Md. Mohiuddin (MDN)

Differentiation

Introduction: The derivative is a mathematical operator, which measures the rate of change of a quantity relative to another quantity. The process of finding a derivative is called differentiation. There are many phenomena related changing quantities such as speed of a particle, inflation of currency, intensity of an earthquake and voltage of an electrical signal etc. in the world. In this chapter we will discuss about various techniques of derivative.

Outcomes: After successful completion of the chapter, the students will be able to:

1. determine the speed, velocity and acceleration of a particle with respect to time.
2. calculate the rate at which the number of bacteria , the population changes with time.
3. measure the rate at which the length of a metal rod changes with temperature.
4. find out the rate at which production cost changes with the quantity of a product .

Derivatives of elementary functions:

1. $\frac{d}{dx}(c)=0,$ *where c is a constant.*

2. $\frac{d}{dx}(x)=1.$

3. $\frac{d}{dx}(x^n)=nx^{n-1}.$

4. $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}.$

5. $\frac{d}{dx}(e^x)=e^x.$

6. $\frac{d}{dx}(a^x)=a^x \ln a.$

7. $\frac{d}{dx}(\ln x)=\frac{1}{x}.$

8. $\frac{d}{dx}(\sin x)=\cos x.$

9. $\frac{d}{dx}(\cos x)=-\sin x.$

10. $\frac{d}{dx}(\tan x)=\sec^2 x.$

11. $\frac{d}{dx}(\sec x)=\sec x \tan x.$

12. $\frac{d}{dx}(\cot x)=-\cos ec^2 x.$

13. $\frac{d}{dx}(\cos ecx)=-\cos ecx \cot x.$

14. $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}.$

15. $\frac{d}{dx}(\cos^{-1} x)=\frac{-1}{\sqrt{1-x^2}}.$

16. $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}.$

17. $\frac{d}{dx}(\cot^{-1} x)=\frac{-1}{1+x^2}.$

18. $\frac{d}{dx}(\sec^{-1} x)=\frac{1}{x\sqrt{x^2-1}}.$

19. $\frac{d}{dx}(\cos ec^{-1} x)=\frac{-1}{x\sqrt{x^2-1}}.$

20. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}.$

21. $\frac{d}{dx}(u^v)=u^v\frac{d}{dx}(v\ln u).$

where u and v are functions of x.

22. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

- ***Find the differential coefficient ($\frac{dy}{dx}$) of the following functions with respect to x.***

1. $y = 5x^8$

Sol: *Given that*, $y = 5x^8$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^8\right)$$
$$= 5\frac{d}{dx}\left(x^8\right)$$
$$= 5 \times 8x^{8-1}$$
$$= 40x^7 \quad (Ans.)$$

2. $y = 3x^7 + 2x + 1$

Sol: *Given that*, $y = 3x^7 + 2x + 1$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^7 + 2x + 1\right)$$
$$= 3\frac{d}{dx}\left(x^7\right) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1)$$
$$= 21x^6 + 2 + 0$$
$$= 21x^6 + 2 \quad (Ans.)$$

3. $y = 4\sin x - \cos x$

Sol: *Given that*, $y = 4\sin x - \cos x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(4\sin x - \cos x\right)$$
$$= 4\frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$$
$$= 4\cos x - (-\sin x)$$
$$= 4\cos x + \sin x \quad (Ans.)$$

4. $y = \sec^2 x - \tan^2 x$

Sol: *Given that*, $y = \sec^2 x - \tan^2 x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sec^2 x - \tan^2 x\right)$$
$$= \frac{d}{dx}\left(\sec^2 x\right) - \frac{d}{dx}\left(\tan^2 x\right)$$
$$= 2\sec x\frac{d}{dx}(\sec x) - 2\tan x\frac{d}{dx}(\tan x)$$
$$= 2\sec x(\sec x\tan x) - 2\tan x\left(\sec^2 x\right)$$
$$= 2\sec^2 x\tan x - 2\sec^2 x\tan x$$
$$= 0 \quad (Ans.)$$

5. $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

Sol: *Given that*, $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln\left(x + \sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{d}{dx}\left(\sqrt{x^2 + a^2}\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{1}{2\sqrt{x^2 + a^2}} \cdot \frac{d}{dx}\left(x^2 + a^2\right)\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$
$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}$$
$$= \frac{1}{\sqrt{x^2 + a^2}} \quad (Ans.)$$

6. $y = \ln(\sec x + \tan x)$

Sol: *Given that*, $y = \ln(\sec x + \tan x)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\ln(\sec x + \tan x)\right\}$$
$$= \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\left(\sec x\tan x + \sec^2 x\right)}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$
$$(Ans.)$$

7. $y=e^{ax^2+bx+c}$

Sol : *Given that,* $y=e^{ax^2+bx+c}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{ax^2+bx+c}\right)$$
$$=e^{ax^2+bx+c}.\frac{d}{dx}\left(ax^2+bx+c\right)$$
$$=e^{ax^2+bx+c}\left(2ax+b+0\right)$$
$$=\left(2ax+b\right)e^{ax^2+bx+c}$$

$(Ans.)$

8. $y=e^{\sqrt{\cot x}}$

Sol : *Given that,* $y=e^{\sqrt{\cot x}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sqrt{\cot x}}\right)$$
$$=e^{\sqrt{\cot x}}.\frac{d}{dx}\left(\sqrt{\cot x}\right)$$
$$=e^{\sqrt{\cot x}}.\frac{1}{2\sqrt{\cot x}}.\frac{d}{dx}\left(\cot x\right)$$
$$=\frac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.\left(-\cos ec^2 x\right)$$
$$=-\frac{e^{\sqrt{\cot x}}\cos ec^2 x}{2\sqrt{\cot x}}$$

$(Ans.)$

9. $y=\sqrt{x^3-2x+5}$

Sol : *Given that,* $y=\sqrt{x^3-2x+5}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x^3-2x+5}\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\frac{d}{dx}\left(x^3-2x+5\right)$$
$$=\frac{1}{2\sqrt{x^3-2x+5}}.\left(3x^2-2+0\right)$$
$$=\frac{3x^2-2}{2\sqrt{x^3-2x+5}}$$

$(Ans.)$

10. $y=\tan\ln\sin\left(e^{x^2}\right)$

Sol : *Given that,* $y=\tan\left(\ln\sin e^{x^2}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\tan\left(\ln\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{d}{dx}\left\{\ln\left(\sin e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\frac{d}{dx}\left\{\sin\left(e^{x^2}\right)\right\}$$
$$=\sec^2\left(\ln\sin e^{x^2}\right).\frac{1}{\sin\left(e^{x^2}\right)}.\cos\left(e^{x^2}\right).\frac{d}{dx}\left(e^{x^2}\right)$$
$$=\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right).e^{x^2}\frac{d}{dx}\left(x^2\right)$$
$$=2xe^{x^2}\cot\left(e^{x^2}\right)\sec^2\left(\ln\sin e^{x^2}\right)$$

$(Ans.)$

11. $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Sol : *Given that,* $y=\cos^{-1}\left(e^{\cot^{-1}x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\cos^{-1}\left(e^{\cot^{-1}x}\right)\right\}$$
$$=-\frac{1}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(e^{\cot^{-1}x}\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}.\frac{d}{dx}\left(\cot^{-1}x\right)$$
$$=-\frac{e^{\cot^{-1}x}}{\sqrt{1-e^{2\cot^{-1}x}}}\left(-\frac{1}{1+x^2}\right)$$
$$=\frac{e^{\cot^{-1}x}}{\left(1+x^2\right)\sqrt{1-e^{2\cot^{-1}x}}}$$

$(Ans.)$

12. $y=e^{\sin^{-1}x}+\tan^{-1}x$

Sol : *Given that,* $y=e^{\sin^{-1}x}+\tan^{-1}x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}+\tan^{-1}x\right)$$
$$=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)+\frac{d}{dx}\left(\tan^{-1}x\right)$$
$$=e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{1}{1+x^2}$$
$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}+\frac{1}{1+x^2}$$

$(Ans.)$

13. $y=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

Sol : *Given that*, $y=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

put, $x=\sin\theta \quad \therefore\ \theta=\sin^{-1}x$

$$\begin{aligned} Now,\ y &= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) \\ &= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right) \\ &= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \tan^{-1}.\tan\theta \\ &= \theta \\ &= \sin^{-1}x \end{aligned}$$

Differentiating with respect to x we get,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sin^{-1}x\right) \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

$(Ans.)$

14. $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Sol : *Given that*, $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

put, $x=\tan\theta \quad \therefore\ \theta=\tan^{-1}x$

$$\begin{aligned} Now,\ y &= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \\ &= \cos^{-1}.\cos 2\theta \\ &= 2\theta \\ &= 2\tan^{-1}x \end{aligned}$$

Differentiating with respect to x we get,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(2\tan^{-1}x\right) \\ &= \frac{2}{1+x^2} \end{aligned}$$

$(Ans.)$

15. $y=\dfrac{\cos x}{1+\sin x}$

Sol : *Given that*, $y=\dfrac{\cos x}{1+\sin x}$

Differentiating with respect to x we get,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right) \\ &= \frac{(1+\sin x)\frac{d}{dx}(\cos x)-\cos x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2} \\ &= \frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2} \\ &= \frac{-\sin x-1}{(1+\sin x)^2} \\ &= -\frac{1}{1+\sin x} \end{aligned}$$

$(Ans.)$

16. $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Sol : *Given that*, $y=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Differentiating with respect to x we get,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right) \\ &= \frac{(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)-(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)}{(\cos x+\sin x)^2} \\ &= \frac{(\cos x+\sin x)(-\sin x-\cos x)-(\cos x-\sin x)(-\sin x+\cos x)}{\cos^2 x+\sin^2 x+2\sin x\cos x} \\ &= \frac{-(\cos x+\sin x)^2-(\cos x-\sin x)^2}{1+\sin 2x} \\ &= \frac{-(1+\sin 2x)-(1-\sin 2x)}{1+\sin 2x} \\ &= -\frac{2}{1+\sin 2x} \end{aligned}$$

$(Ans.)$

17. $y = \tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

Sol: *Given that*, $y = \tan^{-1}\left(\dfrac{\sqrt{(1+x^2)}-1}{x}\right)$

put, $x = \tan\theta \quad \therefore \; \theta = \tan^{-1} x$

$$Now,\; y = \tan^{-1}\left(\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2 \theta/2}{2\sin\theta/2\cos\theta/2}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$= \tan^{-1}.\tan\theta/2$$

$$= \theta/2$$

$$= 1/2\tan^{-1} x$$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(1/2\tan^{-1} x\right)$$

$$= \frac{1}{2(1+x^2)}$$

$(Ans.)$

18. $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Sol : *Given that*, $y=\sin^{-1}\left(\dfrac{a+b\cos x}{b+a\cos x}\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)\right\}$$

$$=\frac{1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}}.\frac{d}{dx}\left(\frac{a+b\cos x}{b+a\cos x}\right)$$

$$=\frac{(b+a\cos x)}{\sqrt{(b+a\cos x)^2-(a+b\cos x)^2}}.\frac{(b+a\cos x)(-b\sin x)-(a+b\cos x)(-a\sin x)}{(b+a\cos x)^2}$$

$$=\frac{1}{\sqrt{b^2-a^2+a^2\cos^2 x-b^2\cos^2 x}}.\frac{-b^2\sin x-ab\sin x\cos x+a^2\sin x+ab\sin x\cos x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)-(b^2-a^2)\cos^2 x}}.\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{1-\cos^2 x}}.\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sqrt{\sin^2 x}}.\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{1}{\sqrt{(b^2-a^2)}\sin x}.\frac{(a^2-b^2)\sin x}{b+a\cos x}$$

$$=\frac{-\sqrt{b^2-a^2}}{b+a\cos x}$$

$(Ans.)$

19. $y=x\sin x$

Sol : *Given that*, $y=x\sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}(x\sin x)$$

$$=x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)$$

$$=x\cos x+\sin x$$

$(Ans.)$

20. $y=e^{ax}\cos bx$

Sol : *Given that*, $y=e^{ax}\cos bx$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{ax}\cos bx\right)$$

$$=e^{ax}\frac{d}{dx}(\cos bx)+\cos bx\frac{d}{dx}\left(e^{ax}\right)$$

$$=e^{ax}(-b\sin bx)+\cos bx\left(ae^{ax}\right)$$

$$=ae^{ax}\cos bx-be^{ax}\sin bx$$

$(Ans.)$

21. $y = x^2 \cot^{-1} x$

Sol : *Given that*, $y = x^2 \cot^{-1} x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \cot^{-1} x\right)$$
$$= x^2 \frac{d}{dx}\left(\cot^{-1} x\right) + \cot^{-1} x \frac{d}{dx}\left(x^2\right)$$
$$= x^2\left(\frac{-1}{1+x^2}\right) + \cot^{-1} x\left(2x\right)$$
$$= 2x\cot^{-1} x - \frac{x^2}{1+x^2}$$

$(Ans.)$

22. $y = x^3 \ln x$

Sol : *Given that*, $y = x^3 \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 \ln x\right)$$
$$= x^3 \frac{d}{dx}\left(\ln x\right) + \ln x \frac{d}{dx}\left(x^3\right)$$
$$= x^3 . \frac{1}{x} + \ln x\left(2x^2\right)$$
$$= x^2 + 2x^2 \ln x$$

$(Ans.)$

23. $y = xe^x \sin x$

Sol : *Given that*, $y = xe^x \sin x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(xe^x \sin x\right)$$
$$= xe^x \frac{d}{dx}\left(\sin x\right) + \sin x \frac{d}{dx}\left(xe^x\right)$$
$$= xe^x \cos x + \sin x\left\{x\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}\left(x\right)\right\}$$
$$= xe^x \cos x + \sin x\left(xe^x + e^x\right)$$
$$= xe^x \cos x + xe^x \sin x + e^x \sin x$$

$(Ans.)$

24. $y = \sqrt{x}e^x \sec x$

Sol : *Given that*, $y = \sqrt{x}e^x \sec x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}e^x \sec x\right)$$
$$= \sqrt{x}e^x \frac{d}{dx}\left(\sec x\right) + \sec x \frac{d}{dx}\left(\sqrt{x}e^x\right)$$
$$= \sqrt{x}e^x \sec x \tan x + \sec x\left(\sqrt{x}e^x + \frac{e^x}{2\sqrt{x}}\right)$$

$(Ans.)$

25. $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Sol : *Given that*, $y = \left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x + e^{\sqrt{1+x^2}}\right\}$$
$$= \frac{d}{dx}\left\{\left(x^2+1\right)\sin^{-1} x\right\} + \frac{d}{dx}\left(e^{\sqrt{1+x^2}}\right)$$
$$= \left(x^2+1\right).\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x.\left(2x\right) + e^{\sqrt{1+x^2}} . \frac{1}{2\sqrt{1+x^2}}.2x$$
$$= \frac{x^2+1}{\sqrt{1-x^2}} + 2x\sin^{-1} x + \frac{xe^{\sqrt{1+x^2}}}{\sqrt{1+x^2}}$$

$(Ans.)$

26. $y = e^{\sin x}\sin\left(a^x\right)$

Sol : *Given that*, $y = e^{\sin x}\sin\left(a^x\right)$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{e^{\sin x}\sin\left(a^x\right)\right\}$$
$$= e^{\sin x}\frac{d}{dx}\left\{\sin\left(a^x\right)\right\} + \sin\left(a^x\right)\frac{d}{dx}\left(e^{\sin x}\right)$$
$$= e^{\sin x}.\cos\left(a^x\right).\frac{d}{dx}\left(a^x\right) + \sin\left(a^x\right).e^{\sin x}.\cos x$$
$$= e^{\sin x}.\cos\left(a^x\right).a^x \ln a + \sin\left(a^x\right).e^{\sin x}.\cos x$$

$(Ans.)$

Homework:- Find $\dfrac{dy}{dx}$ of the following functions:

1. $y=\ln\left(\sqrt{x-a}+\sqrt{x-b}\right)$ Ans: $\dfrac{1}{2\sqrt{(x-a)(x-b)}}$

2. $y=\ln\left(x+\sqrt{x^2\pm b^2}\right)$ Ans: $\dfrac{1}{\sqrt{x^2\pm b^2}}$

3. $y=\cos(\ln x)+\ln(\tan x)$ Ans: $2\cos ec2x-\dfrac{\sin(\ln x)}{x}$

4. $y=e^{ax}\sin^m rx$ Ans: $e^{ax}\sin^m rx\left(a+mr\cot rx\right)$

5. $y=x\sec x\ln\left(xe^x\right)$ Ans: $\sec x\left\{(x+1)+(x\tan x+1)\ln\left(xe^x\right)\right\}$

6. $y=\sin^{-1}x^2-xe^{x^2}$ Ans: $\dfrac{2x}{\sqrt{1-x^4}}-\left(2x^2+1\right)e^{x^2}$

7. $y=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ Ans: $\dfrac{1}{\sqrt{1-x^2}}$

8. $y=\tan^{-1}\left(\dfrac{4\sqrt{x}}{1-4x}\right)$ Ans: $\dfrac{2}{\sqrt{x}(1+4x)}$

9. $y=\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ Ans: $-\dfrac{1}{2}$

10. $y=\sin^{-1}\left(\dfrac{2x^{-1}}{x+x^{-1}}\right)$ Ans: $\dfrac{2}{\sqrt{x}(1+4x)}$

Logarithmic differentiation: If we have functions that are composed of products, quotients and powers, to differentiate such functions it would be convenient first to take logarithm of the function and then differentiate. Such a technique is called the logarithmic differentiation.

1. $y=(\sin x)^{\ln x}$

Sol: *Given that*, $y=(\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$
$$=(\sin x)^{\ln x}\frac{d}{dx}\{\ln x\ln(\sin x)\}$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{d}{dx}\{\ln(\sin x)\}+\ln(\sin x).\frac{d}{dx}(\ln x)\right]$$
$$=(\sin x)^{\ln x}\left[\ln x.\frac{1}{\sin x}.\cos x+\ln(\sin x).\frac{1}{x}\right]$$
$$=(\sin x)^{\ln x}\left[\cot x\ln x+\frac{\ln(\sin x)}{x}\right]$$

$(Ans.)$

2. $y=x^{1+x+x^2}$

Sol: *Given that*, $y=x^{1+x+x^2}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{1+x+x^2}\right)$$
$$=x^{1+x+x^2}\frac{d}{dx}\left\{\left(1+x+x^2\right)\ln x\right\}$$
$$=x^{1+x+x^2}\left[\ln x.\frac{d}{dx}\left(1+x+x^2\right)+\left(1+x+x^2\right).\frac{d}{dx}(\ln x)\right]$$
$$=x^{1+x+x^2}\left[\ln x.(0+1+2x)+\left(1+x+x^2\right).\frac{1}{x}\right]$$
$$=x^{1+x+x^2}\left[(2x+1)\ln x+\frac{\left(1+x+x^2\right)}{x}\right]$$

$(Ans.)$

3. $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Sol: *Given that*, $y=\left(\tan^{-1}x\right)^{\sin x+\cos x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(\tan^{-1}x\right)^{\sin x+\cos x}\right\}$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\frac{d}{dx}\left\{(\sin x+\cos x).\ln\left(\tan^{-1}x\right)\right\}$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[(\sin x+\cos x)\frac{d}{dx}\left\{\ln\left(\tan^{-1}x\right)\right\}+\ln\left(\tan^{-1}x\right).\frac{d}{dx}(\sin x+\cos x)\right]$$
$$=\left(\tan^{-1}x\right)^{\sin x+\cos x}\left[\frac{(\sin x+\cos x)}{\tan^{-1}x}.\frac{1}{\left(1+x^2\right)}+\ln\left(\tan^{-1}x\right).(\cos x-\sin x)\right]$$

$(Ans.)$

4. $y = x^x + (\sin x)^{\ln x}$

Sol : *Given that*, $y = x^x + (\sin x)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{x^x + (\sin x)^{\ln x}\right\}$$

$$= \frac{d}{dx}\left(x^x\right) + \frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$

$$= x^x \frac{d}{dx}(x \ln x) + (\sin x)^{\ln x} \frac{d}{dx}\left\{\ln x \ln(\sin x)\right\}$$

$$= x^x\left(x.\frac{1}{x} + \ln x\right) + (\sin x)^{\ln x}\left\{\ln x.\frac{1}{\sin x}.\cos x + \frac{\ln(\sin x)}{x}\right\}$$

$$= x^x(1 + \ln x) + (\sin x)^{\ln x}\left\{\ln x.\cot x + \frac{\ln(\sin x)}{x}\right\} \quad Ans.$$

5. $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Sol : *Given that*, $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{(\sin x)^{\cos x} + (\cos x)^{\sin x}\right\}$$

$$= \frac{d}{dx}\left\{(\sin x)^{\cos x}\right\} + \frac{d}{dx}\left\{(\cos x)^{\sin x}\right\}$$

$$= (\sin x)^{\cos x}\frac{d}{dx}\left\{\cos x \ln(\sin x)\right\} + (\cos x)^{\sin x}\frac{d}{dx}\left\{\sin x \ln(\cos x)\right\}$$

$$= (\sin x)^{\cos x}\left[\cos x.\frac{1}{\sin x}.\cos x - \sin x \ln(\sin x)\right] + (\cos x)^{\sin x}\left[\sin x.\frac{1}{\cos x}.(-\sin x) + \cos x \ln(\cos x)\right]$$

$$= (\sin x)^{\cos x}\left[\cos x.\cot x - \sin x \ln(\sin x)\right] + (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \sin x.\tan x\right] \quad Ans.$$

6. $y = x^{\cos^{-1} x} - \sin x \ln x$

Sol : *Given that*, $y = x^{\cos^{-1} x} - \sin x \ln x$

Differentiating with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\cos^{-1} x} - \sin x \ln x\right)$$

$$= \frac{d}{dx}\left(x^{\cos^{-1} x}\right) - \frac{d}{dx}(\sin x \ln x)$$

$$= x^{\cos^{-1} x}\frac{d}{dx}\left(\cos^{-1} x \ln x\right) - \left(\frac{\sin x}{x} + \cos x \ln x\right)$$

$$= x^{\cos^{-1} x}\left[\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right] - \left(\frac{\sin x}{x} + \cos x \ln x\right) \quad Ans.$$

7. $y=\left(1+x^2\right)^{\tan x}+\left(2-\sin x\right)^{\ln x}$

Sol: *Given that*, $y=\left(1+x^2\right)^{\tan x}+\left(2-\sin x\right)^{\ln x}$

Differentiating with respect to x we get,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x}+\left(2-\sin x\right)^{\ln x}\right\}$$

$$=\frac{d}{dx}\left\{\left(1+x^2\right)^{\tan x}\right\}+\frac{d}{dx}\left\{\left(2-\sin x\right)^{\ln x}\right\}$$

$$=\left(1+x^2\right)^{\tan x}\frac{d}{dx}\left\{\tan x\ln\left(1+x^2\right)\right\}+\left(2-\sin x\right)^{\ln x}\frac{d}{dx}\left\{\ln x\ln\left(2-\sin x\right)\right\}$$

$$=\left(1+x^2\right)^{\tan x}\left[\frac{2x\tan x}{1+x^2}+\sec^2 x\ln\left(1+x^2\right)\right]+\left(2-\sin x\right)^{\ln x}\left[\frac{\ln\left(2-\sin x\right)}{x}-\frac{\cos x\ln x}{\left(2-\sin x\right)}\right] \textit{ Ans.}$$

Homework:- Find $\frac{dy}{dx}$ of the following functions:

1. $y=x^{\sin^{-1}x}$ Ans: $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right]$
2. $y=\left(\sin x\right)^{\cos^{-1}x}$ Ans: $\left(\sin x\right)^{\cos^{-1}x}\left[\cot x\cos^{-1}x-\frac{\ln\sin x}{\sqrt{1-x^2}}\right]$
3. $y=x^{x^x}$ Ans: $x^{x^x}x^x\left[1+\ln x+\frac{1}{x}\right]$
4. $y=x^{\cos^{-1}x}+\left(\sin x\right)^{\ln x}$ Ans:
$x^{\cos^{-1}x}\left[\frac{\cos^{-1}x}{x}-\frac{\ln x}{\sqrt{1-x^2}}\right]+\left(\sin x\right)^{\ln x}\left[\ln x\cot x+\frac{\ln\sin x}{x}\right]$
5. $y=\left(\tan x\right)^{\cot x}+\left(\cot x\right)^{\tan x}$ Ans:
$\left(\tan x\right)^{\cot x}\operatorname{cosec}^2 x\left(1-\ln\tan x\right)+\left(\cot x\right)^{\tan x}\sec^2 x\left(\ln\cot x-1\right)$
6. $y=x^{\ln x}+x^{\sin^{-1}x}$ Ans: $\frac{2x^{\ln x}\ln x}{x}+x^{\cos^{-1}x}\left(\frac{\cos^{-1}x}{x}-\frac{\ln x}{\sqrt{1-x^2}}\right)$

Parametric Equation: If in the equation of a curve $y = f(x)$, x and y are expressed in terms of a third variable known as parameter i.e, $x = \varphi(t)$, $y = \psi(t)$ then the equations are called a parametric equation.

1. $x = a(t + \sin t)$, $y = a(1 - \cos t)$

$sol:$ *Given that*,

$$x = a(t + \sin t) \cdots \cdots (1)$$

$$and \quad y = a(1 - \cos t) \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to t we get*,

$$\frac{dx}{dt} = a(1 + \cos t)$$

$$and \quad \frac{dy}{dt} = a \sin t$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a \sin t}{a(1 + \cos t)}$$

$$= \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}}$$

$$= \tan \frac{t}{2} \quad (Ans.)$$

2. $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$

$sol:$ *Given that*,

$$x = a(\cos t + t \sin t) \cdots \cdots (1)$$

$$and \quad y = a(\sin t - t \cos t) \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to t we get*,

$$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$= at \cos t$$

$$and \quad \frac{dy}{dt} = a(\cos t + t \sin t - \cos t)$$

$$= at \sin t$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{at \sin t}{at \cos t}$$

$$= \tan t \quad (Ans.)$$

3. $x = a\left(\cos t + \ln \tan \frac{t}{2}\right)$, $y = a \sin t$

$sol:$ *Given that*,

$$x = a\left(\cos t + \ln \tan \frac{t}{2}\right) \cdots \cdots (1)$$

$$and \quad y = a \sin t \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to t we get*,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} . \sec^2 \frac{t}{2} . \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2} \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} . \frac{1}{\cos^2 \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right)$$

$$= a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$and \quad \frac{dy}{dt} = a \cos t$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a \cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

$$= \tan t \quad (Ans.)$$

4. $x = t - \sqrt{1 - t^2}$, $y = e^{\sin^{-1} t}$

$sol:$ *Given that*,

$$x = t - \sqrt{1 - t^2} \cdots \cdots (1)$$

$$and \quad y = e^{\sin^{-1} t} \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to t we get*,

$$\frac{dx}{dt} = 1 - \frac{1}{2\sqrt{1 - t^2}} . (-2t)$$

$$= 1 + \frac{t}{\sqrt{1 - t^2}}$$

$$= \frac{t + \sqrt{1 - t^2}}{\sqrt{1 - t^2}}$$

$$and \quad \frac{dy}{dt} = e^{\sin^{-1} t} . \frac{1}{\sqrt{1 - t^2}}$$

$$= \frac{e^{\sin^{-1} t}}{\sqrt{1 - t^2}}$$

$$Now, \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{e^{\sin^{-1} t}}{\sqrt{1 - t^2}} . \frac{\sqrt{1 - t^2}}{t + \sqrt{1 - t^2}}$$

$$= \frac{e^{\sin^{-1} t}}{t + \sqrt{1 - t^2}} \quad (Ans.)$$

5. *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

$$sol: Let,\ y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \ ; \left[\begin{array}{l} putting\ \ x = \tan\theta \\ \therefore\ \ \theta = \tan^{-1} x \end{array}\right]$$
$$= \tan^{-1}.\tan 2\theta$$
$$= 2\theta$$
$$= 2\tan^{-1} x \cdots \cdots (1)$$
$$and\ \ z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \ ; \left[\begin{array}{l} putting\ \ x = \tan\theta \\ \therefore\ \ \theta = \tan^{-1} x \end{array}\right]$$
$$= \sin^{-1}.\sin 2\theta$$
$$= 2\theta$$
$$= 2\tan^{-1} x \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to* x *we get,*

$$\frac{dy}{dx} = \frac{2}{1+x^2} \quad and \quad \frac{dz}{dx} = \frac{2}{1+x^2}$$
$$Now,\ \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$
$$= \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$
$$= 1 \quad (Ans.)$$

6. *Differentiate* $(\sin x)^x$ *with respect to* $x^{\sin x}$.

$$sol: Let,\ y = (\sin x)^x \cdots \cdots (1)$$
$$and\ \ z = x^{\sin x} \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to* x *we get,*

$$\frac{dy}{dx} = (\sin x)^x \frac{d}{dx}(x\ln\sin x)$$
$$= (\sin x)^x\left(x.\frac{\cos x}{\sin x} + \ln\sin x\right)$$
$$= (\sin x)^x (x\cot x + \ln\sin x)$$
$$and\ \frac{dz}{dx} = x^{\sin x}\frac{d}{dx}(\sin x\ln x)$$
$$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x\ln x\right)$$
$$Now,\ \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$
$$= \frac{(\sin x)^x(x\cot x + \ln\sin x)}{x^{\sin x}\left(\frac{\sin x}{x} + \cos x\ln x\right)} \quad (Ans.)$$

7. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\sin^{-1} x$.

$$sol: Let,\ y = x^{\sin^{-1}x} \cdots \cdots (1)$$
$$and\ \ z = \sin^{-1} x \cdots \cdots (2)$$

Differentiating (1) *and* (2) *with respect to* x *we get,*

$$\frac{dy}{dx} = x^{\sin^{-1}x}\frac{d}{dx}\left(\sin^{-1}x\ln x\right) \ ; \left[\because \frac{d}{dx}(u^v) = u^v\frac{d}{dx}(v\ln u)\right]$$
$$= x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)$$
$$and\ \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$$
$$Now,\ \frac{dy}{dz} = \frac{dy/dx}{dz/dx}$$
$$= \frac{x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)}{\frac{1}{\sqrt{1-x^2}}}$$
$$= x^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}.\sin^{-1}x}{x} + \ln x\right) \quad (Ans.)$$

8. *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1-x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$.

sol : *Let*, $y=\tan^{-1}\left(\frac{\sqrt{1-x^2}-1}{x}\right)$

$$=\tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}-1}{\sin\theta}\right);\left[\begin{array}{r}putting\quad x=\sin\theta\\ \therefore\ \theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sqrt{\cos^2\theta}-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta-1}{\sin\theta}\right)$$

$$=\tan^{-1}\left\{-\frac{(1-\cos\theta)}{\sin\theta}\right\}$$

$$=\tan^{-1}\left\{-\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$=\tan^{-1}\left\{-\tan\frac{\theta}{2}\right\}$$

$$=\tan^{-1}\left\{\tan\left(\pi-\frac{\theta}{2}\right)\right\}$$

$$=\pi-\frac{\theta}{2}$$

$$=\pi-\frac{1}{2}\sin^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}x\cdots\cdots(2)$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{1}{2\sqrt{1-x^2}}\qquad and\ \frac{dz}{dx}=\frac{1}{1+x^2}$$

Now, $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{-\frac{1}{2\sqrt{1-x^2}}}{\frac{1}{1+x^2}}$$

$$=-\frac{1+x^2}{2\sqrt{1-x^2}}\quad(Ans.)$$

9. *Differentiate* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ *with respect to* $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

sol : *Let*, $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right);\left[\begin{array}{r}putting\quad x=\cos\theta\\ \therefore\ \theta=\cos^{-1}x\end{array}\right]$$

$$=\sec^{-1}\left(\frac{1}{\cos2\theta}\right)$$

$$=\sec^{-1}(\sec2\theta)$$

$$=2\theta$$

$$=2\cos^{-1}x\cdots\cdots(1)$$

and $z=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right);\left[\begin{array}{r}putting\quad x=\sin\theta\\ \therefore\ \theta=\sin^{-1}x\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$=\tan^{-1}.\tan\theta$$

$$=\theta$$

$$=\sin^{-1}x\cdots\cdots(2)$$

Differentiating (1) *and* (2) *with respect to x we get*,

$$\frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}\qquad and\ \frac{dz}{dx}=\frac{1}{\sqrt{1-x^2}}$$

Now, $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$

$$=\frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$=-2\quad(Ans.)$$

Homework:-

1. *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *with respect to* $\tan^{-1}x$. Ans: $\frac{1}{2}$
2. *Differentiate* $e^{\sin^{-1}x}$ *with respect to* $\cos 3x$. Ans: $-\frac{e^{\sin^{-1}x}}{3\sqrt{1-x^2}.\sin 3x}$
3. *Differentiate* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *with respect to* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$. Ans: 1
4. *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\ln x$. Ans: $x^{\sin^{-1}x}\left(\sin^{-1}x+\frac{x\ln x}{\sqrt{1-x^2}}\right)$

Successive derivative: If $y = f(x)$ be a function of *x* then the first order derivative of *y* with respect to *x* is denoted by $\frac{dy}{dx},\ f'(x),\ y_1,\ y^{(1)},\ f^{(1)}(x),\ f_x'(x)$ *etc.*

Again the derivative of first ordered derivative of *y* with respect to *x* is called second order derivative and is denoted by $\frac{d^2y}{dx^2},\ f''(x),\ y_2,\ y^{(2)},\ f^{(2)}(x),\ f_x''(x)$ *etc.*

Similarly, the nth derivative of *y* with respect to *x* is denoted by

$\frac{d^ny}{dx^n},\ f^n(x),\ y_n,\ y^{(n)},\ f^{(n)}(x),\ f_x^n(x)$ *etc.*

❖ Find the nth derivative of the following functions:

1. $y = x^n$

$sol: Given\ that,\ y = x^n$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = nx^{n-1}$$
$$\therefore\ y_2 = n(n-1)x^{n-2}$$
$$\therefore\ y_3 = n(n-1)(n-2)x^{n-3}$$

$Similarly,$

$$y_r = n(n-1)(n-2)\cdots\cdots\{n-(r-1)\}x^{n-r}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = n(n-1)(n-2)\cdots\cdots\{n-(n-1)\}x^{n-n}$$
$$= n(n-1)(n-2)\cdots\cdots 3.2.1$$
$$= n!\quad Ans.$$

2. $y = e^{ax}$

$sol: Given\ that,\ y = e^{ax}$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = ae^{ax}$$
$$\therefore\ y_2 = a^2e^{ax}$$
$$\therefore\ y_3 = a^3e^{ax}$$

$Similarly,$

$$y_r = a^re^{ax}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = a^ne^{ax}\quad Ans.$$

3. $y = (ax+b)^m$

$sol: Given\ that,\ y = (ax+b)^m$

$Differentiating\ with\ respect\ to\ x\ we\ get,$

$$y_1 = am(ax+b)^{m-1}$$
$$\therefore\ y_2 = a^2m(m-1)(ax+b)^{m-2}$$
$$\therefore\ y_3 = a^3m(m-1)(m-2)(ax+b)^{m-3}$$

$Similarly,$

$$y_r = a^rm(m-1)(m-2)\cdots\cdots\{m-(r-1)\}(ax+b)^{m-r}\quad ;\ where,\ r<n$$
$$\therefore\ y_n = a^nm(m-1)(m-2)\cdots\cdots\{m-(n-1)\}(ax+b)^{m-n}$$
$$= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}\quad Ans.$$

4. $y=\sin(ax+b)$

$sol: Given\ that,\ y=\sin(ax+b)$

Differentiating with respect to x we get,

$$y_1=a\cos(ax+b)$$
$$=a\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2=a^2\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$=a^2\sin\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$=a^2\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3=a^3\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$=a^3\sin\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$=a^3\sin\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r=a^r\sin\left\{\frac{r\pi}{2}+(ax+b)\right\}\ ;\ where,\ r<n$$
$$\therefore\ y_n=a^n\sin\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

5. $y=\cos(ax+b)$

$sol: Given\ that,\ y=\cos(ax+b)$

Differentiating with respect to x we get,

$$y_1=-a\sin(ax+b)$$
$$=a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_2=-a^2\sin\left\{\frac{\pi}{2}+(ax+b)\right\}$$
$$=a^2\cos\left\{\frac{\pi}{2}+\frac{\pi}{2}+(ax+b)\right\}$$
$$=a^2\cos\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$\therefore\ y_3=-a^3\sin\left\{\frac{2\pi}{2}+(ax+b)\right\}$$
$$=a^3\cos\left\{\frac{\pi}{2}+\frac{2\pi}{2}+(ax+b)\right\}$$
$$=a^3\cos\left\{\frac{3\pi}{2}+(ax+b)\right\}$$

Similarly,

$$y_r=a^r\cos\left\{\frac{r\pi}{2}+(ax+b)\right\}\ ;\ where,\ r<n$$
$$\therefore\ y_n=a^n\cos\left\{\frac{n\pi}{2}+(ax+b)\right\}\quad Ans.$$

6. $y=e^{ax}\sin(bx+c)$

$sol: Given\ that,\ y=e^{ax}\sin(bx+c)$

Differentiating with respect to x we get,

$$y_1=ae^{ax}\sin(bx+c)+be^{ax}\cos(bx+c)$$
$$=e^{ax}\{a\sin(bx+c)+b\cos(bx+c)\}$$

put $a=r\cos\varphi$ *and* $b=r\sin\phi$

$$\therefore\ r=\sqrt{a^2+b^2}\ \ and\ \ \varphi=\tan^{-1}\left(\frac{b}{a}\right)$$

$$Now,\ y_1=e^{ax}\{r\cos\varphi\sin(bx+c)+r\sin\varphi\cos(bx+c)\}$$
$$=re^{ax}\sin(bx+c+\varphi)$$
$$\therefore\ y_2=re^{ax}\{a\sin(bx+c+\varphi)+b\cos(bx+c+\varphi)\}$$
$$=re^{ax}\{r\cos\varphi\sin(bx+c+\varphi)+r\sin\varphi\cos(bx+c+\varphi)\}$$
$$=r^2e^{ax}\sin(bx+c+2\varphi)$$
$$\therefore\ y_3=r^3e^{ax}\sin(bx+c+3\varphi)$$

Similarly,

$$y_n=r^ne^{ax}\sin(bx+c+n\varphi)$$
$$=\left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)\quad Ans.$$

7. $y=\ln(ax+b)$

$sol: Given\ that,\ y=\ln(ax+b)$

Differentiating with respect to x we get,

$$y_1=\frac{a}{(ax+b)}$$
$$\therefore\ y_2=-\frac{1.a^2}{(ax+b)^2}$$
$$\therefore\ y_3=\frac{1.2\,a^3}{(ax+b)^3}$$
$$\therefore\ y_4=-\frac{1.2.3\,a^4}{(ax+b)^4}$$

Similarly,

$$\therefore\ y_n=\frac{(-1)^{n-1}(n-1)!\ a^n}{(ax+b)^n}\quad Ans.$$

8. If $y = \sin nx + \cos nx$ then show that $y_r = n^r\left[1+(-1)^r \sin 2nx\right]^{\frac{1}{2}}$.

sol: Given that, $y = \sin nx + \cos nx$

Differentiating with respect to x we get,

$$y_1 = n\cos nx - n\sin nx$$

$$= n\sin\left(\frac{\pi}{2}+nx\right) + n\cos\left(\frac{\pi}{2}+nx\right)$$

$$\therefore\ y_2 = n^2\cos\left(\frac{\pi}{2}+nx\right) - n^2\sin\left(\frac{\pi}{2}+nx\right)$$

$$= n^2\sin\left(\frac{2\pi}{2}+nx\right) + n^2\cos\left(\frac{2\pi}{2}+nx\right)$$

$$\therefore\ y_3 = n^3\cos\left(\frac{2\pi}{2}+nx\right) - n^3\sin\left(\frac{2\pi}{2}+nx\right)$$

$$= n^3\sin\left(\frac{3\pi}{2}+nx\right) + n^3\cos\left(\frac{3\pi}{2}+nx\right)$$

Similarly,

$$y_r = n^r\sin\left(\frac{r\pi}{2}+nx\right) + n^3\cos\left(\frac{r\pi}{2}+nx\right)$$

$$= n^r\left[\left\{\sin\left(\frac{r\pi}{2}+nx\right) + \cos\left(\frac{r\pi}{2}+nx\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= n^r\left[\sin^2\left(\frac{r\pi}{2}+nx\right) + \cos^2\left(\frac{r\pi}{2}+nx\right) + 2\sin\left(\frac{r\pi}{2}+nx\right)\cos\left(\frac{r\pi}{2}+nx\right)\right]^{\frac{1}{2}}$$

$$= n^r\left[1+\sin 2\left(\frac{r\pi}{2}+nx\right)\right]^{\frac{1}{2}}$$

$$= n^r\left[1+\sin\left(r\pi+2nx\right)\right]^{\frac{1}{2}}$$

$$= n^r\left[1+(-1)^r\sin 2nx\right]^{\frac{1}{2}} \quad showed.$$

9. $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$

sol: Given that, $y = \dfrac{x^2+x-1}{x^3+x^2-6x}$

$$= \frac{x^2+x-1}{x\left(x^2+x-6\right)}$$

$$= \frac{x^2+x-1}{x(x-2)(x+3)}$$

$$= \frac{1}{6}.\frac{1}{x} + \frac{1}{2}.\frac{1}{(x-2)} + \frac{1}{3}.\frac{1}{(x+3)}$$

Differentiating with respect to x we get,

$$y_1 = -\frac{1}{6}.\frac{1}{x^2} - \frac{1}{2}.\frac{1}{(x-2)^2} - \frac{1}{3}.\frac{1}{(x+3)^2}$$

$$\therefore\ y_2 = \frac{1.2}{6}.\frac{1}{x^3} + \frac{1.2}{2}.\frac{1}{(x-2)^3} + \frac{1.2}{3}.\frac{1}{(x+3)^3}$$

$$\therefore\ y_3 = -\frac{1.2.3}{6}.\frac{1}{x^4} - \frac{1.2.3}{2}.\frac{1}{(x-2)^4} - \frac{1.2.3}{3}.\frac{1}{(x+3)^4}$$

Similarly,

$$\therefore\ y_n = (-1)^n n!\left[\frac{1}{6}.\frac{1}{x^{n+1}} + \frac{1}{2}.\frac{1}{(x-2)^{n+1}} + \frac{1}{3}.\frac{1}{(x+3)^{n+1}}\right] Ans.$$

Leibnitz's theorem: If *u* and *v* are two functions of *x*, then the nth derivative of their product is,

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \cdots\cdots + {}^n c_r u_{n-r} v_r + \cdots\cdots + u v_n$$

where the suffixes in *u* and *v* denote the order of differentiations of *u* and *v* with respect to *x*.

- Using Leibnitz's theorem find y_n of the following functions:

1. $y = x^3 \sin x$

Sol : Given that, $y = x^3 \sin x$

Differentiating n times by Leibnitz's theorem we get,

$$\begin{aligned} y_n &= \left(x^3 \sin x\right)_n \\ &= (\sin x)_n x^3 + {}^n c_1 (\sin x)_{n-1} \left(x^3\right)_1 + {}^n c_2 (\sin x)_{n-2} \left(x^3\right)_2 + {}^n c_3 (\sin x)_{n-3} \left(x^3\right)_3 + {}^n c_4 (\sin x)_{n-4} \left(x^3\right)_4 + \cdots\cdots + \sin x \left(x^3\right)_n \\ &= \sin\left(\frac{n\pi}{2} + x\right).x^3 + n \sin\left\{\frac{(n-1)\pi}{2} + x\right\}.3x^2 + \frac{n(n-1)}{2} \sin\left\{\frac{(n-2)\pi}{2} + x\right\}.6x + \frac{n(n-1)(n-2)}{6} \sin\left\{\frac{(n-3)\pi}{2} + x\right\}.6 + 0 \\ &= x^3 \sin\left(\frac{n\pi}{2} + x\right) - 3nx^2 \sin\left\{\frac{\pi}{2} - \left(\frac{n\pi}{2} + x\right)\right\} - 3n(n-1)x \sin\left\{\pi - \left(\frac{n\pi}{2} + x\right)\right\} - n(n-1)(n-2) \sin\left\{\frac{3\pi}{2} - \left(\frac{n\pi}{2} + x\right)\right\} \\ &= x^3 \sin\left(\frac{n\pi}{2} + x\right) - 3nx^2 \cos\left(\frac{n\pi}{2} + x\right) - 3n(n-1)x \sin\left(\frac{n\pi}{2} + x\right) + n(n-1)(n-2) \cos\left(\frac{n\pi}{2} + x\right) \\ &= \left\{x^3 - 3n(n-1)x\right\} \sin\left(\frac{n\pi}{2} + x\right) - \left\{3nx^2 - n(n-1)(n-2)\right\} \cos\left(\frac{n\pi}{2} + x\right) \quad \textit{Ans.} \end{aligned}$$

2. $y = x^2 \ln x$

Sol : Given that, $y = x^2 \ln x$

Differentiating n times by Leibnitz's theorem we get,

$$\begin{aligned} y_n &= \left(x^2 \ln x\right)_n \\ &= (\ln x)_n x^2 + {}^n c_1 (\ln x)_{n-1} \left(x^2\right)_1 + {}^n c_2 (\ln x)_{n-2} \left(x^2\right)_2 + {}^n c_3 (\ln x)_{n-3} \left(x^2\right)_3 + \cdots\cdots + \ln x.\left(x^2\right)_n \\ &= \frac{(-1)^{n-1}(n-1)!}{x^n}.x^2 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.2x + \frac{n(n-1)}{2}.\frac{(-1)^{n-3}(n-2)!}{x^{n-2}}.2 + 0 \\ &= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + \frac{2(-1)^{n-2} n(n-2)!}{x^{n-2}} + \frac{(-1)^{n-3} n(n-1)(n-2)!}{x^{n-2}} \quad \textit{Ans.} \end{aligned}$$

3. $y = e^x \ln x$

Sol : Given that, $y = e^x \ln x$

Differentiating n times by Leibnitz's theorem we get,

$$\begin{aligned} y_n &= \left(e^x \ln x\right)_n \\ &= (\ln x)_n e^x + {}^n c_1 (\ln x)_{n-1} \left(e^x\right)_1 + {}^n c_2 (\ln x)_{n-2} \left(e^{x2}\right)_2 + \cdots\cdots + \ln x.\left(e^x\right)_n \\ &= \frac{(-1)^{n-1}(n-1)!}{x^n}.e^x + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.e^x + \frac{n(n-1)}{2}.\frac{(-1)^{n-3}(n-2)!}{x^{n-2}}.e^x + \cdots\cdots + e^x \ln x. \quad \textit{Ans.} \end{aligned}$$

P−01: *If* $y = \tan^{-1} x$ *then show that* $\left(1+x^2\right)y_{n+2} + 2(n+1)xy_{n+1} + \left(n^2+n\right)y_n = 0$

Sol: *Given that*, $y = \tan^{-1} x$

Differentiating with respect to x we get,

$$y_1 = \frac{1}{1+x^2}$$

$$or, \left(1+x^2\right)y_1 = 1$$

Again, differentiating with respect to x we get,

$$\left(1+x^2\right)y_2 + 2xy_1 = 0$$

By Leibnitz's theorem we get,

$$\left(1+x^2\right)y_{n+2} + {}^nc_1.2x.y_{n+1} + {}^nc_2.2.y_n + 2xy_{n+1} + {}^nc_1.2.y_n = 0$$

$$or,\ \left(1+x^2\right)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}.2y_n + 2xy_{n+1} + 2ny_n = 0$$

$$or,\ \left(1+x^2\right)y_{n+2} + 2nxy_{n+1} + \left(n^2-n\right)y_n + 2xy_{n+1} + 2ny_n = 0$$

$$or,\ \left(1+x^2\right)y_{n+2} + 2(n+1)xy_{n+1} + \left(n^2-n+2n\right)y_n = 0$$

$$or,\ \left(1+x^2\right)y_{n+2} + 2(n+1)xy_{n+1} + \left(n^2+n\right)y_n = 0 \qquad showed.$$

P−02: *If* $y = \left(\sin^{-1} x\right)^2$ *then show that* $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Sol: *Given that*, $y = \left(\sin^{-1} x\right)^2$

Differentiating with respect to x we get,

$$y_1 = 2\sin^{-1} x.\frac{1}{\sqrt{1-x^2}}$$

$$or,\ y_1^{\,2} = 4\left(\sin^{-1} x\right)^2.\frac{1}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^{\,2} = 4y$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2 + (-2x).y_1^{\,2} = 4y_1$$

$$or,\ \left(1-x^2\right)y_2 - xy_1 = 2$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2} + {}^nc_1.(-2x).y_{n+1} + {}^nc_2.(-2).y_n - \left\{xy_{n+1} + {}^nc_1.1.y_n\right\} = 0$$

$$or,\ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n = 0$$

$$or,\ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \left(n^2-n\right)y_n - xy_{n+1} - ny_n = 0$$

$$or,\ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-n+n\right)y_n = 0$$

$$or,\ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \qquad showed.$$

P – 03 : *If* $y = e^{a\sin^{-1}x}$ *then show that* $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0$.

Sol : Given that, $y = e^{a\sin^{-1}x}$

Differentiating with respect to x we get,

$$y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$$

$$or,\ \ y_1^2 = \left(e^{a\sin^{-1}x}\right)^2.\frac{a^2}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^2 = a^2y^2$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2 + (-2x)y_1^2 = 2a^2yy_1$$

$$or,\ \left(1-x^2\right)y_2 - xy_1 = a^2y$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2} + {}^nc_1.(-2x).y_{n+1} + {}^nc_2.(-2).y_n - \left\{xy_{n+1} + {}^nc_1.1.y_n\right\} = a^2y_n$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \left(n^2-n\right)y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-n+n+a^2\right)y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0 \qquad showed.$$

P – 04 : *If* $y = \sin\left(a\sin^{-1}x\right)$ *then show that* $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-a^2\right)y_n = 0$.

Sol : Given that, $y = \sin\left(a\sin^{-1}x\right)$

Differentiating with respect to x we get,

$$y_1 = \cos\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}}$$

$$or,\ \ y_1^2 = \cos^2\left(a\sin^{-1}x\right).\frac{a^2}{\left(1-x^2\right)} \quad ;[Squaring\ both\ sides]$$

$$or, \left(1-x^2\right)y_1^2 = a^2\left\{1-\sin^2\left(a\sin^{-1}x\right)\right\}$$

$$or, \left(1-x^2\right)y_1^2 = a^2\left(1-y^2\right)$$

Again, differentiating with respect to x we get,

$$\left(1-x^2\right).2y_1\,y_2 + (-2x).y_1^2 = a^2(-2yy_1)$$

$$or,\ \left(1-x^2\right)y_2 - xy_1 = -a^2y$$

By Leibnitz's theorem we get,

$$\left(1-x^2\right)y_{n+2} + {}^nc_1.(-2x).y_{n+1} + {}^nc_2.(-2).y_n - \left\{xy_{n+1} + {}^nc_1.1.y_n\right\} = -a^2y_n$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - xy_{n+1} - ny_n + a^2y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \left(n^2-n\right)y_n - xy_{n+1} - ny_n + a^2y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-n+n-a^2\right)y_n = 0$$

$$or,\ \ \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2-a^2\right)y_n = 0 \qquad showed.$$

P – 05 : *If* $y = \cos\left\{\ln\left(1+x\right)\right\}$ *then show that* $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2+1\right)y_n = 0.$

Sol : Given that, $y = \cos\left\{\ln\left(1+x\right)\right\}$

Differentiating with respect to x we get,

$$y_1 = -\sin\left\{\ln\left(1+x\right)\right\}.\frac{1}{\left(1+x\right)}$$

or, $\left(1+x\right)y_1 = -\sin\left\{\ln\left(1+x\right)\right\}$

Again, differentiating with respect to x we get,

$$\left(1+x\right)y_2 + y_1 = -\cos\left\{\ln\left(1+x\right)\right\}.\frac{1}{\left(1+x\right)}$$

or, $\left(1+x\right)^2 y_2 + \left(1+x\right)y_1 = -y$

By Leibnitz's theorem we get,

$$\left(1+x\right)^2 y_{n+2} + {}^nc_1.2\left(1+x\right).y_{n+1} + {}^nc_2.2.y_n + \left(1+x\right)y_{n+1} + {}^nc_1.1.y_n = -y_n$$

or, $\left(1+x\right)^2 y_{n+2} + 2n\left(1+x\right)y_{n+1} + \frac{n\left(n-1\right)}{2}.2y_n + \left(1+x\right)y_{n+1} + ny_n + y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2-n\right)y_n + ny_n + y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2-n+n+1\right)y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2+1\right)y_n = 0$ *showed.*

P – 06 : *If* $y = \left(x^2-1\right)^n$ *then show that* $\left(x^2-1\right)y_{n+2} + 2xy_{n+1} - n\left(n+1\right)y_n = 0.$

Sol : Given that, $y = \left(x^2-1\right)^n$

Differentiating with respect to x we get,

$$y_1 = n\left(x^2-1\right)^{n-1}.2x$$

or, $\left(x^2-1\right)y_1 = 2nx\left(x^2-1\right)^n$;$\left[\text{Multiplying by } \left(x^2-1\right)\right]$

or, $\left(x^2-1\right)y_1 = 2nxy$

Again, differentiating with respect to x we get,

$$\left(x^2-1\right)y_2 + 2xy_1 = 2ny + 2nxy_1$$

or, $\left(x^2-1\right)y_2 + 2\left(1-n\right)xy_1 = 2ny$

By Leibnitz's theorem we get,

$$\left(1+x\right)^2 y_{n+2} + {}^nc_1.2\left(1+x\right).y_{n+1} + {}^nc_2.2.y_n + \left(1+x\right)y_{n+1} + {}^nc_1.1.y_n = -y_n$$

or, $\left(1+x\right)^2 y_{n+2} + 2n\left(1+x\right)y_{n+1} + \frac{n\left(n-1\right)}{2}.2y_n + \left(1+x\right)y_{n+1} + ny_n + y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2-n\right)y_n + ny_n + y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2-n+n+1\right)y_n = 0$

or, $\left(1+x\right)^2 y_{n+2} + \left(2n+1\right)\left(1+x\right)y_{n+1} + \left(n^2+1\right)y_n = 0$ *showed.*

Homework:-

1. Find the nth derivative of the following functions:

   a. $y=\dfrac{1}{x^2+5x+6}$ Ans: $y_n=(-1)^n n!\left[\dfrac{1}{(x+2)^{n+1}}-\dfrac{1}{(x+3)^{n+1}}\right]$

   b. $y=\dfrac{2x+3}{x^2+3x+2}$. Ans: $y_n=(-1)^n n!\left[\dfrac{1}{(x+1)^{n+1}}+\dfrac{1}{(x+2)^{n+1}}\right]$

2. $If\ y=\cot^{-1}x\ then\,show\,that\ (1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0.$

3. $If\ y=a\cos(\ln x)+b\sin(\ln x)\ then\,show\,that\ x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$

4. $If\ y=\sin\{a\ln(x+b)\}\ then\,show\,that\ (x+b)^2y_{n+2}+(2n+1)(x+b)y_{n+1}+(n^2+a^2)y_n=0.$

5. $If\ \ln y=\tan^{-1}x\ then\,show\,that\,(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0.$