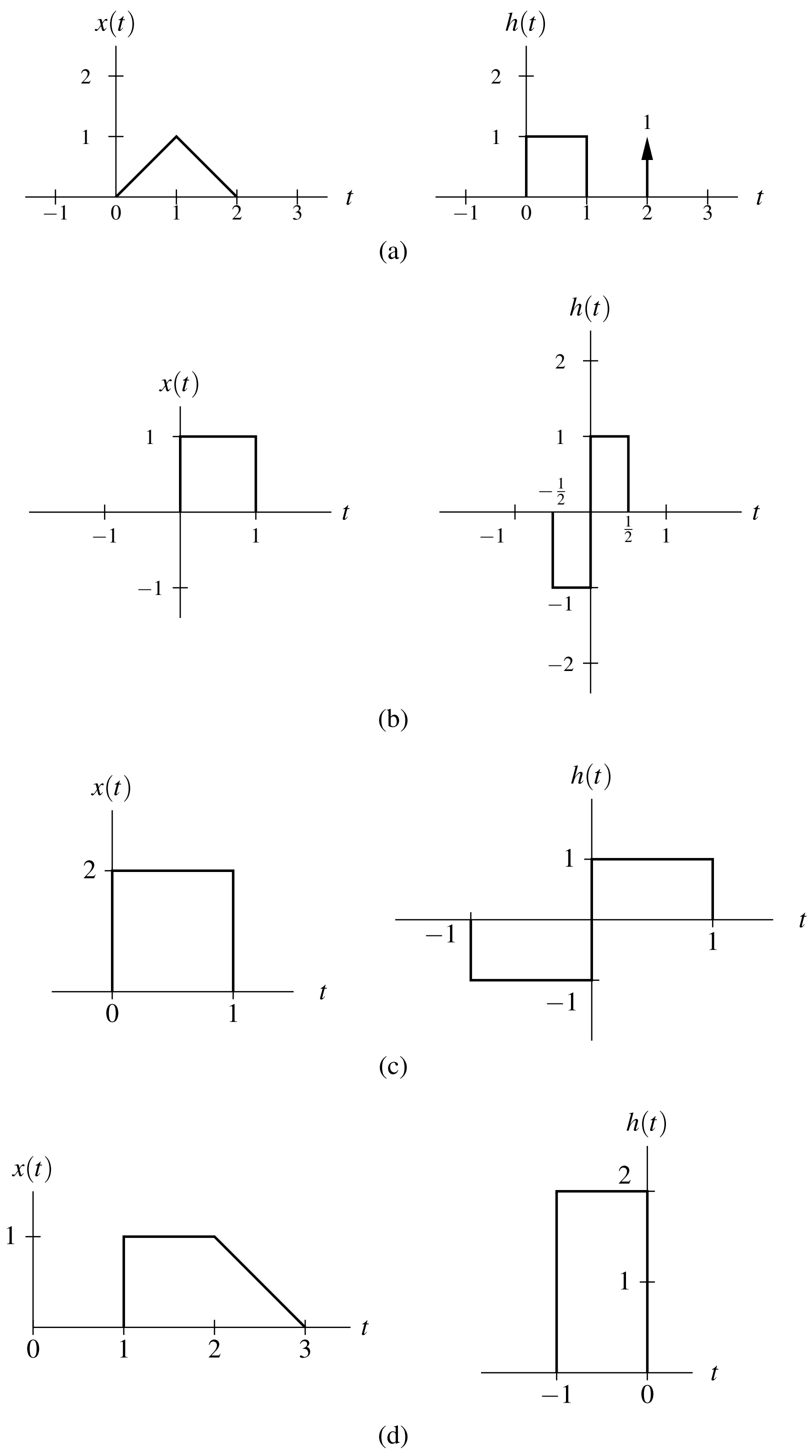
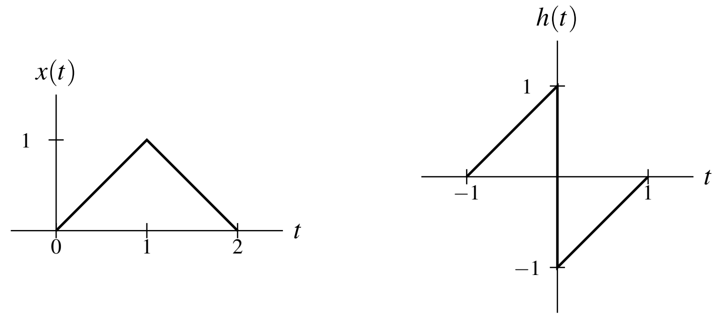
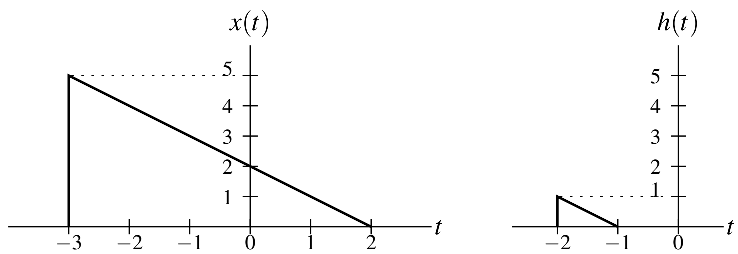
3A **4.1** Using the graphical method, for each pair of functions x and h given in the figures below, directly compute $x * h$. (Do not compute $x * h$ indirectly by instead computing $h * x$ and using the commutative property of convolution.)





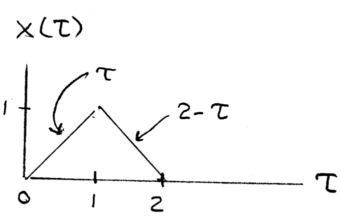
(e)



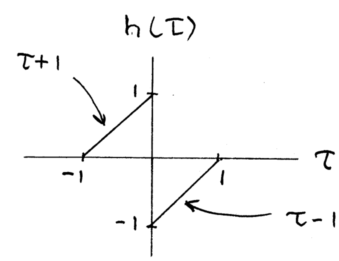
(f)

3A Answer (e).

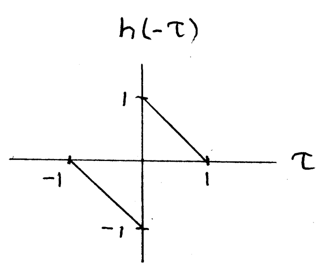
To assist in the convolution computation, we first plot $x(\tau)$ and $h(t-\tau)$ versus τ as shown below.



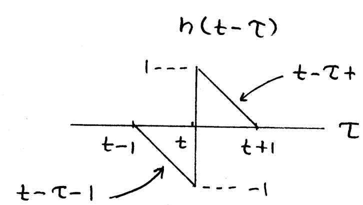
(a)



(b)

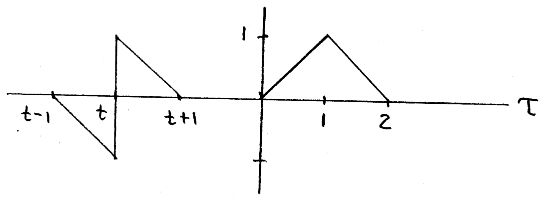


(c)

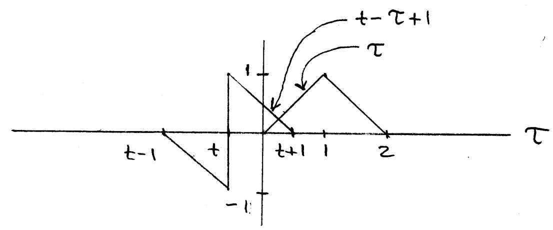


(d)

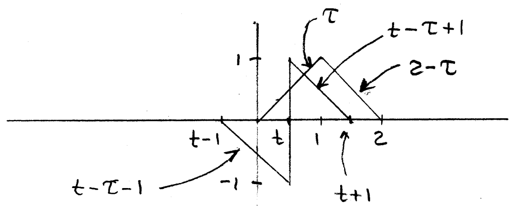
From the above plots, we can deduce that there are six cases (i.e., intervals of t) to be considered, which correspond to the scenarios shown in the graphs below.



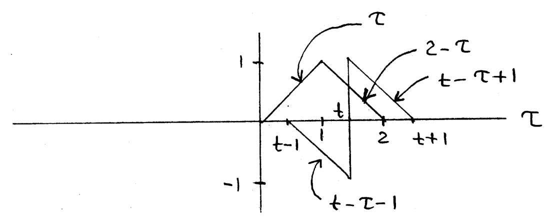
(a)



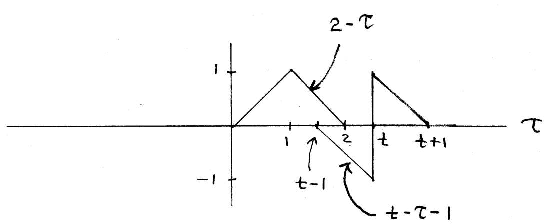
(b)



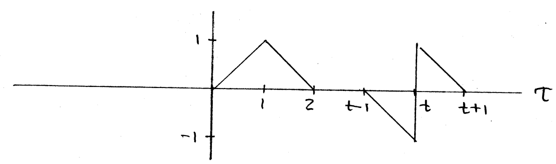
(c)



(d)



(e)



(f)

In the case that $t < -1$, which corresponds to Figure (a), we trivially have

$$x * h(t) = 0.$$

In the case that $-1 \leq t < 0$, which corresponds to Figure (b), we have

$$x * h(t) = \int_0^{t+1} (\tau)(t-\tau+1)d\tau.$$

In the case that $0 \leq t < 1$, which corresponds to Figure (c), we have

$$x * h(t) = \int_0^t (\tau)(t-\tau-1)d\tau + \int_t^1 (\tau)(t-\tau+1)d\tau + \int_1^{t+1} (2-\tau)(t-\tau+1)d\tau.$$

In the case that $1 \leq t < 2$, which corresponds to Figure (d), we have

$$x * h(t) = \int_{t-1}^1 (\tau)(t-\tau-1)d\tau + \int_1^t (2-\tau)(t-\tau-1)d\tau + \int_t^2 (2-\tau)(t-\tau+1)d\tau.$$

In the case that $2 \leq t < 3$, which corresponds to Figure (e), we have

$$x * h(t) = \int_{t-1}^2 (2-\tau)(t-\tau-1)d\tau.$$

In the case that $t \geq 3$, which corresponds to Figure (f), we trivially have

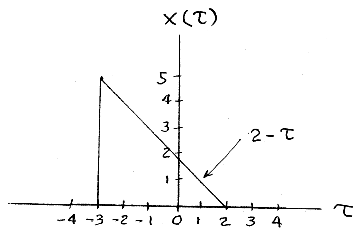
$$x * h(t) = 0.$$

Combining the above results, we have that

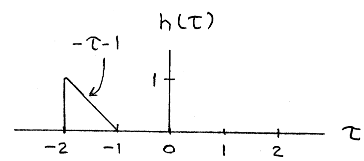
$$x*h(t) = \begin{cases} \int_0^{t+1}(\tau)(t-\tau+1)d\tau & -1 \le t < 0 \\ \int_0^t(\tau)(t-\tau-1)d\tau + \int_t^1(\tau)(t-\tau+1)d\tau + \int_1^{t+1}(2-\tau)(t-\tau+1)d\tau & 0 \le t < 1 \\ \int_{t-1}^1(\tau)(t-\tau-1)d\tau + \int_1^t(2-\tau)(t-\tau-1)d\tau + \int_t^2(2-\tau)(t-\tau+1)d\tau & 1 \le t < 2 \\ \int_{t-1}^2(2-\tau)(t-\tau-1)d\tau & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

3A Answer (f).

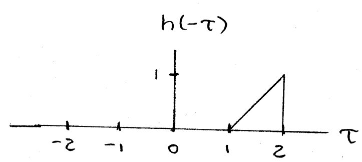
To assist in the convolution computation, we first plot $x(\tau)$ and $h(t-\tau)$ versus τ as shown below.



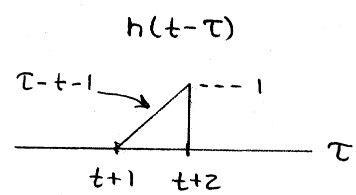
(a)



(b)

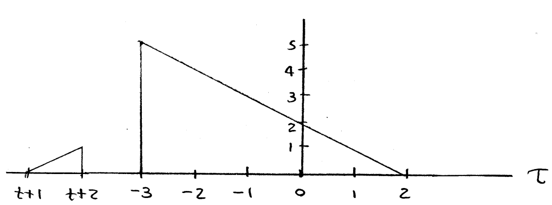


(c)

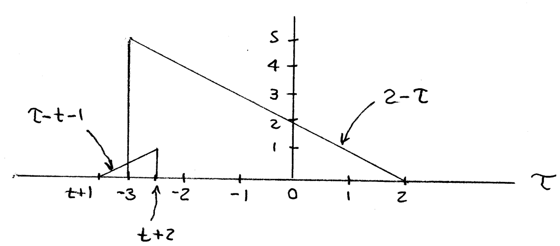


(d)

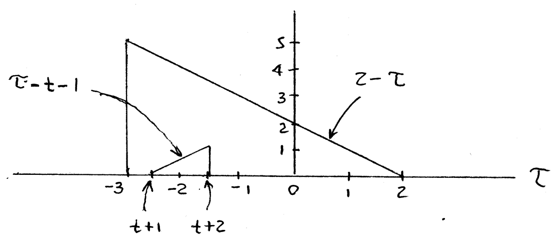
From the above plots, we can deduce that there are five cases (i.e., intervals of t) to be considered, which correspond to the scenarios shown in the graphs below.



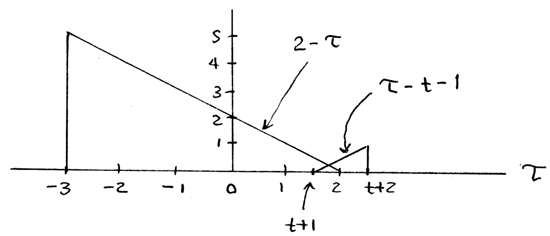
(a)



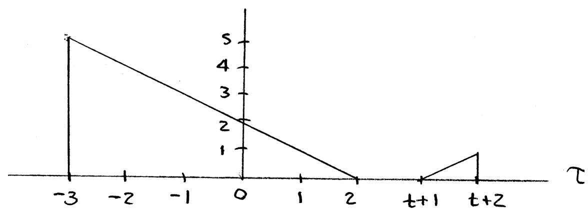
(b)



(c)



(d)



(e)

In the case that $t < -5$, which corresponds to Figure (a), we trivially have

$$x * h(t) = 0.$$

In the case that $-5 \leq t < -4$, which corresponds to Figure (b), we have

$$x * h(t) = \int_{-3}^{t+2} (2-\tau)(\tau - t - 1) d\tau.$$

In the case that $-4 \leq t < 0$, which corresponds to Figure (c), we have

$$x * h(t) = \int_{t+1}^{t+2} (2-\tau)(\tau - t - 1) d\tau.$$

In the case that $0 \leq t < 1$, which corresponds to Figure (d), we have

$$x * h(t) = \int_{t+1}^{2} (2-\tau)(\tau - t - 1) d\tau.$$

In the case that $t \geq 1$, which corresponds to Figure (e), we trivially have

$$x * h(t) = 0.$$

Combining all of the above results, we have

$$x * h(t) = \begin{cases} \int_{-3}^{t+2} (2-\tau)(\tau - t - 1) d\tau & -5 \leq t < -4 \\ \int_{t+1}^{t+2} (2-\tau)(\tau - t - 1) d\tau & -4 \leq t < 0 \\ \int_{t+1}^{2} (2-\tau)(\tau - t - 1) d\tau & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

3A **4.3** Using the graphical method, compute $x*h$ for each pair of functions x and h given below.

(a) $x(t) = e^{t}u(-t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise}; \end{cases}$

(b) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}(\frac{1}{3}[t-\frac{1}{2}])$;

(c) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise}; \end{cases}$

(d) $x(t) = \operatorname{rect}\left(\frac{1}{2}t\right)$ and $h(t) = e^{2-t}u(t-2)$;

(e) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t+2 & -2 \le t < -1 \\ 0 & \text{otherwise}; \end{cases}$

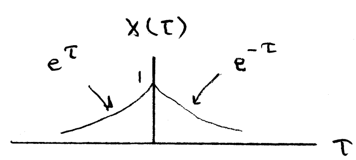
(f) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise}; \end{cases}$

(g) $x(t) = \begin{cases} 1-\frac{1}{4}t & 0 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise}; \end{cases}$

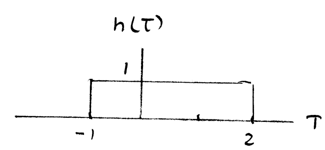
(h) $x(t) = \operatorname{rect}\left(\frac{1}{4}t\right)$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise}; \end{cases}$ and

(i) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-2 & 2 \le t < 4 \\ 0 & \text{otherwise}. \end{cases}$

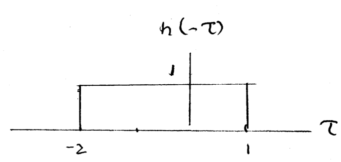
3A **Answer (b).**



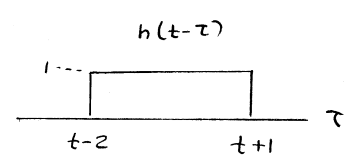
(a)



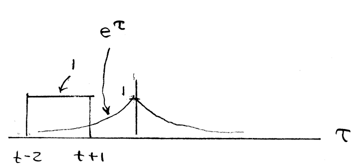
(b)



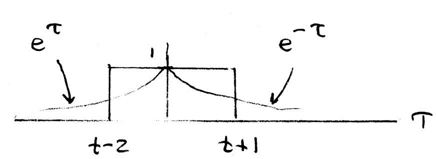
(c)



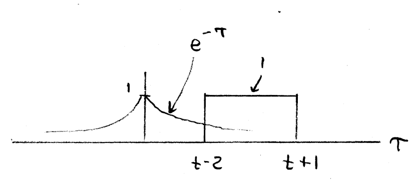
(d)



(e)



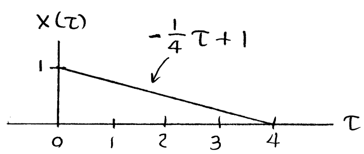
(f)



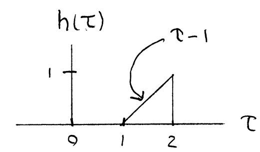
(g)

$$x*h(t) = \begin{cases} \int_{t-2}^{t+1} e^{\tau} d\tau & t < -1 \\ \int_{t-2}^{0} e^{\tau} d\tau + \int_{0}^{t+1} e^{-\tau} d\tau & -1 \leq t < 2 \\ \int_{t-2}^{t+1} e^{-\tau} d\tau & t \geq 2 \end{cases}$$

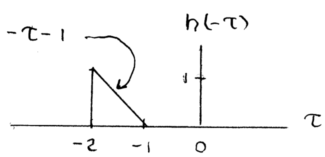
3A Answer (g).



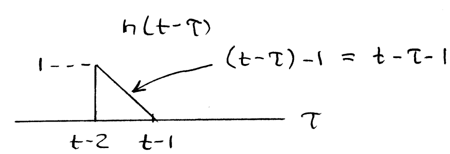
(a)



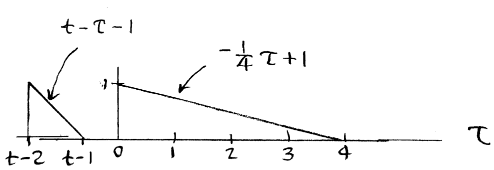
(b)



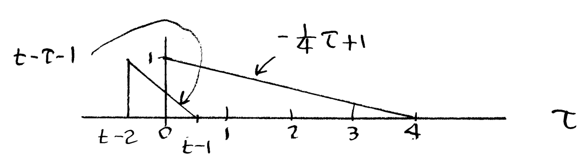
(c)



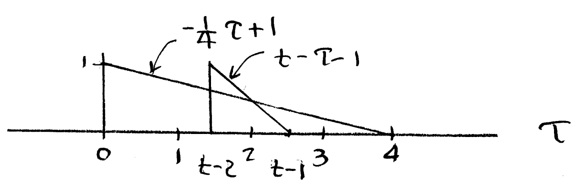
(d)



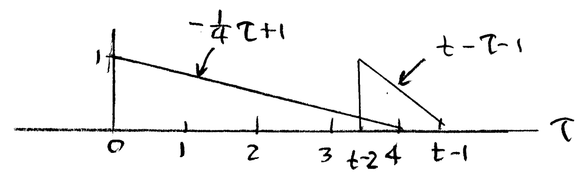
(e)



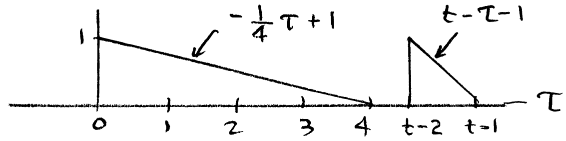
(f)



(g)



(h)



(i)

$$x * h(t) = \begin{cases} \int_0^{t-1} \left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} \left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 2 \le t < 5 \\ \int_{t-2}^{4} \left(-\frac{1}{4}\tau+1\right)(t-\tau-1)d\tau & 5 \le t < 6 \\ 0 & \text{otherwise} \end{cases}$$

3A **4.5** Let x, y, h, and v be functions such that $y = x * h$ and

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where a and b are real constants. Express v in terms of y.

3A **Answer.**

From the definition of v, we have

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau.$$

Now, we employ a change of variable. Let $\lambda = -\tau - b$ so that $\tau = -\lambda - b$ and $d\tau = -d\lambda$. Applying this change of variable and simplifying, we obtain

$$\begin{aligned} v(t) &= \int_{\infty}^{-\infty} x(\lambda)h([-\lambda - b] + at)(-1)d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h(at - b - \lambda)d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h([at - b] - \lambda)d\lambda \\ &= x * h(at - b) \\ &= y(at - b). \end{aligned}$$

Therefore, we have that $v(t) = y(at - b)$.

3A **4.6** Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

3A **Answer (a).**

From the definition of convolution, we have

$$\begin{aligned} y(t) &= x * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \end{aligned}$$

Suppose that x is periodic with period T. Then, we have $x(t) = x(t+T)$ and we can rewrite the above integral as

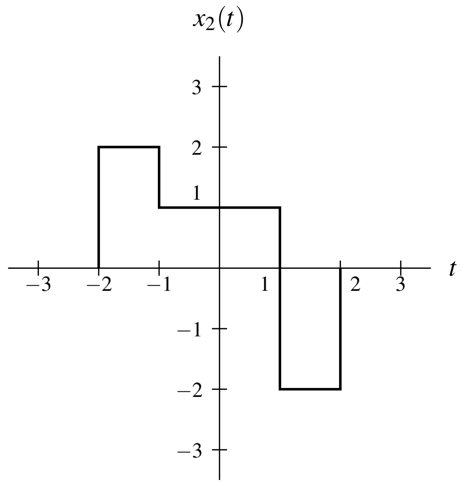
$$y(t) = \int_{-\infty}^{\infty} x(\tau+T)h(t-\tau)d\tau.$$

Now, we employ a change of variable. Let $\lambda = \tau + T$ so that $\tau = \lambda - T$ and $d\lambda = d\tau$. This yields

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\lambda)h(t-[\lambda-T])d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h(t+T-\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda)h([t+T]-\lambda)d\lambda \\ &= x * h(t+T) \\ &= y(t+T). \end{aligned}$$

Therefore, y is periodic with period T.

3A **4.9** Consider a LTI system whose response to the function $x_1(t) = u(t) - u(t-1)$ is the function y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



3A **Answer.**

Let $\mathcal{H}$ denote the operator corresponding to the given LTI system. Let $\mathcal{S}_d$ denote an operator that time shifts a function by d (i.e., $\mathcal{S}_d x(t) = x(t-d)$). Observe that x_1 is a rectangular pulse with a rising edge at 0 and a falling edge at 1. First, we express x_2 in terms of x_1 (using only linear combinations and time shifts). This yields

$$x_2 = 2\mathcal{S}_{-2}x_1 + \mathcal{S}_{-1}x_1 + x_1 - 2\mathcal{S}_1 x_1$$

(i.e., $x_2(t) = 2x_1(t+2) + x_1(t+1) + x_1(t) - 2x_1(t-1)$). Substituting this formula for x_2 into $y_2 = \mathcal{H}x_2$, we have

$$y_2 = \mathcal{H}\{2\mathcal{S}_{-2}x_1 + \mathcal{S}_{-1}x_1 + x_1 - 2\mathcal{S}_1 x_1\}.$$

Since $\mathcal{H}$ is linear (i.e., additive and homogeneous), the preceding equation can be rewritten as

$$\begin{aligned} y_2 &= \mathcal{H}\{2\mathcal{S}_{-2}x_1\} + \mathcal{H}\{\mathcal{S}_{-1}x_1\} + \mathcal{H}x_1 - \mathcal{H}\{2\mathcal{S}_1 x_1\} \\ &= 2\mathcal{H}\mathcal{S}_{-2}x_1 + \mathcal{H}\mathcal{S}_{-1}x_1 + \mathcal{H}x_1 - 2\mathcal{H}\mathcal{S}_1 x_1. \end{aligned}$$

Since $\mathcal{H}$ is time invariant, $\mathcal{H}$ commutes with time shifts, and consequently the preceding equation can be rewritten as

$$y_2 = 2\mathcal{S}_{-2}\mathcal{H}x_1 + \mathcal{S}_{-1}\mathcal{H}x_1 + \mathcal{H}x_1 - 2\mathcal{S}_1\mathcal{H}x_1.$$

Using the fact that $y_1 = \mathcal{H}x_1$, we have

$$\begin{aligned} y_2 &= 2\mathcal{S}_{-2}y_1 + \mathcal{S}_{-1}y_1 + y_1 - 2\mathcal{S}_1 y_1 \\ &= 2y_1(\cdot+2) + y_1(\cdot+1) + y_1 - 2y_1(\cdot-1) \end{aligned}$$

(or equivalently, $y_2(t) = 2y_1(t+2) + y_1(t+1) + y_1(t) - 2y_1(t-1)$).

3A **D.5** Let F denote the complex-valued function of a real variable given by

$$F(\omega) = \frac{1}{j\omega + 1}.$$

Write a program to plot $|F(\omega)|$ and $\arg F(\omega)$ for ω in the interval $[-10, 10]$. Use `subplot` to place both plots on the same figure.

3A **Answer.**

```
w = linspace(-10, 10, 500);
f = (j * w + 1) .^ (-1);
subplot(2, 1, 1);
plot(w, abs(f));
title('Magnitude');
xlabel('\omega');
ylabel('|F(\omega)|');
subplot(2, 1, 2);
plot(w, unwrap(angle(f)));
title('Argument');
xlabel('\omega');
ylabel('arg F(\omega)');
```