ELEC 360 : Control Theory and Systems I

Midterm

February 18th, 2010

Name: ______________________________

Student Number: ______________________________

Mark: ________ /30

Notes:

- Students are permitted a single 8.5 by 11 inch sheet of notes.
- Programmable calculators are allowed.
  - No other aids permitted.
  - Use of cell phones or other electronic devices, except calculators, during the exam will result in a zero grade.

1. From analyzing a real system you have determined that for an input $x(t)$ the system produces an output $y(t)$ that is given by:

$$\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{d^2t} + 4\frac{dy(t)}{dt} - \frac{d^2x}{dt^2} - 12\frac{dx(t)}{dt} - 32x(t) = 0$$

(a) Determine the transfer function of this system. (4 pts)

A linear differential equation has been given that relates $x(t)$ to $y(t)$ - so taking the Laplace transforms gives:

$$\begin{aligned} s^3Y(s) + 2s^2Y(s) + 4s^Y(S) - s^2X(s) - 12sX(s) - 32X(s) = 0 \\ Y(s)s(s^2+2s+4) = X(s)(s^2+12s+32) \\ \frac{Y(s)}{X(s)} = \frac{(s+4)(s+8)}{s(s^2+2s+4)} \end{aligned}$$

(b) Determine $y(t)$ when the input to the system is $x(t)$ is the unit step function. (4 pts)

If $x(t) = u(t)$ then $X(s) = \frac{1}{s}$ So,

$$\begin{aligned} Y(s) &= \left[\frac{(s+4)(s+8)}{s(s^2+2s+4)}\right] \cdot \left[\frac{1}{s}\right] \\ &= \frac{(s+4)(s+8)}{s^2(s^2+2s+4)} \\ &= \frac{s-5}{s^2+2s+4} + \frac{8}{s^2} - \frac{1}{s} \end{aligned}$$

Taking the inverse transform gives:

$$y(t) = 8tu(t) + e^{-t}u(t)\left[\cos(\sqrt{3}t) - 2 * \sqrt{3}\sin(\sqrt{3}t)\right] - u(t)$$

(c) Determine the steady state error of the system as $t \to \infty$ when x(t) is a unit step. (2 pts)

$$\begin{aligned} e_{ss} = \lim_{t\to\infty} e(t) &= \lim_{t\to\infty} [y(t) - x(t)] \\ &= \lim_{t\to\infty} \left[8tu(t) + e^{-t}u(t)\left[\cos(\sqrt{(3)}t) - 2 * \sqrt{(3)}\sin(\sqrt{(3)}t)\right] - u(t) - u(t)\right] \qquad (1) \\ &= \infty \end{aligned}$$

2. You are given a system with the *closed loop* transfer function:

$$\frac{\mathrm{C}(s)}{\mathrm{R}(s)} = \frac{K \cdot 15}{s^2 + 5s + 25}$$

(a) What is the rise time and natural frequency for the unit step response of this system if $K = 2$? (3 pts)

$$\begin{aligned}\frac{\mathrm{C}(s)}{\mathrm{R}(s)} &= \frac{2 \cdot 15}{s^2 + 5s + 25} \\ &= \frac{30}{25}\frac{25}{s^2 + 5s + 25} \\ &= 1.2\left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]\end{aligned}$$

Hence, $\omega_n = 5$ rad/s and, therefore, $\zeta = 0.5$, $\sigma = \zeta \cdot \omega_n = 0.5 \cdot 5 = 2.5$, and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.3301$ rad/s.

We need to take care when computing the rise time t_r. The formula on page B14 of the Lecture notes *does not* take into account that our K value for the standardized second order form is not 1. Instead it is 1.2.

So we need to check that the formula of page B14 actually applies to our case. The definition of the rise time is with respect to the first crossing of the steady state level for the unit step input. For our system this will be the first time we cross 1.2. But, scaling (or amplification) is a linear operator in terms of the formula at the top of page B14 of the Lecture notes. i.e., the 1.2 will appear on both the left and right side of the first line of the rise time derivation on page B14. Therefore, it will cancel out. Hence, from the second line forward the derivation remains the same.

$$\begin{aligned}t_r &= \frac{1}{\omega_d}\left[\pi + \tan^{-1}\left(\frac{-\omega_d}{\sigma}\right)\right] \\ &= \frac{1}{4.3301}\left[\pi - \tan^{-1}\left(\frac{-4.3301}{2.5}\right)\right] \\ &= 0.4837 \text{ sec}\end{aligned}$$

(b) When does the first zero crossing after $t = 0$ occur for the unit impulse response for this system when $K = 2$? (3 pts)

The first zero crossing for the impulse response occurs at $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.4837} = 0.7255$ sec. (page B19 of the Lecture Notes)

(c) What are the steady state errors for this system when $x(t)$ is the unit step, unit ramp, and unit parabola? (2 pts) The system is a type 0 system so,

Unit step: $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{30}{25}} = 0.4545$

Unit ramp: $e_{ss} = \infty$

Unit parabola: $e_{ss} = \infty$

(d) Without changing its damping ratio or natural frequency how can you modify the system such that it will have a faster response? (2 pts)

A zero can be added to the system to increase its response time - but care must be taken in placing the zero if the shape of the step response is to be preserved and adjustments to the scaling factor K will also be required. (by using Matlab you can see the effects that placing different zeros will have on the system's unit step response.)

For a system with an *open loop* transfer function given by $G(s)$ where,

$$G(s) = \frac{K(s^2 + 2s + 10)}{s^4 + 3s^3 + 4s^2 + 25}$$

Determine the values of K for which the *closed loop* system is stable. (5 pts)

Ruth-Hurwitz stability test applied to the *closed loop* transfer function - so need $A(s)$ from the closed loop system,

$$\begin{aligned} A(s) &= (s^4 + 3s^3 + 4s^2 + 25) + K(s^2 + 2s + 10) \\ &= s^4 + 3s^3 + (4 + K)s^2 + 2Ks + (10K + 25) \end{aligned} \tag{2}$$

Writing the Ruth-Hurwitz table:

$$\begin{array}{llll} s^4: & 1 & 4+K & 10K+25 \\ s^3: & 3 & 2K & \\ s^2: & \frac{12+K}{3} & 10K+25 & \\ s^1: & \frac{2K^2-66K-225}{12+K} & & \\ s^0: & 10K+25 & & \end{array}$$

System is stable if there are no sign changes in the first column, therefore we require that:

$$\frac{2K^2 - 66K - 225}{12 + K} \geq 0 \rightarrow K \geq -3.1150 \text{ and } K \geq 36.1150$$

and

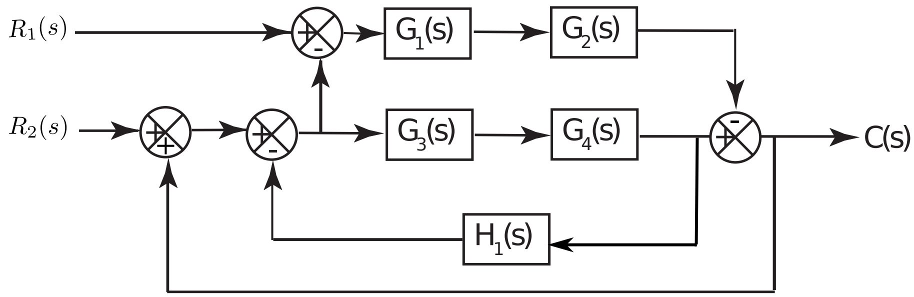
$$\frac{12 + K}{3} \geq 0 \rightarrow K \geq -12$$

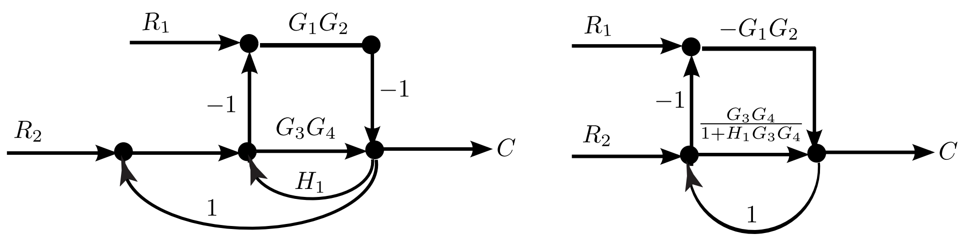
and

$$10K + 25 \geq 0 \rightarrow K \geq -2.5$$

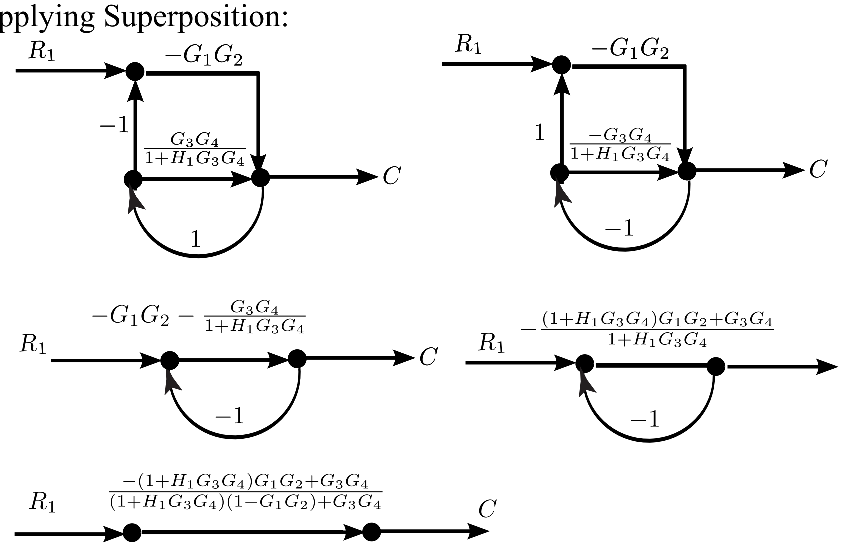
Combining, these conditions gives that $K \geq 36.1150$ if system is to be stable.

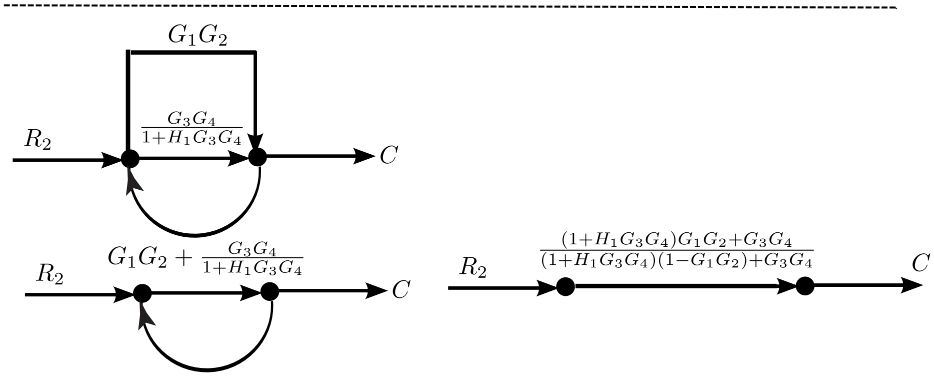
3. For the following block diagram determine the system's transfer function. (5 pts)





Applying Superposition:





From, this point the transfer function can be easily obtained as

$$C(s) = Y_1(s)R_1(s) + Y_2(s)R_2(s)$$

where $Y_1(s)$ and $Y_2(s)$ are given as per the superpositions solutions given above.

END OF EXAM