6.1 Using the Fourier transform analysis equation, find the Fourier transform X of each function x below.

(a) $x(t) = \operatorname{rect}(t - t_0)$, where t_0 is a constant;

(b) $x(t) = e^{-4t}u(t-1)$;

(c) $x(t) = 3[u(t) - u(t-2)]$; and

(d) $x(t) = e^{-|t|}$.

c

$$X(\omega) = \int_{-\infty}^{\infty} 3\left[u(t) - u(t-2)\right] e^{-j\omega t}\, dt$$

$$= 3\int_{-\infty}^{\infty} \left[u(t) - u(t-2)\right] e^{-j\omega t}\, dt$$

$$= 3\int_{0}^{2} e^{-j\omega t}\, dt$$

$$= 3\left[\frac{1}{-j\omega t} e^{-j\omega t}\right]_0^2$$

$$= \frac{3j}{\omega}\left[e^{-j2\omega} - 1\right]$$

$$= \frac{3j}{\omega} e^{-j\omega}\left[-2j\sin\omega\right]$$

$$= \frac{6}{\omega} e^{-j\omega} \sin\omega$$

$$= 6e^{-j\omega} \operatorname{sinc}\omega$$

d

$$X(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{-|t|} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-|t|} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$= \frac{1}{1-j\omega}\left[e^{(1-j\omega)t} \right]_{-\infty}^{0} - \frac{1}{1+j\omega}\left[e^{(-1-j\omega)t} \right]_{0}^{\infty}$$

$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{1+j\omega+1-j\omega}{(1+j\omega)(1-j\omega)} = \frac{2}{1+\omega^2}$$

6.3 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform X of each function x below.

(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t}u(t+2)$;
(c) $x(t) = \cos(t)u(t)$;
(d) $x(t) = 6[u(t) - u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\,\mathrm{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = x_1 * x_2(t)$, where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = te^{-3t}u(t)$; and
(j) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT)$, where a is a complex constant satisfying $|a| < 1$ and T is a strictly-positive real constant; (Hint: Recall the formula for the sum of an infinite geometric sequence (i.e., (F.9)).)

$v_1(t) = \cos(t)$
$v_2(t) = u(t)$

c $\qquad x(t) = \underbrace{\cos(t)}_{v_1(t)}\ \underbrace{u(t)}_{v_2(t)}$

Fourriere Transform yields: $X(\omega) = \frac{1}{2\pi} V_1(\omega) * V_2(\omega)$

$$= \left[\frac{1}{2\pi} \left[\pi \left(\delta(\omega-1) + \delta(\omega+1) \right) \right] * \left[\pi \delta(\omega) + \frac{1}{j\omega} \right] \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \left[\delta(\lambda-1) + \delta(\lambda+1) \right] \left[\pi \delta(\omega-\lambda) + \frac{1}{j(\omega-\lambda)} \right] d\lambda$$

$$= \frac{1}{2} \left[\pi \delta(\omega-1) + \frac{1}{j(\omega-1)} + \pi \delta(\omega+1) + \frac{1}{j(\omega+1)} \right]$$

$$= \frac{1}{2} \left[\pi \delta(\omega-1) + \pi \delta(\omega+1) - \frac{j2\omega}{\omega^2-1} \right]$$

$$= \frac{\pi}{2} \left[\delta(\omega-1) + \delta(\omega+1) \right] - \frac{j\omega}{\omega^2-1}$$

$\underline{\underline{d}}$ $x(t) = 6\left(u(t) - u(t-3)\right)$ $\boxed{x(t) = 6V_3 t}$

$\underline{V_3(t) = V_2(t/3)}$ $\underline{V_2(t) = V_1(t - \frac{1}{2})}$ $\underline{V_1(t) = rect(t)}$

$X(\omega) = 6V_3(\omega)$

$V_3(\omega) = 3V_2(\omega)$

$V_2(\omega) = e^{-j\omega/2} V_1(\omega)$

$V_1(\omega) = sinc(\omega/2)$

$X(\omega) = 6V_3(t)$

$= 6(3) V_2(3\omega)$

$= 18 V_2(3\omega)$

$= 18e^{-j\omega/2} V_1(3\omega)$

$= 18e^{-j\omega/2} sinc\left(\frac{3\omega}{2}\right)$

$= \frac{6}{j\omega}\left[1 - e^{-j3\omega}\right]$

$X(\omega) = 6\left(\pi\delta(\omega) + \frac{1}{j\omega} - e^{-j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right)$

$= 6\left[\pi\delta(\omega) + \frac{1}{j\omega} - \pi\delta(\omega)e^{-j\omega 3} - \frac{1}{j\omega}e^{-j3\omega}\right)$

$= \frac{6}{j\omega}\left[1 - e^{-3j\omega}\right] = \frac{6}{j\omega} e^{-j\frac{3\omega}{2}} (2j) \sin\frac{3\omega}{2} = \frac{3\omega}{2} \cdot \frac{12}{\omega} e^{-\frac{j\omega}{2}} sinc\frac{3\omega}{2}$

$= 18e^{-j\omega/2} sinc\frac{3\omega}{2}$

Q

$x(t) = \frac{1}{t}$ $\quad\quad$ $\text{sgn}\, t \longrightarrow \frac{2}{j\omega}$

$\mathcal{F}\left\{\frac{2}{jt}\right\} = -2\pi\, \text{sgn}\, \omega$ [duality]

$X(\omega) = \mathcal{F}\left\{\frac{1}{t}\right\} = \frac{j}{2}\mathcal{F}\left\{\frac{2}{jt}\right\} = \frac{j}{2}\left[-2\pi\, \text{sgn}\,\omega\right]$

$= -j\pi\, \text{sgn}\,\omega$

f

$x(t) = t\,v_2(t)$ $\qquad V_1(\omega) = \text{sinc}\,\frac{\omega}{2}$

$v_2(t) = v_1(2t)$ $\qquad V_2(\omega) = \frac{1}{2}\,V_1\!\left(\frac{\omega}{2}\right)$

$v_1(t) = \text{rect}(t)$ $\qquad X(\omega) = j\,\frac{d}{d\omega}\,V_2(\omega)$

$$X(\omega) = j\,\frac{d}{d\omega}\,V_2(\omega)$$

$$= j\,\frac{d}{d\omega}\left(\frac{1}{2}\,V_1\frac{\omega}{2}\right) = \frac{j}{2}\,\frac{d}{d\omega}\,V_1\!\left(\frac{\omega}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\omega/4\left(1/4\cos\omega/4\right) - \left(1/4\sin\omega/4\right)}{\omega^2/16}\right]$$

$$= \frac{j}{2}\left[\frac{16\left(\omega/16\cos\omega/4 - 1/4\sin\omega/4\right)}{\omega^2}\right]$$

$$= \frac{j}{2}\left[\frac{1}{\omega}\cos\frac{\omega}{4} - \frac{4}{\omega^2}\sin\frac{\omega}{4}\right]$$

$$= \frac{j}{2\omega}\left[\cos(\omega/4) - \frac{j2}{\omega^2}\cos\frac{\omega}{4}\right]$$

g. $x(t) = e^{-jt+\omega}\sin(5t-2)$

$$x(t) = e^{-j3t}\,v_3(t)$$

$v_3(t) = v_2(5t)$

$v_2(t) = v_1(t-2)$

$v_1(t) = \sin t$

$$V_1(\omega) = \frac{\pi}{j}\left[\delta(\omega-1) - \delta(\omega+1)\right]$$

$$V_2(\omega) = e^{-j\omega 2}\,V_1(\omega)$$

$$V_3(\omega) = \frac{1}{5}\,V_2(\omega/5)$$

$$X(\omega) = V_3(\omega+3)$$

$\boxed{\text{>>> next page}}$

$$X(\omega) = V_3(\omega+3)$$

$$= \frac{1}{5} V_2\left(\frac{\omega+3}{5}\right) = \frac{1}{5} e^{-j2\left(\frac{\omega+3}{5}\right)} V_1\left(\frac{\omega+3}{5}\right)$$

$$= \frac{\pi}{j5} e^{j(\omega+3)/5} \left[\delta\left(\frac{\omega+3}{5}-1\right) - \delta\left(\frac{\omega+3}{5}+1\right) \right]$$

$$= -\frac{j\pi}{5} e^{-j2} \delta\left(\frac{\omega-2}{5}\right) + \frac{j\pi}{5} e^{j2} \delta\left(\frac{\omega+8}{5}\right)$$

$$= -j\pi e^{-j2} \delta(\omega-2) + j\pi e^{j2} \delta(\omega+8)$$

6.4 For each function y given below, find the Fourier transform Y of y in terms of the Fourier transform X of x.

(a) $y(t) = x(at - b)$, where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \mathcal{D}(x*x)(t)$, where $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = tx(2t-1)$;
(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$; and
(h) $y(t) = (\mathcal{D}x)*x_1(t)$, where $x_1(t) = e^{-jt}x(t)$ and $\mathcal{D}$ denotes the derivative operator.

a

$y(t) = x(at-b)$ $\quad y(t) = v_1(at)$ where $v_1(t) = x(t-b)$

$Y(\omega) = \frac{1}{|a|} V_1(\omega/a)$ and $V_1(\omega) = e^{-j\omega b} X(\omega)$

$\therefore Y(\omega) = \frac{1}{|a|} V_1(\omega/a)$

$= \frac{1}{|a|} e^{-j(\omega/a)b} X(\omega/a)$

$= \frac{1}{|a|} e^{-j\left(\frac{-jb\omega}{a}\right)} X\left(\frac{\omega}{a}\right)$

b

$$y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau = v_1(2t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$Y(\omega) = \mathcal{F}\{v_1(2t)\}$$

$$= \frac{1}{2} V_1(\omega/2)$$

$$V_1(\omega) = \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}$$

$$= \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

$$Y(\omega) = \frac{1}{2} V\left(\frac{\omega}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{j(\omega/2)} X\left[\frac{\omega}{2}\right] + \pi X(0)\delta\left(\frac{\omega}{2}\right)\right]$$

$$= \frac{1}{j\omega} X(\omega/2) + \frac{\pi}{2} X(0)\delta\left(\frac{\omega}{2}\right)$$

$\underline{c}$

$$y(t) = \int_{-\infty}^{t} x^2(\tau)\, d\tau$$

let $y(t) = \int_{-\infty}^{t} v_1(\tau)\, d\tau$

where

$v_1(t) = x^2(\tau)$

Fourier Transform from both sides:

$$Y(\omega) = \frac{1}{j\omega} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, X(\omega - \lambda)\, d\lambda \right]$$

$$+ \pi \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) X(-\lambda)\, d\lambda \right] \delta(\omega)$$

$$= \frac{1}{j2\pi\omega} \int_{-\infty}^{\infty} X(\lambda)\, X(\omega - \lambda)\, d\lambda + \frac{\delta(\omega)}{2} \int_{-\infty}^{\infty} X(\lambda)\, X(-\lambda)\, d\lambda$$

$\underline{d}$

$$y(t) = D(\cancel{v_1(t)})(t)$$

$$y(t) = \frac{d}{dt} v_1(t)$$

$$v_1(t) = x(t) * x(t)$$

$$Y(\omega) = \mathcal{F}\left\{ \frac{d}{dt} v_1(t) \right\} = j\omega V(\omega)$$

$$V_1(\omega) = \mathcal{F}\left\{ x(t) * x(t) \right\} = X^2(t)$$

$$Y(\omega) = j\omega(V_1(\omega))$$

$$= j\omega X^2(\omega)$$

e

$y(t) = tx(2t-1)$

$$\boxed{\begin{aligned} y(t) &= t\,v_1(t) \\ \text{where } v_1(t) &= v_2(2t) \\ v_2(t) &= x(t-1) \end{aligned}}$$

Combining these equations, we obtain

$$Y(\omega) = j\frac{d}{d\omega}V_1(\omega) = j\frac{d}{d\omega}\left[\frac{1}{2}V_2\left(\frac{\omega}{2}\right)\right] = \frac{j}{2}\left[\frac{d}{d\omega}e^{-j\omega/2}X\left(\frac{\omega}{2}\right)\right]$$

f

$y(t) = e^{j2t} x(t-1)$

Rewriting $y(t)$ as $\boxed{y(t) = e^{2jt} v_1(t)}$ where $v_1(t) = x(t-1)$

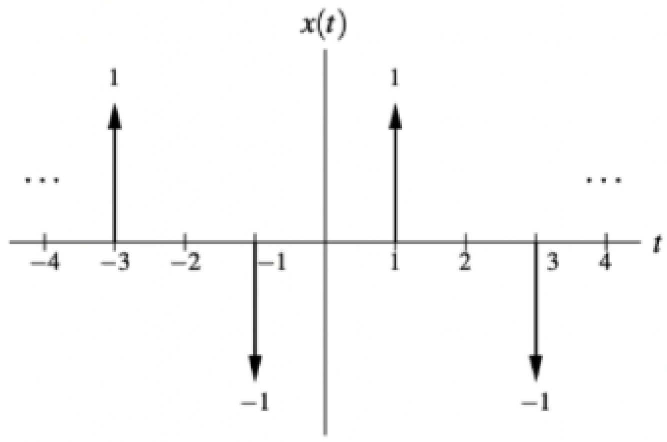
Fourier Transform of both sides:

$$V_1(\omega) = e^{-j\omega} X(\omega)$$

and, $Y(\omega) = V_1(\omega - 2)$

Combining them: $Y(\omega) = V_1(\omega-2) = e^{-j(\omega-2)} X(\omega-2)$

6.5 Find the Fourier transform X of each periodic function x shown below.



(a)

$\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$ For period of $x(t)$ from $-2 \leq t < 2$

We have: $x_T(t) = -\delta(t+1) + \delta(t-1)$

$$x_T(\omega) = \mathcal{F}\{\delta(t-1) - \delta(t+1)\}$$

$$= \mathcal{F}\{\delta(t-1)\} - \mathcal{F}\{\delta(t+1)\}$$

$$= e^{-j\omega} - e^{j\omega} = -[e^{j\omega} - e^{-j\omega}]$$

$$= -2j\sin\omega$$

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\,\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{\pi}{2}\left(-2j\cos k\pi\right)\delta(\omega - k\pi)$$

$$= \sum_{k=-\infty}^{\infty} -\frac{\pi}{2}\left(2j\cos k\pi\right)\delta(\omega - k\pi)$$

6.10 For each function x given below, compute the frequency spectrum of x, and find and plot the corresponding magnitude and phase spectra.

(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and

(b) $x(t) = \operatorname{sinc}\left(\frac{1}{200}t - \frac{1}{200}\right)$.

<u>a</u>

$$X(\omega) = \mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{|1|}{|a+j\omega|} = \frac{1}{\sqrt{a^2+j\omega}}$$

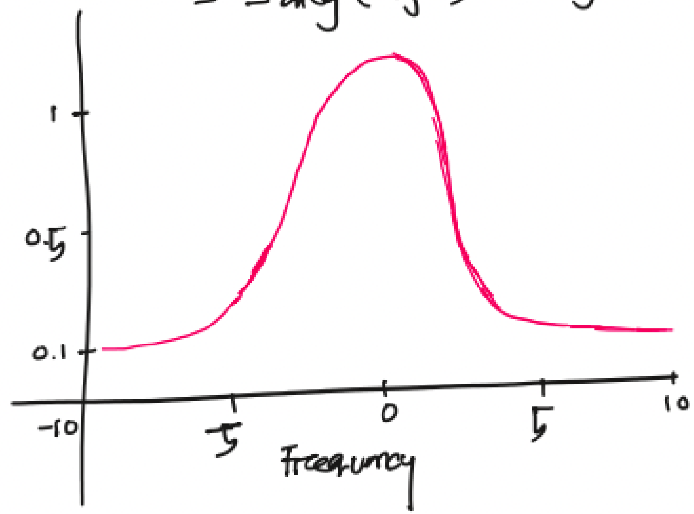
$$\arg X(\omega) = \arg\left[\frac{1}{a+j\omega}\right] = \arg 1 - \arg(a+j\omega)$$

$$= 0 + \arg(a+j\omega)$$

$$= -\arg(a+j\omega) = -\arctan(\omega/a)$$

For $a=1$:

Magnitude:



Phase Spectrum: