Abstract Algebra Meets Music Theory

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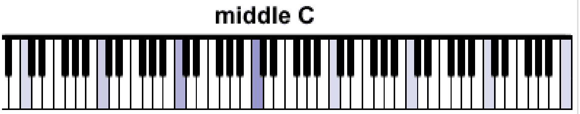
When say that two notes with frequencies in a 2:1 ratio lie one **octave** apart and give **the same name** to both notes.



On the keyboard

$$[C] = \{\ldots\text{Low C}, \text{Middle C}, \text{High C}, \ldots\}$$

In $\mathbb{Z}_{12}$

$$[0] = \{\cdots -24, -12, 0, 12, 24, \cdots\}$$

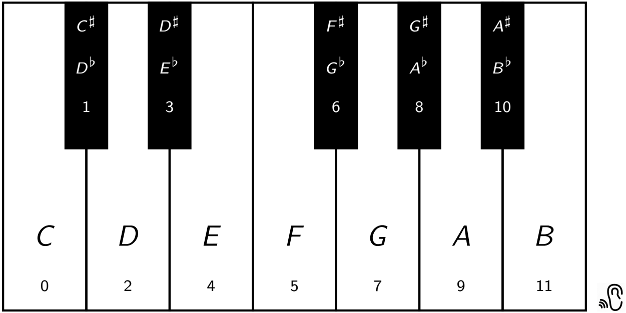
- We organize pitch by octaves $\iff$ frequencies by powers of 2
- Two octaves above middle C $\iff$ $2^2 = 4$ times the frequency of middle C.

We intuitively use a base 2 logarithmic scale!

Major Diatonic Scales

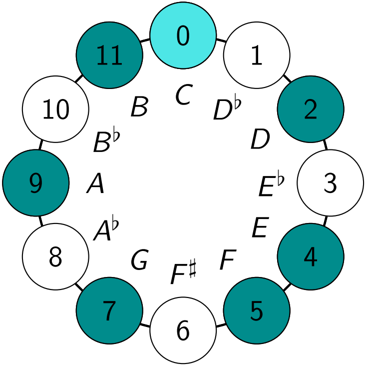
When in a given key a composer selects (most) notes from a seven-note **diatonic scale**.

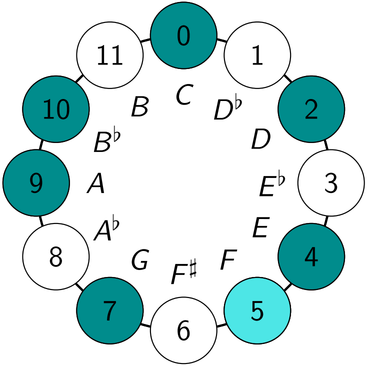
$$C\text{—Major Diatonic Scale}: \{C, D, E, F, G, A, B\} = \{0, 2, 4, 5, 7, 9, 11\}$$



$$k - \text{Major Diatonic Scale} = k + \{0, 2, 4, 5, 7, 9, 11\}$$

$$F - \text{Major Diatonic Scale } (k = 5) = \{5, 7, 9, 10, 0, 2, 4\}$$

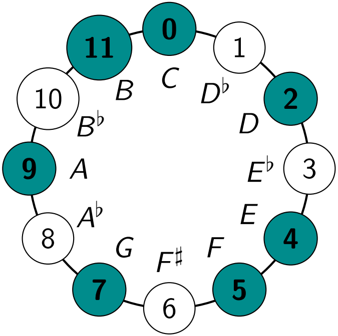




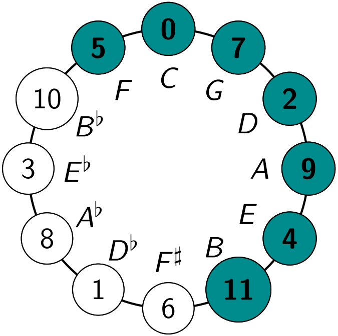
C – Major Diatonic Scale

F – Major Diatonic Scale

C-Major Diatonic Scale



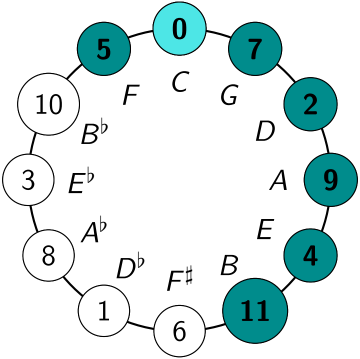
$$\xrightarrow[\pmod{12}]{7 \cdot x}$$



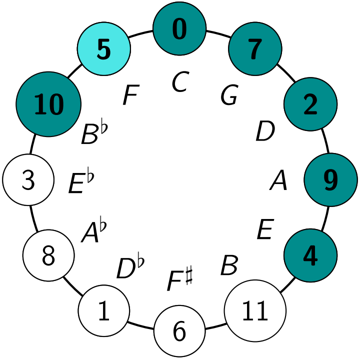
$\mathbb{Z}_{12} = \langle 1 \rangle$, musical clock

$\mathbb{Z}_{12} = \langle 7 \rangle$, circle of fifths

$\mathbb{Z}_{12} = \langle 7 \rangle$, Circle of Fifths



C-Major Diatonic Scale



F-Major Diatonic Scale

F $\to$ F$^\sharp$ Property. When transposing a major scale up by a perfect fifth we obtain a scale with all but one element the same and with the note gained only one semitone above the note lost.

Definition: A generalized n-note microtonal system with generalized perfect fifth $p5$ is a division of the octave into n microtones such that

1. **Reflects equal tempered tuning**: each note has a frequency $2^{\frac{1}{n}}$ times the previous one.
2. **Has a generalized circle of fifths**: $p5$ generates $\mathbb{Z}_n$.
3. Contains generalized diatonic scales appearing as connected regions in the generalized circle of fifths with the $F \to F^\sharp$ property.

Generalized Formula for Microtones of Interest:

- $n = 4k$ microtones per octave.
- $p5 = 2k + 1$ as our generalized perfect fifth.
- Generalize major diatonic scales are connected regions of $2k + 1$ notes in the Cayley graph of $\mathbb{Z}_n = \langle p5 \rangle$.

Major and minor triads

$$
\begin{aligned}
7 &= 4+3 \\
p5 &= M3+m3
\end{aligned}
$$

Major triad : Root, note M3 above root, note p5 above root

$\{0,4,7\} = \{C, E, G\}$

minor triad : Root, note m3 above root, note p5 above root

$\{0,3,7\} = \left\{C, E^{\flat}, G\right\}$

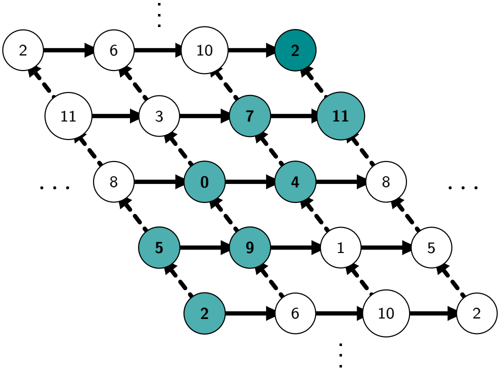
Definition: A generalized n-note microtonal system with generalized perfect fifth $p5$ and generalized major third $M3$ is a division of the octave into n microtones that

1. each note has a frequency $2^{\frac{1}{n}}$ times the previous one.
2. $p5$ generates $\mathbb{Z}_n$.
3. the t-major diatonic scale consists of a connected region of $(p5)^{-1}$ notes in the Cayley graph of $\langle p5 \rangle$, ending at note $t-1$.
4. **the generalized $M3$ and smaller generalized minor third $m3$ partition the generalized perfect fifth $p5$ (i.e. $M3 + m3 = p5$) with $M3$ appearing as an element of the C-major diatonic scale.**

$$\underbrace{7}_{p5} = \underbrace{4}_{M3} + \underbrace{3}_{m3}$$

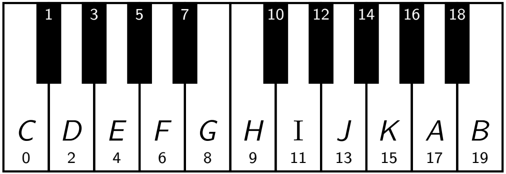
$$\underbrace{2k+1}_{p5} = \underbrace{k+1}_{M3} + \underbrace{k}_{m3}$$

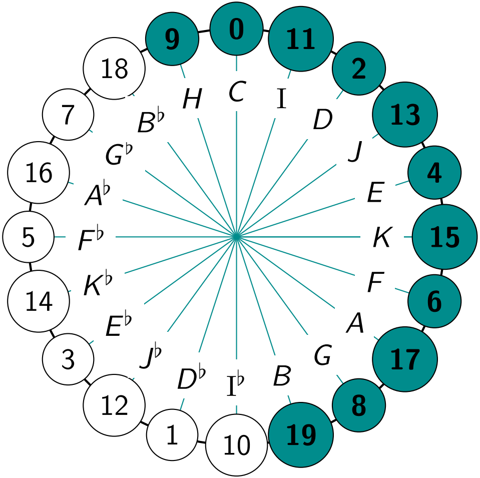
An additional requirement that ensures that $\boxed{M3 = k+1}$ is in the C-major diatonic scale for the $\boxed{n = 4k}$ note microtonal system with $\boxed{p5 = 2k+1}$ is that $\boxed{M3 = k+1}$ is in the C-major diatonic scale only when k is odd. Therefore n-note microtonal systems meeting our definition must have n a multiple of 4 but not a multiple of 8.



$$\mathbb{Z}_{12} = \langle 3, 4 \rangle$$

$$\mathbb{Z}_{20} = \langle 11 \rangle$$

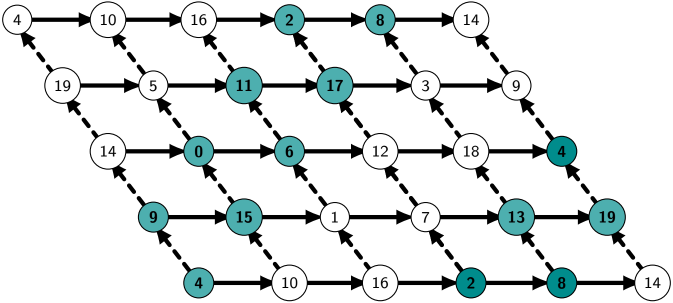




$$\begin{aligned} n &= 4(5) \\ p5 &= 2(5) + 1 \end{aligned}$$

C-Major Triad in $\mathbb{Z}_{20} = \langle 11 \rangle = \{0, 6, 11\}$

C-Minor Triad in $\mathbb{Z}_{20} = \langle 11 \rangle = \{0, 5, 11\}$



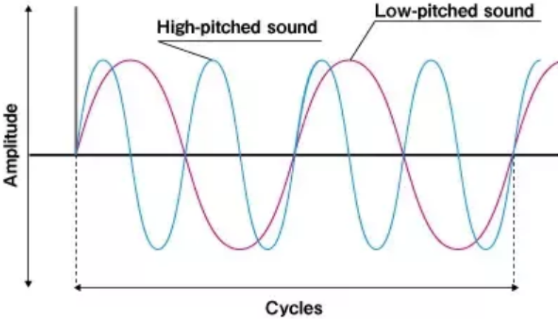
$$\mathbb{Z}_{20} = \langle 5, 6 \rangle$$

Low number ratios sound most consonant to our ears!

Octave : 2:1 ratio

Equal Tempered Perfect Fifth: $\left(2^{\frac{7}{12}}\right) : 1 \approx 2.997 : 2$ ratio

Just Perfect Fifth: 3:2 ratio



Next Semester Plans:

- Chord Structure in n-note Microtonal Systems
- Efficiency in calculating the perfect fifth for an n-note Microtonal Systems
- Exploration and Balancing of Atonal Music