Numerical Solution of Ordinary Differential Equations

The most general form of an ordinary differential equation of *nth* order is given by,

$$\varphi\left(x, y, \frac{dy}{dx}, \cdots \cdots, \frac{d^n y}{dx^n}\right) = 0 \cdots \cdots \cdots (1)$$

The general solution of this equation is a relation between $x,\ y$ & n arbitrary constants and it is of the form,

$$f\left(x, y, c_1,\ c_2 \cdots \cdots, c_n\right) = 0 \cdots \cdots \cdots (2)$$

If particulars values are given to the constants $c_1,\ c_2 \cdots \cdots, c_n$, then the resulting solution is called a particular solution.
To obtain a particular solution from the general solution given by (2), we must be given *n* conditions so that the constants can be determined. If all the *n* conditions are specified at the same value of *x*, then the problem is called an initial value problem.

Although there are many analytical methods for finding the solution of an ordinary differential equation, there exists large number of ordinary differential equations, whose solution cannot be obtained by the known analytical methods. In such cases, we use numerical methods to get an approximate solution of a given differential equation under the prescribed initial condition.

In this chapter we restrict ourselves and develop the numerical methods for finding a solution of an ordinary differential equation of first order and first degree which is of the form,

$$\frac{dy}{dx} = f\left(x, y\right) \cdots \cdots \cdots (3)$$

with the initial condition $y(x_0) = y_0$, which is called initial value problem.
Some numerical methods for finding an approximate solution of an ordinary differential equation are

1. Taylor's series method
2. Euler's method
3. Modified Euler's method
4. Runge - Kutta method
5. Milne's method
6. Adams Bashforth –Moulton method.

Runge - Kutta method: Consider an ordinary differential equation of first order first degree,

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.
Let h be the wide length between equidistant values of x.
If $x_0,\ y_0$ denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{2}\left(k_1 + k_2\right) \cdots \cdots \cdots (1)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

This is called second order Runge- Kutta method.
Again, if $x_0,\ y_0$ denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right) \cdots \cdots \cdots (2)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

This is called third order Runge- Kutta method.

Again, if x_0, y_0 denote the initial values then the first increment Δy in y is computed by,

$$y_1 = y_0 + \Delta y$$

$$or,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \cdots \cdots \cdots (3)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

This is called fourth order Runge- Kutta method.

Problem-01: Find $y(0.1)$ & $y(0.2)$ by Runge-Kutta method of second, third & fourth order for the differential equation

$$\frac{dy}{dx} = -y\ ;\ y(0) = 1.$$

Solution: We have, $\frac{dy}{dx} = -y\ ;\ y(0) = 1.$

$$\therefore\ f(x, y) = -y,\ x_0 = 0\ ,\ y_0 = 1.$$

Let us take $h = 0.1$

(i). By second order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0$ & $y_0 = 1$

$$\begin{aligned} k_1 &= hf(x_0\ ,\ y_0) \\ &= 0.1 \times (-1) \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0 + h\ ,\ y_0 + k_1) \\ &= 0.1 \times f(0 + 0.1\ ,\ 1 - 0.1) \\ &= 0.1 \times f(0.1\ ,\ 0.9) \\ &= 0.1 \times (-0.9) \\ &= -0.09 \end{aligned}$$

$$\begin{aligned} Now,\ y_1 &= y_0 + \frac{1}{2}\left(k_1 + k_2\right) \\ &= 1 + \frac{1}{2}\left(-0.1 - 0.09\right) \\ &= 1 - 0.095 \end{aligned}$$

$\therefore\ y(0.1) = 0.905$ **(As desired)**

2nd Step: For the 2nd approximation, we have $x_1 = 0.1$ & $y_1 = 0.905$

$$k_1 = hf(x_1\ ,\ y_1)$$
$$= 0.1\times f(0.1\ ,\ 0.905)$$
$$= 0.1\times(-0.905)$$
$$= -0.0905$$
$$k_2 = hf(x_1 + h\ ,\ y_1 + k_1)$$
$$= 0.1\times f(0.1+0.1\ ,\ 0.905-0.0905)$$
$$= 0.1\times f(0.2\ ,\ 0.8145)$$
$$= 0.1\times(-0.8145)$$
$$= -0.08145$$
$$Now,\ y_2 = y_1 + \frac{1}{2}\left(k_1 + k_2\right)$$
$$= 0.905 + \frac{1}{2}\left(-0.0905 - 0.08145\right)$$
$$= 0.905 - 0.08598$$
$$\therefore\ y(0.2) = 0.81902$$ **(As desired)**

(ii). By third order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0$ & $y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1\times(-1)$$
$$= -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0 + \frac{0.1}{2}\ ,1 - \frac{0.1}{2}\right)$$
$$= 0.1\times f(0.05\ ,\ 0.95)$$
$$= 0.1\times(-0.95)$$
$$= -0.095$$
$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= 0.1\times f\left(0+0.1\ ,\ 1+2(-0.095)-(-0.1)\right)$$
$$= 0.1\times f(0.1\ ,\ 0.91)$$
$$= 0.1\times(-0.91)$$
$$= -0.091$$
$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$$
$$= 1 + \frac{1}{6}\left(-0.1 + 4(-0.095) + (-0.091)\right)$$
$$= 1 - 0.09517$$
$$\therefore\ y(0.1) = 0.90483$$ **(As desired)**

2nd Step: For the 2nd approximation, we have $x_1 = 0.1$ & $y_1 = 0.90483$

$$k_1 = hf(x_1\ ,\ y_1)$$
$$= 0.1\times f(0.1\ ,\ 0.90483)$$
$$= 0.1\times(-0.90483)$$

$$= -0.090483$$

$$k_2 = hf\left(x_1 + \frac{h}{2}\ ,\ y_1 + \frac{k_1}{2}\right)$$

$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90483 - \frac{0.090483}{2}\right)$$

$$= 0.1 \times f(0.15\ ,\ 0.8596)$$

$$= 0.1 \times (-0.8596)$$

$$= -0.08596$$

$$k_3 = hf(x_1 + h, y_1 + 2k_2 - k_1)$$

$$= 0.1 \times f\left(0.1 + 0.1,\ 0.90483 + 2(-0.08596) - (-0.090483)\right)$$

$$= 0.1 \times f(0.2\ ,\ 0.8234)$$

$$= 0.1 \times (-0.8234)$$

$$= -0.08234$$

$$Now,\ y_2 = y_1 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$$

$$= 0.90483 + \frac{1}{6}\left(-0.090483 + 4(-0.08596) + (-0.08234)\right)$$

$$= 0.90483 - 0.08611$$

$$= 0.81872$$

$\therefore\ y(0.2) = 0.81872$ **(As desired)**

(iii). By fourth order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0\ \&\ y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$

$$= 0.1 \times (-1)$$

$$= -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,\ 1 - \frac{0.1}{2}\right)$$

$$= 0.1 \times f(0.05\ ,\ 0.95)$$

$$= 0.1 \times (-0.95)$$

$$= -0.095$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,\ 1 - \frac{0.095}{2}\right)$$

$$= 0.1 \times f(0.05\ ,\ 0.9525)$$

$$= 0.1 \times (-0.9525)$$

$$= -0.09525$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$= 0.1 \times f\left(0 + 0.1\ ,\ 1 - 0.09525\right)$$

$$= 0.1 \times f(0.1\ ,\ 0.9048)$$

$$= 0.1 \times (-0.9048)$$

$$= -0.09048$$

$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 1 + \frac{1}{6}\left(-0.1 + 2(-0.095) + 2(-0.09525) - 0.09048\right)$$

$$= 1 - 0.09516$$

$\therefore\ y(0.1) = 0.90484$ **(As desired)**

2nd Step: For the 2nd approximation, we have $x_1 = 0.1$ & $y_1 = 0.90484$

$$k_1 = hf(x_1\ ,\ y_1)$$
$$= 0.1 \times f(0.1\ ,\ 0.90484)$$
$$= 0.1 \times (-0.90484)$$
$$= -0.090484$$

$$k_2 = hf\left(x_1 + \frac{h}{2}\ ,\ y_1 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90484 - \frac{0.090484}{2}\right)$$
$$= 0.1 \times f(0.15\ ,\ 0.8596)$$
$$= 0.1 \times (-0.8596)$$
$$= -0.08596$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90484 - \frac{0.08596}{2}\right)$$
$$= 0.1 \times f(0.15\ ,\ 0.8619)$$
$$= 0.1 \times (-0.8619)$$
$$= -0.08619$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$
$$= 0.1 \times f\left(0.1 + 0.1\ ,\ 0.90484 - 0.08619\right)$$
$$= 0.1 \times f(0.2\ ,\ 0.8187)$$
$$= 0.1 \times (-0.8187)$$
$$= -0.08187$$

$$Now,\ y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= 0.90484 + \frac{1}{6}\left(-0.090484 + 2(-0.08596) + 2(-0.08619) - 0.08187\right)$$

$$= 0.90484 - 0.08611$$

$$= 0.81873$$

$\therefore\ y(0.2) = 0.81873$ **(As desired)**

Problem-02: Find $y(0.3)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} + y + xy^2 = 0$; $y(0) = 1$.

Solution: We have, $\frac{dy}{dx} + y + xy^2 = 0$; $y(0) = 1$.

$$\therefore\ f(x,y) = -y - xy^2\ ,\ x_0 = 0\ ,\ y_0 = 1.$$

Let us take $h = 0.1$

By fourth order Runge- Kutta method:

1st Step: For the 1st approximation, we have $x_0 = 0\ \&\, y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1 \times (-1 - 0 \times 1^2)$$
$$= -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,1 - \frac{0.1}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 0.95)$$
$$= 0.1 \times \left(-0.95 - 0.05 \times (0.95)^2\right)$$
$$= -0.09951$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}\ ,1 - \frac{0.09951}{2}\right)$$
$$= 0.1 \times f(0.05\ ,\ 0.95025)$$
$$= 0.1 \times \left(-0.95025 - 0.05 \times (0.95025)^2\right)$$
$$= -0.09954$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.1 \times f\left(0 + 0.1\ ,1 - 0.09954\right)$$
$$= 0.1 \times f(0.1\ ,\ 0.90046)$$
$$= 0.1 \times \left(-0.90046 - 0.1 \times (0.90046)^2\right)$$
$$= -0.09815$$

$$Now,\ y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 1 + \frac{1}{6}\left(-0.1 + 2(-0.09951) + 2(-0.09954) - 0.09815\right)$$
$$= 1 - 0.09938$$

$\therefore\ y(0.1) = 0.90062$ **(As desired)**

2nd Step: For the 2nd approximation, we have $x_1 = 0.1\ \&\, y_1 = 0.90062$

$$k_1 = hf(x_1\ ,\ y_1)$$
$$= 0.1 \times f(0.1\ ,\ 0.90062)$$
$$= 0.1 \times \left(-0.90062 - 0.1 \times (0.90062)^2\right)$$
$$= -0.0982$$
$$k_2 = hf\left(x_1 + \frac{h}{2}\ ,\ y_1 + \frac{k_1}{2}\right)$$
$$= 0.1 \times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90062 - \frac{0.0982}{2}\right)$$

$$= 0.1\times f(0.15\ ,\ 0.8515)$$
$$= 0.1\times\left(-0.8515 - 0.15\times(0.8515)^2\right)$$
$$= -0.096$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0.1 + \frac{0.1}{2}\ ,\ 0.90062 - \frac{0.096}{2}\right)$$
$$= 0.1\times f(0.15\ ,\ 0.8526)$$
$$= 0.1\times\left(-0.8526 - 0.15\times(0.8526)^2\right)$$
$$= -0.0962$$
$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$
$$= 0.1\times f\left(0.1 + 0.1\ ,\ 0.90062 - 0.0962\right)$$
$$= 0.1\times f(0.2\ ,\ 0.8044)$$
$$= 0.1\times\left(-0.8044 - 0.2\times(0.8044)^2\right)$$
$$= -0.0934$$
$$Now,\ y_2 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.90062 + \frac{1}{6}\left(-0.0982 + 2(-0.096) + 2(-0.0962) - 0.0934\right)$$
$$= 0.90062 - 0.096$$
$\therefore\ y(0.2) = 0.80462$ **(As desired)**

3rd Step: For the 3rd approximation, we have $x_2 = 0.2\ \&\ y_2 = 0.80462$

$$k_1 = hf(x_2\ ,\ y_2)$$
$$= 0.1\times f(0.2\ ,\ 0.80462)$$
$$= 0.1\times\left(-0.80462 - 0.2\times(0.80462)^2\right)$$
$$= -0.0934$$
$$k_2 = hf\left(x_2 + \frac{h}{2}\ ,\ y_2 + \frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0.2 + \frac{0.1}{2}\ ,\ 0.80462 - \frac{0.0934}{2}\right)$$
$$= 0.1\times f(0.25\ ,\ 0.7579)$$
$$= 0.1\times\left(-0.7579 - 0.25\times(0.7579)^2\right)$$
$$= -0.0902$$
$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0.2 + \frac{0.1}{2}\ ,\ 0.80462 - \frac{0.0902}{2}\right)$$
$$= 0.1\times f(0.25\ ,\ 0.7595)$$
$$= 0.1\times\left(-0.7595 - 0.25\times(0.7595)^2\right)$$
$$= -0.0904$$

$$k_4 = hf\left(x_2 + h, y_2 + k_3\right)$$
$$= 0.1\times f\left(0.2+0.1\ ,\ 0.80462-0.0904\right)$$
$$= 0.1\times f(0.3\ ,\ 0.7142)$$
$$= 0.1\times\left(-0.7142-0.3\times(0.7142)^2\right)$$
$$= -0.0867$$

$$Now,\ y_3 = y_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 0.80462 + \frac{1}{6}\left(-0.0934 + 2(-0.0902) + 2(-0.0904) - 0.0867\right)$$
$$= 0.80462 - 0.09022$$
$$\therefore\ y(0.3) = 0.7144$$ **(As desired)**

Problem-03: Find $y(0.1)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} = \frac{1}{x+y}$; $y(0) = 1$.

Solution: We have, $\frac{dy}{dx} = \frac{1}{x+y}$; $y(0) = 1$.

$$\therefore\ f(x,y) = \frac{1}{x+y},\ x_0 = 0\ ,\ y_0 = 1.$$

Let us take $h = 0.1$

By fourth order Runge- Kutta method: we have $x_0 = 0$ & $y_0 = 1$

$$k_1 = hf(x_0\ ,\ y_0)$$
$$= 0.1\times\left(\frac{1}{0+1}\right)$$
$$= 0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}\ ,\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\times f\left(0+\frac{0.1}{2}\ ,\ 1+\frac{0.1}{2}\right)$$
$$= 0.1\times f(0.05\ ,\ 1.05)$$
$$= 0.1\times\left(\frac{1}{0.05+1.05}\right)$$
$$= 0.09091$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\times f\left(0+\frac{0.1}{2}\ ,\ 1+\frac{0.09091}{2}\right)$$
$$= 0.1\times f(0.05\ ,\ 1.0455)$$
$$= 0.1\times\left(\frac{1}{0.05+1.04545}\right)$$
$$= 0.09129$$
$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$
$$= 0.1\times f\left(0+0.1\ ,\ 1+0.09129\right)$$

$$= 0.1\times f(0.1\,,\,1.09129)$$
$$= 0.1\times\left(\frac{1}{0.1+1.09129}\right)$$
$$= 0.08394$$

$$Now,\; y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= 1 + \frac{1}{6}\left(0.1 + 2\times 0.09091 + 2\times 0.09129 + 0.08394\right)$$
$$= 1 + 0.09139$$

$\therefore\; y(0.1) = 1.09139$ **(As desired)**

Exercise:

Problem-01: Find $y(0.2)$ by Runge-Kutta method of fourth order for the differential equation $\frac{dy}{dx} + \frac{y}{x} - \frac{1}{x^2} = 0$ with $y(1) = 1$.

Problem-02: Find $y(1)$ & $y(2)$ by Runge-Kutta method of second order for the differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y(0) = 1$.

Problem-03: If $\frac{dy}{dx} = y^2 + 1$ with $y(0) = 0$, then find $y(0.2)$, $y(0.4)$ & $y(0.6)$ by using Runge-Kutta fourth order method.

Problem-04: If $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, then find $y(0.4)$ by using Runge-Kutta fourth order method.

Problem-05: If $\frac{dy}{dx} = x + y$ with $y(1) = 1$, then find $y(1.5)$ by using Runge-Kutta third order method.

Problem-06: If $\frac{dy}{dx} = xy + y^2$ with $y(0) = 1$, then find $y(0.2)$ & $y(0.3)$ by using Runge-Kutta fourth order method.