Chapter 9

Complex Analysis (Appendix A)

A.1 Express each of the following complex numbers in Cartesian form: (a) $2e^{j2\pi/3}$; (b) $\sqrt{2}e^{j\pi/4}$; (c) $2e^{j7\pi/6}$; and (d) $3e^{j\pi/2}$.

Solution.

(a)

$$\begin{aligned} 2e^{j2\pi/3} &= 2\left(\cos\tfrac{2\pi}{3} + j\sin\tfrac{2\pi}{3}\right) \\ &= 2\left(-\tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right) \\ &= -1 + j\sqrt{3}. \end{aligned}$$

(b)

$$\begin{aligned} \sqrt{2}e^{j\pi/4} &= \sqrt{2}\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right) \\ &= \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + j\tfrac{1}{\sqrt{2}}\right) \\ &= 1 + j. \end{aligned}$$

(c)

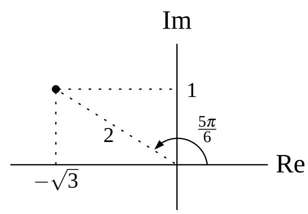
$$\begin{aligned} 2e^{j7\pi/6} &= 2\left(\cos\tfrac{7\pi}{6} + j\sin\tfrac{7\pi}{6}\right) \\ &= 2\left(-\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right) \\ &= -\sqrt{3} - j. \end{aligned}$$

A.2 Express each of the following complex numbers in polar form: (a) $-\sqrt{3} + j$; (b) $-\frac{1}{2} - j\frac{\sqrt{3}}{2}$; (c) $\sqrt{2} - j\sqrt{2}$; (d) $1 + j\sqrt{3}$; (e) $-1 - j\sqrt{3}$; and (f) $-3 + 4j$. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument (i.e., the value θ of the argument that lies in the range $-\pi < \theta \le \pi$).

Solution.

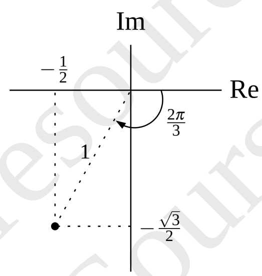
(a)

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$$
$$\arg z = \arctan\left(-\tfrac{1}{\sqrt{3}}\right) + \pi = -\tfrac{\pi}{6} + \pi = \tfrac{5\pi}{6}$$
$$z = 2e^{j5\pi/6}$$



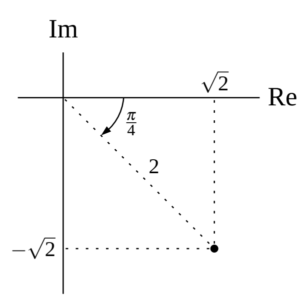
(b)

$$|z| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(-\tfrac{\sqrt{3}}{2}\right)^2} = 1$$
$$\arg z = \arctan\left(\left[-\tfrac{\sqrt{3}}{2}\right] / \left[-\tfrac{1}{2}\right]\right) - \pi = \arctan\left(\sqrt{3}\right) - \pi = \tfrac{\pi}{3} - \pi = -\tfrac{2\pi}{3}$$
$$z = e^{j(-2\pi/3)}.$$



(c)

$$|z| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$$
$$\arg z = \arctan(-\tfrac{\sqrt{2}}{\sqrt{2}}) = \arctan(-1) = -\tfrac{\pi}{4}$$
$$z = 2e^{j(-\pi/4)}.$$



A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument (i.e., choose the argument θ such that $-\pi < \theta \leq \pi$).
(a) $2\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) + j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ (in Cartesian form);
(b) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ (in polar form);

(c) $\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)/(1+j)$ (in polar form);
(d) $e^{1+j\pi/4}$ (in Cartesian form);
(e) $\left(\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^*\right)^8$ (in polar form);
(f) $(1+j)^{10}$ (in Cartesian form); and
(g) $\frac{1+j}{1-j}$ (in polar form).

Solution.

(a)

$$\begin{aligned}2\left(\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right) + j\left(\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right) &= \sqrt{3} - j + \left(-j\tfrac{1}{2} + \tfrac{1}{2}\right)\\ &= \sqrt{3} + \tfrac{1}{2} - j\left(1 + \tfrac{1}{2}\right)\\ &= \tfrac{2\sqrt{3}+1}{2} - j\tfrac{3}{2}.\end{aligned}$$

(b)

$$\begin{aligned}\left(\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right) &= e^{j(-\pi/6)}\tfrac{1}{\sqrt{2}}e^{j(-3\pi/4)}\\ &= \tfrac{1}{\sqrt{2}}e^{j(-11\pi/12)}.\end{aligned}$$

(c)

$$\begin{aligned}\left(\tfrac{\sqrt{3}}{2} - j\tfrac{1}{2}\right)/(1+j) &= \left[e^{j(-\pi/6)}\right]/\left[\sqrt{2}e^{j\pi/4}\right]\\ &= \tfrac{1}{\sqrt{2}}e^{j(-5\pi/12)}.\end{aligned}$$

(d)

$$\begin{aligned}e^{1+j\pi/4} &= ee^{j\pi/4}\\ &= e\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right)\\ &= \tfrac{1}{\sqrt{2}}e + j\tfrac{1}{\sqrt{2}}e.\end{aligned}$$

(e)

$$\begin{aligned}\left(\left(-\tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right)^*\right)^8 &= \left[e^{j(-2\pi/3)}\right]^8\\ &= e^{j(-16\pi/3)}\\ &= e^{j2\pi/3}.\end{aligned}$$

A.4 Show that the following identities hold, where z, z_1, and z_2 are arbitrary complex numbers:
(a) $|z_1/z_2| = |z_1| / |z_2|$ for $z_2 \neq 0$;
(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \neq 0$;
(c) $z + z^* = 2\,\mathrm{Re}\{z\}$;

(d) $zz^* = |z|^2$; and
(e) $(z_1z_2)^* = z_1^* z_2^*$.

Solution.

(a) We rewrite z_1 and z_2 in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}$$

where $r_1, r_2, \theta_1, \theta_2$ are real constants such that $r_1, r_2 \geq 0$. Consider the left-hand side of the given equation, which we can manipulate as follows (assuming that $z_2 \neq 0$):

$$\begin{aligned}\left|\frac{z_1}{z_2}\right| &= \left|\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right| \\ &= \frac{|r_1|\left|e^{j\theta_1}\right|}{|r_2|\left|e^{j\theta_2}\right|} \\ &= \frac{r_1}{r_2} \\ &= \frac{|z_1|}{|z_2|}.\end{aligned}$$

Thus, we have that $|z_1/z_2| = |z_1|/|z_2|$.

(d) We express z in Cartesian form as

$$z = x + jy.$$

Now, we have

$$\begin{aligned}zz^* &= (x+jy)(x-jy) \\ &= x^2 - jxy + jxy + y^2 \\ &= x^2 + y^2 \quad \text{and} \\ |z|^2 &= \left(\sqrt{x^2+y^2}\right)^2 \\ &= x^2 + y^2.\end{aligned}$$

Therefore, $zz^* = |z|^2$.

A.5 Use Euler's relation to prove the following identities:
(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$;
(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$; and
(c) $\cos^2\theta = \frac{1}{2}[1 + \cos 2\theta]$.

Solution.

(a) From Euler's relation, we know

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

Thus, we can write

$$\tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] = \tfrac{1}{2}[\cos\theta + j\sin\theta + \cos(-\theta) + j\sin(-\theta)].$$

Since $\cos\theta$ and $\sin\theta$ are even and odd functions, respectively, we can simplify the above equation to obtain

$$\begin{aligned}\tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] &= \tfrac{1}{2}[\cos\theta + j\sin\theta + \cos\theta - j\sin\theta] \\ &= \tfrac{1}{2}[2\cos\theta] \\ &= \cos\theta.\end{aligned}$$

Therefore, $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$.

A.6 Consider the rational functions given below, where z is a complex variable. For each function, find the value and order of its poles and zeros. Also, plot the poles and zeros in the complex plane.
(a) $F(z) = z^2 + jz + 3$;
(b) $F(z) = z + 3 + 2z^{-1}$;
(c) $F(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z+3z+2)}$;
(d) $F(z) = \dfrac{z^3 - z}{z^2 - 4}$;
(e) $F(z) = \dfrac{z + \frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and
(f) $F(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

Solution.

(d) First, we factor the numerator polynomial.

$$z^3 - z = z(z^2-1) = z(z+1)(z-1).$$

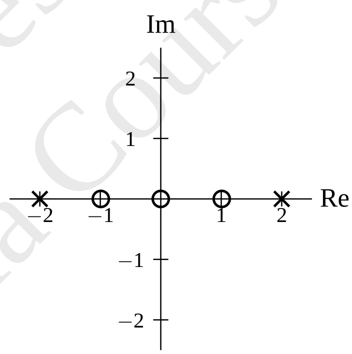
Next, we factor the denominator polynomial.

$$z^2 - 4 = (z+2)(z-2).$$

So, we have

$$F(z) = \frac{z(z+1)(z-1)}{(z+2)(z-2)}.$$

Therefore, $F(z)$ has first order zeros at -1, 0, and 1, and first order poles at -2 and 2.



(f) To find the poles and zeros of a rational function, we must factor the numerator and denominator polynomials. First, we factor $z^2 + 4z + \frac{17}{4}$. The quadratic formula yields

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(\frac{17}{4})}}{2(1)} = -2 \pm j\tfrac{1}{2}.$$

Thus, we have

$$z^2 + 4z + \tfrac{17}{4} = (z+2+j\tfrac{1}{2})(z+2-j\tfrac{1}{2}).$$

Next, we factor $z^2 + 2z + 2$. The quadratic formula yields

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = -1 \pm j.$$

So, we have

$$z^2+2z+2=(z+1+j)(z+1-j).$$

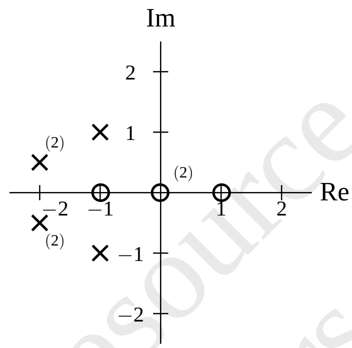
Next, we factor z^2-1 to obtain

$$z^2-1=(z+1)(z-1).$$

Combining the above factorization results, we have

$$F(z)=\frac{z^2(z+1)(z-1)}{(z+2+j\frac{1}{2})^2(z+2-j\frac{1}{2})^2(z+1+j)(z+1-j)}.$$

Therefore, $F(z)$ has first order zeros at 1 and -1, a second order zero at 0, first order poles at $-1+j$ and $-1-j$, and second order poles at $-2+j\frac{1}{2}$ and $-2-j\frac{1}{2}$.



A.7 Determine the values of z for which each of the functions given below is: i) continuous, ii) differentiable, and iii) analytic. Use your knowledge about polynomial and rational functions to deduce the answer. Simply state the final answer along with a short justification (i.e., two or three sentences). (This problem does not require a rigorous proof. In other words, do not use the Cauchy-Riemann equations for this problem.)
(a) $F(z)=3z^3-jz^2+z-\pi$ and
(b) $F(z)=\dfrac{z-1}{(z^2+3)(z^2+z+1)}$.

Solution.

(a) The function $F(z)$ is a polynomial. Polynomials are continuous, differentiable, and analytic everywhere.

(b) The function $F(z)$ is a rational function. Rational functions are continuous, differentiable, and analytic everywhere, except at points where the denominator polynomial becomes zero. So, we find these points. We factor $F(z)$ as

$$F(z)=\frac{z-1}{(z+j\sqrt{3})(z-j\sqrt{3})\left(z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}.$$

Therefore, the denominator becomes zero for

$$z\in\left\{-j\sqrt{3},j\sqrt{3},-\tfrac{1}{2}+j\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2}\right\}.$$

Therefore, $F(z)$ is continuous, differentiable, and analytic everywhere, except at the points: $-j\sqrt{3}$, $j\sqrt{3}$, $-\frac{1}{2}+j\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$.

A.9 Let $H(\omega)$ be a complex-valued function of the real variable ω. For each of the cases below, find $|H(\omega)|$ and $\arg H(\omega)$.

(a) $H(\omega) = \dfrac{1}{(1+j\omega)^{10}}$; and

(b) $H(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$.

Solution.

(a) First, we compute the magnitude of $H(\omega)$ to obtain

$$\begin{aligned}|H(\omega)| &= \frac{|1|}{|(1+j\omega)^{10}|}\\ &= \frac{1}{|1+j\omega|^{10}}\\ &= \frac{1}{(\sqrt{1+\omega^2})^{10}}\\ &= \frac{1}{(1+\omega^2)^5}.\end{aligned}$$

Next, we compute the argument of $H(\omega)$ to obtain

$$\begin{aligned}\arg H(\omega) &= \arg\left(\frac{1}{(1+j\omega)^{10}}\right)\\ &= \arg 1 - \arg\left([1+j\omega]^{10}\right)\\ &= -\arg\left([1+j\omega]^{10}\right)\\ &= -\arg\left([\sqrt{1+\omega^2}e^{j\arctan\omega}]^{10}\right)\\ &= -10\arctan\omega.\end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg H(\omega) = 2\pi k - 10\arctan\omega$$

for all integer k.

(b) First, we compute the magnitude of $H(\omega)$ to obtain

$$\begin{aligned}|H(\omega)| &= \frac{|-2-j\omega|}{|(3+j\omega)^2|}\\ &= \frac{|-2-j\omega|}{|3+j\omega|^2}\\ &= \frac{\sqrt{4+\omega^2}}{(\sqrt{9+\omega^2})^2}\\ &= \frac{\sqrt{4+\omega^2}}{9+\omega^2}.\end{aligned}$$

Next, we calculate the argument of $H(\omega)$ as

$$\begin{aligned}\arg H(\omega) &= \arg(-2-j\omega) - \arg\left([3+j\omega]^2\right)\\ &= \pi + \arctan\omega/2 - \arg\left([\sqrt{9+\omega^2}e^{j\arctan\omega/3}]^2\right)\\ &= \pi + \arctan\omega/2 - 2\arctan\omega/3.\end{aligned}$$

Since the argument is not uniquely determined, in the most general case, we have

$$\arg H(\omega) = (2k+1)\pi + \arctan \omega/2 - 2\arctan \omega/3$$

for all integer k.