x(t)$ as

$$x(t) = e^{-j3t}v_3(t),$$

where

$$\begin{gathered} v_3(t) = v_2(5t), \\ v_2(t) = v_1(t-2), \quad \text{and} \\ v_1(t) = \sin t. \end{gathered}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{gathered} V_1(\omega) = \tfrac{\pi}{j}\left[\delta(\omega-1)-\delta(\omega+1)\right], \\ V_2(\omega) = e^{-j2\omega}V_1(\omega), \\ V_3(\omega) = \tfrac{1}{5}V_2(\tfrac{\omega}{5}), \quad \text{and} \\ X(\omega) = V_3(\omega+3). \end{gathered}$$

Combining the above results, we obtain

$$\begin{aligned} X(\omega) &= V_3(\omega+3) \\ &= \frac{1}{5}V_2\left(\frac{\omega+3}{5}\right) \\ &= \frac{1}{5}e^{-j2(\omega+3)/5}V_1\left(\frac{\omega+3}{5}\right) \\ &= \frac{\pi}{j5}e^{-j2(\omega+3)/5}\left[\delta\left(\frac{\omega+3}{5}-1\right)-\delta\left(\frac{\omega+3}{5}+1\right)\right] \\ &= -\frac{j\pi}{5}e^{-j2(\omega+3)/5}\left[\delta\left(\frac{\omega-2}{5}\right)-\delta\left(\frac{\omega+8}{5}\right)\right] \\ &= -\tfrac{j\pi}{5}e^{-j2(\omega+3)/5}\left[5\delta(\omega-2)-5\delta(\omega+8)\right] \\ &= j\pi e^{-j2(\omega+3)/5}\left[\delta(\omega+8)-\delta(\omega-2)\right] \\ &= j\pi\left[e^{-j2(\omega+3)/5}\delta(\omega+8)-e^{-j2(\omega+3)/5}\delta(\omega-2)\right] \\ &= j\pi\left(\left[e^{-j2(\omega+3)/5}\right]\Big|_{\omega=-8}\delta(\omega+8)-\left[e^{-j2(\omega+3)/5}\right]\Big|_{\omega=2}\delta(\omega-2)\right) \\ &= j\pi\left[e^{j2}\delta(\omega+8)-e^{-j2}\delta(\omega-2)\right]. \end{aligned}$$

(In the above simplification, we used the fact that $\delta(at) = \frac{1}{|a|}\delta(t)$.)

5A **6.4** For each function y given below, find the Fourier transform Y of y in terms of the Fourier transform X of x.
(a) $y(t) = x(at-b)$, where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \mathcal{D}(x*x)(t)$, where $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = tx(2t-1)$;
(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$; and
(h) $y(t) = (\mathcal{D}x)*x_1(t)$, where $x_1(t) = e^{-jt}x(t)$ and $\mathcal{D}$ denotes the derivative operator.

5A **Answer (a).**

We are asked to find the Fourier transform Y of

$$y(t) = x(at-b), \quad \text{where } a,b \in \mathbb{R} \text{ and } a \neq 0.$$

We rewrite $y(t)$ as

$$y(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|}V_1\left(\tfrac{\omega}{a}\right) \quad \text{and} \\ V_1(\omega) &= e^{-j\omega b}X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|}V_1\left(\tfrac{\omega}{a}\right) \\ &= \tfrac{1}{|a|}e^{-j(\omega/a)b}X(\omega/a) \\ &= \tfrac{1}{|a|}e^{-jb\omega/a}X(\omega/a). \end{aligned}$$

5A **Answer (b).**

We are asked to find the Fourier transform Y of

$$y(t) = \int_{-\infty}^{2t} x(\tau)d\tau.$$

We rewrite $y(t)$ as

$$y(t) = v_1(2t)$$

where

$$v_1(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{v_1(2t)\}(\omega) \\ &= \tfrac{1}{2}V_1(\tfrac{\omega}{2}) \quad \text{and} \\ V_1(\omega) &= \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\}(\omega) \\ &= \tfrac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2}V_1(\tfrac{\omega}{2}) \\ &= \tfrac{1}{2}\left(\frac{1}{j(\omega/2)}X(\tfrac{\omega}{2}) + \pi X(0)\delta(\tfrac{\omega}{2})\right) \\ &= \tfrac{1}{j\omega}X(\tfrac{\omega}{2}) + \tfrac{\pi}{2}X(0)\delta(\tfrac{\omega}{2}). \end{aligned}$$

5A **Answer (c).**

We are asked to find the Fourier transform Y of

$$y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau.$$

We rewrite $y(t)$ as

$$y(t) = \int_{-\infty}^{t} v_1(\tau)d\tau$$

where

$$v_1(t) = x^2(t).$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned} V_1(\omega) &= \tfrac{1}{2\pi}X * X(\omega), \quad \text{and} \\ Y(\omega) &= \tfrac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega). \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} Y(\omega) &= \tfrac{1}{j\omega}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda\right] + \pi\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda\right]\delta(\omega) \\ &= \tfrac{1}{j2\pi\omega}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda + \tfrac{1}{2}\delta(\omega)\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda. \end{aligned}$$

5A **Answer (d).**

We are asked to find the Fourier transform Y of

$$y(t) = \mathcal{D}(x*x)(t), \quad \text{where } \mathcal{D} \text{ denotes the derivative operator.}$$

We rewrite $y(t)$ as

$$y(t) = \tfrac{d}{dt}v_1(t)$$

where

$$v_1(t) = x*x(t).$$

Taking the Fourier transform of both sides of these equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{\tfrac{d}{dt}v_1(t)\}(\omega) \\ &= j\omega V_1(\omega) \quad \text{and} \\ V_1(\omega) &= \mathcal{F}\{x * x\}(\omega) \\ &= X^2(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= j\omega V_1(\omega) \\ &= j\omega X^2(\omega). \end{aligned}$$

$\boxed{\text{5A}}$ **Answer (e).**

We are asked to find the Fourier transform Y of

$$y(t) = tx(2t-1).$$

We rewrite $y(t)$ as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t) \quad \text{and} \\ v_2(t) &= x(t-1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{tv_1(t)\}(\omega) \\ &= j\tfrac{d}{d\omega}V_1(\omega), \\ V_1(\omega) &= \mathcal{F}\{v_2(2t)\}(\omega) \\ &= \tfrac{1}{2}V_2(\tfrac{\omega}{2}), \quad \text{and} \\ V_2(\omega) &= \mathcal{F}\{x(t-1)\}(\omega) \\ &= e^{-j\omega}X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= j\tfrac{d}{d\omega}V_1(\omega) \\ &= j\tfrac{d}{d\omega}\left[(\tfrac{1}{2})V_2(\tfrac{\omega}{2})\right] \\ &= \tfrac{j}{2}\left[\tfrac{d}{d\omega}e^{-j\omega/2}X(\tfrac{\omega}{2})\right]. \end{aligned}$$

ALTERNATE SOLUTION. In what follows, we use the prime symbol to denote derivative (i.e., f' denotes the derivative of f). We can rewrite $y(t)$ as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t), \text{ and} \\ v_2(t) &= x(t-1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations, we obtain

$$Y(\omega) = jV_1'(\omega),$$
$$V_1(\omega) = \tfrac{1}{2}V_2(\omega/2), \text{ and}$$
$$V_2(\omega) = e^{-j\omega}X(\omega).$$

In anticipation of what is to come, we compute the quantities:

$$V_1'(\omega) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)V_2'(\omega/2) = \tfrac{1}{4}V_2'(\omega/2) \text{ and}$$
$$V_2'(\omega) = -je^{-j\omega}X(\omega) + X'(\omega)e^{-j\omega}.$$

Combining the above equations, we have

$$\begin{aligned} Y(\omega) &= jV_1'(\omega) \\ &= j\tfrac{1}{4}V_2'(\omega/2) \\ &= \tfrac{j}{4}\left[-je^{-j\omega/2}X(\omega/2) + e^{-j\omega/2}X'(\omega/2)\right]. \end{aligned}$$

5A Answer (f).

We are asked to find the Fourier transform Y of

$$y(t) = e^{j2t}x(t-1).$$

We begin by rewriting $y(t)$ as

$$y(t) = e^{j2t}v_1(t)$$

where

$$v_1(t) = x(t-1).$$

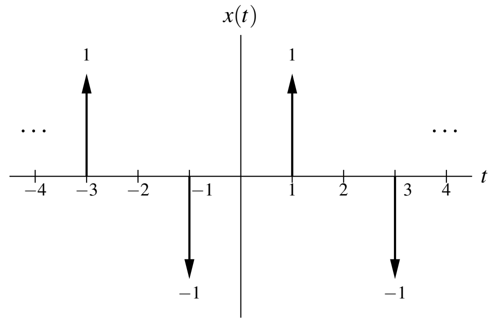
Taking the Fourier transform of both sides of the above equations yields

$$V_1(\omega) = e^{-j\omega}X(\omega) \quad \text{and}$$
$$Y(\omega) = V_1(\omega-2).$$

Combining the above results, we have

$$\begin{aligned} Y(\omega) &= V_1(\omega-2) \\ &= e^{-j(\omega-2)}X(\omega-2). \end{aligned}$$

5A **6.5** Find the Fourier transform X of each periodic function x shown below.



(a)

5A **Answer (a).**

The frequency ω_0 is given by $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$. Consider the period of $x(t)$ for $-2 \leq t < 2$. Let us denote this single period as $x_T(t)$. We have

$$x_T(t) = -\delta(t+1) + \delta(t-1).$$

Taking the Fourier transform of $x_T(t)$, we obtain

$$\begin{aligned} X_T(\omega) &= \mathcal{F}\{\delta(t-1) - \delta(t+1)\}(\omega) \\ &= \mathcal{F}\{\delta(t-1)\}(\omega) - \mathcal{F}\{\delta(t+1)\}(\omega) \\ &= e^{-j\omega} - e^{j\omega} \\ &= -[e^{j\omega} - e^{-j\omega}] \\ &= -2j\sin\omega. \end{aligned}$$

Using the formula for the Fourier transform of a periodic signal, we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}x(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \tfrac{\pi}{2}[-2j\sin\left(k\tfrac{\pi}{2}\right)]\delta(\omega - k\tfrac{\pi}{2}) \\ &= \sum_{k=-\infty}^{\infty} -j\pi[\sin\left(\tfrac{k\pi}{2}\right)]\delta(\omega - \tfrac{k\pi}{2}). \end{aligned}$$

5A **6.10** For each function x given below, compute the frequency spectrum of x, and find and plot the corresponding magnitude and phase spectra.
(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and
(b) $x(t) = \operatorname{sinc}\left(\frac{t-1}{200}\right)$.

5A **Answer (a).**

Taking the Fourier transform of x, we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}\{e^{-at}u(t)\}(\omega) \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

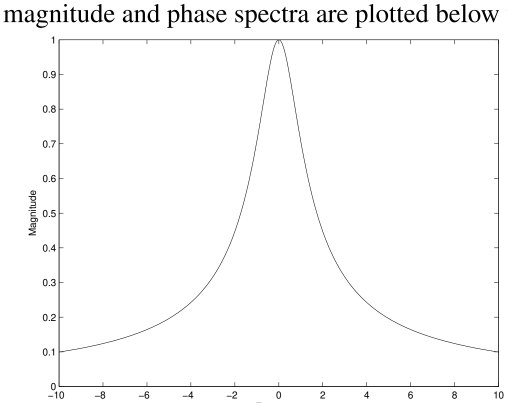
Computing the magnitude spectrum, we obtain

$$\begin{aligned} |X(\omega)| &= \left|\frac{1}{a+j\omega}\right| \\ &= \frac{|1|}{|a+j\omega|} \\ &= \frac{1}{\sqrt{a^2+\omega^2}}. \end{aligned}$$

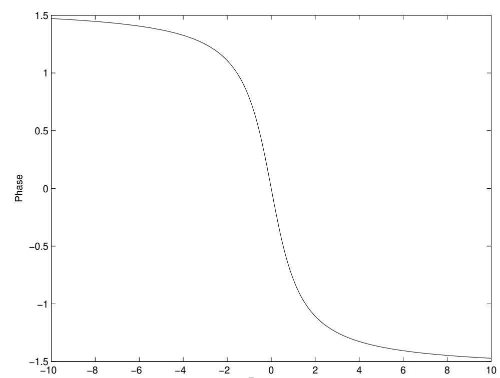
Computing the phase spectrum, we obtain

$$\begin{aligned} \arg X(\omega) &= \arg\left[\frac{1}{a+j\omega}\right] \\ &= \arg 1 - \arg(a+j\omega) \\ &= 0 - \arg(a+j\omega) \\ &= -\arg(a+j\omega) \\ &= -\arctan\tfrac{\omega}{a}. \end{aligned}$$

The magnitude and phase spectra are plotted below for $a = 1$.



(a)



(b)

