6A **7.1** Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.
(a) $x(t) = e^{-at}u(t)$;
(b) $x(t) = e^{-a|t|}$; and
(c) $x(t) = \cos(\omega_0 t)u(t)$. [Note: $\int e^{ax}\cos(bx)dx = \frac{1}{a^2+b^2}(e^{ax}[a\cos(bx)+b\sin(bx)])+C$.]

6A **Answer (c).**

Let $s = \sigma + j\omega$. We have

$$\begin{aligned}\mathcal{L}\{[\cos\omega_0 t]u(t)\}(s) &= \int_{-\infty}^{\infty}[\cos\omega_0 t]u(t)e^{-st}dt \\ &= \int_0^{\infty}[\cos\omega_0 t]e^{-st}dt.\end{aligned}$$

Since this integral does not converge if $s = 0$, we assume that $s \neq 0$. From this assumption, we have

$$\begin{aligned}\mathcal{L}\{[\cos\omega_0 t]u(t)\}(s) &= \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(-s)^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0\sin\omega_0 t]}{s^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-(\sigma+j\omega)t}[-(\sigma+j\omega)\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(\sigma+j\omega)^2+\omega_0^2}\right]\Bigg|_0^{\infty} \\ &= \left[\frac{e^{-\sigma t}e^{-j\omega t}[-(\sigma+j\omega)\cos\omega_0 t + \omega_0\sin\omega_0 t]}{(\sigma+j\omega)^2+\omega_0^2}\right]\Bigg|_0^{\infty}.\end{aligned}$$

The preceding expression only converges to a finite limit if $\sigma > 0$ (i.e., $\mathrm{Re}(s) > 0$). We proceed to compute this limit as follows:

$$\begin{aligned}\mathcal{L}\{[\cos\omega_0 t]u(t)\}(s) &= 0 - \left[\frac{-(\sigma+j\omega)}{(\sigma+j\omega)^2+\omega_0^2}\right] \\ &= \frac{\sigma+j\omega}{(\sigma+j\omega)^2+\omega_0^2} \\ &= \frac{s}{s^2+\omega_0^2} \quad \text{for } \mathrm{Re}(s) > 0.\end{aligned}$$

6A **7.2** Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x below.
(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$, where a and b are real constants and $a > 0$;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

6A **Answer (b).**

$$\begin{aligned}
X(s) &= \mathcal{L}\{3e^{-2t}u(t) + 2e^{5t}u(-t)\}(s) \\
&= 3\mathcal{L}\{e^{-2t}u(t)\}(s) + 2\mathcal{L}\{e^{5t}u(-t)\}(s) \\
&= 3\left(\frac{1}{s+2}\right) - 2\left(\tfrac{1}{s-5}\right) \quad \text{for } \operatorname{Re}(s) > -2 \cap \operatorname{Re}(s) < 5 \\
&= \frac{3(s-5) - 2(s+2)}{(s+2)(s-5)} \\
&= \frac{3s - 15 - 2s - 4}{(s+2)(s-5)} \\
&= \frac{s-19}{(s+2)(s-5)} \quad \text{for } \operatorname{Re}(s) > -2 \cap \operatorname{Re}(s) < 5 \\
&= \frac{s-19}{(s+2)(s-5)} \quad \text{for } -2 < \operatorname{Re}(s) < 5.
\end{aligned}$$

6A **Answer (c).**

To begin, let $v_1(t) = x(t-4)$ so that

$$\begin{aligned}
x(t) &= v_1(t+4) \quad \text{and} \\
v_1(t) &= e^{-2(t-4)}u(t-4+4) \\
&= e^8 e^{-2t}u(t).
\end{aligned}$$

Taking the Laplace transform of these equations yields

$$\begin{aligned}
X(s) &= \mathcal{L}x(s) \\
&= \mathcal{L}\{v_1(t+4)\}(s) \\
&= e^{4s}V_1(s) \quad \text{for ROC of } V_1(s), \quad \text{and} \\
V_1(s) &= \mathcal{L}v_1(s) \\
&= \mathcal{L}\{e^8 e^{-2t}u(t)\}(s) \\
&= e^8\mathcal{L}\{e^{-2t}u(t)\}(s) \\
&= e^8 \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.
\end{aligned}$$

Substituting the above expression for V_1 into the expression for X, we obtain

$$\begin{aligned}
X(s) &= e^{4s}V_1(s) \\
&= e^{4s}\left[e^8 \tfrac{1}{s+2}\right] \\
&= \frac{e^{4s+8}}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.
\end{aligned}$$

6A Answer (d).

We rewrite $x(t)$ as

$$x(t) = \int_{-\infty}^{t} v_1(\tau) d\tau,$$

where

$$\begin{aligned} v_1(t) &= e^{-2t} v_2(t) \quad \text{and} \\ v_2(t) &= u(t). \end{aligned}$$

Let R_X, R_{V_1}, and R_{V_2} denote the ROCs of X, V_1, and V_2, respectively. Taking the Laplace transform of both sides of each of the above equations, we obtain

$$\begin{aligned} X(s) &= \frac{1}{s} V_1(s) \quad \text{for } R_X = R_{V_1} \cap (\operatorname{Re}(s) > 0) \\ V_1(s) &= V_2(s+2) \quad \text{for } R_{V_1} = R_{V_2} - 2 \\ V_2(s) &= \frac{1}{s} \quad \text{for } R_{V_2} = (\operatorname{Re}(s) > 0). \end{aligned}$$

Combining the above equations, we obtain

$$\begin{aligned} X(s) &= \frac{1}{s} \left[V_2(s+2) \right] \\ &= \frac{1}{s} \left(\frac{1}{s+2} \right) \\ &= \frac{1}{s(s+2)} \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

Note that the ROC of X given above was determined as follows:

$$\begin{aligned} R_X &= R_{V_1} \cap (\operatorname{Re}(s) > 0) \\ &= (R_{V_2} - 2) \cap (\operatorname{Re}(s) > 0) \\ &= (\operatorname{Re}(s) > -2) \cap (\operatorname{Re}(s) > 0) \\ &= \operatorname{Re}(s) > 0. \end{aligned}$$

6A Answer (e).

Let us rewrite $x(t)$ as

$$x(t) = v_1(-t),$$

where

$$\begin{aligned} v_1(t) &= v_2(t+b) \quad \text{and} \\ v_2(t) &= -e^{ab} e^{-at} u(t). \end{aligned}$$

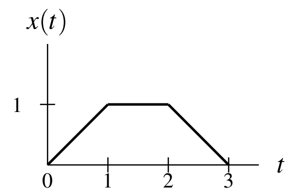
Let R_X, R_{V_1}, and R_{V_2} denote the ROCs of X, V_1, and V_2, respectively. Taking the Laplace transform of both sides of each of the above equations, we obtain

$$\begin{aligned}
X(s) &= \mathcal{L}\{v_1(-t)\}(s) \\
&= V_1(-s) \quad \text{for } R_X = -R_{V_1} \\
V_1(s) &= \mathcal{L}\{v_2(t+b)\}(s) \\
&= e^{bs}V_2(s) \quad \text{for } R_{V_2} \\
V_2(s) &= \mathcal{L}\{-e^{ab}e^{-at}u(t)\}(s) \\
&= -e^{ab}\mathcal{L}\{e^{-at}u(t)\}(s) \\
&= -e^{ab}\frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) > -a.
\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}
X(s) &= V_1(-s) \\
&= e^{-bs}V_2(-s) \\
&= e^{-bs}\left[-e^{ab}\frac{1}{-s+a}\right] \quad \text{for } \operatorname{Re}(s) < a \\
&= e^{-b(s-a)}\frac{1}{s-a} \quad \text{for } \operatorname{Re}(s) < a \\
&= e^{b(a-s)}\frac{1}{s-a} \quad \text{for } \operatorname{Re}(s) < a.
\end{aligned}$$

6A **7.4** Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x shown in the figure below.



(a)

6A **Answer (a).**

We have

$$x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ -t+3 & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

We rewrite $x(t)$ using unit-step functions to obtain

$$\begin{aligned} x(t) &= t\,[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + [-t+3]\,[u(t-2) - u(t-3)] \\ &= tu(t) + (-t+1)u(t-1) + (-t+2)u(t-2) + (t-3)u(t-3) \\ &= tu(t) - (t-1)u(t-1) - (t-2)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Taking the Laplace transform of both sides of this equation, we have

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} \\ &= \frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}. \end{aligned}$$

Since x is of finite duration, the ROC of X is the entire complex plane.

6A **7.5** For each case below, using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x, where the ROCs of X and Y are R_X and R_Y, respectively.
(a) $y(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}[x * x(t-1)]$;
(c) $y(t) = tx(3t-2)$;
(d) $y(t) = \mathcal{D}x_1(t)$, where $x_1(t) = x^*(t-3)$ and $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = e^{-5t}x(3t+7)$; and
(f) $y(t) = e^{-j5t}x(t+3)$.

6A **Answer (e).**

Define

$$v_1(t) = x(t+7) \quad \text{and}$$
$$v_2(t) = v_1(3t),$$

so that we can express $y(t)$ as

$$y(t) = e^{-5t}v_2(t).$$

Taking the Laplace transforms of both sides of the above equations, we obtain

$$\begin{aligned} V_1(s) &= e^{7s}X(s), \quad R_{V_1} = R_X, \\ V_2(s) &= \tfrac{1}{3}V_1\left(\frac{s}{3}\right), \quad R_{V_2} = 3R_{V_1}, \\ Y(s) &= V_2(s+5), \quad R_Y = R_{V_2} - 5, \end{aligned}$$

where R_{V_1} and R_{V_2} denote the ROCs of V_1 and V_2, respectively. Combining the above equations, we have

$$\begin{aligned} Y(s) &= V_2(s+5) \\ &= \tfrac{1}{3}V_1\left(\frac{s+5}{3}\right) \\ &= \tfrac{1}{3}e^{7(s+5)/3}X\left(\frac{s+5}{3}\right). \end{aligned}$$

Also, we have a ROC of

$$\begin{aligned} R_Y &= R_{V_2} - 5 \\ &= 3R_{V_1} - 5 \\ &= 3R_X - 5. \end{aligned}$$

6A **7.6** A causal function x has the Laplace transform

$$X(s) = \frac{-2s}{s^2+3s+2}.$$

(a) Assuming that x has no singularities at 0, find $x(0^+)$.
(b) Assuming that $\lim_{t\to\infty} x(t)$ exists, find this limit.

6A **Answer (a).**

Since x is causal and has no singularities at the origin, we can compute $x(0^+)$ using the initial value theorem as follows:

$$\begin{aligned} x(0^+) &= \lim_{s\to\infty} sX(s) \\ &= \lim_{s\to\infty} \frac{s(-2s)}{s^2+3s+2} \\ &= -2. \end{aligned}$$

6A **Answer (b).**

Since x is causal and we are told that $\lim_{t\to\infty} x(t)$ exists, we can compute this limit using the final value theorem as follows:

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\ &= \left. \frac{s(-2s)}{s^2+3s+2} \right|_{s=0} \\ &= 0. \end{aligned}$$

6A **7.10** Find the inverse Laplace transform x of each function X below.

(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;

(b) $X(s) = \dfrac{2s^2+4s+5}{(s+1)(s+2)}$ for $\operatorname{Re}(s) > -1$;

(c) $X(s) = \dfrac{3s+1}{s^2+3s+2}$ for $-2 < \operatorname{Re}(s) < -1$;

(d) $X(s) = \dfrac{s^2-s+1}{(s+3)^2(s+2)}$ for $\operatorname{Re}(s) > -2$; and

(e) $X(s) = \dfrac{s+2}{(s+1)^2}$ for $\operatorname{Re}(s) < -1$.

6A **Answer (d).**

We are given the function $X(s) = \frac{s^2-s+1}{(s+3)^2(s+2)}$ for $\operatorname{Re}(s) > -2$. The function X has a partial fraction expansion of the form

$$X(s) = \frac{A_{1,1}}{s+3} + \frac{A_{1,2}}{(s+3)^2} + \frac{A_2}{s+2}.$$

Computing the expansion coefficients, we have

$$\begin{aligned}
A_{1,1} &= \tfrac{1}{(2-1)!}\left[\left[\frac{d}{ds}\right]^{2-1}\left[(s+3)^2X(s)\right]\right]\bigg|_{s=-3} = \left[\left[\frac{d}{ds}\right]\left[\frac{s^2-s+1}{s+2}\right]\right]\bigg|_{s=-3} \\
&= \frac{(s+2)(2s-1)-(s^2-s+1)(1)}{(s+2)^2}\bigg|_{s=-3} = \frac{(-1)(-7)-(9+3+1)}{1} = 7-13 = -6, \\
A_{1,2} &= \tfrac{1}{(2-2)!}\left[\left[\frac{d}{ds}\right]^{2-2}\left[(s+3)^2X(s)\right]\right]\bigg|_{s=-3} = \frac{s^2-s+1}{s+2}\bigg|_{s=-3} = \frac{9+3+1}{-1} = -13, \quad \text{and} \\
A_2 &= \left[(s+2)X(s)\right]\big|_{s=-2} = \frac{s^2-s+1}{(s+3)^2}\bigg|_{s=-2} = \frac{4+2+1}{1} = 7.
\end{aligned}$$

Thus, X has the partial fraction expansion

$$X(s) = -\frac{6}{s+3} - \frac{13}{(s+3)^2} + \frac{7}{s+2}.$$

Taking the inverse Laplace transform of X, we have

$$X(s) = -6e^{-3t}u(t) - 13te^{-3t}u(t) + 7e^{-2t}u(t).$$

6A **7.12** Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

6A **Answer.**

Each distinct ROC for H will yield a distinct inverse Laplace transform. Since H is a rational function with poles at -1 and 1, three distinct ROCs are possible: i) $\operatorname{Re}(s) < -1$; ii) $-1 < \operatorname{Re}(s) < 1$; and iii) $\operatorname{Re}(s) > 1$. From the expression for $H(s)$, we have

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}H(t) \\ &= \mathcal{L}^{-1}\left\{\frac{4}{s+1} + \frac{3}{s-1}\right\}(t) \\ &= 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t). \end{aligned}$$

For $\operatorname{Re}(s) < -1$, we have

$$\begin{aligned} h(t) &= 4[-e^{-t}u(-t)] + 3[-e^{t}u(-t)] \\ &= [-4e^{-t} - 3e^{t}]u(-t). \end{aligned}$$

For $-1 < \operatorname{Re}(s) < 1$, we have

$$\begin{aligned} h(t) &= 4[e^{-t}u(t)] + 3[-e^{t}u(-t)] \\ &= 4e^{-t}u(t) - 3e^{t}u(-t). \end{aligned}$$

For $\operatorname{Re}(s) > 1$, we have

$$\begin{aligned} h(t) &= 4[e^{-t}u(t)] + 3[e^{t}u(t)] \\ &= [4e^{-t} + 3e^{t}]u(t). \end{aligned}$$

