CHAPTER 6

B-6-1. The open-loop transfer function for the system is

$$G(s)H(s) = \frac{K(s+1)}{s^2}$$

We first locate the open-loop poles and zero on the complex plane. A root locus exists on the negative real axis between -1 and $-\infty$. Since the open-loop transfer function involves two poles and one zero, there is a possibility that a circular root loci exists.

The equation for the root-locus branches can be obtained from the angle condition

$$\angle \frac{K(s+1)}{s^2} = \pm 180° (2k+1)$$

which can be rewritten as

$$\angle s+1 - 2\angle s = \pm 180° (2k+1)$$

By substituting $s = \sigma + j\omega$, we obtain

$$\angle \sigma + j\omega + 1 - 2\angle \sigma + j\omega = \pm 180° (2k+1)$$

or

$$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - 2\tan^{-1}\frac{\omega}{\sigma} = \pm 180° (2k+1)$$

Rearranging, we obtain

$$\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - \tan^{-1}\frac{\omega}{\sigma} = \tan^{-1}\frac{\omega}{\sigma} \pm 180° (2k+1)$$

Taking the tangents of both sides of this last equation,

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega}{\sigma}\right)\right] = \tan\left[\tan^{-1}\frac{\omega}{\sigma} \pm 180° (2k+1)\right]$$

which can be simplified to

$$\frac{\frac{\omega}{\sigma+1} - \frac{\omega}{\sigma}}{1 + \frac{\omega}{\sigma+1}\frac{\omega}{\sigma}} = \frac{\frac{\omega}{\sigma} \pm 0}{1 + \frac{\omega}{\sigma} \times 0} = \frac{\omega}{\sigma}$$

Hence

$$\frac{\omega}{\sigma+1} - \frac{\omega}{\sigma} = \frac{\omega}{\sigma}\left(1 + \frac{\omega}{\sigma+1}\frac{\omega}{\sigma}\right)$$

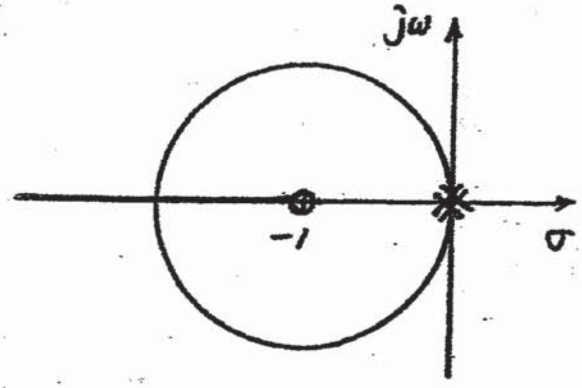
from which we obtain

$$\omega\left[(\sigma+1)^2 + \omega^2 - 1\right] = 0$$

This last equation is equivalent to

$$\omega = 0 \quad \text{or} \quad (\sigma+1)^2 + \omega^2 = 1$$

These two equations are the equations for the root loci for the system. The first equation, $\omega = 0$, is the equation for the real axis. The real axis from $s = -1$ to $s = -\infty$ corresponds to a root locus for $K \geq 0$. (The remaining part of the real axis corresponds to a root locus for $K < 0$.) In the present system, K is positive. The second equation is an equation of a circle with the center at $\sigma = -1$, $\omega = 0$ and the radius equal to 1. The root-locus diagram is shown below.



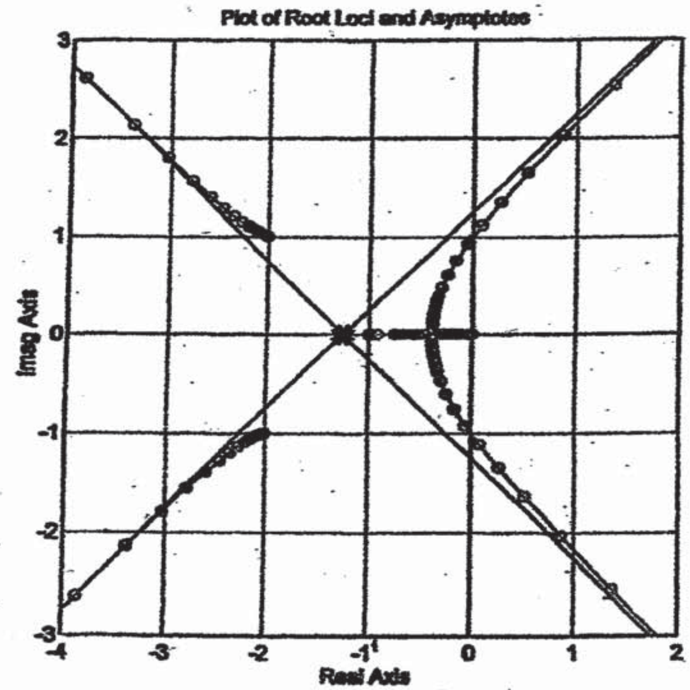
B-6-2. The open-loop transfer function

$$G(s)H(s) = \frac{K}{s(s+1)(s^2+4s+5)}$$

has the poles at $s = 0$, $s = -1$, $s = -2 \pm j1$ and no zeros. The asymptotes have angles $\pm 45°$ and $\pm 135°$. The asymptotes meet on the negative real axis at $\sigma_a = -1.25$. Two branches of the root loci cross the imaginary axis at $s = \pm j1$. The angle of departure from the complex pole in the upper half s plane is +162°.

A MATLAB program to plot the root loci and asymptotes is given below, together with the resulting root-locus plot.

```
% ***** Root-locus plot *****

num = [0  0   0   0   1];
den = [1   5   9   5.  0];
numa = [0  0   0   0   1];
dena = [1   5   9.375   7.8125   2.4414];
r = rlocus(num,den);
plot(r,'-')
hold
Current plot held
plot(r,'o')
rlocus(numa,dena)
v = [-4  2  -3  3]; axis(v); axis('square');
grid
title('Plot of Root Loci and Asymptotes')
```



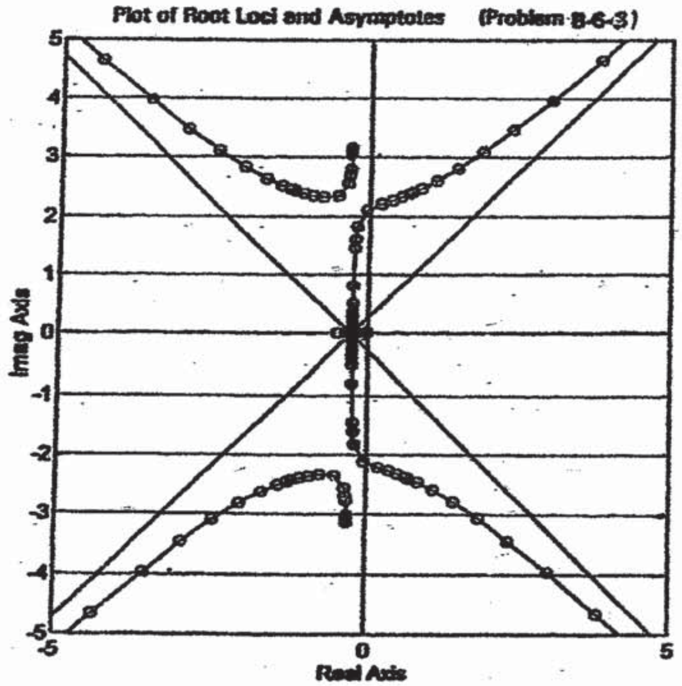
B-6-3. A MATLAB program to plot the root loci and asymptotes for the following system:

$$G(s)H(s) = \frac{K}{s(s+0.5)(s^2+0.6s+10)}$$

is given below and the resulting root-locus plot is shown on the next page.
Note that the equation for the asymptotes is

$$G_a(s)H_a(s) = \frac{K}{(s+0.275)^4} = \frac{K}{s^4+1.1s^3+0.45375s^2+0.0831874s+0.0057191}$$

```
% ***** Root-locus plot *****

num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
numa = [0  0  0  0  1];
dena = [1  1.1  0.45375  0.0831874  0.0057191];
r = rlocus(num,den);
plot(r,'-')
hold
Current plot held
plot(r,'o')
rlocus(numa,dena)

v = [-5  5  -5  5]; axis(v); axis('square');
grid
title('Plot of Root Loci and Asymptotes    (Problem B-6-3)')
```



B-6-4.

$$1+G(s)H(s)=\frac{(1+K)s^2+(2+6K)s+10+10K}{s^2+2s+10}$$

The characteristic equation

$$(1+K)s^2+(2+6K)s+10+10K=0$$

has two roots at

$$s=-\frac{1+3K}{1+K}\pm j\,\frac{\sqrt{K^2+14K+9}}{1+K}$$

If we write $s = X \pm jY$, that is

$$X=-\frac{1+3K}{1+K}\ ,\quad Y=\frac{\sqrt{K^2+14K+9}}{1+K}$$

then

$$X^2+Y^2=\left(\frac{1+3K}{1+K}\right)^2+\frac{K^2+14K+9}{(1+K)^2}=\frac{10(K+1)^2}{(1+K)^2}=10$$

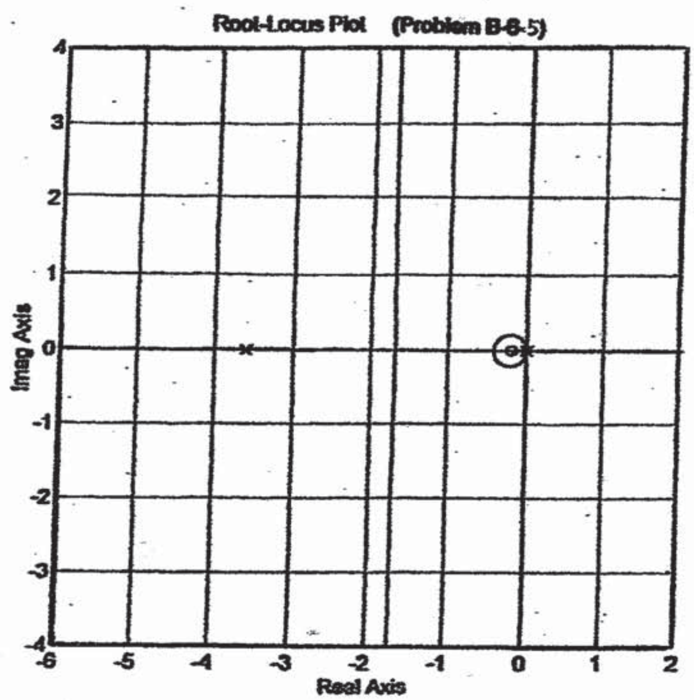
This indicates that the root loci are on a circle about the origin of radius $\sqrt{10}$.

B-6-5. The open-loop transfer function

$$G(s)H(s) = \frac{K(s+0.2)}{s^2(s+3.6)}$$

has the zero at s = −0.2 and the double poles at s = 0 and a single pole at s = −3.6. The asymptotes have angles of ±90°. The asymptotes meet on the real axis at $\sigma_a = -1.7$. The breakaway or break-in points are located at s = 0, s = −0.43155, and s = −1.6685. A MATLAB program to obtain the root locus plot is shown below. The resulting root-locus plot is shown below.

```
% ***** Root-locus plot *****

num = [0  0  1  0.2];
den = [1  3.6  0  0];
rlocus(num,den)
v = [-6  2  -4  4]; axis(v); axis('square')
grid
title('Root-Locus Plot     (Problem B-6-5)')
```



B-6-6. The open-loop transfer function

$$G(s)H(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

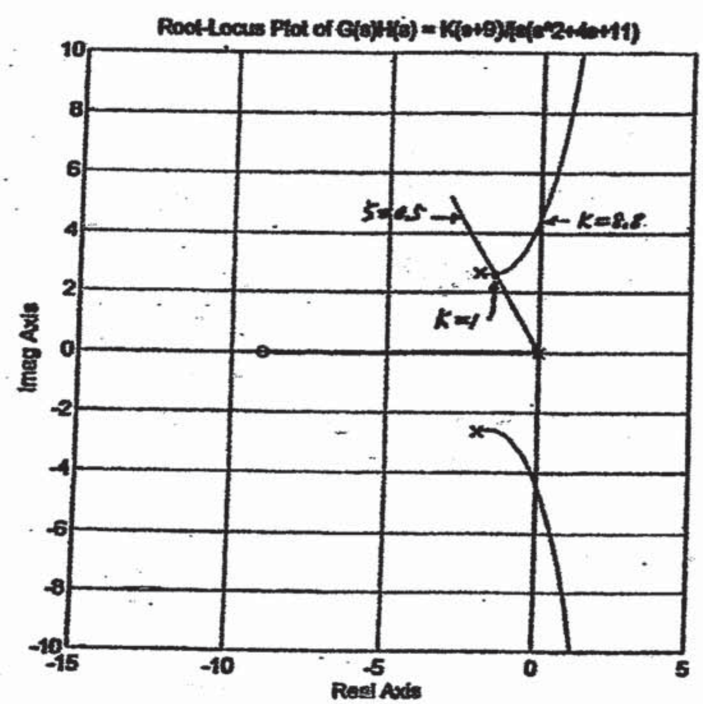
has the poles at $s = 0$, $s = -2 \pm j\sqrt{7}$ and the zero at $s = -9$. The asymptotes have angles $\pm 90°$ and meet the real axis at $\sigma_a = 2.5$. The complex branches cross the imaginary axis at $s = \pm j\,4.45$. The angle of departure from the complex pole in the upper half s plane is $-16.5°$.

The dominant closed-loop poles having the damping ratio $\zeta = 0.5$ can be located as the intersection of the root loci and lines from the origin having angles $\pm 60°$. The desired dominant closed-loop poles are found to be at

$$s = -1.5 \pm j\,2.598$$

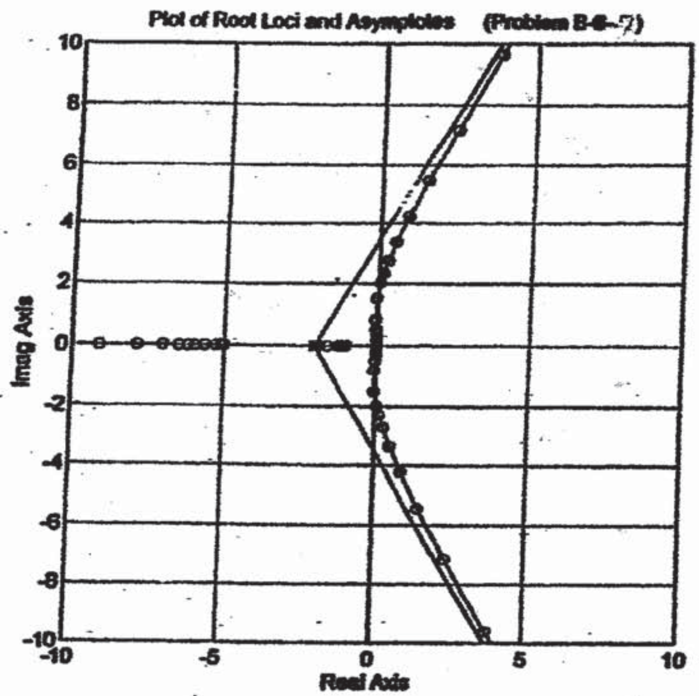
The third pole is at $s = -1$. The gain value corresponding to these dominant closed-loop poles is $K = 1$. A MATLAB program to plot the root-loci is shown below. The resulting root-locus plot is shown on the next page.

```
% ***** Root-locus plot *****

num = [0  0  1   9];
den = [1   4  11   0];
rlocus(num,den)
hold
Current plot held
x = [0,-3]; y = [0,5.196]; line(x,y);
v = [-15  5  -10   10]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s)H(s) = K(s+9)/[s(s^2+4s+11)]')
```



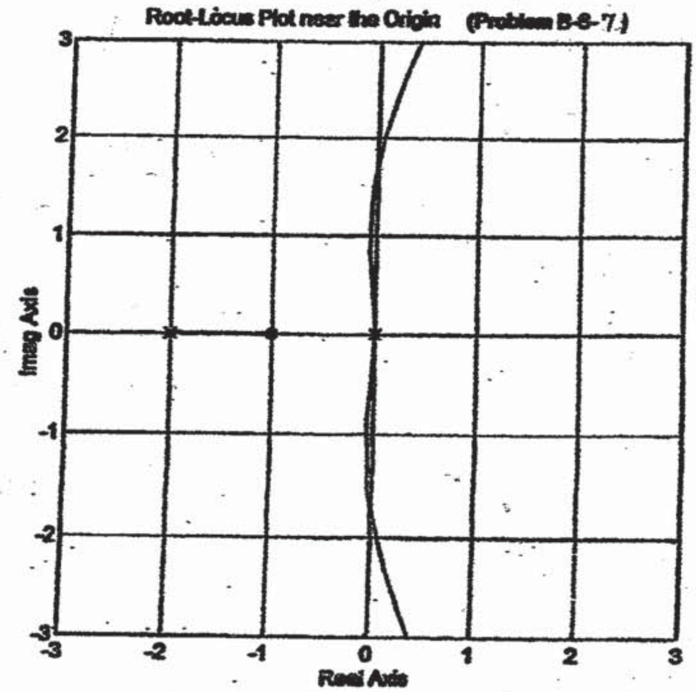
B-6-7. A MATLAB program to obtain a root-locus plot of the given system is shown below. The resulting root-locus plot is shown below.

```
% ***** Root-locus plot *****

num = [0  0  0  2  2];
den = [1  7  10  0  0];
numa = [0  0  0  1];
dena = [0.5  3  6  4];
r = rlocus(num,den);
plot(r,'-')
hold
Current plot held
plot(r,'o')
rlocus(numa,dena)
v = [-10  10  -10  10]; axis(v); axis('square');
grid
title('Plot of Root Loci and Asymptotes    (Problem B-6-7)')
```



A root-locus plot near the origin can be obtained by entering the following MATLAB program into the computer. The resulting root-locus plot near the origin is shown next.

```
% ***** Root-locus plot *****

num = [0  0  0  2  2];
den = [1  7  10  0  0];
rlocus(num,den)
v = [-3  3  -3  3]; axis(v); axis('square');
grid
title('Root-Locus Plot near the Origin    (Problem B-6-7 )')
```



The range of K for stability can be determined by use of Routh stability criterion. Since the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2K(s+1)}{s^4 + 7s^3 + 10s^2 + 2Ks + 2K}$$

the characteristic equation for the system is

$$s^4 + 7s^3 + 10s^2 + 2Ks + 2K = 0$$

The Routh array of coefficients becomes as follows:

$$
\begin{array}{llll}
s^4 & 1 & 10 & 2K \\
s^3 & 7 & 2k & \\
s^2 & \dfrac{70-2K}{7} & 2K & \\
s^1 & \dfrac{\dfrac{(70-2k)2k}{7} - 14K}{\dfrac{70-2k}{7}} & 0 & \\
s^0 & 2K & &
\end{array}
$$

For stability, we require

$$70 > 2K$$

$$42 - 4K > 0$$

$$K > 0$$

Thus, the range of K for stability is

$$10.5 > K > 0$$

B-6-8. The characteristic equation for the system is

$$s^3 + 4s^2 + 8s + K = 0$$

If K is set equal to 2, then the characteristic equation becomes

$$s^3 + 4s^2 + 8s + 2 = 0$$

The closed-loop poles are located as follows:

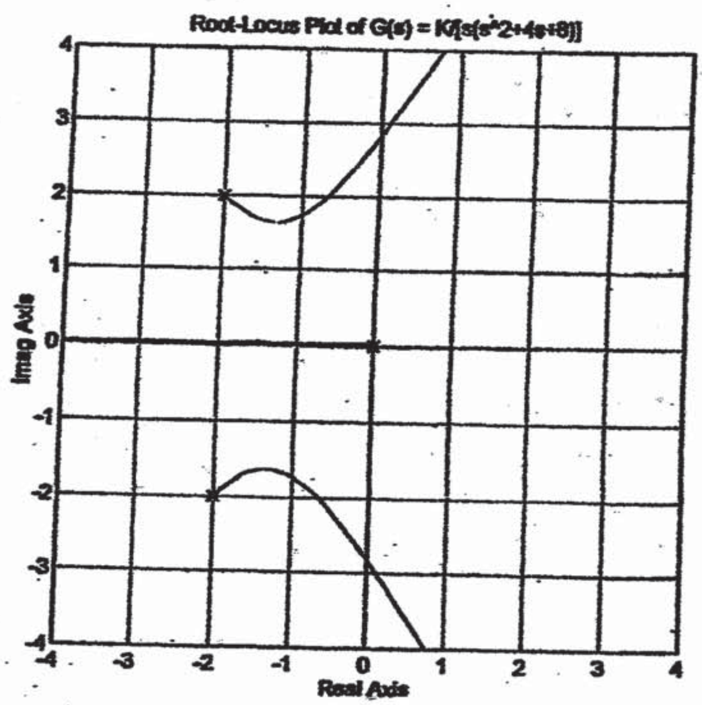
$$s = -1.8557 + j1.8669$$
$$s = -1.8557 - j1.8669$$
$$s = -0.2887$$

See the following MATLAB program for finding the closed-loop poles.

```
p = [1  4  8  2];
roots(p)

ans =

  -1.8557 + 1.8669i
  -1.8557 - 1.8669i
  -0.2887
```

A MATLAB program to plot the root loci is shown below. The resulting root-locus plot is also shown below.

```
% ***** Root-locus plot *****

num = [0  0  0  1];
den = [1  4  8  0];
rlocus(num,den)
axis('square')
grid
title('Root-Locus Plot of G(s) = K/[s(s^2+4s+8)]')
```



B-6-9. The open-loop transfer function is given by

$$G(s)H(s) = \frac{K(s-0.6667)}{s^4+3.3401s^3+7.0325s^2}$$

The equation for the asymptotes may be obtained as

$$G_a(s)H_a(s) = \frac{K}{s^3 + (3.3401 + 0.6667)s^2 + \cdots}$$

$$\doteq \frac{K}{\left(s + \frac{3.3401 + 0.6667}{3}\right)^3}$$

$$= \frac{K}{(s + 1.3356)^3}$$

$$= \frac{K}{s^3 + 4.0068s^2 + 5.3515s + 2.3825}$$

Hence, we enter the following numerators and denominators in the program. For the system,

num = [0 0 0 1 −0.6667]

den = [1 3.3401 7.0325 0 0]

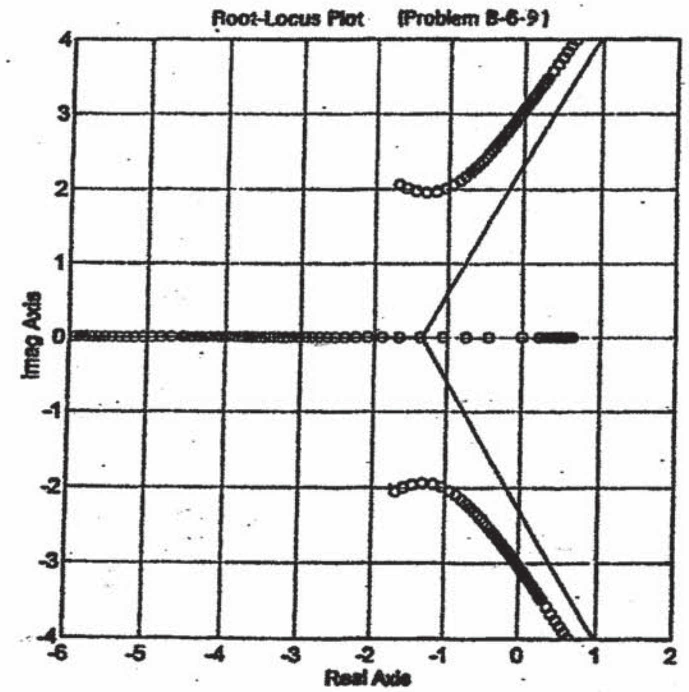
For the asymptotes,

numa = [0 0 0 1]

dena = [1 4.0068 5.3515 2.3825]

A MATLAB program to plot the root loci and asymptotes is given below. The resulting root-locus plot is shown on the next page.

```
% ***** Root-locus plot *****

num = [0  0  0  1  -0.6667];
den = [1  3.3401  7.0325  0  0];
numa = [0  0  0  1];
dena = [1  4.0068  5.3515  2.3825];
K1 = 0:1:50;
K2 = 50:5:200;
K = [K1  K2];
r = rlocus(num,den,K);
a = rlocus(numa,dena,K);
plot(r,'o')
v = [-6  2  -4  4]; axis(v); axis('square')
hold
Current plot held
plot(a,'-')
grid
title('Root-Locus Plot     (Problem B-6-9 )')
xlabel('Real Axis')
ylabel('Imag Axis')
```



B-6-10. By substituting $s = \sigma + j\omega$ into

$$\left|\frac{K}{s(s+1)}\right| = 1$$

and rewriting, we obtain

$$K = \left|(\sigma+j\omega)(\sigma+j\omega+1)\right| = \left|(\sigma+j\omega)^2+\sigma+j\omega\right|$$
$$= \left|\sigma^2+\sigma-\omega^2+j\omega(1+2\sigma)\right|$$

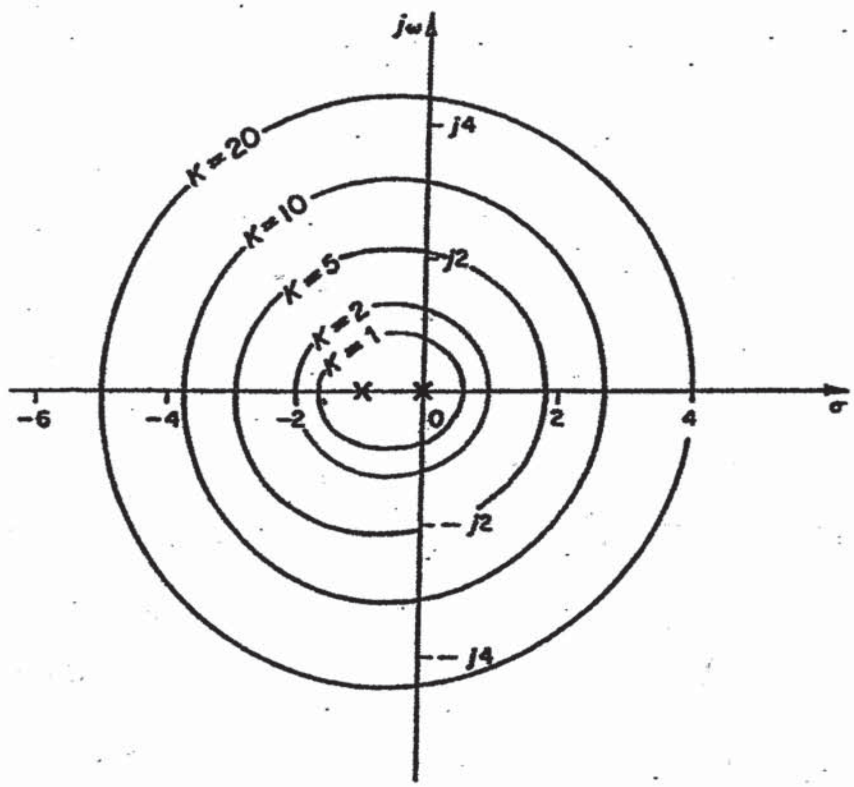
Thus,

$$K^2 = (\sigma^2+\sigma-\omega^2)^2+\omega^2(1+2\sigma)^2$$
$$= [\sigma(\sigma+1)-\omega^2]^2+\omega^2(1+4\sigma+4\sigma^2)$$
$$= [\sigma(\sigma+1)+\omega^2]^2+\omega^2$$

Hence

$$[\sigma(\sigma+1)+\omega^2]^2+\omega^2 = K^2$$

The constant gain loci for K = 1, 2, 5, 10, and 20 on the s plane are shown on the next page.

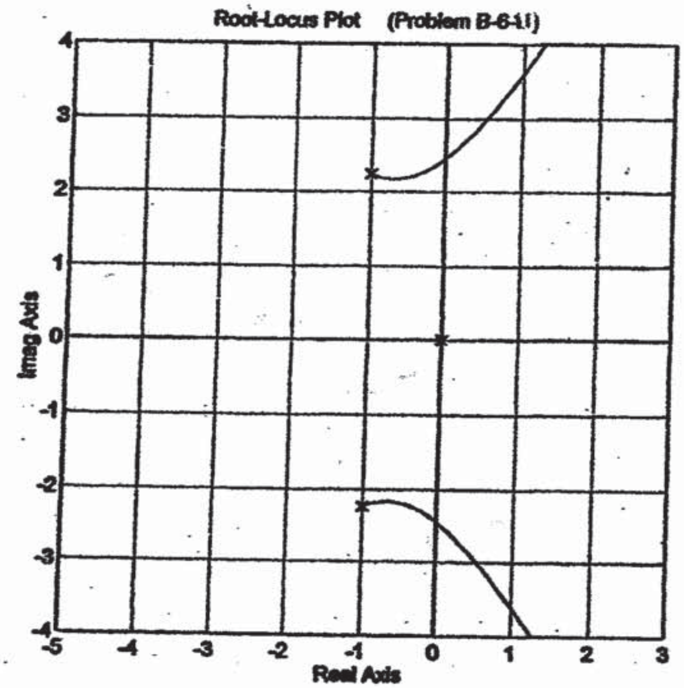


B-6-11. The term (s + 1) in the feedforward transfer function and the term (s + 1) in the feedback transfer function cancel each other. The reduced characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s^2+2s+6)}\,\frac{1}{s+1} = 1 + \frac{K}{s(s^2+2s+6)} = 0$$

The open-loop poles of G(s)H(s) is at s = 0 and $s = -1 \pm j\sqrt{5}$. The following MATLAB program produces the root-locus plot shown on the next page.

```
% ***** Root-locus plot *****

num = [0  0  0  1];
den = [1  2  6  0];
rlocus(num,den)

Warning: Divide by zero
v = [-5  3  -4  4]; axis(v); axis('square')
grid
title('Root-Locus Plot     (Problem B-6-11)')
```



To find the closed-loop poles when the gain K is set equal to 2, we may enter the following MATLAB program into the computer.

```
p = [1  2  6  2];
roots(p)

ans =

  -0.8147 + 2.1754i
  -0.8147 - 2.1754i
  -0.3706
```

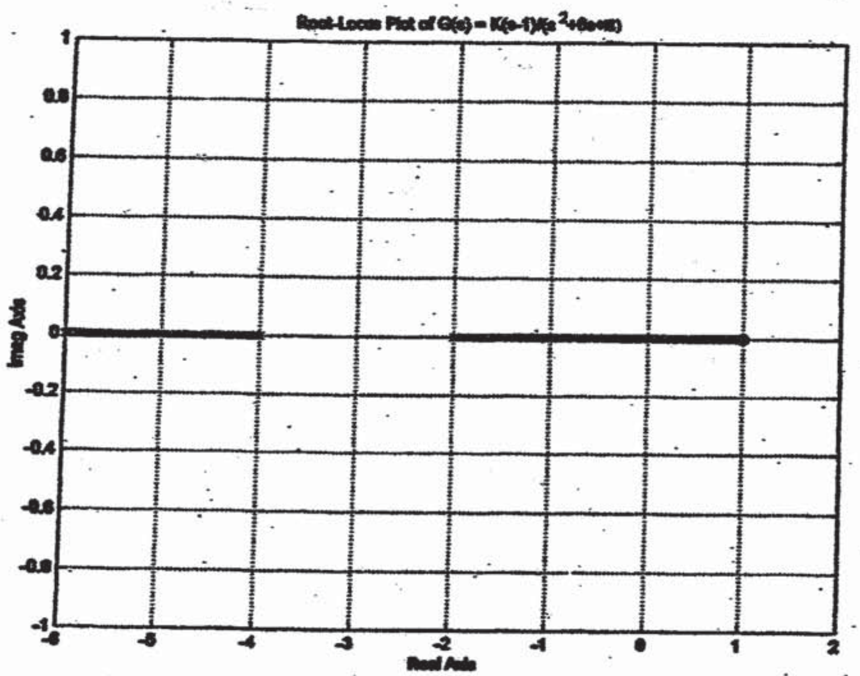
Thus, the closed-loop poles are located at

$$s = -0.8147 \pm j\,2.1754, \qquad s = -0.3706$$

B-6-12. For the system shown in Figure 6-102(a):

A MATLAB program to plot a root-locus diagram for the system shown in Figure 6-102(a) is shown in MATLAB Program (a). The resulting root-locus plot is shown in Figure (a) (see next page).

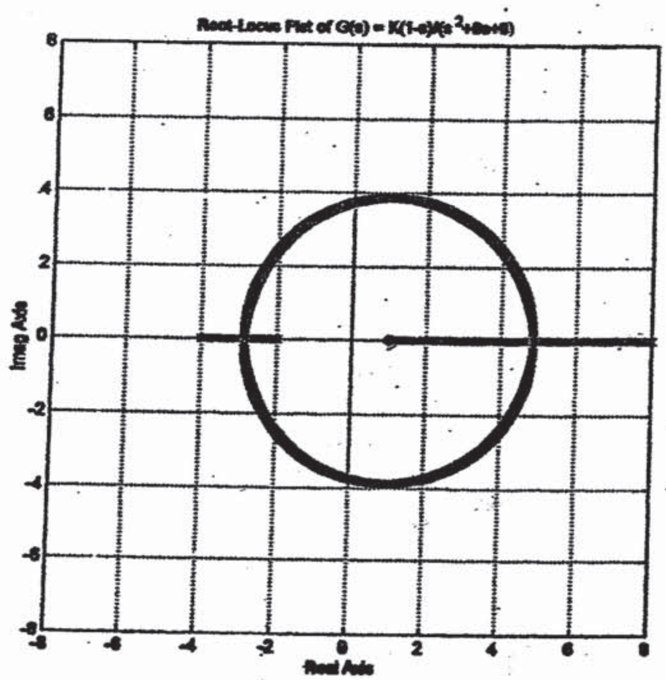
```
% MATLAB Program (a):

num1 = [0  1  -1];
den1 = [1  6  8];
K1 = 0:0.01:50;
K2 = 50:0.5:1000;
K = [K1  K2];
rlocus(num1,den1,K)
grid
title('Root-Locus Plot of G(s) = K(s-1)/(s^2+6s+8)')
xlabel('Real Axis')
ylabel('Imag Axis')
```



<u>For the system shown in Figure 6–102(b):</u>

A MATLAB program to produce a root-locus plot of the system shown in Figure 6–102(b) is given in MATLAB Program (b) The resulting root-locus plot is shown on the next page.

```
% MATLAB Program (b):

num2 = [0  -1  1];
den2 = [1  6  8];
K1 = 0:0.01:50;
K2 = 50:0.5:1000;
K = [K1  K2];
rlocus(num2,den2,K)
v = [-8  8  -8  8]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(1-s)/(s^2+6s+8)')
xlabel('Real Axis')
ylabel('Imag Axis')
```



Note that the equations for the root loci for both systems are the same. They are given by

$$\omega\left[(\sigma-1)^2+\omega^2-15\right]=0$$

This equation is equivalent to

$$\omega=0 \qquad \text{or} \qquad (\sigma-1)^2+\omega^2=15$$

The first equation ($\omega = 0$) is the equation for the real axis. The second equation is the equation for the circle with center at (1,0) and the radius equal to $\sqrt{15}$.

The equation for the break away or break-in points is obtained from $dK/ds = 0$. For both systems, the solutions for $dK/ds = 0$ are

$$s=4.873\ , \qquad s=-2.873$$

For System (a):

$K = -15.746$ for $s = 4.873$

$K = -0.254$ for $s = -2.873$

This means that there are no break away or break-in points for System (a). The root loci exist only on the real axis. (The root loci exist between $s = -2$ and $s = 1$ and between $s = -4$ and $s = -\infty$.)

For System (b):

$K = 15.746$ for $s = 4.873$

$K = 0.254$ for $s = -2.873$

Hence, s = −2.873 and s = 4.873 are actual break away and break-in points, respectively. The root loci involves the circular locus where the center of the circle is at (1,0) and the radius equal to $\sqrt{15}$. The root loci also exist on the real axis, from s = −2 to s = −4 and from s = 1 to s = ∞.

B-6-13. The differential equation for this mechanical system is

$$b_2(\dot{x}_i - \dot{x}_o) + k(x_i - x_o) = b_1 \dot{x}_o$$

Taking the Laplace transforms of both sides of this equation, assuming zero initial conditions and then rewriting, we obtain

$$\frac{X_o(s)}{X_i(s)} = \frac{b_2 s + k}{(b_1+b_2)s + k} = \frac{\frac{b_2}{k}s + 1}{\frac{b_1+b_2}{k}s + 1}$$

If we define

$$\frac{b_2}{k} = T, \qquad \frac{b_1+b_2}{b_2} = \beta > 1$$

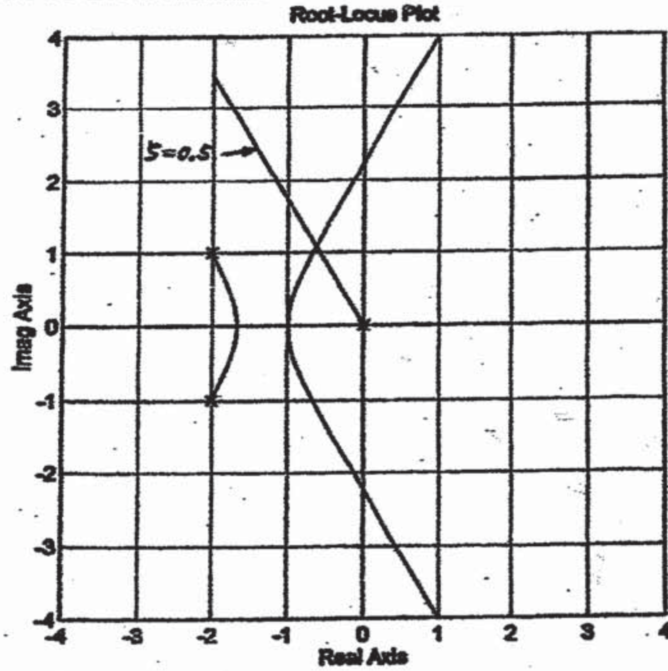
then the transfer function $X_o(s)/X_i(s)$ becomes

$$\frac{X_o(s)}{X_i(s)} = \frac{Ts+1}{\beta Ts+1} = \frac{1}{\beta}\left(\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}\right)$$

This is a lag network, because the pole ($s = -1/\beta T$) is located closer to the origin than the zero ($s = -1/T$).

B-6-14. The following MATLAB program gives a root-locus plot for the system. The plot obtained is shown on the next page.

```
% ***** Root-locus plot *****

num = [0  0  0  1];
den = [1  4  5  0];
rlocus(num,den)
hold
Current plot held
x = [0  -2]; y = [0  3.464]; line(x,y)
axis('square')
grid
title('Root-Locus Plot')
```



Since the dominant closed-loop poles have the damping ratio ζ of 0.5, we may write them as

$$s = x \pm j\sqrt{3}x$$

The characteristic equation for the system is

$$s^3 + 4s^2 + 5s + K = 0$$

By substituting $s = x + j\sqrt{3}x$ into this equation, we obtain

$$(x + j\sqrt{3}x)^3 + 4(x + j\sqrt{3}x)^2 + 5(x + j\sqrt{3}x) + K = 0$$

or

$$-8x^3 - 8x^2 + 5x + K + 2\sqrt{3}j\,(4x^2 + 2.5x) = 0$$

By equating the real part and imaginary part to zero, respectively, we get

$$-8x^3 - 8x^2 + 5x + K = 0 \qquad (1)$$

$$4x^2 + 2.5x = 0 \qquad (2)$$

Noting that $x \neq 0$, from Equation (2), we obtain

$$4x + 2.5 = 0$$

or

$$x = -0.625$$

By substituting x = −0.625 into Equation (1), we get

$$K = 8x^3 + 8x^2 - 5x$$

$$= 8(-0.625)^3 + 8(-0.625)^2 - 5(-0.625)$$

$$= 4.296875$$

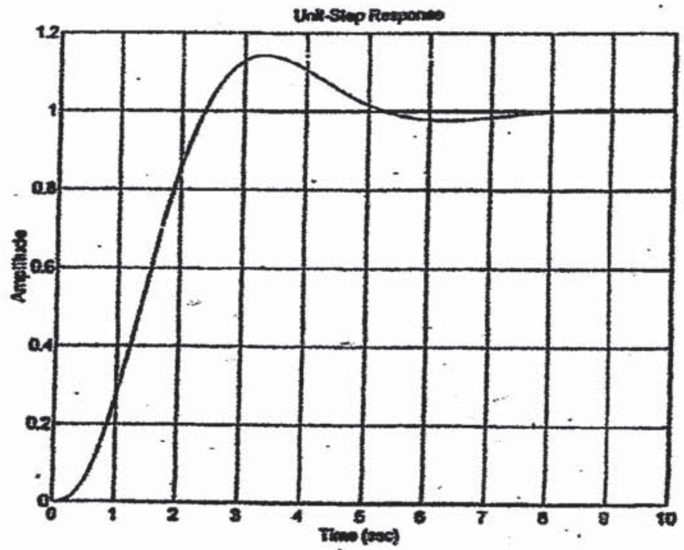
To determine all closed-loop poles, we may enter the following MATLAB program into the computer.

```
p = [1  4  5  4.296875];
roots(p)

ans =

  -2.7500
  -0.6250 + 1.0825i
  -0.6250 - 1.0825i
```

Thus, the closed-loop poles are located at $s = -0.625 \pm j1.0825$ and $s = -2.75$.

The unit-step response curve can be obtained by entering the following MATLAB program into the computer. The resulting unit-step response curve is shown on the next page.

```
% ***** Unit-Step Response *****

num = [0  0  0  4.2969];
den = [1  4  5  4.2969];
step(num,den)
grid
title('Unit-Step Response
```



B-6-15. The solution to such a problem is not unique. We shall present two solutions to the problem in what follows. Note that from the requirement stated in the problem, the dominant closed-loop poles must have $\zeta = 0.5$ and $\omega_n = 3$, or

$$s = -1.5 \pm j\, 2.5981$$

Notice that the angle deficiency is

$$\text{Angle deficiency} = 180^\circ - 120^\circ - 100.894^\circ = -40.894^\circ$$

Method 1: If we choose the zero of the lead compensator at $s = -1$ so that it will cancel the plant pole at $s = -1$, then the compensator pole must be located at $s = -3$, or

$$G_c(s) = K\frac{T_1 s + 1}{T_2 s + 1} = \frac{KT_1}{T_2}\left(\frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}}\right) = \frac{KT_1}{T_2}\,\frac{s+1}{s+3}$$

or

$$G_c(s) = 3K\frac{s+1}{s+3}$$

The value of K can be determined by use of the magnitude condition.

$$\left| 3K\frac{s+1}{s+3}\,\frac{10}{s(s+1)} \right|_{s=-1.5+j2.5981} = 1$$

or

$$K = \left| \frac{s(s+3)}{30} \right|_{s=-1.5+j2.5981} = 0.3$$

Hence

$$G_c(s) = 0.9\frac{s+1}{s+3}$$

The open-loop transfer function is.

$$G_c(s)G(s) = \frac{9}{s(s+3)}$$

The closed-loop transfer function C(s)/R(s) becomes as follows:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2+3s+9}$$

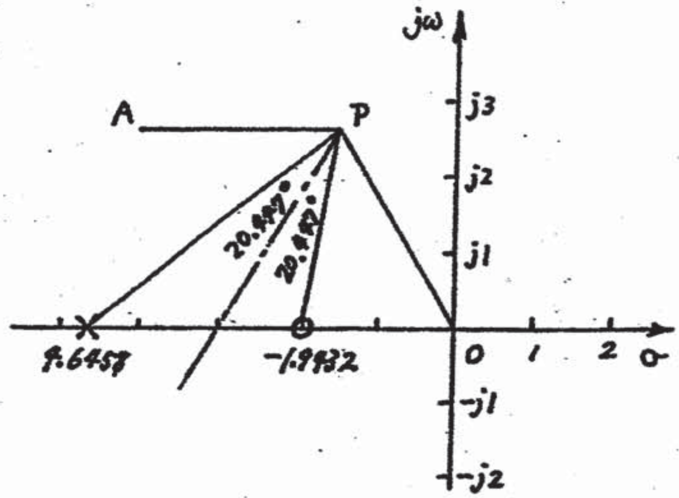
Method 2: Referring to the figure shown below, if we bisect angle OPA and take 20.447° each side, then the locations of the zero and pole are found as follows:

zero at s = -1.9432

pole at s = -4.6458

Thus, $G_c(s)$ can be given as

$$G_c(s) = K\frac{T_1 s+1}{T_2 s+1} = K\frac{T_1}{T_2}\frac{s+1.9432}{s+4.6458} = 2.391K\frac{s+1.9432}{s+4.6458}$$



The value of K can be determined by use of the magnitude condition.

$$\left|2.391K\frac{s+1.9432}{s+4.6458}\frac{10}{s(s+1)}\right|_{s=-1.5+j2.5981} = 1$$

or

$$K = \left|\frac{(s+4.6458)s(s+1)}{23.91(s+1.9432)}\right|_{s=-1.5+j2.5981} = 0.5138$$

Hence, the compensator $G_c(s)$ is given by

$$G_c(s) = 1.2285\frac{s+1.9432}{s+4.6458} = 0.5138\frac{0.5146s+1}{0.2152s+1}$$

Then, the open-loop transfer function becomes as

$$G_c(s)G(s) = 0.5138\left(\frac{0.5146s+1}{0.2152s+1}\right)\frac{10}{s(s+1)}$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{5.138\,(0.5146s+1)}{s(s+1)(0.2152s+1)+5.138\,(0.5146s+1)}$$

$$= \frac{2.644s+5.138}{0.2152s^3+1.2152s^2+3.644s+5.138}$$

It is interesting to compare the static velocity error constants for the two systems designed above.

For the system designed by Method 1:

$$K_v = \lim_{s\to 0} s\,\frac{9}{s(s+3)} = 3$$

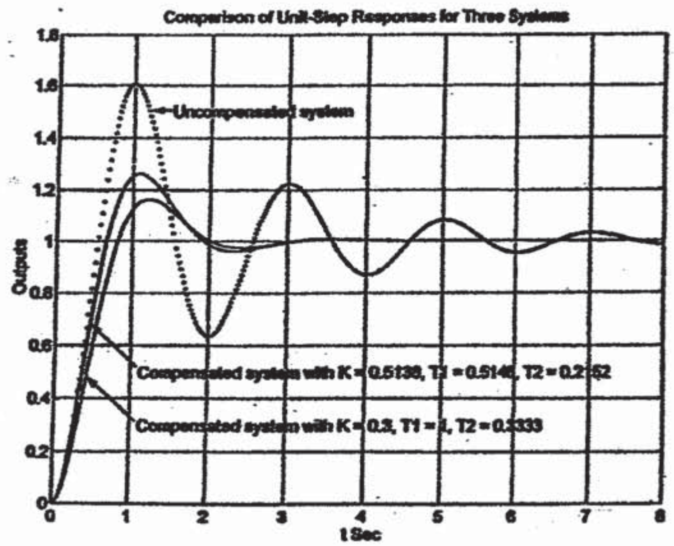
For the system designed by Method 2:

$$K_v = \lim_{s\to 0} s\,(0.5138)\,\frac{0.5146s+1}{0.2152s+1}\,\frac{10}{s(s+1)} = 5.138$$

The system designed by Method 2 gives a larger value of the static velocity error constant. This means that the system designed by Method 2 will give smaller steady-state errors in following ramp inputs than the system designed by Method 1.

In what follows, we compare the unit-step responses of the three systems: the original uncompensated system, the system designed by Method 1, and the system designed by Method 2. The MATLAB program used to obtain the unit-step response curves is given below. The resulting unit-step response curves are shown on the next page.

```
% ***** Comparison of unit-step responses for three systems *****

num = [0  0  10];
den = [1  1  10];
num1 = [0  0  9];
den1 = [1  3  9];
num2 = [0  0  2.644  5.138];
den2 = [0.2152  1.2152  3.644  5.138];
t = 0:0.02:8;
c = step(num,den,t);
c1 = step(num1,den1,t);
c2 = step(num2,den2,t);
plot(t,c,'.',t,c1,'-',t,c2,'-.')
grid
title('Comparison of Unit-Step Responses for Three Systems')
xlabel('t Sec')
ylabel('Outputs')
text(1.5,1.5,'Uncompensated system')
text(1.1,0.5,'Compensated system with K = 0.5138, T1 = 0.5146, T2 = 0.2152')
text(1.1,0.3,'Compensated system with K = 0.3, T1 = 1, T2 = 0.3333')
```



B-6-16. The closed-loop transfer function C(s)/R(s) is given by

$$\frac{C(s)}{R(s)} = \frac{K(Ts+1)}{s(s+2)+K(Ts+1)}$$

Since the closed-loop poles are specified to be

$$s = -2 \pm j2$$

we obtain

$$s(s+2)+K(Ts+1) = (s+2+j2)(s+2-j2)$$

or

$$s^2+(2+KT)s+K = s^2+4s+8$$

Hence, we require

$$2+KT = 4, \qquad K = 8$$

which results in

$$T = 0.25, \qquad K = 8$$

B-6-17. The angle deficiency at the closed-loop pole $s = -2 + j2\sqrt{3}$ is

$$180^\circ - 120^\circ - 90^\circ = -30^\circ$$

The lead compensator must contribute 30°.

Let us choose the zero of the lead compensator at $s = -2$. Then, the pole of the compensator must be located at $s = -4$. Thus,

$$G_c(s) = K\frac{s+2}{s+4}$$

The gain K is determined from the magnitude condition.

$$\left| K \frac{s+2}{s+4} \frac{5}{s(0.5s+1)} \right|_{s=-2+j2\sqrt{3}} = 1$$

or

$$K = \left| \frac{s(s+4)}{10} \right|_{s=-2+j2\sqrt{3}} = 1.6$$

Hence,

$$G_c(s) = 1.6 \frac{s+2}{s+4}$$

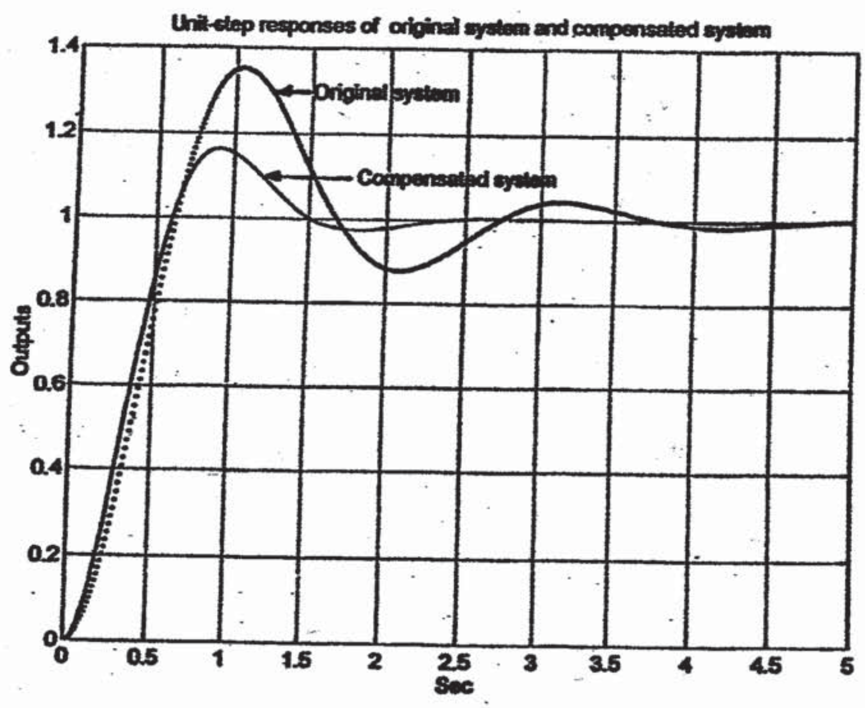
Next, we shall obtain unit-step responses of the original system and the compensated system. The original system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10}$$

The compensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+4s+16}$$

The unit-step response curves of the original system and compensated system are shown below.



B-6-18. The angle deficiency is

$$180° - 135° - 135° = -90°$$

A lead compensator can contribute 90°. Let us choose the zero of the lead compensator at s = -0.5. Then, the pole of the compensator must be at s = -3. Thus,

$$G_c(s) = K \frac{s+0.5}{s+3}$$

The gain K can be determined from the magnitude condition.

$$\left| K \frac{s+0.5}{s+3} \frac{1}{s^2} \right|_{s=-1+j1} = 1$$

or

$$K = \left| \frac{(s+3)\, s^2}{s+0.5} \right|_{s=-1+j1} = 4$$

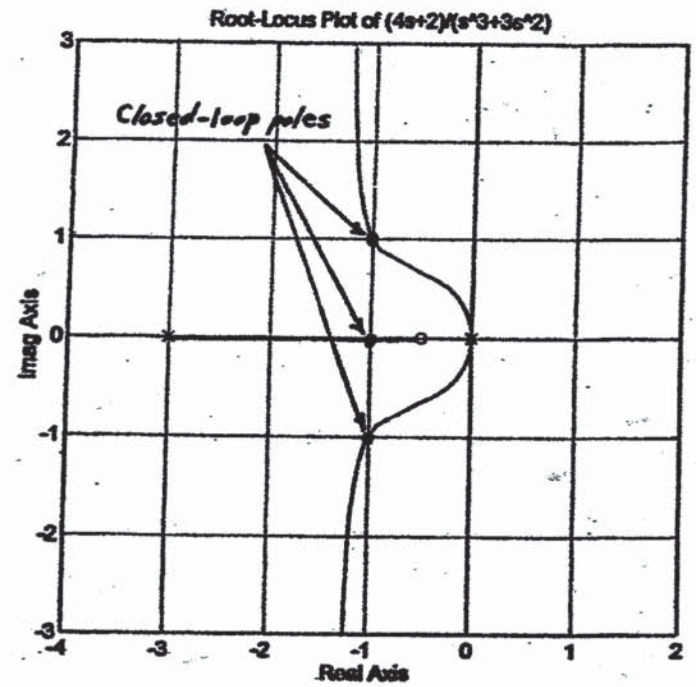
Hence the lead compensator becomes as follows:

$$G_c(s) = 4 \frac{s+0.5}{s+3}$$

The feedforward transfer function is

$$G_c(s)\, G(s) = \frac{4s+2}{s^3+3s^2}$$

A root-locus plot of the system is shown below.



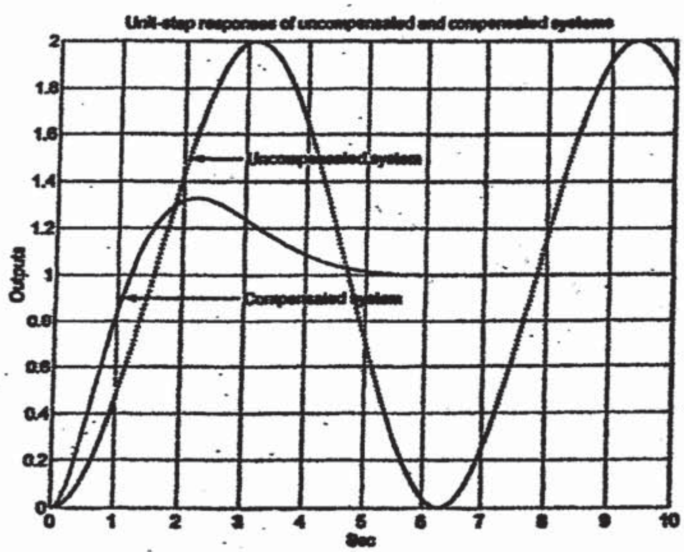
Note that the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{4s+2}{s^3+3s^2+4s+2}$$

The closed-loop poles are located at $s = -1 \pm j1$ and $s = -1$.

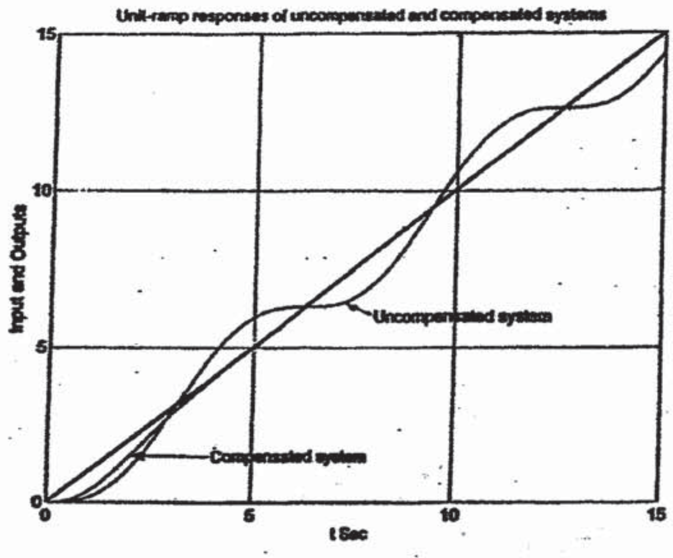
In what follows we shall give the unit-step and unit-ramp responses of the uncompensated system and the compensated system. A MATLAB program to obtain unit-step response curves is given below. The resulting curves are also shown below.

```
% ***** Unit-step responses of uncompensated and compensated systems *****

num = [0  0  1];
den = [1  0  1];
numc = [0  0  4  2];
denc = [1  3  4  2];
t = 0:0.02:10;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,c1,'.',t,c2,'-')
grid
title('Unit-step responses of uncompensated and compensated systems')
xlabel('Sec')
ylabel('Outputs')
text(3,0.9,'Compensated system')
text(3,1.5,'Uncompensated system')
```



A MATLAB program to obtain unit-ramp response curves is given next. The resulting response curves are shown on the next page.

```
% ***** Unit-ramp responses of uncompensated and compensated systems *****

num = [0  0  0  1];
den = [1  0  1  0];
numc = [0  0  0  4  2];
denc = [1  3  4  2  0];
t = 0:0.02:15;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,t,'.',t,c1,'-.',t,c2,'-')
grid
title('Unit-ramp responses of uncompensated and compensated systems ')
xlabel('t Sec')
ylabel('Input and Outputs')
text(4,1.5,'Compensated system')
text(8,6,'Uncompensated system')
```



B-6-19. The original uncompensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+4s+16}$$

The two closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$. Choose a lag compensator of the following form:

$$G_c(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}, \quad (\beta > 1)$$

Then, the static velocity error constant K_v can be given by

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sK_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \frac{16}{s(s+4)} = 4\beta K_c = 20$$

Let us choose $K_c = 1$. Then

$$\beta = 5$$

The pole and zero of the lag compensator must be located close to the origin. Let us choose $T = 20$. Then, the lag compensator becomes

$$G_c(s) = \frac{s+\frac{1}{20}}{s+\frac{1}{100}} = \frac{s+0.05}{s+0.01}$$

Notice that

$$\left|\frac{s+0.05}{s+0.01}\right|_{s=-2+j2\sqrt{3}} = 0.9950$$

$$\left.\angle\frac{s+0.05}{s+0.01}\right|_{s=-2+j2\sqrt{3}} = \angle -1.95+j2\sqrt{3} - \angle -1.99+j2\sqrt{3}$$

$$= -60.6271^\circ + 60.1272^\circ = -0.4999^\circ$$

The angle contribution of this lag network is very small (-0.4999°) and the magnitude of $G_c(s)$ is approximately unity at the desired closed-loop pole. Hence, the designed lag compensator is satisfactory. Thus

$$G_c(s) = \frac{s+0.05}{s+0.01}$$

Let us compare the unit-step response curves of the uncompensated and compensated systems. The closed-loop transfer function of the uncompensated system is

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+4s+16}$$

For the compensated system the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{16(s+0.05)}{(s+0.01)s(s+4)+16(s+0.05)}$$

$$= \frac{16s+0.8}{s^3+4.01s^2+16.04s+0.8}$$

The closed-loop poles can be found by entering the following MATLAB program into the computer.

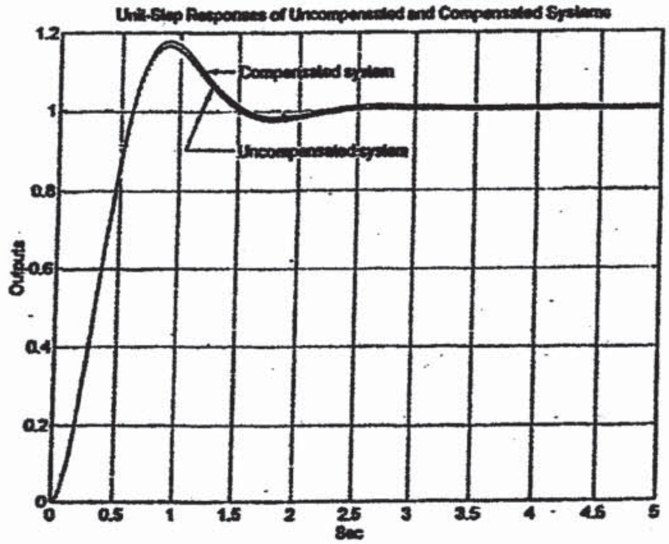
```
p = [1   4.01   16.04   0.8];
roots(p)

ans =

  -1.9797 + 3.4526i
  -1.9797 - 3.4526i
  -0.0505
```

The dominant closed-loop poles are located at $s = -1.9797 \pm j3.4526$. These locations are very close to the original closed-loop poles.

The following MATLAB program produces a plot of unit-step response curves.

```
% ***** Comparison of Unit-Step Responses for Two Systems *****

num = [0   0   16];
den = [1   4   16];
numc = [0   0   16   0.8];
denc = [1   4.01   16.04   0.8];
t = 0:0.02:5;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,c1,'.',t,c2,'-')
grid
title('Unit-Step Responses of Uncompensated and Compensated Systems')
xlabel('Sec')
ylabel('Outputs')
text(1.5,1.1,'Compensated system')
text(1.5,0.9,'Uncompensated system')
```

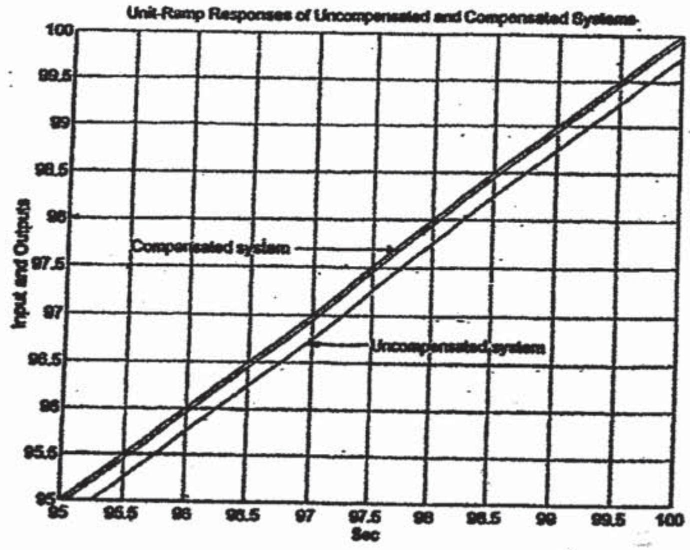
The unit-step response curves obtained are shown below.



Clearly, the unit-step response curves for the two systems are approximately the same.

For the unit-ramp response, the response curves for the two systems differ, because the original uncompensated system gives the steady-state error of 0.25, while the compensated system exhibits the steady-state error of 0.05. The following MATLAB program gives the unit-ramp response curves in the time range 95 sec $\leq t \leq 100$ sec. The resulting unit-ramp response curves are shown on the next page.

```
% ***** Comparison of Unit-Ramp Responses for Two Systems *****

num = [0  0  0  16];
den = [1  4  16  0];
numc = [0  0  0  16  0.8];
denc = [1  4.01  16.04  0.8  0];
t = 0:0.1:100;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,t,'--',t,c1,'-.',t,c2,'-')
v = [95  100  95  100]; axis(v)
grid
title('Unit-Ramp Responses of Uncompensated and Compensated Systems')
xlabel('Sec')
ylabel('Input and Outputs')
text(95.5,97.7,'Compensated system')
text(97.5,96.7,'Uncompensated system')
```



B-6-20. Since the characteristic equation of the uncompensated system is

$$s^3 + 30s^2 + 200s + 820 = 0$$

the uncompensated system has the closed-loop poles at

$$s = -3.60 \pm j4.80, \qquad s = -22.8$$

To increase the static velocity error constant from 4.1 to 41 $\sec^{-1}$ without appreciably changing the location of the dominant closed-loop poles, we need to insert a lag compensator $G_c(s)$ whose pole and zero are located very close to the origin. For example, we may choose

$$G_c(s) = 10 \, \frac{Ts+1}{10Ts+1}$$

where T may be chosen to be 4, or T = 4. Then the lag compensator becomes

$$G_c(s) = 10 \, \frac{4s+1}{40s+1} = \frac{s+0.25}{s+0.025} \qquad (1)$$

The angle contribution of this lag network at $s = -3.60 + j4.80$ is $-1.77°$, which is acceptable in the present problem.

The open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = \frac{820(s+0.25)}{s(s+0.025)(s+10)(s+20)}$$

Clearly, the velocity error constant K_v for the compensated system is

$$K_v = \lim_{s \to 0} s G_c(s) G(s) = 41 \ \sec^{-1}$$

Notice that because of the addition of the lag compensator the compensated system becomes of fourth order. The characteristic equation for the compensated system is

$$s^4 + 30.025s^3 + 200.75s^2 + 825s + 205 = 0$$

The roots of this characteristic equation can be easily obtained by use of MATLAB as shown below.

```
p= [1  30.025  200.75  825  205];
roots(p)

ans =

 -22.7866
  -3.4868 + 4.6697i
  -3.4868 - 4.6697i
  -0.2649
```

Thus, the dominant closed-loop poles are located at

$$s = -3.4868 \pm j\,4.6697$$

The other two closed-loop poles are located at

$$s = -0.2649\,, \qquad s = -22.787$$

The closed-loop pole at s = -0.2648 almost cancels the zero of the lag compensator, s = -0.25. Also, since the closed-loop pole at s = -22.787 is located very farther to the left compared to the complex-conjugate closed-loop poles, the effect of this pole on the system response is very small. Therefore, the closed-loop poles at s = -3.4868 ± j4.6697 are indeed the dominant closed-loop poles.

The undamped natural frequency ω_n of the dominant closed-loop poles is

$$\omega_n = \sqrt{3.4868^2 + 4.6697^2} = 5.828 \text{ rad/sec}$$

Since the original uncompensated system has the undamped natural frequency of 6 rad/sec, the compensated system has an approximately 3% smaller value, which would be acceptable. Hence, the lag compensator given by Equation (1) is satisfactory.

B-6-21. Let us choose a lag-lead compensator as given below.

$$G_c(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \qquad (\beta > 1)$$

The desired closed-loop poles are located at

$$s = -2 \pm j\,2\sqrt{3}$$

and the static velocity error constant K_v is specified as

$$K_v = 50 \text{ sec}^{-1}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \frac{10}{s(s+2)(s+5)}$$

Hence

$$K_v = \lim_{s\to 0} s\,G_c(s)G(s) = \lim_{s\to 0} K_c \frac{10}{2\times 5} = K_c = 50$$

Thus

$$K_c = 50$$

The time constant T_1 and the value of β are determined from the requirements that

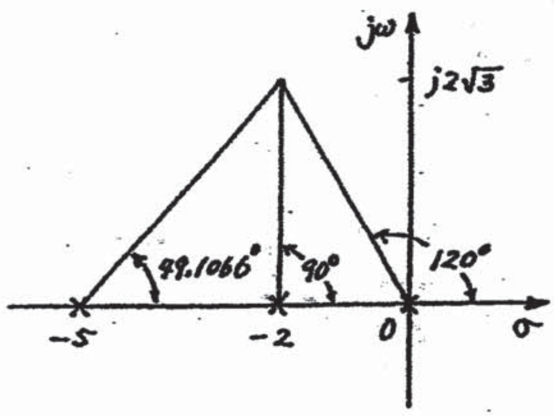
$$\left| \frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} \right| \left| \frac{50\times 10}{s(s+2)(s+5)} \right|_{s=-2+j2\sqrt{3}} = 1$$

$$\left. \angle \frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} \right|_{s=-2+j2\sqrt{3}} = 79.1066^\circ$$

The angle 79.1066° comes from the fact that the lead portion must compensate the angle deficiency which is

Angle deficiency $= 180^\circ - 120^\circ - 90^\circ - 49.1066^\circ = -79.1066^\circ$

See the diagram shown below.



By using trigonometry we find the locations of the zero and pole of the lead portion of the compensator as follows:

$$\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}} = \frac{s+2.2187}{s+27.1111}$$

Hence,

$$T_1 = 0.4507, \qquad \beta = 12.2194$$

For the lag portion, we may choose $T_2 = 10$. Then, the lag portion may be given by

$$\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} = \frac{s+0.1}{s+0.008184}$$

Notice that

$$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right|_{s=-2+j2\sqrt{3}} = \left|\frac{s+0.1}{s+0.008184}\right|_{s=-2+j2\sqrt{3}}$$

$$= 0.9888$$

$$\left.\angle\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right|_{s=-2+j2\sqrt{3}} = -1.1544^\circ$$

The changes caused by the lag portion are small and acceptable. Hence the lag-lead compensator can be given by

$$G_c(s) = 50\,\frac{s+2.2187}{s+27.1111}\,\frac{s+0.1}{s+0.008184}$$

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{50(s+2.2187)(s+0.1)}{(s+27.1111)(s+0.008184)}\cdot\frac{10}{s(s+2)(s+5)}$$

$$= \frac{500s^2 + 1159.35s + 110.935}{s^5+34.1193s^4+200.0570s^3+272.7862s^2+2.2188s}$$

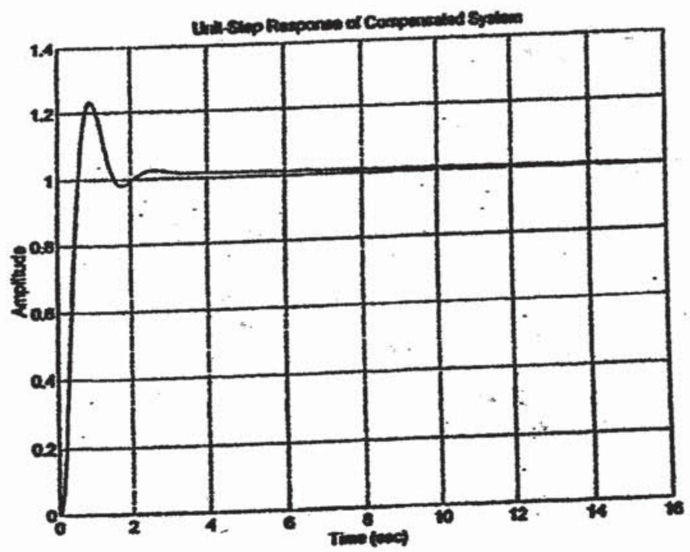
The closed-loop transfer function becomes as follows:

$$\frac{C(s)}{R(s)} = \frac{500s^2 + 1159.35s + 110.935}{s^5+34.1193s^4+200.0570s^3+272.7862s^2+1161.5688s+110.935}$$

The following MATLAB program will give the unit-step response of the compensated system.

```
% ***** Unit-step response *****

num = [0  0  0  500  1159.35  110.935];
den = [1  34.1193  200.0570  772.7462  1161.5688  110.935];
step(num,den)
grid
title('Unit-Step Response of Compensated System')
```

The resulting unit-step response curve is shown below.

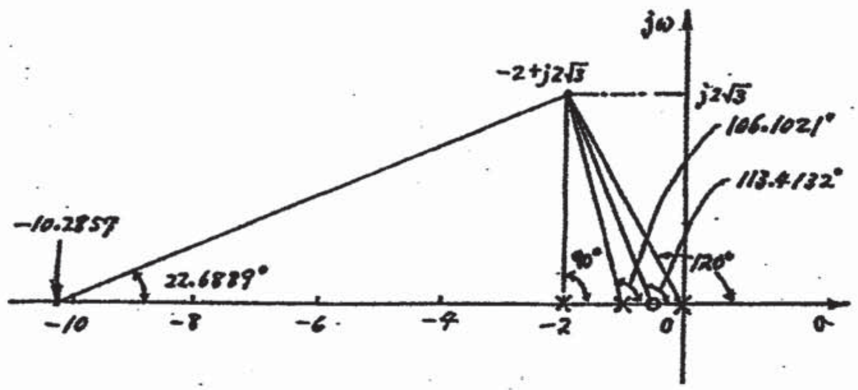


B-6-22. Let us choose the dominant closed-loop poles at $s = -2 \pm j2\sqrt{3}$. Then, the angle deficiency at a closed-loop pole $s = -2 + j2\sqrt{3}$ becomes as follows:

$$\text{Angle deficiency} = 180^\circ - 120^\circ - 90^\circ - 106.1021^\circ + 113.4132^\circ$$

$$= -22.6889^\circ$$

See the following diagram for the computation of the angle deficiency.



From this diagram we find the zero of the compensator to be at $s = -10.2857$. The compensator thus can be written as

$$G_c(s) = K(s+10.2857)$$

The feedforward transfer function becomes

$$G_c(s)G(s) = \frac{K(s+10.2857)(2s+1)}{s(s+1)(s+2)}$$

The gain K can be determined from the magnitude condition:

$$\left| \frac{K(s+10.2857)(2s+1)}{s(s+1)(s+2)} \right|_{s=-2+j2\sqrt{3}} = 1$$

or

$$K = \left| \frac{s(s+1)(s+2)}{(s+10.2857)(2s+1)} \right|_{s=-2+j2\sqrt{3}}$$

The evaluation of this K can be made easily by use of MATLAB. The following MATLAB program produces the value of K.

```
% ***** Determination of gain constant K *****

a = [1   3   2   0];
b = [2   21.5714   10.2857];
s = -2+j*2*sqrt(3);
format long
K = abs(polyval(a,s))/abs(polyval(b,s))

K =

   0.73684318666243
```

Hence, the compensator becomes as follows:

$$G_c(s) = 0.73684(s+10.2857)$$

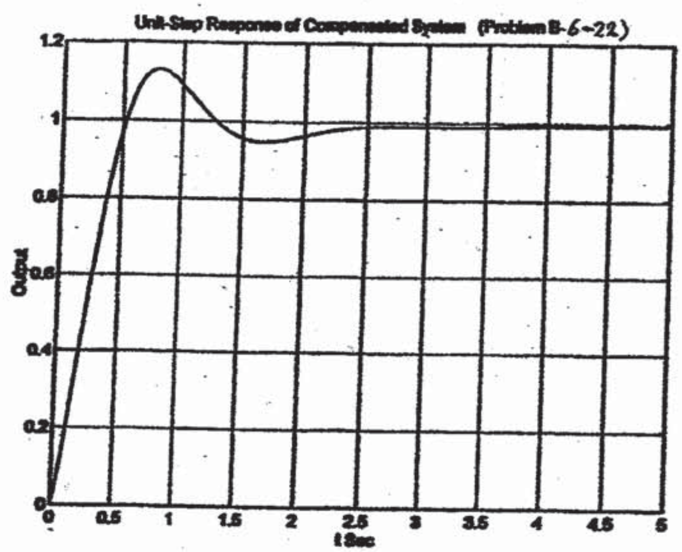
The closed-loop transfer function becomes as

$$\frac{C(s)}{R(s)} = \frac{1.47368 s^2 + 15.89467 s + 7.578915}{s^3 + 4.47368 s^2 + 17.8947 s + 7.578915}$$

The following MATLAB program will produce the unit-step response curve.

```
% ***** Unit-step response *****

numc = [0  1.47368  15.89467  7.578915];
denc = [1  4.47368  17.8947  7.578915];
t = 0:0.01:5;
c = step(numc,denc,t);
plot(t,c)
grid
title('Unit-Step Response of Compensated System   (Problem B-6-22)')
xlabel('t Sec')
ylabel('Output')
```

The resulting unit-step response curve is shown below.



The response curve shows the maximum overshoot of 13% and the settling time of approximately 3 sec. Thus, the designed system satisfies the requirements of the problem.

B-6-23. The first step in the design of the compensator is to choose the desired closed-loop pole locations. Considering the open-loop poles of the plant and the given specifications, we may choose the dominant closed-loop poles to be

$$s = -4 \pm j4$$

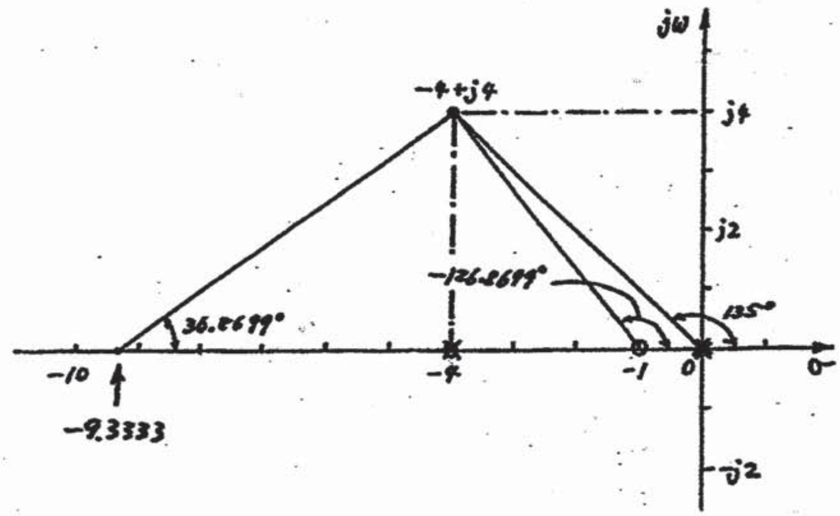
(Of course, other choices can be made.) With the present choice of the dominant closed-loop poles, we may choose the compensator to have a zero at s = -4 so that the plant pole at s = −4 can be canceled. We may also include a zero at s = −1. Thus, we may choose the transfer function of the compensator to be

$$G_c(s) = K\frac{(s+4)(s+1)}{s+b}$$

where the compensator pole at s = −b need be determined based on the angle deficiency. The angle deficiency is

$$\text{Angle deficiency} = 180^\circ - 135^\circ - 135^\circ + 126.8699^\circ = 36.8699^\circ$$

The compensator pole must provide an angle of -36.8699°. From the diagram given below we find b to be −9.3333.



Then, the compensator $G_c(s)$ can be given by

$$G_c(s) = K\frac{(s+4)(s+1)}{s+9.3333}$$

The open-loop transfer function becomes as follows:

$$G_c(s)G(s) = \frac{K(s+4)(s+1)}{(s+9.3333)s^2(s+4)} = \frac{K(s+1)}{s^2(s+9.3333)}$$

The value of gain K can be determined from the magnitude condition:

$$\left|\frac{K(s+1)}{s^2(s+9.3333)}\right|_{s=-4+j4} = 1$$

or

$$K = \left|\frac{s^3+9.3333s^2}{s+1}\right|_{s=-4+j4}$$

The value of gain K can be determined easily by use of MATLAB. See the following MATLAB program:

```
% ***** Determination of gain K *****

a = [1  9.3333  0  0];
b = [1  1];
s = -4+j*4;
K = abs(polyval(a,s))/abs(polyval(b,s))

K =

  42.6665
```

Hence, the compensator becomes as follows:

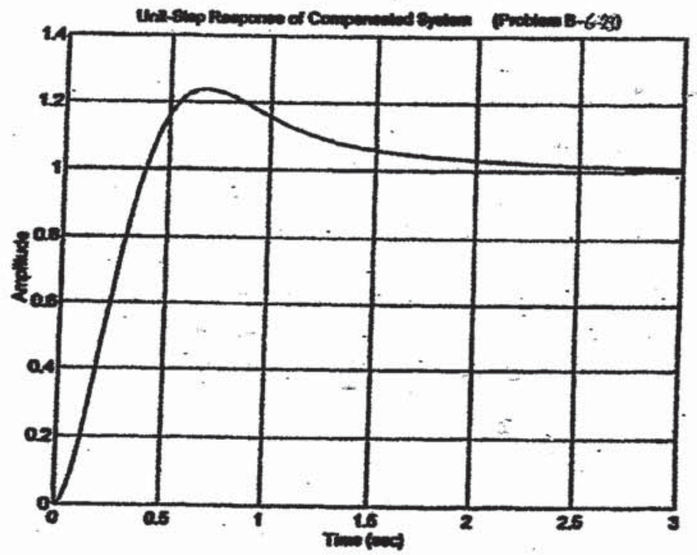
$$G_c(s) = 42.6665 \frac{(s+4)(s+1)}{s+9.3333}$$

The closed-loop transfer function of the designed system becomes as

$$\frac{C(s)}{R(s)} = \frac{42.6665 s + 42.6665}{s^3 + 9.3333 s^2 + 42.6665 s + 42.6665}$$

The following MATLAB program produces the unit-step response curve, which is shown below.

```
% ***** Unit-step response *****

numc = [0  0  42.6665  42.6665];
denc = [1  9.3333  42.6665  42.6665];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System    (Problem B-6-23)')
```



The closed-loop poles are located at s = -4 ± j4 and s = -1.3333 as seen from the following MATLAB output.

```
roots(denc)

ans =

  -4.0000 + 4.0000i
  -4.0000 - 4.0000i
  -1.3333
```

The unit-step response curve shows that the maximum overshoot is approximately 25% and the settling time is approximately 3 sec. Hence, the given specifications are met and the designed system is acceptable.

B-6-24. The closed-loop transfer function for the system is

$$\frac{C(s)}{R(s)} = \frac{K}{2s^2 + s + KK_h s + K} = \frac{\frac{K}{2}}{s^2 + \frac{1+KK_h}{2} s + \frac{K}{2}}$$

From this equation, we obtsin

$$\omega_n = \sqrt{\frac{K}{2}}, \qquad 2\zeta\omega_n = \frac{1+KK_h}{2}$$

Since the damping ratio ζ is specified as 0.5, we get

$$\omega_n = \frac{1+KK_h}{2}$$

Therefore, we have

$$\frac{1+KK_h}{2} = \sqrt{\frac{K}{2}}$$

The settling time is specified as

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{(1+KK_h)/4} = \frac{16}{1+KK_h} \leq 2$$

Since the feedforward transfer function G(s) is

$$G(s) = \frac{\frac{K}{2s+1}}{1+\frac{KK_h}{2s+1}} \frac{1}{s} = \frac{K}{2s+1+KK_h} \frac{1}{s}$$

the static velocity error constant K_v is

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s \frac{K}{2s+1+KK_h} \frac{1}{s} = \frac{K}{1+KK_h}$$

This value must be equal to or greater than 50. Hence,

$$\frac{K}{1+KK_h} \geq 50$$

Thus, the conditions to be satisfied can be summarized as follows:

$$\frac{1+KK_h}{2} = \sqrt{\frac{K}{2}} \tag{1}$$

$$\frac{16}{1+KK_h} \leq 2 \tag{2}$$

$$\frac{K}{1+KK_h} \geq 50 \tag{3}$$

$$0 < K_h < 1$$

From Equations (1) and (2), we get

$$8 \leq 1+KK_h = \sqrt{2K}$$

or

$$32 < K$$

From Equation (3) we obtain

$$\frac{K}{50} \geq 1+KK_h = \sqrt{2K}$$

or

$$K \geq 5000$$

If we choose K = 5000, then we get

$$1+KK_h = \sqrt{2K} = 100$$

or

$$K_h = \frac{99}{5000} = 0.0198$$

Thus, we determined a set of values of K and K_h as follows:

$$K = 5000, \quad K_h = 0.0198$$

With these values of K and K_h, all specifications are satisfied.

B-6-25. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s[(s+1)(s+2)+0.2K]+K} = \frac{K}{s^3+3s^2+2s+0.2Ks+K}$$

The dominant closed-loop poles may be written as

$$s = x \pm j\sqrt{3}\,x$$

Substituting $s = x + j\sqrt{3}x$ into the characteristic equation, we obtain

$$(x+j\sqrt{3}x)^3+3(x+j\sqrt{3}x)^2+2(x+j\sqrt{3}x)+0.2K(x+j\sqrt{3}x)+K=0$$

or

$$-8x^3-6x^2+2x+0.2Kx+K+2\sqrt{3}j(3x^2+x+0.1Kx)=0$$

By equating the real part and imaginary part to zero, respectively, we obtain:

$$-8x^3-6x^2+2x+0.2Kx+K=0 \qquad (1)$$

$$3x^2+x+0.1Kx=0 \qquad (2)$$

From equation (2), noting that $x \neq 0$, we get

$$3x+1+0.1K=0$$

or

$$K=-10(3x+1)$$

By substituting this equation into Equation (1), we obtain

$$8x^3+12x^2+30x+10=0$$

To find the roots of this cubic equation, we may enter the following MATLAB program into the computer:

```
p = [8  12  30  10];
roots(p)

ans =

 -0.5622 + 1.7354i
 -0.5622 - 1.7354i
 -0.3756
```

The value of x must be real. Hence, we take $x = -0.3756$. Thus, the dominant closed-loop poles are located at

$$s=-0.3756 \pm j0.6506$$

The value of K for the dominant closed-loop poles is obtained as

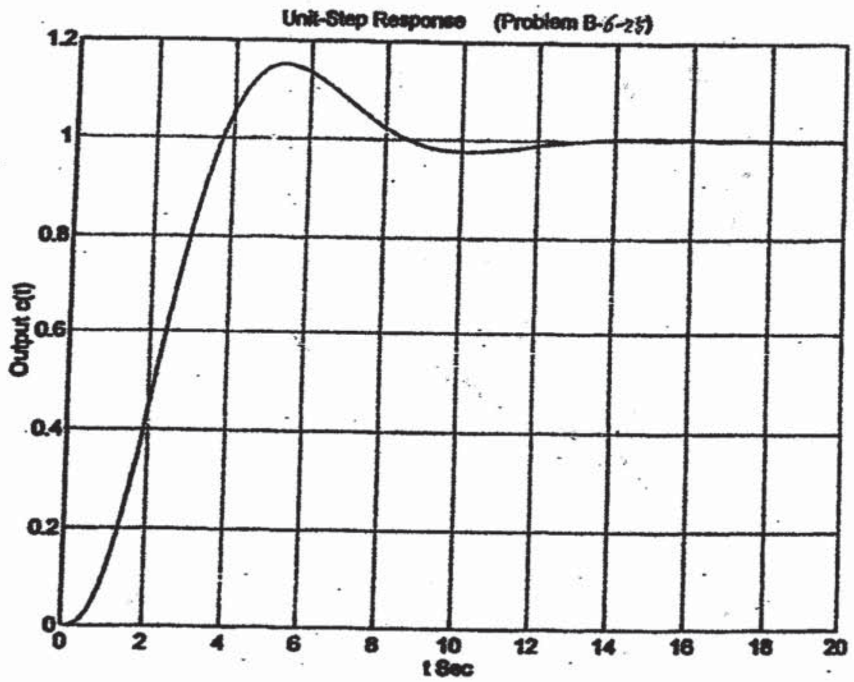
$$K=-10(3x+1)$$

$$=-10(-3\times 0.3756+1)=1.268$$

To obtain the unit-step response of this system, we first substitute $K =$ 1.268 into the closed-loop transfer function and then enter the following MAY-LAB program into the computer:

```
% ***** Unit-step response *****

num = [0  0  0  1.268];
den = [1  3  2.2536  1.268];
t = 0:0.05:20;
c = step(num,den,t);
plot(t,c)
grid
title('Unit-Step Response    (Problem B-6-25)')
xlabel('t Sec')
ylabel('Output c(t)')
```

The resulting unit-step response curve is shown below.



B-6-26. The characteristic equation is

$$(s+\alpha)\frac{2}{s^2(s+2)}+1=0$$

In this case the variable α is not a multiplying factor. Hence, we need to rewrite the charactristic equation such that α becomes a multiplying factor. Since the characteristic equation is

$$s^3+2s^2+2s+2\alpha=0$$

we rewrite it as follows:

$$1+\frac{2\alpha}{s^3+2s^2+2s}=0$$

Define $K=\alpha$. Then, the characteristic equation becomes

$$1 + \frac{2K}{s(s^2+2s+2)} = 0$$

A root-locus plot of this system may be obtained by entering the following MATLAB program into the computer. The resulting root-locus plot is shown below.

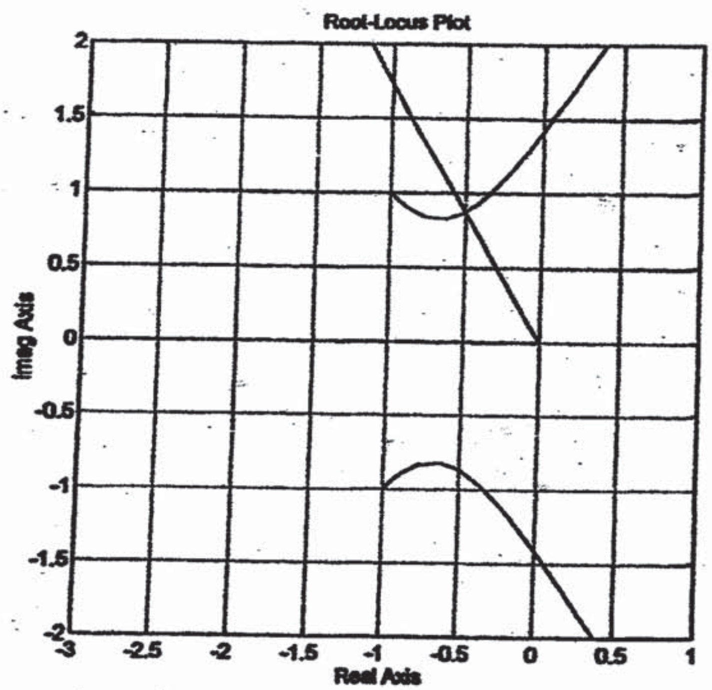
```
% ***** Root-locus plot *****

num = [0  0  0  2];
den = [1  2  2  0];
K1 = 0:0.1:10; K2 = 10:0.5:200;
K = [K1  K2];
r = rlocus(num,den,K);
plot(r,'-')
hold
Current plot held
x = [0  -2]; y = [0  3.464]; line(x,y)
v = [-3  1  -2  2]; axis(v); axis('square')
grid
title('Root-Locus Plot
xlabel('Real Axis');
ylabel('Imag Axis')
```

From the root-locus plot, the dominant closed-loop poles that correspond to the damping ratio ζ of 0.5 are found to be

$$s = -0.5 \pm j0.866$$

The value of K corresponding to the dominant closed-loop poles is obtained as

$$K = \left| \frac{s(s^2+2s+2)}{2} \right|_{s=-0.5+j0.866} = 0.5$$



B-6-27. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{s+1.4}{s+5}\right)\frac{10(s+10)}{s(s+1)(s+10)+10ks}}{1+\left(\frac{s+1.4}{s+5}\right)\frac{10(s+10)}{s(s+1)(s+10)+10ks}}$$

Thus, the characteristic equation is

$$1+\left(\frac{s+1.4}{s+5}\right)\frac{10(s+10)}{s(s+1)(s+10)+10ks} = 0$$

Since the variable k is not a multiplying factor, we rewrite the characteristic equation as

$$(s+5)s(s+1)(s+10) + (s+5)10ks + (s+1.4)10(s+10) = 0$$

which may be rewritten as

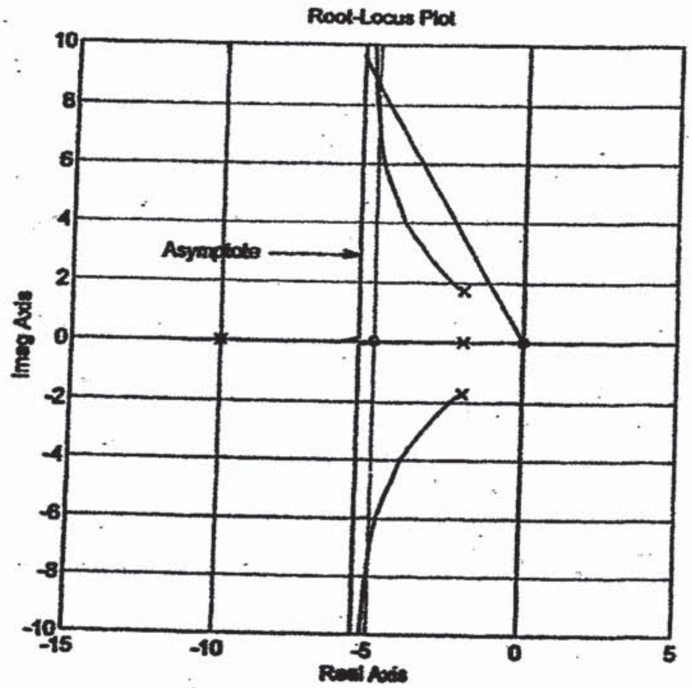
$$1 + \frac{10(s+5)ks}{(s+10)(s^3+6s^2+15s+14)} = 0$$

or

$$1 + \frac{10(s+5)ks}{(s+2)(s+10)(s+2+j1.732)(s+2-j1.732)} = 0$$

Notice that the open-loop poles are at $s = -2$, $s = -10$, and $s = -2 \pm j1.732$. A root locus plot for the system may be obtained by entering the following MATLAB program into the computer. The resulting root-locus plot is shown on the next page.

```
% ***** Root-locus plot *****

num = [0  0  10  50  0];
den = [1  16  75  164  140];
numa = [0  0  10];
dena = [1  11  20];
rlocus(num,den)
hold
Current plot held
a = rlocus(numa,dena);
plot(a,'-')
v = [-15  5  -10  10]; axis(v); axis('square')
x = [0  -5.5]; y = [0  9.5263]; line(x,y)
grid
title('Root-Locus Plot')
text(-12,3,'Asymptote')
```



The dominant closed-loop poles having the damping ratio equal to 0.5 can be determined as the intersections of the root-locus branches and the straight lines from the origin having an angle of 60° or −60° with the negative real axis. The intersections are located at s = −5.14 ± j8.90. The gain value k is obtained from

$$k = \left| \frac{(s+2)(s+10)(s+2+j1.732)(s+2-j1.732)}{10\,(s+5)\,s} \right|_{s=-5.14+j8.90}$$

$$= 9.08$$

With k = 9.08, G(s)H(s) can be given as

$$G(s)H(s) = \left(\frac{s+1.4}{s+5}\right) \frac{10(s+10)}{s(s+1)(s+10) + 90.8s}$$

The static velocity error constant K_v is

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$

$$= \lim_{s \to 0} s \left(\frac{s+1.4}{s+5}\right) \frac{10(s+10)}{s[(s+1)(s+10)+90.8]}$$

$$= 0.2778$$

B-6-28. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + (1+KK_h)s + K}$$

The characteristic equation for the system is

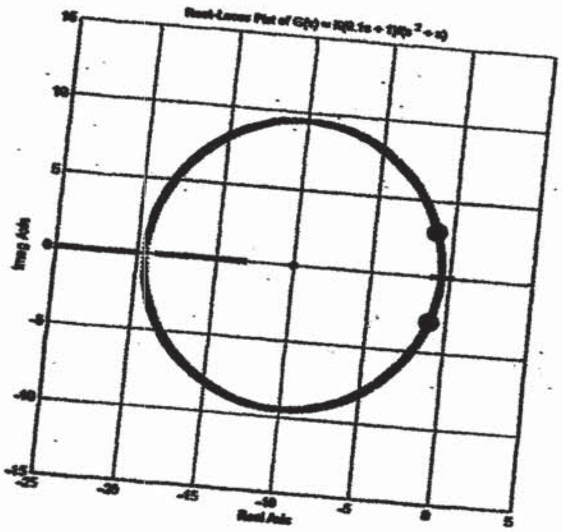
$$s^2 + s + K(K_h s + 1) = 0$$

Divide this characteristic equation by $s^2 + s$ and define

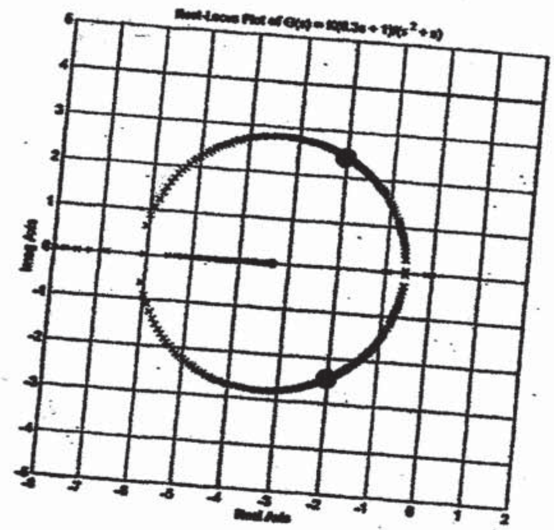
$$G(s) = \frac{K(K_h s + 1)}{s^2 + s}$$

Note that G(s) is in the form suitable for plotting the root loci.

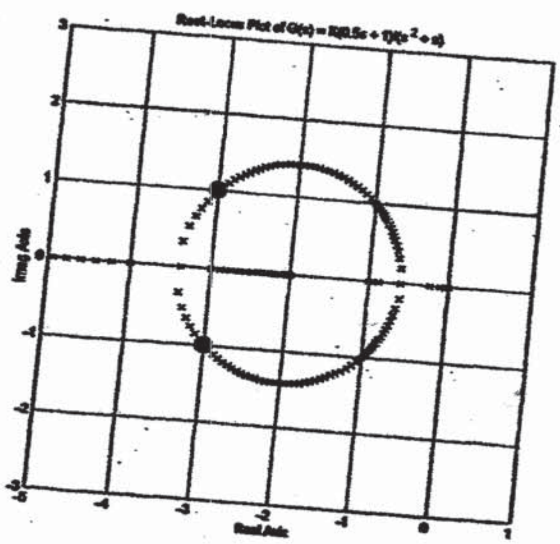
Root-locus plots for G(s) when $K_h = 0.1$, $K_h = 0.3$, and $K_h = 0.5$ are shown in Figures (a), (b), and (c), respectively.



(a)



(b)



(c)

The closed-loop poles when $K = 10$, $K_h = 0.1$; $K = 10$, $K_h = 0.3$; $K = 10$, $K_h = 0.5$ are shown by ● in Figures (a), (b), (c), respectively.

The closed-loop transfer function when K = 10 and K_h = 0.1 becomes as follows:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10}$$

The two closed-loop poles are

$$s = -1 \pm j3$$

The closed-loop transfer function K = 10 and K_h = 0.3 is

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+4s+10}$$

The closed-loop poles are located at

$$s = -2 \pm j\sqrt{6}$$

Similarly, the closed-loop transfer function when K = 10 and K_h = 0.5 is

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+6s+10}$$

The closed-loop poles are located at

$$s = -3 \pm j$$

The unit-step response curves for the above three systems are shown in the figure shown below.