ECE 260

EXAM 3

SOLUTIONS

(FALL 2022)

QUESTION 1

$$c_k = \frac{1}{T}\int_0^T x(t)\, e^{-j(2\pi/T)kt}\, dt$$

$$= \frac{1}{8}\int_0^{8^-} \left[2\delta(t) + \delta(t-3) - \delta(t-5)\right] e^{-j(2\pi/8)kt}\, dt$$

$$= \frac{1}{8}\int_{-\infty}^{\infty} \left[2\delta(t) + \delta(t-3) - \delta(t-5)\right] e^{-j(\pi/4)kt}\, dt$$

$$= \frac{1}{8}\left[\int_{-\infty}^{\infty} 2\delta(t)\, e^{-j(\pi/4)kt}\, dt + \int_{-\infty}^{\infty} \delta(t-3)\, e^{-j(\pi/4)kt}\, dt - \int_{-\infty}^{\infty} \delta(t-5)\, e^{-j(\pi/4)kt}\, dt\right]$$

$$= \frac{1}{8}\left[2e^{-j(\pi/4)k(0)} + e^{-j(\pi/4)k(3)} - e^{-j(\pi/4)k(5)}\right]$$

$$= \frac{1}{8}\left[2 + e^{-j(3\pi/4)k} - e^{-j(5\pi/4)k}\right]$$

$$= \frac{1}{8}\left[2 + e^{-j\pi k}\left(e^{j(\pi/4)k} - e^{-j(\pi/4)k}\right)\right]$$

$$= \frac{1}{8}\left[2 + (-1)^k\, 2j \sin\left(\frac{\pi}{4}k\right)\right]$$

$$= \frac{1}{4}\left[1 + (-1)^k j \sin\left(\frac{\pi}{4}k\right)\right]$$

QUESTION 2

```
function f = evaluate_polynomial(c, z)
    f = 0;
    n = length(c);
    for k = 1 : n
        f = f + c(n - k + 1) * z ^ (k - 1);
    end
end
```

QUESTION 3

$$\boxed{H(\omega) = \frac{1}{4+j\omega} \quad \text{and} \quad x(t) = 8 + \cos(3t)}$$

$$H(\omega) = \frac{1}{4+j\omega} = \frac{1}{|4+j\omega|} e^{j[-\arg(4+j\omega)]} = \frac{1}{\sqrt{16+\omega^2}} e^{-j\arctan(\omega/4)}$$

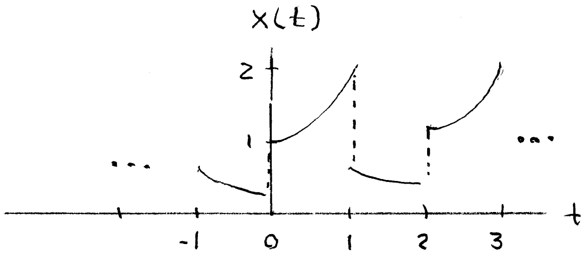
$$\begin{aligned} x(t) &= 8 + \cos(3t) \\ &= 8 + \tfrac{1}{2}\left[e^{j3t} + e^{-j3t}\right] \\ &= 8 + \tfrac{1}{2} e^{j3t} + \tfrac{1}{2} e^{-j3t} \end{aligned}$$

$$\begin{aligned} y(t) &= H(0)[8] + H(3)\left[\tfrac{1}{2} e^{j3t}\right] + H(-3)\left[\tfrac{1}{2} e^{-j3t}\right] \\ &= \tfrac{1}{4}(8) + \tfrac{1}{5} e^{-j\arctan(3/4)}\left[\tfrac{1}{2} e^{j3t}\right] + \tfrac{1}{5} e^{j\arctan(3/4)}\left[\tfrac{1}{2} e^{-j3t}\right] \\ &= 2 + \tfrac{1}{10} e^{-j\arctan(3/4)} e^{j3t} + \tfrac{1}{10} e^{j\arctan(3/4)} e^{-j3t} \\ &= 2 + \tfrac{1}{10}\left[e^{j[3t-\arctan(3/4)]} + e^{-j(3t-\arctan(3/4))}\right] \\ &= 2 + \tfrac{1}{10}\left[2\cos\{3t - \arctan(3/4)\}\right] \\ &= 2 + \tfrac{1}{5}\cos(3t - \arctan(3/4)) \end{aligned}$$

QUESTION 4

$$x(t) = \begin{cases} t^2+1 & 0 \leq t < 1 \\ e^{-t} & 1 \leq t < 2 \end{cases} \quad \text{and} \quad x(t) = x(t+2)$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt}$$



← approximate sketch of x

Since x satisfies the Dirichlet conditions, we have

$$y(0) = \frac{1}{2}\left[x(0^-) + x(0^+)\right]$$
$$= \frac{1}{2}\left[e^{-2}+1\right]$$
$$= \frac{1}{2}\left[\frac{1+e^2}{e^2}\right]$$
$$= \frac{1+e^2}{2e^2}$$

$$y(1) = \frac{1}{2}\left[x(1^-) + x(1^+)\right]$$
$$= \frac{1}{2}\left[(1^2+1) + e^{-1}\right]$$
$$= \frac{1}{2}\left[2+e^{-1}\right]$$
$$= \frac{2e+1}{2e}$$

QUESTION 5

$$\boxed{c_k = \frac{e^{j3k}(jk+3)^2}{(jk-3)^{10}} \quad \text{and} \quad T = 2\pi}$$

(a) $$|c_k| = \left| \frac{e^{j3k}(jk+3)^2}{(jk-3)^{10}} \right| = \frac{|e^{j3k}||(jk+3)^2|}{|(jk-3)^{10}|}$$

$$= \frac{|jk+3|^2}{|jk-3|^{10}} = \frac{(\sqrt{k^2+9})^2}{(\sqrt{k^2+9})^{10}} = \frac{k^2+9}{(k^2+9)^5} = \frac{1}{(k^2+9)^4}$$

(b) The amount of spectral information that x has at the frequency $k\omega_0$ is given by $|c_k|$, where $\omega_0 = \frac{2\pi}{T} = 1$.

So, we need to find the k value that maximizes $|c_k|$.

Clearly, $|c_k|$ is largest at $k=0$.

Therefore, x has the most spectral information at the frequency $0\,\omega_0 = 0$.