Chapter 4

Continuous-Time Fourier Transform (Chapter 5)

5.1 Using the Fourier transform analysis equation, find the Fourier transform of each of the following signals:
(a) $x(t) = A\delta(t - t_0)$ where t_0 and A are real and complex constants, respectively;
(b) $x(t) = \operatorname{rect}(t - t_0)$ where t_0 is a constant;
(c) $x(t) = e^{-4t}u(t-1)$;
(d) $x(t) = 3[u(t) - u(t-2)]$; and
(e) $x(t) = e^{-|t|}$.

Solution.

(d) Let $X(\omega)$ denote the Fourier transform of $x(t)$. From the Fourier transform analysis equation, we can write

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} 3[u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{-\infty}^{\infty} [u(t) - u(t-2)]e^{-j\omega t}dt \\
&= 3\int_{0}^{2} e^{-j\omega t}dt \\
&= 3\left[\tfrac{1}{-j\omega}e^{-j\omega t}\right]\Big|_0^2 \\
&= \tfrac{3}{-j\omega}\left[e^{-j\omega t}\right]\Big|_0^2 \\
&= \tfrac{j3}{\omega}\left[e^{-j2\omega} - 1\right] \\
&= \tfrac{j3}{\omega}[e^{-j\omega}][e^{-j\omega} - e^{j\omega}] \\
&= \tfrac{j3}{\omega}e^{-j\omega}[-2j\sin\omega] \\
&= \tfrac{6}{\omega}e^{-j\omega}\sin\omega \\
&= 6e^{-j\omega}\operatorname{sinc}\omega.
\end{aligned}$$

(e) Let $X(\omega)$ denote the Fourier transform of $x(t)$. From the Fourier transform analysis equation, we have

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{-|t|} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-|t|} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t} dt + \int_{0}^{\infty} e^{(-1-j\omega)t} dt \\
&= \tfrac{1}{1-j\omega} \left[e^{(1-j\omega)t} \right]\Big|_{-\infty}^{0} - \tfrac{1}{1+j\omega} \left[e^{(-1-j\omega)t} \right]\Big|_{0}^{\infty} \\
&= \tfrac{1}{1-j\omega}[1-0] - \tfrac{1}{1+j\omega}[0-1] \\
&= \tfrac{1}{1-j\omega} + \tfrac{1}{1+j\omega} \\
&= \frac{1+j\omega+1-j\omega}{(1+j\omega)(1-j\omega)} \\
&= \frac{2}{1+\omega^2}.
\end{aligned}$$

5.2 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform of each of the signals below.
(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t} u(t+2)$;
(c) $x(t) = [\cos t] u(t)$;
(d) $x(t) = 6[u(t) - u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t \operatorname{rect}(2t)$;
(g) $x(t) = e^{-j3t} \sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = e^{-j2t} \frac{1}{3t+1}$;
(j) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1) d\tau$;
(k) $x(t) = (t+1)\sin(5t-3)$;
(l) $x(t) = (\sin 2\pi t)\delta(t - \frac{\pi}{2})$;
(m) $x(t) = e^{-jt} \frac{1}{3t-2}$;
(n) $x(t) = e^{j5t} (\cos 2t) u(t)$; and
(o) $x(t) = e^{-j2t} \operatorname{sgn}(-t-1)$.

Solution.

(c) We begin by rewriting $x(t)$ as

$$x(t) = v_1(t) v_2(t),$$

where

$$\begin{aligned}
v_1(t) &= \cos t \quad \text{and} \\
v_2(t) &= u(t).
\end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations yields

$$X(\omega) = \tfrac{1}{2\pi} V_1(\omega) * V_2(\omega),$$
$$V_1(\omega) = \pi[\delta(\omega-1)+\delta(\omega+1)], \quad \text{and}$$
$$V_2(\omega) = \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

Combining the above results, we obtain

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2\pi} V_1(\omega) * V_2(\omega) \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} V_1(\lambda) V_2(\omega-\lambda) d\lambda \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left[\delta(\lambda-1)+\delta(\lambda+1)\right] \left[\pi\delta(\omega-\lambda) + \tfrac{1}{j(\omega-\lambda)}\right] d\lambda \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left[\pi\delta(\lambda-1)\delta(\omega-\lambda) + \delta(\lambda-1)\tfrac{1}{j(\omega-\lambda)} + \pi\delta(\lambda+1)\delta(\omega-\lambda) + \delta(\lambda+1)\tfrac{1}{j(\omega-\lambda)}\right] d\lambda \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1) + \tfrac{1}{j(\omega-1)} + \pi\delta(\omega+1) + \tfrac{1}{j(\omega+1)}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1) + \pi\delta(\omega+1) - \tfrac{j}{\omega-1} - \tfrac{j}{\omega+1}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1) + \pi\delta(\omega+1) + \tfrac{-j(\omega-1)-j(\omega+1)}{\omega^2-1}\right] \\
&= \tfrac{1}{2}\left[\pi\delta(\omega-1) + \pi\delta(\omega+1) - \tfrac{j2\omega}{\omega^2-1}\right] \\
&= \tfrac{\pi}{2}\left[\delta(\omega-1)+\delta(\omega+1)\right] - \frac{j\omega}{\omega^2-1}.
\end{aligned}$$

(d) We begin by rewriting $x(t)$ as

$$x(t) = 6v_3(t),$$

where

$$v_3(t) = v_2(t/3),$$
$$v_2(t) = v_1(t-\tfrac{1}{2}), \quad \text{and}$$
$$v_1(t) = \operatorname{rect}(t).$$

Taking the Fourier transform of both sides of each of the above equations yields

$$X(\omega) = 6V_3(\omega),$$
$$V_3(\omega) = 3V_2(3\omega),$$
$$V_2(\omega) = e^{-j\omega/2} V_1(\omega), \quad \text{and}$$
$$V_1(\omega) = \operatorname{sinc} \omega/2.$$

Combining the above results, we have

$$\begin{aligned}
X(\omega) &= 6V_3(\omega) \\
&= 6(3)V_2(3\omega) \\
&= 18V_2(3\omega) \\
&= 18e^{-j3\omega/2} V_1(3\omega) \\
&= 18e^{-j3\omega/2} \operatorname{sinc} \tfrac{3\omega}{2}.
\end{aligned}$$

Alternatively, we can restate this result in a slightly different form (i.e., in terms of complex exponentials) as follows:

$$\begin{aligned}
X(\omega) &= 18e^{-j3\omega/2}\operatorname{sinc}\tfrac{3\omega}{2} \\
&= 18e^{-j3\omega/2}\tfrac{2}{3\omega}\left[\tfrac{1}{2j}\left[e^{j3\omega/2}-e^{-j3\omega/2}\right]\right] \\
&= \tfrac{6}{j\omega}[1-e^{-j3\omega}].
\end{aligned}$$

ALTERNATIVE SOLUTION. We have

$$\begin{aligned}
X(\omega) &= 6\left(\mathscr{F}\{u(t)\}-\mathscr{F}\{u(t-3)\}\right) \\
&= 6\left(\pi\delta(\omega)+\tfrac{1}{j\omega}-e^{-j3\omega}(\pi\delta(\omega)+\tfrac{1}{j\omega})\right) \\
&= 6\left(\pi\delta(\omega)+\tfrac{1}{j\omega}-\pi\delta(\omega)e^{-j3\omega}-\tfrac{1}{j\omega}e^{-j3\omega}\right) \\
&= 6\left(\pi\delta(\omega)+\tfrac{1}{j\omega}-\pi\delta(\omega)-\tfrac{1}{j\omega}e^{-j3\omega}\right) \\
&= \tfrac{6}{j\omega}(1-e^{-j3\omega}) \\
&= \tfrac{6}{j\omega}e^{-j3\omega/2}(e^{j3\omega/2}-e^{-j3\omega/2}) \\
&= \tfrac{6}{j\omega}e^{-j3\omega/2}(2j)\sin 3\omega/2 \\
&= \tfrac{12}{\omega}e^{-j3\omega/2}\sin 3\omega/2 \\
&= \tfrac{3\omega}{2}\tfrac{12}{\omega}e^{-j3\omega/2}(\tfrac{3\omega}{2})^{-1}\sin 3\omega/2 \\
&= 18e^{-j3\omega/2}\operatorname{sinc} 3\omega/2.
\end{aligned}$$

(e) From a table of Fourier transforms, we have

$$\operatorname{sgn} t \stackrel{\mathscr{F}}{\longleftrightarrow} \tfrac{2}{j\omega}.$$

From this transform pair, we can use the duality property of the Fourier transform to deduce

$$\begin{aligned}
\mathscr{F}\{\tfrac{2}{jt}\} &= 2\pi\operatorname{sgn}(-\omega) \\
&= -2\pi\operatorname{sgn}\omega.
\end{aligned}$$

Using this result and the linearity property of the Fourier transform, we can write

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{1/t\} \\
&= \tfrac{j}{2}\mathscr{F}\{\tfrac{2}{jt}\} \\
&= \tfrac{j}{2}[-2\pi\operatorname{sgn}\omega] \\
&= -j\pi\operatorname{sgn}\omega.
\end{aligned}$$

(f) We begin by rewriting $x(t)$ as

$$x(t) = tv_2(t),$$

where

$$\begin{aligned}
v_2(t) &= v_1(2t) \quad \text{and} \\
v_1(t) &= \operatorname{rect}(t).
\end{aligned}$$

Taking the Fourier transform of both sides of each of the above equations, we obtain

$$V_1(\omega) = \operatorname{sinc}\omega/2,$$
$$V_2(\omega) = \tfrac{1}{2}V_1(\tfrac{\omega}{2}), \quad \text{and}$$
$$X(\omega) = j\tfrac{d}{d\omega}V_2(\omega).$$

Combining the above results, we have

$$\begin{aligned}
X(\omega) &= j\tfrac{d}{d\omega}V_2(\omega) \\
&= j\tfrac{d}{d\omega}[\tfrac{1}{2}V_1(\tfrac{\omega}{2})] \\
&= \tfrac{j}{2}\tfrac{d}{d\omega}V_1(\tfrac{\omega}{2}) \\
&= \tfrac{j}{2}\tfrac{d}{d\omega}\operatorname{sinc}\tfrac{\omega}{4} \\
&= \tfrac{j}{2}\left[\frac{\tfrac{\omega}{4}(\tfrac{1}{4}\cos\tfrac{\omega}{4}) - \tfrac{1}{4}\sin\tfrac{\omega}{4}}{\omega^2/16}\right] \\
&= \tfrac{j}{2}\left[\frac{16(\tfrac{\omega}{16}\cos\tfrac{\omega}{4} - \tfrac{1}{4}\sin\tfrac{\omega}{4})}{\omega^2}\right] \\
&= \tfrac{j}{2}\left[\tfrac{1}{\omega}\cos\tfrac{\omega}{4} - \tfrac{4}{\omega^2}\sin\tfrac{\omega}{4}\right] \\
&= \tfrac{j}{2\omega}\cos\tfrac{\omega}{4} - \tfrac{j2}{\omega^2}\sin\tfrac{\omega}{4}.
\end{aligned}$$

(g) We begin by rewriting $x(t)$ as

$$x(t) = e^{-j3t}v_3(t),$$

where

$$v_3(t) = v_2(5t),$$
$$v_2(t) = v_1(t-2), \quad \text{and}$$
$$v_1(t) = \sin t.$$

Taking the Fourier transform of both sides of each of the above equations yields

$$V_1(\omega) = \tfrac{\pi}{j}[\delta(\omega-1) - \delta(\omega+1)],$$
$$V_2(\omega) = e^{-j2\omega}V_1(\omega),$$
$$V_3(\omega) = \tfrac{1}{5}V_2(\tfrac{\omega}{5}), \quad \text{and}$$
$$X(\omega) = V_3(\omega+3).$$

Combining the above results, we obtain

$$\begin{aligned}
X(\omega) &= V_3(\omega+3) \\
&= \tfrac{1}{5}V_2(\tfrac{\omega+3}{5}) \\
&= \tfrac{1}{5}e^{-j2(\omega+3)/5}V_1(\tfrac{\omega+3}{5}) \\
&= \tfrac{\pi}{j5}e^{-j2(\omega+3)/5}\left[\delta(\tfrac{\omega+3}{5}-1) - \delta(\tfrac{\omega+3}{5}+1)\right] \\
&= -\tfrac{j\pi}{5}e^{-j2}\delta(\tfrac{\omega-2}{5}) + \tfrac{j\pi}{5}e^{j2}\delta(\tfrac{\omega+8}{5}).
\end{aligned}$$

(In the above simplification, we used the relationship $f(t)\delta(at-b) = f(b/a)\delta(at-b)$. This identity follows immediately from the fact that $\delta(at-b) = 0$ for all $t \neq b/a$.)

5.5 Given that $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$ and $y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(\omega)$, express $Y(\omega)$ in terms of $X(\omega)$ for each of the following:
(a) $y(t) = x(at-b)$ where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \frac{d}{dt}[x(t) * x(t)]$;
(e) $y(t) = tx(2t-1)$;
(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$;
(h) $y(t) = \left[\frac{d}{dt}x(t)\right] * \left[e^{-jt}x(t)\right]$;
(i) $y(t) = \int_{-\infty}^{3t} x^*(\tau-1)d\tau$;
(j) $y(t) = [\cos(3t-1)]\,x(t)$;
(k) $y(t) = \left[\frac{d}{dt}x(t)\right]\sin(t-2)$;
(l) $y(t) = tx(t)\sin 3t$; and
(m) $y(t) = e^{j7t}\,[x(\lambda) * x(\lambda)]|_{\lambda=t-1}$.

Solution.

(a) We rewrite $y(t)$ as

$$y(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|}V_1(\tfrac{\omega}{a}) \quad \text{and} \\ V_1(\omega) &= e^{-j\omega b}X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|}V_1(\tfrac{\omega}{a}) \\ &= \tfrac{1}{|a|}e^{-j(\omega/a)b}X(\omega/a) \\ &= \tfrac{1}{|a|}e^{-jb\omega/a}X(\omega/a). \end{aligned}$$

(b) We rewrite $y(t)$ as

$$y(t) = v_1(2t)$$

where

$$v_1(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{v_1(2t)\} \\ &= \tfrac{1}{2}V_1(\tfrac{\omega}{2}) \quad \text{and} \\ V_1(\omega) &= \mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} \\ &= \tfrac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2}V_1(\tfrac{\omega}{2}) \\
&= \tfrac{1}{2}\left(\frac{1}{j(\omega/2)}X(\tfrac{\omega}{2}) + \pi X(0)\delta(\tfrac{\omega}{2})\right) \\
&= \tfrac{1}{j\omega}X(\tfrac{\omega}{2}) + \tfrac{\pi}{2}X(0)\delta(\tfrac{\omega}{2}).
\end{aligned}$$

(c) We rewrite $y(t)$ as

$$y(t) = \int_{-\infty}^{t} v_1(\tau)d\tau$$

where

$$v_1(t) = x^2(t).$$

Taking the Fourier transform of both sides of each of the above equations yields

$$\begin{aligned}
V_1(\omega) &= \tfrac{1}{2\pi}X(\omega) * X(\omega), \quad \text{and} \\
Y(\omega) &= \tfrac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega).
\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{j\omega}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda\right] + \pi\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda\right]\delta(\omega) \\
&= \tfrac{1}{j2\pi\omega}\int_{-\infty}^{\infty} X(\lambda)X(\omega-\lambda)d\lambda + \tfrac{1}{2}\delta(\omega)\int_{-\infty}^{\infty} X(\lambda)X(-\lambda)d\lambda.
\end{aligned}$$

(d) We rewrite $y(t)$ as

$$y(t) = \tfrac{d}{dt}v_1(t)$$

where

$$v_1(t) = x(t) * x(t).$$

Taking the Fourier transform of both sides of these equations yields

$$\begin{aligned}
Y(\omega) &= \mathscr{F}\{\tfrac{d}{dt}v_1(t)\} \\
&= j\omega V_1(\omega) \quad \text{and} \\
V_1(\omega) &= \mathscr{F}\{x(t) * x(t)\} \\
&= X^2(\omega).
\end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned}
Y(\omega) &= j\omega V_1(\omega) \\
&= j\omega X^2(\omega).
\end{aligned}$$

(e) We rewrite $y(t)$ as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t) \quad \text{and} \\ v_2(t) &= x(t-1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations yields

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{tv_1(t)\} \\ &= j\tfrac{d}{d\omega}V_1(\omega), \\ V_1(\omega) &= \mathscr{F}\{v_2(2t)\} \\ &= \tfrac{1}{2}V_2(\tfrac{\omega}{2}), \quad \text{and} \\ V_2(\omega) &= \mathscr{F}\{x(t-1)\} \\ &= e^{-j\omega}X(\omega). \end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned} Y(\omega) &= j\tfrac{d}{d\omega}V_1(\omega) \\ &= j\tfrac{d}{d\omega}\left[(\tfrac{1}{2})V_2(\tfrac{\omega}{2})\right] \\ &= \tfrac{j}{2}\left[\tfrac{d}{d\omega}e^{-j\omega/2}X(\tfrac{\omega}{2})\right]. \end{aligned}$$

ALTERNATE SOLUTION. In what follows, we use the prime symbol to denote derivative (i.e., f' denotes the derivative of f). We can rewrite $y(t)$ as

$$y(t) = tv_1(t),$$

where

$$\begin{aligned} v_1(t) &= v_2(2t), \text{ and} \\ v_2(t) &= x(t-1). \end{aligned}$$

Taking the Fourier transform of both sides of the above equations, we obtain

$$\begin{aligned} Y(\omega) &= jV_1'(\omega), \\ V_1(\omega) &= \tfrac{1}{2}V_2(\omega/2), \text{ and} \\ V_2(\omega) &= e^{-j\omega}X(\omega). \end{aligned}$$

In anticipation of what is to come, we compute the quantities:

$$\begin{aligned} V_1'(\omega) &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)V_2'(\omega/2) = \tfrac{1}{4}V_2'(\omega/2) \text{ and} \\ V_2'(\omega) &= -je^{-j\omega}X(\omega) + X'(\omega)e^{-j\omega}. \end{aligned}$$

Combining the above equations, we have

$$\begin{aligned} Y(\omega) &= jV_1'(\omega) \\ &= j\tfrac{1}{4}V_2'(\omega/2) \\ &= \tfrac{j}{4}\left[-je^{-j\omega/2}X(\omega/2) + e^{-j\omega/2}X'(\omega/2)\right]. \end{aligned}$$

(f) We begin by rewriting $y(t)$ as

$$y(t) = e^{j2t} v_1(t)$$

where

$$v_1(t) = x(t-1).$$

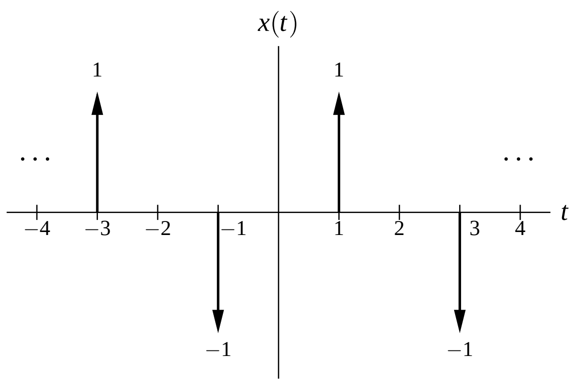
Taking the Fourier transform of both sides of the above equations yields

$$V_1(\omega) = e^{-j\omega} X(\omega) \quad \text{and}$$
$$Y(\omega) = V_1(\omega - 2).$$

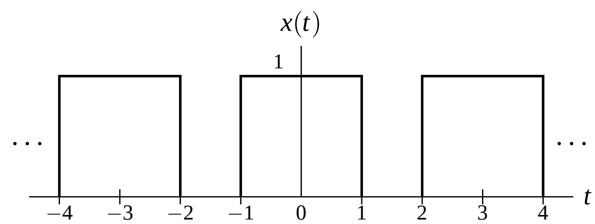
Combining the above results, we have

$$\begin{aligned} Y(\omega) &= V_1(\omega-2) \\ &= e^{-j(\omega-2)} X(\omega-2). \end{aligned}$$

5.6 Find the Fourier transform of each of the periodic signals shown below.



(a)



(b)

Solution.

(a) The frequency ω_0 is given by $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$. Consider the period of $x(t)$ for $-2 \le t < 2$. Let us denote this single period as $x_T(t)$. We have

$$x_T(t) = -\delta(t+1) + \delta(t-1).$$

Taking the Fourier transform of $x_T(t)$, we obtain

$$\begin{aligned} X_T(\omega) &= \mathscr{F}\{\delta(t-1) - \delta(t+1)\} \\ &= \mathscr{F}\{\delta(t-1)\} - \mathscr{F}\{\delta(t+1)\} \\ &= e^{-j\omega} - e^{j\omega} \\ &= -[e^{j\omega} - e^{-j\omega}] \\ &= -2j\sin\omega. \end{aligned}$$

Using the formula for the Fourier transform of a periodic signal, we obtain

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0) \\
&= \sum_{k=-\infty}^{\infty} \tfrac{\pi}{2}[-2j\sin k\tfrac{\pi}{2}]\delta(\omega - k\tfrac{\pi}{2}) \\
&= \sum_{k=-\infty}^{\infty} -j\pi(\sin\tfrac{k\pi}{2})\delta(\omega - \tfrac{k\pi}{2}).
\end{aligned}$$

5.9 Compute the frequency spectrum of each of the signals specified below. In each case, also find and plot the corresponding magnitude and phase spectra.
(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and
(b) $x(t) = \operatorname{sinc}\frac{t-1}{200}$.

Solution.

(a) Taking the Fourier transform of $x(t)$, we obtain

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{e^{-at}u(t)\} \\
&= \frac{1}{a+j\omega}.
\end{aligned}$$

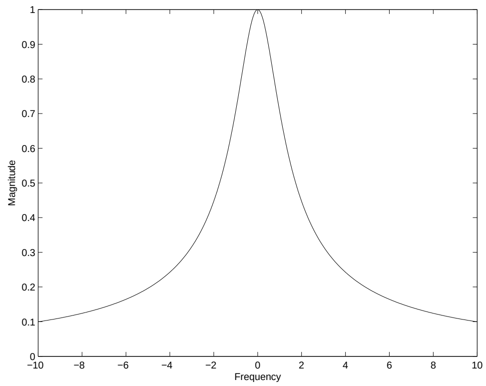
Computing the magnitude spectrum, we obtain

$$\begin{aligned}
|X(\omega)| &= \left|\frac{1}{a+j\omega}\right| \\
&= \frac{|1|}{|a+j\omega|} \\
&= \frac{1}{\sqrt{a^2+\omega^2}}.
\end{aligned}$$

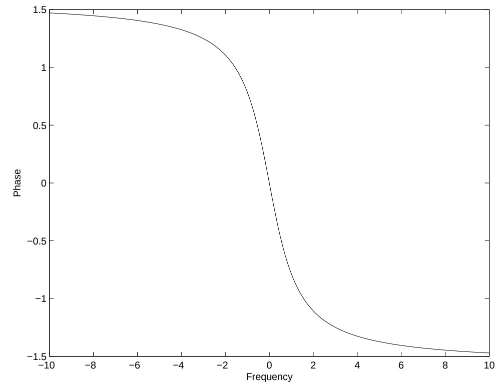
Computing the phase spectrum, we obtain

$$\begin{aligned}
\arg X(\omega) &= \arg\left[\frac{1}{a+j\omega}\right] \\
&= \arg 1 - \arg(a+j\omega) \\
&= 0 - \arg(a+j\omega) \\
&= -\arg(a+j\omega) \\
&= -\arctan\tfrac{\omega}{a}.
\end{aligned}$$

The magnitude and phase spectra are plotted below for $a = 1$.



(a)



(b)

5.10 Suppose that we have the LTI systems defined by the differential/integral equations given below, where $x(t)$ and $y(t)$ denote the system input and output, respectively. Find the frequency response of each of these systems.
(a) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + y(t) + 3\frac{d}{dt}x(t) - x(t) = 0$; and
(b) $\frac{d}{dt}y(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5\frac{d}{dt}x(t) - x(t) = 0$.

Solution.

(b) First, we differentiate the given equation with respect to t. This yields

$$(\tfrac{d}{dt})^2 y(t) + 2\tfrac{d}{dt}y(t) + 3y(t) + 5(\tfrac{d}{dt})^2 x(t) - \tfrac{d}{dt}x(t) = 0.$$

Taking the Fourier transform of both sides of the above equation, we obtain

$$\begin{aligned}
& \mathscr{F}\{(\tfrac{d}{dt})^2 y(t) + 2\tfrac{d}{dt}y(t) + 3y(t) + 5(\tfrac{d}{dt})^2 x(t) - \tfrac{d}{dt}x(t)\} = 0 \\
\Rightarrow \quad & \mathscr{F}\{(\tfrac{d}{dt})^2 y(t)\} + 2\mathscr{F}\{\tfrac{d}{dt}y(t)\} + 3\mathscr{F}\{y(t)\} + 5\mathscr{F}\{(\tfrac{d}{dt})^2 x(t)\} - \mathscr{F}\{\tfrac{d}{dt}x(t)\} = 0 \\
\Rightarrow \quad & (j\omega)^2 Y(\omega) + 2(j\omega)Y(\omega) + 3Y(\omega) + 5(j\omega)^2 X(\omega) - (j\omega)X(\omega) = 0 \\
\Rightarrow \quad & -\omega^2 Y(\omega) + j2\omega Y(\omega) + 3Y(\omega) - 5\omega^2 X(\omega) - j\omega X(\omega) = 0 \\
\Rightarrow \quad & [-\omega^2 + j2\omega + 3]Y(\omega) = [5\omega^2 + j\omega]X(\omega).
\end{aligned}$$

Therefore, the frequency response $H(\omega)$ of the system is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{5\omega^2 + j\omega}{-\omega^2 + 2j\omega + 3}.$$

5.11 Suppose that we have the LTI systems with the frequency responses given below. Find the differential equation that characterizes each of these systems.
(a) $H(\omega) = \dfrac{j\omega}{1 + j\omega}$; and
(b) $H(\omega) = \dfrac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$.

Solution.

(b) From the given frequency response $H(\omega)$, we can write

$$\begin{aligned}
& \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6} \\
\Rightarrow \quad & [-j\omega^3 - 6\omega^2 + 11j\omega + 6]Y(\omega) = [j\omega + \tfrac{1}{2}]X(\omega) \\
\Rightarrow \quad & -j\omega^3 Y(\omega) - 6\omega^2 Y(\omega) + 11j\omega Y(\omega) + 6Y(\omega) = j\omega X(\omega) + \tfrac{1}{2}X(\omega) \\
\Rightarrow \quad & -j\omega^3 Y(\omega) - 6\omega^2 Y(\omega) + 11j\omega Y(\omega) + 6Y(\omega) - j\omega X(\omega) - \tfrac{1}{2}X(\omega) = 0.
\end{aligned}$$

Taking the inverse Fourier transform of both sides of the above equation, we obtain

$$\begin{aligned}
&\mathscr{F}^{-1}\{-j\omega^3 Y(\omega) - 6\omega^2 Y(\omega) + 11 j\omega Y(\omega) + 6Y(\omega) - j\omega X(\omega) - \tfrac{1}{2}X(\omega)\} = 0 \\
\Rightarrow\quad &\mathscr{F}^{-1}\{(j\omega)^3 Y(\omega)\} + 6\mathscr{F}^{-1}\{(j\omega)^2 Y(\omega)\} + 11\mathscr{F}^{-1}\{j\omega Y(\omega)\} + 6\mathscr{F}^{-1}\{Y(\omega)\} \\
&\qquad - \mathscr{F}^{-1}\{j\omega X(\omega)\} - \tfrac{1}{2}\mathscr{F}^{-1}\{X(\omega)\} = 0 \\
\Rightarrow\quad &(\tfrac{d}{dt})^3 y(t) + 6(\tfrac{d}{dt})^2 y(t) + 11\tfrac{d}{dt}y(t) + 6y(t) - \tfrac{d}{dt}x(t) - \tfrac{1}{2}x(t) = 0.
\end{aligned}$$

5.12 Suppose that we have a LTI system with input $x(t)$ and output $y(t)$, and impulse response $h(t)$, where

$$h(t) = \delta(t) - 300\,\mathrm{sinc}\,300\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$, where

$$x_1(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos 200\pi t + \tfrac{1}{2}\cos 400\pi t - \tfrac{1}{4}\cos 600\pi t.$$

Solution.

Let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.

We begin by finding the frequency response $H(\omega)$ of the system. This process yields

$$\begin{aligned}
H(\omega) &= \mathscr{F}\{\delta(t)\} - \mathscr{F}\{\tfrac{300\pi}{\pi}\,\mathrm{sinc}\,300\pi t\} \\
&= 1 - \mathrm{rect}(\tfrac{\omega}{600\pi}) \\
&= \begin{cases} 1 & \text{for } |\omega| > 300\pi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Next, we determine the Fourier transform $X_1(\omega)$ of the input signal. We have

$$\begin{aligned}
X_1(\omega) &= \mathscr{F}\{\tfrac{1}{2}\} + \tfrac{3}{4}\mathscr{F}\{\cos 200\pi t\} + \tfrac{1}{2}\mathscr{F}\{\cos 400\pi t\} - \tfrac{1}{4}\mathscr{F}\{\cos 600\pi t\} \\
&= \pi\delta(\omega) + \tfrac{3}{4}\pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)] + \tfrac{1}{2}\pi[\delta(\omega - 400\pi) + \delta(\omega + 400\pi)] \\
&\quad - \tfrac{1}{4}\pi[\delta(\omega - 600\pi) + \delta(\omega + 600\pi)].
\end{aligned}$$

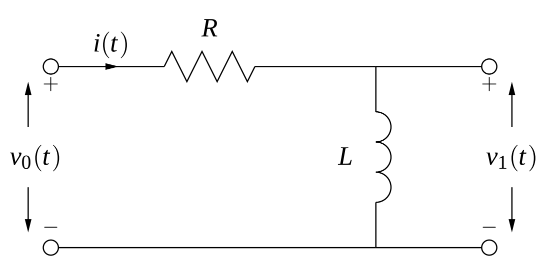
Since the system is LTI, we can write

$$\begin{aligned}
Y(\omega) &= X_1(\omega)H(\omega) \\
&= \tfrac{1}{2}\pi[\delta(\omega - 400\pi) + \delta(\omega + 400\pi)] - \tfrac{1}{4}\pi[\delta(\omega - 600\pi) + \delta(\omega + 600\pi)].
\end{aligned}$$

Taking the inverse Fourier transform of $Y(\omega)$, we obtain

$$\begin{aligned}
y(t) &= \mathscr{F}^{-1}\{Y(\omega)\} \\
&= \tfrac{1}{2}\mathscr{F}^{-1}\{\pi[\delta(\omega - 400\pi) + \delta(\omega + 400\pi)]\} - \tfrac{1}{4}\mathscr{F}^{-1}\{\pi[\delta(\omega - 600\pi) + \delta(\omega + 600\pi)]\} \\
&= \tfrac{1}{2}\cos 400\pi t - \tfrac{1}{4}\cos 600\pi t.
\end{aligned}$$

5.13 Consider the LTI system with input $v_0(t)$ and output $v_1(t)$ as shown in the figure below, where $R = 1$ and $L = 1$.



(a) Find the frequency response $H(\omega)$ of the system.
(b) Determine the magnitude and phase responses of the system.
(c) Find the impulse response $h(t)$ of the system.

Solution.

(a) From basic circuit analysis, we can write

$$\begin{aligned} & v_1(t) = L\tfrac{d}{dt}[\tfrac{1}{R}[v_0(t) - v_1(t)]] \\ \Rightarrow\quad & v_1(t) = \tfrac{L}{R}\tfrac{d}{dt}v_0(t) - \tfrac{L}{R}\tfrac{d}{dt}v_1(t) \end{aligned}$$

Taking the Fourier transform of the preceding equation, we have

$$\begin{aligned} & V_1(\omega) = \tfrac{L}{R}\mathscr{F}\{\tfrac{d}{dt}v_0(t)\} - \tfrac{L}{R}\mathscr{F}\{\tfrac{d}{dt}v_1(t)\} \\ \Rightarrow\quad & V_1(\omega) = \tfrac{L}{R}j\omega V_0(\omega) - \tfrac{L}{R}j\omega V_1(\omega) \\ \Rightarrow\quad & [1 + \tfrac{L}{R}j\omega]V_1(\omega) = \tfrac{L}{R}j\omega V_0(\omega) \end{aligned}$$

$$\begin{aligned} H(\omega) &= \frac{V_1(\omega)}{V_0(\omega)} \\ &= \frac{\frac{L}{R}j\omega}{1 + \frac{L}{R}j\omega} \\ &= \tfrac{j\omega}{1+j\omega}. \end{aligned}$$

(b) Taking the magnitude of the frequency response $H(\omega)$, we obtain

$$\begin{aligned} |H(\omega)| &= \frac{|j\omega|}{|1+j\omega|} \\ &= \frac{|\omega|}{\sqrt{1+\omega^2}}. \end{aligned}$$

Taking the argument of the frequency response $H(\omega)$, we obtain

$$\begin{aligned} \arg H(\omega) &= \arg j\omega - \arg(1 + j\omega) \\ &= \tfrac{\pi}{2}\operatorname{sgn}\omega - \tan^{-1}\omega. \end{aligned}$$

As an aside, we note that

$$\begin{aligned} \arg j\omega &= \begin{cases} \frac{\pi}{2} & \text{for } \omega > 0 \\ -\frac{\pi}{2} & \text{for } \omega < 0 \end{cases} \\ &= \tfrac{\pi}{2}\operatorname{sgn}\omega. \end{aligned}$$

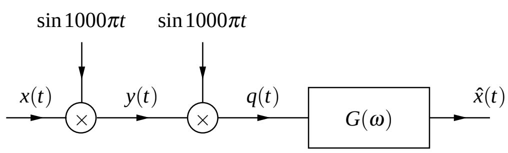
(c) Taking the inverse Fourier transform of the frequency response $H(\omega)$ yields

$$\begin{aligned} h(t) &= \mathscr{F}^{-1}\{j\omega(\tfrac{1}{1+j\omega})\} \\ &= \tfrac{d}{dt}\mathscr{F}^{-1}\{\tfrac{1}{1+j\omega}\} \\ &= \tfrac{d}{dt}[e^{-t}u(t)] \\ &= -e^{-t}u(t) + \delta(t)e^{-t} \\ &= -e^{-t}u(t) + \delta(t). \end{aligned}$$

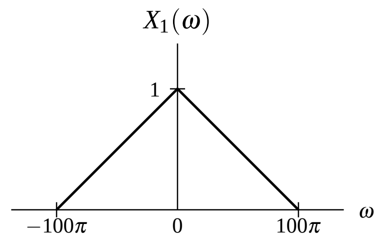
5.18 Consider the system shown below in Figure A with input $x(t)$ and output $\hat{x}(t)$, where

$$G(\omega) = \begin{cases} 2 & \text{for } |\omega| \le 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let $X(\omega)$, $\hat{X}(\omega)$, $Y(\omega)$, and $Q(\omega)$ denote the Fourier transforms of $x(t)$, $\hat{x}(t)$, $y(t)$, and $q(t)$, respectively.



(a)



(b)

(a) Suppose that $X(\omega) = 0$ for $|\omega| > 100\pi$. Find expressions for $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$ in terms of $X(\omega)$.
(b) If $X(\omega) = X_1(\omega)$ where $X_1(\omega)$ is as shown in Figure B, sketch $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$.

Solution.

(a) From the system block diagram, we have

$$\begin{aligned}
Y(\omega) &= \mathscr{F}\{x(t)\sin 1000\pi t\} \\
&= \mathscr{F}\{\tfrac{1}{2j}[e^{j1000\pi t} - e^{-j1000\pi t}]x(t)\} \\
&= \tfrac{1}{2j}\mathscr{F}\{e^{j1000\pi t}x(t)\} - \tfrac{1}{2j}\mathscr{F}\{e^{-j1000\pi t}x(t)\} \\
&= \tfrac{1}{2j}X(\omega - 1000\pi) - \tfrac{1}{2j}X(\omega + 1000\pi), && (4.1) \\
Q(\omega) &= \mathscr{F}\{y(t)\sin 1000\pi t\} \\
&= \mathscr{F}\{\tfrac{1}{2j}[e^{j1000\pi t} - e^{-j1000\pi t}]y(t)\} \\
&= \tfrac{1}{2j}\mathscr{F}\{e^{j1000\pi t}y(t)\} - \tfrac{1}{2j}\mathscr{F}\{e^{-j1000\pi t}y(t)\} \\
&= \tfrac{1}{2j}Y(\omega - 1000\pi) - \tfrac{1}{2j}Y(\omega + 1000\pi), \quad \text{and} && (4.2) \\
\hat{X}(\omega) &= G(\omega)Q(\omega). && (4.3)
\end{aligned}$$

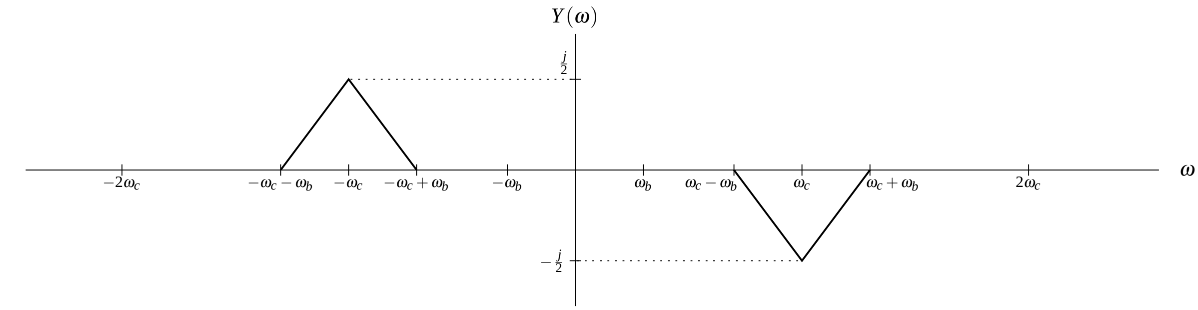
Substituting the expression for $Y(\omega)$ from (4.1) into (4.2), we have

$$\begin{aligned}
Q(\omega) &= \tfrac{1}{2j}Y(\omega - 1000\pi) - \tfrac{1}{2j}Y(\omega + 1000\pi) \\
&= \tfrac{1}{2j}[\tfrac{1}{2j}X([\omega - 1000\pi] - 1000\pi) - \tfrac{1}{2j}X([\omega - 1000\pi] + 1000\pi)] \\
&\quad - \tfrac{1}{2j}[\tfrac{1}{2j}X([\omega + 1000\pi] - 1000\pi) - \tfrac{1}{2j}X([\omega + 1000\pi] + 1000\pi)] \\
&= -\tfrac{1}{4}X(\omega - 2000\pi) + \tfrac{1}{4}X(\omega) + \tfrac{1}{4}X(\omega) - \tfrac{1}{4}X(\omega + 2000\pi) \\
&= \tfrac{1}{2}X(\omega) - \tfrac{1}{4}X(\omega - 2000\pi) - \tfrac{1}{4}X(\omega + 2000\pi). && (4.4)
\end{aligned}$$

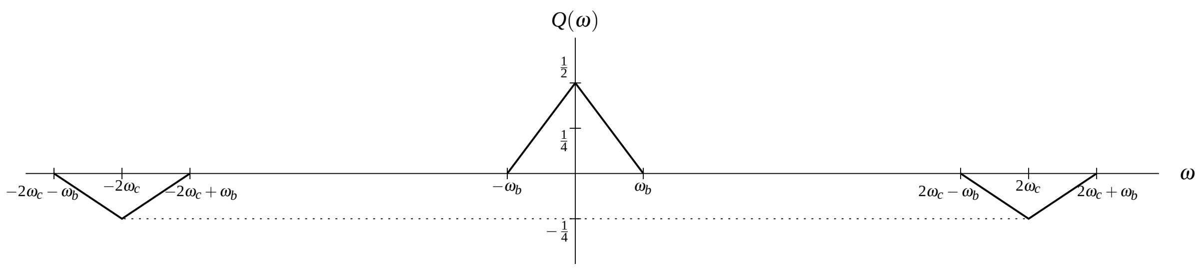
Combining (4.3) and (4.4) and using the fact that $X(\omega) = 0$ for $|\omega| > 100\pi$, we have

$$\begin{aligned}
\hat{X}(\omega) &= G(\omega)Q(\omega) \\
&= 2(\tfrac{1}{2}X(\omega)) \\
&= X(\omega).
\end{aligned}$$

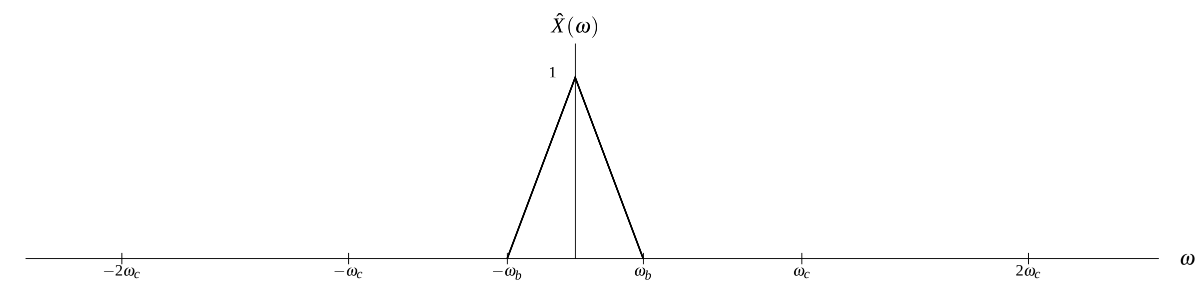
(b) The frequency spectra of the various signals are plotted below, where $\omega_c = 1000\pi$ and $\omega_b = 100\pi$.



(a)



(b)

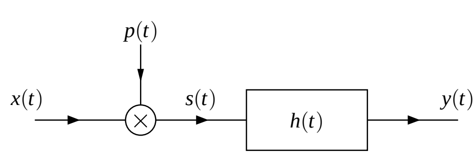


(c)

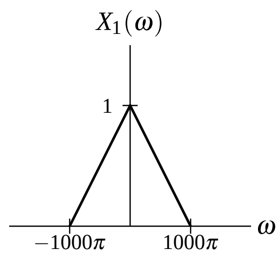
5.19 Consider the system shown below in Figure A with input $x(t)$ and output $y(t)$. Let $X(\omega)$, $P(\omega)$, $S(\omega)$, $H(\omega)$, and $Y(\omega)$ denote the Fourier transforms of $x(t)$, $p(t)$, $s(t)$, $h(t)$, and $y(t)$, respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - \tfrac{n}{1000}) \quad \text{and} \quad H(\omega) = \tfrac{1}{1000} \operatorname{rect}(\tfrac{\omega}{2000\pi}).$$

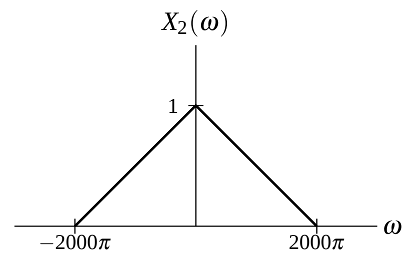
(a) Derive an expression for $S(\omega)$ in terms of $X(\omega)$. Derive an expression for $Y(\omega)$ in terms of $S(\omega)$ and $H(\omega)$.
(b) Suppose that $X(\omega) = X_1(\omega)$, where $X_1(\omega)$ is as shown in Figure B. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.
(c) Suppose that $X(\omega) = X_2(\omega)$, where $X_2(\omega)$ is as shown in Figure C. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.



(a)



(b)



(c)

Solution.

(a) Since $p(t)$ is periodic with period $T = \frac{1}{1000}$ (and frequency $\omega_0 = 2000\pi$), it can be expressed in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2000\pi kt}.$$

Using the Fourier series analysis equation, we compute c_k as

$$\begin{aligned} c_k &= \left(\tfrac{1}{1000}\right)^{-1} \int_{-1/2000}^{1/2000} \delta(t) e^{-j2000\pi kt} dt \\ &= 1000. \end{aligned}$$

Combining the above equations, we have

$$p(t) = 1000 \sum_{k=-\infty}^{\infty} e^{j2000\pi kt}.$$

From the system block diagram, we have

$$\begin{aligned} s(t) &= x(t)p(t) \\ &= x(t)\left(1000 \sum_{k=-\infty}^{\infty} e^{j2000\pi kt}\right) \\ &= 1000 \sum_{k=-\infty}^{\infty} x(t) e^{j2000\pi kt}. \end{aligned}$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

$$S(\omega) = 1000 \sum_{k=-\infty}^{\infty} X(\omega - 2000\pi k).$$

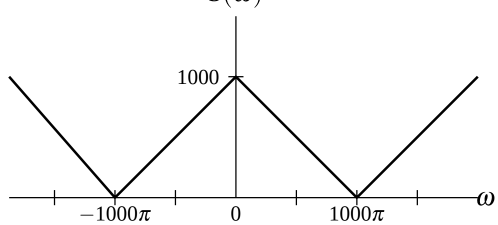
From the system block diagram, we have

$$y(t) = s(t) * h(t).$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

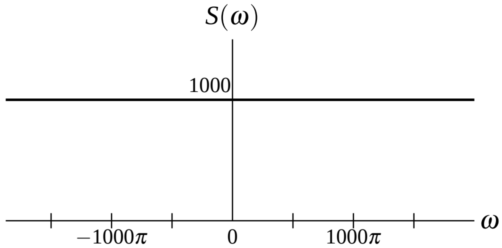
$$\begin{aligned} Y(\omega) &= H(\omega)S(\omega) \\ &= \begin{cases} \frac{1}{1000} S(\omega) & \text{for } |\omega| < 1000\pi \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \sum_{k=-\infty}^{\infty} X(\omega - 2000\pi k) & \text{for } |\omega| < 1000\pi \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

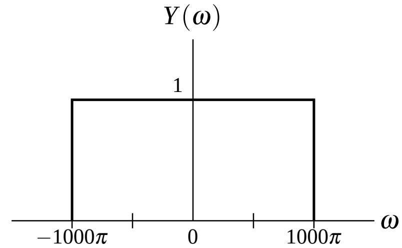
(b) We observe that $X(\omega) = 0$ for $|\omega| < 1000\pi$. Therefore, the copies of the original spectrum of $X(\omega)$ in $S(\omega)$ do not overlap. A plot of $S(\omega)$ is provided below. The plot of $Y(\omega)$ is identical to that of $X(\omega) = X_1(\omega)$ given in the problem statement.



Since $X(\omega) = Y(\omega)$, the input and output of the system are identical.

(c) We observe that $X(\omega) \neq 0$ for some ω satisfying $|\omega| > 1000\pi$. Therefore, the copies of the original spectrum in $S(\omega)$ overlap, resulting in aliasing. The plots of $S(\omega)$ and $Y(\omega)$ are given below.





We can see that $Y(\omega)$ does not in any way resemble $X(\omega)$. The input $x(t)$ and output $y(t)$ are not related, except in the sense that the output is an extremely distorted version of the input, with the distortion being caused by aliasing.

5.101 (a) Consider a frequency response $H(\omega)$ of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k \omega^k}{\sum_{k=0}^{N-1} b_k \omega^k},$$

where a_k and b_k are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments: 1) a vector containing the a_k coefficients, 2) a vector containing the b_k coefficients, 3) a vector containing the values of ω at which to evaluate $H(\omega)$. The function should generate two return values: 1) a vector of function values, and 2) a vector of points at which the function was evaluated. If the function is called with no output arguments (i.e., the `nargout` variable is zero), then the function should plot the magnitude and phase responses before returning. [Hint: The `polyval` function may be helpful.]

(b) Use the function developed in part (a) to plot the magnitude and phase responses of the system with the frequency response

$$H(\omega) = \frac{16.0000}{1.0000\omega^4 - j5.2263\omega^3 - 13.6569\omega^2 + j20.9050\omega + 16.0000}.$$

For each of the plots, use the frequency range $[-5, 5]$.

(c) What type of ideal frequency-selective filter does this system approximate?

Solution.

(a) The `freqw` function can be implemented with the code below.

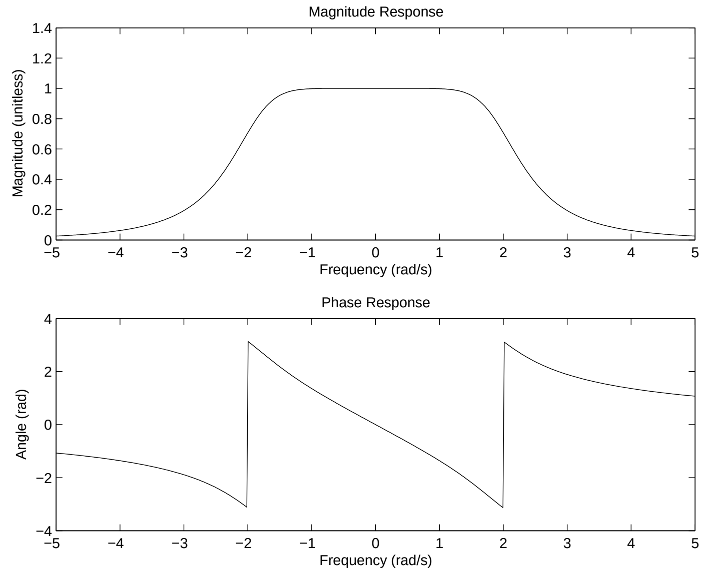
Listing 4.1: `freqw.m`

```
function [freqresp, omega] = freqw(ncoefs, dcoefs, omega)

freqresp = polyval(ncoefs, omega) ./ polyval(dcoefs, omega);

% If no output arguments were specified, plot the frequency response.
if nargout == 0

    % Compute the magnitude response as a unitless quantity.
    magresp = abs(freqresp);

    % Compute the phase response.
    phaseresp = angle(freqresp);

    % On the first of two graphs, plot the magnitude response.
    subplot(2, 1, 1);
    plot(omega, magresp);
    title('Magnitude Response');
    xlabel('Frequency (rad/s)');
    ylabel('Magnitude (unitless)');

    % On the second of two graphs, plot the phase response.
    subplot(2, 1, 2);
    plot(omega, phaseresp);
    title('Phase Response');
    xlabel('Frequency (rad/s)');
    ylabel('Angle (rad)');

end
```

Using the `freqw` function, we can generate the necessary plots with the following few lines of code:

```
ncoefs = [16];
dcoefs = [1.0000 (-j * 5.2263) (-13.6569) (j * 20.9050) 16.0000];
freqw(ncoefs, dcoefs, linspace(-5, 5, 500));
```

(b) The magnitude and phase responses are shown in the figure below.



(c) The system approximates a lowpass filter with a cutoff frequency somewhere in the vicinity of 2 rad/s.

5.103 (a) Use the `butter` and `besself` functions to design a tenth-order Butterworth lowpass filter and tenth-order Bessel lowpass filter, each with a cutoff frequency of 10 rad/s.
(b) For each of the filters designed in part (a), plot the magnitude and phase responses using a linear scale for the frequency axis. In the case of the phase response, plot the unwrapped phase (as this will be helpful later in part (d) of this problem). (Hint: The `freqs` and `unwrap` functions may be helpful.)
(c) Consider the magnitude responses for each of the filters. Recall that an ideal lowpass filter has a magnitude response that is constant in the passband. Which of the two filters more closely approximates this ideal behavior?
(d) Consider the phase responses for each of the filters. An ideal lowpass filter has a phase response that is a linear function. Which of the two filters has a phase response that best approximates a linear (i.e., straight line) function in the passband?

Solution.

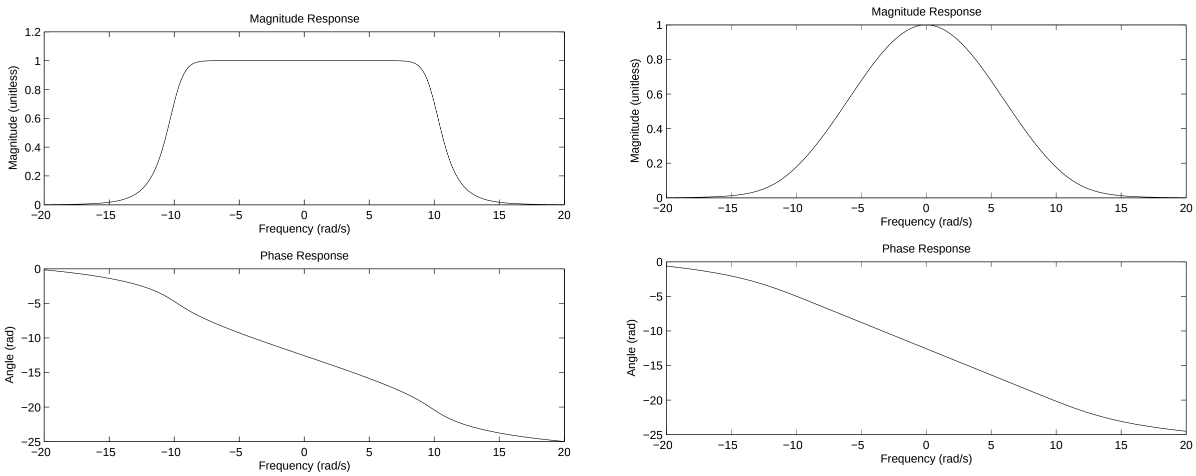
(a) and (b) The frequency responses for the two filters can be plotted using the code shown below.

```
% Choose a filter type.
filtertype = 'butterworth';
% filtertype = 'bessel';

% Set the cutoff frequency for the filter.
wc = 10;

% Calculate the transfer function coefficients for the filter.
switch filtertype
case {'butterworth'}
        [tfnum, tfdenom] = butter(10, wc, 's');
case {'bessel'}
        [tfnum, tfdenom] = besself(10, wc);
end

% Calculate the magnitude and phase responses.
[freqresp, omega] = freqs(tfnum, tfdenom, linspace(-20, 20, 500));
magresp = abs(freqresp);
```



```
phaseresp = unwrap(angle(freqresp));

% Plot the magnitude and phase response (using the unwrapped phase).
clf
subplot(2, 1, 1);
plot(omega, magresp);
title('Magnitude Response');
xlabel('Frequency (rad/s)');
ylabel('Magnitude (unitless)');
subplot(2, 1, 2);
plot(omega, phaseresp);
title('Phase Response');
xlabel('Frequency (rad/s)');
ylabel('Angle (rad)');
```

The above code produces the plots shown below.



(a) (b)

(c) In the passband, the magnitude response of the Butterworth filter is much flatter than that of the Bessel filter. As it turns out, Butterworth filters have very flat magnitude responses in the passband.

(d) The Bessel filter has a phase response that is closer to having linear phase than the Butterworth filter. As it turns out, Bessel filters tend to have phase responses that are approximately linear in the passband.

Chapter 10

Review (Appendix F)

F.1 A communication channel heavily distorts high frequencies but does not significantly affect very low frequencies. Determine which of the following signals would be least distorted by the communication channel:
(a) $x_1(t) = \delta(t)$
(b) $x_2(t) = 5$
(c) $x_3(t) = 10e^{j1000t}$
(d) $x_4(t) = 1/t$

Answer:

$x_2(t)$

Solution. Let $X_1(\omega)$, $X_2(\omega)$, $X_3(\omega)$, and $X_4(\omega)$ denote the Fourier transforms of $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$, respectively. Taking the Fourier transforms of $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$, we obtain:

$$\begin{aligned} X_1(\omega) &= 1, \\ X_2(\omega) &= 5(2\pi\delta(\omega)) \\ &= 10\pi\delta(\omega), \\ X_3(\omega) &= 10(2\pi\delta(\omega-1000)) \\ &= 20\pi\delta(\omega-1000), \quad \text{and} \\ X_4(\omega) &= -j\pi \operatorname{sgn}\omega. \end{aligned}$$

From the above Fourier transforms, we can see that the signals $x_1(t)$, $x_3(t)$, and $x_4(t)$, all contain information at relatively high frequencies, while the signal $x_2(t)$ only has information at the frequency 0. Therefore, $x_2(t)$ will be least distorted. Note that, in computing $X_4(\omega)$, we used the fact that

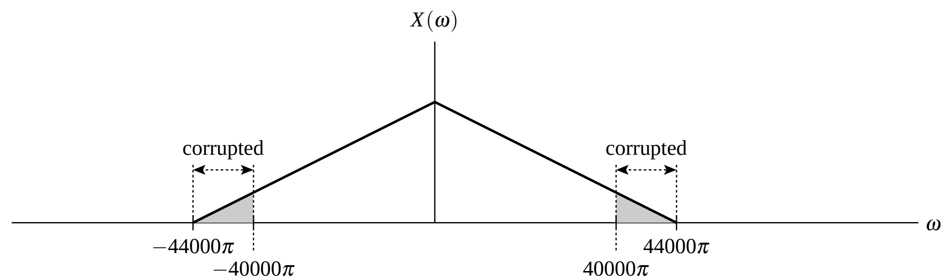
$$\begin{aligned} &\operatorname{sgn} t \stackrel{\mathcal{F}}{\longleftrightarrow} \tfrac{2}{j\omega} \\ \Leftrightarrow\ &\tfrac{2}{jt} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi \operatorname{sgn}(-\omega) \\ \Leftrightarrow\ &1/t \stackrel{\mathcal{F}}{\longleftrightarrow} j\pi \operatorname{sgn}(-\omega) = -j\pi \operatorname{sgn}\omega. \end{aligned}$$

F.37 A signal $x(t)$ is bandlimited to 22 kHz (i.e., only has spectral content for frequencies f in the range $[-22000, 22000]$). Due to excessive noise, the portion of the spectrum that corresponds to frequencies f satisfying $|f| > 20000$ has been badly corrupted and rendered useless. (a) Determine the minimum sampling rate for $x(t)$ that would allow the uncorrupted part of the spectrum to be recovered. (b) Suppose now that the corrupted part of the spectrum were eliminated by filtering prior to sampling. In this case, determine the minimum sampling rate for $x(t)$.

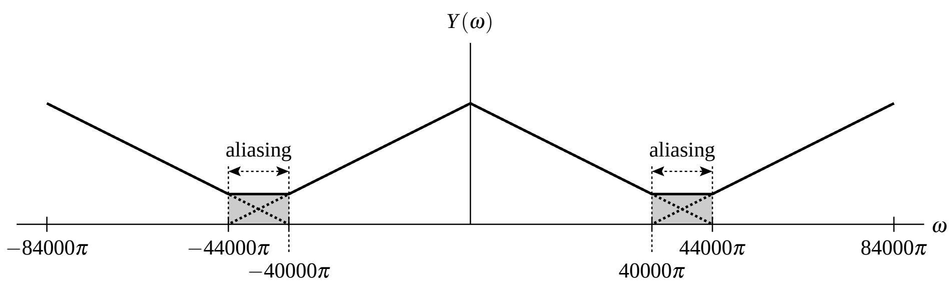
Answer:

(a) 42 kHz; (b) 40 kHz

Solution. Let $y(t)$ denote the signal obtained from $x(t)$ after sampling and reconstruction. Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. The spectrum $X(\omega)$ has a form something like that shown in the figure below.



(a) Since we only wish to be able to recover the uncorrupted part of the spectrum of $x(t)$, it does not matter if aliasing occurs in the range of frequencies where the spectrum has already been corrupted. If we choose the sampling rate to be 84000π, we obtain the spectrum shown in the figure below, after impulse sampling.



From this plot, we can see that aliasing only occurs in the corrupted part of the spectrum. Thus, we can recover the uncorrupted part of the spectrum by lowpass filtering. Using a lower sampling rate would cause aliasing to occur in the uncorrupted part of the spectrum. Thus, the minimum sampling rate required is 84000π rad/s (or equivalently, 42 kHz).

(b) Since the corrupted part of the spectrum has been removed (i.e., set to zero), the new signal is bandlimited to frequencies in the range $[-40000\pi, 40000\pi]$. From the sampling theorem, we must sample the signal at

$$\begin{aligned}\omega_s &= 2(40000\pi)\\ &= 80000\pi.\end{aligned}$$

Thus, a sampling rate of 80000π rad/s (or equivalently, 40 kHz) is required.