Euler's method :

We have so farther yield the solution of a differential equation in the form of a power series. We will now proceed to describe methods which give the solution in the form of a set of tabulated values at equally spaced points.

We consider the differential equation

$$\frac{dy}{dx} = f(x, y) \cdots\cdots (1), \quad \text{with } y(x_0) = y_0 \cdots\cdots (2)$$

Suppose that we want to solve the equation (1) for y at $x = x_r = x_0 + rh, \; r = 1, 2, 3, \cdots$

Integrating (1) we obtain

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx \Rightarrow [y]_{y_0}^{y_1} = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$\therefore y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx \cdots\cdots (3)$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$, we get

$$y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$$

$$\therefore y_1 = y_0 + hf(x_0, y_0) \cdots\cdots (4)$$

Similarly for the range $x_1 \le x \le x_2$, we have

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)\, dx$$

Assuming that $f(x, y) = f(x_1, y_1)$ in $x_1 \le x \le x_2$, we obtain

$$y_2 = y_1 + hf(x_1, y_1)$$

Proceeding in this way, we get the general formula

$$y_{n+1} = y_n + hf(x_n, y_n), \; n = 0, 1, 2, 3, \cdots$$

This formula is known as Euler's formula.

This process is very slow. For practical use, the method is unsuitable because to get reasonable accuracy with this method we need to take a smaller value for h.

Example– Solve by Euler's method of the equation $\frac{dy}{dx} = x + y$, $y(0) = 1$ at the point $x = 0.05$ and $x = 0.10$ taking $h = 0.05$.

Solution : The given differential equation is

$$\frac{dy}{dx} = x + y$$

$\therefore f(x, y) = x + y$

Also we have $x_0 = 0$, $y_0 = 1$ and $h = 0.05$

We know that the Euler's formula for the numerical solution of the differential equation.

$$\frac{dy}{dx} = f(x, y) \text{ is } y_{n+1} = y_n + hf(x_n, y_n) \quad \cdots\cdots (1)$$

Putting $n = 0$ in (1), we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0)$$
$$= 1 + 0.05(0 + 1) = 1.05$$

$\therefore y(0.05) = 1.05$

Now putting $n = 1$ in (1), we get

$$y_2 = y_1 + hf(x_1, y_1)$$
$$\Rightarrow y_2 = y_1 + h(x_1 + y_1)$$
$$= 1.05 + .05(.05 + 1.05)$$
$$= 1.1050$$

$$\because x_1 = x_0 + h = 0 + .05 = .05$$

$\therefore y(0.1) = 1.1050$

Example– Using Euler's method solve $\frac{dy}{dx} = 1 + xy$ with $y(0) = 2$. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$.

Solution : The given differential equation is

$$\frac{dy}{dx} = 1 + xy$$

$\therefore f(x, y) = 1 + xy$

Also we have $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

we know that the Euler's formula for the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ is } y_{n+1} = y_n + hf(x_n, y_n) \cdots\cdots (1)$$

Putting $n = 0$ in (1), we get $y_1 = y_0 + hf(x_0, y_0)$

$$\Rightarrow y_1 = y_0 + h(1 + x_0 y_0) = 2 + 0.1(1 + 0) = 2.1$$

$\therefore y(0.1) = y_1 = 2.1$

Now $x_1 = x_0 + h = 0 + 0.1 = 0.1$

Putting $n = 1$ in (1) we get

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= y_1 + h(1 + x_1 y_1) = 2.1 + 0.1(1 + 0.1 \times 2.1) = 2.221$$

$\therefore y(0.2) = y_2 = 2.221$

Again, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Putting $n = 2$ in (1), we get

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 2.221 + 0.1 \times (1 + 0.2 \times 2.221) = 2.36542$$

$\therefore y(0.3) = y_3 = 2.36542.$

Example- . Use Euler's method with $h = 0.1$ to find the solution of $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$ in the range $0 \le x \le 0.5$.

Solution : The given differential equation is $\frac{dy}{dx} = x^2 + y^2$

$\therefore f(x, y) = x^2 + y^2$

Also we have $x_0 = 0$, $y_0 = 0$, $h = 0.1$

Using $x_r = x_0 + rh$, $r = 1, 2, 3, 4, 5$, we get

$$x_1 = x_0 + h = 0 + 0{,}1 = 0.1$$
$$x_2 = x_0 + 2h = 0 + 0.2 = 0.2$$
$$x_3 = 0.3$$
$$x_4 = 0.4, x = 0.5$$

We know that the Euler's formula for the numerical solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is

$$y_{n+1} = y_n + hf(x_n, y_n) \cdots\cdots (1)$$

Putting $n = 0$ in (1), we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0^2 + y_0^2) \quad [\because f(x, y) = x^2 + y^2]$$

$\Rightarrow y_1 = 0 + 0.1(0 + 0) = 0$

$\therefore y(0.1) = 0$

Putting $n = 0$ in (1), we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0^2 + y_0^2) \quad [\because f(x, y) = x^2 + y^2]$$

$\Rightarrow y_1 = 0 + 0.1(0 + 0) = 0$

$\therefore y(0.1) = 0$

Putting $n = 1$ in (1), we get

$y_2 = y_1 + hf(x_1, y_1)$ $\quad [\because f(x, y) = x^2 + y^2$

$\quad = 0 + 0.1 \times (0.1^2 + 0)$ $\quad \therefore f(x_1, y_1) = x_1^2 + y_1^2]$

$\quad = 0.001$

$\therefore y(0.2) = 0.001$

Again, Putting $n = 2$ in (1), we get

$y_3 = y_2 + hf(x_2, y_2)$ $\quad [\because f(x, y) = x^2 + y^2$

$\quad = 0.001 + 0.1(0.2^2 + 0.001^2)$ $\quad f(x_2, y_2) = x_2^2 + y_2^2]$

$\quad = 0.005 \quad \therefore y(0.3) = 0.005$

Putting $n = 3$ in (1), we get

$y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 0.005 + 0.1(0.3^2 + 0.005^2)$

$\therefore y_4 = 0.014 \quad \therefore y(0.4) = y_4 = 0.014$

Putting $n = 4$ in (1), we get

$y_5 = y_4 + hf(x_4, y_4) = 0.014 + 0.1\,(0.4^2 + 0.014^2) = 0.03002$

$\therefore y(0.5) = 0.03002.$

Modified Euler's Method :

By Euler's method, we have $y_1 = y_0 + hf(x_0, y_0)$ ………………(1)

Instead of approximating f(x, y) by $f(x_0, y_0)$ in (3), we approximate it by $\frac{1}{2}[f(x_0, y_0) + f(x_1, y_1)]$, which is average of the slopes of the tangents at the points corresponding to $x = x_0$ and $x = x_1$.

Thus from (1) we obtain $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

Here $y_1^{(1)}$ is the first modified value of y_1.

Let $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})$

Proceeding in this way, we thus obtain the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \; n = 0, 1, 2, \cdots$$

where $y_1^{(n)}$ is ~~the nth~~ approximation to y_1.

Example- Using modified Euler's method solve $\frac{dy}{dx} = 1 + xy$ with y(0) = 2. Find y(0.1), y(0,2) and y(0.3).

Solution : Given $\frac{dy}{dx} = 1 + xy$ with y(0) = 2

$\therefore f(x, y) = 1 + xy, \; x_0 = 0, \; y_0 = 2, \; h = 0.1$

By Euler's formula, we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h[1 + x_0y_0] = 2 + 0.1\,(1 + 0 \times 2) = 2.1$$

By modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = y_0 + \frac{h}{2}[1 + x_0y_0 + 1 + x_1y_1]$$

$$= 2 + \frac{0.1}{2}[1 + 0 + 1 + 0.1 \times 2.1] = 2.1105$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = y_0 + \frac{h}{2}[1 + x_0y_0 + 1 + x_1 y_1^{(1)}]$$

$$= 2 + \frac{0.1}{2}[1 + 0 + 1 + 0.1 \times 2.1105] = 2.11055$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})] = y_0 + \frac{h}{2}[1 + x_0y_0 + 1 + x_1y_1^{(2)}]$$

$$= 2 + \frac{0.1}{2}[1 + 0 + 1 + 0.1 \times 2.11055] = 2.11055$$

Now we find that $y_1^{(2)} = y_1^{(3)}$. So we stop here. Hence the required value y(0.1) = 2.11055.

Now, starting value of

$y_2 = y_1 + hf(x_0 + h, y_1) = y_1 + h[1 + (x_0 + h)y_1]$

$= 2.11055 + 0.1[1 + (0 + 0.1) \times 2.11055] = 2.23166$

$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$

$= y_1 + \frac{h}{2} [1 + x_0 + h)y_1 + 1 + (x_0 + 2h)y_2]$

$= 2.11055 + \frac{0.1}{2} [1 + (0 + 0.1) \times 2.11055 + 1 + (0 + 2 \times 0.1) \times 2.23266]$

$= 2.24342$

$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_0 + y, y_1) + f(x_0 + 2h, y_2^{(1)})]$

$= y_1 + \frac{h}{2} [1 + (x_0 + h)y_1 + 1 + (x_0 + 2h) y_2^{(1)}]$

$= 2.11055 + \frac{0.1}{2} [1 + (0 + 0.1 \times 2.11055 + 1 + (0 + 2 \times 0.1) \times 2.24342]$

$= 2.24354$

$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_0 + h, y_1) + f(x_0 + 2h, y_2^{(2)}]$

$= y_1 + \frac{h}{2} [1 + (x_0 + h)y_1 + 1 + (x_0 + 2h) y_2^{(2)}]$

$= 2.11055 + \frac{0.1}{2} [1 + (0 + 0.1) \times 2.11055 + 1 + (0 + 2 \times 0.1) \times 2.24354]$

$= 2.24354$

$\therefore$ we find that $y_2^{(2)} = y_2^{(3)}$. So we stop here. Hence the required value of y(0.2) is 2.24354.

Now staring value of

$y_3 = y_2 + hf(x_0 + 2h, y_2) = y_2 + h[1 + (x_0 + 2h)y_2]$

$= 2.24354 + 0.1 \times [1 + (0 + 2 \times 0.1) \times 2.24354] = 2,38841$

$y_3^{(1)} = y_2 + \frac{h}{2} [f(x_0 + 2h, y_2) + f(x_0 + 3h, y_3)]$

$= y_2 + \frac{h}{2} [1 + (x_0 + 2h)y_2 + 1 + x_0 + 3h)y_3]$

$= 2.24354 + \frac{0.1}{2} [1 + (0 + 2 \times 0.1) \times 2.24354 + 1$
$+ (0 + 3 \times 0.1) \times 2.38841]$

$= 2.4018$

$$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_0 + 2h, y_2) + f(x_0 + 3h, y_3^{(1)})]$$

$$= y_2 + \frac{h}{2}[1 + (x_0 + 2h)y_2 + 1 + (x_0 + 3h)\, y_3^{(1)}]$$

$$= 2.24354 + \frac{0.1}{2}[1 + (0 + 2 \times 0.1) \times 2.24354 +$$

$$+ (0 + 3 \times 0.1) \times 2.4018] = 2.4020$$

$$y_3^{(3)} = y_2 + \frac{h}{2}[f(x_0 + 2h, y_2) + f(x_0 + 3h, y_3^{(2)})]$$

$$= y_2 + \frac{h}{2}[1 + (x_0 + 2h)y_2 + 1 + (x_0 + 3h)y_3^{(2)}]$$

$$= 2.24354 + \frac{0.1}{2}[1 + (0 + 2 \times 0.1) \times 2.24354 + 1$$

$$+ (0 + 3 \times 0.1) \times 2.4018] = 2.4020$$

We find that $y_3^{(2)} = y_3^{(3)}$. So we stop here.
Hence the required value of $y(0.3) = 2.4020$

Example- Using modified Euler's method find the value of y when $x = 0.1$ of the equation $y' = x^2 + y$ when $y(0) = 1$.

Solution : Given $y' = x^2 + y$ with $y(0) = 1$

Here $f(x; y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$\therefore f(x_0, y_0) = x_0^2 + y_0 = 0 + 1 = 1$

$\therefore y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1 \times 1 = 1 + 0.1 = 1.1$

By Euler's Modified Formula for first approximation, we get

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$$

For $n = 0$, $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$$= 1 + \frac{0.1}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(0)})]$$

$$= 1 + \frac{0.1}{2}[(0 + 1) + (0.1^2 + 1.1)] = 1.10550$$

For $n = 1$, $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$$= 1 + \frac{0.1}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(1)})]$$

$$= 1 + \frac{0.1}{2}[(0 + 1) + (0.1^2 + 1.10550)] = 1.10578$$

For n = 2, $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$$= 1 + \frac{0.1}{2}[(x_0^2 + y_0) + (x_1^2 + y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}[(0 + 1) + (0.1^2 + 1.10578)] = 1.10579$$

For n = 3, $y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$

$$= 1 + \frac{0.1}{2}[(0 + 1) + (0.1^2 + 1.10579)] = 1.10579$$

Since $y_1^{(3)} = y_1^{(4)} = 1.10579$

So, y(0.1) = 1.10579

EXERCISE-

1. Use Euler's method with h = 0.025 to find the solution of the equation $y' = \frac{y - x}{y + x}$ with y(0) = 1 in the range $0 \le x \le 0.1$

   Ans : 1.0932
2. Use Euler's method to find y(0.4) when $y' = xy$ with y(0) = 1. Ans : 1.061106
3. Use Euler's method to find the solution of the equation $y' = -2xy^2$ with y(0) = 1 for x = 0.25(0.25)1.

   Ans : y(0.25) = 1, y(0.5) = 0.875, y(0.75) = 0.6835, y(0.1) = 0.9107
4. Solve $y' = x + \sqrt{y}$ with y(0) = 1 by modified Euler's method at x = 0.2, 0.4 and 0.6. Ans : y(0.6) = 1.882782
5. Use modified Euler's method to find the value of y for x = 0.2 when $y' = ln(x + y)$ with y(0) = 1 Ans : y(0.2) = 1.0082
6. Compute y(0.3) with h = 0.1 from $\frac{dy}{dx} = y - \frac{2x}{y}$, y(0) = 1. by modified Euler's method. Ans : y(0.3) = 1.2662