4.1 Closed-Loop Transfer Function with Proportional Control (Answers)

The open-loop transfer function of the DC motor is given by:

$$G(s) = \frac{\Omega_m(s)}{U_m(s)} = \frac{K}{\tau s + 1}$$

The control signal for proportional control is:

$$u_m(t) = k_p \left(r(t) - \omega_m(t)\right)$$

In the Laplace domain, this becomes:

$$U_m(s) = k_p \left(R(s) - \Omega_m(s)\right)$$

Substitute U_m(s) into the open-loop transfer function:

$$\Omega_m(s) = G(s)U_m(s) = \frac{K}{\tau s + 1} k_p \left(R(s) - \Omega_m(s)\right)$$

Rearrange to solve for $\Omega_m(s)$:

$$\Omega_m(s)(\tau s + 1) = K k_p \left(R(s) - \Omega_m(s)\right)$$

$$\Omega_m(s)(\tau s + 1 + K k_p) = K k_p R(s)$$

Finally, the closed-loop transfer function $G_p(s)$ is:

$$G_p(s) = \frac{\Omega_m(s)}{R(s)} = \frac{K k_p}{\tau s + 1 + K k_p}$$

4.1.2. Location of Poles as a Function of k_p

The closed-loop transfer function from $4.1.1$ is:

$$G_p(s) = \frac{K k_p}{\tau s + 1 + K k_p}$$

The characteristic equation is given by the denominator of the transfer function:

$$\tau s + 1 + K k_p = 0$$

Solving for s :

$$s = -\frac{1 + K k_p}{\tau}$$

Thus, the location of the pole depends on k_p. As k_p increases, the real part of the pole becomes more negative, meaning the system becomes faster. The system is more stable as k_p increases because the pole moves farther left in the s-plane, increasing the speed of convergence to steady-state.

Unit Step Response

When k_p is small, the system will have a slower response, meaning it takes longer to reach the steady state.

4.1.3. Steady-State Value Using Final Value Theorem

Consider a step input $r(t) = r_0$. The Laplace transform of this input is:

$$R(s) = \frac{r_0}{s}$$

The closed-loop transfer function is:

$$G_p(s) = \frac{Kk_p}{\tau s + 1 + Kk_p}$$

The output $\Omega_m(s)$ is:

$$\Omega_m(s) = G_p(s)R(s) = \frac{Kk_p}{\tau s + 1 + Kk_p} \cdot \frac{r_0}{s}$$

Using the **Final Value Theorem**:

$$\omega_m(\infty) = \lim_{s \to 0} s \cdot \Omega_m(s)$$

Substitute $\Omega_m(s)$:

$$\omega_m(\infty) = \lim_{s \to 0} s \cdot \left(\frac{Kk_p}{\tau s + 1 + Kk_p} \cdot \frac{r_0}{s} \right)$$

At $s = 0$:

$$\omega_m(\infty) = \frac{Kk_p}{1 + Kk_p} \cdot r_0$$

Thus, the steady-state value of the output is:

$$\omega_m(\infty) = \frac{Kk_p}{1 + Kk_p} \cdot r_0$$
