3B **4.11** Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.

(a) $\mathcal{H}x(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$;

(b) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau$;

(c) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)v(t-\tau)d\tau$; and

(d) $\mathcal{H}x(t) = \int_{t-1}^{t} x(\tau)d\tau$.

3B **Answer (a).**

Let h denote the impulse response of the system.

$$\begin{aligned} h(t) &= \int_{-\infty}^{t+1} \delta(\tau)d\tau \\ &= \begin{cases} 1 & t > -1 \\ 0 & t < -1 \end{cases} \\ &= u(t+1). \end{aligned}$$

3B **Answer (b).**

Let h denote the impulse response of the system.

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} \delta(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau \\ &= e^{-5-t+1}u(t-[-5]-2) \\ &= e^{-t-4}u(t+3). \end{aligned}$$

3B **Answer (c).**

Let h denote the impulse response of the system. We have

$$h(t) = \int_{-\infty}^{t} \delta(\tau)v(t-\tau)d\tau.$$

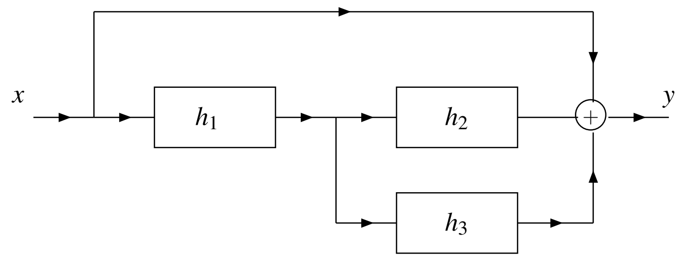
Now, we employ a change of variable. Let $\lambda = \tau - t$ so that $\tau = \lambda + t$ and $d\lambda = d\tau$. Applying this change of variable, we obtain

$$\begin{aligned} h(t) &= \int_{-\infty}^{0} \delta(\lambda+t)v(t-[\lambda+t])d\lambda \\ &= \int_{-\infty}^{0} \delta(\lambda+t)v(-\lambda)d\lambda. \end{aligned}$$

From the equivalence property of the unit-impulse function, we can write

$$\begin{aligned} h(t) &= \int_{-\infty}^{0} \delta(\lambda+t)v(t)d\lambda \\ &= v(t)\int_{-\infty}^{0} \delta(\lambda+t)d\lambda \\ &= v(t)u(t). \end{aligned}$$

3B **4.12** Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



(a) Find the impulse response h of the overall system in terms of h_1, h_2, and h_3.
(b) Determine the impulse response h in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

3B **Answer (a).**

Let v denote the output of the system with impulse response h_1. From the block diagram, we have

$$\begin{aligned} v(t) &= x * h_1(t) \\ y(t) &= v * [h_2 + h_3](t) + x(t). \end{aligned}$$

Combining these equations yields

$$\begin{aligned} y(t) &= v * [h_2 + h_3](t) + x(t) \\ &= [x * h_1] * [h_2 + h_3](t) + x(t) \\ &= x * [h_1 * [h_2 + h_3]](t) + x(t) \\ &= x * [h_1 * h_2 + h_1 * h_3](t) + x * \delta(t) \\ &= x * [h_1 * h_2 + h_1 * h_3 + \delta](t). \end{aligned}$$

Therefore, we have

$$h(t) = h_1 * h_2(t) + h_1 * h_3(t) + \delta(t).$$

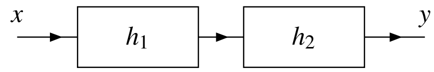
3B **Answer (b).**

Substituting the given expressions for h_1, h_2, and h_3 into the expression for h, we obtain

$$\begin{aligned} h(t) &= \delta(\cdot + 1) * \delta(t) + \delta(\cdot + 1) * \delta(t) + \delta(t) \\ &= \delta(t+1) + \delta(t+1) + \delta(t) \\ &= 2\delta(t+1) + \delta(t). \end{aligned}$$

(Note that $\delta(\cdot + 1)$ simply means the function v defined by the equation $v(t) = \delta(t+1)$.)

3B **4.13** Consider the system shown in the figure below with input x and output y. This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2.



For each pair of h_1 and h_2 given below, find the output y if the input $x(t) = u(t)$.
(a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
(b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$; and
(c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

3B **Answer (b).**

First, we calculate the impulse response h of the overall system. We have

$$\begin{aligned} h(t) &= h_1 * h_2(t) \\ &= (\delta(\cdot+1) * \delta(\cdot+1))(t) \\ &= \int_{-\infty}^{\infty} \delta(\tau+1)\delta(t-\tau+1)d\tau \\ &= \delta(t+2). \end{aligned}$$

(Note that $\delta(\cdot+1)$ is simply an abbreviated notation for the function $v(t) = \delta(t+1)$.) The output y is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= u * \delta(t+2) \\ &= u(t+2). \end{aligned}$$

3B **Answer (c).**

First, we calculate the impulse response h of the overall system. We have that

$$\begin{aligned} h(t) &= h_1 * h_2(t) \\ &= [e^{-3t}u(t)] * \delta(t) \\ &= e^{-3t}u(t). \end{aligned}$$

Now, we compute the convolution $y(t) = x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$. By drawing graphs of $x(\tau)$ and $h(t-\tau)$ and applying the appropriate logic, we conclude that there are two cases to consider for the computation of $y(t)$: $t < 0$ and $t \geq 0$. In the case that $t < 0$, we have that $y(t)$ is trivially zero. So, we now consider the case that $t \geq 0$. For $t \geq 0$, we have that $y(t)$ is given by

$$\begin{aligned} y(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_0^t e^{-3\tau}d\tau \quad \text{for } t > 0 \\ &= [-\tfrac{1}{3}e^{-3\tau}]\big|_0^t \\ &= -\tfrac{1}{3}[e^{-3t} - 1] \\ &= -\tfrac{1}{3}e^{-3t} + \tfrac{1}{3}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} y(t) &= \begin{cases} -\frac{1}{3}e^{-3t} + \frac{1}{3} & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= [-\tfrac{1}{3}e^{-3t} + \tfrac{1}{3}]u(t). \end{aligned}$$

3B **4.14** Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(t) = (t+1)u(t-1)$;
(b) $h(t) = 2\delta(t+1)$;
(c) $h(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$;
(d) $h(t) = e^{-4t}u(t-1)$;
(e) $h(t) = e^{t}u(-1-t)$;
(f) $h(t) = e^{-3|t|}$; and
(g) $h(t) = 3\delta(t)$.

3B **Answer (a–g).**

A LTI system with impulse response h is memoryless if $h(t) = 0$ for all $t \neq 0$. Therefore, the systems in (a), (b), (c), (d), (e), and (f) all have memory, while the system in (g) is memoryless.

A LTI system with impulse response h is causal if $h(t) = 0$ for all $t < 0$. Therefore, the systems in (a), (d), and (g) are causal, while the systems in (b), (c), (e), and (f) are not causal.

3B **4.15** Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
(b) $h(t) = t^{-1}u(t-1)$;
(c) $h(t) = e^{t}u(t)$;
(d) $h(t) = \delta(t-10)$;
(e) $h(t) = \operatorname{rect}(t)$; and
(f) $h(t) = e^{-|t|}$.

3B **Answer (a).**

A LTI system with impulse response h is BIBO stable if and only if h is absolutely integrable. From the given h, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(-t)\right| dt \\
&= \int_{-\infty}^{\infty} e^{at}u(-t)dt \\
&= \int_{-\infty}^{0} e^{at}\,dt \\
&= \left[\tfrac{1}{a}e^{at}\right]\Big|_{-\infty}^{0} \quad \text{for } a \neq 0 \\
&= \tfrac{1}{a}\left[e^{at}\right]\Big|_{-\infty}^{0} \\
&= \tfrac{1}{a}[1-0] \\
&= \tfrac{1}{a} \\
&< \infty.
\end{aligned}$$

Therefore, the system is BIBO stable.

3B **Answer (b).**

A LTI system with impulse response h is BIBO stable if and only if h is absolutely integrable. From the given h, can write

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|t^{-1}u(t-1)\right| dt \\
&= \int_{-\infty}^{\infty} t^{-1}u(t-1)dt \\
&= \int_{1}^{\infty} t^{-1}dt \\
&= [\ln(t)]\big|_{1}^{\infty} \\
&= \ln(\infty) - \ln(1) \\
&= \infty - 0 \\
&= \infty.
\end{aligned}$$

Therefore, the system is not BIBO stable.

3B **4.16** Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \tfrac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

3B **Answer.**

These systems are inverses if $h_1 * h_2(t) = \delta(t)$. We have

$$\begin{aligned}
h_1 * h_2(t) &= \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \tfrac{1}{2}\delta(\tau-1)2\delta(t-\tau+1)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau-1)\delta(t-\tau+1)d\tau \\
&= \delta(t).
\end{aligned}$$

Therefore, the systems are inverses of one another.

3B **4.17** For each case below, find the response y of the LTI system with system function H to the input x.

(a) $H(s) = \dfrac{1}{s+1}$ for $\operatorname{Re}(s) > -1$; and $x(t) = 10 + 4\cos(3t) + 2\sin(5t)$; and

(b) $H(s) = \dfrac{1}{e^s(s+1)}$ for $\operatorname{Re}(s) > -1$; and $x(t) = 10 + 2e^{3t} - e^t$.

3B **Answer (a).**

Rewriting x in terms of eigenfunctions of LTI systems, we obtain

$$\begin{aligned} x(t) &= 10 + 4\left[\tfrac{1}{2}\left(e^{j3t} + e^{-j3t}\right)\right] + 2\left[\tfrac{1}{2j}\left(e^{j5t} - e^{-j5t}\right)\right] \\ &= 10 + 2e^{j3t} + 2e^{-j3t} - je^{j5t} + je^{-j5t}. \end{aligned}$$

Since the system is LTI, we have

$$\begin{aligned} y(t) &= H(0)(10) + H(j3)(2e^{j3t}) + H(-j3)(2e^{-j3t}) + H(j5)(-je^{j5t}) + H(-j5)(je^{-j5t}) \\ &= 10H(0) + 2H(j3)e^{j3t} + 2H(-j3)e^{-j3t} - jH(j5)e^{j5t} + jH(-j5)e^{-j5t}. \end{aligned}$$

Now, we compute the eigenvalues $H(0)$, $H(j3)$, $H(-j3)$, $H(j5)$, and $H(-j5)$. We have

$$\begin{aligned} H(0) &= \frac{1}{0+1} \\ &= 1, \\ H(j3) &= \tfrac{1}{1+j3} = \tfrac{1-j3}{1+9} = \tfrac{1-j3}{10} = \tfrac{1}{10} - j\tfrac{3}{10} = \sqrt{\tfrac{1}{10}}e^{j\arctan(-3)} \\ &= \tfrac{1}{\sqrt{10}}e^{-j\arctan(3)}, \\ H(-j3) &= \tfrac{1}{1-j3} = \tfrac{1+j3}{1+9} = \tfrac{1+j3}{10} = H(j3)^* \\ &= \tfrac{1}{\sqrt{10}}e^{j\arctan(3)}, \\ H(j5) &= \tfrac{1}{1+j5} = \tfrac{1-j5}{1+25} = \tfrac{1-j5}{26} = \tfrac{1}{26} - j\tfrac{5}{26} = \sqrt{\tfrac{26}{676}}e^{j\arctan(-5)} = \sqrt{\tfrac{1}{26}}e^{-j\arctan(5)} \\ &= \tfrac{1}{\sqrt{26}}e^{-j\arctan(5)}, \quad \text{and} \\ H(-j5) &= \tfrac{1}{1-j5} = \tfrac{1+j5}{1+25} = \tfrac{1+j5}{26} = H(j5)^* \\ &= \tfrac{1}{\sqrt{26}}e^{j\arctan(5)}. \end{aligned}$$

Substituting the computed eigenvalues into the above formula for $y(t)$, we have

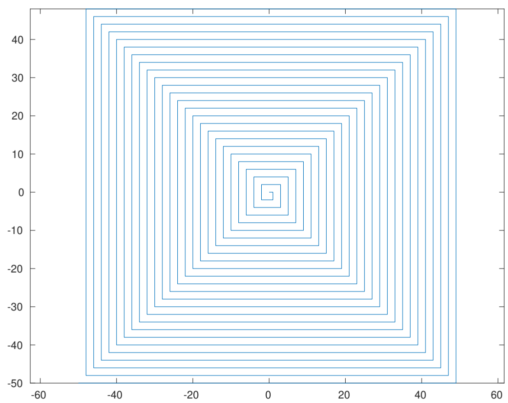
$$\begin{aligned} y(t) &= 10(1) + 2\left(\tfrac{1}{\sqrt{10}}e^{-j\arctan(3)}e^{j3t}\right) + 2\left(\tfrac{1}{\sqrt{10}}e^{j\arctan(3)}e^{-j3t}\right) - j\left(\tfrac{1}{\sqrt{26}}e^{-j\arctan(5)}e^{j5t}\right) + j\left(\tfrac{1}{\sqrt{26}}e^{j\arctan(5)}e^{-j5t}\right) \\ &= 10 + \tfrac{2}{\sqrt{10}}e^{-j\arctan(3)}e^{j3t} + \tfrac{2}{\sqrt{10}}e^{j\arctan(3)}e^{-j3t} - \tfrac{j}{\sqrt{26}}e^{-j\arctan(5)}e^{j5t} + \tfrac{j}{\sqrt{26}}e^{j\arctan(5)}e^{-j5t} \\ &= 10 + \tfrac{2}{\sqrt{10}}\left(e^{j[3t-\arctan(3)]} + e^{-j[3t-\arctan(3)]}\right) - \tfrac{j}{\sqrt{26}}\left(e^{j[5t-\arctan(5)]} - e^{-j[5t-\arctan(5)]}\right) \\ &= 10 + \tfrac{2}{\sqrt{10}}[2\cos(3t-\arctan(3))] - \tfrac{j}{\sqrt{26}}[2j\sin(5t-\arctan(5))] \\ &= 10 + \tfrac{4}{\sqrt{10}}\cos(3t-\arctan(3)) + \tfrac{2}{\sqrt{26}}\sin(5t-\arctan(5)). \end{aligned}$$

3B **D.8** In this exercise, we consider an algorithm for generating a sequence p of n points in the plane (i.e., $p_0, p_1, \ldots, p_{n-1}$). The first point p_0 is chosen as the origin (i.e., $p_0 = [\,0\;0\,]^T$), with the remaining points being given by the formula

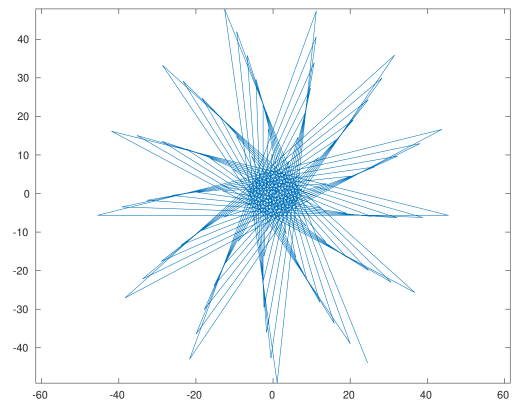
$$p_i = p_{i-1} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{i-1} \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

(a) Using MATLAB, write a function called `drawpattern` that takes n and θ as input arguments (in that order) with θ being specified in degrees, and then computes and plots the points $p_0, p_1, \ldots, p_{n-1}$ connected by straight lines (i.e., draw a line from p_0 to p_1, p_1 to p_2, p_2 to p_3, and so on). When performing the plotting, be sure to use `axis('equal')` in order to maintain the correct aspect ratio for the plot. For illustrative purposes, the plots produced for two sets of θ and n values are shown in Figures (a) and (b) below.

(b) Generate the plots obtained by invoking `drawpattern` with $n = 100$ and θ set to each of the following values: 89°, 144°, and 154°. [Note: In MATLAB, the `sin` and `cos` functions take values in radians, not degrees.]



(a) $\theta = 90°$ and $n = 100$



(b) $\theta = 166°$ and $n = 100$

3B **Answer (a).**

The `drawpattern` function can be implemented using the code below.

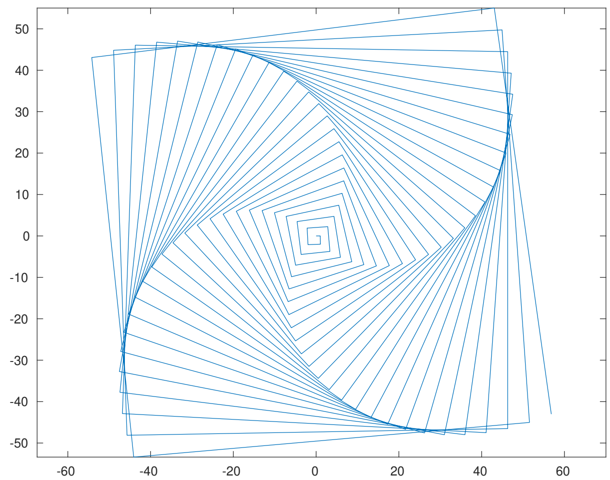
Listing D.6: `drawpattern.m`

```
function drawpattern(n, theta)
    % drawpattern - Draw a pattern
    % This function takes two arguments:
    %     the number of iterations
    %     the rotation angle

    % Convert from degrees to radians.
    t = theta * pi / 180;

    % Generate the list of points.
    p = [0 0]';
    x = p';
    for i = 1 : (n - 1)
        p = p + [cos(t) sin(t); -sin(t) cos(t)] ^ (i - 1) * [i 0]';
        x = [x; p'];
    end

    % Plot the list of points.
    plot(x(:, 1), x(:, 2));
```

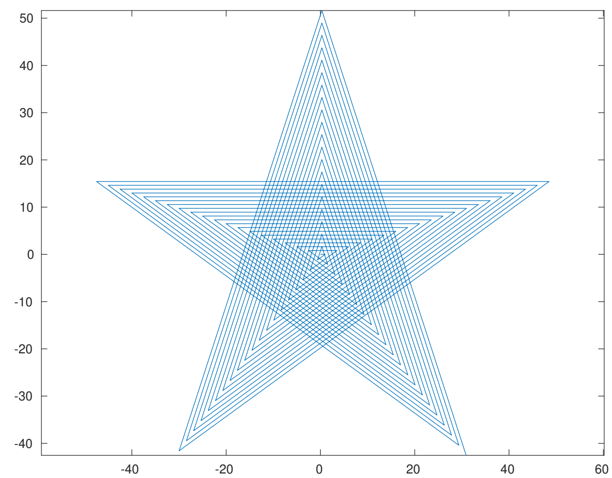
```
    axis('equal');

    % Print the plot to a file.
    % eval(sprintf('print -dps data/drawpattern_%d_%d.ps', theta, n))
end
```

3B Answer (b).

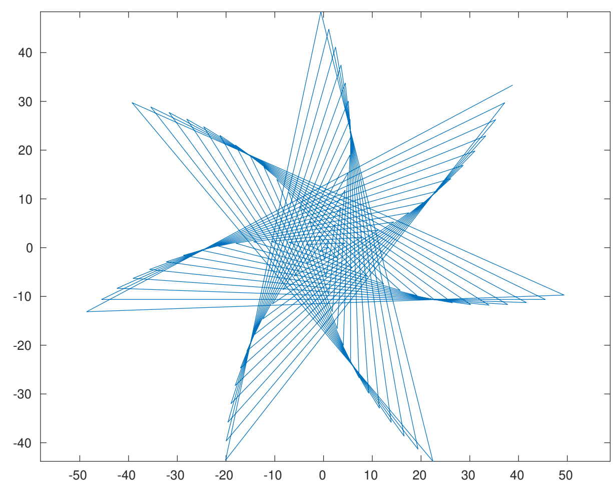
The plots obtained with the `drawpattern` function are shown below.



$\theta = 89°$, $n = 100$



$\theta = 144°$, $n = 100$



$\theta = 154°$, $n = 100$