4.3 Proportional and Integral (PI) Control (Answer)

4.3.1 Closed-Loop Transfer Function with PI Control

1. The control signal for PI control is: $u_m(t) = k_p(b_{sp}r(t) - \omega_m(t)) + k_i \int_0^t (r(\tau) - \omega_m(\tau))d\tau$
2. In the Laplace domain: $U_m(s) = k_p(b_{sp}R(s) - \Omega_m(s)) + \frac{k_i}{s}(R(s) - \Omega_m(s))$
3. Substitute this into the open-loop transfer function: $\Omega_m(s) = \frac{K}{\tau s+1}U_m(s)$
4. This gives: $\Omega_m(s) = \frac{K}{\tau s+1}\left(k_p(b_{sp}R(s) - \Omega_m(s)) + \frac{k_i}{s}(R(s) - \Omega_m(s))\right)$
5. Rearranging: $\Omega_m(s) = \frac{K(k_p b_{sp}R(s) + \frac{k_i}{s}R(s))}{\tau s + 1 + Kk_p + \frac{Kk_i}{s}}$
6. Thus, the closed-loop transfer function $G_{PI}(s)$ is: $G_{PI}(s) = \frac{\Omega_m(s)}{R(s)} = \frac{K(k_p b_{sp}s + k_i)}{s^2(\tau s+1) + sKk_p + Kk_i}$

4.3.2 Designing PI Controller Parameters

We want the closed-loop transfer function to match a second-order system:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1. Equating the characteristic equation of the second-order system: $s^2 + 2\zeta\omega_n s + \omega_n^2$
2. With the denominator of the PI controlled system: $s^2(\tau s + 1) + sKk_p + Kk_i$
3. Comparing coefficients, we get: $2\zeta\omega_n = \frac{Kk_p}{\tau}$ and $\omega_n^2 = \frac{Kk_i}{\tau}$.
4. From these, we can solve for k_p, k_i, and b_{sp}: $k_p = \frac{2\zeta\omega_n\tau}{K}$ and $k_i = \frac{\omega_n^2\tau}{K}$

4.3.3 Determining PI Controller Gains for Given $\omega_n = 16\,\mathrm{rad/sec}$ and $\zeta = 1$

Given $\omega_n = 16\,\mathrm{rad/sec}$ and $\zeta = 1$, we can now calculate the gains. Substituting into the equations for k_p and k_i:

$$k_p = \frac{2 \times 1 \times 16 \times \tau}{K} = \frac{32\tau}{K} k_i = \frac{16^2 \times \tau}{K} = \frac{256\tau}{K}$$

The proportional gain k_p is $\frac{32\tau}{K}$ and the integral gain k_i is $\frac{256\tau}{K}$.

4.3.4 Steady-State Value and Step Response

For a unit step input r(t) = 1, the Laplace transform is: $R(s) = \frac{1}{s}$. Using the Final Value Theorem:

$$\omega_{ss,PI} = \lim_{s\to 0} s \cdot \Omega_m(s) = \lim_{s\to 0} \frac{s \cdot K(k_p b_{sp}s + k_i)}{s^2(\tau s + 1) + sKk_p + Kk_i}$$

At $s = 0$: $\omega_{ss,PI} = 1$. Thus, the steady-state value of the output matches the input, eliminating steady-state error. The 2% settling time T_s is: $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1\times 16} = 0.25\,\mathrm{seconds}$

This gives a fast settling time for the system. The step response will be a smooth curve approaching the steady-state value without overshoot due to the critical damping condition ($\zeta = 1$).
