Sheet 1: Introduction - Exercices

Advanced Algorithms - Master DSC/MLDM/CPS2

Recap

Classic series:

- $\sum_{i=1}^{n} i = 1 + \cdots + n = \frac{n(n+1)}{2}$
- $a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{j=0}^{n-1} ar^j = a\frac{1-r^n}{1-r}$.

1 Exercise

Let f and g be two functions that take non negative values and suppose that f is $O(g)$, show that g is $\Omega(f)$.
Repeat the same process for proving (easy?):

- If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$
- If $f \in \Omega(g)$ and $g \in \Omega(h)$ then $f \in \Omega(h)$
- If $f \in \theta(g)$ and $g \in \theta(h)$, then $f \in \theta(h)$
- If $f \in O(h)$ and $g \in O(h)$ then $f + g \in O(h)$
- If $g \in O(f)$ then $f + g \in \theta(f)$

2 Exercise

Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think is true or false and give a proof or a counter example

1. $\log_2 f(n)$ is $O(\log_2 g(n))$
2. $2^{f(n)}$ is $O(2^{g(n)})$
3. $f(n)^2$ is $O(g(n)^2)$

3 Exercise

Arrange the following list of functions in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list than it should be the case that $f(n)$ is $O(g(n))$.

1. $f_1(n) = 10^n$
2. $f_2(n) = n^{1/3}$
3. $f_3(n) = n^n$
4. $f_4(n) = \log_2 n$
5. $f_5(n) = 2^{\sqrt{\log_2 n}}$

4 Recurrences

Can you find the solution for each recurrence?

1. $T(n) = T(n-1) + n$, $n \geq 2$ and $T(1) = 1$.
2. $T(n) = T(n/2) + n$, $n \geq 2$, $T(1) = 0$ and n a power of 2.
3. $T(n) = 2T(n/2) + n^2$, $n \geq 2$, $T(1) = 0$ and n a power of 2.
4. $T(n) = 2T(\sqrt{n}) + \log_2 n$, with $n \geq 4$ and $T(2) = 1$.
5. $T(n) = T(\sqrt{n}) + \log_2 \log_2 n$, with $n \geq 4$ and $T(2) = 1$.

Bonus recurrence: $T(n) = 3T(n/2) - 2T(n/4) + \log n$, $T(2) = 3$, $T(1) = 3$. This problem is hard and out of the scope of the class: this is a non homogeneous linear relation, for your culture the solution will be given in the solution sheet. You must solve the equation of the form: $F(k) = \alpha \times l_1^k + \beta \times l_2^k + \gamma \times k^2 + \delta \times k$, with l_1 and l_2 solutions of the characteritic equation: $x^2 - 3x + 2 = 0$. Try to see how to come to the formulation $F(k)$, and then to get the parameters of $\alpha, \beta, \gamma, \delta$ and then solve the equation.

5 Exercise

We have a set of electronic chips, and we have a tool for testing if two chips are equivalent in $O(1)$ (*e.g.* they have the same function). We would like to design an algorithm that is able to answer *yes* if in a set of n chips, there are **strictly more** than $n/2$ chips that are equivalent to one another, in other words we are looking for a chip such that it is similar to at least $(n/2) + 1$ other chips. The only possible operation is to pick two chips and use the testing tool.

Propose an algorithm able to answer this question in $O(n \log n)$. Can you propose a better algorithm?

Note on Master theorems

*Simple formulation 1

Theorem 1 (Master Theorem 1)**.** *If $T(n) \leq aT(n/b) + O(n^d)$ for some positive constants a, b, d then*

1. *$T(n) \in O(n^d)$ if $a < b^d$*
2. *$T(n) \in O(n^d \log n)$ if $a = b^d$*
3. *$T(n) \in O(n^{\log_b a})$ if $a > b^d$.*

*More complex formulation 2

Theorem 2 (Master theorem 1)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function and let $T(n)$ be defined on the non negative integers by the recurrence*

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$, then $T(n)$ can be bounded asymptotically as follows:

1. *if $f(n) \in O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) \in \theta(n^{\log_b a})$*
2. *if $f(n) \in \theta(n^{\log_b a} \log^k n)$ with with $k \geq 0$ a constant, then $T(n) \in \theta(n^{\log_b a} \log^{k+1} n)$*
3. *if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) \in \theta(f(n))$.*