RUNGE-KUTTA METHOD

Runge-Kutta method : The use of Euler's method to solve the differential equation numerically is less efficient since it requires h to be small for obtaining reasonable accuracy. But in Runge-Kutta methods, the derivatives of higher order are not required and they are designed to give greater accuracy with the advantage of requiring only the functional values at some selected points on the sub-interval.

Second Order Runge-Kutta Method for First Order Ordinary Differential Equation:-

Consider $\frac{dy}{dx} = f(x, y)$ with initial condition

$$y(x_0) = y_0$$

Let h be the interval between equidistant values of x.

By Taylor's series method, we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + O(h^3) \text{ ...... (1)}$$

$\because$ $y' = f(x, y)$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + ff_y$$

Using these values in (1), we get

$$y(x + h) = y(x) + hf + \frac{1}{2} h^2 [f_x + ff_y] + O(h^3)$$

$$\Rightarrow y(x + h) - y(x) = hf + \frac{1}{2} h^2 [f_x + ff_y] + O(h^3)$$

$$\therefore \Delta y = hf + \frac{1}{2} h^2 [f_x + ff_y] + O(h^3) \text{ ...... (2)}$$

Let $\Delta_1 y = k_1 = f(x, y). \Delta x = hf(x, y)$ ...... (3)

$\Delta_2 y = k_2 = hf(x + ph, y + pk_1)$ ...... (4)

and $\Delta y = p_1 k_1 + p_2 k_2$ ...... (5)

where p_1, p_2 and p are constants to be determined.

From (4), $k_2 = hf(x + ph, y + pk_1)$

$$= h\left[f(x, y) + \left(ph\frac{\partial}{\partial x} + pk_1\frac{\partial}{\partial y}\right)f + \frac{\left(ph\frac{\partial}{\partial x} + pk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots\right]$$

$$= h\left[f + phf_x + phff_y + \frac{\left(ph\frac{\partial}{\partial x} + pk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots\right] \quad [\because (3) \Rightarrow k_1 = hf]$$

$\therefore k_2 = hf + ph^2(f_x + ff_y) + O(h^3)$ ...... (6)

Using (3) and (6) in (5), we get

$\Delta y = p_1 hf + p_2[hf + ph^2(f_x + ff_y) + O(h^3)]$

$= (p_1 + p_2)hf + pp_2 h^2(f_x + ff_y) + O(h^3)$ ...... (7)

From (2) and (7), we get

$p_1 + p_2 = 1, \qquad pp_2 = \frac{1}{2}$

$\Rightarrow p_1 = 1 - p_2, \qquad \Rightarrow p = \frac{1}{2p_2}$

Using these values, (5) becomes

$\Delta y = (1 - p_2)k_1 + p_2 k_2$, where $k_1 = hf(x, y)$

and $k_2 = hf\left(x + \frac{1}{2p_2}h, y + \frac{hf}{2p_2}\right)$

Now $\Delta y = y(x + h) - y(x)$

or $y(x + h) = y(x) + \Delta y$

or $y(x + h) = y(x) + (1 - p_2)\, k_1 + p_2 k_2$

$$= y(x) + (1 - p_2)\, hf + p_2 hf\left(x + \frac{h}{2p_2}, y + \frac{hf}{2p_2}\right)$$

i. e. $y_{n+1} = y_n + (1 - p_2)\, hf(x_n, y_n)$

$$+ p_2 hf\left(x_n + \frac{h}{2p_2}, y_n + \frac{h}{2p_2} f(x_n, y_n)\right) + O(h^3)$$

Which is the general 2nd order Runge-Kutta method.

Put $p_1 = 0$, $p_2 = 1$ and $p = \frac{1}{2}$, we get the following second order Runge-Kutta formula

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{1}{2} k_1\right) \text{ and } \Delta y = k_2 \text{ where } h = \Delta x.$$

The third order Runge-Kutta formula is given by

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf(x + h, y + 2k_2 - k_1)$$

and $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$

✪ The fourth order Runge-Kutta formula is given by

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{1}{2}k_2\right)$$

so, $y_1 = y_0 + \Delta y$

$$k_4 = hf(x + h, y + k_3) \text{ and } \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

This fourth Runge-Kutta method is mostly used in problems unless otherwise mentioned.

Working Rule : To solve the initial value problem $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$.

Calculate $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3) \text{ and } \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where h is the interval between equidistant values of x.

Now $y_1 = y_0 + \Delta y$

Now starting from (x_1, y_1) and repeating this process, we get y_2 etc.

Remark-1. By second order Runge-Kutta Method, we have

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

and $\Delta y_0 = k_2 \Rightarrow y_1 - y_0 = k_2 \Rightarrow y_1 = y_0 + k_2$

$$\therefore y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

This is the Modified Euler method.

So the Runge-Kutta Method is exactly the Modified Euler's method.

Remark-2. If $f(x, y) = f(x)$. Then the fourth order Runge-Kutta method reduces to $k_1 = hf(x_0)$.

Example Compute y(0.2) by Runge-Kutta method of 4th order for the differential equation

$$\frac{dy}{dx} = xy + y^2, \ y(0) = 1.$$

Solution : Given $\frac{dy}{dx} = xy + y^2, \ y(0) = 1$

$\therefore f(x, y) = xy + y^2, \ x_0 = 0, \ y_0 = 1$

Let us take $h = 0.1$

By fourth order Runge-Kutta method, for the first approximation, we have

$$k_1 = hf(x_0, y_0) = h(x_0 y_0 + y_0^2) = 0.1\ (0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left\{\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right\}$$

$$= 0.1\left\{\left(0 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right\} = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left[\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{.1}{2}\right)\left(1 + \frac{0.1155}{2}\right) + \left(1 + \frac{0.1155}{2}\right)^2\right] = 0.11717$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h[(x_0 + h)(y_0 + k_3) + (y_0 + k_3)^2]$$

$$= 0.1\ [(0 + 0.1)(1 + 0.11717) + (1 + 0.11717)^2] = 0.13598$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 2 \times 0.1155 + 2 \times 0.11717 + 0.13598) = 0.11689$$

$\therefore y_1 = y_0 + \Delta y = 1 + 0.11689 = 1.11689.$

For the second approximation we have $x_1 = 0.1$

$$k_1 = hf(x_1, y_1) = h(x_1 y_1 + y_1^2)$$

$$= 0.1(0.1 \times 1.11689 + 1.11689^2) = 0.13591$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_1}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right)\left(1.11689 + \frac{0.13591}{2}\right) + \left(1.11689 + \frac{0.13591}{2}\right)^2\right]$$

$k_2 = 0.15816$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left[\left(x_1 + \frac{h_2}{2}\right)\left(y_1 + \frac{k_2}{2}\right) + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right)\left(1.11689 + \frac{0.15816}{2}\right) + \left(1.11689 + \frac{0.15816}{2}\right)^2\right]$$

$= 0.16097$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h[(x_1 + h)(y_1 + k_3) + (y_1 + k_3)^2]$$

$$= 0.1[(0.1 + 0.1)(1.11689 + 0.16097) + (1.11689 + 0.16097)^2]$$

$= 0.18885.$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.13591 + 2 \times 0.15816 + 2 \times 0.16097 + 0.18885) = 0.1605$$

$\therefore y_2 = y_1 + \Delta y = 1.11689 + 0.1605 = 1.27739$

$\therefore y(0.2) = 1.27739.$

Example y(0.2) by Runge-Kutta method of 4th order for the differential equation

$$\frac{dy}{dx} = x + y, \ y(0) = 1.$$

Solution : Given $\frac{dy}{dx} = x + y; \ y(0) = 1$

$$\therefore f(x, y) = x + y, \ x_0 = 0, \ y_0 = 1$$

Let us take $h = 0.1$

By fourth order Runge-Kutta method, we have

$k_1 = hf(x_0, y_0) = h(x_0 + y_0) = 0.1(0 + 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$$

$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{1}{2}k_1\right]$$

$$= 0.1\left[0 + \frac{0.1}{2} + 1 + \frac{1}{2} \times 0.1\right] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = h\left(x_0 + \frac{h}{2} + y_0 + \frac{1}{2}k_2\right)$$

$$= 0.1\left(0 + \frac{0.1}{2} + 1 + \frac{1}{2} \times 0.11\right) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h(x_0 + h + y_0 + k_3)$$

$$= 0.1(0 + 01 + 1 + 0.1105) = 0.12105$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 2 \times 0.11 + 2 \times 0.1105 + 0.12105) = 0.11034$$

$\therefore y_1 = y_0 + \Delta y \Rightarrow y_1 = 1 + 0.11034$

$\therefore y_1 = 1.11034, \quad \therefore y(0.1) = 1.11034$

For the second approximation we have $x_1 = 0.1$

$k_1 = kf(x_1, y_1) = h(x_1 + y_1) = 0.1(0.1 + 1.11034) = 0.12103$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = h\left(x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2}\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{2} + 1.11034 + \frac{0.12103}{2}\right) = 0.13209$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left(x_1 + \frac{h_2}{2} + y_1 + \frac{k_2}{2}\right)$$

$$= 0.1\left(0.1 + \frac{0.1}{2} + 1.1.034 + \frac{0.13209}{2}\right) = 0.13264$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h(x_1 + h + y_1 + k_3)$$

$$= 0.1(0.1 + 0.1 + 1.11034 + 0.13264) = 0.14430$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.12103 + 2 \times 0.13209 + 2 \times 0.13264 + 0.14430)$$

$$= 0.13247$$

$$\therefore y_2 = y_1 + \Delta y = 1.11034 + 0.13247 = 1.24281, \quad \therefore y(0.2) = 1.24281$$

Example- **Comput $y(0.1)$ and $y(0.2)$ using** Runge-Kutta **method of (i) second order, (ii) third order and (iii)** 4th order for **the differential equation $y' = -y$ with** the initial condition **$y(0) = 1$.**

Solution : Given $\frac{dy}{dx} = -y;\ y(0) = 1, \quad \therefore f(x, y) = -y, x_0 = 0, y_0 = 1$

Let us take $h = 0.1$

(i) By second order Runge-Kutta method, we have

$$k_1 = hf(x_0, y_0) = 0.1 \times (-1) = -0.1 \quad [\because f(x_0, y_0) = -y_0]$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[-y_0 - \frac{k_1}{2}\right]$$

$$= 0.1\left(-1 - \frac{-0.1}{2}\right) = -0.095, \quad \therefore \Delta y = k_2 + \Delta y = -0.095$$

Now, $y_1 = y_0 = 1 - 0.095 = 0.905m \quad \therefore y(0.1) = y_1 = 0.905$

For the second approximation, we have

$$k_1 = hf(x_1, y_1) = h(-y_1) = 0.1(-0.905) = -0.0905$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left(-y_1 - \frac{k_1}{2}\right)$$

$$= 0.1\left(-0.905 + \frac{0.0905}{2}\right) = -0.08598$$

Now $\Delta y = k_2 \Rightarrow \Delta y = -0.08598$

$\therefore y_2 = y(0.2) = y_1 + \Delta y = 0.905 - 0.085598 = 0.81902$

Hence $y(0.2) = 0.81902$

(ii) By Third order Runge-Kutta method, we have

$k_1 = hf(x_0, y_0) = h(-y_0) = 0.1(-1) = -0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left(-y_0 - \frac{k_1}{2}\right)$$

$$= 0.1\left(-1 + \frac{0.1}{2}\right) = -0.095$$

$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$

$= h(-y_0 - 2k_2 + k_1) = 0.1(-1 + 2 \times 0.095 - 0.1) = -0.091$

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= \frac{1}{6}(-0.1 + 4 \times (-0.095) - 0.091) = -0.09517$$

$\therefore y(0.1) = y_1 = y_0 + \Delta y = 1 - 0.09517 = 0.90483$

Hence $y(0.1) = 0.90483$

For the second approximation, we have

$x_1 = 0.1, y_1 = 0.90483$

$\therefore k_1 = hf(x_1, y_1) = h(-y_1) = 0.1(-0.90483) = -0.090483$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left(-y_1 - \frac{k_1}{2}\right)$$

$$= 0.1\left(-0.94483 + \frac{0.090483}{2}\right) = -0.08596$$

$k_3 = hf(x_1 + h, y_1 + 2k_2 - k_1) = h(-y_1 - 2k_2 + k_1)$

$= 0.1(-0.90483 - 2(-0.08596) - 0.090483 = -0.08234$

and $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$

$$= \frac{1}{6}(-0.090483) + 4 \times (-0.08596) - (0.08234) = -0.08611$$

$\therefore y_2 = y_1 + \Delta y = 0.90482 - 0.08611 = 0.81872$

Hence $y(0.2) = y_2 = 0.81872$

(iii) by 4th order Runge-Kutta method, we have

$k_1 = hf(x_0, y_0) = h.(-y_0) = 0.1 \times (-1) = -0.1$

$$h_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left(-y_0 - \frac{k_1}{2}\right)$$

$$= 0.1\left(-1 + \frac{0.1}{2}\right) = -0.095$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left(-y_0 - \frac{k_2}{2}\right)$$

$$= 0.1\left(-1 + \frac{0.095}{2}\right) = -0.09525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h(-y_0 - k_3)$$

$$= 0.1(-1 + 0.09525) = -0.09048$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[(-0.1 + 2 \times (-0.095) + 2(-0.09525) - (0.09048)] = -0.09516$$

$\therefore y_1 = y_0 + \Delta y = 1 - 0.09516 = 0.90484$

Hence $y(0.1) = y_1 = 0.90484$

For the second approximation, we have

$x_1 = 0.1$ and $y = 0.90484$

$\therefore k_1 = hf(x_1, y_1) = h(-y_1) = 0.1(-0.90484) = -0.090484$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= h\left(-y_1 - \frac{k_1}{2}\right) = 0.1\left(-0.90484 + \frac{0.090484}{2}\right) = -0.08596$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= h\left(-y_1 - \frac{k_2}{2}\right) = 0.1\,(-0.90484 + 0.08596 \div 2) = -0.08619$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h(-y_1 - k_3) = 0.1\,(-0.90484 + 0.08619) = -0.08187$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.090484 + 2(-0.08596) + 2(-0.08619) - 0.08187)$$

$$= -0.08611$$

$\therefore y_2 = y_1 + \Delta y = 0.90484 - 0.08611 = 0.81873$

Hence $y(0.2) = y_2 = 0.81873.$

Example—4. Find the value of y(0.3) by 4th order Runge-Kutta method for the following differential equation $\frac{dy}{dx} + y + xy^2 = 0$. $y(0) = 1$.

Solution : Given $\frac{dy}{dx} + y + xy^2 = 0$ with $y(0) = 1$

$$\therefore f(x, y) = -y - xy^2,\ x_0 = 0,\ y_0 = 1$$

Let us take $h = 0.1$

By fourth order Runge-Kutta method for the first approximation, we have

$$k_1 = hf(x_0, y_0) = h(-y_0 - x_0 y_0^2) = 0.1(-1 - 0 \times 1^2) = -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[-y_0 - \frac{k_1}{2} - \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[-1 + \frac{0.1}{2} - \left(0 + \frac{.1}{2}\right)\left(1 + \frac{0.1}{2}\right)^2\right] = -0.09951$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[\left(-y_0 - \frac{k_2}{2}\right) - \left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(-1 + \frac{0.09951}{2}\right) - \left(0 + \frac{0.1}{2}\right)\left(1 + \frac{-0.1}{2}\right)^2\right] = -0.09954$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h[(-y_0 - k_3) - (x_0 + h)(y_0 + k_3)^2]$$

$$= 0.1[(-1 + 0.09954) - (0 + 0.1)(1 - 0.09954)] = -0.09815$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.1 - 2 \times 0.09951 - 2 \times 0.09954 - 0.09815) = -0.09938$$

$$y_1 = y_0 + \Delta y = 1 - 0.09938 = 0.90062$$

$$\therefore y(0.1) = y_1 = 0.90062$$

Again, for the second approximation, we have

$$x_1 = 0.1,\ y_1 = 0.90062.$$

$$k_1 = hf(x_1, y_1) = h(-y_1 - x_1 y_1^2)$$

$$= 0.1(-0.90062 - 0.1 \times 0.90062^2) = -0.0982$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= h\left[-y_1 - \frac{k_1}{2} - \left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[-0.90062 + \frac{0.0982}{2} - \left(0.1 + \frac{0.1}{2}\right) \times \left(0.90062 + \frac{-0.0982}{2}\right)^2\right]$$

$$= -0.096$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left[-\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[-\left(0.90062 - \frac{0.096}{2}\right) - \left(0.1 + \frac{0.1}{2}\right) \left(0.90062 - \frac{0.096}{2}\right)^2\right]$$

$$= -0.0962.$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h[-(y_1 + k_3) - (x_1 + h)(y_1 + k_3)^2]$$

$$= 0.1[-(0.90062 - 0.0962) - (0.1 + 0.1)(0.90062 - 0.0962)^2]$$

$$= -0.0934$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.0982 + 2(-0.096) + 2(-0.0962) + (-0.0934)] = -0.096$$

$$\therefore y_2 = y_1 + \Delta y = 0.90062 - 0.096 = 0.80462$$

Hence $y(0.2) = y_2 = 0.80462$

Again, for the third approximation, we have

$$x_2 = 0.2, y_2 = 0.80462$$

$$k_1 = hf(x_2, y_2) = h[-y_2 - x_2 y_2^2] = -0.0934$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= h\left[-\left(y_2 + \frac{k_1}{2}\right) - \left(x_2 + \frac{h}{2}\right)\left(y_2 + \frac{k_1}{2}\right)^2\right] = -0.0902$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= h\left[-\left(y_2 + \frac{k_2}{2}\right) - \left(x_2 + \frac{h}{2}\right)\left(y_2 + \frac{k_2}{2}\right)^2\right] = -0.0904$$

$$k_4 = hf(x_2 + h, y_2 + k_3)$$

$$= h[-(y_2 + k_3) - (x_2 + h)(y_2 + k_3)^2] = -0.0867$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.0934 - 2 \times 0.0902 - 2 \times 0.0904 - 0.0867) = -0.09022$$

$$\therefore y_3 = y_2 + \Delta y = 0.80462 - 0.09022 = 0.71440$$

Hence, $y(0.3) = y_3 = 0.71440$.

Example- Using Runge–kutta method of fourth order to find $y(0.1)$, given that $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$.

Solution : Given $\frac{dy}{dx} = \frac{1}{x+y}$ with $y(0) = 1$

$\therefore f(x, y) = \frac{1}{x+y}$, $x_0 = 0$ and $y_0 = 1$

Let us take $h = 0.1$

$\therefore$ By fourth order Runge-Kutta method, we have

$$k_1 = hf(x_0, y_0) = h \times \frac{1}{x_0 + y_0} = 0.1\left(\frac{1}{0+1}\right) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h \times \frac{1}{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)}$$

$$= 0.1 \times \frac{1}{\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)}$$

$$\doteq 0.09091$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h \times \frac{1}{\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)}$$

$$= 0.1 \times \frac{1}{\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.09091}{2}\right)} = 0.09129$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h \times \frac{1}{(x_0 + h) + (y_0 + k_3)}$$

$$= 0.1 \times \frac{1}{(0 + 0.1) + (1 + 0.09129)} = 0.08394$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 2 \times 0.0901 + 2 \times 0.09129 + 0.08394) = 0.09139$$

$$\therefore y_1 = y_0 + \Delta y = 1 + 0.09139 = 1.09139$$

Hence $y(0.1) = y_1 = 1.09139$.

Example- Using fourth order Runge–Kutta method, find the value of y when x = 1.1 given that

$$\frac{dy}{dx} + \frac{y}{x} - \frac{1}{x^2} = 0 \text{ with } y(1) = 1$$

Solution : Given that $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$ with $y(1) = 1$

So $f(x, y) = \frac{1}{x^2} - \frac{y}{x}$, $x_0 = 1$ and $y_0 = 1$

Let ustake $h = 0.1$

$\therefore$ By fourth-order Runge-Kutta method, we have

$$k_1 = hf(x_0, y_0) = h\left(\frac{1}{x_0^2} - \frac{y_0}{x_0}\right) = 0.1\left(\frac{1}{1^2} - \frac{1}{1}\right) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[\frac{1}{\left(x_0 + \frac{h}{2}\right)^2} - \frac{y_0 + \frac{k_1}{2}}{x_0 + \frac{h}{2}}\right]$$

$$= 0.1\left[\frac{1}{\left(1 + \frac{0.1}{2}\right)^2} - \frac{1 + 0}{1 + \frac{0.1}{2}}\right] = -0.00454$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[\frac{1}{\left(x_0 + \frac{h}{2}\right)^2} - \frac{y_0 + \frac{k_2}{2}}{x_0 + \frac{h}{2}}\right]$$

$$= 0.1\left[\frac{1}{\left(1 + \frac{0.1}{2}\right)^2} - \frac{1 - \frac{0.00454}{2}}{\left(1 + \frac{0.1}{2}\right)}\right]$$

$$= -0.00432$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h\left[\frac{1}{(x_0 + h)^2} - \frac{y_0 + k_3}{x_0 + h}\right]$$

$$= 0.1\left[\frac{1}{(1 + 0.1)^2} - \frac{1 - 0.00432}{(1 + 0.1)}\right]$$

$$= -0.00787$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0 - 2 \times 0.00454 - 2 \times 0.00432 - 0.00787) = -0.00427$$

Now $y_1 = y_0 + \Delta y = 1 - 0.00427 = 0.99573$

Hence $y_1 = y(0.1) = 0.99573.$

Example- Use Runge-Kutta 4th order method to solve $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$ at the points $x = 1,$ by taking $h = 0.5.$

Solution : Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with $y(0) = 1$ and $h = 0.5$

$\therefore f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0, y_0 = 1$

By fourth order Runge-Kutta method for the first approximation, we have

$$k_1 = hf(x_0, y_0) = h \times \frac{y_0 - x_0}{y_0 + x_0} = 0.5 \times \frac{1-0}{1+0} = 0.5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.5 \times \frac{y_0 + \frac{k_1}{2} - x_0 - \frac{h}{2}}{y_0 + \frac{k_1}{2} + x_0 + \frac{h}{2}}$$

$$\therefore k_2 = 0.5 \times \frac{1 + \frac{0.5}{2} - 0 - \frac{0.5}{2}}{1 + \frac{0.5}{2} + 0 + \frac{.5}{2}} = 0.3333$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.5 \times \frac{y_0 + \frac{k_2}{2} - x_0 - \frac{h}{2}}{y_0 + \frac{k_2}{2} + x_0 + \frac{h}{2}}$$

$$= 0.5 \times \frac{1 + \frac{0.3333}{2} - 0 - \frac{0.5}{2}}{1 + \frac{0.3333}{2} + 0 + \frac{.5}{2}} = 0.3235$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h \times \frac{y_0 + k_3 - x_0 - h}{y_0 + k_3 + x_0 + h} = 0.5 \times \frac{1 + 0.3235 - 0 - 0.5}{1 + 0.3235 + 0 + 0.5}$$

$$= 0.2258$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.5 + 2 \times 0.3333 + 2 \times 0.3235 + 0.2258) = 0.3399$$

$$\therefore y(0.5) = y_1 = y_0 + \Delta y = 1 + 0.3399 = 1.3399$$

For the second approximation, we have

$h = 0.5, x_1 = 0.5, y_1 = 1.3399$

$$k_1 = hf(x_1, y_1) = h \times \frac{y_1 - x_1}{y_1 + x_1} = 0.5 \times \frac{1.3399 - 0.5}{1.3399 + 0.5} = 0.2282$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.5 \times \frac{y_1 + \frac{k_1}{2} - x_1 - \frac{h}{2}}{y_1 + \frac{k_1}{2} + x_1 + \frac{h}{2}}$$

$$= 0.5 \times \frac{1.3399 + \frac{0.2282}{2} - 0.5 - \frac{0.5}{2}}{1.3399 + \frac{0.2282}{2} + 0.5 + \frac{0.5}{2}} = 0.1597$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h \times \frac{y_1 + \frac{k_2}{2} - x_1 - \frac{h}{2}}{y_1 + \frac{k_2}{2} + x_1 + \frac{h}{2}}$$

$$= 0.5 \times \frac{1.3399 + \frac{0.1597}{2} - 0.5 - \frac{0.5}{2}}{1.3399 + \frac{0.1597}{2} + 0.5 + \frac{0.5}{2}} = 0.1543$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h \times \frac{y_1 + k_3 - x_1 - h}{y_1 + k_3 + x_1 + h}$$

$$= 0.5 \times \frac{1.3399 + 0.1543 - 0.5 - 0.5}{1.3399 + 0.1543 + 0.5 + 0.5} = 0.0991$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2282 + 2 \times 0.1597 + 2 \times 0.1543 + 0.0991) = 0.1592$$

$$y(1.0) = y_2 = y_1 + \Delta y = 1.3399 + 0.1592 = 1.4991$$

Example- Solve $\frac{dy}{dx} = \frac{1}{x+y}$ for x = .5 to x = 2, (h = 0.5) by using Runge-Kutta's method with $x_0 = 0$, $y_0 = 1$.

Solution : $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$

$$f(x, y) = \frac{1}{x+y}$$

Here $x_0 = 0$, $y_0 = 1$ and $h = 0.5$

Using fourth order Runge-Kutta's method, we get

$$k_1 = hf(x_0, y_0) = \frac{h}{x_0 + y_0} = \frac{0.5}{0+1} = 0.5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= \frac{h}{x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}} = \frac{0.5}{0 + \frac{.5}{2} + 1 + \frac{.5}{2}} = \frac{0.5}{1.5} = 0.3333$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = \frac{h}{x_0 + \frac{h}{2} + y_0 + \frac{k_2}{2}}$$

$$= \frac{0.5}{0 + \frac{.5}{2} + 1 + \frac{0.3333}{2}} = \frac{0.5}{0.25 + 1 + 0.1666}$$

$$= \frac{0.5}{1.4166} = 0.353$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = \frac{h}{x_0 + h + y_0 + k_3}$$

$$= \frac{0.5}{0 + 0.5 + 1 + 0.353} = \frac{0.5}{1.853} = 0.2698$$

$$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(.5 + 2(0.333) + 2(0.353) + 0.2698)$$

$$= 1 + \frac{.5 + 0.6666 + 0.706 + 0.2698}{6}$$

$$= 1 + \frac{2.1424}{6} = 1 + 0.357 = 1.357$$

$\therefore y(0.5) = 1.537 \Rightarrow x_1 = 0.5$ and $y_1 = 1.537$

For second approximation

$$k_1 = hf(x_1, y_1) = \frac{h}{x_1 + y_1} = \frac{0.5}{0.5 + 1.537} = \frac{0.5}{2.037} = 0.2455$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = \frac{h}{x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2}}$$

$$= \frac{0.5}{0.5 + \frac{0.5}{2} + 1.537 + \frac{0.2455}{2}} = \frac{0.5}{2.4097} = 0.2075$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = \frac{h}{x_1 + \frac{h}{2} + y_1 + \frac{k_2}{2}}$$

$$= \frac{0.5}{0.5 + \frac{0.5}{2} + 1.537 + \frac{0.2075}{2}} = \frac{0.5}{2.3908} = 0.2091$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = \frac{h}{x_1 + h + y_1 + k_3}$$

$$= \frac{0.5}{0.5 + 0.5 + 1.537 + 0.2091} = \frac{0.5}{2.7461} = 0.1821$$

$\therefore k_4 = 0.1421$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.537 + \frac{0.2455 + 2(0.2075) + 2(0.2091) + 0.1421}{6}$$

$$= 1.537 + \frac{1.0133}{6} = 1.537 + 0.16888 = 1.7059$$

$\therefore y(1.0) = 1.7059)$

For third approximation : $x_2 = 1.0$, $y_2 = 1.7059$ and $h = .5$

$$k_1 = hf(x_2, y_2) = \frac{h}{x_2 + y_2} = \frac{0.5}{1 + 1.7059} = 0.1848$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = \frac{h}{x_2 + \frac{h}{2} + y_2 + \frac{k_1}{2}}$$

$$= \frac{0.5}{1 + \frac{.5}{2} + 1.7059 + \frac{.1848}{2}} = \frac{0.5}{3.0483}$$

$$= 0.1640$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = \frac{h}{x_2 + \frac{h}{2} + y_2 + \frac{k_2}{2}}$$

$$= \frac{0.5}{1 + \frac{0.5}{2} + 1.7059 + \frac{0.1640}{2}} = \frac{0.5}{3.0379} = 0.1646$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = \frac{h}{x_2 + h + y_2 + k_3}$$

$$= \frac{0.5}{1 + .5 + 1.7059 + 0.1646}$$

$$= \frac{0.5}{3.3705} = 0.1481$$

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.7059 + \frac{0.1848 + 2(0.1640) + 2(0.1646) + 0.1481}{6}$$

$$= 1.7059 + \frac{0.9901}{6} = 1.7059 + 0.1650 = 1.8709$$

$\therefore y(1.5) = 1.8709$

For fourth approximation : $x_3 = 1.5$ and $y_3 = 1.8709$ and $h = 0.5$

$$\therefore k_1 = hf(x_3, y_3) = \frac{h}{x_3 + y_3} = \frac{0.5}{1.5 + 1.8709}$$

$$= \frac{0.5}{3.3709} = 0.1483$$

$$k_2 = hf\left(x_3 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = \frac{h}{x_3 + \frac{h}{2} + y_3 + \frac{k_1}{2}}$$

$$= \frac{0.5}{1.5 + \frac{0.5}{2} + 1.8709 + \frac{0.1483}{2}} = \frac{0.5}{3.6950}$$

$$= 0.1353$$

$$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right)$$

$$= \frac{h}{x_3 + \frac{h}{2} + y_2 + \frac{k_2}{2}}$$

$$= \frac{0.5}{1.5 + \frac{0.5}{2} + 1.8709 + \frac{0.1353}{2}} = \frac{0.5}{3.6886}$$

$$= 0.1356$$

$$k_4 = hf(x_3 + h, y_3 + k_3) = \frac{h}{x_3 + 3 + y_3 + k_3}$$

$$= \frac{0.5}{1.5 + 0.5 + 1.8709 + 0.1356} = \frac{0.5}{4.0065}$$

$$= 0.1248$$

$$\therefore y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.8709 + \frac{0.1483 + 2(0.1353) + 2(0.1356) + 0.1248}{6}$$

$$= 1.8709 + \frac{0.5418}{6} = 1.8709 + 0.0903 = 1.9612$$

$\therefore y(2.0) = 1.9612$

Hence $y(0.5) = 1.537$

$y(1.0) = 1.7059$

$y(1.5) = 1.8709$

$y(2.0) = 1.9612$

EXERCISE–

Solve $y' = 1 + y^2$ with $y(0) = 0$ for $x = 0.2\ (0.2)\ 0.6$ by Runge-Kutta method of fourth order. **[NUH-1999]**

Ans : $y(0.6) = 0.6841$

Solve the following initial value problem using Runge-Kutta method of fourth order :

(i) $\frac{dy}{dx} = (1 + x)\, y$ with $y(0) = 1$ for $x = 0(0.2)\ 0.6$, $y(0.2) = 1.2247$, $y(0.4) = 1.5240$, $y(0.6) = 1.9581$

(ii) $y' = \frac{1}{x + y}$ with $y(0) = 1$ for $x = 0.5\ (0.5)\ 2$.

Ans : $y(0.5) = 1.3571$, $y(1.0) = 1.5873$, $y(1.5) = 1.7555$, $y(2.0) = 1.8956$

) $y' = \sin x + \cos y$ with $y(0) = 2.5$ for $x = 3(0.5)4$

Ans : $y(3.0) = 0.649$, $y(3.5) = 0.935$, $y(4.0) = 0.941$

Solve $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ with $y(1) = 0$ for $x = 1.2$ and 1.4 by Runge–Kutta method of fourth order.

Ans : $y(1.2) = 0.1402$, $y(1.4) = 0.2705$

Find the value of $y(0.1)$, $y(0.2)$ and $y(0.3)$ by fourth order Runge–Kutta method for the following differential equation

(i) $10y' = x^2 + y^2$ with $y(0) = 1$

(ii) $8y' = x + y^2$ with $y(0) = 0.5$