4 **5.1** For each case below, find the Fourier series representation (in complex exponential form) of the function x, explicitly identifying the fundamental period of x and the Fourier series coefficient sequence c.
(a) $x(t) = 1 + \cos(\pi t) + \sin^2(\pi t)$;
(b) $x(t) = \cos(4t)\sin(t)$; and
(c) $x(t) = |\sin(2\pi t)|$. [Hint: $\int e^{ax}\sin(bx)dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2+b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

4 **Answer (a).**

We can rewrite $x(t)$ in the form of a Fourier series by simple algebraic manipulation as follows:

$$
\begin{aligned}
x(t) &= 1 + \cos(\pi t) + \sin^2(\pi t) \\
&= 1 + \tfrac{1}{2}[e^{j\pi t} + e^{-j\pi t}] + \left[\tfrac{1}{2j}\left[e^{j\pi t} - e^{-j\pi t}\right]\right]^2 \\
&= 1 + \tfrac{1}{2}e^{j\pi t} + \tfrac{1}{2}e^{-j\pi t} - \tfrac{1}{4}[e^{j2\pi t} - 2 + e^{-j2\pi t}] \\
&= -\tfrac{1}{4}e^{-j2\pi t} + \tfrac{1}{2}e^{-j\pi t} + \tfrac{3}{2} + \tfrac{1}{2}e^{j\pi t} - \tfrac{1}{4}e^{j2\pi t}.
\end{aligned}
$$

Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = \pi$ (i.e., $T = 2$) and

$$c_k = \begin{cases} \frac{3}{2} & k = 0 \\ \frac{1}{2} & k = \pm 1 \\ -\frac{1}{4} & k = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

4 **Answer (c).**

The function x is periodic with period $T = \frac{1}{2}$ and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1/2} = 4\pi$. From the Fourier series analysis equation, we have

$$
\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{1/2}\int_0^{1/2} |\sin(2\pi t)|\, e^{-jk4\pi t}dt \\
&= 2\int_0^{1/2} e^{-j4\pi k t}\sin(2\pi t)dt \\
&= 2\left[\frac{e^{-j4\pi k t}[-j4\pi k\sin(2\pi t) - 2\pi\cos(2\pi t)]}{(-j4\pi k)^2 + (2\pi)^2}\right]\Bigg|_0^{1/2} \\
&= \tfrac{2(2\pi)}{-16\pi^2k^2+4\pi^2}\left[e^{-j4\pi k t}\left[-j2k\sin(2\pi t) - \cos(2\pi t)\right]\right]\Big|_0^{1/2} \\
&= \tfrac{1}{\pi(1-4k^2)}\left[e^{-j4\pi k/2}[-j2k\sin(2\pi/2) - \cos(2\pi/2)] - [-\cos 0]\right] \\
&= \tfrac{1}{\pi(1-4k^2)}[e^{-j2\pi k}[-j2k\sin(\pi) - \cos(\pi)] + \cos(0)] \\
&= \tfrac{1}{\pi(1-4k^2)}[2] \\
&= \frac{2}{\pi(1-4k^2)}.
\end{aligned}
$$

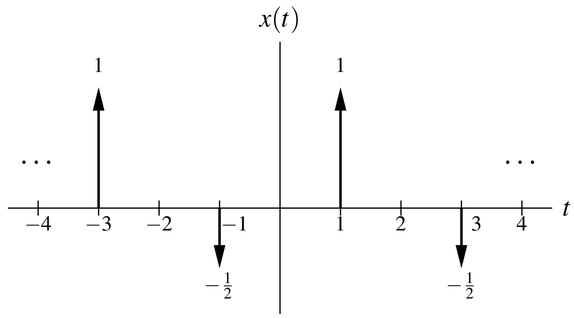
Since the integral table entry given (i.e., for the integral $\int e^{ax} \sin(bx) dx$) is valid for the case of $a = 0$, we did not need to assume that $k \neq 0$ in the above integration. Therefore, the above expression is valid for all k. Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

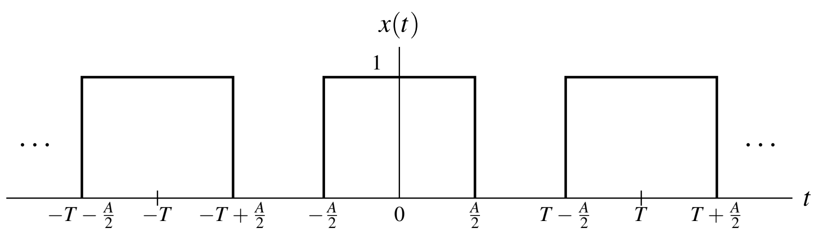
where $\omega_0 = 4\pi$ and

$$c_k = \frac{2}{\pi(1-4k^2)}.$$

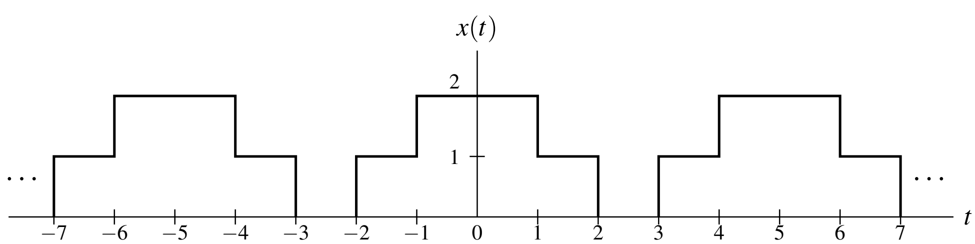
4 **5.2** For each of the periodic functions shown in the figures below, find the corresponding Fourier series coefficient sequence.



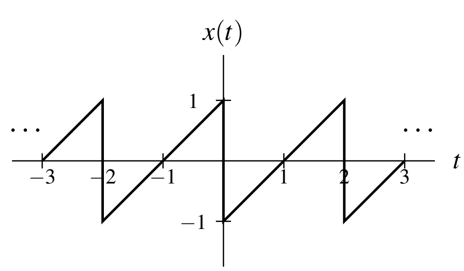
(a)



(b)



(c)



(d)

4 **Answer (a).**

We calculate the fundamental frequency ω_0 as

$$\omega_0 = \tfrac{2\pi}{T} = \tfrac{2\pi}{4} = \tfrac{\pi}{2}.$$

(So, we have $T = \frac{2\pi}{\pi/2} = 4$.) From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{4}\int_{-2}^{2}[\delta(t-1)-\tfrac{1}{2}\delta(t+1)]e^{-j\pi kt/2}dt \\
&= \tfrac{1}{4}\left[\int_{-2}^{2}\delta(t-1)e^{-j\pi kt/2}dt - \tfrac{1}{2}\int_{-2}^{2}\delta(t+1)e^{-j\pi kt/2}dt\right] \\
&= \tfrac{1}{4}[e^{-j\pi k/2} - \tfrac{1}{2}e^{j\pi k/2}] \\
&= \tfrac{1}{4}e^{-j\pi k/2} - \tfrac{1}{8}e^{j\pi k/2} \\
&= \tfrac{1}{4}(-j)^k - \tfrac{1}{8}j^k.
\end{aligned}$$

4 **Answer (c).**

The function x is periodic with period $T = 5$ and frequency $\omega_0 = \frac{2\pi}{5}$. From the Fourier series analysis equation, we can write

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{5}\int_{-5/2}^{5/2}x(t)e^{-j2\pi kt/5}dt \\
&= \tfrac{1}{5}\left[\int_{-2}^{-1}e^{-j2\pi kt/5}dt + \int_{-1}^{1}2e^{-j2\pi kt/5}dt + \int_{1}^{2}e^{-j2\pi kt/5}dt\right] \\
&= \tfrac{1}{5}\left[\int_{-2}^{2}e^{-j2\pi kt/5}dt + \int_{-1}^{1}e^{-j2\pi kt/5}dt\right] \\
&= \tfrac{1}{5}\left[\left[\tfrac{1}{-j2\pi k/5}e^{-j2\pi kt/5}\right]\Big|_{-2}^{2} + \left[\tfrac{1}{-j2\pi k/5}e^{-j2\pi kt/5}\right]\Big|_{-1}^{1}\right] \quad \text{for } k \neq 0 \\
&= \tfrac{1}{-j2\pi k}\left[e^{-j2\pi kt/5}\Big|_{-2}^{2} + e^{-j2\pi kt/5}\Big|_{-1}^{1}\right] \\
&= \tfrac{1}{-j2\pi k}\left[e^{-j4\pi k/5} - e^{j4\pi k/5} + e^{-j2\pi k/5} - e^{j2\pi k/5}\right] \\
&= \tfrac{1}{-j2\pi k}[-2j\sin(4\pi k/5) - 2j\sin(2\pi k/5)] \\
&= \tfrac{1}{\pi k}[\sin(4\pi k/5) + \sin(2\pi k/5)] \\
&= \frac{\sin(4\pi k/5)}{\pi k} + \frac{\sin(2\pi k/5)}{\pi k} \\
&= \tfrac{4}{5}\operatorname{sinc}(4\pi k/5) + \tfrac{2}{5}\operatorname{sinc}(2\pi k/5).
\end{aligned}$$

In the above derivation, we assumed that $k \neq 0$. So, now we must consider the case of $k = 0$. From the Fourier series analysis equation, we have

$$\begin{aligned}
c_0 &= \tfrac{1}{T}\int_T x(t)dt \\
&= \tfrac{1}{5}\int_{-5/2}^{5/2}x(t)dt \\
&= \tfrac{1}{5}\left[\int_{-2}^{-1}dt + \int_{-1}^{1}2dt + \int_{1}^{2}dt\right] \\
&= \tfrac{1}{5}[1+4+1] \\
&= \tfrac{6}{5}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned} c_k &= \begin{cases} \frac{6}{5} & k = 0 \\ \frac{4}{5}\operatorname{sinc}(4\pi k/5) + \frac{2}{5}\operatorname{sinc}(2\pi k/5) & \text{otherwise} \end{cases} \\ &= \frac{4}{5}\operatorname{sinc}(4\pi k/5) + \frac{2}{5}\operatorname{sinc}(2\pi k/5). \end{aligned}$$

The first few coefficients are approximately as follows:

$$c_0 = 1.2, \quad c_1 = c_{-1} \approx 0.489828, \quad \text{and} \quad c_2 = c_{-2} \approx -0.057816.$$

4 **5.6** A periodic function x with period T and Fourier series coefficient sequence c is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if x is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all t, then x is odd harmonic.

4 **Answer (a,b).**

Using the Fourier series synthesis equation, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{5.1}$$

Substituting $t - \frac{T}{2}$ for t in the preceding equation, we obtain

$$\begin{aligned} x(t - \tfrac{T}{2}) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t-T/2)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jk\omega_0 T/2} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-j\pi k} \\ &= \sum_{k=-\infty}^{\infty} (-1)^k c_k e^{jk\omega_0 t}. \end{aligned}$$

(Note that, in simplifying the above equation, we used the fact that $\frac{T}{2} = \frac{\pi}{\omega_0}$ and $e^{-j\pi k} = (-1)^k$.) Thus, we have

$$x(t - \tfrac{T}{2}) = \sum_{k=-\infty}^{\infty} (-1)^k c_k e^{jk\omega_0 t}. \tag{5.2}$$

Using (5.1) and (5.2), we can write

$$\begin{aligned} & x(t) = -x\left(t - \tfrac{T}{2}\right) \\ \Leftrightarrow\quad & \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = -\sum_{k=-\infty}^{\infty} (-1)^k c_k e^{jk\omega_0 t} \\ \Leftrightarrow\quad & \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (-1)^{k+1} c_k e^{jk\omega_0 t} \\ \Leftrightarrow\quad & c_k = (-1)^{k+1} c_k = \begin{cases} c_k & k \text{ odd} \\ -c_k & k \text{ even} \end{cases} \\ \Leftrightarrow\quad & c_k = 0 \text{ for even } k. \end{aligned}$$

Thus, we have shown that x is odd harmonic if and only if $x(t) = -x(t - \frac{T}{2})$ for all t.

4 **Answer (b[alternative]).**

ALTERNATIVE SOLUTION. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} \left[\int_0^{T/2} x(t) e^{-jk\omega_0 t} dt + \int_{T/2}^{T} x(t) e^{-jk\omega_0 t} dt \right]. \end{aligned}$$

Now, we employ a change a variable in the second integral. Let $\lambda = t + T/2$ so that $t = \lambda - T/2$ and $d\lambda = dt$. Applying this change of variable, we obtain

$$\begin{aligned}
c_k &= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \int_T^{3T/2} x(\lambda - \tfrac{T}{2})e^{-jk\omega_0(\lambda - T/2)}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \int_T^{3T/2} -x(\lambda)e^{-jk\omega_0(\lambda - T/2)}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - \int_T^{3T/2} x(\lambda)e^{jk\omega_0 T/2}e^{-jk\omega_0\lambda}d\lambda\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - e^{jk\pi}\int_T^{3T/2} x(\lambda)e^{-jk\omega_0\lambda}d\lambda\right].
\end{aligned}$$

Now, we rename the dummy variable of integration in the second integral from λ to t. This yields

$$\begin{aligned}
c_k &= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - e^{j\pi k}\int_T^{3T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \tfrac{1}{T}\left[\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt - (-1)^k\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \tfrac{1}{T}\left[(1-(-1)^k)\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt\right] \\
&= \begin{cases} \tfrac{2}{T}\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}
\end{aligned}$$

Therefore, $c_k = 0$ for even k.

4 Answer (b[alternative]).

ALTERNATIVE SOLUTION. From the Fourier series analysis equation, we have

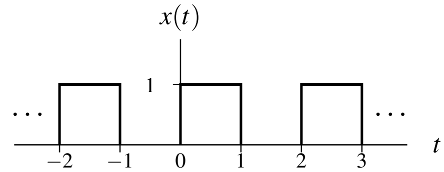
$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{T}\int_T (-x(t-T/2))e^{-jk\omega_0 t}dt \\
&= -\tfrac{1}{T}\int_T x(t-T/2)e^{-jk\omega_0 t}dt \\
&= -\tfrac{1}{T}\int_\alpha^{\alpha+T} x(t-T/2)e^{-jk\omega_0 t}dt.
\end{aligned}$$

Now, we employ a change of variable. Let $v = t - T/2$ so that $t = v + T/2$ and $dv = dt$. Applying the change of variable, we obtain

$$\begin{aligned}
c_k &= -\tfrac{1}{T}\int_{\alpha-T/2}^{\alpha+T/2} x(v)e^{-jk\omega_0(v+T/2)}dv \\
&= -\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}e^{-jk\omega_0 T/2}dv \\
&= -\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}e^{-jk(2\pi/2)}dv \\
&= (-1)^k\left(-\tfrac{1}{T}\int_T x(v)e^{-jk\omega_0 v}dv\right) \\
&= (-1)^k(-c_k) \\
&= (-1)^{k+1}c_k.
\end{aligned}$$

So, we have that $c_k = (-1)^{k+1} c_k$. If k is even, then $c_k = -c_k$. This implies, however, that $c_k = 0$. Therefore, for even k, we have that $c_k = 0$.

4 **5.8** Find the Fourier series coefficient sequence c of the periodic function x shown in the figure below. Plot the frequency spectrum of x, including the first five harmonics.



4 **Answer.**

The signal x is periodic with period $T = 2$ and frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$. From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \tfrac{1}{2}\int_0^2 x(t)e^{-j\pi kt}dt \\
&= \tfrac{1}{2}\int_0^1 e^{-j\pi kt}dt \\
&= \tfrac{1}{2}\left[\tfrac{1}{-j\pi k}e^{-j\pi kt}\right]\Big|_0^1 \quad \text{for } k \neq 0 \\
&= \tfrac{-1}{j2\pi k}\left[e^{-j\pi kt}\right]\Big|_0^1 \\
&= \tfrac{1}{j2\pi k}\left[1 - e^{-j\pi k}\right] \\
&= \tfrac{1}{j2\pi k}\left[1 - (-1)^k\right] \\
&= \begin{cases} -\frac{j}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0 \end{cases}
\end{aligned}$$

Since we assumed that $k \neq 0$ in the derivation above, we must now consider the case of $k = 0$. From the Fourier series analysis equation, we have

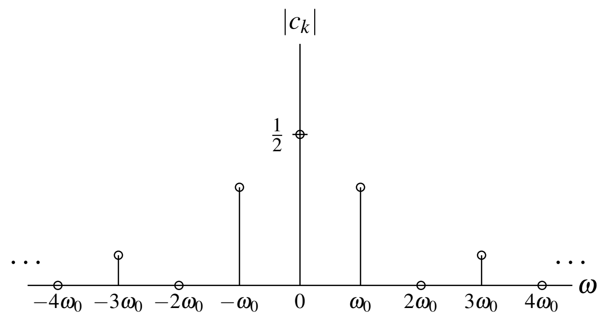
$$\begin{aligned}
c_0 &= \tfrac{1}{T}\int_T x(t)dt \\
&= \tfrac{1}{2}\int_0^2 x(t)dt \\
&= \tfrac{1}{2}\int_0^1 dt \\
&= \tfrac{1}{2}[t]|_0^1 \\
&= \tfrac{1}{2}.
\end{aligned}$$

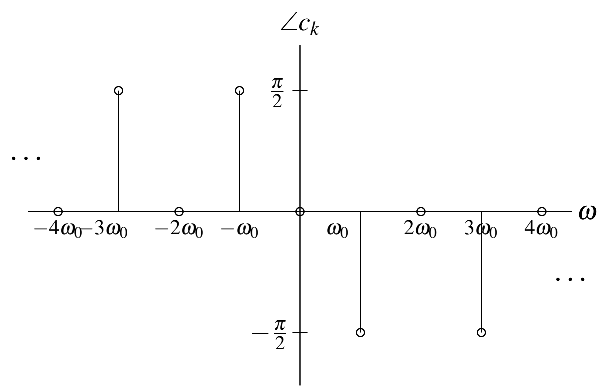
Thus, we have

$$c_k = \begin{cases} \frac{1}{2} & k = 0 \\ -\frac{j}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0 \end{cases}$$

Calculating the first several Fourier series coefficients yields the following:

k	$\lvert c_k \rvert$	$\arg c_k$
0	$\frac{1}{2}$	0
1	$\frac{1}{\pi}$	$-\frac{\pi}{2}$
2	0	0
3	$\frac{1}{3\pi}$	$-\frac{\pi}{2}$
4	0	0
5	$\frac{1}{5\pi}$	$-\frac{\pi}{2}$





4 **5.9** Consider a LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output y of the system if the input x is given by

$$x(t) = 1 + 2\cos(2t) + 2\cos(4t) + \tfrac{1}{2}\cos(6t).$$

4 **Answer.**

We begin by finding the Fourier series representation of x. Using Euler's relation, we can rewrite x as

$$\begin{aligned} x(t) &= 1 + 2\cos(2t) + 2\cos(4t) + \tfrac{1}{2}\cos(6t) \\ &= 1 + 2\left[\tfrac{1}{2}(e^{j2t} + e^{-j2t})\right] + 2\left[\tfrac{1}{2}(e^{j4t} + e^{-j4t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6t} + e^{-j6t})\right] \\ &= 1 + e^{j2t} + e^{-j2t} + e^{j4t} + e^{-j4t} + \tfrac{1}{4}e^{j6t} + \tfrac{1}{4}e^{-j6t}. \end{aligned}$$

Thus, we have that the Fourier series representation of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2$ and

$$a_k = \begin{cases} 1 & k = 0 \\ 1 & k = \pm 1 \\ 1 & k = \pm 2 \\ \tfrac{1}{4} & k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Since the system is LTI, we know that the output y has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $b_k = a_k H(k\omega_0)$. Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned} b_0 &= a_0 H([0][2]) = 0, \\ b_1 &= a_1 H([1][2]) = 0, \\ b_{-1} &= a_{-1} H([-1][2]) = 0, \\ b_2 &= a_2 H([2][2]) = 0, \\ b_{-2} &= a_{-2} H([-2][2]) = 0, \\ b_3 &= a_3 H([3][2]) = \tfrac{1}{4}(1) = \tfrac{1}{4}, \quad \text{and} \\ b_{-3} &= a_{-3} H([-3][2]) = \tfrac{1}{4}(1) = \tfrac{1}{4}. \end{aligned}$$

Thus, we have

$$b_k = \begin{cases} \tfrac{1}{4} & k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the output y is given by

$$\begin{aligned} y(t) &= \tfrac{1}{4}e^{-j6t} + \tfrac{1}{4}e^{j6t} \\ &= \tfrac{1}{4}[e^{-j6t} + e^{j6t}] \\ &= \tfrac{1}{4}[2\cos(6t)] \\ &= \tfrac{1}{2}\cos(6t). \end{aligned}$$

4 **5.101** Consider the periodic function x shown in Figure B of Exercise 5.2, where $T = 1$ and $A = \frac{1}{2}$. We can show that x has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2} \operatorname{sinc}\left(\frac{\pi k}{2}\right)$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N$ converges to x. [Hint: You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Note that the MATLAB `sinc` function does not compute the sinc function as defined herein. Instead, the MATLAB `sinc` function computes the normalized sinc function as defined by (3.21).]
(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N$ converge to x uniformly (i.e., at the same rate everywhere)? If not, where is the rate of convergence slower?
(c) The function x is not continuous everywhere. For example, x has a discontinuity at $\frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

4 **Answer (a,b,c).**

(a) The graphs necessary in this problem can be generated using the code given below.

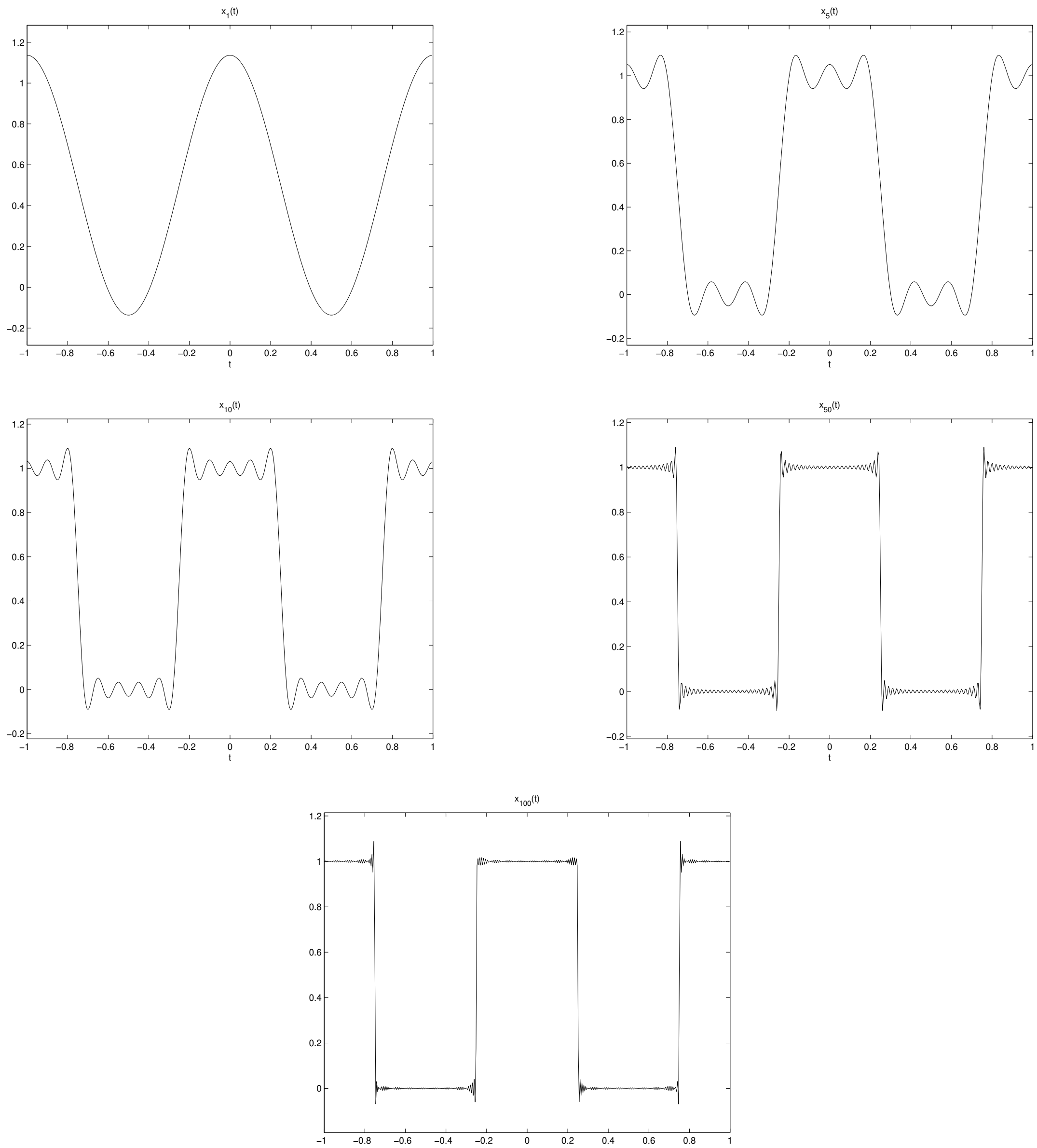
Listing 5.1: main.m

```
clear all

syms k t;

% Define a function that is one at the origin and zero elsewhere.
delta = @(t) 1 - abs(heaviside(-t) - heaviside(t));

% Define the sinc function in a manner that avoids division by zero when
% the function is evaluated at the origin.
mysinc = @(t) (sin(t) + delta(t)) / (t + delta(t));

w = 2 * pi;

for n = [1 5 10 50 100]

    % Sum the appropriate number of terms.
    f = symsum(0.5 * mysinc(pi / 2 * k) * exp(j * k * w * t), k, -n, n);

    % Plot the result.
    ezplot(f, [-1 1]);
    title(sprintf('x_{%d}(t)', n));

    % Pause for a moment so that the graph does not disappear too quickly.
    pause(1);

    % Print the graph to a file.
    eval(sprintf('print -dps data/sqrwav_%d.ps', n));
```



```
end
```

Using the above code, we obtain the graphs given below.



(b) The function $\hat{x}_N(t)$ does not converge to $x(t)$ uniformly (i.e., at the same rate everywhere). The rate of convergence is (relatively) lower at/near the points of discontinuity of $x(t)$.

(c) At the point of discontinuity of $x(t)$ located at $t = \frac{1}{4}$, the function $\hat{x}_N(t)$ appears to converge to the average of the left and right limits of $x(t)$ at that point, namely the value of $\frac{1}{2}$.