CHAPTER 9

B-9-1.

(a) **Controllable canonical form:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) **Observable canonical form:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

B-9-2. The transfer function representation of this system is

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)}$$

The partial-fraction expansion of Y(s)/U(s) is

$$\frac{Y(s)}{U(s)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

Then, a diagonal canonical form of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-9-3. **We shall present two methods to obtain the controllable canonical form of the given system equation.**

Referring to Equation (2-29), we have

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & -2 \\ 4 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+5} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -4 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{3s+1}{s^2+2s+5} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

Hence

$$a_1 = 2, \quad a_2 = 5, \quad b_0 = 0, \quad b_1 = 3, \quad b_2 = 1$$

Then, referring to Equations (9-3) and (9-4), the controllable canonical **form of the state and output equations are obtained as**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The method presented below is a useful approach to obtain the controllable canonical form of state space representation not presented in Chapter 9, but is given in Chapter 10. [Refer to Equations (10-4) through (10-9).] Transform the original state vector $\mathbf{x}$ to a new state vector $\hat{\mathbf{x}}$ by means of the transformation matrix $\mathbf{T}$ such that $\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$, where

$$\mathbf{T} = \mathbf{M}\mathbf{W} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence,

$$\underset{\sim}{T} = \begin{bmatrix} 1 & 5 \\ 2 & -10 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}$$

and

$$\underset{\sim}{T}^{-1} = \begin{bmatrix} 0.1 & -0.05 \\ 0.3 & 0.35 \end{bmatrix}$$

The controllable canonical form of the state equation and output equation are given by

$$\dot{\hat{\underset{\sim}{x}}} = \underset{\sim}{T}^{-1}\underset{\sim}{A}\,\underset{\sim}{T}\,\hat{\underset{\sim}{x}} + \underset{\sim}{T}^{-1}\underset{\sim}{B}u$$

$$y = \underset{\sim}{C}\,\underset{\sim}{T}\,\hat{\underset{\sim}{x}}$$

or

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.05 \\ 0.3 & 0.35 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0.1 & -0.05 \\ 0.3 & 0.35 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$= \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

B-9-4. Referring to Equation (2-29), we have

$$G(s) = \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B}$$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} s+1 & 0 & -1 \\ -1 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\frac{1}{(s+1)(s+2)(s+3)}\begin{bmatrix} (s+2)(s+3) & 0 & s+2 \\ s+3 & (s+1)(s+3) & 1 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{s+3}{(s+1)(s+2)(s+3)} = \frac{s+3}{s^3 + 6s^2 + 11s + 6}$$

Although this is a third-order system, there is a cancellation of (s + 3) in the numerator and denominator. Hence, the reduced transfer function becomes of second order.

The transfer function expression can be easily obtained from the state-space expression if MATLAB command

[num,den] = ss2tf(A,B,C,D)

is used. See the following MATLAB output.

```
A = [-1  0  1;1  -2  0;0  0  -3];
B = [0;0;1];
C = [1  1  0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

      0       0   1.0000   3.0000


den =

    1    6   11    6
```

This output corresponds to the transfer function

$$\frac{s+3}{s^3+6s^2+11s+6}$$

Notice that the MATLAB output does not show the reduced transfer function when cancellation occurs.

B-9-5. The eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = j, \quad \lambda_4 = -j$$

The following transformation matrix $\mathbf{P}$ will give $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$:

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & j & -j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -j & j \end{bmatrix}$$

This can be seen as follows. Since the inverse of matrix $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \\ 1 & j & -1 & -j \end{bmatrix}$$

we have

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & j & -j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -j & j \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & -j \end{bmatrix}$$

B-9-6.

Method 1:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} & -\frac{1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Method 2: **Referring to Equation** (9-46), we have

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\mathbf{P}^{-1}$$

Since the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$, we obtain

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Method 3: Referring to Equation (9–47), we have

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

or

$$\begin{vmatrix} 1 & -1 & e^{-t} \\ 1 & -2 & e^{-2t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

which can be rewritten as

$$-e^{\mathbf{A}t} + (\mathbf{A} + 2\mathbf{I})e^{-t} - e^{-2t}\mathbf{I} = \mathbf{A}e^{-2t}$$

Thus

$$e^{\mathbf{A}t} = (\mathbf{A} + 2\mathbf{I})e^{-t} - e^{-2t}\mathbf{I} - e^{-2t}\mathbf{A}$$

$$= \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}e^{-t} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}e^{-2t} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}e^{-2t}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

B-9-7. The given state matrix is in the Jordan canonical form. The eigenvalues are

$$\lambda_1 = 2\,, \quad \lambda_2 = 2\,, \quad \lambda_3 = 2$$

Since

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

we have

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

B-9-8.

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 3 & s+2 \end{vmatrix} = s^2 + 2s + 3 = (s+1+j\sqrt{2})(s+1-j\sqrt{2})$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)+3}\begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+1+1}{(s+1)^2+\sqrt{2}^2} & \dfrac{1}{(s+1)^2+\sqrt{2}^2} \\ \dfrac{-3}{(s+1)^2+\sqrt{2}^2} & \dfrac{s+1-1}{(s+1)^2+\sqrt{2}^2} \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+1}{(s+1)^2+\sqrt{2}^2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2} & \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2} \\ -\frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2} & \frac{s+1}{(s+1)^2+\sqrt{2}^2} - \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t & \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t \\ -\frac{3}{\sqrt{2}}e^{-t}\sin\sqrt{2}t & e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t \end{bmatrix}$$

Hence

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos\sqrt{2}t \\ -e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t \end{bmatrix}$$

B-9-9. Define

$$\mathbf{A} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Define also the transformation matrix as $\mathbf{P}$ such that $\mathbf{x} = \mathbf{Pz}$.

$$\mathbf{x} = \mathbf{Pz} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Then with this transformation the state equation and output equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

can be written as

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}u$$

$$y = \mathbf{C}\mathbf{P}\mathbf{z}$$

In this problem it is specified that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \qquad \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$\mathbf{B} = \mathbf{P}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

Hence

$$\mathbf{P} = \begin{bmatrix} 2 & p_{12} & p_{13} \\ 6 & p_{22} & p_{23} \\ 2 & p_{32} & p_{33} \end{bmatrix}$$

Since

$$\mathbf{A}\mathbf{P} = \mathbf{P}\begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

we have

$$\begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & p_{12} & p_{13} \\ 6 & p_{22} & p_{23} \\ 2 & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 2 & p_{12} & p_{13} \\ 6 & p_{22} & p_{23} \\ 2 & p_{32} & p_{33} \end{bmatrix}\begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

or

$$\begin{bmatrix} -12+6 & -6p_{12}+p_{22} & -6p_{13}+p_{23} \\ -22+2 & -11p_{12}+p_{32} & -11p_{13}+p_{33} \\ -12 & -6p_{12} & -6p_{13} \end{bmatrix} = \begin{bmatrix} p_{12} & p_{13} & -12-11p_{12}-6p_{13} \\ p_{22} & p_{23} & -36-11p_{22}-6p_{23} \\ p_{32} & p_{33} & -12-11p_{32}-6p_{33} \end{bmatrix}$$

from which we obtain

$$p_{12} = -6, \quad p_{22} = -20, \quad p_{32} = -12$$

and

$$-6p_{12} + p_{22} = p_{13}$$
$$-6p_{13} + p_{23} = -12 - 11p_{12} - 6p_{13}$$
$$-11p_{12} + p_{32} = p_{23}$$

$$-11p_{13} + p_{33} = -36 - 11p_{22} - 6p_{23}$$

$$-6p_{12} = p_{33}$$

$$-6p_{13} = -12 - 11p_{32} - 6p_{33}$$

Solving the last six equations for p_{13}, p_{23}, and p_{33} we find

$$p_{13} = 16, \quad p_{23} = 54, \quad p_{33} = 36$$

Hence

$$\mathbf{P} = \begin{bmatrix} 2 & -6 & 16 \\ 6 & -20 & 54 \\ 2 & -12 & 36 \end{bmatrix}$$

We thus determined the necessary transformation matrix $\mathbf{P}$. The output equation becomes

$$y = \mathbf{C}\mathbf{P}\mathbf{z} = [2 \quad -6 \quad 16]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

<u>Alternative approach:</u> An alternative approach to the solution of this problem is given below. Since the characteristic equation for the system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+6 & -1 & 0 \\ 11 & s & -1 \\ 6 & 0 & s \end{vmatrix} = s^3 + 6s^2 + 11s + 6$$

$$= s^3 + a_1 s^2 + a_2 s + a_3$$

we find

$$a_1 = 6, \quad a_2 = 11, \quad a_3 = 6$$

Define

$$\mathbf{M} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 2 & -6 & 16 \\ 6 & -20 & 54 \\ 2 & -12 & 36 \end{bmatrix}$$

Then

$$\mathbf{M}^{-1} = \begin{bmatrix} 9 & -3 & 0.5 \\ 13.5 & -5 & 1.5 \\ 4 & -1.5 & 0.5 \end{bmatrix}$$

It can be shown that

$$\underset{\sim}{M}^{-1}\underset{\sim}{A}\underset{\sim}{M} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \qquad \underset{\sim}{M}^{-1}\underset{\sim}{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Also

$$\underset{\sim}{C}\underset{\sim}{M} = [1 \quad 0 \quad 0]\begin{bmatrix} 2 & -6 & 16 \\ 6 & -20 & 54 \\ 2 & -12 & 36 \end{bmatrix} = [2 \quad -6 \quad 16]$$

Hence, by use of the following transformation:

$$\underset{\sim}{x} = \underset{\sim}{M}\underset{\sim}{z} = \begin{bmatrix} 2 & -6 & 16 \\ 6 & -20 & 54 \\ 2 & -12 & 36 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

the given system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}u$$
$$y = \underset{\sim}{C}\underset{\sim}{x}$$

can be transformed into

$$\dot{\underset{\sim}{z}} = \underset{\sim}{M}^{-1}\underset{\sim}{A}\underset{\sim}{M}\underset{\sim}{z} + \underset{\sim}{M}^{-1}\underset{\sim}{B}u$$
$$y = \underset{\sim}{C}\underset{\sim}{M}\underset{\sim}{z}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = [2 \quad -6 \quad 16]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

B-9-10. A MATLAB program to obtain a state-space representation is given next.

```
num = [0   10.4   47   160];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =

  -14  -56  -160
    1    0     0
    0    1     0

B =

    1
    0
    0

C =

  10.4000   47.0000  160.0000

D =

    0
```

The state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10.4 & 47 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0\,u$$

B-9-11.

```
A = [0   1   0;-1   -1   0;1   0   0];
B = [0;1;0];
C = [0   0   1];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

       0        0   0.0000   1.0000

den =

   1.0000   1.0000   1.0000        0
```

The transfer function representation of the system is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + s^2 + s}$$

B-9-12.

```
A = [2  1  0;0  2  0;0  1  3];
B = [0  1;1  0;0  1];
C = [1  0  0];
D = [0  0];
[NUM,den] = ss2tf(A,B,C,D,1)

NUM =

   0   0   1   -3


den =

   1   -7   16   -12

[NUM,den] = ss2tf(A,B,C,D,2)

NUM =

   0   1   -5   6


den =

   1   -7   16   -12
```

The transfer function representation of the system consists of two equations:

$$\frac{Y(s)}{U_1(s)} = \frac{s-3}{s^3 - 7s^2 + 16s - 12}$$

$$\frac{Y(s)}{U_2(s)} = \frac{s^2 - 5s + 6}{s^3 - 7s^2 + 16s - 12}$$

B-9-13. The controllability and observability of the system can be determined by examining the rank conditions of

$$\left[\underset{\sim}{B} \quad \underset{\sim}{A}\underset{\sim}{B} \quad \underset{\sim}{A}^2\underset{\sim}{B}\right]$$

and

$$\left[\underset{\sim}{C}^* \quad \underset{\sim}{A}^*\underset{\sim}{C}^* \quad (\underset{\sim}{A}^*)^2\underset{\sim}{C}^*\right]$$

respectively.

```
A = [-1  -2  -2;0  -1  1;1  0  -1];
B = [2;0;1];
C = [1   1   0];
D = [0];
rank([B   A*B   A^2*B])

ans =

    3

rank([C'   A'*C'   A'^2*C'])

ans =

    3
```

Since the rank of [$\underset{\sim}{B}$ $\underset{\sim}{A}\underset{\sim}{B}$ $\underset{\sim}{A}^2\underset{\sim}{B}$] is 3 and the rank of [$\underset{\sim}{C}'$ $\underset{\sim}{A}'\underset{\sim}{C}'$ $\underset{\sim}{A}'^2\underset{\sim}{C}'$] is also 3, the system is completely state controllable and observable.

B-9-14.

```
A = [2   0   0;0  -2   0;0   3   1];
B = [0   1;1   0;0   1];
C = [1   0   0;0   1   0];
D = [0   0;0   0];
rank([B   A*B   A^2*B])

ans =

    3

rank([C'   A'*C'   A'^2*C'])

ans =

    2

rank([C*B   C*A*B   C*A^2*B])

ans =

    2
```

From the rank conditions obtained above, the system is completely state controllable but not completely observable. It is completely output controllable. Note that the condition of the output controllability is that the rank of

$$[\underset{\sim}{C}\underset{\sim}{B} \quad \underset{\sim}{C}\underset{\sim}{A}\underset{\sim}{B} \quad \underset{\sim}{C}\underset{\sim}{A}^2\underset{\sim}{B}]$$

be m (the dimension of the output vector, which is 2 in the present system).

B-9-15.

```
A = [0  1  0;0  0  1;-6  -11  -6];
B = [0;0;1];
C = [20  9  1];
D = [0];
rank([B  A*B  A^2*B])

ans =

   3

rank([C'  A'*C'  A'^2*C'])

ans =

   3
```

Since the rank of $[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]$ is 3 and that of $[\mathbf{C}' \quad \mathbf{A}'\mathbf{C}' \quad \mathbf{A}'^2\mathbf{C}']$ is also 3, the system is completely state controllable and completely observable.

B-9-16.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [\, c_1 \quad c_2 \quad c_3 \,]$$

The observability matrix is

$$[\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^* \quad \mathbf{A}^{*2}\mathbf{C}^*] = \begin{bmatrix} c_1 & -6c_3 & -6(c_2 - 6c_3) \\ c_2 & c_1 - 11c_3 & -11c_2 + 60c_3 \\ c_3 & c_2 - 6c_3 & c_1 - 6c_2 + 25c_3 \end{bmatrix}$$

There are infinitely many sets of c_1, c_2, and c_3 that will make the system unobservable. Examples of such a set of c_1, c_2, and c_3 are

$$\mathbf{C} = [\, 1 \quad 1 \quad 0 \,]$$

$$\mathbf{C} = [\, 1 \quad 1 \quad \tfrac{2}{9} \,]$$

$$\mathbf{C} = [\, 6 \quad 5 \quad 1 \,]$$

$$\mathbf{C} = [\, 1 \quad 1 \quad \tfrac{1}{4} \,]$$

etc.

With any of these matrices $\mathbf{C}$ the rank of the observability matrix becomes less than 3 and the system becomes unobservable.

B-9-17.

(a)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

The rank of

$$\begin{bmatrix} C^* & A^* C^* & A^{*2} C^* \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 13 \\ 1 & 1 & 1 \end{bmatrix}$$

is two, because

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 5 & 13 \\ 1 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = 3$$

Hence, the system is not completely observable.

(b) If the output vector is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \hat{C} x$$

then the rank of

$$\begin{bmatrix} \hat{C}^* & A^* \hat{C}^* & A^{*2} \hat{C}^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 \\ 1 & 2 & 5 & 13 & 13 & 35 \\ 1 & 3 & 1 & 3 & 1 & 3 \end{bmatrix}$$

is three, because the determinant of a 3 x 3 matrix consisting of the first, fourth, and sixth column is

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 13 & 35 \\ 1 & 3 & 3 \end{vmatrix} = -42$$

Since the rank of $[\hat{C}^* \quad A^*\hat{C}^* \quad A^{*2}\hat{C}^*]$ is 3, the system is completely observable. A MATLAB solution to this problem is given on the next page.

```
A = [2  0  0;0  2  0;0  3  1];
C = [1  1  1];
rank([C'  A'*C'  A'^2*C'])

ans =

    2

A = [2  0  0;0  2  0;0  3  1];
C = [1  1  1;1  2  3];
rank([C'  A'*C'  A'^2*C'])

ans =

    3
```