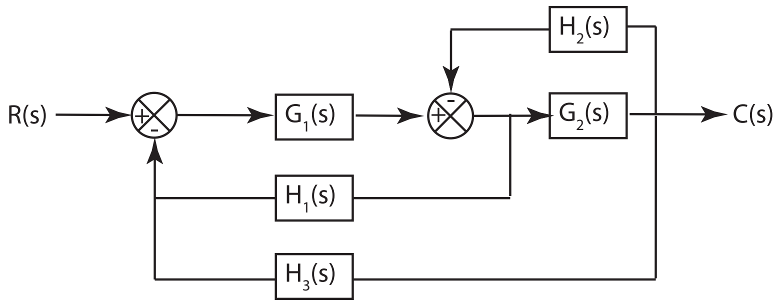
ELEC 360 : Control Theory and Systems I
Midterm
February 26th, 2010

Name: ______________________________
Student Number: ______________________________
Mark: ________ /30

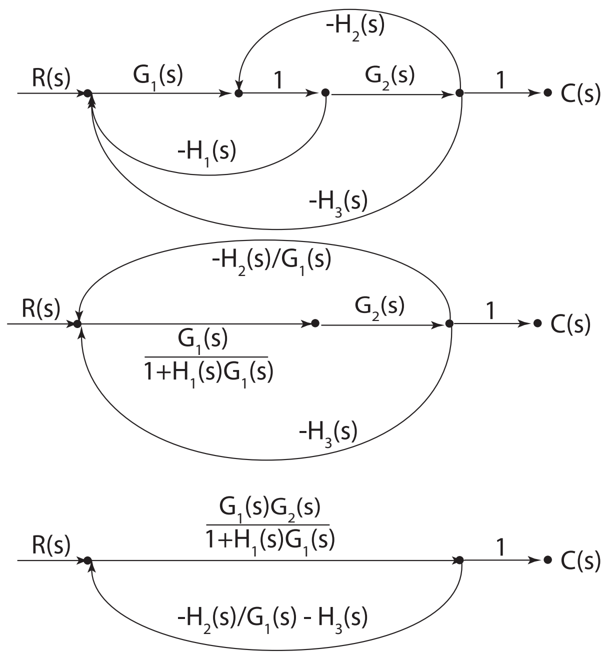
Notes:

- Students are permitted a single 8.5 by 11 inch sheet of notes.
- Programmable calculators are allowed.
  - No other aids permitted.
  - Use of cell phones or other electronic devices, except calculators, during the exam will result in a zero grade.

1. Write the transfer function for the system diagram given below in terms of $G_1(s)$, $G_2(s)$, $H_1(s)$, $H_2(s)$, and $H_3(s)$. (10 pts)



Signal Flow Graph:



R(s) ⟶ C(s)

$$\frac{\dfrac{G_1(s)G_2(s)}{1+H_1(s)G_1(s)}}{1 + [H_2(s)/G_1(s) + H_3(s)]\,\dfrac{G_1(s)G_2(s)}{1+H_1(s)G_1(s)}}$$

R(s) ⟶ C(s)

$$\frac{G_1(s)G_2(s)}{1 + H_1(s)G_1(s) + H_2(s)G_2(s) + H_3(s)G_1(s)G_2(s)}$$

2. We have a system with the transfer function given below:

$$\frac{\mathrm{C}(s)}{\mathrm{R}(s)} = \frac{32}{2s^2 + 1.6s + 32}$$

(a) What is the rise time and natural frequency for the unit step response of this system? (3 pts)
Step 1: Place transfer function into standard form and determine ζ and ω_n,

$$\frac{\mathrm{C}(s)}{\mathrm{R}(s)} = \frac{32}{2s^2 + 1.6s + 32} = \frac{16}{s^2 + 0.8s + 16} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n + {\omega_n}^2}$$

Hence, $\omega_n = 4\mathrm{rad/sec}$ and $\zeta = 0.1$ (now know we are in the underdamped case).

Rise time is given by $t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$ (page B14 of the lecture notes) where $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.1^2} = 3.97995$ and $\sigma = \omega_n \cdot \zeta = (4)(0.1) = 0.4$ (page B8 of the lecture notes).

So the rise time is $t_r = \frac{1}{3.97995}\left[\pi + \tan^{-1}\left(-\frac{3.97995}{0.4}\right)\right] = 0.41985$ sec.

Note: the angle in the formula $t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$ is a quadrant II angle; hence, the correction by π in the computation above. Also this formula is in radians and not degrees.

(b) What is the peak time and maximum overshoot for the unit step response of this system? (4 pts)

Peak time is given by $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4\cdot\sqrt{1-0.1^2}} = 0.7894$ sec (page B14 of the lecture notes).

Maximum overshoot is given by $M_p = e^{-\frac{\sigma\pi}{\omega_d}} = e^{-\frac{0.4\pi}{4\sqrt{1-0.1^2}}} = 0.72925$ (page B15 of the lecture notes).

(c) What is the approximate settling time using the 2 % criterion for the unit step response of this system? (3 pts)

Approximate settling time at the 2 % criterion is given by $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.1\cdot 4} = 10$ sec (page B15 of the lecture notes).

3. We have a system with a closed-loop transfer function of $\frac{\mathrm{C}(s)}{\mathrm{R}(s)} = \frac{K \cdot \mathrm{G}(s)}{1+K \cdot \mathrm{G}(s)}$ where K is a variable gain and the transfer function $\mathrm{G}(s)$ is given by

$$\mathrm{G}(s) = \frac{(s+10)}{(s+5)(s^2+6s+18)}$$

(a) Using the Routh-Hurwitz stability test determine the values of K for which this system is stable. (4 pts)

Closed loop transfer function is given by $\frac{\mathrm{C}(s)}{\mathrm{R}(s)} = \frac{K \cdot \frac{(s+10)}{(s+5)(s^2+6s+18)}}{1+K \cdot \frac{(s+10)}{(s+5)(s^2+6s+18)}}$ - need to simplify to a $\frac{B(s)}{A(s)}$ form so that we can apply the Routh-Hurwitz test.

For Routh-Hurwitz we only need $A(s)$:

Bringing $(s+5)\left(s^2+6s+18\right)$ from the numerator into the denominator gives,

$$A(s) = (s+5)\left(s^2+6s+18\right) + K \cdot (s+10)$$

Multiplying out gives $A(s) = (s^3+6s^2+18s+5s^2+30s+90) + Ks + (10K) = s^3+11s^2+(48+K)s+(90+10K)$

Now building the Routh-Hurwitz table for $A(s)$ gives,

s^3:	1	$48+K$
s^2:	11	$90+10K$
s^1:	$\frac{11\cdot(48+K)-(90+10K)}{11}$	
s^0:	$90+10K$	

Roots of $A(s)$ in the right half plane cause sign changes to occur in first column of Routh-Hurwitz table.

So if there are root of $A(s)$ in the right half plane K's must exist for which $\frac{11\cdot(48+K)-(90+10K)}{11} < 0$ or for which $90+10K < 0$ (a the s^3 and s^2 terms in the 1st column of the table are positive).

Simplifying $\frac{11\cdot(48+K)-(90+10K)}{11} < 0$ gives $48 + K - \frac{90}{11} - \frac{10}{11}K = 48 - \frac{90}{11} + \frac{1}{11}K < 0$.

Hence, $K < 11 \cdot \left(\frac{90}{11} - 48\right) < -438$ will make the system unstable from the s^1 row.

But, from the s^0 row we have that $90+10K < 0$ will make the system unstable, which gives $K < -9$.

Hence, for $K \geq -9$ the system will be stable and for $K < -9$ the system will be unstable.

Note: The root locus approach only covers the cases when $K \geq 0$. It does not cover negative K. Hence, the root locus will show that for $K \geq 0$ the system is always stable. Negative K are perfectly allowable - for example a proportional gain implemented by an op amp circuit can be used to produce a negative K.

(b) Draw the root locus for the system when G(s) is given by,

$$\mathrm{G}(s) = \frac{(s+10)}{(s+5)^2\,(s^2+6s+18)}$$

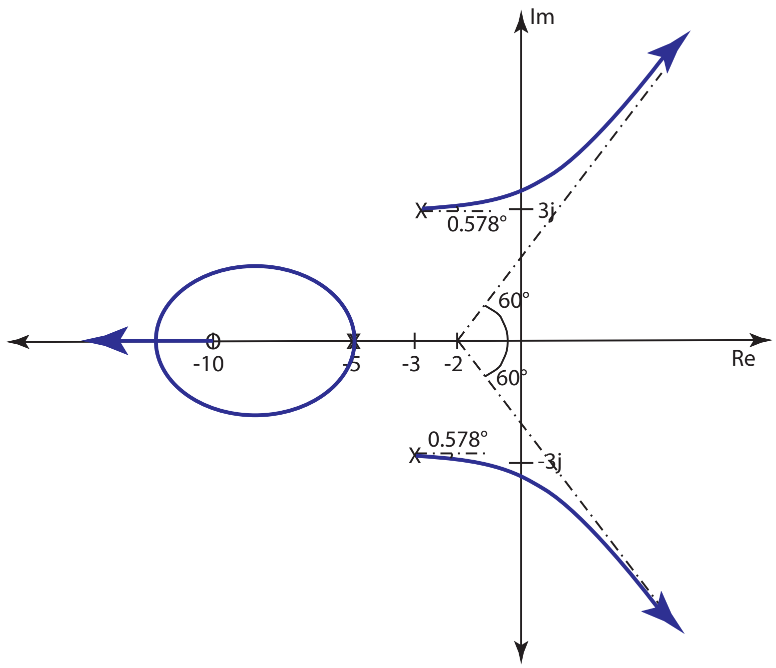
Identify angles and intersections of all asymptotes. Is this new system stable for all values of K? (6 pts)

Asymptotes are at angles of $\frac{\pm 180^\circ(2k+1)}{n-m} = \pm 60^\circ(2k+1)$.

The intersection point for the asymptotes is at: $\frac{(-5-5-3+3j-3-3j)-(-10)}{3} = -2$.

The angle of departure from the $3+3j$ root is $\theta = 180^\circ + \tan^{-1}(3/7) - 2\cdot\tan^{-1}(3/2) - 90^\circ = 0.5787^\circ$.

The sketch of the root locus is: (checkable via Matlab)



END OF EXAM