Basic Concepts of Probability

Experiment: Experiment is an act that can be repeated under given conditions.

Trail: Unit of an experiment is known as trial. This means that trial is a special case of experiment. Experiment may be a trial or two or more trials.

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Outcomes: The result of an experiment is known as outcomes.
Ex: Throwing a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an outcome

Equally likely Outcomes: Outcomes of a trial are said to be equally likely if we have no reason to expect any one rather than the other. Example-1) In tossing a fair coin, the outcomes head and tail are equally likely,2) In throwing a balanced die all the six faces are equally likely.

Mutually Exclusive Outcomes: Outcomes or cases are said to be mutually exclusive if the happening of any one of them precludes the happening of all others. Example-1) In tossing a coin, the outcomes hear and tail are mutually exclusive.2) In throwing a die, the six outcomes which are the different points on the faces of the die is mutually exclusive.

Sample space: The collection or totality of all possible outcomes of a random experiment is called sample space. Sample space is usually denoted by S or Ω .
Example:1) If we toss a coin, the sample space is, $\Omega=\{H,T\}$.
Where H and T denote the hear and tail of the coin,respectively,2)If a six –sided die is thrown, the sample space is, $\Omega=\{1,2,3,4,5,6\}$.

Events: An event is a subset of the sample space. Events are generally denoted by capital letters A, B, C, etc.

Event Space: The class of all events associated with a given experiment is defined to be the event space. It is usually denoted by *A.*

Example: Suppose a fair coin is tossed twice, then the sample space of the experiment will be
Ω = {HH, HT, TH.TT}.
Here the event space will contain $2^4 = 16$ events.

There are three definitions of probability: classical, empirical, and subjective.

Classical or mathematical Probability:
If there are n mutually exclusive, equally likely and exhaustive outcomes of an experiment and if m of these outcomes are favorable to an event A, then the probability of the event A which is denoted by $P(A)$ is defined by

$P(A)$= Favorable outcomes of an event A/Total number of outcomes of the experiments

$$P(A)=\frac{m}{n}$$

Example: After tossing a coin what is the probability that we will get head?

Solution:
We know,

$$P(A)=\frac{m}{n}$$

Here total number of outcome is n= 2 and favorable outcome, m=1

So $P(A)=\frac{m}{n}=\frac{1}{2}=0.5$

Example: Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has

1) An odd number
2) A number 4 or multiple of 4
3) A number which is greater than 70 and
4) A number which is square?

Solution

Since there are 100 tickets, the total number of exclusive mutually exclusive and equally likely case is 100.

1) Let A denote the event that the ticket drawn an odd number. Since there are 50 odd number tickets so the number of cases favorable to the event A is 50.

$$\therefore p(A)=\frac{50}{100}=0.5.$$

2) Let B denote the event that the ticket drawn has a number 4 or multiple of 4. The numbers favorable to event B are $4,\ 8,\ 12,\ 16,\ 20,...,92,\ 96,\ 100$. The total number of cases will be $\frac{100}{4}=25$.

$$\therefore p(B)=\frac{25}{100}=0.25.$$

3) Let C denote the event that the drawn ticket has a number greater than 70. Since the number greater than 70 are $71,\ 72,\ 73,...,\ 100$. Therefore, 30 cases are favorable to the event C.

$$\therefore p(C)=\frac{30}{100}=0.3.$$

4) Let D denote the event that the drawn ticket has a number which is a square. Since the squares between 1 and 100 are $1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81$ and 100. So the cases favorable to event D are 10 in number. Hence,

$$\therefore p(D)=\frac{10}{100}=0.1.$$

Empirical or statistical probability:
If an experiment is repeated a large number of times under the same conditions, then the probability of an event A is the limiting value of the ratio of the number of times that the event A happens to the total number of trails, as the number of trails increases indefinitely large, provided the ratio approaches a finite and unique limit p .

$$p = P(A) = \lim_{n \to \alpha} \frac{m}{n}$$

Example: Throughout her teaching career Professor Jones has awarded 186 A's out of 1,200 students. What is the probability that a student in her section this semester will receive an A?

Solution: To find the probability a selected student earned an A:

$$P(A) = \frac{186}{1200} = 0.155$$

Subjective probability is based on whatever information is available.

Laws of Probability:

There are two laws which are very important.

1. **All probabilities are between 0 and 1 inclusive**
   i.e. $0 \le P(A) \le 1$
2. **The sum of all the probabilities in the sample space is 1**
   i.e. $\sum_{i=1}^{n} P(A_i) = 1$

Probability Rules:

Addition Rule

- **Specific Addition Rule**

If two events are mutually exclusive, then the probability of either occurring is the sum of the probabilities of each occurring.

P (A or B) = P (A) + P (B).

Example on probabilities:

Example: A bag contains 4 white, 6 black balls and 5 green balls. If one ball is drawn at random from the bag, what probability that it is 1) black, 2) white, 3) white or black.

Solution: Total numbers of balls are 15.Since one ball is draw from the bag; there are 15 mutually exclusive, equally likely and exhaustive outcomes of this experiment.

1) Let A be the event that the ball is black, and then the number of outcomes favorable to A is 6.hence

P (A) =Number of black balls/Total number of balls= $\frac{6}{15}$ =0.4.

2) Let B be the event that the ball is white, and then the favorable outcomes corresponding to B is 4.Therefore

P (B) = $\frac{4}{15}$ =0.266

3) Let C be the event that the ball is white or black, then

P(C) =P (A or B) = $\frac{6}{15}+\frac{4}{15}=\frac{10}{15}$ = 0.666

Example
If we toss a coin then what is the probability of head or tail?

Solution
Here there are two events, namely event $A=H$ and event $B=T$. So that

$$p(A \text{ or } B) = p(A) + p(B)$$
$$= \frac{1}{2} + \frac{1}{2} = 1.$$

- **<u>General Addition Rule:</u>**

In events which are not mutually exclusive, there is some overlap. When P (A) and P (B) are added, the probability of the intersection (and) is added twice. To compensate for that double addition, the intersection needs to be subtracted.

P (A or B) = P (A) + P (B) - P (A and B)

Example:
In a sample of 500 students, 320 said they had a stereo, 175 said they had a TV, and 100 said they had both. 5 said they had neither. If a student is selected at random, what is the probability that the student has only a stereo or TV? What is the probability that the student has both a stereo and TV?

Solution: Let S and T be the events that students had stereo and TV, respectively. Then the probability that student has only stereo or TV is-
P(S or T) = *P*(S) + *P* (T) - *P*(S and T)
= 320/500 + 175/500 – 100/500
= .79.
The probability that the student has both a stereo and TV
P(S and TV) = 100/500
= .20

Example

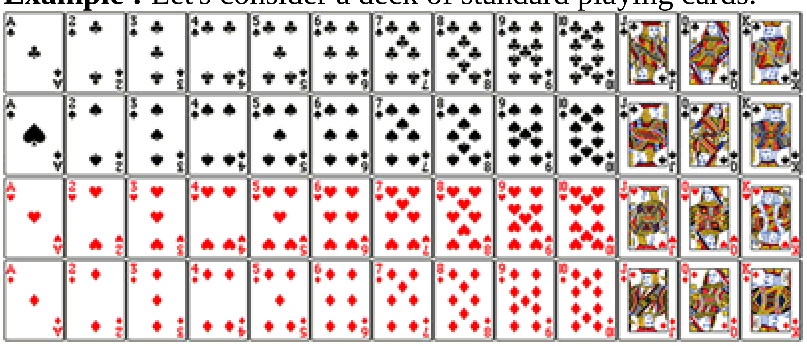
Mr. X feels that the probability that he will pass Mathematics is $\frac{2}{3}$ and Statistics is $\frac{5}{6}$. If the probability that he will pass both the course is $\frac{3}{5}$. What is the probability that he will pass at least one of the course?

Solution

Let M and S be the events that he will pass the courses Mathematics and Statistics respectively. The event $M \cup S$ means that at least one of M or S occurs. Therefore

$$p(M \cup S) = p(M \text{ or } S)$$
$$= p(\text{he pass at least one of the course})$$
$$= p(M) + p(S) - p(M \text{ or } S)$$
$$= \frac{2}{3} + \frac{5}{6} - \frac{3}{5} = \frac{9}{10}.$$

Example : Let's consider a deck of standard playing cards.



1.

Suppose we draw one card at random from the deck and define the following events:
E = the card drawn is an ace
F = the card drawn is a king
Use these definitions to find P(E or F).
Solution:
since E and F have no outcomes in common, we can use the Addition Rule for Disjoint Events:
P(E or F) = P(E) + P(F) = 4/52 + 4/52 = 8/52 = **2/13**

2.

Considering the deck of playing cards, where one is drawn at random. Suppose we define the following events:
F = the card drawn is a king
G = the card drawn is a heart
Use these definitions to find P(F or G).
Solution:
Unlike in the previous example, events F and G, *do* have an outcome in common - the king of hearts - so we'll need to use the General Addition Rule:
P(E or F)
= P(E) + P(F) - P(E and F)

= 4/52 + 13/52 - 1/52
= 16/52 = **4/13**

Example-2: A card is drawn from a pack of 52 cards. Find the probability that it is 1) a red card, 2) a spade, 3) an ace, 4) not a spade and e) a king or queen.

Solution: when a card is draw from a pack of 52 cards, the total number of equally likely, mutually exclusive and exhaustive outcomes are 52.That is, here n=52.

1) Let A be the event of drawing a red card. There are 26 black and 26 red cards in a pack and any one of the red cards can be drawn in 26 ways.

That is, m=26.Then the probability of a red card is

$$P(A) = \frac{26}{52} = \frac{1}{2}.$$

2) Let B be the event of drawing a spade. There are 13 spades in a pack of 52 cards. That is m=13.Then the probability of a spade is

$$P(B) = \frac{13}{52} = \frac{1}{4}$$

3) Let C be the event of drawing an ace. There are four aces in all. That is ,m=4.Then the probability of an ace is

$$P(C) = \frac{4}{52} = \frac{1}{13}$$

4) Let D be the event that the card is not a spade. In 52 cards, only 13 are spades and the remaining 39 are not spades.

That is m=52-13=39

The probability that the card is not spade is

$$P(D) = \frac{39}{52} = \frac{3}{4}$$

Alternately, P (D) =1-P (B)

$$=1-\frac{1}{4} = \frac{3}{4}$$

5) Let E be the event that the card is a king or qreen.Out of 52 cards, there are 4 kings and 4 queens. That is m=8

$$P(E) = \frac{8}{52} = \frac{2}{13}$$

Exercise: A student is taking two courses, history and math. The probability the student will pass the history course is .60, and the probability of passing the math course is .70. The probability of passing both is .50. What is the probability of passing at least one?

$$= \frac{3}{10} * \frac{2}{9} = \frac{6}{90} = 0.07 .$$

Joint probability:

Joint probability is the probability of two events in conjunction. That is, it is the $P(A \cap B), P(AB)$ both events together. The joint probability of *A* and *B* is written P (A and B) or.

Example:

The question, "Do you like watching TV?" was asked of 100 people. Results are shown in the table**.** What is the probability of a randomly selected individual being a male who likes watching TV?

	Yes	No	Total
Male	19	41	**60**
Female	12	28	**40**
Total	**31**	**69**	**100**

Solution: This is just a **joint probability.** The number of "Male and like watching TV" divided by the total = 19/100 = 0.19

Marginal probability:

Marginal probability is the probability of *A*, regardless of whether event *B* did or did not occur. If *B* can be thought of as the event of a random variable *X* having a given outcome, the marginal probability of *A* can be obtained by summing the joint probabilities over all outcomes for *X*.

Example:

The question, "Do you like watching TV?" was asked of 100 people. Results are shown in the table. What is the probability of a randomly selected individual like watching TV?

	Yes	No	Total
Male	19	41	**60**
Female	12	28	**40**
Total	**31**	**69**	**100**

Solution:

Since no mention is made of gender, **this is a marginal probability,** the total who like watching TV divided by the total = 31/100 = 0.31.

Conditional probability

Let A and B be two events. The conditional probability of event A given that B has occurred, is defined by the symbol $p(A|B)$ and is found to be:

$$p(A|B) = \frac{p(A \cap B)}{p(B)}; \text{ provided } p(B) > 0.$$

Similarly, $p(B|A) = \dfrac{p(A \cap B)}{p(A)}$; provided $p(A) > 0$.

Example

A hamburger chain found that 75% of all customers use salad, 80% use ketchup and 65% use both. What are the probabilities that a ketchup user uses salad and that a salad user uses ketchup?

Solution

Let A be the event "customer uses mustard" and B be the event "customer uses ketchup". Thus, we have, $p(A) = 0.75$, $p(B) = 0.80$ and $p(A \cap B) = 0.65$.

The probability that a ketchup user uses mustard is the conditional probability of event A, given event B is

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{0.65}{0.80} = 0.8125.$$

Similarly, the probability that a mustard user use ketchup is

Example: The question, "Do you like watching TV?" was asked of 100 people. Results are shown in the table**.** What is the probability of a randomly selected individual is a male if it is given that he likes watching TV?

	Yes	**No**	**Total**
Male	19	41	**60**
Female	12	28	**40**
Total	**31**	**69**	**100**

Solution: The conditional probability M given Y is

$$\text{P (M|Y)} = \frac{P(M \text{ and Y})}{\text{P(Y)}} = \frac{P(M \cap Y)}{P(Y)} = \frac{\frac{19}{100}}{\frac{31}{100}} = \frac{19}{100} \times \frac{100}{31} = \frac{19}{31} = 0.6$$

A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

Solution:

$$P(Blue|\text{Red}) = \frac{P(\text{Blue and Red})}{P(\text{Red})} = \frac{0.28}{0.5} = 0.56$$

Specific Multiplication Rule:

Specific Multiplication Rule:

The Specific Rule of Multiplication requires that two events *A* and *B* are *independent.*

This rule is written: $P(A \text{ and } B) = P(A)*P(B)$

$$P(A \cap B) = P(A) * P(B)$$

Example:

1. If the probability that person *A* will be alive in 0.7 and the probability that person *B* will be alive in 0.5, what is the probability that they will both be alive in 20 years?

These are independent events, so

$P(E_1 \text{ and } E_2) = P(E_1) \times P(E_2) = 0.7 \times 0.5 = 0.35$

[Note, however, that if person *A* knows person *B*, then they will be **dependent** events, especially if *A* is married to *B*.]

2. A company has two large computers. The probability that the newer one will breakdown on any particular month is 0.05, the probability that the older one will breakdown on any particular month is 0.1. What is the probability that they will both breakdowns in a particular month?

Solution

Let, Event A is the newer one will breakdown and Event B is the older one will breakdown. So that $p(A)=0.05$ and $p(B)=0.1$.

$$\therefore\ p(A \text{ and } B) = p(A) * p(B) = 0.05 * 0.1 = 0.005.$$

General Multiplication Rule

The General Rule of Multiplication requires that two events *A* and *B* are *dependent*

P (A and B) = P (A) * P (B|A)

$P(A \cap B)$ = P (A) * P(B|A)

Example

1. There are 10 rolls of film in a box, 3 of which are defective. Two rolls are to be selected one after another. What is the probability of selecting a defective roll followed by another defective roll?

Solution

The first roll of film selected from the box being found defective is event D_1. $\therefore\ p(D_1)=\frac{3}{10}$.

The second roll selected being found defective is event D_2. Therefore, $p(D_2|D_1)=\frac{2}{9}$. Since, after the first selection was found to be defective, only 2 defective rolls of film remained in the box containing 9 rolls.

So the probability of two defectives is

$$= p(D_1 \text{ and } D_2)$$
$$= p(D_1) * p(D_2|D_1)$$
=3/10*2/9=.066

Complement rule

The complement rule is used to determine the probability of an event occurring by subtracting the probability of the event not occurring from 1 i., e. $p(A)=1-p(\bar{A})$.

Example

Weight	Event	Probability
Underweight	A	0.025
Satisfactory	B	??
Overweight	C	0.075

Find $p(B)$.

Solution

We know that $p(B)=1-p(\bar{B})=1-(0.025+0.075)=0.90$.

Example:

It is known that the probability of obtaining zero defectives in a sample of 40 item is 0.34 while the probability of obtaining 1 defective item in the sample is 0.46. What is the probability of

(a) obtaining not more than 1 defective item in a sample?

(b) obtaining more than 1 defective items in a sample?

"Obtaining not more than one" means we choose either 0 or 1 defective.

Let event E_1 be "obtaining zero defectives" and E_2 be "obtaining 1 defective item".

(a) Events E_1 and E_2 are mutually exclusive, so

$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) = 0.34 + 0.46 = 0.8$

(b) $P(\text{more than } 1)=1-0.8=0.2$

Example

Rahim can hit a target 5 out of 7 chances. Karim can hit a target 6 out of 11.Find the chances that the target is hit once they both try?

Solution:

Suppose

P(A)=Probability that Rahim can hit the target=5/7

P($\overline{A}$)=Probability that Rahim cannot hit the target=1-5/6=2/7

P(B)=Probability that Rahim can hit the target=5/7

P($\overline{B}$)=Probability that Rahim cannot hit the target=1-5/6=2/7

The probability that no one can hit the target is =P($A\cap B$) = $P(\overline{A}).P(\overline{B})=\frac{2}{7}\cdot\frac{5}{11}=\frac{10}{77}$

So,the chances that the target is hit one is =1- P($A\cap B$) =1- $\frac{10}{77}$ =67/77=.87

Example: In a bag containing 29 marbls,5 of the marbles are red,2 are green .What is the probability that the randomly selected marble is neither red nor green?