ECE 260

EXAM 4

SOLUTIONS

(FALL 2022)

QUESTION 1

$$\boxed{x(t) = e^{-j2t} e^{-|3t-1|} \quad \text{and} \quad e^{-|t|} \overset{CTFT}{\longleftrightarrow} \frac{2}{\omega^2+1}}$$

$$v_1(t) = e^{-|t|} \overset{CTFT}{\longleftrightarrow} V_1(\omega) = \frac{2}{\omega^2+1}$$

$$v_2(t) = v_1(t-1) \overset{CTFT}{\longleftrightarrow} V_2(\omega) = e^{-j\omega} V_1(\omega)$$

$$v_3(t) = v_2(3t) \overset{CTFT}{\longleftrightarrow} V_3(\omega) = \frac{1}{3} V_2(\omega/3)$$

$$x(t) = e^{-j2t} v_3(t) \overset{CTFT}{\longleftrightarrow} X(\omega) = V_3(\omega+2)$$

$$X(\omega) = V_3(\omega+2)$$

$$= \frac{1}{3} V_2\left(\frac{\omega+2}{3}\right)$$

$$= \frac{1}{3} e^{-j(\omega+2)/3} V_1\left(\frac{\omega+2}{3}\right)$$

$$= \frac{1}{3} e^{-j(\omega+2)/3} \frac{2}{\left(\frac{\omega+2}{3}\right)^2+1} \longleftarrow \text{This is sufficiently simplified.}$$

$$= \frac{2}{3} e^{-j(\omega+2)/3} \frac{1}{\frac{(\omega+2)^2}{9}+1}$$

$$= \frac{2}{3} e^{-j(\omega+2)/3} \frac{1}{\left(\frac{(\omega+2)^2+9}{9}\right)}$$

$$= \frac{2}{3} e^{-j(\omega+2)/3} \frac{9}{(\omega+2)^2+9}$$

$$= e^{-j(\omega+2)/3} \frac{6}{(\omega+2)^2+9}$$

QUESTION 2

$$\boxed{x(t) = -t^3 e^{-2t} u(t) \xleftrightarrow{\text{CTFT}} X(\omega) = \frac{-6}{(2+j\omega)^4}}$$

(a) $$\|X(\omega)\| = \left\| \frac{-6}{(2+j\omega)^4} \right\| = \frac{\|-6\|}{\|(2+j\omega)^4\|} = \frac{6}{\|2+j\omega\|^4} = \frac{6}{(\sqrt{4+\omega^2})^4}$$

$$= \frac{6}{(4+\omega^2)^2}$$

(b) $$\arg X(\omega) = \arg\left(\frac{-6}{(2+j\omega)^4}\right)$$

$$= \arg(-6) - \arg\left[(2+j\omega)^4\right]$$

$$= \arg(-6) - 4\arg(2+j\omega)$$

$$= \pi - 4\arctan(\omega/2)$$

[In the general case, we have

$\arg X(\omega) = (2k+1)\pi - 4\arctan(\omega/2)$ for all $k \in \mathbb{Z}$.]

(c) The function x has the most spectral content at the frequency that maximizes $|X(\omega)|$.

Clearly, $|X(\omega)|$ has a maximum of $\frac{6}{16} = \frac{3}{8}$ at the frequency $\omega = 0$.

QUESTION 3

$$v_o(t) = 2\,i(t) + 2\int_{-\infty}^{t} i(\tau)\,d\tau + v_1(t)$$

$$i(t) = \frac{1}{2}\int_{-\infty}^{t} v_1(\tau)\,d\tau$$

Taking the derivative of each of the given equations, we have

$$D v_o(t) = 2\,D i(t) + 2\,i(t) + D v_1(t)$$

$$D i(t) = \frac{1}{2} v_1(t)$$

Taking the Fourier transform of each equation we have

$$jw\,V_o(w) = 2jw\,I(w) + 2\,I(w) + jw\,V_1(w)$$

$$jw\,I(w) = \frac{1}{2} V_1(w) \Rightarrow I(w) = \frac{1}{j2w} V_1(w)$$

Combining these equations, we have

$$jw\,V_o(w) = 2jw\left[\frac{1}{j2w} V_1(w)\right] + 2\left[\frac{1}{j2w} V_1(w)\right] + jw\,V_1(w)$$

$$jw\,V_o(w) = V_1(w) + \frac{1}{jw} V_1(w) + jw\,V_1(w)$$

$$jw\,V_o(w) = \left[1 + \frac{1}{jw} + jw\right] V_1(w)$$

$$jw\,V_o(w) = \frac{jw + 1 - w^2}{jw} V_1(w)$$

$$\frac{V_1(w)}{V_o(w)} = \frac{jw\,(jw)}{-w^2 + jw + 1} = \frac{-w^2}{-w^2 + jw + 1}$$

$$H(w) = \frac{w^2}{w^2 - jw - 1}$$

QUESTION 4

$$\boxed{y(t) = [1 + \cos(15t)]\, x(t)}$$

(a) $y(t) = [1 + \frac{1}{2}e^{j15t} + \frac{1}{2}e^{-j15t}]\, x(t)$

$y(t) = x(t) + \frac{1}{2}e^{j15t}x(t) + \frac{1}{2}e^{-j15t}x(t)$

$Y(w) = X(w) + \frac{1}{2}X(w-15) + \frac{1}{2}X(w+15)$

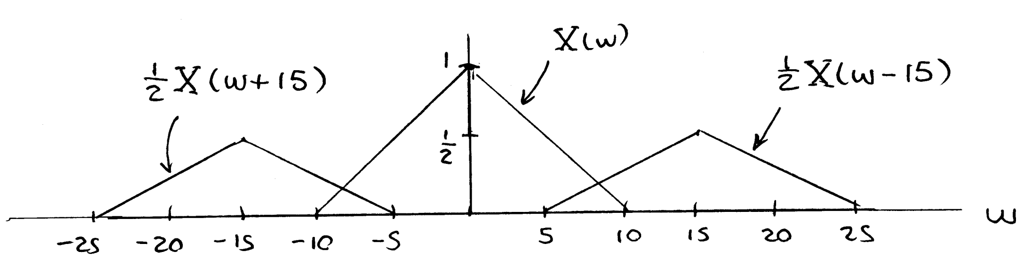
(b) From the sampling theorem, $w_X > 2(10) = 20$.

That is, we must sample at a rate exceeding twice the highest frequency in x.

(c) The frequency spectrum $Y(w)$ is nonzero only in the interval $[-25, 25]$.

So, we must sample such that $w_Y > 2(25) = 50$

That is, we must sample at a rate exceeding twice the highest frequency in y.



QUESTION 5

$$\boxed{x(t) = e^{-2t}\cos(3t)\,u(t) \quad \text{and} \quad h(t) = e^{-2t}u(t)}$$

$$X(\omega) = \mathcal{F}x(\omega) = \mathcal{F}\{e^{-2t}\cos(3t)\,u(t)\}(\omega)$$

$$= \frac{2+j\omega}{(2+j\omega)^2+9}$$

$$H(\omega) = \mathcal{F}h(\omega) = \mathcal{F}\{e^{-2t}u(t)\}(\omega)$$

$$= \frac{1}{2+j\omega}$$

$$Y(\omega) = \mathcal{F}y(\omega) = X(\omega)\,H(\omega)$$

$$= \left[\frac{2+j\omega}{(2+j\omega)^2+9}\right]\left[\frac{1}{2+j\omega}\right]$$

$$= \frac{1}{(2+j\omega)^2+9}$$

$$y(t) = \mathcal{F}^{-1}\left\{\frac{1}{(2+j\omega)^2+9}\right\}(t)$$

$$= \frac{1}{3}\mathcal{F}^{-1}\left\{\frac{3}{(2+j\omega)^2+3^2}\right\}(t)$$

$$= \frac{1}{3}e^{-2t}\sin(3t)\,u(t)$$