Lecture 8: Pumping Lemma

CSC 320: Foundations of Computer Science

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Pumping Lemma

If $\boldsymbol{L}$ is a regular language, then there is a **natural number $\boldsymbol{p}$** (pumping length of $\boldsymbol{L}$) such that for **every string $\boldsymbol{s} \in \boldsymbol{L}$** of **length at least $\boldsymbol{p}$**, $\boldsymbol{s}$ can be **divided into $\boldsymbol{s = xyz}$** satisfying the following:

1. $|\boldsymbol{y}| > \boldsymbol{0}$ (i.e. $\boldsymbol{y} \neq \boldsymbol{\varepsilon}$)
2. $|\boldsymbol{xy}| \leq \boldsymbol{p}$
3. $\boldsymbol{xy^iz} \in \boldsymbol{L}$ for each $\boldsymbol{i} \geq \boldsymbol{0}$

Notes:

- $\boldsymbol{y^i}$ means concatenation of $\boldsymbol{i}$ copies of substring $\boldsymbol{y}$
- Conditions 1 to 3 hold **for all strings** in L that are of length at least $\boldsymbol{p}$
- We only use the pumping lemma to prove that languages are **non-regular**

Pumping Lemma Contradiction Proof

How to use the pumping lemma to prove a **language L is non-regular**:

- Assume for a **contradiction that L is regular**
- Let p be the (hypothetical) pumping length of L (do not assume a number for p)

- Pick **one string s** which $\in L$ and $|s| \geq p$
  - We should be able to **divide s into xyz** such that the three properties of the pumping lemma hold

- Show that it is **impossible** to write $s = xyz$ such that all three properties hold
  - Try to **satisfy** $|y| > 0$ and $|xy| \leq p$
  - Then show that $xy^iz \notin L$ for **some** $i \geq 0$ (violating property 3)

- We have a contradiction, therefore **L is not regular**

Pumping Lemma Example 1

Prove that $\boldsymbol{L} = \{\ \mathbf{0}^{\boldsymbol{n}}\mathbf{1}^{\boldsymbol{n}} \mid \boldsymbol{n} \geq \mathbf{0}\,\}$ is not regular.

Proof:

- Assume for a contradiction that $\boldsymbol{L}$ is regular
- Let $\boldsymbol{p}$ be the pumping length given by the pumping lemma.
- We choose $\boldsymbol{s} = \mathbf{0}^{\boldsymbol{p}}\mathbf{1}^{\boldsymbol{p}}$.
- Since $\boldsymbol{s} \in \boldsymbol{L}$ and $|\boldsymbol{s}| \geq \boldsymbol{p}$, according to the PL, we can rewrite $\boldsymbol{s} = \boldsymbol{xyz}$ satisfying
    1. $|\boldsymbol{y}| > \mathbf{0}$ (i.e. $\boldsymbol{y} \neq \boldsymbol{\varepsilon}$)
    2. $|\boldsymbol{xy}| \leq \boldsymbol{p}$
    3. $\boldsymbol{xy^iz} \in \boldsymbol{L}$ for each $\boldsymbol{i} \geq \mathbf{0}$

Pumping Lemma Example 1

$$L = \{\ 0^n 1^n \mid n \geq 0 \}$$

$$xyz$$

$$s = \underbrace{0 \ldots 0}_{p}\,\underbrace{1 \ldots 1}_{p}$$

- Because of **property 1**, $y \neq \varepsilon$, therefore y can be the following:
  - **Case 1:** y consists of only $\mathbf{0}$**'s**
    We **can't say exactly many $\mathbf{0}$'s**, we just know y is **some non-empty** number of $\mathbf{0}$'s in this case

    $$s = \underbrace{0}_{x}\,\underbrace{\ldots}_{y}\,\underbrace{0\ 1 \ldots 1}_{z}$$

    Example rewriting of $s = xyz$

  - **Case 2:** y consists of only $\mathbf{1}$**'s**
    We **can't say exactly many $\mathbf{1}$'s**, we just know y is **some non-empty** number of $\mathbf{1}$'s in this case

    $$s = \underbrace{0 \ldots 0\ 1}_{x}\,\underbrace{\ldots}_{y}\,\underbrace{1}_{z}$$

    Example rewriting of $s = xyz$

  - **Case 3:** y consists of both $\mathbf{0}$**'s** and $\mathbf{1}$**'s**
    We **can't say exactly many $\mathbf{0}$'s and $\mathbf{1}$'s**, we just know y is **some non-empty** number of $\mathbf{0}$'s and $\mathbf{1}$'s in this case

    $$s = \underbrace{0 \ldots}_{x}\,\underbrace{0\ 1}_{y}\,\underbrace{\ldots 1}_{z}$$

    Example rewriting of $s = xyz$

Pumping Lemma Example 1

$\boxed{L = \{\ 0^n 1^n \mid n \geq 0\}}$

Case 1: y consists of only $\mathbf{0}$**'s**

$$s = \underbrace{0}_{x}\underbrace{\ldots}_{y}\underbrace{0\,1\ldots1}_{z}$$

Example rewriting of $s = xyz$

- By **property 3**, $xy^iz \in L$ for each $i \geq 0$
- Consider the string $xy^2z = xyyz$:

$$\underbrace{0}_{x}\underbrace{\ldots}_{y}\underbrace{\ldots}_{y}\underbrace{0\,1\ldots1}_{z}$$

- $xyyz$ has more $\mathbf{0}$'s than $\mathbf{1}$'s, so $xyyz \notin L$
- This **violates property 3** of the pumping lemma.

Pumping Lemma Example 1

$L = \{\ 0^n 1^n \mid n \geq 0\ \}$

Case 2: y consists of only $\mathbf{1}$'s

$$s = \underbrace{0 \ldots 0\ 1}_{x} \underbrace{\ldots}_{y} \underbrace{1}_{z}$$

Example rewriting of $s = xyz$

- By **property 3**, $xy^iz \in L$ for each $i \geq 0$
- Consider the string $xy^2z = xyyz$:

$$\underbrace{0 \ldots 0\ 1}_{x} \underbrace{\ldots}_{y} \underbrace{\ldots}_{y} \underbrace{1}_{z}$$

- $xyyz$ has more $\mathbf{1}$'s than $\mathbf{0}$'s, so $xyyz \notin L$
- This **violates property 3** of the pumping lemma.

Pumping Lemma Example 1

$\boxed{L = \{\ 0^n 1^n \mid n \geq 0\ \}}$

<u>**Case 3:** y consists of both $\mathbf{0}$**'s** and $\mathbf{1}$**'s**</u>

$$s = \underbrace{0 \ldots}_{x}\underbrace{0\ 1}_{y}\underbrace{\ldots 1}_{z}$$

Example rewriting of $s = xyz$

- By **property 3**, $xy^iz \in L$ for each $i \geq 0$
- Consider the string $xy^2z = xyyz$:

$$\underbrace{0 \ldots}_{x}\underbrace{0\ 1 \ldots}_{y}\underbrace{0\ 1 \ldots}_{y}\underbrace{\ 1}_{z}$$

- $xyyz$ has $\mathbf{0}$'s and $\mathbf{1}$'s out of order, so $xyyz \notin L$
- This **violates property 3** of the pumping lemma.

Pumping Lemma Example 1

$L = \{\ 0^n 1^n \mid n \geq 0 \}$

- **Property 1** of the PL states that $y \neq \varepsilon$
- However, we showed that no matter what y is, we cannot satisfy $xy^iz \in L$ for each $i \geq 0$, which **violates property 3** of the PL
- Hence, it is **impossible** to rewrite $s = 0^p 1^p$ as $s = xyz$ to satisfy all three properties of the PL.
- This is a **contradiction** since **if L was regular**, then we **should be able to** for all $s \in L$ where $|s| \geq p$
- Therefore, L **is not regular**

Common Mistake

$\boxed{L = \{\ 0^n 1^n \mid n \geq 0\ \}}$

In case 3 of the previous proof, y consisted of both 0's and 1's:

$$s = \underbrace{0 \ldots}_{x}\underbrace{0\ 1}_{y}\underbrace{\ldots 1}_{z}$$

Example rewriting of $s = xyz$

- Could we have derived a **contradiction for property 3** of the pumping lemma by saying the string $xy^0z = xz \notin L$?

- **No**, because we never know exactly what y looks like.
    - y could be **equal number** of **0**'s and **1**'s, then $xy^0z \in L$

- So, to conclusively derive a contradiction, we needed to concatenate more y's to show that the resulting string $\notin L$

Pumping Lemma Property 2

- In the first pumping lemma example, we showed a contradiction **by only using properties 1 and 3** of the pumping lemma

- We can make our proof **shorter (fewer cases)** if we also use **property 2**
  - To satisfy property 1, $\boldsymbol{y}$ **is non empty**
  - To satisfy property 2 ($|\boldsymbol{xy}| \leq \boldsymbol{p}$), $\boldsymbol{xy}$ is within the **first $\boldsymbol{p}$ symbols** of $\boldsymbol{s}$

Pumping Lemma Example 2

Prove that $\boldsymbol{L} = \{\, \boldsymbol{w} \in \{\mathbf{0}, \mathbf{1}\}^* \mid \boldsymbol{w} \text{ has an equal number of } \mathbf{0}\text{s and } \mathbf{1}\text{s} \,\}$ is not regular.

Proof:

- Assume for a contradiction that $\boldsymbol{L}$ is regular
- Let $\boldsymbol{p}$ be the pumping length given by the pumping lemma.
- We choose $\boldsymbol{s} = \mathbf{1}^{\boldsymbol{p}}\mathbf{0}^{\boldsymbol{p}}$.
- Since $\boldsymbol{s} \in \boldsymbol{L}$ and $|\boldsymbol{s}| \geq \boldsymbol{p}$, according to the PL, we can rewrite $\boldsymbol{s} = \boldsymbol{xyz}$ satisfying
  1. $|\boldsymbol{y}| > \mathbf{0}$ (i.e. $\boldsymbol{y} \neq \boldsymbol{\varepsilon}$)
  2. $|\boldsymbol{xy}| \leq \boldsymbol{p}$
  3. $\boldsymbol{xy^iz} \in \boldsymbol{L}$ for each $\boldsymbol{i} \geq \mathbf{0}$

Pumping Lemma Example 2

$L = \{\boldsymbol{w} \in \{\mathbf{0}, \mathbf{1}\}^* \mid \boldsymbol{w} \text{ has an equal number of } \mathbf{0}\text{s and } \mathbf{1}\text{s}\}$

$$\boldsymbol{xyz}$$

$$\boldsymbol{s} = \underbrace{\mathbf{1} \ldots \mathbf{1}}_{\boldsymbol{p}}\ \underbrace{\mathbf{0} \ldots \mathbf{0}}_{\boldsymbol{p}}$$

$$\underbrace{\quad}_{\boldsymbol{x}}\underbrace{\quad}_{\boldsymbol{y}}\underbrace{\qquad\qquad}_{\boldsymbol{z}}$$

Example rewriting of $\boldsymbol{s} = \boldsymbol{xyz}$

- By **property 2**, $|\boldsymbol{xy}| \leq \boldsymbol{p}$, so $\boldsymbol{xy}$ must consist of **1**'s
  - We **can't say exactly many 1's**, we just know $\boldsymbol{xy}$ lies within the $\mathbf{1}$'s
- By **property 1**, $\boldsymbol{y} \neq \boldsymbol{\varepsilon}$, so $\boldsymbol{y}$ is some non-empty substring of $\mathbf{1}$'s

Pumping Lemma Example 2

$L = \{ w \in \{0,1\}^* \mid w \text{ has an equal number of } 0\text{s and } 1\text{s} \}$

$$s = \underbrace{1}_{x}\underbrace{\ldots}_{y}\underbrace{1\ 0 \ldots 0}_{z}$$

- By **property 3**, $xy^iz \in L$ for each $i \geq 0$
- Consider the string $xy^0z = xz$

$$\underbrace{1..}_{x}\underbrace{0 \ldots 0}_{z}$$

- The string $xz \notin L$ since it contains less 1's than 0's, which **violates property 3**
- We cannot rewrite $s = xyz$ satisfying all PL properties, so we have a contradiction.
- Therefore, L **is not regular**.

Pumping Lemma Example 3

Prove that $\boldsymbol{L} = \{\ \mathbf{0}^{\boldsymbol{i}}\mathbf{1}^{\boldsymbol{j}} \mid \boldsymbol{i} > \boldsymbol{j}\ \}$ is not regular.