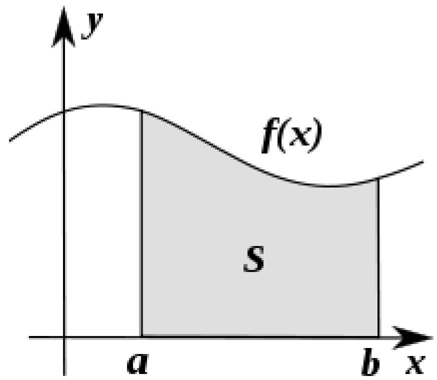
NUMERICAL INTEGRATION

There are two main reasons for you to need to do numerical integration: analytical integration may be impossible or infeasible, or you may wish to integrate tabulated data rather than known functions. In this section, we outline the main approaches to numerical integration.

Numerical integration is the approximate computation of integral using numerical techniques. The numerical computation of an integral is sometimes called quadrature. Therefore, the basic problem in numerical integration is to compute an approximate value to a definite integral

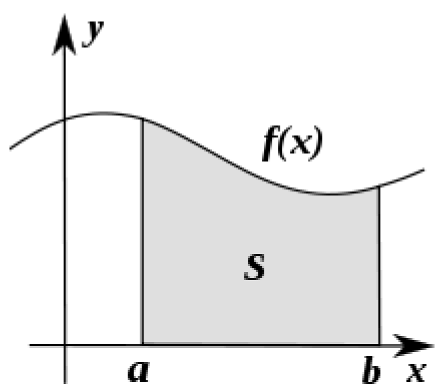
$$\int_a^b f(x)\,dx$$

to a given degree of accuracy.



Numerical Integration consists of finding numerical approximations for the value S.

Meaning of $\int_a^b f(x)\,dx$:



The definite integral $\int_a^b f(x)\,dx = \int_a^b y\,dx$ represents the area between the curve $y = f(x)$,the x-axis and the lines $x = a$ and $x = b$.

Numerical Integration:

Numerical integration is the process by which we can find the value of definite integral $\int_a^b f(x)\,dx$ numerically by using some well-established formulae or rules. The exact value of a definite integral $\int_a^b f(x)\,dx$ can be computed only when the function $f(x)$ is integrable in finite terms ,whenever the function $y = f(x)$ cannot be exactly integrated in finite terms or the evaluation of its integral is too cumbersome, integration can be more conveniently performed by numerical method. Various methods have been derived to find the above area approximately, in this case when $f(x)$ is not easily integrable. Hence these methods of approximating an area are essential methods for approximating a definite integral. The developed approximating methods are as follows:

i) Trapezoidal Rule
ii) Simpson's $\frac{1}{3}$ Rule
iii) Simpson's $\frac{3}{8}$ Rule
iv) Boole's Rule
v) Weddle's Rule
vi) Romberg's Integration Rule etc.

General Formula for Numerical Integration:

Let us consider an integral $\int_a^b f(x)\,dx$ where $f(x)$ is continuous on [a,b] and be given for certain equidistant values of x. Our intention here is to find the approximate value of the definite integral $\int_a^b f(x)\,dx$.

Assume the partition of [a,b] with equal distance h is

$$\text{Partition, } P: x_0 = a, x_1 = a+h, x_2 = a+2h, \ldots\ldots\ldots\ldots\ldots, x_n = a+nh = b$$

That means $y_0, y_1, y_2, \ldots\ldots\ldots\ldots\ldots, y_n$ be a set of (n+1) values of the function $y = f(x)$ corresponding to the equidistant values $x_0, x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n$ of the independent variable x.

Here $x_n = x_0 + nh \Rightarrow h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$, where a is a lower bound of the interval [a,b] and where b is the upper bound of the interval [a,b] and n is the number of intervals.

Now,

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (i)$$

From Newton's Forward Interpolation formula, we have,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ where}$$

$$u = \frac{x - x_0}{h} \Rightarrow x = x_0 + uh \qquad \therefore dx = h\,du$$

Limit Change:

When $x = x_0$ then $u = 0$

When $x = x_n$ then $u = n$

Therefore, above equation (i) takes the form,

$$\int_a^b f(x)\,dx = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] hdu$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^2-u)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right]_0^n$$

$$= h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1)terms \right)$$

This Formula is known as general quadrature formula or General formula for numerical integration and also known as General Gauss -Legendre integration formula for equidistant ordinates.

Note:

1. This formula is used to compute $\int_a^b f(x)\,dx$
2. Putting $n = 1$ in above equation we obtain Trapezoidal rule
3. Putting $n = 2$ in above equation we obtain Simpson's $\frac{1}{3}$ Rule
4. Putting $n = 3$ in above equation we obtain Simpson's $\frac{3}{8}$ Rule
5. Putting $n = 4$ in above equation we obtain Boole's Rule
6. Putting $n = 6$ in above equation we obtain Weddle's Rule

Trapezoidal Rule:

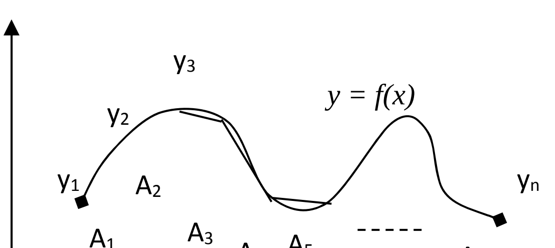
The general integration formula is

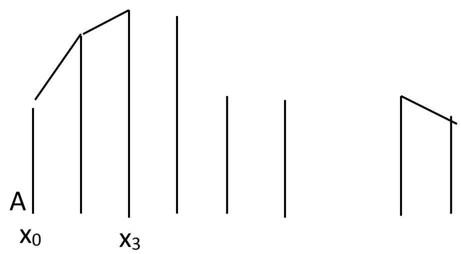
$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1)terms \right)$$

Setting $n = 1$ in above equation we have the interval $[x_0, x_1]$ and neglecting the higher order differences more than one, we get

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$
$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}(y_1 - y_0)\right)$$
$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right)$$
$$\int_{x_0}^{x_1} f(x)\,dx = h\left(\frac{1}{2}y_0 + \frac{1}{2}y_1\right)$$
$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0 + y_1)$$
Similarly, we can get,
$$\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$
$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$
By the rule of definite integral, we can write
$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \cdots\cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots\cdots\cdots + \frac{h}{2}(y_{n-1} + y_n)$$
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots\cdots\cdots + y_{n-1} + y_n)$$
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots\cdots + y_{n-1})]$$
$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$
$$\int_{a}^{b} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$
The above formula is known as the trapezoidal rule for numerical integration.

Geometrical Proof:





Let $y = f(x)$ is a continuous curve on [a,b] and we have to find $\int_a^b f(x)\,dx$ .Here $OA = a$ and $OB = b$.

Therefore $AB = OB - OA = b - a$ .Now divide the line segment AB into n equal parts with distance h so that $h = \dfrac{b-a}{n}\ [say]$.

Then the area,

$$A_1 = \frac{1}{2}(y_0 + y_1) \times h = \frac{h}{2}(y_0 + y_1)$$

Similarly, we find,

$$A_2 = \frac{1}{2}(y_1 + y_2) \times h = \frac{h}{2}(y_1 + y_2)$$

$$A_3 = \frac{1}{2}(y_2 + y_3) \times h = \frac{h}{2}(y_2 + y_3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

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$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_n = \frac{1}{2}(y_{n-1} + y_n) \times h = \frac{h}{2}(y_{n-1} + y_n)$$

Now,

$$\int_a^b f(x)\,dx = A_1 + A_2 + A_3 + \cdots\cdots\cdots\cdots\cdots\cdots + A_n$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \cdots\cdots\cdots\cdots\cdots\cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots\cdots\cdots\cdots\cdots + y_{n-1} + y_n)$$

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots\cdots\cdots\cdots + y_{n-1})]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

This is the well-known trapezoidal rule, so called because it approximates the integral by the sum of n trapezoidals. If the number of points of the line segments AB be increased a better approximation to the area well be obtained.

Simpson's $\frac{1}{3}$ Rule:

The general integration formula is

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1)terms\right)$$

Setting $n = 2$ in above equation we have the interval $[x_0, x_2]$ and neglecting the higher order differences more than two, we get

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{8-6}{3}\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \left(\frac{2}{3}\right)\frac{\Delta^2 y_0}{2}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2\Delta y_0 + \frac{\Delta^2 y_0}{3}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta(\Delta y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta(y_1 - y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\{(y_2 - y_1) - (y_1 - y_0)\}\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = h\left(2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(6y_0 + 6y_1 - 6y_0 + (y_2 - 2y_1 + y_0))$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, we can write,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\int_{x_4}^{x_6} f(x)\,dx = \frac{h}{3}(y_4 + 4y_5 + y_6)$$

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$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

By the rule of definite integral, we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \cdots\cdots\cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \frac{h}{3}\left(y_4 + 4y_5 + y_6\right) + \cdots\cdots\cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \cdots\cdots\cdots + y_{n-2} + 4y_{n-1} + y_n\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + y_5 + \cdots\cdots\cdots + y_{n-1}\right) + 2\left(y_2 + y_4 + y_6 + \cdots\cdots\cdots + y_{n-2}\right)\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

$$\int_{a}^{b} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

This is the Simpson's one third rule for numerical integration.
Note:
This formula is used only when the number of partitions of the interval of integration is even.

Simpson's $\frac{3}{8}$ Rule:

The general integration formula is

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots\cdots + upto\,(n+1)terms\right)$$

Setting $n = 3$ in above equation we have the interval $[x_0, x_3]$ and neglecting the higher order differences more than three, we get

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!}\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \left(9 - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \left(\frac{81}{4} - 18\right)\frac{\Delta^3 y_0}{6}\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \left(\frac{81}{4} - 18\right)\frac{\Delta^3 y_0}{6}\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6\Delta(y_1 - y_0) + \Delta^2(y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(\Delta y_1 - \Delta y_0) + \Delta(\Delta y_1 - \Delta y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - y_1 - y_1 + y_0) + \Delta(y_2 - y_1 - y_1 + y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + \Delta(y_2 - 2y_1 + y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (\Delta y_2 - 2\Delta y_1 + \Delta y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - y_2 - 2(y_2 - y_1) + y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + (y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + (y_3 - 3y_2 + 3y_1 - y_0)\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0\right)$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly, we can write,

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\int_{x_6}^{x_9} f(x)\,dx = \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

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$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

By the rule of definite integral, we can write

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \int_{x_6}^{x_9} f(x)\,dx + \cdots\cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}(y_0+3y_1+3y_2+y_3) + \frac{3h}{8}(y_3+3y_4+3y_5+y_6) + \frac{3h}{8}(y_6+3y_7+3y_8+y_9) + \cdots\cdots + \frac{3h}{8}(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\{(y_0+3y_1+3y_2+y_3)+(y_3+3y_4+3y_5+y_6)+(y_6+3y_7+3y_8+y_9)+\cdots\cdots+(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)\}$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\{y_0+3y_1+3y_2+y_3+y_3+3y_4+3y_5+y_6+y_6+3y_7+3y_8+y_9+\cdots\cdots+y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\}$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\{(y_0+y_n)+3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots\cdots+y_{n-2}+y_{n-1})+2(y_3+y_6+y_9+\cdots\cdots+y_{n-3})\}$$

$$\int_a^b f(x)dx = \frac{3h}{8}\left\{(y_0+y_n)+3\sum_{\substack{k\neq 3,6,9,\ldots\\ k=1}}^{n-1} y_k + 2\sum_{k=3,6,9,\ldots}^{n-3} y_k\right\}$$

This is the Simpson's three-eight rule for integration.

Note: This formula is used only when the number of partitions of the interval of integration is a multiple of the number 3.

Similarly, we can derive Boole's rule and Weddle's rule for numerical Integration for $n=4$ and $n=6$ respectively as follows:

Weddle's Rule:

$$\int_a^b f(x)dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{n} y_k + 5\sum_{k=1,3,5,\ldots}^{n-1} y_k + \sum_{k=3,6,9\ldots}^{n-3} y_k\right\}$$

Note:

1.This formula requires at least seven consecutive values of the function.

2. This formula is used only when the number of partitions of the interval of integration is a multiple of the number 6.

Mathematical Problems

1.Compute the definite integral $\int_{0.2}^{1.4} \left(\sin x - \ln x + e^x\right)\, dx$ by using various rules using 6 equidistant subintervals correct up to three decimal places.

Solution:

Here upper limit is $b = 1.4$, lower limit is $a = 0.2$ and No. of subintervals $n = 6$ and also

$y = f(x) = \sin x - \ln x + e^x$.

Now,

$$h = \frac{1.4-0.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4

y	3.0295	2.7975	2.8975	3.1660	3.5597	4.0698	4.7041

- **Trapezoidal Rule:**

  We know that

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[(y_0 + y_6) + 2\sum_{k=1}^{5} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$\int_a^b f(x)\,dx = \frac{0.2}{2}\left[(3.0295 + 4.7041) + 2(2.7975 + 2.8975 + 3.1660 + 3.5597 + 4.0698)\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{2}\left[7.7326 + 32.981\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{2} \times 40.7136$$

$$\int_{0.2}^{1.4} (\sin x + \ln x - e^x)\,dx = 4.07136$$

(As desired)

- Simpson's $\frac{1}{3}$ Rule:

  We know that

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{0.2}{3}\left[(y_0 + y_6) + 4\sum_{k=1,3,5,\ldots\ldots}^{5} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{4} y_k\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\int_a^b f(x)dx = \frac{0.2}{3}\left[(3.0295 + 4.7041) + 4(2.7975 + 3.1660 + 4.0698) + 2(2.8975 + 3.5597)\right]$$

$$\int_{0.2}^{1.4} (\sin x + \ln x - e^x)dx = \frac{0.2}{3}\left[7.7336 + 40.1332 + 12.9144\right] = 4.05208$$

(As desired)

□ **Simpson's $\frac{3}{8}$ Rule:**

We know that

$$\int_a^b f(x)dx = \frac{3h}{8}\left\{(y_0 + y_n) + 3\sum_{\substack{k\neq 3,6,9,\ldots \\ k=1}}^{n-1} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k\right\}$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\left\{(y_0 + y_6) + 3\sum_{\substack{k\neq 3,6,9,\ldots \\ k=1}}^{5} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{3} y_k\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\left\{(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\left\{(3.0295 + 4.7041) + 3(2.7975 + 2.8975 + 3.5597 + 4.0698) + 2\times 3.1660\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\left\{7.7336 + 3\times 13.3245 + 2\times 3.1660\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\times 54.0391$$

$$\int_{0.2}^{1.4}\left(\sin x + \ln x - e^x\right)dx = \frac{3\times 0.2}{8}\times 54.0391 = 4.0529$$

(As desired)

□ **Weddle's Rule:**

We know that

$$\int_a^b f(x)dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{n} y_k + 5\sum_{k=1,3,5,\ldots}^{n-1} y_k + \sum_{k=3,6,9\ldots}^{n-3} y_k\right\}$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{6} y_k + 5\sum_{k=1,3,5,\ldots}^{5} y_k + \sum_{k=3,6,9\ldots}^{3} y_k\right\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\left\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_3 + y_5) + y_3\right\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\left\{3.0295 + 2.8975 + 3.5597 + 4.7041 + 5(2.7975 + 3.1660 + 4.0698) + 3.1660\right\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\left\{14.1908 + 5\times 10.0333 + 3.1660\right\}$$

$$\int_{0.2}^{1.4}\left(\sin x+\ln x-e^{x}\right)dx=\frac{3\times 0.2}{10}\times 67.5233=4.051398$$ **(As desired)**

2.Evaluate $\int_{0}^{6} f(x)\,dx$ by using trapezoidal rule where the values of $f(x)$ are given by the following table:

x	0	1	2	3	4	5	6
Y=f(x)	0.146	0.161	0.176	0.190	0.204	0.217	0.230

Solution:

Here upper limit is $b=6$, lower limit is $a=0$ and No. of subintervals $n=6$.

Now,

$$h=\frac{6-0}{6}=1$$

The values of the function y at each subinterval are given in the tabular form:

x	0	1	2	3	4	5	6
Y=f(x)	0.146	0.161	0.176	0.190	0.204	0.217	0.230

From trapezoidal rule we have

$$\int_{a}^{b} f(x)\,dx=\frac{h}{2}\left[\left(y_0+y_n\right)+2\sum_{k=1}^{n-1} y_k\right]$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_{a}^{b} f(x)\,dx=\frac{1}{2}\left[\left(y_0+y_6\right)+2\sum_{k=1}^{5} y_k\right]$$

$$\int_{a}^{b} f(x)\,dx=\frac{1}{2}\left[\left(y_0+y_6\right)+2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$\int_{a}^{b} f(x)\,dx=\frac{1}{2}\left[\left(0.146+0.230\right)+2\left(0.161+0.176+0.190+0.204+0.217\right)\right]$$

$$\int_{0}^{6} f(x)\,dx=1.136$$ **(As desired)**

3.Compute $\int_{1}^{2} x^2\,dx$ by Simpson's one third rule and compare with exact value.

Solution:

Here upper limit is $b=2$, lower limit is $a=1$ and No. of subintervals $n=4$ and also $f(x)=x^2$.

Now,

$$h=\frac{2-1}{4}=\frac{1}{4}=0.25$$

The values of the function y at each subinterval are given in the tabular form:

x	1	1.25	1.50	1.75	2
Y=f(x)	1	1.5625	2.25	3.0625	4

From Simpson's $\frac{1}{3}$ Rule we have,

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

Now for $n = 4$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{0.25}{3}\left[(y_0 + y_n) + 4\sum_{k=1,3,5,\ldots\ldots}^{3} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{2} y_k\right]$$

$$\int_a^b f(x)\,dx = \frac{0.25}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$\int_a^b f(x)\,dx = \frac{0.25}{3}[(1+4) + 4(1.5625 + 3.0625) + 2\times 2.25]$$

$$\int_1^2 x^2\,dx = \frac{7}{3}$$

Now exact value is $\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3}$

It is shown that exact result and Simpson's $\frac{1}{3}$ Rule's result are exactly same so there is no error between two results.

(As desired)

4. Determine $\int_4^{5.2} \ln x\,dx$ by Simpson's 3/8 rule and Weddle's rule considering the number of intervals six. Find true value and then compare and comment on it.

Solution:

Here upper limit is $b = 5.2$, lower limit is $a = 4$ and No. of subintervals $n = 6$ and also $f(x) = \ln x$.

Now,

$$h = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

x	4	4.2	4.4	4.6	4.8	5.0	5.2
Y=f(x)	1.3862	1.4350	1.4816	1.5260	1.5686	1.6094	1.6486

Simpson's 3/8 rule:

We know that

$$\int_a^b f(x)dx = \frac{3h}{8}\left\{(y_0 + y_n) + 3\sum_{\substack{k\neq 3,6,9,\ldots \\ k=1}}^{n-1} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k\right\}$$

Now for $n = 6$ the above formula reduces to the following form,

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\left\{(y_0+y_6)+3\sum_{\substack{k\neq 3,6,9,\ldots\\ k=1}}^{5} y_k + 2\sum_{k=3,6,9,\ldots\ldots}^{3} y_k\right\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\{(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\}$$

$$\int_a^b f(x)\,dx = \frac{3\times 0.2}{8}\{(1.3862+1.6486)+3(1.4350+1.4816+1.5686+1.6094)+2\times 1.5260\}$$

$$\int_4^{5.2} \ln x\ dx = \frac{3\times 0.2}{8}\{3.0348+3\times 6.0946+2\times 1.5260\} = 1.827795$$

Weddle's Rule:

We know that

$$\int_a^b f(x)dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{n} y_k + 5\sum_{k=1,3,5,\ldots}^{n-1} y_k + \sum_{k=3,6,9\ldots}^{n-3} y_k\right\}$$

Now for $n=6$ the above formula reduces to the following form,

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\left\{\sum_{k=0,2,4,6,\ldots\ldots}^{6} y_k + 5\sum_{k=1,3,5,\ldots}^{5} y_k + \sum_{k=3,6,9\ldots}^{3} y_k\right\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\{y_0+y_2+y_4+y_6+5(y_1+y_3+y_5)+y_3\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\{1.3862+1.4816+1.5686+1.6486+5(1.4350+1.5260+1.6094)+1.5260\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\{6.085+5\times 4.5704+1.5260\}$$

$$\int_a^b f(x)dx = \frac{3\times 0.2}{10}\{6.085+5\times 4.5704+1.5260\}$$

$$\int_4^{5.2} \ln x\, dx = 1.82778$$

True value is $\displaystyle\int_4^{5.2} \ln x\,dx = [x\ln x]_4^{5.2} - \int_4^{5.2}\left[\frac{d}{dx}(\ln x)\int dx\right]dx$

$$= [x\ln x]_4^{5.2} - \int_4^{5.2}\left[\frac{1}{x}.\ x\right]dx$$

$$= [x\ln x]_4^{5.2} - \int_4^{5.2} dx$$

$$= [x\ln x]_4^{5.2} - [x]_4^{5.2}$$

$$= (5.2\ln 5.2 - 4\ln 4) - (5.2-4)$$

$$= 1.827847409$$

Result on Simpson's 3/8 rule and Weddle rule are closer to one another and also to the true value. That means both methods work well. **(As desired)**

Try Yourself

1.Derive newton's general quadrature formula for numerical integration.

2.Obtain the formula for Simpson's one-third rule from general quadrature formula.

3. Obtain the formula for Simpson's rule and Weddle's rule from general quadrature formula to find $\int_a^b f(x)\,dx$.

4.Using Simpson's 3/8 th rule find the value of $\int_0^3 e^{-2x}\sin 4x\,dx$ taking six sub-intervals.

5.Calculate the value of $\int_{1.2}^{1.8}\left(x+\frac{1}{x}\right)dx$,correct up 5D taking six sub-intervals by Simpson's 3/8 th rule. Also, find the percentage errors and compare to the exact solution.

6. Using Simpson's 3/8 th rule find the value of $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx$

7.Evaluate $\int_0^2 e^{2x}\sin 3x\,dx$,using Simpson's rule and Weddle's rule.

8.Using Simpson's three-eight rule evaluate the integrals $\int_{0.2}^{1.4}\left(\sin x+e^{2x}\right)dx$ and hence find the errors.

9.Discuss the necessity of numerical techniques of integration.

10.Calculate the value of the integral $I=\int_0^1 \frac{x\,dx}{1+x^2}$ by taking seven equidistant ordinates, using the Simpson's 1/3 rule and trapezoidal rule. Find the exact value of I and then compare and comment on it.

11.Find $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's 1/3 and 3/8 rules. Hence obtain the approximate value of π in each case.