From Statistics to Data Mining

Master 1
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Tutorial

- Linear Algebra → Matrix Multiplication

➢ compute the matrix product AB with $A = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

➢ then compute the matrix product BA

➢ what can you conclude?

Tutorial

- Linear Algebra → Matrix Multiplication

➢ results:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

➢ but

$$BA = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$$

➢ we can observe that $AB \neq BA$
which is a result that is most often encountered

Tutorial

- Linear Algebra → Matrix Determinant

➢ compute the matrix determinant of the (3×3) square matrix

$$C = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

Tutorial

- Linear Algebra → Matrix Determinant

➢ **Solution 1:** we extend the matrix with the 2 first columns:

$$C = \left(\begin{array}{ccc|cc} 3 & 2 & -1 & 3 & 2 \\ -1 & 2 & 3 & -1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

➢ we compute the positive sum of the products in NW-SE diag. and the negative sum of the products in SW-NE diag.

➢ the result gives:

$$3 \times 2 \times 1 + 2 \times 3 \times 1 + (-1) \times (-1) \times 1$$
$$-\big((-1) \times 2 \times 1\big) - (3 \times 3 \times 1) - (2 \times (-1) \times 1)$$
$$= 6 + 6 + 1 + 2 - 9 + 2$$
$$= 8$$

Tutorial

- Linear Algebra → Matrix Determinant

➢ **Solution 2:** we make the sum or subtraction (alternatively) of the terms of the 1st row of the (3×3)-matrix C with the determinants of the (2×2)-square matrices extracted from C

$$C = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

➢ $3 \times det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} - 2 \times det \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} + (-1) \times det \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

➢ $= 3 \times (2 \times 1 - 3 \times 1) - 2 \times \big((-1) \times 1 - 3 \times 1\big) - 1 \times \big((-1) \times 1 - 2 \times 1\big)$

➢ $= 3 \times (-1) - 2 \times (-4) - 1 \times (-3)$

➢ $= -3 + 8 + 3 = 8$

Tutorial

- Linear Algebra → Reduced row echelon form

➢ the matrix $R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row-echelon form

➢ The vectors $\begin{cases} r_1 = [1 \ -2 \ 5 \ 0 \ 3] \\ r_2 = [0 \ \ 1 \ \ 3 \ 0 \ 0] \\ r_3 = [0 \ \ 0 \ \ 0 \ 1 \ 0] \end{cases}$ form a basis for the row space of R,

and the vectors $c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

form a basis for the column space of R

Tutorial

- Linear Algebra → Reduced row echelon form

➢ **exercise:** reduce the matrix A to row-echelon form (first step for finding the bases for the row and column spaces)

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

➢ **solution:** reduction A to row-echelon form

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Tutorial

- Linear Algebra → Reduced row echelon form

➢ **exercise:** find a basis for the space spanned by the vectors $v_1 = (1, -2, 0, 0, 3)$, $v_2 = (2, -5, -3, -2, 6)$, $v_3 = (0, 5, 15, 10, 0)$, $v_4 = (2, 6, 18, 8, 6)$.

➢ **solution:** write down the vectors as row vectors first

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

➢ the nonzero row vectors in this matrix are $w_1 = (1, -2, 0, 0, 3)$, $w_2 = (0, 1, 3, 2, 0)$, $w_3 = (0, 0, 1, 1, 0)$

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- Linear Algebra → Reduced row echelon form

➢ keeping in mind that A and R may have different column spaces, we cannot find a basis for the column space of A directly from the column vectors of R

➢ however, if we can find a set of column vectors of R that forms a basis for the column space of R, then the corresponding column vectors of A will form a basis for the column space of A

➢ in the previous example, the basis vectors obtained for the column space of A consisted of column vectors of A, but the basis vectors obtained for the row space of A were not all vectors of A → transposition of the matrix

Tutorial

- Linear Algebra → Reduced row echelon form

➢ find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A

➢ **solution:**

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

➢ the nonzero vectors in this matrix are
$w_1 = (1, 2, 0, 2)$, $w_2 = (0, 1, 5, -10)$ and $w_3 = (0, 0, 0, 1)$

Tutorial

- Convex Optimization

➢ **Exercise 1:** we want to solve:

$$\begin{cases} \max x.y \\ \text{s.t. } x + 3y = 24 \end{cases}$$

➢ **Questions:**

- what are the objective and constraint functions?
- solve this system by using the method of Lagrange multipliers

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- Convex Optimization

➢ **Solution:**

$$\begin{cases} \quad \max x.y \\ \text{s.t. } x + 3y = 24 \end{cases}$$

- objective function: $\max x.y$
- constraint function: $x + 3y = 24 \Leftrightarrow x + 3y - 24 = 0$

- method of Lagrange multipliers:
  → $\mathcal{L}(x, y, \lambda)$ = objective function to optimize $-\lambda(\text{constraint})$
- Lagrange function to optimize:
  $\mathcal{L}(x, y, \lambda) = x.y - \lambda(x + 3y - 24) = x.y - \lambda x - 3\lambda y + 24\lambda$

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- Convex Optimization

- $\mathcal{L}(x, y, \lambda) = x.y - \lambda(x + 3y - 24) = x.y - \lambda x - 3\lambda y + 24\lambda$
- optimizing a function → finding the critical points
  → finding when the derivative of the function is equal to zero
- we have 3 variables: x, y, λ, therefore the function need to be derived 3 times
- there are 3 first order conditions:
- $\frac{\partial \mathcal{L}}{\partial x} = 0 \Leftrightarrow y - \lambda = 0 \Leftrightarrow \lambda = y \quad (1)$
- $\frac{\partial \mathcal{L}}{\partial y} = 0 \Leftrightarrow x - 3\lambda = 0 \Leftrightarrow \lambda = \frac{x}{3} \quad (2)$
- $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow x + 3y - 24 = 0 \quad (3)$

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- Convex Optimization

- $\mathcal{L}(x, y, \lambda) = x.y - \lambda(x + 3y - 24) = x.y - \lambda x - 3\lambda y + 24\lambda$
- (1) and (2): $y = \frac{x}{3} \quad (4)$
- (3) and (4): $x + 3 \times \frac{x}{3} - 24 = 0$
- $\Leftrightarrow x + x - 24 = 0$
- $\Leftrightarrow 2x = 24$
- $\Leftrightarrow x = 12$
- with (4): $y = \frac{x}{3} = \frac{12}{3} = 4$
- with (1): $\lambda = y = 4$

therefore $\begin{cases} x = 12 \\ y = 4 \\ \lambda = 4 \end{cases}$

Tutorial

- Convex Optimization

➢ **Exercise 2:** we want to solve:

$$\begin{cases} \min(2x + 2y) \\ \text{s.t.} \quad x.y = 4 \end{cases}$$

➢ **Solution:** → constraint: $x.y = 4 \Leftrightarrow x.y - 4 = 0$

- $\mathcal{L}(x, y, \lambda) = 2x + 2y - \lambda(x.y - 4)$. FOC:
- $\frac{\partial \mathcal{L}}{\partial x} = 0 \Leftrightarrow 2 - \lambda y = 0 \Leftrightarrow 2 = \lambda y \Leftrightarrow \lambda = \frac{2}{y} \quad (1)$
- $\frac{\partial \mathcal{L}}{\partial y} = 0 \Leftrightarrow 2 - \lambda x = 0 \Leftrightarrow 2 = \lambda x \Leftrightarrow \lambda = \frac{2}{x} \quad (2)$
- $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow x.y - 4 = 0 \quad (3)$

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- Convex Optimization

○ $\mathcal{L}(x, y, \lambda) = 2x + 2y - \lambda(x.y - 4)$

○ (1) and (2): $\frac{2}{y} = \frac{2}{x} \Leftrightarrow x = y \quad (4)$

○ (3) and (4): $x^2 - 4 = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = 2 \quad (5)$

○ (4) and (5): $y = 2$ and $\lambda = 1$

○ therefore $\begin{cases} x = 2 \\ y = 2 \\ \lambda = 1 \end{cases}$

• Convex Optimization

➢ **Exercise 3:** we want to solve:

$$\begin{cases} f(x,y) = \max(x^2y) \\ \text{s.t. } \ x^2 + y^2 = 1 \end{cases}$$

- what is the graphical interpretation of $x^2 + y^2 = 1$?
- what is the graphical interpretation of x^2y?
- therefore, what is the graphical interpretation of $\begin{cases} f(x,y) = \max(x^2y) \\ \text{s.t. } \ x^2 + y^2 = 1 \end{cases}$?

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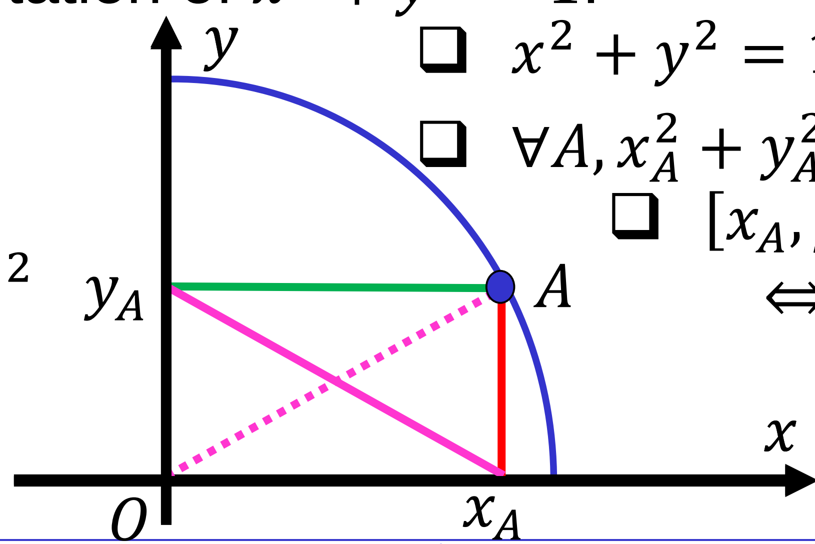
- Convex Optimization

➢ **Solution:** we want to solve:

$$\begin{cases} f(x,y) = \max(x^2 y) \\ \text{s.t. } \; x^2 + y^2 = 1 \end{cases}$$

○ graphical interpretation of $x^2 + y^2 = 1$:

- ❑ value for x: x_A
- ❑ value for y: y_A
- ❑ $x_A^2 + y_A^2 = [x_A, y_A]^2$
- ❑ $[x_A, y_A] = [O, A]$



- ❑ $x^2 + y^2 = 1$
- ❑ $\forall A, x_A^2 + y_A^2 = [x_A, y_A]^2 = 1$
  - ❑ $[x_A, y_A] = 1$

$\Leftrightarrow [O, A] = 1$

→ circle with radius with the size of 1

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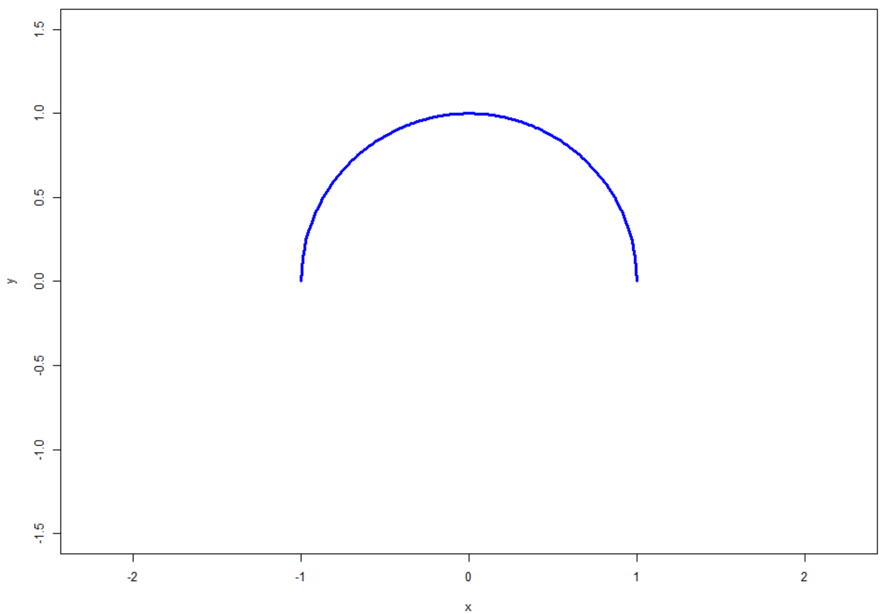
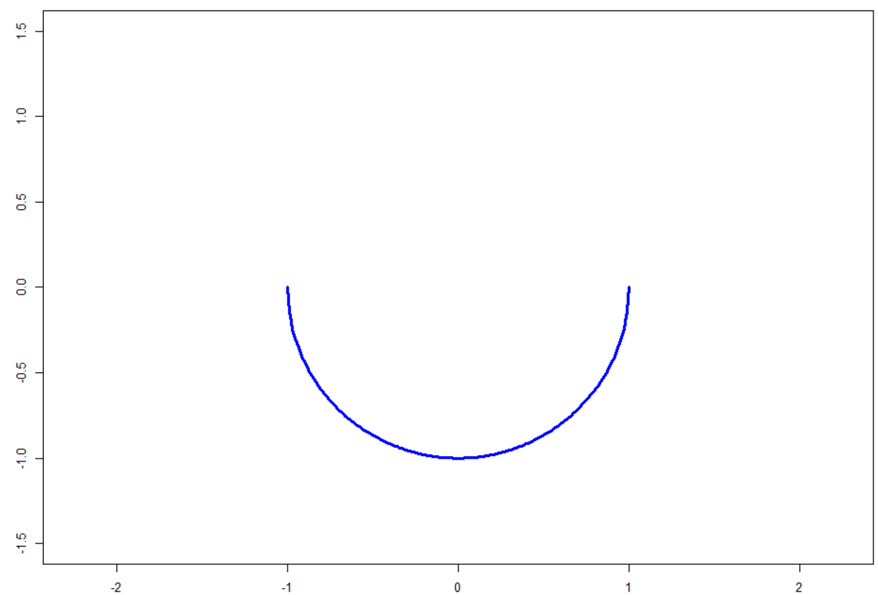
- Convex Optimization

○ graphical interpretation of $x^2 + y^2 = 1$: circle with radius =1

○ $x^2 + y^2 = 1 \Leftrightarrow y^2 = 1 - x^2$

○ $\Leftrightarrow y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$





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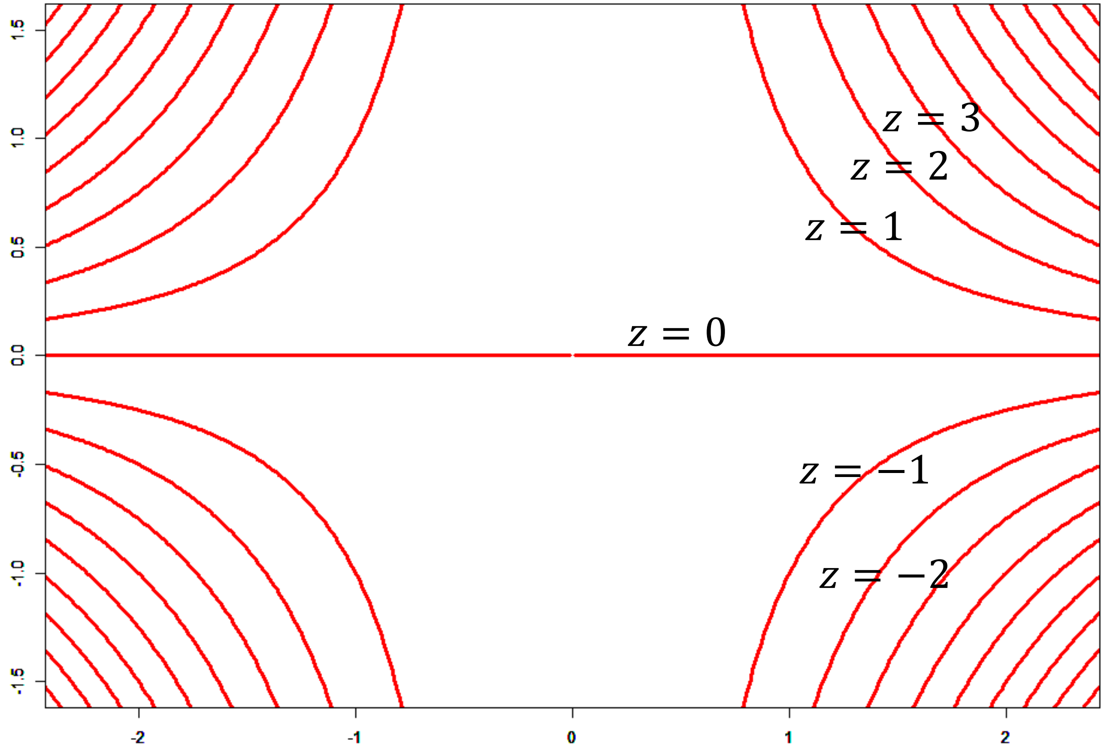
- Convex Optimization

○ graphical interpretation of x^2y: $y = z \times \frac{1}{x^2}$



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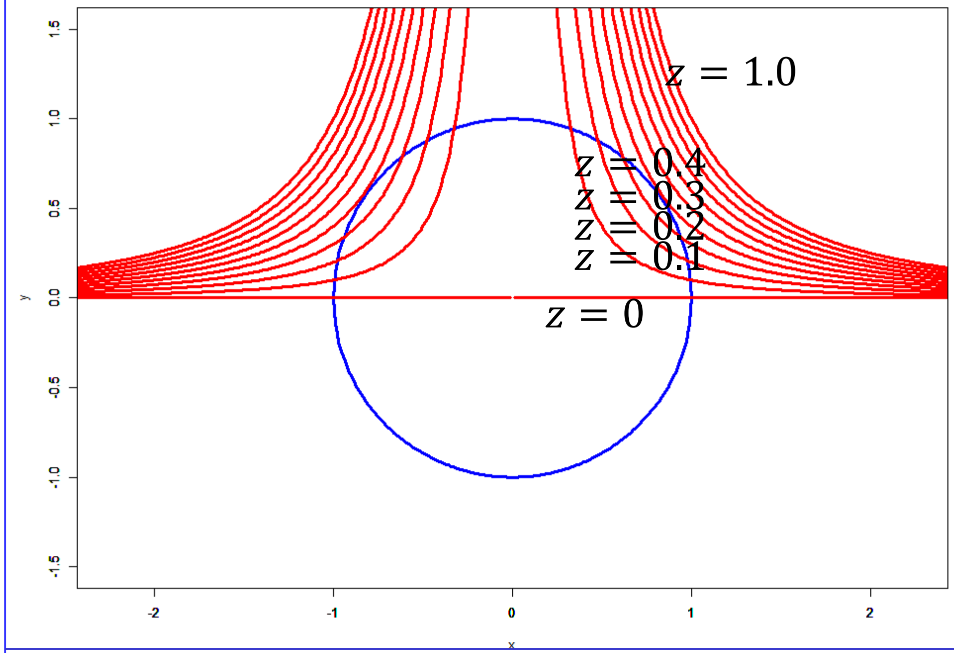
- Convex Optimization

  - graphical interpretation of $\begin{cases} f(x,y) = \max(x^2 y) \Rightarrow y = z \times \frac{1}{x^2} \\ \text{s.t. } x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-x^2} \end{cases}$



  - when $f(x,y) = z \leq 0.3$, the constraint intersects with the circle, some solutions for the 2 conditions exists
  - when $f(x,y) = z \geq 0.4$, no intersection → no solution
  - maxima → tangency points
  - z ? graphically, $z \in [0.3; 0.4]$

Tutorial

- Convex Optimization

➢ **Solution:** we want to solve:

$$\begin{cases} f(x,y) = \max(x^2 y) \\ \quad \text{s.t.} \ \ x^2 + y^2 = 1 \end{cases}$$

➢ constraint: $x^2 + y^2 = 1 \Leftrightarrow x^2 + y^2 - 1 = 0$

- $\mathcal{L}(x,y,\lambda) = x^2 y - \lambda(x^2 + y^2 - 1)$. FOC:
- $\dfrac{\partial \mathcal{L}}{\partial x} = 0 \Leftrightarrow 2x.y - \lambda 2x = 0 \Leftrightarrow 2x.y = \lambda 2x \Leftrightarrow x \neq 0, \lambda = y \quad (1)$
- $\dfrac{\partial \mathcal{L}}{\partial y} = 0 \Leftrightarrow x^2 - \lambda 2y = 0 \Leftrightarrow x^2 = \lambda 2y \Leftrightarrow (1) \Rightarrow x^2 = 2y^2 \ (2)$
- $\dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \Leftrightarrow x^2 + y^2 - 1 = 0 \quad (3)$

Tutorial

- Convex Optimization

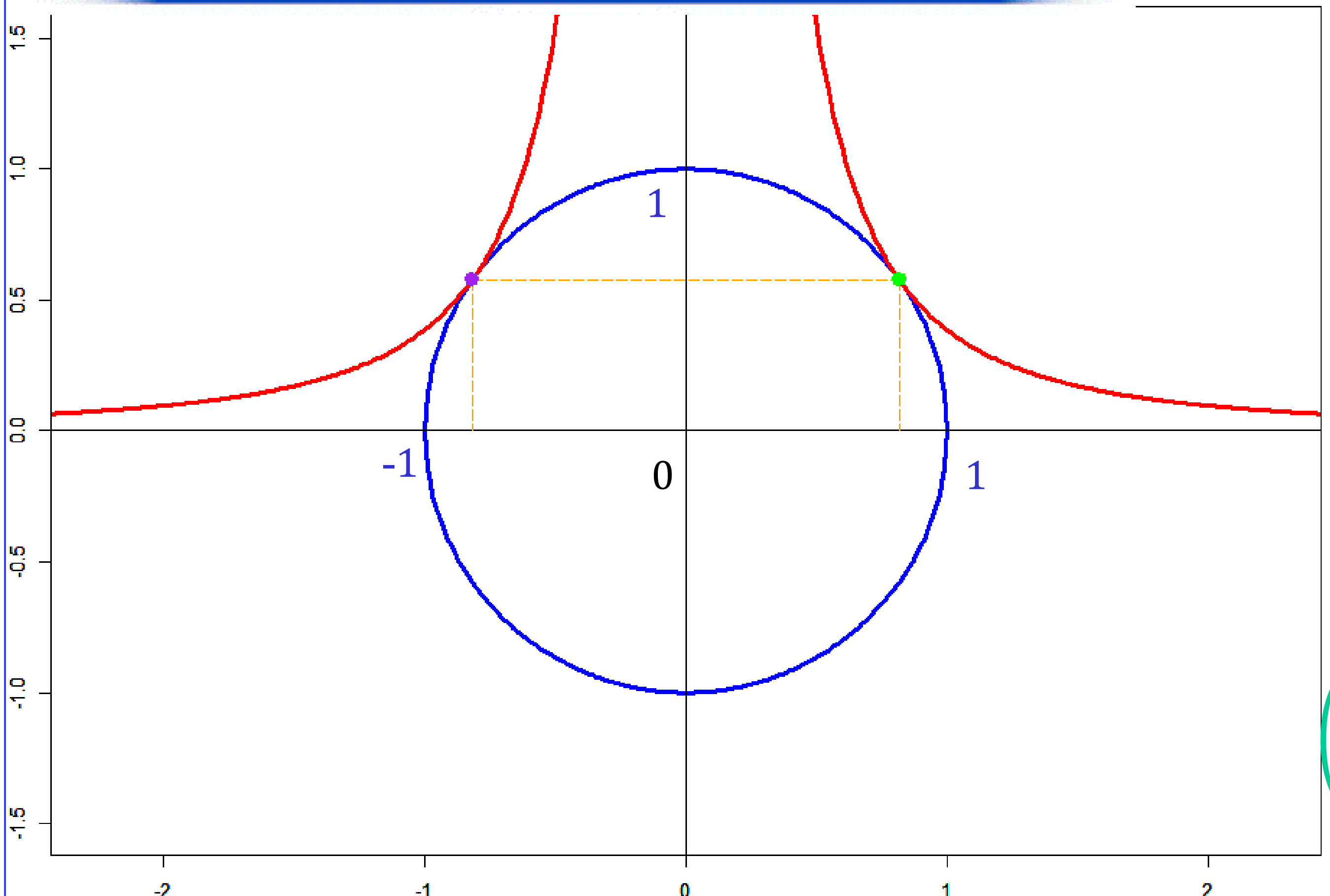
  - $\mathcal{L}(x, y, \lambda) = x^2 y - \lambda(x^2 + y^2 - 1)$
  - (3) and (2): $2y^2 + y^2 - 1 = 0 \Leftrightarrow 3y^2 = 1$
  - $\Leftrightarrow y^2 = \frac{1}{3} \Rightarrow y = \pm\sqrt{\frac{1}{3}}$ (4)
  - (2) and (4): $x^2 = 2y^2 = 2 \times \frac{1}{3} \Leftrightarrow x^2 = \frac{2}{3} \Rightarrow x = \pm\sqrt{\frac{2}{3}}$ (5)
  - (1) and (4): $y = \lambda \Rightarrow \lambda = \pm\sqrt{\frac{1}{3}}$
  - solutions: $\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$

Tutorial

- Convex Optimization

  - $\mathcal{L}(x, y, \lambda) = x^2 y - \lambda(x^2 + y^2 - 1)$
  - solutions: $\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right)$
  - → the possible solutions are the four points where the contour lines are tangent
  - which one maximize the function $f(x, y) = \max(x^2 y)$
  - but y cannot be negative because $x^2 y$ will be negative
  - $f\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = f\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \frac{1}{3}\sqrt{\frac{2}{3}}$ → 2 solutions
  - $z = x^2 y = {}^1/_3 \sqrt{{}^2/_3} \cong 0.3849$ (reminder: $z \in [0.3; 0.4]$)

- Convex Optimization



- function to maximize: $z = {}^{1}/_{3}\sqrt{{}^{2}/_{3}}$
  $\rightarrow y = z \times \frac{1}{x^2}$
- constraint: $y = \pm\sqrt{1 - x^2}$
- solutions: $\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right), \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right)$