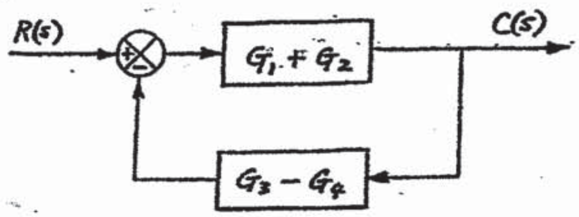
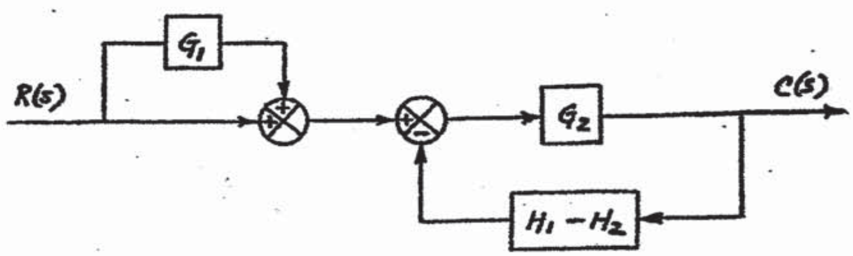
CHAPTER 2

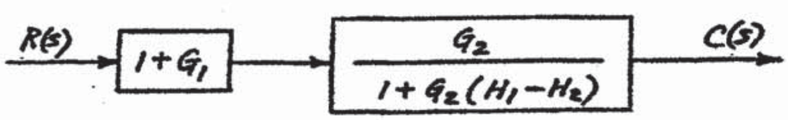
B-2-1.



$$\frac{C(s)}{R(s)} = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_3 - G_4)}$$

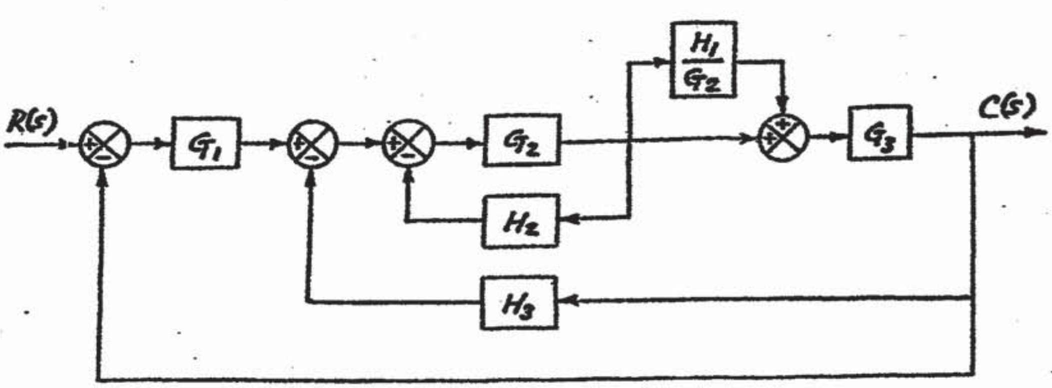
B-2-2.

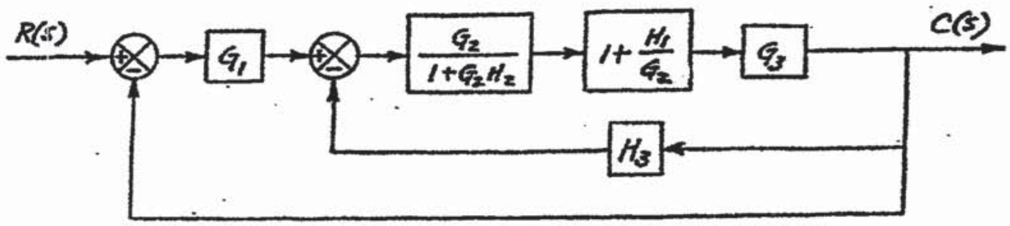


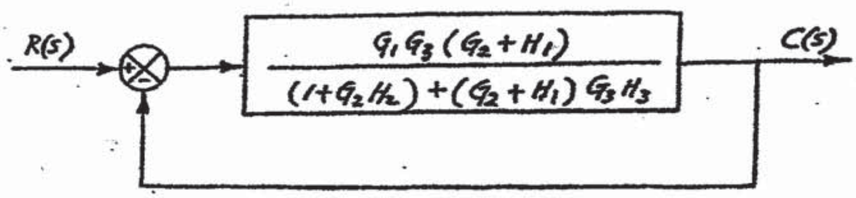


$$\frac{C(s)}{R(s)} = \frac{(1 + G_1)G_2}{1 + G_2(H_1 - H_2)}$$

B-2-3.

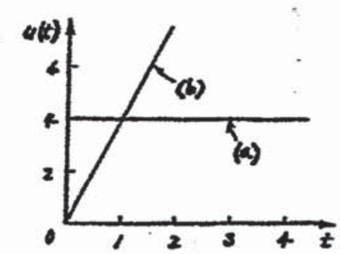




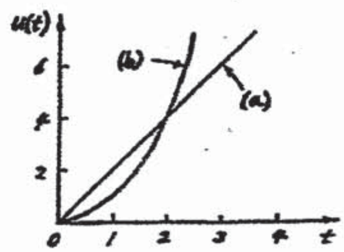


$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_3H_1}{1 + G_2H_2 + G_2G_3H_3 + G_3H_1H_3 + G_1G_2G_3 + G_1G_3H_1}$$

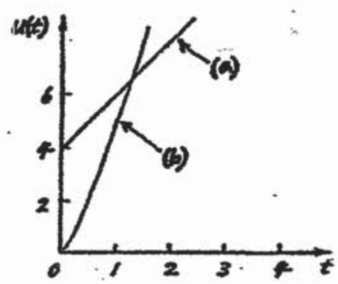
B-2-4. In the following diagrams, (a) denotes the unit-step response and (b) corresponds to the unit-ramp response.



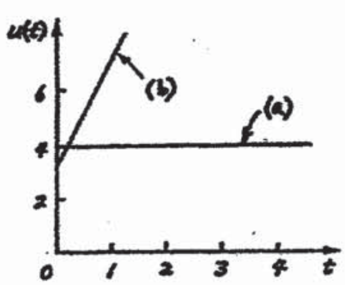
$\frac{U(s)}{E(s)} = K_p = 4$



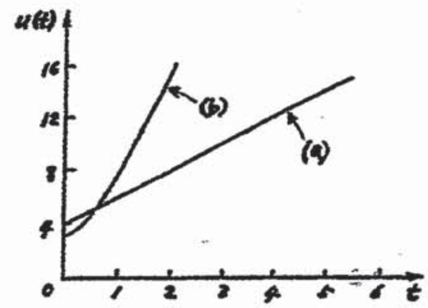
$\frac{U(s)}{E(s)} = \frac{K_i}{s} = \frac{2}{s}$



$\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right) = 4\left(1+\frac{1}{2s}\right)$



$\frac{U(s)}{E(s)} = K_p(1+T_d s) = 4(1+0.8s)$



$$\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}+T_d s\right) = 4\left(1+\frac{1}{2s}+0.8s\right)$$

B-2-5. When D(s) is zero, the closed-loop transfer function $C_R(s)/R(s)$ is

$$\frac{C_R(s)}{R(s)} = \frac{G_c(s)\,G_p(s)}{1+G_c(s)\,G_p(s)}$$

When R(s) = 0, the closed-loop transfer function $C_D(s)/D(s)$ is

$$\frac{C_D(s)}{D(s)} = \frac{1}{1+G_c(s)\,G_p(s)}$$

When both the reference input and disturbance input are present, the output C(s) is the sum of $C_R(s)$ and $C_D(s)$. Hence

$$C(s) = C_R(s) + C_D(s) = \frac{1}{1+G_c(s)\,G_p(s)}\left[G_c(s)\,G_p(s)\,R(s) + D(s)\right]$$

B-2-6. When only the reference input R(s) is present, the output $C_R(s)$ is given by

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1+G_1(s)\,G_2(s)}$$

For the reference input R(s), the desired output is R(s) for the unity-feedback system such as the present system. Thus, the error $E_R(s)$ is the difference between R(s) and the actual output $C_R(s)$. The error $E_R(s)$ is given by

$$E_R(s) = R(s) - C_R(s) = R(s)\left[1 - \frac{C_R(s)}{R(s)}\right]$$

$$= R(s)\left[1 - \frac{G_1(s)\,G_2(s)}{1+G_1(s)\,G_2(s)}\right] = \frac{1}{1+G_1(s)\,G_2(s)}\,R(s)$$

Assuming the system to be stable, the steady-state error $e_{ssR}(t)$ can be given by

$$e_{ssR}(t) = \lim_{t\to\infty} e_R(t) = \lim_{s\to 0} s\,E_R(s) = \lim_{s\to 0} \frac{s\,R(s)}{1+G_1(s)\,G_2(s)}$$

When only the disturbance input D(s) is present, the output $C_D(s)$ is given by

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)}$$

Since the desired output to the disturbance input D(s) is zero, the error $E_D(s)$ can be given by

$$E_D(s) = 0 - C_D(s) = - C_D(s)$$

Hence

$$E_D(s) = -C_D(s) = - \frac{G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

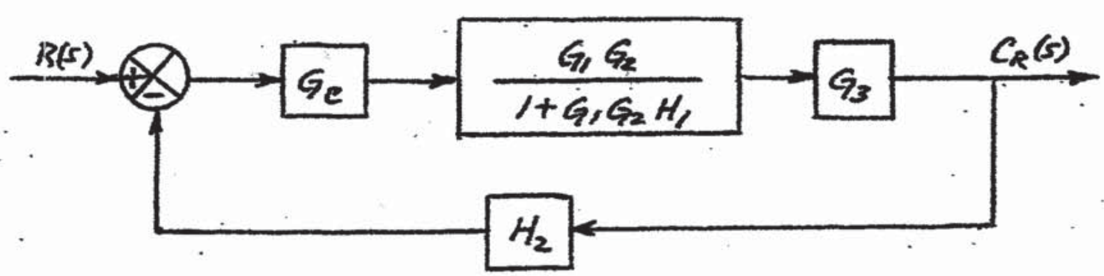
For the stable system, the steady-state error $e_{ssD}(t)$ is given by

$$e_{ssD}(t) = \lim_{t \to \infty} e_D(t) = \lim_{s \to 0} s E_D(s) = \lim_{s \to 0} \frac{-s G_2(s) D(s)}{1 + G_1(s)G_2(s)}$$

The steady-state error when both the reference input R(s) and disturbance input D(s) are present is the sum of $e_{ssR}(t)$ and $e_{ssD}(t)$ and is given by

$$\begin{aligned} e_{ss}(t) &= e_{ssR}(t) + e_{ssD}(t) \\ &= \lim_{s \to 0} \left[\frac{sR(s)}{1+G_1(s)G_2(s)} - \frac{sG_2(s)D(s)}{1+G_1(s)G_2(s)} \right] \\ &= \lim_{s \to 0} \left\{ \frac{s}{1+G_1(s)G_2(s)} \left[R(s) - G_2(s)D(s) \right] \right\} \end{aligned}$$

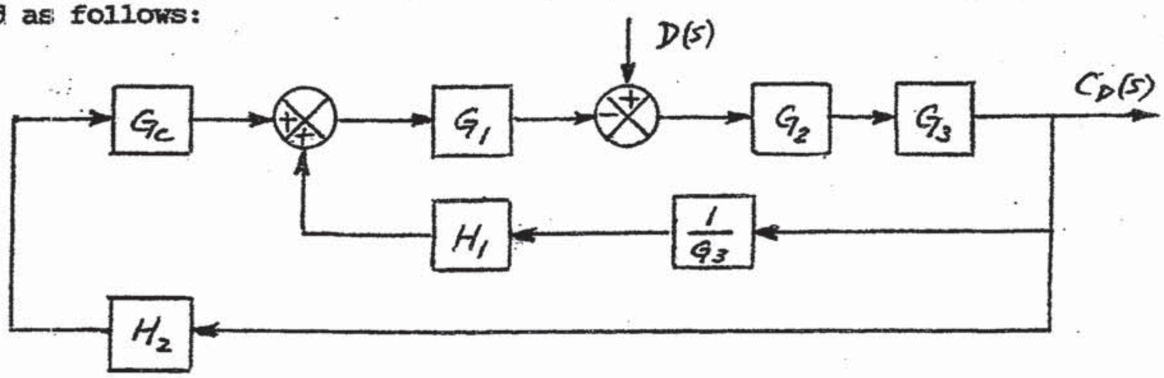
B-2-7. When D(s) = 0, the block diagram of the system can be simplified as follows:

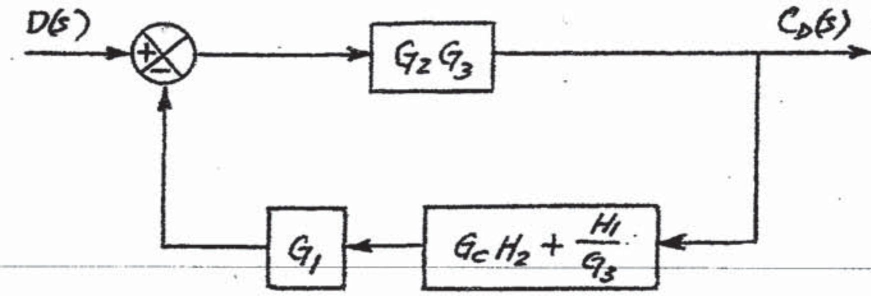


The closed-loop transfer function $C_R(s)/R(s)$ can be given by

$$\frac{C_R(s)}{R(s)} = \frac{\dfrac{G_c G_1 G_2 G_3}{1+G_1 G_2 H_1}}{1 + \dfrac{G_c G_1 G_2 G_3 H_2}{1+G_1 G_2 H_1}} = \frac{G_c G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_c G_1 G_2 G_3 H_2}$$

When $R(s) = 0$, the block diagram of the system shown in Figure 2-34 can be modified as follows:





Hence

$$\frac{C_D(s)}{D(s)} = \frac{G_2 G_3}{1+G_2 G_3 G_1 \left(G_c H_2 + \frac{H_1}{G_3}\right)} = \frac{G_2 G_3}{1+G_1 G_2 G_3 G_c H_2 + G_1 G_2 H_1}$$

B-2-8. There are infinitely many state-space representations for this system. We shall give two of the possible state-space representations.

State-space representation 1: From Figure 2-35, we obtain

$$\frac{Y(s)}{U(s)} = \frac{\frac{s+z}{s+p}\,\frac{1}{s^2}}{1+\frac{s+z}{s+p}\,\frac{1}{s^2}} = \frac{s+z}{s^3+ps^2+s+z}$$

which is equivalent to

$$\dddot{y} + p\ddot{y} + \dot{y} + zy = \dot{u} + zu$$

Comparing this equation with the standard equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we obtain

$$a_1 = p, \quad a_2 = 1, \quad a_3 = z, \quad b_0 = 0, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = z$$

Define

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

where

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1 \beta_0 = 0$$
$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 = 1$$

Also, define

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0 = z - p$$

Then, state-space equations can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z & -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

or

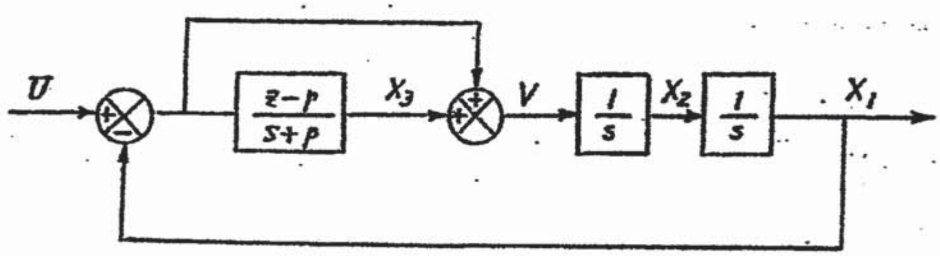
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -z & -1 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z-p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>State-space representation 2:</u> Since

$$\frac{s+z}{s+p} = \frac{s+p+z-p}{s+p} = 1 + \frac{z-p}{s+p}$$

we can redraw the block diagram as shown below.



From this block disgram we get the following equations:

$$V = U - X_1 + X_3$$

$$\frac{X_3}{U - X_1} = \frac{z-p}{s+p}$$

$$\frac{X_2}{U - X_1 + X_3} = \frac{1}{s}$$

$$\frac{X_1}{X_2} = \frac{1}{s}$$

from which we obtain

$$\dot{x}_3 + p x_3 = (z-p)u - (z-p)x_1$$

$$\dot{x}_2 = -x_1 + x_3 + u$$

$$\dot{x}_1 = x_2$$

Rewriting, we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + x_3 + u$$

$$\dot{x}_3 = -(z-p)x_1 - p x_3 + (z-p)u$$

$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ p-z & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z-p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-2-9 .

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = u$$

Define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

Then

$$\dot{x}_3 + 3x_3 + 2x_2 = u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_1 = x_2$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-2-10 .

$$\underset{\sim}{A} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underset{\sim}{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The transfer function G(s) of the system is given by

$$G(s) = \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \frac{1}{(s+4)(s+1)+3} \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+7} [1 \quad 0] \begin{bmatrix} s \\ s+7 \end{bmatrix}$$

$$= \frac{s}{s^2+5s+7}$$

B-2-11.

$$\underset{\sim}{A} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \underset{\sim}{C} = [1 \quad 2]$$

The transfer function G(s) of the system is given by

$$G(s) = \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B} = [1 \quad 2] \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= [1 \quad 2] \frac{1}{(s+5)(s+1)+3} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+8} [1 \quad 2] \begin{bmatrix} 2s-3 \\ 5s+31 \end{bmatrix} = \frac{12s+59}{s^2+6s+8}$$

A MATLAB solution to this problem is given below.

```
A=[-5  -1;3  -1];
B=[2;5];
C=[1  2];
D=0;
[num,den]=ss2tf(A,B,C,D)

num =

    0   12   59


den =

    1   6   8
```

B-2-12.

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -6 \end{bmatrix}, \quad \underset{\sim}{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \underset{\sim}{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The transfer matrix of the system can be given by

$$G(s) = \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 2 & 4 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{1}{s^3+6s^2+4s+2} \begin{bmatrix} s^2+s+4 & s+6 & 1 \\ -2 & s^2+6s & s \\ -2s & -4s-2 & s^2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^3+6s^2+4s+2} \begin{bmatrix} 1 & s+6 \\ s & s^2+6s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s^3+6s^2+4s+2} & \dfrac{s+6}{s^3+6s^2+4s+2} \\ \dfrac{s}{s^3+6s^2+4s+2} & \dfrac{s^2+6s}{s^3+6s^2+4s+2} \end{bmatrix}$$

A MATLAB solution to this problem is given below.

```
A=[0 1 0;0 0 1;-2 -4 -6];
B=[0 0;0 1;1 0];
C=[1 0 0;0 1 0];
D=[0 0;0 0];
[num,den] = ss2tf(A,B,C,D,1)

num =

     0   0.0000   0.0000   1.0000
     0   0.0000   1.0000   0.0000


den =

   1.0000   6.0000   4.0000   2.0000

[num,den] = ss2tf(A,B,C,D,2)

num =

     0   0.0000   1.0000   6.0000
     0   1.0000   6.0000   0.0000


den =

   1.0000   6.0000   4.0000   2.0000
```

B-2-13. Define

$$z = x^2 + 8xy + 3y^2 = f(x, y)$$
$$2 \leq x \leq 4, \quad 10 \leq y \leq 12$$

Let us choose $\bar{x} = 3$ and $\bar{y} = 11$. Then

$$\bar{z} = \bar{x}^2 + 8\bar{x}\bar{y} + 3\bar{y}^2 = 9 + 264 + 363 = 636$$

We shall obtain a linearized equation for the nonlinear equation near the point $\bar{x} = 3$, $\bar{y} = 11$. Expanding the nonlinear equation into a Taylor series about point $x = \bar{x}$, $y = \bar{y}$ and neglecting the higher-order terms, we obtain

$$z - \bar{z} = K_1 (x - \bar{x}) + K_2 (y - \bar{y})$$

where

$$K_1 = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\ y=\bar{y}} = 2\bar{x} + 8\bar{y}\Big|_{\bar{x}=3,\ \bar{y}=11} = 6 + 88 = 94$$

$$K_2 = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\ y=\bar{y}} = 8\bar{x} + 6\bar{y}\Big|_{\bar{x}=3,\ \bar{y}=11} = 24 + 66 = 90$$

Hence the linearized equation is

$$z - 636 = 94(x - 3) + 90(y - 11)$$

or

$$z = 94x + 90y - 636$$

B-2-14. Define

$$y = 0.2x^3 = f(x), \qquad \bar{x} = 2$$

Then

$$y = f(x) = f(\bar{x}) + \frac{\partial f}{\partial x}(x - \bar{x}) + \cdots$$

Since the higher-order terms in this equation are small, neglecting those terms, we obtain

$$y - f(\bar{x}) = 0.6\,\bar{x}^2 (x - \bar{x})$$

or

$$y - 0.2 \times 2^3 = 0.6 \times 2^2 (x - 2)$$

Thus, linear approximation of the given nonlinear equation near the operating point is

$$y = 2.4x - 3.2$$