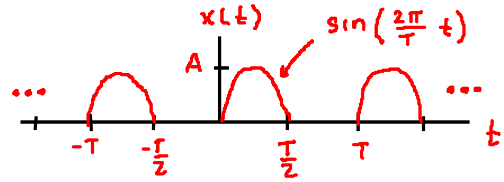
Exercise 5.103



L Answer (d).

From the Fourier series analysis equation, we have

Fourier series analysis equation

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_0^T x(t)e^{-j(2\pi/T)kt}dt \\ &= \tfrac{1}{T}\int_0^{T/2} A\sin\left(\tfrac{2\pi}{T}t\right)e^{-j(2\pi/T)kt}dt \\ &= \tfrac{A}{T}\int_0^{T/2} \sin\left(\tfrac{2\pi}{T}t\right)e^{-j(2\pi/T)kt}dt. \end{aligned}$$

Substitute given x

move A outside integral

To evaluate this integral, we can use (F.4). Since the constraint in (F.4) may be violated (as $-j(2\pi/T)k = \pm j(2\pi/T)$ for some k), we must exercise some care in performing the integration. In particular, we must consider several cases, namely: $k=1$, $k=-1$, and $k \notin \{-1,1\}$ (i.e., the otherwise case). To more clearly see the reason for the constraint in (F.4), we can rewrite the above integral as follows:

$$\begin{aligned} c_k &= \tfrac{A}{T}\int_0^{T/2}\tfrac{1}{j2}\left[e^{j2\pi t/T}-e^{-j2\pi t/T}\right]e^{-j2\pi kt/T}dt \\ &= \tfrac{A}{j2T}\int_0^{T/2}\left[e^{j2\pi(1-k)t/T}-e^{-j2\pi(1+k)t/T}\right]dt \\ &= \begin{cases} \tfrac{A}{j2T}\int_0^{T/2}\left[1-e^{-j4\pi t/T}\right]dt & k=1 \quad (1) \\ \tfrac{A}{j2T}\int_0^{T/2}\left[e^{j4\pi t/T}-1\right]dt & k=-1. \quad (2) \end{cases} \end{aligned}$$

constant term

constant term

These steps show in more detail why k=1 and k=-1 must be handled as special cases. We also use the formulas for k=1 and k=-1 later.

Clearly, if $k=1$ or $k=-1$ one of the exponentials degenerates into a constant function.

First, consider the case of $k \notin \{-1,1\}$. From (F.4), we have

plug into integral table entry (F.4) given in problem statement

$$\begin{aligned} c_k &= \tfrac{A}{T}\left[\frac{e^{-j2\pi kt/T}\left[\frac{-j2\pi k}{T}\sin\left(\frac{2\pi}{T}t\right)-\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}t\right)\right]}{\left(\frac{-j2\pi k}{T}\right)^2+\left(\frac{2\pi}{T}\right)^2}\right]\Bigg|_0^{T/2} \\ &= \tfrac{A}{T}\left[\frac{e^{-j2\pi kt/T}\left[\frac{-j2\pi k}{T}\sin\left(\frac{2\pi}{T}t\right)-\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}t\right)\right]}{\frac{-4\pi^2k^2+4\pi^2}{T^2}}\right]\Bigg|_0^{T/2} \\ &= \tfrac{A}{T}\left[\frac{T^2}{4\pi^2(1-k^2)}\right]\left[e^{-j2\pi kt/T}\left[\frac{-j2\pi k}{T}\sin\left(\frac{2\pi}{T}t\right)-\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}t\right)\right]\right]\Bigg|_0^{T/2} \\ &= \tfrac{A}{T}\left[\frac{T^2}{4\pi^2(1-k^2)}\right]\left[e^{-j\pi k}\left[-\tfrac{2\pi}{T}(-1)\right]-\left(-\tfrac{2\pi}{T}\right)\right] \\ &= \frac{AT}{4\pi^2(1-k^2)}\left[(-1)^k\left(\tfrac{2\pi}{T}\right)+\tfrac{2\pi}{T}\right] \\ &= \frac{AT}{4\pi^2(1-k^2)}\left(\tfrac{2\pi}{T}\right)\left[1+(-1)^k\right] \\ &= \frac{A\left[1+(-1)^k\right]}{2\pi(1-k^2)} \\ &= \begin{cases} \frac{A}{\pi(1-k^2)} & k \text{ even} \\ 0 & k \text{ odd and } k \notin \{-1,1\}. \end{cases} \end{aligned}$$

add terms in denominator

pull out factor from denominator factoring $4\pi^2$ in process

evaluate at 0 and T/2

$e^{j\pi} = -1$

factor out $\frac{2\pi}{T}$

cancel factors of T and 2π

$1+(-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

Next, consider the case of $k=1$. We have

$$
\begin{aligned}
c_k &= \tfrac{A}{j2T}\int_0^{T/2}\left[1-e^{-j4\pi t/T}\right]dt && \text{① from earlier}\\
&= \tfrac{A}{j2T}\left[t-\tfrac{T}{-j4\pi}e^{-j4\pi t/T}\right]\Big|_0^{T/2} && \text{integrate}\\
&= \tfrac{A}{j2T}\left[\tfrac{T}{2}-\tfrac{T}{-j4\pi}e^{-j4\pi(T/2)/T}-\left(\tfrac{T}{j4\pi}\right)\right] && \text{evaluate at 0 and } \tfrac{T}{2}\\
&= \tfrac{A}{j2T}\left[\tfrac{T}{2}+\tfrac{T}{j4\pi}e^{-j2\pi}-\tfrac{T}{j4\pi}\right] && \text{simplify exponent in 2nd term}\\
&= \tfrac{A}{j2T}\left[\tfrac{T}{2}\right] && e^{-j2\pi}=1\\
&= \tfrac{A}{j4} && \text{cancel factor of } T\\
&= \tfrac{-jA}{4}. && \text{move } j \text{ to numerator}
\end{aligned}
$$

Finally, consider the case of $k=-1$. Since x is real, c is conjugate symmetric. Therefore, $c_{-1}=c_1^*=\frac{jA}{4}$. Combining the above results, we conclude

$$
c_k=\begin{cases}\dfrac{A}{\pi(1-k^2)} & k \text{ even}\\[2mm] \dfrac{-jAk}{4} & k\in\{-1,1\}\\[2mm] 0 & \text{otherwise.}\end{cases}
$$