Replacement Problems

Introduction : The replacement problems are concerned with the situations that arise when some items such as machines, electric equipment, need to replace due to their decreased efficiency, failure or breakdown. This decreased efficiency or complete breakdown may be either gradual or sudden.

Following are some situations that demand replacement of certain items :

(i) The old item has become inefficient or required expensive maintenance.

(ii) The old item has failed due to an accident or expected to fail shortly.

(iii) A better design of equipment has been developed, making the older design obsolescent.

Objectives: The problem of replacement is to decide the best policy in determining a time at which the replacement is most economical, instead of continuing at an increased cost. The main objective of replacement is to direct the organization in maximizing its profit (or minimizing the cost).

Replacement Models:

(I) Replacement of items that deteriorate with time.

(a) Value of Money does not change with time.

(b) Value of Money change with time.

(II) Replacement of Equipment that fails suddenly.

(a) Individual Replacement Policy.

(b) Group Replacement Policy.

@ Start Practicing

Replacement Policy : If the running or maintenance cost of the machine for the next year is more than the average annual cost of the selected year, then replace the machine at the end of the selected year.

Some related terms :

C : Capital cost of machine.

S : Scrap value or resale value of machine

N : Number of years the machine would be in use.

F(t) : Maintenance cost

A(n) : Average annual cost

Total cost = Capital Cost − Scrap Value + Maintenance Cost
= Depreciation cost + M.C

Depreciation cost = Capital cost − Scrap Value.

@Start Practicing

Replacement Problem

①
Q: The cost of a machine is ₹61,000 and its scrap value is ₹1000. The maintenance costs found from the past experience are as follows :

Year	1	2	3	4	5	6	7	8
Maintenance Cost in rupees	1000	2500	4000	6000	9000	12000	16000	20000

When should the machine be replaced?

Solution : We are given the maintenance cost in table.

Scrap value = ₹1000 and

Cost of the machine = ₹61000.

In order to determine the optimal time when the machine should be replaced, we calculate an average annual cost.

The total cost per year during the life of the machine is shown in the following table.

Year (1)	Running cost (2)	Cumulative running cost (3)	Depreciation cost (C-S) (4)	Total cost TC = 3+4 (5)	Average annual cost 6 = 5/1 (6)
1	1000	1000	61000-1000 = 60000	61000	61000
2	2500	3500	60000	63500	31750
3	4000	7500	60000	67500	22500
4	6000	13500	60000	73500	18375
5	9000	22500	60000	82500	16500
6	12000	34500	60000	94500	$\boxed{15750}$
7	16000	50500	60000	110500	15785.71
8	20000	70500	60000	130500	16312.5

@start Practicing

From the table, it is noted that the average annual cost is minimum in 6^{th} year (15750).

Running cost for next year > Average annual cost for selected year.

$$16000 > 15750$$

Then we can replace the machine at end of 6^{th} year.

@Start Practicing

Replacement Problem

(2)
Q: A machine cost ₹10,000. Its operating cost and resale values are given below.

Year	1	2	3	4	5	6	7	8
operating cost	1000	1200	1400	1700	2000	2500	3000	3500
Resale value	6000	4000	3200	2600	2500	2400	2000	1600

Determine at what time it should be replaced?

Solution: Given, the cost of the machine = 10 000

To determine the optimal time, when the machine should be replaced, we calculate an average total cost per year as shown in following table.

Year ①	Running cost ②	Cumulative Running cost ③	Resale value (S) ④	Depreciation cost (C-S) ⑤ (10000 - S)	Total cost 6 = 5+3	Average annual cost 7 = 6/1
1	1000	1000	6000	4000	5000	5000
2	1200	2200	4000	6000	8200	4100
3	1400	3600	3200	6800	10400	3466.7
4	1700	5300	2600	7400	12700	3175
5	2000	7300	2500	7500	14800	2960
6	2500	9800	2400	7600	17400	$\boxed{2900}$
7	3000	12800	2000	8000	20800	2971.4
8	3500	16300	1600	8400	24700	3087.5

from this table, it is clear that average annual cost is minimum at the end of 6^{th} year (2900)

Running cost the machine for the next year = 3000

Average annual cost for selected year = 2900

3000 > 2900

We conclude that the equipment should be replace at the end of 6^{th} year.

Replacement Problem

Value of Money changes with Time

Discount rate:

$V = (1+r)^{-1}$, where r is called rate of interest and V is called discount rate.

Discount factor, (V^{n-1}), n = years.

$$W(n) = \frac{C + \sum_{n=1} R_n V^{n-1} - S_n V^n}{\sum_{n=1} V^{n-1}}$$

$W(n)$ = Weighted average cost of all previous n year.

C = Capital cost

R_n = Running Cost.

S_n = Salvage value

V^{n-1} = Discount Factor.

Replacement Problem

Value of Money changes with time

(3)

Q: A machine costs ₹15,000. The running cost for the different years are given below.

Year	1	2	3	4	5	6	7
Running	2500	3000	4000	5000	6500	8000	10000

find the optimal replacement period if the capital is worth 10% per annum and has no salvage value.

Solution: Since money is worth 10% per year.
The discount rate will be, $V = (1+r)^{-1} = \left(1+\frac{10}{100}\right)^{-1} = (1+0.1)^{-1} = \frac{1}{1.1}$
$= 0.9091$

Thus, discount factor will be, V^{n-1}

For $n=1$, $dF = V^{n-1} = (0.909)^{1-1} = (0.9091)^{0} = 1$

For $n=2$, $dF = V^{n-1} = (0.9091)^{2-1} = (0.9091)^{1} = 0.9091$

For $n=3$, $dF = v^{n-1} = (0.9091)^{3-1} = (0.9091)^2 = 0.8264$

For $n=4$, $dF = v^{n-1} = (0.9091)^{4-1} = (0.9091)^3 = 0.7513$

For $n=5$, $dF = v^{n-1} = (0.9091)^{5-1} = (0.9091)^4 = 0.6830$

For $n=6$, $dF = v^{n-1} = (0.9091)^{6-1} = (0.9091)^5 = 0.6209$

For $n=7$, $dF = v^{n-1} = (0.9091)^{7-1} = (0.9091)^6 = 0.5645$.

Year (n)	Running cost (R_n)	Discount Factor (V^{n-1})	Cumulative discount Factor (ΣV^{n-1})	Discount Running cost ($R_n V^{n-1}$)	Cumulative discount Running Cost ($\Sigma R_n V^{n-1}$)	Total cost $C+\Sigma R_n V^{n-1}$	Weighted average cost $\frac{C+\Sigma R_n V^{n-1}}{\Sigma V^{n-1}}$
1	2500	1	1	2500	2500	17500	17500
2	3000	0.9091	1.9091	2727.3	5227.3	20227.3	10595.2
3	4000	0.8265	2.7356	3306	8533.30	23533.3	8602.60
4	5000	0.75134	3.486	3756.68	12289.98	27289.9	7826.34
5	6500	0.6830	4.169	4439.76	16729.7	31729.7	$\boxed{7610.87}$
6	8000	0.6209	4.7899	4967.61	21697.3	36697.3	7661.40
7	10000	0.5645	5.3544	5645	27342.4	42342.4	7907.96

@Start Practicing

We observe that the weighted average cost is minimum at the end of 5^{th} year (7610.87)

Weighted average cost for 5^{th} year $= 7610.87$

Running cost for 6^{th} year $= 8000$.

$$8000 > 7610.87$$

Hence, the optimal replacement period is every 5^{th} year.

@Start Practicing

Replacement Problem (Example-4)

Value of Money changes with time

(4)
Q: A machine costs ₹6000. The running cost and the salvage value at the end of the year is given in the table below

Year	1	2	3	4	5	6	7
Running Cost	1200	1400	1600	1800	2000	2400	3000
Salvage Value	4000	2666	2000	1500	1000	600	600

If the rate of interest is 10% per year, ~~which~~ when should the machine be replaced?

Solution: Since, money is worth 10% per year.

The discount rate will be, $V = (1+r)^{-1} = \left(1+\frac{10}{100}\right)^{-1}$

$$= (1+0.1)^{-1} = (1.1)^{-1} = \frac{1}{1.1}$$

$$= 0.9091$$

The discount factor will be, v^{n-1}

For $n=1$, $dF = v^{n-1} = (0.9091)^{1-1} = (0.9091)^0 = 1.$

For $n=2$, $dF = v^{n-1} = (0.9091)^{2-1} = (0.9091)^1 = 0.9091$

For $n=3$, $dF = v^{n-1} = (0.9091)^{3-1} = (0.9091)^2 = 0.8264$

For $n=4$, $dF = v^{n-1} = (0.9091)^{4-1} = (0.9091)^3 = 0.7513$

For $n=5$, $dF = v^{n-1} = (0.9091)^{5-1} = (0.9091)^4 = 0.6830$

For $n=6$, $dF = v^{n-1} = (0.9091)^{6-1} = (0.9091)^5 = 0.6209$

For $n=7$, $dF = v^{n-1} = (0.9091)^{7-1} = (0.9091)^6 = 0.5645.$

For $n=7$, $v^n = (0.9091)^7 = 0.51319$

@Start Practicing

Year (n) 1	R_n 2	V^{n-1} 3	R_nV^{n-1} 4=2×3	ΣR_nV^{n-1} 5	S_n 6	V^n 7	S_nV^n 8=6×7	ΣV^{n-1} 9	$C+\Sigma R_nV^{n-1} - S_nV^n$ 10=C+5−8	W(n) 11 = $\frac{10}{9}$
1	1200	1	1200	1200	4000	0.9091	3636.4	1	3563.6	3563.6
2	1400	0.9091	1272.7	2472.7	2666	0.8264	2203.34	1.9091	6269.4	3283.95
3	1600	0.8264	1322.34	3795.08	2000	0.7513	1502.67	2.7356	8292.52	3031.33
4	1800	0.7513	1352.4	5147.48	1500	0.6830	1024.56	3.4869	10122.96	2903.14
5	2000	0.6830	1366.01	6513.49	1000	0.6209	620.9	4.1699	11892.57	$\boxed{2852.01}$ ✗
6	2400	0.6209	1490.19	8003.68	600	0.5645	338.7	4.7708	13664.96	$\boxed{2852.33}$ ✓
7	3000	0.5645	1693.38	9697.06	600	0.51319	307.91	5.3552	15389.25	2873.64

@start Practicing

We observe that weighted average cost is minimum at the end of 5th year.

Weighted average cost at 5th year = 2852.01

Running cost for 6th year = 2400.

$2400 > 2852.01$ (False)

Thus, we can choose next minimum weighted cost, ~~at~~ which is at 6th year.

again,

weighted average cost for 6th year = 2852.33

Running cost for 7th year = 3000.

$3000 > 2852.33$ (True).

Hence we can replace the machine at the end of 6th year.

Replacement of Equipment that fails Suddenly

(i) Individual replacement policy : Under this policy, an item is replaced immediatly after it fails.

(ii) Group replacement policy : Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or not, with a provision that if any item fails before the optimal time, it may be individually replaced.

We have optimal policy.

Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the t^{th} period is greater than the average cost per period, through the end of 't' period

@Start Practicing

Replacement of Equipment that fails suddenly

(5) Q: The following mortality rates have been observed for a certain type of light bulbs.

Week	1	2	3	4	5
Percent failing by the end of Week	10	25	50	80	100

There are 1000 bulbs in use and its costs ₹2 to replace an individual bulb, which has burnt out. If all the bulbs were replaced simultaneously, it would cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced?

@Start Practicing

Solution : Let P_i be the probability that a bulb, which was new when placed in position for use, fails during the i^{th} week of its life.

Step-1

$$P_1 = \frac{10}{100} = 0.1$$

$$P_2 = 0.25 - 0.1 = 0.15$$

$$P_3 = 0.5 - 0.25 = 0.25$$

$$P_4 = 0.8 - 0.5 = 0.3$$

$$P_5 = 1 - 0.8 = 0.2$$

Since the sum of probabilities is 1, all the probability beyond P_5 will be taken as zero.

@Start Practicing

Step-2 : Let N_i be the number of replacement at the end of the i^{th} week.

N_0 = Num of items in the beginning.

$= 1000$

$N_1 = N_0 P_1$

$= 1000 \times 0.1 = 100$

$N_2 = N_0 P_2 + N_1 P_1$

$= (1000 \times 0.15) + (100 \times 0.1) = 150 + 10 = 160$

$N_3 = N_0 P_3 + N_1 P_2 + N_2 * P_1$

$= (1000 \times 0.25) + (100 \times 0.15) + (160 \times 0.1) = 250 + 15 + 16 = 281$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= (1000 \times 0.3) + (100 \times 0.25) + (160 \times 0.15) + (281 \times 0.1) = 377.1$

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= (1000 \times 0.2) + (100 \times 0.3) + (160 \times 0.25) + (281 \times 0.15) + (377 \times 0.1) = 350$

Step-3 : We calculate the expected life of each bulb,

$$= \sum_{i=1}^{5} i P_i$$

$$= (1 \times 0.1) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.3) + (5 \times 0.2)$$

$$= 1.1 + 0.3 + 0.75 + 1.2 + 1$$

$$= 3.35$$

Average number of failure per week,

$$= \frac{1000}{3.35} = 298.50$$

$$= 299 \text{ (approx)}$$

Step-~~3~~4:

The cost of individual replacement $= 299 \times 2$

$= ₹ 598$

@ Start Practicing

Now, since the replacement of all the ~~bu~~ 1000 bulbs simultaneously costs 50 paise per bulb and replacement of an individual bulb on failure cost ₹2, the average cost for different group replacement policies is given below.

End of week	Individual Replacement	Total cost (Individual + group)	Average cost
1	100	$(100 \times 2) + (1000 \times 0.5) = 200 + 500 = 700$	$\frac{700}{1} = 700$
2	$100 + 160 = 260$	$(260 \times 2) + (1000 \times 0.5) = 520 + 500 = 1020$	$\frac{1020}{2} = 510$ ←
3	$260 + 281 = 541$	$(541 \times 2) + (1000 \times 0.5) = 1082 + 500 = 1582$	$\frac{1582}{3} = 527.3$
4	$541 + 377 = 918$	$(918 \times 2) + (1000 \times 0.5) = 1836 + 500 = 2336$	$\frac{2336}{4} = 584$
5	$918 + 350 = 1268$	$(1268 \times 2) + (1000 \times 0.5) = 2536 + 500 = 3036$	$\frac{3036}{5} = 607.2$

Since the average cost is minimum in the 2nd week, the optimal replacement period to have a group replacement is after every 2nd week.

Since the average cost is less than ₹598 for individual replacement, the group replacement policy is preferable.

@start Practicing