CHAPTER 7

B-7-1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{10}{s+11}$$

The steady-state outputs of the system when it is subjected to the given inputs are

(a)

$$c_{ss}(t) = 0.905 \sin(t + 24.8^\circ)$$

(b)

$$c_{ss}(t) = 1.79 \cos(2t - 55.3^\circ)$$

(c)

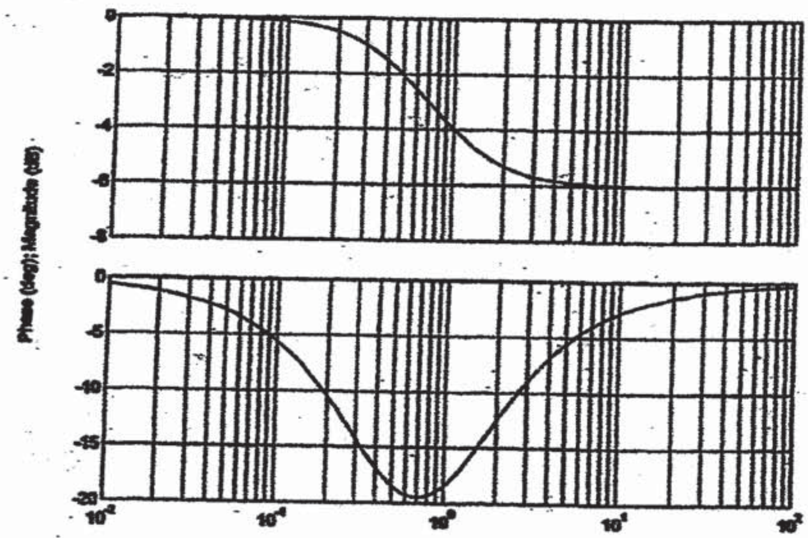
$$c_{ss}(t) = 0.905 \sin(t + 24.8^\circ) - 1.79 \cos(2t - 55.3^\circ)$$

B-7-2. The steady-state output $c_{ss}(t)$ is

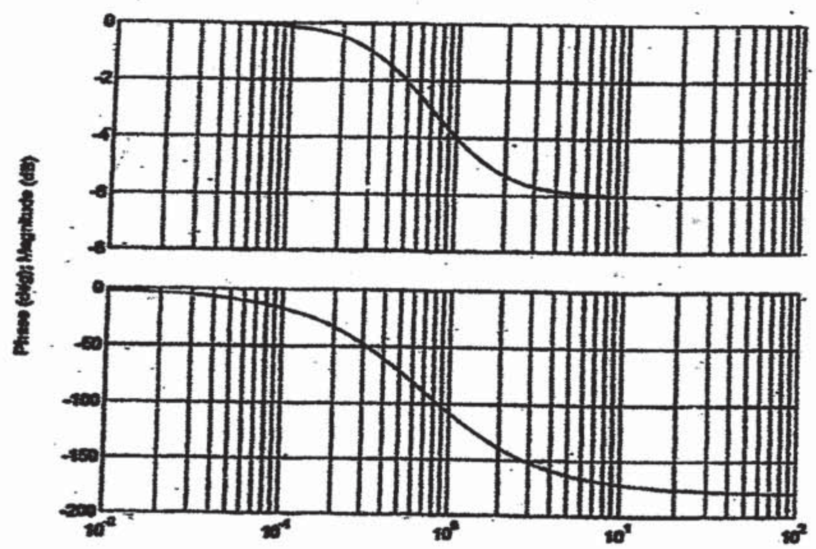
$$c_{ss}(t) = RK\sqrt{\frac{1+T_2^2\omega^2}{1+T_1^2\omega^2}} \sin\left(\omega t + \tan^{-1} T_2\omega - \tan^{-1} T_1\omega\right)$$

B-7-3.

Bode Diagram of G1(s) = (1 + s)/(1 + 2s)

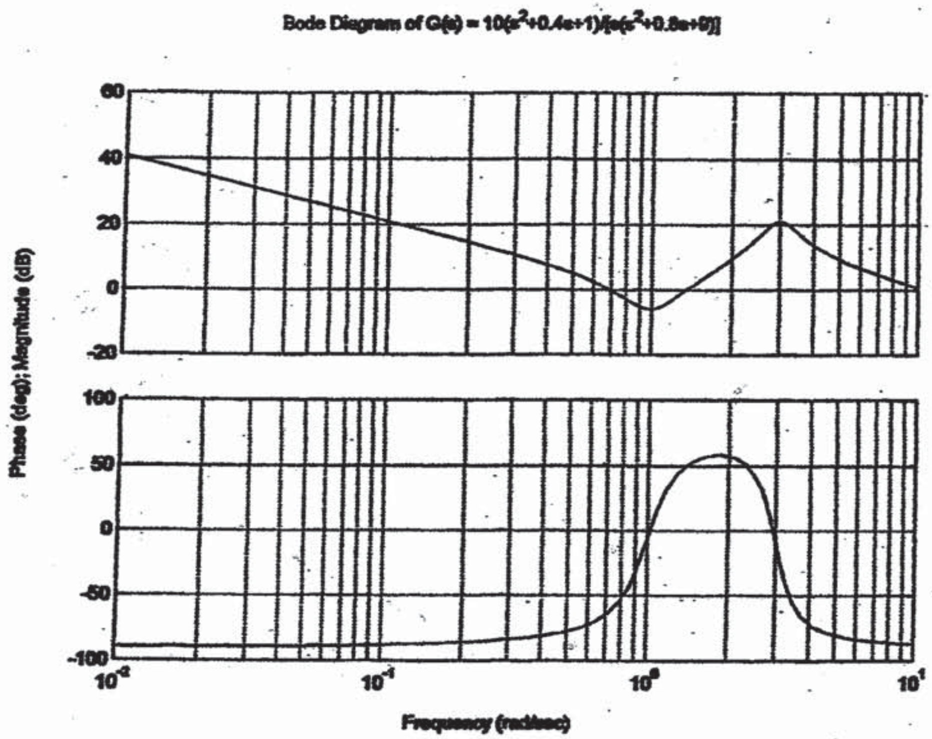


Bode Diagram of G2(s) = (1 - s)/(1 + 2s)



B-7-4. The following MATLAB program produces the Bode diagram shown below.

```
% ***** Bode diagram *****

num = [0  10  4  10];
den = [1  0.8  9  0];
bode(num,den)
title('Bode Diagram of G(s) = 10(s^2+0.4s+1)/[s(s^2+0.8s+9)]')
```



B-7-5. Noting that

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}$$

we have

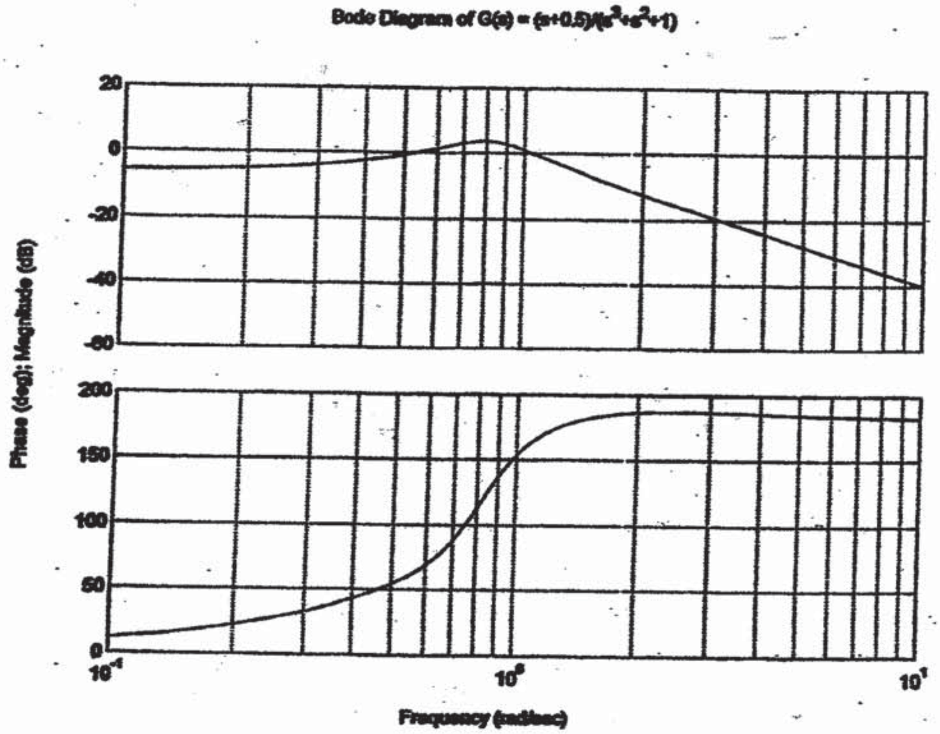
$$|G(j\omega_n)| = \left|\frac{1}{-1 + 2\zeta j + 1}\right| = \frac{1}{2\zeta}$$

B-7-6.

$$G(s) = \frac{s+0.5}{s^3+s^2+1}$$

The following MATLAB program produces the Bode diagram of G(s) shown below. Notice that the phase curve starts from 0° and ends at 180°.

```
% ***** Bode Diagram *****

num = [0 0 1 0.5];
den = [1 1 0 1];
bode(num,den)
title('Bode Diagram of G(s) = (s+0.5)/(s^3+s^2+1)')
```



To verify why the phase angle starts from 0° and ends at 180°, we may compute angles $\angle G(j0)$ and $\angle G(j\infty)$. Since

$$G(s) = \frac{s+0.5}{(s+1.4656)(s-0.2328-j0.7926)(s-0.2328+j0.7926)}$$

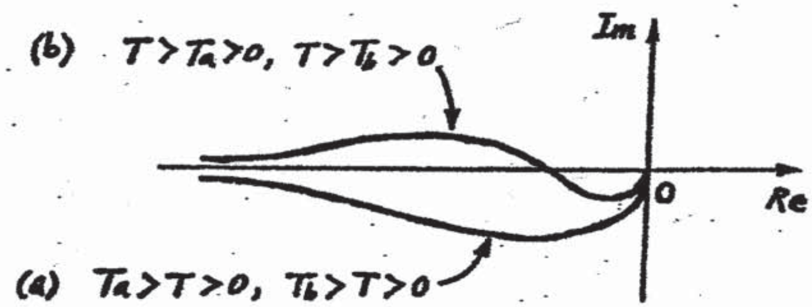
we have

$$\angle G(j0) = \angle 0.5 - \angle 1.4656 - \angle -0.2328-j0.7926 - \angle -0.2328+j0.7926$$

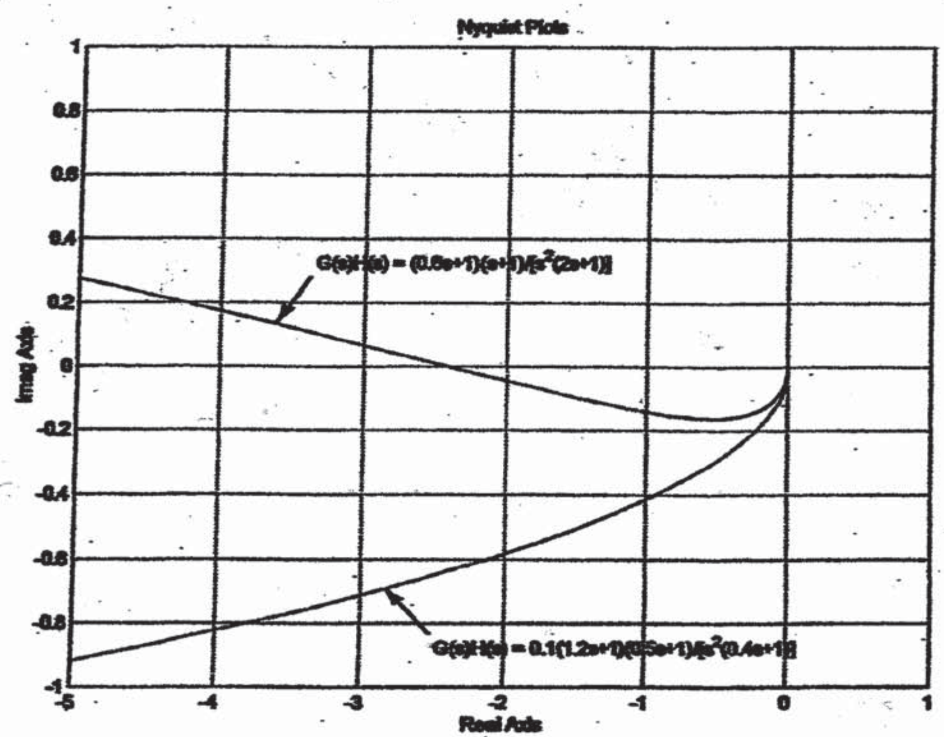
$$= 0° - 0° - \tan^{-1}\frac{0.7926}{0.2328} + \tan^{-1}\frac{0.7926}{0.2328} = 0°$$

and
$$\angle G(j\infty) = 90^\circ - 90^\circ - \tan^{-1}\frac{\infty}{-0.2328} - \tan^{-1}\frac{\infty}{-0.2328}$$
$$= 90^\circ - 90^\circ + 90^\circ + 90^\circ = 180^\circ$$

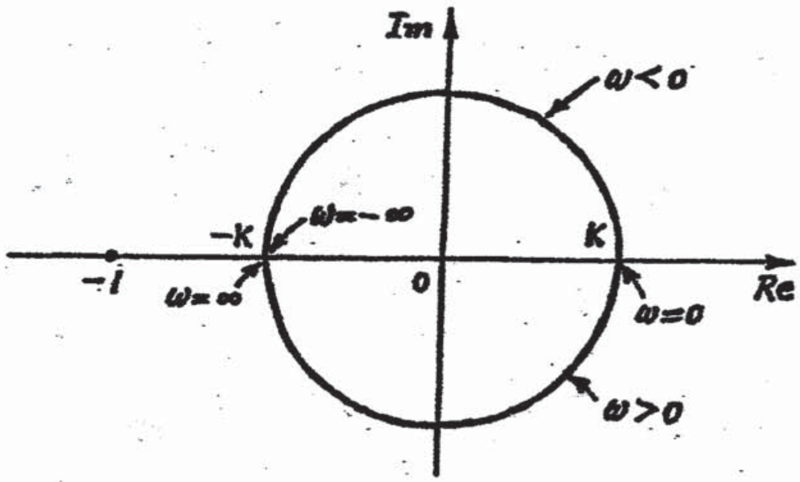
B-7-7. Typical Nyquist curves for the cases (a) and (b) are shown below.



Nyquist plots of example systems that belong to case (a) and case (b) are shown below.



B-7-8.



The stability requirement of the unity feedback control system with

$$G(j\omega) = \frac{K(1-j\omega)}{j\omega+1}$$

is that -K be greater than -1, or

$$K < 1$$

Since we assume that K > 0, the condition for stability is

$$1 > K > 0$$

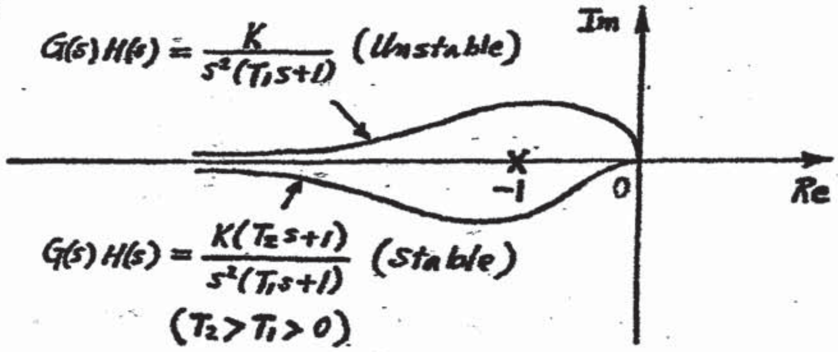
B-7-9. A closed-loop system with the following open-loop transfer function

$$G(s)H(s) = \frac{K}{s^2(T_1 s+1)} \qquad (T_1 > 0)$$

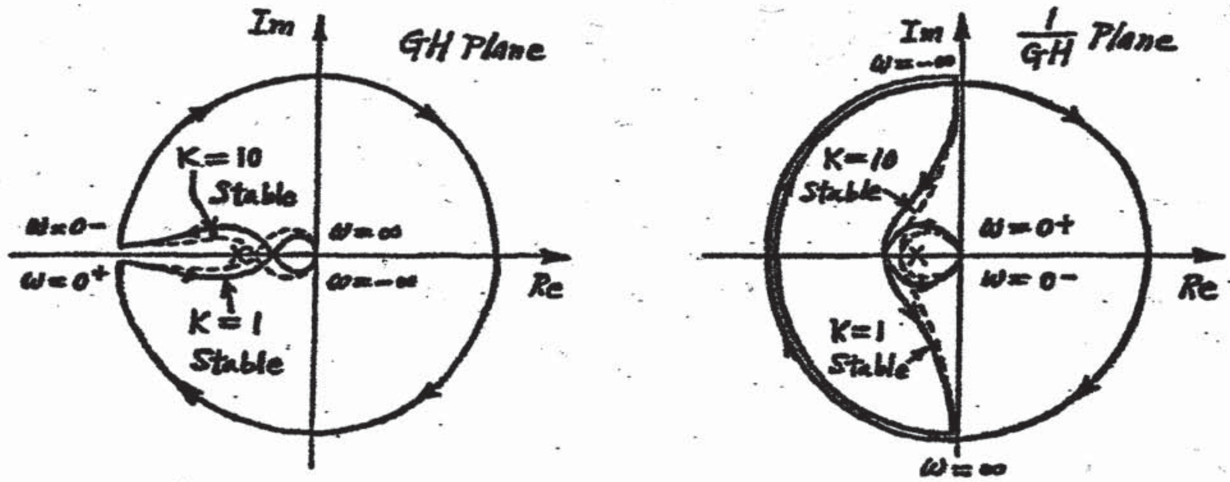
is unstable, while a closed-loop system with the following open-loop transfer function is stable.

$$G(s)H(s) = \frac{K(T_2 s+1)}{s^2(T_1 s+1)} \qquad (T_2 > T_1 > 0)$$

Nyquist plots of these two systems are shown below.



B-7-10.



The system is stable for both K = 1 and K = 10.

B-7-11.

$$G(j\omega)H(j\omega) = \frac{Ke^{-2j\omega}}{j\omega}$$

$$\angle G(j\omega)H(j\omega) = \angle \cos 2\omega - j\sin 2\omega - 90^\circ$$

$$= -2\omega - 90^\circ$$

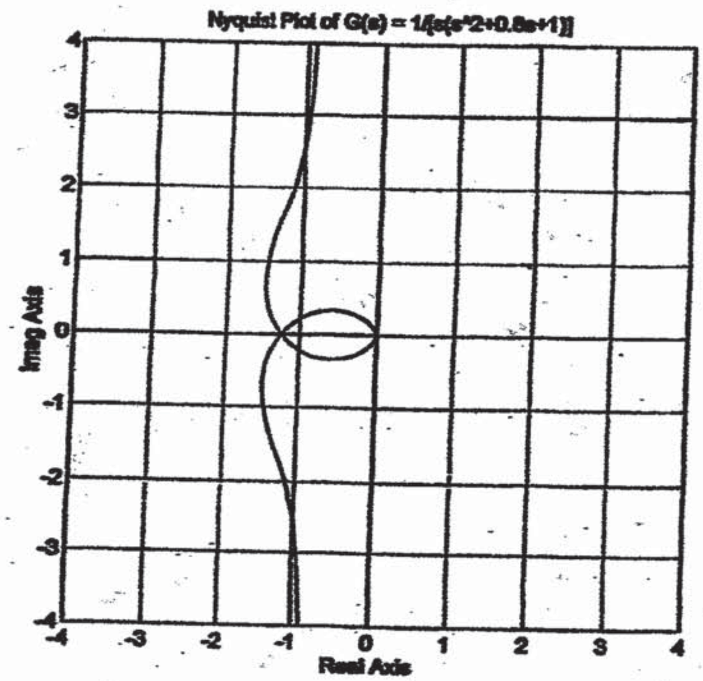
The phase angle becomes equal to $-180°$ at $2\omega = \pi/2$ rad/sec. For stability, the magnitude $|G(j\omega)H(j\omega)|$ at $\omega = \pi/4$ must be less than unity. Hence, noting that

$$\left|G(j\omega)H(j\omega)\right| = \frac{K}{\omega}$$

we require that $K < \pi/4$ for stability.

B-7-12. The following MATLAB program will produce the Nyquist plot shown below.

```
% ***** Nyquist plot *****

num = [0  0  0  1];
den = [1  0.8  1  0];
nyquist(num,den)
v = [-4  4  -4  4]; axis(v); axis('square')
grid
title('Nyquist Plot of G(s) = 1/[s(s^2+0.8s+1)]')
```

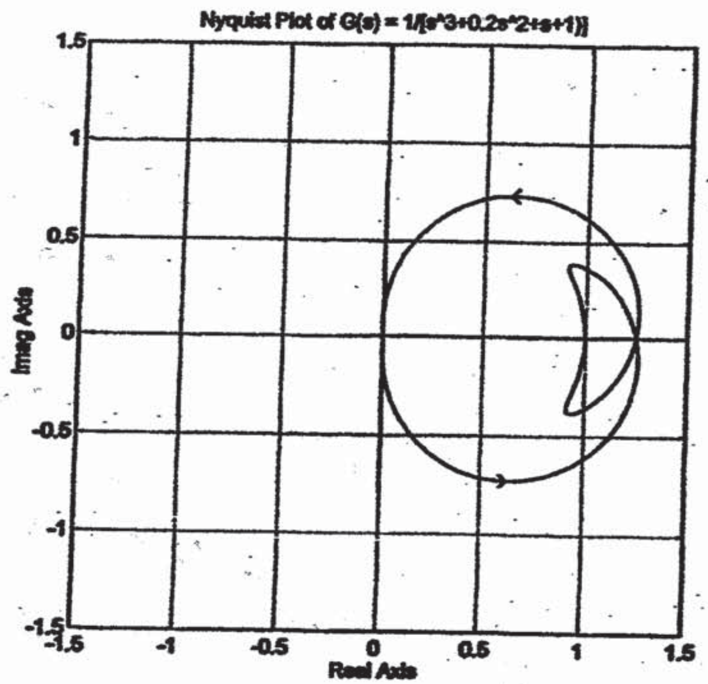


B-7-13. Note that G(s) has two open-loop poles in the right-hals s plane, as seen from the following MATLAB output.

```
p = [1  0.2  1  1];
roots(p)

ans =

  0.2623 + 1.1451i
  0.2623 - 1.1451i
 -0.7246
```

The following MATLAB program produces the Nyquist plot shown below.

```
% ***** Nyquist plot *****

num = [0  0  0  1];
den = [1  0.2  1  1];
nyquist(num,den)
v = [-1.5  1.5  -1.5  1.5]; axis(v); axis('square')
grid
title('Nyquist Plot of G(s) = 1/[s^3+0.2s^2+s+1]]')
```

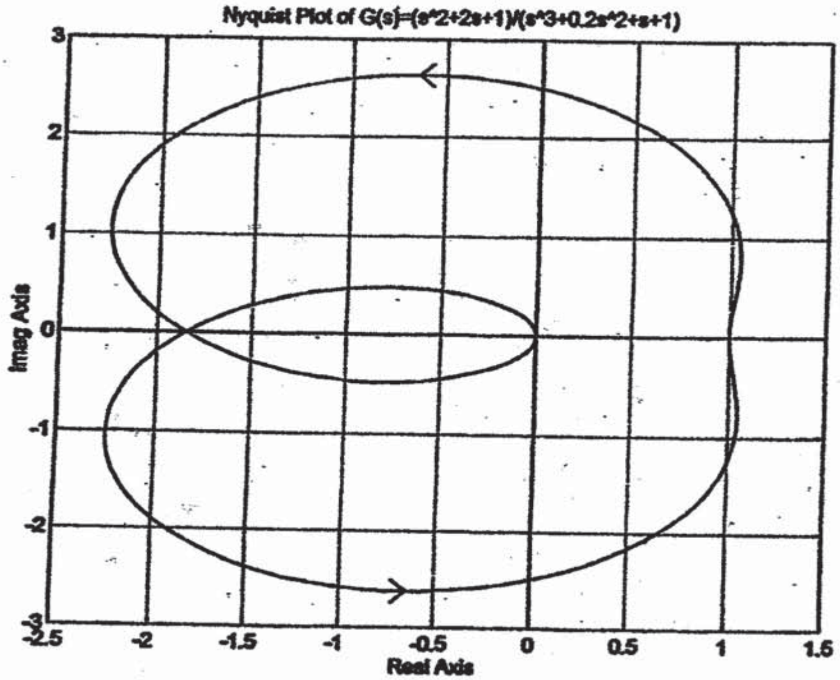


From the plot notice that the critical point (-1+j0) is not encircled. Because there are two open-loop poles in the right-half s plane and no encirclement of the critical point, the closed-loop system is unstable.

B-7-14. The following MATLAB program produces the Nyquist plot shown on the next page.

```
% ***** Nyquist plot *****

num = [0  1  2  1];
den = [1  0.2  1  1];
nyquist(num,den)
grid
title('Nyquist Plot of G(s)=(s^2+2s+1)/(s^3+0.2s^2+s+1)')
```

Since G(s) has two open-loop poles in the right-half s plane (see the solution to Problem B-7-13) and the Nyquist plot encircles the critical point (-1+j0) twice counterclockwise, the system is stable.



B-7-15. The open-loop transfer function is

$$G(s) = \frac{1}{s(s-1)}$$

The points corresponding to $s = j0+$ and $s = j0-$ on the locus of $G(s)$ in the $G(s)$ plane are $j\infty$ and $-j\infty$, respectively. On the semicircular path with radius ε (where $\varepsilon \ll 1$), the complex variable s can be written as

$$s = \varepsilon e^{j\theta}$$

where θ varies from $-90°$ to $+90°$. Then $G(s)$ becomes

$$G(\varepsilon e^{j\theta}) = -\frac{1}{\varepsilon e^{j\theta}} = \frac{1}{\varepsilon} e^{-j(\theta+180°)}$$

The value $1/\varepsilon$ approaches infinity as ε approaches zero, and $-\theta$ varies from $-90°$ to $-270°$ as a representative point s moves along the semicircle in the s plane. Thus the points $G(j0-) = -j\infty$ and $G(j0+) = +j\infty$ are joined by a semicircle of infinite radius in the left-half G plane. The infinitesimal semicircular detour around the origin in the s plane maps into the G plane as a semicircle of infinite radius. Figure (a) shows the G(s) locus in the G plane. [Figure (a) is shown on the next page.]

Since $G(s)$ has one pole in the right-half s plane ($P = 1$) and $G(s)$ locus encircles the $-1 + j0$ point once clockwise ($N = 1$), we have

$$Z = N + P = 2$$

There are two zeros of $1 + G(s)$ in the right-half s plane. Therefore, the system is unstable.

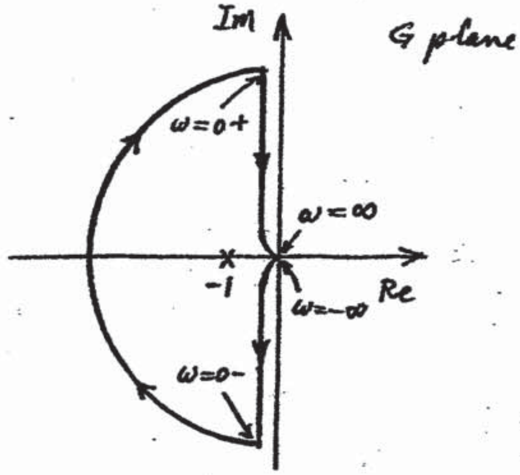
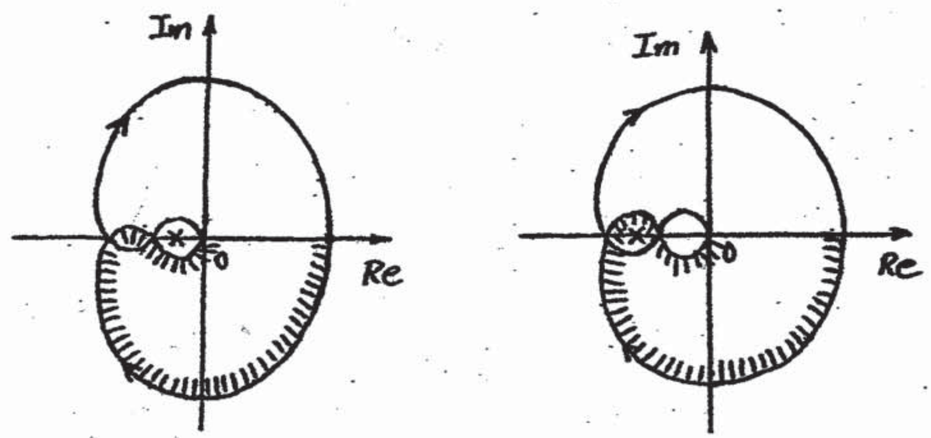


Figure (a)

B-7-16. Since G(s) has no poles in the right-half s plane, the stability of the system can be studied by checking the enclosure of the -1 + j0 point by the Nyquist locus for $0 < \omega < \infty$.

If the Nyquist plot of G(s) is as shown in Figure 7-158(a), then there is no enclosure of the -1 + j0 point. [See Figure (a) below.] Hence, the system is stable.

For the case of the Nyquist plot shown in Figure 7-158(b), the -1 + j0 point is enclosed by the Nyquist plot of $G(j\omega)$ for $0 < \omega < \infty$. [See Figure (b) below.] Hence, the system is unstable.



(a) (b)

B-7-17. Consider the case where G(s) has one pole in the right-half s plane. From the Nyquist plot of $G(j\omega)$ shown on the next page, the -1 + j0 point is encircled by the $G(j\omega)$ locus once clockwise and once counterclockwise. Hence N = 0. Since G(s) has one pole in the right-half s plane, we have P = 1. Since

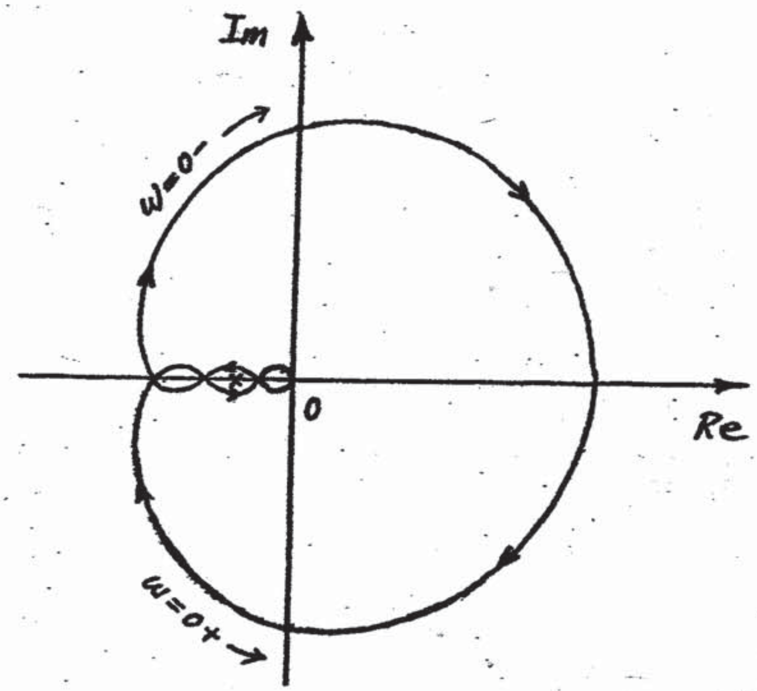
$$Z = N + P = 0 + 1 = 1$$

the system is unstable.

Next, consider the case where G(s) has no pole in the right-half s plane, but has one zero in the right-half s plane. The $-1 + j0$ point is encircled by the $G(j\omega)$ locus once clockwise and once counterclockwise. Hence, N = 0. Since G(s) has no poles in the right-half s plane, we have P = 0. Therefore,

$$Z = N + P = 0 + 0 = 0$$

The system is stable. (Note that the presence of a zero of G(s) in the right-half s plane does not affect the stability of the system.)

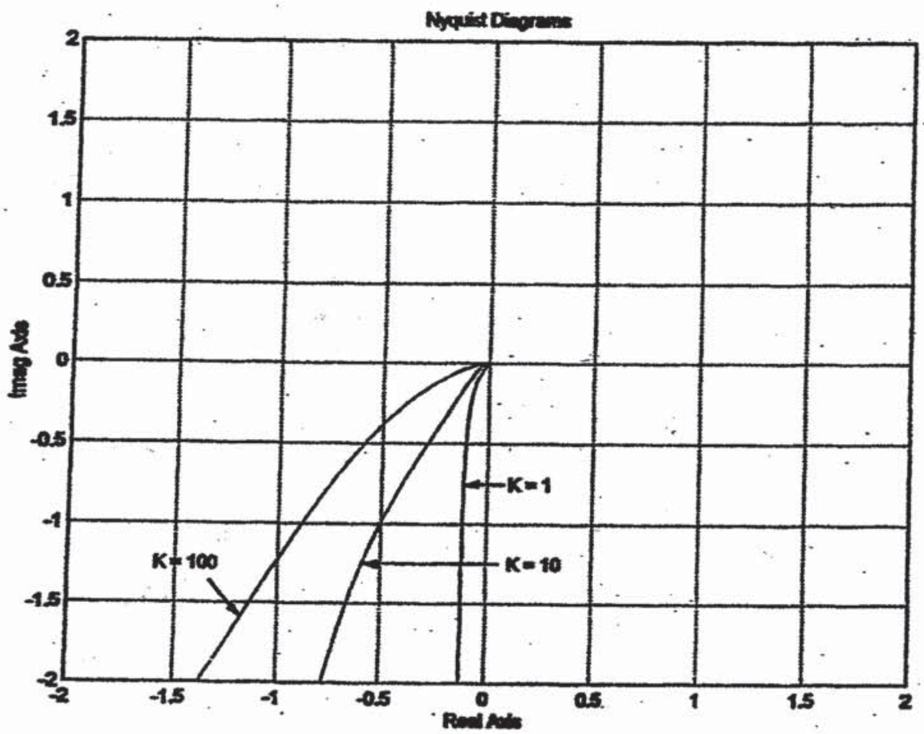


B-7-18.

$$G(s) = \frac{K(s+2)}{s(s+1)(s+10)}$$

A MATLAB program to plot Nyquist diagrams of G(s) for K = 1, K = 10, and K = 100 is shown below. The resulting Nyquist diagrams are shown on the next page.

```
%***** Nyquist Diagrams *****

num = [1  2];
den = [1  11  10  0];
w = 0.1:0.1:100;
[re1,im1,w] = nyquist(num,den,w);
[re2,im2,w] = nyquist(10*num,den,w);
[re3,im3,w] = nyquist(100*num,den,w);
plot(re1,im1,re2,im2,re3,im3)
v = [-2  2  -2  2]; axis(v)
grid
title('Nyquist Diagrams')
xlabel('Real Axis')
ylabel('Imag Axis')
text(0.1,-0.75,'K = 1')
text(0.1,-1.25,'K = 10')
text(-1.6,-1.25,'K = 100')
```

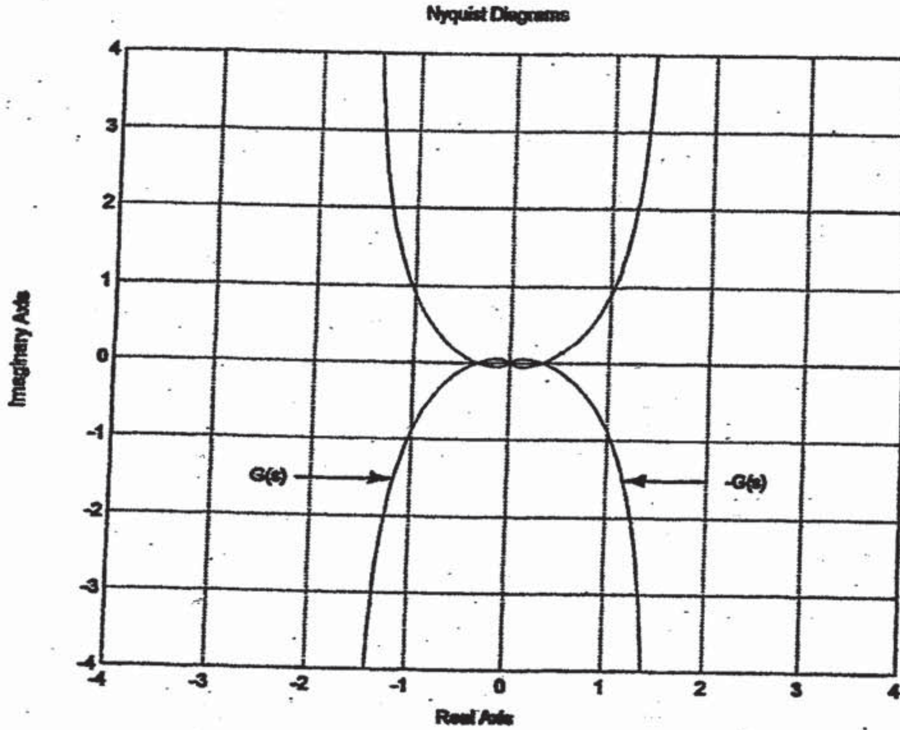


B-7-19.

$$G(s) = \frac{2}{s(s+1)(s+2)}$$

The Nyquist diagrams for G(s) and -G(s) are symmetric about the imaginary axis. A MATLAB program for plotting the Nyquist diagrams for the two cases is shown below. The resulting Nyquist diagrams are shown on the next page.

```
% ***** Nyquist Diagrams of G(s) and -G(s) *****

num1 = [0  0  0  2];
den1 = [1  3  2  0];
num2 = [0  0  0  -2];
den2 = [1  3  2  0];
nyquist(num1,den1)
hold
Current plot held
nyquist(num2,den2)
v = [-4  4  -4  4]; axis(v)
grid
text(-2.6,-1.5,'G(s)')
text(2.2,-1.5,'-G(s)')
```

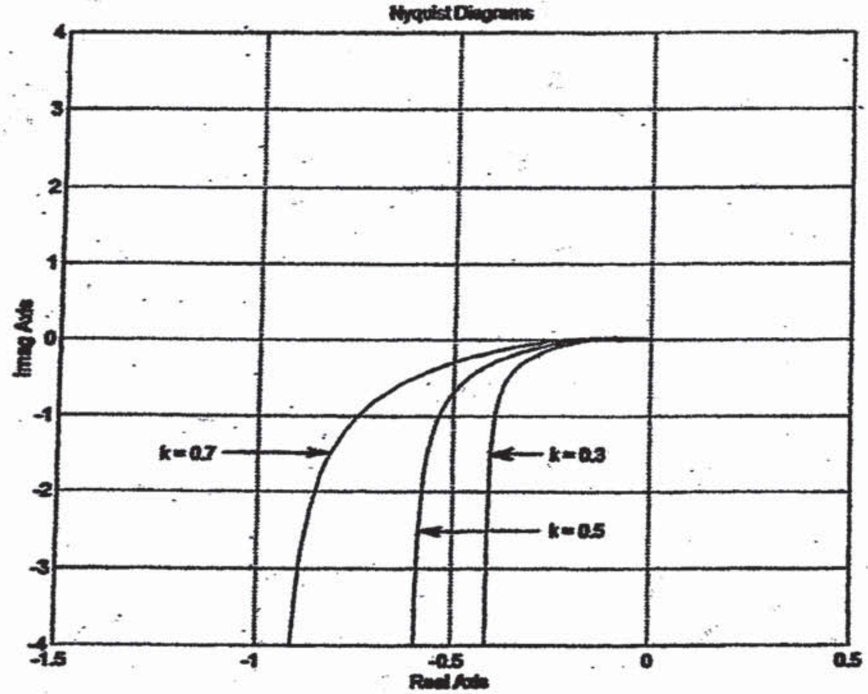


B-7-20.

$$G(s) = \frac{10}{s^3 + 6s^2 + (5+10k)s}$$

A MATLAB program for plotting Nyquist diagrams of G(s) for k = 0.3, k = 0.5, and k = 0.7 is shown below. The resulting Nyquist diagrams are shown on the next page.

```
% ***** Nyquist Diagrams *****

num = [0  0  0  10];
den1 = [1  6  8  0];  %k=0.3
den2 = [1  6  10  0]; %k=0.5
den3 = [1  6  12  0]; %k=0.7
w = 0.1:0.1:100;
[re1,im1,w] = nyquist(num,den1,w);
[re2,im2,w] = nyquist(num,den2,w);
[re3,im3,w] = nyquist(num,den3,w);
plot(re1,im1,re2,im2,re3,im3)
v = [-1.5  0.5  -4  4]; axis(v)
grid
title('Nyquist Diagrams')
xlabel('Real Axis')
ylabel('Imag Axis')
text(-0.25,-1.5,'k=0.3')
text(-0.25,-2.5,'k=0.5')
text(-1.25,-1.5,'k=0.7')
```

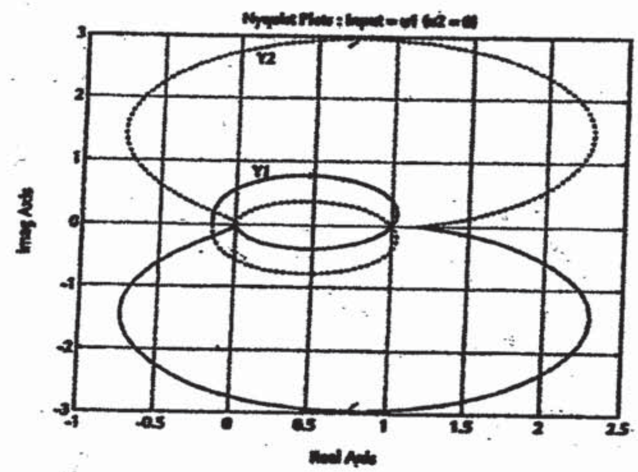


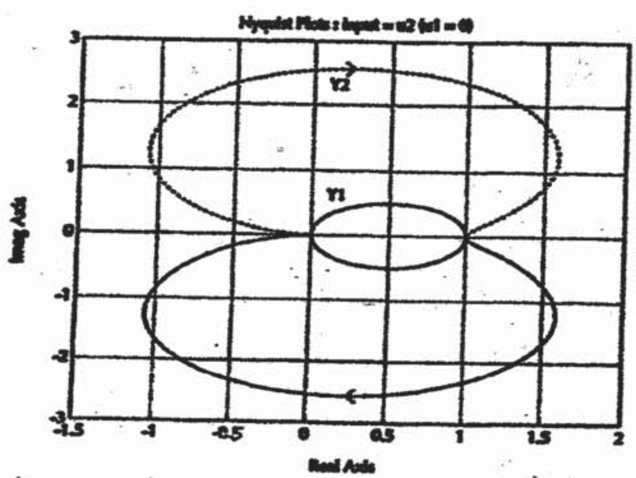
B-7-21. The following MATLAB program produces two Nyquist plots for the input u_1 in one diagram and two Nyquist plots for the input u_2 in another diagram.

```
% ***** Nyquist plots *****

% ***** We shall first obtain Nyquist plots when the input is
% u1. Then we shall obtain Nyquist plots when the input is
% u2 *****

% ***** Enter matrices A, B, C, and D *****

A = [-1  -1;6.5  0];
B = [1  1;1  0];
C = [1  0;0  1];
D = [0  0;0  0];
```

```
% ***** To obtain Nyquist plots when the input is u1, enter
% the command 'nyquist(A,B,C,D,1)' *****

nyquist(A,B,C,D,1)
grid
title('Nyquist Plots : Input = u1 (u2 = 0)')
text(0.1,0.7,'Y1')
text(0.1,2.5,'Y2')

% ***** Next, we shall obtain Nyquist plots when the input is
% u2. Enter the command 'nyquist(A,B,C,D,2)' *****

nyquist(A,B,C,D,2)
grid
title('Nyquist Plots : Input = u2 (u1 = 0)')
text(0.1,0.5,'Y1')
text(0.1,2.2,'Y2')
```

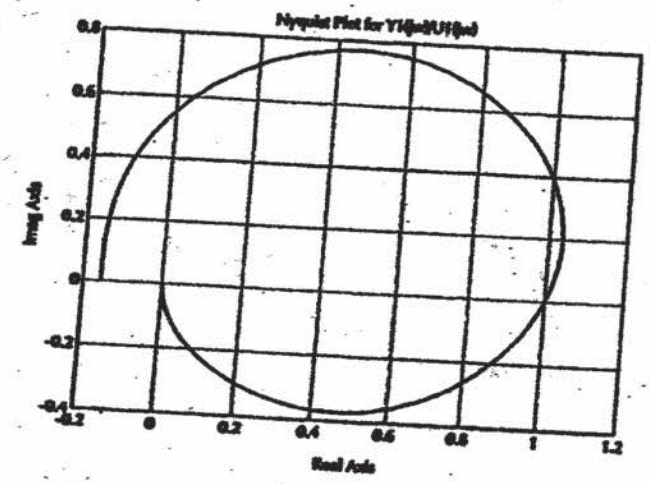
The Nyquist plots obtained by this MATLAB program are shown below.





B-7-22. The following MATLAB program produces the Nyquist plot for $Y_1(j\omega)/U_1(j\omega)$ for $\omega > 0$. The plot obtained is shown below.

```
% ***** Nyquist plot *****
A = [-1  -1;6.5  0];
B = [1  1;1  0];
C = [1  0;0  1];
D = [0  0;0  0];
[re,im,w] = nyquist(A,B,C,D,1);
re1 = re*[1;0];
im1 = im*[1;0];
plot(re1,im1)
grid
title('Nyquist Plot for Y1(jw)/U1(jw)')
xlabel('Real Axis')
ylabel('Imag Axis')
```



To plot the Nyquist locus for $-\infty < \omega < \infty$, replace the plot command plot(re1,im1) in the above MATLAB program by plot(re1,im1,re1,-im1).

B-7-23.

$$|G(j\omega)| = \frac{\sqrt{a^2\omega^2+1}}{\omega^2}, \quad \underline{/G(j\omega)} = \tan^{-1} a\omega - 180^\circ$$

The phase margin of 45° at $\omega = \omega_1$ requires that

$$\frac{\sqrt{a^2\omega_1^2+1}}{\omega_1^2} = 1$$

$$\tan^{-1} a\omega_1 - 180^\circ = 45^\circ - 180^\circ$$

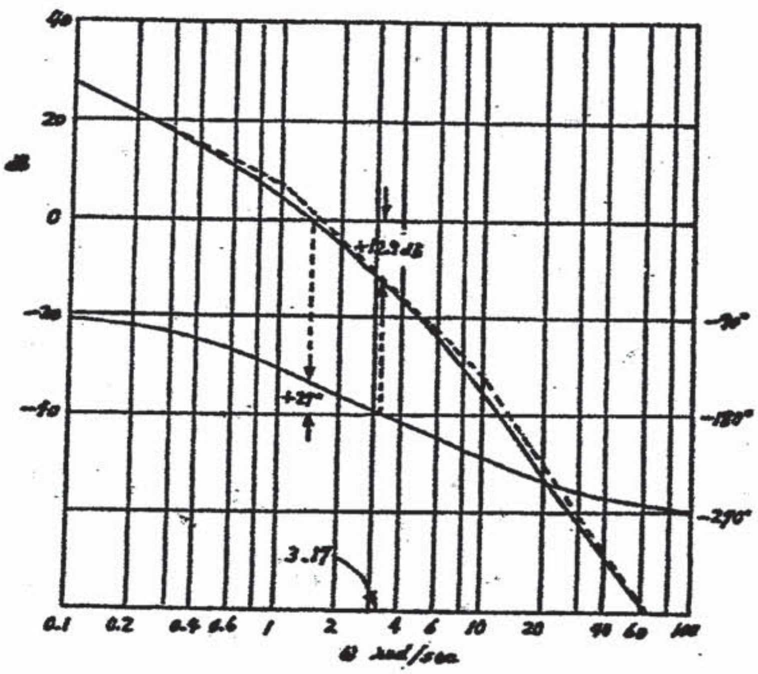
Thus, we have

$$a^2\omega_1^2 + 1 = \omega_1^4 , \quad a\omega_1 = 1$$

Solving for a, we obtain

$$a = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} = 0.841$$

B-7-24. A Bode diagram of the system is shown below.



From this Bode diagram, we find the phase margin and gain margin to be 27° and 13 dB, respectively.

The phase margin, gain margin, phase crossover frequency, and gain crossover frequency can be obtained easily with MATLAB. Use the command

```
[Gm, pm, wcp, wcg] = margin(sys)
```

See Problem B-7-25.

B-7-25.

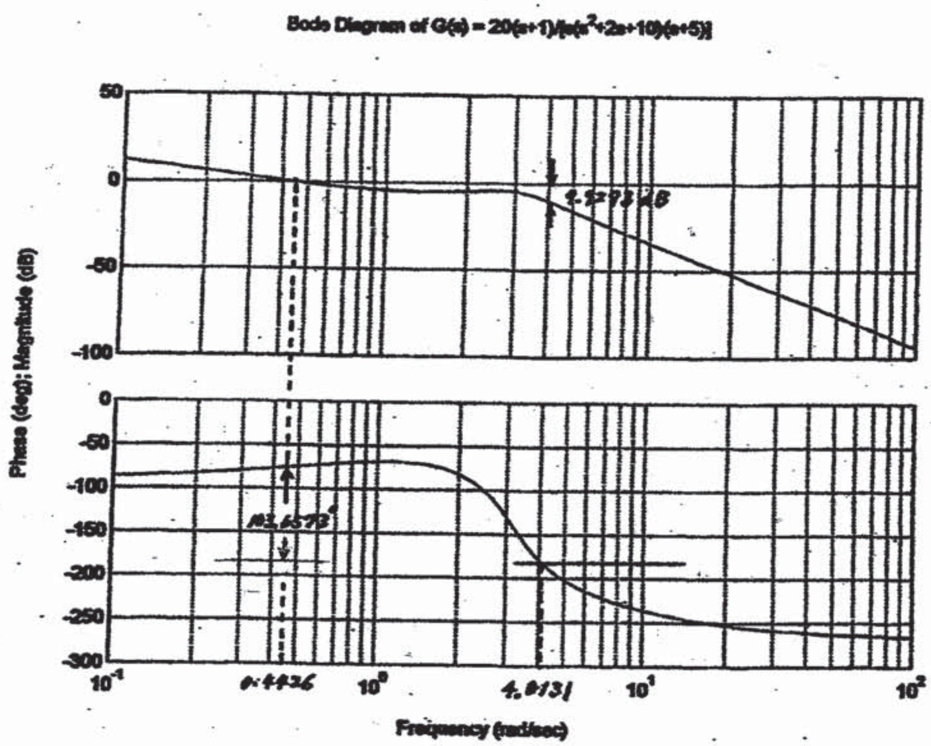
$$G(s) = \frac{20(s+1)}{s(s^2+2s+10)(s+5)}$$

The phase margin, gain margin, phase crossover frequency, and gain crossover frequency are obtained by use of the command

```
[Gm, pm, wcp, wcg] = margin(sys)
```

A MATLAB program to solve this problem is given below. The Bode diagram shown below verifies the phase margin, gain margin, phase crossover frequency, and gain crossover frequency obtained with MATLAB.

```
% ***** Bode Diagram *****

num = [0  0  0  20  20];
den = conv([1  2  10  0],[1  5]);
sys = tf(num,den);
w = logspace(-1,2,100);
bode(sys,w)
title('Bode Diagram of G(s) = 20(s+1)/[s(s^2+2s+10)(s+5)]')
[Gm,pm,wcp,wcg] = margin(sys);
GmdB = 20*log10(Gm);
[GmdB  pm  wcp  wcg]

ans =

   9.9293 103.6573   4.0131   0.4426
```



B-7-26.

$$G(s) = \frac{K}{s(s^2+s+4)} = \frac{0.25K}{s(0.25s^2+0.25s+1)}$$

The quadratic term in the denominator has the undamped natural frequency of 2 rad/sec and the damping ratio of 0.25. Define the frequency corresponding to the angle of $-130°$ to be ω_1.

$$\angle G(j\omega_1) = -\angle j\omega_1 - \angle 1 - 0.25\omega_1^2 + j0.25\omega_1$$

$$= -90° - \tan^{-1}\frac{0.25\omega_1}{1-0.25\omega_1^2} = -130°$$

Solving this last equation for ω_1, we find $\omega_1 = 1.491$. Thus, the phase angle becomes equal to $-130°$ at $\omega = 1.491$ rad/sec. At this frequency, the magnitude must be unity, or $|G(j\omega_1)| = 1$. The required gain K can be determined from

$$|G(j1.491)| = \left|\frac{0.25K}{(j1.491)(-0.555 + j0.3725 + 1)}\right| = 0.2890K$$

Setting $|G(j1.491)| = 0.2890K = 1$, we find

$$K = 3.46$$

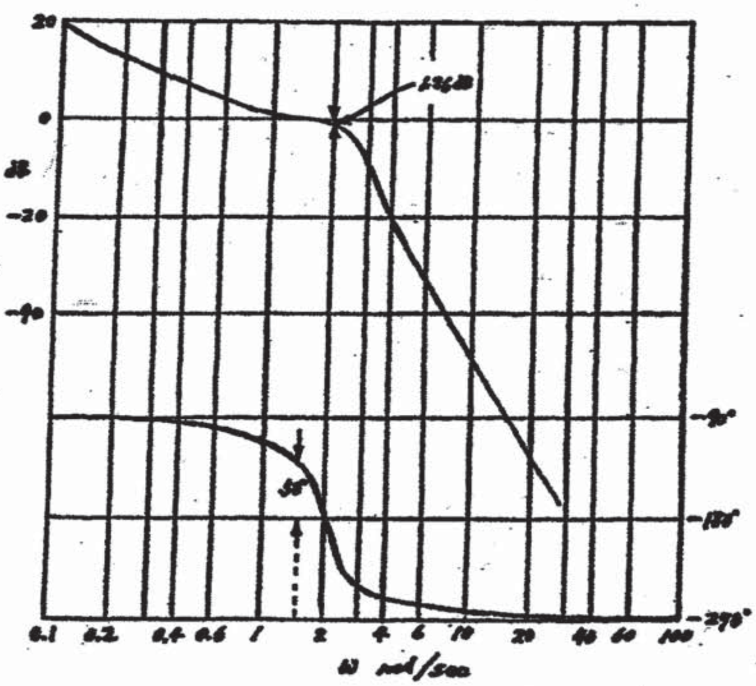
Note that the phase crossover frequency is at $\omega = 2$ rad/sec, since

$$\angle G(j2) = -\angle j2 - \angle -0.25\times 2^2 + 0.25\times j2 + 1 = -90° - 90° = -180°$$

The magnitude $|G(j2)|$ with K = 3.46 becomes

$$|G(j2)| = \left|\frac{0.865}{(j2)(-1+0.5j+1)}\right| = 0.865 = -1.26\text{ dB}$$

Thus, the gain margin is 1.26 dB. The Bode diagram of $G(j\omega)$ with K = 3.46 is shown below.



B-7-27. Note that

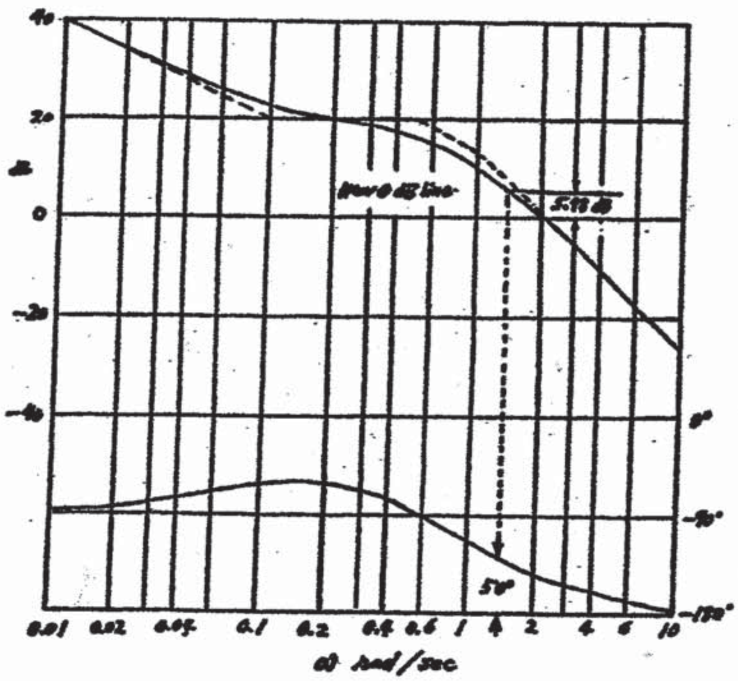
$$K\frac{j\omega+0.1}{j\omega+0.5}\frac{10}{j\omega(j\omega+1)}=\frac{2K(10j\omega+1)}{j\omega(2j\omega+1)(j\omega+1)}$$

We shall plot the Bode diagram when 2K = 1. That is, we plot the Bode diagram of

$$G(j\omega)=\frac{10j\omega+1}{j\omega(2j\omega+1)(j\omega+1)}$$

The diagram is shown below. The phase curve shows that the phase angle is -130° at ω = 1.438 rad/sec. Since we require the phase margin to be 50°, the magnitude of G(1.438) must be equal to 1 or 0 dB. Since the Bode diagram indicates that G(1.438) is 5.48 dB, we need to choose 2K = -5.48 dB, or

$$K=0.266$$



Since the phase curve lies above the -180° line for all ω, the gain margin is $+\infty$ dB.

B-7-28. Note that

$$G(s)=\frac{K}{s(s^2+s+0.5)}=\frac{2K}{s(2s^2+2s+1)}$$

We shall first plot a Bode diagram of G($j\omega$) when K = 0.5. That is, we plot a Bode diagram for

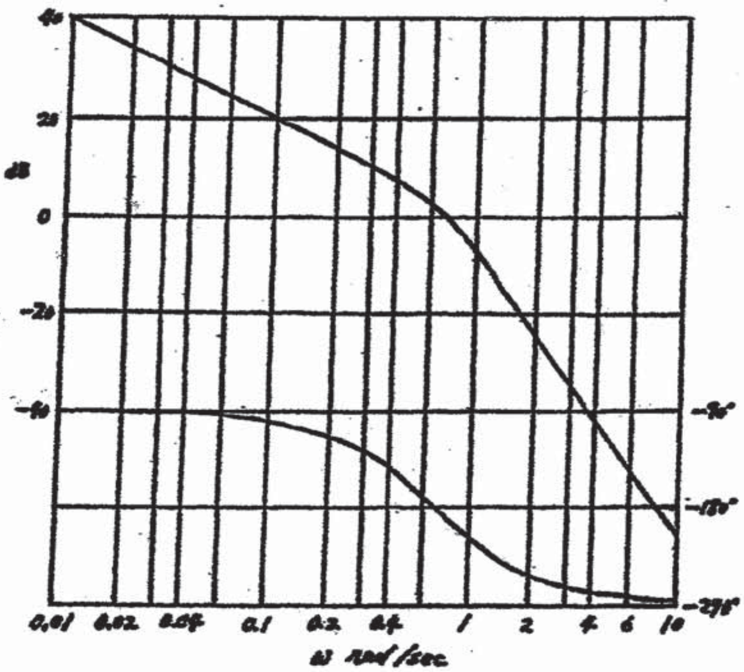
$$G(j\omega)=\frac{1}{j\omega[2(j\omega)^2+2j\omega+1]}$$

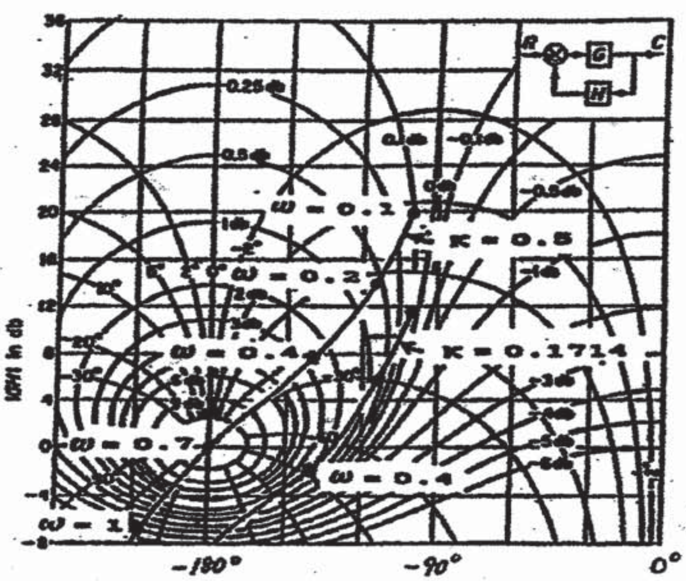
It is shown below. By reading the magnitude and phase angle values at each frequency point condidered, the log-magnitude versus phase curve can be plotted as shown below the Bode diagram. By moving the curve vertically, we can shift the curve to be tangent to the M = 2 dB locus. The vertical shift needed is 9.3 dB. That is, if we lower the curve by 9.3 dB, then it is tangent to the M = 2 dB locus. Therefore, we set

$$2K = -9.3 \text{ dB}$$

Solving this equation for K determines the desired value of K as

$$K = 0.1714$$





B-7-29. Referring to Figure 7-9 and examining the Bode diagram of Figure 7-164, we find the damping ratio ζ and undamped natural frequency ω_n of the quadratic term to be

$$\zeta = 0.1, \qquad \omega_n = 2 \text{ rad/sec}$$

Noting that there is another corner frequency at $\omega = 0.5$ rad/sec and the slope of the magnitude curve in the low-frequency range is -40 dB/decade, $G(j\omega)$ can be tentatively determined as follows:

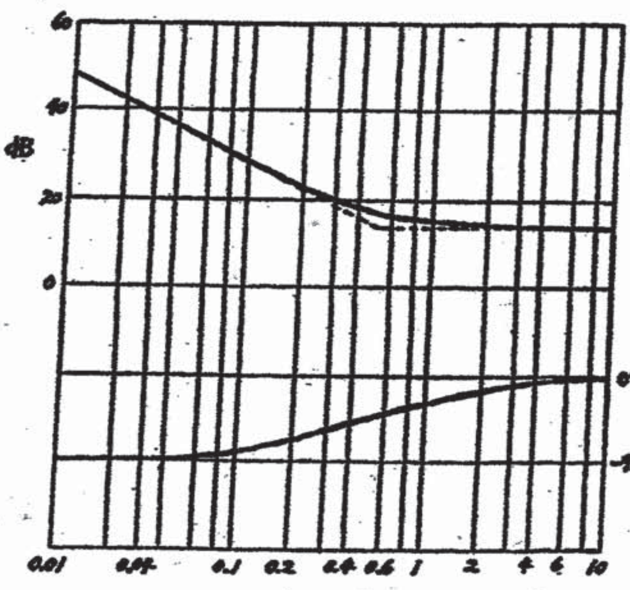
$$G(j\omega) = \frac{K\left(\frac{j\omega}{0.5} + 1\right)}{(j\omega)^2\left[\left(\frac{j\omega}{2}\right)^2 + 0.1(j\omega) + 1\right]}$$

Since, from Figure 7-164, we find $|G(j0.1)| = 40$ dB, the gain value K can be determined to be unity. Also, the calculated phase curve, $\angle G(j\omega)$ versus ω, agrees with the given phase curve. Hence, the transfer function G(s) can be determined to be

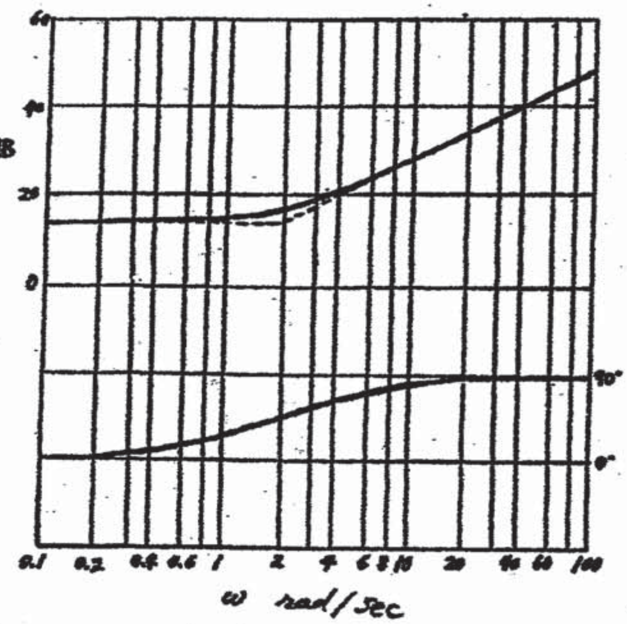
$$G(s) = \frac{4(2s+1)}{s^2(s^2 + 0.4s + 4)}$$

B-7-30.

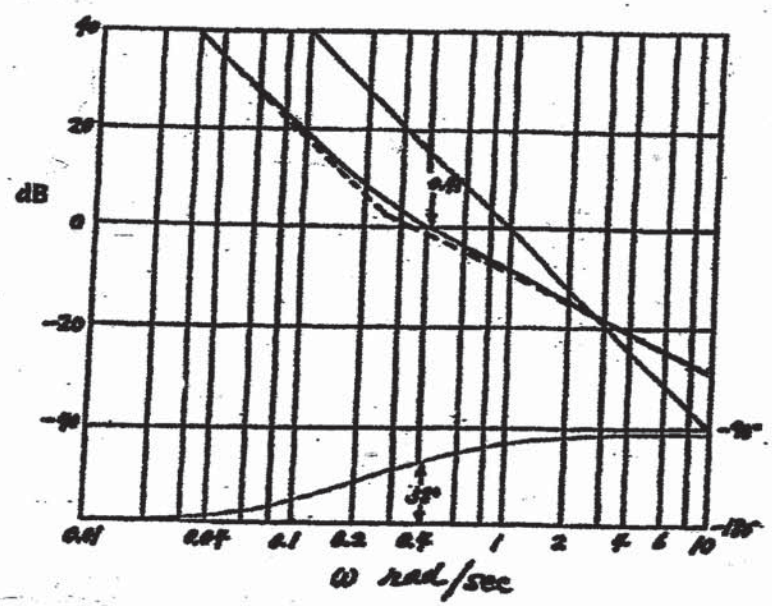
$$G_c(s) = 5\left(1 + \frac{1}{2s}\right)$$



$G_c(s) = 5(1 + 0.5s)$



B-7-31. Choose the gain crossover frequency to be approximately 0.4 rad/sec and the phase margin to be approximately 60°. Draw the high frequency asymptote having the slope of −20 dB/dec to cross the 0 dB line at about $\omega = 0.35$ rad/sec. Choose the corner frequency to be 0.25 rad/sec. Then the low-frequency asymptote can be drawn on Bode diagram. See the Bode diagram shown below.



The actual magnitude curve crosses the 0 dB line at about $\omega = 0.41$ rad/sec and the phase margin is approximately 58°.

Since we have chosen the corner frequency to be 0.25 rad/sec, we get

$$T_d = 4$$

From the Bode diagram, K_d must be chosen to be -21.4 dB, or

$$K_d = -21.4 \text{ dB}$$

$$= 0.0851$$

Thus

$$K_d(1+T_d s) = 0.0851\,(1+4s)$$

Then, the open-loop transfer function becomes

$$G(s) = \frac{0.0851\,(1+4s)}{s^2}$$

The closed-loop transfer function is

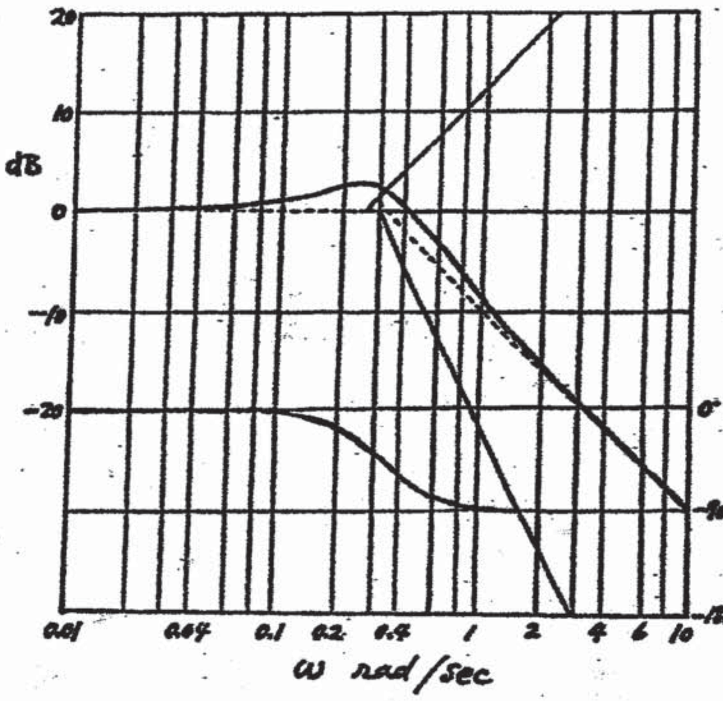
$$\frac{C(s)}{R(s)} = \frac{0.0851\,(1+4s)}{s^2+0.0851\,(1+4s)}$$

$$= \frac{4s+1}{11.751\,s^2+4s+1}$$

A Bode diagram of

$$\frac{C(j\omega)}{R(j\omega)} = \frac{4j\omega+1}{11.751(j\omega)^2+4j\omega+1}$$

is shown on the next page. From this diagram we see that the bandwidth is approximately 0.5 rad/sec.



B-7-32. Let us use the following lead compensator:

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Since K_v is specified as 4.0 $\sec^{-1}$, we have

$$K_v = \lim_{s\to 0} sK_c\alpha \frac{Ts+1}{\alpha Ts+1} \frac{K}{s(0.1s+1)(s+1)} = K_c\alpha K = 4$$

Let us set $K = 1$ and define $K_c\alpha = \hat{K}$. Then

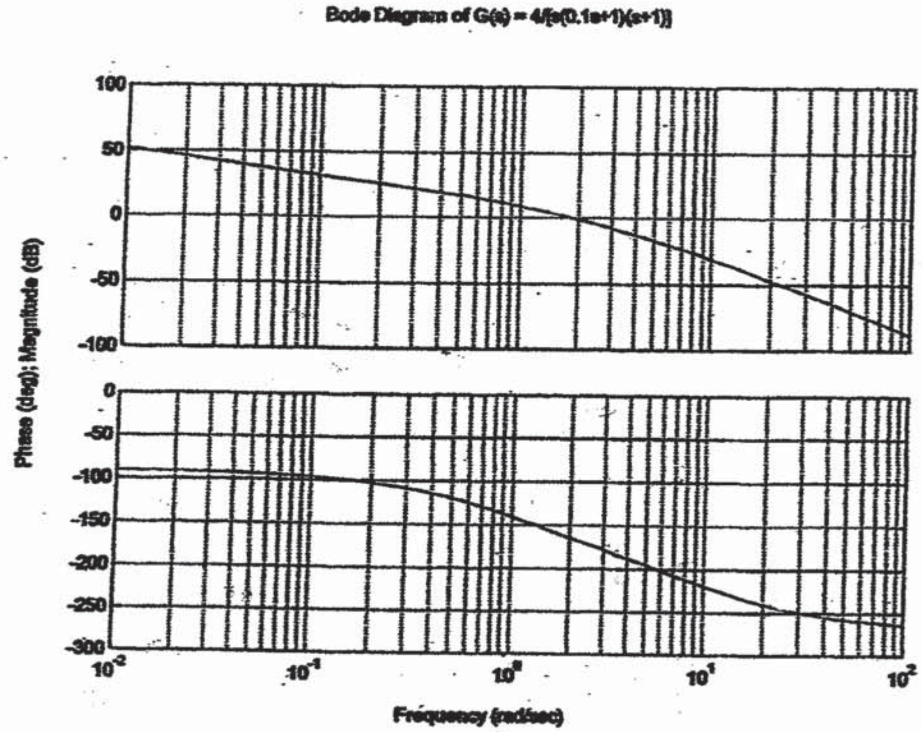
$$\hat{K} = 4$$

Next, plot a Bode diagram of

$$\frac{4}{s(0.1s+1)(s+1)} = \frac{4}{0.1s^3+1.1s^2+s}$$

The following MATLAB program produces the Bode diagram shown on the next page.

```
% ***** Bode diagram *****

num = [0  0  0  4];
den = [0.1  1.1  1  0];
bode(num,den)
title('Bode Diagram of G(s) = 4/[s(0.1s+1)(s+1)]')
```



From this plot, the phase and gain margins are 17° and 8.7 dB, respectively.

Since the specifications call for a phase margin of 45°, let us choose

$$\phi_m = 45° - 17° + 12° = 40°$$

(This means that 12° has been added to compensate for the shift in the gain crossover frequency.) The maximum phase lead is 40°. Since

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \qquad (\phi_m = 40°)$$

α is determined as 0.2174. Let us choose, instead of 0.2174, α to be 0.21, or

$$\alpha = 0.21$$

Next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. Note that the maximum phase-lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha}\,T)$. The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}\,T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left|\frac{1+j\omega T}{1+j\omega\alpha T}\right|_{\omega=\frac{1}{\sqrt{\alpha}T}} = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.21}} = 2.1822 = 6.7778 \text{ dB}$$

We need to find the frequency point where, when the lead compensator is added, the total magnitude becomes 0 dB. The magnitude $G(j\omega)$ is −6.7778 dB corresponds to $\omega = 2.81$ rad/sec. We select this frequency to be the new gain crossover frequency ω_c. Then we obtain

$$\frac{1}{T} = \sqrt{\alpha}\,\omega_c = \sqrt{0.21} \times 2.81 = 1.2877$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{2.81}{\sqrt{0.21}} = 6.1319$$

Hence

$$G_c(s) = K_c \frac{s+1.2877}{s+6.1319}$$

and

$$K_c = \frac{\hat{K}}{\alpha} = \frac{4}{0.21}$$

Thus

$$G_c(s) = \frac{4}{0.21} \frac{s+1.2877}{s+6.1319} = 4\,\frac{0.7766s+1}{0.163085s+1}$$

The open-loop transfer function becomes as

$$G_c(s)G(s) = 4\,\frac{0.77665s+1}{0.163085s+1} \frac{1}{s(0.1s+1)(s+1)}$$

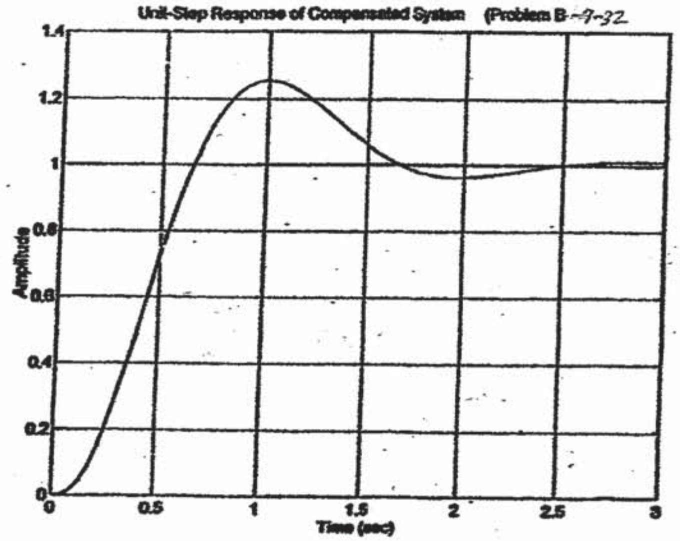
$$= \frac{3.1064s+4}{0.01631s^4 + 0.2794s^3 + 1.2631s^2 + s}$$

This open-loop transfer function yields phase margin of 45° and gain margin of 13 dB. So, the requirements on the phase margin and gain margin are satisfied. The closed-loop transfer function of the designed system is

$$\frac{C(s)}{R(s)} = \frac{3.1064s+4}{0.01631s^4 + 0.2794s^3 + 1.2631s^2 + 4.1064s + 4}$$

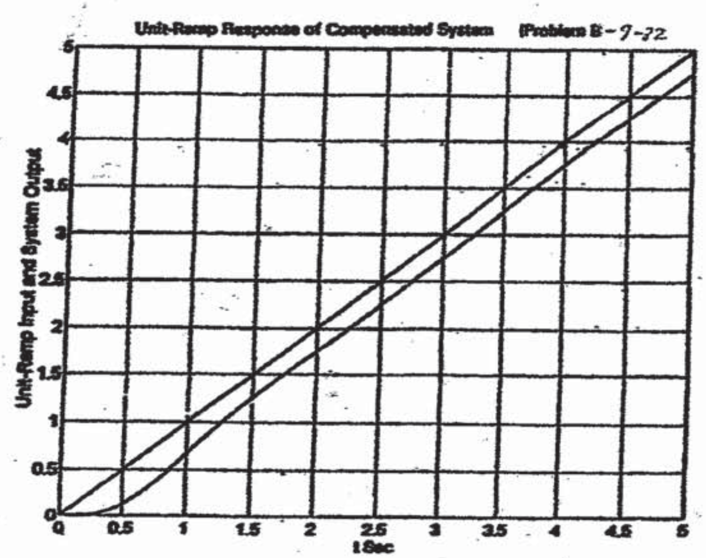
The following MATLAB program produces the unit-step response curve as shown on the next page.

```
% ***** Unit-step response *****

numc = [0  0  0  3.1064  4];
denc = [0.01631  0.2794  1.2631  4.1064  4];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System    (Problem B-7-32)')
```



Similarly, the following MATLAB program produces the unit-ramp response curve as shown below.

```
% ***** Unit-ramp response *****

numc = [0  0  0  0  3.1064  4];
denc = [0.01631   0.2794   1.2631   4.1064  4   0];
t = 0:0.01:5;
c = step(numc, denc,t);
plot(t,c,t,t)
grid
title('Unit-Ramp Response of Compensated System    (Problem B-7-32)')
xlabel('t Sec')
ylabel('Unit-Ramp Input and System Output')
```



B-7-33. The plant transfer function is

$$G(s) = \frac{2s+0.1}{s(s^2+0.1s+4)}$$

The plant involves a quadratic term with $\zeta = 0.025$. This term is quite oscillatory. MATLAB program shown below produces the Bode diagram of G(s) as shown below.

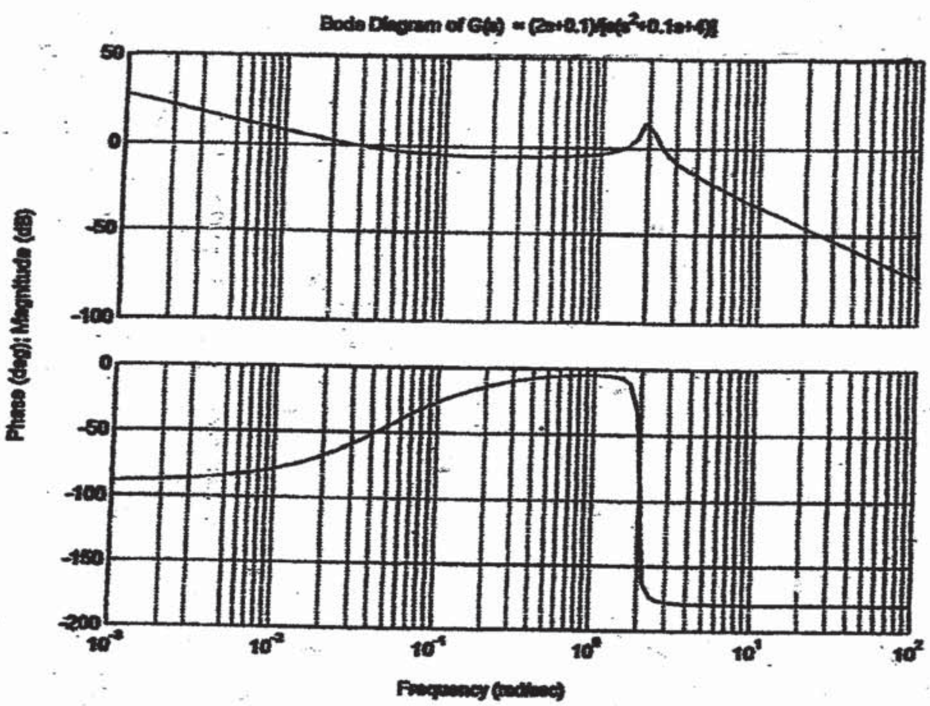
```
% ***** Bode diagram *****

num = [0  0  2  0.1];
den = [1  0.1  4  0];
w = logspace(-3,2,100);
bode(num,den,w);
title('Bode Diagram of G(s) = (2s+0.1)/[s(s^2+0.1s+4)]')
```

The closed-loop transfer function of the original uncompensated system is

$$\frac{C(s)}{R(s)} = \frac{2s+0.1}{s^3+0.1s^2+6s+0.1}$$

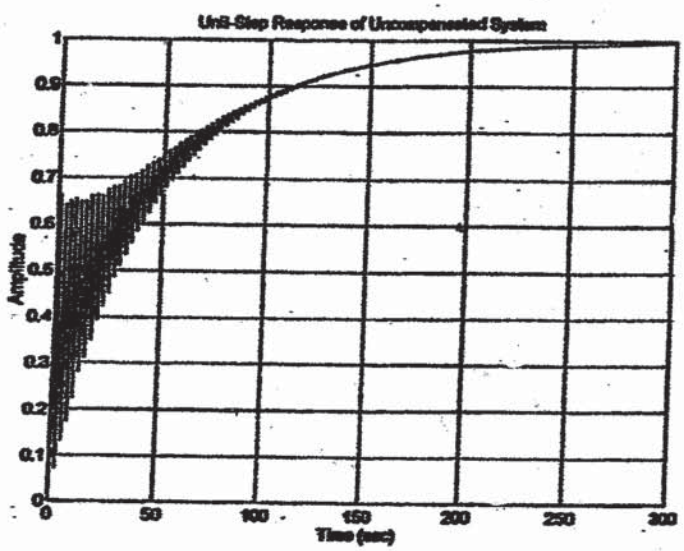
The closed-loop poles of the uncompensated system are

$$s = -0.0417 + j2.4489$$
$$s = -0.0417 - j2.4489$$
$$s = -0.0167$$



The unit-step response of this original, uncompensated system is obtained by entering the following MATLAB program into the computer. The resulting unit-step response curve is shown below.

```
% ***** Unit-step response *****

num = [0  0  2  0.1];
den = [1  0.1  6  0.1];
step(num,den)
grid
title('Unit-Step Response of Uncompensated System')
```



To design a compensator for such a system, it is desirable to cancel the zero of the plant, since it is located very close to the origin. It is sometimes useful to include double zero and double pole in the compensator. So, we may choose the compensator to be

$$G_c(s) = K_c \frac{(s+2)^2}{(s+10)^2} \frac{s+a}{2s+0.1}$$

where we have chosen the double zero at $s = -2$ and double pole at $s = -10$. The value of a is determined later. Since the static velocity error constant is specified as 4 sec^{-1}, we have

$$K_v = \lim_{s \to 0} s G_c(s) G(s)$$

$$= \lim_{s \to 0} s K_c \frac{(s+2)^2}{(s+10)^2} \frac{s+a}{2s+0.1} \frac{2s+0.1}{s(s^2+0.1s+4)}$$

$$= K_c \frac{a}{100} = 4$$

Hence,

$$K_c a = 400$$

By several MATLAB trials we find a = 4 will give a satisfactory result. Therefore, we choose a = 4 and K_c = 100. Then, the transfer function of the compensator becomes

$$G_c(s) = 100 \frac{(s+2)^2}{(s+10)^2} \frac{s+4}{2s+0.1}$$

The open-loop transfer function becomes as follows:

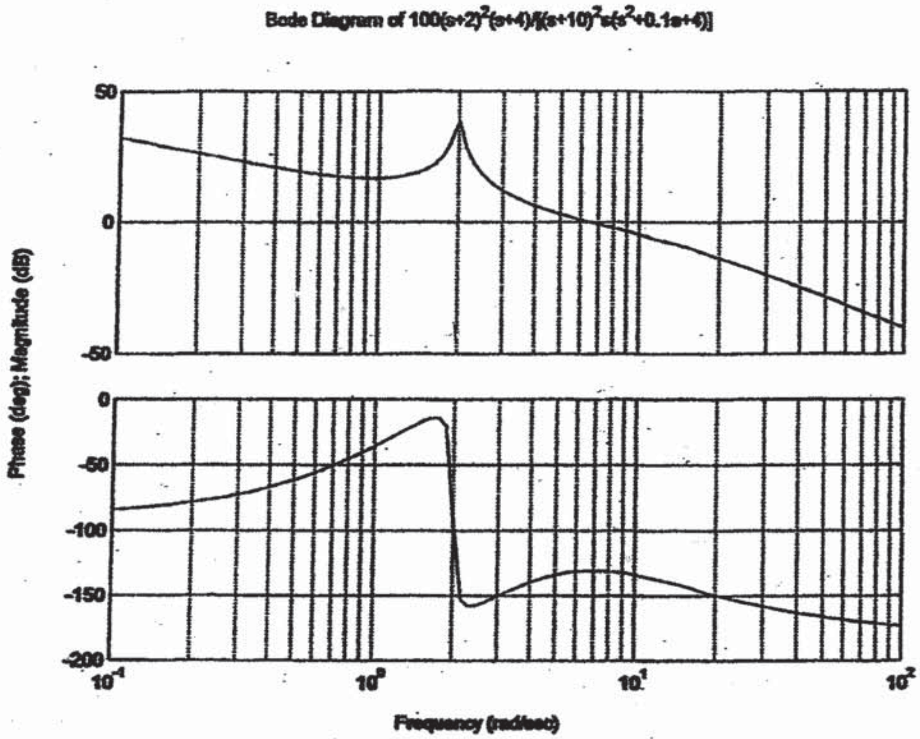
$$G_c(s) G(s) = \frac{100 (s+2)^2 (s+4)}{(s+10)^2 \, s(s^2+0.1s+4)}$$

$$= \frac{100 s^3 + 800 s^2 + 2000 s + 1600}{s^5 + 20.1 s^4 + 106 s^3 + 90 s^2 + 400 s}$$

The following MATLAB program produces a Bode diagram of $G_c(s)G(s)$. The resulting Bode diagram is shown on the next page.

```
% ***** Bode diagram *****

num = [0  0  100  800  2000  1600];
den = [1  20.1  106  90  400  0];
w = logspace(-1,2,100);
bode(num,den,w);
title('Bode Diagram of 100(s+2)^2(s+4)/[(s+10)^2s(s^2+0.1s+4)]')
```

From this Bode diagram, it is seen that $K_v = 4 \text{ sec}^{-1}$, phase margin is approximately 50° and gain margin is $+\infty$ dB. So, all the requirements are met.



The closed-loop transfer function of the compensated system becomes as follows:

$$\frac{C(s)}{R(s)} = \frac{100s^3 + 800s^2 + 2000s + 1600}{s^5 + 20.1s^4 + 206s^3 + 890s^2 + 2400s + 1600}$$

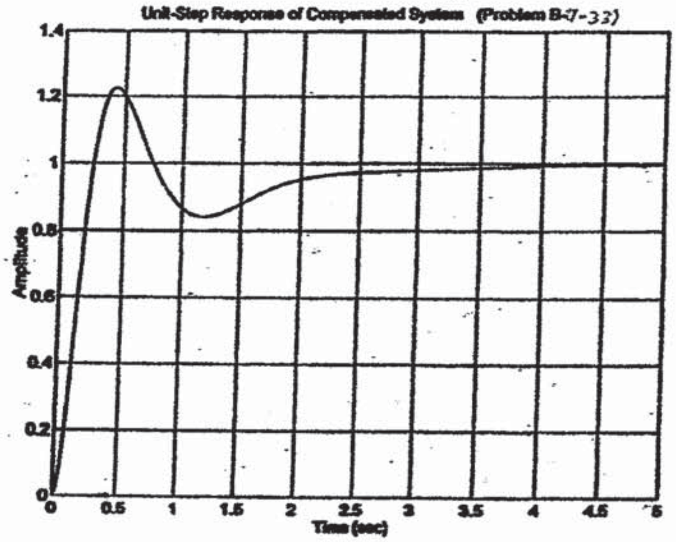
The closed-loop poles of the compensated system can be found as follows.

```
denc = [1   20.1   206   890   2400   1600];
roots(denc)

ans =

 -7.3481 + 7.2145i
 -7.3481 - 7.2145i
 -2.2424 + 3.3751i
 -2.2424 - 3.3751i
 -0.9189
```

The following MATLAB program produces the unit-step response of the designed system.

```
% ***** Unit-step response *****

numc = [0  0  100  800  2000  1600];
denc = [1  20.1  206  890  2400  1600];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System   (Problem B-7-33)')
```

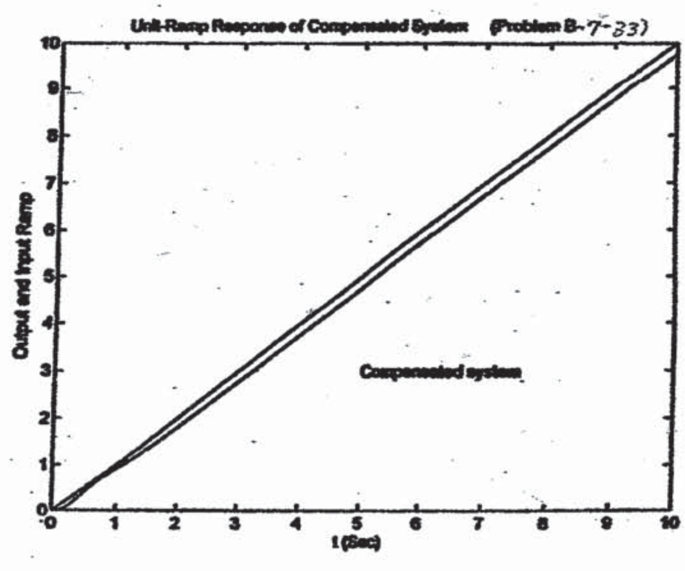
The unit-step response curve is shown below.



The following MATLAB program will produce the unit-ramp response of the compensated system.

```
% ***** Unit-ramp response *****

numc = [0  0  0  100  800  2000  1600];
denc = [1  20.1  206  890  2400  1600  0];
t = 0:0.02:10;
c = step(numc,denc,t);
plot(t,c,'-',t,t,'--')
title('Unit-Ramp Response of Compensated System     (Problem B-7-33)')
xlabel('t (Sec)')
ylabel('Output and Input Ramp')
text(5,3,'Compensated system')
```

The unit-ramp response curve is shown below.



It is noted that there are infinitely many possible compensators for this system. A few possible compensators are shown below.

$$G_c(s) = 400 \frac{(s+1)^2}{(s+25)(2s+0.1)}$$

$$G_c(s) = 320 \frac{(s+1)^2}{(s+20)(2s+0.1)}$$

$$G_c(s) = 160 \frac{s+4}{s+30} \frac{s+1}{s+0.1333}$$

$$G_c(s) = 1212.12 \frac{s+9.81}{s+78.32}$$

B-7-34. Let us assume that the compensator $G_c(s)$ has the following form:

$$G_c(s) = K_c \frac{(T_1 s+1)(T_2 s+1)}{\left(\frac{T_1}{\beta} s+1\right)(\beta T_2 s+1)} = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}$$

Since K_v is specified as 20 $\sec^{-1}$, we have

$$K_v = \lim_{s\to 0} s\, G_c(s) \frac{1}{s(s+1)(s+5)} = K_c \frac{1}{5} = 20$$

Hence

$$K_c = 100$$

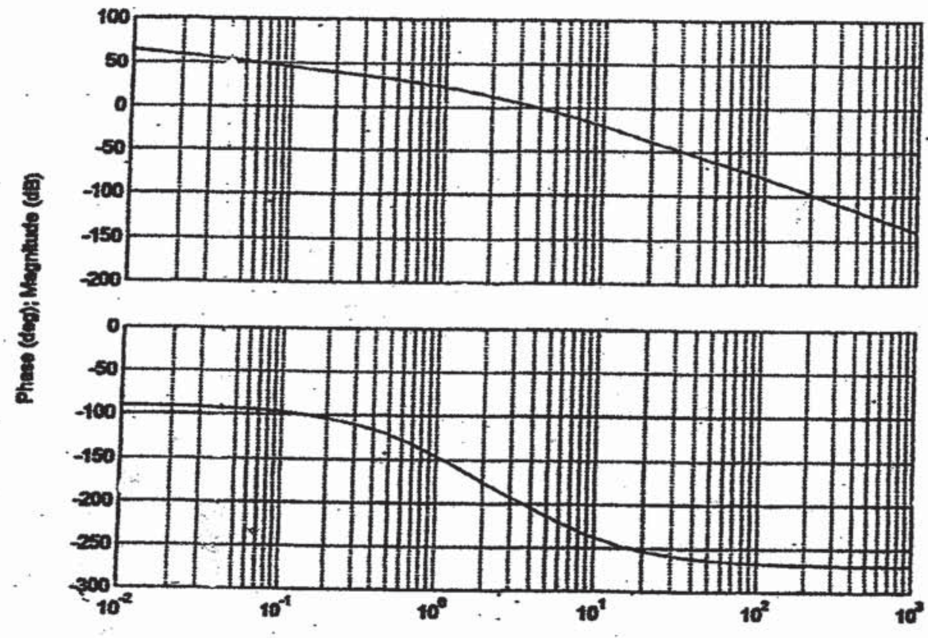
Define

$$G_1(s) = 100\, G(s) = \frac{100}{s(s+1)(s+5)}$$

The following MATLAB program produces the Bode diagram of $G_1(s)$ as shown on the next page.

```
% ***** Bode diagram *****

num=[0 0 0 100];
den=[1 6 5 0];
w=logspace(-2,3,100);
bode(num,den,w);
title('Bode Diagram of G1(s)=100/[s(s+1)(s+5)]')
```

Bode Diagram of G1(s) = 100/[s(s+1)(s+5)]



From this diagram we find the phase crossover frequency to be $\omega = 2.25$ rad/sec. Let us choose the gain crossover frequency of the designed system to be 2.25 rad/sec so that the phase lead angle required at $\omega = 2.25$ rad/sec is 60°.

Once we choose the gain crossover frequency to be 2.25 rad/sec, we can determine the corner frequencies of the phase lag portion of the lag-lead compensator. Let us choose the corner frequency $1/T_2$ to be one decade below the new gain crossover frequency, or $1/T_2 = 0.225$. For the lead portion of the compensator, we first determine the value of β that provides $\phi_m = 65°$. (5° added to 60°.) Since

$$\sin \phi_m = \frac{1 - \frac{1}{\beta}}{1 + \frac{1}{\beta}} = \frac{\beta - 1}{\beta + 1}$$

we find $\beta = 20$ corresponds to 64.7912°. Since we need 65° phase margin, we may choose $\beta = 20$. Thus

$$\beta = 20$$

Then, the corner frequency $1/(\beta T_2)$ of the phase lag portion becomes as follows:

$$\frac{1}{\beta T_2} = \frac{1}{20 \times \frac{1}{0.225}} = \frac{0.225}{20} = 0.01125$$

Hence, the phase lag portion of the compensator becomes as

$$\frac{s + 0.225}{s + 0.01125} = 20\,\frac{4.4444 s + 1}{88.8889 s + 1}$$

For the phase lead portion, we first note that

$$G_1(j2.25) = 10.35 \text{ dB}$$

If the lag-lead compensator contributes -10.35 dB at ω = 2.25 rad/sec, then the new gain crossover frequency will be as desired. The intersections of the line with slope +20 dB/dec [passing through the point (2.25, -10.35 dB)] and the 0 dB line and -26.0206 dB line determine the corner frequencies. Such intersections are found as ω = 0.3704 and ω = 7.4077 rad/sec, respectively. Thus, the phase lead portion becomes

$$\frac{s+0.3704}{s+7.4077} = \frac{1}{20}\left(\frac{2.6998\,s+1}{0.1350\,s+1}\right)$$

Hence the compensator can be written as

$$G_c(s) = 100\left(\frac{4.4444\,s+1}{88.8889\,s+1}\right)\left(\frac{2.6998\,s+1}{0.1350\,s+1}\right)$$

$$= 100\left(\frac{s+0.225}{s+0.01125}\right)\left(\frac{s+0.3704}{s+7.4077}\right)$$

Then the open-loop transfer function $G_c(s)G(s)$ becomes as follows:

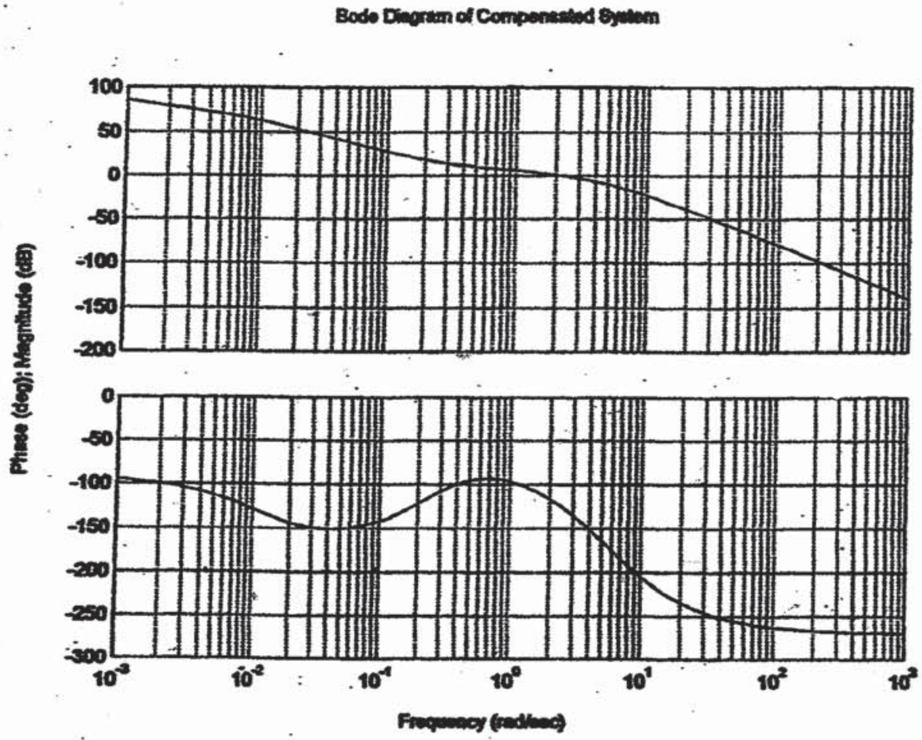
$$G_c(s)\,G(s) = 100\left(\frac{4.4444s+1}{88.8889s+1}\right)\left(\frac{2.6998s+1}{0.1350s+1}\right)\frac{1}{s(s+1)(s+5)}$$

$$= \frac{1199.90\,s^2 + 714.42\,s + 100}{12s^5 + 161.0239\,s^4 + 595.1434\,s^3 + 451.1195\,s^2 + 5s}$$

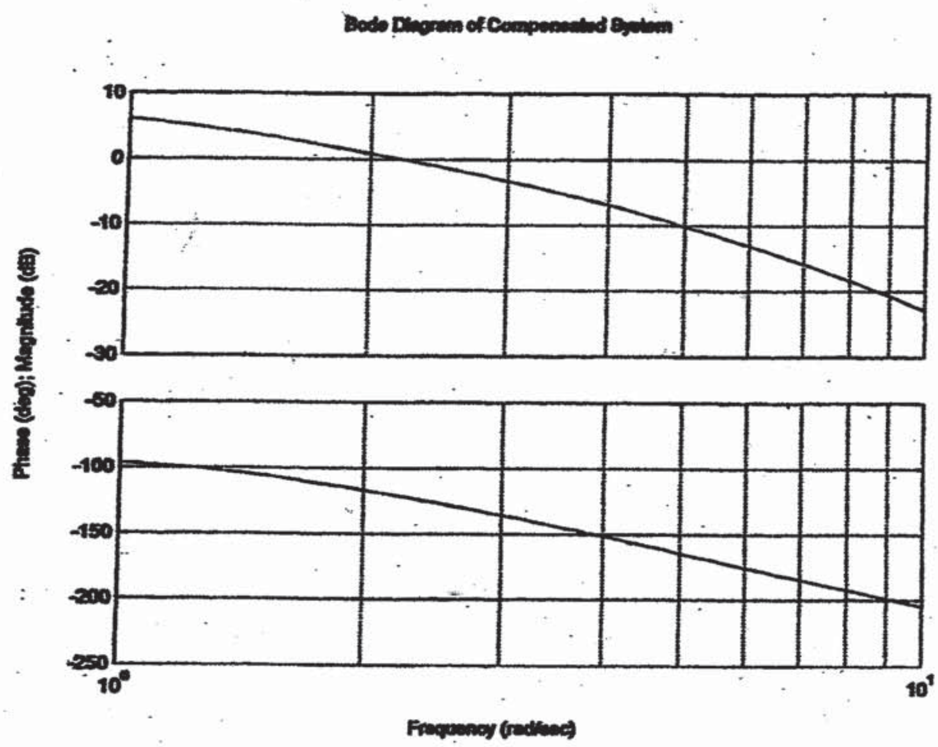
The following MATLAB program produces the Bode diagram of the open-loop transfer function.

```
% ***** Bode diagram *****

num = [0  0  0  1199.90  714.42  100];
den = [12  161.0239  595.1434  451.1195  5  0];
w = logspace(-3,3,100);
bode(num,den,w);
title('Bode Diagram of Compensated System')
```

The resulting Bode diagram is shown on the next page.



To read the phase margin and gain margin precisely, we need to expand the diagram between $\omega = 1$ and $\omega = 10$ rad/sec. This can be done easily by modifying the preceding MATLAB program. [Simply change the command w = logspace (-3,3, 100) to w = logspace(0,1,100).] The resulting Bode diagram is shown below.



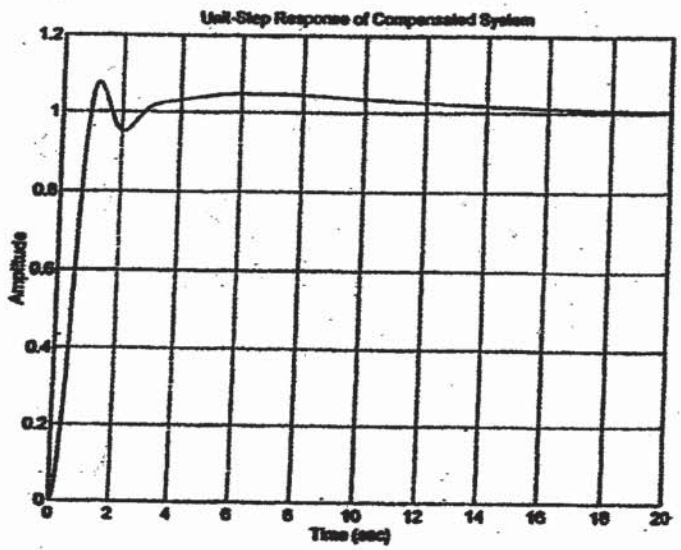
From this diagram we find that the phase margin is approximately 60° and gain margin is 14.35 dB. The static velocity error constant is 20 $\sec^{-1}$.

The closed-loop transfer function of the designed system is

$$\frac{C(s)}{R(s)} = \frac{1199.90\, s^2 + 714.42\, s + 100}{12 s^5 + 161 s^4 + 595.1 s^3 + 1651 s^2 + 719.4 s + 100}$$

The following MATLAB program produces the unit-step response. The resulting unit-step response curve is shown below.

```
% ***** Unit-step response *****

numc = [0  0  0  1199.90  714.42  100];
denc = [12  161  595.1  1651  719.4  100];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System')
```



The closed-loop poles can be obtained by entering the following MATLAB program into the computer.

```
roots(denc)

ans =

 -9.7022
 -1.6110 + 3.0494i
 -1.6110 - 3.0494i
 -0.2463 + 0.1076i
 -0.2463 - 0.1076i
```

Notice that there are two zeros ($s = -0.225$ and $s = -0.4939$) near the closed-loop poles at $s = -0.2463 \pm j0.1076$. Such a pole-zero combination generates a long tail with small amplitude in the unit-step response.

The following MATLAB program will produce the unit-ramp response as shown below.

```
% ***** Unit-ramp response *****

numc = [0  0  0  0  1199.90  714.42  100];
denc = [12  161  595.1  1651  719.4  100  0];
t = 0:0.05:20;
c = step(numc,denc,t);
plot(t,c,'-',t,t,'.')
grid
title('Unit-Ramp Response of Compensated System')
xlabel('t Sec')
ylabel('Output and Ramp Input')
text(11,7,'Output'); text(1,7,'Ramp Input')
```