System Dynamics

Fourth Edition

Katsuhiko Ogata
University of Minnesota

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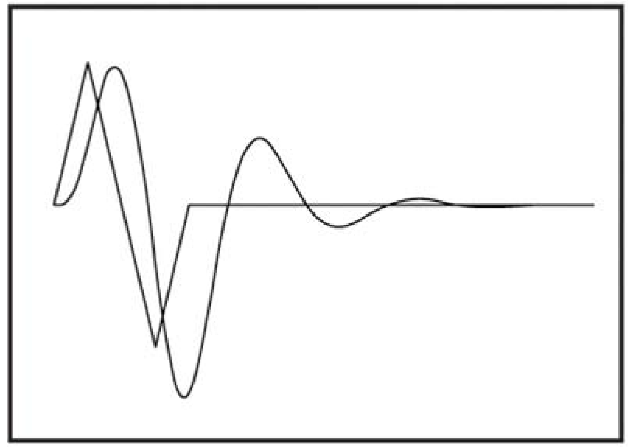
SYSTEM DYNAMICS

Fourth Edition



Katsuhiko Ogata

Contents

PREFACE vii

1 INTRODUCTION TO SYSTEM DYNAMICS 1

1–1 Introduction 1
1–2 Mathematical Modeling of Dynamic Systems 3
1–3 Analysis and Design of Dynamic Systems 5
1–4 Summary 6

2 THE LAPLACE TRANSFORM 8

2–1 Introduction 8
2–2 Complex Numbers, Complex Variables, and Complex Functions 8
2–3 Laplace Transformation 14
2–4 Inverse Laplace Transformation 29
2–5 Solving Linear, Time-Invariant Differential Equations 34
Example Problems and Solutions 36
Problems 49

3 MECHANICAL SYSTEMS 53

3–1 Introduction 53
3–2 Mechanical Elements 57
3–3 Mathematical Modeling of Simple Mechanical Systems 61
3–4 Work, Energy, and Power 73
Example Problems and Solutions 81
Problems 100

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4 TRANSFER-FUNCTION APPROACH TO MODELING DYNAMIC SYSTEMS 106

4–1 Introduction 106
4–2 Block Diagrams 109
4–3 Partial-Fraction Expansion with MATLAB 112
4–4 Transient-Response Analysis with MATLAB 119
Example Problems and Solutions 135
Problems 162

5 STATE-SPACE APPROACH TO MODELING DYNAMIC SYSTEMS 169

5–1 Introduction 169
5–2 Transient-Response Analysis of Systems in State-Space Form with MATLAB 174
5–3 State-Space Modeling of Systems with No Input Derivatives 181
5–4 State-Space Modeling of Systems with Input Derivatives 187
5–5 Transformation of Mathematical Models with MATLAB 202
Example Problems and Solutions 209
Problems 239

6 ELECTRICAL SYSTEMS AND ELECTROMECHANICAL SYSTEMS 251

6–1 Introduction 251
6–2 Fundamentals of Electrical Circuits 254
6–3 Mathematical Modeling of Electrical Systems 261
6–4 Analogous Systems 270
6–5 Mathematical Modeling of Electromechanical Systems 274
6–6 Mathematical Modeling of Operational-Amplifier Systems 281
Example Problems and Solutions 288
Problems 312

7 FLUID SYSTEMS AND THERMAL SYSTEMS 323

7–1 Introduction 323
7–2 Mathematical Modeling of Liquid-Level Systems 324
7–3 Mathematical Modeling of Pneumatic Systems 332
7–4 Linearization of Nonlinear Systems 337
7–5 Mathematical Modeling of Hydraulic Systems 340
7–6 Mathematical Modeling of Thermal Systems 348
Example Problems and Solutions 352
Problems 375

8 TIME-DOMAIN ANALYSIS OF DYNAMIC SYSTEMS 383

8–1 Introduction 383
8–2 Transient-Response Analysis of First-Order Systems 384
8–3 Transient-Response Analysis of Second-Order Systems 388
8–4 Transient-Response Analysis of Higher Order Systems 399
8–5 Solution of the State Equation 400
Example Problems and Solutions 409
Problems 424

9 FREQUENCY-DOMAIN ANALYSIS OF DYNAMIC SYSTEMS 431

9–1 Introduction 431
9–2 Sinusoidal Transfer Function 432
9–3 Vibrations in Rotating Mechanical Systems 438
9–4 Vibration Isolation 441
9–5 Dynamic Vibration Absorbers 447
9–6 Free Vibrations in Multi-Degrees-of-Freedom Systems 453
Example Problems and Solutions 458
Problems 484

10 TIME-DOMAIN ANALYSIS AND DESIGN OF CONTROL SYSTEMS 491

10–1 Introduction 491
10–2 Block Diagrams and Their Simplification 494
10–3 Automatic Controllers 501
10–4 Transient-Response Analysis 506
10–5 Transient-Response Specifications 513
10–6 Improving Transient-Response and Steady-State Characteristics 522
10–7 Stability Analysis 538
10–8 Root-Locus Analysis 545
10–9 Root-Locus Plots with MATLAB 562
10–10 Tuning Rules for PID Controllers 566
Example Problems and Solutions 576
Problems 600

11 FREQUENCY-DOMAIN ANALYSIS AND DESIGN OF CONTROL SYSTEMS 608

11–1 Introduction 608
11–2 Bode Diagram Representation of the Frequency Response 609
11–3 Plotting Bode Diagrams with MATLAB 629
11–4 Nyquist Plots and the Nyquist Stability Criterion 630

11–5 Drawing Nyquist Plots with MATLAB 640
11–6 Design of Control Systems in the Frequency Domain 643
Example Problems and Solutions 668
Problems 690

APPENDIX A SYSTEMS OF UNITS 695
APPENDIX B CONVERSION TABLES 700
APPENDIX C VECTOR–MATRIX ALGEBRA 705
APPENDIX D INTRODUCTION TO MATLAB 720
REFERENCES 757
INDEX 759

Preface

A course in system dynamics that deals with mathematical modeling and response analyses of dynamic systems is required in most mechanical and other engineering curricula. This book is written as a textbook for such a course. It is written at the junior level and presents a comprehensive treatment of modeling and analyses of dynamic systems and an introduction to control systems.

Prerequisites for studying this book are first courses in linear algebra, introductory differential equations, introductory vector-matrix analysis, mechanics, circuit analysis, and thermodynamics. Thermodynamics may be studied simultaneously.

Main revisions made in this edition are to shift the state space approach to modeling dynamic systems to Chapter 5, right next to the transfer function approach to modeling dynamic systems, and to add numerous examples for modeling and response analyses of dynamic systems. All plottings of response curves are done with MATLAB. Detailed MATLAB programs are provided for MATLAB works presented in this book.

This text is organized into 11 chapters and four appendixes. Chapter 1 presents an introduction to system dynamics. Chapter 2 deals with Laplace transforms of commonly encountered time functions and some theorems on Laplace transform that are useful in analyzing dynamic systems. Chapter 3 discusses details of mechanical elements and simple mechanical systems. This chapter includes introductory discussions of work, energy, and power.

Chapter 4 discusses the transfer function approach to modeling dynamic systems. Transient responses of various mechanical systems are studied and MATLAB is used to obtain response curves. Chapter 5 presents state space modeling of dynamic systems. Numerous examples are considered. Responses of systems in the state space form are discussed in detail and response curves are obtained with MATLAB.

Chapter 6 treats electrical systems and electromechanical systems. Here we included mechanical–electrical analogies and operational amplifier systems. Chapter 7 deals with mathematical modeling of fluid systems (such as liquid-level systems, pneumatic systems, and hydraulic systems) and thermal systems. A linearization technique for nonlinear systems is presented in this chapter.

Chapter 8 deals with the time-domain analysis of dynamic systems. Transient-response analysis of first-order systems, second-order systems, and higher order systems is discussed in detail. This chapter includes analytical solutions of state-space equations. Chapter 9 treats the frequency-domain analysis of dynamic systems. We first present the sinusoidal transfer function, followed by vibration analysis of mechanical systems and discussions on dynamic vibration absorbers. Then we discuss modes of vibration in two or more degrees-of-freedom systems.

Chapter 10 presents the analysis and design of control systems in the time domain. After giving introductory materials on control systems, this chapter discusses transient-response analysis of control systems, followed by stability analysis, root-locus analysis, and design of control systems. Finally, we conclude this chapter by giving tuning rules for PID controllers. Chapter 11 treats the analysis and design of control systems in the frequency domain. Bode diagrams, Nyquist plots, and the Nyquist stability criterion are discussed in detail. Several design problems using Bode diagrams are treated in detail. MATLAB is used to obtain Bode diagrams and Nyquist plots.

Appendix A summarizes systems of units used in engineering analyses. Appendix B provides useful conversion tables. Appendix C reviews briefly a basic vector-matrix algebra. Appendix D gives introductory materials on MATLAB. If the reader has no prior experience with MATLAB, it is recommended that he/she study Appendix D before attempting to write MATLAB programs.

Throughout the book, examples are presented at strategic points so that the reader will have a better understanding of the subject matter discussed. In addition, a number of solved problems (A problems) are provided at the end of each chapter, except Chapter 1. These problems constitute an integral part of the text. It is suggested that the reader study all these problems carefully to obtain a deeper understanding of the topics discussed. Many unsolved problems (B problems) are also provided for use as homework or quiz problems. An instructor using this text for his/her system dynamics course may obtain a complete solutions manual for B problems from the publisher.

Most of the materials presented in this book have been class tested in courses in the field of system dynamics and control systems in the Department of Mechanical Engineering, University of Minnesota over many years.

If this book is used as a text for a quarter-length course (with approximately 30 lecture hours and 18 recitation hours), Chapters 1 through 7 may be covered. After studying these chapters, the student should be able to derive mathematical models for many dynamic systems with reasonable simplicity in the forms of transfer function or state-space equation. Also, he/she will be able to obtain computer solutions of system responses with MATLAB. If the book is used as a text for a semester-length course (with approximately 40 lecture hours and 26 recitation hours), then the first nine chapters may be covered or, alternatively, the first seven chapters plus Chapters 10 and 11 may be covered. If the course devotes 50 to 60 hours to lectures, then the entire book may be covered in a semester.

Finally, I wish to acknowledge deep appreciation to the following professors who reviewed the third edition of this book prior to the preparation of this new edition: R. Gordon Kirk (Virginia Institute of Technology), Perry Y. Li (University of Minnesota), Sherif Noah (Texas A & M University), Mark L. Psiaki (Cornell University), and William Singhose (Georgia Institute of Technology). Their candid, insightful, and constructive comments are reflected in this new edition.

Katsuhiko Ogata

1

Introduction to System Dynamics

1–1 INTRODUCTION

System dynamics deals with the mathematical modeling of dynamic systems and response analyses of such systems with a view toward understanding the dynamic nature of each system and improving the system's performance. Response analyses are frequently made through computer simulations of dynamic systems.

Because many physical systems involve various types of components, a wide variety of different types of dynamic systems will be examined in this book. The analysis and design methods presented can be applied to mechanical, electrical, pneumatic, and hydraulic systems, as well as nonengineering systems, such as economic systems and biological systems. It is important that the mechanical engineering student be able to determine dynamic responses of such systems.

We shall begin this chapter by defining several terms that must be understood in discussing system dynamics.

Systems. A *system* is a combination of components acting together to perform a specific objective. A *component* is a single functioning unit of a system. By no means limited to the realm of the physical phenomena, the concept of a system can be extended to abstract dynamic phenomena, such as those encountered in economics, transportation, population growth, and biology.

A system is called *dynamic* if its present output depends on past input; if its current output depends only on current input, the system is known as *static*. The output of a static system remains constant if the input does not change. The output changes only when the input changes. In a dynamic system, the output changes with time if the system is not in a state of equilibrium. In this book, we are concerned mostly with dynamic systems.

Mathematical models. Any attempt to design a system must begin with a prediction of its performance before the system itself can be designed in detail or actually built. Such prediction is based on a mathematical description of the system's dynamic characteristics. This mathematical description is called a *mathematical model*. For many physical systems, useful mathematical models are described in terms of differential equations.

Linear and nonlinear differential equations. Linear differential equations may be classified as linear, time-invariant differential equations and linear, time-varying differential equations.

A *linear, time-invariant differential equation* is an equation in which a dependent variable and its derivatives appear as linear combinations. An example of such an equation is

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 10x = 0$$

Since the coefficients of all terms are constant, a linear, time-invariant differential equation is also called a *linear, constant-coefficient differential equation*.

In the case of a *linear, time-varying differential equation*, the dependent variable and its derivatives appear as linear combinations, but a coefficient or coefficients of terms may involve the independent variable. An example of this type of differential equation is

$$\frac{d^2x}{dt^2} + (1 - \cos 2t)x = 0$$

It is important to remember that, in order to be linear, the equation must contain no powers or other functions or products of the dependent variables or its derivatives.

A differential equation is called *nonlinear* if it is not linear. Two examples of nonlinear differential equations are

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0$$

and

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = \sin \omega t$$

Linear systems and nonlinear systems. For linear systems, the equations that constitute the model are linear. In this book, we shall deal mostly with linear systems that can be represented by linear, time-invariant ordinary differential equations.

The most important property of linear systems is that the *principle of superposition* is applicable. This principle states that the response produced by simultaneous applications of two different forcing functions or inputs is the sum of two individual responses. Consequently, for linear systems, the response to several inputs can be calculated by dealing with one input at a time and then adding the results. As a result of superposition, complicated solutions to linear differential equations can be derived as a sum of simple solutions.

In an experimental investigation of a dynamic system, if cause and effect are proportional, thereby implying that the principle of superposition holds, the system can be considered linear.

Although physical relationships are often represented by linear equations, in many instances the actual relationships may not be quite linear. In fact, a careful study of physical systems reveals that so-called linear systems are actually linear only within limited operating ranges. For instance, many hydraulic systems and pneumatic systems involve nonlinear relationships among their variables, but they are frequently represented by linear equations within limited operating ranges.

For nonlinear systems, the most important characteristic is that the principle of superposition is *not* applicable. In general, procedures for finding the solutions of problems involving such systems are extremely complicated. Because of the mathematical difficulty involved, it is frequently necessary to linearize a nonlinear system near the operating condition. Once a nonlinear system is approximated by a linear mathematical model, a number of linear techniques may be used for analysis and design purposes.

Continuous-time systems and discrete-time systems. Continuous-time systems are systems in which the signals involved are continuous in time. These systems may be described by differential equations.

Discrete-time systems are systems in which one or more variables can change only at discrete instants of time. (These instants may specify the times at which some physical measurement is performed or the times at which the memory of a digital computer is read out.) Discrete-time systems that involve digital signals and, possibly, continuous-time signals as well may be described by difference equations after the appropriate discretization of the continuous-time signals.

The materials presented in this text apply to continuous-time systems; discrete-time systems are not discussed.

1–2 MATHEMATICAL MODELING OF DYNAMIC SYSTEMS

Mathematical modeling. Mathematical modeling involves descriptions of important system characteristics by sets of equations. By applying physical laws to a specific system, it may be possible to develop a mathematical model that describes the dynamics of the system. Such a model may include unknown parameters, which

must then be evaluated through actual tests. Sometimes, however, the physical laws governing the behavior of a system are not completely defined, and formulating a mathematical model may be impossible. If so, an experimental modeling process can be used. In this process, the system is subjected to a set of known inputs, and its outputs are measured. Then a mathematical model is derived from the input–output relationships obtained.

Simplicity of mathematical model versus accuracy of results of analysis. In attempting to build a mathematical model, a compromise must be made between the simplicity of the model and the accuracy of the results of the analysis. It is important to note that the results obtained from the analysis are valid only to the extent that the model approximates a given physical system.

In determining a reasonably simplified model, we must decide which physical variables and relationships are negligible and which are crucial to the accuracy of the model. To obtain a model in the form of linear differential equations, any distributed parameters and nonlinearities that may be present in the physical system must be ignored. If the effects that these ignored properties have on the response are small, then the results of the analysis of a mathematical model and the results of the experimental study of the physical system will be in good agreement. Whether any particular features are important may be obvious in some cases, but may, in other instances, require physical insight and intuition. Experience is an important factor in this connection.

Usually, in solving a new problem, it is desirable first to build a simplified model to obtain a general idea about the solution. Afterward, a more detailed mathematical model can be built and used for a more complete analysis.

Remarks on mathematical models. The engineer must always keep in mind that the model he or she is analyzing is an approximate mathematical description of the physical system; it is not the physical system itself. In reality, no mathematical model can represent any physical component or system precisely. Approximations and assumptions are always involved. Such approximations and assumptions restrict the range of validity of the mathematical model. (The degree of approximation can be determined only by experiments.) So, in making a prediction about a system's performance, any approximations and assumptions involved in the model must be kept in mind.

Mathematical modeling procedure. The procedure for obtaining a mathematical model for a system can be summarized as follows:

1. Draw a schematic diagram of the system, and define variables.
2. Using physical laws, write equations for each component, combine them according to the system diagram, and obtain a mathematical model.
3. To verify the validity of the model, its predicted performance, obtained by solving the equations of the model, is compared with experimental results. (The question of the validity of any mathematical model can be answered only by experiment.) If the experimental results deviate from the prediction

to a great extent, the model must be modified. A new model is then derived and a new prediction compared with experimental results. The process is repeated until satisfactory agreement is obtained between the predictions and the experimental results.

1–3 ANALYSIS AND DESIGN OF DYNAMIC SYSTEMS

This section briefly explains what is involved in the analysis and design of dynamic systems.

Analysis. *System analysis* means the investigation, under specified conditions, of the performance of a system whose mathematical model is known.

The first step in analyzing a dynamic system is to derive its mathematical model. Since any system is made up of components, analysis must start by developing a mathematical model for each component and combining all the models in order to build a model of the complete system. Once the latter model is obtained, the analysis may be formulated in such a way that system parameters in the model are varied to produce a number of solutions. The engineer then compares these solutions and interprets and applies the results of his or her analysis to the basic task.

It should always be remembered that deriving a reasonable model for the complete system is the most important part of the entire analysis. Once such a model is available, various analytical and computer techniques can be used to analyze it. The manner in which analysis is carried out is independent of the type of physical system involved—mechanical, electrical, hydraulic, and so on.

Design. *System design* refers to the process of finding a system that accomplishes a given task. In general, the design procedure is not straightforward and will require trial and error.

Synthesis. By *synthesis*, we mean the use of an explicit procedure to find a system that will perform in a specified way. Here the desired system characteristics are postulated at the outset, and then various mathematical techniques are used to synthesize a system having those characteristics. Generally, such a procedure is completely mathematical from the start to the end of the design process.

Basic approach to system design. The basic approach to the design of any dynamic system necessarily involves trial-and-error procedures. Theoretically, a synthesis of linear systems is possible, and the engineer can systematically determine the components necessary to realize the system's objective. In practice, however, the system may be subject to many constraints or may be nonlinear; in such cases, no synthesis methods are currently applicable. Moreover, the features of the components may not be precisely known. Thus, trial-and-error techniques are almost always needed.

Design procedures. Frequently, the design of a system proceeds as follows: The engineer begins the design procedure knowing the specifications to be met and the dynamics of the components, the latter of which involve design parameters. The specification may be given in terms of both precise numerical values and vague qualitative descriptions. (Engineering specifications normally include statements on such factors as cost, reliability, space, weight, and ease of maintenance.) It is important to note that the specifications may be changed as the design progresses, for detailed analysis may reveal that certain requirements are impossible to meet. Next, the engineer will apply any applicable synthesis techniques, as well as other methods, to build a mathematical model of the system.

Once the design problem is formulated in terms of a model, the engineer carries out a mathematical design that yields a solution to the mathematical version of the design problem. With the mathematical design completed, the engineer simulates the model on a computer to test the effects of various inputs and disturbances on the behavior of the resulting system. If the initial system configuration is not satisfactory, the system must be redesigned and the corresponding analysis completed. This process of design and analysis is repeated until a satisfactory system is found. Then a prototype physical system can be constructed.

Note that the process of constructing a prototype is the reverse of mathematical modeling. The prototype is a physical system that represents the mathematical model with reasonable accuracy. Once the prototype has been built, the engineer tests it to see whether it is satisfactory. If it is, the design of the prototype is complete. If not, the prototype must be modified and retested. The process continues until a satisfactory prototype is obtained.

1–4 SUMMARY

From the point of view of analysis, a successful engineer must be able to obtain a mathematical model of a given system and predict its performance. (The validity of a prediction depends to a great extent on the validity of the mathematical model used in making the prediction.) From the design standpoint, the engineer must be able to carry out a thorough performance analysis of the system before a prototype is constructed.

The objective of this book is to enable the reader (1) to build mathematical models that closely represent behaviors of physical systems and (2) to develop system responses to various inputs so that he or she can effectively analyze and design dynamic systems.

Outline of the text. Chapter 1 has presented an introduction to system dynamics. Chapter 2 treats Laplace transforms. We begin with Laplace transformation of simple time functions and then discuss inverse Laplace transformation. Several useful theorems are derived. Chapter 3 deals with basic accounts of mechanical systems. Chapter 4 presents the transfer-function approach to modeling dynamic systems. The chapter discusses various types of mechanical systems. Chapter 5 examines the state-space approach to modeling dynamic systems. Various types of mechanical systems are considered. Chapter 6 treats electrical systems and electromechanical systems, including operational-amplifier systems. Chapter 7 deals with fluid systems,

such as liquid-level systems, pneumatic systems, and hydraulic systems, as well as thermal systems. A linearization technique for nonlinear systems is explored.

Chapter 8 presents time-domain analyses of dynamic systems—specifically, transient-response analyses of dynamic systems. The chapter also presents the analytical solution of the state equation. Chapter 9 treats frequency-domain analyses of dynamic systems. Among the topics discussed are vibrations of rotating mechanical systems and vibration isolation problems. Also discussed are vibrations in multi-degrees-of-freedom systems and modes of vibrations.

Chapter 10 presents the basic theory of control systems, including transient-response analysis, stability analysis, and root-locus analysis and design. Also discussed are tuning rules for PID controllers. Chapter 11 deals with the analysis and design of control systems in the frequency domain. The chapter begins with Bode diagrams and then presents the Nyquist stability criterion, followed by detailed design procedures for lead, lag, and lag-lead compensators.

Appendix A treats systems of units, Appendix B summarizes conversion tables, and Appendix C gives a brief summary of vector–matrix algebra. Appendix D presents introductory materials for MATLAB.

Throughout the book, MATLAB is used for the solution of most computational problems. Readers who have no previous knowledge of MATLAB may read Appendix D before solving any MATLAB problems presented in this text.

2

The Laplace Transform

2–1 INTRODUCTION

The Laplace transform is one of the most important mathematical tools available for modeling and analyzing linear systems. Since the Laplace transform method must be studied in any system dynamics course, we present the subject at the beginning of this text so that the student can use the method throughout his or her study of system dynamics.

The remaining sections of this chapter are outlined as follows: Section 2–2 reviews complex numbers, complex variables, and complex functions. Section 2–3 defines the Laplace transformation and gives Laplace transforms of several common functions of time. Also examined are some of the most important Laplace transform theorems that apply to linear systems analysis. Section 2–4 deals with the inverse Laplace transformation. Finally, Section 2–5 presents the Laplace transform approach to the solution of the linear, time-invariant differential equation.

2–2 COMPLEX NUMBERS, COMPLEX VARIABLES, AND COMPLEX FUNCTIONS

This section reviews complex numbers, complex algebra, complex variables, and complex functions. Since most of the material covered is generally included in the basic mathematics courses required of engineering students, the section can be omitted entirely or used simply for personal reference.

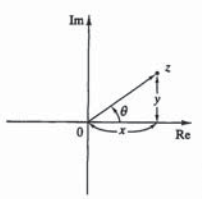


Figure 2–1 Complex plane representation of a complex number z.

Complex numbers. Using the notation $j = \sqrt{-1}$, we can express all numbers in engineering calculations as

$$z = x + jy$$

where z is called a *complex number* and x and jy are its *real* and *imaginary parts*, respectively. Note that both x and y are real and that j is the only imaginary quantity in the expression. The complex plane representation of z is shown in Figure 2–1. (Note also that the real axis and the imaginary axis define the complex plane and that the combination of a real number and an imaginary number defines a point in that plane.) A complex number z can be considered a point in the complex plane or a directed line segment to the point; both interpretations are useful.

The magnitude, or absolute value, of z is defined as the length of the directed line segment shown in Figure 2–1. The angle of z is the angle that the directed line segment makes with the positive real axis. A counterclockwise rotation is defined as the positive direction for the measurement of angles. Mathematically,

$$\text{magnitude of } z = |z| = \sqrt{x^2 + y^2}, \qquad \text{angle of } z = \theta = \tan^{-1}\frac{y}{x}$$

A complex number can be written in rectangular form or in polar form as follows:

$$\left.\begin{aligned} z &= x + jy \\ z &= |z|(\cos\theta + j\sin\theta) \end{aligned}\right\}\text{rectangular forms}$$

$$\left.\begin{aligned} z &= |z|\,\underline{/\theta} \\ z &= |z|\,e^{j\theta} \end{aligned}\right\}\text{polar forms}$$

In converting complex numbers to polar form from rectangular, we use

$$|z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

To convert complex numbers to rectangular form from polar, we employ

$$x = |z|\cos\theta, \qquad y = |z|\sin\theta$$

Complex conjugate. The *complex conjugate* of $z = x + jy$ is defined as

$$\bar{z} = x - jy$$

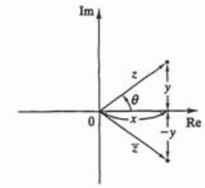


Figure 2–2 Complex number z and its complex conjugate $\bar{z}$.

The complex conjugate of z thus has the same real part as z and an imaginary part that is the negative of the imaginary part of z. Figure 2–2 shows both z and $\bar{z}$. Note that

$$z = x + jy = |z| \angle \theta = |z|(\cos\theta + j\sin\theta)$$
$$\bar{z} = x - jy = |z| \angle -\theta = |z|(\cos\theta - j\sin\theta)$$

Euler's theorem. The power series expansions of $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Thus,

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

it follows that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

This is known as *Euler's theorem.*

Using Euler's theorem, we can express the sine and cosine in complex form. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we find that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex algebra. If the complex numbers are written in a suitable form, operations like addition, subtraction, multiplication, and division can be performed easily.

Equality of complex numbers. Two complex numbers z and w are said to be equal if and only if their real parts are equal and their imaginary parts are equal. So if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then $z = w$ if and only if $x = u$ and $y = v$.

Addition. Two complex numbers in rectangular form can be added by adding the real parts and the imaginary parts separately:

$$z + w = (x + jy) + (u + jv) = (x + u) + j(y + v)$$

Subtraction. Subtracting one complex number from another can be considered as adding the negative of the former:

$$z - w = (x + jy) - (u + jv) = (x - u) + j(y - v)$$

Note that addition and subtraction can be done easily on the rectangular plane.

Multiplication. If a complex number is multiplied by a real number, the result is a complex number whose real and imaginary parts are multiplied by that real number:

$$az = a(x + jy) = ax + jay \qquad (a = \text{real number})$$

If two complex numbers appear in rectangular form and we want the product in rectangular form, multiplication is accomplished by using the fact that $j^2 = -1$. Thus, if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then

$$zw = (x + jy)(u + jv) = xu + jyu + jxv + j^2yv$$
$$= (xu - yv) + j(xv + yu)$$

In polar form, multiplication of two complex numbers can be done easily. The magnitude of the product is the product of the two magnitudes, and the angle of the product is the sum of the two angles. So if two complex numbers are written

$$z = |z| \angle \theta, \qquad w = |w| \angle \phi$$

then

$$zw = |z||w| \angle \theta + \phi$$

Multiplication by j. It is important to note that multiplication by j is equivalent to counterclockwise rotation by 90°. For example, if

$$z = x + jy$$

then

$$jz = j(x + jy) = jx + j^2y = -y + jx$$

or, noting that $j = 1 \angle 90°$, if

$$z = |z| \angle \theta$$

then

$$jz = 1 \angle 90° \, |z| \angle \theta = |z| \angle \theta + 90°$$

Figure 2–3 illustrates the multiplication of a complex number z by j.

Division. If a complex number $z = |z| \angle \theta$ is divided by another complex number $w = |w| \angle \phi$, then

$$\frac{z}{w} = \frac{|z| \angle \theta}{|w| \angle \phi} = \frac{|z|}{|w|} \angle \theta - \phi$$

That is, the result consists of the quotient of the magnitudes and the difference of the angles.

Division in rectangular form is inconvenient, but can be done by multiplying the denominator and numerator by the complex conjugate of the denominator. This procedure converts the denominator to a real number and thus simplifies division. For instance,

$$\frac{z}{w} = \frac{x + jy}{u + jv} = \frac{(x + jy)(u - jv)}{(u + jv)(u - jv)} = \frac{(xu + yv) + j(yu - xv)}{u^2 + v^2}$$
$$= \frac{xu + yv}{u^2 + v^2} + j\frac{yu - xv}{u^2 + v^2}$$

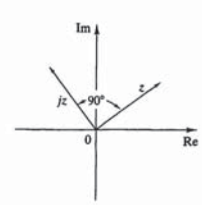


Figure 2–3 Multiplication of a complex number z by j.

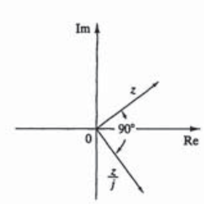


Figure 2–4 Division of a complex number z by j.

Division by j. Note that division by j is equivalent to clockwise rotation by 90°. For example, if $z = x + jy$, then

$$\frac{z}{j} = \frac{x + jy}{j} = \frac{(x + jy)j}{jj} = \frac{jx - y}{-1} = y - jx$$

or

$$\frac{z}{j} = \frac{|z| \angle \theta}{1 \angle 90°} = |z| \angle \theta - 90°$$

Figure 2–4 illustrates the division of a complex number z by j.

Powers and roots. Multiplying z by itself n times, we obtain

$$z^n = (|z| \angle \theta)^n = |z|^n \angle n\theta$$

Extracting the nth root of a complex number is equivalent to raising the number to the $1/n$th power:

$$z^{1/n} = (|z| \angle \theta)^{1/n} = |z|^{1/n} \angle \frac{\theta}{n}$$

For instance,

$$(8.66 - j5)^3 = (10 \angle -30°)^3 = 1000 \angle -90° = 0 - j\,1000 = -j\,1000$$
$$(2.12 - j2.12)^{1/2} = (9 \angle -45°)^{1/2} = 3 \angle -22.5°$$

Comments. It is important to note that

$$|zw| = |z||w|$$

and

$$|z + w| \neq |z| + |w|$$

Complex variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part or the imaginary part (or both) are variables, the complex number is called a *complex variable*. In the Laplace transformation, we use the notation s to denote a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and $j\omega$ is the imaginary part. (Note that both σ and ω are real.)

Complex function. A complex function $F(s)$, a function of s, has a real part and an imaginary part, or

$$F(s) = F_x + jF_y$$

where F_x and F_y are real quantities. The magnitude of $F(s)$ is $\sqrt{F_x^2 + F_y^2}$, and the angle θ of $F(s)$ is $\tan^{-1}(F_y/F_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $F(s)$ is $\bar{F}(s) = F_x - jF_y$.

Complex functions commonly encountered in linear systems analysis are single-valued functions of s and are uniquely determined for a given value of s. Typically, such functions have the form

$$F(s) = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

Points at which $F(s)$ equals zero are called *zeros*. That is, $s = -z_1, s = -z_2, \ldots, s = -z_m$ are zeros of $F(s)$. [Note that $F(s)$ may have additional zeros at infinity; see the illustrative example that follows.] Points at which $F(s)$ equals infinity are called *poles*. That is, $s = -p_1, s = -p_2, \ldots, s = -p_n$ are poles of $F(s)$. If the denominator of $F(s)$ involves k-multiple factors $(s + p)^k$, then $s = -p$ is called a *multiple pole* of order k or *repeated pole* of order k. If $k = 1$, the pole is called a *simple pole*.

As an illustrative example, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2$ and $s = -10$, simple poles at $s = 0, s = -1$, and $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since, for large values of s,

$$G(s) \doteqdot \frac{K}{s^3}$$

it follows that $G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros $(s = -2, s = -10, s = \infty, s = \infty, s = \infty)$ and five poles $(s = 0, s = -1, s = -5, s = -15, s = -15)$.

2-3 LAPLACE TRANSFORMATION

The Laplace transform method is an operational method that can be used advantageously in solving linear, time-invariant differential equations. Its main advantage is that differentiation of the time function corresponds to multiplication of the transform by a complex variable s, and thus the differential equations in time become algebraic equations in s. The solution of the differential equation can then be found by using a Laplace transform table or the partial-fraction expansion technique. Another advantage of the Laplace transform method is that, in solving the differential equation, the initial conditions are automatically taken care of, and both the particular solution and the complementary solution can be obtained simultaneously.

Laplace transformation. Let us define

$f(t)$ = a time function such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathscr{L}$ = an operational symbol indicating that the quantity upon which it operates is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called *inverse Laplace transformation*. The notation for inverse Laplace transformation is $\mathscr{L}^{-1}$. Thus,

$$\mathscr{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace transform. The Laplace transform of a function $f(t)$ exists if the Laplace integral converges. The integral will converge if $f(t)$ is piecewise continuous in every finite interval in the range $t > 0$ and if $f(t)$ is of exponential order as t approaches infinity. A function $f(t)$ is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit of the function $e^{-\sigma t}|f(t)|$ approaches zero for σ greater than σ_c and the limit approaches infinity for σ less than σ_c, the value σ_c is called the *abscissa of convergence*.

It can be seen that, for such functions as t, $\sin \omega t$, and $t \sin \omega t$, the abscissa of convergence is equal to zero. For functions like e^{-ct}, te^{-ct}, and $e^{-ct}\sin \omega t$, the abscissa of convergence is equal to $-c$. In the case of functions that increase faster than the exponential function, it is impossible to find suitable values of the abscissa of convergence. Consequently, such functions as e^{t^2} and te^{t^2} do not possess Laplace transforms.

Nevertheless, it should be noted that, although e^{t^2} for $0 \le t \le \infty$ does not possess a Laplace transform, the time function defined by

$$\begin{aligned} f(t) &= e^{t^2} && \text{for } 0 \le t \le T < \infty \\ &= 0 && \text{for } t < 0, T < t \end{aligned}$$

does, since $f(t) = e^{t^2}$ for only a limited time interval $0 \le t \le T$ and not for $0 \le t \le \infty$. Such a signal can be physically generated. Note that the signals that can be physically generated always have corresponding Laplace transforms.

If functions $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then the Laplace transform of $f_1(t) + f_2(t)$ is given by

$$\mathscr{L}[f_1(t) + f_2(t)] = \mathscr{L}[f_1(t)] + \mathscr{L}[f_2(t)]$$

Exponential function. Consider the exponential function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= Ae^{-\alpha t} && \text{for } t \ge 0 \end{aligned}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathscr{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = A\int_0^\infty e^{-(\alpha+s)t}\,dt = \frac{A}{s + \alpha}$$

In performing this integration, we assume that the real part of s is greater than $-\alpha$ (the abscissa of convergence), so that the integral converges. The Laplace transform $F(s)$ of any Laplace transformable function $f(t)$ obtained in this way is valid throughout the entire s plane, except at the poles of $F(s)$. (Although we do not present a proof of this statement, it can be proved by use of the theory of complex variables.)

Step function. Consider the step function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= A && \text{for } t > 0 \end{aligned}$$

where A is a constant. Note that this is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathscr{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

The step function whose height is unity is called a *unit-step function*. The unit-step function that occurs at $t = t_0$ is frequently written $1(t - t_0)$, a notation that will be used in this book. The preceding step function whose height is A can thus be written $A1(t)$.

The Laplace transform of the unit-step function that is defined by

$$\begin{aligned} 1(t) &= 0 && \text{for } t < 0 \\ &= 1 && \text{for } t > 0 \end{aligned}$$

is $1/s$, or

$$\mathscr{L}[1(t)] = \frac{1}{s}$$

Physically, a step function occurring at $t = t_0$ corresponds to a constant signal suddenly applied to the system at time t equals t_0.

Ramp function. Consider the ramp function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= At && \text{for } t \ge 0 \end{aligned}$$

where A is a constant. The Laplace transform of this ramp function is

$$\mathscr{L}[At] = A\int_0^\infty te^{-st}\,dt$$

To evaluate the integral, we use the formula for integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

In this case, $u = t$ and $dv = e^{-st}\,dt$. [Note that $v = e^{-st}/(-s)$.] Hence,

$$\begin{aligned} \mathscr{L}[At] &= A\int_0^\infty te^{-st}\,dt = A\left(t\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt\right) \\ &= \frac{A}{s}\int_0^\infty e^{-st}\,dt = \frac{A}{s^2} \end{aligned}$$

Sinusoidal function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= A \sin \omega t && \text{for } t \ge 0 \end{aligned}$$

where A and ω are constants, is obtained as follows: Noting that

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

and

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

we can write

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Hence,

$$\begin{aligned} \mathscr{L}[A \sin \omega t] &= \frac{A}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})e^{-st}\,dt \\ &= \frac{A}{2j}\frac{1}{s - j\omega} - \frac{A}{2j}\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2} \end{aligned}$$

Similarly, the Laplace transform of $A \cos \omega t$ can be derived as follows:

$$\mathscr{L}[A \cos \omega t] = \frac{As}{s^2 + \omega^2}$$

Comments. The Laplace transform of any Laplace transformable function $f(t)$ can be found by multiplying $f(t)$ by e^{-st} and then integrating the product from $t = 0$ to $t = \infty$. Once we know the method of obtaining the Laplace transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace transform tables can conveniently be used to find the transform of a given function $f(t)$. Table 2–1 shows Laplace transforms of time functions that will frequently appear in linear systems analysis. In Table 2–2, the properties of Laplace transforms are given.

Translated function. Let us obtain the Laplace transform of the translated function $f(t - \alpha)1(t - \alpha)$, where $\alpha \ge 0$. This function is zero for $t < \alpha$. The functions $f(t)1(t)$ and $f(t - \alpha)1(t - \alpha)$ are shown in Figure 2–5.

By definition, the Laplace transform of $f(t - \alpha)1(t - \alpha)$ is

$$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = \int_0^\infty f(t - \alpha)1(t - \alpha)e^{-st}\,dt$$

TABLE 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

TABLE 2–1 (continued)

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos \omega_1 t - \cos \omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

TABLE 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_{\pm}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{[\int f(t)\,dt]_{t=0\pm}}{s}$
7	$\mathscr{L}_{\pm}\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{[\int f(t)\,dt]_{t=0\pm}}{s^2} + \frac{[\iint f(t)\,dt\,dt]_{t=0\pm}}{s}$
8	$\mathscr{L}_{\pm}\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s \to 0} F(s) \quad$ if $\int_0^\infty f(t)\,dt$ exists
11	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \quad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \quad n = 1, 2, 3, \ldots$
16	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \quad$ if $\lim_{t \to 0}\frac{1}{t}f(t)$ exists
17	$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

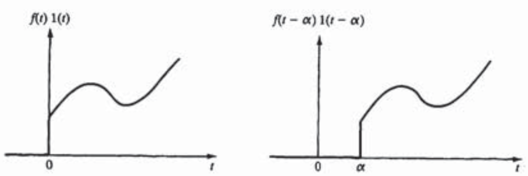


Figure 2–5 Function $f(t)1(t)$ and translated function $f(t-\alpha)1(t-\alpha)$.

By changing the independent variable from t to τ, where $\tau = t - \alpha$, we obtain

$$\int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}\,dt = \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau$$

Noting that $f(\tau)1(\tau) = 0$ for $\tau < 0$, we can change the lower limit of integration from $-\alpha$ to 0. Thus,

$$\begin{aligned} \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau &= \int_0^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}e^{-\alpha s}\,d\tau \\ &= e^{-\alpha s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-\alpha s}F(s) \end{aligned}$$

where

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Hence,

$$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$$

This last equation states that the translation of the time function $f(t)1(t)$ by α (where $\alpha \geq 0$) corresponds to the multiplication of the transform $F(s)$ by $e^{-\alpha s}$.

Pulse function. Consider the pulse function shown in Figure 2–6, namely,

$$\begin{aligned} f(t) &= \frac{A}{t_0} \qquad \text{for } 0 < t < t_0 \\ &= 0 \qquad \text{for } t < 0,\ t_0 < t \end{aligned}$$

where A and t_0 are constants.

The pulse function here may be considered a step function of height A/t_0 that begins at $t = 0$ and that is superimposed by a negative step function of height A/t_0

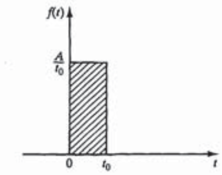


Figure 2–6 Pulse function.

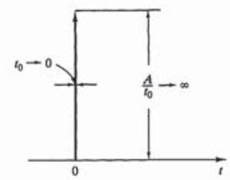


Figure 2–7 Impulse function.

beginning at $t = t_0$; that is,

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$

Then the Laplace transform of $f(t)$ is obtained as

$$\begin{aligned}\mathscr{L}[f(t)] &= \mathscr{L}\left[\frac{A}{t_0}1(t)\right] - \mathscr{L}\left[\frac{A}{t_0}1(t - t_0)\right] \\ &= \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0} \\ &= \frac{A}{t_0 s}(1 - e^{-st_0}) \qquad (2\text{–}1)\end{aligned}$$

Impulse function. The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$\begin{aligned} f(t) &= \lim_{t_0 \to 0} \frac{A}{t_0} && \text{for } 0 < t < t_0 \\ &= 0 && \text{for } t < 0, t_0 < t \end{aligned}$$

Figure 2–7 depicts the impulse function defined here. It is a limiting case of the pulse function shown in Figure 2–6 as t_0 approaches zero. Since the height of the impulse function is A/t_0 and the duration is t_0, the area under the impulse is equal to A. As the duration t_0 approaches zero, the height A/t_0 approaches infinity, but the area under the impulse remains equal to A. Note that the magnitude of the impulse is measured by its area.

From Equation (2–1), the Laplace transform of this impulse function is shown to be

$$\begin{aligned}\mathscr{L}[f(t)] &= \lim_{t_0 \to 0}\left[\frac{A}{t_0 s}(1 - e^{-st_0})\right] \\ &= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)} = \frac{As}{s} = A\end{aligned}$$

Thus, the Laplace transform of the impulse function is equal to the area under the impulse.

The impulse function whose area is equal to unity is called the *unit-impulse function* or the *Dirac delta function*. The unit-impulse function occurring at $t = t_0$ is usually denoted by $\delta(t - t_0)$, which satisfies the following conditions:

$$\begin{aligned} \delta(t - t_0) &= 0 && \text{for } t \neq t_0 \\ \delta(t - t_0) &= \infty && \text{for } t = t_0 \end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$

An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration very short compared with the system time constants, then we can approximate the pulse input by an impulse function. For instance, if a force or torque input $f(t)$ is applied to a system for a very short time duration $0 < t < t_0$, where the magnitude of $f(t)$ is sufficiently large so that $\int_0^{t_0} f(t)\,dt$ is not negligible, then this input can be considered an impulse input. (Note that, when we describe the impulse input, the area or magnitude of the impulse is most important, but the exact shape of the impulse is usually immaterial.) The impulse input supplies energy to the system in an infinitesimal time.

The concept of the impulse function is highly useful in differentiating discontinuous-time functions. The unit-impulse function $\delta(t - t_0)$ can be considered the derivative of the unit-step function $1(t - t_0)$ at the point of discontinuity $t = t_0$, or

$$\delta(t - t_0) = \frac{d}{dt}1(t - t_0)$$

Conversely, if the unit-impulse function $\delta(t - t_0)$ is integrated, the result is the unit-step function $1(t - t_0)$. With the concept of the impulse function, we can differentiate a function containing discontinuities, giving impulses, the magnitudes of which are equal to the magnitude of each corresponding discontinuity.

Multiplication of $f(t)$ by $e^{-\alpha t}$. If $f(t)$ is Laplace transformable and its Laplace transform is $F(s)$, then the Laplace transform of $e^{-\alpha t}f(t)$ is obtained as

$$\mathscr{L}[e^{-\alpha t}f(t)] = \int_0^{\infty} e^{-\alpha t}f(t)e^{-st}\,dt = F(s + \alpha) \qquad (2\text{–}2)$$

We see that the multiplication of $f(t)$ by $e^{-\alpha t}$ has the effect of replacing s by $(s + \alpha)$ in the Laplace transform. Conversely, changing s to $(s + \alpha)$ is equivalent to multiplying $f(t)$ by $e^{-\alpha t}$. (Note that α may be real or complex.)

The relationship given by Equation (2–2) is useful in finding the Laplace transforms of such functions as $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$. For instance, since

$$\mathscr{L}[\sin\omega t] = \frac{\omega}{s^2 + \omega^2} = F(s) \qquad \text{and} \qquad \mathscr{L}[\cos\omega t] = \frac{s}{s^2 + \omega^2} = G(s)$$

it follows from Equation (2–2) that the Laplace transforms of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$ are given, respectively, by

$$\mathscr{L}[e^{-\alpha t}\sin\omega t] = F(s + \alpha) = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

and

$$\mathscr{L}[e^{-\alpha t}\cos\omega t] = G(s + \alpha) = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

Comments on the lower limit of the Laplace integral. In some cases, $f(t)$ possesses an impulse function at $t = 0$. Then the lower limit of the Laplace integral must be clearly specified as to whether it is $0-$ or $0+$, since the Laplace transforms of $f(t)$ differ for these two lower limits. If such a distinction of the lower limit of the Laplace integral is necessary, we use the notations

$$\mathscr{L}_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}\,dt$$

and

$$\mathscr{L}_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \mathscr{L}_+[f(t)] + \int_{0-}^{0+} f(t)e^{-st}\,dt$$

If $f(t)$ involves an impulse function at $t = 0$, then

$$\mathscr{L}_+[f(t)] \neq \mathscr{L}_-[f(t)]$$

since

$$\int_{0-}^{0+} f(t)e^{-st}\,dt \neq 0$$

for such a case. Obviously, if $f(t)$ does not possess an impulse function at $t = 0$ (i.e., if the function to be transformed is finite between $t = 0-$ and $t = 0+$), then

$$\mathscr{L}_+[f(t)] = \mathscr{L}_-[f(t)]$$

Differentiation theorem. The Laplace transform of the derivative of a function $f(t)$ is given by

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \qquad (2\text{–}3)$$

where $f(0)$ is the initial value of $f(t)$, evaluated at $t = 0$. Equation (2–3) is called the differentiation theorem.

For a given function $f(t)$, the values of $f(0+)$ and $f(0-)$ may be the same or different, as illustrated in Figure 2–8. The distinction between $f(0+)$ and $f(0-)$ is important when $f(t)$ has a discontinuity at $t = 0$, because, in such a case, $df(t)/dt$ will

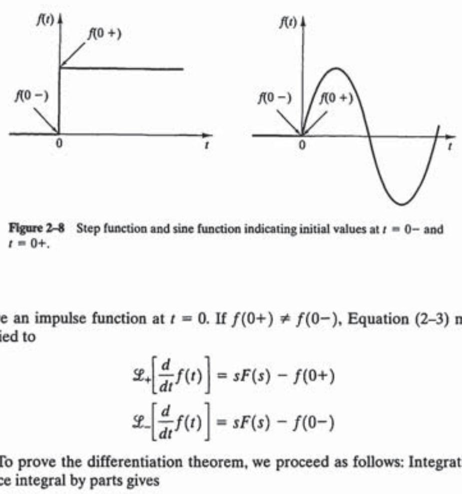


Figure 2–8 Step function and sine function indicating initial values at $t = 0-$ and $t = 0+$.

involve an impulse function at $t = 0$. If $f(0+) \neq f(0-)$, Equation (2–3) must be modified to

$$\mathscr{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

$$\mathscr{L}_-\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0-)$$

To prove the differentiation theorem, we proceed as follows: Integrating the Laplace integral by parts gives

$$\int_0^{\infty} f(t)e^{-st}\,dt = f(t)\frac{e^{-st}}{-s}\bigg|_0^{\infty} - \int_0^{\infty}\left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}\,dt$$

Hence,

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathscr{L}\left[\frac{d}{dt}f(t)\right]$$

It follows that

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

Similarly, for the second derivative of $f(t)$, we obtain the relationship

$$\mathscr{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

where $\dot{f}(0)$ is the value of $df(t)/dt$ evaluated at $t = 0$. To derive this equation, define

$$\frac{d}{dt}f(t) = g(t)$$

Then

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = \mathcal{L}\left[\frac{d}{dt}g(t)\right] = s\mathcal{L}[g(t)] - g(0)$$
$$= s\mathcal{L}\left[\frac{d}{dt}f(t)\right] - \dot{f}(0)$$
$$= s^2F(s) - sf(0) - \dot{f}(0)$$

Similarly, for the nth derivative of $f(t)$, we obtain

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - \overset{(n-1)}{f}(0)$$

where $f(0), \dot{f}(0), \ldots, \overset{(n-1)}{f}(0)$ represent the values of $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t = 0$. If the distinction between $\mathcal{L}_+$ and $\mathcal{L}_-$ is necessary, we substitute $t = 0+$ or $t = 0-$ into $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, depending on whether we take $\mathcal{L}_+$ or $\mathcal{L}_-$.

Note that, for Laplace transforms of derivatives of $f(t)$ to exist, $d^nf(t)/dt^n$ $(n = 1, 2, 3, \ldots)$ must be Laplace transformable.

Note also that, if all the initial values of $f(t)$ and its derivatives are equal to zero, then the Laplace transform of the nth derivative of $f(t)$ is given by $s^nF(s)$.

Final-value theorem. The final-value theorem relates the steady-state behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$. The theorem, however, applies if and only if $\lim_{t\to\infty} f(t)$ exists [which means that $f(t)$ settles down to a definite value as $t \to \infty$]. If all poles of $sF(s)$ lie in the left half s plane, then $\lim_{t\to\infty} f(t)$ exists, but if $sF(s)$ has poles on the imaginary axis or in the right half s plane, $f(t)$ will contain oscillating or exponentially increasing time functions, respectively, and $\lim_{t\to\infty} f(t)$ will not exist. The final-value theorem does not apply to such cases. For instance, if $f(t)$ is a sinusoidal function $\sin \omega t$, then $sF(s)$ has poles at $s = \pm j\omega$, and $\lim_{t\to\infty} f(t)$ does not exist. Therefore, the theorem is not applicable to such a function.

The final-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are Laplace transformable, if $F(s)$ is the Laplace transform of $f(t)$, and if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

To prove the theorem, we let s approach zero in the equation for the Laplace transform of the derivative of $f(t)$, or

$$\lim_{s\to 0}\int_0^\infty \left[\frac{d}{dt}f(t)\right]e^{-st}\,dt = \lim_{s\to 0}[sF(s) - f(0)]$$

Since $\lim_{s\to 0} e^{-st} = 1$, if $\lim_{t\to\infty} f(t)$ exists, then we obtain

$$\int_0^\infty \left[\frac{d}{dt}f(t)\right]dt = f(t)\Big|_0^\infty = f(\infty) - f(0)$$
$$= \lim_{s\to 0} sF(s) - f(0)$$

from which it follows that

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Initial-value theorem. The initial-value theorem is the counterpart of the final-value theorem. Using the initial-value theorem, we are able to find the value of $f(t)$ at $t = 0+$ directly from the Laplace transform of $f(t)$. The theorem does not give the value of $f(t)$ at exactly $t = 0$, but rather gives it at a time slightly greater than zero.

The initial-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are both Laplace transformable and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0+) = \lim_{s\to\infty} sF(s)$$

To prove this theorem, we use the equation for the $\mathcal{L}_+$ transform of $df(t)/dt$:

$$\mathcal{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

For the time interval $0+ \leq t \leq \infty$, as s approaches infinity, e^{-st} approaches zero. (Note that we must use $\mathcal{L}_+$ rather than $\mathcal{L}_-$ for this condition.) Hence,

$$\lim_{s\to\infty}\int_{0+}^\infty \left[\frac{d}{dt}f(t)\right]e^{-st}\,dt = \lim_{s\to\infty}[sF(s) - f(0+)] = 0$$

or

$$f(0+) = \lim_{s\to\infty} sF(s)$$

In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus, the theorem is valid for the sinusoidal function.

Note that the initial-value theorem and the final-value theorem provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

Integration theorem. If $f(t)$ is of exponential order, then the Laplace transform of $\int f(t)\,dt$ exists and is given by

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} \tag{2–4}$$

where $F(s) = \mathcal{L}[f(t)]$ and $f^{-1}(0) = \int f(t)\,dt$, evaluated at $t = 0$. Equation (2–4) is called the integration theorem.

The integration theorem can be proven as follows: Integration by parts yields

$$\mathcal{L}\left[\int f(t)\,dt\right] = \int_0^\infty \left[\int f(t)\,dt\right]e^{-st}\,dt$$
$$= \left[\int f(t)\,dt\right]\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}\,dt$$

$$= \frac{1}{s}\int f(t)\,dt\Big|_{t=0} + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$
$$= \frac{f^{-1}(0)}{s} + \frac{F(s)}{s}$$

and the theorem is proven.

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $f^{-1}(0+) \neq f^{-1}(0-)$. So if $f(t)$ involves an impulse function at $t = 0$, we must modify Equation (2–4) as follows:

$$\mathcal{L}_+\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}$$
$$\mathcal{L}_-\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0-)}{s}$$

We see that integration in the time domain is converted into division in the s domain. If the initial value of the integral is zero, the Laplace transform of the integral of $f(t)$ is given by $F(s)/s$.

The integration theorem can be modified slightly to deal with the definite integral of $f(t)$. If $f(t)$ is of exponential order, the Laplace transform of the definite integral $\int_0^t f(t)\,dt$ can be given by

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2–5}$$

To prove Equation (2–5), first note that

$$\int_0^t f(t)\,dt = \int f(t)\,dt - f^{-1}(0)$$

where $f^{-1}(0)$ is equal to $\int f(t)\,dt$, evaluated at $t = 0$, and is a constant. Hence,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \mathcal{L}\left[\int f(t)\,dt - f^{-1}(0)\right]$$
$$= \mathcal{L}\left[\int f(t)\,dt\right] - \mathcal{L}[f^{-1}(0)]$$

Referring to Equation (2–4) and noting that $f^{-1}(0)$ is a constant, so that

$$\mathcal{L}[f^{-1}(0)] = \frac{f^{-1}(0)}{s}$$

we obtain

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} - \frac{f^{-1}(0)}{s} = \frac{F(s)}{s}$$

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $\int_{0+}^t f(t)\,dt \neq \int_{0-}^t f(t)\,dt$, and the following distinction must be observed:

$$\mathcal{L}_+\left[\int_{0+}^t f(t)\,dt\right] = \frac{\mathcal{L}_+[f(t)]}{s}$$
$$\mathcal{L}_-\left[\int_{0-}^t f(t)\,dt\right] = \frac{\mathcal{L}_-[f(t)]}{s}$$

2–4 INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transformation refers to the process of finding the time function $f(t)$ from the corresponding Laplace transform $F(s)$. Several methods are available for finding inverse Laplace transforms. The simplest of these methods are (1) to use tables of Laplace transforms to find the time function $f(t)$ corresponding to a given Laplace transform $F(s)$ and (2) to use the partial-fraction expansion method. In this section, we present the latter technique. [Note that MATLAB is quite useful in obtaining the partial-fraction expansion of the ratio of two polynomials, $B(s)/A(s)$. We shall discuss the MATLAB approach to the partial-fraction expansion in Chapter 4.]

Partial-fraction expansion method for finding inverse Laplace transforms. If $F(s)$, the Laplace transform of $f(t)$, is broken up into components, or

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

and if the inverse Laplace transforms of $F_1(s), F_2(s), \ldots, F_n(s)$ are readily available, then

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)]$$
$$= f_1(t) + f_2(t) + \cdots + f_n(t)$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are the inverse Laplace transforms of $F_1(s), F_2(s), \cdots, F_n(s)$, respectively. The inverse Laplace transform of $F(s)$ thus obtained is unique, except possibly at points where the time function is discontinuous. Whenever the time function is continuous, the time function $f(t)$ and its Laplace transform $F(s)$ have a one-to-one correspondence.

For problems in systems analysis, $F(s)$ frequently occurs in the form

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are polynomials in s and the degree of $B(s)$ is not higher than that of $A(s)$.

The advantage of the partial-fraction expansion approach is that the individual terms of $F(s)$ resulting from the expansion into partial-fraction form are very simple functions of s; consequently, it is not necessary to refer to a Laplace transform table if we memorize several simple Laplace transform pairs. Note, however, that in applying the partial-fraction expansion technique in the search for the inverse Laplace transform of $F(s) = B(s)/A(s)$, the roots of the denominator polynomial $A(s)$ must be known in advance. That is, this method does not apply until the denominator polynomial has been factored.

Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ are either real or complex quantities, but for each complex p_i or z_i, there will occur the complex conjugate of p_i or z_i, respectively. Here, the highest power of s in $A(s)$ is assumed to be higher than that in $B(s)$.

In the expansion of $B(s)/A(s)$ into partial-fraction form, it is important that the highest power of s in $A(s)$ be greater than the highest power of s in $B(s)$ because if that is not the case, then the numerator $B(s)$ must be divided by the denominator $A(s)$ in order to produce a polynomial in s plus a remainder (a ratio of polynomials in s whose numerator is of lower degree than the denominator). (For details, see **Example 2–2.**)

Partial-fraction expansion when $F(s)$ involves distinct poles only. In this case, $F(s)$ can always be expanded into a sum of simple partial fractions; that is,

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \tag{2–6}$$

where $a_k (k = 1, 2, \ldots, n)$ are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (2–6) by $(s + p_k)$ and letting $s = -p_k$, giving

$$\begin{aligned}\left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1}(s + p_k) + \frac{a_2}{s + p_2}(s + p_k) + \cdots \right.\\ &\left. \quad + \frac{a_k}{s + p_k}(s + p_k) + \cdots + \frac{a_n}{s + p_n}(s + p_k)\right]_{s=-p_k}\\ &= a_k\end{aligned}$$

We see that all the expanded terms drop out, with the exception of a_k. Thus, the residue a_k is found from

$$a_k = \left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{2–7}$$

Note that since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2, need be evaluated, because the other is known automatically.

Since

$$\mathcal{L}^{-1}\left[\frac{a_k}{s + p_k}\right] = a_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t} \qquad t \geq 0$$

Example 2–1

Find the inverse Laplace transform of

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}$$

where a_1 and a_2 are found by using Equation (2–7):

$$a_1 = \left[(s + 1)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-1} = \left[\frac{s + 3}{s + 2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s + 2)\frac{s + 3}{(s + 1)(s + 2)}\right]_{s=-2} = \left[\frac{s + 3}{s + 1}\right]_{s=-2} = -1$$

Thus,

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}[F(s)]\\ &= \mathcal{L}^{-1}\left[\frac{2}{s + 1}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s + 2}\right]\\ &= 2e^{-t} - e^{-2t} \qquad t \geq 0\end{aligned}$$

Example 2–2

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator:

$$G(s) = s + 2 + \frac{s + 3}{(s + 1)(s + 2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is unity and that the Laplace transform of $d\delta(t)/dt$ is s. The third term on the right-hand side of this last equation is $F(s)$ in **Example 2–1.** So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \qquad t \geq 0-$$

Comment. Consider a function $F(s)$ that involves a quadratic factor $s^2 + as + b$ in the denominator. If this quadratic expression has a pair of complex-conjugate roots, then it is better not to factor the quadratic, in order to avoid complex numbers. For example, if $F(s)$ is given as

$$F(s) = \frac{p(s)}{s(s^2 + as + b)}$$

where $a \geq 0$ and $b > 0$, and if $s^2 + as + b = 0$ has a pair of complex-conjugate roots, then expand $F(s)$ into the following partial-fraction expansion form:

$$F(s) = \frac{c}{s} + \frac{ds + e}{s^2 + as + b}$$

(See **Example 2–3** and **Problems A–2–15, A–2–16,** and **A–2–19.**)

Example 2–3

Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

The two roots of the denominator are complex conjugates. Hence, we expand $F(s)$ into the sum of a damped sine and a damped cosine function.

Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t}\sin \omega t$ and $e^{-\alpha t}\cos \omega t$, rewritten as

$$\mathcal{L}[e^{-\alpha t}\sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

and

$$\mathcal{L}[e^{-\alpha t}\cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

we can write the given $F(s)$ as a sum of a damped sine and a damped cosine function:

$$\begin{aligned}F(s) &= \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2}\\ &= 5\frac{2}{(s + 1)^2 + 2^2} + 2\frac{s + 1}{(s + 1)^2 + 2^2}\end{aligned}$$

It follows that

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}[F(s)]\\ &= 5\mathcal{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right]\\ &= 5e^{-t}\sin 2t + 2e^{-t}\cos 2t \qquad t \geq 0\end{aligned}$$

Partial-fraction expansion when $F(s)$ involves multiple poles. Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of $F(s)$. (See also **Problems A–2–17** and **A–2–19.**)

Consider

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms:

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_3}{(s + 1)^3} + \frac{b_2}{(s + 1)^2} + \frac{b_1}{s + 1}$$

where b_3, b_2, and b_1 are determined as follows: Multiplying both sides of this last equation by $(s + 1)^3$, we have

$$(s + 1)^3\frac{B(s)}{A(s)} = b_3 + b_2(s + 1) + b_1(s + 1)^2 \tag{2–8}$$

Then, letting $s = -1$, we find that Equation (2–8) gives

$$\left[(s + 1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

Also, differentiating both sides of Equation (2–8) with respect to s yields

$$\frac{d}{ds}\left[(s + 1)^3\frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s + 1) \tag{2–9}$$

If we let $s = -1$ in Equation (2–9), then

$$\frac{d}{ds}\left[(s + 1)^3\frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

Differentiating both sides of Equation (2–9) with respect to s, we obtain

$$\frac{d^2}{ds^2}\left[(s + 1)^3\frac{B(s)}{A(s)}\right] = 2b_1$$

From the preceding analysis, it can be seen that the values of b_3, b_2, and b_1 are found systematically as follows:

$$\begin{aligned}b_3 &= \left[(s + 1)^3\frac{B(s)}{A(s)}\right]_{s=-1}\\ &= (s^2 + 2s + 3)_{s=-1}\\ &= 2\\ b_2 &= \left\{\frac{d}{ds}\left[(s + 1)^3\frac{B(s)}{A(s)}\right]\right\}_{s=-1}\\ &= \left[\frac{d}{ds}(s^2 + 2s + 3)\right]_{s=-1}\\ &= (2s + 2)_{s=-1}\\ &= 0\\ b_1 &= \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s + 1)^3\frac{B(s)}{A(s)}\right]\right\}_{s=-1}\\ &= \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2 + 2s + 3)\right]_{s=-1}\\ &= \frac{1}{2}(2) = 1\end{aligned}$$

We thus obtain

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] \\
&= t^2e^{-t} + 0 + e^{-t} \\
&= (t^2+1)e^{-t} \qquad t \geq 0
\end{aligned}
$$

2–5 SOLVING LINEAR, TIME-INVARIANT DIFFERENTIAL EQUATIONS

In this section, we are concerned with the use of the Laplace transform method in solving linear, time-invariant differential equations.

The Laplace transform method yields the complete solution (complementary solution and particular solution) of linear, time-invariant differential equations. Classical methods for finding the complete solution of a differential equation require the evaluation of the integration constants from the initial conditions. In the case of the Laplace transform method, however, this requirement is unnecessary because the initial conditions are automatically included in the Laplace transform of the differential equation.

If all initial conditions are zero, then the Laplace transform of the differential equation is obtained simply by replacing d/dt with s, d^2/dt^2 with s^2, and so on.

In solving linear, time-invariant differential equations by the Laplace transform method, two steps are followed:

1. By taking the Laplace transform of each term in the given differential equation, convert the differential equation into an algebraic equation in s and obtain the expression for the Laplace transform of the dependent variable by rearranging the algebraic equation.
2. The time solution of the differential equation is obtained by finding the inverse Laplace transform of the dependent variable.

In the discussion that follows, two examples are used to demonstrate the solution of linear, time-invariant differential equations by the Laplace transform method.

Example 2–4

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \qquad x(0) = a, \qquad \dot{x}(0) = b$$

where a and b are constants.

Writing the Laplace transform of $x(t)$ as $X(s)$, or

$$\mathcal{L}[x(t)] = X(s)$$

we obtain

$$
\begin{aligned}
\mathcal{L}[\dot{x}] &= sX(s) - x(0) \\
\mathcal{L}[\ddot{x}] &= s^2X(s) - sx(0) - \dot{x}(0)
\end{aligned}
$$

The Laplace transform of the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 3[sX(s) - x(0)] + 2X(s) = 0$$

Substituting the given initial conditions into the preceding equation yields

$$[s^2X(s) - as - b] + 3[sX(s) - a] + 2X(s) = 0$$

or

$$(s^2 + 3s + 2)X(s) = as + b + 3a$$

Solving this last equation for $X(s)$, we have

$$X(s) = \frac{as+b+3a}{s^2+3s+2} = \frac{as+b+3a}{(s+1)(s+2)} = \frac{2a+b}{s+1} - \frac{a+b}{s+2}$$

The inverse Laplace transform of $X(s)$ produces

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{2a+b}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{a+b}{s+2}\right] \\
&= (2a+b)e^{-t} - (a+b)e^{-2t} \qquad t \geq 0
\end{aligned}
$$

which is the solution of the given differential equation. Notice that the initial conditions a and b appear in the solution. Thus, $x(t)$ has no undetermined constants.

Example 2–5

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Noting that $\mathcal{L}[3] = 3/s$, $x(0) = 0$, and $\dot{x}(0) = 0$, we see that the Laplace transform of the differential equation becomes

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

Solving this equation for $X(s)$, we obtain

$$
\begin{aligned}
X(s) &= \frac{3}{s(s^2+2s+5)} \\
&= \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5} \\
&= \frac{3}{5}\frac{1}{s} - \frac{3}{10}\frac{2}{(s+1)^2+2^2} - \frac{3}{5}\frac{s+1}{(s+1)^2+2^2}
\end{aligned}
$$

Hence, the inverse Laplace transform becomes

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}[X(s)] \\
&= \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{3}{5}\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] \\
&= \frac{3}{5} - \frac{3}{10}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t \qquad t \geq 0
\end{aligned}
$$

which is the solution of the given differential equation.

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–2–1

Obtain the real and imaginary parts of

$$\frac{2+j1}{3+j4}$$

Also, obtain the magnitude and angle of this complex quantity.

Solution

$$
\begin{aligned}
\frac{2+j1}{3+j4} &= \frac{(2+j1)(3-j4)}{(3+j4)(3-j4)} = \frac{6+j3-j8+4}{9+16} = \frac{10-j5}{25} \\
&= \frac{2}{5} - j\frac{1}{5}
\end{aligned}
$$

Hence,

$$\text{real part} = \frac{2}{5}, \qquad \text{imaginary part} = -j\frac{1}{5}$$

The magnitude and angle of this complex quantity are obtained as follows:

$$\text{magnitude} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}} = 0.447$$

$$\text{angle} = \tan^{-1}\frac{-1/5}{2/5} = \tan^{-1}\frac{-1}{2} = -26.565°$$

Problem A–2–2

Find the Laplace transform of

$$
\begin{aligned}
f(t) &= 0 \qquad &t < 0 \\
&= te^{-3t} \qquad &t \geq 0
\end{aligned}
$$

Solution Since

$$\mathcal{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2–2), we obtain

$$F(s) = \mathcal{L}[te^{-3t}] = G(s+3) = \frac{1}{(s+3)^2}$$

Problem A–2–3

What is the Laplace transform of

$$
\begin{aligned}
f(t) &= 0 \qquad &t < 0 \\
&= \sin(\omega t + \theta) \qquad &t \geq 0
\end{aligned}
$$

where θ is a constant?

Solution Noting that

$$\sin(\omega t + \theta) = \sin\omega t\cos\theta + \cos\omega t\sin\theta$$

we have

$$
\begin{aligned}
\mathcal{L}[\sin(\omega t + \theta)] &= \cos\theta\,\mathcal{L}[\sin\omega t] + \sin\theta\,\mathcal{L}[\cos\omega t] \\
&= \cos\theta\frac{\omega}{s^2+\omega^2} + \sin\theta\frac{s}{s^2+\omega^2} \\
&= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}
\end{aligned}
$$

Problem A–2–4

Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2–9. Also, find the limiting value of $F(s)$ as a approaches zero.

Solution The function $f(t)$ can be written

$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t-a) + \frac{1}{a^2}1(t-2a)$$

Then

$$
\begin{aligned}
F(s) &= \mathcal{L}[f(t)] \\
&= \frac{1}{a^2}\mathcal{L}[1(t)] - \frac{2}{a^2}\mathcal{L}[1(t-a)] + \frac{1}{a^2}\mathcal{L}[1(t-2a)] \\
&= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as} \\
&= \frac{1}{a^2s}(1 - 2e^{-as} + e^{-2as})
\end{aligned}
$$

As a approaches zero, we have

$$
\begin{aligned}
\lim_{a\to 0}F(s) &= \lim_{a\to 0}\frac{1-2e^{-as}+e^{-2as}}{a^2s} = \lim_{a\to 0}\frac{\dfrac{d}{da}(1-2e^{-as}+e^{-2as})}{\dfrac{d}{da}(a^2s)} \\
&= \lim_{a\to 0}\frac{2se^{-as}-2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as}-e^{-2as}}{a}
\end{aligned}
$$

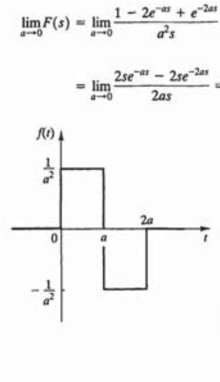


Figure 2–9 Function $f(t)$.

$$= \lim_{a \to 0} \frac{\dfrac{d}{da}(e^{-as} - e^{-2as})}{\dfrac{d}{da}(a)} = \lim_{a \to 0} \frac{-se^{-as} + 2se^{-2as}}{1}$$
$$= -s + 2s = s$$

Problem A-2-5

Obtain the Laplace transform of the function $f(t)$ shown in Figure 2-10.

Solution The given function $f(t)$ can be defined as follows:

$$\begin{aligned} f(t) &= 0 && t \le 0 \\ &= \frac{b}{a}t && 0 < t \le a \\ &= 0 && a < t \end{aligned}$$

Notice that $f(t)$ can be considered a sum of the three functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ shown in Figure 2-11. Hence, $f(t)$ can be written as

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ &= \frac{b}{a}t \cdot 1(t) - \frac{b}{a}(t - a) \cdot 1(t - a) - b \cdot 1(t - a) \end{aligned}$$

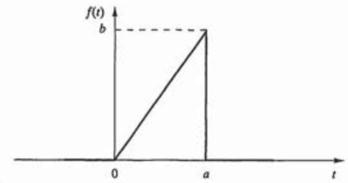


Figure 2-10 Function $f(t)$.

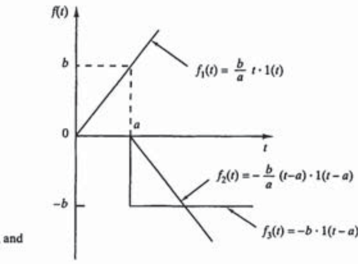


Figure 2-11 Functions $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Then the Laplace transform of $f(t)$ becomes

$$\begin{aligned} F(s) &= \frac{b}{a}\frac{1}{s^2} - \frac{b}{a}\frac{1}{s^2}e^{-as} - b\frac{1}{s}e^{-as} \\ &= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as} \end{aligned}$$

The same $F(s)$ can, of course, be obtained by performing the following Laplace integration:

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^a \frac{b}{a}te^{-st}\,dt + \int_a^\infty 0\,e^{-st}\,dt \\ &= \frac{b}{a}t\frac{e^{-st}}{-s}\bigg|_0^a - \int_0^a \frac{b}{a}\frac{e^{-st}}{-s}\,dt \\ &= b\frac{e^{-as}}{-s} + \frac{b}{as}\frac{e^{-st}}{-s}\bigg|_0^a \\ &= b\frac{e^{-as}}{-s} - \frac{b}{as^2}(e^{-as} - 1) \\ &= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as} \end{aligned}$$

Problem A-2-6

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then, except at poles of $F(s)$,

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$
$$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$$

and in general,

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Solution

$$\begin{aligned} \mathcal{L}[tf(t)] &= \int_0^\infty tf(t)e^{-st}\,dt = -\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = -\frac{d}{ds}F(s) \end{aligned}$$

Similarly, by defining $tf(t) = g(t)$, the result is

$$\begin{aligned} \mathcal{L}[t^2f(t)] &= \mathcal{L}[tg(t)] = -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right] \\ &= (-1)^2\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}F(s) \end{aligned}$$

Repeating the same process, we obtain

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Problem A-2-7

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= t^2 \sin \omega t && t \ge 0 \end{aligned}$$

Solution Since

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

referring to **Problem A-2-6**, we have

$$\mathcal{L}[f(t)] = \mathcal{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

Problem A-2-8

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as) \qquad a > 0$$

Solution If we define $t/a = \tau$ and $as = s_1$, then

$$\begin{aligned} \mathcal{L}\left[f\left(\frac{t}{a}\right)\right] &= \int_0^\infty f\left(\frac{t}{a}\right)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-as\tau}a\,d\tau \\ &= a\int_0^\infty f(\tau)e^{-s_1\tau}\,d\tau = aF(s_1) = aF(as) \end{aligned}$$

Problem A-2-9

Prove that if $f(t)$ is of exponential order and if $\int_0^\infty f(t)\,dt$ exists [which means that $\int_0^\infty f(t)\,dt$ assumes a definite value], then

$$\int_0^\infty f(t)\,dt = \lim_{s \to 0} F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

Solution Note that

$$\int_0^\infty f(t)\,dt = \lim_{t \to \infty}\int_0^t f(t)\,dt$$

Referring to Equation (2-5), we have

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Since $\int_0^\infty f(t)\,dt$ exists, by applying the final-value theorem to this case, we obtain

$$\lim_{t \to \infty}\int_0^t f(t)\,dt = \lim_{s \to 0} s\frac{F(s)}{s}$$

or

$$\int_0^\infty f(t)\,dt = \lim_{s \to 0} F(s)$$

Problem A-2-10

The convolution of two time functions is defined by

$$\int_0^t f_1(\tau)f_2(t - \tau)\,d\tau$$

A commonly used notation for the convolution is $f_1(t)*f_2(t)$, which is defined as

$$f_1(t)*f_2(t) = \int_0^t f_1(\tau)f_2(t - \tau)\,d\tau = \int_0^t f_1(t - \tau)f_2(\tau)\,d\tau$$

Show that if $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then

$$\mathcal{L}\left[\int_0^t f_1(\tau)f_2(t - \tau)\,d\tau\right] = F_1(s)F_2(s)$$

where $F_1(s) = \mathcal{L}[f_1(t)]$ and $F_2(s) = \mathcal{L}[f_2(t)]$.

Solution Noting that $1(t - \tau) = 0$ for $t < \tau$, we have

$$\begin{aligned} \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t - \tau)\,d\tau\right] &= \mathcal{L}\left[\int_0^\infty f_1(\tau)f_2(t - \tau)1(t - \tau)\,d\tau\right] \\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f_1(\tau)f_2(t - \tau)1(t - \tau)\,d\tau\right]dt \\ &= \int_0^\infty f_1(\tau)\,d\tau\int_0^\infty f_2(t - \tau)1(t - \tau)e^{-st}\,dt \end{aligned}$$

Changing the order of integration is valid here, since $f_1(t)$ and $f_2(t)$ are both Laplace transformable, giving convergent integrals. If we substitute $\lambda = t - \tau$ into this last equation, the result is

$$\begin{aligned} \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t - \tau)\,d\tau\right] &= \int_0^\infty f_1(\tau)e^{-s\tau}\,d\tau\int_0^\infty f_2(\lambda)e^{-s\lambda}\,d\lambda \\ &= F_1(s)F_2(s) \end{aligned}$$

or

$$\mathcal{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s)$$

Thus, the Laplace transform of the convolution of two time functions is the product of their Laplace transforms.

Problem A-2-11

Determine the Laplace transform of $f_1(t)*f_2(t)$, where

$$\begin{aligned} f_1(t) &= f_2(t) = 0 && \text{for } t < 0 \\ f_1(t) &= t && \text{for } t \ge 0 \\ f_2(t) &= 1 - e^{-t} && \text{for } t \ge 0 \end{aligned}$$

Solution Note that

$$\mathcal{L}[t] = F_1(s) = \frac{1}{s^2}$$
$$\mathcal{L}[1 - e^{-t}] = F_2(s) = \frac{1}{s} - \frac{1}{s + 1}$$

The Laplace transform of the convolution integral is given by

$$\mathscr{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$
$$= \frac{1}{s^3} - \frac{1}{s^2(s+1)} = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

To verify that the expression after the rightmost equal sign is indeed the Laplace transform of the convolution integral, let us first integrate the convolution integral and then take the Laplace transform of the result. We have

$$f_1(t)*f_2(t) = \int_0^t \tau[1 - e^{-(t-\tau)}]\,d\tau$$
$$= \int_0^t (t-\tau)(1-e^{-\tau})\,d\tau$$
$$= \int_0^t (t - \tau - te^{-\tau} + \tau e^{-\tau})\,d\tau$$

Noting that

$$\int_0^t (t-\tau)\,d\tau = \frac{t^2}{2}$$
$$\int_0^t te^{-\tau}\,d\tau = -te^{-t} + t$$
$$\int_0^t \tau e^{-\tau}\,d\tau = -\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau}\,d\tau = -te^{-t} - e^{-t} + 1$$

we have

$$f_1(t)*f_2(t) = \frac{t^2}{2} - t + 1 - e^{-t}$$

Thus,

$$\mathscr{L}\left[\frac{t^2}{2} - t + 1 - e^{-t}\right] = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

Problem A–2–12

Prove that if $f(t)$ is a periodic function with period T, then

$$\mathscr{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

Solution

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$$

By changing the independent variable from t to $\tau = t - nT$, we obtain

$$\mathscr{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau + nT)e^{-s\tau}\,d\tau$$

Since $f(t)$ is a periodic function with period T, $f(\tau + nT) = f(\tau)$. Hence,

$$\mathscr{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\int_0^T f(\tau)e^{-s\tau}\,d\tau$$

Noting that

$$\sum_{n=0}^{\infty} e^{-nTs} = 1 + e^{-Ts} + e^{-2Ts} + \cdots$$
$$= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \cdots)$$
$$= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)$$

we obtain

$$\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}$$

It follows that

$$\mathscr{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}}$$

Problem A–2–13

What is the Laplace transform of the periodic function shown in Figure 2–12?

Solution Note that

$$\int_0^T f(t)e^{-st}\,dt = \int_0^{T/2} e^{-st}\,dt + \int_{T/2}^T (-1)e^{-st}\,dt$$
$$= \frac{e^{-st}}{-s}\Big|_0^{T/2} - \frac{e^{-st}}{-s}\Big|_{T/2}^{T}$$
$$= \frac{e^{-(1/2)Ts} - 1}{-s} + \frac{e^{-Ts} - e^{-(1/2)Ts}}{s}$$
$$= \frac{1}{s}[e^{-Ts} - 2e^{-(1/2)Ts} + 1]$$
$$= \frac{1}{s}[1 - e^{-(1/2)Ts}]^2$$

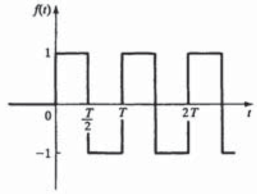


Figure 2–12 Periodic function (square wave).

Consequently,

$$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}} = \frac{(1/s)[1 - e^{-(1/2)Ts}]^2}{1 - e^{-Ts}}$$
$$= \frac{1 - e^{-(1/2)Ts}}{s[1 + e^{-(1/2)Ts}]} = \frac{1}{s}\tanh\frac{Ts}{4}$$

Problem A–2–14

Find the initial value of $df(t)/dt$, where the Laplace transform of $f(t)$ is given by

$$F(s) = \mathscr{L}[f(t)] = \frac{2s+1}{s^2 + s + 1}$$

Solution Using the initial-value theorem, we obtain

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s(2s+1)}{s^2 + s + 1} = 2$$

Since the $\mathscr{L}_+$ transform of $df(t)/dt = g(t)$ is given by

$$\mathscr{L}_+[g(t)] = sF(s) - f(0+)$$
$$= \frac{s(2s+1)}{s^2 + s + 1} - 2 = \frac{-s-2}{s^2 + s + 1}$$

the initial value of $df(t)/dt$ is obtained as

$$\lim_{t\to 0+}\frac{df(t)}{dt} = g(0+) = \lim_{s\to\infty} s[sF(s) - f(0+)]$$
$$= \lim_{s\to\infty}\frac{-s^2 - 2s}{s^2 + s + 1} = -1$$

To verify this result, notice that

$$F(s) = \frac{2(s + 0.5)}{(s + 0.5)^2 + (0.866)^2} = \mathscr{L}[2e^{-0.5t}\cos 0.866t]$$

Hence,

$$f(t) = 2e^{-0.5t}\cos 0.866t$$

and

$$\dot{f}(t) = -e^{-0.5t}\cos 0.866t + 2e^{-0.5t}0.866\sin 0.866t$$

Thus,

$$\dot{f}(0) = -1 + 0 = -1$$

Problem A–2–15

Obtain the inverse Laplace transform of

$$F(s) = \frac{cs + d}{(s^2 + 2as + a^2) + b^2}$$

where a, b, c, and d are real and a is positive.

Solution Since $F(s)$ can be written as

$$F(s) = \frac{c(s+a) + d - ca}{(s+a)^2 + b^2}$$
$$= \frac{c(s+a)}{(s+a)^2 + b^2} + \frac{d - ca}{b}\,\frac{b}{(s+a)^2 + b^2}$$

we obtain

$$f(t) = ce^{-at}\cos bt + \frac{d - ca}{b}e^{-at}\sin bt$$

Problem A–2–16

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Solution Since

$$s^2 + 2s + 2 = (s + 1 + j1)(s + 1 - j1)$$

it follows that $F(s)$ involves a pair of complex-conjugate poles, so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2s + 2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2 + 2s + 2) + (a_2 s + a_3)s$$

By comparing corresponding coefficients of the s^2, s, and s^0 terms on both sides of this last equation respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which it follows that

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{s^2 + 2s + 2}$$
$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+1)^2 + 1^2} - \frac{1}{2}\frac{s+1}{(s+1)^2 + 1^2}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t \qquad t \geq 0$$

Problem A–2–17

Derive the inverse Laplace transform of

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

Solution

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

where

$$a_1 = \frac{5(s+2)}{s^2(s+3)}\bigg|_{s=-1} = \frac{5}{2}$$

$$a_2 = \frac{5(s+2)}{s^2(s+1)}\bigg|_{s=-3} = \frac{5}{18}$$

$$b_2 = \frac{5(s+2)}{(s+1)(s+3)}\bigg|_{s=0} = \frac{10}{3}$$

$$b_1 = \frac{d}{ds}\left[\frac{5(s+2)}{(s+1)(s+3)}\right]_{s=0}$$
$$= \frac{5(s+1)(s+3) - 5(s+2)(2s+4)}{(s+1)^2(s+3)^2}\bigg|_{s=0} = -\frac{25}{9}$$

Thus,

$$F(s) = \frac{10}{3}\frac{1}{s^2} - \frac{25}{9}\frac{1}{s} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{10}{3}t - \frac{25}{9} + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t} \qquad t \geq 0$$

Problem A–2–18

Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

Solution Since the numerator polynomial is of higher degree than the denominator polynomial, by dividing the numerator by the denominator until the remainder is a fraction, we obtain

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)} = s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where

$$a_1 = \frac{2s+5}{s+1}\bigg|_{s=0} = 5$$

$$a_2 = \frac{2s+5}{s}\bigg|_{s=-1} = -3$$

It follows that

$$F(s) = s^2 + s + 2 + \frac{5}{s} - \frac{3}{s+1}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t} \qquad t \geq 0-$$

Problem A–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 6}{s^2(s^2 + 2s + 10)} \tag{2–10}$$

Solution Since the quadratic term in the denominator involves a pair of complex-conjugate roots, we expand $F(s)$ into the following partial-fraction form:

$$F(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10}$$

The coefficient a_1 can be obtained as

$$a_1 = \frac{2s^2 + 4s + 6}{s^2 + 2s + 10}\bigg|_{s=0} = 0.6$$

Hence, we obtain

$$F(s) = \frac{0.6}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10}$$
$$= \frac{(a_2 + b)s^3 + (0.6 + 2a_2 + c)s^2 + (1.2 + 10a_2)s + 6}{s^2(s^2 + 2s + 10)} \tag{2–11}$$

By equating corresponding coefficients in the numerators of Equations (2–10) and (2–11), respectively, we obtain

$$a_2 + b = 0$$
$$0.6 + 2a_2 + c = 2$$
$$1.2 + 10a_2 = 4$$

from which we get

$$a_2 = 0.28, \qquad b = -0.28, \qquad c = 0.84$$

Hence,

$$F(s) = \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28s + 0.84}{s^2 + 2s + 10}$$
$$= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28(s+1) + (1.12/3) \times 3}{(s+1)^2 + 3^2}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = 0.6t + 0.28 - 0.28e^{-t}\cos 3t + \frac{1.12}{3}e^{-t}\sin 3t$$

Problem A–2–20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

Solution

$$F(s) = \frac{1}{s(s^2 + \omega^2)} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega^2}\right)$$
$$= \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2 + \omega^2}$$

Thus, the inverse Laplace transform of $F(s)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{\omega^2}(1 - \cos \omega t) \qquad t \geq 0$$

Problem A–2–21

Obtain the solution of the differential equation

$$\dot{x} + ax = A\sin \omega t, \qquad x(0) = b$$

Solution Laplace transforming both sides of this differential equation, we have

$$[sX(s) - x(0)] + aX(s) = A\frac{\omega}{s^2 + \omega^2}$$

or

$$(s + a)X(s) = \frac{A\omega}{s^2 + \omega^2} + b$$

Solving this last equation for $X(s)$, we obtain

$$X(s) = \frac{A\omega}{(s+a)(s^2 + \omega^2)} + \frac{b}{s+a}$$
$$= \frac{A\omega}{a^2 + \omega^2}\left(\frac{1}{s+a} - \frac{s-a}{s^2 + \omega^2}\right) + \frac{b}{s+a}$$
$$= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)\frac{1}{s+a} + \frac{Aa}{a^2 + \omega^2}\frac{\omega}{s^2 + \omega^2} - \frac{A\omega}{a^2 + \omega^2}\frac{s}{s^2 + \omega^2}$$

The inverse Laplace transform of $X(s)$ then gives

$$x(t) = \mathscr{L}^{-1}[X(s)]$$
$$= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)e^{-at} + \frac{Aa}{a^2 + \omega^2}\sin \omega t - \frac{A\omega}{a^2 + \omega^2}\cos \omega t \qquad t \geq 0$$

PROBLEMS

Problem B–2–1

Derive the Laplace transform of the function

$$f(t) = 0 \qquad t < 0$$
$$= te^{-2t} \qquad t \geq 0$$

Problem B–2–2

Find the Laplace transforms of the following functions:

(a) $f_1(t) = 0 \qquad t < 0$
$\quad = 3\sin(5t + 45°) \qquad t \geq 0$

(b) $f_2(t) = 0 \qquad t < 0$
$\quad = 0.03(1 - \cos 2t) \qquad t \geq 0$

Problem B–2–3

Obtain the Laplace transform of the function defined by

$$f(t) = 0 \qquad t < 0$$
$$= t^2 e^{-at} \qquad t \geq 0$$

Problem B–2–4

Obtain the Laplace transform of the function

$$f(t) = 0 \qquad t < 0$$
$$= \cos 2\omega t \cdot \cos 3\omega t \qquad t \geq 0$$

Problem B–2–5

What is the Laplace transform of the function $f(t)$ shown in Figure 2–13?

Problem B–2–6

Obtain the Laplace transform of the pulse function $f(t)$ shown in Figure 2–14.

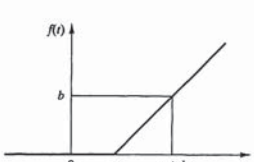


Figure 2–13 Function $f(t)$.

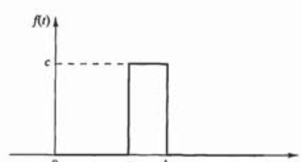


Figure 2–14 Pulse function.

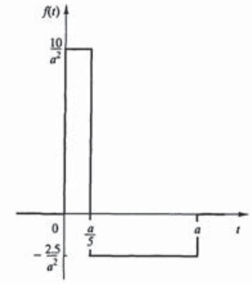


Figure 2–15 Function $f(t)$.

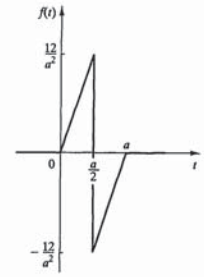


Figure 2–16 Function $f(t)$.

Problem B–2–7

What is the Laplace transform of the function $f(t)$ shown in Figure 2–15? Also, what is the limiting value of $\mathscr{L}[f(t)]$ as a approaches zero?

Problem B–2–8

Find the Laplace transform of the function $f(t)$ shown in Figure 2–16. Also, find the limiting value of $\mathscr{L}[f(t)]$ as a approaches zero.

Problem B–2–9

Given

$$F(s) = \frac{5(s+2)}{s(s+1)}$$

obtain $f(\infty)$. Use the final-value theorem.

Problem B–2–10

Given

$$F(s) = \frac{2(s+2)}{s(s+1)(s+3)}$$

obtain $f(0+)$. Use the initial-value theorem.

Problem B–2–11

Consider a function $x(t)$. Show that

$$\dot{x}(0+) = \lim_{s \to \infty} [s^2 X(s) - sx(0+)]$$

Problem B–2–12

Derive the Laplace transform of the third derivative of $f(t)$.

Problem B–2–13

What are the inverse Laplace transforms of the following functions?

(a) $$F_1(s) = \frac{s+5}{(s+1)(s+3)}$$

(b) $$F_2(s) = \frac{3(s+4)}{s(s+1)(s+2)}$$

Problem B–2–14

Find the inverse Laplace transforms of the following functions:

(a) $$F_1(s) = \frac{6s+3}{s^2}$$

(b) $$F_2(s) = \frac{5s+2}{(s+1)(s+2)^2}$$

Problem B–2–15

Find the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 5}{s(s+1)}$$

Problem B–2–16

Obtain the inverse Laplace transform of

$$F(s) = \frac{s^2 + 2s + 4}{s^2}$$

Problem B–2–17

Obtain the inverse Laplace transform of

$$F(s) = \frac{s}{s^2 + 2s + 10}$$

Problem B–2–18

Obtain the inverse Laplace transform of

$$F(s) = \frac{s^2 + 2s + 5}{s^2(s+1)}$$

Problem B–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s+10}{(s+1)^2(s+4)}$$

Problem B–2–20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)}$$

Problem B–2–21

Obtain the inverse Laplace transform of

$$F(s) = \frac{c}{s^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

where $a > 0$.

Problem B–2–22

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 4x = 0, \qquad x(0) = 5, \qquad \dot{x}(0) = 0$$

Problem B–2–23

Obtain the solution $x(t)$ of the differential equation

$$\ddot{x} + \omega_n^2 x = t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Problem B–2–24

Determine the solution $x(t)$ of the differential equation

$$2\ddot{x} + 2\dot{x} + x = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = 2$$

Problem B–2–25

Obtain the solution $x(t)$ of the differential equation

$$\ddot{x} + x = \sin 3t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

3

Mechanical Systems

3–1 INTRODUCTION

This chapter is an introductory account of mechanical systems. Details of mathematical modeling and response analyses of various mechanical systems are given in Chapters 4, 5, 7, 8, and 9.

We begin with a review of systems of units; a clear understanding of which is necessary for the quantitative study of system dynamics.

Systems of units. Most engineering calculations in the United States are based on the *International System* (abbreviated SI)[1] *of units* and the *British engineering system* (BES) of measurement. The International System is a modified metric system, and, as such, it differs from conventional metric absolute or metric gravitational systems of units. Table 3–1 lists some units of measure from each of the International System, conventional metric systems, and British systems of units. (The table presents only those units necessary to describe the behavior of mechanical systems. Units used in describing the behaviors of electrical systems are given in Chapter 6. For additional details on systems of units, refer to Appendix A.)

The chief difference between "absolute" systems of units and "gravitational" systems of units lies in the choice of mass or force as a primary dimension. In the

[1]This "backward" abbreviation is for the French *Système International*.

TABLE 3–1 Systems of Units

Systems of units / Quantity	Absolute systems				Gravitational systems	
	Metric			British	Metric engineering	British engineering
	SI	mks	cgs			
Length	m	m	cm	ft	m	ft
Mass	kg	kg	g	lb	$\frac{kg_f\text{-}s^2}{m}$	slug $= \frac{lb_f\text{-}s^2}{ft}$
Time	s	s	s	s	s	s
Force	N $= \frac{kg\text{-}m}{s^2}$	N $= \frac{kg\text{-}m}{s^2}$	dyn $= \frac{g\text{-}cm}{s^2}$	poundal $= \frac{lb\text{-}ft}{s^2}$	kg_f	lb_f
Energy	J = N-m	J = N-m	erg = dyn-cm	ft-poundal	kg_f-m	ft-lb_f or Btu
Power	W $= \frac{N\text{-}m}{s}$	W $= \frac{N\text{-}m}{s}$	$\frac{dyn\text{-}cm}{s}$	$\frac{ft\text{-}poundal}{s}$	$\frac{kg_f\text{-}m}{s}$	$\frac{ft\text{-}lb_f}{s}$ or hp

absolute systems (SI and the metric and British absolute systems), mass is chosen as a primary dimension and force is a derived quantity. Conversely, in gravitational systems (metric engineering and British engineering systems) of units, force is a primary dimension and mass is a derived quantity. In gravitational systems, the mass of a body is defined as the ratio of the magnitude of the force to that of acceleration. (Thus, the dimension of mass is force/acceleration.)

Mass. The *mass* of a body is the quantity of matter in it, which is assumed to be constant. Physically, mass is the property of a body that gives it inertia, that is, resistance to starting and stopping. A body is attracted by the earth, and the magnitude of the force that the earth exerts on the body is called its *weight*.

In practical situations, we know the weight w of a body, but not the mass m. We calculate mass m from

$$m = \frac{w}{g}$$

where g is the gravitational acceleration constant. The value of g varies slightly from point to point on the earth's surface. As a result, the weight of a body varies slightly at different points on the earth's surface, but its mass remains constant. For engineering purposes,

$$g = 9.807 \text{ m/s}^2 = 980.7 \text{ cm/s}^2 = 32.174 \text{ ft/s}^2 = 386.1 \text{ in./s}^2$$

Far out in space, a body becomes weightless. Yet its mass remains constant, so the body possesses inertia.

The units of mass are kg, g, lb, kg_f-s^2/m, and slug, as shown in Table 3–1. If mass is expressed in units of kilograms (or pounds), we call it kilogram mass (or pound mass) to distinguish it from the unit of force, which is termed kilogram force (or pound force). In this book, kg is used to denote a kilogram mass and kg_f a kilogram force. Similarly, lb denotes a pound mass and lb_f a pound force.

A slug is a unit of mass such that, when acted on by a 1-pound force, a 1-slug mass accelerates at 1 ft/s^2 (slug = lb_f-s^2/ft). In other words, if a mass of 1 slug is acted on by a 32.174-pound force, it accelerates at 32.174 ft/s^2 ($= g$). Hence, the mass of a body weighing 32.174 lb_f at the earth's surface is 1 slug, or

$$m = \frac{w}{g} = \frac{32.174 \text{ lb}_f}{32.174 \text{ ft/s}^2} = 1 \text{ slug}$$

Force. *Force* can be defined as the cause which tends to produce a change in motion of a body on which it acts. To move a body, force must be applied to it. Two types of forces are capable of acting on a body: *contact* forces and *field* forces. Contact forces are those which come into direct contact with a body, whereas field forces, such as gravitational force and magnetic force, act on a body, but do not come into contact with it.

The units of force are the newton (N), dyne (dyn), poundal, kg_f, and lb_f. In SI units and the mks system (a metric absolute system) of units, the force unit is the newton. One newton is the force that will give a 1-kg mass an acceleration of 1 m/s^2, or

$$1 \text{ N} = 1 \text{ kg-m/s}^2$$

This implies that 9.807 N will give a 1-kg mass an acceleration of 9.807 m/s^2. Since the gravitational acceleration constant is $g = 9.807$ m/s^2, a mass of 1 kg will produce a force of 9.807 N on its support.

The force unit in the cgs system (a metric absolute system) is the dyne, which will give a 1-g mass an acceleration of 1 cm/s^2, or

$$1 \text{ dyn} = 1 \text{ g-cm/s}^2$$

The force unit in the metric engineering (gravitational) system is kg_f, which is a primary dimension in the system. Similarly, in the British engineering system, the force unit is lb_f, a primary dimension in this system of units.

Comments. The SI units of force, mass, and length are the newton (N), kilogram mass (kg), and meter (m). The mks units of force, mass, and length are the same as the SI units. The cgs units for force, mass, and length are the dyne (dyn), gram (g), and centimeter (cm), and those for the BES units are the pound force (lb_f), slug, and foot (ft). Each system of units is consistent in that the unit of force accelerates the unit of mass 1 unit of length per second per second.

A special effort has been made in this book to familiarize the reader with the various systems of measurement. In examples and problems, for instance, calculations are often made in SI units, conventional metric units, and BES units, in order to illustrate how to convert from one system to another. Table 3–2 shows some convenient conversion factors among different systems of units. (Other detailed conversion tables are given in Appendix B.)

TABLE 3–2 Conversion Table

Length	1	1 m = 100 cm	
	2	1 ft = 12 in.	1 in. = 2.54 cm
	3	1 m = 3.281 ft	1 ft = 0.3048 m
Mass	4	1 kg = 2.2046 lb	1 lb = 0.4536 kg
	5	1 kg = 0.10197 kg_f-s^2/m	1 kg_f-s^2/m = 9.807 kg
	6	1 slug = 14.594 kg	1 kg = 0.06852 slug
	7	1 slug = 32.174 lb	1 lb = 0.03108 slug
	8	1 slug = 1.488 kg_f-s^2/m	1 kg_f-s^2/m = 0.6720 slug
Moment of inertia	9	1 slug-ft^2 = 1.356 kg-m^2	1 kg-m^2 = 0.7376 slug-ft^2
	10	1 slug-ft^2 = 0.1383 kg_f-s^2-m	1 kg_f-s^2-m = 7.233 slug-ft^2
	11	1 slug-ft^2 = 32.174 lb-ft^2	1 lb-ft^2 = 0.03108 slug-ft^2
Force	12	1 N = 10^5 dyn	
	13	1 N = 0.10197 kg_f	1 kg_f = 9.807 N
	14	1 N = 7.233 poundals	1 poundal = 0.1383 N
	15	1 N = 0.2248 lb_f	1 lb_f = 4.4482 N
	16	1 kg_f = 2.2046 lb_f	1 lb_f = 0.4536 kg_f
	17	1 lb_f = 32.174 poundals	1 poundal = 0.03108 lb_f
Energy	18	1 N-m = 1 J = 1 W-s	1 J = 0.10197 kg_f-m
	19	1 dyn-cm = 1 erg = 10^{-7} J	1 kg_f-m = 9.807 N-m
	20	1 N-m = 0.7376 ft-lb_f	1 ft-lb_f = 1.3557 N-m
	21	1 J = 2.389 × 10^{-4} kcal	1 kcal = 4186 J
	22	1 Btu = 778 ft-lb_f	1 ft-lb_f = 1.285 × 10^{-3} Btu
Power	23	1 W = 1 J/s	
	24	1 hp = 550 ft-lb_f/s	1 ft-lb_f/s = 1.818 × 10^{-3} hp
	25	1 hp = 745.7 W	1 W = 1.341 × 10^{-3} hp

Outline of the chapter. Section 3–1 has presented a review of systems of units necessary in the discussions of dynamics of mechanical systems. Section 3–2 treats mechanical elements. Section 3–3 discusses mathematical modeling of mechanical systems and analyzes simple mechanical systems. Section 3–4 reviews the concept of work, energy, and power and then presents energy methods for deriving mathematical models of conservative systems (systems that do not dissipate energy).

3–2 MECHANICAL ELEMENTS

Any mechanical system consists of mechanical elements. There are three types of basic elements in mechanical systems: inertia elements, spring elements, and damper elements.

Inertia elements. By *inertia elements*, we mean masses and moments of inertia. *Inertia* may be defined as the change in force (torque) required to make a unit change in acceleration (angular acceleration). That is,

$$\text{inertia (mass)} = \frac{\text{change in force}}{\text{change in acceleration}} \quad \frac{\text{N}}{\text{m/s}^2} \text{ or kg}$$

$$\text{inertia (moment of inertia)} = \frac{\text{change in torque}}{\text{change in angular acceleration}} \quad \frac{\text{N-m}}{\text{rad/s}^2} \text{ or kg-m}^2$$

Spring Elements. A linear *spring* is a mechanical element that can be deformed by an external force or torque such that the deformation is directly proportional to the force or torque applied to the element.

Consider the spring shown in Figure 3–1(a). Here, we consider translational motion only. Suppose that the natural length of the spring is X, the spring is fixed at one end, and the other end is free. Then, when a force f is applied at the free end, the spring is stretched. The elongation of the spring is x. The force that arises in the spring is proportional to x and is given by

$$F = kx \tag{3–1}$$

where k is a proportionality constant called the *spring constant*. The dimension of the spring constant k is force/displacement. At point P, this spring force F acts opposite to the direction of the force f applied at point P.

Figure 3–1(b) shows the case where both ends (denoted by points P and Q) of the spring are deflected due to the forces f applied at each end. (The forces at each

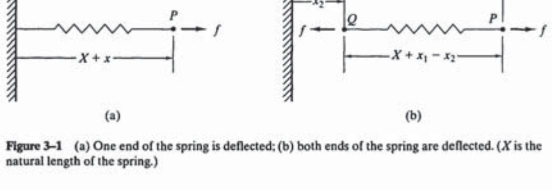


Figure 3–1 (a) One end of the spring is deflected; (b) both ends of the spring are deflected. (X is the natural length of the spring.)

end of the spring are on the same line and are equal in magnitude but opposite in direction.) The natural length of the spring is X. The net elongation of the spring is $x_1 - x_2$. The force acting in the spring is then

$$F = k(x_1 - x_2) \tag{3-2}$$

At point P, the spring force F acts to the left. At point Q, F acts to the right. (Note that the displacements $X + x_1$ and x_2 of the ends of the spring are measured relative to the same frame of reference.)

Next, consider the torsional spring shown in Figure 3–2(a), where one end is fixed and a torque τ is applied to the other end. The angular displacement of the free end is θ. Then the torque T that arises in the torsional spring is

$$T = k\theta \tag{3-3}$$

At the free end, this torque acts in the torsional spring in the direction opposite that of the applied torque τ.

For the torsional spring shown in Figure 3–2(b), torques equal in magnitude, but opposite in direction, are applied to the ends of the spring. In this case, the torque T acting in the torsional spring is

$$T = k(\theta_1 - \theta_2) \tag{3-4}$$

At each end, the spring torque acts in the direction opposite that of the applied torque at that end. The dimension of the torsional spring constant k is torque/angular displacement, where angular displacement is measured in radians.

When a linear spring is stretched, a point is reached in which the force per unit displacement begins to change and the spring becomes a nonlinear spring. If the spring is stretched farther, a point is reached at which the material will either break or yield. For practical springs, therefore, the assumption of linearity may be good only for relatively small net displacements. Figure 3–3 shows the force–displacement characteristic curves for linear and nonlinear springs.

For linear springs, the spring constant k may be defined as follows:

$$\text{spring constant } k \text{ (for translational spring)} = \frac{\text{change in force}}{\text{change in displacement of spring}}\ \frac{\text{N}}{\text{m}}$$

Figure 3–2 (a) A torque τ is applied at one end of torsional spring, and the other end is fixed; (b) a torque τ is applied at one end, and a torque τ, in the opposite direction, is applied at the other end.

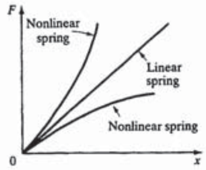


Figure 3–3 Force–displacement characteristic curves for linear and nonlinear springs.

$$\text{spring constant } k \text{ (for torsional spring)} = \frac{\text{change in torque}}{\text{change in angular displacement of spring}}\ \frac{\text{N-m}}{\text{rad}}$$

Spring constants indicate stiffness; a large value of k corresponds to a hard spring, a small value of k to a soft spring. The reciprocal of the spring constant k is called *compliance* or *mechanical capacitance* C. Thus, $C = 1/k$. Compliance or mechanical capacitance indicates the softness of a spring.

Practical spring versus ideal spring. All practical springs have inertia and damping. In our analysis in this book, however, we assume that the effect of the mass of a spring is negligibly small; that is, the inertia force due to acceleration of the spring is negligibly small compared with the spring force. Also, we assume that the damping effect of the spring is negligibly small.

An ideal linear spring, in comparison to a practical spring, will have neither mass nor damping and will obey the linear force–displacement law as given by Equations (3–1) and (3–2) or the linear torque–angular displacement law as given by Equations (3–3) and (3–4).

Damper elements. A *damper* is a mechanical element that dissipates energy in the form of heat instead of storing it. Figure 3–4(a) shows a schematic diagram of a translational damper, or dashpot. It consists of a piston and an oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by oil

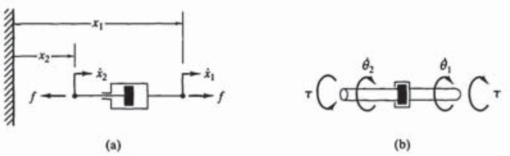


Figure 3–4 (a) Translational damper; (b) torsional (or rotational) damper.

because oil must flow around the piston (or through orifices provided in the piston) from one side to the other. Essentially, the damper absorbs energy, and the absorbed energy is dissipated as heat that flows away to the surroundings.

In Figure 3–4(a), the forces f applied at the ends of the translational damper are on the same line and are of equal magnitude, but opposite in direction. The velocities of the ends of the damper are $\dot{x}_1$ and $\dot{x}_2$. Velocities $\dot{x}_1$ and $\dot{x}_2$ are taken relative to the same frame of reference.

In the damper, the damping force F that arises in it is proportional to the velocity difference $\dot{x}_1 - \dot{x}_2$ of the ends, or

$$F = b(\dot{x}_1 - \dot{x}_2) = b\dot{x} \tag{3-5}$$

where $\dot{x} = \dot{x}_1 - \dot{x}_2$ and the proportionality constant b relating the damping force F to the velocity difference $\dot{x}$ is called the *viscous friction coefficient* or *viscous friction constant*. The dimension of b is force/velocity. Note that the initial positions of both ends of the damper do not appear in the equation.

For the torsional damper shown in Figure 3–4(b), the torques τ applied to the ends of the damper are of equal magnitude, but opposite in direction. The angular velocities of the ends are $\dot{\theta}_1$ and $\dot{\theta}_2$ and they are taken relative to the same frame of reference. The damping torque T that arises in the damper is proportional to the angular velocity difference $\dot{\theta}_1 - \dot{\theta}_2$ of the ends, or

$$T = b(\dot{\theta}_1 - \dot{\theta}_2) = b\dot{\theta} \tag{3-6}$$

where, analogous to the translational case, $\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$ and the proportionality constant b relating the damping torque T to the angular velocity difference $\dot{\theta}$ is called the viscous friction coefficient or viscous friction constant. The dimension of b is torque/angular velocity. Note that the initial angular positions of both ends of the damper do not appear in the equation.

A damper is an element that provides resistance in mechanical motion, and, as such, its effect on the dynamic behavior of a mechanical system is similar to that of an electrical resistor on the dynamic behavior of an electrical system. Consequently, a damper is often referred to as a *mechanical resistance element* and the viscous friction coefficient as the *mechanical resistance*.

Practical damper versus ideal damper. All practical dampers produce inertia and spring effects. In this book, however, we assume that these effects are negligible.

An ideal damper is massless and springless, dissipates all energy, and obeys the linear force–velocity law or linear torque–angular velocity law as given by Equation (3–5) or Equation (3–6), respectively.

Nonlinear friction. Friction that obeys a linear law is called *linear friction*, whereas friction that does not is described as *nonlinear*. Examples of nonlinear friction include static friction, sliding friction, and square-law friction. Square-law friction occurs when a solid body moves in a fluid medium. Figure 3–5 shows a characteristic curve for square-law friction. In this book, we shall not discuss nonlinear friction any further.

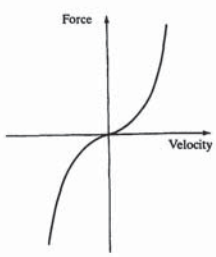


Figure 3–5 Characteristic curve for square-law friction.

3–3 MATHEMATICAL MODELING OF SIMPLE MECHANICAL SYSTEMS

A mathematical model of any mechanical system can be developed by applying Newton's laws to the system. In this section, we shall deal with the problem of deriving mathematical models of simple mechanical systems. More on deriving mathematical models of various mechanical systems and response analyses is presented in Chapters 4, 5, 7, 8, and 9.

Rigid body. When any real body is accelerated, internal elastic deflections are always present. If these internal deflections are negligibly small relative to the gross motion of the entire body, the body is called a *rigid body*. Thus, a rigid body does not deform.

Newton's laws. There are three well-known laws called *Newton's laws*. Newton's first law, which concerns the conservation of momentum, states that the total momentum of a mechanical system is constant in the absence of external forces. Momentum is the product of mass m and velocity v, or mv, for translational or linear motion. For rotational motion, momentum is the product of moment of inertia J and angular velocity ω, or $J\omega$, and is called angular momentum.

Newton's second law gives the force–acceleration relationship of a rigid translating body or the torque–angular acceleration relationship of a rigid rotating body. The third law concerns action and reaction and, in effect, states that every action is always opposed by an equal reaction.

Newton's second law (for translational motion). For translational motion, Newton's second law says that if a force is acting on a rigid body through the center of mass in a given direction, the acceleration of the rigid body in the same direction is directly proportional to the force acting on it and is inversely proportional to the mass of the body. That is,

$$\text{acceleration} = \frac{\text{force}}{\text{mass}}$$

or

$$(\text{mass})(\text{acceleration}) = \text{force}$$

Suppose that forces are acting on a body of mass m. If ΣF is the sum of all forces acting on mass m through the center of mass in a given direction, then

$$ma = \sum F \tag{3–7}$$

where a is the resulting absolute acceleration in that direction. The line of action of the force acting on a body must pass through the center of mass of the body. Otherwise, rotational motion will also be involved. Rotational motion is not defined by Equation (3–7).

Newton's second law (for rotational motion). For a rigid body in pure rotation about a fixed axis, Newton's second law states that

$$(\text{moment of inertia})(\text{angular acceleration}) = \text{torque}$$

or

$$J\alpha = \sum T \tag{3–8}$$

where ΣT is the sum of all torques acting about a given axis, J is the moment of inertia of a body about that axis, and α is the angular acceleration of the body.

Torque or moment of force. *Torque*, or moment of force, is defined as any cause that tends to produce a change in the rotational motion of a body on which it acts. Torque is the product of a force and the perpendicular distance from a point of rotation to the line of action of the force. The units of torque are force times length, such as N-m, dyn-cm, kg_f-m, and lb_f-ft.

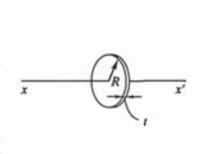
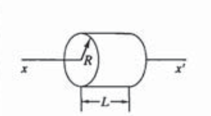
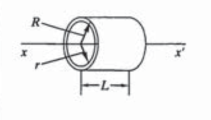
Moments of inertia. The *moment of inertia* J of a rigid body about an axis is defined by

$$J = \int r^2 dm$$

where dm is an element of mass, r is distance from the axis to dm, and integration is performed over the body. In considering moments of inertia, we assume that the rotating body is perfectly rigid. Physically, the moment of inertia of a body is a measure of the resistance of the body to angular acceleration.

Table 3–3 gives the moments of inertia of rigid bodies with common shapes.

TABLE 3–3 Moments of Inertia

	$t \ll R$ m = mass of disk $J_x = \frac{1}{2}mR^2$
	m = mass of solid cylinder $J_x = \frac{1}{2}mR^2$
	m = mass of hollow cylinder $J_x = \frac{1}{2}m(R^2 + r^2)$

Example 3–1

Figure 3–6 shows a homogeneous cylinder of radius R and length L. The moment of inertia J of this cylinder about axis AA' can be obtained as follows: Consider a ring-shaped mass element of infinitesimal width dr at radius r. The mass of this ring-shaped element is $2\pi rL\rho\, dr$, where ρ is the density of the cylinder. Thus,

$$dm = 2\pi rL\rho\, dr$$

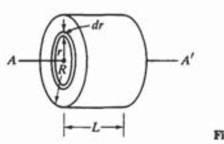


Figure 3–6 Homogeneous cylinder.

Consequently,

$$J = \int_0^R r^2 2\pi rL\rho\, dr = 2\pi L\rho \int_0^R r^3\, dr = \frac{\pi L\rho R^4}{2}$$

Since the entire mass m of the cylinder body is $m = \pi R^2 L\rho$, we obtain

$$J = \frac{1}{2}mR^2$$

Moment of inertia about an axis other than the geometrical axis. Sometimes it is necessary to calculate the moment of inertia of a homogeneous rigid body about an axis other than its geometrical axis. If the axes are parallel, the calculation can be done easily. The moment of inertia about an axis that is a distance x from the geometrical axis passing through the center of gravity of the body is the sum of the moment of inertia about the geometrical axis and the moment of inertia about the new axis when the mass of the body is considered concentrated at the center of gravity.

Example 3–2

Consider the system shown in Figure 3–7, where a homogeneous cylinder of mass m and radius R rolls on a flat surface. Find the moment of inertia, J_x, of the cylinder about its line of contact (axis xx') with the surface.

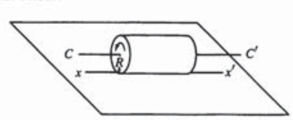


Figure 3–7 Homogeneous cylinder rolling on a flat surface.

The moment of inertia of the cylinder about axis CC' is

$$J_C = \frac{1}{2}mR^2$$

The moment of inertia of the cylinder about axis xx' when mass m is considered concentrated at the center of gravity is mR^2. Thus, the moment of inertia J_x of the cylinder about axis xx' is

$$J_x = J_C + mR^2 = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2$$

Forced response and natural response. The behavior determined by a forcing function is called a *forced response*, and that due to initial conditions (initial energy storages) is called a *natural response*. The period between the initiation of a response and the ending is referred to as the *transient period*. After the response has become negligibly small, conditions are said to have reached a *steady state*.

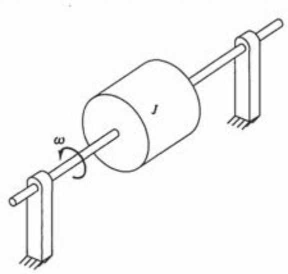


Figure 3–8 Rotor mounted in bearings.

Rotational system. A schematic diagram of a rotor mounted in bearings is shown in Figure 3–8. The moment of inertia of the rotor about the axis of rotation is J. Let us assume that at $t = 0$ the rotor is rotating at the angular velocity $\omega(0) = \omega_0$. We also assume that the friction in the bearings is viscous friction and that no external torque is applied to the rotor. Then the only torque acting on the rotor is the friction torque $b\omega$ in the bearings.

Applying Newton's second law, Equation (3–8), we obtain the equation of motion,

$$J\dot{\omega} = -b\omega, \qquad \omega(0) = \omega_0$$

or

$$J\dot{\omega} + b\omega = 0 \tag{3–9}$$

Equation (3–9) is a mathematical model of the system. (Here, the output of the system is considered to be the angular velocity ω, not the angular displacement.)

To find $\omega(t)$, we take the Laplace transform of Equation (3–9); that is,

$$J[s\Omega(s) - \omega(0)] + b\Omega(s) = 0$$

where $\Omega(s) = \mathcal{L}[\omega(t)]$. Simplifying, we obtain

$$(Js + b)\Omega(s) = J\omega(0) = J\omega_0$$

Hence,

$$\Omega(s) = \frac{\omega_0}{s + \dfrac{b}{J}} \tag{3–10}$$

The denominator, $s + (b/J)$, is called the *characteristic polynomial*, and

$$s + \frac{b}{J} = 0$$

is called the *characteristic equation*.

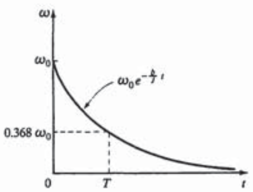


Figure 3–9 Curve of angular velocity ω versus time t for the rotor system shown in Figure 3–8.

The inverse Laplace transform of $\Omega(s)$, the solution of the differential equation given by Equation (3–9), is

$$\omega(t) = \omega_0 e^{-(b/J)t}$$

The angular velocity decreases exponentially, as shown in Figure 3–9.

Since the exponential factor $e^{-(b/J)t}$ approaches zero as t increases without limit, mathematically the response lasts forever. In dealing with such an exponentially decaying response, it is convenient to depict the response in terms of a *time constant*: that value of time which makes the exponent equal to -1. For this system, the time constant T is equal to J/b, or $T = J/b$. When $t = T$, the value of the exponential factor is

$$e^{-T/T} = e^{-1} = 0.368$$

In other words, when the time t in seconds is equal to the time constant, the exponential factor is reduced to approximately 36.8% of its initial value, as shown in Figure 3–9.

Spring–mass system. Figure 3–10 depicts a system consisting of a mass and a spring. Here, the mass is suspended by the spring. For the vertical motion, two

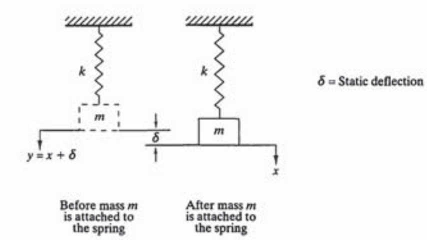


Figure 3–10 Spring–mass system.

forces are acting on the mass: the spring force ky and the gravitational force mg. (In the diagram, the positive direction of displacement y is defined downward.) If the mass is pulled downward by an external force and then released, the spring force acts upward and tends to pull the mass upward. The gravitational force pulls the mass downward. So, by applying Newton's second law to this system, we obtain the equation of motion

$$m\ddot{y} = \sum \text{forces} = -ky + mg$$

or

$$m\ddot{y} + ky = mg \tag{3–11}$$

The gravitational force is opposed statically by the equilibrium spring deflection δ. If we measure the displacement from this equilibrium position, then the term mg can be dropped from the equation of motion. By substituting $y = x + \delta$ into Equation (3–11) and noting that δ = constant, we have

$$m\ddot{x} + k(x + \delta) = mg \tag{3–12}$$

Since the spring force $k\delta$ and the gravitational force mg balance, or $k\delta = mg$, Equation (3–12) simplifies to

$$m\ddot{x} + kx = 0 \tag{3–13}$$

which is a mathematical model of the system.

In this book, unless otherwise stated, when writing equations of motion for systems involving the gravitational force, we measure the displacement of the mass from the equilibrium position in order to eliminate the term mg and simplify the mathematical model.

Free vibration. For the spring–mass system of Figure 3–10, suppose that the mass is pulled downward and then released with arbitrary initial conditions $x(0)$ and $\dot{x}(0)$. In this case, the mass will oscillate and the motion will be periodic. (We assume that the magnitude of the displacement is such that the spring remains a linear spring.) The periodic motion that is observed as the system is displaced from its static equilibrium position is called *free vibration*. It is a natural response due to the initial condition.

To find the mathematical form of the periodic motion, let us solve Equation (3–13). By taking the Laplace transforms of both sides of that equation, we obtain

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + kX(s) = 0$$

or

$$(ms^2 + k)X(s) = m\dot{x}(0) + msx(0)$$

Hence,

$$X(s) = \frac{\dot{x}(0)}{s^2 + \frac{k}{m}} + \frac{sx(0)}{s^2 + \frac{k}{m}}$$

This last equation may be rewritten so that the inverse Laplace transform of each term can be easily identified:

$$X(s) = \sqrt{\frac{m}{k}}\,\dot{x}(0)\frac{\sqrt{k/m}}{s^2 + (\sqrt{k/m})^2} + x(0)\frac{s}{s^2 + (\sqrt{k/m})^2}$$

Noting that

$$\mathscr{L}[\sin\sqrt{k/m}\,t] = \frac{\sqrt{k/m}}{s^2 + (\sqrt{k/m})^2}$$

$$\mathscr{L}[\cos\sqrt{k/m}\,t] = \frac{s}{s^2 + (\sqrt{k/m})^2}$$

we obtain the inverse Laplace transform of $X(s)$ as

$$x(t) = \sqrt{\frac{m}{k}}\,\dot{x}(0)\sin\sqrt{\frac{k}{m}}\,t + x(0)\cos\sqrt{\frac{k}{m}}\,t \tag{3–14}$$

Periodic motion such as that described by Equation (3–14) is called *simple harmonic motion*.

If the initial conditions were given as $x(0) = x_0$ and $\dot{x}(0) = 0$, then, by substituting these initial conditions into Equation (3–14), the displacement of the mass would be given by

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}\,t$$

The period and frequency of simple harmonic motion can now be defined as follows: The *period* T is the time required for a periodic motion to repeat itself. In the present case,

$$\text{period } T = \frac{2\pi}{\sqrt{\frac{k}{m}}} \text{ seconds}$$

The *frequency* f of periodic motion is the number of cycles per second (cps), and the standard unit of frequency is the hertz (Hz); that is, 1 Hz is 1 cps. In the present case of harmonic motion,

$$\text{frequency } f = \frac{1}{T} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \text{ Hz}$$

The natural frequency, or undamped natural frequency, is the frequency in the free vibration of a system having no damping. If the natural frequency is measured in Hz or cps, it is denoted by f_n. If it is measured in radians per second (rad/s), it is denoted by ω_n. In the present system,

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

It is important to remember that, when Equation (3–13) is written in the form

$$\ddot{x} + \frac{k}{m}x = 0$$

where the coefficient of the $\ddot{x}$ term is unity, the square root of the coefficient of the x term is the natural frequency ω_n. This means that a mathematical model for the system shown in Figure 3–10 can be put in the form

$$\ddot{x} + \omega_n^2 x = 0$$

where $\omega_n = \sqrt{k/m}$.

Experimental determination of moment of inertia. It is possible to calculate moments of inertia for homogeneous bodies having geometrically simple shapes. However, for rigid bodies with complicated shapes or those consisting of materials of various densities, such calculation may be difficult or even impossible; moreover, calculated values may not be accurate. In these instances, experimental determination of moments of inertia is preferable. The process is as follows: We mount a rigid body in frictionless bearings so that it can rotate freely about the axis of rotation around which the moment of inertia is to be determined. Next, we attach a torsional spring with known spring constant k to the rigid body. (See Figure 3–11.) The spring is then twisted slightly and released, and the period of the resulting simple harmonic motion is measured. Since the equation of motion for this system is

$$J\ddot{\theta} + k\theta = 0$$

or

$$\ddot{\theta} + \frac{k}{J}\theta = 0$$

the natural frequency is

$$\omega_n = \sqrt{\frac{k}{J}}$$

and the period of vibration is

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{J}}}$$

The moment of inertia is then determined as

$$J = \frac{kT^2}{4\pi^2}$$

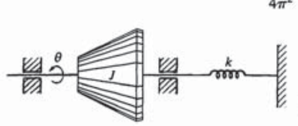


Figure 3–11 Setup for the experimental determination of moment of inertia.

Similarly, in the spring–mass system of Figure 3–10, if the spring constant k is known and the period T of the free vibration is measured, then the mass m can be calculated from

$$m = \frac{kT^2}{4\pi^2}$$

Spring–mass–damper system. Most physical systems involve some type of damping—viscous damping, magnetic damping, and so on. Such damping not only slows the motion of (a part of) the system, but also causes the motion to stop eventually. In the discussion that follows, we shall consider a simple mechanical system involving viscous damping. Note that a typical viscous damping element is a damper or dashpot.

Figure 3–12 is a schematic diagram of a spring–mass–damper system. Suppose that the mass is pulled downward and then released. If the damping is light, vibratory motion will occur. (The system is then said to be underdamped.) If the damping is heavy, vibratory motion will not occur. (The system is then said to be overdamped.) A critically damped system is a system in which the degree of damping is such that the resultant motion is on the borderline between the underdamped and overdamped cases. Regardless of whether a system is underdamped, overdamped, or critically damped, the free vibration or free motion will diminish with time because of the presence of damper. This free vibration is called *transient motion.*

In the system shown in Figure 3–12, for the vertical motion, three forces are acting on the mass: the spring force, the damping force, and the gravitational force. As noted earlier, if we measure the displacement of the mass from a static equilibrium position (so that the gravitational force is balanced by the equilibrium spring deflection), the gravitational force will not enter into the equation of motion. So, by measuring the displacement x from the static equilibrium position, we obtain the equation of motion,

$$m\ddot{x} = \sum \text{forces} = -kx - b\dot{x}$$

or

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{3–15}$$

Equation (3–15), which describes the motion of the system, is a mathematical model of the system.

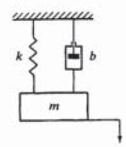


Figure 3–12 Spring–mass–damper system.

Only the underdamped case is considered in our present analysis. (A more complete analysis of this system for the underdamped, overdamped, and critically damped cases is given in Chapter 8.)

Let us solve Equation (3–15) for a particular case. Suppose that $m = 0.1$ slug, $b = 0.4$ lb_f-s/ft, and $k = 4$ lb_f/ft. Then Equation (3–15) becomes

$$0.1\ddot{x} + 0.4\dot{x} + 4x = 0$$

or

$$\ddot{x} + 4\dot{x} + 40x = 0 \tag{3–16}$$

Let us obtain the motion $x(t)$ when the mass is pulled downward at $t = 0$, so that $x(0) = x_0$, and is released with zero velocity, or $\dot{x}(0) = 0$. (We assume that the magnitude of the downward displacement is such that the system remains a linear system.) Taking the Laplace transform of Equation (3–16), we obtain

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 40X(s) = 0$$

Simplifying this last equation and noting that $x(0) = x_0$ and $\dot{x}(0) = 0$, we get

$$(s^2 + 4s + 40)X(s) = sx_0 + 4x_0$$

or

$$X(s) = \frac{(s + 4)x_0}{s^2 + 4s + 40} \tag{3–17}$$

The characteristic equation for the system

$$s^2 + 4s + 40 = 0$$

has a pair of complex-conjugate roots. This implies that the inverse Laplace transform of $X(s)$ is a damped sinusoidal function. Hence, we may rewrite $X(s)$ in Equation (3–17) as a sum of the Laplace transforms of a damped sine function and a damped cosine function:

$$\begin{aligned} X(s) &= \frac{2x_0}{s^2 + 4s + 40} + \frac{(s + 2)x_0}{s^2 + 4s + 40} \\ &= \frac{1}{3}x_0\frac{6}{(s + 2)^2 + 6^2} + x_0\frac{s + 2}{(s + 2)^2 + 6^2} \end{aligned}$$

Noting that

$$\frac{6}{(s + 2)^2 + 6^2} = \mathscr{L}[e^{-2t}\sin 6t], \qquad \frac{s + 2}{(s + 2)^2 + 6^2} = \mathscr{L}[e^{-2t}\cos 6t]$$

we can obtain the inverse Laplace transform of $X(s)$ as

$$\begin{aligned} x(t) &= \frac{1}{3}x_0e^{-2t}\sin 6t + x_0e^{-2t}\cos 6t \\ &= e^{-2t}\left(\frac{1}{3}\sin 6t + \cos 6t\right)x_0 \end{aligned} \tag{3–18}$$

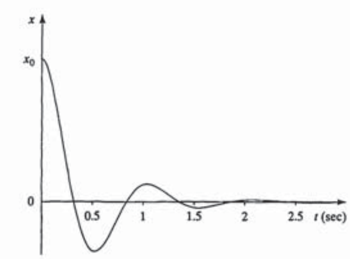


Figure 3–13 Free vibration of the spring–mass–damper system described by $\ddot{x} + 4\dot{x} + 40x = 0$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$.

Equation (3–18) depicts the free vibration of the spring–mass–damper system with the given numerical values. The free vibration here is a damped sinusoidal vibration, as shown in Figure 3–13.

Comments. The numerical values in the preceding problem were stated in BES units. Let us convert these values into units of other systems.

1. *SI units (refer to Tables 3–1 and 3–2):*

$$\begin{aligned} m &= 0.1 \text{ slug} = 1.459 \text{ kg} \\ b &= 0.4 \text{ lb}_f\text{-s/ft} = 0.4 \times 4.448/0.3048 \text{ N-s/m} = 5.837 \text{ N-s/m} \\ k &= 4 \text{ lb}_f\text{/ft} = 4 \times 4.448/0.3048 \text{ N/m} = 58.37 \text{ N/m} \end{aligned}$$

Hence, Equation (3–15) becomes

$$1.459\ddot{x} + 5.837\dot{x} + 58.37x = 0$$

or

$$\ddot{x} + 4\dot{x} + 40x = 0$$

which is the same as Equation (3–16).

2. *Metric engineering (gravitational) units (refer to Tables 3–1 and 3–2):*

$$\begin{aligned} m &= 0.1 \text{ slug} = 0.1488 \text{ kg}_f\text{-s}^2\text{/m} \\ b &= 0.4 \text{ lb}_f\text{-s/ft} = 0.4 \times 0.4536/0.3048 \text{ kg}_f\text{-s/m} = 0.5953 \text{ kg}_f\text{-s/m} \\ k &= 4 \text{ lb}_f\text{/ft} = 4 \times 0.4536/0.3048 \text{ kg}_f\text{/m} = 5.953 \text{ kg}_f\text{/m} \end{aligned}$$

Therefore, Equation (3–15) becomes

$$0.1488\ddot{x} + 0.5953\dot{x} + 5.953x = 0$$

or

$$\ddot{x} + 4\dot{x} + 40x = 0$$

which, again, is the same as Equation (3–16).

Note that as long as we use consistent units, the differential equation (mathematical model) of the system remains the same.

3–4 WORK, ENERGY, AND POWER

In this section, we discuss work, energy, and power. We also discuss energy methods for deriving equations of motion or undamped natural frequencies of certain conservative systems.

If force is considered a measure of effort, then *work* is a measure of accomplishment and *energy* is the ability to do work. The concept of work makes no allowance for a time factor. When a time factor is considered, the concept of power must be introduced. *Power* is work per unit time.

Work. The *work* done in a mechanical system is the product of a force and the distance (or a torque and the angular displacement) through which the force is exerted, with both force and distance measured in the same direction. For instance, if a body is pushed with a horizontal force of F newtons along a horizontal floor for a distance of x meters, the work done in pushing the body is

$$W = Fx \text{ N-m}$$

Units of work. Different systems have different units of work.

SI units and mks (metric absolute) system of units. Force is measured in newtons and distance in meters. Thus, the unit of work is the N-m. Note that

$$1 \text{ N-m} = 1 \text{ joule} = 1 \text{ J}$$

British engineering system of units. In this system, force is measured in pounds and distance in feet. Hence, the unit of work is the ft-lb_f, and

$$1 \text{ ft-lb}_f = 1.3557 \text{ J} = 1.285 \times 10^{-3} \text{ Btu}$$

$$1 \text{ Btu} = 778 \text{ ft-lb}_f$$

cgs (metric absolute) system of units. Here, the unit of work is the dyn-cm, or erg. Note that

$$10^7 \text{ erg} = 10^7 \text{ dyn-cm} = 1 \text{ J}$$

Metric engineering (gravitational) system of units. The unit of work in the metric engineering system is the kg_f-m. Note that

$$1 \text{ kg}_f\text{-m} = 9.807 \times 10^7 \text{ dyn-cm} = 9.807 \text{ J}$$

$$1 \text{ J} = 0.10197 \text{ kg}_f\text{-m}$$

Energy. In a general way, *energy* can be defined as the capacity or ability to do work. Energy is found in many different forms and can be converted from one form into another. For instance, an electric motor converts electrical energy into mechanical energy, a battery converts chemical energy into electrical energy, and so forth.

A system is said to possess energy when it can do work. When a system does mechanical work, the system's energy decreases by the amount equal to the energy required for the work done. Units of energy are the same as units for work, that is, newton-meter, joule, kcal, Btu, and so on.

According to the law of conservation of energy, energy can be neither created nor destroyed. This means that the increase in the total energy within a system is equal to the net energy input to the system. So if there is no energy input, there is no change in total energy of the system.

The energy that a body possesses because of its position is called *potential* energy, whereas the energy that a body has as a result of its velocity is called *kinetic* energy.

Potential energy. In a mechanical system, only mass and spring elements can store potential energy. The change in the potential energy stored in a system equals the work required to change the system's configuration. Potential energy is always measured with reference to some chosen level and is relative to that level.

Potential energy is the work done by an external force. For a body of mass m in the gravitational field of the earth, the potential energy U measured from some reference level is mg times the altitude h measured from the same reference level, or

$$U = \int_0^h mg\,dx = mgh$$

Notice that the body, if dropped, has the capacity to do work, since the weight mg of the body causes it to travel a distance h when released. (The weight is a force.) Once the body is released, the potential energy decreases. The lost potential energy is converted into kinetic energy.

For a translational spring, the potential energy U is equal to the net work done on the spring by the forces acting on its ends as it is compressed or stretched. Since the spring force F is equal to kx, where x is the net displacement of the ends of the spring, the total energy stored is

$$U = \int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

If the initial and final values of x are x_1 and x_2, respectively, then

$$\text{change in potential energy } \Delta U = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} kx\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

Note that the potential energy stored in a spring does not depend on whether it is compressed or stretched.

Similarly, for a torsional spring,

$$\text{change in potential energy } \Delta U = \int_{\theta_1}^{\theta_2} T\,d\theta = \int_{\theta_1}^{\theta_2} k\theta\,d\theta = \frac{1}{2}k\theta_2^2 - \frac{1}{2}k\theta_1^2$$

Kinetic energy. Only inertial elements can store kinetic energy in mechanical systems. A mass m in pure translation with velocity v has kinetic energy $T = \frac{1}{2}mv^2$, whereas a moment of inertia J in pure rotation with angular velocity $\dot{\theta}$

has kinetic energy $T = \frac{1}{2}J\dot{\theta}^2$. The change in kinetic energy of the mass is equal to the work done on it by an applied force as the mass accelerates or decelerates. Thus, the change in kinetic energy T of a mass m moving in a straight line is

$$\begin{aligned}\text{change in kinetic energy} = \Delta T = \Delta W &= \int_{x_1}^{x_2} F\,dx = \int_{t_1}^{t_2} F\frac{dx}{dt}\,dt \\ &= \int_{t_1}^{t_2} Fv\,dt = \int_{t_1}^{t_2} m\dot{v}v\,dt = \int_{v_1}^{v_2} mv\,dv \\ &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\end{aligned}$$

where $x(t_1) = x_1$, $x(t_2) = x_2$, $v(t_1) = v_1$, and $v(t_2) = v_2$. Notice that the kinetic energy stored in the mass does not depend on the sign of the velocity v.

The change in kinetic energy of a moment of inertia in pure rotation at angular velocity $\dot{\theta}$ is

$$\text{change in kinetic energy } \Delta T = \frac{1}{2}J\dot{\theta}_2^2 - \frac{1}{2}J\dot{\theta}_1^2$$

where J is the moment of inertia about the axis of rotation, $\dot{\theta}_1 = \dot{\theta}(t_1)$, and $\dot{\theta}_2 = \dot{\theta}(t_2)$.

Dissipated energy. Consider the damper shown in Figure 3–14, in which one end is fixed and the other end is moved from x_1 to x_2. The dissipated energy ΔW of the damper is equal to the net work done on it:

$$\Delta W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} b\dot{x}\,dx = b\int_{t_1}^{t_2} \dot{x}\frac{dx}{dt}\,dt = b\int_{t_1}^{t_2} \dot{x}^2\,dt$$

The energy of the damper element is always dissipated, regardless of the sign of $\dot{x}$.

Power. *Power* is the time rate of doing work. That is,

$$\text{power} = P = \frac{dW}{dt}$$

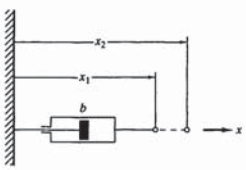


Figure 3–14 Damper.

where dW denotes work done during time interval dt. The average power during a duration of $t_2 - t_1$ seconds can be determined by measuring the work done in $t_2 - t_1$ seconds, or

$$\text{average power} = \frac{\text{work done in } (t_2 - t_1) \text{ seconds}}{(t_2 - t_1) \text{ seconds}}$$

In SI units or the mks (metric absolute) system of units, the work done is measured in newton-meters and the time in seconds. The unit of power is the newton-meter per second, or watt:

$$1 \text{ N-m/s} = 1 \text{ W}$$

In the British engineering system of units, the work done is measured in ft-lb$_f$ and the time in seconds. The unit of power is the ft-lb$_f$/s. The power 550 ft-lb$_f$/s is called 1 horsepower (hp). Thus,

$$1 \text{ hp} = 550 \text{ ft-lb}_f\text{/s} = 33\,000 \text{ ft-lb}_f\text{/min} = 745.7 \text{ W}$$

In the metric engineering system of units, the work done is measured in kg$_f$-m and the time in seconds. The unit of power is the kg$_f$-m/s, where

$$1 \text{ kg}_f\text{-m/s} = 9.807 \text{ W}$$
$$1 \text{ W} = 1 \text{ J/s} = 0.10197 \text{ kg}_f\text{-m/s}$$

Example 3–3

Find the power required to raise a body of mass 500 kg at a rate of 20 m/min.

Let us define displacement per second as x. Then

$$\text{work done in 1 second} = mgx = 500 \times 9.807 \times \frac{20}{60}\frac{\text{kg-m}^2}{\text{s}^2} = 1635 \text{ N-m}$$

and

$$\text{power} = \frac{\text{work done in 1 second}}{1 \text{ second}} = \frac{1635 \text{ N-m}}{1 \text{ s}} = 1635 \text{ W}$$

Thus, the power required is 1635 W.

An energy method for deriving equations of motion. Earlier in this chapter, we presented Newton's method for deriving equations of motion of mechanical systems. Several other approaches for obtaining equations of motion are available, one of which is based on the law of conservation of energy. Here we derive such equations from the fact that the total energy of a system remains the same if no energy enters or leaves the system.

In mechanical systems, friction dissipates energy as heat. Systems that do not involve friction are called *conservative* systems. Consider a conservative system in which the energy is in the form of kinetic or potential energy (or both). Since energy enters and leaves the conservative system in the form of mechanical work, we obtain

$$\Delta(T + U) = \Delta W$$

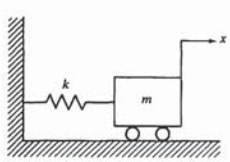


Figure 3–15 Mechanical system.

where $\Delta(T + U)$ is the change in the total energy and ΔW is the net work done on the system by an external force. If no external energy enters the system, then

$$\Delta(T + U) = 0$$

which results in

$$T + U = \text{constant}$$

If we assume no friction, then the mechanical system shown in Figure 3–15 can be considered conservative. The kinetic energy T and potential energy U are given by

$$T = \frac{1}{2}m\dot{x}^2, \qquad U = \frac{1}{2}kx^2$$

Consequently, in the absence of any external energy input,

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}$$

The equation of motion for the system can be obtained by differentiating the total energy with respect to t:

$$\frac{d}{dt}(T + U) = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} = 0$$

Since $\dot{x}$ is not always zero, we have

$$m\ddot{x} + kx = 0$$

which is the equation of motion for the system.

Let us look next at the mechanical system of Figure 3–16. Here, no damping is involved; therefore, the system is conservative. In this case, since the mass is suspended by a spring, the potential energy includes that due to the position of the mass element. At the equilibrium position, the potential energy of the system is

$$U_0 = mgx_0 + \frac{1}{2}k\delta^2$$

where x_0 is the equilibrium position of the mass element above an arbitrary datum line and δ is the static deflection of the spring when the system is in the equilibrium position, or $k\delta = mg$. (For the definition of δ, see Figure 3–10.)

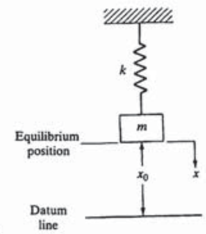


Figure 3–16 Mechanical system.

The instantaneous potential energy U is the instantaneous potential energy of the weight of the mass element, plus the instantaneous elastic energy stored in the spring. Thus,

$$U = mg(x_0 - x) + \frac{1}{2}k(\delta + x)^2$$
$$= mgx_0 - mgx + \frac{1}{2}k\delta^2 + k\delta x + \frac{1}{2}kx^2$$
$$= mgx_0 + \frac{1}{2}k\delta^2 - (mg - k\delta)x + \frac{1}{2}kx^2$$

Since $mg = k\delta$, it follows that

$$U = U_0 + \frac{1}{2}kx^2$$

Note that the increase in the total potential energy of the system is due to the increase in the elastic energy of the spring that results from its deformation from the equilibrium position. Note also that, since x_0 is the displacement measured from an arbitrary datum line, it is possible to choose the datum line such that $U_0 = 0$. Finally, note that an increase (decrease) in the potential energy is offset by a decrease (increase) in the kinetic energy.

The kinetic energy of the system is $T = \frac{1}{2}m\dot{x}^2$. Since the total energy is constant, we obtain

$$T + U = \frac{1}{2}m\dot{x}^2 + U_0 + \frac{1}{2}kx^2 = \text{constant}$$

By differentiating the total energy with respect to t and noting that U_0 is a constant, we have

$$\frac{d}{dt}(T + U) = m\dot{x}\ddot{x} + kx\dot{x} = 0$$

or

$$(m\ddot{x} + kx)\dot{x} = 0$$

Since $\dot{x}$ is not always zero, it follows that

$$m\ddot{x} + kx = 0$$

This is the equation of motion for the system.

Example 3–4

Figure 3–17 shows a homogeneous cylinder of radius R and mass m that is free to rotate about its axis of rotation and that is connected to the wall through a spring. Assuming that the cylinder rolls on a rough surface without sliding, obtain the kinetic energy and potential energy of the system. Then derive the equations of motion from the fact that the total energy is constant. Assume that x and θ are measured from respective equilibrium positions.

The kinetic energy of the cylinder is the sum of the translational kinetic energy of the center of mass and the rotational kinetic energy about the axis of rotation:

$$\text{kinetic energy} = T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy of the system is due to the deflection of the spring:

$$\text{potential energy} = U = \frac{1}{2}kx^2$$

Since the total energy $T + U$ is constant in this conservation system (which means that the loss in potential energy equals the gain in kinetic energy), it follows that

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}kx^2 = \text{constant} \tag{3–19}$$

The cylinder rolls without sliding, which means that $x = R\theta$. Rewriting Equation (3–19) and noting that the moment of inertia J is equal to $\frac{1}{2}mR^2$, we have

$$\frac{3}{4}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}$$

Differentiating both sides of this last equation with respect to t yields

$$\frac{3}{2}m\dot{x}\ddot{x} + kx\dot{x} = 0$$

or

$$\left(m\ddot{x} + \frac{2}{3}kx\right)\dot{x} = 0$$

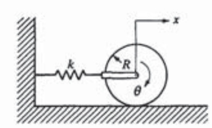


Figure 3–17 Homogeneous cylinder connected to a wall through a spring.

Note that $\dot{x}$ is not always zero, so $m\ddot{x} + \frac{2}{3}kx$ must be identically zero. Therefore,

$$m\ddot{x} + \frac{2}{3}kx = 0$$

or

$$\ddot{x} + \frac{2k}{3m}x = 0$$

This equation describes the horizontal motion of the cylinder. For the rotational motion, we substitute $x = R\theta$ to get

$$\ddot{\theta} + \frac{2k}{3m}\theta = 0$$

In either of the equations of motion, the natural frequency of vibration is the same, $\omega_n = \sqrt{2k/(3m)}$ rad/s.

An energy method for determining natural frequencies. The natural frequency of a conservative system can be obtained from a consideration of the kinetic energy and the potential energy of the system.

Let us assume that we choose the datum line so that the potential energy at the equilibrium state is zero. Then, in such a conservative system, the maximum kinetic energy equals the maximum potential energy, or

$$T_{max} = U_{max}$$

Using this relationship, we are able to determine the natural frequency of a conservative system, as presented in **Example 3–5.**

Example 3–5

Consider the system shown in Figure 3–18. The displacement x is measured from the equilibrium position. The kinetic energy of this system is

$$T = \frac{1}{2}m\dot{x}^2$$

If we choose the datum line so that the potential energy U_0 at the equilibrium state is zero, then the potential energy of the system is given by

$$U = \frac{1}{2}kx^2$$

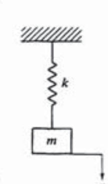


Figure 3–18 Conservative mechanical system.

Let us assume that the system is vibrating about the equilibrium position. Then the displacement is given by

$$x = A \sin \omega t$$

where A is the amplitude of vibration. Consequently,

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega^2(\cos \omega t)^2$$
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2(\sin \omega t)^2$$

Hence, the maximum values of T and U are given by

$$T_{max} = \frac{1}{2}mA^2\omega^2, \qquad U_{max} = \frac{1}{2}kA^2$$

Since $T_{max} = U_{max}$, we have

$$\frac{1}{2}mA^2\omega^2 = \frac{1}{2}kA^2$$

from which we get

$$\omega = \sqrt{\frac{k}{m}}$$

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–3–1

Calculate the moment of inertia about axis xx' of the hollow cylinder shown in Figure 3–19.

Solution The moment of inertia about axis xx' of the solid cylinder of radius R is

$$J_R = \frac{1}{2}m_1R^2$$

where

$$m_1 = \pi R^2 L\rho \qquad (\rho = \text{density})$$

The moment of inertia about axis xx' of the solid cylinder of radius r is

$$J_r = \frac{1}{2}m_2r^2$$

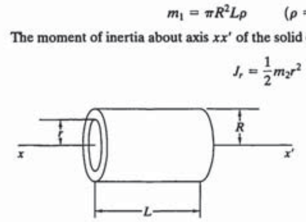


Figure 3–19 Hollow cylinder.

where

$$m_2 = \pi r^2 L\rho$$

Then the moment of inertia about axis xx' of the hollow cylinder shown in the figure is

$$\begin{aligned} J &= J_R - J_r = \frac{1}{2}m_1 R^2 - \frac{1}{2}m_2 r^2 \\ &= \frac{1}{2}[(\pi R^2 L\rho)R^2 - (\pi r^2 L\rho)r^2] \\ &= \frac{1}{2}\pi L\rho\,(R^4 - r^4) \\ &= \frac{1}{2}\pi L\rho\,(R^2 + r^2)(R^2 - r^2) \end{aligned}$$

The mass of the hollow cylinder is

$$m = \pi(R^2 - r^2)L\rho$$

Hence,

$$\begin{aligned} J &= \frac{1}{2}(R^2 + r^2)\pi(R^2 - r^2)L\rho \\ &= \frac{1}{2}m(R^2 + r^2) \end{aligned}$$

(See the third item of Table 3–3.)

Problem A–3–2

A rotating body whose mass is m is suspended by two vertical wires, each of length h, a distance $2a$ apart. The center of gravity is on the vertical line that passes through the midpoint between the points of attachment of the wires. (See Figure 3–20.)

Assume that the body is turned through a small angle about the vertical axis through the center of gravity and is then released. Define the period of oscillation as T. Show that moment of inertia J of the body about the vertical axis that passes through the center of gravity is

$$J = \left(\frac{T}{2\pi}\right)^2 \frac{a^2 mg}{h}$$

Solution Let us assume that, when the body rotates a small angle θ from the equilibrium position, the force in each wire is F. Then, from Figure 3–20, the angle ϕ that each wire makes with the vertical is small. Angles θ and ϕ are related by

$$a\theta = h\phi$$

Thus,

$$\phi = \frac{a\theta}{h}$$

Notice that the vertical component of force F in each wire is equal to $mg/2$. The horizontal component of F is $mg\phi/2$. The horizontal components of F of both wires produce a torque $mg\phi a$ to rotate the body. Thus, the equation of motion for the oscillation is

$$J\ddot{\theta} = -mg\phi a = -mg\frac{a^2\theta}{h}$$

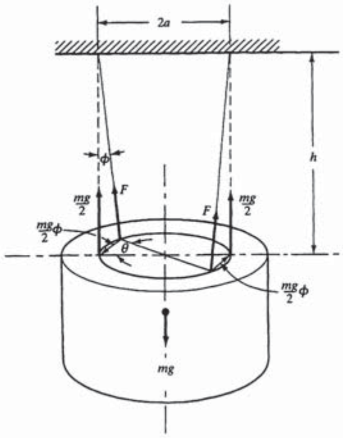


Figure 3–20 Experimental setup for measuring the moment of inertia of a rotating body.

or

$$\ddot{\theta} + \frac{a^2 mg}{Jh}\theta = 0$$

from which the period of the oscillation is found to be

$$T = \frac{2\pi}{\sqrt{\dfrac{a^2 mg}{Jh}}}$$

Solving this last equation for J gives

$$J = \left(\frac{T}{2\pi}\right)^2 \frac{a^2 mg}{h}$$

Problem A–3–3

A brake is applied to a car traveling at a constant speed of 90 km/h. If the deceleration α caused by the braking action is 5 m/s^2, find the time and distance before the car stops.

Solution Note that

$$90 \text{ km/h} = 25 \text{ m/s}$$

The equation of motion for the car is

$$m\ddot{x} = -m\alpha$$

where m is the mass of the car and x is the displacement of the car, measured from the point where the brake is first applied. Integrating this last equation, we have

$$\dot{x}(t) = -\alpha t + v(0)$$

and

$$x(t) = -\frac{1}{2}\alpha t^2 + v(0)t + x(0)$$

where $x(0) = 0$ and $v(0) = 25$ m/s.

Assume that the car stops at $t = t_1$. Then $\dot{x}(t_1) = 0$. The value of t_1 is determined from

$$\dot{x}(t_1) = -\alpha t_1 + v(0) = 0$$

or

$$t_1 = \frac{v(0)}{\alpha} = \frac{25}{5} = 5 \text{ s}$$

The distance traveled before the car stops is

$$\begin{aligned} x(t_1) &= -\frac{1}{2}\alpha t_1^2 + v(0)t_1 = -\frac{1}{2}\times 5 \times 5^2 + 25 \times 5 \\ &= 62.5 \text{ m} \end{aligned}$$

Problem A–3–4

Consider a homogeneous cylinder with radius 1 m. The mass of the cylinder is 100 kg. What will be the angular acceleration of the cylinder if it is acted on by an external torque of 10 N-m about its axis? Assume no friction in the system.

Solution The moment of inertia is

$$J = \frac{1}{2}mR^2 = \frac{1}{2}\times 100 \times 1^2 = 50 \text{ kg-m}^2$$

The equation of motion for this system is

$$J\ddot{\theta} = T$$

where $\ddot{\theta}$ is the angular acceleration. Therefore,

$$\ddot{\theta} = \frac{T}{J} = \frac{10 \text{ N-m}}{50 \text{ kg-m}^2} = 0.2 \text{ rad/s}^2$$

(Note that, in examining the units of this last equation, we see that the unit of $\ddot{\theta}$ is not s^{-2}, but rad/s^2. This usage occurs because writing rad/s^2 indicates that the angle θ is measured in radians. The radian is the ratio of a length of arc to the radius of a circle. That is, in radian measure, the angle is a pure number. In the algebraic handling of units, the radian is added as necessary.)

Problem A–3–5

Suppose that a disk is rotated at a constant speed of 100 rad/s and we wish to stop it in 2 min. Assuming that the moment of inertia J of the disk is 6 kg-m^2, determine the torque T necessary to stop the rotation. Assume no friction in the system.

Solution The necessary torque T must act so as to reduce the speed of the disk. Thus, the equation of motion is

$$J\dot{\omega} = -T, \qquad \omega(0) = 100$$

Noting that the torque T is a constant and taking the Laplace transform of this last equation, we obtain

$$J[s\Omega(s) - \omega(0)] = -\frac{T}{s}$$

Substituting $J = 6$ and $\omega(0) = 100$ into this equation and solving for $\Omega(s)$, we get

$$\Omega(s) = \frac{100}{s} - \frac{T}{6s^2}$$

The inverse Laplace transform of $\Omega(s)$ gives

$$\omega(t) = 100 - \frac{T}{6}t$$

At $t = 2$ min $= 120$ s, we want to stop, so $\omega(120)$ must equal zero. Therefore,

$$\omega(120) = 0 = 100 - \frac{T}{6}\times 120$$

Solving for T, we get

$$T = \frac{600}{120} = 5 \text{ N-m}$$

Problem A–3–6

Obtain the equivalent spring constant for the system shown in Figure 3–21.

Solution For the springs in parallel, the equivalent spring constant k_{eq} is obtained from

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

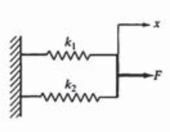


Figure 3–21 System consisting of two springs in parallel.

Problem A–3–7

Find the equivalent spring constant for the system shown in Figure 3–22(a), and show that it can also be obtained graphically as in Figure 3–22(b).

Solution For the springs in series, the force in each spring is the same. Thus,

$$k_1 y = F, \qquad k_2(x - y) = F$$

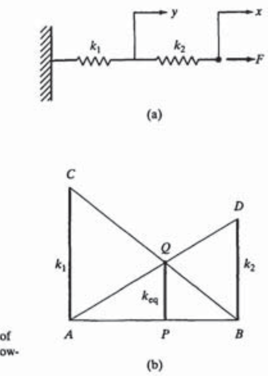


Figure 3–22 (a) System consisting of two springs in series; (b) diagram showing the equivalent spring constant.

Eliminating y from these two equations yields

$$k_2\left(x - \frac{F}{k_1}\right) = F$$

or

$$k_2 x = F + \frac{k_2}{k_1}F = \frac{k_1 + k_2}{k_1}F$$

The equivalent spring constant for this case is then found to be

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

For the graphical solution, notice that

$$\frac{\overline{AC}}{\overline{PQ}} = \frac{\overline{AB}}{\overline{PB}}, \qquad \frac{\overline{BD}}{\overline{PQ}} = \frac{\overline{AB}}{\overline{AP}}$$

from which it follows that

$$\overline{PB} = \frac{\overline{AB}\cdot\overline{PQ}}{\overline{AC}}, \qquad \overline{AP} = \frac{\overline{AB}\cdot\overline{PQ}}{\overline{BD}}$$

Since $\overline{AP} + \overline{PB} = \overline{AB}$, we have

$$\frac{\overline{AB}\cdot\overline{PQ}}{\overline{BD}} + \frac{\overline{AB}\cdot\overline{PQ}}{\overline{AC}} = \overline{AB}$$

or

$$\frac{\overline{PQ}}{\overline{BD}} + \frac{\overline{PQ}}{\overline{AC}} = 1$$

Solving for $\overline{PQ}$, we obtain

$$\overline{PQ} = \frac{1}{\dfrac{1}{\overline{AC}} + \dfrac{1}{\overline{BD}}}$$

So if lengths $\overline{AC}$ and $\overline{BD}$ represent the spring constants k_1 and k_2, respectively, then length $\overline{PQ}$ represents the equivalent spring constant k_{eq}. That is,

$$\overline{PQ} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = k_{eq}$$

Problem A–3–8

In Figure 3–23, the simple pendulum shown consists of a sphere of mass m suspended by a string of negligible mass. Neglecting the elongation of the string, find a mathematical model of the pendulum. In addition, find the natural frequency of the system when θ is small. Assume no friction.

Solution The gravitational force mg has the tangential component $mg \sin\theta$ and the normal component $mg\cos\theta$. The torque due to the tangential component is $-mgl\sin\theta$, so the equation of motion is

$$J\ddot{\theta} = -mgl\sin\theta$$

where $J = ml^2$. Therefore,

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

For small θ, $\sin\theta \doteqdot \theta$, and the equation of motion simplifies to

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

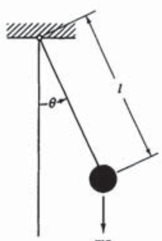


Figure 3–23 Simple pendulum.

This is a mathematical model of the system. The natural frequency is then obtained as

$$\omega_n = \sqrt{\frac{g}{l}}$$

Problem A–3–9

Consider the spring-loaded pendulum system shown in Figure 3–24. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical ($\theta = 0$). Assume also that the friction involved is negligible and the angle of oscillation, θ, is small. Obtain a mathematical model of the system.

Solution Two torques are acting on this system, one due to the gravitational force and the other due to the spring force. Applying Newton's second law, we find that the equation of motion for the system becomes

$$J\ddot{\theta} = -mgl\sin\theta - 2(ka\sin\theta)(a\cos\theta)$$

where $J = ml^2$. Rewriting this last equation, we obtain

$$ml^2\ddot{\theta} + mgl\sin\theta + 2ka^2\sin\theta\cos\theta = 0$$

For small θ, we have $\sin\theta = \theta$ and $\cos\theta = 1$. So the equation of motion can be simplified to

$$ml^2\ddot{\theta} + (mgl + 2ka^2)\theta = 0$$

or

$$\ddot{\theta} + \left(\frac{g}{l} + 2\frac{ka^2}{ml^2}\right)\theta = 0$$

This is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{g}{l} + 2\frac{ka^2}{ml^2}}$$

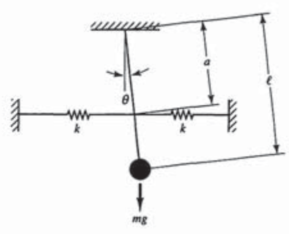


Figure 3–24 Spring-loaded pendulum system.

Problem A–3–10

Consider the rolling motion of the ship shown in Figure 3–25. The force due to buoyancy is $-w$ and that due to gravity is w. These two forces produce a couple that causes rolling motion of the ship. The point where the vertical line through the center of buoyancy, C, intersects the symmetrical line through the center of gravity, which is in the ship's centerline plane, is called the *metacenter* (point M). Define

R = distance of the metacenter to the center of gravity of the ship = $\overline{MG}$

J = moment of inertia of the ship about its longitudinal centroidal axis

Derive the equation of rolling motion of the ship when the rolling angle θ is small.

Solution From Figure 3–25, we obtain

$$J\ddot{\theta} = -wR\sin\theta$$

or

$$J\ddot{\theta} + wR\sin\theta = 0$$

For small θ, we have $\sin\theta \doteqdot \theta$. Hence, the equation of rolling motion of the ship is

$$J\ddot{\theta} + wR\theta = 0$$

The natural frequency of the rolling motion is $\sqrt{wR/J}$. Note that the distance $R(= \overline{MG})$ is considered positive when the couple of weight and buoyancy tends to rotate the ship toward the upright position. That is, R is positive if point M is above point G, and R is negative if point M is below point G.

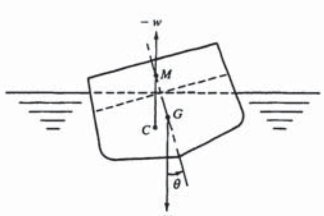


Figure 3–25 Rolling motion of a ship.

Problem A–3–11

In Figure 3–26, a homogeneous disk of radius R and mass m that can rotate about the center of mass of the disk is hung from the ceiling and is spring preloaded. (Two springs are connected by a wire that passes over a pulley as shown.) Each spring is prestretched by an amount x. Assuming that the disk is initially rotated by a small angle θ and then released, obtain both a mathematical model of the system and the natural frequency of the system.

Solution If the disk is rotated by an angle θ as shown in Figure 3–26, then the right spring is stretched by $x + R\theta$ and the left spring is stretched by $x - R\theta$. So, applying Newton's second law to the rotational motion of the disk gives

$$J\ddot{\theta} = -k(x + R\theta)R + k(x - R\theta)R$$

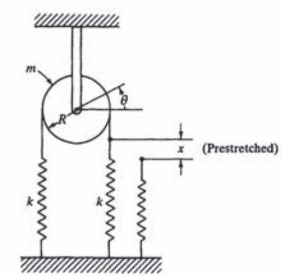


Figure 3-26 Spring-pulley system.

where the moment of inertia J is $\frac{1}{2}mR^2$. Simplifying the equation of motion, we have

$$\ddot{\theta} + \frac{4k}{m}\theta = 0$$

This is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{4k}{m}}$$

Problem A-3-12

For the spring-mass-pulley system of Figure 3-27, the moment of inertia of the pulley about the axis of rotation is J and the radius is R. Assume that the system is initially at equilibrium. The gravitational force of mass m causes a static deflection of the spring such that $k\delta = mg$. Assuming that the displacement x of mass m is measured from the equilibrium position, obtain a mathematical model of the system. In addition, find the natural frequency of the system.

Solution Applying Newton's second law, we obtain, for mass m,

$$m\ddot{x} = -T \tag{3-20}$$

where T is the tension in the wire. (Note that since x is measured from the static equilibrium position the term mg does not enter into the equation.) For the rotational motion of the pulley,

$$J\ddot{\theta} = TR - kxR \tag{3-21}$$

If we eliminate the tension T from Equations (3-20) and (3-21), the result is

$$J\ddot{\theta} = -m\ddot{x}R - kxR \tag{3-22}$$

Noting that $x = R\theta$, we can simplify Equation (3-22) to

$$(J + mR^2)\ddot{\theta} + kR^2\theta = 0$$

or

$$\ddot{\theta} + \frac{kR^2}{J + mR^2}\theta = 0$$

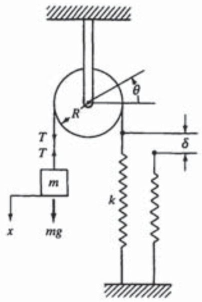


Figure 3-27 Spring-mass-pulley system.

This is a mathematical model of the system. The natural frequency is

$$\omega_n = \sqrt{\frac{kR^2}{J + mR^2}}$$

Problem A-3-13

In the mechanical system of Figure 3-28, one end of the lever is connected to a spring and a damper, and a force F is applied to the other end of the lever. Derive a mathematical model of the system. Assume that the displacement x is small and the lever is rigid and massless.

Solution From Newton's second law, for small displacement x, the rotational motion about pivot P is given by

$$Fl_1 - (b\dot{x} + kx)l_2 = 0$$

or

$$b\dot{x} + kx = \frac{l_1}{l_2}F$$

which is a mathematical model of the system.

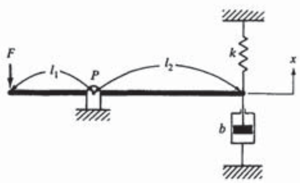


Figure 3-28 Lever system.

Problem A-3-14

Consider the mechanical system shown in Figure 3-29(a). The massless bar AA' is displaced 0.05 m by a constant force of 100 N. Suppose that the system is at rest before the force is abruptly released. The time-response curve when the force is abruptly released at $t = 0$ is shown in Figure 3-29(b). Determine the numerical values of b and k.

Solution Since the system is at rest before the force is abruptly released, the equation of motion is

$$kx = F \qquad t \le 0$$

Note that the effect of the force F is to give the initial condition

$$x(0) = \frac{F}{k}$$

Since $x(0) = 0.05$ m, we have

$$k = \frac{F}{x(0)} = \frac{100}{0.05} = 2000 \text{ N/m}$$

At $t = 0$, F is abruptly released, so, for $t > 0$, the equation of motion becomes

$$b\dot{x} + kx = 0 \qquad t > 0$$

Taking the Laplace transform of this last equation, we have

$$b[sX(s) - x(0)] + kX(s) = 0$$

Substituting $x(0) = 0.05$ and solving the resulting equation for $X(s)$, we get

$$X(s) = \frac{0.05}{s + \frac{k}{b}}$$

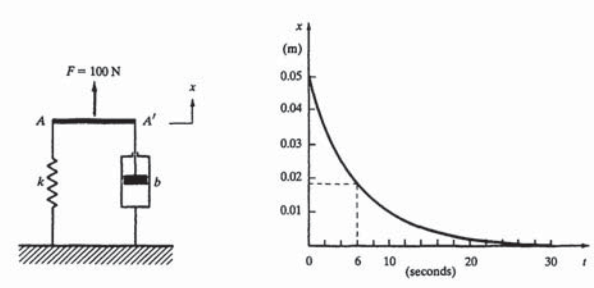


Figure 3-29 (a) Mechanical system; (b) response curve.

The inverse Laplace transform of $X(s)$, using the value of $k = 2000$ just obtained, is

$$x(t) = 0.05e^{-(2000/b)t}$$

Since the solution is an exponential function, at t = time constant = $b/2000$ the response becomes

$$x\left(\frac{b}{2000}\right) = 0.05 \times 0.368 = 0.0184 \text{ m}$$

From Figure 3-29(b), $x = 0.0184$ m occurs at $t = 6$ s. Hence,

$$\frac{b}{2000} = 6$$

from which it follows that

$$b = 12{,}000 \text{ N-s/m}$$

Problem A-3-15

In the rotating system shown in Figure 3-30, assume that the torque T applied to the rotor is of short duration, but large amplitude, so that it can be considered an impulse input. Assume also that initially the angular velocity is zero, or $\omega(0-) = 0$. Given the numerical values

$$J = 10 \text{ kg-m}^2$$

and

$$b = 2 \text{ N-s/m}$$

find the response $\omega(t)$ of the system. Assume that the amplitude of torque T is 300 N-m/s and that the duration of T is 0.1 s; that is, the magnitude of the impulse input is $300 \times 0.1 = 30$ N-m. Show that the effect of an impulse input on a first-order system that is at rest is to generate a nonzero initial condition at $t = 0+$.

Solution The equation of motion for the system is

$$J\dot{\omega} + b\omega = T, \qquad \omega(0-) = 0$$

Let us define the impulsive torque of magnitude 1 N-m as $\delta(t)$. Then, by substituting the given numerical values into this last equation, we obtain

$$10\dot{\omega} + 2\omega = 30\delta(t)$$

Taking the $\mathscr{L}_-$ transform of this last equation, we have

$$10[s\Omega(s) - \omega(0-)] + 2\Omega(s) = 30$$

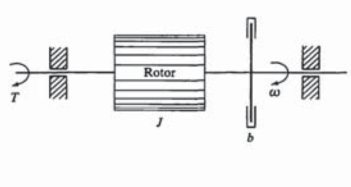


Figure 3-30 Mechanical rotating system.

or

$$\Omega(s) = \frac{30}{10s + 2} = \frac{3}{s + 0.2}$$

The inverse Laplace transform of $\Omega(s)$ is

$$\omega(t) = 3e^{-0.2t} \tag{3–23}$$

Note that $\omega(0+) = 3$ rad/s. The angular velocity of the rotor is thus changed instantaneously from $\omega(0-) = 0$ to $\omega(0+) = 3$ rad/s.

If the system is subjected only to the initial condition $\omega(0) = 3$ rad/s and there is no external torque ($T = 0$), then the equation of motion becomes

$$10\dot{\omega} + 2\omega = 0, \qquad \omega(0) = 3$$

Taking the Laplace transform of this last equation, we obtain

$$10[s\Omega(s) - \omega(0)] + 2\Omega(s) = 0$$

or

$$\Omega(s) = \frac{10\omega(0)}{10s + 2} = \frac{30}{10s + 2} = \frac{3}{s + 0.2}$$

The inverse Laplace transform of $\Omega(s)$ gives

$$\omega(t) = 3e^{-0.2t}$$

which is identical to Equation (3–23).

From the preceding analysis, we see that the response of a first-order system that is initially at rest to an impulse input is identical to the motion from the initial condition at $t = 0+$. That is, the effect of the impulse input on a first-order system that is initially at rest is to generate a nonzero initial condition at $t = 0+$.

Problem A–3–16

A mass $M = 8$ kg is supported by a spring with spring constant $k = 400$ N/m and a damper with $b = 40$ N-s/m, as shown in Figure 3–31. When a mass $m = 2$ kg is gently placed on the top of mass M, the system exhibits vibrations. Assuming that the displacement x of the masses is measured from the equilibrium position before mass m is placed on mass M, determine the response $x(t)$ of the system. Determine also the static deflection δ—the deflection of the spring when the transient response died out. Assume that $x(0) = 0$ and $\dot{x}(0) = 0$.

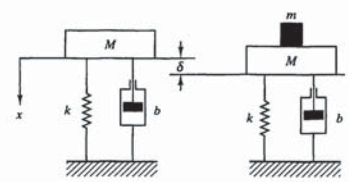


Figure 3–31 Mechanical system.

Notice that the numerical values of M, m, b, and k are given in the SI system of units. If the units are changed to BES units, how does the mathematical model change? How will the solution be changed?

Solution We shall first solve this problem using SI units. The input to the system is a constant force mg that acts as a step input to the system. The system is at rest before $t = 0$, and at $t = 0+$ the masses start to move up and down. A mathematical model, or equation of motion, is

$$(M + m)\ddot{x} + b\dot{x} + kx = mg$$

where $M + m = 10$ kg, $b = 40$ N-s/m, $k = 400$ N/m, and $g = 9.807$ m/s^2.

Substituting the numerical values into the equation of motion, we find that

$$10\ddot{x} + 40\dot{x} + 400x = 2 \times 9.807$$

or

$$\ddot{x} + 4\dot{x} + 40x = 1.9614 \tag{3–24}$$

Equation (3–24) is a mathematical model for the system when the units used are SI units. To obtain the response $x(t)$, we take the Laplace transform of Equation (3–24) and substitute the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ into the Laplace-transformed equation as follows:

$$s^2X(s) + 4sX(s) + 40X(s) = \frac{1.9614}{s}$$

Solving for $X(s)$ yields

$$\begin{aligned} X(s) &= \frac{1.9614}{(s^2 + 4s + 40)s} \\ &= \frac{1.9614}{40}\left(\frac{1}{s} - \frac{s + 4}{s^2 + 4s + 40}\right) \\ &= 0.04904\left[\frac{1}{s} - \frac{2}{6}\,\frac{6}{(s + 2)^2 + 6^2} - \frac{s + 2}{(s + 2)^2 + 6^2}\right] \end{aligned}$$

The inverse Laplace transform of this last equation gives

$$x(t) = 0.04904\left(1 - \frac{1}{3}e^{-2t}\sin 6t - e^{-2t}\cos 6t\right)\text{m}$$

This solution gives the up-and-down motion of the total mass $(M + m)$. The static deflection δ is 0.04904 m.

Next, we shall solve the same problem using BES units. If we change the numerical values of M, m, b, and k given in the SI system of units to BES units, we obtain

$$\begin{aligned} M &= 8 \text{ kg} = 0.54816 \text{ slug} \\ m &= 2 \text{ kg} = 0.13704 \text{ slug} \\ b &= 40 \text{ N-s/m} = 2.74063 \text{ lb}_f\text{-s/ft} \\ k &= 400 \text{ N/m} = 27.4063 \text{ lb}_f/\text{ft} \\ mg &= 0.13704 \text{ slug} \times 32.174 \text{ ft/s}^2 = 4.4091 \text{ slug-ft/s}^2 \\ &= 4.4091 \text{ lb}_f \end{aligned}$$

Then the equation of motion for the system becomes

$$0.6852\ddot{x} + 2.74063\dot{x} + 27.4063x = 4.4091$$

which can be simplified to

$$\ddot{x} + 4\dot{x} + 40x = 6.4348 \tag{3–25}$$

Equation (3–25) is a mathematical model for the system. Comparing Equations (3–24) and (3–25), we notice that the left-hand sides of the equations are the same, which means that the characteristic equation remains the same. The solution of Equation (3–25) is

$$x(t) = 0.1609\left(1 - \frac{1}{3}e^{-2t}\sin 6t - e^{-2t}\cos 6t\right)\text{ft}$$

The static deflection δ is 0.1609 ft. (Note that 0.1609 ft = 0.04904 m.) Notice that, whenever consistent systems of units are used, the results carry the same information.

Problem A–3–17

Consider the spring-loaded inverted pendulum shown in Figure 3–32. Assume that the spring force acting on the pendulum is zero when the pendulum is vertical ($\theta = 0$). Assume also that the friction involved is negligible. Obtain a mathematical model of the system when the angle θ is small, that is, when $\sin\theta \doteqdot \theta$ and $\cos\theta \doteqdot 1$. Also, obtain the natural frequency ω_n of the system.

Solution Suppose that the inverted pendulum is given an initial angular displacement $\theta(0)$ and released with zero initial angular velocity. Then, from Figure 3–32, for small θ such that $\sin\theta \doteqdot \theta$ and $\cos\theta \doteqdot 1$, the left-hand side spring is stretched by $h\theta$ and the right-hand side spring is compressed by $h\theta$. Hence, the torque acting on the pendulum in the counterclockwise direction is $2kh^2\theta$. The torque due to the gravitational force is $mgl\theta$, which acts in the clockwise direction. The moment of inertia of the pendulum is ml^2. Thus, the equation of motion of the system for small θ is

$$ml^2\ddot{\theta} = mgl\theta - 2kh^2\theta$$

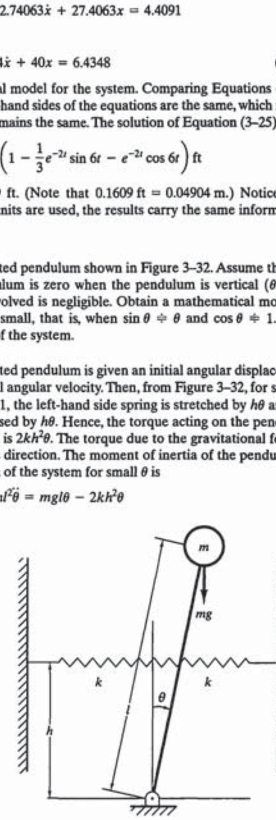


Figure 3–32 Spring-loaded inverted pendulum.

or

$$\ddot{\theta} + \left(\frac{2kh^2}{ml^2} - \frac{g}{l}\right)\theta = 0$$

This is a mathematical model of the system for small θ. If $2kh^2 > mgl$, the torques acting in the system cause it to vibrate. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{2kh^2}{ml^2} - \frac{g}{l}}$$

If, however, $2kh^2 < mgl$, then, starting with a small disturbance, the angle θ increases and the pendulum will fall down or hit the vertical wall and stop. The vibration will not occur.

Problem A–3–18

Consider the spring–mass–pulley system of Figure 3–33(a). If the mass m is pulled downward a short distance and released, it will vibrate. Obtain the natural frequency of the system by applying the law of conservation of energy.

Solution Define x, y, and θ as the displacement of mass m, the displacement of the pulley, and the angle of rotation of the pulley, measured respectively from their corresponding equilibrium positions. Note that $x = 2y$, $R\theta = x - y = y$, and $J = \frac{1}{2}MR^2$.

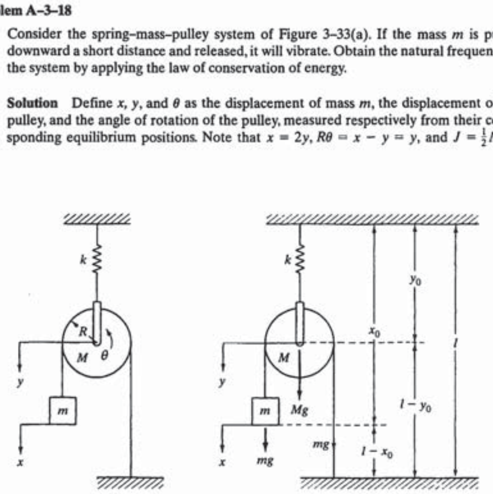


Figure 3–33 (a) Spring–mass–pulley system; (b) diagram for figuring out potential energy of the system.

The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{8}M\dot{x}^2 + \frac{1}{4}MR^2\left(\frac{\dot{y}}{R}\right)^2$$
$$= \frac{1}{2}m\dot{x}^2 + \frac{3}{16}M\dot{x}^2$$

The potential energy U of the system can be obtained from Figure 3–33(b). At the equilibrium state, the potential energy is

$$U_0 = \frac{1}{2}ky_\delta^2 + Mg(l - y_0) + mg(l - x_0)$$

where y_δ is the static deflection of the spring due to the hanging masses M and m. When masses m and M are displaced by x and y, respectively, the instantaneous potential energy can be obtained as

$$U = \frac{1}{2}k(y_\delta + y)^2 + Mg(l - y_0 - y) + mg(l - x_0 - x)$$
$$= \frac{1}{2}ky_\delta^2 + ky_\delta y + \frac{1}{2}ky^2 + Mg(l - y_0) - Mgy + mg(l - x_0) - mgx$$
$$= U_0 + \frac{1}{2}ky^2 + ky_\delta y - Mgy - mgx$$

Again from Figure 3–33(b), the spring force ky_δ must balance with $Mg + 2mg$, or

$$ky_\delta = Mg + 2mg$$

Therefore,

$$ky_\delta y = Mgy + 2mgy = Mgy + mgx$$

and

$$U = U_0 + \frac{1}{2}ky^2 = U_0 + \frac{1}{8}kx^2$$

where U_0 is the potential energy at the equilibrium state.

Applying the law of conservation of energy to this conservative system gives

$$T + U = \frac{1}{2}m\dot{x}^2 + \frac{3}{16}M\dot{x}^2 + U_0 + \frac{1}{8}kx^2 = \text{constant}$$

and differentiating this last equation with respect to t yields

$$m\dot{x}\ddot{x} + \frac{3}{8}M\dot{x}\ddot{x} + \frac{1}{4}kx\dot{x} = 0$$

or

$$\left[\left(m + \frac{3}{8}M\right)\ddot{x} + \frac{1}{4}kx\right]\dot{x} = 0$$

Since $\dot{x}$ is not always zero, we must have

$$\left(m + \frac{3}{8}M\right)\ddot{x} + \frac{1}{4}kx = 0$$

or

$$\ddot{x} + \frac{2k}{8m + 3M}x = 0$$

The natural frequency of the system, therefore, is

$$\omega_n = \sqrt{\frac{2k}{8m + 3M}}$$

Problem A–3–19

If, for the spring–mass system of Figure 3–34, the mass m_s of the spring is small, but not negligibly small, compared with the suspended mass m, show that the inertia of the spring can be allowed for by adding one-third of its mass m_s to the suspended mass m and then treating the spring as a massless spring.

Solution Consider the free vibration of the system. The displacement x of the mass is measured from the static equilibrium position. In free vibration, the displacement can be written as

$$x = A\cos\omega t$$

Since the mass of the spring is comparatively small, we can assume that the spring is stretched uniformly. Then the displacement of a point in the spring at a distance ξ from the top is given by $(\xi/l)A\cos\omega t$.

In the mean position, where $x = 0$ and the velocity of mass m is maximum, the velocity of the suspended mass is $A\omega$ and that of the spring at the distance ξ from the

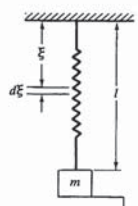


Figure 3–34 Spring–mass system.

top is $(\xi/l)A\omega$. The maximum kinetic energy is

$$T_{\max} = \frac{1}{2}m(A\omega)^2 + \int_0^l \frac{1}{2}\left(\frac{m_s}{l}\right)\left(\frac{\xi}{l}A\omega\right)^2 d\xi$$
$$= \frac{1}{2}mA^2\omega^2 + \frac{1}{2}\left(\frac{m_s}{l}\right)\left(\frac{A^2\omega^2}{l^2}\right)\frac{1}{3}l^3$$
$$= \frac{1}{2}\left(m + \frac{m_s}{3}\right)A^2\omega^2$$

Note that the mass of the spring does not affect the change in the potential energy of the system and that, if the spring were massless, the maximum kinetic energy would have been $\frac{1}{2}mA^2\omega^2$. Therefore, we conclude that the inertia of the spring can be allowed for simply by adding one-third of mass m_s to the suspended mass m and then treating the spring as a massless spring, provided that m_s is small compared with m.

PROBLEMS

Problem B–3–1

A homogeneous disk has a diameter of 1 m and mass of 100 kg. Obtain the moment of inertia of the disk about the axis perpendicular to the disk and passing through its center.

Problem B–3–2

Figure 3–35 shows an experimental setup for measuring the moment of inertia of a rotating body. Suppose that the moment of inertia of a rotating body about axis AA' is known. Describe a method to determine the moment of inertia of any rotating body, using this experimental setup.

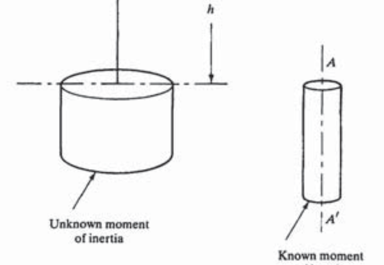


Figure 3–35 Experimental setup for measuring the moment of inertia of a rotating body.

Problem B–3–3

A ball is dropped from a point 100 m above the ground with zero initial velocity. How long will it take until the ball hits the ground? What is the velocity when the ball hits the ground?

Problem B–3–4

A flywheel of $J = 50$ kg-m^2 initially standing still is subjected to a constant torque. If the angular velocity reaches 20 Hz in 5 s, find the torque given to the flywheel.

Problem B–3–5

A brake is applied to a flywheel rotating at an angular velocity of 100 rad/s. If the angular velocity reduces to 20 rad/s in 15 s, find (a) the deceleration produced by the brake and (b) the total angle the flywheel rotates in the 15-s period.

Problem B–3–6

Consider the series-connected springs shown in Figure 3–36(a). Referring to Figure 3–36(b), show that the equivalent spring constant k_{eq} can be graphically obtained as the length $\overline{OC}$ if lengths $\overline{OA}$ and $\overline{OB}$ represent k_1 and k_2, respectively.

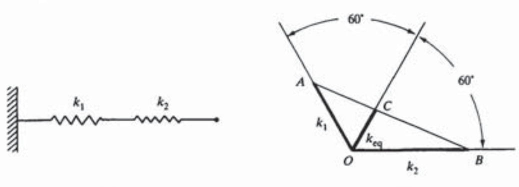


Figure 3–36 (a) System consisting of two springs in series; (b) diagram showing the equivalent spring constant.

Problem B–3–7

Obtain the equivalent spring constant k_{eq} for the system shown in Figure 3–37.

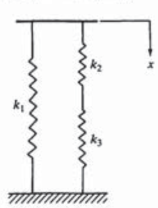


Figure 3–37 System consisting of three springs.

Problem B–3–8

Obtain the equivalent viscous-friction coefficient b_{eq} for each of the systems shown in Figure 3–38 (a) and (b).

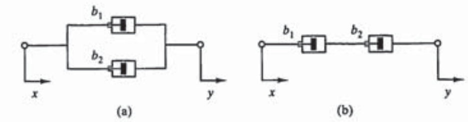


Figure 3–38 (a) Two dampers connected in parallel; (b) two dampers connected in series.

Problem B–3–9

Obtain the equivalent viscous-friction coefficient b_{eq} of the system shown in Figure 3–39.

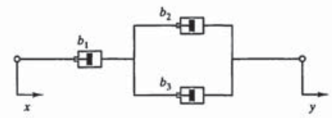


Figure 3–39 Damper system.

Problem B–3–10

Find the natural frequency of the system shown in Figure 3–40.

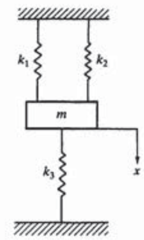


Figure 3–40 Mechanical system.

Problem B–3–11

Consider the U-shaped manometer shown in Figure 3–41. The liquid partially fills the U-shaped glass tube. Assuming that the total mass of the liquid in the tube is m, the

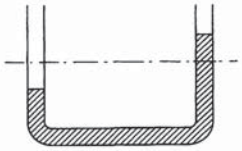


Figure 3–41 U-shaped manometer system.

total length of liquid in the tube is L, and the viscosity of the liquid is negligible, what is the equation of motion of the liquid? Find the frequency of oscillation.

Problem B–3–12

In the mechanical system shown in Figure 3–42, assume that the rod is massless, perfectly rigid, and pivoted at point P. The displacement x is measured from the equilibrium position. Assuming that x is small, that the weight mg at the end of the rod is 5 N, and that the spring constant k is 400 N/m, find the natural frequency of the system.

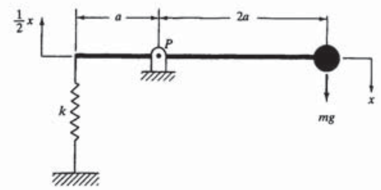


Figure 3–42 Mechanical system.

Problem B–3–13

Obtain a mathematical model of the system shown in Figure 3–43. The input to the system is the angle θ_i and the output is the angle θ_o.

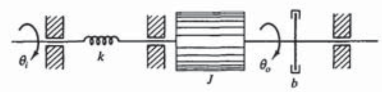


Figure 3–43 Mechanical system.

Problem B–3–14

Obtain a mathematical model for the system shown in Figure 3–44.

Problem B–3–15

Consider the system shown in Figure 3–45, where $m = 2$ kg, $b = 4$ N-s/m, and $k = 20$ N/m. Assume that $x(0) = 0.1$ m and $\dot{x}(0) = 0$. [The displacement $x(t)$ is measured from the equilibrium position.]

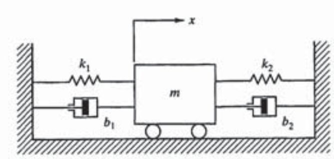


Figure 3–44 Mechanical system.

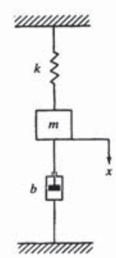


Figure 3–45 Mechanical system.

Derive a mathematical model of the system. Then find $x(t)$ as a function of time t.

Problem B–3–16

By applying Newton's second law to the spring–mass–pulley system of Figure 3–33(a), obtain the motion of mass m when it is pulled down a short distance and then released. The displacement x of a hanging mass m is measured from the equilibrium position. (The mass, the radius, and the moment of inertia of the pulley are M, R, and $J = \frac{1}{2}MR^2$, respectively.)

Problem B–3–17

Consider the mechanical system shown in Figure 3–46. Two pulleys, small and large, are bolted together and act as one piece. The total moment of inertia of the pulleys is J. The mass m is connected to the spring k_1 by a wire wrapped around the large pulley. The gravitational force mg causes static deflection of the spring such that $k_1\delta = mg$. Assume that the displacement x of mass m is measured from the equilibrium position. Two springs (denoted by k_2) are connected by a wire that passes over the small pulley as shown in the figure. Each of the two springs is prestretched by an amount y.

Obtain a mathematical model of the system. Also, obtain the natural frequency of the system.

Problem B–3–18

A disk of radius 0.5 m is subjected to a tangential force of 50 N at its periphery and is rotating at an angular velocity of 100 rad/s. Calculate the torque and power of the disk shaft.

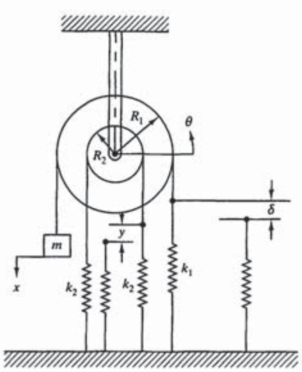


Figure 3–46 Mechanical system.

Problem B–3–19

Referring to the spring-loaded inverted pendulum system shown in Figure 3–32, obtain the natural frequency ω_n of the system, using the energy method that equates the maximum kinetic energy T_{max} and the maximum potential energy U_{max}. (Choose the potential energy at the equilibrium state to be zero.)

Problem B–3–20

Assuming that mass m of the rod of the pendulum shown in Figure 3–47 is small, but not negligible, compared with mass M, find the natural frequency of the pendulum when the angle θ is small. (Include the effect of m in the expression of the natural frequency.)

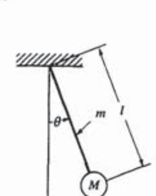


Figure 3–47 Pendulum system.

4

Transfer-Function Approach to Modeling Dynamic Systems

4–1 INTRODUCTION

In this chapter, we present the transfer-function approach to modeling and analyzing dynamic systems. We first define the transfer function and then introduce block diagrams. Since MATLAB plays an important role in obtaining computational solutions of transient response problems, we present a detailed introduction to writing MATLAB programs to obtain response curves for time-domain inputs such as the step, impulse, ramp, and others.

In the field of system dynamics, transfer functions are frequently used to characterize the input–output relationships of components or systems that can be described by linear, time-invariant differential equations. We begin this section by defining the transfer function and deriving the transfer function of a mechanical system. Then we discuss the impulse response function, or the weighting function, of the system.

Transfer Function. The *transfer function* of a linear, time-invariant differential-equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the differential equation

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace-transformed output to the Laplace-transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$
$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (4\text{–}1)$$

By using the concept of a transfer function, it is possible to represent system dynamics by algebraic equations in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an *nth-order system*.

Comments on the Transfer Function. The applicability of the concept of the transfer function is limited to linear, time-invariant differential-equation systems. Still, the transfer-function approach is used extensively in the analysis and design of such systems. The following list gives some important comments concerning the transfer function of a system described by a linear, time-invariant differential equation:

1. The transfer function of a system is a mathematical model of that system, in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
2. The transfer function is a property of a system itself, unrelated to the magnitude and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)
4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.
5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

Example 4–1

Consider the mechanical system shown in Figure 4–1. The displacement x of the mass m is measured from the equilibrium position. In this system, the external force $f(t)$ is the input and x is the output.

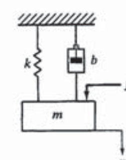


Figure 4–1 Mechanical system.

The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

Taking the Laplace transform of both sides of this equation and assuming that all initial conditions are zero yields

$$(ms^2 + bs + k)X(s) = F(s)$$

where $X(s) = \mathscr{L}[x(t)]$ and $F(s) = \mathscr{L}[f(t)]$. From Equation (4–1), the transfer function for the system is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Impulse-Response Function. The transfer function of a linear, time-invariant system is

$$G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ is the Laplace transform of the input and $Y(s)$ is the Laplace transform of the output and where we assume that all initial conditions involved are zero. It follows that the output $Y(s)$ can be written as the product of $G(s)$ and $X(s)$, or

$$Y(s) = G(s)X(s) \qquad (4\text{–}2)$$

Now, consider the output (response) of the system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, or $X(s) = 1$, the Laplace transform of the output of the system is

$$Y(s) = G(s) \qquad (4\text{–}3)$$

The inverse Laplace transform of the output given by Equation (4–3) yields the impulse response of the system. The inverse Laplace transform of $G(s)$, or

$$\mathscr{L}^{-1}[G(s)] = g(t)$$

is called the *impulse-response function*, or the *weighting function*, of the system.

The impulse-response function $g(t)$ is thus the response of a linear system to a unit-impulse input when the initial conditions are zero. The Laplace transform of $g(t)$ gives the transfer function. Therefore, the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of a system by exciting it with an impulse input and measuring the response. (In practice, a large pulse input with a very short duration compared with the significant time constants of the system may be considered an impulse.)

Outline of the Chapter. Section 4–1 has presented the concept of the transfer function and impulse-response function. Section 4–2 discusses the block diagram. Section 4–3 sets forth the MATLAB approach to the partial-fraction expansion of a ratio of two polynomials, $B(s)/A(s)$. Section 4–4 details the MATLAB approach to the transient response analysis of transfer-function systems.

4–2 BLOCK DIAGRAMS

Block diagrams of dynamic systems. A block diagram of a dynamic system is a pictorial representation of the functions performed by each component of the system and of the flow of signals within the system. Such a diagram depicts the interrelationships that exist among the various components. Differing from a purely abstract mathematical representation, a block diagram has the advantage of indicating the signal flows of the actual system more realistically.

In a block diagram, all system variables are linked to each other through functional blocks. The *functional block*, or simply *block*, is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that a signal can pass only in the direction of the arrows. Thus, a block diagram of a dynamic system explicitly shows a unilateral property.

Figure 4–2 shows an element of a block diagram. The arrowhead pointing toward the block indicates the input to the block, and the arrowhead leading away from the block represents the output of the block. As mentioned, such arrows represent signals.

Note that the dimension of the output signal from a block is the dimension of the input signal multiplied by the dimension of the transfer function in the block.

The advantages of the block diagram representation of a system lie in the fact that it is easy to form the overall block diagram for the entire system merely by connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

In general, the functional operation of a system can be visualized more readily by examining a block diagram of the system than by examining the physical system itself. A block diagram contains information concerning dynamic behavior, but it

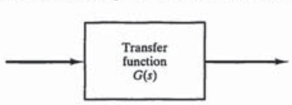


Figure 4–2 Element of a block diagram.

does not include any information about the physical construction of the system. Consequently, many dissimilar and unrelated systems can be represented by the same block diagram.

Note that in a block diagram the main source of energy is not explicitly shown and that the block diagram of a given system is not unique. A number of different block diagrams can be drawn for a system, depending on the point of view of the analysis. (See **Example 4-2.**)

Summing point. Figure 4-3 shows a circle with a cross, the symbol that stands for a summing operation. The plus or minus sign at each arrowhead indicates whether the associated signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

Branch point. A *branch point* is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block diagram of a closed-loop system. Figure 4-4 is a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point, where it is compared with the input $R(s)$. The closed-loop nature of the system is indicated clearly by the figure. The output $C(s)$ of the block is obtained by multiplying the transfer function $G(s)$ by the input to the block, $E(s)$.

Any linear system can be represented by a block diagram consisting of blocks, summing points, and branch points. When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. This conversion is accomplished by the feedback element whose transfer function is $H(s)$, as shown in Figure 4-5. Another important role of the feedback element is to modify the output before it is compared with the input. In the figure, the feedback signal that is fed back to the summing point for comparison with the input is $B(s) = H(s)C(s)$.

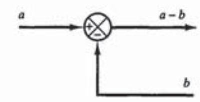


Figure 4-3 Summing point.

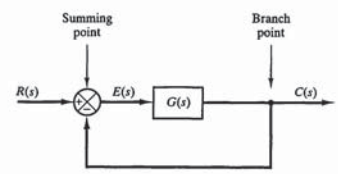


Figure 4-4 Block diagram of a closed-loop system.

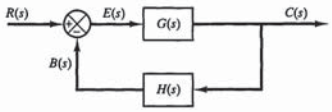


Figure 4-5 Block diagram of a closed-loop system with feedback element.

Simplifying complex block diagrams and obtaining overall transfer functions from such block diagrams are discussed in Chapter 5.

Example 4-2

Consider again the mechanical system shown in Figure 4-1. The transfer function of this system (see **Example 4-1**) is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{4-4}$$

A block diagram representation of the system is shown in Figure 4-6(a).

Notice that Equation (4-4) can be written as

$$(ms^2 + bs + k)X(s) = F(s) \tag{4-5}$$

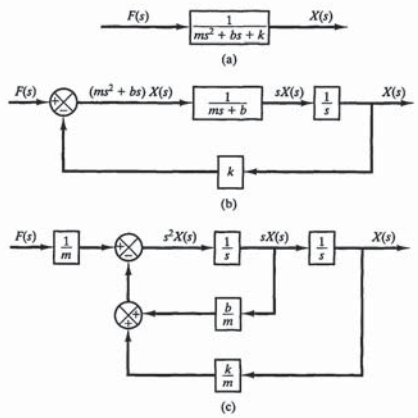


Figure 4-6 Block diagrams of the system shown in Figure 4-1. (a) Block diagram based on Equation (4-4); (b) block diagram based on Equation (4-6); (c) block diagram based on Equation (4-7).

Rewriting the latter equation as

$$(ms^2 + bs)X(s) = F(s) - kX(s) \tag{4-6}$$

we can obtain a different block diagram for the same system, as shown in Figure 4-6(b).

Equation (4-5) can also be rewritten as

$$F(s) - [kX(s) + bsX(s)] = ms^2X(s)$$

or

$$\frac{1}{m}F(s) - \frac{k}{m}X(s) - \frac{b}{m}sX(s) = s^2X(s) \tag{4-7}$$

A block diagram for the system based on Equation (4-7) is shown in Figure 4-6(c).

Figures 4-6(a), (b), and (c) are thus block diagrams for the same system—that shown in Figure 4-1. (Many different block diagrams are possible for any given system.)

4-3 PARTIAL-FRACTION EXPANSION WITH MATLAB

We begin this section, with an examination of the partial-fraction expansion of the transfer function $B(s)/A(s)$ with MATLAB. Then we discuss how to obtain the system response analytically. Computational solutions (response curves) for the system responses to time-domain inputs are given in Section 4-4.

MATLAB representation of transfer functions. The transfer function of a system is represented by two arrays of numbers. For example, consider a system defined by

$$\frac{Y(s)}{U(s)} = \frac{25}{s^2 + 4s + 25}$$

This system is represented as two arrays, each containing the coefficients of the polynomials in decreasing powers of s as follows:

```
num = [25]
den = [1   4   25]
```

Partial-fraction expansion with MATLAB. MATLAB allows us to obtain the partial-fraction expansion of the ratio of two polynomials,

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b(1)s^h + b(2)s^{h-1} + \cdots + b(h)}{a(1)s^n + a(2)s^{n-1} + \cdots + a(n)}$$

where $a(1) \neq 0$, some of $a(i)$ and $b(j)$ may be zero, and num and den are row vectors that specify the coefficients of the numerator and denominator of $B(s)/A(s)$. That is,

```
num = [b(1)  b(2)  ···  b(h)]
den = [a(1)  a(2)  ···  a(n)]
```


The command

```
[r,p,k] = residue(num,den)
```

finds the residues, poles, and direct terms of a partial-fraction expansion of the ratio of the two polynomials $B(s)$ and $A(s)$. The partial-fraction expansion of $B(s)/A(s)$ is given by

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \cdots + \frac{r(n)}{s - p(n)}$$

As an example, consider the function

$$\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$

For this function,

```
num = [1   8   16   9   6]
den = [1   6   11   6]
```

Entering the command

```
[r,p,k] = residue(num,den)
```

as shown in MATLAB Program 4-1, we obtain the residues (r), poles (p), and direct terms (k).

MATLAB Program 4-1

```
>> num = [1   8   16   9   6];
>> den = [1   6   11   6];
>> [r,p,k] = residue(num,den)

r =

   -6.0000
   -4.0000
    3.0000

p =

   -3.0000
   -2.0000
   -1.0000

k =

    1   2
```

MATLAB Program 4–1 is the MATLAB representation of the partial-fraction expansion of $B(s)/A(s)$:

$$\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$
$$= s + 2 + \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1}$$

Note that MATLAB first divides the numerator by the denominator and produces a polynomial in s (denoted as row vector k) plus a remainder (a ratio of polynomials in s, where the numerator is of lower degree than the denominator). Then MATLAB expands this remainder into partial fractions and returns the residues as column vector r and the pole locations as column vector p.

The command

```
[num,den] = residue(r,p,k)
```

where r, p, and k are outputs in MATLAB Program 4–1, converts the partial-fraction expansion back to the polynomial ratio $B(s)/A(s)$, as shown in MATLAB Program 4–2.

MATLAB Program 4–2

```
>> r = [-6  -4  3];
>> p = [-3  -2  -1];
>> k = [1  2];
>> [num, den] = residue(r,p,k)

num =

     1     8    16     9     6

den =

     1     6    11     6
```

Example 4–3

Consider the spring–mass–dashpot system mounted on a massless cart as shown in Figure 4–7. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy, which is dissipated as heat. The dashpot, also called a *damper*, does not store any kinetic or potential energy.

Let us obtain a mathematical model of this system by assuming that both the cart and the spring–mass–dashpot system on it are standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and the input to the system. The displacement $y(t)$ of the mass relative to the ground is the output. Also, m denotes the mass, b denotes the viscous friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is linear; that is, the spring force is proportional to $y - u$.

After a mathematical model of the system is obtained, we determine the output $y(t)$ analytically when $m = 10$ kg, $b = 20$ N-s/m, and $k = 100$ N/m. The input is assumed to be a unit-step input.

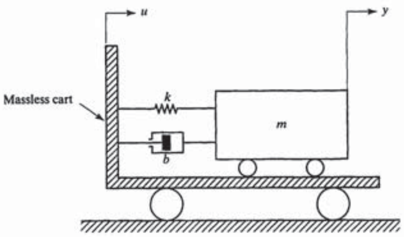


Figure 4–7 Spring–mass–dashpot system mounted on a cart.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and ΣF is the sum of the forces acting on the mass in the direction of the acceleration. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

The latter equation represents a mathematical model of the system under consideration. Taking the Laplace transform of the equation, assuming zero initial conditions, gives

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k} \tag{4–8}$$

Next, we shall obtain an analytical solution of the response to the unit-step input. Substituting the given numerical values into Equation (4–8) gives

$$\frac{Y(s)}{U(s)} = \frac{20s + 100}{10s^2 + 20s + 100} = \frac{2s + 10}{s^2 + 2s + 10}$$

Since the input u is a unit-step function,

$$U(s) = \frac{1}{s}$$

Then the output $Y(s)$ becomes

$$Y(s) = \frac{2s + 10}{s^2 + 2s + 10}\frac{1}{s} = \frac{2s + 10}{s^3 + 2s^2 + 10s}$$

To obtain the inverse Laplace transform of $Y(s)$, we need to expand $Y(s)$ into partial fractions.

Applying MATLAB and noting that num and den of the system are

```
num = [2  10]
den = [1  2  10  0]
```

we may use the residue command

```
[r,p,k] = residue(num,den)
```

to find the residues (r), poles (p), and direct term (k) as shown in MATLAB Program 4–3. MATLAB Program 4–3 is the MATLAB representation of the partial-fraction expansion of $Y(s)$:

$$Y(s) = \frac{-0.5 - j0.1667}{s + 1 - j3} + \frac{-0.5 + j0.1667}{s + 1 + j3} + \frac{1}{s}$$

MATLAB Program 4–3

```
>> num = [2  10];
>> den = [1  2  10  0];
>> [r,p,k] = residue(num,den)

r =

   -0.5000 - 0.1667i
   -0.5000 + 0.1667i
    1.0000

p =

   -1.0000 + 3.0000i
   -1.0000 - 3.0000i
         0

k =

     []
```

Since $Y(s)$ involves complex-conjugate poles, it is convenient to combine two complex-conjugate terms into one as follows:

$$\frac{-0.5 - j0.1667}{s + 1 - j3} + \frac{-0.5 + j0.1667}{s + 1 + j3} = \frac{-s}{(s+1)^2 + 3^2}$$

Then $Y(s)$ can be expanded as

$$Y(s) = \frac{1}{s} - \frac{s}{(s+1)^2 + 3^2}$$
$$= \frac{1}{s} - \frac{s + 1 - 1}{(s+1)^2 + 3^2}$$
$$= \frac{1}{s} - \frac{s + 1}{(s+1)^2 + 3^2} + \frac{1}{3}\frac{3}{(s+1)^2 + 3^2}$$

The inverse Laplace transform of $Y(s)$ is obtained as

$$y(t) = 1 - e^{-t}\cos 3t + \frac{1}{3}e^{-t}\sin 3t$$

where $y(t)$ is measured in meters and t in seconds. This equation is an analytical solution to the problem.

Note that a plot of $y(t)$ versus t can be obtained easily with MATLAB from the information on num, den, and $u(t)$ without using the partial-fraction expansion. (See **Example 4–5**.)

Example 4–4

Consider the mechanical system shown in Figure 4–8. The system is at rest initially. The displacements x and y are measured from their respective equilibrium positions. Assuming that $p(t)$ is a step force input and the displacement $x(t)$ is the output, obtain the transfer function of the system. Then, assuming that $m = 0.1$ kg, $b_2 = 0.4$ N-s/m, $k_1 = 6$ N/m, $k_2 = 4$ N/m, and $p(t)$ is a step force of magnitude 10 N, obtain an analytical solution $x(t)$.

The equations of motion for the system are

$$m\ddot{x} + k_1x + k_2(x - y) = p$$
$$k_2(x - y) = b_2\dot{y}$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(ms^2 + k_1 + k_2)X(s) = k_2Y(s) + P(s) \tag{4–9}$$
$$k_2X(s) = (k_2 + b_2s)Y(s) \tag{4–10}$$

Solving Equation (4–10) for $Y(s)$ and substituting the result into Equation (4–9), we get

$$(ms^2 + k_1 + k_2)X(s) = \frac{k_2^2}{k_2 + b_2s}X(s) + P(s)$$

or

$$[(ms^2 + k_1 + k_2)(k_2 + b_2s) - k_2^2]X(s) = (k_2 + b_2s)P(s)$$

from which we obtain the transfer function

$$\frac{X(s)}{P(s)} = \frac{b_2s + k_2}{mb_2s^3 + mk_2s^2 + (k_1 + k_2)b_2s + k_1k_2} \tag{4–11}$$

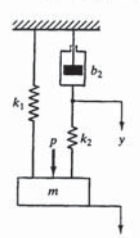


Figure 4–8 Mechanical system.

Substituting the given numerical values for m, k_1, k_2, and b_2 into Equation (4–11), we have

$$\frac{X(s)}{P(s)} = \frac{0.4s + 4}{0.04s^3 + 0.4s^2 + 4s + 24} = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600} \qquad (4\text{–}12)$$

Since $P(s)$ is a step force of magnitude 10 N,

$$P(s) = \frac{10}{s}$$

Then, from Equation (4–12), $X(s)$ can be written as

$$X(s) = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600}\frac{10}{s}$$

To find an analytical solution, we need to expand $X(s)$ into partial fractions. For this purpose, we may use MATLAB Program 4–4, which produces the residues, poles, and direct term.

MATLAB Program 4–4

```
>> num = [100  1000];
>> den = [1  10  100  600  0];
>> [r,p,k] = residue(num, den)

r =

  -0.6845 + 0.2233i
  -0.6845 - 0.2233i
  -0.2977
   1.6667

p =

  -1.2898 + 8.8991i
  -1.2898 - 8.8991i
  -7.4204
        0

k =

     []
```

On the basis of the MATLAB output, $X(s)$ can be written as

$$X(s) = \frac{-0.6845 + j0.2233}{s + 1.2898 - j8.8991} + \frac{-0.6845 - j0.2233}{s + 1.2898 + j8.8991} + \frac{-0.2977}{s + 7.4204} + \frac{1.6667}{s}$$
$$= \frac{-1.3690(s + 1.2898) - 3.9743}{(s + 1.2898)^2 + 8.8991^2} - \frac{0.2977}{s + 7.4204} + \frac{1.6667}{s}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = -1.3690e^{-1.2898t}\cos(8.8991t) - 0.4466e^{-1.2898t}\sin(8.8991t) - 0.2977e^{-7.4204t} + 1.6667$$

where $x(t)$ is measured in meters and time t in seconds. This is the analytical solution to the problem. [For the response curve $x(t)$ versus t, see **Example 4–6**.]

From the preceding examples, we have seen that once the transfer function $X(s)/U(s) = G(s)$ of a system is obtained, the response of the system to any input can be determined by taking the inverse Laplace transform of $X(s)$, or

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[G(s)U(s)]$$

Finding the inverse Laplace transform of $G(s)U(s)$ may be time consuming if the transfer function $G(s)$ of the system is complicated, even though the input $U(s)$ may be a simple function of time. Unless, for some reason, an analytical solution is needed, we should use a computer to get a numerical solution. Throughout this book, we use MATLAB to obtain numerical solutions to many problems. Obtaining numerical solutions and presenting them in the form of response curves is the subject discussed in the next section.

4–4 TRANSIENT-RESPONSE ANALYSIS WITH MATLAB

This section presents the MATLAB approach to obtaining system responses when the inputs are time-domain inputs such as the step, impulse, and ramp functions. The system response to the frequency-domain input (e.g., a sinusoidal input) is presented in Chapters 9 and 11.

MATLAB representation of transfer-function systems. Figure 4–9 shows a block with a transfer function. Such a block represents a system or an element of a system. To simplify our presentation, we shall call the block with a transfer function a system. MATLAB uses sys to represent such a system. The statement

sys = tf(num, den) (4–13)

represents the system. For example, consider the system

$$\frac{Y(s)}{X(s)} = \frac{2s + 25}{s^2 + 4s + 25}$$

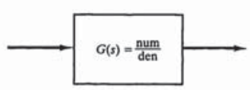


Figure 4–9 Block diagram of transfer-function system.

This system can be represented as two arrays, each containing the coefficients of the polynomials in decreasing powers of s as follows:

```
num = [2  25]
den = [1  4  25]
```

Entering MATLAB Program 4–5 into a computer produces the transfer function of the system.

MATLAB Program 4–5

```
>> num = [2  25];
>> den = [1  4  25];
>> sys = tf(num,den)

Transfer function:
  2 s + 25
-------------------
s^2 + 4 s + 25
```

In this book, we shall use Equation (4–13) to represent the transfer function system.

Step response. If num and den (the numerator and denominator of a transfer function) are known, we may define the system by

sys = tf(num,den)

Then, a command such as

step(sys) or step(num,den)

will generate a plot of a unit-step response and will display a response curve on the screen. The computation interval Δt and the time span of the response are determined by MATLAB.

If we wish MATLAB to compute the response every Δt seconds and plot the response curve for $0 \le t \le T$ (where T is an integer multiple of Δt), we enter the statement

t = 0 : Δt : T;

in the program and use the command

step(sys,t) or step(num,den,t)

where t is the user-specified time.

If step commands have left-hand arguments, such as

y = step(sys,t) or y = step(num,den,t)

and

[y,t] = step(sys,t) or [y,t] = step(num,den,t)

MATLAB produces the unit-step response of the system, but displays no plot on the screen. It is necessary to use a plot command to see response curves.

The next two examples demonstrate the use of step commands.

Example 4–5

Consider again the spring–mass–dashpot system mounted on a cart as shown in Figure 4–7. (See **Example 4–3**.) The transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Assuming that $m = 10$ kg, $b = 20$ N-s/m, $k = 100$ N/m, and the input $u(t)$ is a unit-step input (a step input of 1 m), obtain the response curve $y(t)$.

Substituting the given numerical values into the transfer function, we have

$$\frac{Y(s)}{U(s)} = \frac{20s + 100}{10s^2 + 20s + 100} = \frac{2s + 10}{s^2 + 2s + 10}$$

MATLAB Program 4–6 will produce the unit-step response $y(t)$. The resulting unit-step response curve is shown in Figure 4–10.

MATLAB Program 4–6

```
>> num = [2  10];
>> den = [1  2  10];
>> sys = tf(num,den);
>> step(sys)
>> grid
```

In this plot, the duration of the response is automatically determined by MATLAB. The title and axis labels are also automatically determined by MATLAB.

If we wish to compute and plot the curve every 0.01 sec over the interval $0 \le t \le 8$, we need to enter the following statement in the MATLAB program:

t = 0:0.01:8;

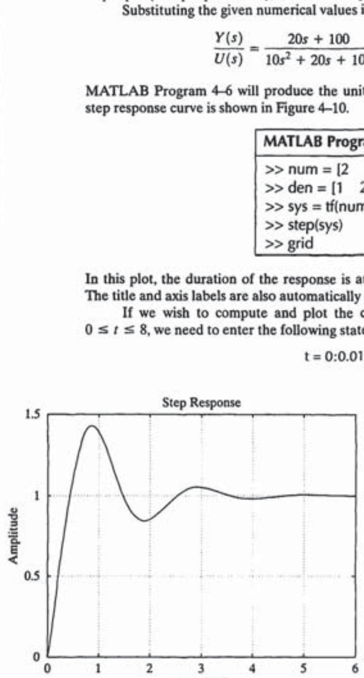


Figure 4–10 Unit-step response curve.

Also, if we wish to change the title and axis labels, we enter the desired title and desired labels as shown in MATLAB Program 4–7.

MATLAB Program 4–7

```
>> t = 0:0.01:8;
>> num = [2  10];
>> den = [1  2  10];
>> sys = tf(num,den);
>> step(sys,t)
>> grid
>> title('Unit-Step Response','Fontsize',20')
>> xlabel('t','Fontsize',20')
>> ylabel ('Output y','Fontsize',20')
```

Note that if we did not enter the desired title and desired axis labels in the program, the title, x-axis label, and y-axis label on the plot would have been "Step Response", "Time (sec)", and "Amplitude", respectively. (This statement applies to MATLAB version 6 and not to versions 3, 4, and 5.) When we enter the desired title and axis labels as shown in MATLAB Program 4–7, MATLAB erases the predetermined title and axis labels, except "(sec)" in the x-axis label, and replaces them with the ones we have specified. If the font sizes are too small, they can be made larger. For example, entering

'Fontsize', 20

in the title, xlabel, and ylabel variables as shown in MATLAB Program 4–7 results in that size text appearing in those places. Figure 4–11 is a plot of the response curve obtained with MATLAB Program 4–7.

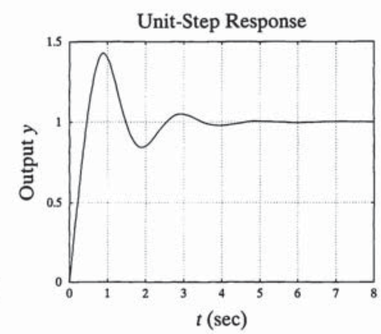


Figure 4–11 Unit-step response curve. Font sizes for title, xlabel, and ylabel are enlarged.

Example 4–6

Consider again the mechanical system shown in Figure 4–8. (See **Example 4–4**.) The transfer function $X(s)/P(s)$ was found to be

$$\frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{m b_2 s^3 + m k_2 s^2 + (k_1 + k_2) b_2 s + k_1 k_2} \tag{4–14}$$

The transfer function $Y(s)/X(s)$ is obtained from Equation (4–10):

$$\frac{Y(s)}{X(s)} = \frac{k_2}{b_2 s + k_2}$$

Hence,

$$\frac{Y(s)}{P(s)} = \frac{Y(s)}{X(s)}\frac{X(s)}{P(s)} = \frac{k_2}{m b_2 s^3 + m k_2 s^2 + (k_1 + k_2) b_2 s + k_1 k_2} \tag{4–15}$$

Assuming that $m = 0.1$ kg, $b_2 = 0.4$ N-s/m, $k_1 = 6$ N/m, $k_2 = 4$ N/m, and $p(t)$ is a step force of magnitude 10 N, obtain the responses $x(t)$ and $y(t)$.

Substituting the numerical values for m, b_2, k_1, and k_2 into the transfer functions given by Equations (4–14) and (4–15), we obtain

$$\frac{X(s)}{P(s)} = \frac{0.4s + 4}{0.04s^3 + 0.4s^2 + 4s + 24} = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600} \tag{4–16}$$

and

$$\frac{Y(s)}{P(s)} = \frac{4}{0.04s^3 + 0.4s^2 + 4s + 24} = \frac{100}{s^3 + 10s^2 + 100s + 600} \tag{4–17}$$

Since $p(t)$ is a step force of magnitude 10 N, we may define $p(t) = 10u(t)$, where $u(t)$ is a unit-step input of magnitude 1 N. Then Equations (4–16) and (4–17) can be written as

$$\frac{X(s)}{U(s)} = \frac{100s + 1000}{s^3 + 10s^2 + 100s + 600} \tag{4–18}$$

and

$$\frac{Y(s)}{U(s)} = \frac{1000}{s^3 + 10s^2 + 100s + 600} \tag{4–19}$$

Since $u(t)$ is a unit-step input, $x(t)$ and $y(t)$ can be obtained from Equations (4–18) and (4–19) with the use of a step command. (Step commands assume that the input is the unit-step input.)

In this example, we shall demonstrate the use of the commands

y = step(sys,t)

and

plot(t,y)

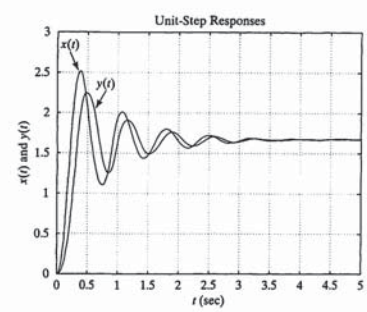


Figure 4–12 Step-response curves $x(t)$ and $y(t)$.

MATLAB Program 4–8 produces the responses $x(t)$ and $y(t)$ of the system on one diagram.

MATLAB Program 4–8

```
>> t = 0:0.01:5;
>> num1 = [100  1000];
>> num2 = [1000];
>> den = [1  10  100  600];
>> sys1 = tf(num1,den);
>> sys2 = tf(num2,den);
>> y1 = step(sys1,t);
>> y2 = step(sys2,t);
>> plot(t,y1,t,y2)
>> grid
>> title ('Unit-Step Responses')
>> xlabel ('t (sec)')
>> ylabel ('x(t) and y(t)')
>> text(0.07,2.8,'x(t)')
>> text(0.7,2.35,'y(t)')
```

The response curves $x(t)$ and $y(t)$ are shown in Figure 4–12.

Writing text on the graph. When we plot two or more curves on one diagram, we may need to write text on the graph to distinguish the curves. For example, to write the text 'x(t)' horizontally, beginning at the point (0.07, 2.8) on the graph, we use the command

text(0.07,2.8,'x(t)')

Impulse response. The unit-impulse response of a dynamic system defined in the form of the transfer function may be obtained by use of one of the following

MATLAB commands:

```
impulse(sys)          or    impulse(num,den)
impulse(sys,t)        or    impulse(num,den,t)
y = impulse(sys)      or    y = impulse(num,den)
[y,t] = impulse (sys,t)  or  [y,t] = impulse(num,den,t)
```

The command impulse(sys) will generate a plot of the unit-impulse response and will display the impulse-response curve on the screen. If the command has a left-hand argument, such as y = impulse(sys), no plot is shown on the screen. It is then necessary to use a plot command to see the response curve on the screen.

Before discussing computational solutions of problems involving impulse inputs, we present some necessary background material.

Impulse input. The impulse response of a mechanical system can be observed when the system is subjected to a very large force for a very short time, for instance, when the mass of a spring–mass–dashpot system is hit by a hammer or a bullet. Mathematically, such an impulse input can be expressed by an impulse function.

The impulse function is a mathematical function without any actual physical counterpart. However, as shown in Figure 4–13(a), if the actual input lasts for a short time (Δt s) but has a large amplitude (h), so that the area ($h\Delta t$) in a time plot is not negligible, it can be approximated by an impulse function. The impulse input is usually denoted by a vertical arrow, as shown in Figure 4–13(b), to indicate that it has a very short duration and a very large height.

In handling impulse functions, only the magnitude (or area) of the function is important; its actual shape is immaterial. In other words, an impulse of amplitude $2h$ and duration $\Delta t/2$ can be considered the same as an impulse of amplitude h and duration Δt, as long as Δt approaches zero and $h\Delta t$ is finite.

We next briefly discuss a review of the law of conservation of momentum, which is useful in determining the impulse responses of mechanical systems.

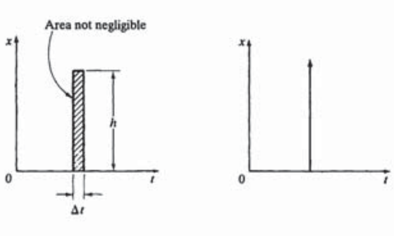


Figure 4–13 Impulse inputs.

Law of conservation of momentum. The momentum of a mass m moving at a velocity v is mv. According to Newton's second law,

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv)$$

Hence,

$$F\,dt = d(mv) \tag{4–20}$$

Integrating both sides of Equation (4–20), we have

$$\int_{t_1}^{t_2} F\,dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1 \tag{4–21}$$

where $v_1 = v(t_1)$ and $v_2 = v(t_2)$. Equation (4–21) states that a change in momentum equals the time integral of force between $t = t_1$ and $t = t_2$.

Momentum is a vector quantity, with magnitude, direction, and sense. The direction of the change in momentum is the direction of the force.

In the absence of any external force, Equation (4–20) becomes

$$d(mv) = 0$$

or

$$mv = \text{constant}$$

Thus, the total momentum of a system remains unchanged by any action that may take place within the system, provided that no external force is acting on the system. This principle is called the *law of conservation of momentum*.

The angular momentum of a rotating system is $J\omega$, where J is the moment of inertia of a body and ω is the angular velocity of the body. In the absence of an external torque, the angular momentum of a body remains unchanged. This principle is the *law of conservation of angular momentum*.

Example 4–7

A bullet is fired horizontally into a wood block resting on a horizontal, frictionless surface. If the mass m_1 of the bullet is 0.02 kg and the velocity is 600 m/s, what is the velocity of the wood block after the bullet is embedded in it? Assume that the wood block has a mass m_2 of 50 kg.

If we consider the bullet and wood block as constituting a system, no external force is acting on the system. Consequently, its total momentum remains unchanged. Thus, we have

$$\text{momentum before impact} = m_1v_1 + m_2v_2$$

where v_1, the velocity of the bullet before the impact, is equal to 600 m/s and v_2, the velocity of the wood block before the impact, is equal to zero. Also,

$$\text{momentum after impact} = (m_1 + m_2)v$$

where v is the velocity of the wood block after the bullet is embedded. (Velocities v_1 and v are in the same direction.)

The law of conservation of momentum states that

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v$$

Substituting the given numerical values into this last equation, we obtain

$$0.02 \times 600 + 50 \times 0 = (0.02 + 50)v$$

or

$$v = 0.24 \text{ m/s}$$

Hence, the wood block after the bullet is embedded will move at the velocity of 0.24 m/s in the same direction as the original velocity v_1 of the bullet.

Example 4–8

Consider the mechanical system shown in Figure 4–14. A bullet of mass m is shot into a block of mass M (where $M \gg m$). Assume that when the bullet hits the block, it becomes embedded there. Determine the response (displacement x) of the block after it is hit by the bullet. The displacement x of the block is measured from the equilibrium position before the bullet hits it. Suppose that the bullet is shot at $t = 0-$ and that the initial velocity of the bullet is $v(0-)$. Assuming the following numerical values for M, m, b, k, and $v(0-)$, draw a curve $x(t)$ versus t:

$$M = 50 \text{ kg}, \quad m = 0.01 \text{ kg}, \quad b = 100 \text{ N-s/m},$$
$$k = 2500 \text{ N/m}, \quad v(0-) = 800 \text{ m/s}$$

The input to the system in this case can be considered an impulse, the magnitude of which is equal to the rate of change of momentum of the bullet. At the instant the bullet hits the block, the velocity of the bullet becomes the same as that of the block, since the bullet is assumed to be embedded in it. As a result, there is a sudden change in the velocity of the bullet. [See Figure 4–15(a).] Since the change in the velocity of the bullet occurs instantaneously, $\dot{v}$ has the form of an impulse. (Note that $\dot{v}$ is negative.)

For $t > 0$, the block and the bullet move as a combined mass $M + m$. The equation of motion for the system is

$$(M + m)\ddot{x} + b\dot{x} + kx = F(t) \tag{4–22}$$

where $F(t)$, an impulse force, is equal to $-m\dot{v}$. [Note that $-m\dot{v}$ is positive; the impulse force $F(t)$ is in the positive direction of x.] From Figure 4–15(b), the impulse force can be written as

$$F(t) = -m\dot{v} = A\,\Delta t\,\delta(t)$$

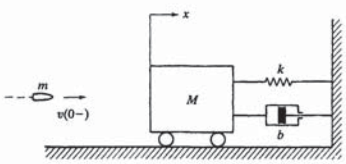


Figure 4–14 Mechanical system subjected to an impulse input.

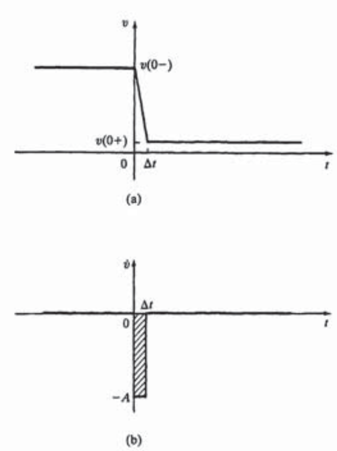


Figure 4–15 (a) Change in velocity of the bullet when it hits the block; (b) change in acceleration of the bullet when it hits the block.

where $A\,\Delta t$ is the magnitude of the impulse input. Thus,

$$\int_{0-}^{0+} A\,\Delta t\,\delta(t)\,dt = -m\int_{0-}^{0+} \dot{v}\,dt$$

or

$$A\,\Delta t = mv(0-) - mv(0+) \tag{4–23}$$

The momentum of the bullet is changed from $mv(0-)$ to $mv(0+)$. Since

$$v(0+) = \dot{x}(0+) = \text{initial velocity of combined mass } M + m$$

we can write Equation (4–23) as

$$A\,\Delta t = mv(0-) - m\dot{x}(0+)$$

Then Equation (4–22) becomes

$$(M + m)\ddot{x} + b\dot{x} + kx = F(t) = [mv(0-) - m\dot{x}(0+)]\delta(t)$$

Taking the $\mathcal{L}_-$ transform of both sides of this last equation, we see that

$$(M + m)[s^2X(s) - sx(0-) - \dot{x}(0-)] + b[sX(s) - x(0-)] + kX(s) = mv(0-) - m\dot{x}(0+)$$

Also, noting that $x(0-) = 0$ and $\dot{x}(0-) = 0$, we have

$$X(s) = \frac{mv(0-) - m\dot{x}(0+)}{(M + m)s^2 + bs + k} \tag{4–24}$$

To determine the value of $\dot{x}(0+)$, we apply the initial-value theorem:

$$\begin{aligned}\dot{x}(0+) &= \lim_{t\to 0+}\dot{x}(t) = \lim_{s\to\infty} s[sX(s)] \\ &= \lim_{s\to\infty}\frac{s^2[mv(0-) - m\dot{x}(0+)]}{(M + m)s^2 + bs + k} \\ &= \frac{mv(0-) - m\dot{x}(0+)}{M + m}\end{aligned}$$

from which we get

$$mv(0-) - m\dot{x}(0+) = (M + m)\dot{x}(0+)$$

or

$$\dot{x}(0+) = \frac{m}{M + 2m}v(0-)$$

So Equation (4–24) becomes

$$\begin{aligned}X(s) &= \frac{(M + m)\dot{x}(0+)}{(M + m)s^2 + bs + k} \\ &= \frac{1}{(M + m)s^2 + bs + k}\,\frac{(M + m)m\,v(0-)}{M + 2m}\end{aligned} \tag{4–25}$$

The inverse Laplace transform of Equation (4–25) gives the impulse response $x(t)$.

Substituting the given numerical values into Equation (4–25), we obtain

$$\begin{aligned}X(s) &= \frac{1}{50.01s^2 + 100s + 2500}\,\frac{50.01 \times 0.01 \times 800}{50.02} \\ &= \frac{7.9984}{50.01s^2 + 100s + 2500} \\ &= 0.02285\frac{6.9993}{(s + 0.9998)^2 + (6.9993)^2}\end{aligned}$$

Taking the inverse Laplace transform of this last equation yields

$$x(t) = 0.02285e^{-0.9998t}\sin 6.9993t$$

Thus, the response $x(t)$ is a damped sinusoidal motion.

Example 4–9

Referring to **Example 4–8**, obtain the impulse response of the system shown in Figure 4–14 with MATLAB. Use the same numerical values for M, m, b, k, and $v(0-)$ as in **Example 4–8**.

The response $X(s)$ was obtained in **Example 4–8**, as given by Equation (4–25). This is the response to the impulse input $[mv(0-) - m\dot{x}(0+)]\delta(t)$. Note that the magnitude of the impulse input is

$$mv(0-) - m\dot{x}(0+) = (M + m)\dot{x}(0+) = \frac{m(M + m)}{M + 2m}v(0-)$$

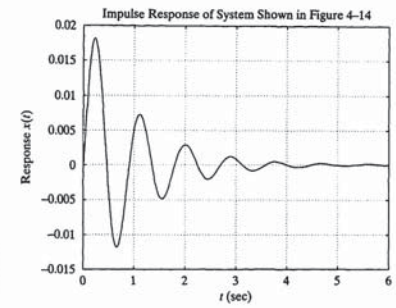


Figure 4–16 Impulse-response curve of the system shown in Figure 4–14 with $M = 50$ kg, $m = 0.01$ kg, $b = 100$ N-s/m, $k = 2500$ N/m, and $v(0-) = 800$ m/s.

Hence, the impulse input can be written as

$$F(t) = \frac{m(M+m)}{M+2m} v(0-)\delta(t) = 7.9984\,\delta(t)$$

The system equation is

$$(M+m)\ddot{x} + b\dot{x} + kx = F(t) = 7.9984\,\delta(t)$$

so that

$$\frac{X(s)}{F(s)} = \frac{1}{(M+m)s^2 + bs + k} \tag{4–26}$$

To find the response of the system to $F(t)$ (which is an impulse input whose magnitude is not unity), we modify Equation (4–26) to the following form:

$$\frac{X(s)}{\mathcal{L}[\delta(t)]} = \frac{1}{(M+m)s^2 + bs + k} \, \frac{m(M+m)\,v(0-)}{M+2m}$$
$$= \frac{7.9984}{50.01s^2 + 100s + 2500} \tag{4–27}$$

If we define

```
num = [7.9984];
den = [50.01   100   2500];
sys = tf(num,den)
```

then the command

```
impulse(sys)
```

will produce the unit-impulse response of the system defined by Equation (4–27), which is the same as the response of the system of Equation (4–26) to the impulse input $F(t) = 7.9984\,\delta(t)$. MATLAB Program 4–9 produces the response of the system subjected to the impulse input $F(t)$. The impulse response obtained is shown in Figure 4–16.

MATLAB Program 4–9

```
>> num = [7.9984];
>> den = [50.01   100   2500];
>> sys = tf(num,den);
>> impulse (sys)
>> grid
>> title ('Impulse Response of System Shown in Figure 4–14')
>> xlabel('t')
>> ylabel('Response x(t)')
```

Obtaining response to arbitrary input. The command lsim produces the response of linear, time-invariant systems to arbitrary inputs. If the initial conditions of the system are zero, then

```
lsim(sys,u,t)      or      lsim(num,den,u,t)
```

produces the response of the system to the input u. Here, u is the input and t represents the times at which responses to u are to be computed. (The response time span and the time increment are stated in t; an example of how t is specified is t = 0:0.01:10). If the initial conditions are nonzero, use the state-space approach presented in Section 5–2.

If the initial conditions of the system are zero, then any of the commands

```
y = lsim(sys,u,t)      or      y = lsim(num,den,u,t)
```

and

```
[y,t] = lsim(sys,u,t)      or      [y,t] = lsim(num,den,u,t)
```

returns the output response y. No plot is drawn. To plot the response curve, it is necessary to use the command plot(t,y).

Note that the command

```
lsim(sys1,sys2, ...,u,t)
```

plots the responses of systems sys1, sys2, ... on a single diagram. Note also that, by using lsim commands, we are able to obtain the response of the system to ramp inputs, acceleration inputs, and any other time functions that we can generate with MATLAB.

Ramp response. The next example plots the unit-ramp response curve with the use of the lsim command

```
lsim(sys,u,t)
```

where u = t.

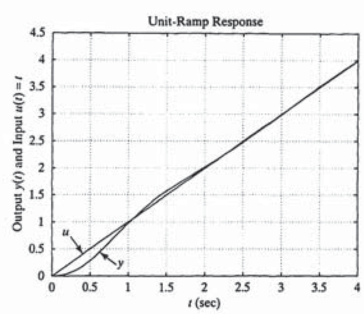


Figure 4–17 Plots of unit-ramp response curve $y(t)$ and input ramp function $u(t)$.

Example 4–10

Consider once again the system shown in Figure 4–7. (See **Example 4–3.**) Assume that $m = 10$ kg, $b = 20$ N-s/m, $k = 100$ N/m, and $u(t)$ is a unit-ramp input—that is, the displacement u increases linearly, or $u = \alpha t$, where $\alpha = 1$. We shall obtain the unit-ramp response using the command

```
lsim(sys,u,t)
```

The transfer function of the system, derived in **Example 4–3**, is

$$\frac{Y(s)}{U(s)} = \frac{2s + 10}{s^2 + 2s + 10}$$

MATLAB Program 4–10 produces the unit-ramp response. The resulting response curve $y(t)$ versus t and the input ramp function $u(t)$ versus t are shown in Figure 4–17.

MATLAB Program 4–10

```
>> num = [2   10];
>> den = [1   2   10];
>> sys = tf(num, den);
>> t = 0:0.01:4;
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response')
>> xlabel('t')
>> ylabel('Output y(t) and Input u(t) = t')
>> text(0.8,0.25,'y')
>> text(0.15,0.8,'u')
```


[Note that the command lsim(sys,u,t) produces plots of both $y(t)$ versus t and $u(t)$ versus t.]

In some cases it is desired to plot multiple curves on one graph. This can be done by using a plot command with multiple arguments, for example,

```
plot(t,y1, t,y2, ..., t,yn)
```

MATLAB Program 4–11 uses the command

```
plot(t,y,t,u)
```

to plot a curve $y(t)$ versus t and a line $u(t)$ versus t. The resulting plots are shown in Figure 4–18.

MATLAB Program 4–11

```
>> num = [2   10];
>> den = [1   2   10];
>> sys = tf(num,den);
>> t = 0:0.01:4;
>> u = t;
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> grid
>> title('Unit-Ramp Response')
>> xlabel('t (sec)')
>> ylabel('Output y(t) and Input u(t) = t')
>> text(0.85,0.25,'y')
>> text(0.15,0.8,'u')
```

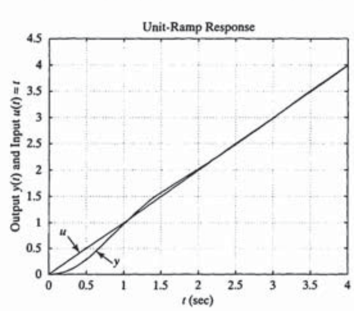


Figure 4–18 Plots of unit-ramp response curve $y(t)$ and input ramp function $u(t)$. (Plots are obtained with the use of the command plot(t,y,t,u).)

Response to initial condition (transfer-function approach). The next example obtains the response of a transfer-function system subjected to an initial condition.

Example 4–11

Consider the mechanical system shown in Figure 4–19, where $m = 1$ kg, $b = 3$ N-s/m, and $k = 2$ N/m. Assume that the displacement x of mass m is measured from the equilibrium position and that at $t = 0$ the mass m is pulled downward such that $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/s. Obtain the motion of the mass subjected to the initial condition. (Assume no external forcing function.)

The system equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

with the initial conditions $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/s. The Laplace transform of the system equation gives

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + kX(s) = 0$$

or

$$(ms^2 + bs + k)X(s) = mx(0)s + m\dot{x}(0) + bx(0)$$

Solving this last equation for $X(s)$ and substituting the given numerical values into $x(0)$ and $\dot{x}(0)$, we obtain

$$X(s) = \frac{mx(0)s + m\dot{x}(0) + bx(0)}{ms^2 + bs + k} = \frac{0.1s + 0.35}{s^2 + 3s + 2}$$

This equation can be written as

$$X(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}\frac{1}{s}$$

Hence, the motion of the mass m is the unit-step response of the following system:

$$G(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}$$

MATLAB Program 4–12 produces a plot of the motion of the mass when the system is subjected to the initial condition. The plot is shown in Figure 4–20.

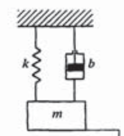


Figure 4–19 Mechanical system.

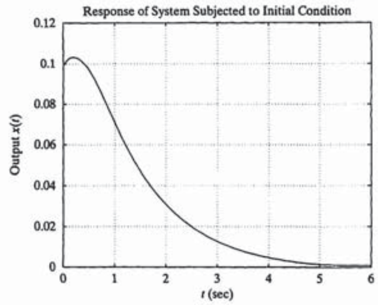


Figure 4–20 Response of system subjected to initial condition.

MATLAB Program 4–12

```
>> % ----------- Response to initial condition ------------
>>
>> % System response to initial condition is converted to a
>> % unit-step response by modifying the numerator polynomial.
>>
>> num = [0.1   0.35   0];
>> den = [1   3   2];
>> sys = tf(num,den);
>> step(sys)
>> grid
>> title('Response of System Subjected to Initial Condition')
>> xlabel('t')
>> ylabel('Output x(t)')
```

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–4–1

Consider the satellite attitude control system depicted in Figure 4–21. The diagram shows the control of only the yaw angle θ. (In the actual system, there are controls about three axes.) Small jets apply reaction forces to rotate the satellite body into the desired attitude. The two skew symmetrically placed jets denoted by A and B operate in pairs. Assume that each jet thrust is $F/2$ and a torque $T = Fl$ is applied to the system. The jets are turned on for a certain length of time, so the torque can be written as $T(t)$. The moment of inertia about the axis of rotation at the center of mass is J.

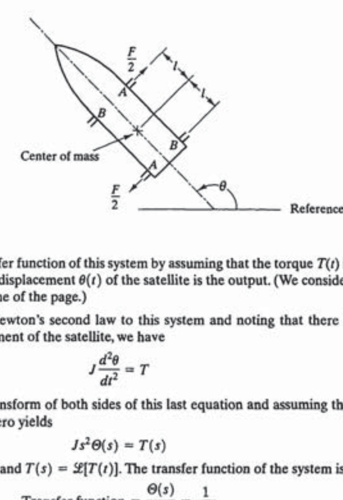


Figure 4–21 Schematic diagram of a satellite attitude control system.

Obtain the transfer function of this system by assuming that the torque $T(t)$ is the input, and the angular displacement $\theta(t)$ of the satellite is the output. (We consider the motion only in the plane of the page.)

Solution Applying Newton's second law to this system and noting that there is no friction in the environment of the satellite, we have

$$J\frac{d^2\theta}{dt^2} = T$$

Taking the Laplace transform of both sides of this last equation and assuming that all initial conditions are zero yields

$$Js^2\Theta(s) = T(s)$$

where $\Theta(s) = \mathcal{L}[\theta(t)]$ and $T(s) = \mathcal{L}[T(t)]$. The transfer function of the system is thus

$$\text{Transfer function} = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$

Problem A–4–2

Consider the mechanical system shown in Figure 4–22. Displacements x_i and x_o are measured from their respective equilibrium positions. Derive the transfer function of the system wherein x_i is the input and x_o is the output. Then obtain the response $x_o(t)$ when input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$. Assume that $x_o(0-) = 0$.

Solution The equation of motion for the system is

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2\dot{x}_o$$

Taking the $\mathcal{L}_-$ transform of this equation and noting that $x_i(0-) = 0$ and $x_o(0-) = 0$, we have

$$(b_1s + k_1)X_i(s) = (b_1s + k_1 + b_2s)X_o(s)$$

The transfer function $X_o(s)/X_i(s)$ is

$$\frac{X_o(s)}{X_i(s)} = \frac{b_1s + k_1}{(b_1 + b_2)s + k_1} \qquad (4\text{–}28)$$

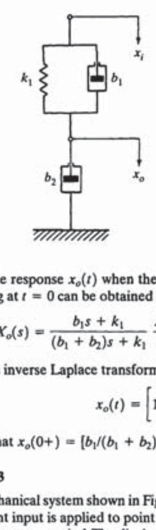


Figure 4–22 Mechanical system.

The response $x_o(t)$ when the input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$ can be obtained from Equation (4–28). First we have

$$X_o(s) = \frac{b_1s + k_1}{(b_1 + b_2)s + k_1}\frac{X_i}{s} = \left\{\frac{1}{s} - \frac{b_2}{b_1 + b_2}\frac{1}{s + [k_1/(b_1 + b_2)]}\right\}X_i$$

Then the inverse Laplace transform of $X_o(s)$ gives

$$x_o(t) = \left[1 - \frac{b_2}{b_1 + b_2}e^{-k_1t/(b_1+b_2)}\right]X_i$$

Notice that $x_o(0+) = [b_1/(b_1 + b_2)]X_i$.

Problem A–4–3

The mechanical system shown in Figure 4–23 is initially at rest. At $t = 0$, a unit-step displacement input is applied to point A. Assume that the system remains linear throughout the response period. The displacement x is measured from the equilibrium position. If $m = 1$ kg, $b = 10$ N-s/m, and $k = 50$ N/m, find the response $x(t)$ as well as the values of $x(0+)$, $\dot{x}(0+)$, and $x(\infty)$.

Solution The equation of motion for the system is

$$m\ddot{x} + b(\dot{x} - \dot{y}) + kx = 0$$

or

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}$$

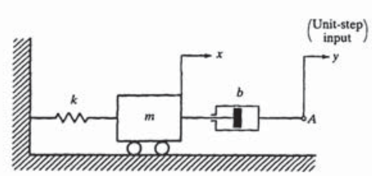


Figure 4–23 Mechanical system.

Noting that $x(0-) = 0$, $\dot{x}(0-) = 0$, and $y(0-) = 0$, we take the $\mathcal{L}_-$ transform of this last equation and obtain

$$(ms^2 + bs + k)X(s) = bsY(s)$$

Thus,

$$\frac{X(s)}{Y(s)} = \frac{bs}{ms^2 + bs + k}$$

Since the input y is a unit step, $Y(s) = 1/s$. Consequently,

$$X(s) = \frac{bs}{ms^2 + bs + k}\frac{1}{s} = \frac{b}{ms^2 + bs + k}$$

Substituting the given numerical values for m, b, and k into this last equation, we get

$$X(s) = \frac{10}{s^2 + 10s + 50} = \frac{10}{(s+5)^2 + 5^2}$$

The inverse Laplace transform of $X(s)$ is

$$x(t) = 2e^{-5t}\sin 5t$$

The values of $x(0+)$, $\dot{x}(0+)$, and $x(\infty)$ are found from the preceding equation and are

$$x(0+) = 0, \qquad \dot{x}(0+) = 10, \qquad x(\infty) = 0$$

Thus, the mass m returns to the original position as time elapses.

Problem A–4–4

Find the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 4–24. Obtain the response $x_o(t)$ when the input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$. Assume that the system is initially at rest $[x_o(0-) = 0$ and $y(0-) = 0]$. Assume also that x_i and x_o are measured from their respective equilibrium positions. The numerical values of b_1, b_2, k_1, and k_2 are as follows:

$$b_1 = 5\ \text{N-s/m}, \qquad b_2 = 20\ \text{N-s/m}, \qquad k_1 = 5\ \text{N/m}, \qquad k_2 = 10\ \text{N/m}$$

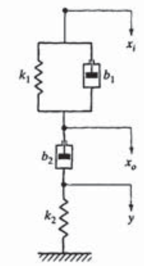


Figure 4–24 Mechanical system.

Solution The equations of motion for the mechanical system are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$
$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

Taking the $\mathcal{L}_-$ transform of these two equations, with the initial conditions $x_i(0-) = 0$, $x_o(0-) = 0$ and $y(0-) = 0$, we get

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$
$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

If we eliminate $Y(s)$ from the last two equations, the transfer function $X_o(s)/X_i(s)$ becomes

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

Substitution of the given numerical values into the transfer function yields

$$\frac{X_o(s)}{X_i(s)} = \frac{(s+1)(2s+1)}{(s+1)(2s+1) + 4s} = \frac{s^2 + 1.5s + 0.5}{s^2 + 3.5s + 0.5}$$

For an input $x_i(t) = X_i \cdot 1(t)$, the response $x_o(t)$ can be obtained as follows: Since

$$X_o(s) = \frac{s^2 + 1.5s + 0.5}{s^2 + 3.5s + 0.5}\frac{X_i}{s}$$
$$= \left(\frac{0.6247}{s + 3.3508} - \frac{0.6247}{s + 0.1492} + \frac{1}{s}\right)X_i$$

we find that

$$x_o(t) = (0.6247e^{-3.3508t} - 0.6247e^{-0.1492t} + 1)X_i$$

Notice that $x_o(0+) = X_i$.

Problem A–4–5

Obtain the transfer function $X(s)/U(s)$ of the system shown in Figure 4–25, where u is the force input. The displacement x is measured from the equilibrium position.

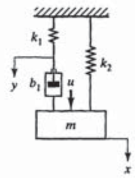


Figure 4–25 Mechanical system.

Solution The equations of motion for the system are

$$m\ddot{x} = -k_2 x - b_1(\dot{x} - \dot{y}) + u$$
$$b_1(\dot{x} - \dot{y}) = k_1 y$$

Laplace transforming these two equations and assuming initial conditions equal to zero, we obtain

$$ms^2X(s) = -k_2X(s) - b_1sX(s) + b_1sY(s) + U(s)$$
$$b_1sX(s) - b_1sY(s) = k_1Y(s)$$

Eliminating $Y(s)$ from the last two equations yields

$$(ms^2 + b_1s + k_2)X(s) = b_1s\frac{b_1s}{b_1s + k_1}X(s) + U(s)$$

Simplifying, we obtain

$$[(ms^2 + b_1s + k_2)(b_1s + k_1) - b_1^2s^2]X(s) = (b_1s + k_1)U(s)$$

from which we get the transfer function $X(s)/U(s)$ as

$$\frac{X(s)}{U(s)} = \frac{b_1s + k_1}{mb_1s^3 + mk_1s^2 + b_1(k_1 + k_2)s + k_1k_2}$$

Problem A–4–6

Figure 4–26(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires excite the automobile suspension system, whose motion consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

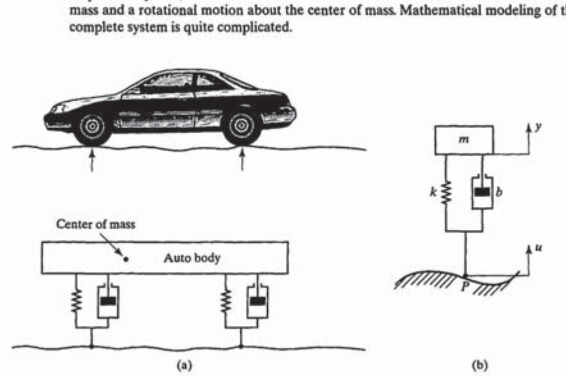


Figure 4–26 (a) Automobile suspension system; (b) simplified suspension system.

A highly simplified version of the suspension system is shown in Figure 4–26(b). Assuming that the motion u at point P is the input to the system and the vertical motion y of the body is the output, obtain the transfer function $Y(s)/U(s)$. (Consider the motion of the body only in the vertical direction.) The displacement y is measured from the equilibrium position in the absence of the input u.

Solution The equation of motion for the system shown in Figure 4–26(b) is

$$m\ddot{y} + b(\dot{y} - \dot{u}) + k(y - u) = 0$$

or

$$m\ddot{y} + b\dot{y} + ky = b\dot{u} + ku$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Hence, the transfer function $Y(s)/U(s)$ is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Problem A–4–7

Obtain the transfer function $Y(s)/U(s)$ of the system shown in Figure 4–27. The vertical motion u at point P is the input. (Similar to the system of **Problem A–4–6**, this system is also a simplified version of an automobile or motorcycle suspension system. In Figure 4–27, m_1 and k_1 represent the wheel mass and tire stiffness, respectively.) Assume that the displacements x and y are measured from their respective equilibrium positions in the absence of the input u.

Solution Applying Newton's second law to the system, we get

$$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$$
$$m_2\ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

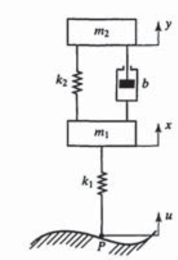


Figure 4–27 Suspension system.

Hence, we have

$$m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2 y + k_1 u$$
$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1 U(s)$$
$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1 s^2 + bs + k_1 + k_2)\frac{m_2 s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1 U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + [k_1 m_2 + (m_1 + m_2)k_2]s^2 + k_1 bs + k_1 k_2}$$

Problem A–4–8

Expand the function

$$\frac{B(s)}{A(s)} = \frac{3s^3 + 5s^2 + 10s + 40}{s^4 + 16s^3 + 69s^2 + 94s + 40}$$

into partial fractions with MATLAB.

Solution A MATLAB program for obtaining the partial-fraction expansion is given in MATLAB Program 4–13.

MATLAB Program 4–13

```
>> num = [3  5  10  40];
>> den = [1  16  69  94  40];
>> [r, p, k] = residue(num,den)
r =

    5.2675
   -2.0741
   -0.1934
    1.1852
p =

  -10.0000
   -4.0000
   -1.0000
   -1.0000
k =
     []
```

From the results of the program, we get the following expression:

$$\frac{B(s)}{A(s)} = \frac{5.2675}{s + 10} + \frac{-2.0741}{s + 4} + \frac{-0.1934}{s + 1} + \frac{1.1852}{(s + 1)^2}$$

Note that the row vector k is zero, because the degree of the numerator is lower than that of the denominator.

Problem A–4–9

Expand the function

$$\frac{B(s)}{A(s)} = \frac{2s^2 + 5s + 7}{s^3 + 3s^2 + 7s + 5}$$

into partial fractions with MATLAB.

Solution A MATLAB program for obtaining the partial-fraction expansion is shown in MATLAB Program 4–14.

MATLAB Program 4–14

```
>> num = [2  5  7];
>> den = [1  3  7  5];
>> [r, p, k] = residue(num,den)
r =
   0.5000 - 0.2500i
   0.5000 + 0.2500i
   1.0000
p =
  -1.0000 + 2.0000i
  -1.0000 - 2.0000i
  -1.0000
k =
     []
```

From the MATLAB output, we get the following expression:

$$\frac{B(s)}{A(s)} = \frac{0.5 - j0.25}{s + 1 - j2} + \frac{0.5 + j0.25}{s + 1 + j2} + \frac{1}{s + 1}$$
$$= \frac{(0.5 - j0.25)(s + 1 + j2) + (0.5 + j0.25)(s + 1 - j2)}{(s + 1 - j2)(s + 1 + j2)} + \frac{1}{s + 1}$$
$$= \frac{s + 2}{s^2 + 2s + 5} + \frac{1}{s + 1}$$

Note that, because the row vector k is zero, there is no constant term in this partial-fraction expansion.

Problem A–4–10

Consider the mechanical system shown in Figure 4–28. The system is initially at rest, and the displacement x is measured from the equilibrium position. Assume that $m = 1$ kg, $b = 12$ N-s/m, and $k = 100$ N/m.

Obtain the response of the system when 10 N of force (a step input) is applied to the mass m. Also, plot a response curve with the use of MATLAB.

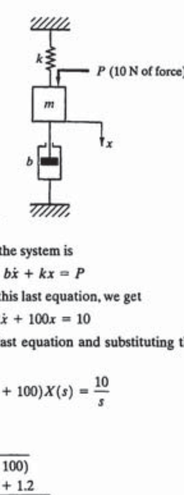


Figure 4–28 Mechanical system.

Solution The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = P$$

Substituting the numerical values into this last equation, we get

$$\ddot{x} + 12\dot{x} + 100x = 10$$

Taking the Laplace transform of this last equation and substituting the initial conditions [$x(0) = 0$ and $\dot{x}(0) = 0$] yields

$$(s^2 + 12s + 100)X(s) = \frac{10}{s}$$

Solving for $X(s)$, we obtain

$$X(s) = \frac{10}{s(s^2 + 12s + 100)}$$
$$= \frac{0.1}{s} - \frac{0.1s + 1.2}{s^2 + 12s + 100}$$
$$= \frac{0.1}{s} - \frac{0.1(s + 6)}{(s + 6)^2 + 8^2} - \left(\frac{0.6}{8}\right)\frac{8}{(s + 6)^2 + 8^2}$$

The inverse Laplace transform of this last equation gives

$$x(t) = 0.1 - 0.1e^{-6t}\cos 8t - 0.075e^{-6t}\sin 8t$$

The response exhibits damped vibration.

A MATLAB program to plot the response curve is given in MATLAB Program 4–15. The resulting response curve is shown in Figure 4–29.

MATLAB Program 4–15

```
>> t = 0:0.01:2;
>> num = [10];
>> den = [1   12   100];
>> sys = tf(num,den);
>> step(sys,t)
>> grid
>> title ('Step Response')
>> xlabel('t')
>> ylabel('Output x(t)')
```

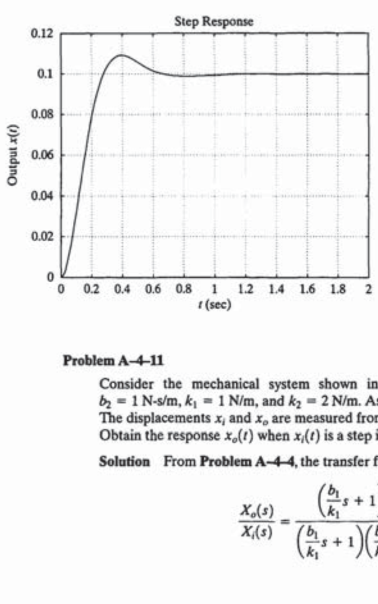



Figure 4–29 Step response of mechanical system.

Problem A–4–11

Consider the mechanical system shown in Figure 4–30, where $b_1 = 0.5$ N-s/m, $b_2 = 1$ N-s/m, $k_1 = 1$ N/m, and $k_2 = 2$ N/m. Assume that the system is initially at rest. The displacements x_i and x_o are measured from their respective equilibrium positions. Obtain the response $x_o(t)$ when $x_i(t)$ is a step input of magnitude 0.1 m.

Solution From **Problem A–4–4**, the transfer function $X_o(s)/X_i(s)$ is

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

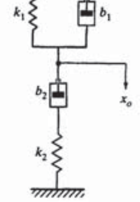


Figure 4–30 Mechanical system.

Substitution of the given numerical values yields

$$\frac{X_o(s)}{X_i(s)} = \frac{(0.5s + 1)(0.5s + 1)}{(0.5s + 1)(0.5s + 1) + s} = \frac{0.25s^2 + s + 1}{0.25s^2 + 2s + 1} = \frac{s^2 + 4s + 4}{s^2 + 8s + 4}$$

Since $x_i(t) = (0.1)1(t)$, we have

$$X_i(s) = \frac{0.1}{s}$$

Hence,

$$X_o(s) = \frac{s^2 + 4s + 4}{s^2 + 8s + 4}\frac{0.1}{s} = \frac{0.1s^2 + 0.4s + 0.4}{s^2 + 8s + 4}\frac{1}{s}$$

MATLAB Program 4–16 is used to obtain the step response, which is shown in Figure 4–31.

MATLAB Program 4–16

```
>> t = 0:0.02:12;
>> num = [0.1  0.4  0.4];
>> den = [1  8  4];
>> sys = tf(num,den);
>> [x_o,t] = step(sys,t);
>> plot(t,x_o)
>> grid
>> title('Step Response of (0.1s^2 + 0.4s + 0.4) / (s^2 + 8s + 4)')
>> xlabel('t (sec)')
>> ylabel('x_o(t)')
```

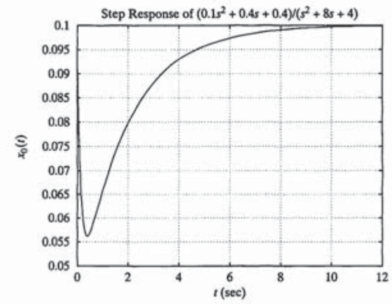


Figure 4–31 Step response of system shown in Figure 4–30.

Figure 4–31 shows the response curve from $t = 0+$ to $t = 12$. Note that $x_o(0+) = 0.1$. If we wish to plot the curve from $x_o(0-) = 0$ to $x_o(12) = 0.1$, we may add the axis command

v = [-2 12 -0.02 0.12]; axis(v)

to the program, as shown in MATLAB Program 4–17. Then the xy domain of the plot becomes $-2 \le x \le 12$, $-0.02 \le y \le 0.12$. The plot of the response curve produced by MATLAB Program 4–17 is shown in Figure 4–32.

MATLAB Program 4–17

```
>> t = 0:0.02:12;
>> num = [0.1  0.4  0.4];
>> den = [1  8  4];
>> sys = tf(num,den);
>> [x_o, t] = step(sys,t);
>> plot(t,x_o)
>> v = [-2  12  -0.02  0.12]; axis(v)
>> grid
>> title('Step Response of (0.1s^2 + 0.4s + 0.4) / (s^2 + 8s + 4)')
>> xlabel('t (sec)')
>> ylabel('x_o(t)')
>> text (1.5, 0.007, 'These two lines are manually drawn.')
```

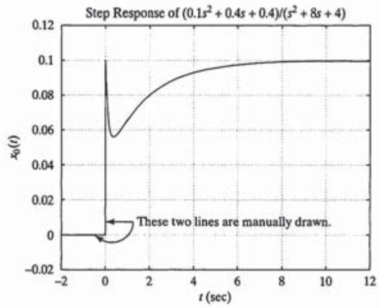


Figure 4–32 Step response of system, shown in the region $-2 \le x \le 12$, $-0.02 \le y \le 0.12$.

Problem A–4–12

Plot the unit-step response curves of the two systems defined by the transfer functions

$$\frac{X(s)}{U(s)} = \frac{25}{s^2 + 5s + 25}$$

and

$$\frac{Y(s)}{U(s)} = \frac{5s + 25}{s^2 + 5s + 25}$$

in one diagram. Then plot each curve in a separate diagram, using the subplot command.

Solution MATLAB Program 4–18 produces the unit-step response curves of the two systems. The curves are shown in Figure 4–33.

Multiple curves on one diagram can be split into multiple windows with the use of the subplot command. MATLAB Program 4–19 uses the subplot command to plot two curves in two subwindows, one curve per subwindow. Figure 4–34 shows the resulting plot.

MATLAB Program 4–18

```
>> t = 0:0.05:3;
>> sys1 = tf([25], [1  5  25]);
>> sys2 = tf([5  25], [1  5  25]);
>> [x, t] = step(sys1,t);
>> [y, t] = step(sys2,t);
>> plot(t,x,t,y,t,y,'o')
>> grid
>> title('Unit-Step Responses')
>> xlabel('t (sec)')
>> ylabel('Outputs x(t) and y(t)')
>> text(1.25,1.15,'x')
>> text(1.25,1.3,'y')
```

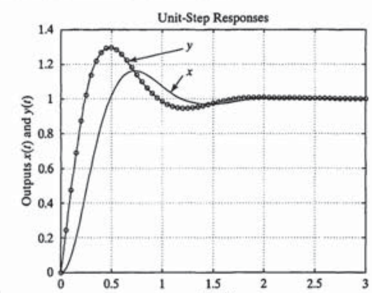


Figure 4–33 Two unit-step response curves shown in one diagram.

Problem A–4–13

Consider the system shown in Figure 4–35. The system is initially at rest. Suppose that the cart is set into motion by an impulsive force whose strength is unity. Can it be stopped by another such impulsive force?

Solution When the mass m is set into motion by a unit-impulse force, the system equation becomes

$$m\ddot{x} + kx = \delta(t)$$

MATLAB Program 4–19

```
>> t = 0:0.05:3;
>> sys1 = tf([25], [1  5  25]);
>> sys2 = tf([5  25], [1  5  25]);
>> [x, t] = step(sys1,t);
>> [y, t] = step(sys2,t);
>> subplot(121), plot(t,x), grid
>> title('Unit-Step Response')
>> xlabel('t (sec)')
>> ylabel('Output x(t)')
>> subplot(122), plot(t,y), grid
>> title('Unit-Step Response')
>> xlabel('t (sec)')
>> ylabel('Output y(t)')
```

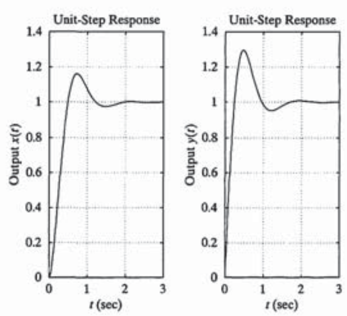


Figure 4–34 Plots of two unit-step response curves in two subwindows, one in each subwindow.

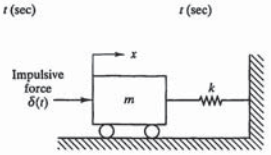


Figure 4–35 Mechanical system.

We define another impulse force to stop the motion as $A\delta(t - T)$, where A is the undetermined magnitude of the impulse force and $t = T$ is the undetermined instant that this impulse is to be given to the system to stop the motion. Then, the equation for the system when the two impulse forces are given is

$$m\ddot{x} + kx = \delta(t) + A\delta(t - T), \qquad x(0-) = 0, \qquad \dot{x}(0-) = 0$$

The $\mathscr{L}_-$ transform of this last equation with $x(0-) = 0$ and $\dot{x}(0-) = 0$ gives

$$(ms^2 + k)X(s) = 1 + Ae^{-sT}$$

Solving for $X(s)$, we obtain

$$X(s) = \frac{1}{ms^2 + k} + \frac{Ae^{-sT}}{ms^2 + k}$$
$$= \frac{1}{\sqrt{km}} \frac{\sqrt{\frac{k}{m}}}{s^2 + \frac{k}{m}} + \frac{A}{\sqrt{km}} \frac{\sqrt{\frac{k}{m}}\, e^{-sT}}{s^2 + \frac{k}{m}}$$

The inverse Laplace transform of $X(s)$ is

$$x(t) = \frac{1}{\sqrt{km}} \sin \sqrt{\frac{k}{m}}\, t + \frac{A}{\sqrt{km}} \left[\sin \sqrt{\frac{k}{m}}(t - T) \right] 1(t - T)$$

If the motion of the mass m is to be stopped at $t = T$, then $x(t)$ must be identically zero for $t \geq T$, a condition we can achieve if we choose

$$A = 1, \qquad T = \frac{\pi}{\sqrt{\frac{k}{m}}}, \quad \frac{3\pi}{\sqrt{\frac{k}{m}}}, \quad \frac{5\pi}{\sqrt{\frac{k}{m}}}, \quad \cdots$$

Thus, the motion of the mass m can be stopped by another impulse force, such as

$$\delta\left(t - \frac{\pi}{\sqrt{\frac{k}{m}}}\right), \quad \delta\left(t - \frac{3\pi}{\sqrt{\frac{k}{m}}}\right), \quad \delta\left(t - \frac{5\pi}{\sqrt{\frac{k}{m}}}\right), \quad \cdots$$

Problem A–4–14

Consider the mechanical system shown in Figure 4–36. Suppose that a person drops a steel ball of mass m onto the center of mass M from a height d and catches the ball on the first bounce. Assume that the system is initially at rest. The ball hits mass M at $t = 0$. Obtain the motion of mass M for $0 < t$. Assume that the impact is perfectly elastic. The displacement x of mass M is measured from the equilibrium position before the ball hits it. The initial conditions are $x(0-) = 0$ and $\dot{x}(0-) = 0$.

Assuming that $M = 1$ kg, $m = 0.015$ kg, $b = 2$ N-s/m, $k = 50$ N/m, and $d = 1.45$ m, plot the response curve with MATLAB.

Solution The input to the system can be taken to be an impulse, the magnitude of which is equal to the change in momentum of the steel ball. At $t = 0$, the ball hits mass M. Assume that the initial velocity of the ball is $v(0-)$. At $t = 0$, the ball bounces back with velocity $v(0+)$. Since the impact is assumed to be perfectly elastic, $v(0+) = -v(0-)$. Figure 4–37(a) shows a sudden change in the velocity of the ball. Define the downward velocity to be positive. Since the change in velocity of the ball occurs instantaneously, $\dot{v}$ has the form of an impulse, as shown in Figure 4–37(b). Note that $\dot{v}(0+)$ is negative.

The equation of motion for the system is

$$M\ddot{x} + b\dot{x} + kx = F(t) \tag{4–29}$$

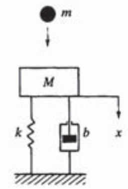


Figure 4–36 Mechanical system subjected to an impulse input.

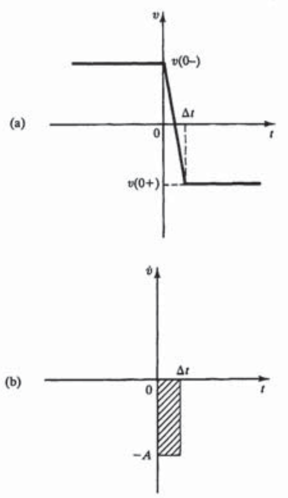


Figure 4–37 (a) Sudden change in the velocity v of steel ball; (b) plot of $\dot{v}$ versus t.

where $F(t)$, an impulse force, is equal to $-m\dot{v}$. [Note that $-m\dot{v}$ is positive; the impulse force $F(t)$ is in the positive direction of x.] From Figure 4–37(b), the impulse force can be written as

$$F(t) = A\,\Delta t\,\delta(t)$$

where $A\,\Delta t$ is the magnitude of the impulse input. Thus,

$$F(t) = A\,\Delta t\,\delta(t) = -m\dot{v}$$

from which we can get

$$\int_{0-}^{0+} A\,\Delta t\,\delta(t)dt = -m\int_{0-}^{0+} \dot{v}\,dt$$

or

$$A\Delta t = mv(0-) - mv(0+) \tag{4–30}$$

The momentum of the steel ball is changed from $mv(0-)$ (downward at $t = 0-$) to $mv(0+)$ (upward at $t = 0+$). Since $v(0+) = -v(0-)$, Equation (4–30) can be written as

$$A\,\Delta t = 2mv(0-)$$

Then Equation (4–29) becomes

$$M\ddot{x} + b\dot{x} + kx = F(t) = 2mv(0-)\,\delta(t)$$

Taking the $\mathscr{L}_-$ transforms of both sides of this last equation, we get

$$M[s^2X(s) - sx(0-) - \dot{x}(0-)] + b[sX(s) - x(0-)] + kX(s) = 2mv(0-)$$

Noting that $x(0-) = 0$ and $\dot{x}(0-) = 0$, we have

$$(Ms^2 + bs + k)X(s) = 2mv(0-)$$

Solving for $X(s)$, we obtain

$$X(s) = \frac{2mv(0-)}{Ms^2 + bs + k}$$

Since the velocity of the steel ball after falling a distance d is $\sqrt{2gd}$, we have

$$v(0-) = \sqrt{2gd}$$

It follows that

$$X(s) = \frac{2m\sqrt{2gd}}{Ms^2 + bs + k} \tag{4–31}$$

Substituting the given numerical values into Equation (4–31), we obtain

$$X(s) = \frac{2 \times 0.015\sqrt{2 \times 9.807 \times 1.45}}{s^2 + 2s + 50}$$
$$= \frac{0.15999}{s^2 + 2s + 50}$$
$$= \frac{0.15999}{7}\,\frac{7}{(s + 1)^2 + 7^2}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = 0.02286e^{-t} \sin 7t \text{ m}$$

Thus, the response of the mass M is a damped sinusoidal motion.

A MATLAB program to produce the response curve is given in MATLAB Program 4–20. The resulting curve is shown in Figure 4–38.

MATLAB Program 4–20

```
>> num = [0.15999];
>> den = [1  2  50];
>> sys = tf(num,den);
>> impulse(sys)
>> grid
>> title('Impulse Response of Mechanical System')
>> xlabel('t')
>> ylabel('Output x(t)')
```

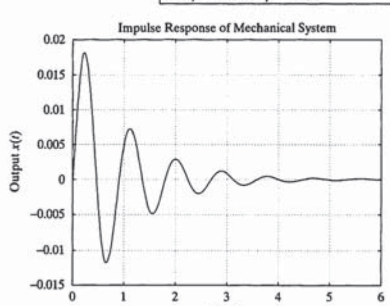


Figure 4–38 Response of mass M subjected to impulse input.

Problem A–4–15

Figure 4–39 shows a mechanism used for a safety seat belt system. Under normal operating conditions, the reel rotates freely and it is possible to let out more slack in the belt, allowing the passenger to move forward even with the belt fastened. However, if the car decelerates rapidly in a collision or sudden stop, the pendulum is subjected to an impulsive torque that causes it to swing forward and also causes the bar to engage the ratchet, locking the reel and safety belt. Thus, the passenger is restrained in place.

Referring to Figure 4–40, assume that the car is moving at a speed of 10 m/s before a sudden stop. The stopping time Δt is 0.3 s. The pendulum length is 0.05 m. Find the time needed for the pendulum to swing forward by 20°.

Solution From Figure 4–41, the moment of inertia of the pendulum about the pivot is $J = ml^2$. The angle of rotation of the pendulum is θ rad. Define the force that acts on the pendulum at the instant the car stops suddenly as $F(t)$. Then the torque that acts on the pendulum due to the force $F(t)$ is $F(t)l \cos\theta$. The equation for the pendulum system is

$$ml^2\ddot{\theta} = F(t)l\cos\theta - mgl\sin\theta \tag{4–32}$$

We linearize this nonlinear equation by assuming that the angle θ is small. (Although $\theta = 20°$ is not quite small, the resulting linearized equation will give an

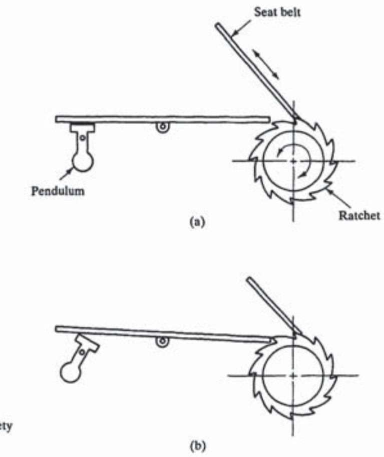


Figure 4–39 Mechanism used for a safety seat belt system. (a) Normal operating condition; (b) emergency condition.

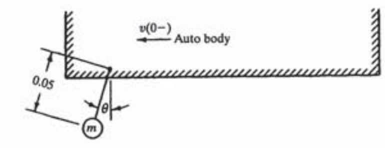


Figure 4–40 Pendulum attached to auto body.

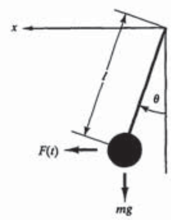


Figure 4–41 Pendulum system.

approximate solution.) Approximating $\cos\theta \doteq 1$ and $\sin\theta \doteq \theta$, we can write Equation (4–32) as

$$ml^2\ddot{\theta} = F(t)l - mgl\,\theta$$

or

$$ml\ddot{\theta} + mg\theta = F(t) \tag{4–33}$$

Since the velocity of the car at $t = 0-$ is 10 m/s and the car stops in 0.3 s, the average deceleration is 33.3 m/s^2.

Under the assumption that a constant acceleration of magnitude 33.3 m/s^2 acts on the pendulum mass for 0.3 sec, $F(t)$ may be given by

$$F(t) = m\ddot{x} = 33.3m[1(t) - 1(t - 0.3)]$$

Then, Equation (4–33) may be written as

$$ml\ddot{\theta} + mg\theta = 33.3m[1(t) - 1(t - 0.3)]$$

or

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{33.3}{l}[1(t) - 1(t - 0.3)]$$

Since $l = 0.05$ m, this last equation becomes

$$\ddot{\theta} + 196.14\theta = 666[1(t) - 1(t - 0.3)]$$

Taking $\mathscr{L}_-$ transforms of both sides of the preceding equation, we obtain

$$(s^2 + 196.14)\Theta(s) = 666\left(\frac{1}{s} - \frac{1}{s}e^{-0.3s}\right) \tag{4–34}$$

where we used the initial conditions that $\theta(0-) = 0$ and $\dot{\theta}(0-) = 0$. Solving Equation (4–34) for $\Theta(s)$ yields

$$\Theta(s) = \frac{666}{s(s^2 + 196.14)}(1 - e^{-0.3s})$$
$$= \left(\frac{1}{s} - \frac{s}{s^2 + 196.14}\right)\frac{666}{196.14}(1 - e^{-0.3s})$$

The inverse Laplace transform of $\Theta(s)$ gives

$$\theta(t) = 3.3955(1 - \cos 14t)$$
$$- 3.3955\{1(t - 0.3) - [\cos 14(t - 0.3)]1(t - 0.3)\} \tag{4–35}$$

Note that $1(t - 0.3) = 0$ for $0 \le t < 0.3$.

Now assume that at $t = t_1$, $\theta = 20° = 0.3491$ rad. Then, tentatively assuming that t_1 occurs before $t = 0.3$, we seek to solve the following equation for t_1:

$$0.3491 = 3.3955(1 - \cos 14t_1)$$

Simplifying yields

$$\cos 14t_1 = 0.8972$$

and the solution is

$$t_1 = 0.0326 \text{ s}$$

Since $t_1 = 0.0326 < 0.3$, our assumption was correct. The terms involving $1(t - 0.3)$ in Equation (4–35) do not affect the value of t_1. It thus takes approximately 33 milliseconds for the pendulum to swing 20°.

Problem A–4–16

Consider the mechanical system shown in Figure 4–42(a). The cart has the mass of m kg. Assume that the wheels have negligible masses and there is no friction involved in the system. The force $u(t)$ applied to the cart is increased linearly from 0 to 5 N for the period $0 \le t \le 10$, as shown in Figure 4–42(b). At $t = 10+$, the force $u(t)$ is disengaged, or

$$u(t) = 0 \qquad \text{for } 10 < t$$

Assuming that $m = 100$ kg, obtain the displacement $x(t)$ of the cart for $0 \le t \le 30$ with MATLAB. The cart is at rest for $t < 0$, and the displacement x is measured from the rest position.

Solution The equation of motion for the system is

$$m\ddot{x} = u$$

Hence, the transfer function of the system is

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2} = \frac{1}{100s^2}$$

The input $u(t)$ is a ramp function for $0 \le t \le 10$ and is zero for $10 < t$, as shown in Figure 4–42(b). (At $t = 10$, $u = 5$ N.) Thus, in MATLAB, we define

u1 = 0.5*[0:0.02:10] for $0 \le t \le 10$
u2 = 0*[10.02:0.02:30] for $10 < t \le 30$

where u is either u1 or u2, depending on which interval u is in. Then the input force $u(t)$ for $0 \le t \le 30$ can be given by the MATLAB array

u = [u1 u2]

MATLAB Program 4–21

```
>> t = 0:0.02:30;
>> u1 = 0.5*[0:0.02:10];
>> u2 = 0*[10.02:0.02:30];
>> u = [u1  u2];
>> num = [1]; den = [100  0  0];
>> sys = tf(num,den);
>> x = lsim(sys,u,t);
>> subplot(211),plot(t,x)
>> grid
>> ylabel('Output x(t)')
>> subplot (212), plot(t,u)
>> v = [0  30  -1  6]; axis (v)
>> grid
>> xlabel('t (sec)')
>> ylabel('Input Force u(t) newton')
```


MATLAB Program 4–21 produces the response curve. The curves of $x(t)$ versus t and the input force $u(t)$ versus t are shown in Figure 4–43.

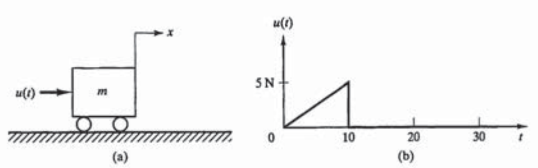


Figure 4–42 (a) Mechanical system; (b) force $u(t)$ applied to the cart.

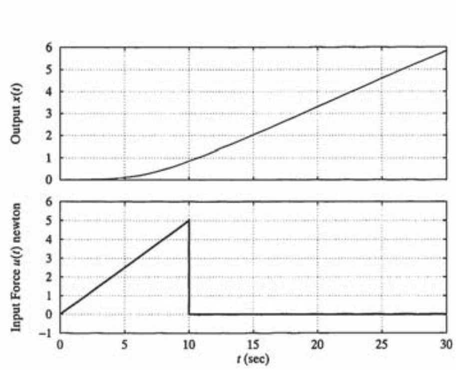


Figure 4–43 Response curve $x(t)$ versus t and input curve $u(t)$ versus t.

Problem A–4–17

Using MATLAB, generate a triangular wave as shown in Figure 4–44.

Solution There are many ways to generate the given triangular wave. In MATLAB Program 4–22, we present one simple way to do so. The resulting wave is shown in Figure 4–45.

MATLAB Program 4–22

```
>> t = 0:1:8;
>> u1 = [0:0.5:1];
>> u2 = [0.5:−0.5:−1];
>> u3 = [−0.5:0.5:0];
>> u = [u1   u2   u3];
>> plot(t,u)
>> v = [−2   10   −1.5   1.5]; axis(v)
>> grid
>> title('Triangular Wave')
>> xlabel('t (sec)')
>> ylabel('Displacement u(t)')
```

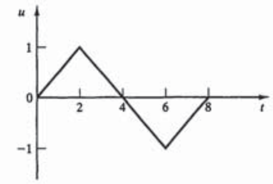


Figure 4–44 Triangular wave.

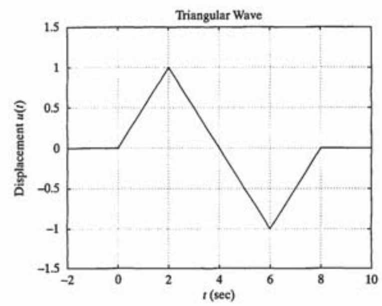


Figure 4–45 Triangular wave generated with MATLAB.

Problem A–4–18

Consider the spring–mass–dashpot system shown in Figure 4–46(a). Assume that the displacement u of point P is the input to the system. Assume also that the input $u(t)$ is a small bump, as shown in Figure 4–46(b). Obtain the response $y(t)$ of the mass m. The displacement y is measured from the equilibrium position in the absence of the input $u(t)$. To obtain the response curve, assume that $m = 100$ kg, $b = 400$ N-s/m, and $k = 800$ N/m.

Solution From **Problem A–4–6**, the transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Substituting the given numerical values for m, b, and k into this transfer function, we obtain

$$\frac{Y(s)}{U(s)} = \frac{400s + 800}{100s^2 + 400s + 800} = \frac{4s + 8}{s^2 + 4s + 8}$$

The input $u(t)$ is a triangular wave for $0 \le t \le 4$ and is zero for $4 < t \le 8$. (For the generation of a triangular wave, see **Problem A–4–17**.)

As in **Problem A–4–17**, the input $u(t)$ can be generated by first defining, in MATLAB,

```
u1 = [0:0.02:1];
u2 = [0.98:-0.02:-1];
u3 = [-0.98:0.02:0];
u4 = 0*[4.02:0.02:8];
```

and then defining

```
u = [u1   u2   u3   u4]
```

MATLAB Program 4–23 produces the response $y(t)$ of the system. The response curve $y(t)$ versus t and the input curve $u(t)$ versus t are shown in Figure 4–47.

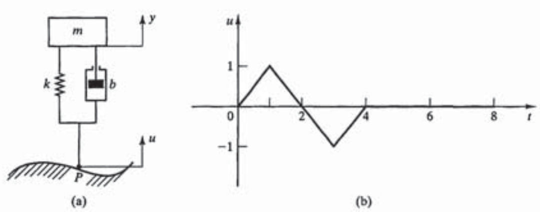


Figure 4–46 (a) Spring–mass–dashpot system; (b) input $u(t)$ versus t.

MATLAB Program 4–23

```
>> t = 0:0.02:8;
>> num = [4   8];
>> den = [1   4   8];
>> sys = tf(num,den);
>> u1 = [0:0.02:1];
>> u2 = [0.98:-0.02:-1];
>> u3 = [-0.98:0.02:0];
>> u4 = 0*[4.02:0.02:8];
>> u = [u1   u2   u3   u4];
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> grid
>> title('Response of Spring-Mass-Dashpot System and Input u(t)')
>> xlabel('t (sec)')
>> ylabel('Output y(t) and Input u(t)')
>> text(2.2,0.72,'y')
>> text(1.05,0.1,'u')
```

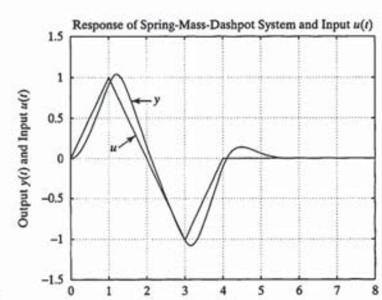


Figure 4–47 Response of spring–mass–dashpot system subjected to the input shown in Figure 4–46(b).

Problem A–4–19

Consider the spring–mass system shown in Figure 4–48. The displacement x is measured from the equilibrium position. The system is initially at rest. Assume that at $t = 0$ the mass is pulled downward by 0.1 m [i.e., $x(0) = 0.1$] and released with the initial velocity of 0.5 m/s [i.e., $\dot{x}(0) = 0.5$]. Obtain the response curve $x(t)$ versus t with MATLAB. Assume that $m = 1$ kg and $k = 9$ N/m.

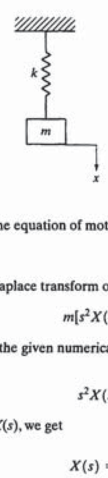


Figure 4–48 Spring–mass system.

Solution The equation of motion for the system is

$$m\ddot{x} = -kx$$

Taking the Laplace transform of the preceding equation, we obtain

$$m[s^2X(s) - sx(0) - \dot{x}(0)] = -kX(s)$$

Substituting the given numerical values for m, k, $x(0)$, and $\dot{x}(0)$ into this last equation, we have

$$s^2X(s) - 0.1s - 0.5 + 9X(s) = 0$$

Solving for $X(s)$, we get

$$X(s) = \frac{0.1s + 0.5}{s^2 + 9} = \frac{0.1s^2 + 0.5s}{s^2 + 9}\,\frac{1}{s}$$

Hence, the response $x(t)$ can be obtained as the unit-step response of

$$G(s) = \frac{0.1s^2 + 0.5s}{s^2 + 9}$$

MATLAB Program 4–24 produces the response curve $x(t)$ versus t. The curve is shown in Figure 4–49.

MATLAB Program 4–24

```
>> t = 0:0.001:4;
>> num = [0.1   0.5   0];
>> den = [1   0   9];
>> sys = tf(num,den);
>> x = step(sys,t);
>> plot(t,x)
>> v = [-1   4   -0.4   0.4]; axis(v)
>> grid
>> title ('Response of System Subjected to Initial Condition')
>> xlabel('t (sec)')
>> ylabel('x(t) meter')
```

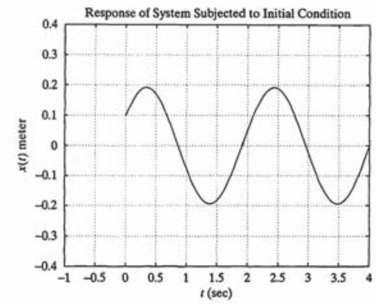


Figure 4–49 Response of spring–mass system subjected to the initial condition $x(0) = 0.1$ m and $\dot{x}(0) = 0.5$ m/s.

PROBLEMS

Problem B–4–1

Find the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 4–50. The displacements x_i and x_o are measured from their respective equilibrium positions. Obtain the displacement $x_o(t)$ when the input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$. Assume that $x_o(0-) = 0$.

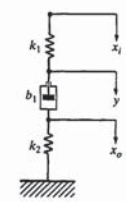


Figure 4–50 Mechanical system.

Problem B–4–2

Derive the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 4–51. The displacements x_i and x_o are measured from their respective equilibrium positions. Obtain the response $x_o(t)$ when the input $x_i(t)$ is the pulse

$$\begin{aligned} x_i(t) &= X_i \qquad 0 < t < t_1 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

Assume that $x_o(0-) = 0$.

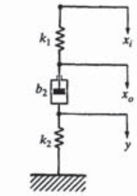


Figure 4–51 Mechanical system.

Problem B–4–3

Consider the mechanical system shown in Figure 4–52. Assume that $u(t)$ is the force applied to the cart and is the input to the system. The displacement x is measured from the equilibrium position and is the output of the system. Obtain the transfer function $X(s)/U(s)$ of the system.

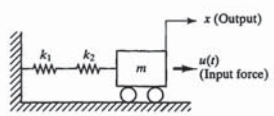


Figure 4–52 Mechanical system.

Problem B–4–4

In the mechanical system shown in Figure 4–53, the force u is the input to the system and the displacement x, measured from the equilibrium position, is the output of the system, which is initially at rest. Obtain the transfer function $X(s)/U(s)$.

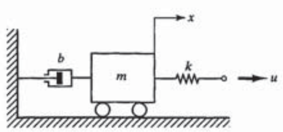


Figure 4–53 Mechanical system.

Problem B–4–5

The system shown in Figure 4–54 is initially at rest, and the displacement x is measured from the equilibrium position. At $t = 0$, a force u is applied to the system. If u is the input to the system and x is the output, obtain the transfer function $X(s)/U(s)$.

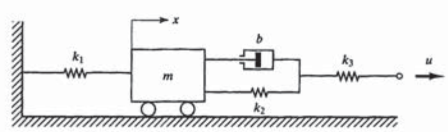


Figure 4–54 Mechanical system.

Problem B–4–6

Consider the mechanical system shown in Figure 4–55. The system is at rest for $t < 0$. The input force u is given at $t = 0$. The displacement x is the output of the system and is measured from the equilibrium position. Obtain the transfer function $X(s)/U(s)$.

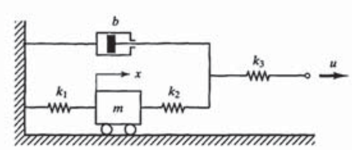


Figure 4–55 Mechanical system.

Problem B–4–7

In the system of Figure 4–56, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for x and θ are zeros, or $x(0-) = 0$ and $\theta(0-) = 0$. Show that this system is a differentiating element. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.

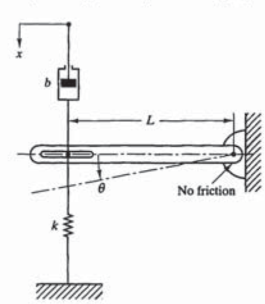


Figure 4–56 Mechanical system.

Problem B–4–8

Consider the mechanical system shown in Figure 4–57. The system is initially at rest. Assume that u is the displacement of point P and x is the displacement of mass m. The displacement x is measured from the equilibrium position when $u = 0$. Draw four different block diagrams for the system.

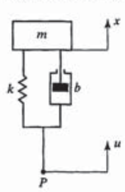


Figure 4–57 Mechanical system.

Problem B–4–9

Using MATLAB, obtain the partial-fraction expansion of

$$\frac{B(s)}{A(s)} = \frac{1}{s^4 + s^3 + 81s^2 + 81s}$$

Problem B–4–10

Using MATLAB, obtain the partial-fraction expansion of

$$\frac{B(s)}{A(s)} = \frac{5(s+2)}{s^5 + 5s^4 + 19s^3 + 12s^2}$$

Problem B–4–11

Consider the mechanical system shown in Figure 4–58. Plot the response curve $x(t)$ versus t with MATLAB when the mass m is pulled slightly downward, generating the initial conditions $x(0) = 0.05$ m and $\dot{x}(0) = 1$ m/s, and released at $t = 0$. The displacement x is measured from the equilibrium position before m is pulled downward. Assume that $m = 1$ kg, $b_1 = 4$ N-s/m, $k_1 = 6$ N/m, and $k_2 = 10$ N/m.

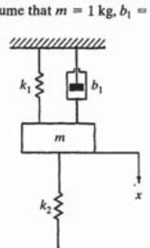


Figure 4–58 Mechanical system.

Problem B–4–12

Consider the mechanical system shown in Figure 4–59. The system is initially at rest. The displacements x_1 and x_2 are measured from their respective equilibrium positions before the input u is applied. Assume that $b_1 = 1$ N-s/m, $b_2 = 10$ N-s/m, $k_1 = 4$ N/m, and $k_2 = 20$ N/m. Obtain the displacement $x_2(t)$ when u is a step force input of 2 N. Plot the response curve $x_2(t)$ versus t with MATLAB.

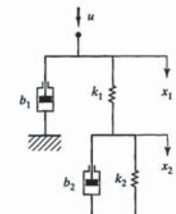


Figure 4–59 Mechanical system.

Problem B–4–13

Figure 4–60 shows a mechanical system that consists of a mass and a damper. The system is initially at rest. Find the response $x(t)$ when the system is set into motion by an impulsive force whose strength is unity. Determine the initial velocity of mass m. Plot the response curve $x(t)$ versus t when $m = 100$ kg and $b = 200$ N-s/m.

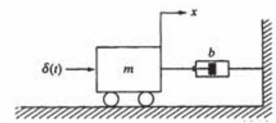


Figure 4–60 Mechanical system.

Problem B–4–14

Consider the mechanical system shown in Figure 4–61. Suppose that the system is initially at rest $[x(0-) = 0, \dot{x}(0-) = 0]$ and at $t = 0$ it is set into motion by a unit-impulse force. Obtain the transfer function of the system. Then obtain an analytical solution $x(t)$. What is the initial velocity $\dot{x}(0+)$ after the unit-impulse force is given to the cart?

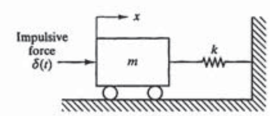


Figure 4–61 Mechanical system.

Problem B–4–15

The mechanical system shown in Figure 4–62 is initially at rest. The displacement x of mass m is measured from the rest position. At $t = 0$, mass m is set into motion by an impulsive force whose strength is unity. Using MATLAB, plot the response curve $x(t)$ versus t when $m = 10$ kg, $b = 20$ N-s/m, and $k = 50$ N/m.

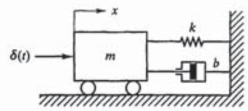


Figure 4–62 Mechanical system.

Problem B–4–16

A mass m of 1 kg is vibrating initially in the mechanical system shown in Figure 4–63. At $t = 0$, we hit the mass with an impulsive force $p(t)$ whose strength is 10 N. Assuming that the spring constant k is 100 N/m, that $x(0-) = 0.1$ m, and that $\dot{x}(0-) = 1$ m/s, find the displacement $x(t)$ as a function of time t. The displacement $x(t)$ is measured from the equilibrium position in the absence of the excitation force.

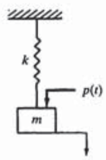


Figure 4–63 Mechanical system.

Problem B–4–17

Consider the system shown in Figure 4–64. The system is at rest for $t < 0$. Assume that the displacement x is the output of the system and is measured from the equilibrium position. At $t = 0$, the cart is given initial conditions $x(0) = x_o$ and $\dot{x}(0) = v_o$. Obtain the output motion $x(t)$. Assume that $m = 10$ kg, $b_1 = 50$ N-s/m, $b_2 = 70$ N-s/m, $k_1 = 400$ N/m, and $k_2 = 600$ N/m.

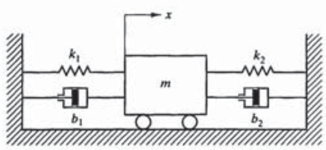


Figure 4–64 Mechanical system.

Problem B–4–18

Referring to **Problem B–4–17**, assume that $m = 100$ kg, $b_1 = 120$ N-s/m, $b_2 = 80$ N-s/m, $k_1 = 200$ N/m, and $k_2 = 300$ N/m. The initial conditions are $x(0) = 0$ m and $\dot{x}(0) = 0.5$ m/s. Obtain the response curve $x(t)$ versus t with MATLAB.

5

State-Space Approach to Modeling Dynamic Systems

5–1 INTRODUCTION

The modern trend in dynamic systems is toward greater complexity, due mainly to the twin requirements of complex tasks and high accuracy. Complex systems may have multiple inputs and multiple outputs. Such systems may be linear or nonlinear and may be time invariant or time varying. A very powerful approach to treating such systems is the state-space approach, based on the concept of state. This concept, by itself, is not new; it has been in existence for a long time in the field of classical dynamics and in other fields. What is new is the combination of the concept of state and the capability of high-speed solution of differential equations with the use of the digital computer.

This chapter presents an introductory account of modeling dynamic systems in state space and analyzing simple dynamic systems with MATLAB. (More on the state-space analysis of dynamic systems is given in Chapter 8.) If the dynamic system is formulated in the state space, it is very easy to simulate it on the computer and find the computer solution of the system's differential equations, because the state-space formulation is developed precisely with such computer solution in mind. Although we treat only linear, time-invariant systems in this chapter, the state-space approach can be applied to both linear and nonlinear systems and to both time-invariant and time-varying systems.

In what follows, we shall first give definitions of state, state variables, state vector, and state space. Then we shall present the outline of the chapter.

State. The *state* of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Thus, the state of a dynamic system at time t is uniquely determined by the state at time t_0 and the input $t \geq t_0$ and is independent of the state and input before t_0. In dealing with linear time-invariant systems, we usually choose the reference time t_0 to be zero.

State variables. The *state variables* of a dynamic system are the variables making up the smallest set of variables that determines the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that, once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then those n variables are a set of state variables. It is important to note that variables that do not represent physical quantities can be chosen as state variables.

State vector. If n state variables are needed to completely describe the behavior of a given system, then those state variables can be considered the n components of a vector $\mathbf{x}$ called a *state vector*. A state vector is thus a vector that uniquely determines the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $\mathbf{u}(t)$ for $t \geq t_0$ is specified.

State space. The n-dimensional space whose coordinate axes consist of the x_1-axis, x_2-axis, $\ldots$, x_n-axis is called a *state space*. Any state can be represented by a point in the state space.

State-space equations. In state-space analysis, we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. As we shall see later, the state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

If a system is linear and time invariant and if it is described by n state variables, r input variables, and m output variables, then the state equation will have the form

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1r}u_r \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2r}u_r \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r
\end{aligned}
$$

and the output equation will have the form

$$
\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1r}u_r \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2r}u_r \\
&\vdots \\
y_m &= c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots + d_{mr}u_r
\end{aligned}
$$

where the coefficients a_{ij}, b_{ij}, c_{ij}, and d_{ij} are constants, some of which may be zero. If we use vector–matrix expressions, these equations can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5–1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5–2}$$

where

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}
$$

$$
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}
$$

Matrices **A**, **B**, **C**, and **D** are called the *state matrix*, *input matrix*, *output matrix*, and *direct transmission matrix*, respectively. Vectors **x**, **u**, and **y** are the *state vector*, *input vector*, and *output vector*, respectively. (In control systems analysis and design, the input matrix **B** and input vector **u** are called the control matrix and control vector, respectively.) The elements of the state vector are the state variables. The elements of the input vector **u** are the input variables. (If the system involves only one input variable, then u is a scalar.) The elements of the output vector **y** are the output variables. (The system may involve one or more output variables.) Equation (5–1) is called the *state equation*, and Equation (5–2) is called the *output equation*. [In this book, whenever we discuss state-space equations, they are described by Equations (5–1) and (5–2).]

A block diagram representation of Equations (5–1) and (5–2) is shown in Figure 5–1. (In the figure, double-line arrows are used to indicate that the signals are vector quantities.)

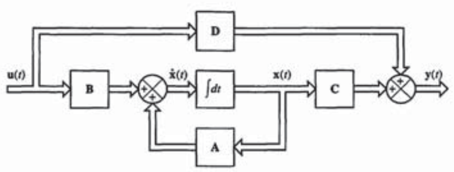


Figure 5–1 Block diagram of the linear, continuous-time system represented in state space.

Example 5–1

Consider the mechanical system shown in Figure 5–2. The displacement y of the mass is the output of the system, and the external force u is the input to the system. The displacement y is measured from the equilibrium position in the absence of the external force. Obtain a state-space representation of the system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \tag{5–3}$$

This system is of second order. (This means that the system involves two integrators.) Thus, we need two state variables to describe the system dynamics. Since $y(0)$, $\dot{y}(0)$, and $u(t) \geq 0$ completely determine the system behavior for $t \geq 0$, we choose $y(t)$ and $\dot{y}(t)$ as state variables, or define

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y}
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ddot{y} = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u
\end{aligned}
$$

or

$$\dot{x}_1 = x_2 \tag{5–4}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{5–5}$$

The output equation is

$$y = x_1 \tag{5–6}$$

In vector–matrix form, Equations (5–4) and (5–5) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{5–7}$$

The output equation, Equation (5–6), can be written as

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5–8}$$

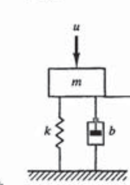


Figure 5–2 Mechanical system.

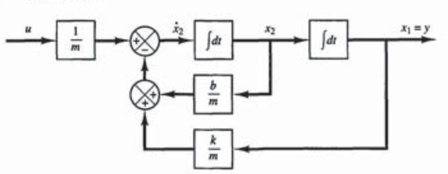


Figure 5–3 Block diagram of the mechanical system shown in Figure 5–2.

Equation (5–7) is a state equation, and Equation (5–8) is an output equation for the system. Equations (5–7) and (5–8) are in the standard form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\
y &= \mathbf{C}\mathbf{x} + Du
\end{aligned}
$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Note that Equation (5–3) can be modified to

$$\frac{u}{m} - \frac{k}{m}y - \frac{b}{m}\dot{y} = \ddot{y}$$

or

$$\frac{1}{m}u - \frac{k}{m}x_1 - \frac{b}{m}x_2 = \dot{x}_2$$

On the basis of this last equation, we can draw the block diagram shown in Figure 5–3. Notice that the outputs of the integrators are state variables.

In a state-space representation, a system is represented by a state equation and an output equation. In this representation, the internal structure of the system is described by a first-order vector–matrix differential equation. This fact indicates that the state-space representation is fundamentally different from the transfer-function representation, in which the dynamics of the system are described by the input and the output, but the internal structure is put in a black box.

Outline of the chapter. Section 5–1 has defined some terms that are necessary for the modeling of dynamic systems in state space and has derived a state-space model of a simple dynamic system. Section 5–2 gives a transient-response analysis of systems in state-space form with MATLAB. Section 5–3 discusses the state-space modeling of systems wherein derivative terms of the input function do not appear in the system differential equations. Numerical response analysis is done with MATLAB. Section 5–4 presents two methods for obtaining state-space models of systems in which derivative terms of the input function appear explicitly in the system differential equations. Section 5–5 treats the transformation of system models from transfer-function representation to state-space representation and vice versa. The section also examines the transformation of one state-space representation to another.

5–2 TRANSIENT-RESPONSE ANALYSIS OF SYSTEMS IN STATE-SPACE FORM WITH MATLAB

This section presents the MATLAB approach to obtaining transient-response curves of systems that are written in state-space form.

Step response. We first define the system with

```
sys = ss(A,B,C,D)
```

For a unit-step input, the MATLAB command

```
step(sys)     or     step(A,B,C,D)
```

will generate plots of unit-step responses. The time vector is automatically determined when t is not explicitly included in the step commands.

Note that when step commands have left-hand arguments, such as

```
y = step(sys,t),          [y,t,x] = step(sys,t),
[y,x,t] = step(A,B,C,D,iu),    [y,x,t] = step(A,B,C,D,iu,t)
```

no plot is shown on the screen. Hence, it is necessary to use a plot command to see the response curves. The matrices y and x contain the output and state response of the system, respectively, evaluated at the computation time points t. (Matrix y has as many columns as outputs and one row for each element in t. Matrix x has as many columns as states and one row for each element in t.)

Note also that the scalar iu is an index into the inputs of the system and specifies which input is to be used for the response; t is the user-specified time. If the system involves multiple inputs and multiple outputs, the step commands produces a series of step response plots, one for each input and output combination of

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

(For details, see **Example 5–2**.)

Transfer matrix. Next, consider a multiple-input–multiple-output system. Assume that there are r inputs $u_1, u_2, \ldots, u_r$, and m outputs $y_1, y_2, \ldots, y_m$. Define

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

The transfer matrix $\mathbf{G}(s)$ relates the output $\mathbf{Y}(s)$ to the input $\mathbf{U}(s)$, or

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

where

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{5–9}$$

[The derivation of Equation (5–9) is given in **Example 5–2**, to follow.] Since the input vector $\mathbf{u}$ is r dimensional and the output vector $\mathbf{y}$ is m dimensional, the transfer matrix $\mathbf{G}(s)$ is an $m \times r$ matrix.

Example 5–2

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves.

Although it is not necessary to obtain the transfer-function expression for the system in order to obtain the unit-step response curves with MATLAB, we shall derive such an expression for reference purposes. For the system defined by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

the transfer matrix $\mathbf{G}(s)$ is a matrix that relates $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ through the formula

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s) \tag{5–10}$$

Taking Laplace transforms of the state-space equations, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX}(s) + \mathbf{BU}(s) \tag{5–11}$$
$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s) \tag{5–12}$$

In deriving the transfer matrix, we assume that $\mathbf{x}(0) = \mathbf{0}$. Then, from Equation (5–11), we get

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s)$$

Substituting this equation into Equation (5–12) yields

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

Upon comparing this last equation with Equation (5–10), we see that

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

The transfer matrix $\mathbf{G}(s)$ for the given system becomes

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2 + s + 6.5}\begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \frac{1}{s^2 + s + 6.5}\begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2+s+6.5} & \dfrac{s}{s^2+s+6.5} \\ \dfrac{s+7.5}{s^2+s+6.5} & \dfrac{6.5}{s^2+s+6.5} \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Since this system involves two inputs and two outputs, four transfer functions can be defined, depending on which signals are considered as input and output. Note that, when considering the signal u_1 as the input, we assume that signal u_2 is zero, and vice versa. The four transfer functions are

$$\frac{Y_1(s)}{U_1(s)} = \frac{s-1}{s^2+s+6.5}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s}{s^2+s+6.5}$$
$$\frac{Y_2(s)}{U_1(s)} = \frac{s+7.5}{s^2+s+6.5}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2+s+6.5}$$

The four individual step-response curves can be plotted with the use of the command

```
step(A,B,C,D)
```

or

```
sys = ss(A,B,C,D);     step(sys)
```

MATLAB Program 5–1 produces four individual unit-step response curves, shown in Figure 5–4.

MATLAB Program 5–1

```
>> A = [-1  -1;6.5  0];
>> B = [1  1;1  0];
>> C = [1  0;0  1];
>> D = [0  0;0  0];
>> sys = ss(A,B,C,D);
>> step(sys)
>> grid
>> title('Unit-Step Responses')
>> xlabel('t')
>> ylabel('Outputs')
```

To plot two step-response curves for the input u_1 in one diagram and two step-response curves for the input u_2 in another diagram, we may use the commands

```
step(A,B,C,D,1)
```

and

```
step(A,B,C,D,2)
```

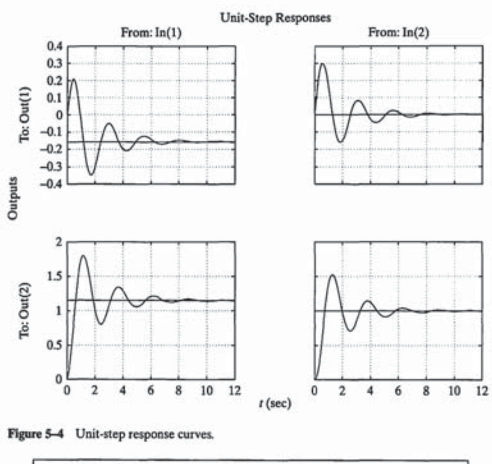



Figure 5–4 Unit-step response curves.

MATLAB Program 5–2

```
>> % ----- In this program, we first plot step-response curves
>> % when the input is u1. Then we plot response curves when
>> % the input is u2. -----
>>
>> A = [-1  -1;6.5  0];
>> B = [1  1;1  0];
>> C = [1  0;0  1];
>> D = [0  0;0  0];
>>
>> step(A,B,C,D,1)
>> grid
>> title('Step-Response Plots (u_1 = Unit-Step Input, u_2 = 0)')
>> xlabel('t'); ylabel('Outputs')
>>
>> step(A,B,C,D,2)
>> grid
>> title('Step-Response Plots (u_1 = 0, u_2 = Unit-Step Input)')
>> xlabel('t'); ylabel('Outputs')
```

respectively. MATLAB Program 5–2 does just that. Figures 5–5 and 5–6 show the two diagrams produced, each consisting of two unit-step response curves.

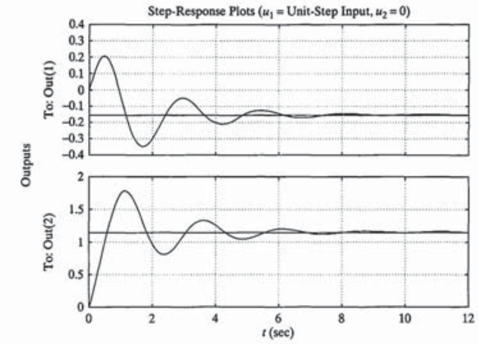


Figure 5–5 Unit-step response curves when u_1 is the input and $u_2 = 0$.

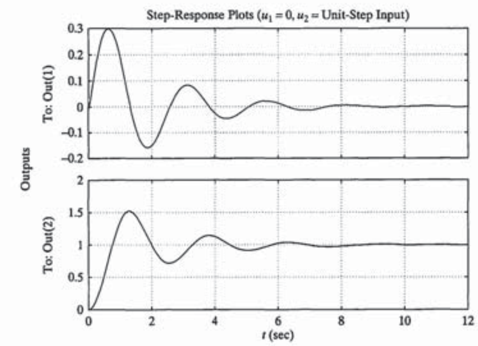


Figure 5–6 Unit-step response curves when u_2 is the input and $u_1 = 0$.

Impulse response. The unit-impulse response of a dynamic system defined in a state space may be obtained with the use of one of the following MATLAB commands:


```
sys = ss(A,B,C,D);        impulse(sys),        y = impulse(sys, t),
[y,t,x] = impulse(sys),        [y,t,x] = impulse(sys,t),
impulse(A,B,C,D),        [y,x,t] = impulse(A,B,C,D),
[y,x,t] = impulse(A,B,C,D,iu),        [y,x,t] = impulse(A,B,C,D,iu,t)
```

The command impulse(sys) or impulse(A,B,C,D) produces a series of unit-impulse response plots, one for each input–output combination of the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

with the time vector automatically determined. If the right-hand side of a command includes the scalar iu (an index into the inputs of the system), then that scalar specifies which input to use for the impulse response.

Note that if a command includes t, it is the user-supplied time vector, which specifies the times at which the impulse response is to be computed.

If MATLAB is invoked with the left-hand argument [y,t,x], as in the case of [y,t,x] = impulse(sys,t), the command returns the output and state responses of the system and the time vector t. No plot is drawn on the screen. The matrices y and x contain the output and state responses of the system, evaluated at the time points t. (Matrix y has as many columns as outputs and one row for each element in t. Matrix x has as many columns as state variables and one row for each element in t.)

Response to arbitrary input. The command lsim produces the response of linear time-invariant systems to arbitrary inputs. If the initial conditions of the system in state-space form are zero, then

lsim(sys,u,t)

produces the response of the system to an arbitrary input u with user-specified time t.

If the initial conditions are nonzero in a state-space model, the command

lsim(sys,u,t,x_0)

where x_0 is the initial state, produces the response of the system, subject to the input u and the initial condition x_0.

The command

[y,t] = lsim(sys,u,t,x_0)

returns the output response y. No plot is drawn. To plot the response curves, it is necessary to use the command plot(t,y).

Response to initial condition. To find the response to the initial condition x_0 given to a system in a state-space form, the following command may be used:

[y,t] = lsim(sys,u,t,x_0)

Here, u is a vector consisting of zeros having length size(t). Alternatively, if we choose $\mathbf{B} = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$, then u can be *any* input having length size(t).

Another way to obtain the response to the initial condition given to a system in a state-space form is to use the command

initial(A,B,C,D,x_0,t)

Example 5–3 is illustrative.

Example 5–3

Consider the system shown in Figure 5–7. The system is at rest for $t < 0$. At $t = 0$, the mass is pulled downward by 0.1 m and is released with an initial velocity of 0.05 m/s. That is, $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/s. The displacement x is measured from the equilibrium position. There is no external input to this system.

Assuming that $m = 1$ kg, $b = 3$ N-s/m, and $k = 2$ N/m, obtain the response curves $x(t)$ versus t and $\dot{x}(t)$ versus t with MATLAB. Use the command initial.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the given numerical values for m, b, and k yields

$$\ddot{x} + 3\dot{x} + 2x = 0$$

If we define the state variables as

$$x_1 = x$$
$$x_2 = \dot{x}$$

and the output variables as

$$y_1 = x_1$$
$$y_2 = x_2$$

then the state equation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}u$$

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}u$$

Thus,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}$$

Using the command

initial(A,B,C,D,x_0,t)

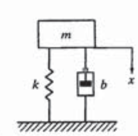


Figure 5–7 Mechanical system.

we can obtain the responses $x(t) = y_1(t)$ versus t and $\dot{x}(t) = y_2(t)$ versus t. MATLAB Program 5–3 produces the response curves, which are shown in Figure 5–8.

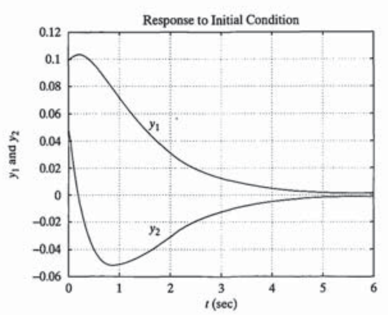


Figure 5–8 Response curves to initial condition.

MATLAB Program 5–3

```
>> t = 0:0.01:6;
>> A = [0  1;-2  -3];
>> B = [0;0];
>> C = [1  0;0  1];
>> D = [0;0];
>> [y, x, t] = initial(A,B,C,D,[0.1; 0.05],t);
>> y1 = [1      0] *y';
>> y2 = [0      1] *y';
>> plot(t,y1,t,y2)
>> grid
>> title('Response to Initial Condition')
>> xlabel('t (sec)')
>> ylabel('y_1 and y_2')
>> text(1.6, 0.05,'y_1')
>> text(1.6, -0.026,'y_2')
```

5–3 STATE-SPACE MODELING OF SYSTEMS WITH NO INPUT DERIVATIVES

In this section, we present two examples of the modeling of dynamic systems in state-space form. The systems used are limited to the case where derivatives of the input functions do not appear explicitly in the equations of motion. In each example, we first derive state-space models and then find the response curves with MATLAB, given the numerical values of all of the variables and the details of the input functions.

Example 5–4

Consider the mechanical system shown in Figure 5–9. The system is at rest for $t < 0$. At $t = 0$, a unit-impulse force, which is the input to the system, is applied to the mass. The displacement x is measured from the equilibrium position before the mass m is hit by the unit-impulse force.

Assuming that $m = 5$ kg, $b = 20$ N-s/m, and $k = 100$ N/m, obtain the response curves $x(t)$ versus t and $\dot{x}(t)$ versus t with MATLAB.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = u$$

The response of such a system depends on the initial conditions and the forcing function u. The variables that provide the initial conditions qualify as state variables. Hence, we choose the variables that specify the initial conditions as state variables x_1 and x_2. Thus,

$$x_1 = x$$
$$x_2 = \dot{x}$$

The state equation then becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(u - kx - b\dot{x}) = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

For the output variables, we choose

$$y_1 = x$$
$$y_2 = \dot{x}$$

Rewriting the state equation and output equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

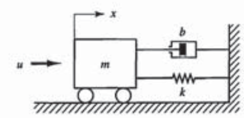


Figure 5–9 Mechanical system.

Substituting the given numerical values for m, b, and k into the state space equations yields

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -20 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB Program 5–4 produces the impulse-response curves $x(t)$ versus t and $\dot{x}(t)$ versus t, shown in Figure 5–10.

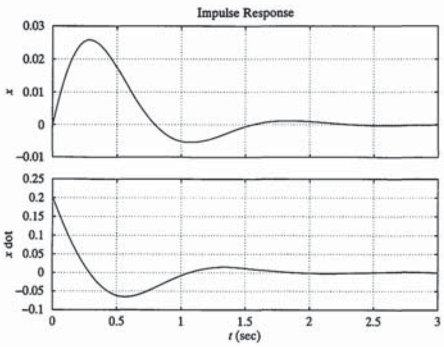


Figure 5–10 Impulse-response curves.

MATLAB Program 5–4

```
>> t = 0:0.01:3;
>> A = [0  1; −20  −4];
>> B = [0;0.2];
>> C = [1  0;0  1];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y, t] = impulse(sys,t);
>> y1 = [1  0] *y';
>> y2 = [0  1] *y';
>> subplot(211); plot(t,y1); grid
>> title('Impulse Response')
>> ylabel('x')
>> subplot(212); plot(t,y2); grid
>> xlabel('t (sec)'); ylabel('x dot')
```


Example 5–5

Consider the mechanical system shown in Figure 5–11. The system is at rest for $t < 0$. At $t = 0$, a step force f of α newtons is applied to mass m_2. [The force $f = \alpha u$, where u is a step force of 1 newton.] The displacements z_1 and z_2 are measured from the respective equilibrium positions of the carts before f is applied. Derive a state-space representation of the system. Assuming that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m, $k_1 = 30$ N/m, $k_2 = 60$ N/m, and $\alpha = 10$, obtain the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_1(t) - z_2(t)$ versus t with MATLAB. Also, obtain $z_1(\infty)$ and $z_2(\infty)$.

The equations of motion for this system are

$$m_1\ddot{z}_1 = -k_1z_1 - k_2(z_1 - z_2) - b(\dot{z}_1 - \dot{z}_2) \tag{5–13}$$

$$m_2\ddot{z}_2 = -k_2(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1) + \alpha u \tag{5–14}$$

In the absence of a forcing function, the initial conditions of any system determine the response of the system. The initial conditions for this system are $z_1(0)$, $\dot{z}_1(0)$, $z_2(0)$, and $\dot{z}_2(0)$. Hence, we choose z_1, $\dot{z}_1$, z_2, and $\dot{z}_2$ as state variables for the system and thus define

$$x_1 = z_1$$
$$x_2 = \dot{z}_1$$
$$x_3 = z_2$$
$$x_4 = \dot{z}_2$$

Then Equation (5–13) can be rewritten as

$$\dot{x}_2 = -\frac{k_1 + k_2}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{b}{m_1}x_4$$

and Equation (5–14) can be rewritten as

$$\dot{x}_4 = \frac{k_2}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}\alpha u$$

The state equation now becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k_1 + k_2}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{b}{m_1}x_4$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{k_2}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}\alpha u$$

Note that z_1 and z_2 are the outputs of the system; hence, the output equations are

$$y_1 = z_1$$
$$y_2 = z_2$$

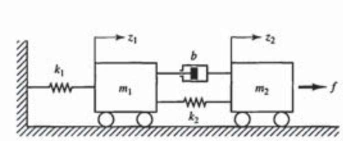


Figure 5–11 Mechanical system.

In terms of vector–matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b}{m_2} & -\frac{k_2}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\alpha}{m_2} \end{bmatrix} u \tag{5–15}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{5–16}$$

Equations (5–15) and (5–16) represent the system in state-space form.

Next, we substitute the given numerical values for m_1, m_2, b, k_1, and k_2 into Equation (5–15). The result is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u \tag{5–17}$$

From Equations (5–17) and (5–16), we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB Program 5–5 produces the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_1(t) - z_2(t)$ versus t. The curves are shown in Figure 5–12.

MATLAB Program 5–5

```
>> t = 0:0.1:200;
>> A = [0  1  0  0;−9  −2  6  2;0  0  0  1;3  1  −3  −1];
>> B = [0;0;0;0.5];
>> C = [1  0  0  0;0  0  1  0];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y,t] = step(sys,t);
>> y1 = [1 0]*y';
>> y2 = [0 1]*y';
>> z1 = y1; subplot(311); plot(t,z1); grid
>> title('Responses z_1 Versus t, z_2 Versus t, and z_1 − z_2 Versus t')
>> ylabel('Output z_1')
>> z2 = y2; subplot(312); plot(t,z2); grid
>> ylabel('Output z_2')
>> subplot(313); plot(t,z1−z2); grid
>> xlabel('t (sec)'); ylabel('z_1 − z_2')
```

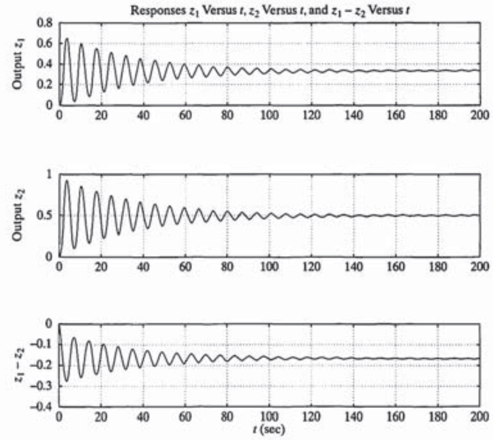


Figure 5–12 Step-response curves.

To obtain $z_1(\infty)$ and $z_2(\infty)$, we set all derivatives of z_1 and z_2 in Equations (5–13) and (5–14) equal to zero, because all derivative terms must approach zero at steady state in the system. Then, from Equation (5–14), we get

$$k_2[z_2(\infty) - z_1(\infty)] = \alpha u$$

from which it follows that

$$z_2(\infty) - z_1(\infty) = \frac{\alpha u}{k_2} = \frac{10}{60} = \frac{1}{6}$$

From Equation (5–13), we have

$$k_1 z_1(\infty) = k_2[z_2(\infty) - z_1(\infty)]$$

Hence,

$$z_1(\infty) = \frac{k_2}{k_1}[z_2(\infty) - z_1(\infty)] = \frac{60}{30}\frac{1}{6} = \frac{1}{3}$$

and

$$z_2(\infty) = \frac{1}{6} + z_1(\infty) = \frac{1}{2}$$

Thus,

$$z_1(\infty) = \frac{1}{3}\,\text{m}, \qquad z_2(\infty) = \frac{1}{2}\,\text{m}$$

The final values of $z_1(t)$ and $z_2(t)$ obtained with MATLAB (see the response curves in Figure 5–12) agree, of course, with the result obtained here.

5–4 STATE-SPACE MODELING OF SYSTEMS WITH INPUT DERIVATIVES

In this section, we take up the case where the equations of motion of a system involve one or more derivatives of the input function. In such a case, the variables that specify the initial conditions do not qualify as state variables. The main problem in defining the state variables is that they must be chosen such that they will eliminate the derivatives of the input function u in the state equation.

For example, consider the mechanical system shown in Figure 5–13. The displacements y and u are measured from their respective equilibrium positions. The equation of motion for this system is

$$m\ddot{y} = -ky - b(\dot{y} - \dot{u})$$

or

$$\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}$$

If we choose the state variables

$$x_1 = y$$
$$x_2 = \dot{y}$$

then we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u} \tag{5–18}$$

The right-hand side of Equation (5–18) involves the derivative term $\dot{u}$. Note that, in formulating state-space representations of dynamic systems, we constrain the input function to be any function of time of order up to the impulse function, but not any higher order impulse functions, such as $d\delta(t)/dt$, $d^2\delta(t)/dt^2$, etc.

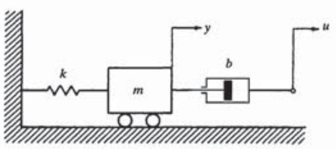


Figure 5–13 Mechanical system.

To explain why the right-hand side of the state equation should not involve the derivative of the input function u, suppose that u is the unit-impulse function $\delta(t)$. Then the integral of Equation (5–18) becomes

$$x_2 = -\frac{k}{m}\int y\,dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

Notice that x_2 includes the term $(k/m)\,\delta(t)$. This means that $x_2(0) = \infty$, which is not acceptable as a state variable. We should choose the state variables such that the state equation will not include the derivative of u.

Suppose that we try to eliminate the term involving $\dot{u}$ from Equation (5–18). One possible way to accomplish this is to define

$$x_1 = y$$
$$x_2 = \dot{y} - \frac{b}{m}u$$

Then

$$\begin{aligned}\dot{x}_2 &= \ddot{y} - \frac{b}{m}\dot{u}\\ &= -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} - \frac{b}{m}\dot{u}\\ &= -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right)\\ &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u\end{aligned}$$

Thus, we have eliminated the term that involves $\dot{u}$. The acceptable state equation can now be given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{b}{m}\\ -\left(\frac{b}{m}\right)^2\end{bmatrix}u$$

If equations of motion involve u, $\dot{u}$, $\ddot{u}$, etc., the choice of state variables becomes more complicated. Fortunately, there are systematic methods for choosing state variables for a general case of equations of motion that involve derivatives of the input function u. In what follows we shall present two systematic methods for eliminating derivatives of the input function from the state equations. Note that MATLAB can also be used to obtain state-space representations of systems involving derivatives of the input function u. (See Section 5–5.)

State-space representation of dynamic systems in which derivatives of the input function appear in the system differential equations. We consider the case where the input function u is a scalar. (That is, only one input function u is involved in the system.)

The differential equation of a system that involves derivatives of the input function has the general form

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0\overset{(n)}{u} + b_1\overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \tag{5–19}$$

To apply the methods presented in this section, it is necessary that the system be written as a differential equation in the form of Equation (5–19) or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

We examine two methods when $n = 2$; for an arbitrary $n = 1, 2, 3, \ldots$, see **Problems A–5–12 and A–5–13.**

Method 1. Consider the second-order system

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u \tag{5–20}$$

As a set of state variables, suppose that we choose

$$x_1 = y - \beta_0 u \tag{5–21}$$
$$x_2 = \dot{x}_1 - \beta_1 u \tag{5–22}$$

where

$$\beta_0 = b_0 \tag{5–23}$$
$$\beta_1 = b_1 - a_1\beta_0 \tag{5–24}$$

Then, from Equation (5–21), we have

$$y = x_1 + \beta_0 u \tag{5–25}$$

Substituting this last equation into Equation (5–20), we obtain

$$\ddot{x}_1 + \beta_0\ddot{u} + a_1(\dot{x}_1 + \beta_0\dot{u}) + a_2(x_1 + \beta_0 u) = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

Noting that $\beta_0 = b_0$ and $\beta_1 = b_1 - a_1\beta_0$, we can simplify the preceding equation to

$$\ddot{x}_1 + a_1\dot{x}_1 + a_2 x_1 = \beta_1\dot{u} + (b_2 - a_2\beta_0)u \tag{5–26}$$

From Equation (5–22), we have

$$\dot{x}_1 = x_2 + \beta_1 u \tag{5–27}$$

Substituting Equation (5–27) into Equation (5–26), we obtain

$$\dot{x}_2 + \beta_1\dot{u} + a_1(x_2 + \beta_1 u) + a_2 x_1 = \beta_1\dot{u} + (b_2 - a_2\beta_0)u$$

which can be simplified to

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u \tag{5–28}$$

where

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \tag{5–29}$$

From Equations (5–27) and (5–28), we obtain the state equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}u \tag{5–30}$$

From Equation (5–25), we get the output equation:

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_0 u \tag{5–31}$$

Equations (5–30) and (5–31) represent the system in a state space.

Note that if $\beta_0 = b_0 = 0$, then the state variable x_1 is the output signal y, which can be measured, and, in this case, the state variable x_2 is the output velocity $\dot{y}$ minus $b_1 u$.

Note that, for the case of the nth-order differential-equation system

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

the state equation and output equation can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

and

$$y = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1\beta_0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\ &\vdots \\ \beta_n &= b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0 \end{aligned}$$

Method 2. Consider the second-order system

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \tag{5–32}$$

Equation (5–32) can be split into two equations as follows:

$$\frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2}, \qquad \frac{Y(s)}{Z(s)} = b_0 s^2 + b_1 s + b_2$$

We then have

$$\ddot{z} + a_1\dot{z} + a_2 z = u \tag{5–33}$$

$$b_0\ddot{z} + b_1\dot{z} + b_2 z = y \tag{5–34}$$

If we define

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{z} \end{aligned} \tag{5–35}$$

then Equation (5–33) can be written as

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + u \tag{5–36}$$

and Equation (5–34) can be written as

$$b_0\dot{x}_2 + b_1 x_2 + b_2 x_1 = y$$

Substituting Equation (5–36) into this last equation, we obtain

$$b_0(-a_2 x_1 - a_1 x_2 + u) + b_1 x_2 + b_2 x_1 = y$$

which can be rewritten as

$$y = (b_2 - a_2 b_0)x_1 + (b_1 - a_1 b_0)x_2 + b_0 u \tag{5–37}$$

From Equations (5–35) and (5–36), we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_2 x_1 - a_1 x_2 + u \end{aligned}$$

These two equations can be combined into the vector–matrix differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5–38}$$

Equation (5–37) can be rewritten as

$$y = [b_2 - a_2 b_0 \;\; \vdots \;\; b_1 - a_1 b_0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u \tag{5–39}$$

Equations (5–38) and (5–39) are the state equation and output equation, respectively. Note that the state variables x_1 and x_2 in this case may not correspond to any physical signals that can be measured.

If the system equation is given by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

then the state equation and the output equation obtained with the use of Method 2 are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \tag{5–40}$$

and

$$y = [b_n - a_n b_0 \;\; \vdots \;\; b_{n-1} - a_{n-1}b_0 \;\; \vdots \;\; \cdots \;\; \vdots \;\; b_1 - a_1 b_0]\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u \tag{5–41}$$

Examples 5–6 and **5–7** illustrate the use of the preceding two analytical methods for obtaining state-space representations of a differential-equation system involving derivatives of the input signal.

Example 5–6

Consider the spring–mass–dashpot system mounted on a cart as shown in Figure 5–14. Assume that the cart is standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u}$ = constant. The displacement y of the mass is the output. (y is measured

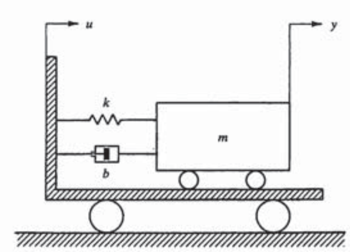


Figure 5–14 Spring–mass–dashpot system mounted on a cart.

from the rest position and is relative to the ground.) In this system, m denotes the mass of the small cart on the large cart (assume that the large cart is massless), b denotes the viscous-friction coefficient, and k is the spring constant. We assume that the entire system is a linear system.

Obtain a state-space representations of the system based on methods 1 and 2 just presented. Assuming that $m = 10$ kg, $b = 20$ N-s/m, $k = 100$ N/m, and the input is a ramp function such that $\dot{u} = 1$ m/s, obtain the response curve $y(t)$ versus t with MATLAB.

First, we shall obtain the system equation. Applying Newton's second law, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku \tag{5–42}$$

Equation (5–42) is the differential equation (mathematical model) of the system. The transfer function is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Method 1. We shall obtain a state-space model of this system based on Method 1. We first compare the differential equation of the system,

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

and identify

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

From Equations (5–23), (5–24), and (5–29), we have

$$\begin{aligned} \beta_0 &= b_0 = 0 \\ \beta_1 &= b_1 - a_1\beta_0 = \frac{b}{m} \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{aligned}$$

From Equations (5–21) and (5–22), we define

$$\begin{aligned} x_1 &= y - \beta_0 u = y \\ x_2 &= \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u \end{aligned} \tag{5–43}$$

From Equations (5–43) and (5–28), we obtain

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u \tag{5–44}$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u \tag{5–45}$$

and the output equation is

$$y = x_1 \tag{5-46}$$

Combining Equations (5–44) and (5–45) yields the state equation, and from Equation (5–46), we get the output equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{b}{m} \\ \dfrac{k}{m} - \left(\dfrac{b}{m}\right)^2 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a state-space representation of the system.

Next, we shall obtain the response curve $y(t)$ versus t for the unit-ramp input $\dot{u} = 1$ m/s. Substituting the given numerical values for m, b, and k into the state equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} u$$

and the output equation is

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

MATLAB Program 5–6 produces the response $y(t)$ of the system to the ramp input $\dot{u} = 1$ m/s. The response curve $y(t)$ versus t and the unit-ramp input are shown in Figure 5–15.

MATLAB Program 5–6

```
>> % ----- The response y(t) is obtained by use of the
>> % state-space equation obtained by Method 1. -----
>>
>> t = 0:0.01:4;
>> A = [0  1;-10  -2];
>> B = [2;6];
>> C = [1  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response (Method 1)')
>> xlabel('t')
>> ylabel('Output y and Unit-Ramp Input u')
>> text(0.85, 0.25,'y')
>> text(0.15,0.8,'u')
```

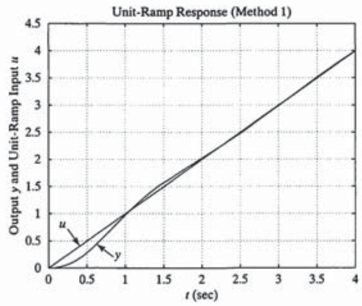



Figure 5–15 Unit-ramp response obtained with the use of Method 1.

Method 2. Since

$$b_0 = 0$$

$$b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} \times 0 = \frac{k}{m}$$

$$b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} \times 0 = \frac{b}{m}$$

from Equations (5–38) and (5–39), we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \dfrac{k}{m} & \dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The last two equations give another state-space representation of the same system.

Substituting the given numerical values for m, b, and k into the state equation, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and the output equation is

$$y = [10 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

MATLAB Program 5–7 produces the response $y(t)$ to the unit-ramp input $\dot{u} = 1$ m/s. The resulting response curve $y(t)$ versus t and the unit-ramp input are shown in Figure 5–16. Notice that the response curve here is identical to that shown in Figure 5–15.

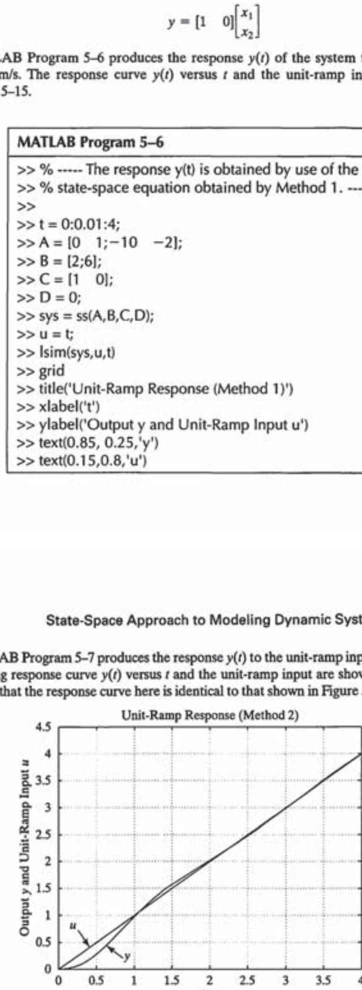


Figure 5–16 Unit-ramp response obtained with the use of Method 2.

MATLAB Program 5–7

```
>> % ----- The response y(t) is obtained by use of the
>> % state-space equation obtained by Method 2. -----
>>
>> t = 0:0.01:4;
>> A = [0  1;-10  -2];
>> B = [0;1];
>> C = [10  2];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response (Method 2)')
>> xlabel('t')
>> ylabel('Output y and Unit-Ramp Input u')
>> text(0.85,0.25,'y')
>> text(0.15,0.8,'u')
```

Example 5–7

Consider the front suspension system of a motorcycle. A simplified version is shown in Figure 5–17(a). Point P is the contact point with the ground. The vertical displacement u of point P is the input to the system. The displacements x and y are measured from their respective equilibrium positions before the input u is given to the system. Assume that

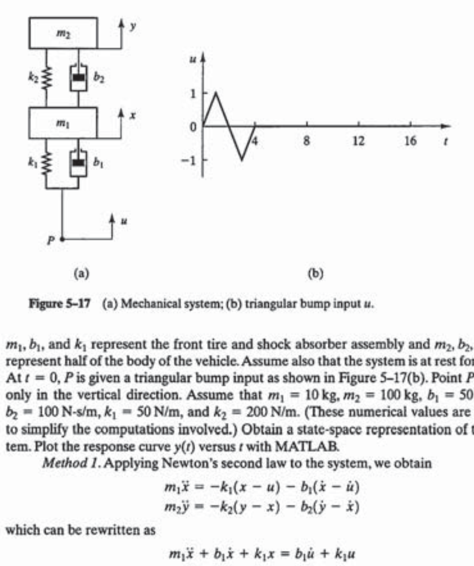


Figure 5–17 (a) Mechanical system; (b) triangular bump input u.

m_1, b_1, and k_1 represent the front tire and shock absorber assembly and m_2, b_2, and k_2 represent half of the body of the vehicle. Assume also that the system is at rest for $t < 0$. At $t = 0$, P is given a triangular bump input as shown in Figure 5–17(b). Point P moves only in the vertical direction. Assume that $m_1 = 10$ kg, $m_2 = 100$ kg, $b_1 = 50$ N-s/m, $b_2 = 100$ N-s/m, $k_1 = 50$ N/m, and $k_2 = 200$ N/m. (These numerical values are chosen to simplify the computations involved.) Obtain a state-space representation of the system. Plot the response curve $y(t)$ versus t with MATLAB.

Method 1. Applying Newton's second law to the system, we obtain

$$m_1\ddot{x} = -k_1(x - u) - b_1(\dot{x} - \dot{u})$$

$$m_2\ddot{y} = -k_2(y - x) - b_2(\dot{y} - \dot{x})$$

which can be rewritten as

$$m_1\ddot{x} + b_1\dot{x} + k_1 x = b_1\dot{u} + k_1 u$$

$$m_2\ddot{y} + b_2\dot{y} + k_2 y = b_2\dot{x} + k_2 x$$

If we substitute the given numerical values for m_1, m_2, b_1, b_2, k_1, and k_2, the equations of motion become

$$10\ddot{x} + 50\dot{x} + 50x = 50\dot{u} + 50u$$

$$100\ddot{y} + 100\dot{y} + 200y = 100\dot{x} + 200x$$

which can be simplified to

$$\ddot{x} + 5\dot{x} + 5x = 5\dot{u} + 5u \tag{5-47}$$

$$\ddot{y} + \dot{y} + 2y = \dot{x} + 2x \tag{5-48}$$

Laplace transforming Equations (5–47) and (5–48), assuming the zero initial conditions, we obtain

$$(s^2 + 5s + 5)X(s) = (5s + 5)U(s)$$

$$(s^2 + s + 2)Y(s) = (s + 2)X(s)$$

Eliminating $X(s)$ from these two equations, we get

$$(s^2 + 5s + 5)(s^2 + s + 2)Y(s) = 5(s + 1)(s + 2)U(s)$$

or

$$(s^4 + 6s^3 + 12s^2 + 15s + 10)Y(s) = (5s^2 + 15s + 10)U(s) \qquad (5\text{–}49)$$

Equation (5–49) corresponds to the differential equation

$$\overset{....}{y} + 6\dddot{y} + 12\ddot{y} + 15\dot{y} + 10y = 5\ddot{u} + 15\dot{u} + 10u$$

Comparing this last equation with the standard fourth-order differential equation

$$\overset{....}{y} + a_1\dddot{y} + a_2\ddot{y} + a_3\dot{y} + a_4 y = b_0\overset{....}{u} + b_1\dddot{u} + b_2\ddot{u} + b_3\dot{u} + b_4 u$$

we find that

$$a_1 = 6, \quad a_2 = 12, \quad a_3 = 15, \quad a_4 = 10$$
$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 5, \quad b_3 = 15, \quad b_4 = 10$$

Next, we define the state variables as follows:

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u \\ x_4 &= \dot{x}_3 - \beta_3 u \end{aligned}$$

where

$$\begin{aligned} \beta_0 &= b_0 = 0 \\ \beta_1 &= b_1 - a_1\beta_0 = 0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = 5 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = 15 - 6 \times 5 = -15 \end{aligned}$$

Hence,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + 5u \\ \dot{x}_3 &= x_4 - 15u \\ \dot{x}_4 &= -a_4x_1 - a_3x_2 - a_2x_3 - a_1x_4 + \beta_4 u \\ &= -10x_1 - 15x_2 - 12x_3 - 6x_4 + \beta_4 u \end{aligned}$$

where

$$\begin{aligned} \beta_4 &= b_4 - a_1\beta_3 - a_2\beta_2 - a_3\beta_1 - a_4\beta_0 \\ &= 10 + 6 \times 15 - 12 \times 5 - 15 \times 0 - 10 \times 0 = 40 \end{aligned}$$

Thus,

$$\dot{x}_4 = -10x_1 - 15x_2 - 12x_3 - 6x_4 + 40u$$

and the state equation and output equation become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ -15 \\ 40 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–8 produces the response $y(t)$ to the triangular bump input shown in Figure 5–17(b). The resulting response curve $y(t)$ versus t, as well as the input $u(t)$ versus t, is shown in Figure 5–18.

MATLAB Program 5–8

```
>> t = 0:0.01:16;
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;5;-15;40];
>> C = [1  0  0  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u1 = [0:0.01:1];
>> u2 = [0.99:-0.01:-1];
>> u3 = [-0.99:0.01:0];
>> u4 = 0*[4.01:0.01:16];
>> u = [u1  u2  u3  u4];
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> v = [0  16  -1.5  1.5]; axis(v)
>> grid
>> title('Response to Triangular Bump (Method 1)')
>> xlabel('t (sec)')
>> ylabel('Triangular Bump and Response')
```

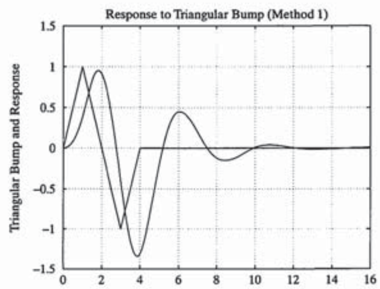


Figure 5–18 Response curve $y(t)$ and triangular bump input $u(t)$.

Method 2. From Equation (5–49), the transfer function of the system is given by

$$\frac{Y(s)}{U(s)} = \frac{5s^2 + 15s + 10}{s^4 + 6s^3 + 12s^2 + 15s + 10}$$

Figure 5–19 shows a block diagram in which the transfer function is split into two parts. If we define the output of the first block as $Z(s)$, then

$$\begin{aligned} \frac{Z(s)}{U(s)} &= \frac{1}{s^4 + 6s^3 + 12s^2 + 15s + 10} \\ &= \frac{1}{s^4 + a_1s^3 + a_2s^2 + a_3s + a_4} \end{aligned}$$

and

$$\frac{Y(s)}{Z(s)} = 5s^2 + 15s + 10 = b_0s^4 + b_1s^3 + b_2s^2 + b_3s + b_4$$

from which we get

$$a_1 = 6, \quad a_2 = 12, \quad a_3 = 15, \quad a_4 = 10,$$
$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 5, \quad b_3 = 15, \quad b_4 = 10$$

Next, we define the state variables as follows:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 \\ x_4 &= \dot{x}_3 \end{aligned}$$

From Equation (5–40), noting that $a_1 = 6, a_2 = 12, a_3 = 15$, and $a_4 = 10$, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Similarly, from the output equation given by Equation (5–41), we have

$$y = \begin{bmatrix} b_4 - a_4b_0 & \vdots & b_3 - a_3b_0 & \vdots & b_2 - a_2b_0 & \vdots & b_1 - a_1b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_0 u$$

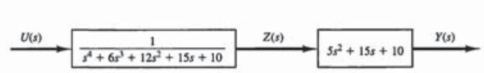


Figure 5–19 Block diagram of $Y(s)/U(s)$.

or

$$y = \begin{bmatrix} 10 & 15 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–9 produces the response $y(t)$ to the triangular bump input. The response curve is shown in Figure 5–20. (This response curve is identical to that shown in Figure 5–19.)

MATLAB Program 5–9

```
>> t = 0:0.01:16;
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;0;0;1];
>> C = [10  15  5  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u1 = [0:0.01:1];
>> u2 = [0.99:-0.01:-1];
>> u3 = [-0.99:0.01:0];
>> u4 = 0*[4.01:0.01:16];
>> u = [u1  u2  u3  u4];
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> v = [0  16  -1.5  1.5]; axis(v)
>> grid
>> title('Response to Triangular Bump (Method 2)')
>> xlabel('t (sec)')
>> ylabel('Triangular Bump and Response')
```

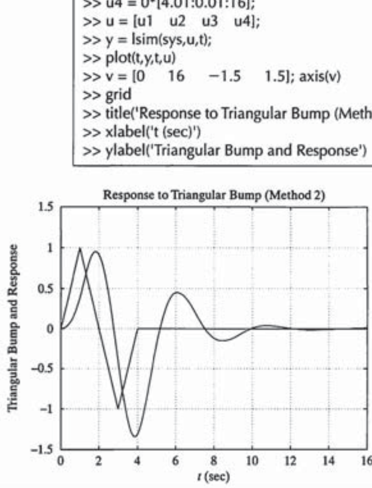


Figure 5–20 Response $y(t)$ to the triangular bump input $u(t)$.

5-5 TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful in transforming a system model from transfer function to state space and vice versa. We shall begin our discussion with the transformation from transfer function to state space.

Let us write the transfer function $Y(s)/U(s)$ as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

```
[A, B, C, D] = tf2ss(num,den)
```

will give a state-space representation. Note that the command can be used when the system equation involves one or more derivatives of the input function. (In such a case, the transfer function of the system involves a numerator polynomial in s.)

It is important to note that the state-space representation of any system is not unique. There are many (indeed, infinitely many) state-space representations of the same system. The MATLAB command gives one possible such representation.

Transformation from transfer function to state space. Consider the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160} \tag{5-50}$$

Of the infinitely many possible state-space representations of this system, one is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Another is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{5-51}$$

$$y = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \tag{5-52}$$

MATLAB transforms the transfer function given by Equation (5-50) into the state-space representation given by Equations (5-51) and (5-52). For the system considered here, MATLAB Program 5-10 will produce matrices **A**, **B**, **C**, and D.

MATLAB Program 5-10

```
>> % ----- Transforming transfer-function model to
>> %       state-space model -----
>>
>> num = [0  0  1  0];
>> den = [1  14  56  160];
>>
>> % ----- Enter the following transformation command -----
>>
>> [A, B, C, D] = tf2ss(num,den)
A =

   -14   -56   160
     1     0     0
     0     1     0

B =

     1
     0
     0

C =

     0     1     0

D =

     0
```

Transformation from state space to transfer function. To obtain the transfer function from state-space equations, use the command

```
[num,den] = ss2tf(A,B,C,D,iu)
```

Note that iu must be specified for systems with more than one input. For example, if the system has three inputs ($u1$, $u2$, $u3$), then iu must be either 1, 2, or 3, where 1 implies $u1$, 2 implies $u2$, and 3 implies $u3$.

If the system has only one input, then either

```
[num,den] = ss2tf(A,B,C,D)
```

or

```
[num,den] = ss2tf(A,B,C,D,1)
```

may be used. (For the case where the system has multiple inputs and multiple outputs, see **Example 5-9**.)

Example 5-8

Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ -15 \\ 40 \end{bmatrix} u$$

$$Y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5-11 produces the transfer function of the system, namely,

$$\frac{Y(s)}{U(s)} = \frac{5s^2 + 15s + 10}{s^4 + 6s^3 + 12s^2 + 15s + 10}$$

MATLAB Program 5-11

```
>> % ----- Transforming state-space model to
>> %       transfer function model -----
>>
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;5;-15;40];
>> C = [1  0  0  0];
>> D = 0;
>>
>> % ----- Enter the following transformation command —
>>
>> [num,den] = ss2tf(A,B,C,D)
num =
        0         0    5.0000   15.0000   10.0000

den =
   1.0000    6.0000   12.0000   15.0000   10.0000
```

Example 5-9

Consider a system with multiple inputs and multiple outputs. When the system has more than one output, the command

```
[NUM,den] = ss2tf(A,B,C,D,iu)
```

produces transfer functions for all outputs to each input. (The numerator coefficients are returned to matrix NUM with as many rows as there are outputs.)

Let the system be defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved: $Y_1(s)/U_1(s)$, $Y_2(s)/U_1(s)$, $Y_1(s)/U_2(s)$, and $Y_2(s)/U_2(s)$. (When considering input u_1, we assume that input u_2 is zero, and vice versa.)

MATLAB Program 5-12 produces representations of the following four transfer functions:

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25}$$

$$\frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

MATLAB Program 5-12

```
>> A = [0   1;-25   -4];
>> B = [1   1;0   1];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> [NUM,den] = ss2tf(A,B,C,D,1)
NUM =

         0    1.0000     4.0000
         0         0   -25.0000

den =

    1.0000    4.0000    25.0000
>> [NUM,den] = ss2tf(A,B,C,D,2)
NUM =

         0    1.0000     5.0000
         0    1.0000   -25.0000

den =

    1.0000    4.0000    25.0000
```

Nonuniqueness of a set of state variables. A set of state variables is not unique for a given system. Suppose that $x_1, x_2, \ldots, x_n$ are a set of state variables. Then we may take as another set of state variables any set of functions

$$\hat{x}_1 = X_1(x_1, x_2, \ldots, x_n)$$
$$\hat{x}_2 = X_2(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$\hat{x}_n = X_n(x_1, x_2, \ldots, x_n)$$

provided that, for every set of values $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, there corresponds a unique set of values $x_1, x_2, \ldots, x_n$, and vice versa. Thus, if **x** is a state vector, then

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

is also a state vector, provided that the matrix $\mathbf{P}$ is nonsingular. (Note that a square matrix $\mathbf{P}$ is nonsingular if the determinant $|\mathbf{P}|$ is nonzero.) Different state vectors convey the same information about the system behavior.

Transformation of a state-space model into another state-space model. A state-space model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5–53}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5–54}$$

can be transformed into another state-space model by transforming the state vector $\mathbf{x}$ into state vector $\hat{\mathbf{x}}$ by means of the transformation

$$\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$$

where $\mathbf{P}$ is nonsingular. Then Equations (5–53) and (5–54) can be written as

$$\mathbf{P}\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{P}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

or

$$\dot{\hat{\mathbf{x}}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\hat{\mathbf{x}} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \tag{5–55}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \tag{5–56}$$

Equations (5–55) and (5–56) represent another state-space model of the same system. Since infinitely many $n \times n$ nonsingular matrices can be used as a transformation matrix $\mathbf{P}$, there are infinitely many state-space models for a given system.

Eigenvalues of an $n \times n$ matrix A. The *eigenvalues* of an $n \times n$ matrix $\mathbf{A}$ are the roots of the characteristic equation

$$|\lambda\mathbf{I} - \mathbf{A}| = 0 \tag{5–57}$$

The eigenvalues are also called the *characteristic roots*.

Consider, for example, the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

The characteristic equation is

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} \\ &= \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\ &= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation, or -1, -2, and -3.

It is sometimes desirable to transform the state matrix into a diagonal matrix. This may be done by choosing an appropriate transformation matrix $\mathbf{P}$. In what follows, we shall discuss the diagonalization of a state matrix.

Diagonalization of state matrix A. Consider an $n \times n$ state matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{5–58}$$

We first consider the case where matrix $\mathbf{A}$ has distinct eigenvalues only. If the state vector $\mathbf{x}$ is transformed into another state vector $\mathbf{z}$ with the use of a transformation matrix $\mathbf{P}$, or

$$\mathbf{x} = \mathbf{P}\mathbf{z}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \tag{5–59}$$

in which $\lambda_1, \lambda_2, \ldots,$ and λ_n are n distinct eigenvalues of $\mathbf{A}$, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ becomes a diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{5–60}$$

Note that each column of the transformation matrix $\mathbf{P}$ in Equation (5–59) is an eigenvector of the matrix $\mathbf{A}$ given by Equation (5–58). (See **Problem A–5–18** for details.)

Next, consider the case where matrix $\mathbf{A}$ involves multiple eigenvalues. In this case, diagonalization is not possible, but matrix $\mathbf{A}$ can be transformed into a Jordan canonical form. For example, consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that $\mathbf{A}$ has eigenvalues λ_1, λ_1, and λ_3, where $\lambda_1 \neq \lambda_3$. In this case, the transformation $\mathbf{x} = \mathbf{S}\mathbf{z}$, where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix} \tag{5–61}$$

will yield

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{5–62}$$

This matrix is in Jordan canonical form.

Example 5–10

Consider a system with the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du \end{aligned} \tag{5–63}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0], \quad D = 0$$

The eigenvalues of the state matrix $\mathbf{A}$ are -1, -2, and -3, or

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3$$

We shall show that Equation (5–63) is not the only possible state equation for the system. Suppose we define a set of new state variables z_1, z_2, and z_3 by the transformation

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{aligned}$$

or

$$\mathbf{x} = \mathbf{P}\mathbf{z} \tag{5–64}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \tag{5–65}$$

Then, substituting Equation (5–64) into Equation (5–63), we obtain

$$\mathbf{P}\dot{\mathbf{z}} = \mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{B}u$$

Premultiplying both sides of this last equation by $\mathbf{P}^{-1}$, we get

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}u \tag{5–66}$$

or

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = &\begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &+ \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u \end{aligned}$$

Simplifying gives

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \tag{5–67}$$

Equation (5–67) is a state equation that describes the system defined by Equation (5–63).

The output equation is modified to

$$\begin{aligned} y &= [1 \quad 0 \quad 0]\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= [1 \quad 1 \quad 1]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{aligned} \tag{5–68}$$

Notice that the transformation matrix $\mathbf{P}$ defined by Equation (5–65) changes the coefficient matrix of $\mathbf{z}$ into the diagonal matrix. As is clearly seen from Equation (5–67), the three separate state equations are uncoupled. Notice also that the diagonal elements of the matrix $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ in Equation (5–66) are identical to the three eigenvalues of $\mathbf{A}$. (For a proof, see **Problem A–5–20**.)

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–5–1

Consider the pendulum system shown in Figure 5–21. Assuming angle θ to be the output of the system, obtain a state-space representation of the system.

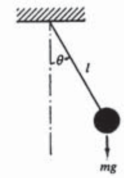


Figure 5–21 Pendulum system.

Solution The equation for the pendulum system is

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This is a second-order system; accordingly, we need two state variables, x_1 and x_2, to completely describe the system dynamics. If we define

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

then we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

(There is no input u to this system.) The output y is angle θ. Thus,

$$y = \theta = x_1$$

A state-space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\frac{\sin x_1}{x_1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that the state equation just obtained is a nonlinear differential equation.

If the angle θ is limited to be small, then the system can be linearized. For small angle θ, we have $\sin\theta = \sin x_1 \doteqdot x_1$ and $(\sin x_1)/x_1 \doteqdot 1$. A state-space representation

of the linearized model is then given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem A–5–2

Obtain a state-space representation of the mechanical system shown in Figure 5–22. The external force $u(t)$ applied to mass m_2 is the input to the system. The displacements y and z are measured from their respective equilibrium positions and are the outputs of the system.

Solution Applying Newton's second law to this system, we obtain

$$m_2\ddot{y} + b_1(\dot{y} - \dot{z}) + k_1(y - z) + k_2 y = u \tag{5–69}$$
$$m_1\ddot{z} + b_1(\dot{z} - \dot{y}) + k_1(z - y) = 0 \tag{5–70}$$

If we define the state variables

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = z$$
$$x_4 = \dot{z}$$

then, from Equation (5–69), we get

$$m_2\dot{x}_2 = -(k_1 + k_2)x_1 - b_1 x_2 + k_1 x_3 + b_1 x_4 + u$$

Also, from Equation (5–70), we obtain

$$m_1\dot{x}_4 = k_1 x_1 + b_1 x_2 - k_1 x_3 - b_1 x_4$$

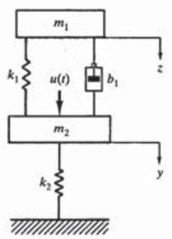


Figure 5–22 Mechanical system.

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_2} & -\frac{b_1}{m_2} & \frac{k_1}{m_2} & \frac{b_1}{m_2} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & \frac{b_1}{m_1} & -\frac{k_1}{m_1} & -\frac{b_1}{m_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_2} \\ 0 \\ 0 \end{bmatrix} u \tag{5–71}$$

The outputs of the system are y and z. Consequently, if we define the output variables as

$$y_1 = y$$
$$y_2 = z$$

then we have

$$y_1 = x_1$$
$$y_2 = x_3$$

The output equation can now be put in the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5–72}$$

Equations (5–71) and (5–72) give a state-space representation of the mechanical system shown in Figure 5–22.

Problem A–5–3

Obtain a state-space representation of the system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{5–73}$$

where u is the input and y is the output of the system.

Solution Since the initial conditions $y(0), \dot{y}(0), \ldots, \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of n state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher order derivative terms are inaccurate due to the noise effects that are inherent in any practical system, this choice of state variables may not be desirable.)

Let us define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y}$$

Then Equation (5–73) can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{5–74}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \tag{5–75}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

Equation (5–74) is the state equation and Equation (5–75) is the output equation.

Note that the state-space representation of the transfer function of the system,

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is also given by Equations (5–74) and (5–75).

Problem A–5–4

Consider a system described by the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and output equation

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.125 & -1.375 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.25 \\ 0.34375 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 1$$

Obtain the transfer function of this system.

Solution From Equation (5–9), the transfer function $G(s)$ can be given in terms of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D as

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 0.125 & s + 1.375 \end{bmatrix}$$

we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 1.375s + 0.125}\begin{bmatrix} s + 1.375 & 1 \\ -0.125 & s \end{bmatrix}$$

Therefore, the transfer function of the system is

$$\begin{aligned} G(s) &= [1 \quad 0]\frac{1}{s^2 + 1.375s + 0.125}\begin{bmatrix} s + 1.375 & 1 \\ -0.125 & s \end{bmatrix}\begin{bmatrix} -0.25 \\ 0.34375 \end{bmatrix} + 1 \\ &= \frac{-0.25(s + 1.375) + 0.34375}{s^2 + 1.375s + 0.125} + 1 \\ &= \frac{s^2 + 1.125s + 0.125}{s^2 + 1.375s + 0.125} \\ &= \frac{8s^2 + 9s + 1}{8s^2 + 11s + 1} \end{aligned}$$

Problem A–5–5

Consider the following state equation and output equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The system involves two inputs and two outputs, so there are four input–output combinations. Obtain the impulse-response curves of the four combinations. (When u_1 is a unit-impulse input, we assume that $u_2 = 0$, and vice versa.)

Next, find the outputs y_1 and y_2 when both inputs, u_1 and u_2, are given at the same time (i.e., $u_1 = u_2 =$ unit-impulse function occurring at the same time $t = 0$).

Solution The command

```
sys = ss(A,B,C,D1),        impulse(sys,t)
```

produces the impulse-response curves for the four input–output combinations. (See MATLAB Program 5–13; when u_1 is a unit-impulse function, we assume that $u_2 = 0$, and vice versa.) The resulting curves are shown in Figure 5–23.

When both unit-impulse inputs $u_1(t)$ and $u_2(t)$ are given at the same time $t = 0$, the responses are

$$y_1(t) = y_{11}(t) + y_{21}(t)$$
$$y_2(t) = y_{12}(t) + y_{22}(t)$$

MATLAB Program 5–13

```
>> t = 0:0.01:10;
>> A = [-1   -1;6.5   0];
>> B = [1   1;1   0];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> sys = ss(A,B,C,D);
>> impulse(sys,t)
>> grid
>> title('Impulse-Response Curves')
>> xlabel('t'); ylabel('Outputs')
```

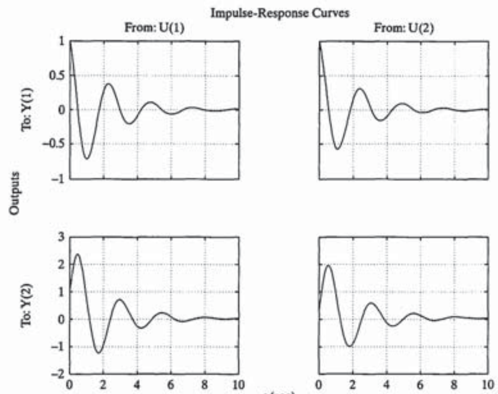


Figure 5–23 Unit-impulse response curves. (The left column corresponds to $u_1 =$ unit-impulse input and $u_2 = 0$. The right column corresponds to $u_1 = 0$ and $u_2 =$ unit-impulse input.)

where

$$\begin{aligned} y_{11} &= y_1 \qquad \text{when } u_1 = \delta(t),\ u_2 = 0 \\ y_{12} &= y_2 \qquad \text{when } u_1 = \delta(t),\ u_2 = 0 \\ y_{21} &= y_1 \qquad \text{when } u_1 = 0,\ u_2 = \delta(t) \\ y_{22} &= y_2 \qquad \text{when } u_1 = 0,\ u_2 = \delta(t) \end{aligned}$$

MATLAB Program 5–14 produces the responses $y_1(t) = y_{11}(t) + y_{21}(t)$ and $y_2(t) = y_{12}(t) + y_{22}(t)$. The resulting response curves are shown in Figure 5–24.

MATLAB Program 5–14

```
>> t = 0:0.01:10;
>> A = [-1   -1;6.5   0];
>> B = [1   1;1   0];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> sys = ss(A,B,C,D);
>> [y,t,x] = impulse(sys,t);
>> y11 = [1   0]*y(:,:,1)';
>> y12 = [0   1]*y(:,:,1)';
>> y21 = [1   0]*y(:,:,2)';
>> y22 = [0   1]*y(:,:,2)';
>> subplot(211); plot(t,y11+y21); grid
>> title('Impulse Response when Both u_1 and u_2 are given at t = 0')
>> ylabel('y_1')
>> subplot(212); plot(t,y12+y22); grid
>> xlabel('t (sec)'); ylabel('y_2')
```

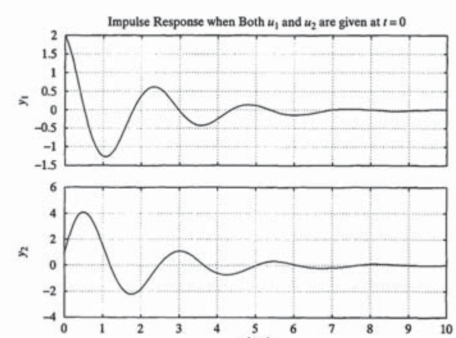


Figure 5–24 Response curves $y_1(t)$ versus t and $y_2(t)$ versus t when $u_1(t)$ and $u_2(t)$ are given at the same time. [Both $u_1(t)$ and $u_2(t)$ are unit-impulse inputs occurring at $t = 0$.]

Problem A–5–6

Obtain the unit-step response and unit-impulse response of the following system with MATLAB:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.01 & -0.1 & -0.5 & -1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.04 \\ -0.012 \\ 0.008 \end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The initial conditions are zeros.

Solution To obtain the unit-step response of this system, the following command may be used:

```
[y, x, t] = step(A, B, C, D)
```

Since the unit-impulse response is the derivative of the unit-step response, the derivative of the output (y = x1) will give the unit-impulse response. From the state equation, we see that the derivative of y is

```
x2 = [0  1  0  0]*x'
```

Hence, x2 versus t will give the unit-impulse response.

MATLAB Program 5–15 produces both the unit-step and unit-impulse responses. The resulting unit-step response curve and unit-impulse curve are shown in Figure 5–25.

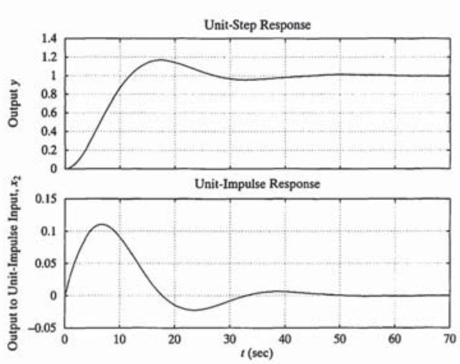


Figure 5–25 Unit-step response curve and unit-impulse response curve.

MATLAB Program 5–15

```
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-0.01  -0.1  -0.5  -1.5];
>> B = [0;0.04;-0.012;0.008];
>> C = [1  0  0  0];
>> D = 0;
>>
>> % To get the step response, enter, for example, the following
>> % command:
>>
>> [y,x,t] = step(A,B,C,D);
>> subplot(211); plot(t,y); grid
>> title('Unit-Step Response')
>> ylabel('Output y')
>>
>> % The unit-impulse response of the system is the same as the
>> % derivative of the unit-step response. (Note that x_1dot
>> % = x_2 in this system.) Hence, the unit-impulse response
>> % of this system is given by ydot = x_2. To plot the unit-
>> % impulse response curve, enter the following command:
>>
>> x2 = [0  1  0  0]*x'; subplot(212); plot(t,x2); grid
>> title('Unit-Impulse Response')
>> xlabel('t (sec)'); ylabel('Output to Unit-Impulse Input, x_2')
```

Problem A–5–7

Two masses m_1 and m_2 are connected by a spring with spring constant k, as shown in Figure 5–26. Assuming no friction, derive a state-space representation of the system, which is at rest for $t < 0$. The displacements y_1 and y_2 are the outputs of the system and are measured from their rest positions relative to the ground.

Assuming that $m_1 = 40$ kg, $m_2 = 100$ kg, $k = 40$ N/m, and f is a step force input of magnitude of 10 N, obtain the response curves $y_1(t)$ versus t and $y_2(t)$ versus t with MATLAB. Also, obtain the relative motion between m_1 and m_2. Define $y_2 - y_1 = x$ and plot the curve $x(t)$ versus t. Assume that we are interested in the period $0 \le t \le 20$.

Solution Let us define a step force input of magnitude 1 N as u. Then the equations of motion for the system are

$$m_1\ddot{y}_1 + k(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + k(y_2 - y_1) = f$$

We choose the state variables for the system as follows:

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$

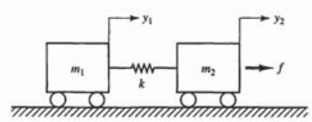


Figure 5–26 Mechanical system.

$$x_3 = y_2$$
$$x_4 = \dot{y}_2$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m_1}x_1 + \frac{k}{m_1}x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}f$$

Noting that $f = 10u$ and substituting the given numerical values for m_1, m_2, and k, we obtain the state equation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0.4 & 0 & -0.4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 0.1\end{bmatrix}u$$

The output equation is

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + 0u$$

MATLAB Program 5–16 produces the outputs y_1 and y_2 and the relative motion $x(= y_2 - y_1 = x_3 - x_1)$. The resulting response curves $y_1(t)$ versus t, $y_2(t)$ versus t, and $x(t)$ versus t are shown in Figure 5–27. Notice that the vibration between m_1 and m_2 continues forever.

MATLAB Program 5–16

```
>> t = 0:0.02:20;
>> A = [0  1  0  0;-1  0  1  0;0  0  0  1;0.4  0  -0.4  0];
>> B = [0;0;0;0.1];
>> C = [1  0  0  0;0  0  1  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> [y,t,x] = step(sys,t);
>> y1 = [1  0]*y';
>> y2 = [0  1]*y';
>> subplot(311); plot(t,y1), grid
>> title('Step Response')
>> ylabel('Output y_1')
>> subplot(312); plot(t,y2), grid
>> ylabel('Output y_2')
>> subplot(313); plot(t,y2 - y1), grid
>> xlabel('t (sec)'); ylabel('x = y_2 - y_1')
```

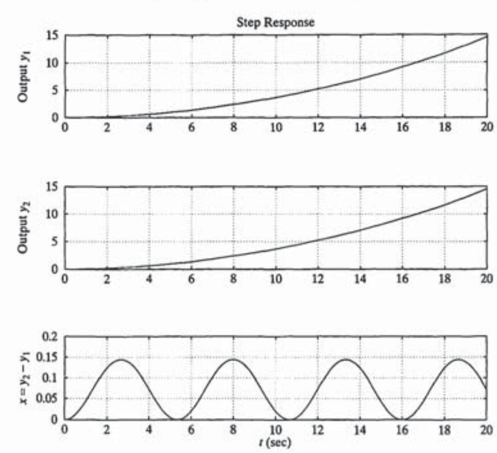



Figure 5–27 Response curves $y_1(t)$ versus t, $y_2(t)$ versus t, and $x(t)$ versus t.

Problem A–5–8

Obtain the unit-ramp response of the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -0.4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]u$$

The system is initially at rest.

Solution Noting that the unit-ramp input is defined by

$$u = t \qquad (0 \le t)$$

we may use the command

```
lsim(sys, u, t)
```

as shown in MATLAB Program 5–17. The unit-ramp response curve and the unit-ramp input are shown in Figure 5–28.

MATLAB Program 5–17

```
>> t = 0:0.01:18;
>> A = [0   1;-1   -0.4];
>> B = [0;1];
>> C = [1   0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response')
>> xlabel('t')
>> ylabel('Output y')
>> text(3.5,0.6,'y')
>> text(0.5,3.2,'u')
```

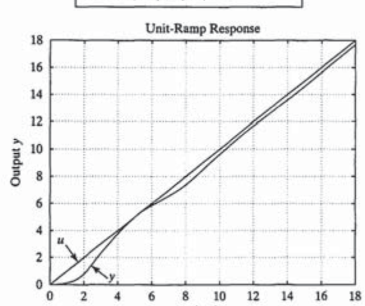


Figure 5–28 Plot of unit-ramp response curve, together with unit-ramp input.

Problem A–5–9

A mass M (where $M = 8$ kg) is supported by a spring (where $k = 400$ N/m) and a damper (where $b = 40$ N-s/m), as shown in Figure 5–29. At $t = 0$, a mass $m = 2$ kg is gently placed on the top of mass M, causing the system to exhibit vibrations. Assuming that the displacement x of the combined mass is measured from the equilibrium position before m is placed on M, obtain a state-space representation of the system. Then plot the response curve $x(t)$ versus t. (For an analytical solution, see **Problem A–3–16.**)

Solution The equation of motion for the system is

$$(M + m)\ddot{x} + b\dot{x} + kx = mg \qquad (0 < t)$$

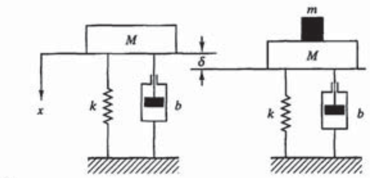


Figure 5–29 Mechanical system.

Substituting the given numerical values for M, m, b, k, and $g = 9.807$ m/s^2 into this last equation, we obtain

$$10\ddot{x} + 40\dot{x} + 400x = 2 \times 9.807$$

or

$$\ddot{x} + 4\dot{x} + 40x = 1.9614$$

The input here is a step force of magnitude 1.9614 N.

Let us define a step force input of magnitude 1 N as u. Then we have

$$\ddot{x} + 4\dot{x} + 40x = 1.9614u$$

If we now choose state variables

$$x_1 = x$$
$$x_2 = \dot{x}$$

then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -40x_1 - 4x_2 + 1.9614u$$

The state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -40 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.9614 \end{bmatrix}u$$

and the output equation is

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

MATLAB Program 5–18 produces the response curve $y(t)$ [$= x(t)$] versus t, shown in Figure 5–30. Notice that the static deflection $x(\infty) = y(\infty) \doteqdot y(600)$ is 0.049035 m.

Problem A–5–10

Consider the system shown in Figure 5–31. The system is at rest for $t < 0$. The displacements z_1 and z_2 are measured from their respective equilibrium positions relative to the ground. Choosing z_1, $\dot{z}_1$, z_2, and $\dot{z}_2$ as state variables, derive a state-space representation of the system. Assuming that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m, $k = 60$ N/m, and f is a step force input of magnitude 10 N, plot the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, $z_2(t) - z_1(t)$ versus t, and $\dot{z}_2(t) - \dot{z}_1(t)$ versus t. Also, obtain the steady-state values of $\ddot{z}_1$, $\ddot{z}_2$, and $z_2 - z_1$.

MATLAB Program 5–18

```
>> t = 0:0.01:6;
>> A = [0   1;-40   -4];
>> B = [0;1.9614];
>> C = [1   0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> [y,t] = step(sys,t);
>> plot(t,y)
>> grid
>> title('Step Response')
>> xlabel('t (sec)'); ylabel('Output y')
>>
>> format long;
>> y(600)

ans =

   0.04903515818520
```

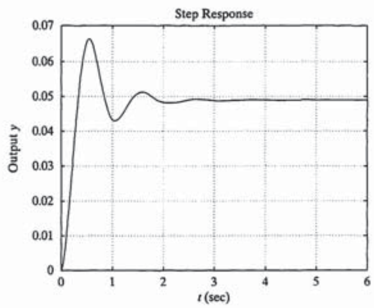


Figure 5–30 Step-response curve.

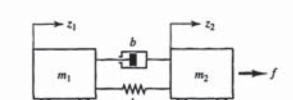


Figure 5–31 Mechanical system.

Solution The equations of motion for the system are

$$m_1\ddot{z}_1 = k(z_2 - z_1) + b(\dot{z}_2 - \dot{z}_1) \qquad (5\text{–}76)$$
$$m_2\ddot{z}_2 = -k(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1) + f \qquad (5\text{–}77)$$

Since we chose state variables as

$$x_1 = z_1$$
$$x_2 = \dot{z}_1$$
$$x_3 = z_2$$
$$x_4 = \dot{z}_2$$

Equations (5–76) and (5–77) can be written as

$$m_1\dot{x}_2 = k(x_3 - x_1) + b(x_4 - x_2)$$
$$m_2\dot{x}_4 = -k(x_3 - x_1) - b(x_4 - x_2) + f$$

We thus have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3 + \frac{b}{m_1}x_4$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{k}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}f$$

Let us define z_1 and z_2 as the system outputs. Then

$$y_1 = z_1 = x_1$$
$$y_2 = z_2 = x_3$$

After substitution of the given numerical values and $f = 10u$ (where u is a step force input of magnitude 1 N occurring at $t = 0$), the state equation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}u$$

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}u$$

MATLAB Program 5–19 produces the response curves z_1 versus t, z_2 versus t, $z_2 - z_1$ versus t, and $\dot{z}_2 - \dot{z}_1$ versus t. The resulting curves are shown in Figure 5–32.

Note that at steady state $\ddot{z}_1(t)$ and $\ddot{z}_2(t)$ approach a constant value, or

$$\ddot{z}_1(\infty) = \ddot{z}_2(\infty) = \alpha$$

Also, at steady state the value of $z_2(t) - z_1(t)$ approaches a constant value, or

$$z_2(\infty) - z_1(\infty) = \beta$$

MATLAB Program 5–19

```
>> t = 0:0.01:15;
>>A = [0  1  0  0;-6  -2  6  2;0  0  0  1;3  1  -3  -1];
>> B = [0;0;0;0.5];
>>C = [1   0   0   0;0   0   1   0];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y,t,x] = step(sys, t);
>> x1 = [1  0  0  0]*x';
>> x2 = [0  1  0  0]*x';
>> x3 = [0  0  1  0]*x';
>> x4 = [0  0  0  1]*x';
>> subplot(221); plot(t,x1); grid
>> xlabel('t (sec)'); ylabel('Output z_1')
>> subplot(222); plot(t,x3); grid
>> xlabel('t (sec)'); ylabel('Output z_2')
>> subplot(223); plot(t,x3 - x1); grid
>> xlabel('t (sec)'); ylabel('Output z_2 - z_1')
>> subplot(224); plot(t,x4 - x2); grid
>> xlabel('t (sec)'); ylabel('z_2dot - z_1dot')
```

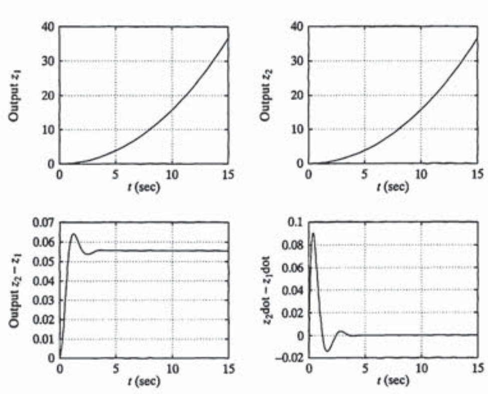


Figure 5–32 Response curves z_1 versus t, z_2 versus t, $z_2 - z_1$ versus t, and $\dot{z}_2 - \dot{z}_1$ versus t.

The steady-state value of $\dot{z}_2(t) - \dot{z}_1(t)$ is zero, or

$$\dot{z}_2(\infty) - \dot{z}_1(\infty) = 0$$

For $t = \infty$, Equation (5–76) becomes

$$m_1\ddot{z}_1(\infty) = k[z_2(\infty) - z_1(\infty)] + b[\dot{z}_2(\infty) - \dot{z}_1(\infty)]$$

or

$$10\alpha = k\beta + b \times 0$$

Also, Equation (5–77) becomes

$$m_2\ddot{z}_2(\infty) = -k[z_2(\infty) - z_1(\infty)] - b[\dot{z}_2(\infty) - \dot{z}_1(\infty)] + f$$

or

$$20\alpha = -k\beta - b \times 0 + f$$

Hence,

$$\begin{aligned} 10\alpha &= 60\beta \\ 20\alpha &= -60\beta + f \end{aligned}$$

from which we get

$$\alpha = \frac{f}{30} = \frac{10}{30} = \frac{1}{3}$$

and

$$\beta = \frac{10\alpha}{60} = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$$

Thus,

$$\ddot{z}_1(\infty) = \ddot{z}_2(\infty) = \alpha = \frac{1}{3}\ \text{m/s}^2$$

$$z_2(\infty) - z_1(\infty) = \beta = \frac{1}{18}\ \text{m}$$

Problem A–5–11

Obtain two state-space representations of the mechanical system shown in Figure 5–33 where u is the input displacement and y is the output displacement. The system is initially at rest. The displacement y is measured from the rest position before the input u is given.

Solution The equation of motion for the mechanical system shown in Figure 5–33 is

$$f_1(\dot{u} - \dot{y}) + k_1(u - y) = f_2\dot{y}$$

Rewriting, we obtain

$$(f_1 + f_2)\dot{y} + k_1 y = f_1\dot{u} + k_1 u$$

or

$$\dot{y} + \frac{k_1}{f_1 + f_2}y = \frac{f_1}{f_1 + f_2}\dot{u} + \frac{k_1}{f_1 + f_2}u \tag{5–78}$$

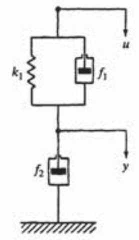


Figure 5–33 Mechanical system.

Comparing this last equation with

$$\dot{y} + a_1 y = b_0\dot{u} + b_1 u \tag{5–79}$$

we get

$$a_1 = \frac{k_1}{f_1 + f_2}, \qquad b_0 = \frac{f_1}{f_1 + f_2}, \qquad b_1 = \frac{k_1}{f_1 + f_2}$$

We shall obtain two state-space representations of the system, based on Methods 1 and 2 presented in Section 5–4.

Method 1. First calculate β_0 and β_1:

$$\beta_0 = b_0 = \frac{f_1}{f_1 + f_2}$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{k_1 f_2}{(f_1 + f_2)^2}$$

Define the state variable x by

$$x = y - \beta_0 u = y - \frac{f_1}{f_1 + f_2}u$$

Then the state equation can be obtained from Equation (5–78) as follows:

$$\dot{x} = -\frac{k_1}{f_1 + f_2}x + \frac{k_1 f_2}{(f_1 + f_2)^2}u \tag{5–80}$$

The output equation is

$$y = x + \frac{f_1}{f_1 + f_2}u \tag{5–81}$$

Equations (5–80) and (5–81) give a state-space representation of the system.

Method 2. From Equation (5–79), we have

$$\frac{Y(s)}{U(s)} = \frac{b_0 s + b_1}{s + a_1}$$

If we define

$$\frac{Z(s)}{U(s)} = \frac{1}{s + a_1}, \qquad \frac{Y(s)}{Z(s)} = b_0 s + b_1$$

then we get

$$\dot{z} + a_1 z = u \tag{5–82}$$

$$b_0\dot{z} + b_1 z = y \tag{5–83}$$

Next, we define the state variable x by

$$x = z$$

Then Equation (5–82) can be written as

$$\dot{x} = -a_1 x + u$$

or

$$\dot{x} = -\frac{k_1}{f_1 + f_2}x + u \tag{5–84}$$

and Equation (5–83) becomes

$$b_0\dot{x} + b_1 x = y$$

or

$$y = \frac{k_1}{f_1 + f_2}x + \frac{f_1}{f_1 + f_2}\dot{x} \tag{5–85}$$

Substituting Equation (5–84) into Equation (5–85), we get

$$y = \frac{k_1 f_2}{(f_1 + f_2)^2}x + \frac{f_1}{f_1 + f_2}u \tag{5–86}$$

Equations (5–84) and (5–86) give a state-space representation of the system.

Problem A–5–12

Show that, for the differential-equation system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u \tag{5–87}$$

state and output equations can be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}u \tag{5–88}$$

and

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u \tag{5–89}$$

where the state variables are defined by

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\ x_3 &= \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \end{aligned}$$

The constants, β_0, β_1, β_2, and β_3 are defined by

$$\begin{aligned} \beta_0 &= b_0 \\ \beta_1 &= b_1 - a_1\beta_0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \end{aligned}$$

Solution From the definition of the state variables x_2 and x_3, we have

$$\dot{x}_1 = x_2 + \beta_1 u \tag{5–90}$$

$$\dot{x}_2 = x_3 + \beta_2 u \tag{5–91}$$

To derive the equation for $\dot{x}_3$, we note that

$$\dddot{y} = -a_1\ddot{y} - a_2\dot{y} - a_3 y + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Since

$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u$$

we have

$$\begin{aligned} \dot{x}_3 &= \dddot{y} - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\ &= (-a_1\ddot{y} - a_2\dot{y} - a_3 y) + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\ &= -a_1(\ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u) - a_1\beta_0\ddot{u} - a_1\beta_1\dot{u} - a_1\beta_2 u \\ &\quad - a_2(\dot{y} - \beta_0\dot{u} - \beta_1 u) - a_2\beta_0\dot{u} - a_2\beta_1 u - a_3(y - \beta_0 u) - a_3\beta_0 u \\ &\quad + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\ &= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_0 - \beta_0)\dddot{u} + (b_1 - \beta_1 - a_1\beta_0)\ddot{u} \\ &\quad + (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)\dot{u} + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\ &= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\ &= -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u \end{aligned}$$

Hence, we get

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u \tag{5–92}$$

Combining Equations (5–90), (5–91), and (5–92) into a vector-matrix differential equation, we obtain Equation (5–88). Also, from the definition of state variable x_1, we get the output equation given by Equation (5–89).

Note that the derivation presented here can be easily extended to the general case of an nth-order system.

Problem A–5–13

Show that, for the system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

or

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

state and output equations may be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

and

$$y = [b_3 - a_3b_0 \;\vdots\; b_2 - a_2b_0 \;\vdots\; b_1 - a_1b_0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0u$$

Solution Let us define

$$\frac{Z(s)}{U(s)} = \frac{1}{s^3 + a_1s^2 + a_2s + a_3}, \qquad \frac{Y(s)}{Z(s)} = b_0s^3 + b_1s^2 + b_2s + b_3$$

Then we obtain

$$\dddot{z} + a_1\ddot{z} + a_2\dot{z} + a_3z = u$$
$$b_0\dddot{z} + b_1\ddot{z} + b_2\dot{z} + b_3z = y$$

Now we define

$$x_1 = z$$
$$x_2 = \dot{x}_1 \tag{5-93}$$
$$x_3 = \dot{x}_2 \tag{5-94}$$

Then, noting that $\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1 = \dddot{z}$, we obtain

$$\dot{x}_3 = -a_3z - a_2\dot{z} - a_1\ddot{z} + u$$

or

$$\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + u \tag{5-95}$$

Also,

$$\begin{aligned} y &= b_0\dddot{z} + b_1\ddot{z} + b_2\dot{z} + b_3z \\ &= b_0\dot{x}_3 + b_1x_3 + b_2x_2 + b_3x_1 \\ &= b_0[(-a_3x_1 - a_2x_2 - a_1x_3) + u] + b_1x_3 + b_2x_2 + b_3x_1 \\ &= (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u \end{aligned} \tag{5-96}$$

From Equations (5-93), (5-94), and (5-95), we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

which is the state equation. From Equation (5-96), we get

$$y = [b_3 - a_3b_0 \;\vdots\; b_2 - a_2b_0 \;\vdots\; b_1 - a_1b_0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0u$$

which is the output equation.

Note that the derivation presented here can be easily extended to the general case of an nth-order system.

Problem A-5-14

Consider the mechanical system shown in Figure 5-34. The system is initially at rest. The displacements u, y, and z are measured from their respective rest positions.

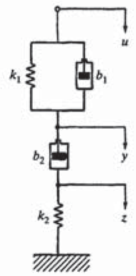


Figure 5-34 Mechanical system.

Assuming that u is the input and y is the output, obtain the transfer function $Y(s)/U(s)$ of the system. Then obtain a state-space representation of the system.

Solution The equations of motion for the system are

$$b_1(\dot{u} - \dot{y}) + k_1(u - y) = b_2(\dot{y} - \dot{z})$$
$$b_2(\dot{y} - \dot{z}) = k_2z$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$b_1[sU(s) - sY(s)] + k_1[U(s) - Y(s)] = b_2[sY(s) - sZ(s)]$$
$$b_2[sY(s) - sZ(s)] = k_2Z(s)$$

Eliminating $Z(s)$ from the last two equations yields

$$(b_1s + k_1)U(s) = \left(b_1s + k_1 + b_2s - \frac{b_2^2s^2}{b_2s + k_2}\right)Y(s)$$

Multiplying both sides of this last equation by $(b_2s + k_2)$, we get

$$(b_1s + k_1)(b_2s + k_2)U(s) = [(b_1s + k_1)(b_2s + k_2) + b_2k_2s]Y(s)$$

The transfer function of the system then becomes

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{(b_1s + k_1)(b_2s + k_2)}{(b_1s + k_1)(b_2s + k_2) + b_2k_2s} \\ &= \frac{s^2 + \left(\dfrac{k_1}{b_1} + \dfrac{k_2}{b_2}\right)s + \dfrac{k_1k_2}{b_1b_2}}{s^2 + \left(\dfrac{k_1}{b_1} + \dfrac{k_2}{b_2} + \dfrac{k_2}{b_1}\right)s + \dfrac{k_1k_2}{b_1b_2}} \end{aligned} \tag{5-97}$$

Next, we shall obtain a state-space representation of the system. The differential equation corresponding to Equation (5-97) is

$$\ddot{y} + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2} + \frac{k_2}{b_1}\right)\dot{y} + \frac{k_1k_2}{b_1b_2}y = \ddot{u} + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2}\right)\dot{u} + \frac{k_1k_2}{b_1b_2}u$$

Comparing this equation with the standard second-order differential equation given by Equation (5-20), namely,

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

we find that

$$a_1 = \frac{k_1}{b_1} + \frac{k_2}{b_2} + \frac{k_2}{b_1}, \qquad a_2 = \frac{k_1k_2}{b_1b_2}$$
$$b_0 = 1, \qquad b_1 = \frac{k_1}{b_1} + \frac{k_2}{b_2}, \qquad b_2 = \frac{k_1k_2}{b_1b_2}$$

From Equations (5-23), (5-24), and (5-29), we have

$$\beta_0 = b_0 = 1$$
$$\beta_1 = b_1 - a_1\beta_0 = -\frac{k_2}{b_1}$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k_1k_2}{b_1^2} + \frac{k_2^2}{b_1b_2} + \frac{k_2^2}{b_1^2}$$

From Equations (5-21) and (5-22), we define the state variables x_1 and x_2 as

$$x_1 = y - \beta_0u = y - u$$
$$x_2 = \dot{x}_1 - \beta_1u = \dot{x}_1 + \frac{k_2}{b_1}u$$

The state equation is given by Equation (5-30) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}u$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_1k_2}{b_1b_2} & -\left(\dfrac{k_1}{b_1} + \dfrac{k_2}{b_2} + \dfrac{k_2}{b_1}\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\dfrac{k_2}{b_1} \\ \dfrac{k_1k_2}{b_1^2} + \dfrac{k_2^2}{b_1b_2} + \dfrac{k_2^2}{b_1^2} \end{bmatrix}u \tag{5-98}$$

The output equation is given by Equation (5-31) as

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_0u$$

or

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \tag{5-99}$$

Equations (5-98) and (5-99) constitute a state-space representation of the system.

Problem A-5-15

Consider the mechanical system shown in Figure 5-35, in which $m = 0.1$ kg, $b = 0.4$ N-s/m, $k_1 = 6$ N/m, and $k_2 = 4$ N/m. The displacements y and z are measured from their respective equilibrium positions. Assume that force u is the input to the system. Considering that displacement y is the output, obtain the transfer function $Y(s)/U(s)$. Also, obtain a state-space representation of the system.

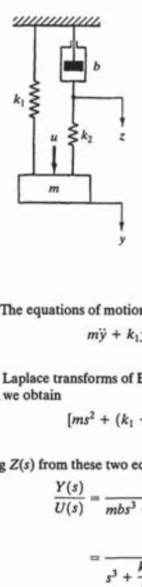


Figure 5-35 Mechanical system.

Solution The equations of motion for the system are

$$m\ddot{y} + k_1y + k_2(y - z) = u \tag{5-100}$$
$$k_2(y - z) = b\dot{z} \tag{5-101}$$

Taking the Laplace transforms of Equations (5-100) and (5-101), assuming zero initial conditions, we obtain

$$[ms^2 + (k_1 + k_2)]Y(s) = k_2Z(s) + U(s)$$
$$k_2Y(s) = (k_2 + bs)Z(s)$$

Eliminating $Z(s)$ from these two equations yields

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{k_2 + bs}{mbs^3 + mk_2s^2 + (k_1 + k_2)bs + k_1k_2} \\ &= \frac{\dfrac{1}{m}s + \dfrac{k_2}{mb}}{s^3 + \dfrac{k_2}{b}s^2 + \dfrac{k_1 + k_2}{m}s + \dfrac{k_1k_2}{mb}} \end{aligned}$$

Substituting numerical values for m, b, k_1, and k_2 into this last equation results in

$$\frac{Y(s)}{U(s)} = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600} \tag{5-102}$$

This is the transfer function of the system.

Next, we shall obtain a state-space representation of the system using Method 1 presented in Section 5-4. From Equation (5-102), we obtain

$$\dddot{y} + 10\ddot{y} + 100\dot{y} + 600y = 10\dot{u} + 100u$$

Comparing this equation with the standard third-order differential equation, namely,

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3u$$

we find that

$$a_1 = 10, \qquad a_2 = 100, \qquad a_3 = 600$$
$$b_0 = 0, \qquad b_1 = 0, \qquad b_2 = 10, \qquad b_3 = 100$$

Referring to **Problem A–5–12**, define

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

where

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 10$$

Also, note that

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = 100 - 10 \times 10 = 0$$

Then the state equation for the system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} u \qquad (5\text{–}103)$$

and the output equation becomes

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (5\text{–}104)$$

Equations (5–103) and (5–104) give a state-space representation of the system.

Problem A–5–16

Consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u \qquad (5\text{–}105)$$

Obtain a state-space representation of the system by the partial-fraction expansion technique.

Solution First, rewrite Equation (5–105) in the form of a transfer function:

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)}$$

Next, expanding this transfer function into partial fractions, we get

$$\frac{Y(s)}{U(s)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

from which we obtain

$$Y(s) = \frac{3}{s+1}U(s) + \frac{-6}{s+2}U(s) + \frac{3}{s+3}U(s) \qquad (5\text{–}106)$$

Let us define

$$X_1(s) = \frac{3}{s+1}U(s)$$
$$X_2(s) = \frac{-6}{s+2}U(s)$$
$$X_3(s) = \frac{3}{s+3}U(s)$$

Then, rewriting these three equations, we have

$$sX_1(s) = -X_1(s) + 3U(s)$$
$$sX_2(s) = -2X_2(s) - 6U(s)$$
$$sX_3(s) = -3X_3(s) + 3U(s)$$

The inverse Laplace transforms of the last three equations give

$$\dot{x}_1 = -x_1 + 3u \qquad (5\text{–}107)$$
$$\dot{x}_2 = -2x_2 - 6u \qquad (5\text{–}108)$$
$$\dot{x}_3 = -3x_3 + 3u \qquad (5\text{–}109)$$

Since Equation (5–106) can be written as

$$Y(s) = X_1(s) + X_2(s) + X_3(s)$$

we obtain

$$y = x_1 + x_2 + x_3 \qquad (5\text{–}110)$$

Combining Equations (5–107), (5–108), and (5–109) into a vector-matrix differential equation yields the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \qquad (5\text{–}111)$$

From Equation (5–110), we get the following output equation:

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (5\text{–}112)$$

Equations (5–111) and (5–112) constitute a state-space representation of the system given by Equation (5–105). (Note that this representation is the same as that obtained in **Example 5–10.**)

Problem A–5–17

Show that the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

has two distinct eigenvalues and that the eigenvectors are linearly independent of each other.

Solution The eigenvalues, obtained from

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0$$

are

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 2$$

Thus, matrix $\mathbf{A}$ has two distinct eigenvalues.

There are two eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ associated with λ_1 and λ_2, respectively. If we define

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$$

then the eigenvector $\mathbf{x}_1$ can be found from

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$

or

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{x}_1 = \mathbf{0}$$

Noting that $\lambda_1 = 1$, we have

$$\begin{bmatrix} 1-1 & -1 \\ 0 & 1-2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$x_{11} = \text{arbitrary constant} \qquad \text{and} \qquad x_{21} = 0$$

Hence, eigenvector $\mathbf{x}_1$ may be written as

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

where $c_1 \neq 0$ is an arbitrary constant.

Similarly, for the eigenvector $\mathbf{x}_2$, we have

$$\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$$

or

$$(\lambda_2\mathbf{I} - \mathbf{A})\mathbf{x}_2 = \mathbf{0}$$

Noting that $\lambda_2 = 2$, we obtain

$$\begin{bmatrix} 2-1 & -1 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from which we get

$$x_{12} - x_{22} = 0$$

Thus, the eigenvector associated with $\lambda_2 = 2$ may be selected as

$$\mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

where $c_2 \neq 0$ is an arbitrary constant.

The two eigenvectors are therefore given by

$$\mathbf{x}_1 = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

That eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent can be seen from the fact that the determinant of the matrix $[\mathbf{x}_1\,\mathbf{x}_2]$ is nonzero:

$$\begin{vmatrix} c_1 & c_2 \\ 0 & c_2 \end{vmatrix} \neq 0$$

Problem A–5–18

Obtain the eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that the eigenvalues are λ_1, λ_2, and λ_3; that is,

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda + a_1 \end{vmatrix} \\ &= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \end{aligned}$$

Assume also that λ_1, λ_2, and λ_3 are distinct.

Solution The eigenvector $\mathbf{x}_i$ associated with an eigenvalue λ_i is a vector that satisfies the equation

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i \qquad (5\text{–}113)$$

which can be written as

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} = \lambda_i \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix}$$

Simplifying this last equation, we obtain

$$x_{i2} = \lambda_i x_{i1}$$
$$x_{i3} = \lambda_i x_{i2}$$
$$-a_3x_{i1} - a_2x_{i2} - a_1x_{i3} = \lambda_i x_{i3}$$

Thus,

$$\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} = \begin{bmatrix} x_{i1} \\ \lambda_i x_{i1} \\ \lambda_i^2 x_{i1} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \end{bmatrix} x_{i1}$$

Hence, the eigenvectors are

$$\begin{bmatrix} x_{11} \\ \lambda_1 x_{11} \\ \lambda_1^2 x_{11} \end{bmatrix}, \qquad \begin{bmatrix} x_{21} \\ \lambda_2 x_{21} \\ \lambda_2^2 x_{21} \end{bmatrix}, \qquad \begin{bmatrix} x_{31} \\ \lambda_3 x_{31} \\ \lambda_3^2 x_{31} \end{bmatrix} \qquad (5\text{–}114)$$

Note that if $\mathbf{x}_i$ is an eigenvector, then $a\mathbf{x}_i$ (where a = scalar $\neq 0$) is also an eigenvector, because Equation (5–113) can be written as

$$a(\mathbf{A}\mathbf{x}_i) = a(\lambda_i\mathbf{x}_i)$$

or

$$\mathbf{A}(a\mathbf{x}_i) = \lambda_i(a\mathbf{x}_i)$$

Thus, by dividing the eigenvectors given by (5–114) by x_{11}, x_{21}, and x_{31}, respectively, we obtain

$$\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{bmatrix}$$

These are also a set of eigenvectors.

Problem A–5–19

Consider a matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that λ_1, λ_2, and λ_3 are distinct eigenvalues of matrix **A**.

Show that if a transformation matrix **P** is defined by

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Solution First note that

$$\mathbf{AP} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ -a_3 - a_2\lambda_1 - a_1\lambda_1^2 & -a_3 - a_2\lambda_2 - a_1\lambda_2^2 & -a_3 - a_2\lambda_3 - a_1\lambda_3^2 \end{bmatrix} \tag{5–115}$$

Since λ_1, λ_2, and λ_3 are eigenvalues, they satisfy the characteristic equation, or

$$\lambda_i^3 + a_1\lambda_i^2 + a_2\lambda_i + a_3 = 0$$

Thus,

$$\lambda_i^3 = -a_3 - a_2\lambda_i - a_1\lambda_i^2$$

Hence,

$$-a_3 - a_2\lambda_1 - a_1\lambda_1^2 = \lambda_1^3$$
$$-a_3 - a_2\lambda_2 - a_1\lambda_2^2 = \lambda_2^3$$
$$-a_3 - a_2\lambda_3 - a_1\lambda_3^2 = \lambda_3^3$$

Consequently, Equation (5–115) can be written as

$$\mathbf{AP} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix} \tag{5–116}$$

Next, define

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Then

$$\mathbf{PD} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix} \tag{5–117}$$

Comparing Equations (5–116) and (5–117), we have

$$\mathbf{AP} = \mathbf{PD}$$

Thus, we have shown that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Problem A–5–20

Prove that the eigenvalues of a square matrix **A** are invariant under a linear transformation.

Solution To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials $|\lambda\mathbf{I} - \mathbf{A}|$ and $|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}|$ are identical.

Since the determinant of a product is the product of the determinants, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}| \\ &= |\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Noting that the product of the determinants $|\mathbf{P}^{-1}|$ and $|\mathbf{P}|$ is the determinant of the product $|\mathbf{P}^{-1}\mathbf{P}|$, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\mathbf{P}^{-1}\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \\ &= |\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Thus, we have proven that the eigenvalues of **A** are invariant under a linear transformation.

PROBLEMS

Problem B–5–1

Obtain state-space representations of the mechanical systems shown in Figures 5–36(a) and (b).

Problem B–5–2

For the spring–mass–pulley system of Figure 5–37, the moment of inertia of the pulley about the axis of rotation is J and the radius is R. Assume that the system is initially in equilibrium. The gravitational force of mass m causes a static deflection of the spring such that $k\delta = mg$. Assuming that the displacement y of mass m is measured from the equilibrium position, obtain a state-space representation of the system. The external force u applied to mass m is the input and the displacement y is the output of the system.

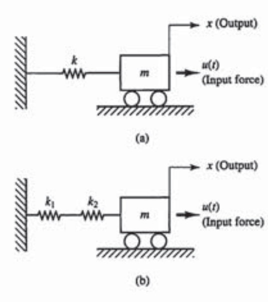


Figure 5–36 (a) and (b) Mechanical systems.

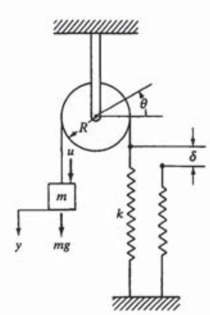


Figure 5–37 Spring–mass–pulley system.

Problem B–5–3

Obtain a state-space representation of the mechanical system shown in Figure 5–38. The force $u(t)$ applied to mass m_1 is the input to the system. The displacements y and z are the outputs of the system. Assume that y and z are measured from their respective equilibrium positions.

Problem B–5–4

Obtain a state-space representation of the mechanical system shown in Figure 5–39, where u_1 and u_2 are the inputs and y_1 and y_2 are the outputs. The displacements y_1 and y_2 are measured from their respective equilibrium positions.

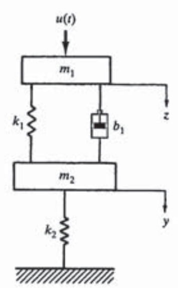


Figure 5–38 Mechanical system.

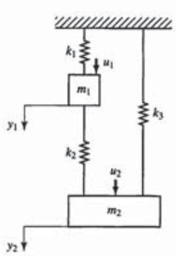


Figure 5–39 Mechanical system.

Problem B–5–5

Given the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u$$

and output equation

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

obtain the corresponding scalar differential equation in terms of y and u.

Problem B–5–6

Consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

where y is the output and u is the input of the system. Obtain a state-space representation for the system.

Problem B–5–7

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function of the system.

Problem B–5–8

Consider a system described by the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and the output equation

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 1$$

Obtain the transfer function of the system.

Problem B–5–9

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0$$

Obtain the transfer function of the system.

Problem B–5–10

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves with MATLAB.

Problem B–5–11

Obtain the unit-step response curve and unit-impulse response curve of the following system with MATLAB:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & -25 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 25 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Problem B–5–12

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x}(0) = \mathbf{x}_0$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Obtain the response to the initial condition

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Use MATLAB command initial(A,B,C,D,[initial condition],t).

Problem B–5–13

Consider the system

$$\dddot{y} + 8\ddot{y} + 17\dot{y} + 10y = 0$$

subjected to the initial condition

$$y(0) = 2, \qquad \dot{y}(0) = 1, \qquad \ddot{y}(0) = 0.5$$

(No external forcing function is present.) Obtain the response curve $y(t)$ to the given initial condition with MATLAB. Use command lsim.

Problem B–5–14

Consider the mechanical system shown in Figure 5–40(a). The system is at rest for $t < 0$. The displacement y is measured from the equilibrium position for $t < 0$. At $t = 0$, an input force

$$u(t) = 1\text{ N} \qquad \text{for } 0 \le t \le 5$$
$$= 0 \qquad \text{for } 5 < t$$

is given to the system. [See Figure 5–40(b).] Derive a state-space representation of the system. Plot the response curve $y(t)$ versus t (where $0 < t < 10$) with MATLAB. Assume that $m = 5$ kg, $b = 8$ N-s/m, and $k = 20$ N/m.

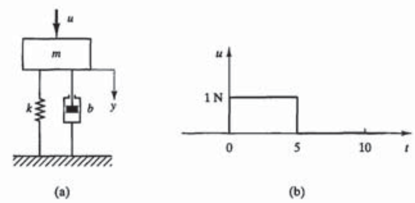


Figure 5–40 (a) Mechanical system; (b) input force u.

Problem B–5–15

Consider the mechanical system shown in Figure 5–41(a). Assume that at $t = 0$ mass m is placed on the massless bar AA'. [See Figure 5–41(b).] Neglecting the mass of the spring–damper device, what is the subsequent motion $y(t)$ of the bar AA'? The displacement $y(t)$ is measured from the equilibrium position before the mass is placed on the bar AA'. Assume that $m = 1$ kg, the viscous-friction coefficient $b = 4$ N-s/m, and the spring constant $k = 40$ N/m. Derive a state-space representation of the system, and plot the response curve $y(t)$ versus t with MATLAB.

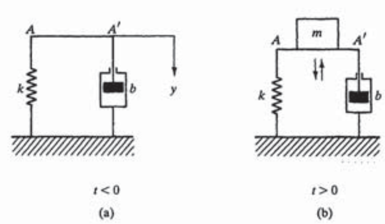


Figure 5–41 (a) Mechanical device; (b) vibration caused by placement of mass m on bar AA'.

Problem B–5–16

Consider the system shown in Figure 5–42. The system is at rest for $t < 0$. Assume that the input and output are the displacements u and y, respectively, measured from the rest positions. Assume that $m = 10$ kg, $b = 20$ N-s/m, and $k = 40$ N/m. The input u is a step displacement input of 0.2 m. Assume also that the system remains linear throughout the transient period. Obtain a state-space representation of the system. Plot the response curve $y(t)$ versus t with MATLAB.

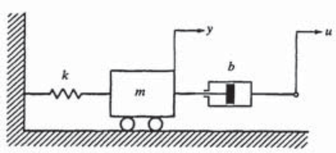


Figure 5–42 Mechanical system.

Problem B–5–17

Referring to **Problem A–5–10**, consider the system shown in Figure 5–31. The system is at rest for $t < 0$. The displacements z_1 and z_2 are measured from their respective equilibrium positions relative to the ground. Define $z_2 - z_1 = z$. Derive a state-space equation when z, $\dot{z}$, z_1, and $\dot{z}_1$ are chosen as state variables. Assuming that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m, $k = 60$ N/m, and f is a step force input of magnitude 10 N, plot the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z(t)$ versus t.

Problem B–5–18

Consider the system shown in Figure 5–43(a). The system is at rest for $t < 0$. The displacements z_1 and z_2 are measured from their respective equilibrium positions before the input force

$$f = t\text{ N} \qquad (0 < t \le 10)$$
$$= 0 \qquad (10 < t)$$

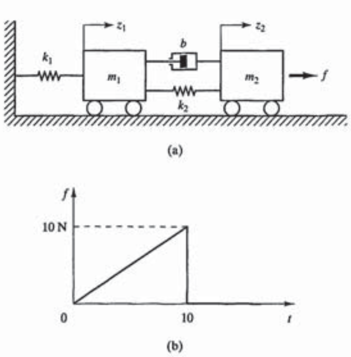


Figure 5–43 (a) Mechanical system; (b) input force f.

[see Figure 5–43(b)] is given to the system. Assume that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m. $k_1 = 30$ N/m, and $k_2 = 60$ N/m, and derive a state-space representation of the system. Then plot the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_2(t) - z_1(t)$ versus t.

Problem B–5–19

Consider the system equation given by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

By choosing appropriate state variables, derive the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \tag{5–118}$$

and output equation

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u \tag{5–119}$$

Problem B–5–20

Consider the system defined by the following transfer function:

$$\frac{Y(s)}{U(s)} = \frac{160(s + 4)}{s^3 + 18s^2 + 192s + 640}$$

Using Methods 1 and 2 presented in Section 5–4, obtain two state-space representations of the system.

Problem B–5–21

Using the partial-fraction expansion approach, obtain a state-space representation for the following system:

$$\frac{Y(s)}{U(s)} = \frac{5}{(s + 1)^2(s + 2)}$$

Problem B–5–22

Consider the mechanical system shown in Figure 5–44. The system is at rest for $t < 0$. The force u is the input to the system and the displacement y, measured from the equilibrium position before u is given at $t = 0$, is the output of the system. Obtain a state-space representation of the system.

Problem B–5–23

Consider the system shown in Figure 5–45. The system is at rest for $t < 0$. The displacements y_1 and y_2 are measured from their respective equilibrium positions before the input force u is given at $t = 0$. Obtain a state-space representation of the system.

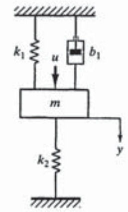


Figure 5–44 Mechanical system.

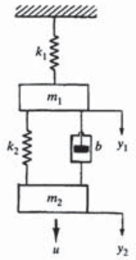


Figure 5–45 Mechanical system.

Assuming that $m_1 = 10$ kg, $m_2 = 5$ kg, $b = 10$ N-s/m, $k_1 = 40$ N/m, and $k_2 = 20$ N/m and that input force u is a constant force of 5 N, obtain the response of the system. Plot the response curves $y_1(t)$ versus t and $y_2(t)$ versus t with MATLAB.

Problem B–5–24

Consider the mechanical system shown in Figure 5–46. The system is initially at rest. The displacement u is the input to the system, and the displacements y and z, measured

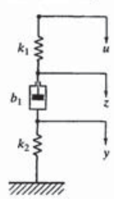


Figure 5–46 Mechanical system.

from their respective rest positions before the input displacement u is given to the system, are the outputs of the system. Obtain a state-space representation of the system.

Problem B–5–25

Consider the mechanical system shown in Figure 5–47. The system is at rest for $t < 0$. The force u is the input to the system and the displacements z_1 and z_2, measured from their respective equilibrium positions before u is applied at $t = 0$, are the outputs of the system. Obtain a state-space representation of the system.

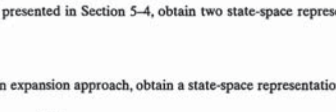


Figure 5–47 Mechanical system.

Problem B–5–26

Consider the system shown in Figure 5–48. The system is at rest for $t < 0$. The force u is the input to the system and the displacements z_1 and z_2, measured from their respective equilibrium positions before u is applied at $t = 0$, are the outputs of the system. Obtain a state-space representation of the system.

Assume that $m_1 = 100$ kg, $m_2 = 200$ kg, $b = 25$ N-s/m, $k_1 = 50$ N/m, and $k_2 = 100$ N/m. The input force u is a step force of magnitude 10 N. Plot the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_2(t) - z_1(t)$ versus t.

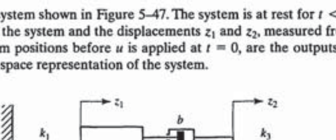


Figure 5–48 Mechanical system.

Problem B–5–27

Consider the system

$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

Obtain a state-space representation of the system with MATLAB.

Problem B–5–28

Consider the system

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 2s^2 + 15s + 10}{s^3 + 4s^2 + 8s + 10}$$

Obtain a state-space representation of the system with MATLAB.

Problem B–5–29

Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved: $Y_1(s)/U_1(s)$, $Y_2(s)/U_1(s)$, $Y_1(s)/U_2(s)$, and $Y_2(s)/U_2(s)$. (When considering input u_1, we assume that input u_2 is zero, and vice versa.)

Obtain the transfer matrix (consisting of the preceding four transfer functions) of the system.

Problem B–5–30

Obtain the transfer matrix of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem B–5–31

Consider a 3×3 matrix having a triple eigenvalue of λ_1. Then any one of the following Jordan canonical forms is possible:

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Each of the three matrices has the same characteristic equation $(\lambda - \lambda_1)^3 = 0$. The first corresponds to the case where there exists only one linearly independent eigenvector. This fact can be seen by denoting the first matrix by $\mathbf{A}$ and solving the following equation for $\mathbf{x}$:

$$\mathbf{A}\mathbf{x} = \lambda_1 \mathbf{x}$$

That is,

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which can be rewritten as

$$\begin{aligned} \lambda_1 x_1 + x_2 &= \lambda_1 x_1 \\ \lambda_1 x_2 + x_3 &= \lambda_1 x_2 \\ \lambda_1 x_3 &= \lambda_1 x_3 \end{aligned}$$

which, in turn, gives

$$x_1 = \text{arbitrary constant}, \qquad x_2 = 0, \qquad x_3 = 0$$

Hence,

$$\mathbf{x} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

where a is a nonzero constant. Thus, there is only one linearly independent eigenvector.

Show that the second and third of the three matrices have, respectively, two and three linearly independent eigenvectors.

6

Electrical Systems and Electromechanical Systems

6–1 INTRODUCTION

This chapter is concerned with mathematical modeling and the response analysis of electrical systems and electromechanical systems. Electrical systems and mechanical systems (as well as other systems, such as fluid systems) are very often described by analogous mathematical models. Therefore, we present brief discussions on analogous systems in the chapter.

In this section, we first review three types of basic elements of electrical systems: resistance, capacitance, and inductance elements. (These elements are passive elements, because, although they can store or dissipate energy that is already present in the circuit, they cannot introduce additional energy into the circuit.) Then we briefly discuss voltage and current sources. (These are active elements, because they can introduce energy into the circuit.) Finally, we provide an outline of the chapter.

Resistance elements. The *resistance* R of a linear resistor is given by

$$R = \frac{e_R}{i}$$

where e_R is the voltage across the resistor and i is the current through the resistor. The unit of resistance is the ohm (Ω), where

$$\text{ohm} = \frac{\text{volt}}{\text{ampere}}$$

Resistors do not store electric energy in any form, but instead dissipate it as heat. Note that real resistors may not be linear and may also exhibit some capacitance and inductance effects.

Capacitance elements. Two conductors separated by a nonconducting medium form a capacitor, so two metallic plates separated by a very thin dielectric material form a capacitor. The *capacitance* C is a measure of the quantity of charge that can be stored for a given voltage across the plates. The capacitance C of a capacitor can thus be given by

$$C = \frac{q}{e_C}$$

where q is the quantity of charge stored and e_C is the voltage across the capacitor. The unit of capacitance is the farad (F), where

$$\text{farad} = \frac{\text{ampere-second}}{\text{volt}} = \frac{\text{coulomb}}{\text{volt}}$$

Note that, since $i = dq/dt$ and $e_C = q/C$, we have

$$i = C\frac{de_C}{dt}$$

or

$$de_C = \frac{1}{C}i\,dt$$

Therefore,

$$e_C(t) = \frac{1}{C}\int_0^t i\,dt + e_C(0)$$

Although a pure capacitor stores energy and can release all of it, real capacitors exhibit various losses. These energy losses are indicated by a *power factor*, which is the ratio of the energy lost per cycle of ac voltage to the energy stored per cycle. Thus, a small-valued power factor is desirable.

Inductance elements. If a circuit lies in a time-varying magnetic field, an electromotive force is induced in the circuit. The inductive effects can be classified as self-inductance and mutual inductance.

Self-inductance is that property of a single coil that appears when the magnetic field set up by the current in the coil links to the coil itself. The magnitude of the induced voltage is proportional to the rate of change of flux linking the circuit. If the circuit does not contain ferromagnetic elements (such as an iron core), the rate of change of flux is proportional to di/dt. Self-inductance, or simply inductance, L, is the proportionality constant between the induced voltage e_L volts and the rate of change of current (or change in current per second) di/dt amperes per second; that is,

$$L = \frac{e_L}{di/dt}$$

The unit of inductance is the henry (H). An electrical circuit has an inductance of 1 henry when a rate of change of 1 ampere per second will induce an emf of 1 volt:

$$\text{henry} = \frac{\text{volt}}{\text{ampere/second}} = \frac{\text{weber}}{\text{ampere}}$$

The voltage e_L across the inductor L is given by

$$e_L = L\frac{di_L}{dt}$$

where i_L is the current through the inductor. The current $i_L(t)$ can thus be given by

$$i_L(t) = \frac{1}{L}\int_0^t e_L dt + i_L(0)$$

Because most inductors are coils of wire, they have considerable resistance. The energy loss due to the presence of resistance is indicated by the *quality factor* Q, which denotes the ratio of stored to dissipated energy. A high value of Q generally means that the inductor contains small resistance.

Mutual inductance refers to the influence between inductors that results from the interaction of their fields. If two inductors are involved in an electrical circuit, each may come under the influence of the magnetic field of the other inductor. Then the voltage drop in the first inductor is related to the current flowing through the first inductor, as well as to the current flowing through the second inductor, whose magnetic field influences the first. The second inductor is also influenced by the first in exactly the same manner. When a change in current of 1 ampere per second in either of the two inductors induces an electromotive force of 1 volt in the other inductor, their mutual inductance M is 1 henry. (Note that it is customary to use the symbol M to denote mutual inductance, to distinguish it from self-inductance L.)

Voltage and current sources. A *voltage source* is a device that causes a specified voltage to exist between two points in a circuit. The voltage may be time varying or time invariant (for a sufficiently long time). Figure 6–1(a) is a schematic diagram of a voltage source. Figure 6–1(b) shows a voltage source that has a constant value for an indefinite time. Often the voltage is denoted by E. A battery is an example of this type of voltage source.

A *current source* causes a specified current to flow through a wire containing this source. Figure 6–1(c) is a schematic diagram of a current source.

Outline of the chapter. Section 6–1 has presented introductory material. Section 6–2 reviews the fundamentals of electrical circuits that are presented in college physics courses. Section 6–3 deals with mathematical modeling and the response analysis of electrical systems. The complex-impedance approach is included. Section 6–4 discusses analogous systems. Section 6–5 offers brief discussions of electromechanical systems. Finally, Section 6–6 treats operational-amplifier systems.

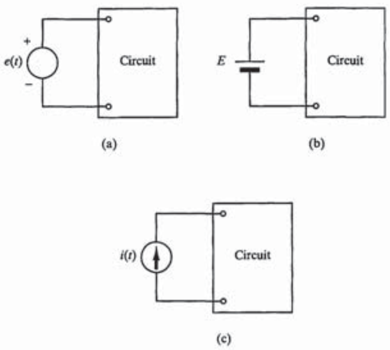


Figure 6–1 (a) Voltage source; (b) constant voltage source; (c) current source.

6–2 FUNDAMENTALS OF ELECTRICAL CIRCUITS

In this section, we review Ohm's law, series and parallel circuits, and Kirchhoff's current and voltage laws.

Ohm's law. *Ohm's law* states that the current in a circuit is proportional to the total electromotive force (emf) acting in the circuit and inversely proportional to the total resistance of the circuit. Ohm's law can be expressed as

$$i = \frac{e}{R}$$

where i is the current (amperes), e is the emf (volts), and R is the resistance (ohms).

Series circuits. The combined resistance of series-connected resistors is the sum of the separate resistances. Figure 6–2 shows a simple series circuit. The voltage between points A and B is

$$e = e_1 + e_2 + e_3$$

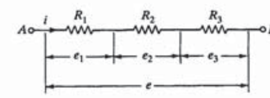


Figure 6–2 Series circuit.

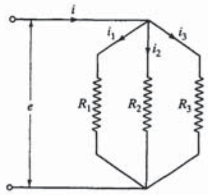


Figure 6–3 Parallel circuit.

where

$$e_1 = iR_1, \qquad e_2 = iR_2, \qquad e_3 = iR_3$$

Thus,

$$\frac{e}{i} = R_1 + R_2 + R_3$$

The combined resistance is then given by

$$R = R_1 + R_2 + R_3$$

Parallel circuits. For the parallel circuit shown in Figure 6–3,

$$i_1 = \frac{e}{R_1}, \qquad i_2 = \frac{e}{R_2}, \qquad i_3 = \frac{e}{R_3}$$

Since $i = i_1 + i_2 + i_3$, it follows that

$$i = \frac{e}{R_1} + \frac{e}{R_2} + \frac{e}{R_3} = \frac{e}{R}$$

where R is the combined resistance. Hence,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

or

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} = \frac{R_1R_2R_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

Resistance of combined series and parallel resistors. Consider the circuit shown in Figure 6–4(a). The combined resistance between points B and C is

$$R_{BC} = \frac{R_2R_3}{R_2 + R_3}$$

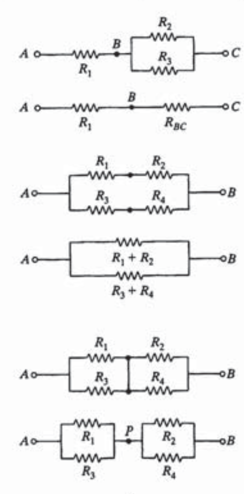


Figure 6–4 Combined series and parallel resistors.

The combined resistance R between points A and C is

$$R = R_1 + R_{BC} = R_1 + \frac{R_2R_3}{R_2 + R_3}$$

The circuit shown in Figure 6–4(b) can be considered a parallel circuit consisting of resistances $(R_1 + R_2)$ and $(R_3 + R_4)$. So the combined resistance R between points A and B is

$$\frac{1}{R} = \frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}$$

or

$$R = \frac{(R_1 + R_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4}$$

Next, consider the circuit shown in Figure 6–4(c). Here, R_1 and R_3 are parallel and R_2 and R_4 are parallel, and the two parallel pairs of resistances are connected in

series. Redrawing this circuit as shown in Figure 6–4(c), therefore, we obtain

$$R_{AP} = \frac{R_1R_3}{R_1 + R_3}, \qquad R_{PB} = \frac{R_2R_4}{R_2 + R_4}$$

As a result, the combined resistance R becomes

$$R = R_{AP} + R_{PB} = \frac{R_1R_3}{R_1 + R_3} + \frac{R_2R_4}{R_2 + R_4}$$

Kirchhoff's laws. In solving circuit problems that involve many electromotive forces, resistances, capacitances, inductances, and so on, it is often necessary to use *Kirchhoff's laws*, of which there are two: the *current law* (node law) and the *voltage law* (loop law).

Kirchhoff's current law (node law). A *node* in an electrical circuit is a point where three or more wires are joined together. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of all the currents entering a node is equal to the sum of all the currents leaving the same node.) In applying the law to circuit problems, the following rules should be observed: Currents going toward a node should be preceded by a plus sign; currents going away from a node should be preceded by a minus sign. As applied to Figure 6–5, Kirchhoff's current law states that

$$i_1 + i_2 + i_3 - i_4 - i_5 = 0$$

Kirchhoff's voltage law (loop law). Kirchhoff's voltage law states that at any given instant of time the algebraic sum of the voltages around any loop in an electrical circuit is zero. This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop. In applying the law to circuit problems, the following rules should be observed: A rise in voltage [which occurs in going through a source of electromotive force from the negative to the positive terminal, as shown in Figure 6–6(a), or in going through a resistance in opposition to the current flow, as shown in Figure 6–6(b)] should be preceded by a plus sign. A drop in voltage [which occurs in going through a source of electromotive force from the positive to the negative terminal, as shown in Figure 6–6(c), or in going through a resistance in the direction of the current flow, as shown in Figure 6–6(d)] should be preceded by a minus sign.

Figure 6–7 shows a circuit that consists of a battery and an external resistance. Here, E is the electromotive force, r is the internal resistance of the battery, R is the external resistance, and i is the current. If we follow the loop in the clockwise direction

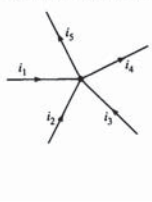


Figure 6–5 Node. $i_1 + i_2 + i_3 - i_4 - i_5 = 0$.

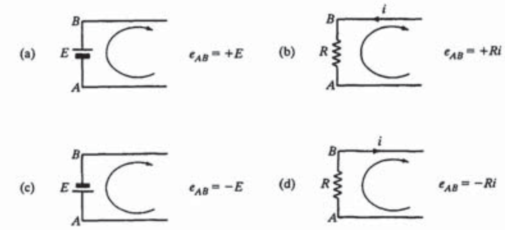


Figure 6–6 Diagrams showing voltage rises and voltage drops in circuits. (*Note:* Each circular arrow shows the direction one follows in analyzing the respective circuit.)

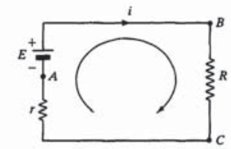


Figure 6–7 Electrical circuit.

$(A \rightarrow B \rightarrow C \rightarrow A)$ as shown, then we have

$$e_{\overrightarrow{AB}} + e_{\overrightarrow{BC}} + e_{\overrightarrow{CA}} = 0$$

or

$$E - iR - ir = 0$$

from which it follows that

$$i = \frac{E}{R + r}$$

A circuit consisting of two batteries and an external resistance appears in Figure 6–8(a), where E_1 and r_1 (E_2 and r_2) are the electromotive force and internal resistance of battery 1 (battery 2), respectively, and R is the external resistance. By assuming the direction of the current i as shown and following the loop clockwise as shown, we obtain

$$E_1 - iR - E_2 - ir_2 - ir_1 = 0$$

or

$$i = \frac{E_1 - E_2}{r_1 + r_2 + R} \tag{6–1}$$

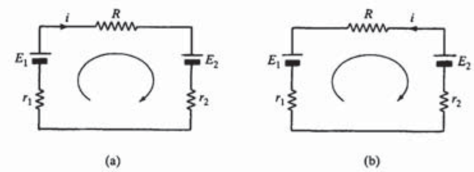


Figure 6–8 Electrical circuits.

If we assume that the direction of the current i is reversed [Figure 6–8(b)], then, by following the loop clockwise, we obtain

$$E_1 + iR - E_2 + ir_2 + ir_1 = 0$$

or

$$i = \frac{E_2 - E_1}{r_1 + r_2 + R} \tag{6–2}$$

Note that, in solving circuit problems, if we assume that the current flows to the right and if the value of i is calculated and found to be positive, then the current i actually flows to the right. If the value of i is found to be negative, the current i actually flows to the left. For the circuits shown in Figure 6–8, suppose that $E_1 > E_2$. Then Equation (6–1) gives $i > 0$, which means that the current i flows in the direction assumed. Equation (6–2), however, yields $i < 0$, which means that the current i flows opposite to the assumed direction.

Note that the direction used to follow the loop is arbitrary, just as the direction of current flow can be assumed to be arbitrary. That is, the direction used in following the loop can be clockwise or counterclockwise; the final result is the same in either case.

Circuits with two or more loops. For circuits with two or more loops, both Kirchhoff's current law and voltage law may be applied. The first step in writing the circuit equations is to define the directions of the currents in each wire. The second is to determine the directions that we follow in each loop.

Consider the circuit shown in Figure 6–9, which has two loops. Let us find the current in each wire. Here, we can assume the directions of currents as shown in the diagram. (Note that these directions are arbitrary and could differ from those shown

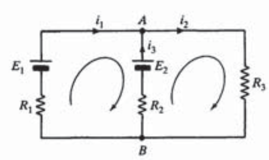


Figure 6–9 Electrical circuit.

in the diagram.) Suppose that we follow the loops clockwise, as is shown in the figure. (Again, the directions could be either clockwise or counterclockwise.) Then we obtain the following equations:

$$\begin{aligned} \text{At point } A\text{:} \quad & i_1 + i_3 - i_2 = 0 \\ \text{For the left loop:} \quad & E_1 - E_2 + i_3R_2 - i_1R_1 = 0 \\ \text{For the right loop:} \quad & E_2 - i_2R_3 - i_3R_2 = 0 \end{aligned}$$

Eliminating i_2 from the preceding three equations and then solving for i_1 and i_3, we find that

$$i_1 = \frac{E_1(R_2 + R_3) - E_2R_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$i_3 = \frac{E_2(R_1 + R_3) - E_1R_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

Hence,

$$i_2 = i_1 + i_3 = \frac{E_1R_2 + E_2R_1}{R_1R_2 + R_2R_3 + R_3R_1}$$

Writing equations for loops by using cyclic currents. In this approach, we assume that a cyclic current exists in each loop. For instance, in Figure 6–10, we assume that clockwise cyclic currents i_1 and i_2 exist in the left and right loops, respectively, of the circuit.

Applying Kirchhoff's voltage law to the circuit results in the following equations:

$$\begin{aligned} \text{For left loop:} \quad & E_1 - E_2 - R_2(i_1 - i_2) - R_1i_1 = 0 \\ \text{For right loop:} \quad & E_2 - R_3i_2 - R_2(i_2 - i_1) = 0 \end{aligned}$$

Note that the net current through resistance R_2 is the difference between i_1 and i_2. Solving for i_1 and i_2 gives

$$i_1 = \frac{E_1(R_2 + R_3) - E_2R_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$i_2 = \frac{E_1R_2 + E_2R_1}{R_1R_2 + R_2R_3 + R_3R_1}$$

(By comparing the circuits shown in Figures 6–9 and 6–10, verify that i_3 in Figure 6–9 is equal to $i_2 - i_1$ in Figure 6–10.)

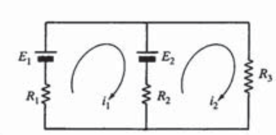


Figure 6–10 Electrical circuit.

6–3 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

The first step in analyzing circuit problems is to obtain mathematical models for the circuits. (Although the terms *circuit* and *network* are sometimes used interchangeably, *network* implies a more complicated interconnection than *circuit*.) A mathematical model may consist of algebraic equations, differential equations, integrodifferential equations, and similar ones. Such a model may be obtained by applying one or both of Kirchhoff's laws to a given circuit. The variables of interest in the circuit analysis are voltages and currents at various points along the circuit.

In this section, we first present the mathematical modeling of electrical circuits and obtain solutions of simple circuit problems. Then we review the concept of complex impedances, followed by derivations of mathematical models of electrical circuits.

Example 6–1

Consider the circuit shown in Figure 6–11. Assume that the switch S is open for $t < 0$ and closed at $t = 0$. Obtain a mathematical model for the circuit and obtain an equation for the current $i(t)$.

By arbitrarily choosing the direction of the current around the loop as shown in the figure, we obtain

$$E - L\frac{di}{dt} - Ri = 0$$

or

$$L\frac{di}{dt} + Ri = E \tag{6–3}$$

This is a mathematical model for the given circuit. Note that at the instant switch S is closed the current $i(0)$ is zero, because the current in the inductor cannot change from zero to a finite value instantaneously. Thus, $i(0) = 0$.

Let us solve Equation (6–3) for the current $i(t)$. Taking the Laplace transforms of both sides, we obtain

$$L[sI(s) - i(0)] + RI(s) = \frac{E}{s}$$

Noting that $i(0) = 0$, we have

$$(Ls + R)I(s) = \frac{E}{s}$$

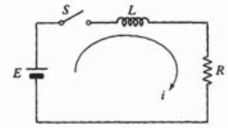


Figure 6–11 Electrical circuit.

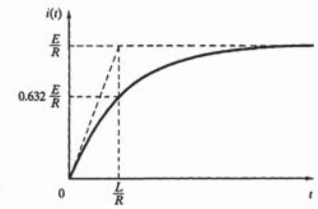


Figure 6–12 Plot of $i(t)$ versus t for the circuit shown in Figure 6–11 when switch S is closed at $t = 0$.

or

$$I(s) = \frac{E}{s(Ls + R)} = \frac{E}{R}\left[\frac{1}{s} - \frac{1}{s + (R/L)}\right]$$

The inverse Laplace transform of this last equation gives

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t}] \tag{6–4}$$

A typical plot of $i(t)$ versus t appears in Figure 6–12.

Example 6–2

Consider again the circuit shown in Figure 6–11. Assume that switch S is open for $t < 0$, it is closed at $t = 0$, and is open again at $t = t_1 > 0$. Obtain a mathematical model for the system, and find the current $i(t)$ for $t \geq 0$.

The equation for the circuit is

$$L\frac{di}{dt} + Ri = E \qquad i(0) = 0 \qquad t_1 > t \geq 0 \tag{6–5}$$

From Equation (6–4), the solution of Equation (6–5) is

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t}] \qquad t_1 > t \geq 0 \tag{6–6}$$

At $t = t_1$, the switch is opened. The equation for the circuit for $t \geq t_1$ is

$$L\frac{di}{dt} + Ri = 0 \qquad t \geq t_1 \tag{6–7}$$

where the initial condition at $t = t_1$ is given by

$$i(t_1) = \frac{E}{R}[1 - e^{-(R/L)t_1}] \tag{6–8}$$

(Note that the instantaneous value of the current at the switching instant $t = t_1$ serves as the initial condition for the transient response for $t \geq t_1$.) Equations (6–5), (6–7), and (6–8) constitute a mathematical model for the system.

Now we shall obtain the solution of Equation (6–7) with the initial condition given by Equation (6–8). The Laplace transform of Equation (6–7), with $t = t_1$ the initial time, gives

$$L[sI(s) - i(t_1)] + RI(s) = 0$$

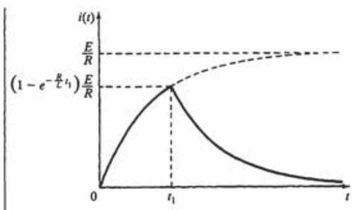


Figure 6–13 Plot of $i(t)$ versus t for the circuit shown in Figure 6–12 when switch S is closed at $t = 0$ and opened at $t = t_1$.

or

$$(Ls + R)I(s) = Li(t_1)$$

Hence,

$$I(s) = \frac{Li(t_1)}{Ls + R} = \frac{E}{R}[1 - e^{-(R/L)t_1}]\frac{1}{s + (R/L)} \tag{6–9}$$

The inverse Laplace transform of Equation (6–9) gives

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t_1}]e^{-(R/L)(t-t_1)} \qquad t \geq t_1 \tag{6–10}$$

Consequently, from Equations (6–6) and (6–10), the current $i(t)$ for $t \geq 0$ can be written

$$\begin{aligned} i(t) &= \frac{E}{R}[1 - e^{-(R/L)t}] && t_1 > t \geq 0 \\ &= \frac{E}{R}[1 - e^{-(R/L)t_1}]e^{-(R/L)(t-t_1)} && t \geq t_1 \end{aligned}$$

A typical plot of $i(t)$ versus t for this case is given in Figure 6–13.

Example 6–3

Consider the electrical circuit shown in Figure 6–14. The circuit consists of a resistance R(in ohms) and a capacitance C (in farads). Obtain the transfer function $E_o(s)/E_i(s)$. Also, obtain a state-space representation of the system.

Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$Ri + \frac{1}{C}\int i\,dt = e_i \tag{6–11}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{6–12}$$

The transfer-function model of the circuit can be obtained as follows: Taking the Laplace transforms of Equations (6–11) and (6–12), assuming zero initial conditions, we get

$$RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

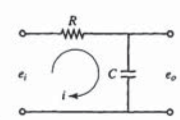


Figure 6–14 RC circuit.

Assuming that the input is e_i and the output is e_o, the transfer function of the system is

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C}\frac{1}{s}I(s)}{\left(R + \frac{1}{C}\frac{1}{s}\right)I(s)} = \frac{1}{RCs + 1} \tag{6–13}$$

This system is a first-order system.

A state-space model of the system may be obtained as follows: First, note that, from Equation (6–13), the differential equation for the circuit is

$$RC\dot{e}_o + e_o = e_i$$

If we define the state variable

$$x = e_o$$

and the input and output variables

$$u = e_i, \qquad y = e_o = x$$

then we obtain

$$\dot{x} = -\frac{1}{RC}x + \frac{1}{RC}u$$

$$y = x$$

These two equations give a state-space representation of the system.

Example 6–4

Consider the electrical circuit shown in Figure 6–15. The circuit consists of an inductance L (in henrys), a resistance R (in ohms), and a capacitance C (in farads). Obtain the transfer function $E_o(s)/E_i(s)$. Also, obtain a state-space representation of the system.

Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{6–14}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{6–15}$$

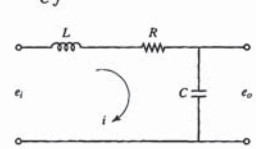


Figure 6–15 Electrical circuit.

The transfer-function model of the circuit can be obtained as follows: Taking the Laplace transforms of Equations (6–14) and (6–15), assuming zero initial conditions, we get

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

Then the transfer function $E_o(s)/E_i(s)$ becomes

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{6–16}$$

A state-space model of the system may be obtained as follows: First, note that, from Equation (6–16), the differential equation for the system is

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then, by defining state variables

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix}u$$

and

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Nonloading Cascaded Elements. The transfer function of a system consisting of two nonloading cascaded elements can be obtained by eliminating the intermediate input and output. For example, consider the system shown in Figure 6–16(a). The transfer functions of the elements are

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \qquad \text{and} \qquad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element. Then the transfer function of the whole system becomes

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)} = G_1(s)G_2(s)$$

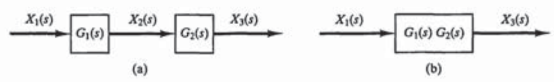


Figure 6–16 (a) System consisting of two nonloading cascaded elements; (b) an equivalent system.

The transfer function of the whole system is thus the product of the transfer functions of the individual elements. This is shown in Figure 6–16(b).

As an example, consider the system shown in Figure 6–17. The insertion of an isolating amplifier between the circuits to obtain nonloading characteristics is frequently used in combining circuits. Since amplifiers have very high input impedances, an isolation amplifier inserted between the two circuits justifies the nonloading assumption.

The two simple RC circuits, isolated by an amplifier as shown in Figure 6–17, have negligible loading effects, and the transfer function of the entire circuit equals the product of the individual transfer functions. Thus, in this case,

$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{R_1C_1s+1}\right)(K)\left(\frac{1}{R_2C_2s+1}\right) = \frac{K}{(R_1C_1s+1)(R_2C_2s+1)}$$

Transfer functions of cascaded elements. Many feedback systems have components that load each other. Consider the system shown in Figure 6–18. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially. Let us show that the second stage of the circuit (the R_2C_2 portion) produces a loading effect on the first stage (the R_1C_1 portion). The equations for the system are

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \tag{6–17}$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2 dt = 0 \tag{6–18}$$

$$\frac{1}{C_2}\int i_2 dt = e_o \tag{6–19}$$

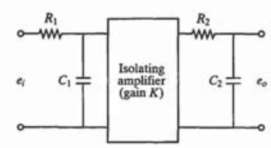


Figure 6–17 Electrical system.

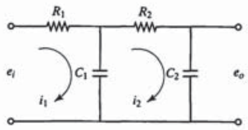


Figure 6–18 Electrical system.

Taking the Laplace transforms of Equations (6–17) through (6–19), respectively, assuming zero initial conditions, we obtain

$$\frac{1}{C_1s}[I_1(s) - I_2(s)] + R_1I_1(s) = E_i(s) \tag{6–20}$$

$$\frac{1}{C_1s}[I_2(s) - I_1(s)] + R_2I_2(s) + \frac{1}{C_2s}I_2(s) = 0 \tag{6–21}$$

$$\frac{1}{C_2s}I_2(s) = E_o(s) \tag{6–22}$$

Eliminating $I_1(s)$ from Equations (6–20) and (6–21) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s+1)(R_2C_2s+1)+R_1C_2s} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1} \tag{6–23}$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since $(R_1C_1 + R_2C_2 + R_1C_2)^2 > 4R_1C_1R_2C_2$, the two roots of the denominator of Equation (6–23) are real.

The analysis just presented shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is *not* the product of $1/(R_1C_1s+1)$ and $1/(R_2C_2s+1)$. The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

Complex impedances. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations. Consider the system shown in Figure 6–19. In this system, Z_1 and Z_2 represent complex impedances. The complex impedance

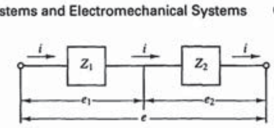


Figure 6–19 Electrical circuit.

$Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals, to $I(s)$, the Laplace transform of the current through the element, under the assumption that the initial conditions are zero, so that $Z(s) = E(s)/I(s)$. If the two-terminal element is a resistance R, a capacitance C, or an inductance L, then the complex impedance is given by R, $1/Cs$, or Ls, respectively. If complex impedances are connected in series, the total impedance is the sum of the individual complex impedances.

The general relationship

$$E(s) = Z(s)I(s)$$

corresponds to Ohm's law for purely resistive circuits. (Note that, like resistances, impedances can be combined in series and in parallel.)

Remember that the impedance approach is valid only if the initial conditions involved are all zero. Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit. This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Deriving transfer functions of electrical circuits with the use of complex impedances. The transfer function of an electrical circuit can be obtained as a ratio of complex impedances. For the circuit shown in Figure 6–20, assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the transfer function of this circuit can be obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)I(s)}{Z_1(s)I(s) + Z_2(s)I(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

where $I(s)$ is the Laplace transform of the current $i(t)$ in the circuit.

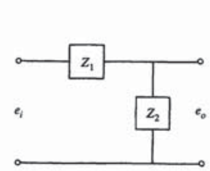


Figure 6–20 Electrical circuit.

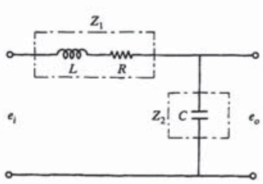


Figure 6–21 Electrical circuit.

For the circuit shown in Figure 6–21,

$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

Hence, the transfer function $E_o(s)/E_i(s)$ is

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{Cs}}{Ls + R + \dfrac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Example 6–5

Consider the system shown in Figure 6–22. Obtain the transfer function $E_o(s)/E_i(s)$ by the complex-impedance approach. (Capacitances C_1 and C_2 are not charged initially.)

The circuit shown in Figure 6–22 can be redrawn as that shown in Figure 6–23(a), which can be further modified to Figure 6–23(b).

In the system shown in Figure 6–23(b), the current I is divided into two currents I_1 and I_2. Noting that

$$Z_2I_1 = (Z_3 + Z_4)I_2, \qquad I_1 + I_2 = I$$

we obtain

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4}I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4}I$$

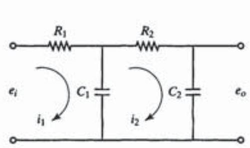


Figure 6–22 Electrical circuit.

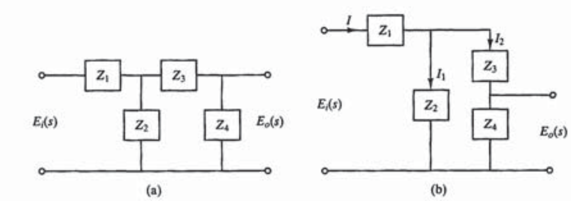


Figure 6–23 (a) The circuit of Figure 6–22 shown in terms of impedances; (b) equivalent circuit diagram.

Observing that

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}\right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we get

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation yields

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C_1 s}\frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

which is the transfer function of the system. [Notice that it is the same as that given by Equation (6–23).]

6–4 ANALOGOUS SYSTEMS

Systems that can be represented by the same mathematical model, but that are physically different, are called *analogous* systems. Thus, analogous systems are described by the same differential or integrodifferential equations or transfer functions.

The concept of analogous systems is useful in practice, for the following reasons:

1. The solution of the equation describing one physical system can be directly applied to analogous systems in any other field.

2. Since one type of system may be easier to handle experimentally than another, instead of building and studying a mechanical system (or a hydraulic system, pneumatic system, or the like), we can build and study its electrical analog, for electrical or electronic systems are, in general, much easier to deal with experimentally.

This section presents analogies between mechanical and electrical systems.

Mechanical–electrical analogies. Mechanical systems can be studied through their electrical analogs, which may be more easily constructed than models of the corresponding mechanical systems. There are two electrical analogies for mechanical systems: the force–voltage analogy and the force–current analogy.

Force–voltage analogy. Consider the mechanical system of Figure 6–24(a) and the electrical system of Figure 6–24(b). In the mechanical system p is the external force, and in the electrical system e is the voltage source. The equation for the mechanical system is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = p \tag{6–24}$$

where x is the displacement of mass m, measured from the equilibrium position. The equation for the electrical system is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e$$

In terms of the electric charge q, this last equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e \tag{6–25}$$

Comparing Equations (6–24) and (6–25), we see that the differential equations for the two systems are of identical form. Thus, these two systems are analogous systems.

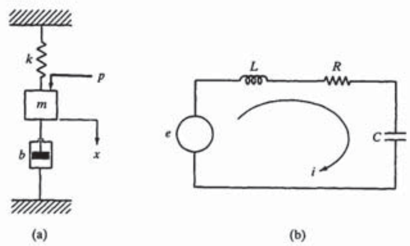


Figure 6–24 Analogous mechanical and electrical systems.

TABLE 6–1 Force–Voltage Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Voltage e
Mass m (moment of inertia J)	Inductance L
Viscous-friction coefficient b	Resistance R
Spring constant k	Reciprocal of capacitance, $1/C$
Displacement x (angular displacement θ)	Charge q
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Current i

The terms that occupy corresponding positions in the differential equations are called *analogous quantities*, a list of which appears in Table 6–1. The analogy here is called the *force–voltage analogy* (or *mass–inductance analogy*).

Force–current analogy. Another analogy between mechanical and electrical systems is based on the force–current analogy. Consider the mechanical system shown in Figure 6–25(a), where p is the external force. The system equation is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = p \tag{6–26}$$

where x is the displacement of mass m, measured from the equilibrium position.

Consider next the electrical system shown in Figure 6–25(b), where i_s is the current source. Applying Kirchhoff's current law gives

$$i_L + i_R + i_C = i_s \tag{6–27}$$

where

$$i_L = \frac{1}{L}\int e\,dt, \qquad i_R = \frac{e}{R}, \qquad i_C = C\frac{de}{dt}$$

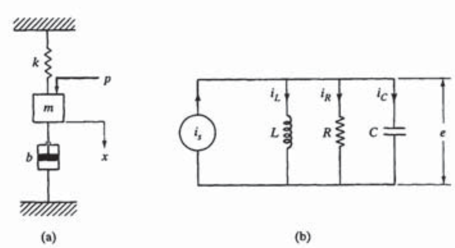


Figure 6–25 Analogous mechanical and electrical systems.

Thus, Equation (6–27) can be written as

$$\frac{1}{L}\int e\,dt + \frac{e}{R} + C\frac{de}{dt} = i_s \tag{6–28}$$

Since the magnetic flux linkage ψ is related to the voltage e by the equation

$$\frac{d\psi}{dt} = e$$

Equation (6–28) can be written in terms of ψ as

$$C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi = i_s \tag{6–29}$$

Comparing Equations (6–26) and (6–29), we find that the two systems are analogous. The analogous quantities are listed in Table 6–2. The analogy here is called the *force–current analogy* (or *mass–capacitance analogy*).

Comments. Analogies between two systems break down if the regions of operation are extended too far. In other words, since the mathematical models on which the analogies are based are only approximations to the dynamic characteristics of physical systems, the analogy may break down if the operating region of one system is very wide. Nevertheless, even if the operating region of a given mechanical system is wide, it can be divided into two or more subregions, and analogous electrical systems can be built for each subregion.

Analogy, of course, is not limited to mechanical–electrical analogy, but includes any physical or nonphysical system. Systems having an identical transfer function (or identical mathematical model) are analogous systems. (The transfer function is one of the simplest and most concise forms of mathematical models available today.)

Analogous systems exhibit the same output in response to the same input. For any given physical system, the mathematical response can be given a physical interpretation.

The concept of analogy is useful in applying well-known results in one field to another. It proves particularly useful when a given physical system (mechanical, hydraulic, pneumatic, and so on) is complicated, so that analyzing an analogous electrical circuit first is advantageous. Such an analogous electrical circuit can be built physically or can be simulated on the digital computer.

TABLE 6–2 Force–Current Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Current i
Mass m (moment of inertia J)	Capacitance C
Viscous-friction coefficient b	Reciprocal of resistance, $1/R$
Spring constant k	Reciprocal of inductance, $1/L$
Displacement x (angular displacement θ)	Magnetic flux linkage ψ
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Voltage e

Example 6–6

Obtain the transfer functions of the systems shown in Figures 6–26(a) and (b), and show that these systems are analogous.

For the mechanical system shown in Figure 6–26(a), the equation of motion is

$$b(\dot{x}_i - \dot{x}_o) = kx_o$$

or

$$b\dot{x}_i = kx_o + b\dot{x}_o$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$bsX_i(s) = (k + bs)X_o(s)$$

Hence, the transfer function between $X_o(s)$ and $X_i(s)$ is

$$\frac{X_o(s)}{X_i(s)} = \frac{bs}{bs + k} = \frac{\frac{b}{k}s}{\frac{b}{k}s + 1}$$

For the electrical system shown in Figure 6–26(b), we have

$$\frac{E_o(s)}{E_i(s)} = \frac{RCs}{RCs + 1}$$

Comparing the transfer functions obtained, we see that the two systems are analogous. (Note that both b/k and RC have the dimension of time and are time constants of the respective systems.)

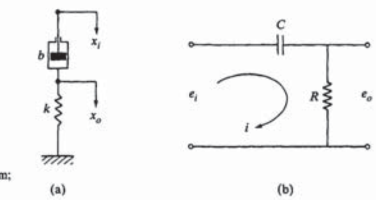


Figure 6–26 (a) Mechanical system; (b) analogous electrical system.

6–5 MATHEMATICAL MODELING OF ELECTROMECHANICAL SYSTEMS

In this section, we obtain mathematical models of dc servomotors. To control the motion or speed of dc servomotors, we control the field current or armature current or we use a servodriver as a motor–driver combination. There are many different

types of servodrivers. Most are designed to control the speed of dc servomotors, which improves the efficiency of operating servomotors. Here, however, we shall discuss only armature control of a dc servomotor and obtain its mathematical model in the form of a transfer function.

Armature control of dc servomotors. Consider the armature-controlled dc servomotor shown in Figure 6–27, where the field current is held constant. In this system,

R_a = armature resistance, Ω
L_a = armature inductance, H
i_a = armature current, A
i_f = field current, A
e_a = applied armature voltage, V
e_b = back emf, V
θ = angular displacement of the motor shaft, rad
T = torque developed by the motor, N-m
J = moment of inertia of the motor and load referred to the motor shaft, kg-m^2
b = viscous-friction coefficient of the motor and load referred to the motor shaft, N-m/rad/s

The torque T developed by the motor is proportional to the product of the armature current i_a and the air gap flux ψ, which in turn is proportional to the field current, or

$$\psi = K_f i_f$$

where K_f is a constant. The torque T can therefore be written as

$$T = K_f i_f K_1 i_a$$

where K_1 is a constant.

For a constant field current, the flux becomes constant and the torque becomes directly proportional to the armature current, so

$$T = Ki_a$$

where K is a motor-torque constant. Notice that if the sign of the current i_a is reversed, the sign of the torque T will be reversed, which will result in a reversal of the direction of rotor rotation.

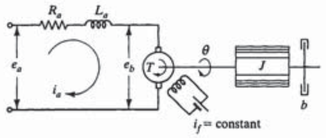


Figure 6–27 Armature-controlled dc servomotor.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage e_b is directly proportional to the angular velocity $d\theta/dt$, or

$$e_b = K_b\frac{d\theta}{dt} \tag{6–30}$$

where e_b is the back emf and K_b is a back-emf constant.

The speed of an armature-controlled dc servomotor is controlled by the armature voltage e_a. The differential equation for the armature circuit is

$$L_a\frac{di_a}{dt} + R_a i_a + e_b = e_a \tag{6–31}$$

The armature current produces the torque that is applied to the inertia and friction; hence,

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = T = Ki_a \tag{6–32}$$

Assuming that all initial conditions are zero and taking the Laplace transforms of Equations (6–30), (6–31), and (6–32), we obtain the following equations:

$$K_b s\Theta(s) = E_b(s) \tag{6–33}$$

$$(L_a s + R_a)I_a(s) + E_b(s) = E_a(s) \tag{6–34}$$

$$(Js^2 + bs)\Theta(s) = T(s) = KI_a(s) \tag{6–35}$$

Considering $E_a(s)$ as the input and $\Theta(s)$ as the output and eliminating $I_a(s)$ and $E_b(s)$ from Equations (6–33), (6–34), and (6–35), we obtain the transfer function for the dc servomotor:

$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s[L_aJs^2 + (L_ab + R_aJ)s + R_ab + KK_b]} \tag{6–36}$$

The inductance L_a in the armature circuit is usually small and may be neglected. If L_a is neglected, then the transfer function given by Equation (6–36) reduces to

$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s(R_aJs + R_ab + KK_b)} = \frac{\dfrac{K}{R_aJ}}{s\left(s + \dfrac{R_ab + KK_b}{R_aJ}\right)} \tag{6–37}$$

Notice that the term $(R_ab + KK_b)/(R_aJ)$ in Equation (6–37) corresponds to the damping term. Thus, the back emf increases the effective damping of the system. Equation (6–37) may be rewritten as

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_m}{s(T_ms + 1)} \tag{6–38}$$

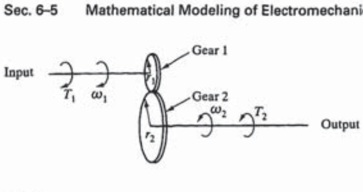


Figure 6–28 Gear train system.

where

$K_m = K/(R_ab + KK_b)$ = motor gain constant
$T_m = R_aJ/(R_ab + KK_b)$ = motor time constant

Equation (6–38) is the transfer function of the dc servomotor when the armature voltage $e_a(t)$ is the input and the angular displacement $\theta(t)$ is the output. Since the transfer function involves the term $1/s$, this system possesses an integrating property. (Notice that the time constant T_m of the motor becomes smaller as the resistance R_a is reduced and the moment of inertia J is made smaller.)

Gear train. Gear trains are frequently used in mechanical systems to reduce speed, to magnify torque, or to obtain the most efficient power transfer by matching the driving member to the given load. Figure 6–28 illustrates a simple gear train system in which the gear train transmits motion and torque from the input member to the output member. If the radii of gear 1 and gear 2 are r_1 and r_2, respectively, and the numbers of teeth on gear 1 and gear 2 are n_1 and n_2, respectively, then

$$\frac{r_1}{r_2} = \frac{n_1}{n_2}$$

Because the surface speeds at the point of contact of the two gears must be identical, we have

$$r_1\omega_1 = r_2\omega_2$$

where ω_1 and ω_2 are the angular velocities of gear 1 and gear 2, respectively. Therefore,

$$\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

If we neglect friction loss, the gear train transmits the power unchanged. In other words, if the torque applied to the input shaft is T_1 and the torque transmitted to the output shaft is T_2, then

$$T_1\omega_1 = T_2\omega_2$$

Example 6–7

Consider the system shown in Figure 6–29. Here, a load is driven by a motor through the gear train. Assuming that the stiffness of the shafts of the gear train is infinite, that there is neither backlash nor elastic deformation, and that the number of teeth on each gear is

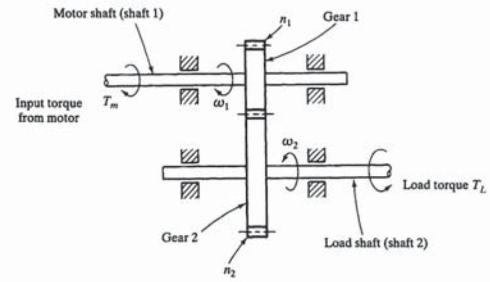


Figure 6–29 Gear train system.

proportional to the radius of the gear, find the equivalent inertia and equivalent friction referred to the motor shaft (shaft 1) and those referred to the load shaft (shaft 2). The numbers of teeth on gear 1 and gear 2 are n_1 and n_2, respectively, and the angular velocities of shaft 1 and shaft 2 are ω_1 and ω_2, respectively. The inertia and viscous friction coefficient of each gear train component are denoted by J_1, b_1 and J_2, b_2, respectively.

By applying Newton's second law to this system, the following two equations can be derived: For the motor shaft (shaft 1),

$$J_1\dot{\omega}_1 + b_1\omega_1 + T_1 = T_m \tag{6–39}$$

where T_m is the torque developed by the motor and T_1 is the load torque on gear 1 due to the rest of the gear train. For the load shaft (shaft 2),

$$J_2\dot{\omega}_2 + b_2\omega_2 + T_L = T_2 \tag{6–40}$$

where T_2 is the torque transmitted to gear 2 and T_L is the load torque. Since the gear train transmits the power unchanged, we have

$$T_1\omega_1 = T_2\omega_2$$

or

$$T_1 = T_2\frac{\omega_2}{\omega_1} = T_2\frac{n_1}{n_2}$$

If $n_1/n_2 < 1$, the gear ratio reduces the speed in addition to magnifying the torque. Eliminating T_1 and T_2 from Equations (6–39) and (6–40) yields

$$J_1\dot{\omega}_1 + b_1\omega_1 + \frac{n_1}{n_2}(J_2\dot{\omega}_2 + b_2\omega_2 + T_L) = T_m \tag{6–41}$$

Since $\omega_2 = (n_1/n_2)\omega_1$, eliminating ω_2 from Equation (6–41) gives

$$\left[J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\right]\dot{\omega}_1 + \left[b_1 + \left(\frac{n_1}{n_2}\right)^2 b_2\right]\omega_1 + \left(\frac{n_1}{n_2}\right)T_L = T_m \tag{6–42}$$

Thus, the equivalent inertia and equivalent viscous friction coefficient of the gear train referred to shaft 1 are given by

$$J_{1\,\mathrm{eq}} = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2, \qquad b_{1\,\mathrm{eq}} = b_1 + \left(\frac{n_1}{n_2}\right)^2 b_2$$

The effect of J_2 on the equivalent inertia $J_{1\,\mathrm{eq}}$ is determined by the gear ratio n_1/n_2. For speed-reducing gear trains, the ratio n_1/n_2 is much smaller than unity. If $n_1/n_2 \ll 1$, then the effect of J_2 on the equivalent inertia $J_{1\,\mathrm{eq}}$ is negligible. Similar comments apply to the equivalent friction of the gear train.

In terms of the equivalent inertia $J_{1\,\mathrm{eq}}$ and equivalent viscous friction coefficient $b_{1\,\mathrm{eq}}$, Equation (6–42) can be simplified to give

$$J_{1\,\mathrm{eq}}\dot{\omega}_1 + b_{1\,\mathrm{eq}}\omega_1 + nT_L = T_m$$

where $n = n_1/n_2$.

The equivalent inertia and equivalent viscous friction coefficient of the gear train referred to shaft 2 are

$$J_{2\,\mathrm{eq}} = J_2 + \left(\frac{n_2}{n_1}\right)^2 J_1, \qquad b_{2\,\mathrm{eq}} = b_2 + \left(\frac{n_2}{n_1}\right)^2 b_1$$

So the relationship between $J_{1\,\mathrm{eq}}$ and $J_{2\,\mathrm{eq}}$ is

$$J_{1\,\mathrm{eq}} = \left(\frac{n_1}{n_2}\right)^2 J_{2\,\mathrm{eq}}$$

and that between $b_{1\,\mathrm{eq}}$ and $b_{2\,\mathrm{eq}}$ is

$$b_{1\,\mathrm{eq}} = \left(\frac{n_1}{n_2}\right)^2 b_{2\,\mathrm{eq}}$$

and Equation (6–42) can be modified to give

$$J_{2\,\mathrm{eq}}\dot{\omega}_2 + b_{2\,\mathrm{eq}}\omega_2 + T_L = \frac{1}{n}T_m$$

Example 6–8

Consider the dc servomotor system shown in Figure 6–30. The armature inductance is negligible and is not shown in the circuit. Obtain the transfer function between the output θ_2 and the input e_a. In the diagram,

R_a = armature resistance, Ω
i_a = armature current, A

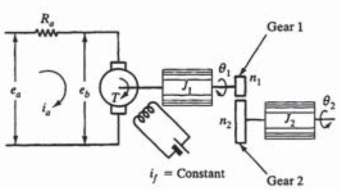


Figure 6–30 DC servomotor system.

i_f = field current, A
e_a = applied armature voltage, V
e_b = back emf, V
θ_1 = angular displacement of the motor shaft, rad
θ_2 = angular displacement of the load element, rad
T = torque developed by the motor, N-m
J_1 = moment of inertia of the rotor of the motor, kg-m²
J_2 = moment of inertia of the load, kg-m²
n_1 = number of teeth on gear 1
n_2 = number of teeth on gear 2

The torque T developed by the dc servomotor is

$$T = Ki_a$$

where K is the motor torque constant. The induced voltage e_b is proportional to the angular velocity $\dot{\theta}_1$, or

$$e_b = K_b\frac{d\theta_1}{dt} \tag{6–43}$$

where K_b is the back-emf constant.

The equation for the armature circuit is

$$R_a i_a + e_b = e_a \tag{6–44}$$

The equivalent moment of inertia of the motor rotor plus the load inertia referred to the motor shaft is

$$J_{1\,\mathrm{eq}} = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2$$

The armature current produces the torque that is applied to the equivalent moment of inertia $J_{1\,\mathrm{eq}}$. Thus,

$$J_{1\,\mathrm{eq}}\frac{d^2\theta_1}{dt^2} = T = Ki_a \tag{6–45}$$

Assuming that all initial conditions are zero and taking the Laplace transforms of Equations (6–43), (6–44), and (6–45), we obtain

$$E_b(s) = K_b s\Theta_1(s) \tag{6–46}$$

$$R_a I_a(s) + E_b(s) = E_a(s) \tag{6–47}$$

$$J_{1\,\mathrm{eq}}s^2\Theta_1(s) = KI_a(s) \tag{6–48}$$

Eliminating $E_b(s)$ and $I_a(s)$ from Equations (6–46), (6–47), and (6–48), we obtain

$$\left(J_{1\,\mathrm{eq}}s^2 + \frac{KK_b}{R_a}s\right)\Theta_1(s) = \frac{K}{R_a}E_a(s)$$

Noting that $\Theta_1(s)/\Theta_2(s) = n_2/n_1$, we can write this last equation as

$$\left(J_{1\,\mathrm{eq}}s^2 + \frac{KK_b}{R_a}s\right)\frac{n_2}{n_1}\Theta_2(s) = \frac{K}{R_a}E_a(s)$$

Hence, the transfer function $\Theta_2(s)/E_a(s)$ is given by

$$\frac{\Theta_2(s)}{E_a(s)} = \frac{\dfrac{n_1}{n_2}K}{\left\{R_a\left[J_1 + \left(\dfrac{n_1}{n_2}\right)^2 J_2\right]s + KK_b\right\}s}$$

6–6 MATHEMATICAL MODELING OF OPERATIONAL-AMPLIFIER SYSTEMS

In this section, we briefly discuss operational amplifiers. We present several examples of operational-amplifier systems and obtain their mathematical models.

Operational amplifiers, often called *op-amps*, are important building blocks in modern electronic systems. They are used in filters in control systems and to amplify signals in sensor circuits.

Consider the operational amplifier shown in Figure 6–31. There are two terminals on the input side, one with a minus sign and the other with a plus sign, called the *inverting* and *noninverting* terminals, respectively. We choose the ground as 0 volts and measure the input voltages e_1 and e_2 relative to the ground. (The input e_1 to the minus terminal of the amplifier is inverted; the input e_2 to the plus terminal is not inverted.) The total input to the amplifier is $e_2 - e_1$. The ideal operational amplifier has the characteristic

$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

where the inputs e_1 and e_2 may be dc or ac signals and K is the differential gain or voltage gain. The magnitude of K is approximately 10^5 to 10^6 for dc signals and ac signals with frequencies less than approximately 10 Hz. (The differential gain K decreases with the frequency of the signal and becomes about unity for frequencies of 1 MHz to about 50 MHz.) Note that the operational amplifier amplifies the difference in voltages e_1 and e_2. Such an amplifier is commonly called a *differential amplifier*. Since the gain of the operational amplifier is very high, the device is inherently unstable. To stabilize it, it is necessary to have negative feedback from the output to the input (feedback from the output to the inverted input).

In the ideal operational amplifier, no current flows into the input terminals and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In

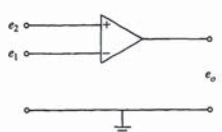


Figure 6–31 Operational amplifier.

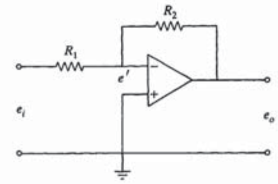


Figure 6–32 Operational-amplifier system.

an actual operational amplifier, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, however, we make the assumption that the operational amplifiers are ideal.

Inverting amplifier. Consider the operational-amplifier system shown in Figure 6–32. Assume that the magnitudes of the resistances R_1 and R_2 are of comparable order.

Let us obtain the voltage ratio e_o/e_i. In the derivation, we assume the voltage gain to be $K \gg 1$. Let us define the voltage at the minus terminal as e'. Ignoring the current flowing into the amplifier, we have

$$\frac{e_i - e'}{R_1} + \frac{e_o - e'}{R_2} = 0$$

from which we get

$$\frac{e_i}{R_1} + \frac{e_o}{R_2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e'$$

Thus,

$$e' = \frac{\dfrac{e_i}{R_1} + \dfrac{e_o}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2}} \tag{6–49}$$

Also,

$$e_o = -Ke' \tag{6–50}$$

Eliminating e' from Equations (6–49) and (6–50), we obtain

$$-\frac{e_o}{K} = \frac{\dfrac{e_i}{R_1} + \dfrac{e_o}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

or

$$e_o\left(-\frac{1}{KR_1} - \frac{1}{KR_2} - \frac{1}{R_2}\right) = \frac{e_i}{R_1}$$

Hence,

$$\frac{e_o}{e_i} = \frac{-\dfrac{R_2}{R_1}}{1 + \dfrac{1 + \dfrac{R_2}{R_1}}{K}}$$

Since $K \gg 1 + (R_2/R_1)$, we have

$$\frac{e_o}{e_i} = -\frac{R_2}{R_1} \tag{6–51}$$

Equation (6–51) gives the relationship between the output voltage e_o and the input voltage e_i. From Equations (6–49) and (6–51) we have

$$e' = \frac{\dfrac{e_i}{R_1} + \dfrac{e_o}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2}} = 0$$

In an operational-amplifier circuit, when the output signal is fed back to the minus terminal, the voltage at the minus terminal becomes equal to the voltage at the plus terminal. This is called an *imaginary short*. If we use the concept of an imaginary short, the ratio e_o/e_i can be obtained much more quickly than the way we just found it, as the following analysis shows:

Consider again the amplifier system shown in Figure 6–32, and define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to the current i_2. Thus,

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

Because the output signal is fed back to the minus terminal, the voltage at the minus terminal and the voltage at the plus terminal become equal, or $e' = 0$. Hence, we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2}$$

or

$$e_o = -\frac{R_2}{R_1}e_i$$

This is a mathematical model relating voltages e_o and e_i. We obtained the same result as we got in the previous analysis [see Equation (6–51)], but much more quickly.

Note that the sign of the output voltage e_o is the negative of that of the input voltage e_i. Hence, this operational amplifier is called an *inverted amplifier*. If $R_1 = R_2$, then the circuit is a sign inverter.

Obtaining mathematical models of physical operational-amplifier systems by means of equations for idealized operational-amplifier systems. In the remaining part of this section, we derive mathematical models of operational-amplifier systems, using the following three conditions that apply to idealized operational-amplifier systems:

1. From Figure 6–31, the output voltage e_o is the differential input voltage $(e_2 - e_1)$ multiplied by the differential gain K. That is,

   $$e_o = K(e_2 - e_1)$$

   where K is infinite. In designing active filters, we construct the circuit such that the negative feedback appears in the operational amplifier like the system shown in Figure 6–32. As a result, the differential input voltage becomes zero, and we have

   Voltage at negative terminal = Voltage at positive terminal

2. The input impedance is infinite.
3. The output impedance is zero.

The use of these three conditions simplifies the derivation of transfer functions of operational-amplifier systems. The derived transfer functions are, of course, not exact, but are approximations that are sufficiently accurate.

In what follows, we shall derive the characteristics of circuits consisting of operational amplifiers, resistors, and capacitors.

Example 6–9

Consider the operational-amplifier circuit shown in Figure 6–33. Obtain the relationship between e_o and e_i.

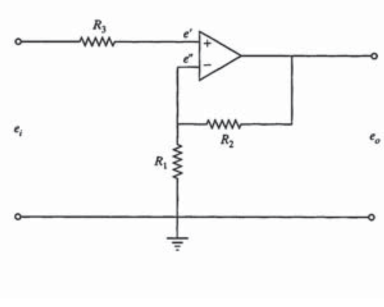


Figure 6–33 Operational-amplifier circuit.

If the operational amplifier is an ideal one, then the output voltage e_o is limited and the differential input voltage becomes zero, or voltage $e'(= e_i)$ and voltage e'', which is equal to $[R_1/(R_1 + R_2)]e_o$ are equal. Thus,

$$e_i = \frac{R_1}{R_1 + R_2}e_o$$

from which it follows that

$$e_o = \left(1 + \frac{R_2}{R_1}\right)e_i$$

This operational-amplifier circuit is a noninverting circuit. If we choose $R_1 = \infty$, then $e_o = e_i$, and the circuit is called a *voltage follower*.

Example 6–10

Consider the operational-amplifier circuit shown in Figure 6–34. Obtain the relationship between the output e_o and the inputs e_1, e_2, and e_3.

We define

$$i_1 = \frac{e_1 - e'}{R_1}, \qquad i_2 = \frac{e_2 - e'}{R_2}, \qquad i_3 = \frac{e_3 - e'}{R_3}, \qquad i_4 = \frac{e' - e_o}{R_4}$$

Noting that the current flowing into the amplifier is negligible, we have

$$\frac{e_1 - e'}{R_1} + \frac{e_2 - e'}{R_2} + \frac{e_3 - e'}{R_3} + \frac{e_o - e'}{R_4} = 0 \tag{6–52}$$

Since the amplifier involves negative feedback, the voltage at the minus terminal and that at the plus terminal become equal. Thus, $e' = 0$, and Equation (6–52) becomes

$$\frac{e_1}{R_1} + \frac{e_2}{R_2} + \frac{e_3}{R_3} + \frac{e_o}{R_4} = 0$$

or

$$e_o = -\frac{R_4}{R_1}e_1 - \frac{R_4}{R_2}e_2 - \frac{R_4}{R_3}e_3$$

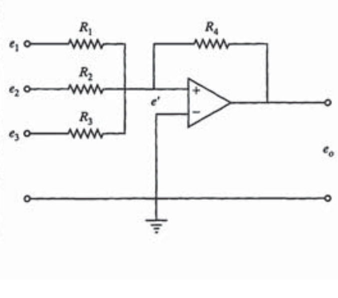


Figure 6–34 Operational-amplifier circuit.

If we choose $R_1 = R_2 = R_3 = R_4$, then

$$e_o = -(e_1 + e_2 + e_3)$$

The circuit is an *inverting adder*.

Example 6–11

Consider the operational-amplifier system shown in Figure 6–35. Letting $e_i(t)$ be the input and $e_o(t)$ be the output of the system, obtain the transfer function for the system. Then obtain the response of the system to a step input of a small magnitude.

Let us define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d(e' - e_o)}{dt}, \qquad i_3 = \frac{e' - e_o}{R_2}$$

Noting that the current flowing into the amplifier is negligible, we have

$$i_1 = i_2 + i_3$$

Hence,

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e' - e_o}{R_2} \tag{6–53}$$

Since the operational amplifier involves negative feedback, the voltage at the minus terminal and that at the plus terminal become equal. Hence, $e' = 0$. Substituting $e' = 0$ into Equation (6–53), we obtain

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2}$$

Taking the Laplace transform of this last equation, assuming a zero initial condition, we have

$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s)$$

which can be written as

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1} \tag{6–54}$$

Equation (6–54) is the transfer function for the system, which is a first-order lag system.

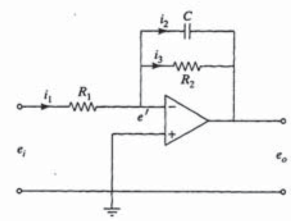


Figure 6–35 First-order lag circuit using an operational amplifier.

Next, we shall find the response of the system to a step input. Suppose that the input $e_i(t)$ is a step function of E volts; that is,

$$\begin{aligned} e_i(t) &= 0 \qquad \text{for } t < 0 \\ &= E \qquad \text{for } t > 0 \end{aligned}$$

where we assume $0 < (R_2/R_1)E < 10$ V. The output $e_o(t)$ can be determined from

$$\begin{aligned} E_o(s) &= -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}E_i(s) \\ &= -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}\frac{E}{s} \\ &= -\frac{R_2E}{R_1}\left[\frac{1}{s} - \frac{1}{s + 1/(R_2C)}\right] \end{aligned}$$

The inverse Laplace transform of $E_o(s)$ gives

$$e_o(t) = -\frac{R_2E}{R_1}[1 - e^{-t/(R_2C)}]$$

The output voltage reaches $-(R_2/R_1)E$ volts as t increases to infinity.

Example 6–12

Consider the operational-amplifier circuit shown in Figure 6–36. Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit.

The voltage at point A is

$$e_A = \frac{1}{2}(e_i + e_o)$$

The Laplace-transformed version of this last equation is

$$E_A(s) = \frac{1}{2}[E_i(s) + E_o(s)]$$

The voltage at point B is

$$E_B(s) = \frac{\dfrac{1}{Cs}}{R_2 + \dfrac{1}{Cs}}E_i(s) = \frac{1}{R_2Cs + 1}E_i(s)$$

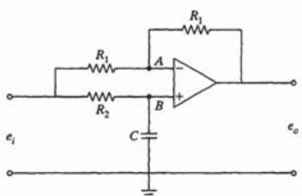


Figure 6–36 Operational-amplifier circuit.

Since the operational amplifier involves negative feedback, the voltage at the minus terminal and that at the plus terminal become equal. Thus,

$$E_A(s) = E_B(s)$$

and it follows that

$$\frac{1}{2}[E_i(s) + E_o(s)] = \frac{1}{R_2Cs + 1}E_i(s)$$

or

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2Cs - 1}{R_2Cs + 1} = -\frac{s - \dfrac{1}{R_2C}}{s + \dfrac{1}{R_2C}}$$

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–6–1

Obtain the resistance between points A and B of the circuit shown in Figure 6–37.

Solution This circuit is equivalent to the one shown in Figure 6–38(a). Since $R_1 = R_4 = 10\ \Omega$ and $R_2 = R_3 = 20\ \Omega$, the voltages at points C and D are equal, and there is no current flowing through R_5. Because resistance R_5 does not affect the value of the total resistance between points A and B, it may be removed from the circuit, as shown in Figure 6–38(b). Then

$$\frac{1}{R_{AB}} = \frac{1}{R_1 + R_4} + \frac{1}{R_2 + R_3} = \frac{1}{20} + \frac{1}{40} = \frac{3}{40}$$

and

$$R_{AB} = \frac{40}{3} = 13.3\ \Omega$$

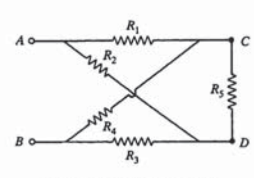


Figure 6–37 Electrical circuit.

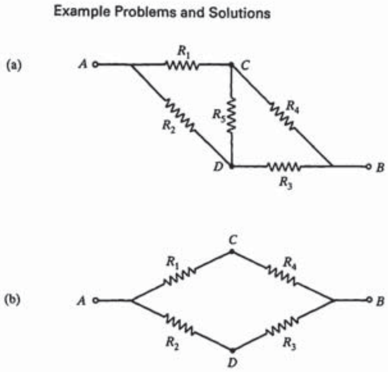


Figure 6–38 Equivalent circuits to the one shown in Figure 6–37.

Problem A–6–2

Given the circuit of Figure 6–39, calculate currents i_1, i_2, and i_3.

Solution The circuit can be redrawn as shown in Figure 6–40. The combined resistance R of the path in which current i_2 flows is

$$R = 100 + \frac{1}{\dfrac{1}{10} + \dfrac{1}{40}} + 50 = 158\ \Omega$$

The combined resistance R_0 as seen from the battery is

$$\frac{1}{R_0} = \frac{1}{40} + \frac{1}{158}$$

or

$$R_0 = 31.92\ \Omega$$

Consequently,

$$i_1 + i_2 = \frac{12}{R_0} = \frac{12}{31.92} = 0.376\ \text{A}$$

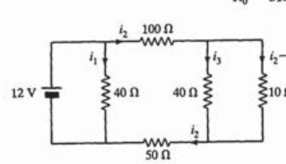


Figure 6–39 Electrical circuit.

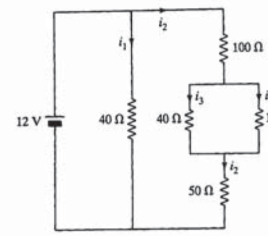


Figure 6–40 Equivalent circuit to the one shown in Figure 6–39.

Noting that $40i_1 = 158i_2$, we obtain

$$i_1 = 0.300 \text{ A}, \qquad i_2 = 0.076 \text{ A}$$

To determine i_3, note that

$$40i_3 = 10(i_2 - i_3)$$

Then

$$i_3 = \frac{10}{50}i_2 = 0.0152 \text{ A}$$

Problem A–6–3

Obtain the combined resistance between points A and B of the circuit shown in Figure 6–41, which consists of an infinite number of resistors connected in the form of a ladder.

Solution We define the combined resistance between points A and B as R_0. Now, let us separate the first three resistors from the rest. [See Figure 6–42(a).] Since the circuit consists of an infinite number of resistors, the removal of the first three resistors does not affect the combined resistance value. Therefore, the combined resistance between points C and D is the same as R_0. Then the circuit shown in Figure 6–41 may be redrawn as shown in Figure 6–42(b), and R_0, the resistance between points A and B, can be obtained as

$$R_0 = 2R + \frac{1}{\dfrac{1}{R} + \dfrac{1}{R_0}} = 2R + \frac{RR_0}{R_0 + R}$$

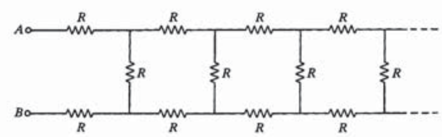


Figure 6–41 Electrical circuit consisting of an infinite number of resistors connected in the form of a ladder.

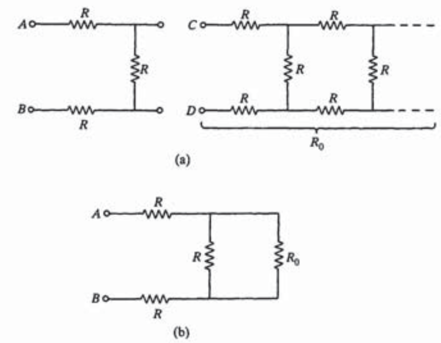


Figure 6–42 Equivalent circuits to the one shown in Figure 6–41.

Rewriting, we get

$$R_0^2 - 2RR_0 - 2R^2 = 0$$

Solving for R_0, we find that

$$R_0 = R \pm \sqrt{3}R$$

Finally, neglecting the negative value for resistance, we obtain

$$R_0 = R + \sqrt{3}R = 2.732R$$

Problem A–6–4

Find currents i_1, i_2, and i_3 for the circuit shown in Figure 6–43.

Solution Applying Kirchhoff's voltage law and current law to the circuit, we have

$$12 - 10i_1 - 5i_3 = 0$$
$$8 - 15i_2 - 5i_3 = 0$$
$$i_1 + i_2 - i_3 = 0$$

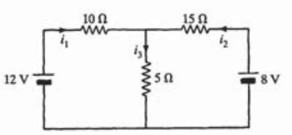


Figure 6–43 Electrical circuit.

Solving for i_1, i_2, and i_3 gives

$$i_1 = \frac{8}{11} \text{ A}, \qquad i_2 = \frac{12}{55} \text{ A}, \qquad i_3 = \frac{52}{55} \text{ A}$$

Since all i values are found to be positive, the currents actually flow in the directions shown in the diagram.

Problem A–6–5

Given the circuit shown in Figure 6–44, obtain a mathematical model. Here, currents i_1 and i_2 are cyclic currents.

Solution Applying Kirchhoff's voltage law gives

$$R_1 i_1 + \frac{1}{C}\int (i_1 - i_2)\, dt = E$$

$$L\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C}\int (i_2 - i_1)\, dt = 0$$

These two equations constitute a mathematical model for the circuit.

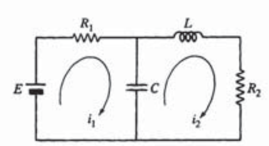


Figure 6–44 Electrical circuit.

Problem A–6–6

In the circuit of Figure 6–45, assume that, for $t < 0$, switch S is connected to voltage source E, and the current in coil L is in a steady state. At $t = 0$, S disconnects the voltage source and simultaneously short-circuits the coil. What is the current $i(t)$ for $t > 0$?

Solution For $t > 0$, the equation for the circuit is

$$L\frac{di}{dt} + Ri = 0, \qquad i(0) = \frac{E}{R}$$

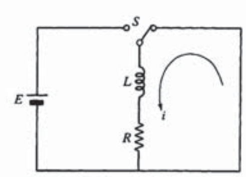


Figure 6–45 Electrical circuit.

[Note that there is a nonzero initial current $i(0-) = E/R$. Since inductance L stores energy, the current in the coil cannot be changed instantaneously. Hence, $i(0+) = i(0-) = i(0) = E/R$.]

Taking the Laplace transform of the system equation, we obtain

$$L[sI(s) - i(0)] + RI(s) = 0$$

or

$$(Ls + R)I(s) = Li(0) = \frac{LE}{R}$$

Thus,

$$I(s) = \frac{E}{R}\frac{L}{Ls + R}$$

The inverse Laplace transform of this last equation gives

$$i(t) = \frac{E}{R}e^{-(R/L)t}$$

Problem A–6–7

Consider the circuit shown in Figure 6–46, and assume that capacitor C is initially charged to q_0. At $t = 0$, switch S is disconnected from the battery and simultaneously connected to inductor L. The capacitance has a value of 50 μF. Calculate the value of the inductance L that will make the oscillation occur at a frequency of 200 Hz.

Solution The equation for the circuit for $t > 0$ is

$$L\frac{di}{dt} + \frac{1}{C}\int i\, dt = 0$$

or, by substituting $i = dq/dt$ into this last equation,

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$

where $q(0) = q_0$ and $\dot{q}(0) = 0$. The frequency of oscillation is

$$\omega_n = \sqrt{\frac{1}{LC}}$$

Since

$$200 \text{ Hz} = 200 \text{ cps} = 200 \times 6.28 \text{ rad/s} = 1256 \text{ rad/s}$$

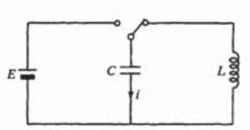


Figure 6–46 Electrical circuit.

we obtain

$$\omega_n = 1256 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{L \times 50 \times 10^{-6}}}$$

Thus,

$$L = \frac{1}{1256^2 \times 50 \times 10^{-6}} = 0.0127 \text{ H}$$

Problem A–6–8

In Figure 6–47(a), suppose that switch S is open for $t < 0$ and that the system is in a steady state. Switch S is closed at $t = 0$. Find the current $i(t)$ for $t \geq 0$.

Solution Notice that, for $t < 0$, the circuit resistance is $R_1 + R_2$. There is a nonzero initial current

$$i(0-) = \frac{E}{R_1 + R_2}$$

For $t \geq 0$, the circuit resistance becomes R_1. Because of the presence of inductance L, there is no instantaneous change in the current in the circuit when switch S is closed. Hence,

$$i(0+) = i(0-) = \frac{E}{R_1 + R_2} = i(0)$$

Therefore, the equation for the circuit for $t \geq 0$ is

$$L\frac{di}{dt} + R_1 i = E \tag{6–55}$$

where

$$i(0) = \frac{E}{R_1 + R_2}$$

Taking the Laplace transforms of both sides of Equation (6–55), we obtain

$$L[sI(s) - i(0)] + R_1 I(s) = \frac{E}{s}$$

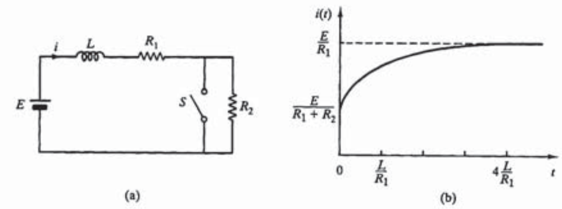


Figure 6–47 (a) Electrical circuit; (b) plot of $i(t)$ versus t of the circuit when switch S is closed at $t = 0$.

Substituting the initial condition $i(0)$ into this last equation and simplifying, we get

$$(Ls + R_1)I(s) = \frac{E}{s} + \frac{LE}{R_1 + R_2}$$

Hence,

$$\begin{aligned} I(s) &= \frac{E}{s(Ls + R_1)} + \frac{E}{R_1 + R_2}\frac{L}{Ls + R_1} \\ &= \frac{E}{R_1}\left(\frac{1}{s} - \frac{L}{Ls + R_1}\right) + \frac{E}{R_1 + R_2}\frac{L}{Ls + R_1} \\ &= \frac{E}{R_1}\left(\frac{1}{s} - \frac{R_2}{R_1 + R_2}\frac{L}{Ls + R_1}\right) \end{aligned}$$

Taking the inverse Laplace transform of this equation, we obtain

$$i(t) = \frac{E}{R_1}\left[1 - \frac{R_2}{R_1 + R_2}e^{-(R_1/L)t}\right]$$

A typical plot of $i(t)$ versus t is shown in Figure 6–47(b).

Problem A–6–9

In the electrical circuit shown in Figure 6–48, there is an initial charge q_0 on the capacitor just before switch S is closed at $t = 0$. Find the current $i(t)$.

Solution The equation for the circuit when switch S is closed is

$$Ri + \frac{1}{C}\int i\,dt = E$$

Taking the Laplace transform of this last equation yields

$$RI(s) + \frac{1}{C}\frac{I(s) + \int i(t)\,dt\Big|_{t=0}}{s} = \frac{E}{s}$$

Since

$$\int i(t)\,dt\Big|_{t=0} = q(0) = q_0$$

we obtain

$$RI(s) + \frac{1}{C}\frac{I(s) + q_0}{s} = \frac{E}{s}$$

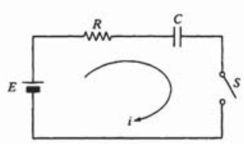


Figure 6–48 Electrical circuit.

or

$$RCsI(s) + I(s) + q_0 = CE$$

Solving for $I(s)$, we have

$$I(s) = \frac{CE - q_0}{RCs + 1} = \left(\frac{E}{R} - \frac{q_0}{RC}\right)\frac{1}{s + \frac{1}{RC}}$$

The inverse Laplace transform of this last equation gives

$$i(t) = \left(\frac{E}{R} - \frac{q_0}{RC}\right)e^{-t/RC}$$

Problem A–6–10

Obtain the impedances of the circuits shown in Figures 6–49(a) and (b).

Solution Consider the circuit shown in Figure 6–49(a). From

$$E(s) = E_L(s) + E_R(s) + E_C(s) = \left(Ls + R + \frac{1}{Cs}\right)I(s)$$

where $I(s)$ is the Laplace transform of the current $i(t)$ in the circuit, the complex impedance is

$$Z(s) = \frac{E(s)}{I(s)} = Ls + R + \frac{1}{Cs}$$

For the circuit shown in Figure 6–49(b),

$$I(s) = \frac{E(s)}{Ls} + \frac{E(s)}{R} + \frac{E(s)}{1/(Cs)} = E(s)\left(\frac{1}{Ls} + \frac{1}{R} + Cs\right)$$

Consequently,

$$Z(s) = \frac{E(s)}{I(s)} = \frac{1}{\frac{1}{Ls} + \frac{1}{R} + Cs}$$

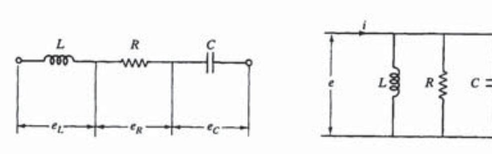


Figure 6–49 Electrical circuits.

Problem A–6–11

Find the transfer function $E_o(s)/E_i(s)$ of the electrical circuit shown in Figure 6–50. Obtain the voltage $e_o(t)$ when the input voltage $e_i(t)$ is a step change of voltage E_i occurring at $t = 0$. Assume that $e_i(0-) = 0$. Assume also that the initial charges in the capacitors are zero. [Thus, $e_o(0-) = 0$.]

Solution With the complex-impedance method, the transfer function $E_o(s)/E_i(s)$ can be obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{(1/R_2) + C_2 s}}{\dfrac{1}{C_1 s} + \dfrac{1}{(1/R_2) + C_2 s}} = \frac{R_2 C_1 s}{R_2(C_1 + C_2)s + 1}$$

Next, we determine $e_o(t)$. For the input $e_i(t) = E_i \cdot 1(t)$, we have

$$\begin{aligned} E_o(s) &= \frac{R_2 C_1 s}{R_2(C_1 + C_2)s + 1}\frac{E_i}{s} \\ &= \frac{R_2 C_1 E_i}{R_2(C_1 + C_2)s + 1} \end{aligned}$$

Inverse Laplace transforming $E_o(s)$, we get

$$e_o(t) = \frac{C_1 E_i}{C_1 + C_2}e^{-t/[R_2(C_1 + C_2)]}$$

from which it follows that $e_o(0+) = C_1 E_i/(C_1 + C_2)$.

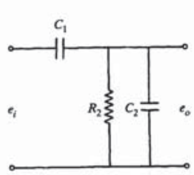


Figure 6–50 Electrical circuit.

Problem A–6–12

Derive the transfer function $E_o(s)/E_i(s)$ of the electrical circuit shown in Figure 6–51. The input voltage is a pulse signal given by

$$\begin{aligned} e_i(t) &= 10 \text{ V} \qquad 0 \leq t \leq 5 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

Obtain the output $e_o(t)$. Assume that the initial charges in the capacitors C_1 and C_2 are zero. Assume also that $C_2 = 1.5\,C_1$ and $R_1 C_1 = 1$ s.

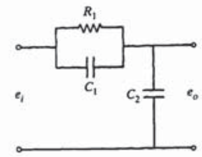


Figure 6–51 Electrical circuit.

Solution By the use of the complex-impedance method, the transfer function $E_o(s)/E_i(s)$ can be obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{C_2 s}}{\dfrac{R_1}{R_1 C_1 s + 1} + \dfrac{1}{C_2 s}} = \frac{R_1 C_1 s + 1}{R_1(C_1 + C_2)s + 1} = \frac{s + 1}{2.5\,s + 1}$$

For the given input $e_i(t)$, we have

$$E_i(s) = \frac{10}{s}(1 - e^{-5s})$$

Thus, the response $E_o(s)$ can be given by

$$\begin{aligned} E_o(s) &= \frac{s+1}{2.5\,s+1}\,\frac{10}{s}(1 - e^{-5s}) \\ &= \left(\frac{10}{s} - \frac{15}{2.5\,s + 1}\right)(1 - e^{-5s}) \end{aligned}$$

The inverse Laplace transform of $E_o(s)$ gives

$$\begin{aligned} e_o(t) &= (10 - 6\,e^{-0.4t}) \\ &\quad - [10 - 6\,e^{-0.4(t-5)}]\,1(t - 5) \end{aligned}$$

Figure 6–52 shows a possible response curve $e_o(t)$ versus t.

Problem A–6–13

Obtain the transfer functions $E_o(s)/E_i(s)$ of the bridged T networks shown in Figures 6–53(a) and (b).

Solution The bridged T networks shown can both be represented by the network of Figure 6–54(a), which uses complex impedances. This network may be modified to that shown in Figure 6–54(b), in which

$$I_1 = I_2 + I_3, \qquad I_2 Z_1 = (Z_3 + Z_4) I_3$$

Hence,

$$I_2 = \frac{Z_3 + Z_4}{Z_1 + Z_3 + Z_4} I_1, \qquad I_3 = \frac{Z_1}{Z_1 + Z_3 + Z_4} I_1$$

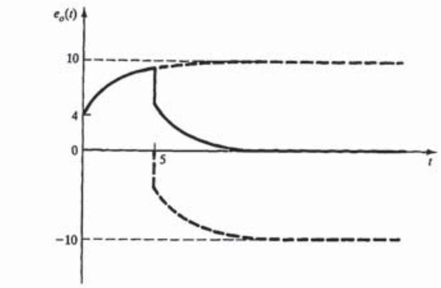


Figure 6–52 Response curve $e_o(t)$ versus t.

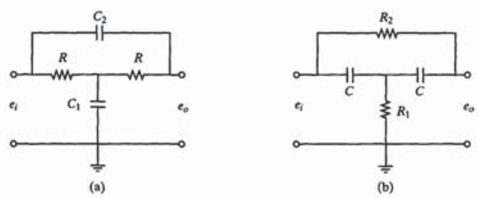


Figure 6–53 Bridged T networks.

Then the voltages $E_i(s)$ and $E_o(s)$ can be obtained as

$$\begin{aligned} E_i(s) &= Z_1 I_2 + Z_2 I_1 \\ &= \left[Z_2 + \frac{Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4}\right] I_1 \\ &= \frac{Z_2(Z_1 + Z_3 + Z_4) + Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \\ E_o(s) &= Z_3 I_3 + Z_2 I_1 \\ &= \frac{Z_3 Z_1}{Z_1 + Z_3 + Z_4} I_1 + Z_2 I_1 \\ &= \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \end{aligned}$$

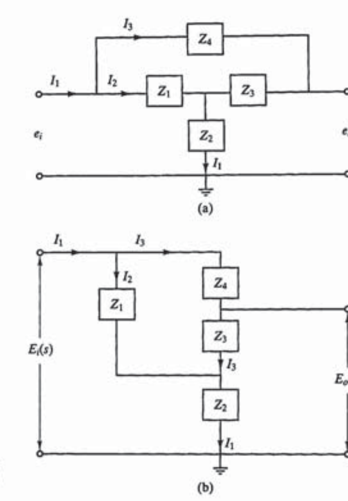


Figure 6–54 (a) Bridged T network in terms of complex impedances; (b) equivalent network.

Thus, the transfer function of the network shown in Figure 6–54(a) is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_2(Z_1 + Z_3 + Z_4) + Z_1 Z_3 + Z_1 Z_4} \tag{6–56}$$

For the bridged T network shown in Figure 6–53(a), we substitute

$$Z_1 = R, \qquad Z_2 = \frac{1}{C_1 s}, \qquad Z_3 = R, \qquad Z_4 = \frac{1}{C_2 s}$$

into Equation (6–56). Then we obtain the transfer function

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R^2 + \dfrac{1}{C_1 s}\left(R + R + \dfrac{1}{C_2 s}\right)}{\dfrac{1}{C_1 s}\left(R + R + \dfrac{1}{C_2 s}\right) + R^2 + R\dfrac{1}{C_2 s}} \\ &= \frac{RC_1 RC_2 s^2 + 2RC_2 s + 1}{RC_1 RC_2 s^2 + (2RC_2 + RC_1)s + 1} \end{aligned}$$

Similarly, for the bridged T network shown in Figure 6–53(b), we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_1, \qquad Z_3 = \frac{1}{Cs}, \qquad Z_4 = R_2$$

into Equation (6–56). Then we obtain the transfer function

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\dfrac{1}{Cs}\dfrac{1}{Cs} + R_1\left(\dfrac{1}{Cs} + \dfrac{1}{Cs} + R_2\right)}{R_1\left(\dfrac{1}{Cs} + \dfrac{1}{Cs} + R_2\right) + \dfrac{1}{Cs}\dfrac{1}{Cs} + R_2\dfrac{1}{Cs}} \\ &= \frac{R_1 C R_2 C s^2 + 2R_1 C s + 1}{R_1 C R_2 C s^2 + (2R_1 C + R_2 C)s + 1} \end{aligned}$$

Problem A–6–14

Consider the electrical circuit shown in Figure 6–55. Assume that voltage e_i is the input and voltage e_o is the output of the circuit. Derive a state equation and an output equation.

Solution The transfer function for the system is

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R_2 + \dfrac{1}{C_2 s}}{\left(\dfrac{R_1}{R_1 C_1 s + 1}\right) + \left(R_2 + \dfrac{1}{C_2 s}\right)} \\ &= \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_1 C_2 s + (R_1 C_1 s + 1)(R_2 C_2 s + 1)} \\ &= \frac{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2)s + 1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1} \end{aligned} \tag{6–57}$$

Hence, we have

$$\begin{aligned} &[R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1]E_o(s) \\ &\quad = [R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2)s + 1]E_i(s) \end{aligned}$$

The inverse Laplace transform of this last equation gives

$$\begin{aligned} &R_1 C_1 R_2 C_2 \ddot{e}_o + (R_1 C_1 + R_2 C_2 + R_1 C_2)\dot{e}_o + e_o \\ &\quad = R_1 C_1 R_2 C_2 \ddot{e}_i + (R_1 C_1 + R_2 C_2)\dot{e}_i + e_i \end{aligned}$$

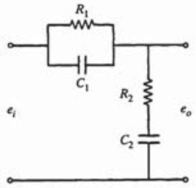


Figure 6–55 Electrical circuit.

By dividing each term of the preceding equation by $R_1C_1R_2C_2$ and defining $e_o = y$ and $e_i = u$, we obtain

$$\ddot{y} + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)\dot{y} + \frac{1}{R_1C_1R_2C_2}y = \ddot{u} + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)\dot{u} + \frac{1}{R_1C_1R_2C_2}u \tag{6–58}$$

To derive a state equation and an output equation based on Method 1 given in Section 5–4, we first compare Equation (6–58) with the following standard second-order equation:

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

We then identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}$$
$$a_2 = \frac{1}{R_1C_1R_2C_2}$$
$$b_0 = 1$$
$$b_1 = \frac{1}{R_1C_1} + \frac{1}{R_2C_2}$$
$$b_2 = \frac{1}{R_1C_1R_2C_2}$$

From Equations (5–23), (5–24), and (5–29), we have

$$\beta_0 = b_0 = 1$$
$$\beta_1 = b_1 - a_1\beta_0 = b_1 - a_1 = -\frac{1}{R_2C_1}$$
$$\beta_2 = b_2 - a_2\beta_0 - a_1\beta_1 = \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)\frac{1}{R_2C_1}$$

If we define state variables x_1 and x_2 as

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$

then, from Equations (5–30) and (5–31), the state-space representation for the system can be given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{R_1C_1R_2C_2} & -\frac{1}{R_1C_1} - \frac{1}{R_2C_2} - \frac{1}{R_2C_1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix} -\frac{1}{R_2C_1} \\ \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)\frac{1}{R_2C_1}\end{bmatrix}u$$

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + u$$

Problem A–6–15

Show that the mechanical and electrical systems illustrated in Figure 6–56 are analogous. Assume that the displacement x in the mechanical system is measured from the equilibrium position and that mass m is released from the initial displacement $x(0) = x_0$ with zero initial velocity, or $\dot{x}(0) = 0$. Assume also that in the electrical system the capacitor has the initial charge $q(0) = q_0$ and that the switch is closed at $t = 0$. Note that $\dot{q}(0) = i(0) = 0$. Obtain $x(t)$ and $q(t)$.

Solution The equation of motion for the mechanical system is

$$m\ddot{x} + kx = 0 \tag{6–59}$$

For the electrical system,

$$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0$$

or, by substituting $i = dq/dt = \dot{q}$ into this last equation,

$$L\ddot{q} + \frac{1}{C}q = 0 \tag{6–60}$$

Since Equations (6–59) and (6–60) are of the same form, the two systems are analogous (i.e., they satisfy the force–voltage analogy).

The solution of Equation (6–59) with the initial condition $x(0) = x_0$, $\dot{x}(0) = 0$ is a simple harmonic motion given by

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}t$$

Similarly, the solution of Equation (6–60) with the initial condition $q(0) = q_0$, $\dot{q}(0) = 0$ is

$$q(t) = q_0 \cos\sqrt{\frac{1}{LC}}t$$

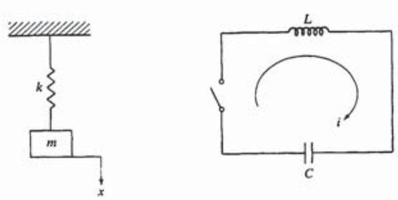


Figure 6–56 Analogous mechanical and electrical systems.

Problem A–6–16

Obtain mathematical models for the systems shown in Figures 6–57(a) and (b), and show that they are analogous systems. In the mechanical system, displacements x_1 and x_2 are measured from their respective equilibrium positions.

Solution For the mechanical system shown in Figure 6–57(a), the equations of motion are

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$
$$b_2\dot{x}_2 + k_2(x_2 - x_1) = 0$$

These two equations constitute a mathematical model for the mechanical system.

For the electrical system shown in Figure 6–57(b), the loop-voltage equations are

$$L_1\frac{di_1}{dt} + \frac{1}{C_2}\int (i_1 - i_2)\,dt + R_1i_1 + \frac{1}{C_1}\int i_1\,dt = 0$$
$$R_2i_2 + \frac{1}{C_2}\int (i_2 - i_1)\,dt = 0$$

Let us write $i_1 = \dot{q}_1$ and $i_2 = \dot{q}_2$. Then, in terms of q_1 and q_2, the preceding two equations can be written

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + \frac{1}{C_1}q_1 + \frac{1}{C_2}(q_1 - q_2) = 0$$
$$R_2\dot{q}_2 + \frac{1}{C_2}(q_2 - q_1) = 0$$

These two equations constitute a mathematical model for the electrical system.

Comparing the two mathematical models, we see that the two systems are analogous. (i.e., they satisfy the force–voltage analogy).

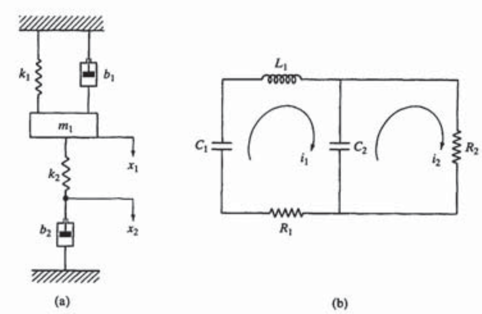


Figure 6–57 Analogous mechanical and electrical systems.

Problem A–6–17

Using the force–voltage analogy, obtain an electrical analog of the mechanical system shown in Figure 6–58. Assume that the displacements x_1 and x_2 are measured from their respective equilibrium positions.

Solution The equations of motion for the mechanical system are

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + b_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = 0$$
$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

With the use of the force–voltage analogy, the equations for an analogous electrical system may be written

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + \frac{1}{C_1}q_1 + R_2(\dot{q}_1 - \dot{q}_2) + \frac{1}{C_2}(q_1 - q_2) = 0$$
$$L_2\ddot{q}_2 + R_2(\dot{q}_2 - \dot{q}_1) + \frac{1}{C_2}(q_2 - q_1) = 0$$

Substituting $\dot{q}_1 = i_1$ and $\dot{q}_2 = i_2$ into the last two equations gives

$$L_1\frac{di_1}{dt} + R_1i_1 + \frac{1}{C_1}\int i_1\,dt + R_2(i_1 - i_2) + \frac{1}{C_2}\int (i_1 - i_2)\,dt = 0 \tag{6–61}$$
$$L_2\frac{di_2}{dt} + R_2(i_2 - i_1) + \frac{1}{C_2}\int (i_2 - i_1)\,dt = 0 \tag{6–62}$$

These two equations are loop-voltage equations. From Equation (6–61), we obtain the diagram shown in Figure 6–59(a). Similarly, from Equation (6–62), we obtain the one given in Figure 6–59(b). Combining these two diagrams produces the desired analogous electrical system (Figure 6–60).

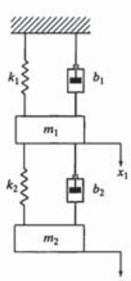


Figure 6–58 Mechanical system.

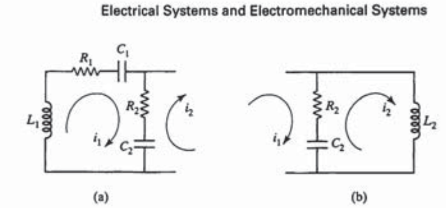


Figure 6–59 (a) Electrical circuit corresponding to Equation (6–61); (b) electrical circuit corresponding to Equation (6–62).

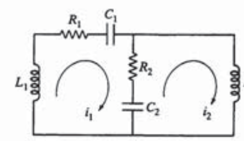


Figure 6–60 Electrical system analogous to the mechanical system shown in Figure 6–58 (force–voltage analogy).

Problem A–6–18

Figure 6–61 shows an inertia load driven by a dc servomotor by means of pulleys and a belt. Obtain the equivalent moment of intertia, J_{eq}, of the system with respect to the motor shaft axis. Assume that there is no slippage between the belt and the pulleys.

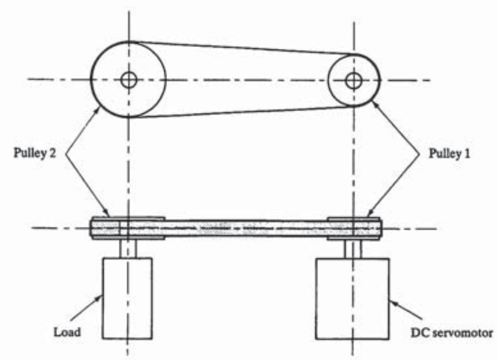


Figure 6–61 Inertia load driven by a dc servomotor by means of pulleys and belt.

Assume also that the diameters of pulleys 1 and 2 are d_1 and d_2, respectively. The moment of inertia of the rotor of the motor is J_r and that of the load element is J_L. The moments of inertia of pulleys 1 and 2 are J_1 and J_2, respectively. Neglect the moment of inertia of the belt.

Solution The given system uses a belt and two pulleys as a drive device. The system acts similarly to a gear train system. Since we assume no slippage between the belt and pulleys, the work done by the belt and pulley 1$(T_1\theta_1)$ is equal to that done by the belt and pulley 2 $(T_2\theta_2)$, or

$$T_1\theta_1 = T_2\theta_2 \tag{6–63}$$

where T_1 is the load torque on the motor shaft, θ_1 is the angular displacement of pulley 1, T_2 is the torque transmitted to the load shaft, and θ_2 is the angular displacement of pulley 2. Note that

$$\frac{\theta_2}{\theta_1} = \frac{d_1}{d_2} \tag{6–64}$$

For the servomotor system,

$$(J_1 + J_r)\ddot{\theta}_1 + T_1 = T_m \tag{6–65}$$

where T_m is the torque developed by the motor. For the load shaft,

$$(J_L + J_2)\ddot{\theta}_2 = T_2 \tag{6–66}$$

From Equations (6–63) and (6–64), we have

$$T_2 = T_1\frac{\theta_1}{\theta_2} = T_1\frac{d_2}{d_1}$$

Then Equation (6–66) becomes

$$(J_L + J_2)\ddot{\theta}_2 = T_1\frac{d_2}{d_1} \tag{6–67}$$

From Equations (6–65) and (6–67), we obtain

$$(J_1 + J_r)\ddot{\theta}_1 + \frac{d_1}{d_2}(J_L + J_2)\ddot{\theta}_2 = T_m$$

Since $\theta_2 = (d_1/d_2)\theta_1$, this last equation can be written as

$$(J_1 + J_r)\ddot{\theta}_1 + \left(\frac{d_1}{d_2}\right)^2(J_L + J_2)\ddot{\theta}_1 = T_m$$

or

$$\left[(J_1 + J_r) + (J_L + J_2)\left(\frac{d_1}{d_2}\right)^2\right]\ddot{\theta}_1 = T_m$$

The equivalent moment of inertia of the system with respect to the motor shaft axis is thus given by

$$J_{eq} = J_1 + J_r + (J_L + J_2)\left(\frac{d_1}{d_2}\right)^2$$

Problem A–6–19

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–62.

Solution Define the voltage at point A as e_A. Then

$$\frac{E_A(s)}{E_i(s)} = \frac{R_1}{\frac{1}{Cs} + R_1} = \frac{R_1Cs}{R_1Cs + 1}$$

Define the voltage at point B as e_B. Then

$$E_B(s) = \frac{R_3}{R_2 + R_3}E_o(s)$$

In this operational-amplifier system, negative feedback appears in the operational amplifier. As a result, the differential input voltage becomes zero, and we have $E_A(s) = E_B(s)$. Hence,

$$E_A(s) = \frac{R_1Cs}{R_1Cs + 1}E_i(s) = E_B(s) = \frac{R_3}{R_2 + R_3}E_o(s)$$

from which we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + R_3}{R_3}\,\frac{R_1Cs}{R_1Cs + 1}$$

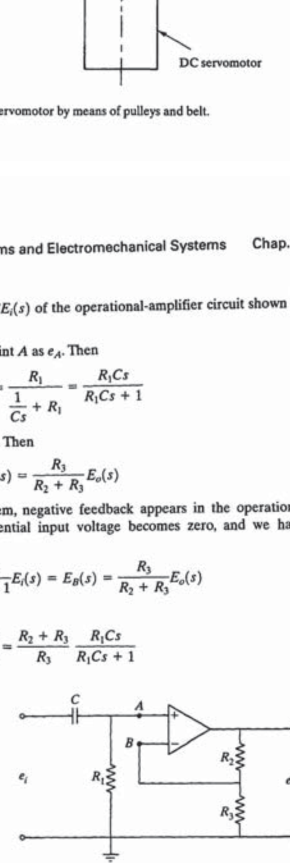


Figure 6–62 Operational-amplifier circuit.

Problem A–6–20

Consider the operational-amplifier circuit shown in Figure 6–63. Obtain the transfer function of this circuit by the complex-impedance approach.

Solution For the circuit shown, we have

$$\frac{E_i(s) - E'(s)}{Z_1} = \frac{E'(s) - E_o(s)}{Z_2}$$

Since the operational amplifier involves negative feedback, the differential input voltage becomes zero. Hence, $E'(s) = 0$. Thus,

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

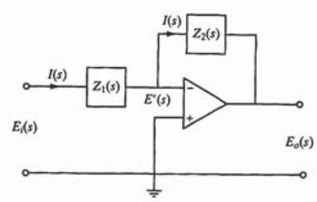


Figure 6–63 Operational-amplifier circuit.

Problem A–6–21

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–64 by the complex-impedance approach.

Solution The complex impedances for this circuit are

$$Z_1(s) = R_1 \qquad \text{and} \qquad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

From **Problem A–6–20**, the transfer function of the system is

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\,\frac{1}{R_2Cs + 1}$$

Notice that the circuit considered here is the same as that discussed in **Example 6–11**. Accordingly, the transfer function $E_o(s)/E_i(s)$ obtained here is, of course, the same as the one obtained in that example.

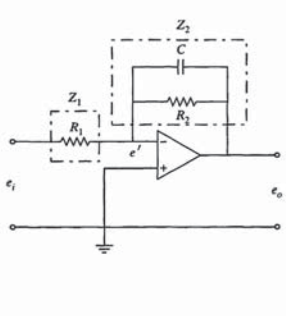


Figure 6–64 Operational-amplifier circuit.

Problem A–6–22

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–65.

Solution We shall first obtain currents i_1, i_2, i_3, i_4, and i_5. Then we shall use the node equation at nodes A and B. The currents are

$$i_1 = \frac{e_i - e_A}{R_1}, \qquad i_2 = \frac{e_A - e_o}{R_3}, \qquad i_3 = C_1\frac{de_A}{dt}$$

$$i_4 = \frac{e_A - 0}{R_2}, \qquad i_5 = C_2\frac{d(0 - e_o)}{dt}$$

At node A, we have $i_1 = i_2 + i_3 + i_4$, or

$$\frac{e_i - e_A}{R_1} = \frac{e_A - e_o}{R_3} + C_1\frac{de_A}{dt} + \frac{e_A}{R_2} \tag{6–68}$$

At node B, we have $e_B = 0$, and no current flows into the amplifier. Thus, we get $i_4 = i_5$, or

$$\frac{e_A}{R_2} = C_2\frac{-de_o}{dt} \tag{6–69}$$

Rewriting Equation (6–68), we have

$$C_1\frac{de_A}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)e_A = \frac{e_i}{R_1} + \frac{e_o}{R_3} \tag{6–70}$$

From Equation (6–69), we get

$$e_A = -R_2C_2\frac{de_o}{dt} \tag{6–71}$$

Substituting Equation (6–71) into Equation (6–70), we obtain

$$C_1\left(-R_2C_2\frac{d^2e_o}{dt^2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)\frac{de_o}{dt} = \frac{e_i}{R_1} + \frac{e_o}{R_3}$$

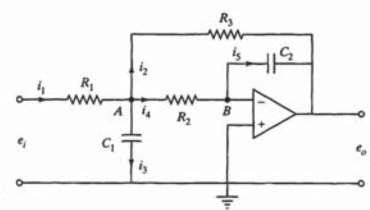


Figure 6–65 Operational-amplifier circuit.

Laplace transforming this last equation, assuming zero initial conditions, yields

$$-C_1C_2R_2s^2E_o(s) + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)(-R_2C_2)sE_o(s) - \frac{1}{R_3}E_o(s) = \frac{E_i(s)}{R_1}$$

from which we get the transfer function $E_o(s)/E_i(s)$:

$$\frac{E_o(s)}{E_i(s)} = -\frac{1}{R_1C_1R_2C_2s^2 + [R_2C_2 + R_1C_2 + (R_1/R_3)R_2C_2]s + (R_1/R_3)}$$

Problem A–6–23

Obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 6–66 in terms of the complex impedances Z_1, Z_2, Z_3, and Z_4. Using the equation derived, obtain the transfer function $E_o(s)/E_i(s)$ of the op-amp circuit shown in Figure 6–36.

Solution From Figure 6–66, we find that

$$\frac{E_i(s) - E_A(s)}{Z_3} = \frac{E_A(s) - E_o(s)}{Z_4}$$

or

$$E_i(s) - \left(1 + \frac{Z_3}{Z_4}\right)E_A(s) = -\frac{Z_3}{Z_4}E_o(s) \tag{6–72}$$

Since the system involves negative feedback, we have $E_A(s) = E_B(s)$, or

$$E_A(s) = E_B(s) = \frac{Z_1}{Z_1 + Z_2}E_i(s) \tag{6–73}$$

By substituting Equation (6–73) into Equation (6–72), we obtain

$$\left[\frac{Z_4Z_1 + Z_4Z_2 - Z_4Z_1 - Z_3Z_1}{Z_4(Z_1 + Z_2)}\right]E_i(s) = -\frac{Z_3}{Z_4}E_o(s)$$

from which we get the transfer function

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_4Z_2 - Z_3Z_1}{Z_3(Z_1 + Z_2)} \tag{6–74}$$

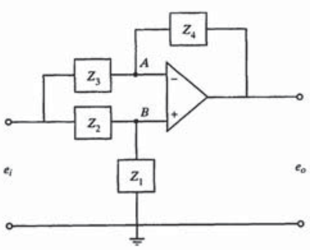


Figure 6–66 Operational-amplifier circuit.

To find the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 6–36, we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_2, \qquad Z_3 = R_1, \qquad Z_4 = R_1$$

into Equation (6–74). The result is

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_1R_2 - R_1\dfrac{1}{Cs}}{R_1\left(\dfrac{1}{Cs} + R_2\right)} = -\frac{R_2Cs - 1}{R_2Cs + 1}$$

which is, as a matter of course, the same as that obtained in **Example 6–12**.

PROBLEMS

Problem B–6–1

Three resistors R_1, R_2, and R_3 are connected in a triangular shape (Figure 6–67). Obtain the resistance between points A and B.

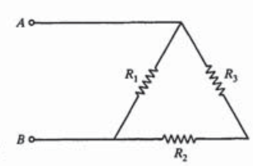


Figure 6–67 Three resistors connected in a triangular shape.

Problem B–6–2

Calculate the resistance between points A and B for the circuit shown in Figure 6–68.

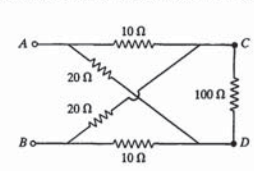


Figure 6–68 Electrical circuit.

Problem B–6–3

In the circuit of Figure 6–69, assume that a voltage E is applied between points A and B and that the current i is i_0 when switch S is open. When switch S is closed, i becomes equal to $2i_0$. Find the value of the resistance R.

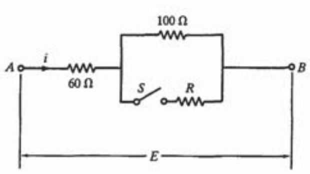


Figure 6–69 Electrical circuit.

Problem B–6–4

Obtain a mathematical model of the circuit shown in Figure 6–70.

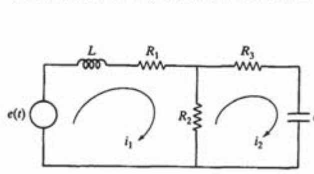


Figure 6–70 Electrical circuit.

Problem B–6–5

Consider the circuit shown in Figure 6–71. Assume that switch S is open for $t < 0$ and that capacitor C is initially charged so that the initial voltage $q(0)/C = e_0$ appears on the capacitor. Calculate cyclic currents i_1 and i_2 when switch S is closed at $t = 0$.

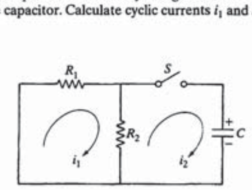


Figure 6–71 Electrical circuit.

Problem B–6–6

The circuit shown in Figure 6–72 is in a steady state with switch S closed. Switch S is then opened at $t = 0$. Obtain $i(t)$.

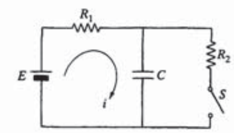


Figure 6–72 Electrical circuit.

Problem B–6–7

Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 6–73.

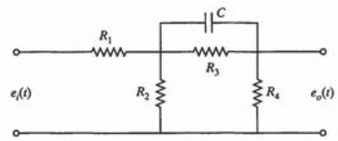


Figure 6–73 Electrical circuit.

Problem B–6–8

Obtain the transfer function $E_o(s)/E_i(s)$ of the system shown in Figure 6–74.

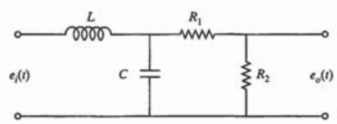


Figure 6–74 Electrical circuit.

Problem B–6–9

Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 6–75.

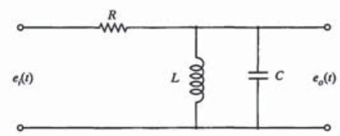


Figure 6–75 Electrical circuit.

Problem B–6–10

Obtain the transfer function $E_o(s)/E_i(s)$ of the electrical circuit shown in Figure 6–76.

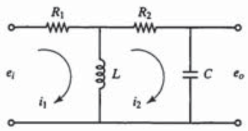


Figure 6–76 Electrical circuit.

Problem B–6–11

Determine the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 6–77. Use the complex-impedance method.

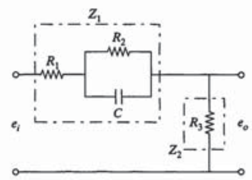


Figure 6–77 Electrical circuit.

Problem B–6–12

Obtain the transfer function $E_o(s)/E_i(s)$ of the circuit shown in Figure 6–78. Use the complex-impedance method.

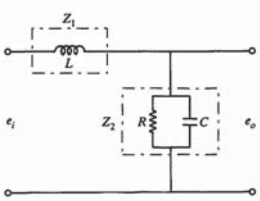


Figure 6–78 Electrical circuit.

Problem B–6–13

Obtain a state-space representation for the electrical circuit shown in Figure 6–79. Assume that voltage e_i is the input and voltage e_o is the output of the system.

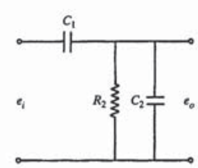


Figure 6–79 Electrical circuit.

Problem B–6–14

In the circuit shown in Figure 6–80, define $i_1 = \dot{q}_1$ and $i_2 = \dot{q}_2$, where q_1 and q_2 are charges in capacitors C_1 and C_2, respectively. Write equations for the circuit. Then obtain a state equation for the system by choosing state variables x_1, x_2, and x_3 as follows:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$

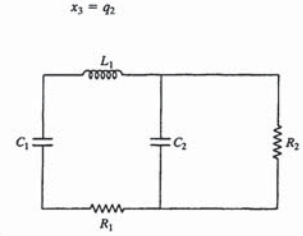


Figure 6–80 Electrical circuit.

Problem B–6–15

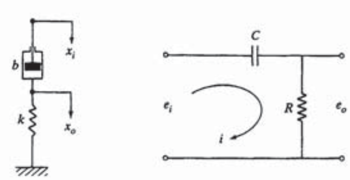
Show that the mechanical system illustrated in Figure 6–81(a) is analogous to the electrical system depicted in Figure 6–81(b).

Problem B–6–16

Derive the transfer function of the electrical circuit shown in Figure 6–82. Draw a schematic diagram of an analogous mechanical system.

Problem B–6–17

Obtain a mechanical system analogous to the electrical system shown in Figure 6–83.



(a) (b)

Figure 6–81 (a) Mechanical system; (b) analogous electrical system.

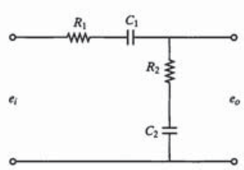


Figure 6–82 Electrical circuit.

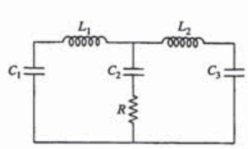


Figure 6–83 Electrical system.

Problem B–6–18

Determine an electrical system analogous to the mechanical system shown in Figure 6–84, where $p(t)$ is the force input to the system. The displacements x_1 and x_2 are measured from their respective equilibrium positions.

Problem B–6–19

Consider the dc servomotor shown in Figure 6–85. Assume that the input of the system is the applied armature voltage e_a and the output is the load shaft position θ_2. Assume also the following numerical values for the constants:

$$R_a = \text{armature winding resistance} = 0.2\ \Omega$$
$$L_a = \text{armature winding inductance} = \text{negligible}$$
$$K_b = \text{back-emf constant} = 5.5 \times 10^{-2}\ \text{V-s/rad}$$

K = motor-torque constant = 6×10^{-5} lb$_f$-ft/A
J_r = moment of inertia of the rotor of the motor = 1×10^{-5} lb$_f$-ft-s^2
b_r = viscous-friction coefficient of the rotor of the motor = negligible
J_L = moment of inertia of the load = 4.4×10^{-3} lb$_f$-ft-s^2
b_L = viscous-friction coefficient of the load = 4×10^{-2} lb$_f$-ft/rad/s
n = gear ratio = N_1/N_2 = 0.1

Obtain the transfer function $\Theta_2(s)/E_a(s)$.

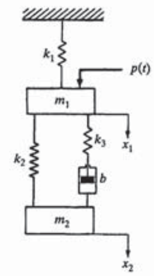


Figure 6–84 Mechanical system.

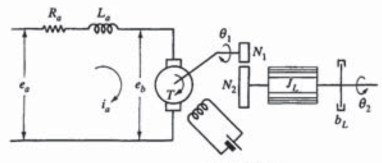


Figure 6–85 DC servomotor.

Problem B–6–20

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–86.

Problem B–6–21

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–87.

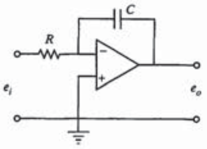


Figure 6–86 Operational-amplifier circuit.

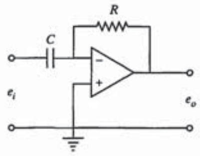


Figure 6–87 Operational-amplifier circuit.

Problem B–6–22

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–88.

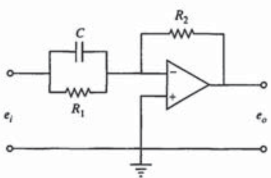


Figure 6–88 Operational-amplifier circuit.

Problem B–6–23

Obtain a state-space representation of the operational-amplifier circuit shown in Figure 6–89.

Problem B–6–24

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–90.

Problem B–6–25

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–91.

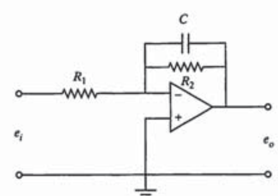


Figure 6–89 Operational-amplifier circuit.

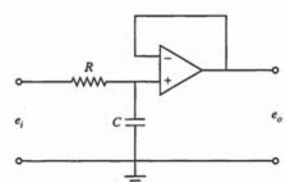


Figure 6–90 Operational-amplifier circuit.

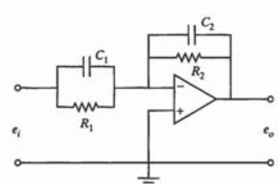


Figure 6–91 Operational-amplifier circuit.

Problem B–6–26

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–92.

Problem B–6–27

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–93.

Problem B–6–28

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–94.

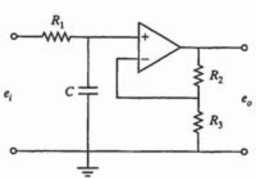


Figure 6–92 Operational-amplifier circuit.

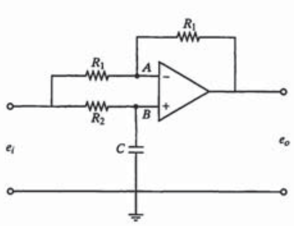


Figure 6–93 Operational-amplifier circuit.

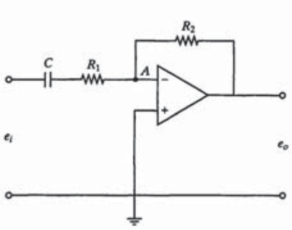


Figure 6–94 Operational-amplifier circuit.

Problem B–6–29

Obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–95.

Problem B–6–30

Using the impedance approach, obtain the transfer function $E_o(s)/E_i(s)$ of the operational-amplifier circuit shown in Figure 6–96.

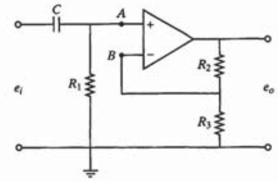


Figure 6–95 Operational-amplifier circuit.

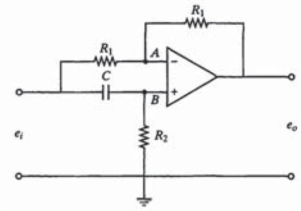


Figure 6–96 Operational-amplifier circuit.

Problem B–6–31

Obtain the output voltage e_o of the operational-amplifier circuit shown in Figure 6–97 in terms of the input voltages e_1 and e_2.

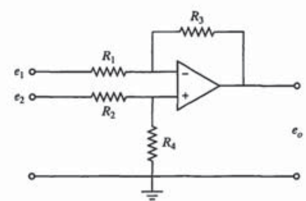


Figure 6–97 Operational-amplifier circuit.