4.4 Closed-Loop System's Response to Disturbances (Answers)

4.4.1 Transfer Function for Disturbance Response with PI Controller

1. In the case where the reference signal $r(t) = 0$, we focus on the disturbance torque T_d applied to the inertial load, and the PI controller is used. The control signal is:

$$u_m(t) = k_p(0 - \omega_m(t)) + k_i \int_0^t (0 - \omega_m(\tau))d\tau$$

2. In the Laplace domain: $U_m(s) = -k_p\Omega_m(s) - \frac{k_i}{s}\Omega_m(s)$
3. The open-loop dynamics of the motor are given by:

$$\Omega_m(s) = \frac{K}{\tau s + 1}U_m(s) + \frac{1}{J_{eq}}T_d(s)$$

4. Substitute the expression for $U_m(s)$ into this equation:

$$\Omega_m(s) = \frac{K}{\tau s + 1}\left(-k_p\Omega_m(s) - \frac{k_i}{s}\Omega_m(s)\right) + \frac{1}{J_{eq}}T_d(s)$$

5. Rearrange this to express $\Omega_m(s)$ as a function of $T_d(s)$:

$$\Omega_m(s)\left(1 + \frac{K}{\tau s + 1}\left(k_p + \frac{k_i}{s}\right)\right) = \frac{1}{J_{eq}}T_d(s)$$

$$or,\ \Omega_m(s)\left(\tau s^2 + s + Kk_p s + Kk_i\right) = \frac{s(\tau s + 1)}{J_{eq}}T_d(s)$$

The transfer function $G_D(s) = \frac{\Omega_m(s)}{T_d(s)}$ is:

$$G_D(s) = \frac{s(\tau s + 1)}{J_{eq}(\tau s^2 + s + Kk_p s + Kk_i)}$$

This expresses how the motor speed $\Omega_m(s)$ responds to the disturbance torque $T_d(s)$, and depends on the parameters k_p, k_i, K, τ, and J_{eq}.

4.4.2 Steady-State Velocity with Proportional Control

When a proportional controller is used (i.e., $k_i = 0$ and $k_p \neq 0$), the control signal is:

$$u_m(t) = -k_p\omega_m(t)$$

The closed-loop transfer function becomes: $\Omega_m(s) = \frac{K}{\tau s + 1 + Kk_p}U_m(s) + \frac{1}{J_{eq}}T_d(s)$

Using the Final Value Theorem for a step disturbance torque $T_d(s) = \frac{T_{d0}}{s}$:

$\omega_{ss,P} = \lim_{s\to 0} s \cdot \Omega_m(s) = \lim_{s\to 0} \frac{s \cdot \frac{1}{J_{eq}} \cdot \frac{T_{d0}}{s}}{\tau s^2 + (1 + Kk_p)s}$. At steady-state: $\omega_{ss,P} = \frac{T_{d0}}{J_{eq}(1 + Kk_p)}$.

Thus, the steady-state velocity $\omega_{ss,P}$ is directly proportional to the disturbance torque T_{d0}, and proportional control alone cannot eliminate the steady-state error caused by the disturbance.

4.4.3 Steady-State Velocity with Integral Control

When an integral controller is used (i.e., $k_p = 0$ and $k_i \neq 0$), the control signal is:

$$u_m(t) = -\frac{k_i}{s}\Omega_m(s)$$

The closed-loop transfer function becomes:

$$\Omega_m(s) = \frac{K}{\tau s + 1 + \frac{Kk_i}{s}}U_m(s) + \frac{1}{J_{eq}}T_d(s)$$

Using the Final Value Theorem for a step disturbance torque $T_d(s) = \frac{T_{d0}}{s}$:

$$\omega_{ss,I} = \lim_{s\to 0} s \cdot \Omega_m(s) = \lim_{s\to 0} \frac{s \cdot \frac{1}{J_{eq}} \cdot \frac{T_{d0}}{s}}{\tau s^2 + s + \frac{Kk_i}{s}}$$

Since the denominator has $\frac{Kk_i}{s}$, the steady-state velocity $\omega_{ss,I} = 0$. Thus, integral control ensures that the steady-state velocity error due to the disturbance torque is zero, meaning that integral control can eliminate steady-state error from disturbances.