B-2-8

There are infinitely many state-space representations for this system. Here are the two possible state space representations.

$$\frac{Y(s)}{U(s)} = \frac{\frac{s+z}{s+p}\,\frac{1}{s^2}}{1+\frac{s+z}{s+p}\,\frac{1}{s^2}} = \frac{s+z}{s^3+ps^2+s+z}$$

Which is equivalent to

$$\dddot{y} + p\ddot{y} + \dot{y} + zy = \dot{u} + zu$$

Comparing with the standard equation:

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

$$\Rightarrow \begin{cases} a_1 = p \\ a_2 = 1 \\ a_3 = z \end{cases} \quad \& \quad \begin{cases} b_0 = 0 \\ b_1 = 0 \\ b_2 = 1 \\ b_3 = z \end{cases}$$

Now, we define,

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \dot{x}_2 - \beta_2 u$$

Where,

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = 0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 1$$

Also. We define,

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = z - p$$

Then, state-space equations can be given by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z & -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

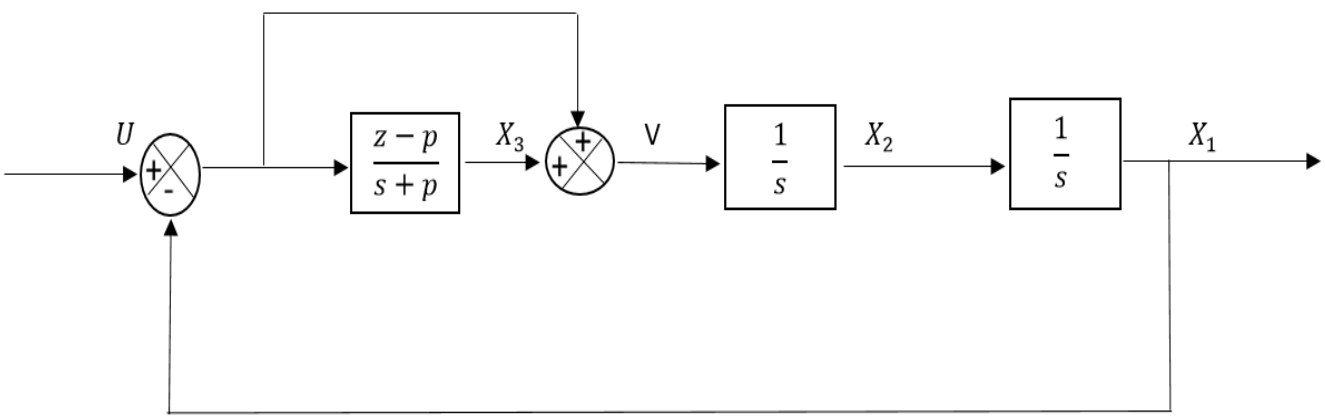
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

A different state-space representations for the same system can be obtained using a different set of state variables.

From,

$$\frac{s+z}{s+p} = 1 + \frac{z-p}{s+p}$$

we have,



From this block diagram we get the following equations (Capital letters represent Laplace transforms):

$$V = U - X_1 + X_3 \qquad \& \qquad X_2 = \frac{1}{s}(U - X_1 + X_3)$$

$$X_3 = \frac{z-p}{s+p}\,(U - X_1) \qquad\qquad X_1 = \frac{1}{s} X_2$$

from which we obtain,

$$\dot{x}_3 + px_3 = (z-p)\,u - (z-p)\,x_1$$

$$\dot{x}_2 = -x_1 + x_3 + u$$

$$\dot{x}_1 = x_2$$

Rewriting we have,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + x_3 + u$$

$$\dot{x}_3 = -(z-p)x_1 - px_3 + (z-p)u$$

$$y = x_1$$

or, $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ p-z & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z-p \end{bmatrix} u$$

<u>B-3-13</u>

Define the current in the armature circuit to be i_a, we have,

$$L\,\frac{di_a}{dt} + Ri_a + K_b\frac{d\theta_m}{dt} = e_i$$

Or,

$$(Ls+R)I_a(s) + K_b\, s\, \Theta_m(s) = E_i(s) \qquad (1)$$

Where K_b is the back emf constant of the motor. We also have,

$$J_m\ddot{\theta}_m + T = T_m = Ki_a \qquad (2)$$

$$\& \; J_L\ddot{\theta} = T_L$$

$$T = \frac{\theta}{\theta_m}T_L \xRightarrow{\theta\ = n\,\theta_m} T = nT_L$$

Where K is the motor torque constant and i_a is the armature current. Equation (2) can be rewrite as,

$$J_m \frac{\ddot{\theta}}{n} + n\, J_L \ddot{\theta} = K\, i_a$$

$$\Rightarrow\ (J_m + n^2\, J_L)\ddot{\theta} = n\, K\, i_a$$

Or,

$$(J_m + n^2\, J_L)\, s^2\, \Theta(s) = n\, K\, I_a(s) \qquad (3)$$

By substituting the equation (3) into equation (1), we obtain,

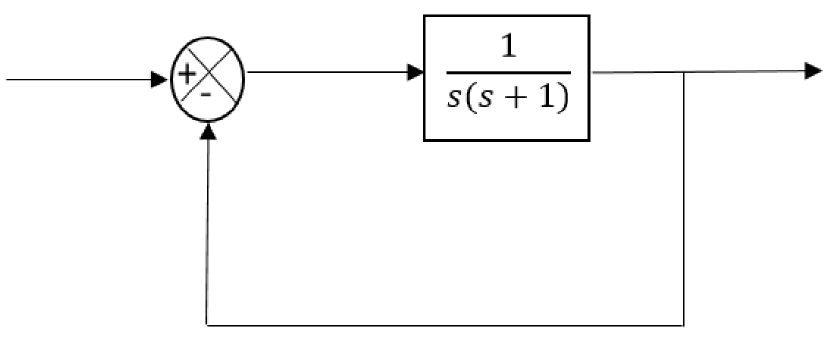
$$(Ls + R)\frac{(J_m + J_L n^2)s^2}{nK}\ \Theta(s) + K_b s\ \frac{\Theta(s)}{n} = E_i(s)$$

$$\Rightarrow [(Ls+R)(J_m + n^2\, J_L)\, s^2 + K_b K_s\,]\ \Theta(s) = n\, K\, E_i(s)$$

$$\Rightarrow \frac{\Theta(s)}{E_i(s)} = \frac{nK}{s[(LS+R)(J_m + n^2 J_L)s + KK_b]}$$

B-5-2

From,



We have,

$$G(s) = \frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}} = \frac{1}{s^2 + s + 1}$$

And,

$$\omega_n^2 = 1 \Rightarrow \omega_n = 1$$

$$2\zeta\omega_n = 1 \Rightarrow \zeta = 0.5$$

Eq. (5-19) → Rise time = 2.42 sec using atan2( ) in (5-19)

Eq. (5-20) → Peak time = 3.63 sec

Eq. (5-21) → Maximum overshoot = 0.163

Eq. (5-22) → Setting time = 8 sec (2% criterion)

B-5-3

From Eq.(5-21)

$$M_P = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \Rightarrow \zeta = 0.69$$

Then,

$$\omega_n = \frac{T_{set}}{\zeta} = \frac{2\text{sec}}{0.69} = 2.90\ \frac{\text{rad}}{\text{sec}}$$