Finite Element Method (FEM) For Moltres

Finite Element Method (FEM) For Moltres

FEM is a method for numerically approximating the solution to partial differential equations (PDEs).

FEM finds a solution function that is made up of "shape functions" multiplied by coefficients and added together, just like in polynomial fitting.

The Galerkin Finite Element method is different from finite difference and finite volume methods because it finds a piecewise continuous function which is an approximate solution to the governing PDEs.

Just as in polynomial fitting you can evaluate a finite element solution anywhere in the domain.

FEM is widely applicable for a large range of PDEs and domains.

Strong Form to Weak Form

Using FEM to find the solution to a PDE starts with forming a "weighted residual" or "variational statement" or "weak form", this processes if referred to here as generating a weak form.

The weak form provides flexibility, both mathematically and numerically and it is needed by MOOSE to solve a problem.

Generating a weak form generally involves these steps:

Step#01: Write down strong form of PDE.

Step#02: Rearrange terms so that zero is on the right of the equals sign.

Step#03: Multiply the whole equation by a "test" function ψ.

Step#04: Integrate the whole equation over the domain Ω.

Step#05: Integrate by parts and use the divergence theorem to get the desired derivative order on your functions and simultaneously generate boundary integrals.

Finite Element Method (FEM) For Moltres

Step#01: Write Strong Form of Equations

Neutron Diffusion Equation :

$$\frac{1}{\upsilon_g}\frac{\partial \phi_g}{\partial t} + \Sigma_g^{r}\phi_g - \nabla \cdot D_g \nabla \phi_g = \sum_{g \neq g'}^{G} \Sigma_{g' \rightarrow g}^{s}\phi_{g'} + \chi_g^{p}\sum_{g'=1}^{G}(1-\beta)\nu\Sigma_{f,g'}\phi_{g'} + \chi_g^{d}\sum_{i}^{I}\lambda_i C_i$$

Heat Transfer Equation :

$$\rho_{C_p}\frac{\partial T}{\partial t} + \rho_{C_p}\vec{u}\cdot\nabla T - k\nabla^2 T = \sum_{g=1}^{G}\epsilon_{f,g}\Sigma_{f,g}\phi_g$$

Finite Element Method (FEM) For Moltres

Step#02: Rearrange terms so that zero is on the right side

Neutron Diffusion Equation :

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} + \Sigma_g^r \phi_g - \nabla \cdot D_g \nabla \phi_g - \sum_{g \neq g'}^{G} \Sigma_{g' \rightarrow g}^s \phi_{g'} - \chi_g^p \sum_{g'=1}^{G} (1 - \beta) \nu \Sigma_{f,g'} \phi_{g'} - \chi_g^d \sum_{i}^{I} \lambda_i C_i = 0$$

Heat Transfer Equation :

$$\rho_{C_p} \frac{\partial T}{\partial t} + \rho_{C_p} \vec{u} \cdot \nabla T - k \nabla^2 T - \sum_{g=1}^{G} \epsilon_{f,g} \Sigma_{f,g} \phi_g = 0$$

Finite Element Method (FEM) For Moltres

Step#03: Multiply by the test function ψ_i

Neutron Diffusion Equation :

$$\psi_i\left(\frac{1}{\upsilon_g}\frac{\partial\phi_g}{\partial t}+\Sigma_g^r\phi_g-\nabla\cdot D_g\nabla\phi_g-\sum_{g\neq g'}^{G}\Sigma_{g'\rightarrow g}^{s}\phi_{g'}-\chi_g^p\sum_{g'=1}^{G}(1-\beta)\nu\Sigma_{f,g'}\phi_{g'}-\chi_g^d\sum_{i}^{I}\lambda_i C_i\right)=0$$

Heat Transfer Equation :

$$\psi_i\left(\rho C_p\frac{\partial T}{\partial t}+\rho C_p\vec{u}\cdot\nabla T-k\nabla^2 T-\sum_{g=1}^{G}\epsilon_{f,g}\Sigma_{f,g}\phi_g\right)=0$$

Finite Element Method (FEM) For Moltres

Step#04: Integrate Over the Domain

Neutron Diffusion Equation :

$$\int_{\Omega} \psi_i \left(\frac{1}{v_g} \frac{\partial \phi_g}{\partial t} + \Sigma_g^r \phi_g - \nabla \cdot D_g \nabla \phi_g - \sum_{g \neq g'}^{G} \Sigma_{g' \to g}^s \phi_{g'} - \chi_g^p \sum_{g'=1}^{G} (1 - \beta) \nu \Sigma_{f,g'} \phi_{g'} - \chi_g^d \sum_{i}^{I} \lambda_i C_i \right) d\Omega = 0$$

Heat Transfer Equation :

$$\int_{\Omega} \psi_i \left(\rho c_p \frac{\partial T}{\partial t} + \rho c_p \vec{u} \cdot \nabla T - k \nabla^2 T - \sum_{g=1}^{G} \epsilon_{f,g} \Sigma_{f,g} \phi_g \right) d\Omega = 0$$

Finite Element Method (FEM) For Moltres

Step#04: Integrate Over the Domain Continued..

Neutron Diffusion Equation :

$$\int_{\Omega} \psi_i \left(\frac{1}{v_g} \frac{\partial \phi_g}{\partial t} \right) d\Omega + \int_{\Omega} \psi_i \left(\Sigma_g^r \phi_g \right) d\Omega - \int_{\Omega} \psi_i \left(\nabla \cdot D_g \nabla \phi_g \right) d\Omega - \int_{\Omega} \psi_i \left(\sum_{g \neq g'}^{G} \Sigma_{g' \to g}^{s} \phi_{g'} \right) d\Omega -$$

$$\int_{\Omega} \psi_i \left(\chi_g^p \sum_{g'=1}^{G} (1 - \beta) \nu \Sigma_{f,g'} \phi_{g'} \right) d\Omega - \int_{\Omega} \psi_i \left(\chi_g^d \sum_{i}^{I} \lambda_i C_i \right) d\Omega = 0$$

Heat Transfer Equation :

$$\int_{\Omega} \psi_i \left(\rho C_p \frac{\partial T}{\partial t} \right) d\Omega + \int_{\Omega} \psi_i \left(\rho C_p \vec{u} \cdot \nabla T \right) d\Omega - \int_{\Omega} \psi_i \left(k \nabla^2 T \right) d\Omega - \int_{\Omega} \psi_i \left(\sum_{g=1}^{G} \epsilon_{f,g} \Sigma_{f,g} \phi_g \right) d\Omega = 0$$

Finite Element Method (FEM) For Moltres

Step#05: Integrate by parts and apply the divergence theorem

$$\int_{\Omega} \vec{\nabla} \cdot \left(\psi_i \, \vec{v}\right) \, d\Omega = \int_{\Omega} \psi_i \left(\vec{\nabla} \cdot \vec{v}\right) d\Omega + \int_{\Omega} \vec{\nabla} \psi_i \cdot \left(\vec{v}\right) d\Omega$$

$$\int_{\Omega} \psi_i \left(\vec{\nabla} \cdot \vec{v}\right) d\Omega = \int_{\Omega} \vec{\nabla} \cdot \left(\psi_i \, \vec{v}\right) d\Omega - \int_{\Omega} \vec{\nabla} \psi_i \cdot \left(\vec{v}\right) d\Omega$$

$$\int_{\Omega} \vec{\nabla} \cdot \left(\psi_i \, \vec{v}\right) d\Omega = \int_{d\Omega} \left(\psi_i \, \vec{v}\right) \cdot \mathbf{n} \, ds$$

$$\int_{\Omega} \psi_i \left(\vec{\nabla} \cdot \vec{v}\right) d\Omega = \int_{d\Omega} \left(\psi_i \, \vec{v}\right) \cdot \mathbf{n} \, ds - \int_{\Omega} \vec{\nabla} \psi_i \cdot \left(\vec{v}\right) d\Omega$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

Galerkin Finite Element Method :

Test Function = Basis Function

$$\psi = (\phi_j)_{j=1}^{N}$$

$$\phi_g = \sum_{j=1}^{N} u_j \phi_j \quad , \quad \vec{\nabla}\phi_g = \sum_{j=1}^{N} u_j \vec{\nabla}\phi_j$$

$$T = \sum_{j=1}^{N} u_j \phi_j \quad , \quad \vec{\nabla} T = \sum_{j=1}^{N} u_j \vec{\nabla}\phi_j$$

$$C_I = \sum_{j=1}^{N} u_j \phi_j \quad ,$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

Numerical Integration :

$$\int_{\Omega} f(x)\,dx =$$

$$\sum_{e}\int_{\Omega e} f(x)\,dx = \sum_{e}\int_{\Omega e} f(\xi)\,|\mathcal{J}_e|\,d\xi = \sum_{e}[\mathcal{J}_e]\int_{\Omega e} f(\xi)\,d\xi = \sum_{e}\sum_{q} w_q[\mathcal{J}_e]\,f(x_q)$$

$[\mathcal{J}_e]$ = Jacobian Matrix

Gaussian Quadrature :

Approximate refrence element integral

$$\int_{\Omega e} f(\xi)\,d\xi = \sum_{q} w_q\,f(x_q)$$

x_q : qth quadrature point

w_q : Associated Weight

$$t = \frac{(b-a)x + b + a}{2} \quad \text{and} \quad dt = \left(\frac{b-a}{2}\right)dx$$

$$\int_a^b f(t)dt = \int_{-1}^{1} f(t)\left(\frac{dt}{dx}\right)dx$$

$$= \left(\frac{b-a}{2}\right)\int_{-1}^{1} f\left\{\frac{(b-a)x + b + a}{2}\right\}dx$$

The mapping can be achieved conveniently by using the nodal shape functions as follows:

$$x = P(\xi,\eta) = \sum_{i=1}^{3} x_i N_i(\xi,\eta) = x_1 N_1(\xi,\eta) + x_2 N_2(\xi,\eta) + x_3 N_3(\xi,\eta),$$

$$y = Q(\xi,\eta) = \sum_{i=1}^{3} y_i N_i(\xi,\eta) = y_1 N_1(\xi,\eta) + y_2 N_2(\xi,\eta) + y_3 N_3(\xi,\eta).$$

Then we have

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{T_{st}} F(P(\xi,\eta),Q(\xi,\eta))|J(\xi,\eta)|\,\mathrm{d}\xi\mathrm{d}\eta,$$

where $J(\xi,\eta)$ is the Jacobian of the transformation, namely,

$$J(\xi,\eta) = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} = 2A_K.$$

Here A_K represents the area of the triangle K, which can be evaluated by

$$A_K = \frac{|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|}{2}.$$

Therefore, we have

$$\iint_K F(x,y)\,\mathrm{d}x\mathrm{d}y = 2A_k \iint_{T_{st}} F(P(\xi,\eta),Q(\xi,\eta))\,\mathrm{d}\xi\mathrm{d}\eta.$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

NtTimeDerivative : $$\int_{\Omega} \psi_i \left(\frac{1}{\upsilon_g} \frac{\partial \phi_g}{\partial t} \right) d\Omega = \sum_{e} \sum_{q} w_q [\mathcal{J}_e] \left(\psi_i . \frac{1}{\upsilon_g} \frac{\partial \phi_g}{\partial t} \right) (x_q)$$

SigmaR : $$\int_{\Omega} \psi_i \left(\Sigma_g^r \phi_g \right) d\Omega = \sum_{e} \sum_{q} w_q [\mathcal{J}_e] \left(\psi_i \left(\Sigma_g^r \phi_g \right) \right) (x_q)$$

GroupDiffusion : $$\int_{\Omega} \psi_i \left(\nabla \cdot D_g \nabla \phi_g \right) d\Omega = -\sum_{e} \sum_{q} w_q [\mathcal{J}_e] \left(\vec{\nabla} \psi_i \cdot \left(D_g \nabla \phi_g \right) \right) (x_q)$$

InScatter : $$\int_{\Omega} \psi_i \left(\Sigma_{g' \to g}^s \phi_g \right) d\Omega = \sum_{e} \sum_{q} w_q [\mathcal{J}_e] \left(\psi_i \left(\Sigma_{g' \to g}^s \phi_g \right) \right) (x_q)$$

DelayedNeutronSource : $$\int_{\Omega} \psi_i \left(\chi_g^d \sum_{i}^{I} \lambda_i C_i \right) d\Omega = \sum_{e} \sum_{q} w_q [\mathcal{J}_e] \left(\psi_i \left(\chi_g^d \sum_{i}^{I} \lambda_i C_i \right) \right) (x_q)$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

$$\textbf{CoupledFissionKernel}: \int_{\Omega} \psi_i \left(\chi_g^p \sum_{g'=1}^{G} (1 - \beta)\, \nu\Sigma_{f,g'}\, \phi_{g'} \right) d\Omega =$$

$$\sum_{e}\sum_{q} \mathbf{w}_q [\mathcal{J}_e] \left(\psi_i \left(\chi_g^p \sum_{g'=1}^{G} (1 - \beta)\, \nu\Sigma_{f,g'}\, \phi_{g'} \right)\right)(\mathbf{x}_q)$$

$$\textbf{MatINSTemperatureTimeDerivative}: \int_{\Omega} \psi_i \left(\rho C_p \frac{\partial T}{\partial t} \right) d\Omega = \sum_{e}\sum_{q} \mathbf{w}_q [\mathcal{J}_e] \left(\psi_i \left(\rho C_p \frac{\partial T}{\partial t} \right)\right)(\mathbf{x}_q)$$

$$\textbf{ConservativeTemperatureAdvection}: \int_{\Omega} \psi_i\, \nabla \cdot (\rho C_p\, \vec{u}\, T)\, d\Omega =$$

$$\int_{\Omega} (\nabla \psi_i \cdot (\rho_f C_{p,f}\, \vec{v_f})\, T)\, d\Omega = \sum_{e}\sum_{q} \mathbf{w}_q [\mathcal{J}_e] (\nabla \psi_i \cdot (\rho C_p\, \vec{u})\, T)(\mathbf{x}_q)$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

$$\textbf{MatDiffusion}: \int_{\Omega} \psi_i \left(\nabla \cdot k\, \nabla \mathrm{T}\right) \mathrm{d}\Omega =$$

$$\int_{\mathrm{d}\Omega} (\psi_i\, k\, \nabla \mathrm{T}) \cdot \mathbf{n}\mathrm{ds} - \int_{\Omega} \vec{\nabla} \psi_i \cdot (k\, \nabla \mathrm{T})\, \mathrm{d}\Omega = -\sum_{\mathrm{e}} \sum_{\mathrm{q}} \mathrm{w_q} [\mathcal{J}_\mathrm{e}] \left(\vec{\nabla} \psi_i \cdot (k\, \nabla \mathrm{T})\right) (\mathrm{x_q})$$

$$\textbf{TransientFissionHeatSource}: \int_{\Omega} \psi_i \left(\sum_{g=1}^{G} \epsilon_{f,g}\, \Sigma_{f,g}\, \phi_g \right) \mathrm{d}\Omega =$$

$$\sum_{\mathrm{e}} \sum_{\mathrm{q}} \mathrm{w_q} [\mathcal{J}_\mathrm{e}] \left(\psi_i \left(\sum_{g=1}^{G} \epsilon_{f,g}\, \Sigma_{f,g}\, \phi_g \right) \right) (\mathrm{x_q})$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

Ressidual_Neutron Diffusion Equation =

$$\sum_{e}\sum_{q} w_q[\mathcal{J}_e]\left(\left(\psi_i.\frac{1}{\upsilon_g}\frac{\partial \phi_g}{\partial t}\right)+\left(\psi_i\left(\Sigma_g^r\phi_g\right)\right)-\left(\vec{\nabla}\psi_i\cdot\left(D_g\nabla\phi_g\right)\right)-\right.$$

$$\left.\left(\psi_i\left(\Sigma_{g'\to g}^{s}\phi_g\right)\right)-\left(\psi_i\left(\chi_g^p\sum_{g'=1}^{G}(1-\beta)\nu\Sigma_{f,g'}\phi_{g'}\right)\right)-\left(\psi_i\left(\chi_g^d\sum_{i}^{I}\lambda_i C_i\right)\right)\right)(x_q)$$

Ressidual_Heat Transfer Equation =

$$\sum_{e}\sum_{q} w_q[\mathcal{J}_e]\left(\left(\psi_i\left(\rho\, C_p\frac{\partial T}{\partial t}\right)\right)+\left(\nabla\psi_i\cdot\left(\rho\, C_p\,\vec{u}\right)T\right)+\left(\vec{\nabla}\psi_i\cdot(k\,\nabla T)\right)-\left(\psi_i\left(\sum_{g=1}^{G}\epsilon_{f,g}\Sigma_{f,g}\phi_g\right)\right)\right)(x_q)$$

Finite Element Method (FEM) For Moltres

Step#05: Residual Formation

Newton's Method :

$$\text{Ressidual\_net}|(u_n) = 0, \; i = 1, \; \ldots \ldots, \; N$$

$$\text{Jacbian} \; (u_n)\, \delta u_{n+1} = -R\,(u_n)$$

$$u_{n+1=}\, u_n \; + \; \delta u_{n+1}$$

$$\text{Jacbian}_{ij} \; (u_n) = \frac{\partial R_i\,(u_n)}{\partial u_j}$$

THANKS