ELEC 360 : Control Theory and Systems I
Midterm
February 26th, 2013

Name: ______________________________

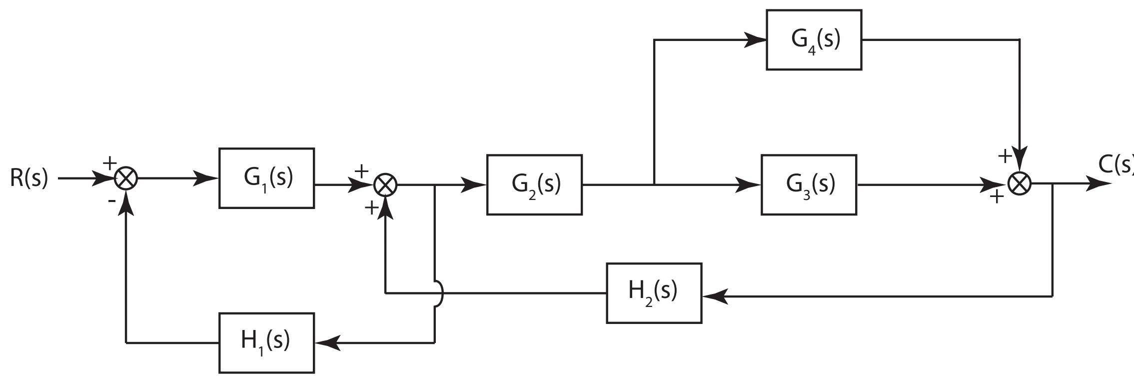
Student Number: ______________________________

Mark: ________ /40

Notes:

- Students are permitted a one page single-side 8.5 by 11 inch crib sheet.
- Calculators are allowed.
  - No other aids permitted.
  - Use of cell phones or other electronic devices, except calculators, during the exam will result in a zero grade.

1. Determine the transfer function of the following system via applying Mason's gain formula. (10 pts)



Solution:

Two forward paths:

$P_1 = G_1G_2G_3$

$P_2 = G_1G_2G_4$

Three feedback loops:

$L_1 = -G_1H_1$

$L_2 = G_2G_3H_2$

$L_3 = G_2G_4H_2$

Mason's Gain Formula:

$$P = \frac{1}{\Delta}\sum_{k=1,2} P_k\Delta_k$$

$$\begin{aligned}\Delta &= 1 - (L_1 + L_2 + L_3)\\ &= 1 - (-G_1H_1) - (G_2G_3H_2) - (G_2G_4H_2)\\ &= 1 + G_1H_1 - G_2G_3H_2 - G_2G_4H_2\end{aligned}$$

Note: L_1 and L_2, L_1 and L_3, and L_2 and L_3 each touch, so there are *no pairs* of two non-touching loops.

$\Delta_1 = 1$

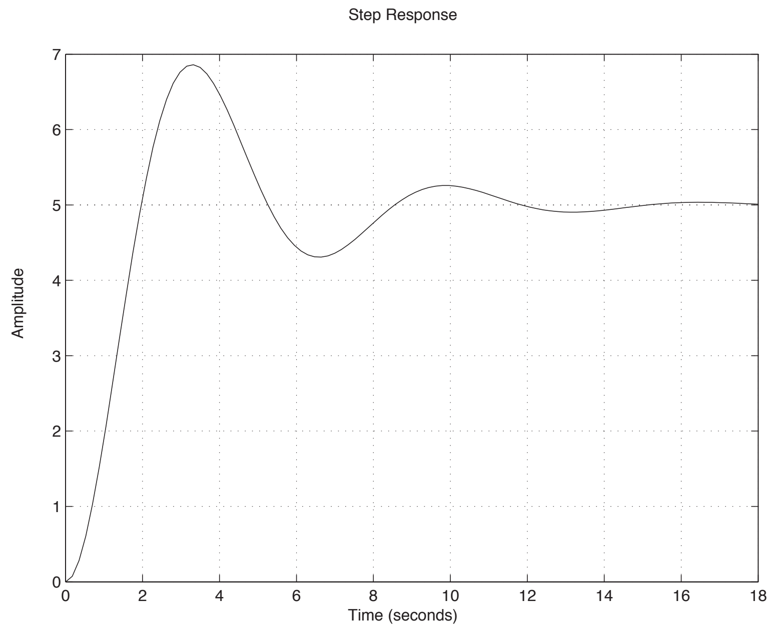
$\Delta_2 = 1$

Which gives:

$P = \frac{G_1G_2G_3+G_1G_2G_4}{1+G_1H_1-G_2G_3H_2-G_2G_4H_2}$

2. Given the following time domain unit step response of a second order system determine its transfer function: (10 pts)

The from the response the following can be measured: $t_r = 1.3286$ s, $M_p = 37.2169\%$, $t_p = 3.3224$ s.



Solution:

As the unit step response settles out to 5 we known that $K = 5$, so we just need to find the damping ration ζ and the natural frequency w_n as,

$$G(s) = \frac{Kw_n^2}{s^2 + 2\zeta w_n + w_n^2}$$

We know that $t_p = \frac{\pi}{w_d}$. Solving for w_d we get $w_d = \frac{\pi}{3.3224}$.

We also know that $t_r = \frac{1}{w_d}\tan^{-1}1\left(-\frac{w_d}{\sigma}\right)$.

Substituting in the values for t_r and w_d gives σ.

Finally, we known that $\zeta = \frac{\sigma}{w_n}$ and that $w_d = w_n\sqrt{1-\zeta^2}$.

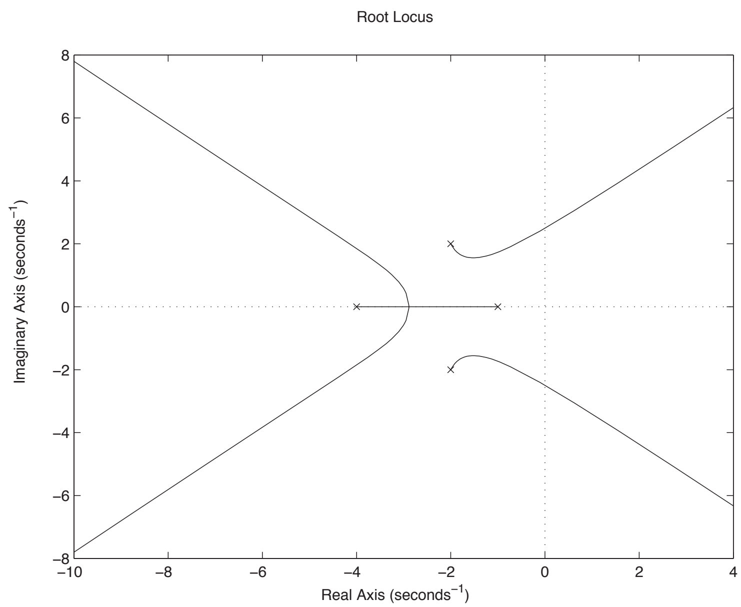
Given that we have computed values for σ and w_d this gives us two equations in the two unknowns ζ and w_n to solve.

Solving gives $\zeta = 0.3$ and $w_n = 1$ and producing the overall transfer function:

$$G(s) = \frac{5}{s^2 + 0.6s + 1}$$

3. Given the following system plot its root locus diagram. Compute all break-in break-out points, complex pole departure angles, asymptotic slopes and intersections. Sketch the unit step response of the system as $K \to 0$ and as $K \to \infty$? (10 pts)

$$\mathrm{G}\,(s) = \frac{5}{(s+4)(s+1)(s^2+4s+8)}$$



Break-out point:

Characteristic equation: $f(s) = A(s) + KB(s) = 0$

$A(s) = (s+4)(s+1)(s^2+4s+8)$ and $B(s) = 1$ and $K = 5$.

To find the break-out point we need to solve for a real root of,

$$\begin{aligned}
\frac{dK}{ds} &= -\frac{A'(s)B(s) - A(s)B'(s)}{B^2(s)} = 0 \\
&= -A'(s) = 0 \\
&= -\frac{d}{ds}[(s^2+5s+4)(s^2+4s+8)] \\
&= (2s+5)(s^2+4s+8) + (s^2+5s+4)(2s+4) \\
&= (2s^3+5s^2+8s^2+20s+16s+40) + (2s^3+10s^2+8s+4s^2+20s+16) \\
&= 4s^3+27s^2+64s+56
\end{aligned}$$

We know that this equation has only one real root, which must be in the interval $[-4,-1]$ and 2 complex conjugate roots. The real root is at $s \approx -2.9$ [Netwon's method can be used to approximate this root as you do not want to solve for all of the roots only find the single real root which must exist within the $[-4,-1]$ interval].

The angle of the asymptotes are at $\frac{\pm 180^\circ}{4} = \pm 45^\circ$.

The intersection of the asymptotes with the real line is at $\sigma_a = \frac{(-4)+(-2-2j)+(-2+2j)+(-1)}{4} = -2.25$

Angle of departure from the complex poles $\theta = 180^\circ - ([180^\circ - tan^{-1}(2/1)] - 90^\circ - 45^\circ) = -71.56^\circ$

As $K \to 0$ the unit step response will look similar to an underdamped second order response, while for large K's, i.e., $K \to \infty$, the unit step response will still contain sinusoidal oscillation but these will have amplitudes that grow exponentially over time producing an unstable system.

4. Apply the Routh-Hurwith approach to determine when the following transfer function is stable, (10 pts)

$$\mathrm{G}\left(s\right)=\frac{\left(s^{3}-5s^{2}+3s+1\right)}{\left(3s^{3}-bs^{2}\right)\left(3s^{2}+s+1\right)}$$

Solution:

$$\begin{aligned}A(s)&=9s^5+3s^4+3s^3-3bs^4-bs^3-bs^2\\&=9s^5+(3-3b)s^4+(3-b)s^3-bs^2\end{aligned}$$

s^5:	9	$3-b$
s^4:	$3-3b$	$-b$
s^3:	$\frac{[(3-3b)(3-b)-(9)(-b)]}{(3-3b)}$	
s^2:	$-b$	
s^1:	0	
s^0:	0	

No sign changes are allowable in 1^{st} column if $G(s)$ is to be stable.

Comparing s^5: and s^4: columns gives: $b \leq 1$.

Comparing s^2: and s^1: columns gives: $b \leq 0$.

Simplifying s^3: gives,

$$\begin{aligned}\frac{[(3-3b)(3-b)]-(-9b)}{(3-3b)}&=\frac{9-9b-3b+3b^2+9b}{3-3b}\\&=\frac{3b^2-3b+9}{3-3b}\\&=\frac{b^2-b+3}{1-b}\\&\geq 0\end{aligned}$$

Solving,

The denominator $1-b$ is negative for all $b>1$ and positive for all $b<1$.

The numerator b^2-b+3 is positive for all b.

So if $\frac{b^2-b+3}{1-b} \geq 0$ then $b<1$.

Combining with the above means that $b<1$ and $b \leq 0$ if there are to be no sign changes in the 1^{st} column.

Hence, the transfer function will be stable for as long as $b \leq 0$.

END OF EXAM