Lecture Slides

ECE 360 Control Theory and Systems I

Prepared by P. Agathoklis

Based on
Ogata K., 'Modern Control Engineering', 5th Edition, Prentice Hall, 2010.

University of Victoria
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ECE 360 Control Theory and Systems I

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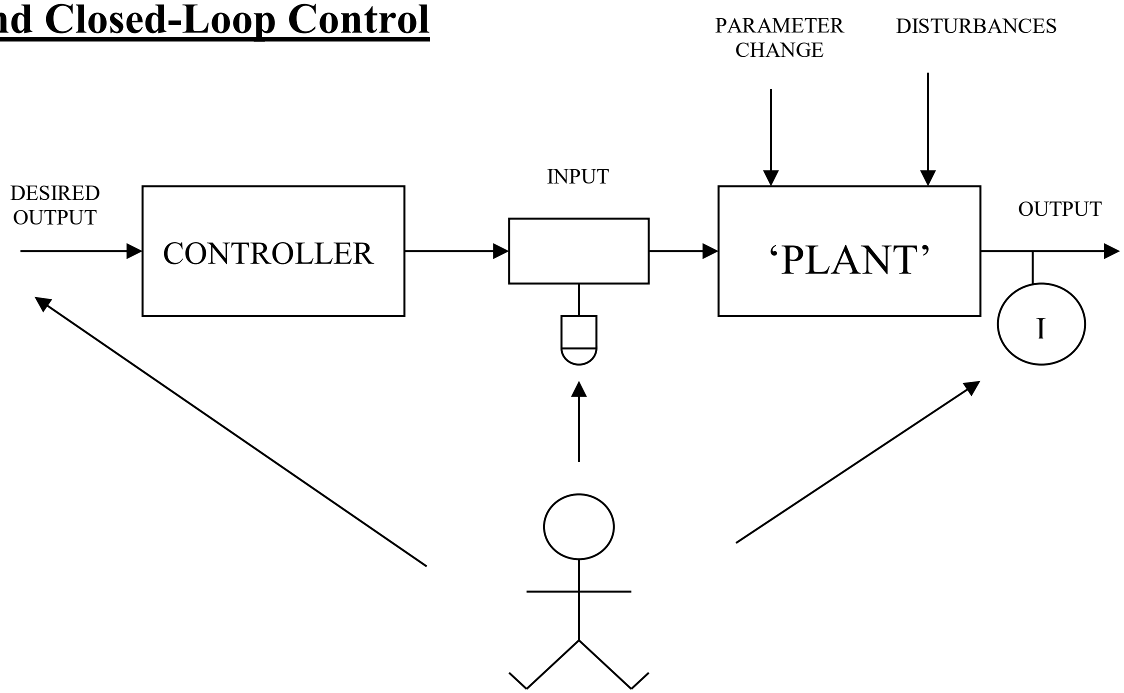
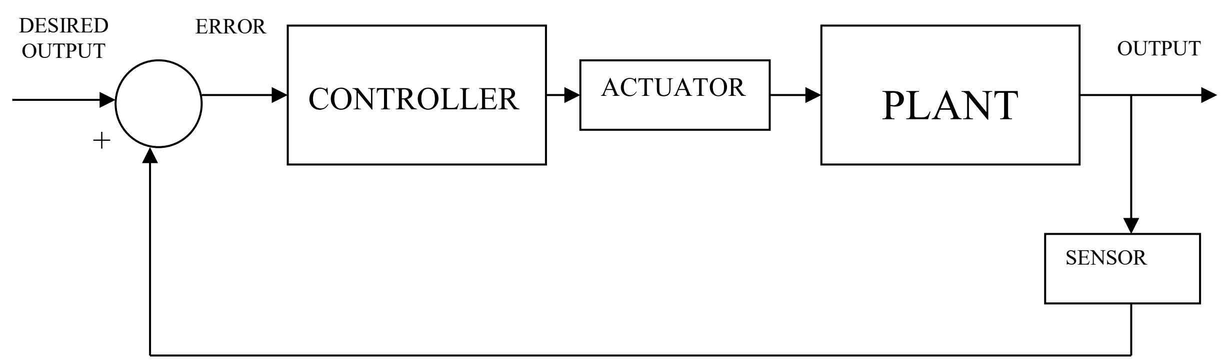
INTRODUCTION

Open-Loop and Closed-Loop Control

Open-Loop:



Closed-Loop:



Control of congested TCP networks:

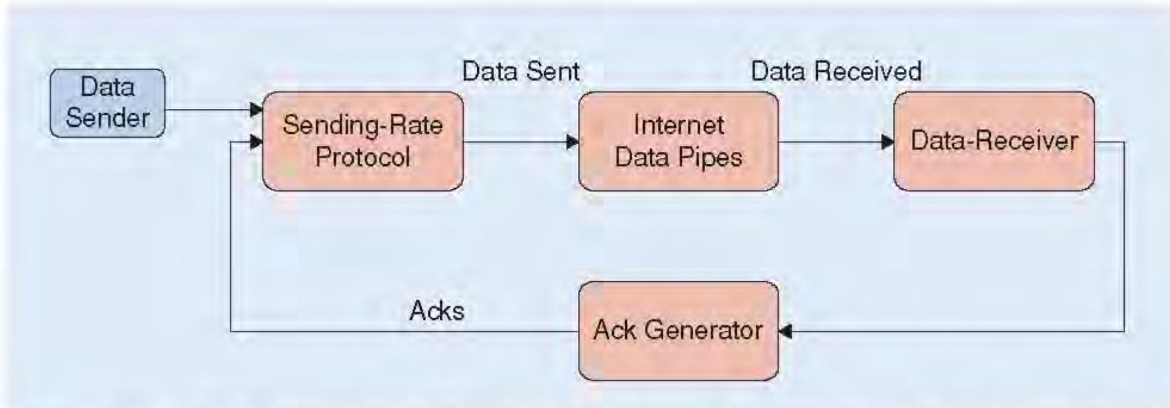


FIGURE 1 This block diagram depicts the Internet's sending-rate protocol as a feedback controller. The sending-rate protocol injects data into the Internet's data pipes. Congestion may occur in these pipes and cause data to be dropped. As data is received (or dropped), positive and negative acks are generated, and a new round of feedback signals are delivered to the sending-rate protocol. If the data sender receives a negative ack, the protocol decreases the data rate, whereas, if a positive ack is received, data rates increase. Thus, the Internet reacts, in negative feedback fashion, to congestion.

From IEEE Control Systems Magazine, Vol. 28, Is. 5, October 2008

Feedback Control of Computing Systems

Self-Adaptive software development

Brief History of Control

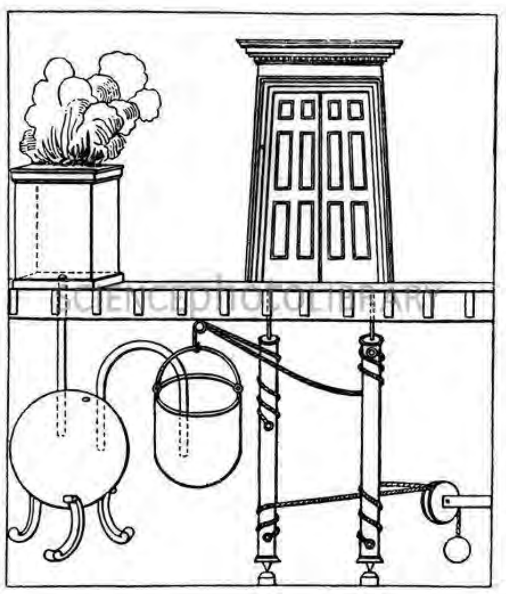
Date	*Technology*	*Problem*	*People*	*Method*
300 – 0 BC		Water clocks Oil lamps Pneumatica	Ktesibios Philon Heron of Alexandria	
16th – 17P$^{th\ P}$ century		Temperature and Pressure regulators Speed regulators	Cornelis Drebbel D. Papin J. Watt I. Polzunov	
19 century	Steam engine	Stability Stability	Maxwell Ruth Hurwitz	Differential Equations (DE)
1920	Ship Steering	Stab/design	Minorsky	DE
1927-32	Feedback Amps	Stability/design	Bode Nyquist	Laplace Transforms(LT)
1930's	Power drives	Stability/design	Brown	Laplace Transforms
1940's	Gun & radar systems	Stability/design	Many	Laplace Transforms

1950's	Aircraft control	Stab/time resp.	Evans	Root locus
	General theory	'Optimal' control	Wiener Pontryakin	Calculus of Variations
1960s	Aerospace	Multivariable State space Optimal Control	Kalman Bellman Russian work	State-Space
1970	Industrial Control	Disturbance rejection Computational methods & many others	Many	State-Space
1980	Digital Control	Worst-case design Plant changes Robust control	Many	

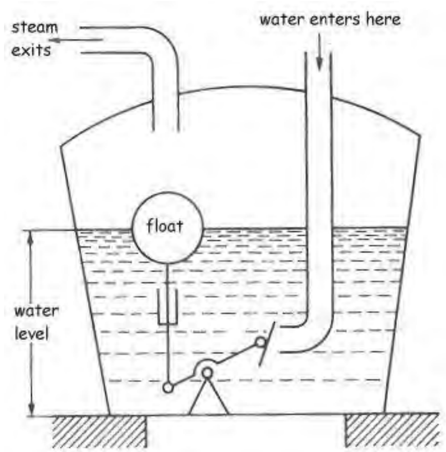
Comments:

- This is "main stream" – many other branches (computer control, chemical process control, etc.).
- In recent decades control theory has been used in many different areas of applications such as robotics, manufacturing, cybernetics, embedded systems, traffic control etc.
- Recent work has confirmed that the *techniques developed for SISO systems by Bode, Nyquist, Evans & others* are (when intelligently used) capable of producing excellent designs.

Hero's machine for opening doors

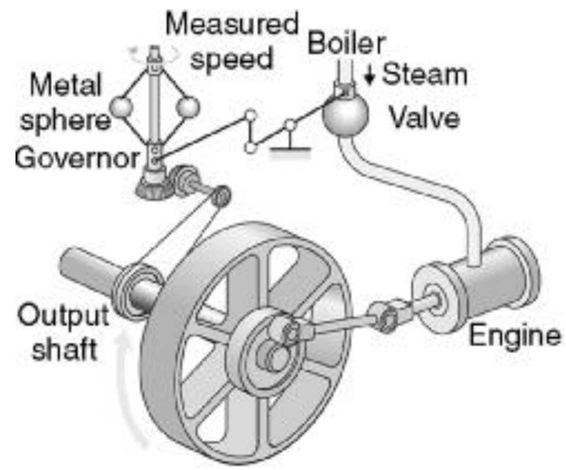


Polzunov's water level regulator (ca 1765)



http://www.uh.edu/engines/epi1616.htm

James Watt's flyball governor (ca 1769)



From Dorf+Bishop, Modern Control Systems

The Controller Design Process

Modeling

- Model the system to be controlled. Simplify the model, if necessary, so that it's tractable.
- Analyze the model & simulate if necessary; decide what sensors and actuators are needed and where they should be placed.
- Identify – usually by experiment – the values of model parameters.
- Verify (by simulation and comparison with plant behavior) that the model adequately represents the plant – if not, repeat from 1.

Design

- Decide on performance specifications.
- Decide on the type of controller to be used.
- Design a controller to meet the specs; if impossible or overly complex, repeat from 6.

Verification

Simulate the closed-loop controlled system (computer model – step 1 – or pilot plant). If unsatisfactory, repeat from appropriate point.

Implementation

- Choose hardware & software; implement and test the controller.
- Tune the controller on-line if needed; train operators and maintainers to get best use out of the system.

MATHEMATICAL BACKGROUND

Complex Number $s = \sigma + j\omega$

Complex Function $G(s) = G_R + jG_I$

$$\frac{dG(s)}{ds} = \lim_{\Delta s \to 0} \frac{G(s+\Delta s) - G(s)}{\Delta s}$$

G(s) is *analytic* in a region if G(s) and all its derivatives exist in this region.

Rational function:

$$G(s) = \frac{A(s)}{B(s)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$-z_i$: zeros of G(s), $-p_j$: poles of G(s), $m \le n$

Rational functions are analytic in the s-plane except at isolated points called *singularities*. *Poles* are singularities of G(s).

Laplace Transforms

Consider $f(t)$, such that $f(t) = 0$ for $t < 0$

$$£\left[f(t)\right] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$£^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

Pierre-Simon Laplace (1749–1827)
(http://www.photo.rmn.fr)

$£[f(t)]$ exists if:

1. $f(t)$ is sectionally continuous in every finite interval in the range $t > 0$
2. $f(t)$ is of exponential order as t approaches infinity, i.e., there exists a real and positive constant σ such that

$$e^{-\sigma t} \left| f(t) \right| \rightarrow 0 \quad \text{for } t \rightarrow \infty \text{ and } \sigma > \sigma_c$$

where σ_c is the *abscissa of convergence*.

<u>Remark 1:</u> $f(t) = e^{t^2}$ for $0 \le t \le \infty$

→ <u>does not</u> have a Laplace transform

$f(t) = e^{t^2}$ for $0 \le t \le T < \infty$ and $f(t) = 0$ for $t > T$

→ <u>do</u> have a Laplace transform

<u>Remark 2:</u>

$$£_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}dt$$

$$£_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$$

$$£_-[f(t)] = £_+[f(t)] + \int_{0-}^{0+} f(t)e^{-st}dt$$

$£_+$ and $£_-$ are equal iff $\int_{0-}^{0+} e^{-st} f(t)dt = 0$

<u>Remark 3:</u> $£_+[\delta(t)] = \int_{0+}^{\infty} \delta(t)e^{-st}dt = 0$ $£_-[\delta(t)] = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1$

Laplace Transform Tables

Table 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Laplace Transform Properties

f(t) is a Laplace transformable function and *F(s)* its Laplace trasform

- $\pounds\left[\mathrm{f}(\mathrm{t}-\alpha)\cdot u(t-\alpha)\right]=e^{-\alpha s}F(s)$ *u(t)* : step
- $\pounds\left[\mathrm{e}^{-\alpha\mathrm{t}}f(t)\right]=F(s+\alpha)$
- $\pounds\left[\mathrm{f}\left(\frac{\mathrm{t}}{\alpha}\right)\right]=\alpha F(\alpha s)$
- $\pounds\left[\frac{\mathrm{d}}{\mathrm{dt}}f(t)\right]=sF(s)-f(0)$

Derivation:

$$\int_0^\infty f(t)e^{-st}dt=f(t)\frac{e^{-st}}{-s}\Big|_0^\infty-\int_0^\infty\left(\frac{d}{dt}f(t)\right)\frac{e^{-st}}{-s}dt$$

$$F(s)=\frac{f(0)}{s}+\frac{1}{s}\pounds\left[\frac{\mathrm{d}}{\mathrm{dt}}f(t)\right]$$

- $£\left[\frac{\mathrm{d^n}}{\mathrm{dt^n}} f(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots s f^{n-2}(0) - f^{n-1}(0)$

- $£\left[\int \mathrm{f}(\mathrm{t}) dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t) dt\right]_{t=0}}{s}$

- **Final Value Theorem:**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt} f(t)$ are Laplace transformable
- $\lim_{t \to \infty} f(t)$ exists
- *F(s)* is analytic in ***Re(s)*** ≥ *0* except for at most a single pole at *s = 0*

Derivation of the Final Value Theorem:

Based on the Laplace Transform theorems:

$$\lim_{s\to 0}\int_0^\infty \left[\frac{d}{dt}f(t)\right]e^{-st}dt = \lim_{s\to 0}[sF(s)-f(0)]$$

since $\lim_{s\to 0} e^{-st} = 1$

$$\rightarrow \quad \int_0^\infty \frac{d}{dt}f(t)dt = f(\infty)-f(0) = \lim_{s\to 0}[sF(s)-f(0)]$$

$$\rightarrow \quad f(\infty) = \lim_{s\to 0} sF(s)$$

- **Initial value Theorem**

$$f(0+) = \lim_{s\to\infty} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt}f(t)$ are both Laplace transformable
- $\lim_{s\to\infty} sF(s)$ exists

From the Laplace Transform Theorems:

$$\lim_{s\to\infty} £_+\left[\frac{d}{dt}f(t)\right] = \lim_{s\to\infty}\left[\int_{0+}^{\infty}\left[\frac{d}{dt}f(t)\right]e^{-st}dt\right] = 0$$

and

$$= \lim_{s\to\infty}\left[sF(s) - f(0+)\right] = 0$$

- **Convolution**

Given:

$$f_1(t) \circ\!-\!-\!\bullet F_1(s)$$

$$f_2(t) \circ\!-\!-\!\bullet F_2(s)$$

$$f_3(t) = f_1(t) * f_2(t) = \int_0^t \mathrm{f}_1(t-\tau) f_2(\tau) d\tau = \pounds^{-1}\left[F_1(s) \cdot F_2(s)\right]$$

Example:

Consider

$$f_1(t) = \begin{cases} 1 & for \quad 0 \le t \le T \\ 0 & else \end{cases}$$

$$f_2(t) = \begin{cases} 1 & for \quad 0 \le t \le \frac{T}{2} \\ 0 & else \end{cases}$$

Find $f_3(t) = f_1(t) * f_2(t) = \int_0^t \mathrm{f}_1(t-\tau) f_2(\tau) d\tau = \pounds^{-1}\left[F_1(s) \cdot F_2(s)\right]$

From the definitions of $f_1(t)$ and $f_2(t)$ we have:

$$f_3(t)=\int_0^t d\tau = t \qquad \text{for} \qquad 0 \le t \le \frac{T}{2}$$

$$f_3(t)=\int_0^{\frac{T}{2}} d\tau = \frac{T}{2} \qquad \text{for} \qquad \frac{T}{2} \le t \le T$$

$$f_3(t)=\int_{t-T}^{\frac{T}{2}} d\tau = \frac{3T}{2} - t \qquad \text{for} \qquad T \le t \le \frac{3T}{2}$$

$$f_3(t)=0 \qquad \text{else}$$

Using $f_3(t)=\pounds^{-1}\left[F_1(s)\cdot F_2(s)\right]$, $F_1(s)=\dfrac{1-e^{-sT}}{s}$ and $F_2(s)=\dfrac{1-e^{\frac{-sT}{2}}}{s}$

Follows $F_3(s)=\dfrac{\left(1-e^{-sT}\right)\left(1-e^{\frac{-sT}{2}}\right)}{s^2}$ and

$$f_3(t)=t-\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)-(t-T)u(t-T)+\left(t-\frac{3T}{2}\right)u\left(t-\frac{3T}{2}\right)$$

for $t \ge 0$ *u(t): step*

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

Consider

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)}$$

1. All poles are <u>*distinct and real*</u>:

$$F(s) = \frac{a_1}{s+p_1} + \ldots + \frac{a_n}{s+p_n}$$

$$a_k = \left[\frac{B(s)}{A(s)}(s+p_k)\right]_{s=-p_k}$$

$$£^{-1}\left[\frac{a_k}{s+p_k}\right] = a_k e^{-p_k t}$$

2. <u>*Complex conjugate*</u> poles:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1 s + a_2}{(s+p_1)(s+p_2)} + \frac{a_3}{s+p_3} + \ldots$$

$$p_1 = p_2\,{}^*$$

$$(a_1 s + a_2)_{s=-p_1} = \left[\frac{B(s)}{A(s)}(s+p_1)(s+p_2)\right]_{s=-p_1}$$

equating real and imaginary parts → α_1, α_2

Obtain *f(t)* using α_1, α_2 and the following Laplace Transform table entries:

$$\pounds[e^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\pounds[e^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2}$$

3. *<u>Multiple real</u>* poles:

$$F(s)=\frac{B(s)}{A(s)}=\frac{B(s)}{(s+p_1)^r(s+p_{r+1})\ldots}$$

$$=\frac{b_r}{(s+p_1)^r}+\frac{b_{r-1}}{(s+p_1)^{r-1}}+\ldots+\frac{b_1}{s+p_1}+\frac{a_{r+1}}{s+p_{r+1}}\ldots$$

where

$$b_r=\left[\frac{B(s)}{A(s)}(s+p_1)^r\right]_{s=-p_1} \qquad b_{r-1}=\left\{\frac{d}{ds}\left[\frac{B(s)}{A(s)}(s+p_1)^r\right]\right\}_{s=-p_1}$$

$$b_{r-j}=\frac{1}{j!}\left\{\frac{d^j}{ds^j}\left[\frac{B(s)}{A(s)}(s+p_1)^r\right]\right\}_{s=-p_1} \quad \ldots\ldots \quad b_1=\frac{1}{(r-1)!}\left\{\frac{d^{r-1}}{ds^{r-1}}\left[\frac{B(s)}{A(s)}(s+p_1)^r\right]\right\}_{s=-p_1}$$

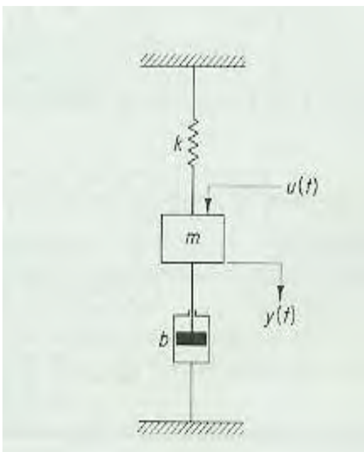
and

$$£^{-1}\left[\frac{1}{(s+p_1)^n}\right]=\frac{t^{n-1}}{(n-1)!}e^{-p_1t}$$

Solution of Linear Differential Equations Using the Laplace Transform

- Take the Laplace transform of each term and convert the differential equation into an algebraic equation.
- Obtain the Laplace transform of the dependent variable.
- The solution is obtained after inverse Laplace transform of the dependent variable.

Example: Simple mechanical system



For this system we have: $m\ddot{y} + b\dot{y} + ky = u(t)$

$m\ddot{y} + ky = u(t)$ for *b=0,* *m, k > 0*

$$m\left[s^2 Y(s) - sy(0) - \dot{y}(0)\right] + kY(s) = U(s)$$

$$Y(s) = \frac{U(s)}{ms^2 + k} + \frac{msy(0) + m\dot{y}(0)}{ms^2 + k}$$ and

$$y(t) = \left(\frac{1}{k} - \frac{1}{k}\cos\sqrt{\frac{k}{m}}t\right) + \left(y(0)\cos\sqrt{\frac{k}{m}}t + \dot{y}(0)\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t\right) \qquad t \geq 0$$ for *U(s)=1/s*

Transfer function

Consider the differential equation

$$a_0 \overset{n}{y} + \ldots + a_n y = b_0 \overset{m}{u} + \ldots b_m u \qquad m \leq n$$

Taking the Laplace transforms and assuming zero initial conditions we obtain:

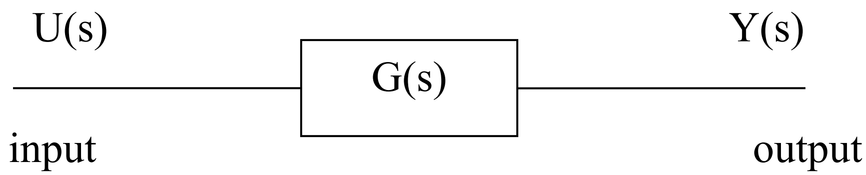
$$\frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} = \frac{b_0 s^m + \ldots + b_m}{a_0 s^n + \ldots + a_n} = G(s)$$

G(s) is the *transfer function* from *U(s)* to *Y(s)*

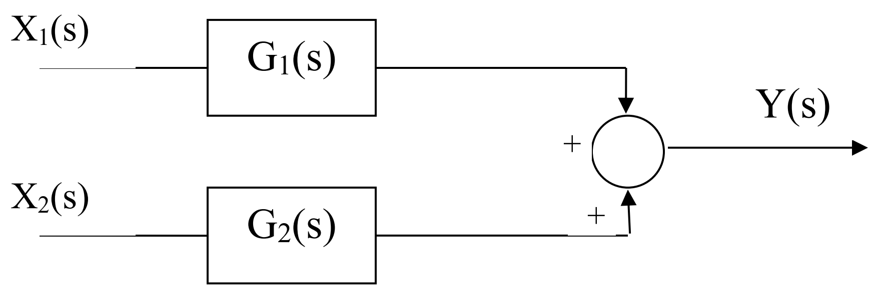
BLOCK DIAGRAMS

Basic Structures:

Open-loop:

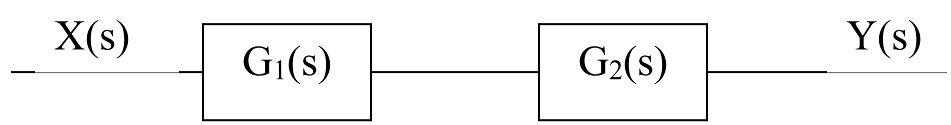


Parallel:



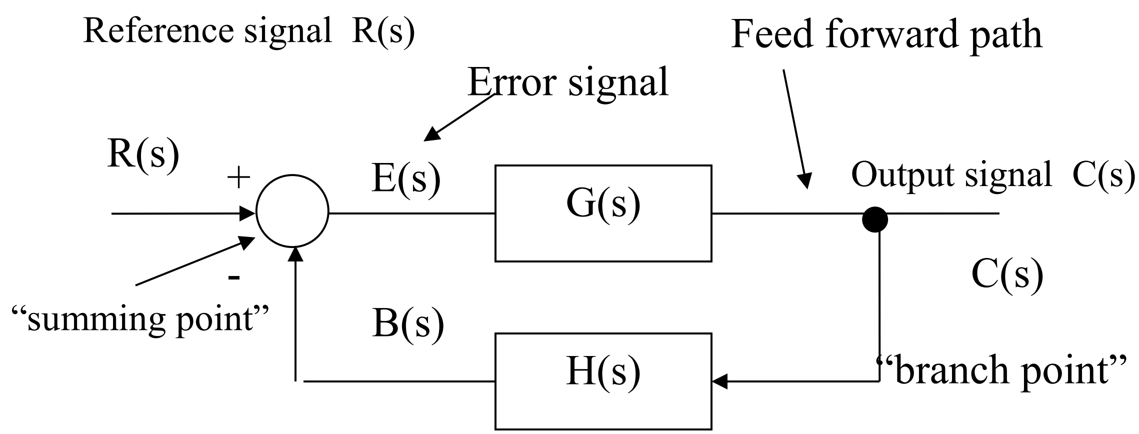
$$Y(s) = G_1(s)X_1(s) + G_2(s)X_2(s)$$

Series (Cascade):



$$Y(s) = G_1(s)G_2(s)X(s)$$

Closed-loop:



Feedforward transfer function: $$\frac{C(s)}{E(s)} = G(s)$$

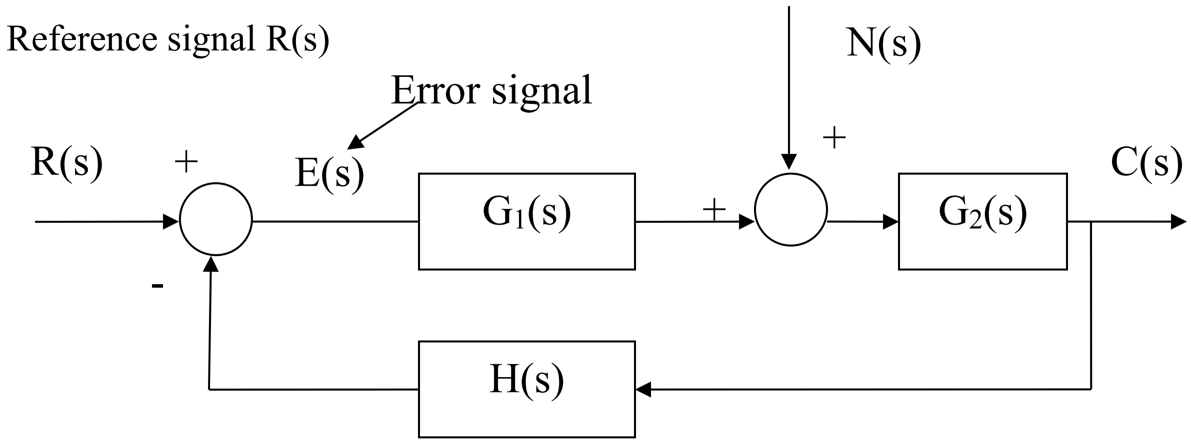
Feedback transfer function: $$\frac{B(s)}{C(s)} = H(s)$$

Open-loop transfer function: $$\frac{B(s)}{E(s)} = G(s)H(s)$$

Closed-loop transfer function: $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$C(s) = G(s)E(s) = G(s)\left[R(s) - B(s)\right] = G(s)R(s) - G(s)H(s)C(s) \quad \rightarrow \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Closed-loop subjected to disturbance:



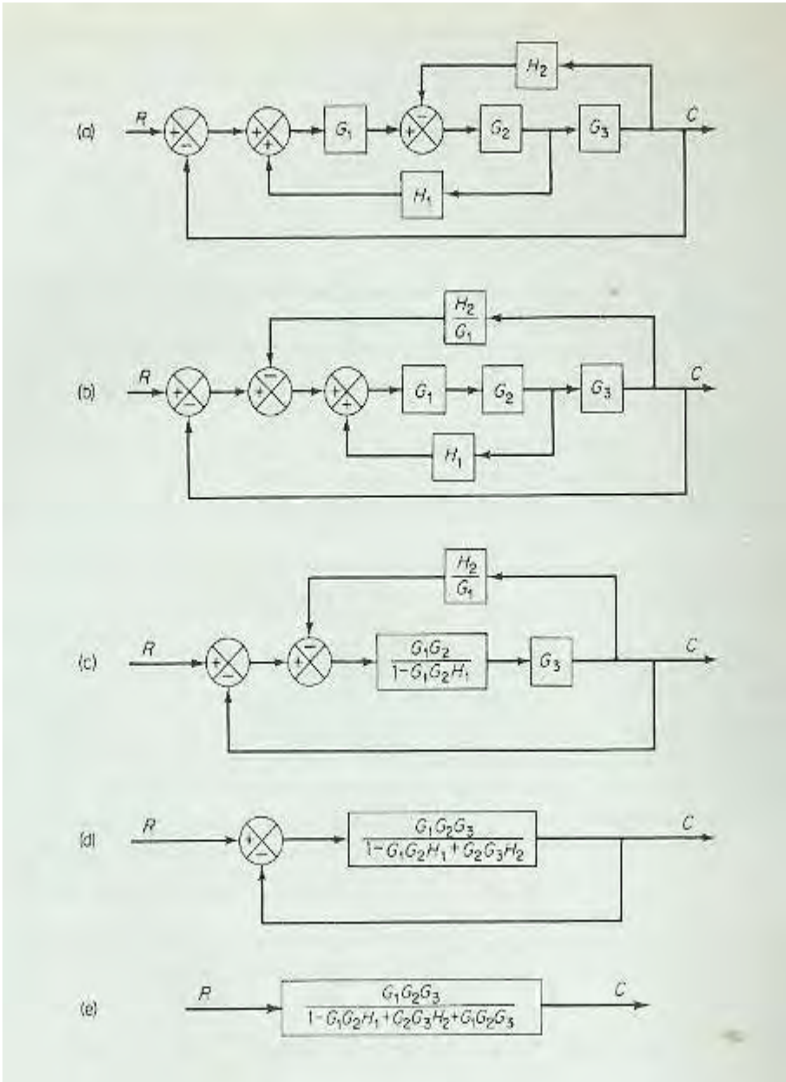
$$\frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} \qquad \text{assuming } R(s) = 0$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} \qquad \text{assuming } N(s) = 0$$

$$C(s) = C_N(s) + C_R(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + N(s)\right]$$

$$C(s) = \left[\frac{G_1(s)\cdot G_2(s)}{1+G_1(s)G_2(s)H(s)}, \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]\begin{bmatrix} R(s) \\ N(s) \end{bmatrix}$$

Block Diagram Reduction



Block Diagram Algebra

(a) to (b) Move the summing point outside the loop of the top feedforward path;

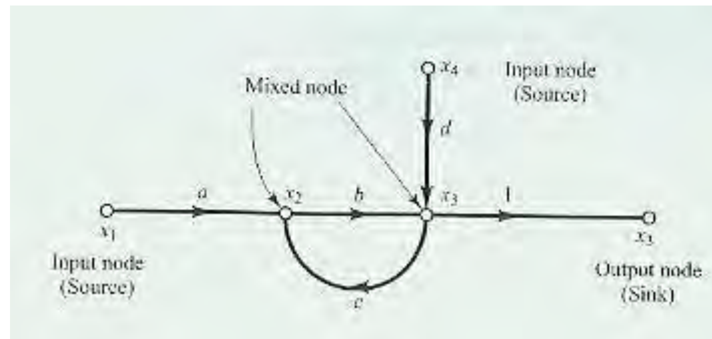
(c) - (e) successive reduction of loops in the block diagram shown in (b)

Signal Flow Graphs (SFG)

A *Signal Flow Graph (SFG)* is a diagram which represents a set of simultaneous linear equations.

It consists of a network in which nodes are connected by directed branches.

Example:



$$x_2 = ax_1 + cx_3 \qquad x_3 = bx_2 + dx_4$$

and

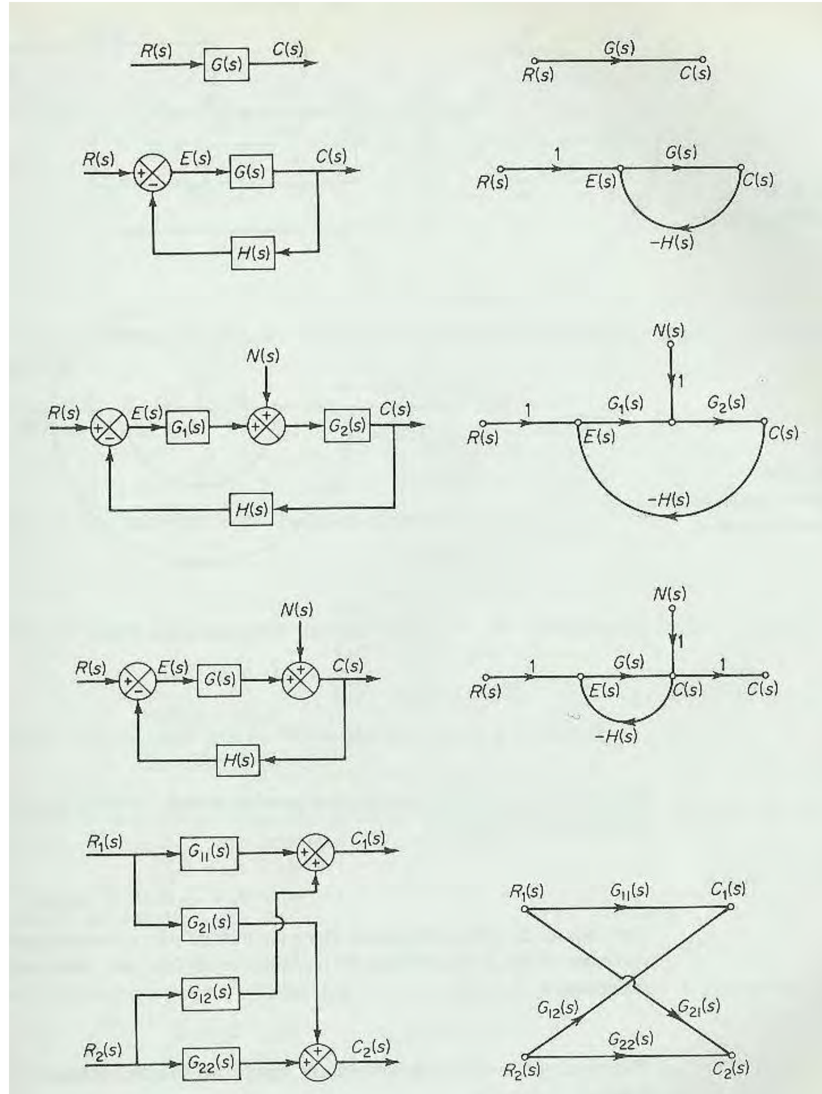
$$x_3 = abx_1 + dx_4 + bcx_3$$

Note that summations are taken over all possible paths from input to output

<u>Definitions:</u>

Node:	Point representing a variable or a signal.
Branch:	Gain between two nodes
Source:	A node without incoming branches (input node)
Sink:	A node with incoming branches only (output nodes)
Mixed node:	Both incoming + outgoing branches
Path:	Connected branches in the direction of the branch arrows. If a node is crossed more than once, it is *closed*.
Forward path:	From input node to output node without crossing nodes more than once.
Loop:	Closed path.
Non touching loops:	Loops that do not possess common nodes.

Block diagrams
and corresponding
signal flow graphs:



Properties of Signal Flow Graphs

1. A branch indicates the functional dependence of one signal on another. A signal passes through only in the direction specified by the arrow of the branch.
2. A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.
3. A mixed node, which has both incoming and outgoing branches, may be treated as an output node (sink) by adding an outgoing branch of unity transmittance. However, we can not change a mixed node to a source by this method.
4. For a given system, the signal flow graph is not unique. Many different signal flow graphs can be drawn for a given system by writing the system equations differently.

Mason's Formula for Signal Flow Graphs

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

P: Overall transmittance between input and output node.

Δ*: determinant of graph* =

1 – (sum of all different loop gains)

+ (sum of gain products of all combinations of *2 non-touching loops*)

- (sum of gain products of all combinations of *3 non-touching loops*)

+…..

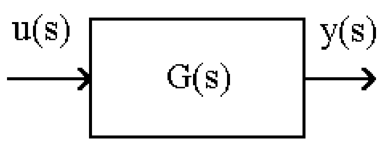
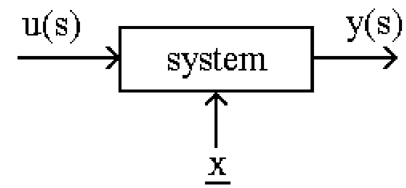
$$\Delta = 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \ldots$$

P_k: Gain of k^{th} forward path

Δ_k: Cofactor of the k^{th} forward path.

Δ_k is determined as the determinant Δ but with the loops touching the k^{th} forward path removed.

STATE-SPACE DESCRIPTION

Input - Output	**State - Space Description**
$\updownarrow$	$\updownarrow$
Transfer Function	State vector $\underline{x} = [x_1, \ldots, x_n]^T$
	
differential equation of order n	n differential equations of first order $\dot{\underline{x}} = A\underline{x} + \underline{b}u$ $y = \underline{c}\underline{x} + du$

State: A set of variables $\underline{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ such that the knowledge of these variables at $t = t_0$ together with the input u(t) for $t \geq t_0$ determines the behavior of the system for any time $t \geq t_0$.

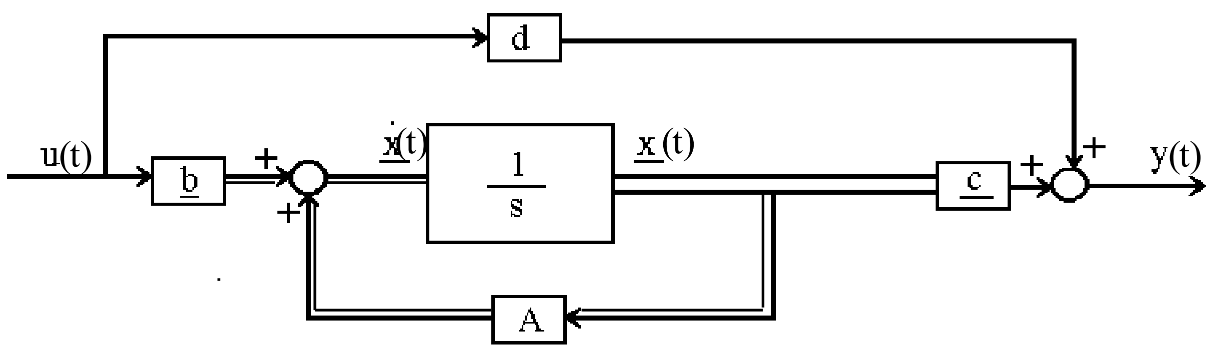
Minimal state: A set of variables $\underline{x}(t)$ with n minimal.

Continuous SISO (Single Input Single Output) system description:

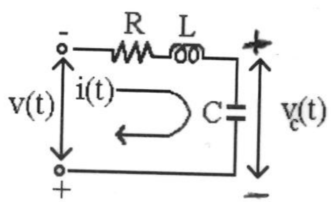
$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

$\underline{x}$: state vector
u: input
y: output

A: system matrix,
$\underline{b}$: input vector
$\underline{c}$: output vector,
d: direct input/output transmission



Example: State-space Description of a single circuit



Circuit described by:

$$L\frac{di}{dt}+Ri+v_c=v \qquad C\frac{dv_c}{dt}=i$$

Choosing:

$$\underline{x}=[i(t), \quad v_c(t)]^T \qquad y=v_C(t) \qquad u=v(t)$$

We obtain the following state-space description

$$\dot{x}=\begin{bmatrix} di/dt \\ dv_C/dt \end{bmatrix}=\underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A}\begin{bmatrix} i \\ v_c \end{bmatrix}+\underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\underline{b}}v \qquad y=v_c=\underbrace{[0 \quad 1]}_{\underline{c}}\begin{bmatrix} i \\ v_C \end{bmatrix}+\underbrace{[0]}_{d}v$$

Input-Output description:

$$G(s)=\frac{V_C(s)}{V(s)}=G(s)=d+\underline{c}\,(sI-A)^{-1}\,\underline{b}$$ from the above equations.

Remark: One could also choose for state variables

$$z_1=v_C(t)+Ri(t)$$
$$z_2=v_C(t)$$

Choice of state variables is not unique.

Minimal number of state variables = order of system = order of differential equation is unique

Nonuniqueness of State-Space Realizations:

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

$\underline{x} = T\underline{z}$ $\quad$ $\underline{z}$: new state

T : similarity transformation $\quad$ det $T \neq 0$

This gives:

$$\dot{\underline{z}} = T^{-1}AT\underline{z} + T^{-1}\underline{b}u$$
$$y = \underline{c}T\underline{z} + du$$

or

$$\dot{\underline{z}} = A_1\underline{z} + b_1 u$$
$$y = \underline{c}_1\underline{z} + du$$

with

$$A_1 = T^{-1}AT$$
$$\underline{b}_1 = T^{-1}\underline{b} \qquad \underline{c}_1 = \underline{c}T$$

Example (from previous page):

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} R & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^{-1}\underline{x}$$

$$T = \frac{1}{R}\begin{bmatrix} 1 & -1 \\ 0 & R \end{bmatrix}$$

Canonical State-Space Representations of Continuous Systems

Observable Canonical Form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{b_0 + b_1 s^{-1} + \ldots + b_n s^{-n}}{1 + a_1 s^{-1} + \ldots + a_n s^{-n}}$$

$$Y(s) = b_0 U(s) - s^{-1}[a_1 Y(s) - b_1 U(s)] - s^{-2}[a_2 Y(s) - b_2 U(s)] - \ldots - s^{-n}[a_n Y(s) - b_n U(s)]$$

$$Y(s) = b_0 U(s) + s^{-1}\{b_1 U(s) - a_1 Y(s) + s^{-1}(b_2 U(s) - a_2 Y(s) + s^{-1}(b_3 \ldots))\}$$

$$Y(s) = b_0 U(s) + X_n(s)$$

Choosing:

$$X_n(s) = s^{-1}[b_1 U(s) - a_1 Y(s) + X_{n-1}(s)], \qquad X_{n-1}(s) = s^{-1}[b_2 U(s) - a_2 Y(s) + X_{n-2}(s)]$$

……………..

$$X_2(s) = s^{-1}[b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)], \qquad X_1(s) = s^{-1}[b_n U(s) - a_n Y(s)]$$

replacing Y(s) in the above equations with $Y(s) = b_0 U(s) + X_n(s)$ and transform them in the time domain

$$\dot{x}_n = x_{n-1} - a_1 x_n + (b_1 - a_1 b_0)\, u, \quad \dot{x}_{n-1} = x_{n-2} - a_2 x_n + (b_2 - a_2 b_0)\, u, \ldots\ldots, \quad \dot{x}_1 = -a_n x_n + (b_n - a_n b_0)\, u$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \\ \vdots \\ \\ \dot{x}_x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & -a_{n-1} \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ \\ \\ b_1 - a_1 b_0 \end{bmatrix} u \qquad y = [0,\ldots,0,1] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Observability Canonical Form {Special case b_i=0, i=0,...,(n-1)}

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = u$$

Define the following state variables:

$$x_1 = y, \quad x_2 = \dot{y}, \quad \ldots, \quad x_n = y^{(n-1)}$$

Then

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \ldots, \quad \dot{x}_{n-1} = x_n$$

$$\dot{x}_n = y^{(n)} = -a_n x_1 - \ldots - a_1 x_n + u$$

and

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & & : \\ : & & & . & : \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \ldots & \ldots & -a_1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ : \\ : \\ 1 \end{bmatrix}$$

$\underline{c} = [1 \quad 0 \quad .. \quad 0]$ and $d = 0$

Observability Canonical Form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots + b_{n-1}\dot{u} + b_n u$$

Define states:

$$x_1 = y - \beta_0 u \; , \qquad x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \quad \rightarrow \quad \dot{x}_1 = x_2 + \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \; , \ldots\ldots \quad x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \ldots - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

From $\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \ldots - a_1 x_n + \beta_n u$

and $\beta_0 = b_0 \; , \quad \beta_1 = b_1 - a_1\beta_0 \; , \quad \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \; , \ldots \quad \beta_n = b_n - a_1\beta_{n-1} - \ldots - a_{n-1}\beta_1 - a_n\beta_0$

follows:

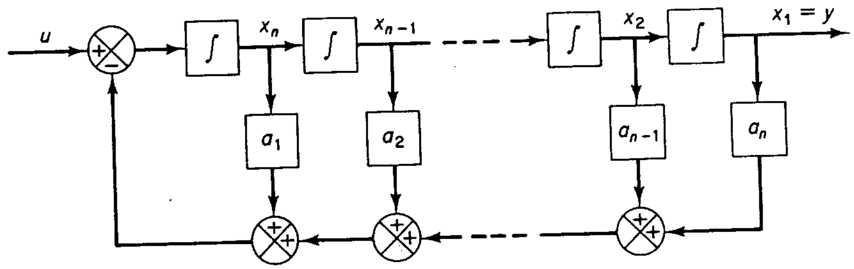
$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{cx} + du$$

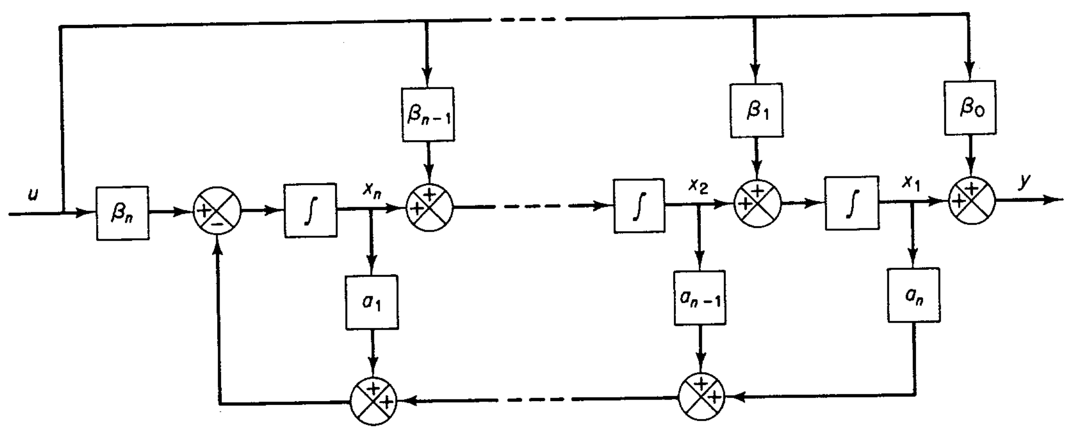
where $\underline{c}$ and A as in the previous case, i.e. $\underline{c} = \begin{bmatrix} 1 & 0 & .. & 0 \end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & & : \\ : & & & . & : \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \ldots & \ldots & -a_1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} \beta_1 \\ : \\ \beta_n \end{bmatrix} \qquad d = [\beta_0]$$

Block Diagram for the Observability Canonical Form (Special case):



Block Diagram for the Observability Canonical Form:



Comments:

- Choice of state variables is not unique. Infinite many possibilities for a state-space description of a dynamic system described by a differential equation.
- Forms with special structure (like the above) are called canonical forms.

Input-Output Description from State-Space Description

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

$$sX(s)=AX(s)+\underline{b}U(s)$$

$$(sI-A)X(s)=\underline{b}U(s)$$

$$X(s) = (sI - A)^{-1}\underline{b}U(s)$$

$$\frac{Y(s)}{U(s)} = \underline{c}(sI - A)^{-1}\underline{b} + d$$

$$Y(s) = [\underline{c}(sI - A)^{-1}\underline{b} + d]U(s)$$

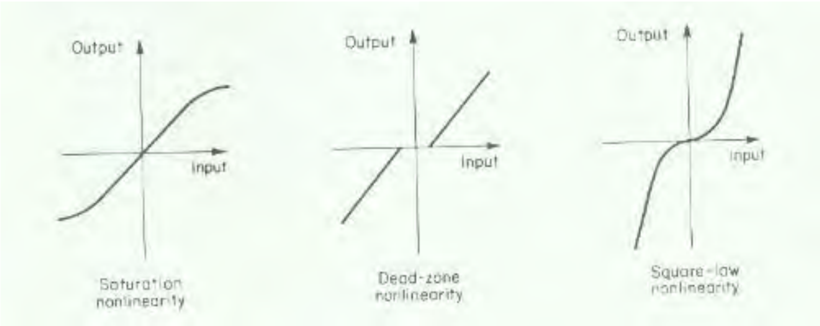
Matrix inversion, special case n=2

use $\quad A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

MATHEMATICAL MODELS

Model: The mathematical description of the dynamic characteristics of a system.

- Simplicity versus accuracy
- Time-variant versus time-invariant
- Linear versus nonlinear
  - Linear systems are described by linear differential equations.
  - Nonlinear, can be mathematically more complex. $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + (x-1)\frac{dx}{dt} = A\sin x$



Examples of mathematical modes of nonlinear dynamic systems

Remember: Real Systems are nonlinear and time-variant. However,

linear time-invariant approximations

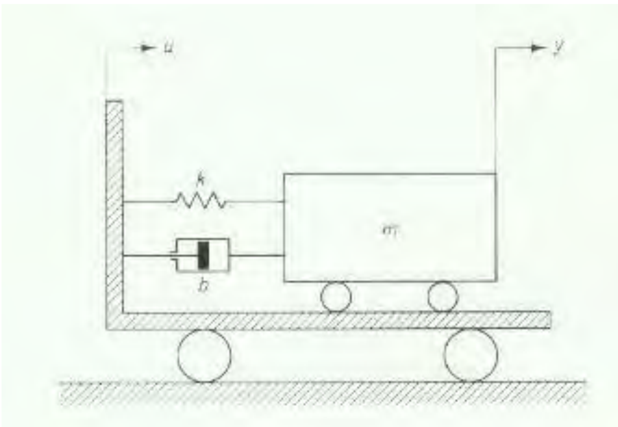
around an operating point

are usually good approximations.

Models of Simple Mechanical Systems

Translational Mechanical Systems

Newton's Law: mass x acceleration = $\sum$ Forces



Spring-mass-dashpot system mounted on a cart

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt}-\frac{du}{dt}\right)-k(y-u)$$

$$m\frac{d^2y}{dt}+b\frac{dy}{dt}+ky = b\frac{du}{dt}+ku$$

which leads to the following *transfer function*:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{bs+k}{ms^2+bs+k}$$

or to the following *state-space model:*

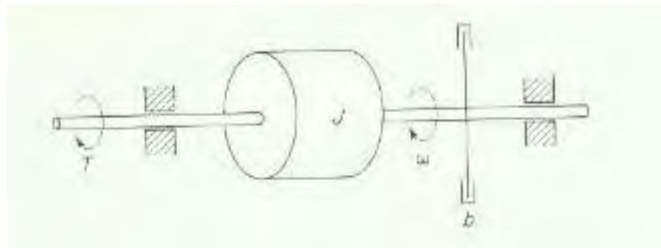
Using

$$\beta_0 = b_0 = 0,\quad \beta_1 = b_1 - a_1\beta_0 = \frac{b}{m},\quad \beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m}-\left(\frac{b}{m}\right)^2 \Rightarrow x_1 = y-\beta_0 u = y,\quad x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

and in observability canonical form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\frac{b}{m}\\ \frac{k}{m}-\left(\frac{b}{m}\right)^2\end{bmatrix}\cdot u \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Mechanical Rotational Systems



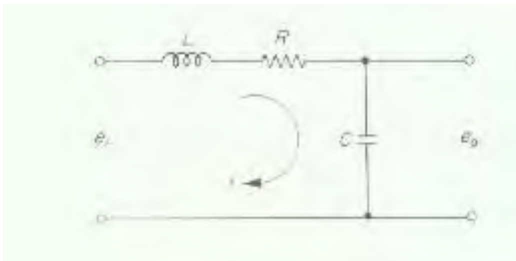
$$J\alpha = \sum T$$

Angular Acceleration
Inertia

$$J\dot{\omega} + f\omega = T \qquad \frac{\Omega(s)}{T(s)} = \frac{1}{Js + f}$$

Models of Simple Electrical Systems

RLC-Circuit



The *transfer function* can be obtained from

$$\ddot{e}_0 + \frac{R}{L}\dot{e}_0 + \frac{1}{LC}e_0 = \frac{1}{LC}e_i$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

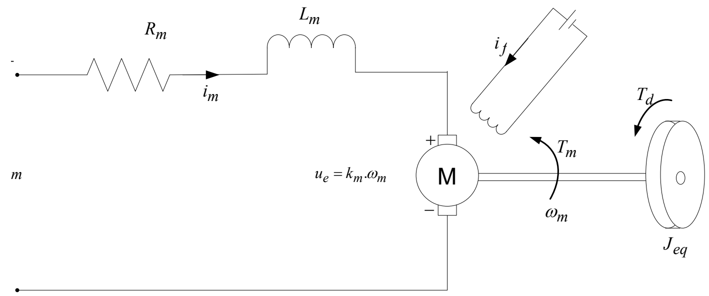
A *state-space model* can be obtained using:

$$x_1 = e_0 \quad x_2 = \dot{e}_0 \quad u = e_i \quad y = e_0 = x_1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\dfrac{1}{LC} & -\dfrac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \dfrac{1}{LC}\end{bmatrix}\cdot u \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Models of Simple Electromechanical Systems

Armature controlled DC Motors



Schematic diagram of an armature-controlled DC motor

Rotor: Armature
Stator: Field

u_m: Input Voltage
T_m: Torque
T_d: Disturbance Torque
ω_m, θ_m: Angular Velocity and Position
u_e: Voltage induced in the armature due to rotation
i_m, i_f: Armature and field currents
k_t,k_m : Motor Torque Constant and Back-Electro-Motive-Constant (equal when using SI units)

For the derivation of the transfer function consider the Electrical Part:

$$u_e(t) = k_m \omega_m(t) \qquad u_m(t) = R_m i_m(t) + L_m\left(\frac{\partial}{\partial t} i_m(t)\right) + u_e(t)$$

and the Mechanical Part:

$$T_m(t) = k_m i_m(t) \qquad J_{eq}\dot{\omega}_m(t) = k_m i_m(t) + T_d(t) \qquad J_{eq} = J_m + J_e = J_m + \frac{1}{2} M_e r_e^2$$

They give in the Laplace Domain:

$$U_m(s) - k_m \Omega_m(s) = R_m I_m(s) + sL_m I_m(s) \qquad sJ_{eq}\Omega_m(s) = k_m I_m(s) + T_d(s)$$

For $T_d(s) = 0$ and after eliminating $I_m(s)$, we have:

$$\frac{\Omega_m(s)}{U_m(s)} = \frac{k_m}{L_m J_{eq} s^2 + R_m J_{eq} s + k_m^2}$$

For most small motors $L_m \approx 0$ → ignore it. This gives:

$$\frac{\Omega_m(s)}{U_m(s)} = \frac{k_m}{R_m\left(J_{eq}s + \frac{k_m^2}{R_m}\right)} = \frac{1}{k_m\left(\frac{J_{eq}R_m s}{k_m^2} + 1\right)} = \frac{K_m}{1 + s\tau_m} \quad \text{where} \quad K_m = \frac{1}{k_m}, \qquad \tau_m = \frac{J_{eq}R_m}{k_m^2}$$

For $U_m(s) = 0, \quad L_m = 0$

and the Laplace Equations for the DC Motor:

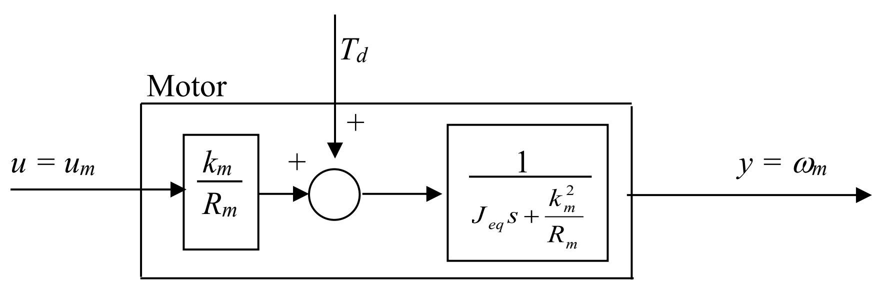
$$U_m(s) - k_m\,\Omega_m(s) = R_m I_m(s) + sL_m I_m(s)$$

$$sJ_{eq}\Omega_m(s) = k_m I_m(s) + T_d(s)$$

we obtain

$$\frac{\Omega_m(s)}{T_d(s)} = \frac{1}{J_{eq}s + \dfrac{k_m^2}{R_m}}$$

The two transfer functions can be combined to give



Block diagram of the armature controlled DC Motor (using $L_m = 0$)

<u>State-space description of armature controlled DC Motors</u>

Form

$$\frac{\Theta_m(s)}{U_m(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

we have

$$\ddot{\theta}_m + \frac{1}{\tau_m}\dot{\theta}_m = \frac{K_m}{\tau_m}u_m$$

Using the state variables

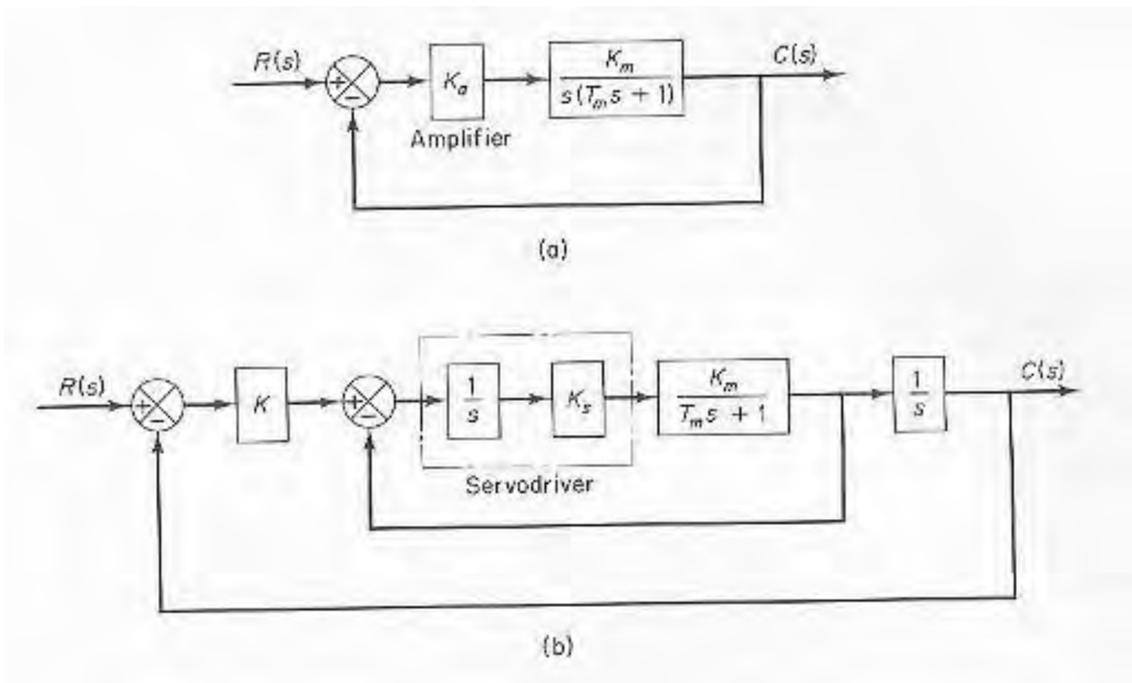
$$x_1 = \theta_m, \quad x_2 = \dot{\theta}_m$$ and

$u = u_m$ and $y = \theta_m = x_1$

we obtain the following state-space model:

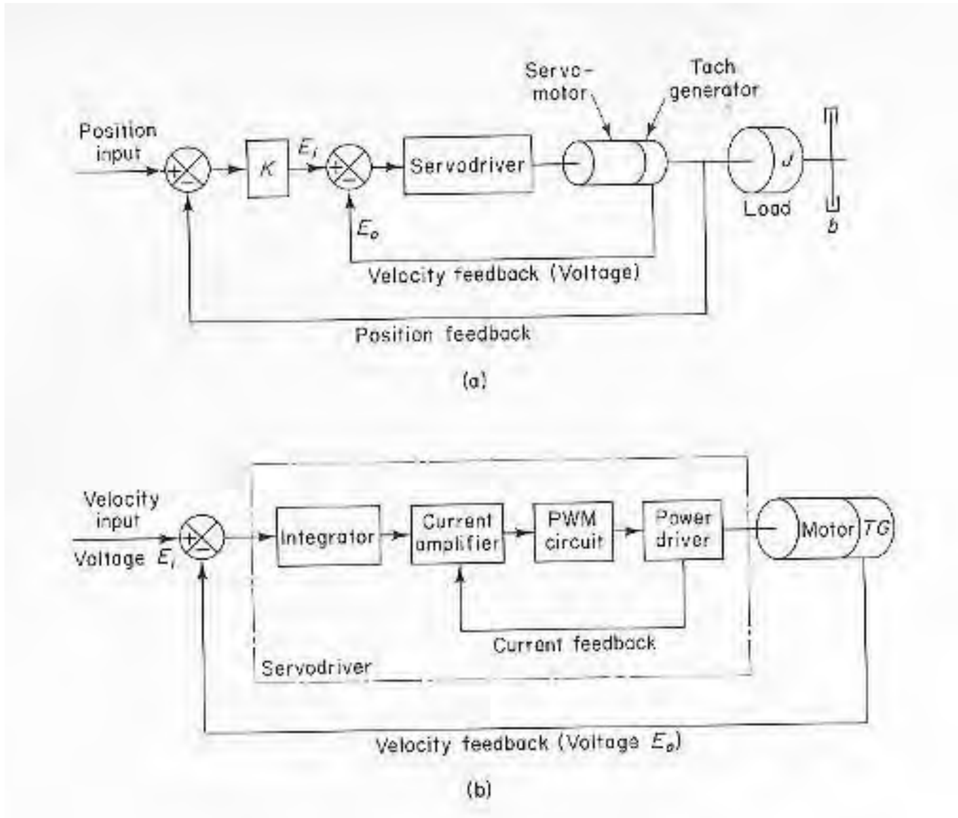
$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -\frac{1}{\tau_m}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{K_m}{\tau_m}\end{bmatrix}u_m \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

Various Positional DC servo system



a) Simple, low-cost positional servo system.
b) High-speed, high-precision positional servo system.

Electronic motion control of DC servomotors



a) High-speed, high precision postional servo system with speed control using a servodriver, servomotor combination.
b) Functional diagram of a servodriver.

Linear Approximation of Non-Linear Systems

Consider the system described by:

$y = f(x)$	where	y:	output
		x:	input
		$(\bar{x}, \bar{y})$:	operating point

The Taylor series expansion of y around $(x - \bar{x})$ gives

$$y = f(\bar{x}) + \left[\frac{df(x)}{dx}\right]_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2!}\left[\frac{d^2 f(x)}{dx^2}\right]_{x=\bar{x}} (x - \bar{x})^2 + \ldots$$

For small $(x - \bar{x})$ higher order derivatives can be considered 0 and

$$y = \bar{y} + k(x - \bar{x})$$

is the linear approximation of the non-linear system

where

$$\bar{y} = f(\bar{x}) \qquad k = \left.\frac{df(x)}{dx}\right|_{x=\bar{x}}$$

Consider the system

$y = f(x_1,x_2)$ where y: output

x_1,x_2: inputs

$((\bar{x}_1,\bar{x}_2)\bar{y})$: operating point

The Taylor series expansion gives

$$y = f(\bar{x}_1,\bar{x}_2) + \left[\frac{\partial f(x_1,x_2)}{\partial x_1}\right]_{\substack{x_1=\bar{x}_1\\x_2=\bar{x}_2}} (x_1-\bar{x}_1) + \left|\frac{\partial f(x_1,x_2)}{\partial x_2}\right|_{\substack{x_1=\bar{x}_1\\x_2=\bar{x}_2}} (x_2-\bar{x}_2) + \ldots\ldots$$

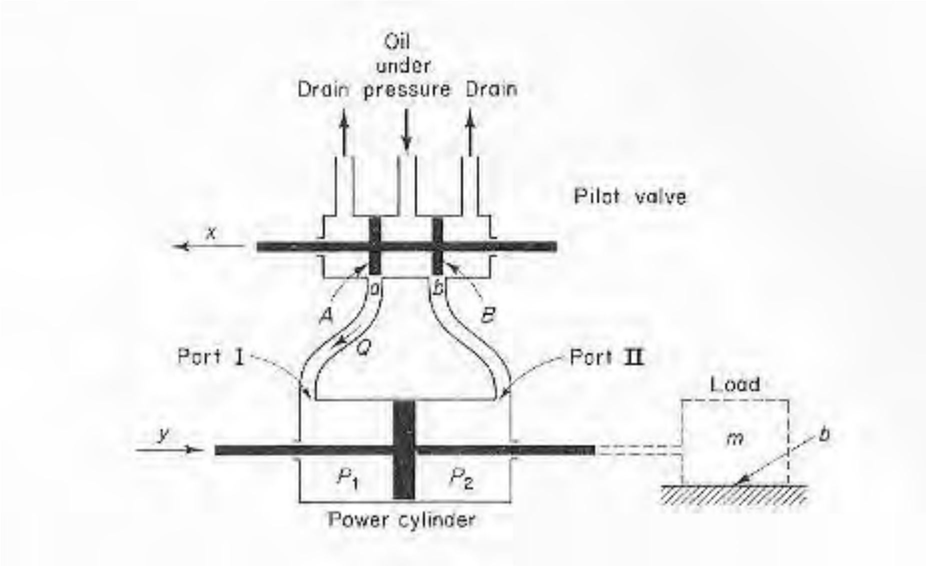
Since higher order terms can be considered 0, the linear approximation can be given by:

$$y = \bar{y} + k_1(x_1-\bar{x}_1) + k_2(x_2-\bar{x}_2)$$

where

$$\bar{y} = f(\bar{x}_1,\bar{x}_2) \qquad k_1 = \left.\frac{\partial f(x_1,x_2)}{\partial x_1}\right|_{\substack{x_1=\bar{x}_1\\x_2=\bar{x}_2}} \qquad k_2 = \left.\frac{\partial y(x_1,x_2)}{\partial x_2}\right|_{\substack{x_1=\bar{x}_1\\x_2=\bar{x}_2}}$$

Example:



Schematic diagram of a hydraulic servomotor

Q : Rate of flow to the power cylinder

$\Delta p = p_2 - p_1$: Pressure difference in the two power cylinders

x : Displacement of pilot valve

The relationship among the above variables is given by the nonlinear equation:

$$Q = f(x, \Delta p)$$

The linearized equation at $(\overline{Q}, \overline{x}, \Delta\overline{p}) = (0,0,0)$ is given by $Q - \overline{Q} = q = k_1(x - \overline{x}) - k_2(\Delta p - \Delta\overline{p})$

where

$$k_1 = \left.\frac{\partial Q(x, \Delta p)}{\partial x}\right|_{\substack{x=\overline{x} \\ \Delta p = \Delta\overline{p}}} \qquad k_2 = \left.\frac{\partial Q(x, \Delta p)}{\partial \Delta p}\right|_{\substack{x=\overline{x} \\ \Delta p = \Delta\overline{p}}}$$

This gives $$Q = k_1 x - k_2 \Delta p \qquad k_1, k_2 > 0$$

Using $Q\,dt = A\,p\,dy$ where
A: Piston area
p: Oil density
dy: Displacement of mass

follows $$\Delta p = \frac{1}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

The force developed by the power piston is given by: $$A\Delta p = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

This force is applied to the mass m and including friction gives: $$m\ddot{y} + f\,\dot{y} = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

where f: friction coefficient

This gives: $$m\ddot{y} + \left(f + \frac{A^2 p}{k_2}\right)\dot{y} = \frac{Ak_1}{k_2}x$$ and $$\frac{Y(s)}{X(s)} = \frac{\frac{Ak_1}{k_2}}{s^2 m + \left(\frac{k_2 f + A^2 p}{k_2}\right)s} = \frac{k}{s(Ts+1)}$$

where $X(s)$ is the input, $Y(s)$ is the output and

$$k = \frac{Ak_1}{k_2 f + A^2 p} \qquad T = \frac{mk_2}{k_2 f + A^2 p}$$

TRANSIENT AND STEADY-STATE RESPONSE ANALYSIS

Transient Response: for t between 0 and T

Steady-state Response: for $t \rightarrow \infty$

Testing the performance of the system with respect to the following input signals:

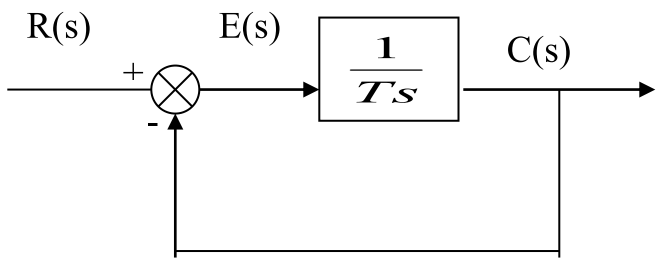
- Impulse
- Step
- Ramp
- sin and/or cos

System Characteristics:

- Stability → transient
- Relative stability → transient
- Steady-state error → steady-state

First order systems

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$



Unit step response:

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{T}{sT+1}$$

$$c(t) = 1 - e^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = e^{-t/T} \qquad e(\infty) = 0$$

$$c(T) = 1 - e^{-1} = 0.632$$

$$\frac{dc(t)}{dt}\Big|_{t=0} = \frac{1}{T} e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$

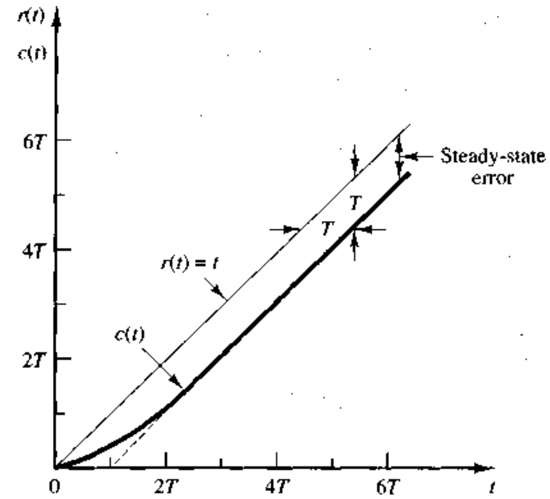
Unit ramp response:

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$c(t) = t - T + Te^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = T\left(1 - e^{-t/T}\right) \qquad t \geq 0$$

$$e(\infty) = T$$



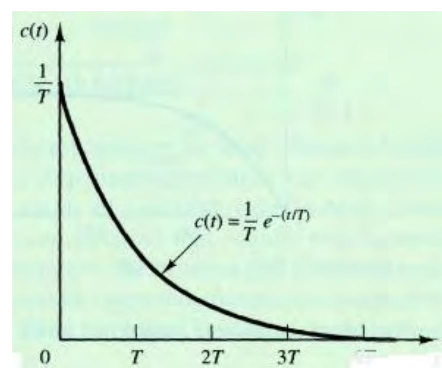
Unit-ramp response of the system

Impulse response:

$$R(s) = 1 \qquad r(t) = \delta(t)$$

$$C(s) = \frac{1}{sT+1}$$

$$c(t) = \frac{e^{-t/T}}{T} \qquad t \geq 0$$



Unit-impulse response of the system

Summary

Input		Output
Ramp	r(t) = t $t \geq 0$	$c(t) = t - T + Te^{-t/T}$ $t \geq 0$
Step	r(t) = 1 $t \geq 0$	$c(t) = 1 - e^{-t/T}$ $t \geq 0$
Impulse	r(t) = δ(t)	$c(t) = \dfrac{e^{-t/T}}{T}$ $t \geq 0$

Observation:

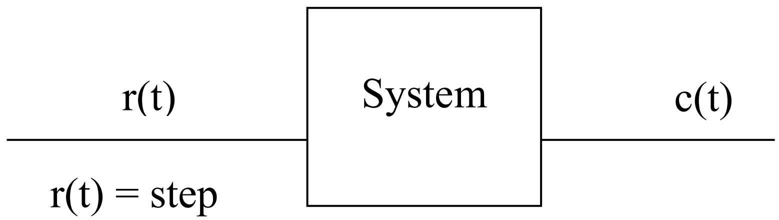
Response to the derivative of an input equals to derivative of the response to the original signal.

$Y(s) = G(s)\, U(s)$ U(s): input

$U_1(s) = s\, U(s)$ $Y_1(s) = s\, Y(s)$ Y(s): output

$G(s)\, U_1(s) = G(s)\, s\, U(s) = s\, Y(s) = Y_1(s)$

How can we recognize if a system is 1st order ?



Plot $ln\,|c(t) - c(\infty)|$

If the plot is linear, then the system is 1st order

Explanation:

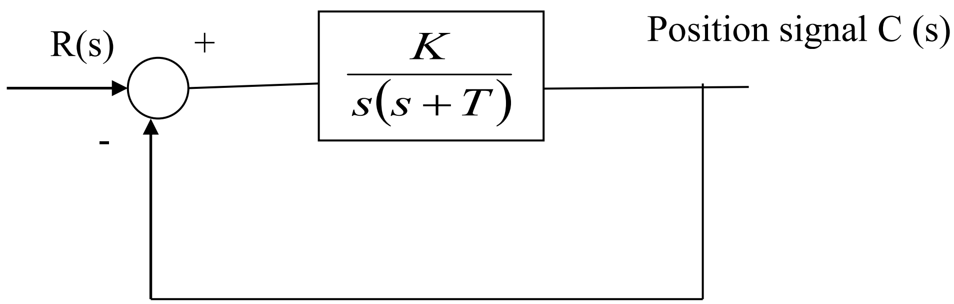
$$c(t) = 1 - e^{-t/T} \qquad\qquad c(\infty) = 1$$

$$ln\,|c(t) - c(\infty)| = ln\,|e^{-t/T}| = \frac{t}{T}$$

Second Order Systems

Example of a second order system: DC Motor with position feedback

Block Diagram



Transfer function:

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Ts + K} = \frac{K}{\left[s + \frac{T}{2} + \sqrt{\left(\frac{T}{2}\right)^2 - K}\right]\left[s + \frac{T}{2} - \sqrt{\left(\frac{T}{2}\right)^2 - K}\right]}$$

Substitute in the transfer function:

$K = \omega_n^2$	ω_n:	undamped natural frequency
$T = 2\,\zeta\omega_n = 2\sigma$	σ:	real part of denominator root
$\zeta = \frac{T}{2\sqrt{K}}$	ζ:	damping ratio

With these substitutions, we obtain ***<u>the general form of a second order system:</u>***

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

- ***Underdamped*** case: $0 < \zeta < 1$

  (in example : $T^2 - 4K < 0$) two *complex conjugate* poles

- ***Critically damped*** case: $\zeta = 1$

  (in example: $T^2 - 4K = 0$) two *equal real* poles

- ***Overdamped*** case: $\zeta > 1$

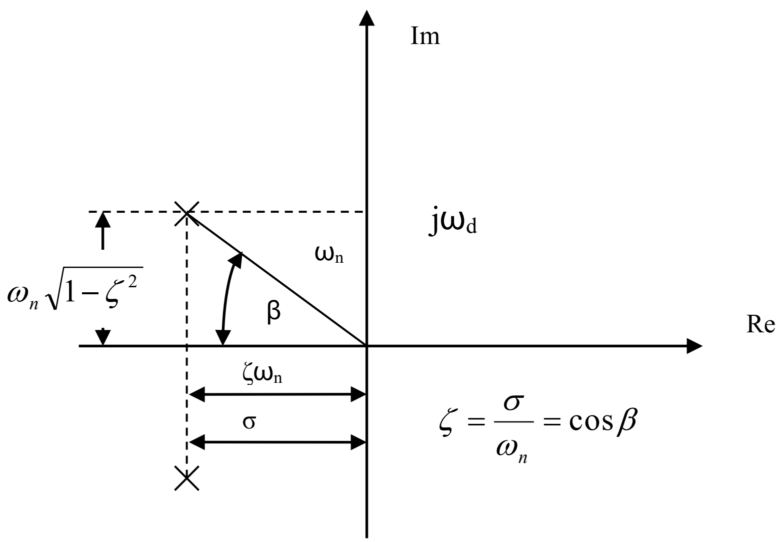
  (in example: $T^2 - 4K > 0$) two *real* poles

- ***Unstable*** case: $\zeta \leq 0$

Step Response of second order systems

Underdamped case *(0 < ζ < 1)*:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{(s+\zeta\omega_n + j\omega_d)(s+\zeta\omega_n - j\omega_d)}$$



ζ : damping ratio

ω_n : undamped natural frequency

ω_d : damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Unit step response:

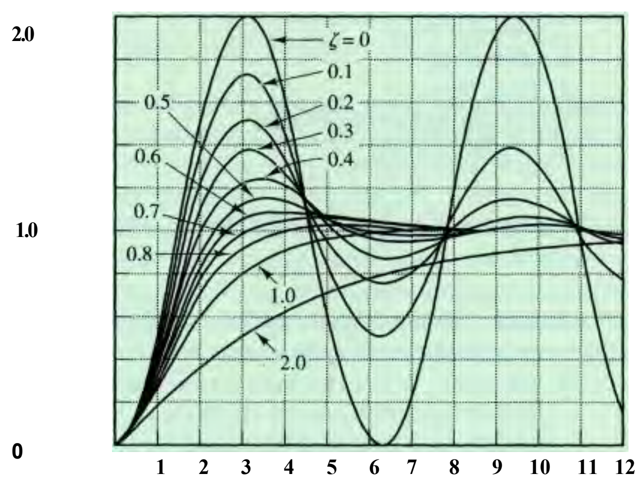
With R(s) = 1/s as input $\Rightarrow$ $$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^{\ 2}} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^{\ 2}}$$

Which gives after inverse Laplace transform

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \quad t \geq 0$$ or $$c(t) = 1 - \frac{1}{\gamma}e^{-\zeta\omega_n t}\sin(\omega_d t + \beta) \quad t \geq 0$$

with $\gamma = \sqrt{1-\zeta^2}$ $\quad \beta = \tan^{-1}\left[\frac{\gamma}{\zeta}\right]$

$$\mathrm{e(t)} = \mathrm{r(t)} - \mathrm{c(t)} = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \quad t \geq 0$$



Unit step response curves of a second order system

Undamped case (*ζ = 0*):

Unit step response:

$$c(t) = 1 - \cos\omega_n t \qquad t \geq 0$$

Critically damped case (*ζ = 1*):

Unit step response:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} = \frac{\omega_n^{\,2}}{(s+\omega_n)^2}$$

and with R(s) = 1/s as input we obtain:

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \qquad t \geq 0$$

Overdamped case *(ζ >1)*:

Unit step response: With R(s) = 1/s as input we obtain:

$$C(s) = \frac{\omega_n^{\,2}}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)}\cdot\frac{1}{s}$$

and

$$c(t) = 1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1}-\frac{e^{-s_2 t}}{s_2}\right) \qquad t \geq 0$$

with $s_1 = \left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n$ and $s_2 = \left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$

if $|s2| << |s1|$, the transfer function can be approximated by

$$\frac{C(s)}{R(s)} = \frac{s_2}{s+s_2}$$

and for R(s) = 1/s

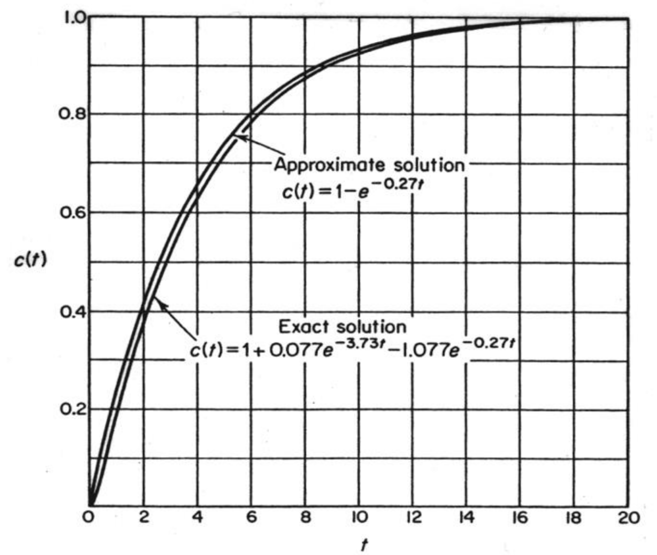
$$c(t) = 1-e^{-s_2 t} \qquad t \geq 0$$

with

$$s_2 = \left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$

Example:

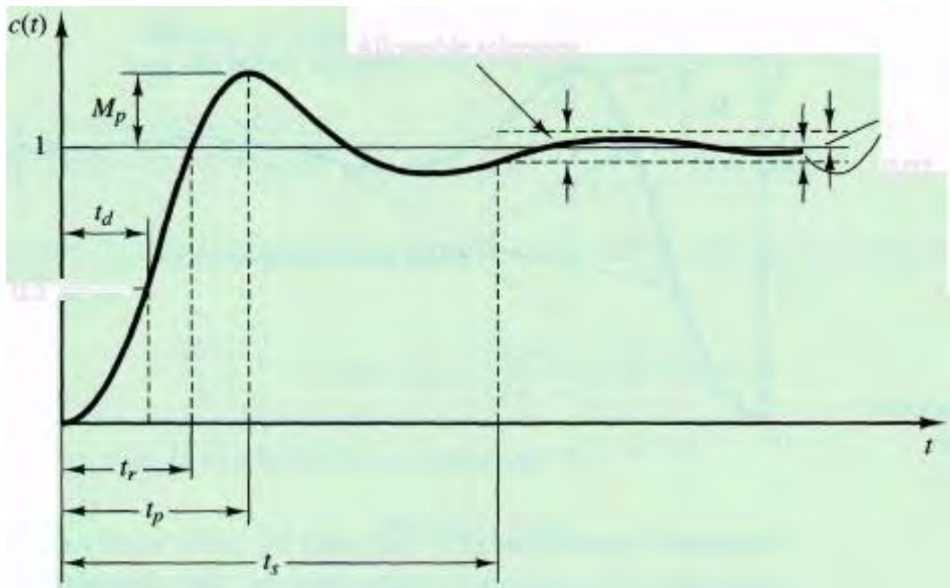
Second order system with $s_1 = -3.73$ and $s_2 = -0.27$



Unit step response curves of a critically damped system.

Definitions of Transient Response Specifications

Unit step response of a 2nd order underdamped system:



t_d : *delay time*, time to reach 50% of c(∞) for the first time.

t_r : *rise time*, time to rise from 0 to 100% of c(∞).

t_p: *peak time*, time required to reach the first peak.

M_p: *maximum overshoot* : $\frac{c(t_p)-c(\infty)}{c(\infty)}\cdot 100\%$

t_s: *settling time*. time to reach and stay within a 2% (or 5%) tolerance of the final value $c(\infty)$.

For a 'good' step response of an underdamped system choose: $0.4 < \zeta < 0.8$

<u>Rise time t_r:</u> time from 0 to 100% of *c(∞)*

From $c(t_r)=1 \;\Rightarrow\; 1-e^{-\zeta\omega_n t_r}(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r)=1$ we obtain

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$ and $$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$$

<u>Peak time t_p:</u> time to reach the first peak of c(t)

From $\left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0 \;\Longrightarrow\; (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$

$$\sin\omega_d t_p = 0$$ and $$t_p = \frac{\pi}{\omega_d}$$

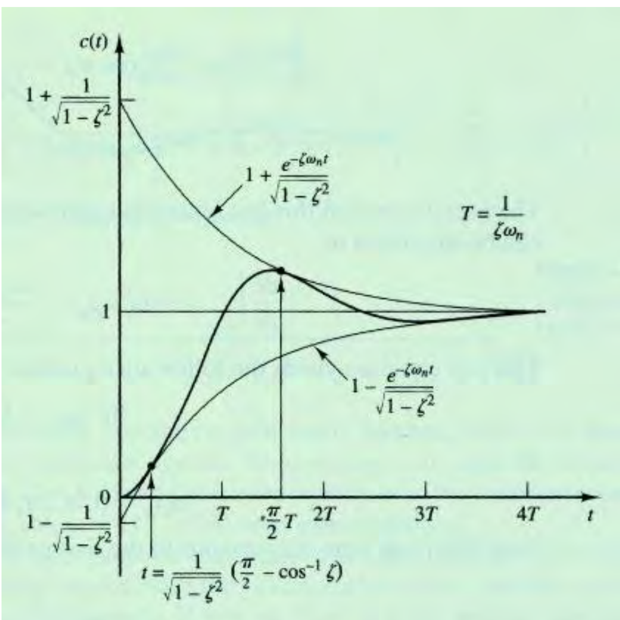
Maximum overshoot M_p:

$$M_p = c(t_p) - 1 = e^{-\zeta\omega_n(\pi/\omega_d)}(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi)$$

Using $t = t_p = \dfrac{\pi}{\omega_d}$ $\Rightarrow$

$$M_p = e^{-\frac{\zeta\omega_n\pi}{\omega_d}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\sigma\pi}{\omega_d}}$$

Settling time t_s:



Using $env(t) = 1 \pm \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$, the envelope curves of

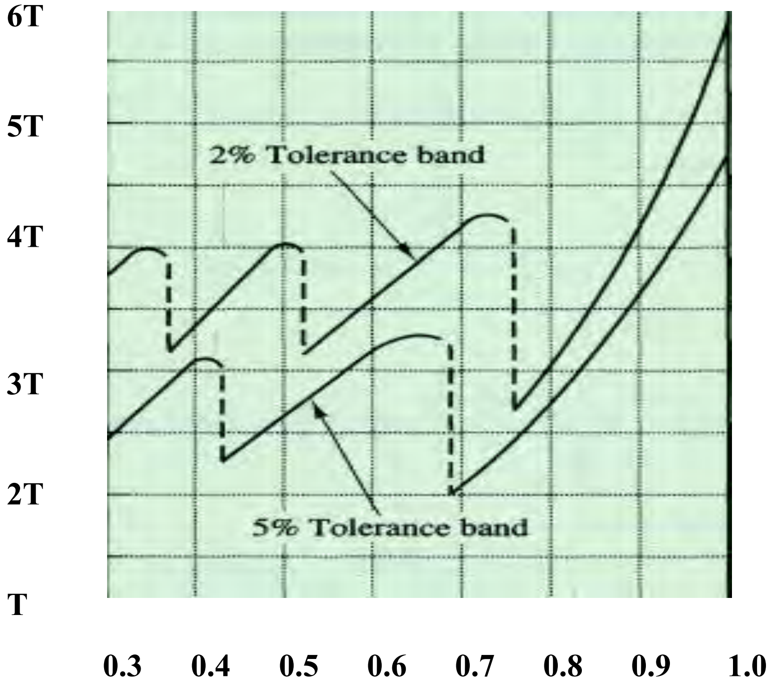
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

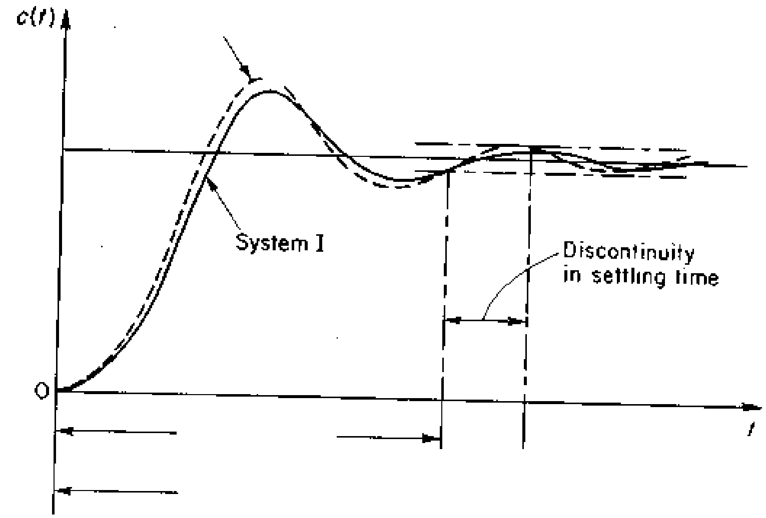
t_s can be approximated as:

2% band: $t_s = \dfrac{4}{\sigma} = \dfrac{4}{\zeta\omega_n}$ or

5% band $t_s = \dfrac{3}{\sigma} = \dfrac{3}{\zeta\omega_n}$

Settling time t_s versus ζ curves $\{T = 1/(\zeta\omega_n)\}$





Impulse response of second-order systems

$$C(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

R(s) = 1

underdamped case (0< ζ < 1):

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \qquad t \ge 0$$

the first peak occurs at $t = t_0$

$$t_0 = \frac{\tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n \sqrt{1-\zeta^2}}$$

and the maximum peak is

$$c(t_0) = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

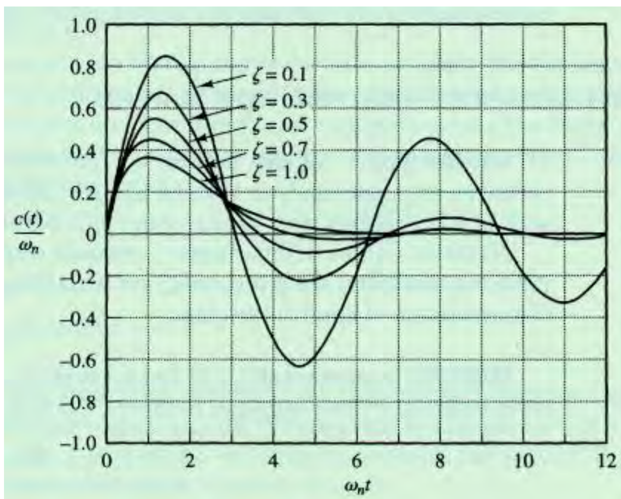
critically damped case (*ζ = 1*):

$$c(t) = \omega_n{}^2 t e^{-\omega_n t} \qquad t \geq 0$$

overdamped case (*ζ >1*):

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_2 t} \qquad t \geq 0$$

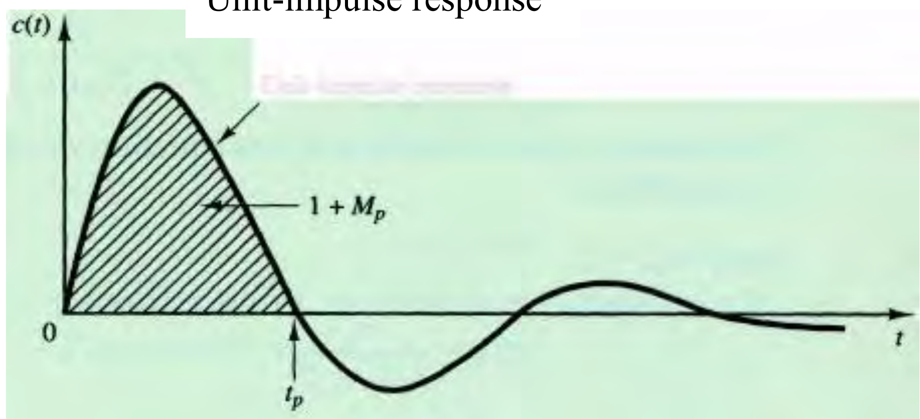
where $s_1 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$ and $s_2 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$



Unit-impulse response for 2nd order systems

Remark: Impulse Response = *d/dt* (Step Response)

Unit-impulse response



Relationship between t_p, M_p and the unit-impulse response curve of a system

Unit ramp response of a second order system

$$C(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n + \omega_n^{\ 2}} \cdot \frac{1}{s^2}$$ R(s) = 1/ s^2

for an underdamped system $(0 < \zeta < 1)$ the response is

$$c(t) = t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t + \frac{2\zeta^2 - 1}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$

and the error:

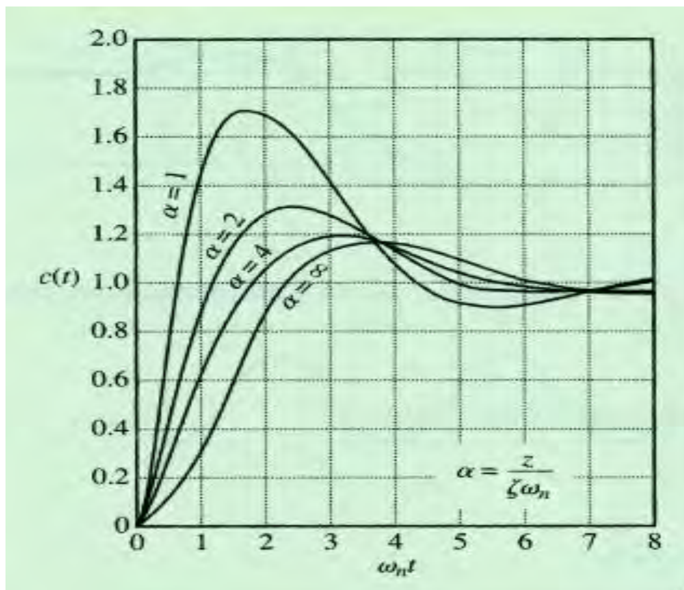
$$e(t) = r(t) - c(t) = t - c(t)$$

at steady-state:

$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n}$$

Effect of a zero in the step response of a 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{z} \cdot \frac{s+z}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} \qquad \zeta = 0.5$$



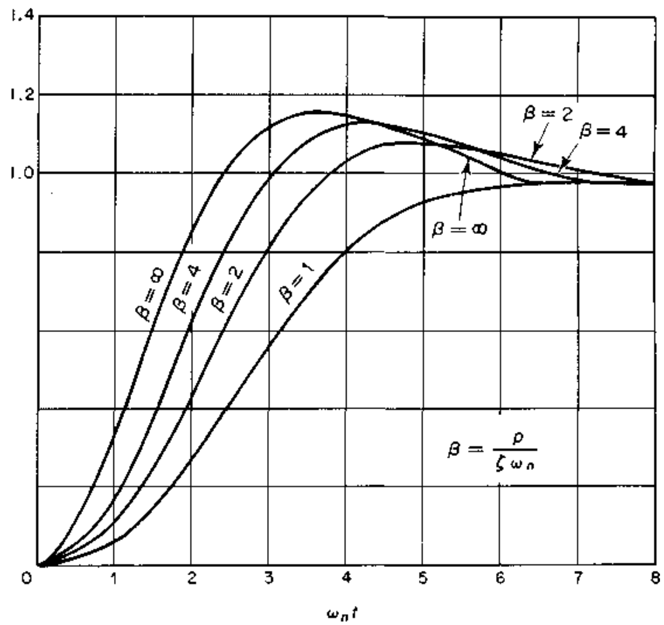
Unit-step response curves of 2nd order systems

Unit step response of 3rd order systems

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 p}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+p)} \qquad 0 < \zeta < 1 \qquad R(s)=1/s$$

$$c(t) = 1 - \frac{e^{-pt}}{\beta\zeta^2(\beta-2)+1} - \frac{e^{-\zeta\omega_n t}}{\beta\zeta^2(\beta-2)+1} \bullet$$

$$\left\{\beta\zeta^2(\beta-2)\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\beta\zeta\left[\zeta^2(\beta-2)+1\right]}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\} \qquad \text{where} \qquad \beta = \frac{p}{\zeta\omega_n}$$



Unit-step response curves of the third-order system, $\zeta = 0.5$

The effect of the pole at $s = -p$ is:

- Reducing the maximum overshoot
- Increasing settling time

Transient response of higher-order systems

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + \ldots + b_{m-1}s + b_m}{s^n + \ldots + a_{n-1}s + a_n} = \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)} \qquad n > m$$

Unit step response:

$$C(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}\left(s^2+2\zeta_k\omega_k s+\omega_k^{\,2}\right)}\cdot\frac{1}{s}, \quad 0 < \zeta_k < 1 \quad k=1,\ldots,r \quad \text{and} \quad q + 2r = n$$

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k) + c_k\omega_k\sqrt{1-\zeta_k^{\,2}}}{s^2+2\zeta_k\omega_k+\omega_k^{\,2}}$$

$$c(t) = a + \sum_{j=1}^{q}a_j e^{-p_j t} + \sum_{k=1}^{r}b_k e^{-\zeta_n\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k^{\,2}}\,t\right)$$

$$+\sum_{k=1}^{r}c_k e^{-\zeta_k\omega_k t}\sin\left(\omega_k\sqrt{1-\zeta_k^{\,2}}\,t\right) \qquad t \geq 0$$

Dominant poles: the poles closest to the imaginary axis

STABILITY ANALYSIS

Consider

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\sum_{i=0}^{n} a_i s^{n-i}}$$

Conditions for Stability:

A. ***Necessary*** condition for stability:

All coefficients of A(s) have the same sign.

B. ***Necessary and sufficient*** condition for stability:

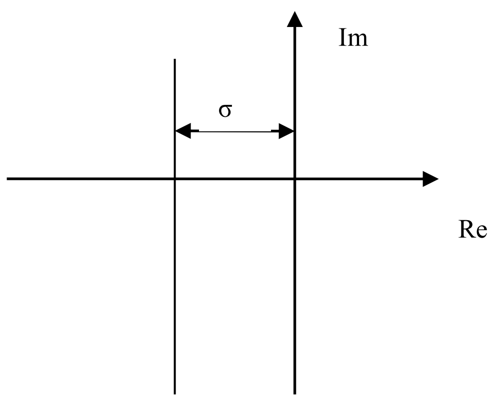
$A(s) \neq 0$ for $\mathrm{Re}[s] \geq 0$

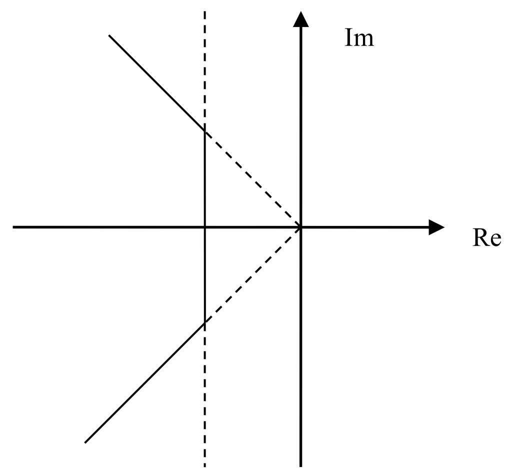
or, equivalently

All poles of G(s) in the left-half-plane (LHP)

Relative stability:

The system is stable and further, all the poles of the system are located in a sub-area of the left-half-plane (LHP).





Necessary condition for stability:

$$
\begin{aligned}
A(s) &= a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \\
&= a_0 (s + p_1)(s + p_2)\ldots(s + p_n) \\
&= a_0 s^n + a_0 (p_1 + p_2 + \ldots p_n) s^{n-1} \\
&\quad + a_0 (p_1 p_2 + \ldots + p_{n-1} p_n) s^{n-2} \\
&\quad\quad \vdots \\
&\quad + a_0 (p_1 p_2 \ldots p_n)
\end{aligned}
$$

$-p_1$ *to* $-p_n$ are the poles of the system.

If the system is stable → all poles have negative real parts

→ the coefficients of a stable polynomial have the same sign.

Examples:

$A(s) = s^3 + s^2 + s + 1$ can be stable or unstable

$A(s) = s^3 - s^2 + s + 1$ is unstable

Necessary and sufficient condition for stability:

$$A(s) \neq 0 \qquad \text{for} \qquad \text{Re}[s] \geq 0$$

or, equivalently

All poles of G(s) in the left-half-plane (LHP)

Stability testing

Test whether all poles of G(s) (roots of A(s)) have *negative real parts*.

Find all roots of A(s) $\rightarrow$ too many computations

Easier Stability test?

Routh-Hurwitz Stability Test

Consider $A(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + \alpha_n$

$$
\begin{array}{llll}
s^n & a_0 & a_2 & a_4 \ \ldots \\
s^{n-1} & a_1 & a_3 & a_5 \ \ldots \\
s^{n-2} & b_1 & b_2 & b_3 \\
 & c_1 & c_2 & \\
\ldots\ldots & & & \\
s^2 & e_1 & e_2 & \\
s^1 & f_1 & & \\
s^0 & g_1 & &
\end{array}
$$

$$b_1 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{1}{-b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = \frac{a_3 b_1 - a_1 b_2}{b_1} \quad \text{etc}$$

Properties of the Ruth-Hurwitz table:

1. Polynomial A(s) is stable (i.e. all roots of A(s) have negative real parts) if there is *no sign change in the first column*.
2. The *number of sign changes in the first column* is equal to the number of roots of A(s) with positive real parts.

Examples:

Consider

$$A(s) = a_0 s^2 + a_1 s + a_2 \qquad \begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & \\ s^0 & a_2 & \end{array}$$

$a_0 > 0, \quad a_1 > 0, \quad a_2 > 0$ or
$a_0 < 0, \quad a_1 < 0, \quad a_2 < 0$

Thus, for 2nd order systems, the *necessary and sufficient condition for stability* is that all coefficients of $A(s)$ have the same sign.

$$A(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 \qquad \begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

$a_0 > 0, \quad a_1 > 0, \quad a_3 > 0, \qquad a_1 a_2 - a_0 a_3 > 0$

(or all first column entries are negative)

Special cases:

1. The properties of the table do not change when all the coefficients of a row are multiplied by the same positive number.
2. If the first-column term becomes zero, replace 0 by ε (a small positive number) and continue.
   - If the signs above and below ε are the same, then there is a pair of (complex) imaginary roots.
   - If there is a sign change, then there are roots with positive real parts.

Examples:

$$A(s) = s^3 + 2s^2 + s + 2$$

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \to \varepsilon & \\ s^0 & 2 & \end{array}$$

→ pair of imaginary roots ($s = \pm j$)

where ε is a small positive number

Consider $A(s) = s^3 - 3s + 2$

From the necessary stability condition we already know that it is unstable.

By generating the Routh-Hurwitz table we have:

$$\begin{array}{lll} s^3 & 1 & -3 \\ s^2 & 0 \approx \varepsilon & 2 \\ s^1 & -3-\dfrac{2}{\varepsilon} & \\ s^0 & 2 & \end{array}$$

→ two roots with positive real parts

Polynomial factorization confirms: $A(s) = s^3 - 3s + 2 = (s-1)^2(s+2)$

3. If all coefficients in a line become 0, then A(s) has roots of equal magnitude radially opposed on the real or imaginary axis. Such roots can be obtained from the roots of the auxiliary polynomial.

Example: Consider $A(s) = s^5 + 2s^4 +24\, s^3 + 48s^2 -25s -50$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

→ auxiliary polynomial $p(s) = 2s^4 + 48s^2 - 50$

$$\frac{dp(s)}{ds} = 8s^3 + 96s \rightarrow$$

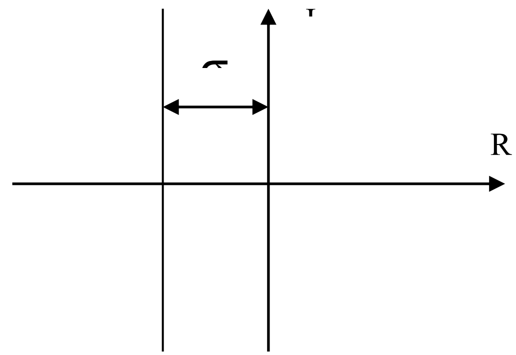
s^3	8	96
s^2	24	-50
s^1	112.7	0
s^0	-50	

- A(s) has two radially opposed root pairs (+1,-1) and (+5j,-5j) which can be obtained from the roots of p(s).
- One sign change indicates A(s) has one root with positive real part.

Note: $A(s) = (s+1)\,(s-1)(s+5j)(s-5j)(s+2)$ and
$p(s) = 2(s^2-1)\,(s^2 +25)$

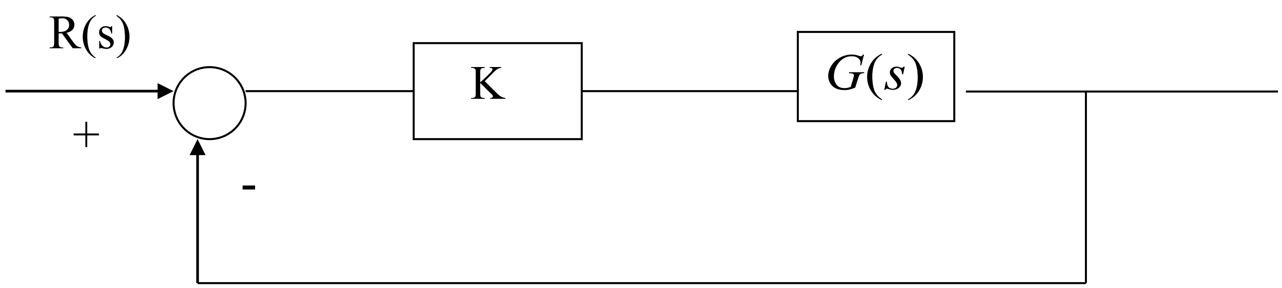
Testing Relative Stability using Routh-Hurwitz

Question: Have all the roots of A(s) a distance of at least σ from the imaginary axis?



Substitute s with $s = z - \sigma$ in $A(s)$ and apply the Routh-Hurwitz test to $A(z)$

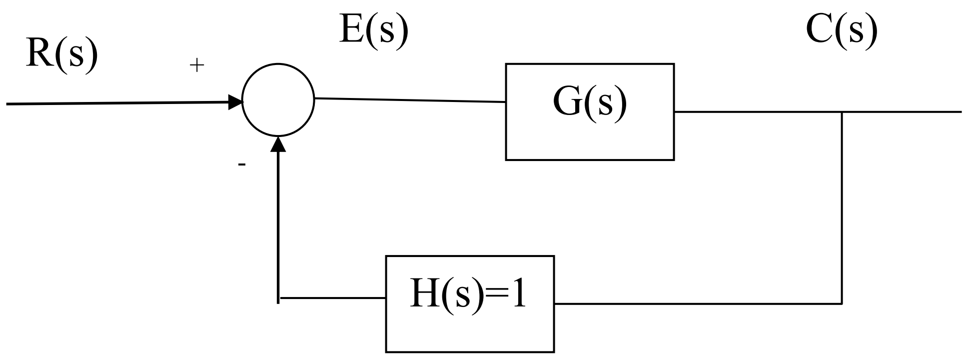
Closed-loop System Stability Analysis



Question: For what value of K is the closed-loop system stable?

Apply the Routh-Hurwitz test to the denominator polynomial of the closed-loop transfer function $\frac{KG(s)}{1+KG(s)}$.

STEADY-STATE ERROR ANALYSIS FOR UNITY FEEDBACK SYSTEMS



Evaluate the steady-state performance of the closed-loop system using the steady-state error e_{ss} given by

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \quad \text{with} \quad E(s) = \frac{1}{1+G(s)H(s)} \cdot R(s) = \frac{1}{1+G(s)} \cdot R(s) \quad (H(s)=1)$$

for the following input signals:
- Unit step input
- Unit ramp input
- Unit parabolic input

<u>*Assumption:*</u> the closed-loop system is stable

<u>*Question:*</u> How can we obtain the steady-state error e_{ss} of the closed-loop system from the open-loop transfer function *G(s)* ?

Classification of systems:

For an open-loop transfer function

$$G(s)H(s) = G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N (T_1 s + 1)(T_2 s + 1)\cdots} \qquad (H(s) = 1)$$

Type of system: Number of poles at the origin, i.e., N

Static Error Constants: K_p, K_v, K_a

Open-loop transfer function: G(s)H(s) = G(s) (H(s)=1)

Closed-loop transfer function: $G_{tot}(s) = \dfrac{G(s)}{1 + G(s)}$

Static Position Error Constant: K_p

Unit step input to the closed-loop system shown in fig, p. B40.

R(s) = 1/s $\qquad e_{ss} = \lim_{s \to 0} sE(s) = \frac{1}{1+G(0)}$

Define: $\qquad K_p = \lim_{s \to 0} G(s)H(s) = G(0)$

Type 0 system	$K_p = K$	$e_{ss} = \frac{1}{1+K_p}$
Type 1 and higher	$K_p = \infty$	$e_{ss} = 0$

Static Velocity Error Constant: K_v

Unit ramp input to the closed-loop system shown if fig, p. B40.

R(s) = 1/s^2 $\quad e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)}$

Define: $\qquad K_v = \lim_{s \to 0} sG(s)$

Type 0 system	$K_v = 0$	$e_{ss} = \infty$
Type 1 system	$K_v = K$	$e_{ss} = 1/K_v$
Type 2 and higher	$K_v = \infty$	$e_{ss} = 0$

Static Acceleration Error Constant: K_a

Unit parabolic input to the closed-loop system shown in fig, p. B33

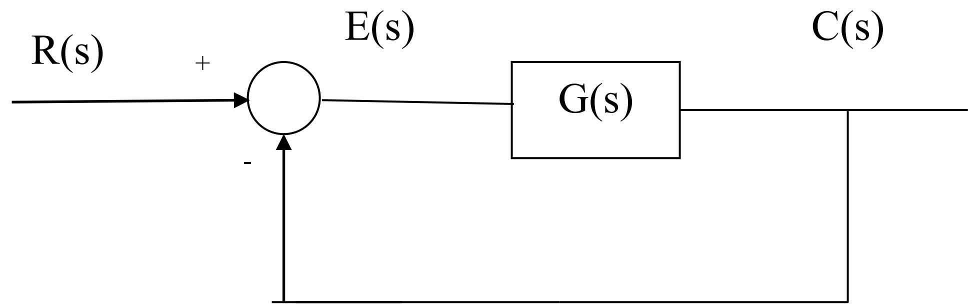
R(s) = 1/s^3 $e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)}$

Define: $K_a = \lim_{s\to 0} s^2 G(s)$

Type 0 system	$K_a = 0$	$e_{ss} = \infty$
Type 1 system	$K_a = 0$	$e_{ss} = \infty$
Type 2 system	$K_a = K$	$e_{ss} = 1/K_a$
Type 3 and higher	$K_a = \infty$	$e_{ss} = 0$

Summary:

Consider a closed-loop system:



with an open-loop transfer function:

$$G(s) = \frac{K(T_a s + 1)\cdot(T_b s + 1)\ldots}{s^N (T_1 s + 1)\cdot(T_2 s + 1)\ldots}$$

and static error constants defined as:

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

$$K_v = \lim_{s\to 0} sG(s)$$

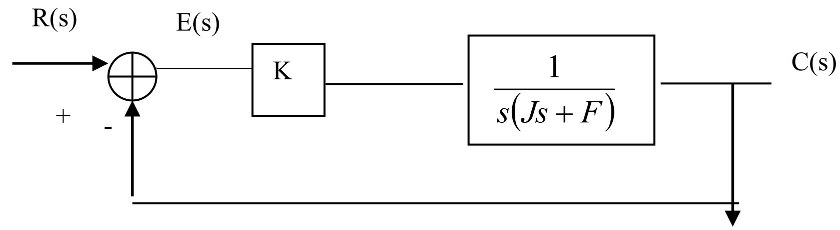
$$K_a = \lim_{s\to 0} s^2 G(s)$$

The steady-state error e_{ss} is given by:

	Unit step $r(t) = 1$	Unit ramp $r(t) = t$	Unit parabolic $r(t) = t^2/2$
Type 0	$e_{ss} = \frac{1}{1+K_p} (= \frac{1}{1+K})$	$e_{ss} = \infty$	$e_{ss} = \infty$
Type 1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_v} (= \frac{1}{K})$	$e_{ss} = \infty$
Type 2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_a} (= \frac{1}{K})$

Examples:

A. Proportional Control



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

with

$$\frac{K}{J} = \omega_n{}^2, \qquad \frac{F}{J} = 2\zeta\omega_n = 2\sigma, \qquad \zeta = \frac{F}{2\sqrt{JK}}$$

Choose K to obtain ‘good’ performance for the closed-loop system

For good *transient response*:

$0.4 < \zeta < 0.8$ → acceptable overshoot
ω_n sufficiently large → good settling time

For small *stead- state error in ramp response*:

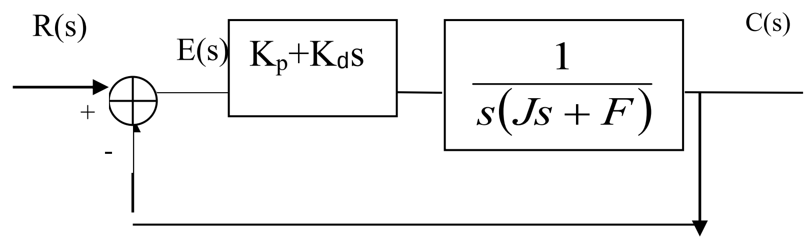
$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n} = \frac{2F}{2\sqrt{KJ}} \cdot \sqrt{\frac{J}{K}} = \frac{F}{K} \quad \rightarrow \quad \text{large K}$$

Large K reduces $e(\infty)$ but also leads to small ζ and large M_p

→compromise necessary

B. *Proportional plus Derivative Control:*

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (F + K_d)s + K_p}$$



with

$$\varsigma = \frac{F + K_d}{2\sqrt{K_p J}} \qquad \omega_n = \sqrt{\frac{K_p}{J}}$$

The error for a ramp response is:

$$E(s) = \frac{s^2 J + sF}{s^2 J + s(F + K_d) + K_p} \cdot R(s)$$

and at steady-state:

$$e(\infty) = \lim_{s \to 0} sE(s) = \frac{F}{K_p}$$

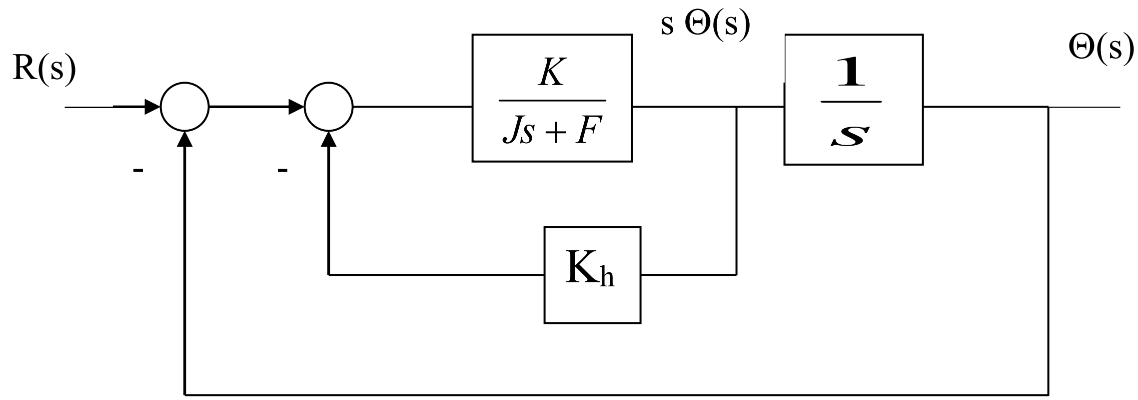
using $z = \frac{K_p}{K_d}$ → $\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{z} \cdot \frac{s + z}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$

Choose K_p, K_d to obtain 'good' performance of the closed-loop system

For small *steady-state error in ramp response* → K_p large

For good *transient response* → K_d so that $0.4 < \zeta < 0.8$

C. *Servo Mechanism with Velocity Feedback*



$$\frac{\Theta(s)}{R(s)} = \frac{K}{Js^2 + (F + KK_h)s + K}$$

where

$$\varsigma = \frac{F + KK_h}{2\sqrt{KJ}}$$

$$\omega_n = \sqrt{\frac{K}{J}} \qquad \text{(not affected by velocity feedback)}$$

$$e(\infty) = \frac{F}{K} \qquad \text{for a ramp}$$

Choose K, K_h to obtain 'good' performance for the closed-loop system

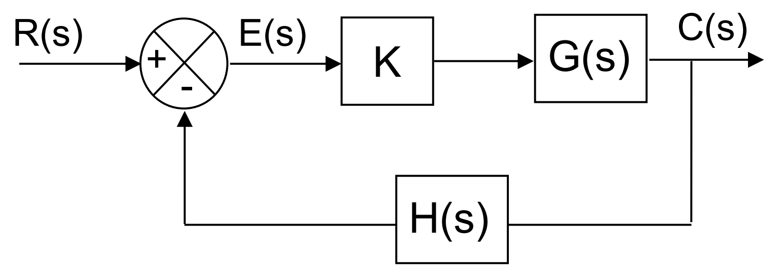
For small *steady-state error in ramp response* → K large

For good *transient response* → K_h so that $0.4 < \zeta < 0.8$

Remark: The damping ratio ζ can be increased without affecting the natural frequency ω_n in this case.

ROOT LOCUS

Consider the system



$$\frac{C(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K \cdot G(s) \cdot H(s)}$$

Root locus represents the *poles of the closed-loop system* when the gain K changes from 0 to ∞

$$1 + K \cdot G(s) \cdot H(s) = 0 \Rightarrow \begin{cases} |K \cdot G(s) \cdot H(s)| = 1 & \textit{Magnitude Condition} \\ \angle G(s) \cdot H(s) = \pm 180^\circ \cdot (2k+1) & \textit{Angle Condition} \end{cases}$$

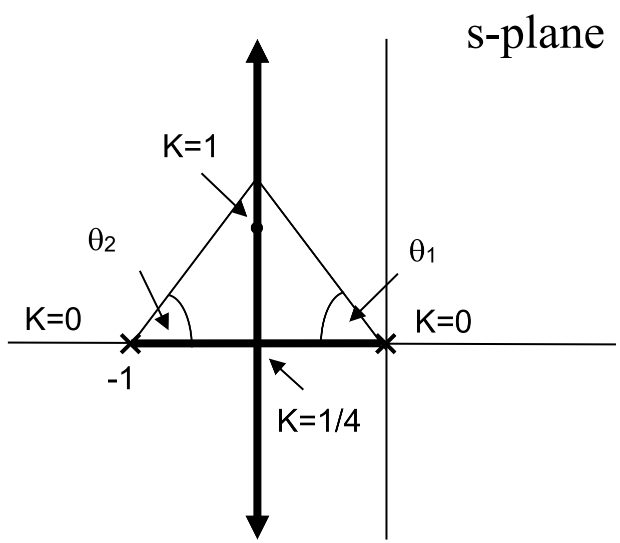
$$k = 0, 1, 2, \ldots$$

Example:

$$K \cdot G(s) \cdot H(s) = \frac{K}{s \cdot (s+1)}$$

$$1 + \mathrm{K} \cdot \mathrm{G(s)} \cdot \mathrm{H(s)} = 0 \Longrightarrow s^2 + s + K = 0 \Longrightarrow s_{1,2} = -\frac{1}{2} \pm \frac{1}{2} \cdot \sqrt{1 - 4 \cdot K}$$

Angle condition: $\angle\left(\frac{K}{s \cdot (s+1)}\right) = -\angle s - \angle s+1 = -(180 - \theta_1) - \theta_2 = \pm 180^\circ$

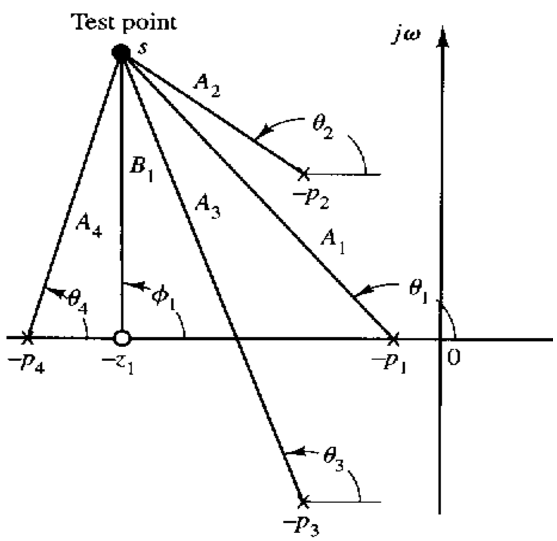


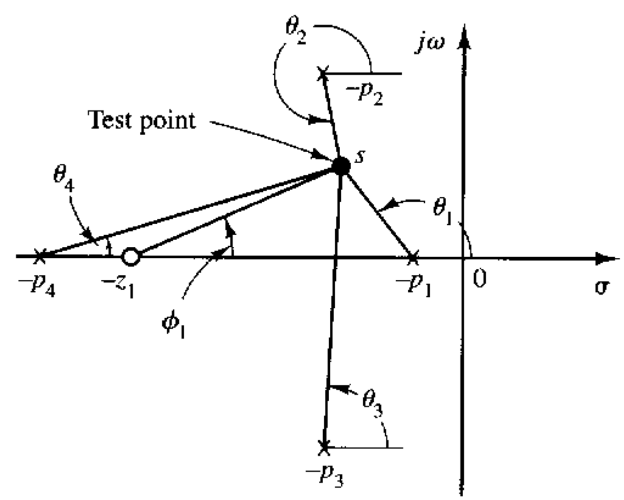
Magnitude and Angle Conditions in s-plane

$$K \cdot G(s) \cdot H(s) \quad = \quad \frac{K \cdot (s + z_1)}{(s + p_1) \cdot (s + p_2) \cdot (s + p_3) \cdot (s + p_4)}$$

$$\left| K \cdot G(s) \cdot H(s) \right| \quad = \quad \frac{K \cdot B_1}{A_1 \cdot A_2 \cdot A_3 \cdot A_4} \quad = \quad 1$$

$$\angle G(s) \cdot H(s) \quad = \quad \varphi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 \quad = \quad \pm 180^\circ \cdot (2 \cdot k + 1) \qquad for \quad k = 0,\ 1,\ 2,\ \dots$$





Construction Rules for Root Locus

Given open-loop transfer function:

$$K\,H(s)\cdot G(s) \quad = \quad K\,\frac{B(s)}{A(s)}$$

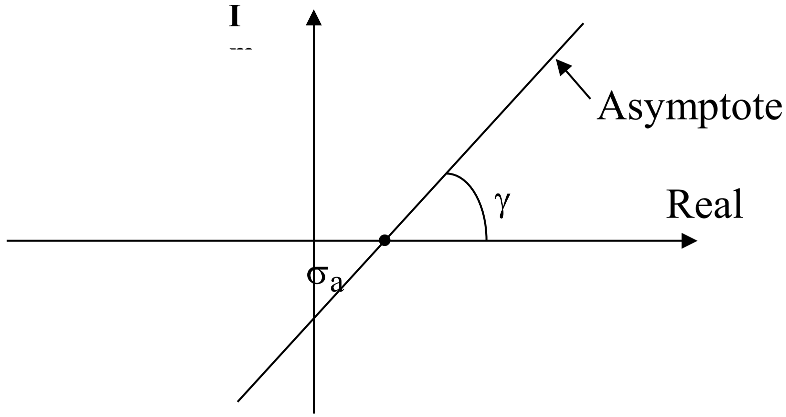
m: order of open-loop numerator polynomial
n: order of open-loop denominator polynomial

Rule 1: Number of branches
The number of branches is equal to the number of poles of the open-loop transfer function.

Rule 2: Real-axis root locus
If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

Rule 3: Root locus end-points
The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=∞) are at the open loop zeros and n-m branches terminate at infinity.

Rule 4: **Slope of asymptotes of root locus as s approaches infinity**



$$\gamma = \frac{\pm 180^\circ \cdot (2k+1)}{n-m}, \quad k = 0, 1, 2, \ldots$$

Rule 5: **Abscissa of the intersection between asymptotes of root locus and real-axis**

$$\sigma_a = \frac{\sum_{i=1}^{n} (-p_i) - \sum_{i=1}^{m} (-z_i)}{n-m}$$

($-p_i$) = poles of open-loop transfer function
($-z_i$) = zeros of open-loop transfer function

<u>Rule 6:</u> **Break-away and break-in points**

From the characteristic equation

$$f(s) = A(s) + K \cdot B(s) = 0 \quad and \quad K = -\frac{A(s)}{B(s)}$$

the break-away and -in points can be found from:

$$\frac{dK}{ds} = -\frac{A^{'}(s) \cdot B(s) - A(s) \cdot B^{'}(s)}{B^{2}(s)} = 0$$

<u>Rule 7:</u> **Angle of departure from complex poles or zeros**

Subtract from 180° the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

<u>Rule 8:</u> **Imaginary-axis crossing points**

Find these points by solving the characteristic equation for $s=j\omega$ or by using the Routh's table.

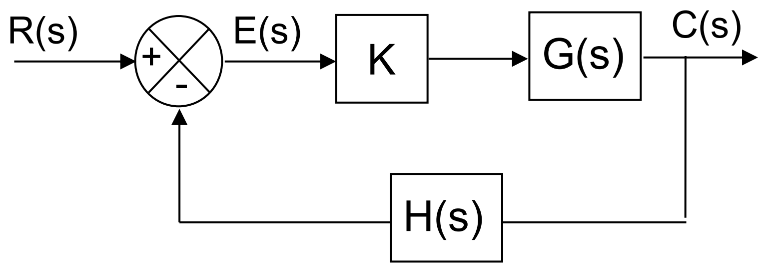
<u>Rule 9:</u> Conservation of the sum of the system roots

If the *order of numerator is lower than the order of denominator by two or more*, then the sum of the roots of the characteristic equation is constant.

Therefore, if some of the roots move towards the left as K is increased, the other roots must move toward the right as K in increased.

Discussion of Root Locus Construction Rules

Consider:



$$K \cdot H(s) \cdot G(s) = K \cdot \frac{B(s)}{A(s)} = K \cdot \frac{\sum_{i=0}^{m} b_i \cdot s^{m-i}}{\sum_{i=0}^{n} \alpha_i \cdot s^{n-i}}$$

m: number of zeros of open-loop $KH(s)G(s)$

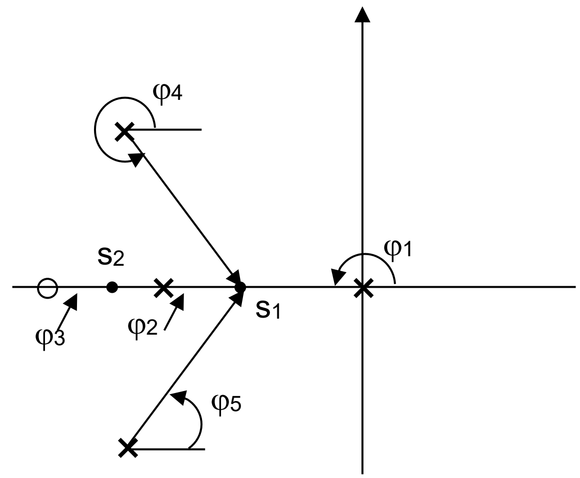
n: number of poles of open-loop $KH(s)G(s)$

Characteristic Equation: $f(s) = A(s) + K \cdot B(s) = 0$

Rule 1: **Number of branches**

The characteristic equation has n zeros $\Longrightarrow$ the root locus has n branches.

Rule 2: **Real-axis root locus**



Consider two points s_1 and s_2 :

$$s_1 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 0 = \varphi_3\,, \quad \varphi_4 + \varphi_5 = 180 \cdot 2 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 3 \cdot 180 \end{cases}$$

$$s_2 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 180, \quad \varphi_3 = 0\,, \quad \varphi_4 + \varphi_5 = 360 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 4 \cdot 180 \end{cases}$$

Therefore, s_1 is on the root locus; s_2 is not.

<u>Rule 3:</u> **Root locus end-points**

Consider the Magnitude Condition: $$\left|\frac{B(s)}{A(s)}\right| = \frac{1}{K} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

This implies that for $K \to 0$ we obtain the starting points at the n open loop poles,
for $K \to \infty$ we obtain endpoints at the m open loop zeros
and the remaining *n-m* branches approach infinity

<u>Rule 4:</u> **Slope of asymptotes of root locus as s approaches infinity**

$$K \cdot \frac{B(s)}{A(s)} = -1 \Longrightarrow \lim_{s\to\infty} \frac{K \cdot B(s)}{A(s)} = \lim_{s\to\infty} \frac{K}{s^{n-m}} = -1 \Longrightarrow s^{n-m} = -K \;\; for \;\; s \to \infty$$

Using the angle condition:

$$\angle s^{n-m} = \angle -K = \pm 180^{\circ} \cdot (2 \cdot k + 1), \text{ for } k = 0,1,2,\ldots \quad \text{or} \quad (n-m)\cdot \angle s = \pm 180^{\circ} \cdot (2 \cdot k + 1)$$

leads to $$\angle s = \gamma = \frac{\pm 180^{\circ} \cdot (2 \cdot k + 1)}{n-m}$$

<u>Rule 5:</u> Abscissa of the intersection between asymptotes of root-locus and real axis

$$\frac{A(s)}{B(s)} = \frac{s^n + s^{n-1}\cdot\sum_{i=1}^{n} p_i + \ldots + \prod_{i=1}^{m} p_i}{s^m + s^{m-1}\cdot\sum_{i=1}^{m} z_i + \ldots + \prod_{i=1}^{n} z_i} = -K$$

Dividing numerator by denominator yields:

$$s^{n-m} - \left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right)\cdot s^{n-m-1} + \ldots = -K$$

For large values of s this can be approximated by:

$$\left(s - \frac{\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i}{n-m}\right)^{n-m} = -K$$

The equation for the asymptote (for $s \to \infty$) was found in Rule 4 as

$$s^{n-m} = -K \quad \text{which implies} \quad \sigma_a = -\frac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} = \frac{\sum_{i=1}^{n} -p_i - \sum_{i=1}^{m} -z_i}{n-m}$$

<u>Rule 6</u>: Break-away and break-in points

At break-away (and break-in) points the characteristic equation:

$$f(s) \quad = \quad A(s) + K \cdot B(s) \quad = \quad 0$$

has multiple roots such that:

$$\frac{df(s)}{ds} = 0 \quad \Rightarrow \quad A'(s) + K \cdot B'(s) = 0 \qquad \left(A'(s) = \frac{dA(s)}{ds} \right)$$

(for s = break-in or break-away point)

$\Rightarrow$ for $K = -\dfrac{A'(s)}{B'(s)}$, $f(s)$ has multiple roots

Substituting the above equation into $f(s)$ gives: $A(s) \cdot B'(s) - A'(s) \cdot B(s) = 0$

Another approach is using: $K = -\dfrac{A(s)}{B(s)}$ from $f(s) = 0$

This gives:

$$\frac{dK}{ds} = -\frac{A'(s) \cdot B(s) - A(s) \cdot B'(s)}{B^2(s)}$$

and break-away, break-in points are obtained from: $\dfrac{dK}{ds} = 0$

<u>Extended Rule 6:</u>

Consider $$f(s) = A(s) + K \cdot B(s) = 0$$

and

$$K = -\frac{A(s)}{B(s)}$$

If the first *(y-1)* derivatives of $K = -\dfrac{A(s)}{B(s)}$ vanish at a given point s_m on the root locus,

then there will be *y* branches approaching and *y* branches leaving this point.

The angle between two adjacent approaching branches is given by:

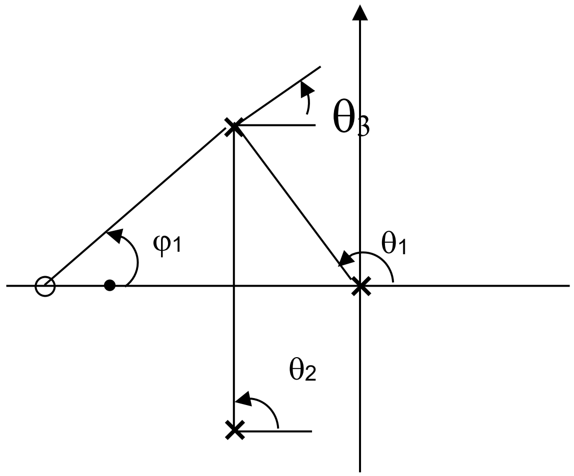
$$\theta_y = \pm\frac{360^\circ}{y}$$

The angle between a leaving branch and an adjacent approaching branch is:

$$\theta_y = \pm\frac{180^\circ}{y}$$

Further, s_m is a solution of $A(s) \cdot B'(s) - A'(s) \cdot B(s) = 0$ with multiplicity *(y-1)*

Rule 7: Angle of departure from complex pole or zero



$$\theta_2 = 90^\circ$$
$$\theta_3 = 180^\circ - (\theta_1 + \theta_2 - \varphi_1)$$

Rule 8: Imaginary-axis crossing points

Example: $f(s) = s^3 + b\cdot s^2 + c\cdot s + K\cdot d = 0$

s^3	1	c
s^2	b	Kd
s^1	(bc-Kd)/b	
s^0	Kd	

For crossing points on the Imaginary axis:

$$b\cdot c - K\cdot d = 0 \Rightarrow \quad K = \frac{bc}{d}$$

Further, $b\cdot s^2 + K\cdot d = 0$ leading to

$$s_{1,2} = \pm j\cdot\sqrt{\frac{K\cdot d}{b}} = \pm j\omega$$

The same result is obtained by solving $f(j\omega) = 0$

<u>Rule 9:</u> Conservation of the sum of the system roots

From

$$A(s)+K\cdot B(s)=\prod_{i=1}^{n}(s+r_i)$$

we have

$$\prod_{i=1}^{n}(s+p_i)+K\cdot\prod_{i=1}^{m}(s+r_i)=\prod_{i=1}^{n}(s+r_i) \quad \text{with} \quad A(s)=\prod_{i=1}^{n}(s+p_i) \quad \text{and} \quad B(s)=\prod_{i=1}^{m}(s+z_i)$$

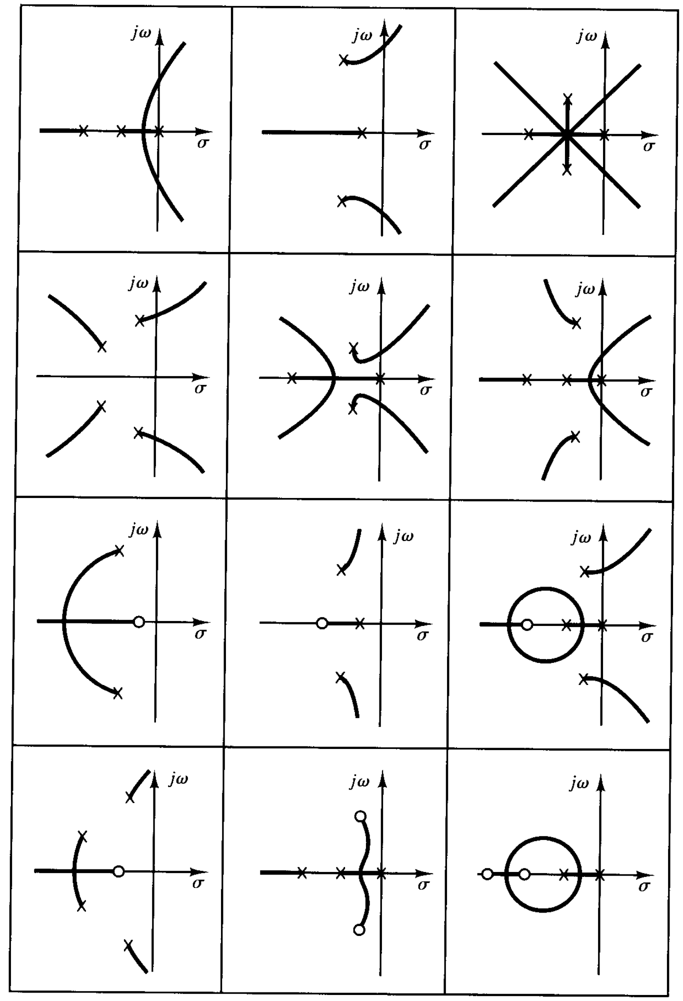
By equating coefficients of s^{n-1} for $n \geq m+2,$ we obtain the following:

Sum of open-loop poles $$\sum_{i=1}^{n}-p_i=\sum_{i=1}^{n}-r_i$$ Sum of closed-loop poles

i.e. the sum of closed-loop poles is independent of K !

Examples of Root Locuses:

Table 6–1 Open-Loop Pole–Zero Configurations and the Corresponding Root Loci



Section 6–3 / Summary of General Rules for Constructing Root Loci

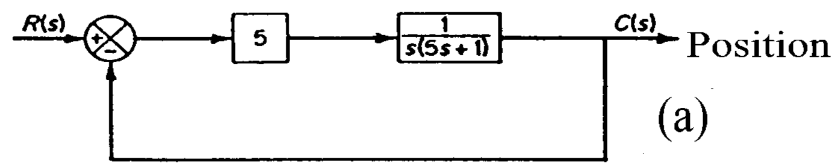
Analysis of Closed-Loop System Performance using the Root Locus

- The root locus shows the location of the closed-loop poles in the s-plane and it can be used to evaluate the type of response of the closed-loop system for some value of K.
- However, the actual response depends on the poles and zeros and the root locus doens not contain any information about the closed-loop zeros.

Example: Effect of Derivative Control and Velocity Feedback

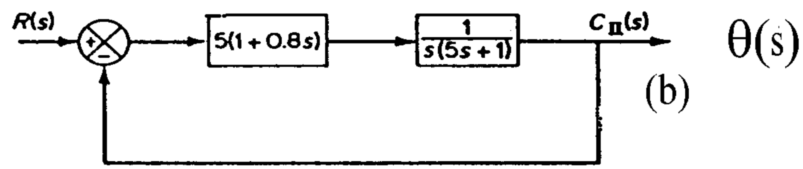
Consider the following three systems:

Positional servo.
Closed-loop poles: $s = -0.1 \pm j \cdot 0.995$



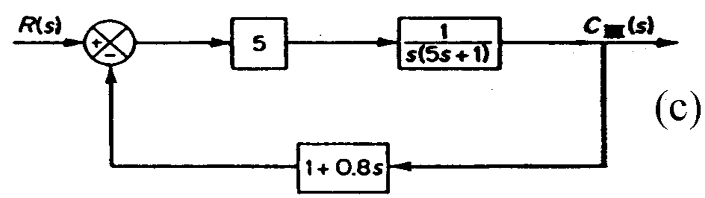
System I

Positional servo with derivative control.
Closed-loop poles: $s = -0.5 \pm j \cdot 0.866$



System II

Positional servo with velocity feedback.
Closed-loop poles: $s = -0.5 \pm j \cdot 0.866$

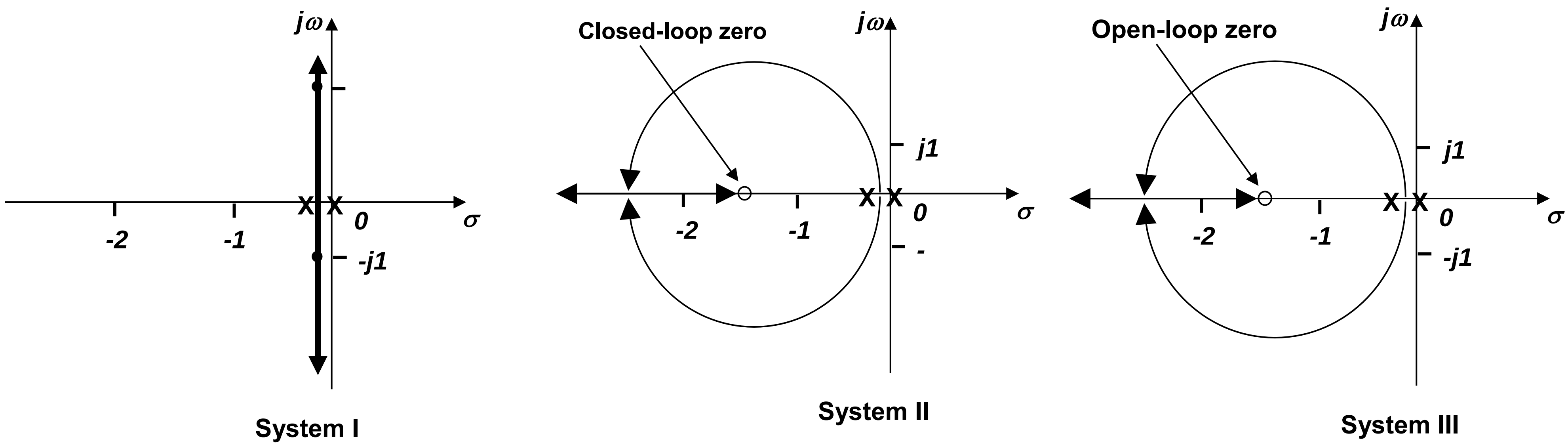


System III

Open-loop transfer functions of system I: $G_I(s) = \dfrac{5}{s\cdot(5\cdot s+1)}$

Open-loop transfer functions of systems II and III: $G(s) = \dfrac{5\cdot(1+0.8s)}{s\cdot(5\cdot s+1)}$

Root locus for the all three systems (open loop is $KG_I(s)$, $KG(s)$)



Closed-loop transfer functions of system I: $G_{cI}(s) = \dfrac{5}{s\cdot(5\cdot s+1)+5}$

Closed-loop transfer functions of systems II: $G_{cII}(s) = \dfrac{5\cdot(1+0.8s)}{s\cdot(5\cdot s+1)+5(1+0.8s)}$

Closed-loop transfer functions of systems III: $G_{cIII}(s) = \dfrac{5}{s\cdot(5\cdot s+1)+5(1+0.8s)}$

Closed-loop zeros: System I: none

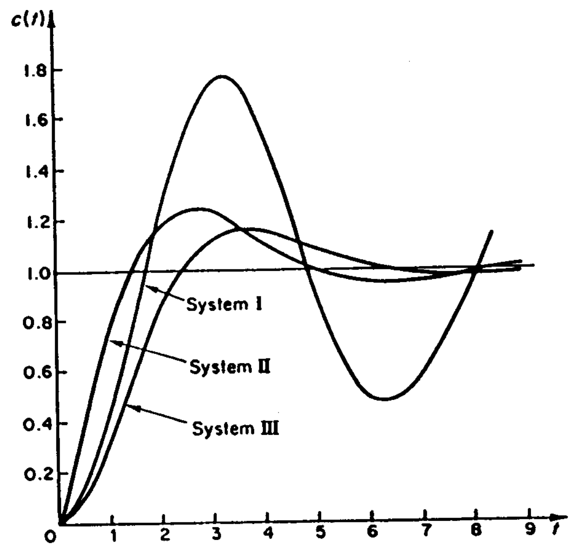
System II: 1+0.8s=0

System III: none

Observations:

- The root locus gives the *closed loop poles* but gives no information about *closed-loop zeros*.
- Two system with same root locus (same closed-loop poles) may have *different responses due to different closed-loop zeros (see next page)*.

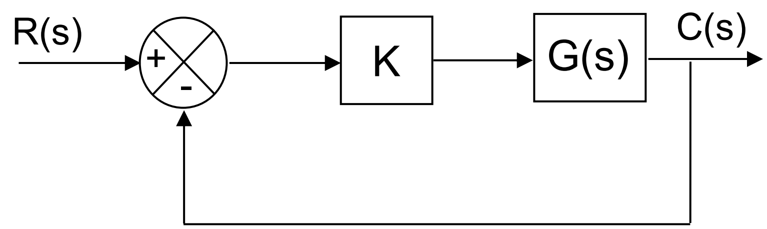
Unit-step response curves for systems I,II and III (same closed-loop poles for II and III):

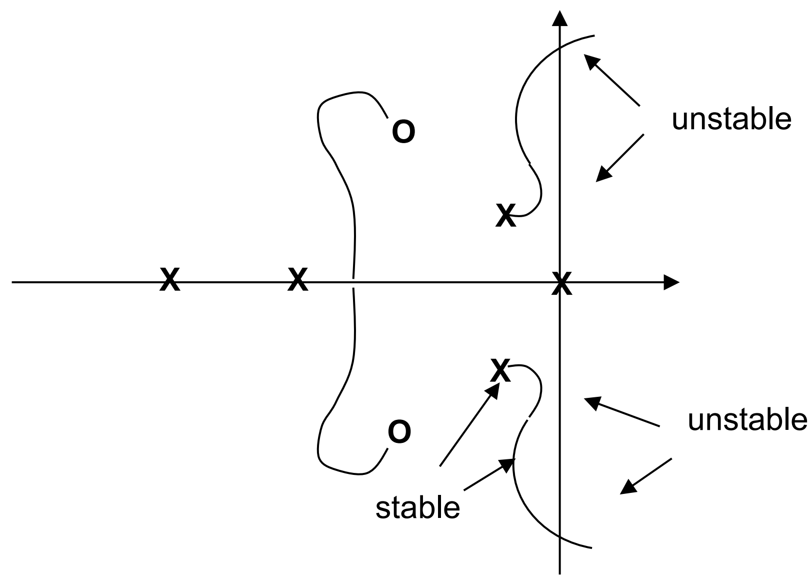


- The unit-step response of system II is faster then that of system III.
- This is due to the fact that derivative control responds to the rate of change of the error signal. Thus, it can produce a correction signal before the error becomes large and can lead to a faster response.

Conditionally Stable Systems:

System which can be stable or unstable depending on the value of gain K.



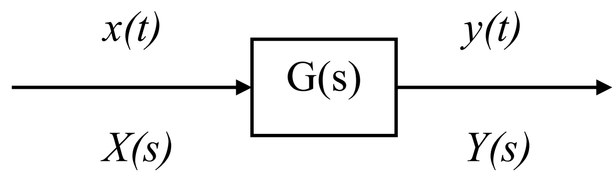


Minimum Phase Systems:

All poles and zeros of the system are in the left half plane.

FREQUENCY RESPONSE

Consider a stable system:



with the input: $x(t) = X\sin(\omega t)$ $\quad X(s) = \dfrac{\omega X}{s^2 + \omega^2}$ and output:

$$Y(s) = G(s)\cdot X(s) = G(s)\cdot\frac{\omega X}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \sum_i \frac{b_i}{s+s_i}$$

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + \sum_i b_i e^{-s_i t}$$

Since the system is assumed stable, $\Rightarrow$ $Re(-s_i) < 0$ for all i

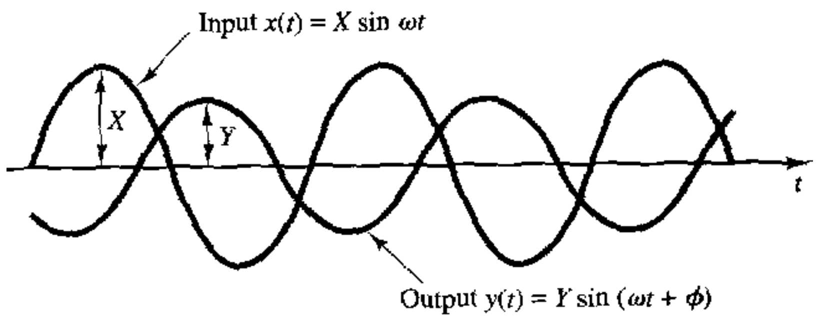
and for $t \to \infty \Rightarrow y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$ where

$$a = G(s)\cdot\frac{\omega X}{s^2+\omega^2}\cdot(s+j\omega)\Big|_{(s=-j\omega)} = -\frac{XG(-j\omega)}{2j}, \quad \bar{a} = G(s)\cdot\frac{\omega X}{s^2+\omega^2}(s-j\omega)\Big|_{(s=j\omega)} = \frac{XG(j\omega)}{2j}$$

Using $G(j\omega) = |G(j\omega)| \cdot e^{j\varphi}$ and $G(-j\omega) = |G(j\omega)| \cdot e^{-j\varphi}$

where $\varphi = \tan^{-1}\left(\dfrac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$

$$y(t) = X \cdot |G(j\omega)| \cdot \frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j} = Y \cdot sin(\omega t + \varphi)$$



$\varphi > 0$ phase lead

$\varphi < 0$ phase lag

Frequency response: $G(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$

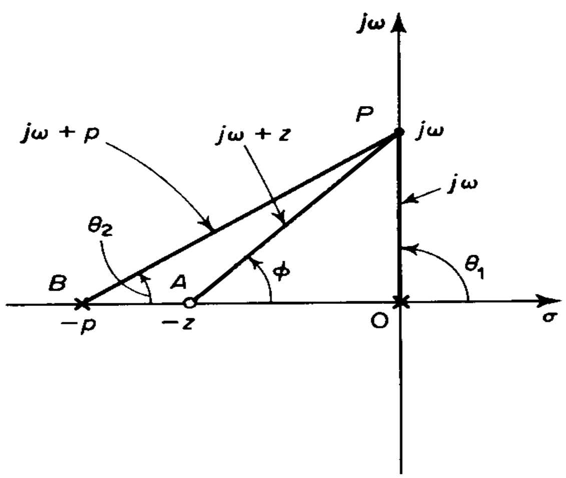
Magnitude response: $|G(j\omega)| = \left|\dfrac{Y(j\omega)}{X(j\omega)}\right|$

Phase response: $\varphi = \angle(G(j\omega)) = \angle\left(\dfrac{Y(j\omega)}{X(j\omega)}\right)$

Frequency Response Plots:

- Bode Diagrams
- Polar Plots (Nyquist Plots)
- Log-Magnitude-Versus-Phase Plots (Nichols Plots)

Connection between pole locations and Frequency Response:



Determination of the frequency response in the complex plane.

$$G(s) \;=\; \frac{K(s+z)}{s(s+p)}$$

$$|\,G(j\omega)| = \frac{|\,K|\cdot|\,j\omega+z\,|}{|\,j\omega\,|\cdot|\,j\omega+p\,|}$$

$$\angle G(j\omega) \;=\; \varphi-\theta_1-\theta_2$$

Bode Diagrams

- Magnitude response: $20\log|G(j\omega)|$ in dB
- Phase response $\angle G(j\omega)$ in degrees

Basic Factors of *G(jω)*:

- Gain K
- Integral or derivative factors $(j\omega)^{\pm 1}$
- First-order factors $(1+j\omega T)^{\pm 1}$
- Quadratic factors $\left(1+2\varsigma\dfrac{j\omega}{\omega_n}+\left(\dfrac{j\omega}{\omega_n}\right)^2\right)^{\pm 1}$

1. Gain Factor K

Horizontal straight line at magnitude *20 log(K) dB*

Phase is zero

2. Integral or Derivative Factors: $(j\omega)^{\pm 1}$

- $(j\omega)^{-1}$ $\quad 20\log\left|\dfrac{1}{j\omega}\right| = -20\log\omega$

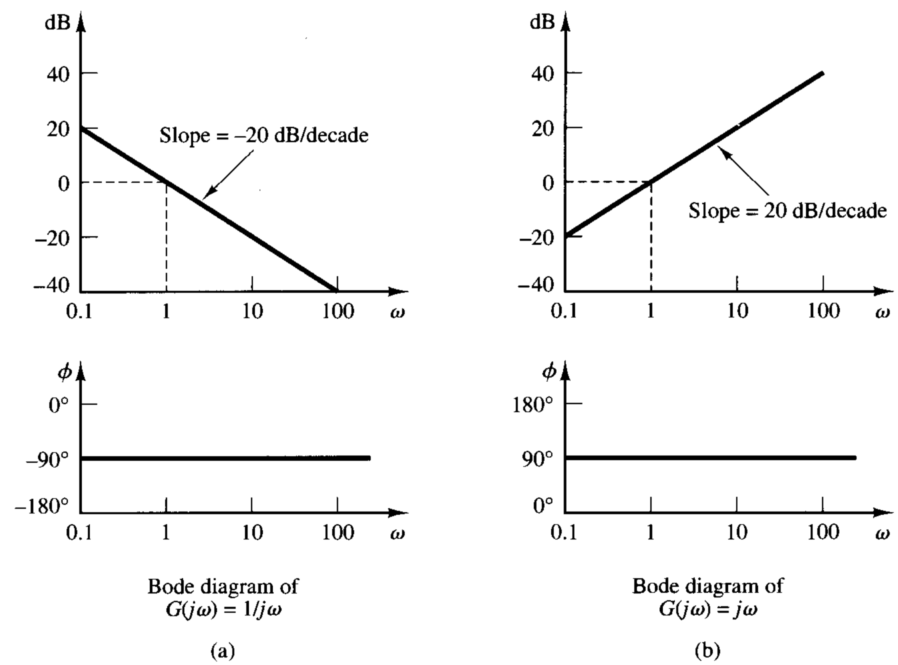
Magnitude: straight line with slope –20 dB/decade

Phase: -90°

- $(j\omega)$ $\quad 20\log|j\omega| = 20\log\omega$

Magnitude: straight line with slope 20 dB/decade

Phase: +90°



(a) Bode diagram of $G(j\omega) = 1/j\omega$; (b) Bode diagram of $G(j\omega) = j\omega$.

3. First Order Factors: $(1 + j\omega T)^{\pm 1}$

- $(1 + j\omega T)^{-1}$

Magnitude: $$20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2T^2}\ dB$$

Approximation of the Magnitude:

- for $\omega \ll T^{-1}$ magnitude tends to $0\ db$

  $\Rightarrow$ for ω between 0 and $\omega = \frac{1}{T}$ 0 dB

- for $\omega \gg T^{-1}$ magnitude tends to $-20\log(\omega T)dB$

  $\Rightarrow$ for $\omega \gg \frac{1}{T}$ straight line with slope –20 dB /decade

Phase: $$\angle(1 + j\omega T)^{-1} = -\tan^{-1}(\omega T)$$

For $\omega = 0$ $\varphi = 0^\circ$

For $\omega = \frac{1}{T} \Rightarrow -\tan^{-1}\left(\frac{T}{T}\right) = \tan^{-1}(1)$ $\varphi = -45^\circ$

For $\omega = \infty$ $\varphi = -90^\circ$

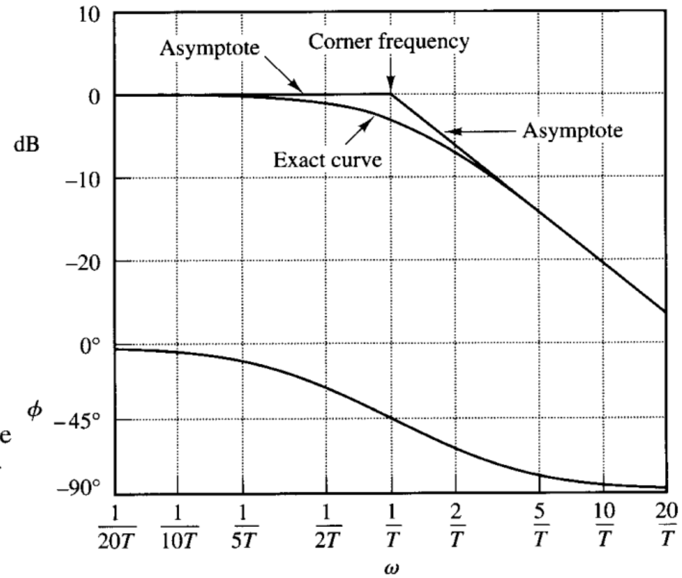
- $(1 + j\omega T)^{+1}$

Magnitude:

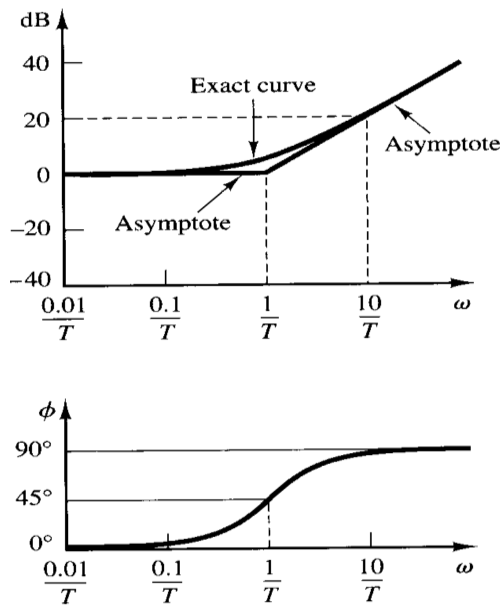
Using $\quad 20\log\left|1 + j\omega T\right| = -20\log\left|\dfrac{1}{1 + j\omega T}\right|$ and the previous discussion

Phase:

Using $\quad \angle(1 + j\omega T) = \tan^{-1}(\omega T) = -\angle\left(\dfrac{1}{1 + j\omega T}\right)$ and the previous discussion



Log-magnitude curve together with the asymptotes and phase angle curve of $1/(1 + j\omega T)$.



Log-magnitude curve together with the asymptotes and phase-angle curve for $1 + j\omega T$.

4. **Quadratic Factors:**

$$G(j\omega)=\frac{1}{1+2\varsigma j\left(\dfrac{\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2} \qquad 0<\varsigma<1$$

Magnitude: $$20\log|G(j\omega)|=-20\log\sqrt{\left(1-\frac{\omega^2}{\omega^2{}_n}\right)^2+\left(2\varsigma\left(\frac{\omega}{\omega_n}\right)\right)^2}$$

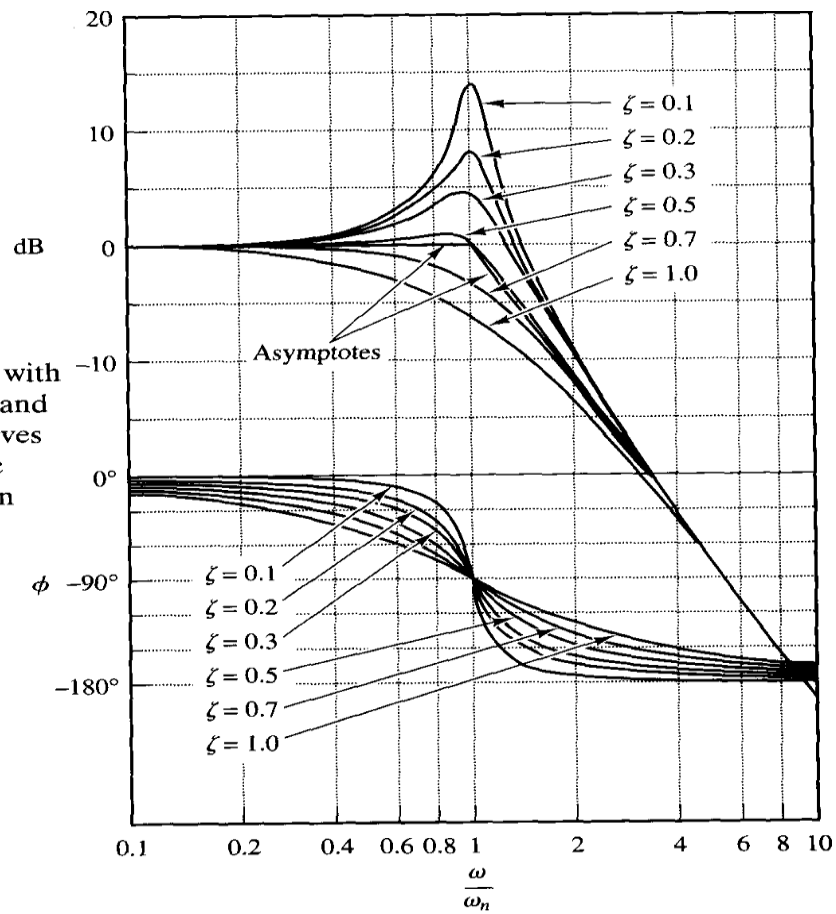
Asymptotes: for $\omega \ll \omega_n \;\Rightarrow\; 0 \;\; dB$

for $\omega \gg \omega_n \;\Rightarrow\; -20\log\left(\dfrac{\omega^2}{\omega^2{}_n}\right) = -40\log\left(\dfrac{\omega}{\omega_n}\right) \;\; dB$

Phase: $$\varphi=\tan^{-1}\angle G(j\omega)=-\tan^{-1}\left[\frac{2\varsigma\dfrac{\omega}{\omega_n}}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}\right]$$

Resonant Frequency: $\omega_r=\omega_n\sqrt{1-2\varsigma^2}$ for $0\le\varsigma\le 0.707$

Resonant Peak Value: $M_r=|G(j\omega)|_{\max}=\dfrac{1}{2\varsigma\sqrt{1-\varsigma^2}}$ for $0\le\varsigma\le 0.707$



Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic transfer function

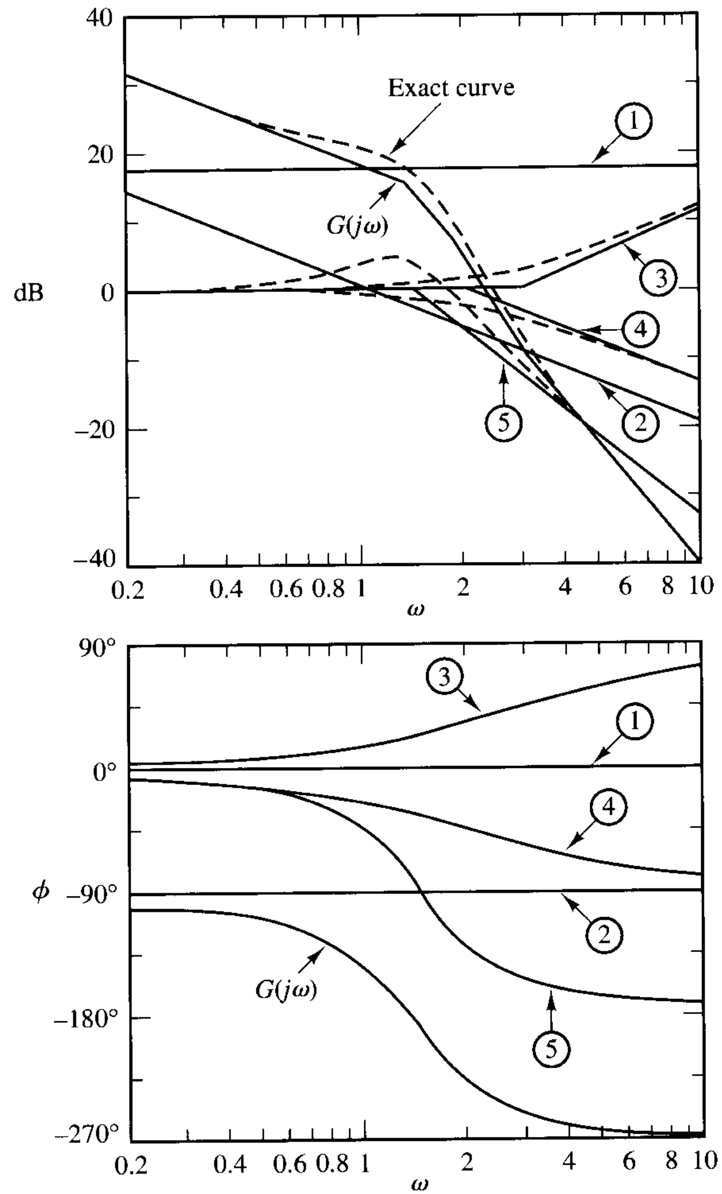
Example:

Consider

$$G(j\omega) = \frac{10(j\omega+3)}{(j\omega)\cdot(j\omega+2)\cdot\left((j\omega)^2+j\omega+2\right)}$$

and sketch the Bode Diagram using the basic factors

$$G(j\omega)=\frac{7.5\cdot\left(\frac{(j\omega)}{3}+1\right)}{j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{(j\omega)^2}{2}+\frac{j\omega}{2}+1\right)}$$

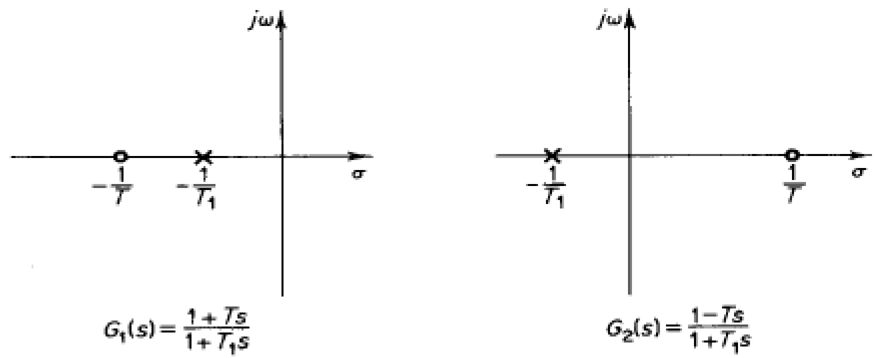


Bode diagram of the system considered in Example 8–1.

Frequency Response of non-Minimum Phase Systems

Minimum phase systems have all poles and zeros in the left half s-plane.

Consider these two systems $G_1(s)$, minimum phase, and $G_2(s)$, non-minimum phase"



Pole-zero configurations of a minimum phase system $G_1(s)$ and nonminimum phase system $G_2(s)$.

Using $\quad A_1(s) = 1+Ts \quad A_2(s) = 1-Ts \quad A_3(s) = Ts-1$

We have: $\quad |A_1(j\omega)| = |A_2(j\omega)| = |A_3(j\omega)|$

$$\angle A_2(j\omega) = -\angle A_1(j\omega)$$

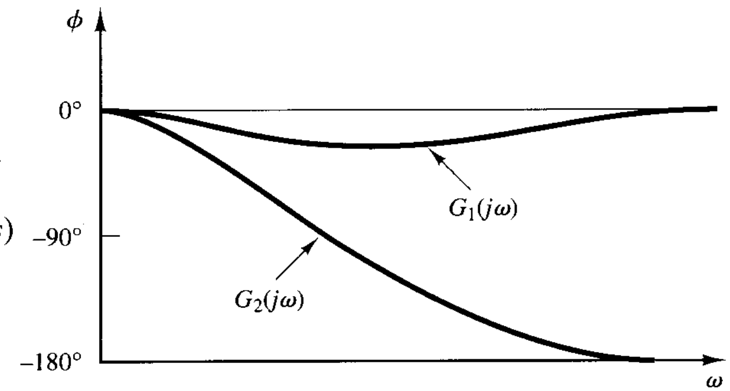
$$\angle A_3(j\omega) = 180 - \angle A_1(j\omega)$$

Since we know that phase of $A_1(j\omega)$ from 0^o to $+90^o$

$\rightarrow$ phase of $A_2(j\omega)$ from 0^o to -90^o

phase of $A_3(j\omega)$ from 180^o to $+90^o$

Phase-angle characteristics of the systems $G_1(s)$ and $G_2(s)$



Phase-angle characteristic of the two systems $G_1(s)$ and $G_2(s)$ having the same magnitude response but $G_1(s)$ is minimum phase while $G_2(s)$ is not.

Frequency Response of Unstable Systems

Consider

$$G_1(s) = \frac{1}{1+Ts} \qquad G_2(s) = \frac{1}{1-Ts} \qquad G_3(s) = \frac{1}{Ts-1}$$

then

$$|G_1(j\omega)| = |G_2(j\omega)| = |G_3(j\omega)|$$

and

$$\angle G_2(j\omega) = -\angle G_1(j\omega)$$

$$\angle G_3(j\omega) = -180^o - \angle G_1(j\omega)$$

We know that phase of $G_1(j\omega)$ from 0^o to -90^o

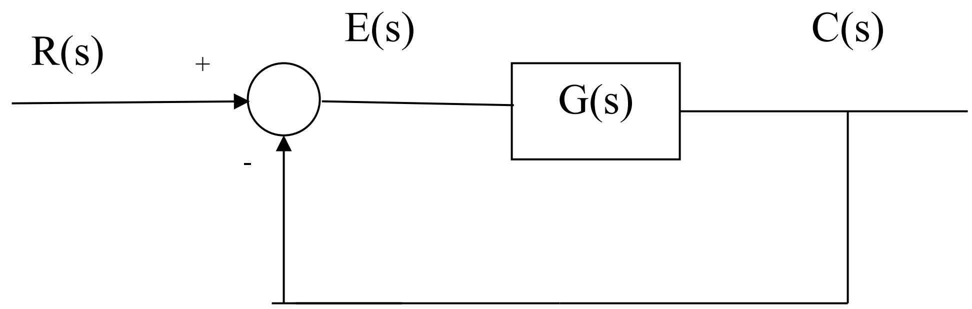
→ phase of $G_2(j\omega)$ from 0^o to $+90^o$

phase of $G_3(j\omega)$ from -180^o to -90^o

Relationship between System Type and Bode Diagram

Consider a closed-loop system:

with an open-loop transfer function:



$$G(s) = \frac{K(T_a s + 1)\cdot(T_b s + 1)\ldots}{s^N (T_1 s + 1)\cdot(T_2 s + 1)\ldots}$$

and static error constants defined as:

$$K_p = \lim_{s\to 0} G(s) = G(0)$$
$$K_v = \lim_{s\to 0} sG(s)$$
$$K_a = \lim_{s\to 0} s^2 G(s)$$

Question: Given the frequency response of the open-loop system *G(s)* can we find the **Type of the system** and the corresponding **Error Constant**?

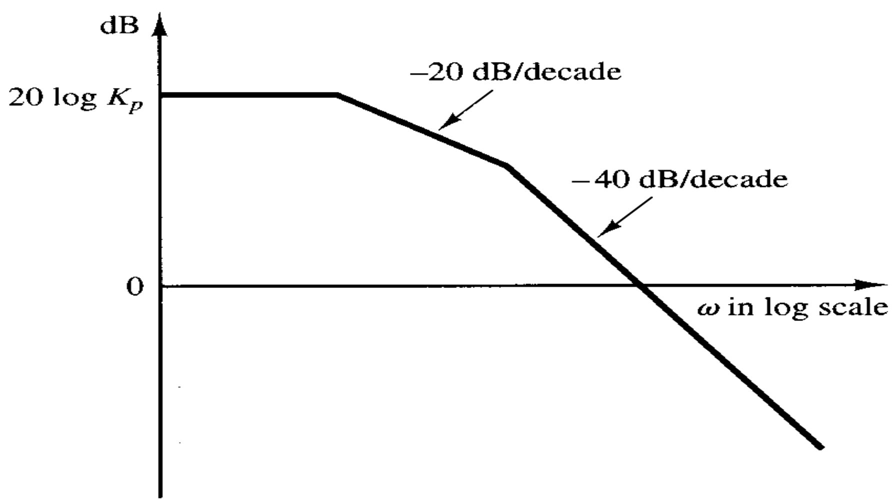
Type of system is the number of integrators in the open-lop system. Thus is equal to:

- the slope of the log-magnitude curve at low frequencies
- for minimum phase, also the phase at low frequencies

The corresponding error constant can be obtained as follows:

Type 0:

Position Error Constant $K_p \neq 0$

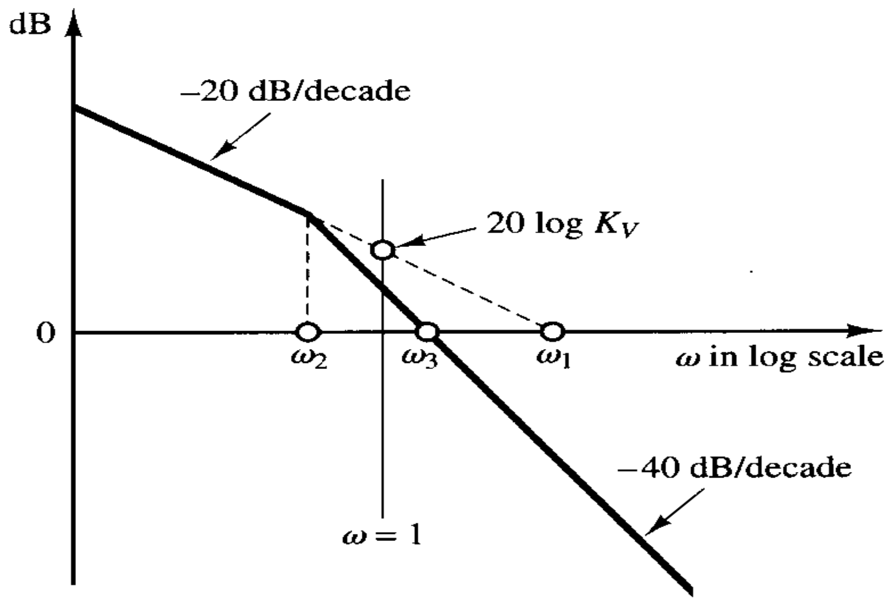


- Slope at low frequencies: 0 db/decade
- Phase at low frequencies (minimum phase): 0^o

$$\lim_{\omega \to 0} G(j\omega) = K_p$$

Type 1:

Velocity Error Constant $K_v \neq 0 \quad (K_p = \infty)$



- Slope at low frequencies:

  -20 db/decade

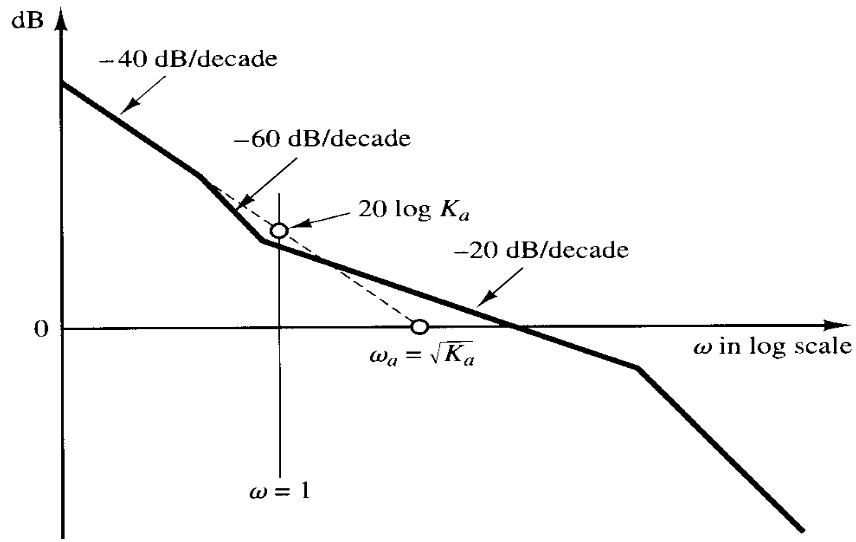
- Phase at low frequencies (minimum phase): -90°

Since $G(j\omega) = \dfrac{K_v}{j\omega}$ for $\omega << 1$ and $K_v = \lim\limits_{\omega \to 0} j\omega\, G(j\omega)$

$$20 log K_v = 20 log \left| \frac{K_v}{j\omega} \right| \quad for \quad \omega = 1$$

Type 2:

Acceleration Error Constant $K_a \neq 0 \quad (K_p = K_v = \infty)$

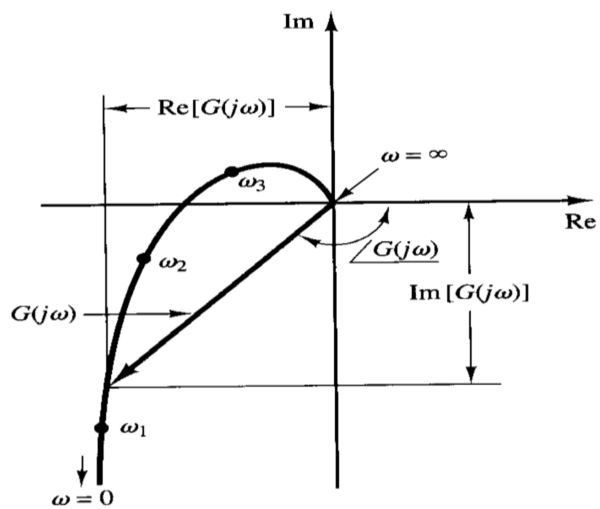


- Slope at low frequencies: -40 db/decade
- Phase at low frequencies (minimum phase): -180°

Since $G(j\omega) = \dfrac{K_a}{(j\omega)^2}$ for $\omega \ll 1$ and $K_a = \lim\limits_{\omega \to 0} (j\omega)^2 G(j\omega) \neq 0$

$$20\log K_a = 20\log\left|\frac{K_a}{(j\omega)^2}\right| \quad for \quad \omega = 1$$

Polar Plots (Nyquist Plots)



$$G(j\omega) = \mathrm{Re}[G(j\omega)] + \mathrm{Im}[G(j\omega)]$$

In Polar Plots the axes are:

$\mathrm{Re}[G(j\omega)]$ and $\mathrm{Im}[G(j\omega)]$

Advantage over Bode plots: Only one plot.

Disadvantage : Polar plot of $G(j\omega) = G_1(j\omega) \cdot G_2(j\omega)$ is more difficult to sketch than its Bode plot.

Basic Factors of *G(jω)*:

1. Integral or Derivative Factors: $(j\omega)^{\pm 1}$

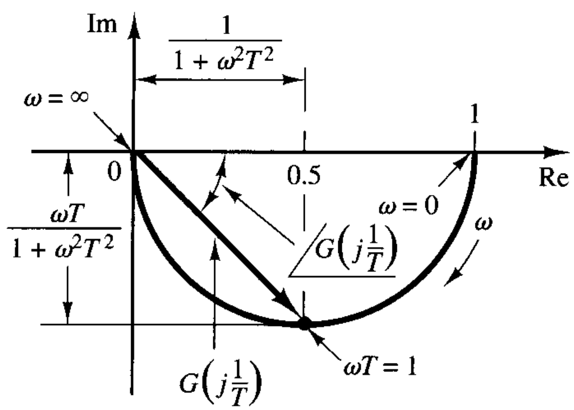
$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ \qquad G(j\omega) = j\omega = \omega\angle 90^\circ$$

2. First Order Factors: $(1 + j\omega T)^{\pm 1}$

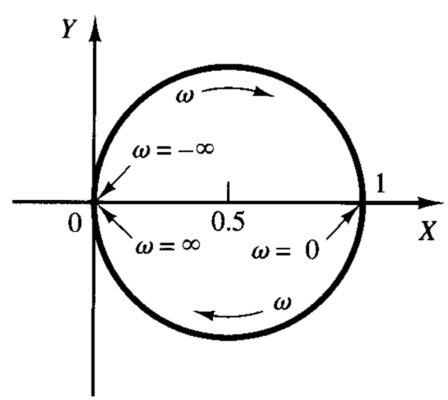
$$G(j\omega) = \frac{1}{1+j\omega T} = X + jY \quad \text{where} \quad X = \frac{1}{1+\omega^2 T^2} \quad \text{and} \quad Y = \frac{-\omega T}{1+\omega^2 T^2}$$

It can be show that $(X-0.5)^2 + Y^2 = (0.5)^2$

$\Rightarrow$ Polar plot is a circle with Center at (1/2, 0) and Radius 0.5.



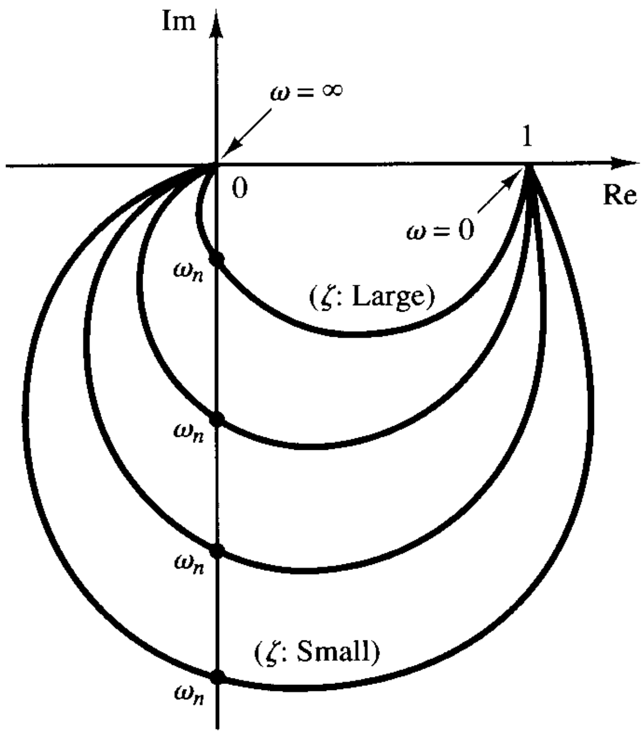
(a)



(b)

3. Quadratic Factors

$$G(j\omega)=\frac{1}{1+2\zeta j\left(\dfrac{\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2} \qquad 1>\zeta>0$$

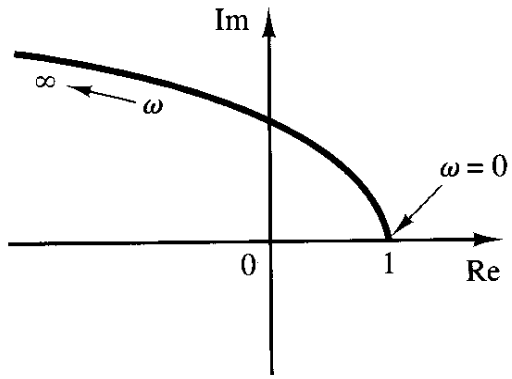


$$\lim_{\omega\to 0} G(j\omega) = 1\angle 0^\circ$$

$$\lim_{\omega\to\infty} G(j\omega) = 0\angle -180^\circ$$

$$G(j\omega_n)=\frac{1}{2\varsigma}\angle -90^\circ$$

$$G(j\omega) = \left(1 + 2\varsigma\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2\right)^{+1}$$



$$\lim_{\omega \to 0} G(j\omega) = 1\angle 0^\circ$$

$$\lim_{\omega \to \infty} G(j\omega) = \infty \angle 180^\circ$$

General shapes of polar plots

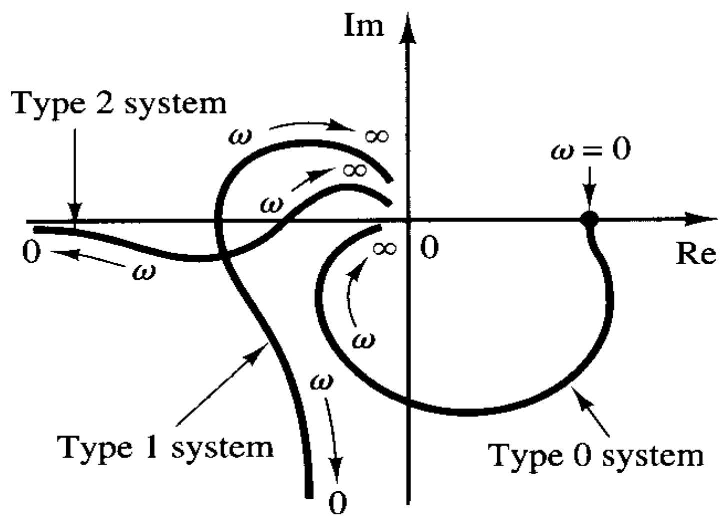
Consider the minimum phase system $G(j\omega)$ given by:

$$G(j\omega)=\frac{K(1+j\omega\bar{T}_1)\ldots(1+j\omega\bar{T}_m)}{(j\omega)^{\lambda}(1+j\omega T_{\lambda+1})\ldots(1+j\omega T_n)}$$

with n = order of the system (denominator)

λ = type of system $\lambda > 0$

m = order of numerator n > m



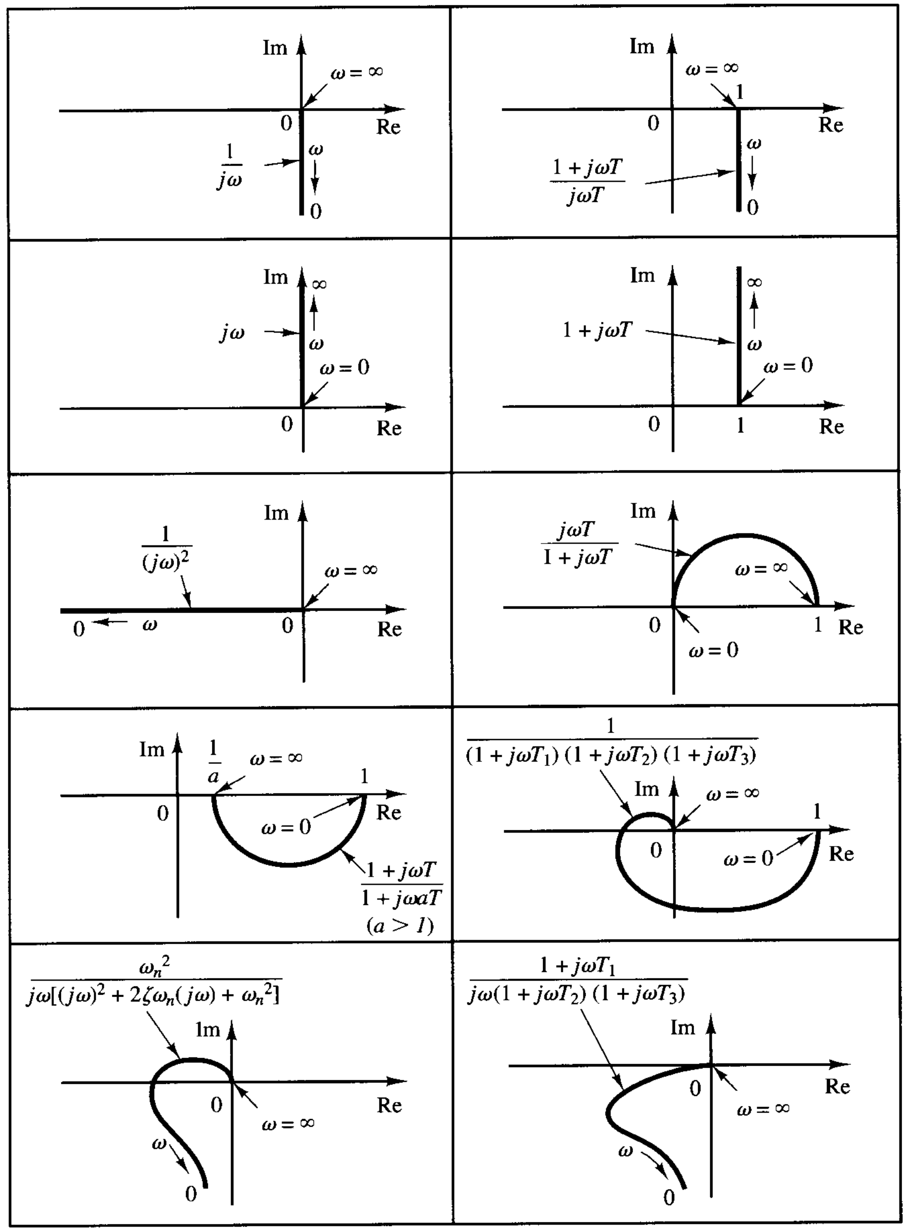
For low frequencies:

- The phase at $\omega \to 0$ **is** $\lambda\left(-90^\circ\right)$
- For system type 1, the low frequency asymptote is obtained by taking:

$$\mathrm{Re}\left[G(j\omega)\right] \text{ for } \omega \to 0$$

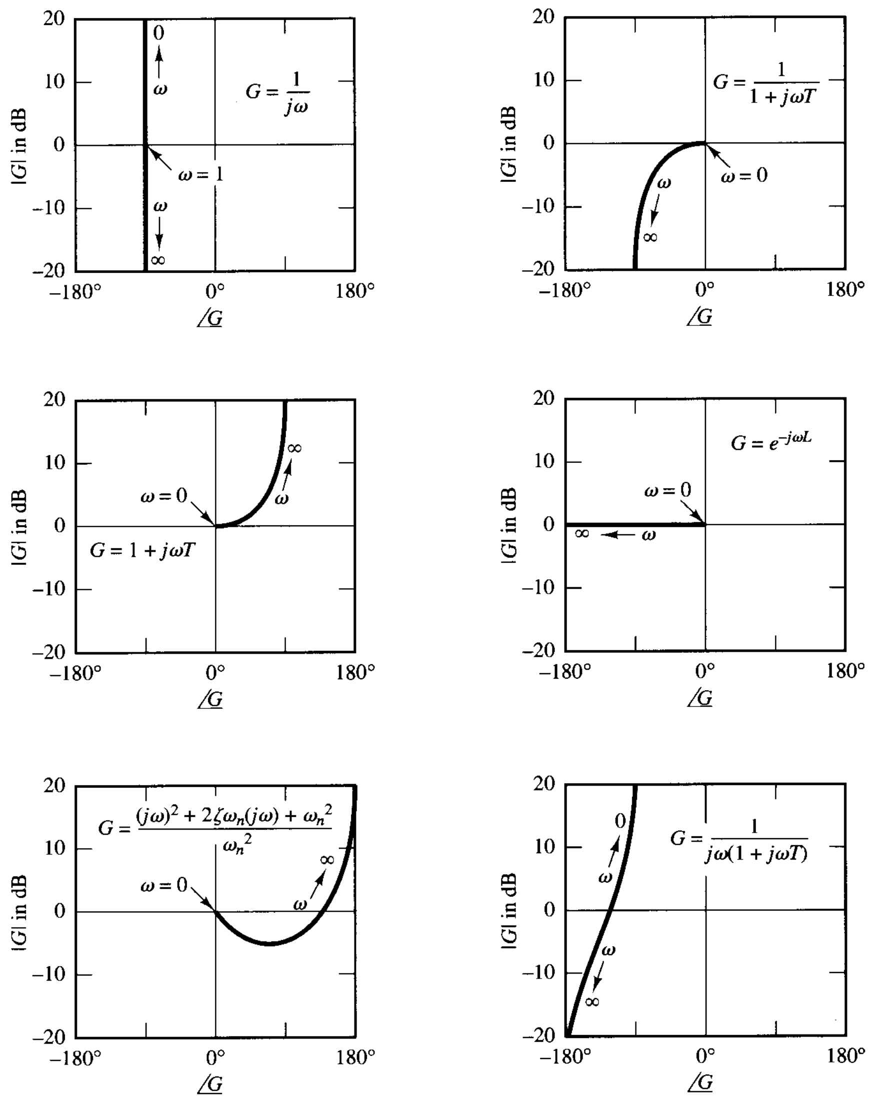
For high frequencies:

The phase is: (n - m) (- 90^o)



Im
Re
$\omega = \infty$
$\frac{1}{j\omega}$
ω
0
$\omega = \infty$
1
$\frac{1 + j\omega T}{j\omega T}$
$j\omega$
∞
$\omega = 0$
$1 + j\omega T$
$\frac{1}{(j\omega)^2}$
$\frac{j\omega T}{1 + j\omega T}$
$\frac{1}{a}$
$\frac{1 + j\omega T}{1 + j\omega a T}$
$(a > 1)$
$\frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)(1 + j\omega T_3)}$
$\frac{\omega_n^2}{j\omega[(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2]}$
$\frac{1 + j\omega T_1}{j\omega(1 + j\omega T_2)(1 + j\omega T_3)}$

Log-Magnitude-Versus-Phase Plots (Nichols Plots)

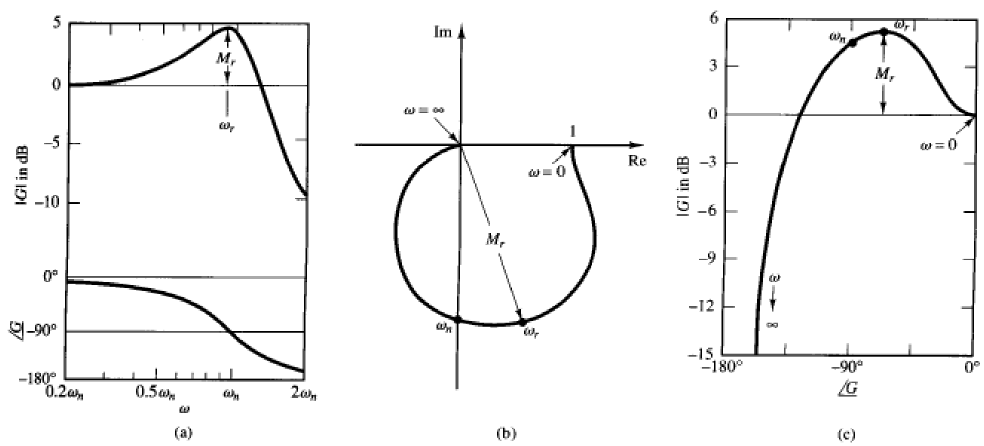


Example:

Frequency Response of a quadratic factor

The same information presented in three different ways:

- Bode Diagram
- Polar Plot
- Log-Magnitude-Versus-Phase Plots

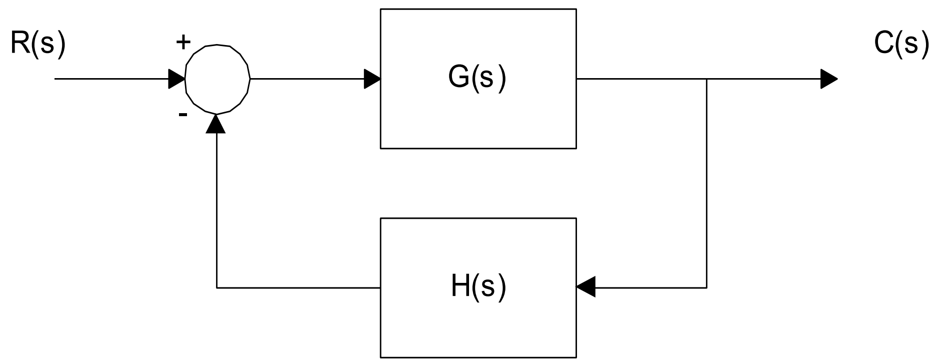


Three representations of the frequency response of $\dfrac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2}$, for $\zeta > 0$.

(a) Bode diagram; (b) polar plot; (c) log-magnitude versus phase plot.

NYQUIST STABILITY CRITERION

The Nyquist stability criterion relates the ***stability of the closed loop*** system to the ***frequency response of the open-loop*** system.



Open-loop: $G(s)\cdot H(s)$ Closed-loop: $\dfrac{G(s)}{1+G(s)\cdot H(s)}$

Advantages of the Nyquist Stability Criterion:

- Simple graphical procedure to determine closed-loop stability from open-loop frequency response.
- Relative stability (transient response) of closed loop system can also be easily obtained.
- The frequency response of the open-loop system can be easily obtained from measurements

Preview

Mathematical Background

- Mapping theorem
- Nyquist path

Nyquist stability criterion

$$Z=N+P$$

Z: Number of zeros of (1+H(s)G(s)) in the right half plane = number of unstable poles of the closed-loop system
N: Number of clockwise encirclements of the point $-1+j0$
P: Number of poles of G(s)H(s) in the right half plane

Application of the Nyquist Stability Criterion

- Sketch the Nyquist plot for $\omega \in (0^+,+\infty)$
- Extend to $\omega \in (0^-,-\infty)$
- Extend to $\omega \in (0^-,0^+)$
- Apply the stability criterion (find N and P and compute Z).

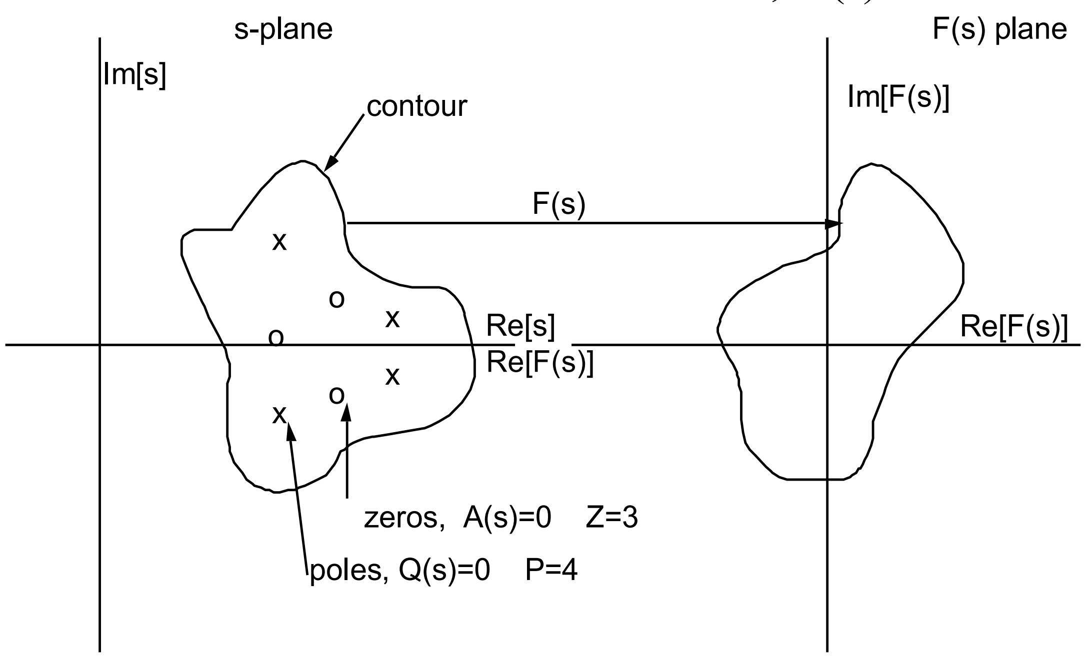
Mapping Theorem

The total number **N** of clockwise encirclements of the origin of the F(s) plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to **Z – P.**

where $F(s) = \dfrac{A(s)}{Q(s)}$

P: Number of poles, Q(s) = 0

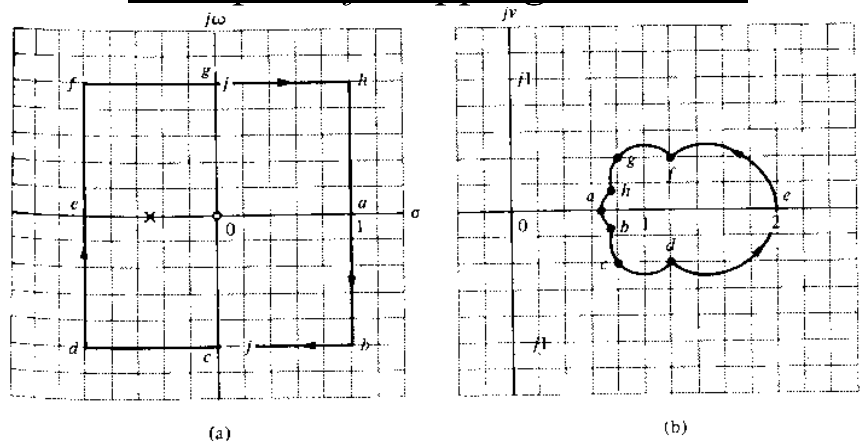
Z: Number of zeros, A(s) = 0



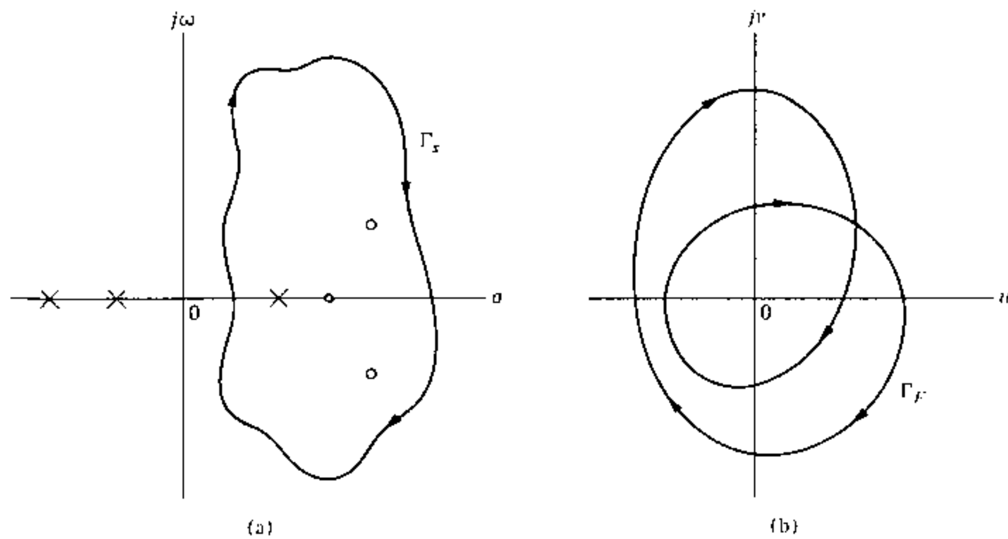
N=Z – P = 3 – 4 =-1

Clockwise contour in s-plane → Counterclockwise contour in F(s)-plane

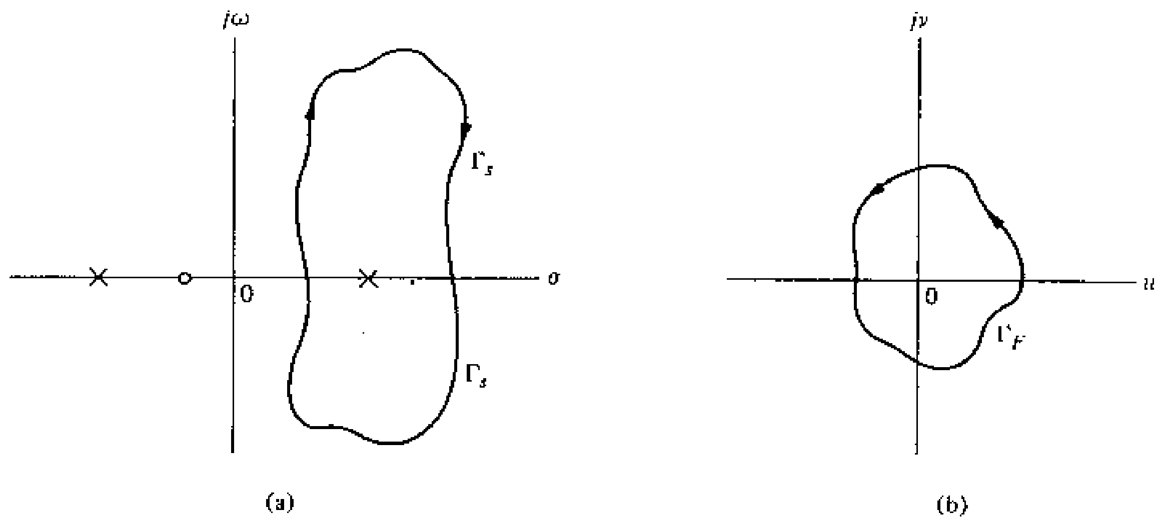
Examples of Mapping Theorem:



Mapping for F(s) = s/(s+0.5), (Z = P =1, N = Z – P = 0)

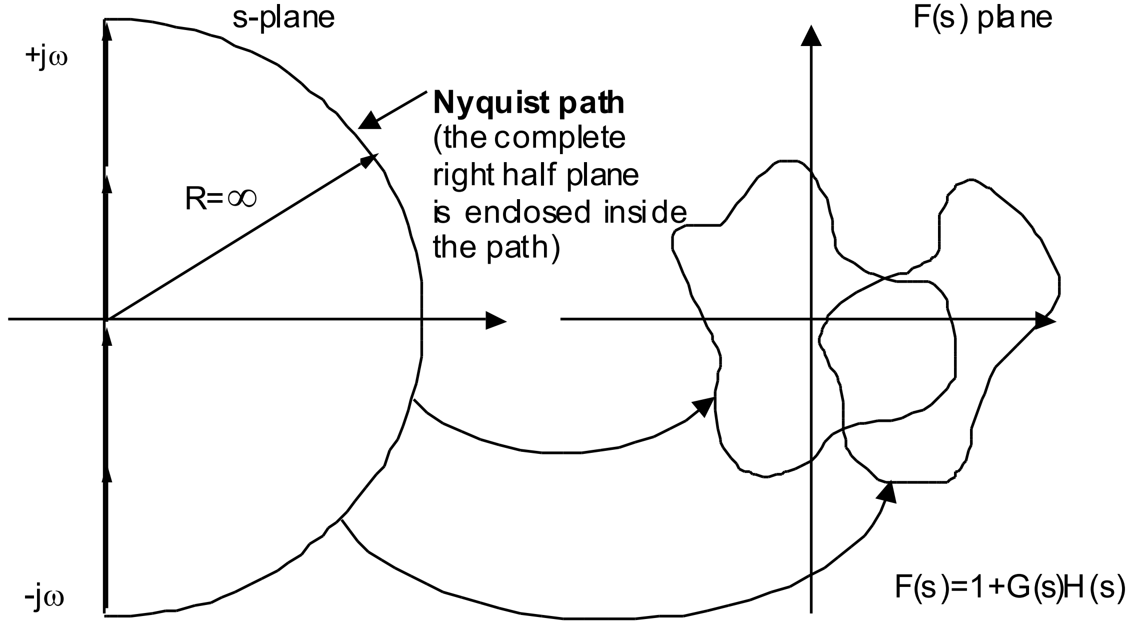


Example of Mapping theorem (N = Z – P = 2).



Example of Mapping theorem (N = Z – P = - 1).

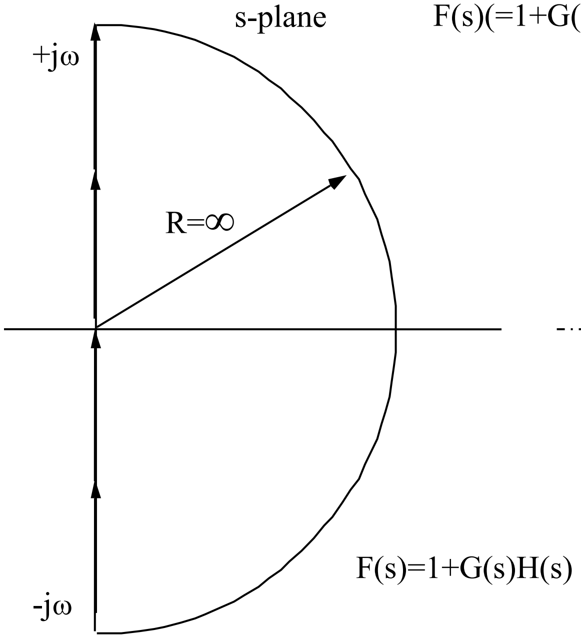
Application of the mapping theorem to stability analysis:

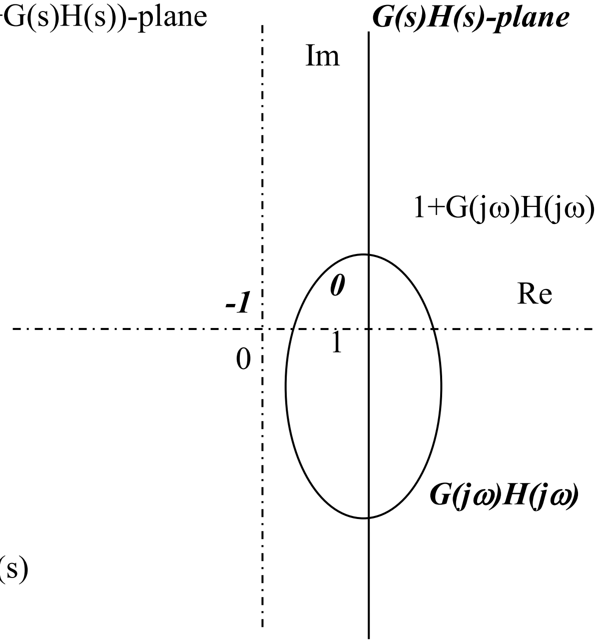


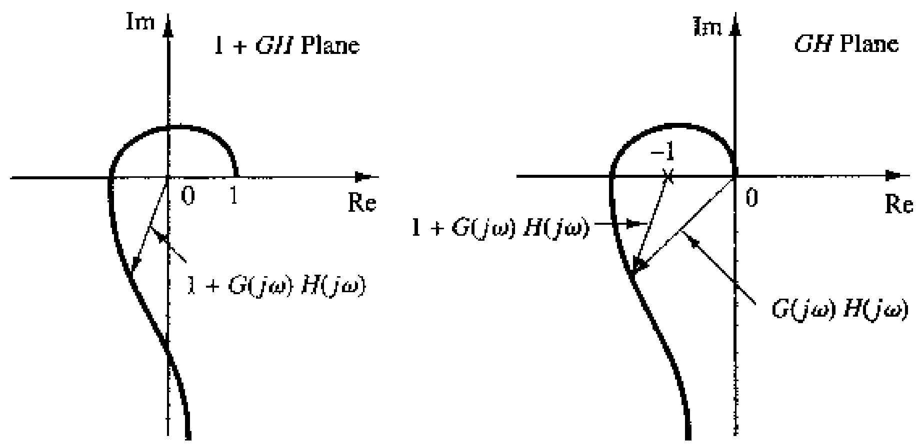
Mapping theorem: The number of clockwise encirclements of the origin is equal to the difference between the zeros and poles of F(s) = 1+ G(s)H(s).

Zeros of F(s) = **Z** = poles of closed-loop system
Poles of F(s) = **P** = poles of open-loop system

Consider the frequency response of an open-loop system: $G(j\omega)H(j\omega)$

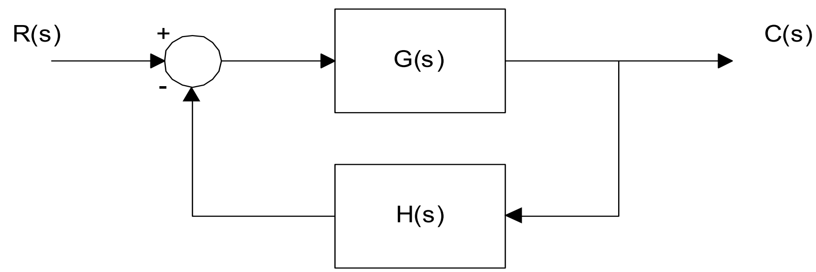






Nyquist stability criterion

Consider the Polar Plot of $G(j\omega)H(j\omega)$



Then the Nyquist stability criterion for the stability of the closed-loop system states:

$$\mathbf{Z = N + P}$$

Z: Number of zeros of **1+H(s)G(s)** in the right half s-plane = number of poles of closed-loop system in right half s-plane.

N: Number of clockwise encirclements of the point **–1+j0** by the open-loop polar plot (when tracing from $\omega = -\infty$ to $\omega = +\infty$).

P: Number of poles of **G(s)H(s)** in the right half s-plane

Thus: if $\mathbf{Z} = 0 \rightarrow$ closed-loop system is stable

if $\mathbf{Z} > 0 \rightarrow$ closed-loop system has Z unstable poles

if $\mathbf{Z} < 0 \rightarrow$ impossible, a mistake has been made

Alternative form for the Nyquist stability criterion:

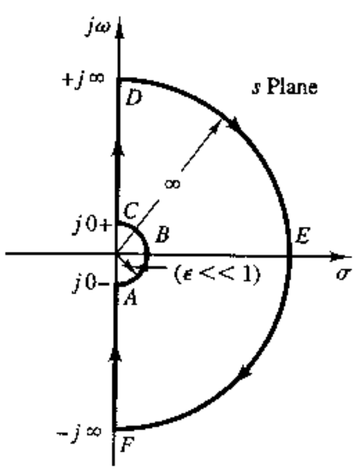
Special case: G(s)H(s) has no poles or zeros on the $j\omega$ axis:

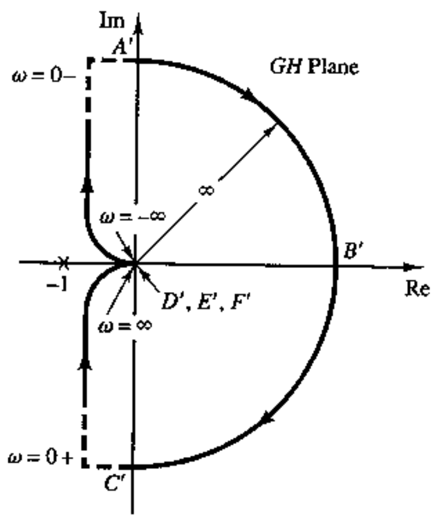
If the open-loops system G(s)H(s) has k poles in the right half s-plane, then the closed-loop system is stable if and only if the G(s)H(s) locus for a representative point s tracing the modified Nyquist path, encircles the $-1+j0$ point k times in the counterclockwise direction.

Application of the Nyquist Stability Criterion

1. Sketch the Frequency Response of $G(j\omega)H(j\omega)$ for ω : $(-\infty,+\infty)$

- $\omega : (0^{+},+\infty)$: using the rules for polar plots discussed earlier
- $\omega : (0^{-},-\infty)$: $G(-j\omega)H(-j\omega)$ is symmetric with $G(j\omega)H(j\omega)$ (real axis is symmetry axis)
- $\omega : (0^{-},0^{+})$: Depends on the poles of $G(s)H(s)$ at the origin : $G(s)\cdot H(s) = \dfrac{(\ldots)}{s^{\lambda}(\ldots)}$





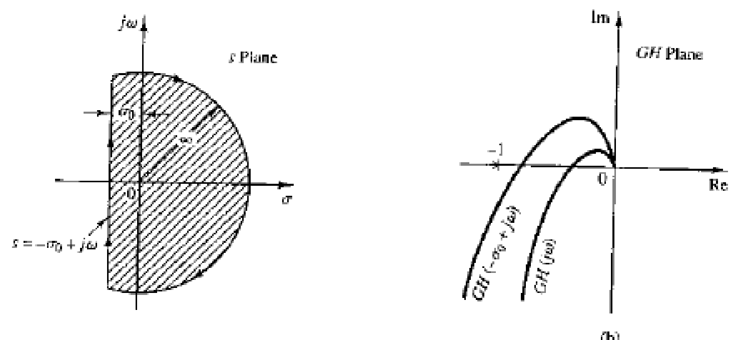
If $G(s)H(s)$ involves a factor $\dfrac{1}{s^{\lambda}}$, then the plot of $G(j\omega)H(j\omega)$, for ω between 0^{-} and 0^{+}, has λ **clockwise semicircles** of infinite radius around the origin in the GH plane.

These semicircles correspond to a representative point s moving along the Nyquist path with a semicircle of radius ε around the origin in the s plane.

2. Count the number of <u>clockwise encirclements</u> of the point $-1+j0$ and $Z = Z = N + P$

Relative Stability

Consider a modified Nyquist path which ensures that the closed-loop system has no poles with real part larger than $-\sigma_0$:



Another possible modified Nyquist path:

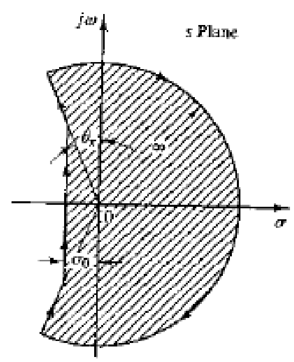


Figure 8–69
Modified Nyquist path.

PHASE AND GAIN MARGINS

A measure for relative stability of the closed-loop system is how close *G(jω)*, the frequency response of the open-loop system, comes to –1+j0 point.
This is represented by **phase and gain margins**.

Phase margin: $\gamma = 180° + \angle G(j\omega_o) = 180° + \phi$

The amount of additional phase lag at the *Gain Crossover Frequency* ω_o required to bring the system to the verge of instability.

Gain Crossover Frequency: ω_o where $\left|G(j\omega_o)\right| = 1$

Gain margin: $K_g = \dfrac{1}{\left|G(j\omega_1)\right|}$ In dB: $K_g \text{ in dB} = -20\log\left|G(j\omega_1)\right|$

The reciprocal of the magnitude $\left|G(j\omega_1)\right|$ at the *Phase Crossover Frequency* ω_1 required to bring the system to the verge of instability.

Phase Crossover Frequency: ω_1 where $\angle G(j\omega_1) = -180°$

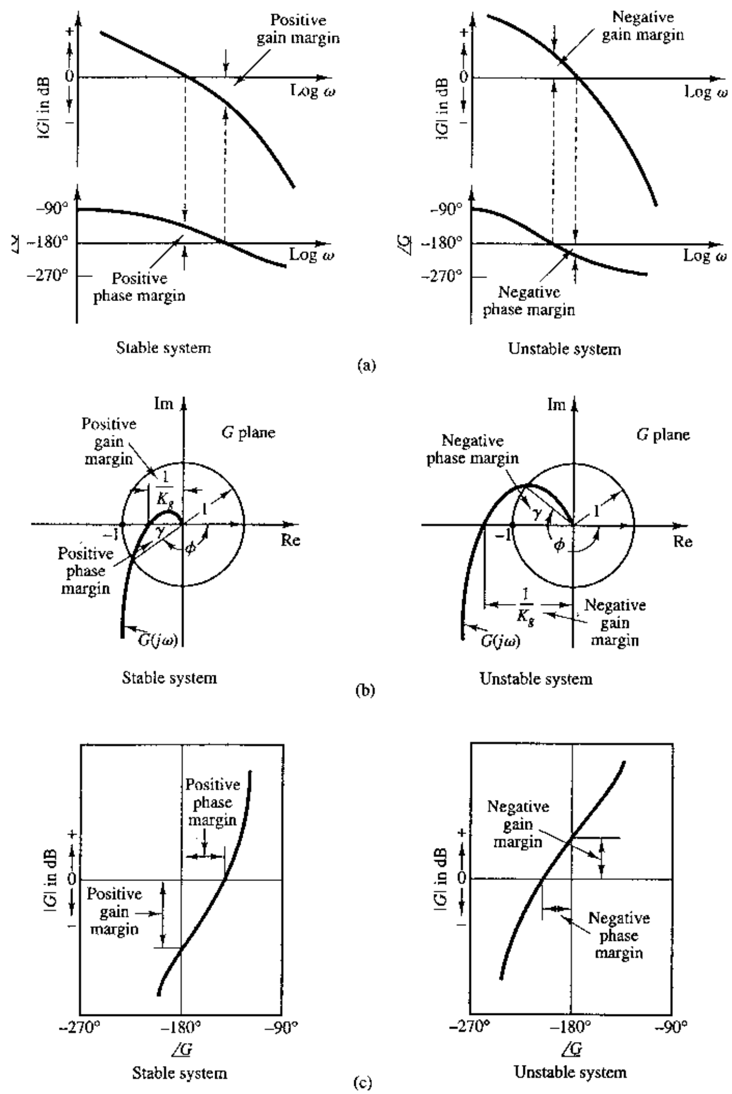


Figure: Phase and **Gain margins** of stable and unstable systems
(a) Bode diagrams; (b) Polar plots; (c) Log-magnitude-versus-phase plots.

For systems with **minimum phase** open-loop transfer function:

Phase Margin: $\gamma > 0$ → then the closed-loop system is stable

$\gamma < 0$ → then the closed-loop system is unstable

Gain Margin: K_g in dB > 0 → then the closed-loop system is stable

K_g in dB < 0 → then the closed-loop system is unstable

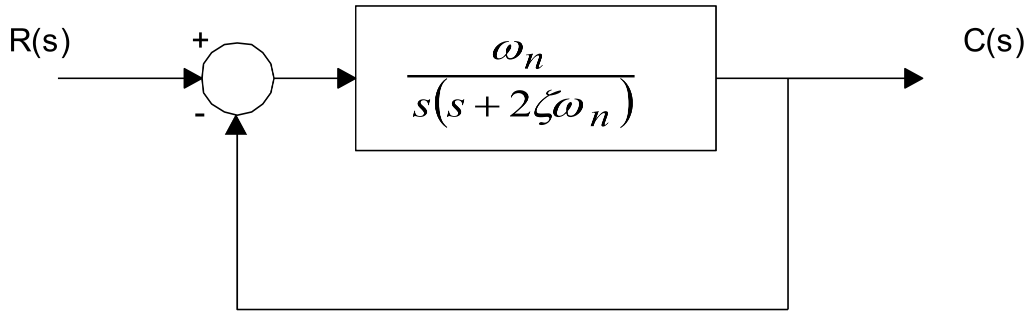
For a satisfactory transient response (good relative stability) both margins are required to above certain values. Good values for minimum phase system are:

Phase margin: 40° – 60°

Gain margin: above 6 dB

Correlation between damping ratio and frequency response for 2nd order systems

Consider



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad\qquad \frac{C(j\omega)}{R(j\omega)} = M(\omega)\cdot e^{j\alpha(\omega)}$$

The **phase margin** can be computed using $\gamma = 180^o + \angle G(j\omega)$, $G(j\omega)$ is the open loop

$$|G(j\omega)| = \left| \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)} \right|$$ becomes unity for $\omega_1 = \omega_n \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$

(ω_1 = Gain crossover frequency)

and $$\gamma = \tan^{-1}\left[\frac{2\zeta\omega_n}{\omega_1}\right] = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right]$$ → γ depends only on ζ

Observations:

Second order systems:

The following depends only on the damping ratio ζ:

- Resonant Peak $M_r = \max|G(j\omega)|$ (see page C30)
- Phase margin
- Overshoot of step response

→ By observing one of them (such as M_r in frequency response) we can deduct the others.

Resonant frequency ω_r (see page C30) is related to ω_n and therefore, also to the rise time t_r.

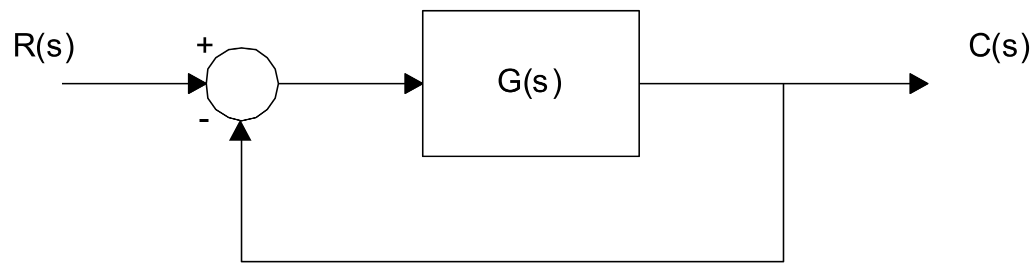
→ When ω_r increases (~bandwith increases), the rise time t_r decreases, i.e the system becomes faster.

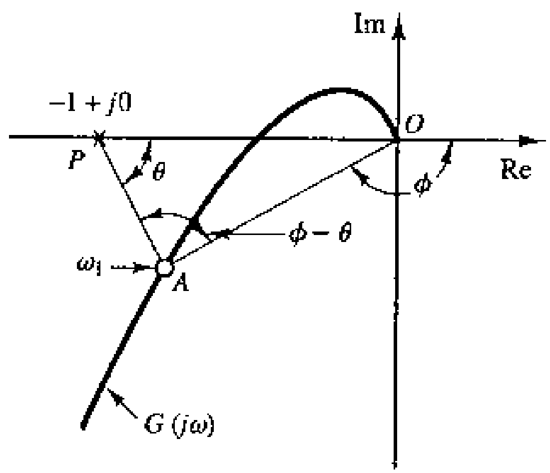
Higher order systems: If the system has a pair of complex conjugate poles as dominant poles, then these observations can be extended to them.

CLOSED-LOOP FREQUENCY RESPONSE

Sketch the Polar Plot of a stable closed-loop system $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Using the given the Polar Plot of the open-loop system: G(s)





$$\frac{|G(s)|}{|1+G(s)|} = \frac{\left|\overrightarrow{OA}\right|}{\left|\overrightarrow{PA}\right|}$$

Closed-Loop Frequency Response: $$\frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}=M\cdot e^{j\alpha}$$

Open-Loop Frequency Response: $$G(j\omega)=X+jY$$

Constant Magnitude Loci: $$M=\frac{|X+jY|}{|1+X+jY|}=const$$

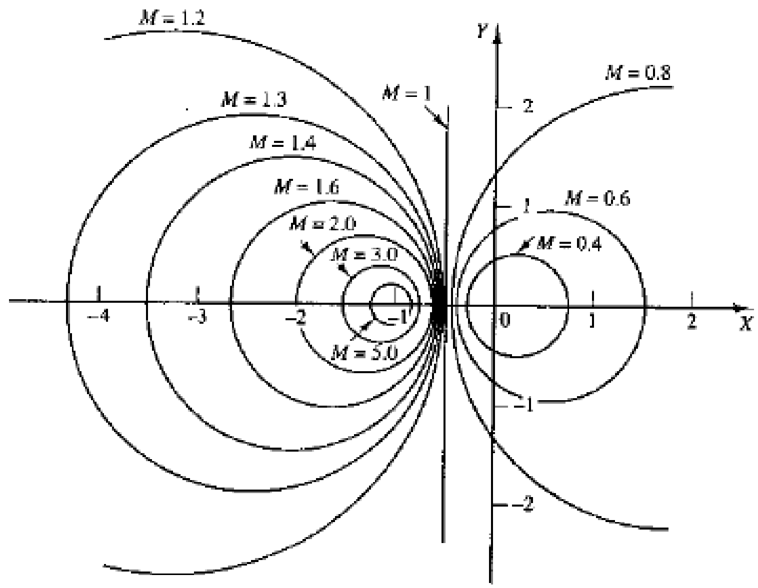
$$\left(X+\frac{M^2}{M^2-1}\right)^2+Y^2=\frac{M^2}{\left(M^2-1\right)^2}$$

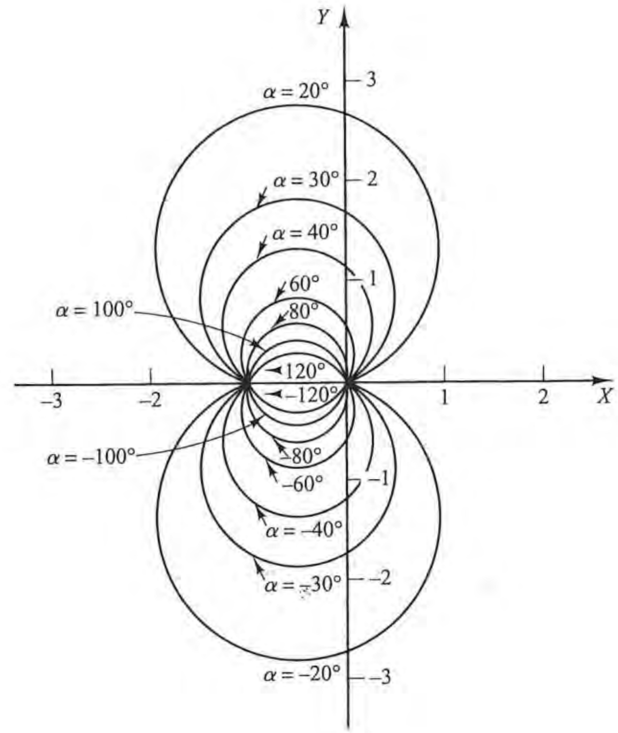
Constant Phase-Angle Loci: $$\angle e^{j\alpha}=\angle\frac{X+jY}{1+X+jY}=const$$

$$\left(X+\tfrac{1}{2}\right)^2+\left(Y+\tfrac{1}{2N}\right)^2=\frac{1}{4}+\left(\frac{1}{2N}\right)^2,\quad N=\tan\alpha$$

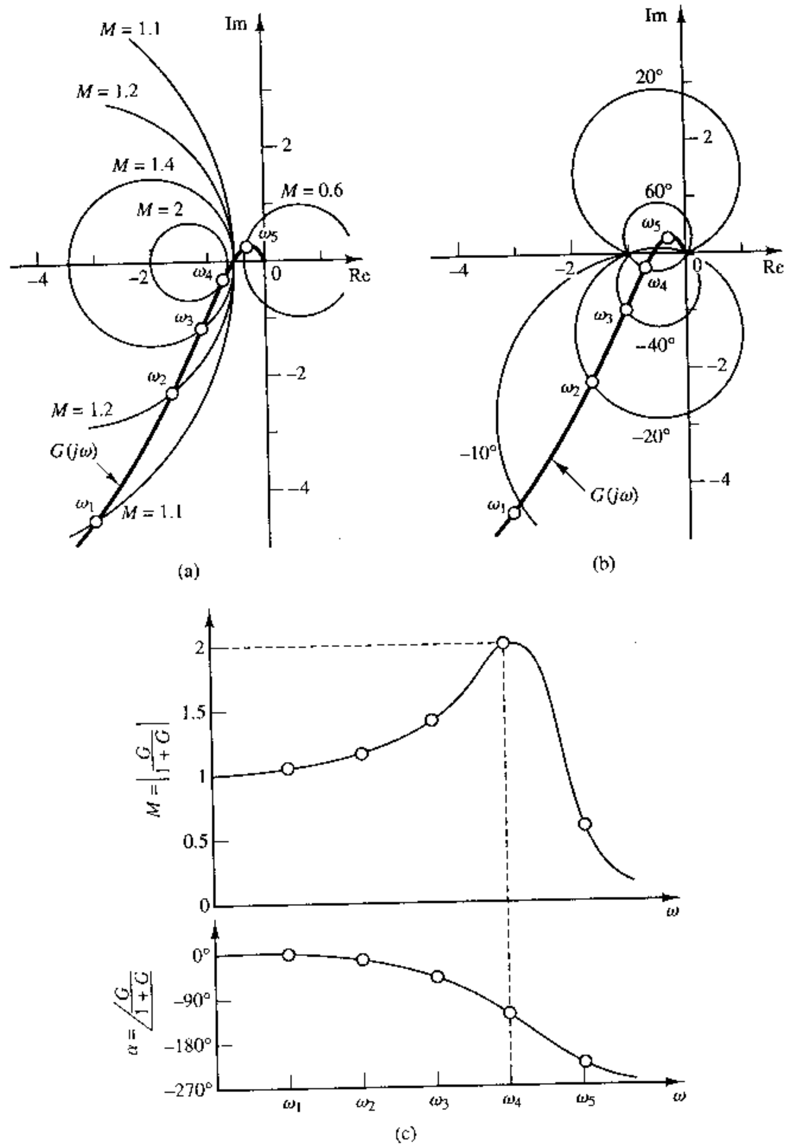
Figures: A family of constant M circles.

A family of constant M circles





Example: Obtain closed loop Polar Plot from open-loop Polar Plot



EXPERIMENTAL DETERMINATION OF TRANSFER FUNCTION

- Derivation of mathematical model is often difficult and involves approximation.

- Frequency response can be obtained using sinusoidal signal generators.

  Measure the output and obtain:
  - Magnitudes (quite accurate)
  - Phase (not as accurate)

Use the Magnitude data and asymptotes to find:

- Type and error coefficients
- Corner frequencies
- Orders of numerator and denominator
- If second order terms are involved, ζ is obtained from the resonant peak.

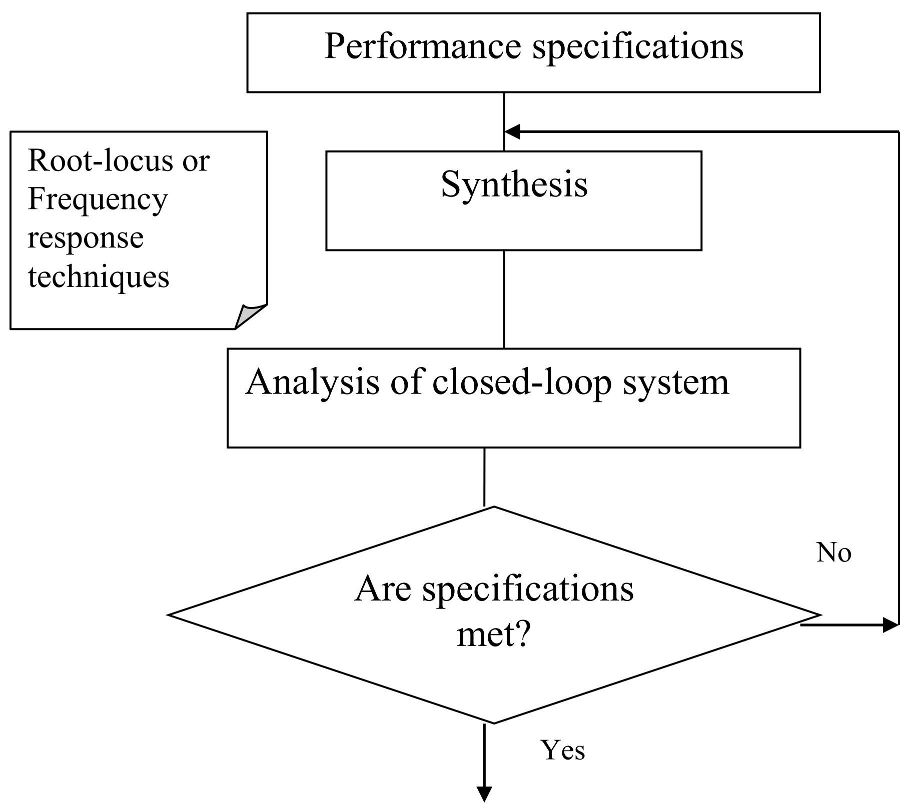
Use phase to determine if system is minimum phase or not:

Minimum phase: $\omega \rightarrow \infty$ phase = -90 (n - m)
(n-m) difference in the order of denominator and numerator.

CONTROLLER DESIGN TECHNIQUES

Performance specifications for the closed-loop system:

- Stability
- Transient response → T_s, M_s (settling time, overshoot) or phase and gain margins
- Steady-state response → e_{ss} (steady state error)

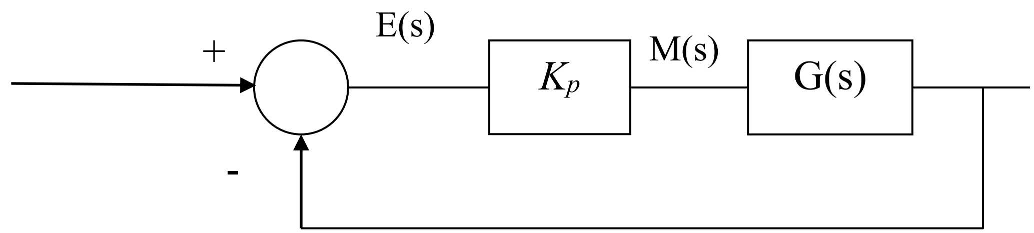


Basic Controls

1. Proportional Control

$$\frac{M(s)}{E(s)} = K_p$$

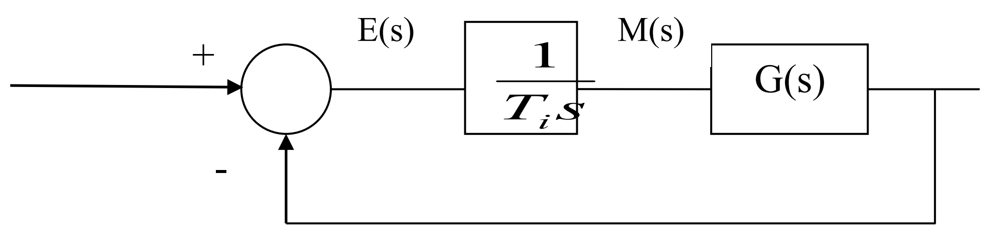
$$m(t) = K_p \cdot e(t)$$



2. Integral Control

$$\frac{M(s)}{E(s)} = \frac{1}{T_i s}$$

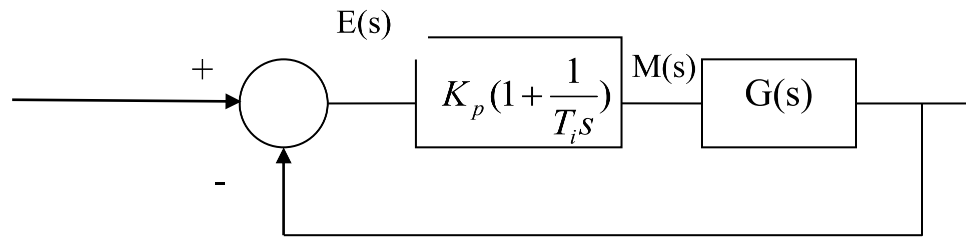
$$m(t) = \frac{1}{T_i}\int e(t)dt$$



Integral control adds a pole at the origin for the open-loop:

- Type of system increased, better steady-state performance.
- Root-locus is "pulled" to the right tending to lower the system's relative stability.

3. Proportional + Integral Control

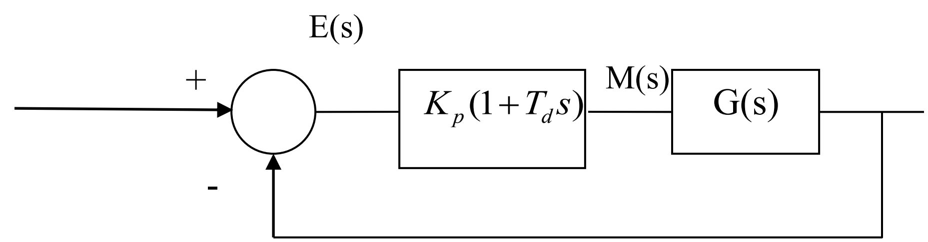


$$\frac{M(s)}{E(s)} = K_p(1+\frac{1}{T_i s}) = \frac{K_p(T_i s+1)}{T_i s}$$

$$m(t) = K_p\left[e(t)+\frac{1}{T_i}\int e(t)dt\right]$$

A pole at the origin and a zero at $-\frac{1}{T_i}$ are added.

4. Proportional + Derivative Control

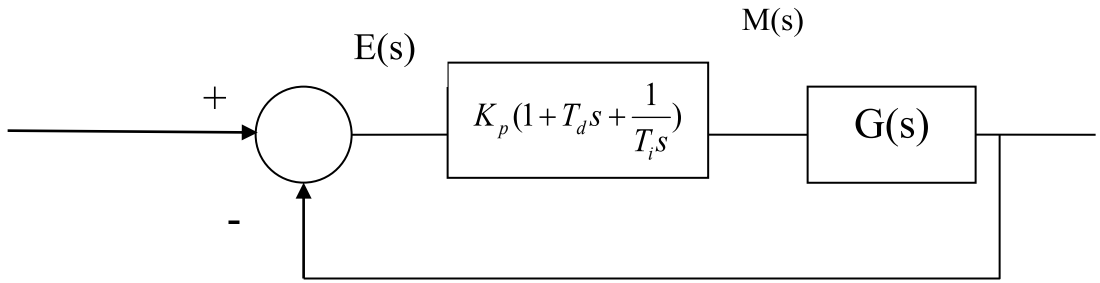


$$\frac{M(s)}{E(s)} = K_p(1+T_d s)$$

$$m(t) = K_p\left[e(t)+T_d\frac{de(t)}{dt}\right]$$

- Root-locus is “pulled” to the left, system becomes more stable and response is speed up.
- Differentiation makes the system sensitive to noise.

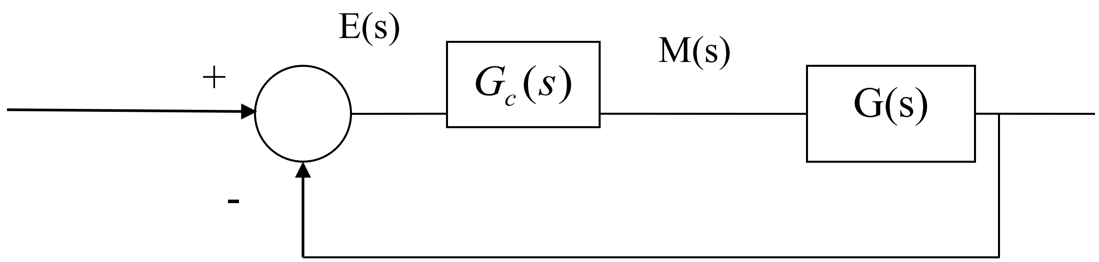
5. Proportional + Derivative + Integral (PID) Control



$$\frac{M(s)}{E(s)} = K_p(1+T_d s+\frac{1}{T_i s}) \qquad m(t) = K_p\left[e(t)+T_d\frac{de(t)}{dt}+\frac{1}{T_i}\int e(t)dt\right]$$

- More than 50% of industrial controls are PID.
- More than 80% in process control industry.
- When G(s) of the system is not known, then initial values for K_p, T_i and T_d can be obtained experimentally and than fine-tuned to give the desired response (Ziegler-Nichols).

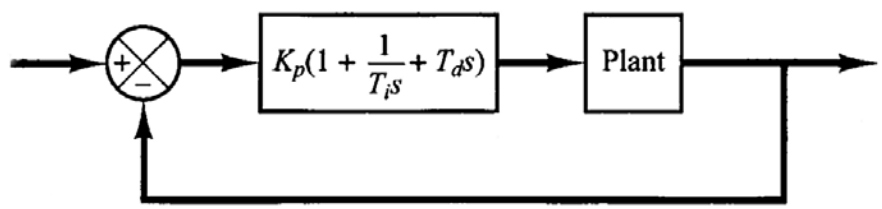
6. Feed-forward compensator



Design $G_c(s)$ can be done using Root-Locus or Frequency Response techniques.

Ziegler–Nichols Tuning Rules for PID Controllers

Given a plant with unknown transfer function,

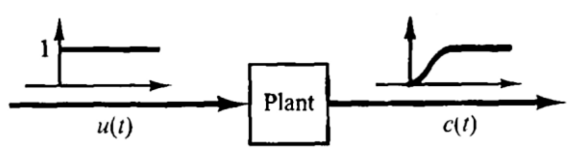


the *Ziegler–Nichols Tuning Rules* can be used to obtain values for K_p, T_i and T_d

First Method

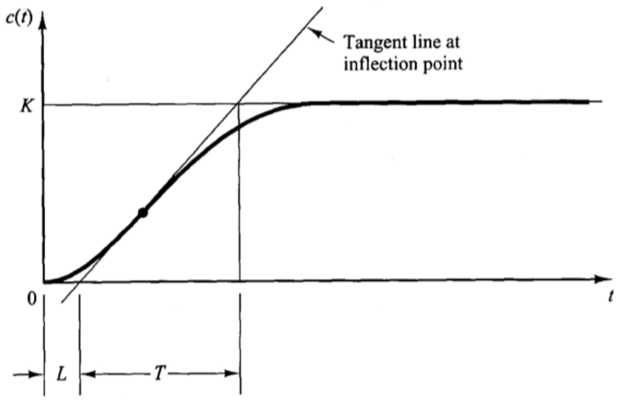
Assumption: Plant does not have integrators, nor dominant complex conjugate poles

- Obtain the Step Response of the Plant



- From the step response obtain K, L and T, i.e. model the plant as:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1}$$



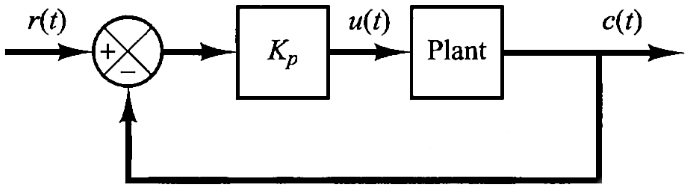
- The *Ziegler–Nichols tuning rules* suggest:

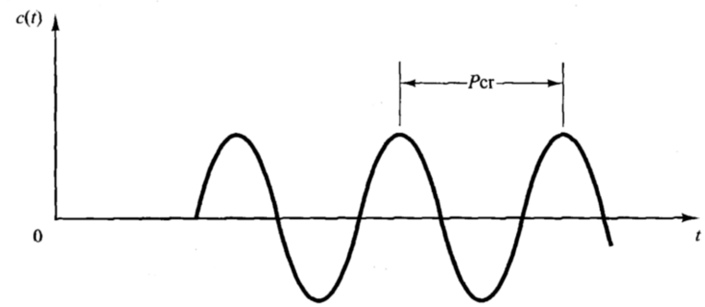
Type of Contoller	K_p	T_i	T_d
P	T / L	n/a	n/a
PI	$0.9\, T / L$	$L / 0.3$	n/a
PID	$1.2\, T / L$	$2\, L$	$0.5\, L$

→ The PID controller has:
a pole at $s=0$ and double zeros at $s = -1/L$

Second Method

- Determine experimentally the K_{cr}, the critical gain K_p and P_{cr}, the corresponding period at this gain, for the closed loop system with only proportional control.





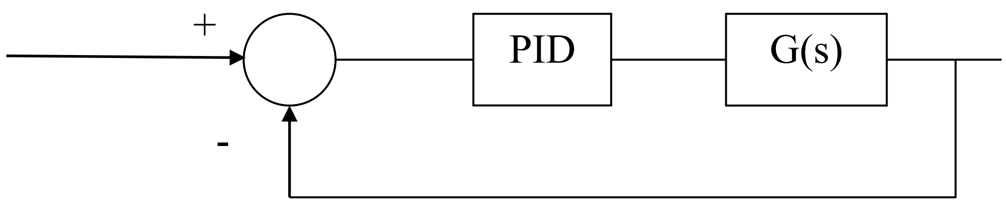
- The *Ziegler–Nichols tuning rules* suggest:

Type of Contoller	K_p	T_i	T_d
P	$0.5\ K_{cr}$	n/a	n/a
PI	$0.45\ K_{cr}$	$P_{cr} / 1.2$	n/a
PID	$0.6\ K_{cr}$	$0.5\ P_{cr}$	$0.125\ P_{cr}$

→ The PID controller has:
a pole at $s = 0$ and double zeros at $s = -4/P_{cr}$

Example:

Use *Ziegler–Nichols tuning rules* to find values for the parameters of the PID controller:



where

$$G(s) = \frac{1}{s(s+1)(1+5)}$$

Since the plant has an integrator, the second method will be used.

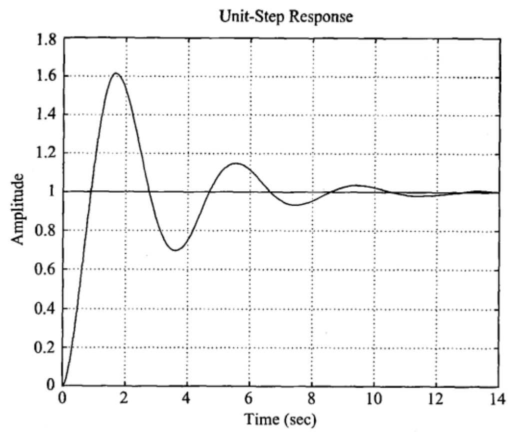
Using the root locus of *G(s)* we obtain that the closed loop system will have poles on the imaginary axis for

$K_{cr} = 30$ at $\omega = \sqrt{5}$ and $P_{cr} = \frac{2\pi}{\omega} = 2.8099$

This leads according to the table at:

$$K_p = 0.6K_{cr} = 18$$
$$T_i = 0.5P_{cr} = 1.405$$
$$T_d = 0.125P_{cr} = 0.35124$$

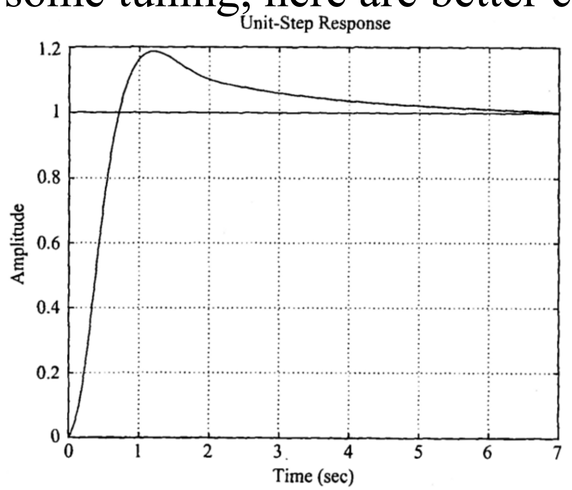
The resulting closed-loop step response is:



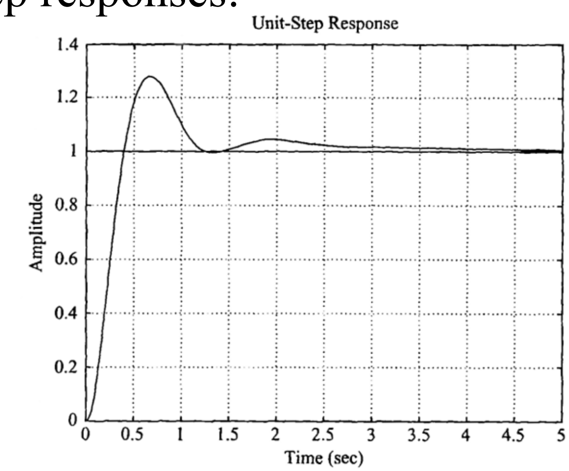
Controller parameters:
$K_p=18,\ T_i=1.405,\ T_d=0.35124$
(double zero at -1.4235)

Overshoot is 62% (too much).
Note $K_v=\infty$ (type-2 system).

After some tuning, here are better closed-loop step responses:



$K_p=18,\ T_i=3.077,\ T_d=0.7692$
(double zero moved to -0.65)



$K_p=39.42,\ T_i=3.077,\ T_d=0.7692$

Frequency response approach to compensator design

Information about the *performance of the closed-loop system*, obtained from the *open-loop frequency response*:

- *Low frequency* region indicates the steady-state behavior.
- *Medium frequency* (around -1 in polar plot, around gain and phase crossover frequencies in Bode plots) indicates relative stability.
- *High frequency* region indicates complexity.

Requirements on open-loop frequency response

- The gain at low frequency should be large enough to give a high value for error constants.
- At medium frequencies the phase and gain margins should be large enough.
- At high frequencies, the gain should be attenuated as rapidly as possible to minimize noise effects.

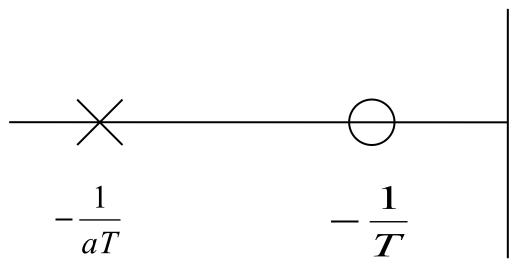
Compensators

- *lead*:improves the transient response.
- *lag*: improves the steady-state performance at the expense of slower settling time.
- *lead-lag*: combines both

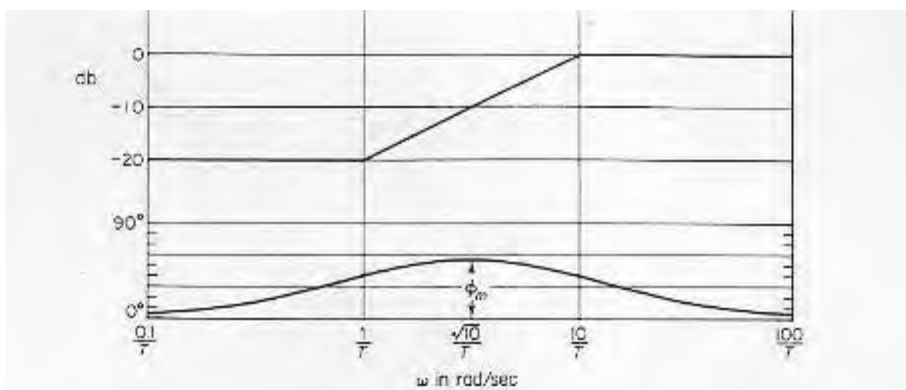
Lead compensators:

$$G_c(s) = K_c a \frac{Ts+1}{aTs+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{aT}} \quad T > 0 \quad \text{and} \quad 0 < \alpha < 1$$

- Poles and zeros of the lead compensator:



- Frequency response of $G_c(j\omega)$:



The maximum phase-lead angle ϕ_m occurs at ω_m, where:

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

and

$$\log\omega_m = \frac{1}{2}\left[\log T + \log\frac{1}{aT}\right] \rightarrow \omega_m = \frac{1}{\sqrt{a}\,T}$$

Since

$$\left|\frac{1+j\omega T}{1+j\omega aT}\right|_{\omega=\omega_m} = \frac{1}{\sqrt{a}}$$

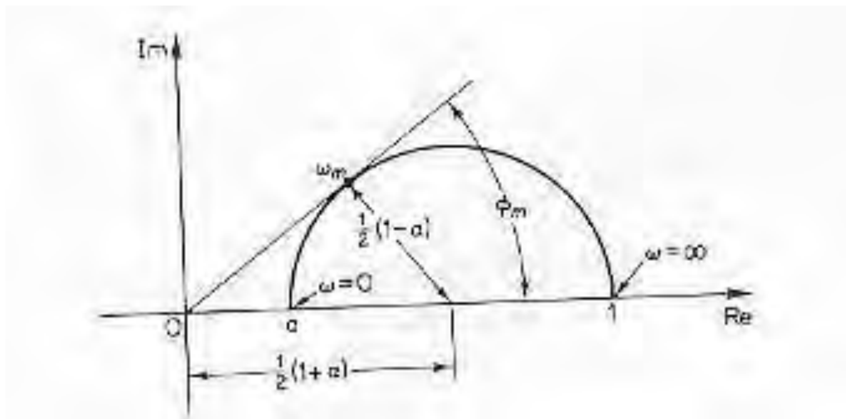
the magnitude of $G_c(j\omega)$ at ω_m is given by:

$$\left|G_c(j\omega_m)\right| = K_c\sqrt{a}$$

Polar plot of a lead compensator:

$$\frac{a(j\omega T + 1)}{(j\omega a T + 1)} \qquad \text{where} \quad 0 < a < 1$$

is given by



General effect of lead compensator:

- Addition of phase lead near gain crossover frequency.
- Increase of gain at higher frequencies.
- Increase of gain crossover frequency (bandwidth).

Lead compensation based on the frequency response

Procedure:

1. Determine the compensator gain $K_c\alpha$ satisfying the given error constant.
2. Determined the additional phase lead ϕ_m required (+ 10%~15%) for the gain adjusted ($K_c\alpha G(s)$) open-loop system.
3. Obtain α from $\sin\phi_m = \dfrac{1-a}{1+a}$
4. Find the new gain cross over frequency ω_c from

$$20\log\left[K_c a\left|G\left(j\omega_c\right)\right|\right] = 10\log a$$

5. Find T from ω_c and transfer function of $G_c(s)$

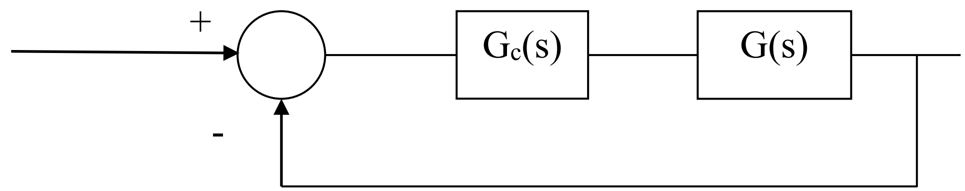
$$T = \frac{1}{\sqrt{a}\,\omega_c}$$

and

$$G_c(s) = K_c a\frac{Ts+1}{aTs+1}$$

Example:

Consider



where $G(s) = \dfrac{4}{s(s+2)}$

Performance requirements for the system:

Steady-state: $K_v = 20$
Transient response: phase margin >50°
gain margin >10 dB

Analysis of the system with $G_c(s) = K$

For $K_v = 20 \rightarrow K = 10$

This leads to: phase margin ≈ 17°
gain margin ≈ +∞ dB

Design of a lead compensator:

$$G_c(s) = K_c a \frac{Ts+1}{aTs+1}$$

1. $K_v = \lim_{s\to 0}[sG_c(s)G(s)] = \frac{4K_c a}{2} = 2K_c a = 20$ → $K_c\alpha = 10$

2. From the Bode plot of $K_c\alpha G(j\omega)$, we obtain that the additional phase-lead required is: 50° - 17° = 33°.

   → we choose 38° (~33° + 15%)

3. $\sin\phi_m = \sin 38^\circ = \frac{1-a}{1+a}$ → $\alpha = 0.24$

4. Since for ω_m, the frequency with the maximum phase-lead angle, we have:

$$\left|\frac{1+j\omega_m T}{1+j\omega_m aT}\right| = \frac{1}{\sqrt{a}} \quad \rightarrow \quad 10\log(a) = 6.2\, dB$$

we choose ω_c, the new gain crossover frequency so that

$$\omega_m = \omega_c \quad \text{and} \quad \left|G_c(s)G(s)\right|_{s=j\omega_c} = 1$$

This gives that:

$$\left| K_c a \frac{1+j\omega_m T}{1+j\omega_m aT} \frac{4}{j\omega_c\left(j\omega_c+2\right)} \right| = 1$$

and using $\omega_m = \omega_c$; $K_c\alpha$=10 and the equation for $1/\sqrt{a}$

$$\left| \frac{10}{\sqrt{a}} \frac{4}{j\omega_c\left(j\omega_c+2\right)} \right| = 1$$

From the Bode plot of $K_c\alpha G(j\omega)$ we obtain that

$$20\log\left[\left|\frac{40}{j\omega_c\left(j\omega_c+2\right)}\right|\right] = -6.2dB \quad \text{at } \omega_c = 9 \text{ rad/sec}$$

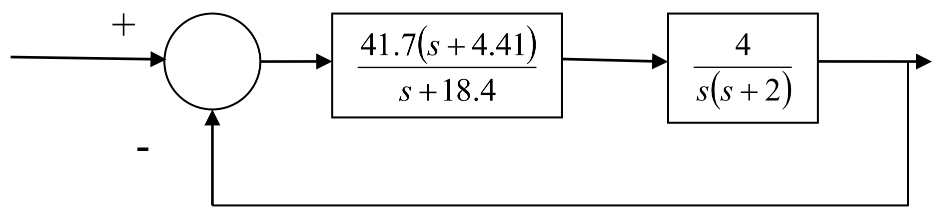
5. This implies for T

$$\omega_c = \frac{1}{\sqrt{a}T} = \frac{1}{\sqrt{0.24}T} = 9rad/\sec \rightarrow \frac{1}{T} = 4.41$$

and

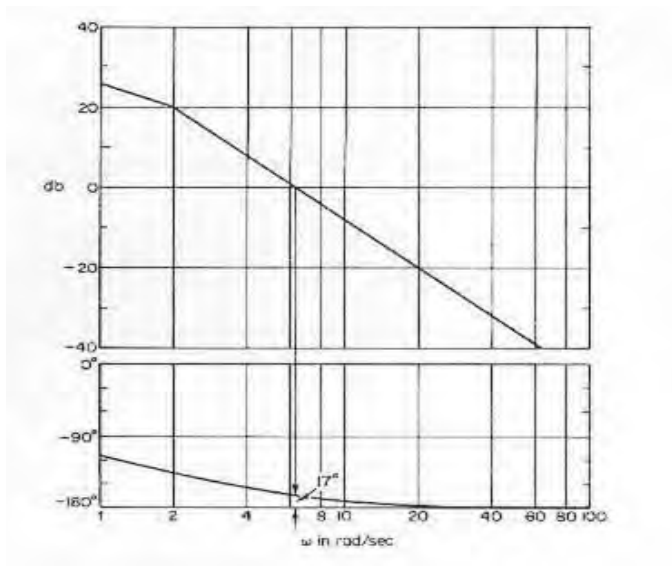
$$K_c = \frac{20}{2a} = 41.7 \rightarrow G_c(s) = 41.7\frac{s+4.41}{s+18.4}$$

The compensated system is given by:

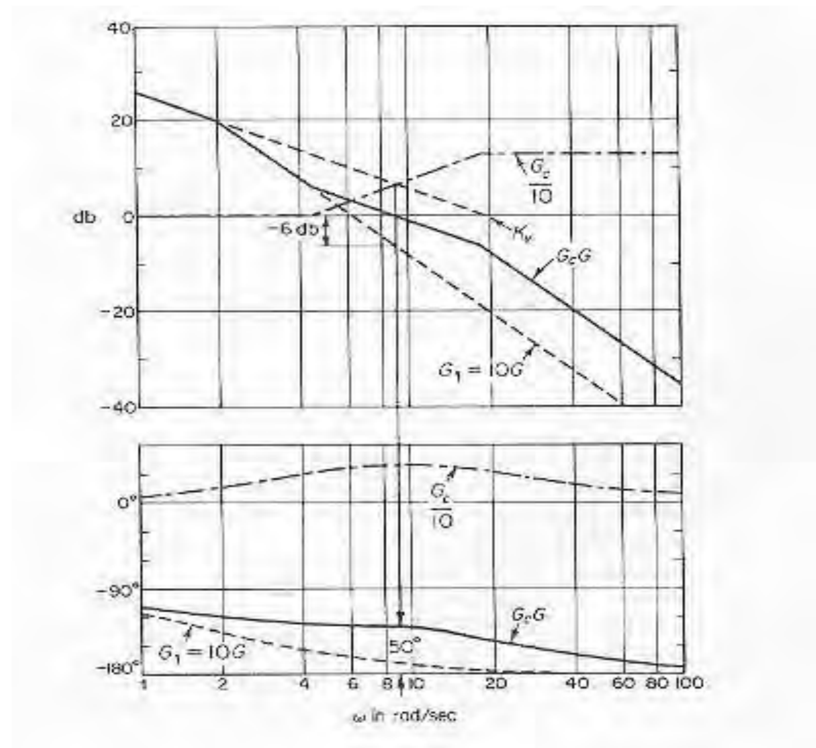


The effect of the lead compensator is:

- Phase margin: from 17° to 50° → better transient response with less overshoot.
- ω_c : from 6.3rad/sec to 9 rad/sec → the system response is faster.
- Gain margin remains ∞ .
- K_v is 20, as required → acceptable steady-state response.



Bode diagram for $K_c \cdot a \cdot G(j\omega) = \dfrac{40}{j\omega(j\omega+2)}$



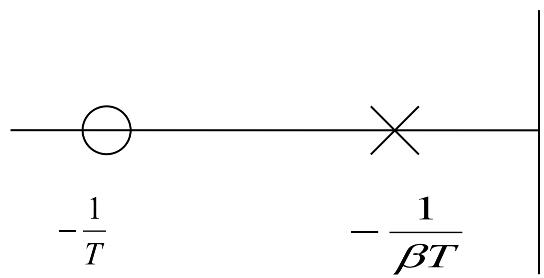
Bode diagram for the compensated system

$$G_c(j\omega)G(j\omega) = 41.7\,\frac{j\omega+4.41}{j\omega+18.4}\cdot\frac{4}{j\omega(j\omega+2)}$$

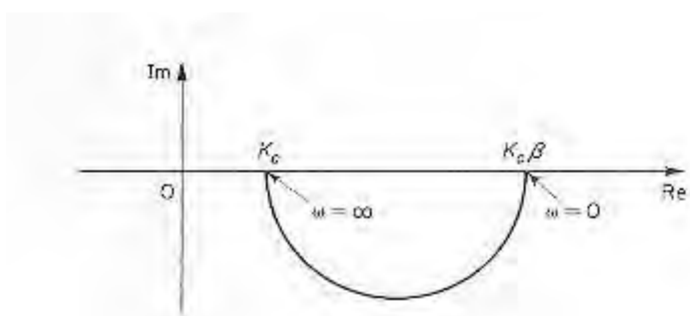
Lag compensators

$$G_c(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad T > 0\,,\ \beta > 1$$

Poles and zeros:

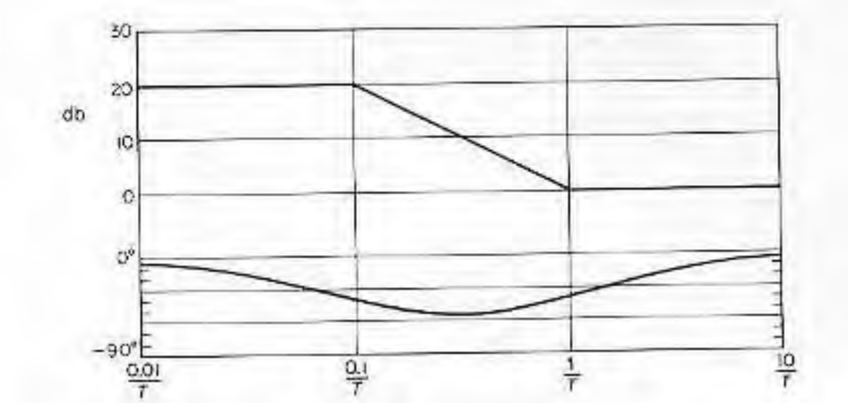


Polar plot:

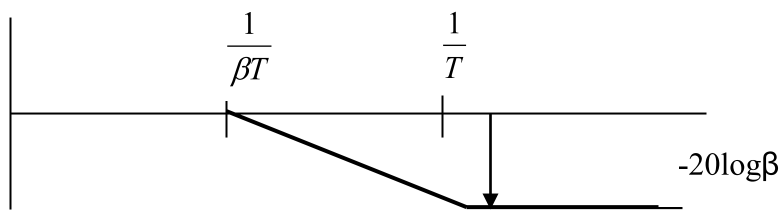


Frequency response:

Bode diagram of a lag compensator with $K_c=1$, $\beta = 10$



Magnitude of $(j\omega T+1)/(j\omega\beta T+1)$



<u>**General effect of lag compensation**</u>:

- Decrease gain at high frequencies.
- Move the gain crossover frequency lower to obtain the desired phase margin.

Lag compensation based on the frequency response

Procedure:

2. Determine the compensator gain $K_c\beta$ to satisfy the requirement for the given error constant.

3. Find the frequency point where the phase of the gain adjusted open-loop system ($K_c\beta G(s)$) is equal to -180° + the required phase margin + 5°~ 12°.

   This will be the new gain crossover frequency ω_c.

4. Choose the zero of the compensator s = -1/T at about 1 octave to 1 decade below ω_c .

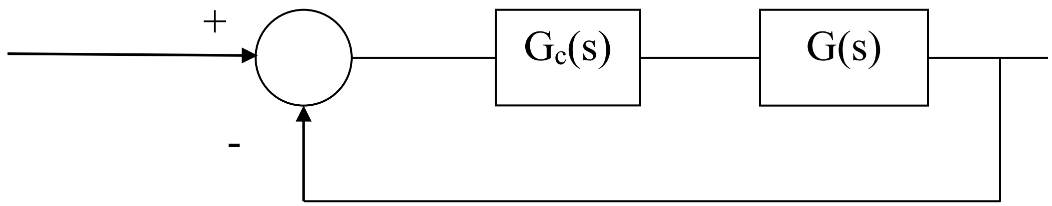
5. Determine the attenuation necessary to bring the magnitude curve down to 0dB at the new gain crossover frequency

   $$-20\log\left[K_c\beta\left|G(j\omega_c)\right|\right] = -20\log\beta \quad \rightarrow \beta$$

6. Find the transfer function $G_c(s)$.

Example:

Consider



where

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

Performance requirements for the system:

Steady state: K_v =5

Transient response: Phase margin > 40°

Gain margin > 10 dB

Analysis of the system with $G_c(s) = K$

$$K_v = \lim_{s\to 0} sKG(s) = K = 5$$

for K = 5, the closed-loop system is unstable

Design of a lag compensator

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts + 1}{\beta Ts + 1}$$

1. $K_v = \lim_{s \to 0} sG_c(s)G(s) = K_c\beta = 5$

2. Phase margin of the system *5G(s)* is -13°
   → the closed-loop system is unstable.

   From the Bode diagram of $5G(j\omega)$ we obtain that the additional required phase margin of 40° + 12° = 52° is obtained at ω = 0.5 rad/sec.

   The new gain crossover frequency will be:

   ω_c = 0.5 rad/sec

3. Place the zero of the lag compensator at ω = 1/T = 0.1 rad/sec(at about 1/5 of ω_c).

4. The magnitude of $5G(j\omega)$ at the new gain crossover frequency ω_c =0.5 rad/sec is 20 dB. In order to have ω_c as the new gain crossover frequency, the lag compensator must give an attenuation of -20db at ω_c.

   Therefore $- 20\log\beta = - 20$ dB → $\beta = 10$

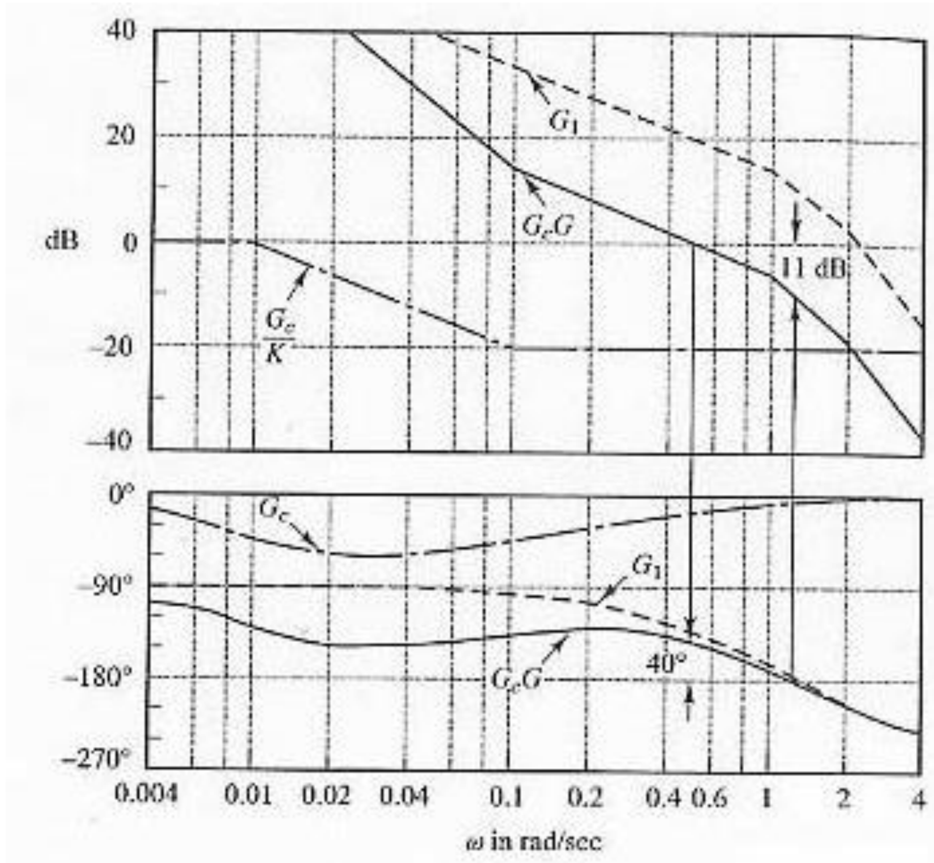
5. $K_c = \frac{5}{\beta} = 0.5$, $\quad pole: \quad \frac{1}{\beta T} = 0.01$

and

$$G_c(s) = 0.5\frac{s + 0.1}{s + 0.01}$$

The effect of the lag compensator is:

- The original unstable closed-loop system is now stable.
- The phase margin ≈ 40° → acceptable transient response.
- The gain margin ≈ 11dB → acceptable transient response.
- K_v is 5 as required → acceptable steady-state response.
- The gain at high frequencies has been decreased.



Bode diagrams for:

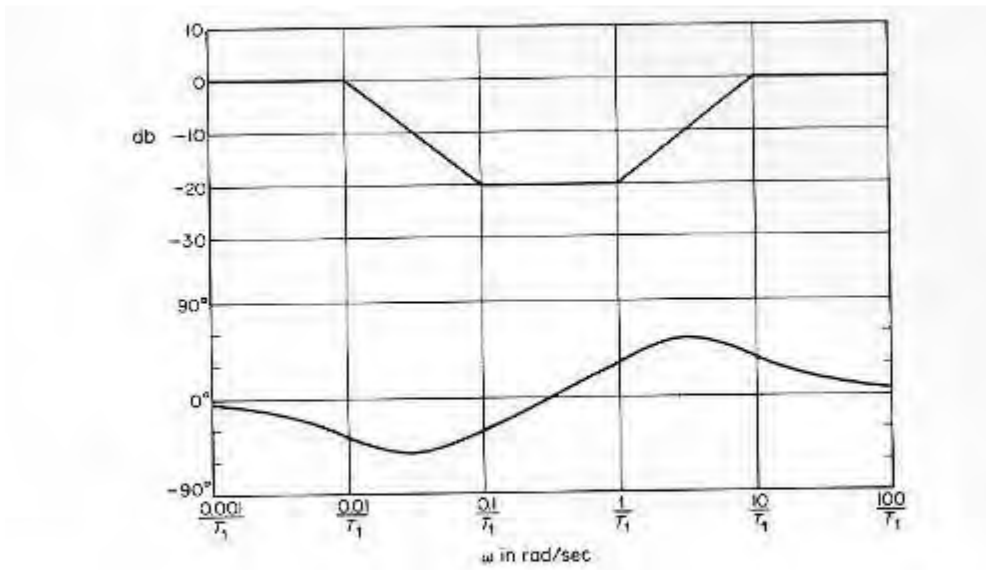
- $G_1(j\omega) = 5G(j\omega)$ (gain-adjusted $K_c\beta G(j\omega)$ open-loop transfer function),
- $G_c(j\omega)/K = G_c(j\omega)/5$ (compensator divided by gain $K_c\beta = 5$),
- $G_c(j\omega)G(j\omega)$ (compensated open-loop transfer function)

Lead-lag compensators

$$G_c(s) = K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{aT_1}} \cdot \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} = K_c \frac{\beta}{\gamma} \frac{sT_1+1}{s\frac{T_1}{\gamma}+1} \cdot \frac{sT_2+1}{s\beta T_2+1}$$

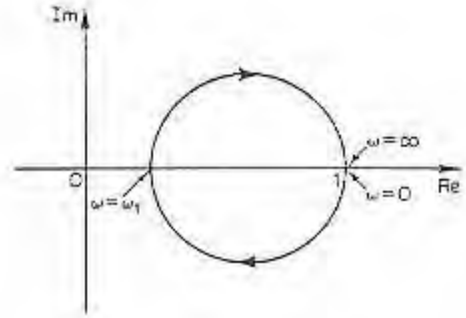
$T_1,\ T_2 > 0,\quad \beta > 1 \quad and \quad \gamma > 1$

Frequency response:



Bode diagram of a lag-lead compensator with $K_c = 1$, $\gamma = \beta = 10$ and $T_2 = 10\,T_1$

Polar Plot:



Polar plot of a lag-lead compensator with $K_c = 1$ and $\gamma = \beta$

Comparison between lead and lag compensators

Lead compensator	**Lag** compensator
o High pass	o Low pass
o Approximates derivative plus proportional control	o Approximates integral plus proportional control
o Contributes phase lead	o Attenuation at high frequencies
o Increases the gain crossover frequency	o Moves the gain-crossover frequency lower
o Increases bandwidth	o Reduces bandwidth

CONTROLLER IMPLEMENTATION

Issues:

- Analog versus digital
- Real-time systems
- Implementation errors
    - Component inaccuracies
    - Measurement and roundoff errors

Options:

- Building Specialized Hardware

  (Simple control, no need for changes, high volume)
- Microcontroller Based Systems

  (More sophisticated functions, relative inflexible, medium volume)
- Programmable Logic Controllers (PLCs)

  (Control and monitoring functions, flexible, easy to change)

PLCs

- Extensively used in industry (automotive, process control, etc)
- Capabilities:
  - Monitoring sensors
  - Switching on/off motors, valves, etc
  - Simple PID controller implementation
- Consist of processor, I/O capabilities, network interface
- Reprogrammable as control/monitoring needs change
- Programming (IEC 61131-3 Standard):
  - Function Block Diagram (FBD)
  - Ladder Diagram (LD)
  - Structured Text (ST)
  - Sequential Function Chart (SFC)

PLC s from Siemens, Allen Bradley, etc

from http://www.ab.com/programmablecontrol/plc/