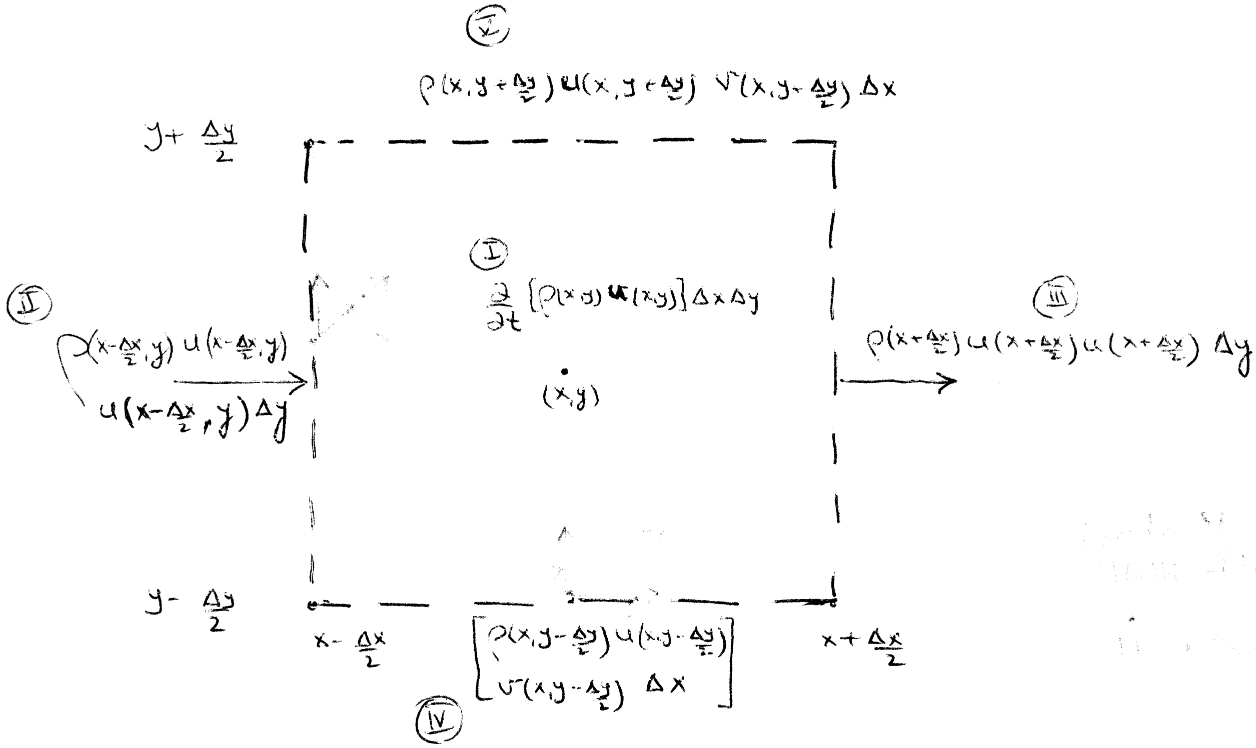


write all terms as Taylor expansion about (x,y)

$$\text{II} = \rho(x,y)\,u(x,y)\,u(x,y)\,\Delta y + \frac{\partial}{\partial x}\Big(\rho(x,y)\,u(x,y)\,u(x,y)\,\Delta y\Big)\cdot\left(-\frac{\Delta x}{2}\right) + \dots$$

$$\text{III} = \rho(x,y)\,u(x,y)\,u(x,y)\,\Delta y + \frac{\partial}{\partial x}\Big(\rho(x,y)\,u(x,y)\,u(x,y)\,\Delta y\Big)\left(\frac{\Delta x}{2}\right) + \dots$$

$$\text{IV} = \rho(x,y)\,u(x,y)\,v(x,y)\,\Delta x + \frac{\partial}{\partial y}\Big[\rho(x,y)\,u(x,y)\,v(x,y)\,\Delta x\Big]\left(-\frac{\Delta y}{2}\right) + \dots$$

$$\text{V} = \rho(x,y)\,u(x,y)\,v(x,y)\,\Delta x + \frac{\partial}{\partial y}\Big[\rho(x,y)\,u(x,y)\,v(x,y)\,\Delta x\Big]\left(\frac{\Delta y}{2}\right) + \dots$$

plugging in the mom. balance x-direction

$$\frac{\partial}{\partial t}\big[\rho(x,y,t)\,u(x,y,t)\big]\,\Delta x\,\Delta y = (\text{II} + \text{IV}) - (\text{III} + \text{V}) + \text{forces}_x$$

$$= (\text{II} - \text{III}) + (\text{IV} - \text{V}) + \text{forces}_x$$

$$= -\frac{\partial}{\partial x}\big[\rho(x,y,t)\,u(x,y,t)\,u(x,y,t)\big]\,\Delta y\,\Delta x - \frac{\partial}{\partial y}\big[\rho(x,y,t)\,u(x,y,t)\,v(x,y,t)\big]\,\Delta x\,\Delta y + \text{forces}$$

divide by $\Delta x \, \Delta y$

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + uu \frac{\partial \rho}{\partial x} + u\rho \frac{\partial u}{\partial x} + \rho u \frac{\partial u}{\partial x} + uv \frac{\partial \rho}{\partial y} + u\rho \frac{\partial v}{\partial y} + \rho v \frac{\partial u}{\partial y} = \frac{\text{forces}_x}{\text{Vol}}$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] + u \underbrace{\left\{ \frac{\partial \rho}{\partial t} + \left[u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right] + \rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \right\}}_{\text{continuity equ.} = 0} = \frac{\text{forces}_x}{\text{Volume}}$$

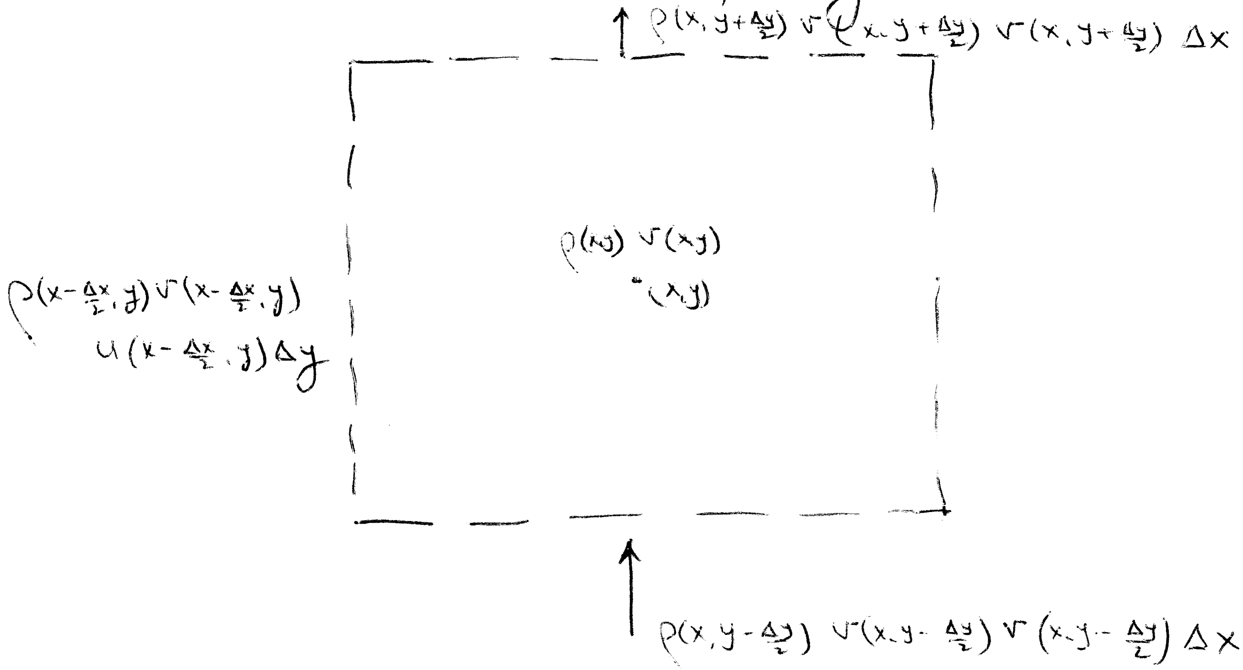
$$\therefore \quad \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{\text{forces}_x}{\text{Volume}}$$

$$\rho \frac{Du}{Dt} = \frac{\text{forces}_x}{\text{Vol}}$$

$$\rho \left[\frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u \right] = \frac{\text{forces}_x}{\text{Vol}}$$

or $$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) u = \frac{\text{forces}_x}{\text{Vol}}$$

Same control volume, y direction



$\vdots$

$$\rho \frac{Dv}{Dt} = \frac{\text{forces}_y}{\text{Vol}}$$

also

$$\rho \frac{Dw}{Dt} = \frac{\text{forces}_z}{\text{Vol}}$$

Now lets look at forces

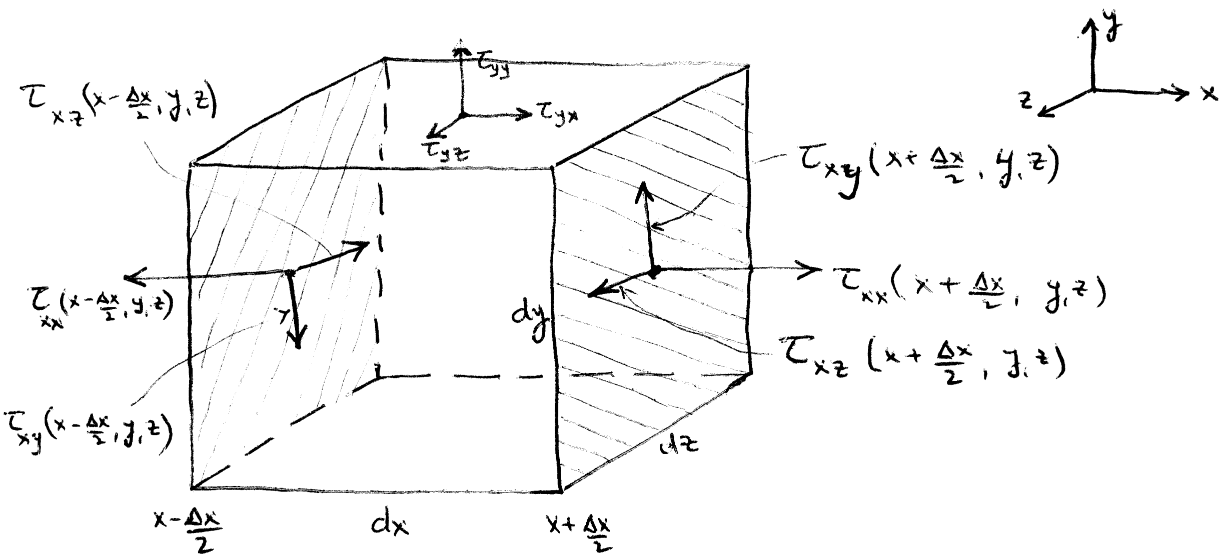
There is body force, such as gravity: $\Phi \Delta x \Delta y \Delta z$

and there are surface forces acting by contact

In terms of the stress tensor τ

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

each component is force per unit area in a certain direction



Similar forces also exist on y = const. & z = const. surfaces forces in x-direction $= \left[\tau_{xx}(x+\frac{\Delta x}{2}, y, z) - \tau_{xx}(x-\frac{\Delta x}{2}, y, z)\right] \Delta y \, \Delta z$

$$+ \left[\tau_{yx}(x, y+\tfrac{\Delta y}{2}, z) - \tau_{yx}(x, y-\tfrac{\Delta y}{2}, z)\right] \Delta x \, \Delta z$$

$$+ \left[\tau_{zx}(x, y, z+\tfrac{\Delta z}{2}) - \tau_{zx}(x, y, z-\tfrac{\Delta z}{2})\right] \Delta y \, \Delta x$$

$$= \frac{\partial}{\partial x} \tau_{xx}(x,y) \, \Delta x \, \Delta y \, \Delta z$$

$$+ \frac{\partial}{\partial y} \tau_{yx} \, \Delta x \, \Delta y \, \Delta z$$

$$+ \frac{\partial}{\partial z} \tau_{zx} \, \Delta x \, \Delta y \, \Delta z$$

Similarly,
forces in y direction $= \frac{\partial}{\partial x} \tau_{xy}(x,y) \, \Delta x \, \Delta y \, \Delta z$

$$+ \frac{\partial}{\partial y} \tau_{yy}(x,y,z) \, \Delta x \, \Delta y \, \Delta z$$

$$+ \frac{\partial}{\partial z} \tau_{zy}(x,y,z) \, \Delta x \, \Delta y \, \Delta z$$

Hence, the fundamental force balance is

$$\rho \frac{Du(x,y,z,t)}{Dt} = \rho \Phi_x + \frac{\partial \tau_{xx}(x,y,z,t)}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho(x,y,z,t) \frac{Dv(x,y,z,t)}{Dt} = \rho \Phi_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$\rho(x,y,z,t) \frac{Dw(x,y,z,t)}{Dt} = \rho \Phi_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

but what are the τ_{ij} ?

we are interested in ρ, u, v, ω. Alongwith the continuity eqn, we have 4 eqns (scalar) and 4 + 9 unknowns.

Get rid of the 9 unknowns (the τ_{ij} terms) in favor of the other four (ρ, u, v, ω), using constitutive relations.

Recall, heat conduction eqn: the heat fluxes q_x, q_y, q_z were eliminated using Fourier's law $q_x = -k \frac{\partial T}{\partial x}$
a constitutive relation based on empirical evidence (defines k)

Now τs are forces per unit area. τ_{ii} are <u>normal</u> to the surface in direction i, & τ_{ij} $(i \neq j)$ are <u>parallel</u> to the surface (shear stress)

Physics behind shear stress: adjacent layers w/ different velocitys

Constitutive Relation are

$$\tau_{xx} = -P + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{yy} = -P + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{zz} = -P + 2\mu \frac{\partial w}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

for a fluid at rest $(\vec{v} = 0)$

$$P = -\frac{1}{3}\left(\tau_{xx} + \tau_{yy} + \tau_{zz}\right)$$

Stokes hypothesis says that even for a fluid in motion, the above holds

For $\boxed{\text{relations}}$ $\boxed{\text{constitutive}}$ for off diagonal elements

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_{zx}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

μ is the dynamic viscosity

(λ : bulk coeff. of viscosity)

$3\lambda + 2\mu = 0$

Substituting τ_{ij} in the force balance

x-direction :

$$\rho \frac{Du}{Dt} = \rho \Phi_x + \frac{\partial}{\partial x}\left[-P + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \, \nabla\cdot\vec{v}\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

for constant viscosity μ

$$= \rho \Phi_x - \frac{\partial P}{\partial x} + \mu\left\{2 \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\frac{\partial}{\partial x}\nabla\cdot\vec{v} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right.$$
$$\left. + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial x}\right\}$$

$$= \rho \Phi_x - \frac{\partial P}{\partial x} + \mu \nabla^2 u + \mu \frac{\partial}{\partial x}\left\{-\frac{2}{3}\nabla\cdot\vec{v} + \overbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}^{\nabla\cdot\vec{v}}\right\}$$

$$\rho \frac{Du}{Dt} = \rho \Phi_x - \frac{\partial P}{\partial x} + \mu \nabla^2 u + " \frac{1}{3}\frac{\partial}{\partial x}(\nabla\cdot\vec{v})$$

$\vdots$

$$\rho \frac{Dv}{Dt} = \rho \Phi_y - \frac{\partial P}{\partial y} + \mu \nabla^2 v + " \frac{1}{3}\frac{\partial}{\partial y}(\nabla\cdot\vec{v})$$

$$\rho \frac{Dw}{Dt} = \rho \Phi_z - \frac{\partial P}{\partial z} + \mu \nabla^2 w + \frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\vec{v})$$

for incompressible flow, ($\rho \simeq$ const)
Continuity eqn $\Rightarrow$ $\nabla \cdot \vec{V} = 0$

$\therefore$

$$\rho \frac{Du}{Dt} = \rho \Phi_x - \frac{\partial P}{\partial x} + \mu \nabla^2 u$$

$$\rho \frac{Dv}{Dt} = \rho \Phi_y - \frac{\partial P}{\partial y} + \mu \nabla^2 v$$

$$\rho \frac{Dw}{Dt} = \rho \Phi_z - \frac{\partial P}{\partial z} + \mu \nabla^2 w$$

each is a scalar eqn.

Vector form is

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{\Phi} - \nabla P + \mu \nabla^2 \vec{V}$$

or

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \vec{V} \cdot \nabla \vec{V} = \rho \vec{\Phi} - \nabla P + \mu \nabla^2 \vec{V}$$

const viscosity and density.

See form of Navier-Stokes eqn in other orthogonal curvilinear coordinate system.