Solution to Differential Equation

The Question

We are given the following differential equation:

$$2\ddot{x} + 7\dot{x} + 3x = u(t),$$

with initial conditions $x(0) = 3$ and $\dot{x}(0) = 0$, where $u(t)$ is the unit step function.

Step 1: Apply Laplace Transform

To solve this differential equation, we begin by applying the Laplace transform to both sides. The Laplace transform of a derivative is:

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0),$$
$$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0).$$

Applying the Laplace transform to the entire equation:

$$2\mathcal{L}\{\ddot{x}(t)\} + 7\mathcal{L}\{\dot{x}(t)\} + 3\mathcal{L}\{x(t)\} = \mathcal{L}\{u(t)\},$$

where $\mathcal{L}\{u(t)\} = \frac{1}{s}$.
Using the initial conditions $x(0) = 3$ and $\dot{x}(0) = 0$, we substitute into the transformed equation:

$$2(s^2X(s) - 3s) + 7(sX(s) - 3) + 3X(s) = \frac{1}{s}.$$

Step 2: Simplify the Equation

Now, simplify the Laplace-transformed equation:

$$2(s^2X(s) - 3s) + 7(sX(s) - 3) + 3X(s) = \frac{1}{s}.$$

Expanding both terms:

$$2s^2X(s) - 6s + 7sX(s) - 21 + 3X(s) = \frac{1}{s}.$$

Collect all terms involving $X(s)$ on the left-hand side:

$$(2s^2 + 7s + 3)X(s) = \frac{1}{s} + 6s + 21.$$

Step 3: Solve for $X(s)$

We now solve for $X(s)$:

$$X(s) = \frac{\frac{1}{s} + 6s + 21}{2s^2 + 7s + 3}.$$

Multiply the numerator by s to clear the fraction:

$$X(s) = \frac{1 + 6s^2 + 21s}{s(2s^2 + 7s + 3)}.$$

Step 4: Partial Fraction Decomposition

To proceed, we perform partial fraction decomposition on $X(s)$. The poles of the denominator are obtained by factoring $2s^2 + 7s + 3$. The roots of the quadratic are found using the quadratic formula:

$$s = \frac{-7 \pm \sqrt{49 - 4(2)(3)}}{2(2)} = \frac{-7 \pm 1}{4}.$$

So the poles are $s = -\frac{1}{2}$ and $s = -3$. Thus, we can express $X(s)$ as:

$$X(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{2}} + \frac{C}{s + 3}.$$

Step 5: Solve for Constants A, B, and C

To find A, B, and C, we multiply both sides of the equation by $s(s + \frac{1}{2})(s + 3)$ and match coefficients. After solving, we get the values:

$$A = \frac{1}{3}, \quad B = \frac{16}{5}, \quad C = -\frac{8}{15}.$$

Thus, we can write:

$$X(s) = \frac{1}{3s} + \frac{16}{5(s + \frac{1}{2})} - \frac{8}{15(s + 3)}.$$

Step 6: Apply Inverse Laplace Transform

Now, apply the inverse Laplace transform to each term separately:

$$\mathcal{L}^{-1}\left\{\frac{1}{3s}\right\} = \frac{1}{3}u(t),$$

$$\mathcal{L}^{-1}\left\{\frac{16}{5(s + \frac{1}{2})}\right\} = \frac{16}{5}e^{-\frac{1}{2}t},$$

$$\mathcal{L}^{-1}\left\{\frac{8}{15(s + 3)}\right\} = -\frac{8}{15}e^{-3t}.$$

Thus, the solution for $x(t)$ is:

$$x(t) = \frac{1}{3}u(t) + \frac{16}{5}e^{-\frac{1}{2}t} - \frac{8}{15}e^{-3t}, \quad t \geq 0.$$

Final Answer

The solution to the differential equation is:

$$x(t) = \frac{1}{3}u(t) + \frac{16}{5}e^{-\frac{1}{2}t} - \frac{8}{15}e^{-3t}, \quad t \geq 0.$$