ECE 260

EXAM 3

SOLUTIONS

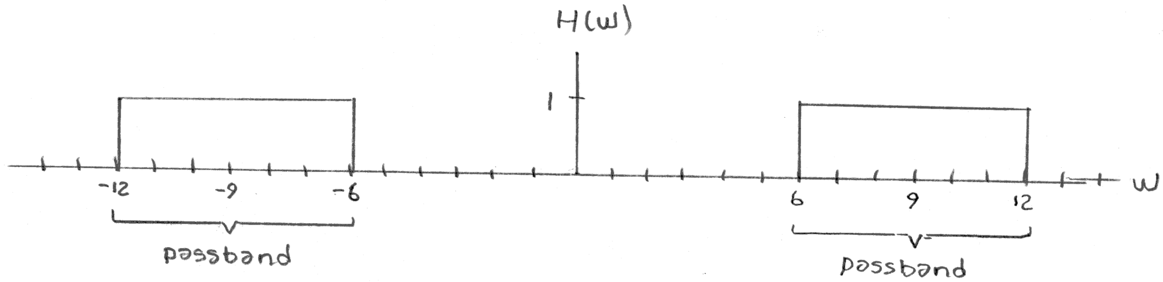
(FALL 2024)

QUESTION 1(A)

$$\boxed{h(t) = \frac{6}{\pi}\cos(9t)\,\mathrm{sinc}(3t)}$$

$$
\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \frac{6}{\pi}\cos(9t)\,\mathrm{sinc}(3t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \frac{6}{\pi}\left(\frac{1}{2}\right)\left(e^{j9t} + e^{-j9t}\right)\mathrm{sinc}(3t)\, e^{-j\omega t}\, dt \\
&= \frac{3}{\pi}\int_{-\infty}^{\infty} \left(e^{j9t} + e^{-j9t}\right)\mathrm{sinc}(3t)\, e^{-j\omega t}\, dt \\
&= \frac{3}{\pi}\int_{-\infty}^{\infty} e^{j9t}\,\mathrm{sinc}(3t)\, e^{-j\omega t}\, dt + \frac{3}{\pi}\int_{-\infty}^{\infty} e^{-j9t}\,\mathrm{sinc}(3t)\, e^{-j\omega t}\, dt \\
&= \frac{3}{\pi}\int_{-\infty}^{\infty} \mathrm{sinc}(3t)\, e^{-j(\omega-9)t}\, dt + \frac{3}{\pi}\int_{-\infty}^{\infty} \mathrm{sinc}(3t)\, e^{-j(\omega+9)t}\, dt \\
&= \frac{3}{\pi}\left(\frac{\pi}{3}\right)\mathrm{rect}\left(\frac{\omega-9}{6}\right) + \frac{3}{\pi}\left(\frac{\pi}{3}\right)\mathrm{rect}\left(\frac{\omega+9}{6}\right) \\
&= \mathrm{rect}\left(\frac{\omega-9}{6}\right) + \mathrm{rect}\left(\frac{\omega+9}{6}\right)
\end{aligned}
$$

QUESTION 1(B)



from the plot of H, we can see that the system is an ideal bandpass filter with a passband corresponding to $|\omega| \in [6, 12]$

QUESTION 2(A)

$$T = 10 \,, \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk(2\pi/T)t}\, dt$$

$$= \frac{1}{10}\int_{-5}^{5} x(t)\, e^{-jk\pi t/5}\, dt$$

$$= \frac{1}{10}\int_{-4}^{4} e^{-j\pi k t/5}\, dt \quad ①$$

consider ① if $k \neq 0$

$$C_k = \frac{1}{10}\left[\frac{1}{-j\pi k/5}\, e^{-j\pi k t/5}\right]\Bigg|_{-4}^{4}$$

$$= \frac{1}{-j2\pi k}\left[e^{-j\pi k t/5}\right]\Bigg|_{-4}^{4}$$

$$= \frac{j}{2\pi k}\left[e^{-j\pi k 4/5} - e^{-j\pi k(-4)/5}\right]$$

$$= \frac{j}{2\pi k}\left[e^{-j\pi 4k/5} - e^{j\pi 4k/5}\right]$$

$$= \frac{j}{2\pi k}\left[2j \sin(-4\pi k/5)\right]$$

$$= \frac{-1}{\pi k}(-1)\sin(4\pi k/5)$$

$$= \frac{1}{\pi k}\sin(4\pi k/5)$$

$$= \frac{1}{\pi k}\left[\frac{4\pi k/5 \,\sin(4\pi k/5)}{4\pi k/5}\right]$$

$$= \frac{4}{5}\,\text{sinc}\,(4\pi k/5)$$

consider ① if $k = 0$

$$C_0 = \frac{1}{10}\int_{-4}^{4} 1\, dt = \frac{1}{10}[t]\Big|_{-4}^{4} = \frac{1}{10}(8) = \frac{8}{10} = \frac{4}{5}$$

combining the above results, we have

$$C_k = \begin{cases} \frac{4}{5}\,\text{sinc}\,(4\pi k/5) & k \neq 0 \\ \frac{4}{5} & k = 0 \end{cases}$$

$$= \frac{4}{5}\,\text{sinc}(4\pi k/5)$$

QUESTION 2(B)

The function x has the most information at the frequency corresponding to the value of k that maximizes $|c_k|$.

Since the sinc function has a maximum magnitude (of 1) at the origin, $|c_k| = \left|\frac{4}{5}\operatorname{sinc}(4\pi k/5)\right|$ has a maximum at $k = 0$.

Since c_k corresponds to the frequency $k\omega_0$, the function x has the most information at $k\omega_0 = 0\omega_0 = 0$.

QUESTION 3(A)

$$\boxed{v(t) = 4 + 2\cos(3t)}$$

$T = \frac{2\pi}{3}$, $\omega_o = 3$

$$\begin{aligned} v(t) &= 4 + 2\cos(3t) \\ &= 4 + 2\left[\tfrac{1}{2}\left(e^{j3t} + e^{-j3t}\right)\right] \\ &= 4 + e^{j3t} + e^{-j3t} \\ &= e^{j(-1)(3)t} + 4e^{j(0)(3)t} + e^{j(1)(3)t} \end{aligned}$$

Therefore, we have

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}$$ where

$$c_k = \begin{cases} 1 & k \in \{-1, 1\} \\ 4 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

QUESTION 3(B)

From the eigenfunction properties of LTI systems, we have

$$y(t) = \sum_{K=-\infty}^{\infty} b_K e^{jKw_0 t} \quad \text{where} \quad b_K = H(Kw_0)\,c_K$$

we have

$$b_{-1} = H(-w_0)\,c_{-1} = [H(-3)](1) = (-1)(1) = -1$$

$$b_1 = H(w_0)\,c_1 = [H(3)](1) = (1)(1) = 1$$

$$b_0 = H(0)\,c_0 = (1)(4) = 4$$

and all other b_K are zero

So, we have

$$y(t) = b_{-1} e^{j(-1)(3)t} + b_1 e^{j(1)(3)t} + b_0 e^{j(0)(3)t}$$

$$= -e^{-j3t} + e^{j3t} + 4$$

$$= 4 + e^{j3t} - e^{-j3t}$$

$$= 4 + 2j\sin(3t)$$

QUESTION 4

```
function xn = evaluate_fs(c, T, t)
    n = length(c);
    xn = c(1);
    for k = 1 : n - 1
        ck = c(k + 1);
        xn = xn + 2 * abs(ck) * cos(2 * pi * k * t / T + angle(ck));
    end
end
```