CSC 320 - Tutorial 7

1. Pumping Lemma for CFL

Pumping Lemma (Context-Free)

If L is a context-free language, them there exists a number p (pumping length) where any string $w \in L$ where $|w| \geq p$ the string can be divided into five pieces $w = uvxyz$ where the following conditions are satisfied:

1. $|vxy| \leq p$
2. $|vy| > 0$
3. $uv^i xy^i z \in L$ for all $i \geq 0$

Questions

1. Show that the following languages are not Context-Free

   a. A = $\{a^n b^n c^n \mid 0 \leq n\}$

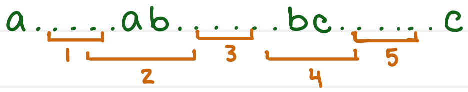
(again the pumping lemma is a proof by contradiction)

Assume context free — Assume A is context free, then the PL for CFL must hold.

present counter example — Consider $w = a^p b^p c^p$ we have $w \in A$ and $|w| \geq p$

prove properties — Then the pumping lemma states $w = uvxyz$ where the following properties hold:

1) $|vxy| \leq p$



5 cases?

these questions are LONG! it helps if you can generalize your cases

we can generalize our cases.

Case 1: vxy consists entirely of one kind of symbol (1,3 and 5)

Case 2: vxy consists of 2 kinds of symbols (2 and 4)

2) $|vy| > 0$ ∴ vy is at least one symbol long.

3) $uv^i xy^i z \in L$ for $i \geq 0$

Case 1: vxy consists of one kind of symbol
then vy consists of one kind of symbol

consider pumping up to obtain uv^2xy^2z
then our string would not have equal number as bs cs

ie. if vxy consists of only as and $vy = a^n$ $0 < n \leq p$
then $uv^2xy^2z = a^{p+n} b^p c^p$

Case 2: vxy consists of 2 kinds of symbols

lots of cases to consider :

$v = \varepsilon$ $x = as\,bs$ $y = bs$ $v = as$ $x = \varepsilon$ $b = bs$

$v = \varepsilon$ $x = as$ $y = bs$ $v = as\,bs$ $x = bs$ $y = bs$

$v = \varepsilon$ $x = as$ $y = as\,bs$ ... (there is a lot of options)

try to visualize what happens after pumping each case

consider pumping up to obtain uv^2xy^2z then vy is at least one symbol long ∴ the new string would not have an equal occurence of as bs and Cs

Also, if either v or y consist of 2 kinds symbols, pumping up disrupts the pattern of the string

all as, followed by all bs, followed by all Cs

ie. $v = \varepsilon$ $x = as$ $y = bs$ ⇒ uv^2xy^2z would have too many bs

$v = as$ $x = bs$ $y = \varepsilon$ ⇒ uv^2xy^2z would have too many a

$v = as\,bs$ $x = \varepsilon$ $y = \varepsilon$ ⇒ uv^2xy^2z would not follow order as bs Cs

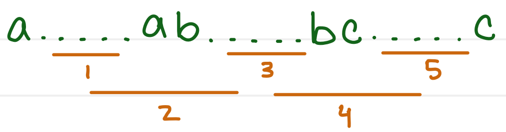
In either case after pumping the new string is not in the language ∴ the PL for context-free languages doesn't hold ∴ the language A is not context-free.

b. B = $\{a^ib^jc^k \mid 0 \leq i \leq j \leq k\}$

Assume B is context free ∴ PL should hold.

Consider string $w = a^p b^p c^p$ $w \in B$ $|w| \geq p$
then $w = uvxyz$ s.t.

1) $|vxy| \leq p$



How can we generalize this?

Case 1: vxy consists of one kind of symbol (1, 3 & 5)
Case 2: vxy consists of 2 kinds of symbols (2 & 4)

2) $|vy| > 0$ ∴ vy consists of @ least one symbol

3) $uv^0xy^0z \in B$ for all $i \geq 0$

Case 1.1: vxy consists of all as or all bs (1 & 3)
pumping up to uv^2xy^2z we will have more as than bs or more bs than Cs respectively.

ie. $vxy = a^m \Rightarrow uv^2xy^2z = a^{p+m}b^pc^p$ where $p+m > p$
$vxy = b^m \Rightarrow uv^2xy^2z = a^pb^{p+m}c^p$

Case 1.2: vxy consists of all cs
pumping down to uv^0xy^0z we will have more as and bs than Cs

ie. $vxy = c^m \Rightarrow uv^0xy^0z = a^pb^pc^{p-m}$ $p > p-m$

★ notice if we pumped up the string would be in the language still

Case 2 : vxy consists of 2 kinds of symbols

there are many cases to consider here...

essentially we only care about 2

Case 2.1 : v consists of 2 kinds of symbols or

y consists of 2 kinds of symbols

pumping up to uv^2xy^2z we break the pattern

all as followed by all bs then all cs

ie. if $v = b\ldots bc\ldots c$

then uv^2xy^2z is $a\ldots\ldots a\,b\ldots\ldots b\underbrace{b\ldots bc\ldots c}_{v}\underbrace{b\ldots bc\ldots c}_{v}c\ldots\ldots c$

Case 2.2: v consists of one kind of symbol and

y consists of another

(option 1) if $v = a^i$ $y = b^j$ then pumping up

$uv^2xy^2z = a^{p+i}\ b^{p+j}\ c^p$ where $p+j > p$

have more bs than Cs

(option 2) if $v = b^i$ $y = c^j$ then pumping down

$uv^0xy^0z = a^p\ b^{p-i}\ c^{p-j}$ where $p > p-i$

have more as than bs

c. $C = \{0^n \# 0^{2n} \# 0^{3n} \mid n \geq 0\}$

Assume C is context-free ∴ PL for Context-free must hold.

Consider string $w = 0^p \# 0^{2p} \# 0^{3p}$ $w \in C$ and $|w| \geq p$

then $w = uvxyz$ s.t.

1) $|vxy| \leq p$ $0\underbrace{\ldots 0}_{1}\underbrace{\# 0 \ldots}_{2}.\underbrace{00 \ldots}_{3} \underbrace{0 \# 0 \ldots}_{4} \underbrace{00 \ldots}_{5} 00 \ldots 0$

Generalize!

case 1.1 : vxy consists of only 0s (1, 3 and 5)

case 1.2 : vxy consists of some 0s and exactly one # (2 & 4)

2) $|vy| > 0$ ∴ vy consists of at least one symbol

Case 2.1 : vy consists of one or more 0s

Case 2.2 : vy consists of one # and zero or more 0s

note if we have the case $\underbrace{0 \ldots 0}_{v} \underbrace{\#}_{x} \underbrace{0 \ldots 0}_{y}$

this is covered in case 1.2

and then case 2.1

3) $uv^i x y^i z \in C$ for all $i \geq 0$

Case 2.1: vy consists of one or more 0s

if we pump down we will have too few 0s in at least one of the intervals

ie. if $vy = 0^m$ then uv^0xy^0z is $0^{p-m} \# 0^{2p} \# 0^{3p}$

or $0^p \# 0^{2p-m} \# 0^{3p}$ or $0^p \# 0^{2p} \# 0^{3p-m}$

if $v = 0^i$ $x = \#$ $y = 0^j$ then uv^0xy^0z is $0^{p-i} \# 0^{2p-j} \# 0^{3p}$

or $0^p \# 0^{2p-i} \# 0^{3p-j}$

Case 2.2: vy consists of one # and zero or more 0s pumping down to uv^0xy^0z removes one of the three # breaking the pattern

In both cases the new string uv^0xy^0z is not in C ∴ the language is not context free

d. (Extra Practice) D = $\{0^n 1^n 0^n \mid n \geq 0\}$