ECE 260 A4

**5.1** For each case below, find the Fourier series representation (in complex exponential form) of the function $x$, explicitly identifying the fundamental period of $x$ and the Fourier series coefficient sequence $c$.
(a) $x(t) = 1 + \cos(\pi t) + \sin^2(\pi t)$;
(b) $x(t) = \cos(4t)\sin(t)$; and
(c) $x(t) = |\sin(2\pi t)|$. [Hint: $\int e^{ax}\sin(bx)dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2+b^2} + C$, where $a$ and $b$ are arbitrary complex and nonzero real constants, respectively.]

a

$$x(t) = 1 + \cos\pi t + \sin^2\pi t$$

$$= 1 + \frac{1}{2}\left[e^{j\pi t} + e^{-j\pi t}\right] + \left(\frac{1}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right]\right)^2$$

$$= 1 + \frac{1}{2}e^{j\pi t} + \frac{1}{2}e^{-j\pi t} - \frac{1}{4}e^{j(2\pi t)} + \frac{1}{2} - \frac{1}{4}e^{-j(2\pi t)}$$

$$= -\tfrac{1}{4}e^{-j2\pi t} + \frac{1}{2}e^{-j\pi t} + \tfrac{3}{2} + \frac{1}{2}e^{j\pi t} - \tfrac{1}{4}e^{j2\pi t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$\omega_0 = \pi$

$$c_k = \begin{cases} 3/2 & \text{for } k=0 \\ 1/2 & \text{for } k=\pm 1 \\ -1/4 & \text{for } k=\pm 2 \\ 0 & \text{otherwise} \end{cases}$$

c

$x(t)$ is periodic w/ period $T = 1/2$ and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1/2} = 4\pi$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = 2\int_0^{1/2} e^{-j4\pi k t}\sin 2\pi t\, dt$$

$$= \frac{2(2\pi)}{16\pi^2k^2 + 4\pi^2}\left[e^{-4j\pi k t}\left[-j2k\sin 2\pi t - \cos 2\pi t\right]\right]_0^{1/2}$$

$$= \frac{1}{\pi(1-4k^2)}\left[e^{\left(-j4\pi k/2\right)}\left[-j2k\sin\frac{2\pi}{2} - \cos\frac{2\pi}{2}\right] - \left(-\cos 0\right)\right]$$

$$= \frac{1}{\pi(1-\text{[illegible]})}(2) = \frac{2}{\pi(1-16k^2)}$$

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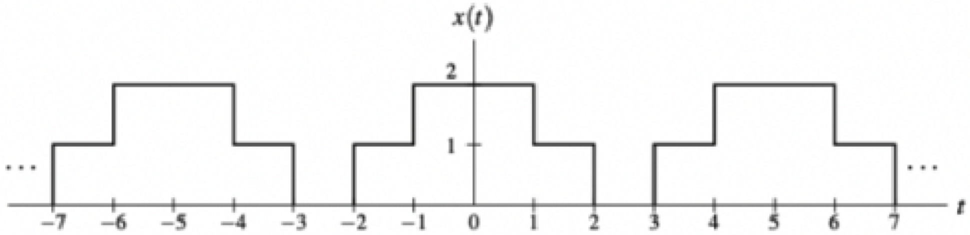
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$w_0 = 4\pi$$

$$c_k = \frac{2}{\pi(1-16k^2)}$$

**5.2** For each of the periodic functions shown in the figures below, find the corresponding Fourier series coefficient sequence.




(c)

c $x(t)$ is periodic w/ $T = 5$ ~~&~~ frequency $\omega_0 = \frac{2\pi}{5}$

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{5}\int_{-5/2}^{5/2} x(t)\, e^{\frac{-j2\pi k t}{5}}\, dt$$

$$= \frac{1}{5}\left[\int_{-2}^{-1} e^{-j2\pi k t/5}\, dt + \int_{-1}^{1} 2e^{-j2\pi k t/5}\, dt + \int_{1}^{2} e^{-j2\pi k t/5}\, dt\right]$$

$$= \frac{1}{5}\left[\int_{-2}^{2} e^{-j2\pi k t/5}\, dt + \int_{-1}^{1} e^{-j2\pi k t/5}\, dt\right]$$

$$= \frac{1}{-j2\pi k}\left[e^{-j4\pi k/5} - e^{j4\pi k/5}\right]_{-2}^{2} + \left[e^{-j4\pi k/5}\right]_{-1}^{1}$$

$$= \frac{1}{-j2\pi k}\left[e^{-j4\pi k/5} - e^{j4\pi k/5} + e^{-j2\pi k/5}\right]$$

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$$= \frac{1}{\pi k}\left(-2j\sin 4\pi k/5 - 2j\sin\frac{2\pi k}{5}\right) = \frac{\sin 4\pi k}{5\pi k} + \frac{\sin 2\pi k}{5\pi k}$$

$$= \frac{4}{5}\sin\frac{4\pi k}{5} + \frac{2}{5}\sin\frac{2\pi k}{5}$$

$$c_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{5}\int_{-5/2}^{5/2} x(t)\,dt = \frac{1}{5}\left[\int_{-2}^{-1} dt + \int_{-1}^{1} 2\,dt + \int_{1}^{2} dt\right]$$

$$= \frac{1}{5}(1+4+1) = \frac{6}{5}$$

$$c_k = \begin{cases} 6/5 & \ldots k=0 \\ \frac{4}{5}\sin\left(\frac{4\pi k}{5}\right) + \frac{2}{5}\sin\left(\frac{2\pi k}{5}\right) & \ldots \text{otherwise} \end{cases}$$

$c_0 = 1.2$ $\quad$ $c_1 = c_{-1} \simeq 0.489828$ and $c_2 = c_{-2} \simeq -057816$

**5.3** Find the Fourier series coefficient sequence $c$ of each periodic function $x$ given below with fundamental period $T$.

(a) $x(t) = 2\delta(t-3) + 2\delta(t-5) + \delta(t-7) - \delta(t-9) + 3\delta(t-12)$ and $T = 16$; express $c$ in terms of sin and cos to whatever extent is possible.

$$x(t) = 2\delta(t-3) + 2\delta(t-5) + \delta(t-7) - \delta(t-9) + 3\delta(t-12) \text{ and } T=16$$

$$c_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\left(\frac{2\pi}{T}\right)kt}\, dt$$

$$= \frac{1}{16}\int_0^{16} \left[2\delta(t-3) + (t-5)2\delta + \delta(t-7) + \delta(t-9) + 3\delta(t-12)\right] e^{-j\left(\frac{2\pi}{16}\right)kt}$$

$$= \frac{1}{16}\left[2e^{-j\left(\frac{\pi}{8}\right)k(3)} + 2e^{-j\left(\frac{\pi}{8}\right)k(5)} + e^{-j\left(\frac{\pi}{8}\right)k(7)} - e^{-j\left(\frac{\pi}{8}\right)k(9)} + 3e^{-j\frac{\pi}{8}k(12)}\right]$$

$$= \frac{1}{16}\left[2e^{-j(4\pi/8)k}\left(e^{j(\pi/8)k} + e^{-j\pi\left(\frac{1}{8}\right)k}\right) + e^{-j\pi k}\left(e^{j\pi/8 k} - e^{-j\frac{\pi}{8}k}\right) + 3e^{-j\left(\frac{3\pi}{2}k\right)}\right]$$

$$= \frac{1}{16}\left[2(-j)^k\left[2\cos\left(\frac{\pi}{8}k\right)\right] + (-1)^k\left[2j\sin\left(\frac{\pi}{8}k\right)\right] + 3j^k\right]$$

$$= \frac{1}{16}\left[4(-j)^k\cos\left(\frac{\pi}{8}k\right) + 2j(-1)^k\sin\left(\frac{\pi}{8}k\right) + 3j^k\right]$$

$$= \frac{1}{4}(-j)^k\cos\left(\frac{\pi}{8}k\right) + \frac{j}{8}(-1)^k\sin\left(\frac{\pi}{8}\right)k + \frac{3}{16}j^k$$

**5.7** A periodic function $x$ with period $T$ and Fourier series coefficient sequence $c$ is said to be odd harmonic if $c_k = 0$ for all even $k$.

(a) Show that if $x$ is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all $t$.

(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all $t$, then $x$ is odd harmonic.

$$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt = \frac{1}{T}\left[\int_0^{T/2} x(t) e^{-j\omega_0 kt}\,dt + \int_{T/2}^{T} x(t) e^{-jk\omega_0 t}\,dt\right]$$

$$= \frac{1}{T}\left[\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt + \int_T^{3T/2} x(\lambda - \tfrac{T}{2}) e^{-jk\omega_0(\lambda - \frac{T}{2})}\,d\lambda\right]$$

$$= \frac{1}{T}\left[\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt + \int_T^{3T/2} -x(\lambda) e^{-jk\omega_0(\lambda - \frac{T}{2})}\,d\lambda\right]$$

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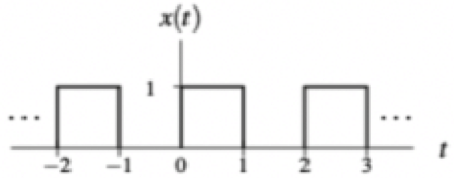
$$= \frac{1}{T}\left[\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt - e^{jk\pi}\int_T^{3T/2} x(\lambda) e^{-jk\omega_0 \lambda}\,d\lambda\right]$$

$$= \frac{1}{T}\left[\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt - (-1)^{k}\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt\right]$$

$$= \frac{1}{T}\left[\left(1 - (-1)^k\right)\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt\right] = \begin{cases} \frac{2}{T}\int_0^{T/2} x(t) e^{-jk\omega_0 t}\,dt & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

**5.9** Find the Fourier series coefficient sequence $c$ of the periodic function $x$ shown in the figure below. Plot the frequency spectrum of $x$, including the first five harmonics.




$x(t)$ is periodic w/ period $T=2$ and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^2 x(t)\,e^{-j\pi k t}\,dt$$

$$= \frac{1}{2}\left[\frac{1}{-j\pi k}e^{-j\pi k t}\right]\Bigg|_0^1$$

$$= \frac{1}{j2\pi k}\left(1-(-1)^k\right)$$

$$= \begin{cases} \left(-\,{j}/{\pi k}\right) & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

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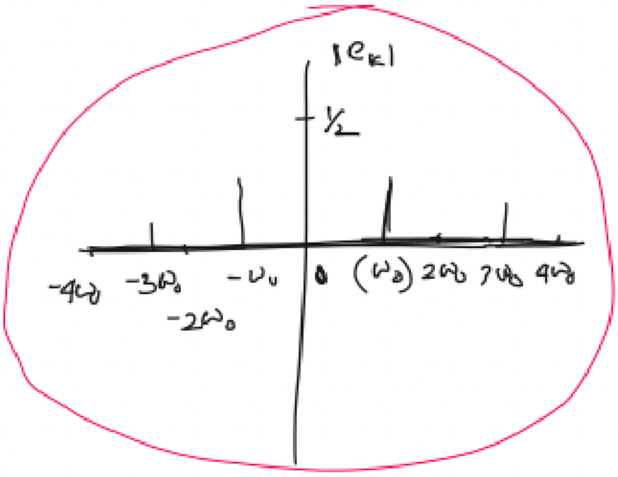
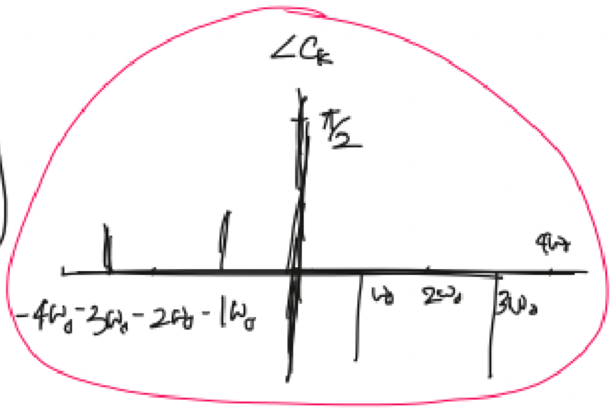
$$c_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\int_0^2 x(t)\,dt$$

$$= \frac{1}{2}\int_0^1 dt = \frac{1}{2}(t)\Big|_0^1 = \frac{1}{2}$$

Therefore, $c_k = \begin{cases} 1/2 & \ldots\ k=0 \\ -\,j/\pi k & \ldots\ \text{odd } k \\ 0 & \ldots\ \text{even } k,\ k \neq 0 \end{cases}$

| $k$ | $\lvert c_k \rvert$ | $\arg c_k$ |
|---|---|---|
| 0 | $1/2$ | 0 |
| 1 | $1/\pi$ | $-\pi/2$ |
| 2 | 0 | 0 |
| 3 | $3/\pi$ | $-\pi/2$ |
| 4 | 0 | 0 |
| 5 | $1/5\pi$ | $-\pi/2$ |



**5.10** Consider a LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output $y$ of the system if the input $x$ is given by

$$x(t) = 1 + 2\cos(2t) + 2\cos(4t) + \tfrac{1}{2}\cos(6t).$$

$$x(t) = 1 + 2\cos 2t + 2\cos 4t + \tfrac{1}{2}\cos 6t = 1 + 2\left[\tfrac{1}{2} e^{j2t} + e^{-j2t}\right]$$

$$+ 2\left[\tfrac{1}{2}\left(e^{j4t} + e^{-j4t}\right)\right] + \tfrac{1}{2}\left(\tfrac{1}{2}\left(e^{j6t} + e^{-j6t}\right)\right)$$

$$= 1 + e^{j2t} + e^{-j2t} + e^{j4t} + e^{-j4t} + \tfrac{1}{4} e^{j6t} + \tfrac{1}{4} e^{-j6t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2$$

$$a_k = \begin{cases} 1 & \ldots k = 0 \\ 1 & \ldots k = \pm 1 \\ 1 & \ldots k = \pm 2 \\ 1/4 & \ldots k = \pm 3 \\ 0 & \ldots \text{otherwise} \end{cases}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\ldots \quad b_0 = a_0 H(j[0][2]) = 0$$

$$b_1 = a_1 H(j[1][2]) = 0$$

$$b_{-1} = a_{-1} H(j[-1][2]) = 0$$

$$b_2 = a_2 H(j[2][2]) = 0$$

$$b_{-2} = a_{-2} H(j[-2][2]) = 0$$

$$b_3 = a_3 H(j[3][2]) = \tfrac{1}{4}(1) = \tfrac{1}{4}$$

$$b_{-3} = a_{-3} H(j[-3][2])$$

$$= \tfrac{1}{4}(1) = \tfrac{1}{4}$$

$$y(t) = \frac{1}{4} e^{-j6t} + \frac{1}{4} e^{j6t}$$

$$= \frac{1}{4}\left(e^{-j6t} + e^{j6t}\right)$$

$$= \frac{1}{4}\left(2\cos 6t\right) = \frac{1}{2}\cos 6t$$

main.m

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```
% Clear all variables and close all figures.
clear all;
close all;

% Define the time variable.
t = linspace(-1, 1, 1000);

% Values for n.
n_values = [1, 5, 10, 50, 100];

for n = n_values
    % Symbolic expression for the square wave.
    syms k w;
    f = symsum(0.5 * mysinc(pi * k / 2) * exp(j * k * w * t), 'k', -n, n);
    f = subs(f, w, 2 * pi);

    % Plot the result.
    figure;
    plot(t, real(f), 'LineWidth', 1.5);
    title(['x_{', num2str(n), '}(t)']);
    xlabel('Time');
    ylabel('Amplitude');
    grid on;

    % Save the plot to a file.
    print(['data/sqrwav_', num2str(n)], '-depsc');

end

function y = mysinc(x)
    y = ones(size(x));
    i = find(x);
    y(i) = sin(x(i)) ./ x(i);
end
```

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b. The function $\hat{m}_{K,I}(t)$ does not converge to $m(t)$ as the rate of convergence is lower at the point of discontinuity of $m(t)$

c. Point of discontinuity located at $t=\frac{1}{4}$ the function will converge around left & right limits of $m(t)$, namely the value of $\frac{1}{2}$.