Solution of system of linear Equations

Chapter- 02

System of linear equation:

A system of linear equations is a collection of equations involving the same set of variables. That is the equations of the form

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots\cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots\cdots + a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + \cdots\cdots\cdots\cdots + a_{3n}x_n = b_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots\cdots\cdots + a_{mn}x_n = b_m
\end{aligned}
$$

We can write this system of equations as matrix form as follows

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} \\
a_{21} & a_{23} & a_{23} & \cdots & \cdots & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & \cdots & a_{mn}
\end{bmatrix}
.
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}
$$

We simply write AX=B

Where $A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} \\
a_{21} & a_{23} & a_{23} & \cdots & \cdots & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & \cdots & a_{mn}
\end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$

Here A is coefficient matrix while

$$
(A|B) = \left[\begin{array}{ccccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} & b_1 \\
a_{21} & a_{23} & a_{23} & \cdots & \cdots & \cdots & a_{2n} & b_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & \cdots & a_{mn} & b_n
\end{array}\right]
$$
is called an augmented matrix.

Solution of linear systems:

A solution of a system of linear equation s is an n-tuple that satisfies all equations in the system. Here the word system indicates that the equations are to be considered collectively. In other way we define it as a solution to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied. The solution of linear system of equations can be accomplished by a numerical method which falls in one of two categories:

Direct method:

These methods require no knowledge of an initial approximation and are used for solving polynomial equations. Direct methods involve a certain amount of fixed computation and they are exact solutions. In practice while using a computer direct methods lead to very poor and useless results. This is because of the various types of errors involved in numerical approximations.

Name of Some direct methods are,

1. **Gauss elimination Method**

2. Gauss Jordan Method
3. LU Decomposition Method (Method of Factorization/Triangularisation Method)
4. Choleski's Decomposition Method
5. Crout's Method
6. Cramer's Rule / Determinant Method
7. Matrix Inverse Method

Iterative/Indirect method:

Iterative methods are those in which the solution is got by successive approximations. But the method of iteration is not applicable to all system of equations. Some iterative methods may actually diverge and some others may converge so slowly that they are computationally useless. The iterative methods are suited for use in computers because of simplicity and uniformity of the operations to be performed.

Name of Some Indirect methods are,

1. Gauss Jacobi's Method
2. Gauss Seidel Method
3. Relaxaton Method

There are many such methods such as Newton-Raphson method, Iterative method, Bisection method etc.

Generally, solutions are two types according to solving system:

1. **Analytical solution:**
   A solution which is obtained by using direct method is called analytical solution. Analytical solutions are calculated using techniques that provide exact solutions. It may not always be possible to calculate the solution using analytical techniques. This is when a solution can be approximated using numerical techniques.
2. **Numerical solution:**
   A solution which is obtained by using approximation or iterative method is called numerical solution. In this case repeat the process till you get as close as you want to the required solution. Numerical methods sometimes are the only way to solve problems such as: sin(x) + x - 0.5 = 0

Direct Methods

In this Chapter we describe four direct methods namely Gauss elimination Method, Gauss Jordan Method, LU Decomposition Method, Choleski's Decomposition Method and also two iterative methods namely Gauss Jacobi's Method, Gauss Seidel Method.

Definition of strictly diagonally dominant matrix:

A matrix $A=(a_{ij})_{n\times n}$ is called strictly diagonally dominant if $\sum_{\substack{j\neq i\\ j=1}}^{n}|a_{ij}|<|a_{ii}|\quad ,i=1,2,......,n.$

Problem:

Check the system whether it is strictly diagonally dominant.

$$20x_1+2x_2+6x_3=28$$
$$x_1+20x_2+9x_3=-23$$
$$2x_1-7x_2-20x_3=-57$$

Solution:

Here $|a_{11}|=20\ ,|a_{22}|=20\ ,|a_{33}|=20.$

We have

$$|a_{12}|+|a_{13}|=2+6=8<20=|a_{11}|$$
$$|a_{21}|+|a_{23}|=1+9=10<20=|a_{22}|$$
$$|a_{31}|+|a_{32}|=2+7=9<20=|a_{33}|$$

So $$\sum_{\substack{j\neq i\\ j=1}}^{3}|a_{ij}|<|a_{ii}| \quad ,i=1\,,2,3.$$

Hence the system is strictly diagonally dominant.

Gauss elimination method:

The objective of gauss elimination method is to transform (reduce) the given system in to an equivalent upper triangular system with unit diagonal elements by forward elimination from which the solution can easily be obtained by back substitution.

Consider the system of n linear equations in n unknowns as

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots\cdots\cdots\cdots\cdots\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots\cdots\cdots\cdots\cdots\cdots+a_{2n}x_n=b_2\\ a_{31}x_1+a_{32}x_2+\cdots\cdots\cdots\cdots\cdots\cdots+a_{3n}x_n=b_3\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots\cdots\cdots\cdots\cdots\cdots+a_{mn}x_n=b_m \end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(i)$$

First stage:
We transform the given system into an upper-triangular form by forward elimination. This is done in the following steps.
Step-1: suppose that $a_{11}\neq 0$. Now eliminate x_1 from all but the first equation.
This is done as follows:
Divide the first equation by a_{11} and then multiply this first equation successively by $-a_{21},-a_{31},\cdots\cdots,-a_{n1}$ and add respectively with second, third, … , n^{th} equations of the system(i).Then the new system would be of the following form:

$$\begin{array}{l} x_1+a_{12}^{(1)}x_2+a_{13}^{(1)}x_3+\cdots\cdots+a_{1n}^{(1)}x_n=b_1^1\\ \qquad a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\cdots\cdots+a_{2n}^{(1)}x_n=b_2^1\\ \qquad a_{32}^{(1)}x_2+a_{33}^{(1)}x_3+\cdots\cdots+a_{3n}^{(1)}x_n=b_3^1 \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(\text{ii})\\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \qquad a_{n2}^{(1)}x_2+a_{n3}^{(1)}x_3+\ldots\ldots\ldots+a_{nn}^{(1)}x_n=b_n^1 \end{array}$$

Where the new coefficients and constants in $k^{th},k\geq 2$ equation are given by

$$\left.\begin{array}{l} a_{kj}^{(1)}=a_{kj}+a_{1j}^{(1)}\times(-a_{k1}),\ where\ a_{1j}^{(1)}=\dfrac{a_{1j}}{a_{11}}\\ b_k^{(1)}=b_k+b_1^{(1)}\times(-a_{k1}),\ where\, b_1^{(1)}=\dfrac{b_1}{a_{11}} \end{array}\right\}\ k\geq 2,\ j=1,2,\ldots.,n$$

Step-2: Let us assume that $a_{22}^{(1)} \neq 0$. Eliminate x_2 from the third equation through to the last equation in the new system (ii).
This is done as follows:
Normalize the 2nd equation by dividing it by $a_{22}^{(1)}$ and then multiply this normalized equation successively by - $a_{32}^{(1)}, -a_{42}^{(1)}, \ldots\ldots, -a_{n2}^{(1)}$ and add respectively with third, fourth, ………, n^{th} equations of the system (ii). Then the new system would be of the following form:

$$\left.\begin{aligned} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots\cdots + a_{1n}^{(1)}x_n &= b_1^{(1)} \\ x_2 + a_{23}^{(2)}x_3 + \cdots\cdots\cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\ a_{33}^{(2)}x_3 + \cdots\cdots\cdots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n3}^{(2)}x_3 + \cdots\cdots\cdots + a_{nn}^{(2)}x_n &= b_n^{(2)} \end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(iii)$$

Where the new coefficients and constants in $k^{th}(k \geq 3)$ equation given by

$$\left.\begin{aligned} a_{kj}^{(2)} &= a_{kj}^{(1)} + a_{2j}^{(2)} \times \left(-a_{k2}^{(1)}\right), \text{where } a_{2j}^{(2)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \\ b_k^{(2)} &= b_k^{(1)} + b_2^{(2)} \times \left(-a_{k2}^{(1)}\right), where\ b_2^{(2)} = \frac{b_2^{(1)}}{a_{22}^{(1)}} \end{aligned}\right\}\ k \geq 3\, and\ j = 2,3,4,\ldots\ldots,n.$$

Step-3: Assume that $a_{33}^{(2)} \neq 0$. now repeating the same process, eliminate x_3 from 4th equation through to the n^{th} equation. Then we can find another new system of equations.

Last step of elimination:

This elimination process is repeated until we get a triangular system of equations or the n^{th} equation containing only one unknown, namely x_n. Then the triangular form of equations will be as follows:

$$\left.\begin{aligned} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots + a_{1n}^{(1)}x_n &= b_1^{(1)} \\ x_2 + a_{23}^{(2)}x_3 + \cdots\cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\ x_3 + \cdots\cdots + a_{3n}^{(3)}x_n &= b_3^{(3)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= b_n^{(n)} \end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(iv)$$

Second stage (*Backward substitution*):

We obtain a unique solution for $x_n, x_{n-1}, \ldots\ldots, x_3, x_2, x_1$ from the above triangular system by back substitution. The solution is as follows:

$$x_k = b_k^{(k)}, \text{for k=n and } x_k = \left[b_k^{(k)} - \sum_{j=k+1}^{n} a_{kj}^{(k)}x_j\right], \text{ for k=n-1, n-2… 3,2,1.}$$

Restriction:

We now come to the important case to ensure that small values (especially zeros) do not appear on the diagonal and, if they do, to remove them by rearranging the matrix and vectors. If the pivot is zero, the entire process fails and if it is close to zero, round off errors may occur. These problems can be avoided by adopting a procedure called pivoting. If a_{11} is either zero or very small compared to the other coefficients of the equation, then we find the largest available coefficient in the columns below the pivot equation and then interchange the two rows. In this way, we obtain a new pivot equation with a nonzero pivot. Such a process is called partial pivoting. If we search both columns and rows for the largest element, the procedure is called complete pivoting. It is obvious that complete pivoting involves more complexity in computations since interchange of columns means change of order of unknowns which invariably requires more programming effort. In comparison, partial pivoting, i.e. row interchanges, is easily adopted in programming. Due to this reason, complete pivoting is rarely used.

Advantage:

We can use this process for many variables compare to the above Method. When we perform the elimination there is a lot of writing. It is especially inconvenient to carry on the symbols of variables.

Problem01:

Solve the following system of linear equations with the help of Gaussian elimination method.

$$3x + y + 2z = 13$$
$$2x + 3y + 4z = 19$$
$$x + 4y + 3z = 15$$

Solution:

Let us consider the following system

$$3x + y + 2z = 13$$
$$2x + 3y + 4z = 19$$
$$x + 4y + 3z = 15$$

We see that in the last equation the x coefficient is 1. It is very convenient because it would be an equation to use to eliminate x from the other equations. Let us move it to the front by changing the order of equations.

$$x + 4y + 3z = 15$$
$$3x + y + 2z = 13$$
$$2x + 3y + 4z = 19$$

Now the elimination starts. We add the first equation multiplied by (-3) to the second equation.

$$x + 4y + 3z = 15$$
$$-11y - 7z = -32$$
$$2x + 3y + 4z = 19$$

We add the first equation multiplied by (-2) to the third equation.

$$x + 4y \ + 3z = 15$$
$$-11y - 7z = -32$$
$$-5y \ - 2z = -11$$

The system still does not fit to back-substitution. We need to eliminate the variable y from the last equation. In order to do it we need first to multiply the third equation by (-11).

$$x + 4y \ + 3z = 15$$
$$-11y - 7z = -32$$
$$55y \ + 22z = 121$$

Then we add the second equation multiplied by 5 to third equation.

$$x + 4y \ + \ 3z = 15$$
$$-11y - \ 7z = -32$$
$$-22z = 121$$

Now we can use the back-substitution Method. It gives us the solution (2,1 ,3).

Problem02:

Solve the following system of linear equations

$$\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}$$

Solution: The given system of linear equations is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (i)$$

The system (i) can be written as following Matrices form

$$\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}.$$

The augmented matrix is,

$$[A, B] = \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 5 & -17 & \vdots & -24 \end{pmatrix} \begin{matrix} R_2' \to 2R_2 + R_1 \\ R_3' \to 2R_3 - 3R_1 \end{matrix}$$

$$\approx \begin{pmatrix} 2 & -1 & 3 & \vdots & 8 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & -76 & \vdots & -152 \end{pmatrix} R_3' \to 3R_3 - 5R_2$$

Above Augmented matrix is in row echelon form.

Therefore, the reduced system is,

$$\left.\begin{aligned} 2x - y + 3z &= 8 \\ 3y + 5z &= 16 \\ -76z &= -152 \end{aligned}\right\}$$

By back substitution we get, $z = 2$, $y = 2$, $x = 2$.

Hence the given system is consistent and the solution is,

$$x = 2, \ y = 2, z = 2.$$

Problem03:

Solve by the Gauss Elimination Method the equations

$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Solution: The given system of linear equations is,

$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

Let us rearrange the given system to make it diagonally dominant,

$$8x_1 - 3x_2 + 4x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$2x_1 + x_2 + 4x_3 = 12$$

Converting the given system of linear equations into upper triangular form by using successive operations we get,

$$\approx \left.\begin{array}{l} 8x_1 - 3x_2 + 4x_3 = 20 \\ 0 \;\; + 25x_2 - 6x_3 = 46 \\ 0 \;\; + 7x_2 + 12x_3 = 28 \end{array}\right\} \qquad \begin{array}{l} L_2' = 2L_2 - L_1 \\ L_3' = 4L_3 - L_1 \end{array}$$

$$\approx \left.\begin{array}{l} 8x_1 - 3x_2 + 4x_3 = 20 \\ 0 \;\; + 25x_2 - 6x_3 = 46 \\ 0 \;\; + \; 0 \;\; + 342x_3 = 378 \end{array}\right\} \qquad L_3' = 25L_3 - 7L_2$$

Finally using backward substitution, we get

$$x_3 = \frac{378}{342} = \frac{19}{21} \quad , \; x_2 = \frac{40}{19} \text{ and } x_1 = \frac{52}{19}$$

(As desired)

Try Yourself
Mathematical Problems

1. Use Gaussian elimination Method to Solve the following system

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

2. Solve the system

$$
\begin{aligned}
2x+3y+z&=9\\
x+2y+3z&=6 \text{ by the Gauss elimination method.}\\
3x+y+2z&=8
\end{aligned}
$$

3. Solve the following system of linear equations

$$
\begin{aligned}
3x+5y-7z&=13\\
4x+y-12z&=6\\
2x+9y-3z&=20
\end{aligned}
$$

4.Solve the following system of linear equations

$$
\begin{aligned}
x+y+z&=3\\
x+2y+3z&=4\\
x+4y+9z&=6
\end{aligned}
$$

5.Solve the following system of linear equations

$$
\begin{aligned}
2x-3y+4z&=1\\
3x+4y-5z&=1\\
5x-7y+2z&=3
\end{aligned}
$$

6.Using the Gauss Elimination Method solve the following system of equations

$$
\begin{aligned}
5x_1-2x_2+x_3&=4\\
7x_1+x_2-5x_3&=8\\
3x_1+7x_2+4x_3&=10
\end{aligned}
$$

Retain the results in the form of $\frac{p}{q}$ if necessary.

7. Using the Gauss Elimination Method solve the following system of equations

$$
\begin{aligned}
3x_1+2x_2+4x_3&=7\\
2x_1+x_2+x_3&=7\\
x_1+3x_2+5x_3&=2
\end{aligned}
$$

8.Solve the following system of equations by Gauss Elimination Method

$$
\begin{aligned}
x_1+3x_2+2x_3&=5\\
2x_1-x_2+x_3&=-1\\
x_1+2x_2+3x_3&=2
\end{aligned}
$$

9.Solve by Gauss Elimination Method

$$
\begin{aligned}
2x_1+2x_2+4x_3&=18\\
x_1+3x_2+2x_3&=13\\
3x_1+x_2+3x_3&=14
\end{aligned}
$$

10. Using the Gauss Elimination Method solve

$$
\begin{aligned}
x+\frac{y}{2}+\frac{z}{3}&=1\\
\frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=0\\
\frac{x}{3}+\frac{y}{4}+\frac{z}{5}&=0
\end{aligned}
$$

11.Explain/Discuss the Gauss Elimination Method to solve a set of n linear algebraic equations with n unknowns considering the case of without zero division.

12.When does a system said to be Strictly diagonally dominant?
13.Write an algorithm for the method you discussed in above.

Modified Gauss Elimination Method (Jordan Method)
The objective of Gauss-Jordan elimination method is to transform the given system into an equivalent unit diagonal system by elimination method from which the solution can easily be obtained.
Consider a general system of n equations with n unknowns:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots\cdots\cdots + a_{3n}x_n = b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + a_{nn}x_n = b_n \end{array}\right\} \cdots\cdots\cdots\cdots (i)$$

First stage:
we transform the given system into a unit diagonal from by elimination method. This is done in the following steps:
Step-1: Suppose $a_{11} \neq 0$. Now eliminate x_1 from all but first equation. Divide the first equation by a_{11} and hence multiply this new first equation successively by $-a_{21}, -a_{31}, \dots\dots, -a_{n1}$ and then add respectively with second third, … , n^{th} equation of the system (1) . Then the new system is of the following form

$$\left.\begin{array}{l} x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 + \cdots\cdots\cdots + a_{1n}^{(1)}x_n = b_1^1 \\ 0 + a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots\cdots\cdots + a_{2n}^{(1)}x_n = b_2^1 \\ 0 + a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \cdots\cdots\cdots + a_{3n}^{(1)}x_n = b_3^1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 + a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots\cdots\cdots + a_{nn}^{(1)}x_n = b_n^1 \end{array}\right\} \cdots\cdots\cdots\cdots (ii)$$

where the new coefficient and constant in k th $(k \neq 1)$ equation are given by

$$a_{kj}^{(1)} = a_{kj} + a_{1j}^{(1)} \times (-a_{k1}),\ where\ a_{1j}^{(1)} = \frac{a_{1j}}{a_{11}}\ and\ b_k^{(1)} = b_k + b_1^{(1)} \times (-a_{k1})\, where\, b_1^{(1)} = \frac{b_1}{a_{11}}, k \geq 2, j = 1,2,\dots,n$$

Step-2: we assume that $a_{22}^{(1)} \neq 0$. Now eliminate x_2 from all but second equation in the new system (2). Divide the second equation by $a_{22}^{(1)}$ and hence multiply this new second equation successively by $-a_{12}^{(1)}, -a_{32}^{(1)}, \dots\dots\dots\dots\dots, -a_{n2}^{(1)}$ and then add respectively with first, third, ……….., n^{th} equation of the system (2).
Then the system would be form:

$$\left.\begin{array}{l} x_1+0+a_{13}^{(2)}x_3+\cdots\cdots\cdots+a_{1n}^{(2)}x_n=b_1^{(2)} \\ 0+x_2+a_{23}^{(2)}x_3+\cdots\cdots\cdots+a_{2n}^{(2)}x_n=b_2^{(2)} \\ 0+0+\ a_{33}^{(2)}x_3+\cdots\cdots\cdots+a_{3n}^{(2)}x_n=b_3^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0+0+\ a_{n3}^{(2)}x_3+\cdots\cdots\cdots+a_{nn}^{(2)}x_n=b_n^{(2)} \end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(iii)$$

where the new coefficient and constant in $k^{th}, k \neq 2$ equation are given by

$a_{kj}^{(2)} = a_{kj}^{(1)} + a_{2j}^{(2)} \times \left(-a_{k2}^{(1)}\right)$,

where $a_{2j}^{(2)} = \dfrac{a_{2j}^{(1)}}{a_{22}^{(1)}}$ *and* $b_k^{(2)} = b_k^{(1)} + b_2^{(2)} \times \left(-a_{k2}^{(1)}\right)$, *where* $b_2^{(2)} = \dfrac{b_2^{(1)}}{a_{22}^{(1)}}, k \geq 3$ *and* $j = 2,3,4,........,n.$

Step-3: we assume that $a_{33}^{(2)} \neq 0$. Now repeating the same process eliminate x_3 from all but third equation of the system (3) . Divide the third equation by $a_{33}^{(2)}$ and hence multiply this new third equation successively by $-a_{13}^{(2)}, -a_{23}^{(2)}, -a_{43}^{(2)}, ..., -a_{n3}^{(2)}$ and then add respectively with first, third, … … , n^{th} equation of the system (3).

Then the system would as follows:

$$\left.\begin{array}{ccccccccccccc} x_1 & + & 0 & + & 0 & + & a_{14}^{(3)}x_4 & + & \cdots & + & a_{1n}^{(3)}x_n & = & b_1^{(3)} \\ 0 & + & x_2 & + & 0 & + & a_{24}^{(3)}x_4 & + & \cdots & + & a_{2n}^{(3)}x_n & = & b_2^{(3)} \\ 0 & + & 0 & + & x_3 & + & a_{34}^{(3)}x_4 & + & \cdots & + & a_{3n}^{(3)}x_n & = & b_3^{(3)} \\ \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots. & \ldots. & \ldots & \ldots. & \ldots. & \ldots & \ldots & \ldots \\ 0 & + & 0 & + & 0 & + & a_{n4}^{(3)} & + & \cdots & + & a_{nn}^{(3)}x_n & = & b_n^{(3)} \end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(iv)$$

Where the new coefficients and constants in $k^{th}(k \neq 3)$ equation are given by

$$a_{kj}^{(3)} = a_{kj}^{(2)} - a_{3j}^{(3)} \times a_{k3}^{2}\ where\ a_{3j}^{(3)} = \frac{a_{3j}^{(2)}}{a_{33}^{2}}\ and\ b_k^{(3)} = b_k^{(2)} - b_3^{(3)} \times a_{k3}^{(2)}\ where$$

$$b_3^{(3)} = \frac{b_3^{(2)}}{a_{33}^{(2)}}, k \neq 3,\ j = 3,4,...,n$$

Last step elimination:

This elimination process repeated until we get a until we get a unit diagonal system of equations, that is, first, second, third, ……, n^{th} equation containing the corresponding unknown only. Then the final form of equations will be as follows:

$$
\left.\begin{array}{ccccccccccc}
x_1 & + & 0 & + & 0 & + & \cdots & + & 0 & = & b_1^{(n)} \\
0 & + & x_2 & + & 0 & + & \cdots & + & 0 & = & b_2^{(n)} \\
0 & + & 0 & + & x_3 & + & \cdots & + & 0 & = & b_3^{(n)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & = & \cdots \\
0 & + & 0 & + & 0 & + & \cdots & + & x_n & = & b_n^{(n)}
\end{array}\right\} \cdots\cdots\cdots\cdots\cdots(v)
$$

Second stage: The system (5) has unique solutions for $x_1, x_2, x_3, \dots\dots\dots\dots\dots, x_n$ and can easily be obtained from the system (5). The solution is:

$$x_k = b_k^{(n)} \quad , \quad k = 1, 2, \dots\dots\dots..n.$$

Problem 01:

Solve the following system of linear equations with the help of Gauss-Jordan method

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &= 6 \\
4x_1 + 3x_2 + 2x_3 &= 9 \\
7x_1 + 5x_2 - x_3 &= 11
\end{aligned}
$$

Solution: Given that

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &= 6 \\
4x_1 + 3x_2 + 2x_3 &= 9 \\
7x_1 + 5x_2 - x_3 &= 11
\end{aligned}
$$

Reduce the given system to canonical form we get the desired solution of the system

$$
\approx \left.\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{x_3}{3} &= 2 \\
4x_1 + 3x_2 + 2x_3 &= 9 \\
7x_1 + 5x_2 - x_3 &= 11
\end{aligned}\right\} \qquad L_1' = \frac{L_1}{3}
$$

$$
\approx \left.\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 2 \\
0 + \frac{1}{3}x_2 + \frac{2}{3}x_3 &= 1 \\
0 + \frac{1}{3}x_2 - \frac{10}{3}x_3 &= -3
\end{aligned}\right\} \qquad \begin{aligned} L_2' &= L_2 - 4L_1 \\ L_3' &= L_3 - 7L_1 \end{aligned}
$$

$$
\approx \left.\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 2 \\
0 + x_2 + 2x_3 &= 3 \\
0 + \frac{1}{3}x_2 - \frac{10}{3}x_3 &= -3
\end{aligned}\right\} \qquad L_2' = 3L_2
$$

$$\approx \left.\begin{array}{l} x_1+0-x_3=0 \\ 0+x_2+2x_3=3 \\ 0+0-4x_3=-4 \end{array}\right\} \quad \begin{array}{l} L_1' = L_1 - \frac{2}{3}L_2 \\ \\ L_3' = L_3 - \frac{1}{3}L_2 \end{array}$$

$$\approx \left.\begin{array}{l} x_1+0-x_3=0 \\ 0+x_2+2x_3=3 \\ 0+0+x_3=1 \end{array}\right\} \quad L_3' = -\frac{1}{4}L_3$$

$$\approx \left.\begin{array}{l} x_1+0+0=1 \\ 0+x_2+0=1 \\ 0+0+x_3=1 \end{array}\right\} \begin{array}{l} L_1' = L_1 + L_3 \\ L_2' = L_2 - 2L_3 \\ \end{array}$$

From above canonical form we have $x_3=1$, $x_2=1$ and $x_1=1$. **(As desired)**

Try Yourself
Mathematical Problems

1.Use Gaussian Jordan Method to Solve the following system

$$\begin{array}{l} 2x+y+z=10 \\ 3x+2y+3z=18 \\ x+4y+9z=16 \end{array}$$

2.Solve the system

$$\begin{array}{l} 2x+3y+z=9 \\ x+2y+3z=6 \\ 3x+y+2z=8 \end{array}$$ by the Gauss Jordan method.

3. Solve the following system of linear equations

$$\begin{array}{l} 3x+5y-7z=13 \\ 4x+y-12z=6 \\ 2x+9y-3z=20 \end{array}$$

4.Solve the following system of linear equations

$$\begin{array}{l} 2x+2y+4z=14 \\ 3x-y+2z=2 \\ 5x+2y-2z=2 \end{array}$$

5.Solve the following system of linear equations

$$\begin{array}{l} 2x-3y-5z=11 \\ 5x+2y-7z=-12 \\ -4x+3y+z=5 \end{array}$$

6.Using the Gauss Jordan Method solve the following system of equations

$$5x_1 - 2x_2 + x_3 = 4$$
$$7x_1 + x_2 - 5x_3 = 8$$
$$3x_1 + 7x_2 + 4x_3 = 10$$

Retain the results in the form of $\frac{p}{q}$ if necessary.

7. Using the Gauss Jordan Method solve the following system of equations

$$2x_1 - 3x_2 + x_3 = -1$$
$$x_1 + 4x_2 + 5x_3 = 25$$
$$3x_1 - 4x_2 + x_3 = 2$$

8.Solve the following system of equations by Gauss Jordan Method

$$x_1 + 3x_2 + 2x_3 = 5$$
$$2x_1 - x_2 + x_3 = -1$$
$$x_1 + 2x_2 + 3x_3 = 2$$

9.Solve by Gauss Jordan Method

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

10. Using the Gauss Jordan Method solve

$$x + \frac{y}{2} + \frac{z}{3} = 1$$
$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 0$$
$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 0$$

11.Using the Gauss Jordan Method solve

$$x_1 + 3x_2 + 2x_3 = 5$$
$$2x_1 - x_2 + x_3 = -1$$
$$x_1 + 2x_2 + 3x_3 = 2$$

12.Explain/Discuss the Gauss Jordan Method to solve a set of n linear algebraic equations with n unknowns considering the case of without zero division.

13.Write an algorithm for the method you discussed in above.

Definition of LU Decomposition:

The nonsingular matrix **A** has an LU-factorization if it can be expressed as the product of a lower-triangular matrix **L** and an upper triangular matrix **U: A=LU**

When this is possible we say that **A** has an **LU**-decomposition. It turns out that this factorization (when it exists) is not unique. If **L** has 1's on its diagonal, then it is called a Doolittle factorization. If **U** has 1's on its diagonal, then it is called a Crout factorization. When $U=L^T$ (or $L=U^T$), it is called a Cholesky decomposition.

Theorem (A = LU; Factorization with NO Pivoting):

Assume that A has a Doolittle, Crout or Cholesky factorization. The solution X to the linear system AX=B is found in three steps:

1. Construct the matrices L and U, if possible.
2. Solve LY=B for Y using forward substitution.
3. Solve UX=Y for X using back substitution.

LU Decomposition Method (Factorization Method), Doolittle Factorization Method or Method of Triangularisation:

In this method we use the fact that a square matrix A can be factorized into the form A=LU where L is unit lower triangular matrix and U is upper triangular matrix, if all the principal minors of A are non-singular, i.e.,

if $a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, etc.$

Let us consider a system of linear equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \cdots(1)$$

This can be put in the form AX=B … (2)

Let A=LU

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

From (2) we have LUX=B …(3)

Setting UX=Y, then equation (3) becomes

LY=B. …(4)

The equation (4) is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\} \quad ....(5)$$

By forward substitution we get the values y_1, y_2, y_3.

When we know Y, the system UX=Y gives:

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \quad ....(5)$$

By the backward substitution we get the values of $x_1, x_2 and\ x_3$.

Now we shall discuss the procedure of computing the matrices L and U . From the relation A= LU, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating the corresponding components, we get

$$u_{11} = a_{11} \qquad u_{12} = a_{12} \qquad u_{13} = a_{13} \qquad \text{.......(1)}$$

$$l_{21}u_{11} = a_{21} \qquad l_{21}u_{12} + u_{22} = a_{22} \qquad l_{21}u_{13} + u_{23} = a_{23} \qquad \text{.........(2)}$$

$$l_{31}u_{11} = a_{31} \qquad l_{31}u_{12} + l_{32}u_{22} = a_{32} \qquad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \qquad \text{..........(3)}$$

From (1), we get

$$u_{11} = a_{11} \qquad u_{12} = a_{12} \qquad u_{13} = a_{13}$$

From (2), we get

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}} \qquad l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12}$$

$$and \quad l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - l_{21}u_{13} = a_{23} - \frac{a_{21}}{a_{11}}.a_{13}$$

From (3), we get

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}} \qquad l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{(a_{32} - l_{31}u_{12})}{u_{22}} = \frac{(a_{32} - \frac{a_{31}}{a_{11}}.a_{12})}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}} = \frac{a_{32}.a_{11} - a_{31}.a_{12}}{a_{22}.a_{11} - a_{21}.a_{12}}$$

and

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \Rightarrow u_{33} = a_{33} - (l_{31}u_{13} + l_{32}u_{23}) = a_{33} - \{\frac{a_{31}}{a_{11}}.a_{13} + \frac{a_{32}.a_{11} - a_{31}.a_{12}}{a_{22}.a_{11} - a_{21}.a_{12}}.(a_{23} - \frac{a_{21}}{a_{11}}.a_{13})\}$$

Hence in a systematic way the elements of L and U can be evaluated.

Problem:

Solve the equations

$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$

by the Factorization Method.

Solution:

Given System
$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$

Here $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$. and

Let A=LU

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ $and\ U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then form A=LU we have

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Now equating the like component from both-sides of the matrices, we get

$$u_{11}=2 \quad , \quad u_{12}=3 \quad , \quad u_{13}=1$$

$$l_{21}u_{11}=1 \Rightarrow l_{21}=\frac{1}{u_{11}}=\frac{1}{2} \quad , \quad l_{21}u_{12}+u_{22}=2 \Rightarrow 0.5\times 3+u_{22}=2 \;\Rightarrow u_{22}=2-1.5=0.5$$

$$l_{21}u_{13}+u_{23}=3 \Rightarrow 0.5\times 1+u_{23}=3 \Rightarrow u_{23}=3-0.5=2.5, \quad l_{31}u_{11}=3 \Rightarrow l_{31}=\frac{3}{u_{11}}=\frac{3}{2}=1.5$$

$$l_{31}u_{12}+l_{32}u_{22}=1 \Rightarrow 1.5\times 3+l_{32}\times 0.5=1 \Rightarrow l_{32}\times 0.5=1-4.5=-3.5 \Rightarrow l_{32}=-\frac{3.5}{0.5}=-7$$

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2 \Rightarrow 1.5\times 1+(-7)\times 2.5+u_{33}=2 \Rightarrow u_{33}=2-1.5+17.5=18$$

Therefore

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \; and \; U=\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Now calculation for $LY=B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}.$$

$$\begin{bmatrix} y_1 \\ \frac{1}{2}y_1+y_2 \\ \frac{3}{2}y_1-7y_2+y_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

This system is equivalent to

$$y_1=9$$

$$\tfrac{1}{2}y_1+y_2=6 \Rightarrow y_2=\tfrac{3}{2}$$

$$\tfrac{3}{2}y_1-7y_2+y_3=8 \Rightarrow y_3=5.$$

Now calculation for $UX=Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ 0+\frac{1}{2}y+\frac{5}{2}z \\ 0+0+18z \end{bmatrix}=\begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3y+z \\ \frac{1}{2}y+\frac{5}{2}z \\ 18z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

This system is equivalent to

$$\left.\begin{aligned} 2x+3y+z&=9 \\ \frac{1}{2}y+\frac{5}{2}z&=\frac{3}{2} \\ 18z&=5 \end{aligned}\right\}$$

Solving this system by back substitution, we get $x = 35/18, y = 29/18, z = 5/18$.

Try Yourself
Mathematical Problems

1.Use LU Method to Solve the following system

$$\begin{aligned} 2x+y+z&=10 \\ 3x+2y+3z&=18 \\ x+4y+9z&=16 \end{aligned}$$

2.Solve the system

$$\begin{aligned} x+5y+\ \ z&=21 \\ 2x+y+3z&=20 \\ 3x+y+4z&=26 \end{aligned}$$ by LU method.

3. Solve the following system of linear equations

$$\begin{aligned} 3x+5y-7z&=13 \\ 4x+y-12z&=6 \\ 2x+9y-3z&=20 \end{aligned}$$

4.Solve the following system of linear equations

$$\begin{aligned} 2x+2y+4z&=14 \\ 3x-\ \ y+2z&=2 \\ 5x+2y-2z&=2 \end{aligned}$$

5. Apply Triangularization(Factorization) Method to solve the equations

$$\begin{aligned} 2x-3y-5z&=11 \\ 5x+2y-7z&=-12 \\ -4x+3y+z&=5 \end{aligned}$$

6.Explain the technique of LU factorization Method to solve a system of linear equations $AX = B$.

7.What are the decomposition methods for solving simultaneous algebraic equations? Illustrate any one of these with the help of suitable example.

8.Decompose the matrix $\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU and hence solve the system $AX = B$ where $B = \begin{bmatrix} 4 & 8 & 10 \end{bmatrix}^T$ .Determine also L^{-1} and U^{-1} and hence find A^{-1}.

9.Discuss the method to solve the tridiagonal system.

Iterative Method

Definition of iteration:

A computational procedure in which a cycle of operations is repeated, often to approximate the desired result more closely.

Gauss-Jacobi iteration method:

We consider the system Ax=b where A= (a_{ij}) is non-singular and x= $(x_i)_{n\times 1};b=(b_i)_{n\times 1}$

Now we shall write the system in detail:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ...+ a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ...+ a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ...+ a_{3n}x_n = b_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ...+ a_{nn}x_n = b_n
\end{aligned}$$

If necessary, we re-arrange the given system (1) making strictly diagonally dominant, such that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j=i}}^{n} |a_{ij}|, (i = 1,2,3,..........., n).$$

Suppose that the system (1) is (strictly) diagonally dominant. Now we re-write the system as follows:

$$\begin{aligned}
&x_1 = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3 + \ ....+ a_{1n}x_n\right)\right] \\
&x_2 = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1 + a_{23}x_3 + ...+ a_{2n}x_n\right)\right] \\
&x_3 = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1 + a_{32}x_2 + ...+ a_{3n}x_n\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&x_n = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1 + a_{n2}x_2 + ...+ a_{n(n-1)}x_{n-1}\right)\right]
\end{aligned}$$

The set of equations in (2) can be written as :

$$x_i = \frac{1}{a_{ij}}\left[b_i - \left(\sum_{\substack{j=1 \\ j=i}}^{n} a_{ij}x_j\right)\right]. \ (i = 1,2,3,.........., n).$$

Then the solution of system (1) by Gauss-Jacobi iteration method is given by following iterative formulae:

$$
\begin{aligned}
x_1^{(k)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + \dots\dots\dots + a_{1n}x_n^{(k-1)}\right)\right] \\
x_2^{(k)} &= \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k-1)} + a_{23}x_3^{(k-1)} + \dots\dots\dots + a_{2n}x_n^{(k-1)}\right)\right] \\
x_3^{(k)} &= \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(k-1)} + a_{32}x_2^{(k-1)} + \dots\dots\dots + a_{3n}x_n^{(k-1)}\right)\right] \qquad \dots\dots\dots(3) \\
\dots & \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
x_n^{(k)} &= \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k-1)} + a_{n2}x_2^{(k-1)} + \dots\dots\dots + a_{n(n-1)}x_{n-1}^{(k-1)}\right)\right]
\end{aligned}
$$

Where $x_i^{(k)}(i=1,2,3,\dots\dots,n)$ denote the values of x_i at k^{th} iteration, and $x_i^{(0)}(i=1,2,3,\dots\dots\dots,n)$ are the initial guesses being taken arbitrarily.

We shall continue the until the values of x_i at two successive iteration are approximately equal that is until $x_i^{(k)} \cong x_i^{(k-1)}$ for any values of k.

The equations in system (3) can be written as follows:

$$
x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j\neq 1 \\ j=1}} a_{ij}x_j^{(k-1)}\right] i = 1,2,3,\dots\dots\dots,n.
$$

Remarks:

The sufficient condition for the convergence for Gauss-Jacobi method is that the system of equations must be strictly diagonally dominant that is the coefficient matrix $A = \left(a_{kj}\right)_{n-1}$ be such that

$$
\left|a_{kk}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right|, (i = 1,2,3,\dots\dots\dots,\ n).
$$

Gauss-Seidel iteration method:

We consider the system Ax=b where A= $\left(a_{ij}\right)$ is non-singular and x= $\left(x_i\right)_{n\times 1}; b = \left(b_i\right)_{n\times 1}$

Now we shall write the system in detail:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots\dots\dots + a_{3n}x_n &= b_3 \qquad \dots\dots\dots\dots\dots\dots\dots\dots (1) \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad & \dots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n &= b_n
\end{aligned}
$$

If necessary, we re-arrange the given system (1) making strictly diagonally dominant, such that

$$
\left|a_{ii}\right| > \sum_{\substack{j=1 \\ j=i}}^{n} \left|a_{ij}\right|, (i = 1,2,3,\dots\dots\dots\dots,n).
$$

Suppose that the system (1) is (strictly) diagonally dominant. Now we re-write the system as follows:

$$x_1 = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2 + a_{13}x_3 + \text{..........} + a_{1n}x_n\right)\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1 + a_{23}x_3 + \text{..........} + a_{2n}x_n\right)\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1 + a_{32}x_2 + \text{..........} + a_{3n}x_n\right)\right] \qquad \text{.....................} \ (2)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1 + a_{n2}x_2 + \text{..........} + a_{n(n-1)}x_{n-1}\right)\right]$$

The set of equations in (2) can be written as :

$$x_i = \frac{1}{a_{ij}}\left[b_i - \left(\sum_{\substack{j=1\\ j=i}}^{n} a_{ij}x_j\right)\right]. \ (i = 1,2,3,\text{..........},n).$$

Then the solution of system (1) by Gauss-Seidel iteration method is given by following iterative formulae:

$$x_1^{(k)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + \text{.........} + a_{1n}x_n^{(k-1)}\right)\right]$$

$$x_2^{(k)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k)} + a_{23}x_3^{(k-1)} + \text{........} + a_{2n}x_n^{(k-1)}\right)\right]$$

$$x_3^{(k)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(k)} + a_{32}x_2^{(k)} + \text{........} + a_{3n}x_n^{(k-1)}\right)\right] \qquad \text{...................} \ (3)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n^{(k)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \text{........} + a_{n(n-1)}x_{n-1}^{(k)}\right)\right]$$

Where $x_i^{(k)}(i = 1,2,3,\text{.......},n)$ denote the values of x_i at k^{th} iteration and $x_i^{(0)}$ $(i = 1,2,3,\text{.......},n)$ are the initial guesses being taken arbitrarily.

We shall continue the iteration until the values of x_i at two successive iteration are approximately equal that is until $x_i^{(k)} \cong x_i^{(k-1)}$ for any *i.*

The equations in system (3) can be written as follows:

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \left(\sum_{\substack{i>j\\ i}}^{n} a_{ij}x_j^{(k)} + \sum_{i<j} a_{ij}x_j^{(k-1)}\right]i,\ j = 1,2,3,\text{.........},n.$$

Remarks:

The sufficient condition for the convergence for Gauss-Seidel method is that the system of equations must be strictly diagonally dominant that is the coefficient matrix $A = \left(a_{kj}\right)_{n-1}$ be such that

$$|a_{kk}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, (i = 1,2,3,\text{........},\ n).$$

Convergence of Gauss-Jacobi and Gauss-Seidel:

The Gauss-Jacobi and Gauss-Seidel methods converge for any choice of the initial guess $x_i^{(0)}(i = 1,2,3,\text{.........},n)$ if every equation of the system (2) satisfies the condition that the some of the absolute values of the co-efficients a_{ij}/a_{ii} is almost equal to or in at least one equation less than unity that is , provided that

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left|\frac{a_{ij}}{a_{ii}}\right| \le 1, i = 1,2,\dots\dots, n.$$

Where the` <' sign should be valid in the case of at least one equation.

Problem:

Solve the following system of equations by Gauss-Jacobi and Gauss-Seidel method.

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution: The given systems of equations are

$$\begin{aligned} 28x + 4y - z &= 32 \\ x + 3y + 10z &= 24 \quad \dots\dots\dots\dots\dots\dots\dots (1) \\ 2x + 17y + 4z &= 35 \end{aligned}$$

We observe that the coefficient matrix of (1) is not diagonally dominant because $|3| > |1| + |10|$

Now $A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix} R_2 \leftrightarrow R_3 \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix}$

This is diagonally dominant.

So Gauss-Jacobi and Gauss-Seidel iteration method is applicable.

$\therefore$ The given system (1) becomes

$$28x + 4y - z = 32\dots\dots\dots\dots\dots(1)$$
$$2x + 17y + 4z = 35\dots\dots\dots\dots\dots(2)$$
$$x + 3y + 10z = 24\dots\dots\dots\dots\dots\dots(3)$$

$$\therefore (1) \Rightarrow x = \frac{1}{28}(-4y + z + 32)\dots\dots\dots\dots(14)$$

$$(2) \Rightarrow y = \frac{1}{17}(-2x - 4z + 36)\dots\dots\dots\dots\dots(5)$$

$$(3) \Rightarrow z = \frac{1}{10}(-x - 3y + 24)\dots\dots\dots\dots\dots\dots(6)$$

Gauss-Jacobi method:

First iteration: Let $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ be the initial values **of** x,y,z respectively. Using these values in (4) (5) and (6), we get

$$x^{(1)} = \frac{1}{28}(0 + 0 + 32) = 1.1429$$

$$y^{(1)} = \frac{1}{27}(00 + 35) = 2.0588$$

$$z^{(1)} = \frac{1}{10}(-0 - 9 + 24) = 2.4$$

Second iteration:

$$x^{(2)} = \frac{1}{28}\left(-4\times 2.0588 + 2.4 + 32\right) = 0.9345$$

$$y^{(2)} = \frac{1}{17}\left(-2\times 1.1429 - 4\times 2.4 + 35\right) = 1.3597$$

$$z^{(2)} = \frac{1}{10}\left(-1.1429 - 3\times 2.0588 + 24\right) = 1.6681$$

Third iteration:

$$x^{(3)} = \frac{1}{28}\left(-4\times 1.3597 + 1.6681 + 32\right) = 1.0082$$

$$y^{(3)} = \frac{1}{17}\left(-2\times 0.9345 - 4\times 1.6681 + 35\right) = 1.5564$$

$$z^{(3)} = \frac{1}{10}\left(-0.9345 - 2\times 1.3597 + 24\right) = 1.8986$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}\left(-4\times 1.5564 + 1.8986 + 32\right) = 0.9883$$

$$y^{(4)} = \frac{1}{17}\left(-2\times 1.0082 - 4\times 1.8986 + 35\right) = 1.4935$$

$$z^{(4)} = \frac{1}{10}\left(-1.0082 - 3\times 1.5564 + 24\right) = 1.8323$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}\left(-4\times 1.4935 + 1.8323 + 32\right) = 0.9949$$

$$y^{(5)} = \frac{1}{17}\left(-2\times 0.9883 - 3\times 1.8323 + 35\right) = 1.5114$$

$$z^{(5)} = \frac{1}{10}\left(-0.9883 - 3\times 1.4935 + 24\right) = 1.8531$$

Sixth iteration:

$$x^{(6)} = \frac{1}{28}\left(-4\times 1.5114 + 1.8531 + 32\right) = 0.9931$$

$$y^{(6)} = \frac{1}{17}\left(-2\times 0.9949 - 4\times 1.8531 + 35\right) = 1.5058$$

$$z^{(6)} = \frac{1}{10}\left(-0.9949 - 3\times 1.5114 + 24\right) = 1.8471$$

Seventh iteration:

$$x^{(7)} = \frac{1}{28}(-4\times1.5058+1.8471+32) = 0.9937$$
$$y^{(7)} = \frac{1}{17}(-2\times0.9931-4\times1.8471+35) = 1.5074$$
$$z^{(7)} = \frac{1}{10}(-0.9931-3\times1.5058+24) = 1.8490$$

Eighth iteration:

$$x^{(8)} = \frac{1}{28}(-4\times1.5074+1.8490+32) = 0.9936$$
$$y^{(8)} = \frac{1}{17}(-2\times0.9937-4\times1.8490+35) = 1.5069$$
$$z^{(8)} = \frac{1}{10}(-0.9937-3\times1.5074+24) = 1.8484$$

Ninth iteration:

$$x^{(9)} = \frac{1}{28}(-4\times1.5069+1.8484+32) = 0.9936$$
$$y^{(9)} = \frac{1}{17}(-2\times0.9936-4\times1.8484+35) = 1.5070$$
$$z^{(9)} = \frac{1}{10}(-0.9936-3\times1.5069+24) = 1.8486$$

After ninth iteration the difference between 8^{th} and 9^{th} iteration is very negligible. Hence the required solution is x=0.9936, y=1.5070 , z=1.8486.

Gauss-Seidel Method:

Let $y^{(0)} = 0$ *and* $z^{(0)} = 0$ be the initial values of y and z respectively.

First iteration:

$$x^{(1)} = \frac{1}{28}\left(-4y^{(0)} + z^{(0)} + 32\right) = \frac{1}{28}(-4.0+0+32) = 1.1429$$
$$y^{(1)} = \frac{1}{27}\left(-2x^{(1)} - 4z^{(0)} + 35\right) = \frac{1}{27}(-2\times1.1429-4.0+35) = 1.2116$$
$$z^{(1)} = \frac{1}{10}\left(-x^{(1)} - 3y^{(1)} + 24\right) = \frac{1}{10}(-1.1429-3\times1.2116+24) = 1.9322$$

Second iteration:

$$x^{(2)} = \frac{1}{28}\left(-4\times1.2116+1.9322+32\right)=1.0388$$

$$y^{(2)} = \frac{1}{17}\left(-2\times1.0388-4\times1.9322+35\right)=1.4820$$

$$z^{(2)} = \frac{1}{10}\left(-1.0388-3\times1.4820+24\right)=1.8515$$

Third iteration:

$$x^{(3)} = \frac{1}{28}\left(-4\times1.4820+1.8515+32\right)=0.9973$$

$$y^{(3)} = \frac{1}{17}\left(-2\times0.9973-4\times1.8515+35\right)=1.5059$$

$$z^{(3)} = \frac{1}{10}\left(-0.9973-2\times1.5059+24\right)=1.8485$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}\left(-4\times1.5059+1.8485+32\right)=0.9938$$

$$y^{(4)} = \frac{1}{17}\left(-2\times0.8838-4\times4\times1.8485+35\right)=1.5070$$

$$z^{(4)} = \frac{1}{10}\left(-0.9938-3\times1.5070+24\right)=1.8485$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}\left(-4\times1.5070+1.8485+32\right)=0.9936$$

$$y^{(5)} = \frac{1}{17}\left(-2\times0.9936-4\times1.8485+35\right)=1.5070$$

$$z^{(5)} = \frac{1}{10}\left(-0.9936-3\times1.5070+24\right)=1.8485$$

After five iterations the difference between 4th and 5th iteration is very negligible. Hence the solution of the given system of equations by Gauss-Seidel method is x=0.994, y=1.507, z=1.849 correct up to three decimal places.

Try Yourself
Mathematical Problems

1.Find the solution to three decimal places of the system

$$\begin{aligned}83x+11y-4z&=95\\ 7x+52y+13z&=104\\ 3x+8y+29z&=71\end{aligned}$$

using Jacobi and Gauss Seidel methods.

2. Solve by Gauss Jacobi's Method

$$x+5y+z=21$$
$$2x+y+3z=20$$
$$3x+y+4z=26$$

3. Solve by Gauss Jacobi's Method

$$5x+2y+z=12$$
$$x+4y+2z=15$$
$$x+2y+5z=20$$

4.Solve by Gauss-Seidel Method of Iteration the equations

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14$$

5. Solve by Gauss-Seidel Method of Iteration the equations

$$x+10y+z=6$$
$$10x+y+z=6$$
$$x+y+10z=6$$

6.Derive the technique of Gauss Seidel Method for Numerical solution of the system of linear equations $AX=B$.

7. Derive the technique of Gauss Jacobi Method for Numerical solution of the system of linear equations $AX=B$.

8.Discuss the advantage and disadvantage of the Method over Jacobi's iterative Method.

9.Make a comparative study of Gauss Jacobi and Gauss Seidel Methods and both of them which method is more accurate and why?