ECE 260

EXAM 2

SOLUTIONS

(FALL 2024)

QUESTION 1A

For a series interconnection, the impulse responses convolve.

For a parallel interconnection, the impulse responses add.

Let v denote the output of the leftmost adder.

We have

$$
\begin{aligned}
v(t) &= x(t) + x * h_1(t) \\
&= x * \delta(t) + x * h_1(t) \\
&= x * \{\delta + h_1\}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
y(t) &= v * h_2(t) + x * h_3(t) \\
&= x * (\delta + h_1) * h_2(t) + x * h_3(t) \\
&= x * [\underbrace{(\delta + h_1) * h_2 + h_3}_{h}](t)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
h &= (\delta + h_1) * h_2 + h_3 \\
&= h_2 + h_1 * h_2 + h_3
\end{aligned}
$$

QUESTION 1B

$$
\begin{aligned}
h(t) &= h_1 * h_2(t) + h_2(t) + h_3(t) \\
&= h_1 * h_1(t) + h_1(t) + h_3(t) \\
&= \int_{-\infty}^{\infty} h_1(\tau)\, h_1(t-\tau)\, d\tau + h_1(t) + h_3(t) \\
&= \int_{-\infty}^{\infty} \delta(\tau-1)\, \delta(t-\tau-1)\, d\tau + \delta(t-1) + \delta(t) \\
&= \left[\delta(t-\tau-1)\right]\Big|_{\tau=1} + \delta(t-1) + \delta(t) \\
&= \delta(t-2) + \delta(t-1) + \delta(t)
\end{aligned}
$$

sifting property

QUESTION 2A

A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

(i.e., h is absolutely integrable).

QUESTION 2B

$$\boxed{h(t) = e^{-|at|}, \quad a \in \mathbb{R}}$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

$$= \int_{-\infty}^{\infty} \left| e^{-|at|} \right| dt$$

$$= \int_{-\infty}^{\infty} e^{-|at|}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-|a||t|}\, dt$$

$$= \int_{-\infty}^{0} e^{-|a|(-t)}\, dt + \int_{0}^{\infty} e^{-|a|t}\, dt$$

if $a \neq 0$

$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

$$= \frac{1}{|a|} e^{|a|t} \Big|_{-\infty}^{0} + \frac{-1}{|a|} e^{-|a|t} \Big|_{0}^{\infty}$$

$$= \frac{1}{|a|}[1-0] - \frac{1}{|a|}[0-1]$$

$$= \frac{1}{|a|} + \frac{1}{|a|}$$

$$= \frac{2}{|a|} < \infty$$

if $a = 0$

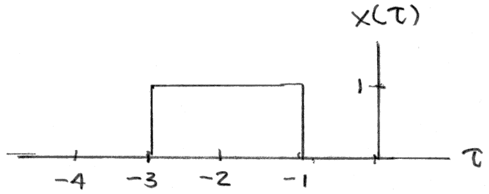
$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

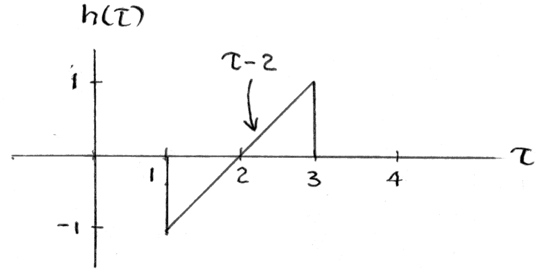
$$= \int_{-\infty}^{\infty} 1\, dt$$

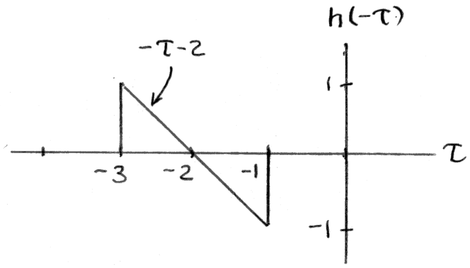
$$= \infty$$

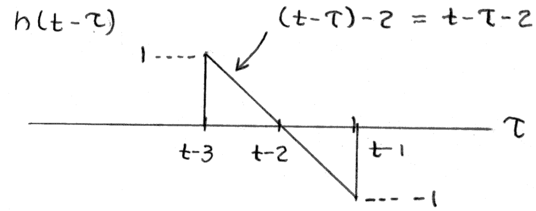
$\therefore$ the system is BIBO stable if and only if $a \neq 0$

QUESTION 3

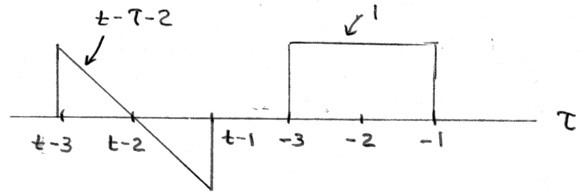






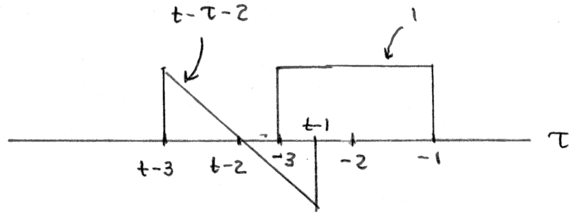


QUESTION 3 (CONTINUED)



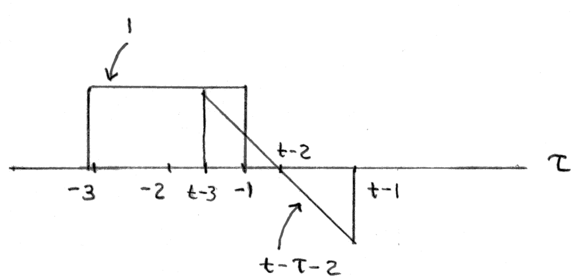
$t-1 < -3 \Rightarrow t < -2$

$x * h(t) = 0$



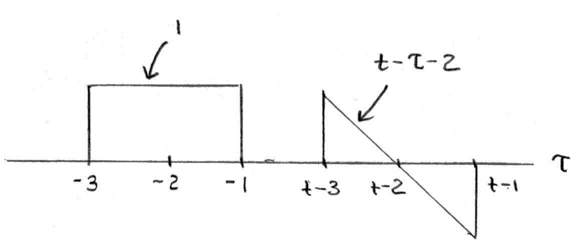
$-3 \leq t-1 < -1 \Rightarrow -2 \leq t < 0$

$x * h(t) = \int_{-3}^{t-1} (1)(t-\tau-2)\, d\tau$



$-3 \leq t-3 < -1 \Rightarrow 0 \leq t < 2$

$x * h(t) = \int_{t-3}^{-1} (1)(t-\tau-2)\, d\tau$



$t-3 > -1 \Rightarrow t \geq 2$

$x * h(t) = 0$

QUESTION 4

Let S_{t_0} denote an operator that time shifts by t_0 (i.e., $S_{t_0} x(t) = x(t-t_0)$).

$$x_2(t) = S_{-1} x_1(t) + 2 S_1 x_1(t)$$

$$\begin{aligned}
y_2(t) &= \mathcal{H} x_2(t) \\
&= \mathcal{H}\{ S_{-1} x_1 + 2 S_1 x_1 \}(t) && \text{linearity of } \mathcal{H} \\
&= \mathcal{H} S_{-1} x_1(t) + 2 \mathcal{H} S_1 x_1(t) && \text{time invariance of } \mathcal{H} \\
&= S_{-1} \mathcal{H} x_1(t) + 2 S_1 \mathcal{H} x_1(t) && \text{by definition of } y_1, y_2 \\
&= S_{-1} y_1(t) + 2 S_1 y_1(t) \\
&= y_1(t+1) + 2 y_1(t-1)
\end{aligned}$$