ECE 260

EXAM 4

SOLUTIONS

(SUMMER 2022)

QUESTION 1A

$i(t) = \frac{1}{2} v_1(t)$ (1)

$D v_0(t) = 2D^2 i(t) + 2i(t) + D v_1(t)$ (2)

Taking the FT of (1) and (2), we have

$I(w) = \frac{1}{2} V_1(w)$ (3)

$jw V_0(w) = 2(jw)^2 I(w) + 2I(w) + (jw) V_1(w)$ (4)

Substituting (3) into (4), we have

$jw V_0(w) = -2w^2 \left[\frac{1}{2} V_1(w)\right] + 2\left[\frac{1}{2} V_1(w)\right] + jw V_1(w)$

$jw V_0(w) = -w^2 V_1(w) + V_1(w) + jw V_1(w)$

$[-w^2 + jw + 1] V_1(w) = jw V_0(w)$

$H(w) = \frac{V_1(w)}{V_0(w)} = \frac{jw}{-w^2 + jw + 1}$

QUESTION 1B

$$|H(\omega)| = \left|\frac{j\omega}{-\omega^2+j\omega+1}\right| = \frac{|j\omega|}{|-\omega^2+1+j\omega|} = \frac{|\omega|}{\sqrt{(1-\omega^2)^2+\omega^2}}$$

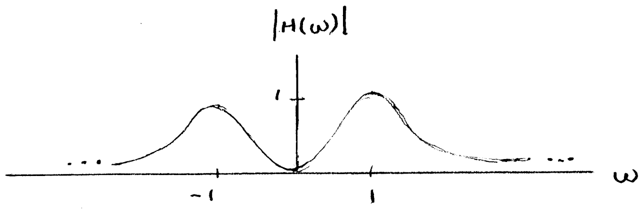
$$= \frac{|\omega|}{\sqrt{1-2\omega^2+\omega^4+\omega^2}} = \frac{|\omega|}{\sqrt{\omega^4-\omega^2+1}}$$

$$\lim_{\omega\to 0} |H(\omega)| = \frac{0}{1} = 0$$

$$\lim_{|\omega|\to\infty} |H(\omega)| = \lim_{\omega\to\infty} \frac{|\omega|}{\sqrt{\omega^4}} = 0$$

Clearly, $|H(\omega)|$ is nonzero and has some maximum value for $|\omega|$ between 0 and ∞. [In particular, $|H(\omega)|$ has a maximum of 1 at $|\omega| = 1$.]

Therefore, the system best approximates a bandpass filter. A rough sketch of $|H(\omega)|$ is shown below.



QUESTION 2A

$$y(t) = 4x(t)\cos\left(8t - \frac{\pi}{3}\right)$$

$$= 4\left[\frac{1}{2}\left(e^{j(8t-\pi/3)} + e^{-j(8t-\pi/3)}\right)\right]x(t)$$

$$= 2e^{j8t}e^{-j\pi/3}x(t) + 2e^{-j8t}e^{j\pi/3}x(t)$$

$$= 2e^{-j\pi/3}e^{j8t}x(t) + 2e^{j\pi/3}e^{-j8t}x(t)$$

$$Y(w) = 2e^{-j\pi/3}X(w-8) + 2e^{j\pi/3}X(w+8)$$

QUESTION 2B

Let m_x and m_y denote the highest (in magnitude) frequencies in x and y, respectively.

Then, we have (from the sampling theorem):

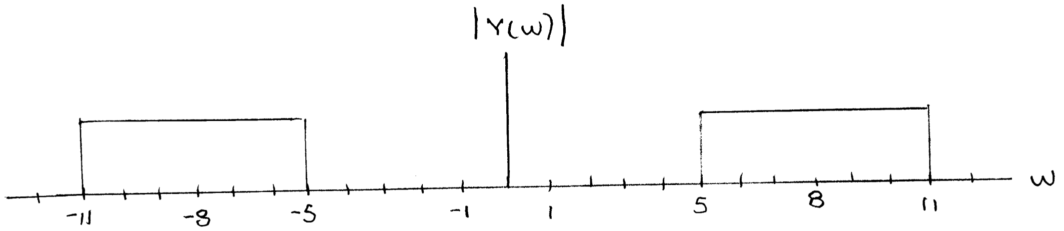
$w_x > 2m_x$ and

$w_y > 2m_y$.

The function X is zero outside the range $[-3, 3]$.
Therefore, $m_x = 3$.

So, $w_x > 2(3) = 6$. Thus, we conclude $w_x > 6$.

A plot of $|Y(w)|$ versus w is sketched below.



From the plot of $|Y(w)|$, $m_y = 11$.

So, $w_y > 2m_y = 2(11) = 22$. Thus, we conclude $w_y > 22$.

QUESTION 3

Let $x_T(t) = \begin{cases} e^{-t} & t \in [0,1) \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
X_T(\omega) &= \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt \\
&= \int_0^1 e^{-t} e^{-j\omega t}\, dt \\
&= \int_0^1 e^{-(1+j\omega)t}\, dt \\
&= \left[\frac{-1}{1+j\omega} e^{-(1+j\omega)t} \right] \Bigg|_0^1 \\
&= \left(\frac{-1}{1+j\omega} \right) \left[e^{-(1+j\omega)} - 1 \right] \\
&= \frac{1 - e^{-(1+j\omega)}}{1+j\omega} \\
&= \frac{1 - e^{-1} e^{-j\omega}}{1+j\omega}
\end{aligned}$$

$$\begin{aligned}
X_T\left(k \tfrac{2\pi}{T}\right) &= X_T(2\pi k) \\
&= \frac{1 - e^{-1} e^{-j2\pi k}}{1 + j2\pi k} \\
&= \frac{1 - e^{-1}}{1 + j2\pi k}
\end{aligned}$$

$$\begin{aligned}
X(\omega) &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} X_T\left(\tfrac{2\pi}{T} k\right) \delta\left(\omega - \tfrac{2\pi}{T} k\right) \\
&= \sum_{k=-\infty}^{\infty} 2\pi \left(\frac{1 - e^{-1}}{1 + j2\pi k} \right) \delta(\omega - 2\pi k)
\end{aligned}$$

QUESTION 4

$$v_1(t) = e^{-t^2/8} \iff V_1(\omega) = 2\sqrt{2\pi}\, e^{-4\omega^2/2}$$

$$v_2(t) = v_1(t-2) \iff V_2(\omega) = e^{-j2\omega} V_1(\omega)$$

$$x(t) = v_2(3t) \iff X(\omega) = \frac{1}{3} V_2\left(\frac{\omega}{3}\right)$$

$$X(\omega) = \frac{1}{3} V_2\left(\frac{\omega}{3}\right)$$

$$= \frac{1}{3}\left[e^{-j2(\omega/3)}\, V_1\left(\frac{\omega}{3}\right)\right]$$

$$= \frac{1}{3} e^{-j2\omega/3}\left[2\sqrt{2\pi}\, e^{-4(\omega/3)^2/2}\right]$$

$$= \frac{2\sqrt{2\pi}}{3} e^{-j2\omega/3}\, e^{-2(\omega/3)^2}$$

$$= \frac{2\sqrt{2\pi}}{3} e^{-j2\omega/3}\, e^{-2\omega^2/9}$$

QUESTION 5

$$h(t) = -D\delta(t)$$

$$x(t) = 10 + \cos(2t) + \sin(6t)$$

$$H(\omega) = -j\omega \, F\{\delta\}(\omega) = -j\omega$$

$$X(\omega) = F\{10\}(\omega) + F\{\cos(2\cdot)\}(\omega) + F\{\sin(6\cdot)\}(\omega)$$

$$= 20\pi\,\delta(\omega) + \pi\left[\delta(\omega-2) + \delta(\omega+2)\right] + \frac{\pi}{j}\left[\delta(\omega-6) - \delta(\omega+6)\right]$$

$$= 20\pi\,\delta(\omega) + \pi\,\delta(\omega-2) + \pi\,\delta(\omega+2) + \frac{\pi}{j}\delta(\omega-6) - \frac{\pi}{j}\delta(\omega+6)$$

$$Y(\omega) = X(\omega)H(\omega)$$

$$= -20\pi j\omega\,\delta(\omega) - \pi j\omega\,\delta(\omega-2) - \pi j\omega\,\delta(\omega+2)$$

$$- \frac{\pi}{j} j\omega\,\delta(\omega-6) + \frac{\pi}{j} j\omega\,\delta(\omega+6)$$

$$= -20\pi j\omega\,\delta(\omega) - \pi j\omega\,\delta(\omega-2) - \pi j\omega\,\delta(\omega+2) - \pi\omega\,\delta(\omega-6) + \pi\omega\,\delta(\omega+6)$$

$$= 0 - \pi j(2)\delta(\omega-2) - \pi j(-2)\delta(\omega+2) - \pi(6)\delta(\omega-6) + \pi(-6)\delta(\omega+6)$$

$$= -j2\pi\,\delta(\omega-2) + j2\pi\,\delta(\omega+2) - 6\pi\,\delta(\omega-6) - 6\pi\,\delta(\omega+6)$$

$$= +\frac{2\pi}{j}\delta(\omega-2) - \frac{2\pi}{j}\delta(\omega+2) - 6\pi\left[\delta(\omega-6) + \delta(\omega+6)\right]$$

$$= \frac{2\pi}{j}\left[\delta(\omega-2) - \delta(\omega+2)\right] - 6\pi\left[\delta(\omega-6) + \delta(\omega+6)\right]$$

$$y(t) = 2\sin(2t) - 6\cos(6t)$$