Assignment 7

ECE 360

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B-7-1

Consider the unity-feedback system with the open-loop transfer function:

$$G(s) = \frac{10}{s+1}$$

Obtain the steady-state output of the system when it is subjected to each of the following inputs:

(a) $r(t) = \sin(t + 30^\circ)$

(b) $r(t) = 2\cos(2t - 45^\circ)$

(c) $r(t) = \sin(t + 30^\circ) - 2\cos(2t - 45^\circ)$

S-7-1

For a unity feedback system with open-loop transfer function $G(s) = \frac{10}{s+1}$, we'll analyze the steady-state response to sinusoidal inputs.

Preliminary Analysis

The closed-loop transfer function is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{10}{s+1+10} = \frac{10}{s+11}$$

For sinusoidal inputs, we utilize the frequency response approach. The general form of $T(j\omega)$ is:

$$T(j\omega) = \frac{10}{j\omega + 11} = \frac{10}{\sqrt{\omega^2 + 121}} \angle - \tan^{-1}\left(\frac{\omega}{11}\right)$$

Part (a): $r(t) = \sin(t + 30)$

Given Input:

- Angular frequency: $\omega = 1$ rad/s
- Initial phase: $\theta_1 = 30$
- Amplitude: $A_1 = 1$

Analysis:

$$\begin{aligned}
\text{Magnitude Response:} \quad |T(j1)| &= \frac{10}{\sqrt{1^2 + 11^2}} \\
&= \frac{10}{\sqrt{122}} \\
&= 0.905
\end{aligned}$$

$$\begin{aligned}
\text{Phase Response:} \quad \phi_1 &= -\tan^{-1}\left(\frac{1}{11}\right) \\
&= -5.2
\end{aligned}$$

Steady-State Output: $\boxed{C_{ss,a}(t) = 0.905\sin(t + 30 - 2.2) = 0.905\sin(t + 27.8)}$

Part (b): $r(t) = 2\cos(2t - 45)$

Given Input:

- Angular frequency: $\omega = 2$ rad/s
- Initial phase: $\theta_2 = -45$
- Amplitude: $A_2 = 2$

Analysis:

$$\begin{aligned}\text{Magnitude Response:} \quad |T(j2)| &= \frac{10}{\sqrt{2^2 + 11^2}} \\ &= \frac{10}{\sqrt{125}} \\ &= 0.895\end{aligned}$$

$$\begin{aligned}\text{Phase Response:} \quad \phi_2 &= -\tan^{-1}\left(\frac{2}{11}\right) \\ &= -10.3\end{aligned}$$

Steady-State Output: $\boxed{C_{ss,b}(t) = 2(0.895)\cos(2t - 45 - 10.3) = 1.79\cos(2t - 55.3)}$

Part (c): $r(t) = \sin(t + 30) - 2\cos(2t - 45)$

Application of Superposition Principle: Since the system is linear, we can apply the principle of superposition. The steady-state response to the combined input is the sum of the individual responses found in parts (a) and (b).

Steady-State Output: $\boxed{C_{ss,c}(t) = 0.905\sin(t + 27.8) - 1.79\cos(2t - 55.3)}$

Important Observations

For any sinusoidal input with frequency ω, the steady-state output exhibits the following characteristics:

1. **Magnitude Relationship:**
$$|T(j\omega)| = \frac{10}{\sqrt{\omega^2 + 121}}$$
2. **Phase Relationship:**
$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{11}\right)$$
3. **Output Properties:**
   - Frequency remains unchanged from input to output
   - Amplitude is scaled by $|T(j\omega)|$
   - Phase is shifted by $\phi(\omega)$

B-7-4

Plot the Bode diagram of

$$G(s) = \frac{10(s^2 + 0.4s + 1)}{s(s^2 + 0.8s + 9)}$$

S-7-4

The following MATLAB program produces the Bode diagram shown below.

```
% ***** Bode diagram *****
num = [0 10 4 10];
den = [1 0.8 9 0];
bode(num, den)
title('Bode Diagram of G(s) = 10(s^2 + 0.4s + 1)/[s(s^2 + 0.8s + 9)]')
```

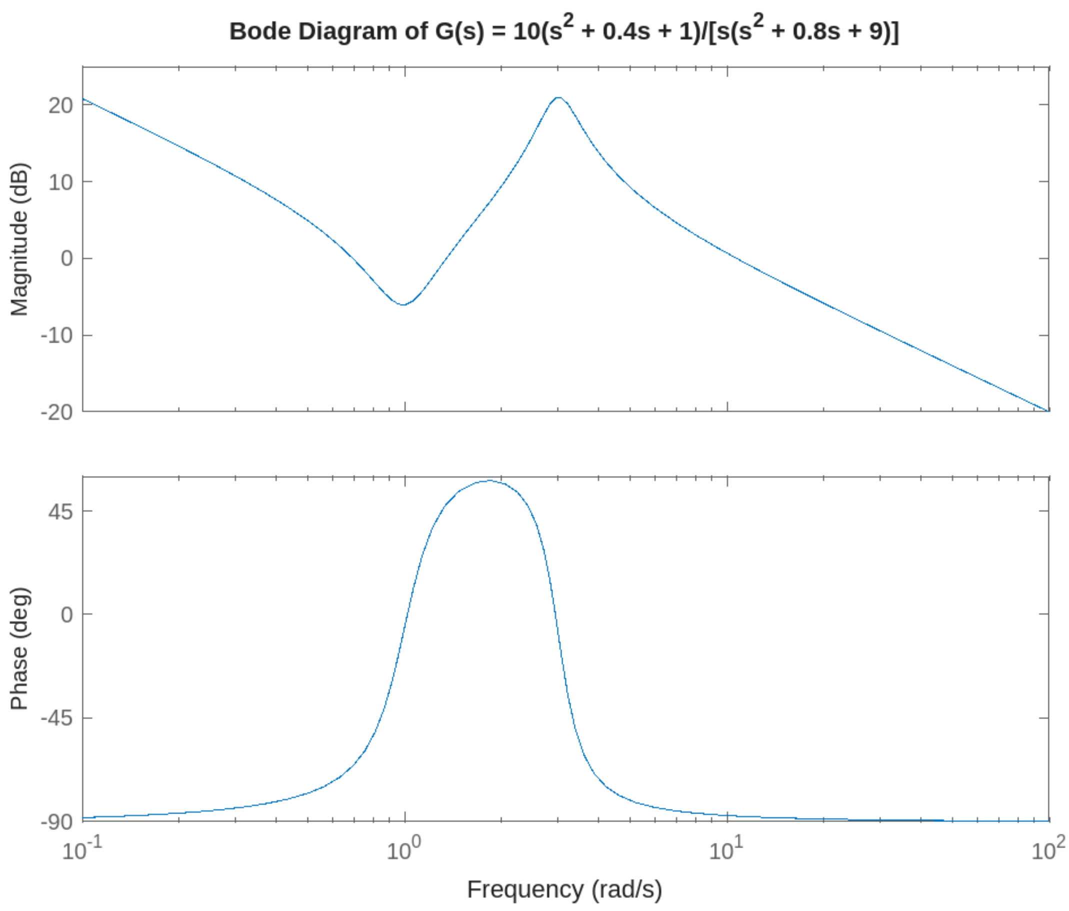

Figure 1: Bode Diagram of $G(s) = \frac{10(s^2+0.4s+1)}{s(s^2+0.8s+9)}$

B-7-5

Given $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ show that $|G(j\omega_n)| = \frac{1}{2\zeta}$

S-7-5

Let's solve this step by step:

1. First, we substitute $s = j\omega_n$ into $G(s)$:

$$G(j\omega_n) = \frac{\omega_n^2}{(j\omega_n)^2 + 2\zeta\omega_n(j\omega_n) + \omega_n^2}$$

2. Simplify the denominator:

$$G(j\omega_n) = \frac{\omega_n^2}{-\omega_n^2 + 2\zeta\omega_n^2 j + \omega_n^2}$$
$$= \frac{\omega_n^2}{2\zeta\omega_n^2 j}$$
$$= \frac{1}{2\zeta j}$$

3. Taking the magnitude:

$$|G(j\omega_n)| = \left|\frac{1}{2\zeta j}\right| = \frac{1}{|2\zeta j|} = \frac{1}{2\zeta}$$

Therefore, we have proven that $|G(j\omega_n)| = \frac{1}{2\zeta}$.

B-7-7

Sketch the polar plots of the open-loop transfer function

$$G(s)H(s) = \frac{K(T_a s + 1)(T_b s + 1)}{s^2(Ts + 1)}$$

for the following two cases:

(a) $T_a > T > 0, \quad T_b > T > 0$

(b) $T > T_a > 0, \quad T > T_b > 0$

S-7-7

The Nyquist plots for cases (a) and (b) are shown below. The key difference between the two cases lies in the relationship between the time constants, which affects the shape of the curves.

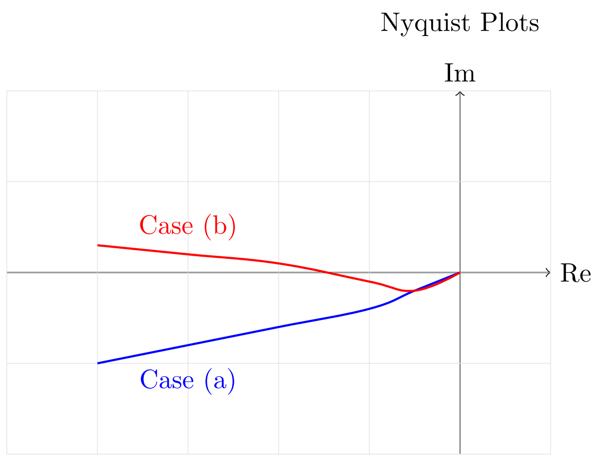

Figure 2: Nyquist plots for cases (a) and (b)

Analysis of the plots:

- **Case (a)** $T_a > T > 0$, $T_b > T > 0$:
  - The curve starts from the negative real axis

  - Shows a more pronounced curvature in the lower half-plane
  - Approaches the origin from below

- **Case (b)** $T > T_a > 0$, $T > T_b > 0$:
  - The curve starts in the upper half-plane
  - Has a gentler curvature
  - Crosses the real axis and approaches the origin from below

The differences in the curves arise from how the zeros (determined by T_a and T_b) interact with the poles (determined by T) in each case. In case (a), the zeros are slower than the non-zero pole, while in case (b), the zeros are faster than the non-zero pole, leading to the distinct shapes observed.