INTERPOLATION

Md. Mehedi Hasan, Lecturer @ DIU

Meaning of Interpolation:

In our daily life we are sometimes confronted with the problem where we become interested in finding some unknown values with help of a given set of observations. For example, if we are find out the population of Bangladesh in 1978 when we know the population of Bangladesh in the year 1971, 1975, 1979, 1984, 1988, 1992 and so on. i,e. the figure of population are available for 1971, 1975, 1979, 1984, 1988, 1992 etc, then the process of finding the population of 1978 is known as interpolation.

Definition of Interpolation:

In the mathematical field of numerical analysis,interpolation is a method of constructing new data points within the range of a discrete set of known data points.

Mathematically, let the observations of a function $y = f(x)$ be $f(a), f(a+h), f(a+2h) \dots \dots \dots \dots f(a+nh)$for $x = a, a+h, a+2h, \dots \dots a+nh$ respectively. Then the method of finding $f(x)$ for $x = \alpha$, where α lies in the range$a$ and $a+nh$ is known as **interpolation** and if the value of $x = \alpha$ lies outside this range it is called **extrapolation.**

For example, let as suppose we are given the following data.

$x =$	0	3	6	9	12	15
$f(x) =$	1	2	7	12	17	22

Then the method of finding $f(10)$ or $f(4)$ with help of the given data will be called **interpolation** and that for $f(27)$ or $f(-2)$ will be known as **extrapolation**.

Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *differences* of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0, \qquad \Delta y_1 = y_2 - y_1, \ldots, \qquad \Delta y_{n-1} = y_n - y_{n-1},$$

where Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \ldots,$ are called *first forward differences*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$. Similarly, one can define *third forward differences*, *fourth forward differences*, etc. Thus,

$$\begin{aligned}
\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0, \\
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0) \\
&= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.
\end{aligned}$$

It is therefore clear that any higher-order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

Table 3.1 shows how the forward differences of all orders can be formed:

Table 3.1 Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called *first backward differences* if they are denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, so that $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$, where ∇ is called the *backward difference operator*. In a similar way, one can define backward differences of higher orders. Thus we obtain

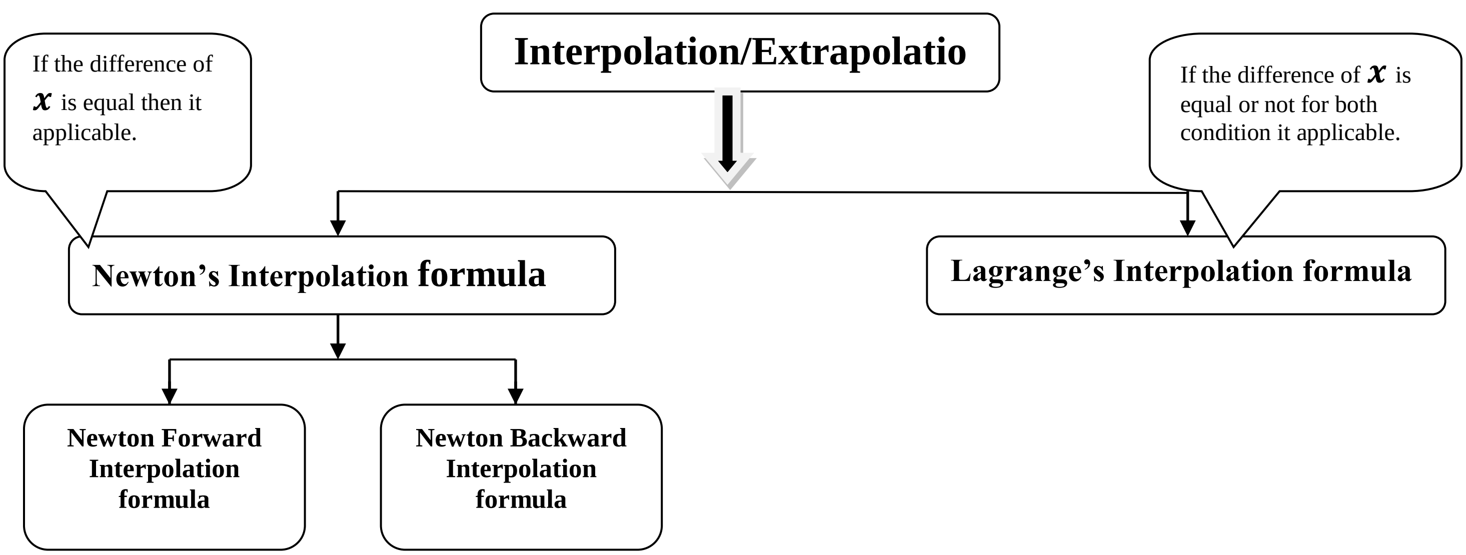
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0, \text{ etc.}$$

With the same values of x and y as in Table 3.1, a backward difference table can be formed:

Table 3.2 Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$



Newton's Forward Difference:

If the given data is,

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_3	....	y_n

Then the Newton's forward interpolation formula will be,

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots\ldots\ldots(u-n+1)}{n!}\Delta^n y_0$$

Where, $u = \frac{x - x_0}{h}$; h = difference of x which is always equal interval.

Newton's Backward Difference:

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x	x_0	x_1	x_2	x_3	....	x_n
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Then the Newton's backward interpolation formula will be,

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots\ldots(u+n-1)}{n!}\nabla^n y_n$$

Where, $u = \frac{x - x_n}{h}$; h = difference of x which is always equal interval.

Example– Find the value of y at $x = 21$ and $x = 28$ from the following data.

x :	20	23	26	29
y :	0.342	0.3907	0.4384	0.4848

Solution : We construct the difference table for the given data is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
23	0.3907		–0.0010	
		0.0477		–0.0003
26	0.4384		–0.0013	
		0.0464		
29	0.4848			

we have to find y(21). Here $u = \dfrac{21-20}{3} = 0.3333$ since $x = 21$ is nearer to the beginning of the table, so we use Newton's forward formula.

Now, by Newton's forward interpolation formula, we get

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$\therefore y(21) = 0.342 + 0.3333 \times (0.0487) + \frac{0.3333(0.3333-1)}{21}(-0.0010)$$
$$+ \frac{0.3333(0.3333-1)(0.3333-2)}{3!}(-0.0003) = 0.3583$$

Again we have to find y(28). Since $x = 28$ is nearer to end value. So we use Newton's backward interpolation formula. The lower most diagonal of the table gives the backward differences of y_n.

In this case $u = \dfrac{x - x_n}{h} = \dfrac{28-29}{3} = -0.3333$.

Now by Newton's backward interpolation formula, we have

$$y(x) = y_0 + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)(u+3)}{3!}\nabla^3 y_n$$

$$\therefore y(28) = 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(-0.3333+1)}{2}(-0.0013) + \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{6}(-0.0003)$$

$$= 0.46946$$

$\therefore$ The values of y at x = 21 and 28 are 0.3583 and 0.46946 respectively.

Example- The population of a town in the decennial census was as given below. Estimate the population for the year 1895.

Year	x	:	1891	1901	1911	1921	1931
Population (in thousands)	y	:	46	66	81	93	101

Solution : We construct the difference table for the given data is as follows :

Year (x)	Population (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

we have to find y(1895). Since x = 1895 is nearer to the beginning of the table, so we use Newton's forward interpolation formula. In this case

$$u = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4.$$

$\therefore$ By Newton's forward interpolation formula, we have

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$\therefore y(1895) = 46 + 0.4 \times 20 + \frac{0.4(0.4-1)}{2}(-5) + \frac{0.4(0.4-1)(0.4-2)}{3!} \times 2 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \cdot (-3)$$

= 54.853 thousands (approx.)

$\therefore$ The population for the year 1895 is 54.853 thousands (approx.)

Example Find the annual premium at the age of 30 from the following table.

Age :	21	25	29	33
Premium :	14.27	15.81	17.72	19.96

Solution : Since Age = 30 is nearer to the ending of the table. So we will use Newton's backward interpolation formula.

Now, we construct the difference table for the given data is as follows :

Age (x)	Premium y	∇y	$\nabla^2 y$	$\nabla^3 y$
21	14.27			
		1.54		
25	15.81		0.37	
		1.91		-0.04
29	17.72		0.33	
		2.24		
33	19.96			

Now, by Newton's backward interpolation formula, we have

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

where $u = \frac{x - x_n}{h}$

In our case $u = \frac{30 - 33}{4} = -0.75$

$$\therefore y(30) = 19.96 + (-0.75) \times 2.24 + \frac{-0.75(-0.75+1)}{2} 0.33$$

$$+ \frac{-0.75(-0.75+1)(-0.75+2)}{6}(-0.04) = 18.2506$$

Hence the annual premium at the age of 30 is 18.2506.

Example-12 From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46 and 63.

Age	x	:	45	50	55	60	65
Premium	y	:	114.84	96.16	83.32	74.48	68.48

Answer: Try yourself.

Example– A second degree polynomial passes through (0, 1), (1, 3), (2, 7) and (3, 13). Find the polynomial.

Solution : We construct the difference table for the given data is as follows :

x	y = f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

By Newton's forward interpolation formula, we get

$$f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2}\Delta^2 f(a), \quad \text{where } u = \frac{x-a}{h}$$

In this problem $u = \dfrac{x-0}{1}$ $[\because a = 0, h = 1]$

$= x$

$$\therefore f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0)$$

$$\Rightarrow f(x) = 1 + x \times 2 + \frac{x(x-1)}{2} \cdot 2 \Rightarrow f(x) = 1 + 2x + x^2 - x$$

$\therefore f(x) = x^2 + x + 1$, which is the required polynomial

Example– The following table is given

x	:	0	1	2	3	4
f(x)	:	3	6	11	18	27

find the function f(x)

Solution : We construct the difference table for the given data is as follows :

x	f(x)	Δf(x)	$\Delta^2 f(x)$	$f^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

By Newton's forward interpolation formula, we have

$$f(a + uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a), \text{ where } u = \frac{x-a}{h}$$

In this problem, a = 0, h = 1

$$\therefore u = \frac{x-0}{1} = x$$

$$\therefore f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0)$$

$$= 3 + x.3 + \frac{x(x-1)}{2} \times 2$$

$$= 3 + 3x + x^2 - x$$

$$\therefore f(x) = x^2 + 2x + 3$$

Example 3.4 Find the cubic polynomial which takes the following values: $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x-1)/2$. Substituting this value of p in Eq. (3.10), we obtain

$$y(x) = 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48)$$

$$= x^3 + 6x^2 + 11x + 6.$$

To determine $y(8)$, we observe that $p = 7/2$. Hence, formula (3.10) gives:

$$y(8) = 24 + \frac{7}{2}(96) + \frac{(7/2)(7/2-1)}{2}(120) + \frac{(7/2)(7/2-1)(7/2-2)}{6}(48) = 990.$$

Direct substitution in $y(x)$ also yields the same value.

For Practice:

1. Population was recorded as follows in a village.

Year	1941	1951	1961	1971	1981	1991
Population	2500	2800	3200	3700	4350	5225

Estimate the population for the year 1945.

Ans. 2607

3. From the table given below find sin 52^0 by using Newton's forward interpolation formula.

x	45	50	55	60
sin x	0.7071	0.7660	0.8192	0.8660

Ans. 0.7880032

4. The value of annuities are given for the following ages. Find the value of annuity at the age of 28.5.

Age	25	26	27	28	29
Annuity	16.2	15.9	15.6	15.3	15.0

Ans. 15.15

5. Evalute log 6237 from the following table.

x	45	50	55	60	65
$y_x = \log x$	1.65321	1.74897	1.74036	1.77815	1.81291

Ans. log 6237 = 3.74484

6. Estimate $e^{-1.9}$ from the given data.

x	1.00	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

Ans. 0.1496

7. Find y(1.02) given

x	1.00	1.05	1.10	1.15	1.20
y	0.3413	0.2531	0.3643	0.2749	0.3849

Ans. 0.34614

8. The population of a town in the decemmial census was as given below. Estimate the population for the year 1895 :

Years : x	1891	1901	1911	1921	1931
Population : y (in thousands)	46	66	81	93	101

Lagrange's Interpolation Formula :

Statement:

Given (n + 1) values of the function f(x) for $x = x_0, x_1, x_2, \cdots, x_n$ namely $f(x_0), f(x_1), f(x_2), \cdots, f(x_n)$ respectively, the formula states

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2) + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n).$$

Q: Derive Lagrange's interpolation formula for degree 3.

Answer: Let the function is $y = f(x)$.

For degree 3 we choose the value of x are x_0, x_1, x_2 and the corresponding values of the function $f(x)$ be $f(x_0), f(x_1),$ and $f(x_2)$.

Now we fit the polynomial of $f(x)$ of degree 3.

Let,
$$f(x) = a_0(x-x_1)(x-x_2) + a_1(x-x_0)(x-x_2) + a_2(x-x_0)(x-x_1) \quad \text{------} (1)$$

To find the value of a_0, a_1 and a_2 we put $x = x_0, x_1, x_2$ respectively in ① we get,

At $x = x_0$,

$$f(x_0) = a_0 (x - x_1)(x - x_2)$$

$$\therefore a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

At $x = x_1$ we get,

$$f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2)$$

$$\therefore a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)}$$

Again, at $x = x_2$ we get,

$$f(x_2) = a_2 (x_2 - x_0)(x_2 - x_1)$$

$$\therefore a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

Substituting these values a_0, a_1 and a_2 in ① we get

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Which is the required Lagrange's interpolation formula for degree 3.

Problem Solving:

Example- Apply Lagrange's formula to find $\log_{10}656$, using the following values of the function f(x)

x	654	658	659	661
f(x)	$\log_{10}654 = 2.8156$	$\log_{10}658 = 2.8182$	$\log_{10}659 = 2.8189$	$\log_{10}661 = 2.8202$

Solution : Here $x_0 = 654$, $x_1 = 658$, $x_2 = 659$ and $x_3 = 661$.

By Lagrange's formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Substituting the values of x_0, x_1, x_2, x_3 and corresponding values of f(x) in the above equation, we get

$$\therefore \log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$
$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$
$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$
$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$\Rightarrow \log_{10}656 = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{2(-3)(-5)}{4(-1)(-3)} \times 2.8182$$
$$+ \frac{2.(-2)(-5)}{5.1.(-2)} \times 2.8189 + \frac{2(-2)(-3)}{7.3.2} \times 2.8202$$
$$= 0.60334 + 7.0455 - 5.6378 + 0.80577$$

$\therefore \log_{10}656 = 2.81681$. Ans

Example 5. Estimate $\sqrt{(155)}$ using Lagrange's interpolation formula from the table given below :

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

Solution : Here $x = 155$, $x_0 = 150$, $x_1 = 152$, $x_2 = 154$ and $x_3 = 156$, $y_0 = 12.247$, $y_1 = 12.329$, $y_2 = 12.410$, $y_3 = 12.490$.

$x - x_0 = 155 - 150 = 5$	$x_0 - x_1 = 150 - 152 = -2$	$x_0 - x_2 = 150 - 154 = -4$	$x_0 - x_3 = 150 - 156 = -6$
$x_1 - x_0 = 152 - 150 = 2$	$x - x_1 = 155 - 152 = 3$	$x_1 - x_2 = 152 - 154 = -2$	$x_1 - x_3 = 152 - 156 = -4$
$x_2 - x_0 = 154 - 150 = 4$	$x_2 - x_1 = 154 - 152 = 2$	$x - x_2 = 155 - 154 = 1$	$x_2 - x_3 = 154 - 156 = -2$
$x_3 - x_0 = 156 - 150 = 6$	$x_3 - x_1 = 156 - 152 = 4$		

We know Lagrange's Interpolation formula is

$$y(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \cdot y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \cdot y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \cdot y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \cdot y_3 + \cdots$$

$$\therefore y(155) = \sqrt{155} = \frac{(3)(1)(-1)}{(-2)(-4)(-6)} \times 12.247 + \frac{(5)(1)(-1)}{(2)(-2)(-4)} \times 12.329 + \frac{(5)(3)(-1)}{(4)(2)(-2)} \times 12.410 + \frac{(5)(3)(1)}{(6)(4)(2)} \times 12.490$$

$$= 12.45.$$

Example– : Given the following data satisfying $y = f(x)$:

x	−1	0	2	5
y	9	5	3	15

Find the Lagrange polynomial.

Solution : Here $x_0 = -1$, $x_1 = 0$, $x_2 = 2$, $x_3 = 5$

$y_0 = 9$, $y_1 = 5$, $y_2 = 3$, $y_3 = 15$

$x - x_0 = x + 1$	$x_0 - x_1 = -1$	$x_0 - x_2 = -3$	$x_0 - x_3 = -6$
$x_1 - x_0 = 1$	$x - x_1 = x$	$x_1 - x_2 = -2$	$x_1 - x_3 = -5$
$x_2 - x_0 = 3$	$x_2 - x_1 = 2$	$x - x_2 = x - 2$	$x_2 - x_3 = -3$
$x_3 - x_0 = 6$	$x_3 - x_1 = 5$	$x_3 - x_2 = 3$	$x - x_3 = x - 5$

Lagrange's interpolation polynomial is :

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$\Rightarrow y(x) = \frac{x(x-2)(x-5)}{(-1)(-3)(-6)} \times 9 + \frac{(x+1)(x-2)(x-5)}{(1)(-2)(-5)} \times 5$$

$$+ \frac{(x+1)x(x-5)}{(3)(2)(-3)} \times 3 + \frac{(x+1)x(x-2)}{(6)(5)(3)} \times 15$$

$$= x^2 - 3x + 5.$$

Example– From the data in the following table find by Lagrange's formula the value of y when x = 27.

x :	22.0	23.5	25.2	28.7
y :	2.8	3.5	4.6	5.3

Solution : Here $x_0 = 22.0$, $x_1 = 23.5$, $x_2 = 25.2$, $x_3 = 28.7$

$y_0 = 2.8$, $y_1 = 3.5$, $y_2 = 4.6$, $y_3 = 5.3$

By Lagrange's interpolation formula, we have

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore y(27) = \frac{(3.5)(1.8)(-1.7)}{(-1.5)(-3.2)(-6.7)} \times (2.8) + \frac{(5.0)(1.8)(-1.7)}{(1.5)(-1.7)(-5.2)} \times (3.5)$$

$$+ \frac{(5.0)(3.5)(-1.7)}{(3.2)(1.7)(-3.5)} \times (4.6) + \frac{(5.0)(3.5)(1.8)}{(6.7)(5.2)(3.5)} \times (5.3)$$

$$= 0.93246 - 4.03846 + 7.18750 + 1.36912$$

$$= 5.45062$$

Hence the required value of y at x = 27 is 5.45062.

For Practice:

1 Estimate $\sqrt{155}$ using Lagrange's inter pollution's formula from the table given below :

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

2 Apply Lagrange's interpolation formula to find the values of f(8) and f(15) from the following table :

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Ans. 448, 3150

3 Apply Lagrange's interpolation formula to find the value of y when x = 10

x	5	6	9	11
y	12	13	14	16

Ans. 14.66

Another Problem:

Q: Given the data of the values

x	50	52	54	56
$\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Use Lagrange's formula to find x when $\sqrt[3]{x} = 3.756$

Solution: Here, given that $y = \sqrt[3]{x} = 3.756$ and we find the value of x.

$\therefore\ x_0 = 50,\ x_1 = 52,\ x_2 = 54,\ x_3 = 56$

$y_0 = 3.684,\ y_1 = 3.732,\ y_2 = 3.779,\ y_3 = 3.825$

and $y = 3.756$

We know the formula of Lagrange's interpolation is,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \times x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \times x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \times x_3$$

$$= \frac{(3.756-3.732)(3.756-3.779)(3.756-3.825)}{(3.684-3.732)(3.684-3.779)(3.684-3.825)} \times 50$$

$$+ \frac{(3.756-3.684)(3.756-3.779)(3.756-3.825)}{(3.732-3.684)(3.732-3.779)(3.732-3.825)} \times 52$$

$$+ \frac{(3.752-3.684)(3.752-3.732)(3.752-3.825)}{(3.779-3.684)(3.779-3.732)(3.779-3.825)} \times 54$$

$$+ \frac{(3.756-3.684)(3.756-3.732)(3.756-3.779)}{(3.825-3.684)(3.825-3.732)(3.825-3.779)} \times 56$$

$$\therefore x = 52.9879$$

(Answer).

For Practice:

Q1 From the table of values

x	y
1.8	2.9422
2.0	3.6269
2.2	4.4571
2.4	5.4662
2.6	6.6947

find x when $y = 5.0$ using the method of successive approximations.

Q2 From the following table of values, find x for which $\sinh x = 5$:

x	$\sinh x$
2.2	4.457
2.4	5.466
2.6	6.695
2.8	8.198
3.0	10.018

3.A.15. Merits of the Lagrange's formula :

(1) This formula is simple and easy to remember.

(2) There is no need to construct the difference table.

(3) We can directly find out the unknown value with the help of the given set of observations.

Demerits of the Lagrange's formula :

(1) The application of this formula is very slow.

(2) The calculations in the formula are more complicated than in the divided difference formula.

(3) The calculations provide no check whether the function values used are taken correctly or not.

(4) Due to a number of positive and negative signs in th denominatior of each term, the chance of committin some error is always there.

Comparisons Between Lagrange and Newton Interpolation:

The Lagrange and Newton interpolating formulas provide two different forms for an interpolating polynomial, even though the interpolating polynomial is unique.

- Lagrange method is numerically unstable but Newton's method is usually numerically stable and computationally efficient.
- Newton formula is much better for computation than the Lagrange formula.
- Lagrange form is most often used for deriving formulas for approximating derivatives and integrals

- Lagrange's form is more efficient then the Newton's formula when you have to interpolate several data sets on the same data points.