7.1 Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.

(a) $x(t) = e^{-at}u(t)$;

(b) $x(t) = e^{-a|t|}$; and

(c) $x(t) = \cos(\omega_0 t)u(t)$. [Note: Use (F.3).]

c

$$\boxed{s = \sigma + \omega}$$

$$\mathcal{L}\{[\cos\omega_0 t]u(t)\}(s) = \int_{-\infty}^{\infty} [\cos\omega_0 t]\, u(t)\, e^{-st}\,dt = \int_0^{\infty} [\cos\omega_0 t] e^{-st}\,dt$$

Since this integral does not converge if $s = 0$, we assume that $s \neq 0$.

$$\mathcal{L}\{[\cos\omega_0 t]\,u(t)\}(s) = \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0 \sin\omega_0 t]}{(-s)^2 + \omega_0^2}\right]_0^{\infty}$$

$$= \left[\frac{e^{-st}[-s\cos\omega_0 t + \omega_0 \sin\omega_0 t]}{s^2 + \omega_0^2}\right]_0^{\infty}$$

$$= \left[\frac{e^{-\sigma t} e^{-j\omega t}[-(\sigma + j\omega)\cos\omega_0 t + \omega_0 \sin\omega_0 t]}{(\sigma + j\omega)^2 + \omega_0^2}\right]_0^{\infty}$$

This converges to finite limit if $\sigma > 0$.

$$\mathcal{L}\{[\cos \omega_0 t]u(t)\}(s) = 0 - \left[\frac{-(\sigma+j\omega)}{(\sigma+j\omega)^2+\omega_0^2}\right] = \frac{\sigma+j\omega}{(\sigma+j\omega)^2+\omega_0^2} = \frac{s}{s^2+\omega_0^2} \quad \text{for } \mathrm{Re}(s) > 2$$

7.2 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x below.

(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$, where a and b are real constants and $a > 0$;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

b. $X(s) = \mathcal{L}\{3e^{-2t}u(t) + 2e^{5t}u(-t)\}(s)$

$$= 3\mathcal{L}\{e^{-2t}u(t)\}(s) + 2\mathcal{L}\{e^{5t}u(-t)\}(s)$$

$$= 3\left(\tfrac{1}{s+2}\right) - 2\left(\tfrac{1}{s-5}\right) \quad \text{for } (\mathrm{Re}(s) > -2) \cap (\mathrm{Re}(s) < 5)$$

$$= \frac{3s-15-2s-4}{(s+2)(s-5)} = \frac{s-19}{(s+2)(s-5)} \quad \text{for } -2 < \mathrm{Re}(s) < 5$$

c.

$v_1(t) = x(t-4) \cdots \qquad x(t) = v_1(t+4)$

$v_1(t) = e^{-2(t-4)} u(t-4+4)$

$= e^{8} e^{-2t} u(t)$

$X(s) = \mathcal{L}\,x(t) = \mathcal{L}\{v_1(t+4)\}(s) = e^{4s} V_1(s)$ for $\underline{\text{ROC of } V_1(s)}$ &

$V_1(s) = \mathcal{L}\,v_1(s) = \mathcal{L}\{e^{8} e^{-2t} u(t)\}(s) = e^{8}\,\mathcal{L}\{e^{-2t} u(t)\}(s) = e^{8} \cdot \frac{1}{s+2}$ for $\underline{Re(s) > -2}$

Substituting V_1 into X...

$X(s) = e^{4s} V_1(s) = e^{4s} e^{8} \cdot \frac{1}{s+2} = \frac{e^{4s+8}}{s+2}$ $\underline{\text{for } Re(s) > -2}$

d.

$$x(t) = \int_{-\infty}^{t} v_1(\tau)\, d\tau$$ where $v_1(t) = e^{-2t} v_2(t)$ & $v_2(t) = u(t)$

letting
R_x ...... ROC of X
R_{v_1} .... ROC of V_1
R_{v_2} ..... ROC of V_2

Laplace for both sides... $X(s) = \frac{1}{s} V_1(s)$ for $R_x = R_{v_1} \cap (Re(s) > 0)$

$V_1(s) = V_2(s+2)$ for $R_{v_1} = R_{v_2} - 2$

Combining the above equations we obtain...

$$X(s) = \frac{1}{s}\left[V_2(s+2)\right] = \frac{1}{s}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{s(s+2)} \quad \ldots \quad Re(s) > 0$$

Note that the ROC of X given above...

$$R_x = R_{v_1} \cap (Re(s) > 0)$$

$$= (R_{v_2} - 2) \cap (Re(s) > 0)$$

$$= (Re(s) > -2) \cap (Re(s) > 0)$$

$$= Re(s) > 0$$

e. $x(t) = v_1(t)$ ... $v_1(t) = v_2(t+b)$ & $v_2(t) = -e^{ab} e^{-at} u(t)$

letting R_x ...... ROC of X

R_{v_1} .... ROC of V_1

R_{v_2} ..... ROC of V_2

Laplace from both sides... $X(s) = L\{v_1(-t)\}(s)$

$= V_1(-s)$ for $R_x = -R_{v_1}$

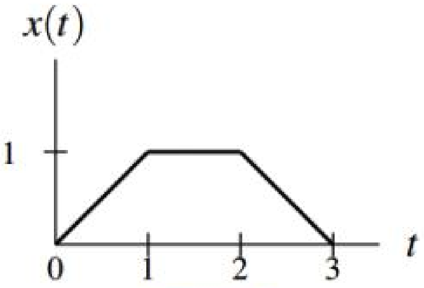
$V_1(s) = L\{v_2(t+b)\}(s) = e^{bs} V_2(s)$ ... R_{v_2}

$V_2(s) = L\{-e^{ab} e^{-at} u(t)\}(s)$

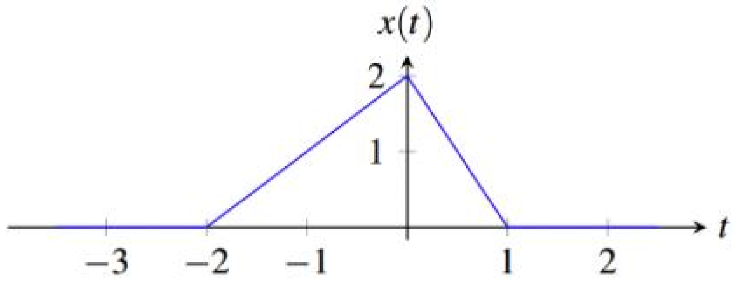
$= -e^{ab} \cdot \frac{1}{s+a}$ for $Re(s) > -a$

$$
\begin{aligned}
X(s) &= V_1(-s) \\
&= e^{-bs} V_2(-s) \\
&= e^{-bs}\left[-e^{ab} \frac{1}{-s+a}\right] \quad \ldots\ Re(s) < a \\
&= e^{-b(s-a)} \cdot \frac{1}{s-a} \quad \ldots\ Re(s) < a \\
&= e^{b(a-s)} \cdot \frac{1}{s-a} \quad \ldots\ Re(s) < a
\end{aligned}
$$

7.4 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x shown in the figure below.



(a)



(b)

(a) $x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ -t+3 & 2 \le t < 3 \\ 0 & — \end{cases}$

$= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)]\cdot 1$

$+ (-t+3)[u(t-2) - u(t-3)]$

$= tu(t) + (-t+1)u(t-1) + (-t+2)u(t-2)$

$+ (t-3)u(t-3)$

$X(s) = \frac{1}{s^2} - \frac{e}{s^2} - \frac{e}{s^2} + \frac{}{s^2}$

$= \frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}$

x is of finite duration...

ROC of X is in complex plane.

7.6 A causal function x has the Laplace transform

$$X(s) = \frac{-2s}{s^2+3s+2}.$$

(a) Assuming that x has no singularities at 0, find $x(0^+)$.
(b) Assuming that $\lim_{t\to\infty} x(t)$ exists, find this limit.

<u>a</u>

x is causal... has no singularities at the origin

we can compute $x(0^+)$ using initial value theorem ...

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \frac{s(-2)}{s^2+3s+2}$$

$$= -2$$

<u>b</u>

x is causal... $\lim_{t\to\infty} x(t)$ exists, so...

we can compute $x(0^+)$ using initial value theorem ...

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

$$= \left.\frac{s(-2s)}{s^2+3s+2}\right|_{s=0} = 0$$

7.10 Find the inverse Laplace transform x of each function X below.

(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;

(b) $X(s) = \dfrac{2s^2+4s+5}{(s+1)(s+2)}$ for $\operatorname{Re}(s) > -1$;

(c) $X(s) = \dfrac{3s+1}{s^2+3s+2}$ for $-2 < \operatorname{Re}(s) < -1$;

(d) $X(s) = \dfrac{s^2-s+1}{(s+3)^2(s+2)}$ for $\operatorname{Re}(s) > -2$; and

(e) $X(s) = \dfrac{s+2}{(s+1)^2}$ for $\operatorname{Re}(s) < -1$.

d. $X(s) = \dfrac{s^2-s+1}{(s+3)^2(s+2)} = \dfrac{A_{1,1}}{s+3} + \dfrac{A_{1,2}}{(s+3)^2} + \dfrac{A_2}{s+2}$

$$A_{1,1} = \frac{1}{(2-1)!}\left[\left[\frac{d}{ds}\right]^{2-1}(s+3)^2 X(s)\right]_{s=-3}$$

$$= \frac{-1\cdot -7 - (9+3+1)}{1} = 7-13 = -6$$

$$A_{1,2} = \frac{1}{(2-2)!}\left[\left[\frac{d}{ds}\right]^{2-2}[s+3]^2 X(s)\right]_{s=-3}$$

$$= \left.\frac{s^2-s+1}{s+2}\right|_{s=-3} = \frac{9+3+1}{-1} = -13$$

$$A_2 = \left[(s+2)X(s)\right]_{s=-2} = \left.\frac{s^2-s+1}{(s+3)^2}\right|_{s=-2} = \frac{4+2+1}{1} = 7$$

$$X(s) = -\frac{6}{s+3} - \frac{13}{(s+3)^2} + \frac{7}{s+2}$$

$$= -6e^{-3t}u(t) - 13te^{-3t}u(t) + 7e^{-2t}u(t)$$

7.12 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

Each distinct ROC for H will yield a distinct inverse Laplace transform.

Since H is rational function w/ poles at 1 and -1, three ROCs are possible.

$$h(t) = \mathcal{L}^{-1} H(t) = \mathcal{L}^{-1}\left\{\frac{4}{s+1} + \frac{3}{s-1}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$

$$+ 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)$$

For $\boxed{R(s) < -1}$ we have

$$h(t) = 4\left[-e^{-t} u(-t)\right] + 3\left[-e^{t} u(-t)\right]$$

$$= \left[-4e^{-t} - 3e^{t}\right] u(-t)$$

$\boxed{-1 < Re(s) < 1}$ we have

$$h(t) = 4\left[e^{-t} u(t)\right] + 3\left[-e^{t} u(-t)\right]$$

$$= 4e^{-t} u(t) - 3e^{t} u(-t)$$

$$\boxed{R(s) > 1}$$

$$x(t) = 4\left[e^{-t}u(t)\right] + 3\left[e^{t}u(t)\right] = \left[4e^{-t} + 3e^{t}\right]u(t)$$

7.5 For each case below, using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x, where the ROCs of X and Y are R_X and R_Y, respectively.

(a) $y(t) = x(at - b)$, where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}[x * x(t-1)]$;
(c) $y(t) = tx(3t-2)$;
(d) $y(t) = \mathcal{D}x_1(t)$, where $x_1(t) = x^*(t-3)$ and $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = e^{-5t}x(3t+7)$; and
(f) $y(t) = e^{-j5t}x(t+3)$.

e.

$v_1(t) = x(t+7)$ ... $y(t) = e^{5t} v_2(t)$

$v_2(t) = v_1(3t)$

$V_1(s) = e^{7s} X(s)$ ... $R_{V_1} = R_X$ (ROC of V_1)

$V_2(s) = \frac{1}{3} V_1(s/3)$ ... $R_{V_2} = 3R_{V_1}$ (ROC of V_2)

$Y(s) = V_2(s+5)$ ... $R_Y = R_{V_2} - 5$

... $Y(s) = V_2(s+5) = \frac{1}{3} V_1\left(\frac{s+5}{3}\right) = \frac{1}{3} e^{7\left(\frac{s+5}{3}\right)} X\left(\frac{s+5}{3}\right)$

Also, we've a ROC of ... $\boxed{R_Y = R_{V_2} - 5} = \boxed{3R_{V_1} - 5} = 3R_X - 5$