Complexity

The DOMINATING SET problem is defined as follows:

DOMINATING SET (DS)
INSTANCE: a graph $G = (V, E)$ and an integer $J \leq |V|$
QUESTION: is there a subset $V' \subseteq V$ with $|V'| \leq J$ and such that $\forall u \in V \setminus V'$ ($u \in V$ but $u \notin V'$) there is $v \in V'$ with $(u, v) \in E$?

Note that this definition implies that every vertex in V is either in V' or is adjacent to some vertex in V'. When such a V' exists, with $|V'| = k$, we will say that G has a *dominating set* V' of size k. We say a vertex is *isolated* if it does not share any edge with another vertex (in other words, $u \in V$ is isolated if there is no edge $(u, v) \in E$ for any $v \in V$).

Question 1 *Provide one positive and one negative instance of DOMINATING SET, with* $|V| \geq 5$*. Show that in a positive instance of DOMINATING SET, all isolated vertices must be in* V'*.*

We want to show that DOMINATING SET is NP-complete. In the following we will denote (G, J) an instance of DOMINATING SET. We proceed by a reduction from the problem VERTEX COVER which is known to be NP-complete:

VERTEX COVER INSTANCE : a graph $G = (V, E)$ and an integer $J \leq |V|$
QUESTION : is there a subset $V' \subseteq V$ with $|V'| \leq J$ and such that $\forall (u, v) \in E$ we have $u \in V'$ or $v \in V'$

When such a V' exists, with $|V'| = k$, we will say that G has a *vertex cover* V' of size k. As previously, we will denote (G, J) an instance of VERTEX COVER.

Question 2 *Provide one positive and one negative instance of VERTEX COVER, with* $|V| \geq 5$*. First assume that G is connected. Show that if G has a vertex cover* V' *of size* k*, then G has a dominating set* V' *of size* k*. Show that converse is not true (you can use the following graph* $G = (V, E)$ *with* $V = \{1, 2, 3, 4, 5, 6\}$ *and* $E = \{(1, 2), (1, 3), (2, 3), (3, 4), (3, 5), (5, 6)\}$*). Provide an example of a type of graphs for which finding a small dominating set is easy. Provide an example of a kind of graphs for which finding a small dominating set is difficult.*

In the following, we now assume that **graphs are not necessarily connected**. Reduction from VERTEX COVER is defined as follows. Provided (G, J) an instance of VERTEX COVER, we build an instance (G_R, J_R) of DOMINATING SET:

1. for all $u \in V$, create a vertex $u_R \in V_R$ (set of vertices of G_R)
2. for all $(u, v) \in E$, create a vertex $w_{uv} \in V_R$
3. for all $(u, v) \in E$, create $(u_R, v_R) \in E_R$
4. for all $w_{uv} \in V_R$, create $(w_{uv}, u_R) \in E_R$ and $(w_{uv}, v_R) \in E_R$
5. We denote n_I the number of isolated vertices in G. Then, we set $J_R = J + n_I$.

Question 3 *Show that the reduction is polynomial. We first start with an example. Assume the instance of VERTEX COVER (G, J) is defined by $G = (V, E)$ with*

$$V = \{1, 2, 3, 4, 5, 6, 7\} \text{ and } E = \{(1, 4), (1, 5), (3, 4), (3, 5), (7, 6)\}$$

You can draw these vertices forming a circle. Set $J = 3$. Define (and draw) (G_R, J_R) the corresponding instance of DOMINATING SET obtained by the reduction defined above. Prove that both instances are positive.

Provided an instance (G, J) of VERTEX COVER, we denote V^I the set of isolated vertices in V. For any subset $X \subseteq V$ of G, we will denote $X_R \in V_R$ the set of corresponding vertices of G_R, obtained by step 1. of the reduction defined above.

Question 4 *We now turn to the general proof. Assuming (G, J) is a positive instance of VERTEX COVER where G has a vertex cover V', we consider the set $V'' \subseteq V_R$, with $V'' = V'_R \cup V^I_R$. Provide an upper bound for $|V''|$. Show that G_R has a dominating set V'' by:*

- *considering the case of isolated vertices of V^I_R*
- *considering all edges in G_R corresponding to one edge (u, v) in G.*

Conclude about the instance of DOMINATING SET.

Question 5 *Conversely, assuming (G_R, J_R) is a positive instance of DOMINATING SET: G_R has a dominating set V'_R, with $|V'_R| \leq J_R$. First consider V^I_R the set of isolated vertices of G_R. What can you say about these vertices ? Now consider $V''_R = V'_R \setminus V^I_R$. Note that V''_R may contain some vertices w_{uv}. Show that a dominating set V'_R for G_R remains a dominating set if we replace every vertex $w_{uv} \in V'_R$ by vertex u (or v). Show that this allows to define a vertex cover for graph G. Conclude about the instance of VERTEX COVER.*

Question 6 *Show that DOMINATING SET is in NP. Conclude.*

Q1: Take a graph G with 6 vertices numbered from 1 to 6 and forming a circle. With $J = 3$ it forms a positive instance, with e.g. $V' = \{1, 3, 5\}$. With $J = 2$ a negative one.

Q2: Consider the preceding graph G and add a vertex numbered 7, sharing and edge with the others 6 vertices. Note that with $V' = \{7\}$, we have a dominating set for this graph (positive instance of the corresponding problem). Still with $J = 2$ this is a negative instance of VERTEX COVER, and a positive one with $J = 4$.
If G has a vertex cover V', then for every edge $(u, v) \in E$ we have $u \in V'$ or $v \in V'$. Thus for every vertex $x \in V$:

- $x \in V'$
- or $X \notin V'$ but, as G is connected, there must be a vertex $v \in V$ such that $(x, v) \in E$ and, as V' is a vertex cover, we must have $v \in V'$.

Thus, V' is also a dominating set. Considering the graph defined in the question: it has a dominating set of size 2 (e.g. $V' = \{3, 6\}$), but no vertex cover of this size.
A "star graph" has a dominating set of size 1. A "line graph" requires more vertices...

Q3: By steps

1. $O(|V|)$
2. $O(|E|)$
3. $O(|E|)$
4. $O(|V_R|)$ and by definition we have $|V_R| = |V| + |E|$
5. $O(|V|)$

In the end we have a complexity $O(|E|) = O(|V|^2)$
For G: you can choose $V' = \{4, 5, 7\}$. For G_R you can choose $V' = \{2, 4, 5, w_{76}\}$ (recall $J_R = J + n_I = 3 + 1$).

Q4: As (G, J) is a positive instance of VERTEX COVER, we have $|V'| \leq J$. Thus $|V'_R| \leq J$ and $|V'_R \cup V^I_R| \leq J + n_I$, that is $|V''| \leq J + n_I = J'$. Note that some vertices of V^I can be in V', and thus some vertices of V^I_R can be in V'_R, but $|V'_R \cup V^I_R| \leq J + n_I$ still holds. G_R has a dominating set V'' because:

- isolated vertices are in V'' by definition $V'' = V'_R \cup V^I_R$ (and they have to be, by question 1)
- for all edge $(u, v) \in G$ we have $u \in V'$ or $v \in V'$. Thus, for all edge $(u_R, v_R) \in G_R$ we have $u_R \in V'_R$ or $v_R \in V'_R$, and consequently $u_R \in V''$ or $v_R \in V''$. Thus, for all non-isolated vertex $x \in V_R$, we have $x \in V''$ or there is a vertex $v \in V''$ such that $(x, v) \in E_R$. The instance of DOMINATING SET is positive.

Q5: From question 1, we know that all isolated vertices must be in the dominating set. Considering a vertex $w_{uv} \in V'_R$, we note that only u and v share an edge with w_{uv}. Now:

- if we assume that vertices u and v are not in V'_R (both of them), they are the only vertices $x \in V_R$ for which there is a vertex $w_{uv} \in V'_R$ such that $(x, w_{uv}) \in E_R$. Then, we note that if we replace w_{uv} in V'_R by u (or equivalently v) this property stills holds for the two vertices w_{uv} and v (resp. u). By doing this, we obtain a vertex cover V' for graph G, as every edge $(u, v) \in E$ will be such that there is $u \in V'$ or $v \in V'$

- if one vertex u or v is in V'_R then the preceding argument holds.

Note that in VERTEX COVER we do not take isolated vertices into account, as VERTEX COVER is defined by a question on edges and not vertices. Thus V''_R allows to define a VERTEX COVER in G, by the preceding procedure. Concerning its size, we have $|V'_R| \leq J_R = J + n_I$. With $V''_R = V'_R \setminus V^I_R$ we obtain $|V''_R| \leq |V'_R| - |V^I_R| \leq J + n_I - n_I = J$.

From the two steps, we conclude that there is a polynomial reduction from VERTEX COVER to DOMINATING SET.

Q6: a certificate could be a set $V' \subseteq V$. Checking that $\forall u \in V \setminus V'$ $(O(|V|))$ there is a $v \in V'$ s.t. $(u, v) \in E$ is $O(|V|)$ for each of them, thus $O(|V|^2)$ globally. This is polynomial. Then DOMINATING SET is in NP. Finally, we have proved that DOMINATING SET is NP-complete.