Complexity

This exam contains 3 pages.

We consider the following game Solitaire Game. You are given an $m \times k$ board (m rows and k columns) where each one of the mk positions may be empty or occupied by either a red stone or a blue stone. Initially, some configuration of stones is placed on the board. This will be called the initial configuration of the Solitaire game. Now, for each column:

- if it contains both red and blue stones, then you **must** remove either all of the red stones in that column or all of the blue stones in that column;
- if it already contains only blue or only red stones, then you do not have to remove any further stones from that column.

The objective is to leave at least one stone in each row. If this objective can be obtained, we say that the game has a solution. In other words, the game has a solution if there is a set of decisions to be taken for every columns (between "remove red stones", "remove blue stones" and "remove nothing") which is such that in the end there is at least one stone per row. Finding a solution that achieves this objective may or may not be possible depending upon the initial configuration.

Question 1 *We consider the initial configuration in a 2 × 3 board (R and B respectively stand for Red and Blue stone, columns are numbered from 1 to 3, left to right):*

R	R	R
B	B	B

What if you decide to remove the red stone in every column? What if you decide to remove the red stone in two columns and a blue one in the remaining column?
What can we say if the initial configuration is:

	R	R
	B	B

Considering a 4 × 3 board with the initial configuration defined below. Is there a solution?

R		
B	B	R
B	R	
	B	B

We now define the corresponding decision problem:
SOLITAIRE
INSTANCE: an initial configuration of a Solitaire Game
QUESTION: does this game have a solution?

Question 2 *Prove that* **SOLITAIRE** *is in NP.*

We now proceed by a reduction form **3SAT**. For any instance ϕ of **3SAT** with m clauses $C_1, \ldots, C_m$ and k variables $x_1, \ldots, x_k$, we create an $m \times k$ Solitaire game as follows:

- each column corresponds to a variable and each row to a clause;
- for every cell (i, j), we place a red stone in the cell if variable x_j occurs in C_i, and we place a blue stone in the cell if variable $\neg x_j$ occurs in clause C_i. Else, leave the cell empty.

Question 3 *What happens if one column of the initial configuration contains stones of the same color (all blue or all red) on every row? Explain why we can assume in the following that we do not consider this case anymore.*
Show that the construction is polynomial in m and k.

Question 4 *Create the instance from* **SOLITAIRE** *built from the following instance of* **3SAT***: $\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_4 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$. Propose a truth assignment satisfying all clauses of ϕ.*
Write the instance ϕ' of **3SAT** *from which the following instance of* **SOLITAIRE** *was built.*

R	R	R
R	R	B
R	B	R
R	B	B
B	R	R
B	R	B
B	B	R
B	B	B

We now consider a positive instance of **3SAT** and the instance of **SOLITAIRE** which was created from it. If in a truth assignment which satisfies all clauses, x_j is set to True then remove all blue stones in the corresponding column, if x_j is set to False then remove all red stones in the column.

Question 5 *What can you say about the colors of the stone in one column? What is the boolean value of the* **literals** *(recalling that a red stone encodes an occurence of a literal x_j and a blue stone, an occurence of a literal $\neg x_j$) corresponding to the removed stones? Consider one arbitrary clause C_i and the correponding row i in the Solitaire game. Show that there is at least one stone on this row. Conclude.*

Conversely, we now assume that the instance of **SOLITAIRE** is positive. Recall that this means that all the stones in one column have the same color. We now define values for the boolean variable according to the following rule: Considering column j, if it has only red stones then wet set x_j to True, if it has only blue stones we set x_j to False.

Question 6 *Find a solution for the instance of* **SOLITAIRE** *built from* ϕ *in Question 4. Now set the boolean values to* $x_1, \ldots, x_4$ *according to the preceding rule. What is the result for* ϕ*? Show that, in general, this rule defines a truth assignment which satisfy all clauses. Conclude about* **SOLITAIRE**.

Q1: If we decide to remove the red stone in every column then there are only blue stones remaining on the second row. This does not prove that the Game has a solution. BUT, if we decide to remove the red stones in the (e.g.) first two columns and the blue stone in the last column we obtain:

		R
B	B	

There is at least one stone per row, thus, this proves that the initial configuration has a solution.

Considering the second 2×3 initial configuration, we can remove the red stone of the column 2 and the blue stone of column 3, there is at least one stone per row and the Game has a solution (despite the fact that one column is empty).

Finally, concerning the 4×3 board: the column 1 forces the player to remove the blue stones. Thus, we can not remove the red stones of columns 2 (or the row 3 would contain no stone). Thus we remove the blue stone in column 2. If we remove the red in column 3 then there is no stone on row 2, if we remove the blue, then there is no stone on row 4. This proves that this initial configuration has no solution.

Q2: a certificate could be the sequence of the decisions taken row per row (size $O(k)$). Then we have to:

- check that the decisions are correct, that is, we do not remove blue stones in a column full of blue stones, or red stones in a column full of red stones. This is $O(km)$;
- check there is at least one stone per row after applying the decisions. This is $O(mk)$.

In the end, the checking phase's complexity is $O(mk)$.

Q3: when a column indexed by x_j contains only red stone, this means that all clauses contain the literal x_j. It suffices to set it to True to satisfy all clauses. If it contains only blue stones, then this is $\neg x_j$ which appears in all clauses and we set x_j to False. Thus, we will not consider such cases which are evidently positive instances of 3SAT.

From an instance of 3SAT, we build a table of size $m \times k$ and fill every cell with R or B or nothing considering every clause of ϕ one by one. This takes $O(mk)$ steps.

Q4: From ϕ we build the following initial configuration:

R	R	B	
	B	B	R
B	R	R	

To satisfy all clauses we can propose $F(x_1) = True, F(x_2) = True, F(x_3) = False, F(x_4) = True$.

$\phi' = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$ Notice that this is a negative instance of 3SAT, and that the corresponding instance of SOLITAIRE is also negative...

Q5: all literals in a column are True because we exactly **remove** those which are False, that is: blue stones (occurence of $\neg x_j$ in a clause) when x_j is set to True, and red stones (occurence of x_j in a clause) when it is set to False. Because the instance is positive, every clause contains at least one literal which is True, thus there is at least one stone per row, and the instance of SOLITAIRE is positive.

Q6: a solution to the instance built from ϕ could be: remove blue stone in column 1, blue stones in column 2, red stone in column 3, nothing in column 4 (or alternatively, blue stones...). Following the rule, this gives : $F(x_1) = True, F(x_2) = True, F(x_3) = False, F(x_4) = True$ (the one we already proposed) and ϕ is True. Generally: the instance of SOLITAIRE being positive there is at least one stone per row. Consider this stone: if it is red, set the corresponding variable x_j to True, if it is blue, set it to False. Notice first that there is no contradiction in this assignment because all stones in a column have the same color. It is clear that all clauses are True because the stone we considered makes it True.

Conclusion: we have a polynomial reduction from 3SAT to SOLITAIRE, and SOLITAIRE is in NP, thus SOLITAIRE is NP-complete.