ECE-260

Tutorial 05

Topics covered

1) Fourier series
2) Properties of Fourier series
3) Convergence of Fourier series

Review Concepts

- **Continuous-Time Fourier Series representation**

A periodic signal/function x(t) is represented in the complex exponential form as:

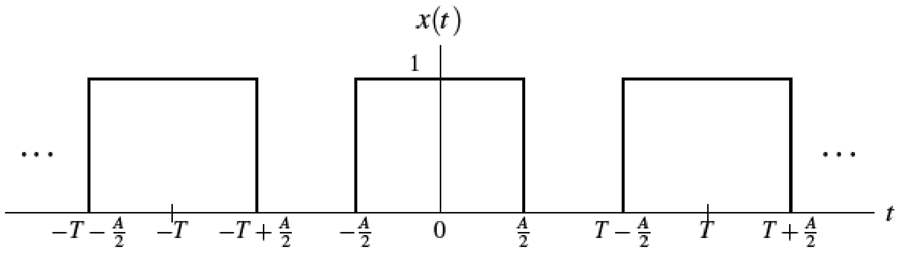
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

where,

$$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt,$$

Question 01(a)

For the periodic function shown in the figure, find the corresponding Fourier series representation of x(t).



Let us consider a single period of $x(t)$ for

$$-\frac{A}{2} \le t < \frac{A}{2}$$

$$x(t) = \begin{cases} 1 & , -\frac{A}{2} \le t \le \frac{A}{2} \\ 0 & , \text{elsewhere} \end{cases}$$

Let $w_0 = \frac{2\pi}{T}$. From Fourier series analysis eqn., we have

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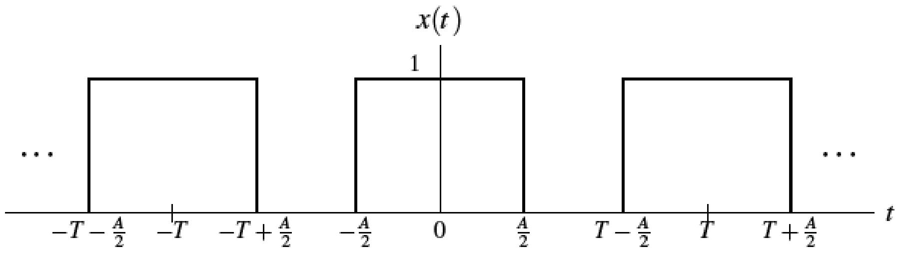


$$C_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$T = \cancel{\tfrac{A}{2}}\, T$; Active period:

Duration $\Rightarrow$ $D = \frac{A}{2} - \left(-\frac{A}{2}\right) = A$

$$C_k = \frac{1}{T}\int_{-A/2}^{A/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k = \frac{1}{T}\left[\frac{1}{-jk\omega_0}\cdot e^{-jk\omega_0 t}\Big|_{-A/2}^{A/2}\right]$$

$$C_k = \frac{1}{T}\left[\frac{1}{-jk\omega_0}\cdot\left(e^{-jk\omega_0 A/2} - e^{+jk\omega_0 A/2}\right)\right]$$

$$C_k = \frac{1}{T}\left[\frac{1}{jk\omega_0}\left(e^{jk\omega_0 A/2} - e^{-jk\omega_0 A/2}\right)\right]$$

$$C_k = \frac{1}{T}\left[\frac{2}{k\omega_0}\cdot\left(\frac{e^{jk\omega_0 A/2} - e^{-jk\omega_0 A/2}}{2j}\right)\right]$$

$$C_k = \frac{1}{T}\left[\frac{2}{k\omega_0}\cdot \sin(k\omega_0 A/2)\right]$$

$$C_k = \frac{1}{T}\left[\frac{2}{k\omega_0}\cdot\frac{A}{A}\cdot \sin(k\omega_0 A/2)\right]$$

$$C_k = \frac{1}{T}\left[A\cdot\frac{\sin(k\omega_0 A/2)}{k\omega_0 A/2}\right]$$

$$C_k = \frac{1}{T}\left[A\cdot \mathrm{Sinc}\left(k\frac{A}{T}\right)\right]$$

$$\Downarrow$$

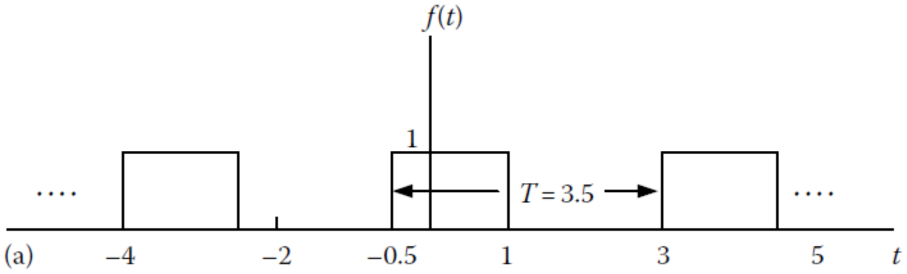
$x(t) \xleftrightarrow{\text{CTFS}} C_k$ $\boxed{\text{ANSWER}}$

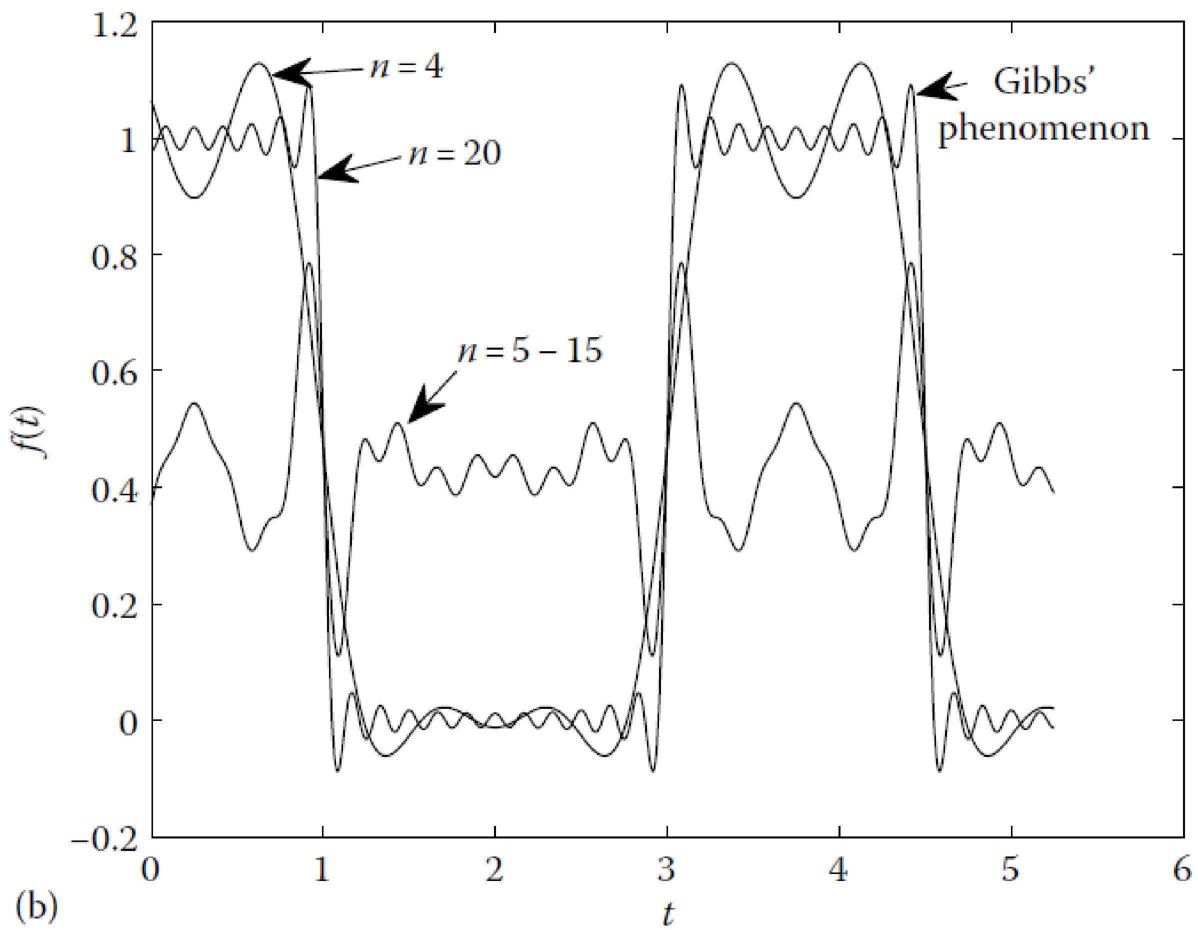
Alternatively,

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} A\, \mathrm{Sinc}\left(k\frac{A}{T}\right) e^{jk\omega_0 t}$$

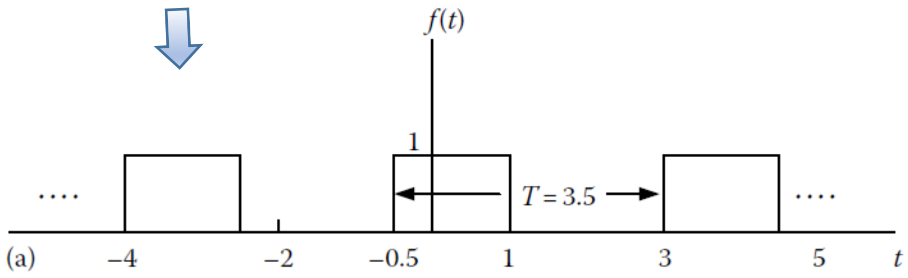
Question 01(b)

For the periodic function shown in the figure, find the corresponding Fourier series representation of $f(t)$ in sinusoidal form.





Question 01(b)

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% Tutorial 05_ECE 260
%To compute any Fourier series we must
%supply a0,an,bn,n,T;

a0 = 3/7;     % DC Component
T = 3.5;      % Time-Period
w0 = 2*pi*T; % Fundamental Frequency

k = 1:2000;  % Vary to see the behavior of the signal,e.g., k = 1:100 and k = 1:10000

t = 0:0.005:1.5*T; % Time-steps

an = (4./(3.5*k*w0)).*(sin(0.75*k*w0).*cos(0.25*k*w0)); % Even components of Fr-Series

bn = (4./(3.5*k*w0)).*(sin(0.75*k*w0).*sin(0.25*k*w0)); % Odd components of Fr-Series

cn = (an.^k + bn.^k).^(1/2); % Fourier series component coefficients

x = a0 + cos(t'*k*w0)*an' + sin(t'*k*w0)*bn'; % function representation
```

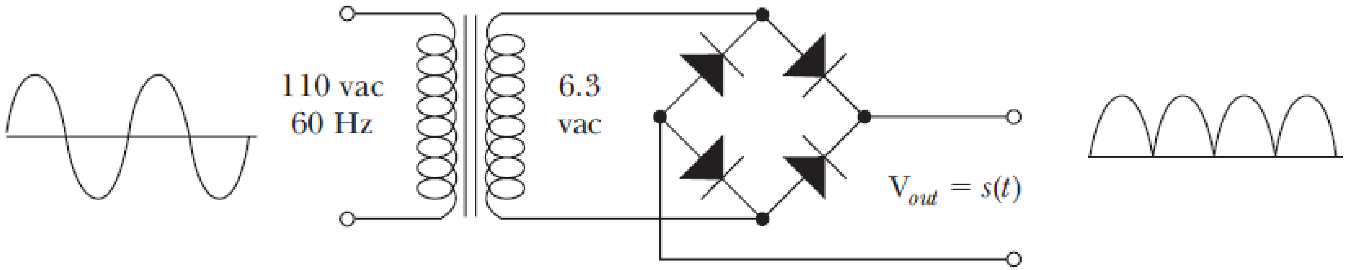
Question 02(a)

Let's examine a practical example.

Given in the figure is the "*AC-DC Converter*".

Q1. What is the Fourier series expansion of s(t)?

Q2. What is the DC component of this waveform?



$$s_{in}(t) = \mathrm{A.}\cos(2\pi f_0 t)$$

$$s(t) = \mathrm{A.}\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

The first component to be computed in the Fourier series is the zero-frequency or DC component, given by:

$$A_0 = \frac{1}{T}\int_{-T/2}^{+T/2} s(t)\ dt$$

$$A_0 = \frac{1}{T}\int_{-T/2}^{+T/2} A\cos(2\pi(1/2T)t)\ dt$$

$$A_0 = \frac{2}{T}\int_{0}^{+T/2} A\cos(2\pi(1/2T)t)\ dt$$

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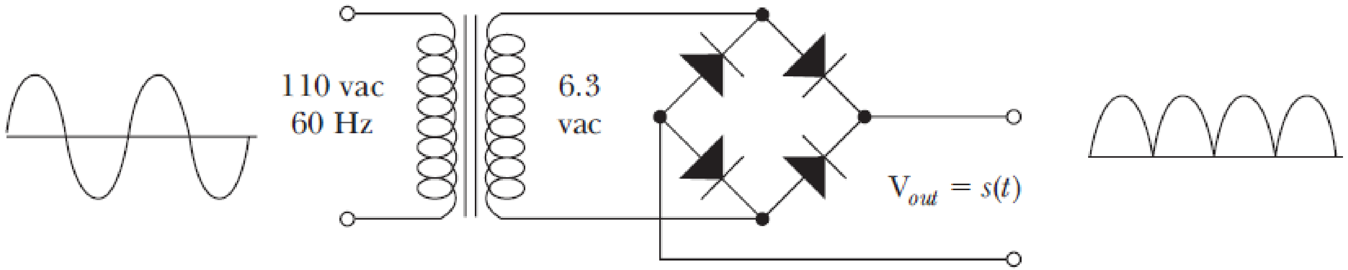


$$s_{in}(t) = \mathrm{A}.\cos(2\pi f_0 t) \qquad \Downarrow \qquad s(t) = \mathrm{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

$$= \frac{2A}{T}\int_0^{+T/2} \cos(\pi(1/T)t)\;dt$$

$$= \frac{2A}{\pi}\sin(\pi(1/T)t)\Big|_{t=0}^{T/2}$$

$$= \frac{2A}{\pi}\sin\left(\frac{\pi}{2}\right)$$

$$A_0 = \frac{2A}{\pi}$$

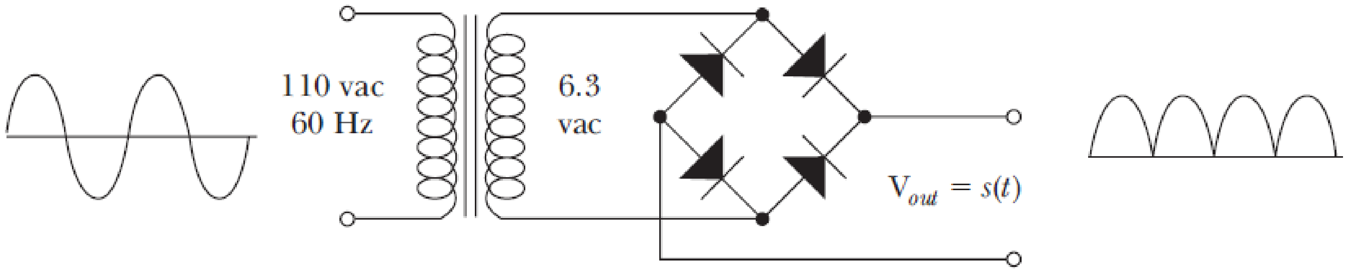
Question 02(b)

Let's examine a practical example.

Given in the figure is the "*AC-DC Converter*".

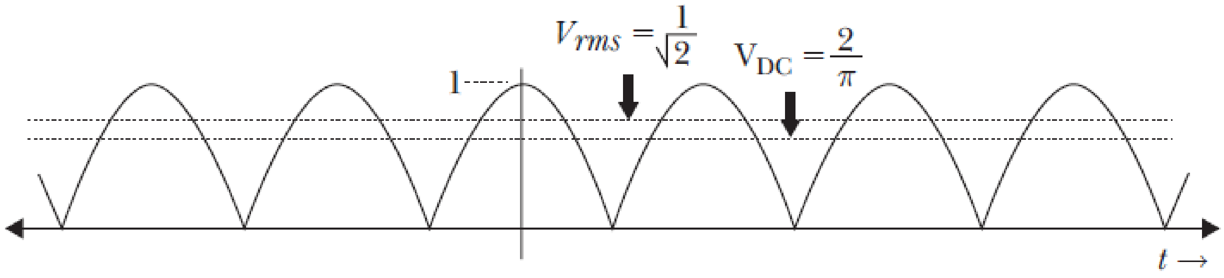
Q3. What is the V_{rms} value of this signal (s(t))?

Q4. How is the V_{rms} related to the DC component?



$$s_{in}(t) = \mathrm{A.}\cos(2\pi f_0 t) \quad \Downarrow \quad s(t) = \mathrm{A.}\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

- The Vrms is the square root of the total power in all components, including DC. By inspection, therefore, $V_{rms} \geq V_{dc}$. For A = 1, Vrms = (1/sqrt(2)).



- Computation of the power in the individual Fourier series components can be used to determine what fraction of the overall waveform power is actually contained in the DC component.

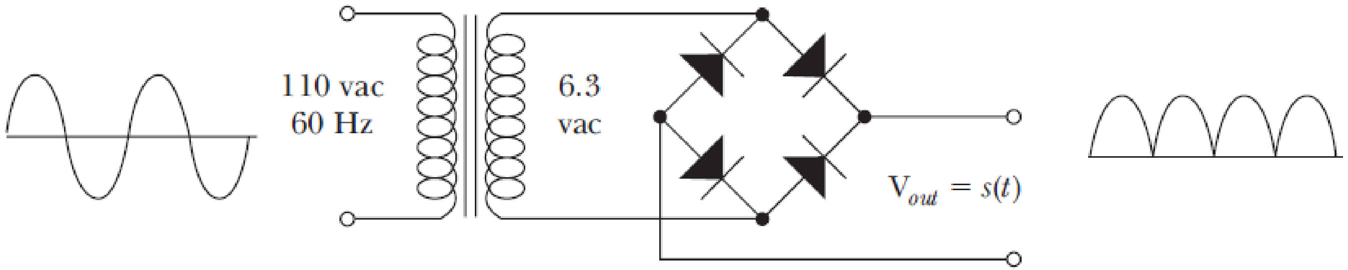
Question 02(c)

Let's examine a practical example.

Given in the figure is the "*AC-DC Converter*".

Q5. What fraction of the total power is contained in the DC component?

Q6. How can the DC component be usefully extracted from this signal?



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$$s(t) = \mathrm{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

THEOREM *(Parseval's Theorem)*

If a periodic signal $\mathrm{s(t)}$ *is expressed as complex Fourier series components* $\{\mathrm{C_n}\}$, *then the power in* $\mathrm{s(t)}$ *is given by:*

$$\text{power} = \int_{-\infty}^{\infty} |s(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

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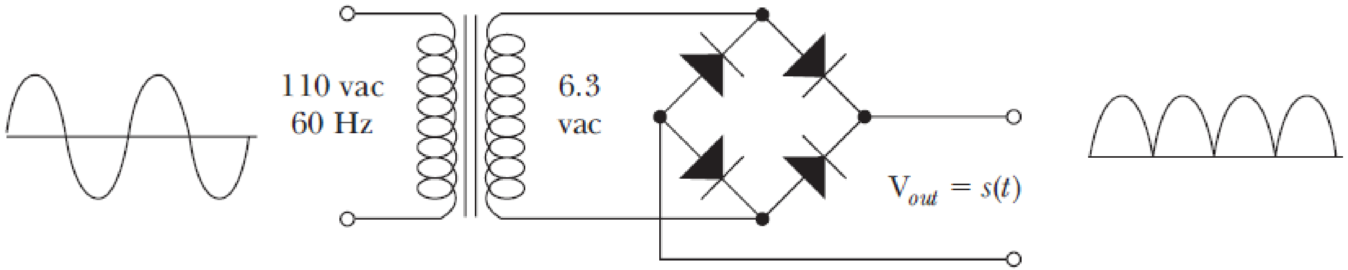

$$s_{in}(t) = \mathrm{A}.\cos(2\pi f_0 t)$$

$$s(t) = \mathrm{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

The total power in this waveform is given by $(V_{rms})^2 = 1/2$. The power in the DC component alone is given by $(2/\pi)^2 = 4/\pi^2$. The fraction of total signal power found in the DC component is therefore:

$$\frac{\text{Power in DC}}{\text{Total Power}} = \frac{4/\pi^2}{1/2} = 0.811$$

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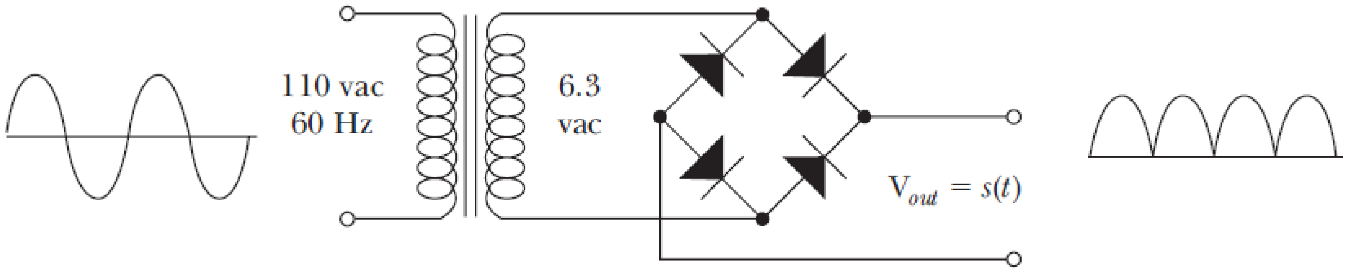

$$s_{in}(t) = \mathrm{A}.\cos(2\pi f_0 t) \quad \Downarrow \quad s(t) = \mathrm{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

Since the full-wave rectified signal is even, the simple Fourier series components $\{A_n, B_n\}$ will be found.

Odd Fourier series components B_n—Since the full-wave rectified cosine waveform is an even function, there will be no odd components in the series. By inspection, $B_n = 0$, for all n.

Even Fourier series components A_n—The fundamental frequency $f_0 = 120$ Hz $= 1/T$, and $s(t) = A\cos(\pi t/T)$ as when computing the DC component. This leaves:

$$A_n = \frac{2}{T}\int_{-T/2}^{+T/2} s(t)\cos(2\pi n(1/T)t)\,dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{+T/2} A\cos(2\pi(1/2T)t)\cos(2\pi n(1/T)t)\,dt$$

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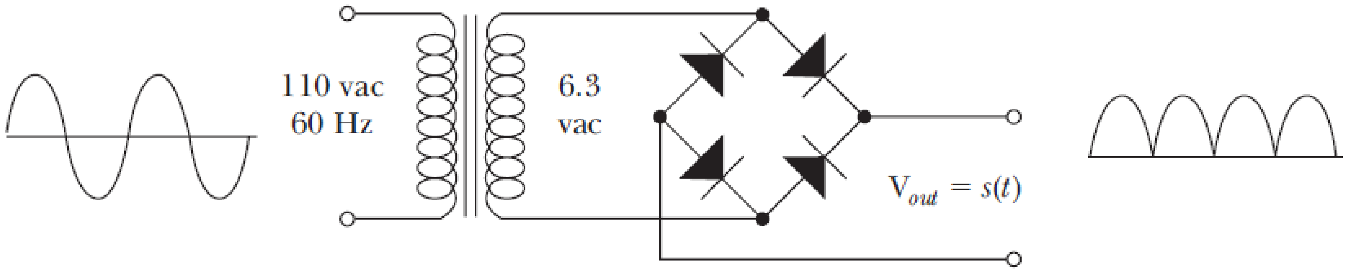

$$s_{in}(t) = \mathrm{A}.\cos(2\pi f_0 t) \qquad \Downarrow \qquad s(t) = \mathrm{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

Recognizing that the integral is an even function, this simplifies to:

$$A_n = \frac{4}{T}\int_0^{+T/2} A\cos(2\pi(1/2T)t)\cos(2\pi n(1/T)t)\ dt$$

Next, recall the identity $2\cos\mathrm{A}\cos\mathrm{B} = \cos(A - B) + \cos(A + B)$, leaving:

$$A_n = \frac{2A}{T}\int_0^{+T/2} \cos\left(2\pi\frac{1-2n}{2T}t\right) + \cos\left(2\pi\frac{1+2n}{2T}t\right)\ dt$$

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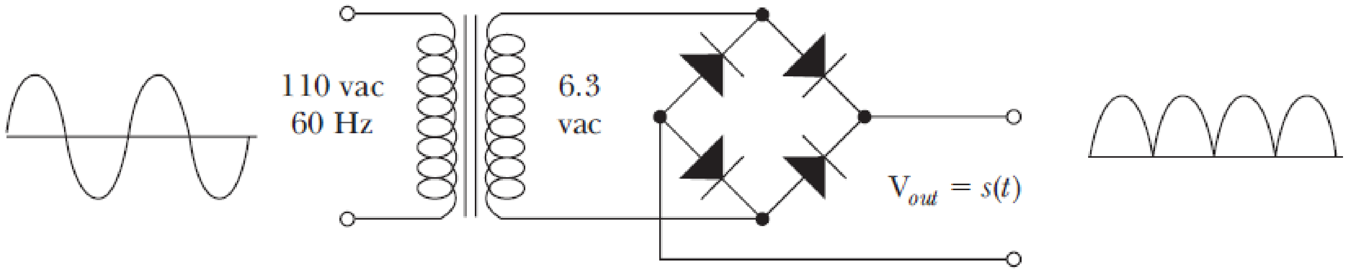

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$$A_n = \frac{2A}{T}\left[\int_0^{+T/2} \cos\left(\pi\frac{1-2n}{T}t\right)dt + \int_0^{+T/2} \cos\left(\pi\frac{1+2n}{T}t\right)dt\right]$$

$$A_n = \frac{2A}{T}\left[\frac{T}{\pi(1-2n)}\sin\left(\pi\frac{1-2n}{T}t\right)\Big|_{t=0}^{T/2} + \frac{T}{\pi(1+2n)}\sin\left(\pi\frac{1+2n}{T}t\right)\Big|_{t=0}^{T/2}\right]$$

$$A_n = \frac{2A}{\pi}\left[\frac{1}{1-2n}\sin\left(\pi\frac{1-2n}{T}t\right)\Big|_{t=0}^{T/2} + \frac{1}{1+2n}\sin\left(\pi\frac{1+2n}{T}t\right)\Big|_{t=0}^{T/2}\right]$$

$$A_n = \frac{2A}{\pi}\left[\frac{1}{1-2n}\sin\left(\pi\frac{1-2n}{2}\right) + \frac{1}{1+2n}\sin\left(\pi\frac{1+2n}{2}\right)\right]$$

$$A_n = \frac{2A}{\pi}\left[\frac{1}{1-2n}\sin\left(\frac{\pi}{2}(1-2n)\right) + \frac{1}{1+2n}\sin\left(\frac{\pi}{2}(1+2n)\right)\right]$$

Question 02(c)

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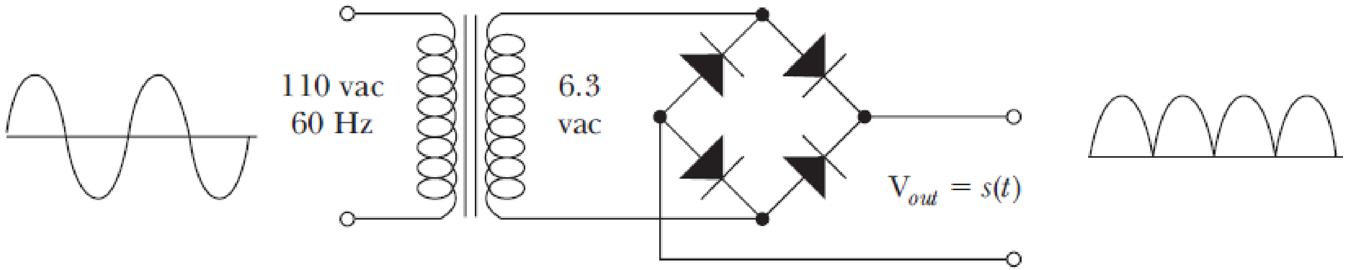
Q5. What fraction of the total power is contained in the DC component?

Q6. How can the DC component be usefully extracted from this signal?



$$s_{in}(t) = A.\cos(2\pi f_0 t)$$

$$s(t) = A.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

Both the sine expressions above simply alternate sign with successive values of n; both can be replaced by $(-1)^n$ found by substitution in this equation. Let the amplitude $A = 1$, then:

$$A_1 = \frac{2A}{\pi}\left[\frac{1}{1-2}\sin\left(\frac{\pi}{2}(1-2)\right) + \frac{1}{1+2}\sin\left(\frac{\pi}{2}(1+2)\right)\right]$$

$$= \frac{2A}{\pi}\left[\frac{1}{-1}\sin\left(-1\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(3\frac{\pi}{2}\right)\right] = \frac{4}{\pi}\cdot\frac{1}{3}$$

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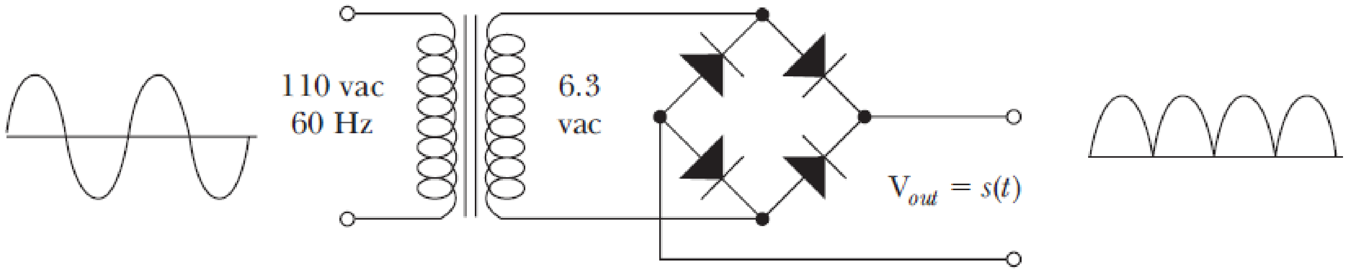

$$s_{in}(t) = A.\cos(2\pi f_0 t)$$

$$s(t) = A.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

Other terms follow as:

$$A_1 = +\frac{4}{\pi}\cdot\frac{1}{1\cdot 3}$$

$$A_2 = -\frac{4}{\pi}\cdot\frac{1}{3\cdot 5}$$

$$A_3 = +\frac{4}{\pi}\cdot\frac{1}{5\cdot 7}$$

$$A_4 = -\frac{4}{\pi}\cdot\frac{1}{7\cdot 9}$$

From these coefficients, the frequency-domain representation of this signal includes components of cosine spaced at 120 Hz intervals, alternating sign, with a DC component of $2/\pi$ as shown in Figure

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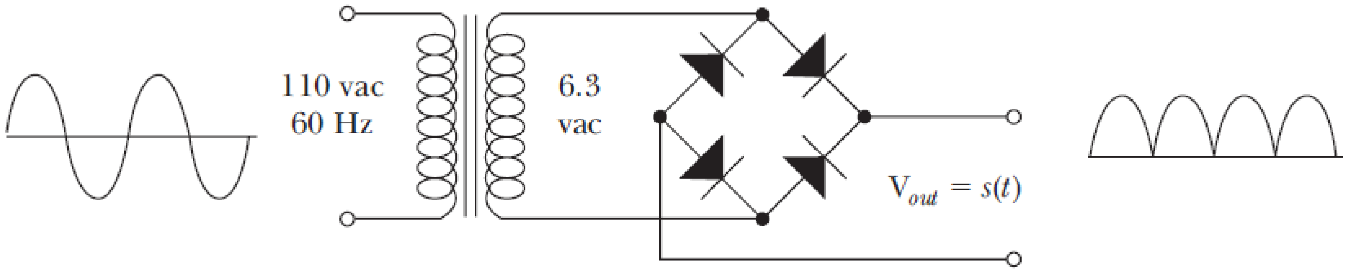

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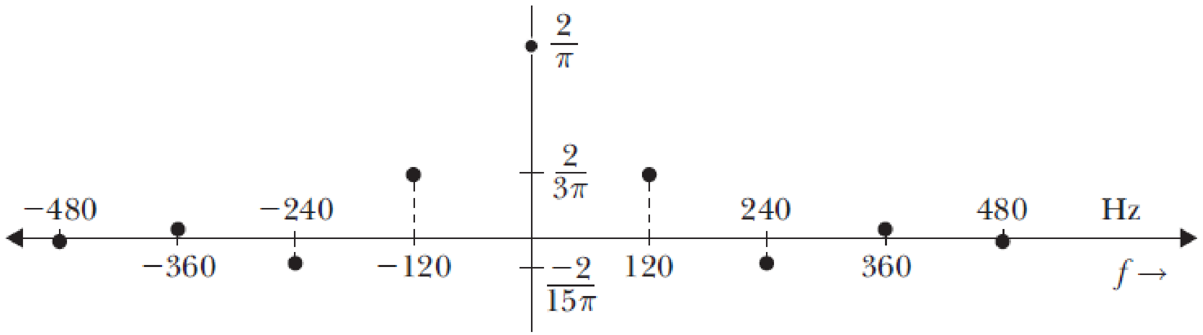

FIGURE Fourier Series Expansion of the full-wave rectified cosine $s(t) = |\cos(2\pi f_0 t)|$.

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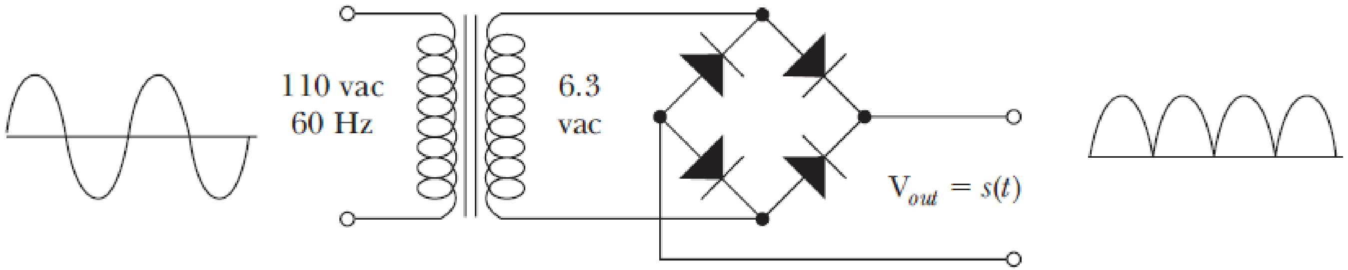

$$s_{in}(t) = \text{A}.\cos(2\pi f_0 t) \qquad s(t) = \text{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

By inspection, the power in the non-zero frequency components will fall off rapidly with frequency. Each cosine component A_n has power equal to $A_n^2/2$. Now, from the previous calculation, $A_1 = 4A/(3\pi)$. Therefore, the power in A_1 is given by:

$$A_1^2/2 = 8A^2/(9\pi^2) = 0.090A^2$$

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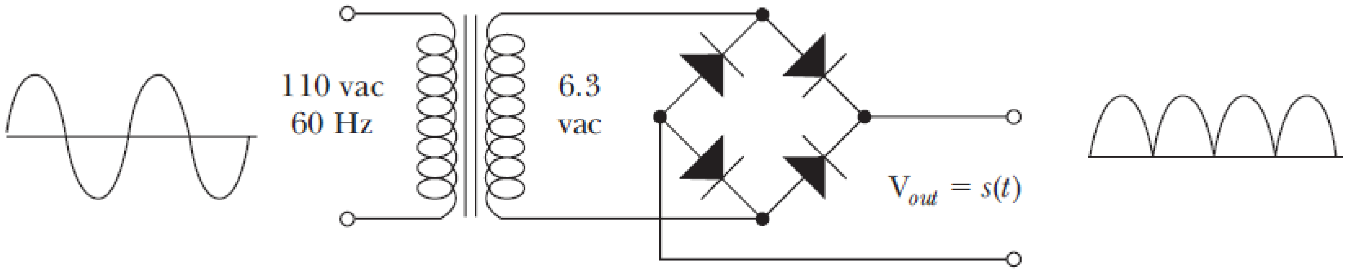
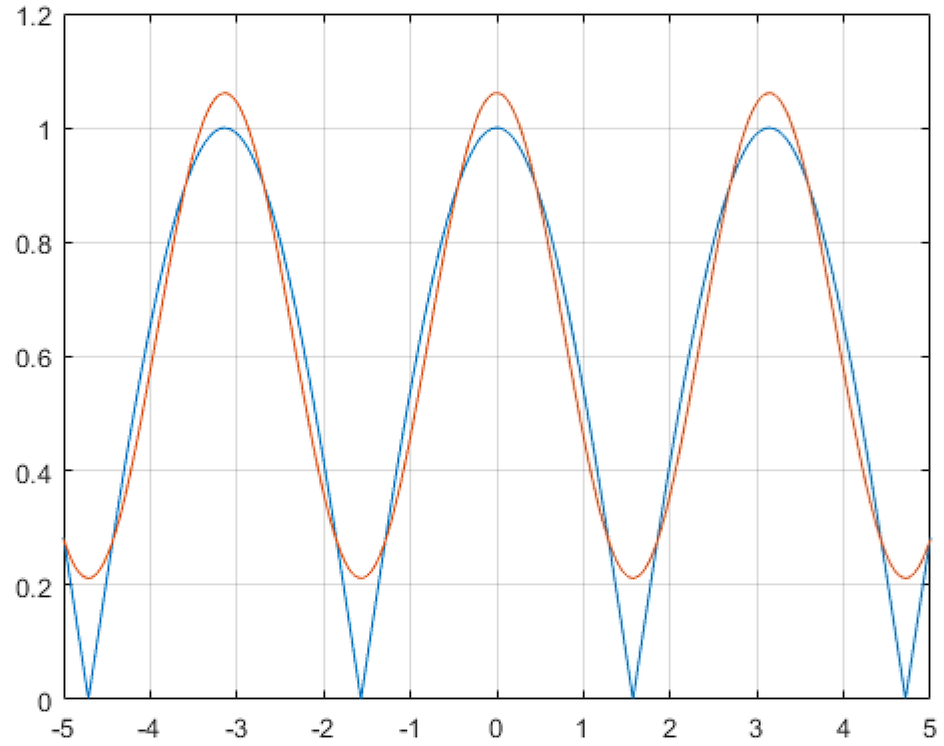


$$s_{in}(t) = \text{A}.\cos(2\pi f_0 t)$$

$$s(t) = \text{A}.\cos\left(2\pi\left(\frac{1}{2T}\right)t\right)$$

```
t = -5 : 0.01 : 5;
a = abs(cos(t)); % full-wave rectified cosine
f0= 2/pi; % define DC component
f1= (1/3)*(4/pi)*cos(2*t); % define fundamental component
plot(t,a,t,f0+f1); % plot DC and the 1st term
grid on;
```



End of Tutorial 05