UNIVERSITÉ JEAN MONNET SAINT-ÉTIENNE

Optimization and Operational Research: Linear Problems

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Linear Programming

Optimization is an important and fascinating area of management science and operations research. It helps to do less work, but gain more.

Linear programming (LP) is a central topic in optimization. It provides a powerful tool in modeling many applications. LP has attracted most of its attention in optimization during the last six decades for two main reasons:

- Applicability: There are many real-world applications that can be modeled as linear programming;
- Solvability: There are theoretically and practically efficient techniques for solving large-scale problems.

Basic Components of an LP:

Each optimization problem consists of three elements:

- **decision variables**: describe our choices that are under our control;
- **objective function**: describes a criterion that we wish to minimize (e.g., cost) or maximize (e.g., profit);
- **constraints**: describe the limitations that restrict our choices for decision variables.

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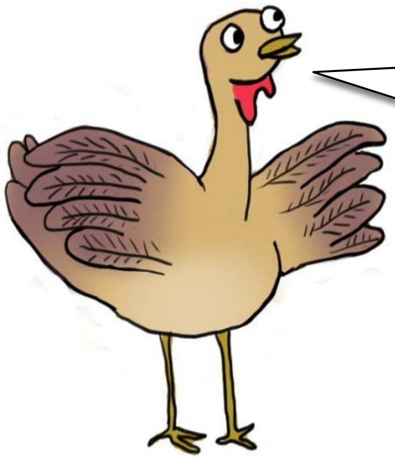
Formally, we use the term "**linear programming (LP)**" to refer to an optimization problem in which the objective function is linear and each constraint is a linear inequality or equality. I'll discuss these features soon.

An Introductory Example



I am a bit confused about the LP elements. Can you give me more details.

Let's start with an example. I'll describe it first in words, and then we'll translate it into a linear program.

An Introductory Example

Problem Statement: A company makes two products (say, P and Q) using two machines (say, A and B). Each unit of P that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Q that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

Machine A is going to be available for 40 hours and machine B is available for 35 hours. The profit per unit of P is \$25 and the profit per unit of Q is \$30. Company policy is to determine the production quantity of each product in such a way as to maximize the total profit given that the available resources should not be exceeded

Task: The aim is to formulate the problem of deciding how much of each product to make in the current week as an LP.

Step 1: Defining the Decision Variables

We often start with identifying ***decision variables*** (i.e., what we want to determine among those things which are under our control).

Tom! Can you identify the decision variables for our example?

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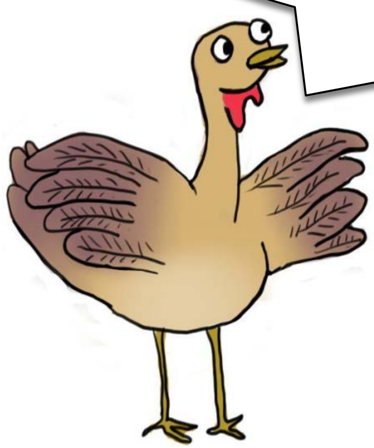
The company wants to determine the optimal product to make in the current week. So there are two decision variables:

x: the number of units of P
y: the number of units of Q



Good job! Let's move on to the second step.

Step 2: Choosing an Objective Function

We usually seek a criterion (or a measure) to compare alternative solutions. This yields the objective function.

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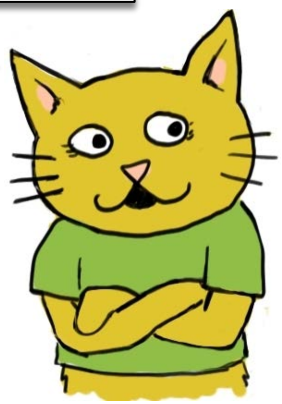
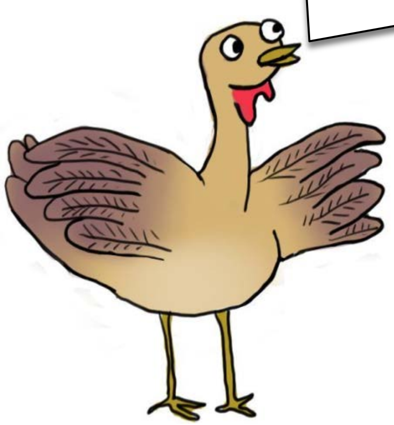


We want to maximize the total profit. The profit per each unit of product P is \$25 and profit per each unit of product Q is \$30. Therefore, the total profit is $25x+30y$ if we produce x units of P and y units of Q. This leads to the following objective function:

$$\max 40x+35y$$



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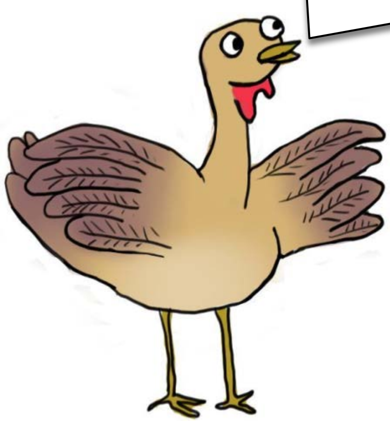

Note that:

1: The objective function is linear in terms of decision variables x and y (i.e., it is of the form $ax + by$, where a and b are constant).

2: We typically use the variable z to denote the value of the objective. So the objective function can be stated as:

$$\max z=25x+30y$$

Step 3: Identifying the Constraints

In many practical problems, there are limitations (such as resource / physical / strategic / economical) that restrict our decisions. We describe these limitations using mathematical constraints.

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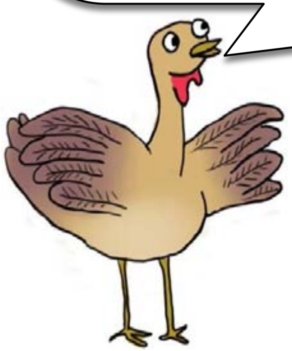
The amount of time that machine A is available restricts the quantities to be manufactured. If we produce x units of P and y units of Q, machine A should be used for 50x+24y minutes since each unit of P requires 50 minutes processing time on machine A and each unit of Q requires 24 minutes processing time on machine A. On the other hand, machine A is available for 40 hours or equivalently for 2400 minutes. This imposes the following constraint:

$$50x + 24y \leq 2400.$$

Similarly, the amount of time that machine B is available imposes the following constraint:

$$30x + 33y \leq 2100.$$



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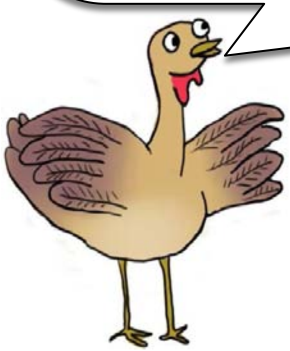

These constrains are linear inequalities since in each constraint the left-hand side of the inequality sign is a linear function in terms of the decision variables x and y and the right hand side is constant.

Step 3: Identifying the Constraints

Note: In most problems, the decision variables are required to be ***nonnegative***, and this should be explicitly included in the formulation. This is the case here. So you need to include the following two non-negativity constraints as well:

$$x \geq 0 \text{ and } y \geq 0$$

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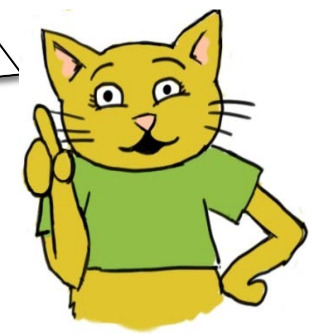
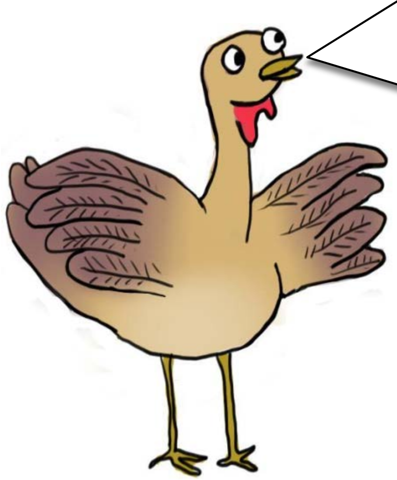


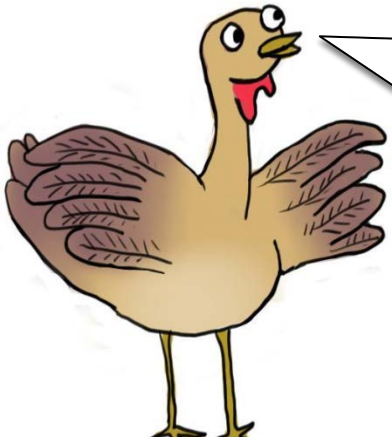


I see your point. So the constraints we are subject to (s.t.) are :

$$\begin{aligned} 50x + 24y &\leq 2400, && \text{(machine A time)} \\ 30x + 33y &\leq 2100, && \text{(machine B time)} \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

LP for the Example

Well done Tom! You did a good job. Now, write the LP by putting all the LP elements together.



Here is the LP:

$$
\begin{aligned}
\max \quad & z = 25x + 30y \\
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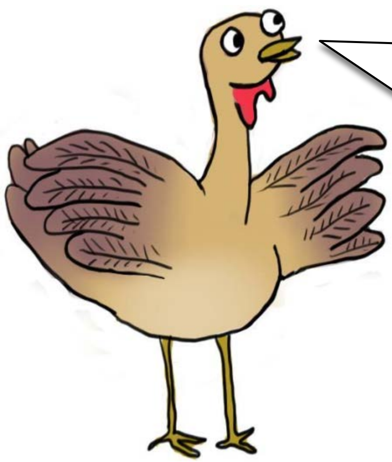

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$$

Note: To be realistic, we would require ***integrality*** for the decision variables x and y. It will lead to an integer program if we include integrality. Integer programs are harder to solve and we will consider them in later lectures. For the moment, we leave out integrality and consider this LP in this tutorial.

Now let's consider a generalization of this example in order to test whether you have understood the concepts.

A Manufacturing Example

Problem Statement: An operations manager is trying to determine a production plan for the next week. There are three products (say, P, Q, and Q) to produce using four machines (say, A and B, C, and D). Each of the four machines performs a unique process. There is one machine of each type, and each machine is available for 2400 minutes per week. The unit processing times for each machine is given in Table 1.

Table 1: Machine Data

Unit Processing Time (min)				
Machine	Product P	Product Q	Product R	Availability (min)
A	20	10	10	2400
B	12	28	16	2400
C	15	6	16	2400
D	10	15	0	2400
Total processing time	57	59	42	9600

A Manufacturing Example

Problem Statement (cont.): The unit revenues and maximum sales for the week are indicated in Table 2. Storage from one week to the next is not permitted. The operating expenses associated with the plant are $6000 per week, regardless of how many components and products are made. The $6000 includes all expenses except for material costs.

Table 2: Product Data

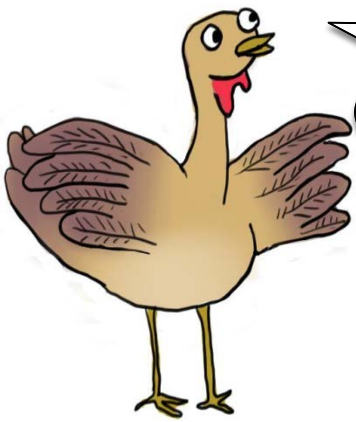
Item	Product P	Product Q	Product R
Revenue per unit	$90	$100	$70
Material cost per unit	$45	$40	$20
Profit per unit	$45	$60	$50
Maximum sales	100	40	60

Task: Here we seek the "optimal" product mix-- that is, the amount of each product that should be manufactured during the present week in order to maximize profits. Formulate this as an LP.

Step 1: Defining the Decision Variables

Tom! You are supposed to do this example by yourself. Remember that the first step is to define the decision variables. Try to come up with the correct definition.

If you need help or want to check your solution, click here to see the answer. Otherwise, you can continue by identifying the objective function.

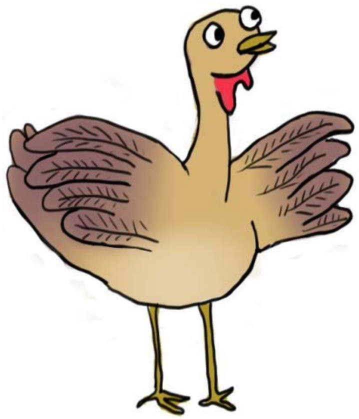


Good idea! Let me try. It should not be a difficult task.

We are trying to select the optimal product mix, so we define three decision variables as follows:

p: number of units of product P to produce,
q: number of units of product Q to produce,
r: number of units of product R to produce.

Step 2: Choosing an Objective Function



Let me review the problem statement to write the constraints

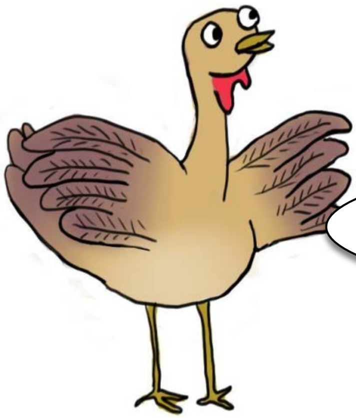
Our objective is to maximize profit:

$$\text{Profit} = (90\text{-}45)p + (100\text{-}40)q + (70\text{-}20r - 6000$$
$$= 45p + 60q + 50r - 6000$$

Note: The operating costs are not a function of the variables in the problem. If we were to drop the \$6000 term from the profit function, we would still obtain the same optimal mix of products. Thus, the objective function is

$$z = 45p + 60q + 50r$$

Step 3: Identifying the Constraints



It is good to see the complete model now to see my answer.

The amount of time a machine is available and the maximum sales potential for each product restrict the quantities to be manufactured. Since we know the unit processing times for each machine, the constraints can be written as linear inequalities as follows:

$$
\begin{array}{rl}
20p+10q+10r \le 2400 & \text{(Machine A)} \\
12p+28q+16r \le 2400 & \text{(Machine B)} \\
15p+6q+16r \le 2400 & \text{(Machine C)} \\
10p+15q+0r \le 2400 & \text{(Machine D)}
\end{array}
$$

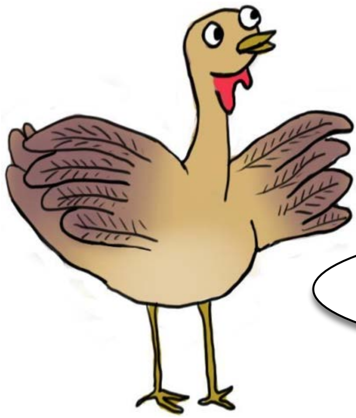
Observe that the unit for these constraints is minutes per week. Both sides of an inequality must be in the same unit. The market limitations are written as simple upper bounds.

Market constraints: $P \le 100$, $Q \le 40$, $R \le 60$.

Logic indicates that we should also include nonnegativity restrictions on the variables .

Nonnegativity constraints: $P \ge 0$, $Q \ge 0$, $R \ge 0$.

A Manufacturing Example



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By combining the objective function and the constraints, we obtain the LP model as follows:

$$
\begin{aligned}
\max \quad & z = 45p + 60q + 50r \\
\text{s.t.} \quad & 20p + 10q + 10r \leq 2400 \\
& 12p + 28q + 16r \leq 2400 \\
& 15p + 6q + 16r \leq 2400 \\
& 10p + 15q + 0r \leq 2400 \\
& 0 \leq p \leq 100 \\
& 0 \leq q \leq 40 \\
& 0 \leq r \leq 60
\end{aligned}
$$

A Manufacturing Example: Optimal Solution

The optimal solution to this LP is

$$P=81.82, Q=16.36, R=60$$

with the corresponding objective value z=$7664. To compute the profit for the week, we reduce this value by $6000 for operating expenses and get $1664. By setting the production quantities to the amount specified by the solution, we find machine usage as shown in the following table.

Machine	**Available usage**(min)	**Actual usage** (min)
A	2400	2400
B	2400	2400
C	2400	2285
D	2400	1064
Product	**Maximum sales**	**Units produced**
P	100	81.82
Q	40	16.63
R	60	60

Post Office Problem:

Let's consider one more example.

Problem Statement: A post office requires different numbers of full-time employees on different days of the week. Each full-time employee must work five consecutive days and then receive two days off.

In the following table, the number of employees required on each day of the week is specified.

Day	Number of full-time Employees Required
1=Monday	17
2=Tuesday	13
3=Wednesday	15
4=Thursday	19
5=Friday	14
6=Saturday	16
7=Sunday	11

Task: Try to formulate an LP that the post office can use to minimize the number of full-time employees who are needed to satisfy these constraints.

Step 1: Defining the Decision Variables

Remember that we often start with identifying decision variables. There might be several ways for defining decision variables, and you should be careful to pick the right one.

In our example, we want to decide home many employees we must hire.

One may think of defining decision variables as follows:

y_i: number of employees who work on day i.

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One may think of defining decision variables as follows:

$$y_i\text{: number of employees who work on day i.}$$

You can check this is not the right choice. To see why, notice that we cannot enforce the constraint that each employee works five consecutive days. In this case, there is a choice of decision variables that ensures we satisfy the constraint that each employee works five consecutive days.

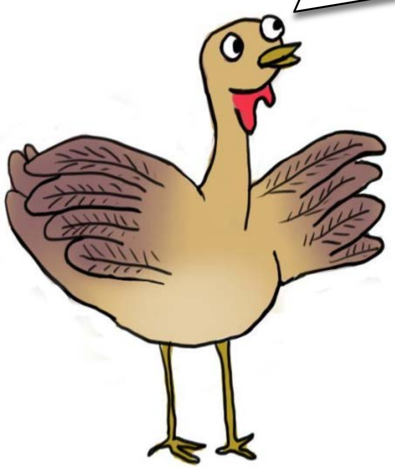
Tom! Can you come up with decision variables that can be used to formulate this problem?

Step 1: Defining the Decision Variables

Cathy, I think I see what you mean. I think we need to define seven decision variables as follows:

$\mathbf{x_i}$**:** number of employees whose five consecutive days of work begin on day i. (They work on days i, i+1, i+2, i+3, and i+4). We define it for i=1,2,…,7.

For example, x_1 is the number of people beginning work on Monday.



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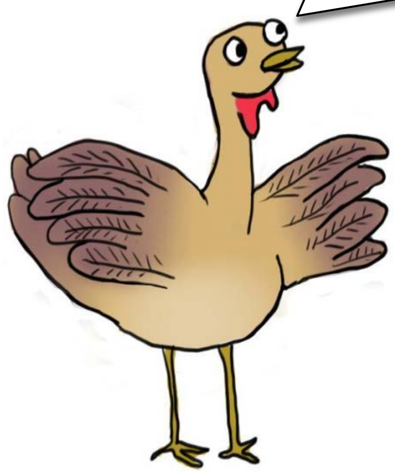
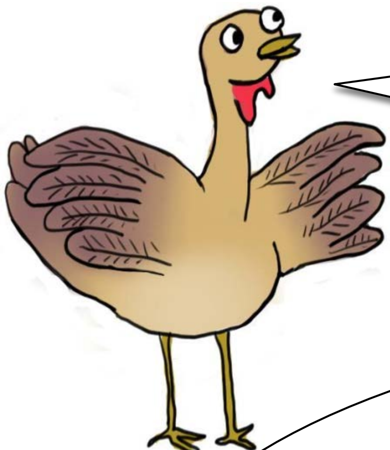


Great choice! That's exactly what I was looking for.

Now, try to formulate the objective function and the constraints.

Step 2 & 3: Choosing an Objective and Identifying the Constraints



Our aim is to minimize the number of hired employees. The objective function is

$$\textbf{min} \quad z = x_1+x_2+x_3+x_4+x_5+x_6+x_7$$

The post office must ensure that enough employees are working on each day of the week. For example, at least 17 employees must be working on Monday. To ensure that at least 17 employees are working on Monday ,we require that the constraint

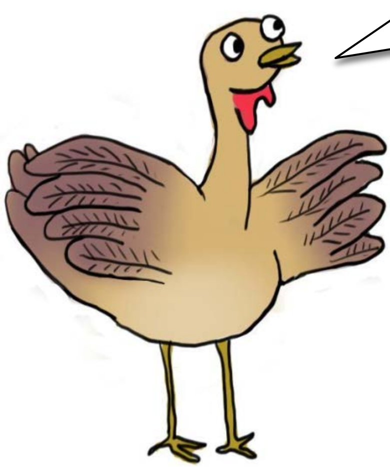
$$x_1+x_4+x_5+x_6+x_7 \geq 17$$

be satisfied. We have similar constraints for other days.

Well done, Tom. You can now put together all the constraints and the objective function to have an LP for the post office problem.

The LP Model for the Post Office Problem

$$
\begin{array}{llll}
\textbf{min } z = x_1+x_2+x_3+x_4+x_5+x_6+x_7 & & & \\
\text{s.t.} & x_1 \qquad\qquad +x_4+x_5+x_6+x_7 \geq 17 & & \text{(Monday constraint)} \\
& x_1+x_2 \qquad\qquad +x_5+x_6+x_7 \geq 13 & & \text{(Tuesday constraint)} \\
& x_1+x_2+x_3 \qquad\qquad +x_6+x_7 \geq 15 & & \text{(Wednesday constraint)} \\
& x_1+x_2+x_3+x_4 \qquad\qquad +x_7 \geq 19 & & \text{(Thursday constraint)} \\
& x_1+x_2+x_3+x_4+x_5 \qquad\qquad \geq 14 & & \text{(Friday constraint)} \\
& \quad x_2+x_3+x_4+x_5+x_6 \qquad \geq 16 & & \text{(Saturday constraint)} \\
& \qquad x_3+x_4+x_5+x_6+x_7 \geq 11 & & \text{(Sunday constraint)} \\
& \qquad x_i \geq 0 \quad \text{for } i=1,2,\ldots,7 & & \text{(Nonnegativity constraints)}
\end{array}
$$



Having formulated the LP as above, I am curious what the optimal solution is.

Optimal Solution for the Post Office Problem

The optimal solution to this LP is

$$z=67/3,\quad x_1=4/3,\quad x_2=10/3,\quad x_3=2,$$
$$x_4=22/3,\quad x_5=0,\quad x_6=10/3,\quad x_7=5.$$

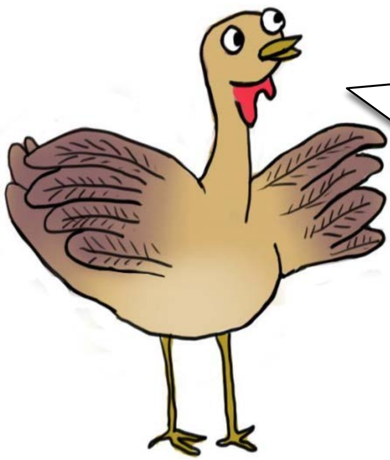
Because we are only allowing full-time employees, however, the variables must be integers (to be realistic). If we include integrality, we get an integer program.

Integer programming techniques can be used to show that an optimal solution to the post office problem is

$$z=23,\ x_1=4,\ x_2=4,\ x_3=2,\ x_4=6,\ x_5=0,\ x_6=6,\ x_7=3.$$

Note: Bartholdi, Orlin and Ratliff (1980) have developed an efficient technique to determine the minimum number of employees required when each worker receives two consecutive days off.

Creating a Fair Schedule for Employees



The optimal solution we found requires 4 workers to start on Monday, 4 on Tuesday, and so on. The workers who starts on Saturday will be unhappy because they never receive a weekend day off. This is not ***fair***.

Good point Tom! By rotating the schedules of the employees over a 23-week period, a fairer schedule can be obtained. To see how this is done, consider the following schedule:

- Weeks 1-4: start on Monday
- Weeks 5-8: start on Tuesday
- Weeks 9-10: start on Wednesday
- Weeks 11-16: start on Thursday
- Weeks 17- 20: start on Saturday
- Weeks 21-23: start on Sunday

Creating a Fair Schedule for Employees (cont.)

Employee 1 follows this schedule for a 23-week period. Employee 2 starts with week 2 of the schedule (starting on Monday for 3 weeks, then on Tuesday for 4 weeks, and closing with 3 weeks starting on Sunday and 1 week on Monday). We continue in this fashion to generate a 23-week schedule for each employee.

For example, employee 13 will have the following schedule :

- Weeks 1-4 : start on Thursday
- Weeks 5-8: start on Saturday
- Weeks 9-11: start on Sunday
- Weeks 12-15: start on Monday
- Weeks 16-19 :start on Tuesday
- Weeks 20-21:start on Wednesday
- Weeks 22-23 :start on Thursday

This method of scheduling treats each employee equally.

How to write down the Optimization pb?

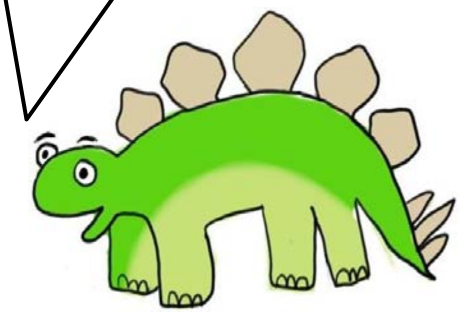
- Standard / Canonical form
  - Problem transformation
- Algorithms (implementation)
  - Constraints

Linear Programs in Standard Form

We say that a linear program is *in standard form* if the following are all true:

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints.
3. The right hand side vector, b, is non-negative.

I think it is really cool that when Ella speaks, some of her words are in red, and some are underlined. I wish I could do that.



Stan

Ella

An LP not in Standard Form

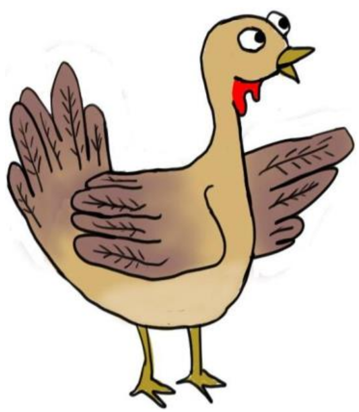
$$\text{max} \quad z = 3x_1 + 2x_2 - x_3 + x_4$$

$x_1 + 2x_2 + x_3 - x_4 \leq 5$; **not equality**

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$; **not equality, and negative RHS**

$x_1 \geq 0,\ x_2 \leq 0$ **x_2 is required to be nonpositive; x_3 and x_4 may be positive or negative.**

Why do students need to know how to convert a linear program to standard form? What's so special about standard form?



Tom

The main reason that we care about standard form is that this form is the starting point for the simplex method, which is the primary method for solving linear programs. Students will learn about the simplex algorithm very soon.

In addition, it is good practice for students to think about transformations, which is one of the key techniques used in mathematical modeling.

Next we will show some techniques (or tricks) for transforming an LP into standard form.

Converting a "≤" constraint into standard form

We first consider a simple inequality constraint. The first inequality constraint of the previous LP is

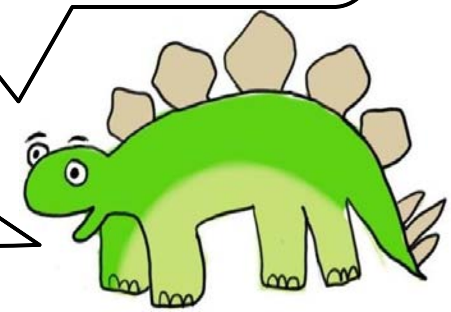
$$\mathbf{x_1 + 2x_2 + x_3 - x_4 \le 5}$$

To convert a "≤" constraint to an equality, add a slack variable. In this case, the inequality constraint becomes the equality constraint:

$$\mathbf{x_1 + 2x_2 + x_3 - x_4 + s_1 = 5.}$$

We also require that the slack variable is non-negative. That is $s_1 \ge 0$.

Nooz can speak in red, just like Ella. **How does he do that?**



Wow! I just spoke in boldface. Cool!

s_1 is called a *slack variable*, which measures the amount of "unused resource." Note that

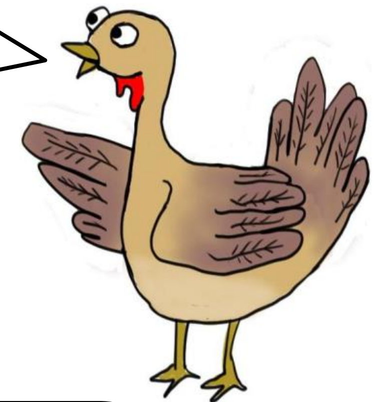
$s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$.

Converting a "≥" constraint into standard form, and converting inequalities with a negative RHS.

We next consider the constraint

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$

I know how to do that one. Just add a slack variable, like we did on the last slide.



Nice try, Tom, but incorrect. First we have to multiply the inequality by -1 in order to obtain a positive RHS. Then we get

$2x_1 + 4x_2 - x_3 - x_4 \geq 1.$

Then we add a surplus variable and get

$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1.$

s_2 is called a *surplus variable*, which measures the amount by which the LHS exceeds the RHS. Note that

$s_2 = 2x_1 + 4x_2 - x_3 - x_4 - 1$

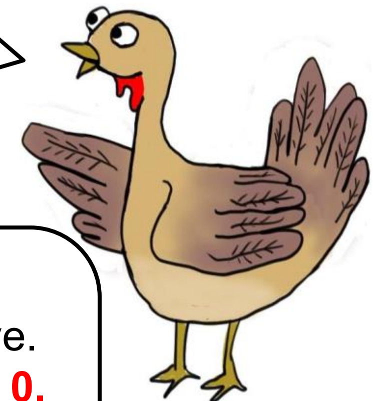
To convert a "≤" constraint to an equality, add a slack variable.

To convert a "≥" constraint to an equality, add a surplus variable.

Getting Rid of Negative Variables

Next, I'll show you how to transform the constraint constraint: $x_2 \le 0$ into standard form.

Can't we just write: $x_2 + s_3 = 0$ and $s_3 \ge 0$?



Tom, what you wrote is correct, but it doesn't help. Standard form requires all variables to be non-negative. But after your proposed change, it is still true that $x_2 \le 0$. The solution in this case is a substitution of variables. We let $y_2 = -x_2$. Then $y_2 \ge 0$. And we substitute $-y_2$ for x_2 wherever x_2 appears in the LP. The resulting LP is given below. (after you click.)

$$
\begin{aligned}
\max \quad z = \ & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \ge 0, \ y_2 \ge 0 \quad s_1 \ge 0, s_2 \ge 0
\end{aligned}
$$

Getting Rid of Variables that are Unconstrained in Sign

Next, we'll show you how to get rid of a variable that is unconstrained in sign. That is, it can be positive or negative.

Actually, we'll show you two ways. The first way is substitution. For example, x_3 below is **unconstrained in sign**. (Sometimes we call this a **free** variable.) Notice that the second constraint can be rewritten as: $\mathbf{x_3 = 2x_1 - 4y_2 - x_4 - s_2 - 1}$.

Now substitute $2x_1 - 4y_2 - x_4 - s_2 - 1$ for x_3 into the current linear program. Notice that you get an equivalent linear program without x_3. You can see it on the next slide.

$$
\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0,\; y_2 \geq 0, \quad s_1 \geq 0,\; s_2 \geq 0
\end{aligned}
$$

Getting Rid of Free Variables by Substitution

When we substitute $2x_1 - 4y_2 - x_4 - s_2 - 1$ for x_3 here is what we get. (Click now.)

The variable x_4 is also unconstrained in sign. You can substitute for it as well. After this substitution, all that will remain is an objective function and non-negativity constraints for x_1, y_2, s_1 and s_2.

This trick only works for variables that are unconstrained in sign. If you tried eliminating x_1 instead of x_3 by substitution, the optimal solution for the resulting LP would not necessarily satisfy the original constraint $x_1 \geq 0$. So eliminating x_1 in this manner would not create an equivalent LP.

$$
\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad z = \; & 1x_1 + 2y_2 + 2x_4 + s_2 + 1 \\
& 3x_1 - 6y_2 - 2x_4 + s_1 + s_2 = 5; \\
& x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0
\end{aligned}
$$

Cathy

Getting Rid of Free Variables: Version 2

There is an even simpler way of getting rid of free variables. We replace a free variable by the difference of two non-negative variables. For example, we replace x_3 by $y_3 - w_3$, and require y_3 and w_3 to be non-negative. (Click now.) You can then substitute $y_4 - w_4$ for x_4.

After solving this new linear program, we can find the solution to the original linear program. For example,
$x_3 = y_3 - w_3$ and
$x_4 = y_4 - w_4$.

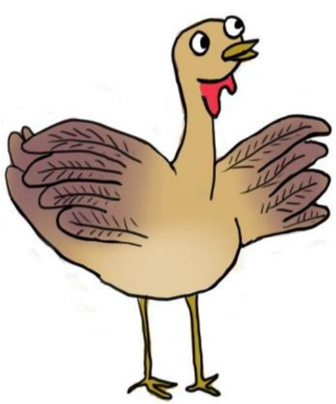
$$
\begin{array}{lrl}
\max & z = & 3x_1 - 2y_2 - x_3 + x_4 \\
 & & x_1 - 2y_2 + x_3 - x_4 + s_1 = 5; \\
 & & 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
 & & x_1 \geq 0,\ y_2 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0
\end{array}
$$

$$
\begin{array}{lrl}
\max & z = & 3x_1 - 2y_2 - y_3 + w_3 + x_4 \\
 & & x_1 - 2y_2 + y_3 - w_3 - x_4 + s_1 = 5; \\
 & & 2x_1 - 4y_2 - y_3 + w_3 - x_4 - s_2 = 1; \\
 & & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ w_3 \geq 0, \quad s_1 \geq 0,\ s_2 \geq 0
\end{array}
$$

Getting Rid of Free Variables: Version 2

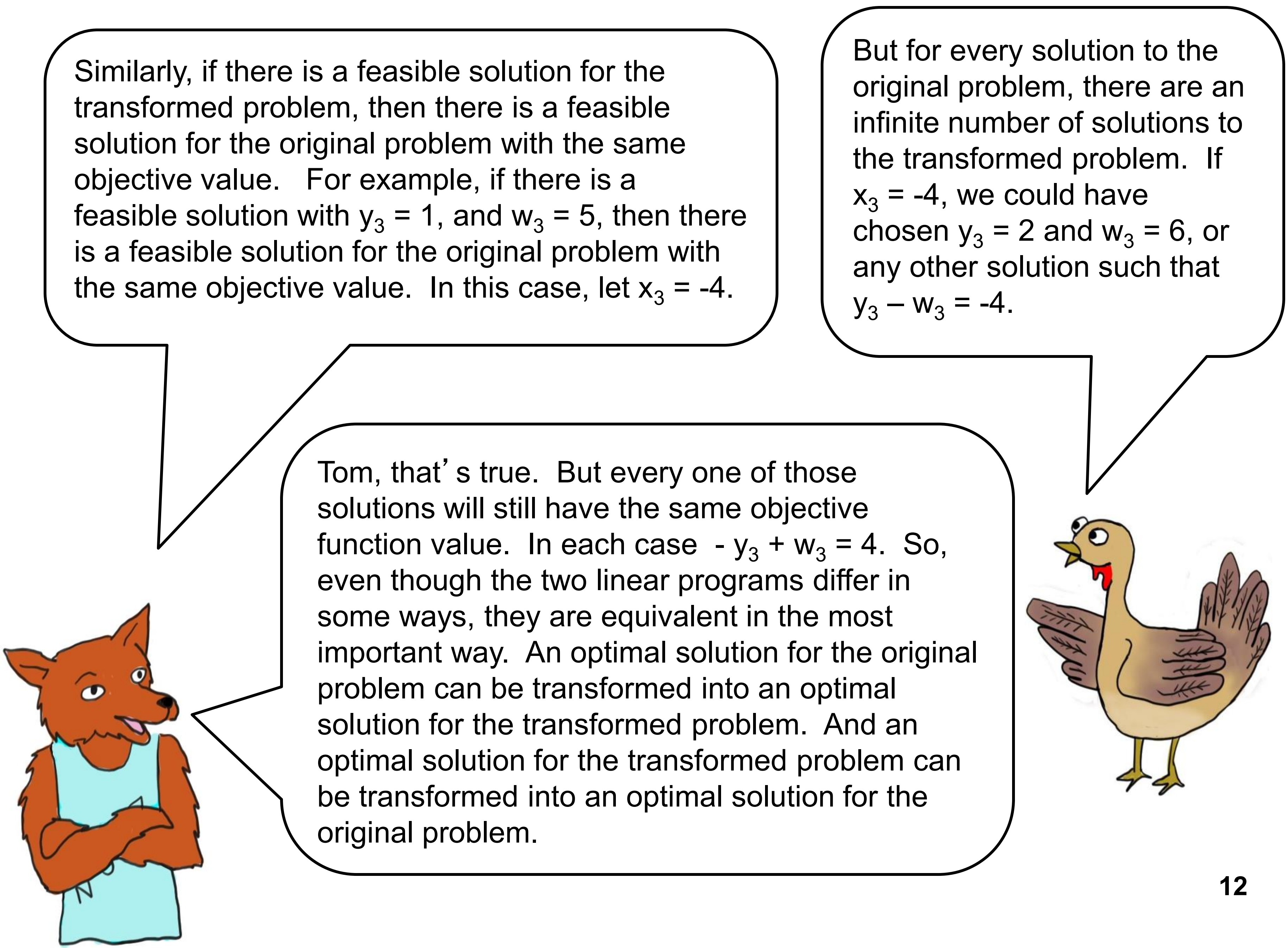
This doesn't make sense to me. Before we had a variable x_3, and now we have two variables y_3 and w_3. How can two variables be the same as a single variable?

It depends on what you mean by "the same." Here is what we mean. For every solution to the original LP, there is a solution to the transformed LP with the same objective value. For example, if there is a feasible solution with x_3 = -4, then there is a feasible solution to the transformed problem with the same objective value. In this case, let $y_3 = 0$ and $w_3 = 4$.



$$\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - x_3 + x_4 \\
& x_1 - 2y_2 + x_3 - x_4 + s_1 = 5 ; \\
& 2x_1 - 4y_2 - x_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0, \; y_2 \geq 0, \quad s_1 \geq 0, \; s_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
\max \quad z = \; & 3x_1 - 2y_2 - y_3 + w_3 + x_4 \\
& x_1 - 2y_2 + y_3 - w_3 - x_4 + s_1 = 5 ; \\
& 2x_1 - 4y_2 - y_3 + w_3 - x_4 - s_2 = 1; \\
& x_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0, \; w_3 \geq 0, \; s_1 \geq 0, \; s_2 \geq 0
\end{aligned}$$

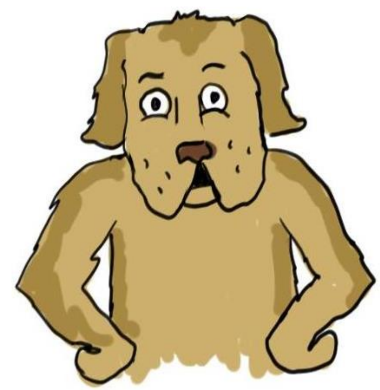

Similarly, if there is a feasible solution for the transformed problem, then there is a feasible solution for the original problem with the same objective value. For example, if there is a feasible solution with y_3 = 1, and w_3 = 5, then there is a feasible solution for the original problem with the same objective value. In this case, let x_3 = -4.
But for every solution to the original problem, there are an infinite number of solutions to the transformed problem. If x_3 = -4, we could have chosen y_3 = 2 and w_3 = 6, or any other solution such that $y_3 – w_3$ = -4.
Tom, that's true. But every one of those solutions will still have the same objective function value. In each case - y_3 + w_3 = 4. So, even though the two linear programs differ in some ways, they are equivalent in the most important way. An optimal solution for the original problem can be transformed into an optimal solution for the transformed problem. And an optimal solution for the transformed problem can be transformed into an optimal solution for the original problem.

Transforming Max to Min

We still have one last pair of transformations. We will show you how to transform a maximization problem into a minimization problem, and how to transform a minimization problem into a maximization problem. This is not part of converting to standard form, but it is still useful.

We illustrate with our original linear program, which is given below. All you need to know is that if we maximize z, then we are minimizing –z, and vice versa. See if you can use this hint to figure out how to change the problem to a minimization problem. Then click to see if you are right.

$$\begin{aligned} \min \quad -z = -3x_1 - 2x_2 + x_3 - x_4 & \\ x_1 + 2x_2 + x_3 - x_4 &\le 5\,; \\ -2x_1 - 4x_2 + x_3 + x_4 &\le -1; \\ x_1 \ge 0,\ x_2 \le 0 & \end{aligned}$$



McGraph

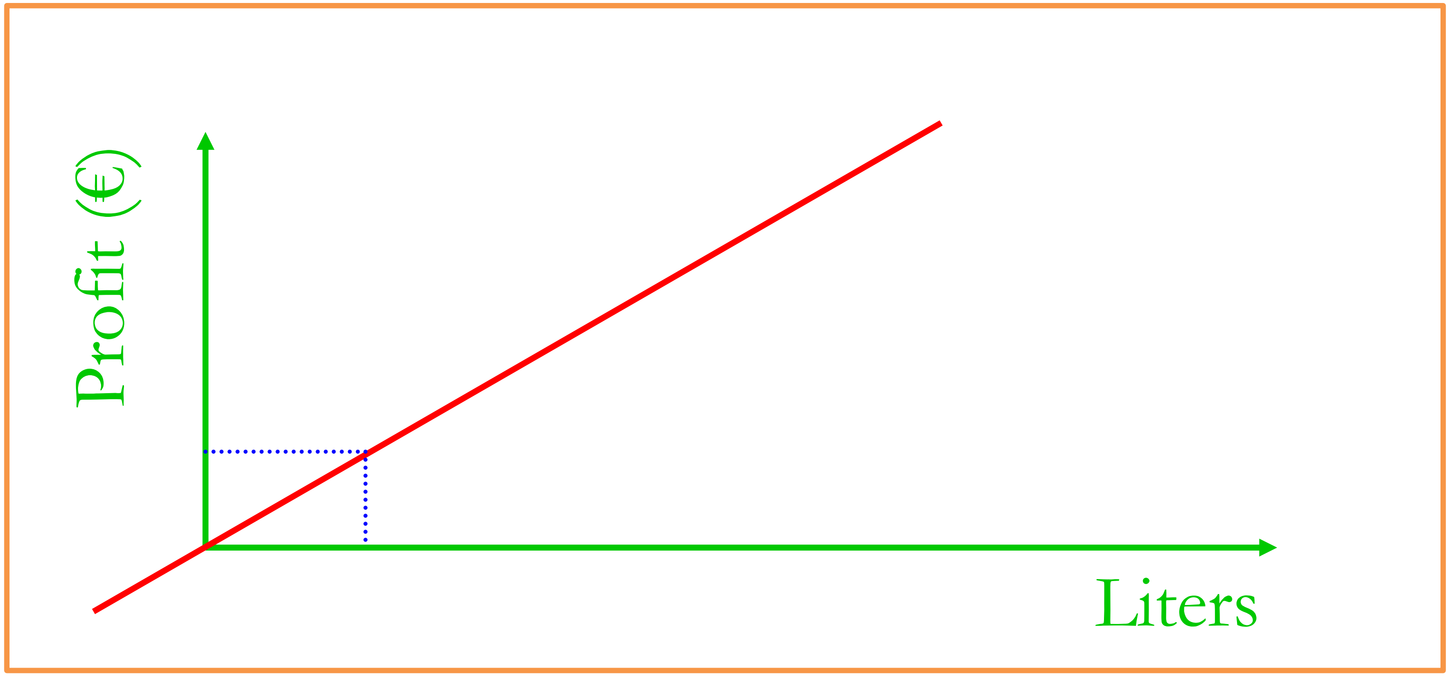
Here is an example for which you can test out these techniques. Consider the LP to the right. See if you can transform it to standard form, with maximization instead of minimization.

$$
\begin{aligned}
\min \quad & z = x_1 - x_2 + x_3 \\
& x_1 + 2x_2 - x_3 \leq 3 \\
& -x_1 + x_2 + x_3 \geq 2 \\
& x_1 - x_2 = 10 \\
& x_1 \geq 0,\ x_2 \leq 0
\end{aligned}
$$

Some examples

Optimization: trivial example

A compagny gains 5€ each time it sells one litter of chemical product. She wants to maximize its profit.

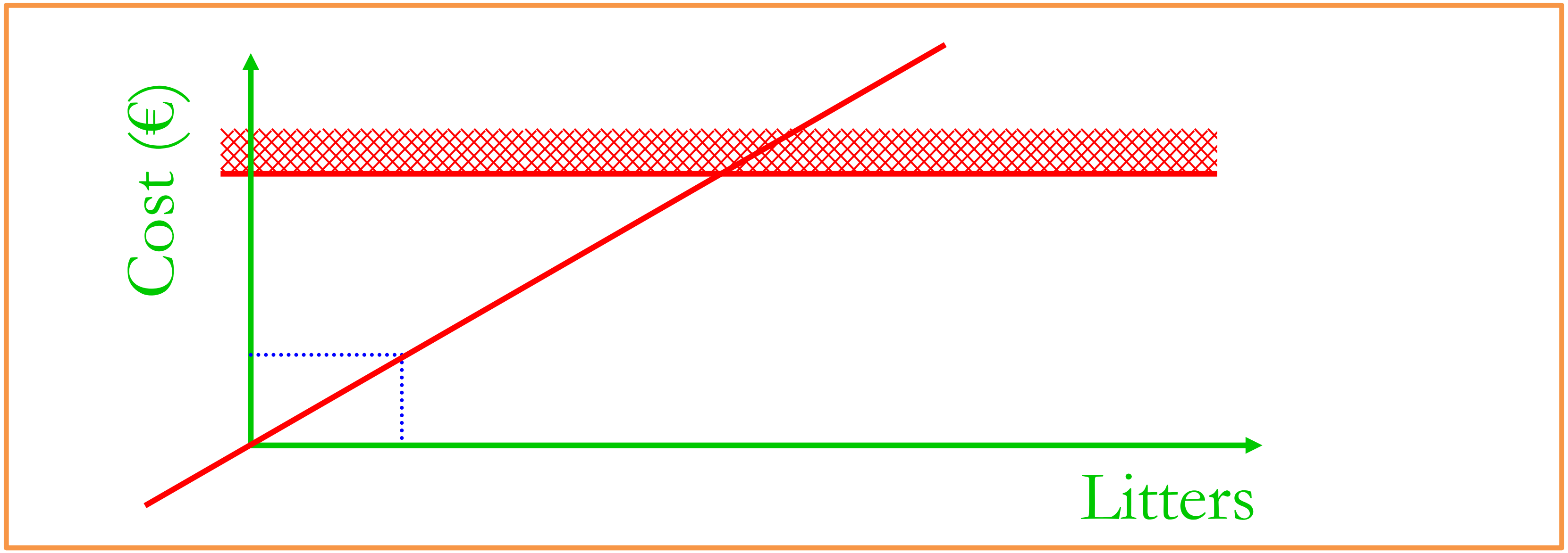


Trivial example: analysis

- Observations:
  - Linear objective function
  - No constraint (except positivity)
  - Infinite solution
- Remarks:
  - The solution is always infinite when the objective function is linear and there is no constraint.

Optimization: example 2

A lab buys chemical products 30€/litter. Its max budget is 1000$. How much product can it buy?

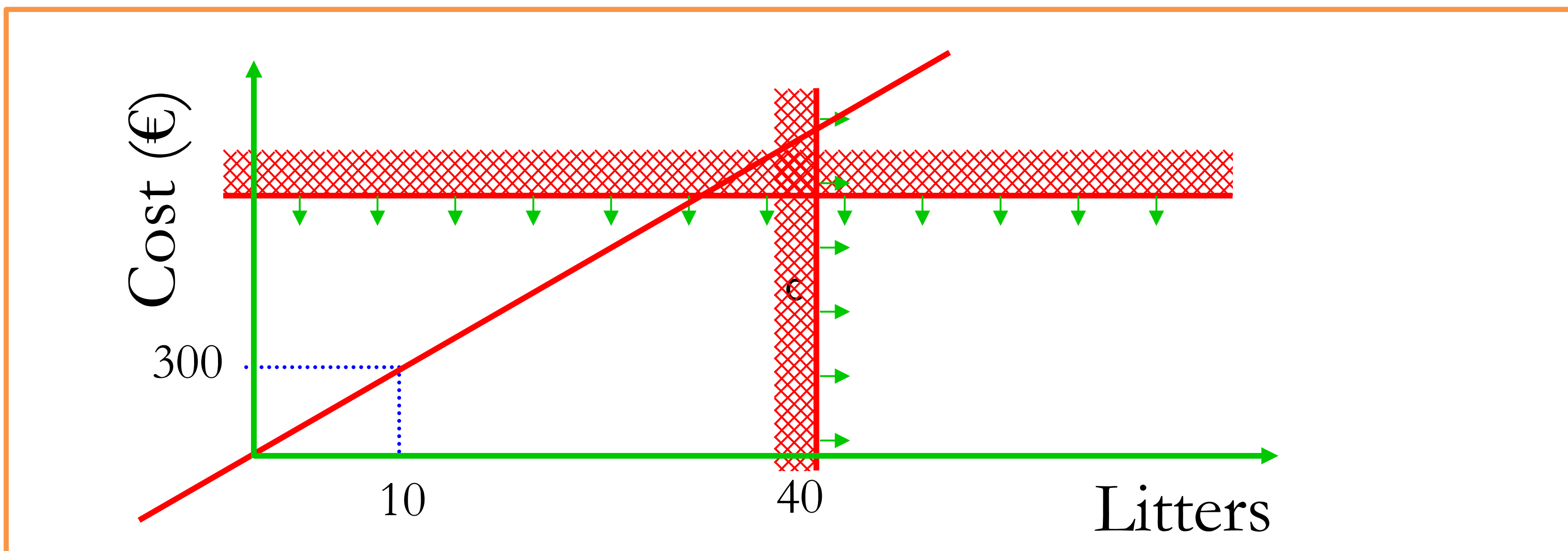


Example 2: analysis

- Observations:
  - Obvious answer: 1000 / 30 litters
  - The lab could spend 1000€ or less but chose to spend exactly 1000€
- Remarks:
  - A constraint $g(x) \leq 0$ is active in x* iff **g(x*)=0**
  - If the objective function and the constraints are linear, there is at least one active constraint at the solution

Optimization: example 3

A lab buys chemical products 30€/litter. Its max budget is 1000€ but it needs to buy at least 40 litters. How much product can it buy?

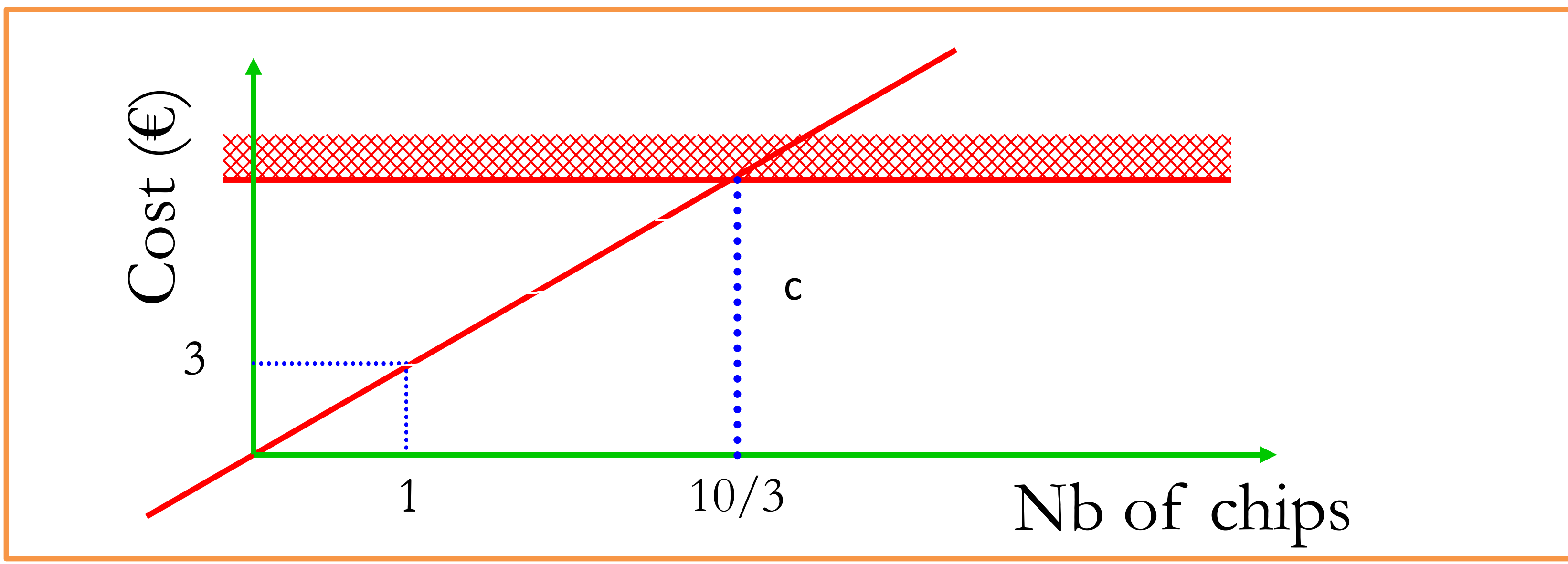


Example 3: analysis

- Observations:
  - Unsolvable problem
  - Incompatible constraints
- Remarks:
  - We say that this problem does not have any admissible solution.

Optimization: example 4

A lab buys each chip 3€. Its max budget is 10€. How many chips can it buy?

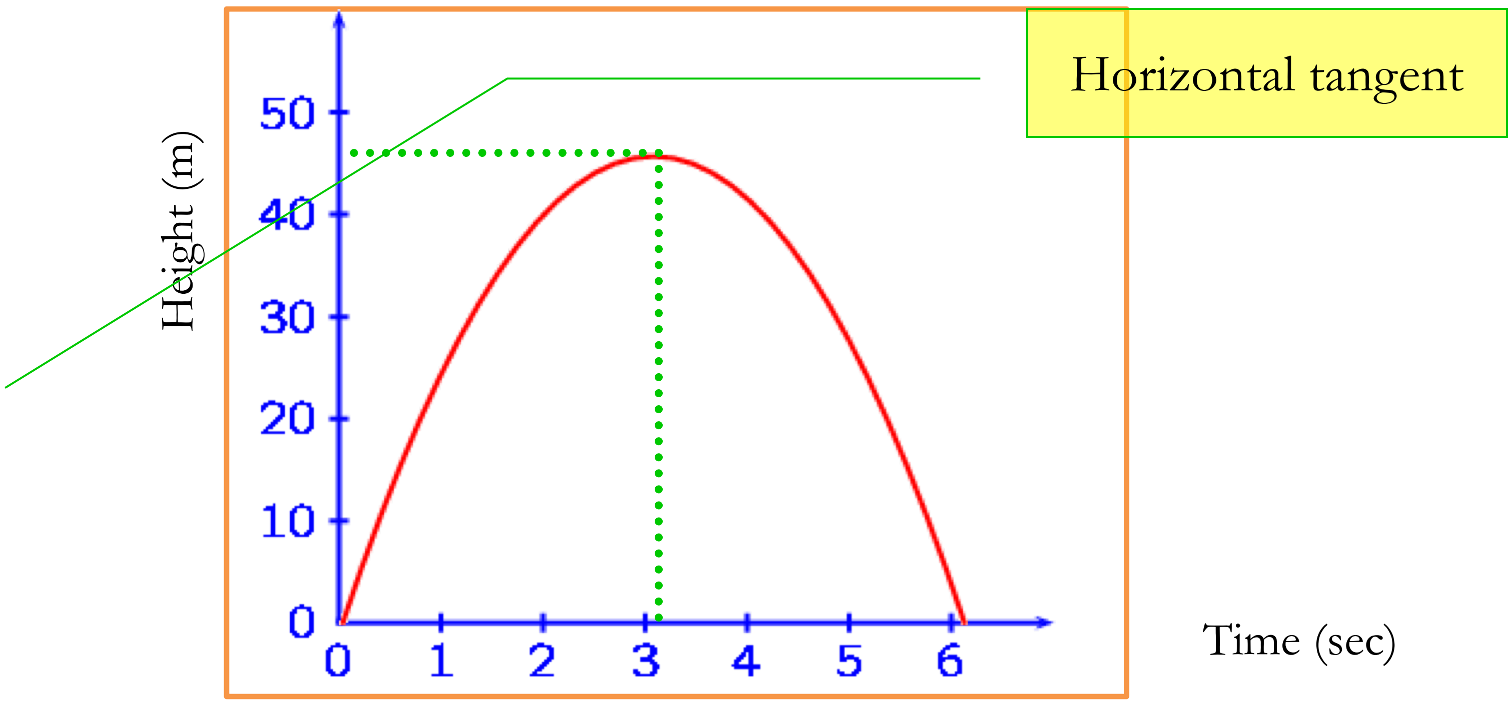


Example 4: analysis

- Observations:
  - One cannot buy parts of a chip (only the entire component)
  - Even if the objective function AND the constraints are linear, the budget will not be entirely spent.
- Remarks:
  - When variables are integer, the theoretical results can be different.

Optimization: example 5

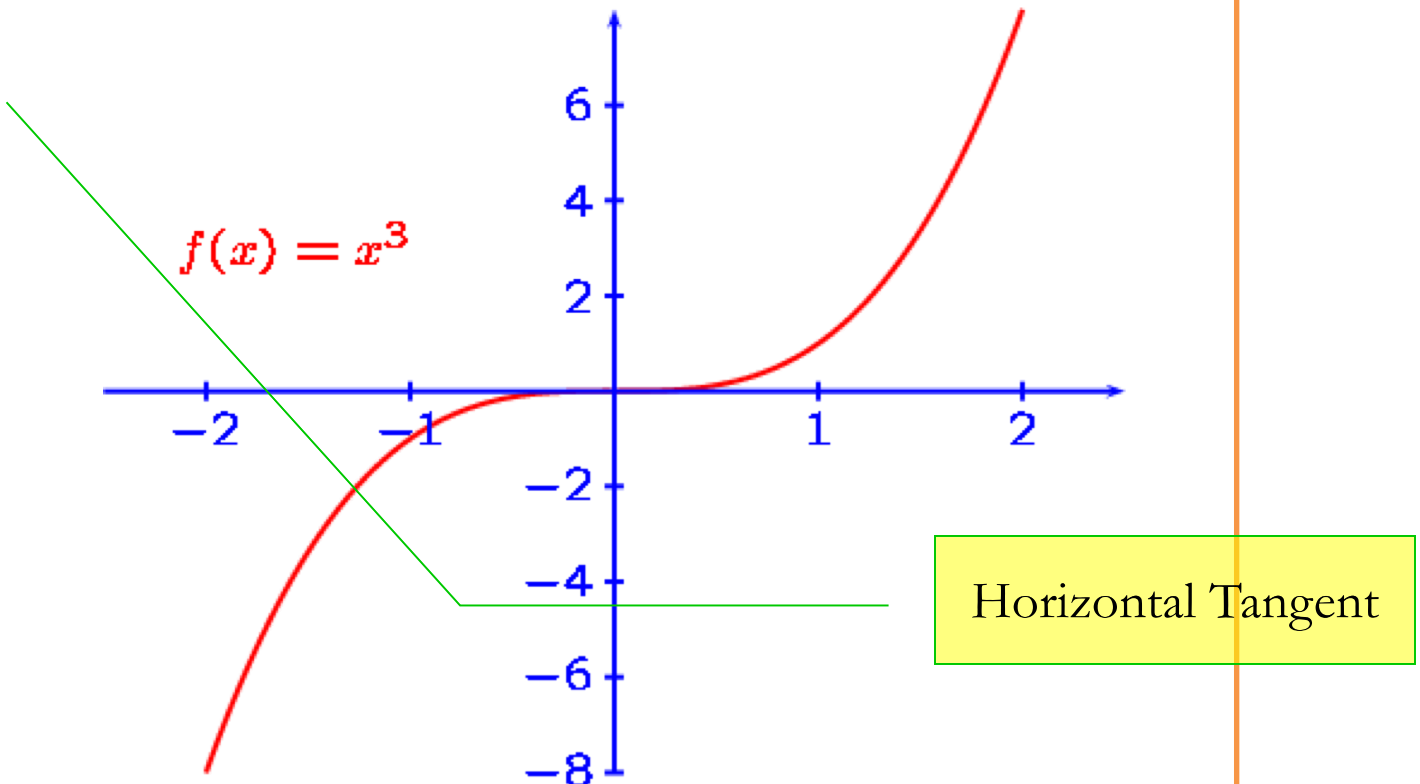
An object is thrown up at a speed of 50 m/s. When does it reach is maximum level?



Example 5: analysis

- Observations
  - Objective function is NOT linear
  - No constraint
  - Finite solution
- Remarks:
  - If the objective function is NOT linear, there might be a finite solution even if there is no constraint.
  - When we reach the solution, the tangent to the curve is horizontal (i.e, the derivative is null)

Optimization: example 6



Not a maximum, not a minimum…

Example 6: analysis

- Observations
  - No finite solution
  - The horizontal tangent exists
- Remarks:
  - A non linear objective function does not guaranty a finite solution.
  - A horizontal tangent does not necessarily give us a solution.

Problem categories (1/2)

Linear vs non-linear

- Definition:

  A function $f(x_1,x_2,\ldots,x_n)$ de $x_1,x_2,\ldots,x_n$ is **linear** if and only if there exists a number of constants $c_1,c_2,\ldots,c_n$ such that: $\mathbf{f(x_1,x_2,\ldots,x_n) = c_1\, x_1+c_2\, x_2+\ldots+c_n x_n}$

- Objective function
- Constraints

Problem categories (2/2)

Summary of the criteria:

- linear / non-linear
- Some constraints / no constraint
- convex / non convex
- concave / non concave (opposite function –f convex)
- differentiable / non differentiable
- continuous variables/ discrete

Linear Programming (LP)

The conditions of LP problems are:

1. Objective function must be a linear function of decision variables
2. Constraints should be linear function of decision variables

The GRAPHICAL Method

Restaurant owner problem

Seafoods available:

30 sea-urchins
24 shrimps
18 oysters

Two types of seafood plates to be offered:

8€ : including 5 sea-urchins, 2 shrimps et 1 oyster
6€ : including 3 sea-urchins, 3 shrimps et 3 oysters

- **Problem:** determine the number of each type of plates to be offered by the owner in order to maximize his revenue according to the seafoods available

Model the problem

What are the variables ?

x:

y:

What is the objective function ?

$$
\begin{aligned}
&\max \quad 8x + 6y \\
&\text{Subject to} \\
&\quad 5x + 3y \leq 30 \\
&\quad 2x + 3y \leq 24 \\
&\quad 1x + 3y \leq 18 \\
&\quad x, y \geq 0
\end{aligned}
$$

Solving a problem graphically

Method to solve problem having only two variables

Consider the restaurant owner problem transformed into a min problem:

$$\max \quad 8x + 6y$$

Subject to

$$5x + 3y \le 30$$
$$2x + 3y \le 24$$
$$1x + 3y \le 18$$
$$x, y \ge 0$$

$$\min \quad -(8x + 6y)$$

Subject to

$$5x + 3y \le 30$$
$$2x + 3y \le 24$$
$$1x + 3y \le 18$$
$$x, y \ge 0$$

$$\min \quad z = -8x - 6y$$

Subject to

$$5x + 3y \le 30$$
$$2x + 3y \le 24$$
$$1x + 3y \le 18$$
$$x, y \ge 0$$

Feasible Domain (1/4)

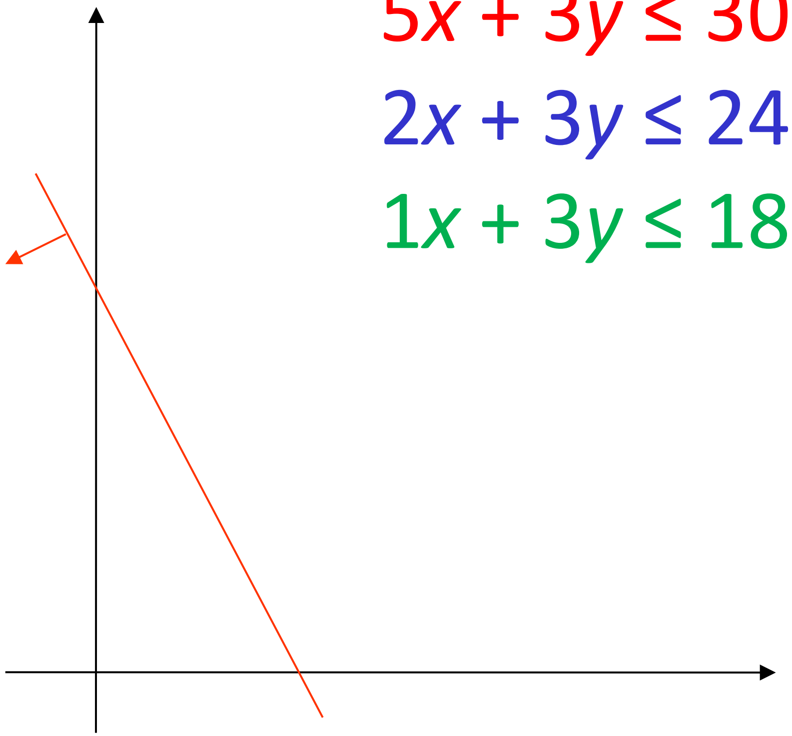
1. Draw the line

$$5x + 3y = 30$$

2. The set of points satisfying the constraint

$$5x + 3y \leq 30$$

is under the line since the origin satisfies the constraint



$5x + 3y \leq 30$

$2x + 3y \leq 24$

$1x + 3y \leq 18$

Feasible Domain (2/4)

1. Draw the line

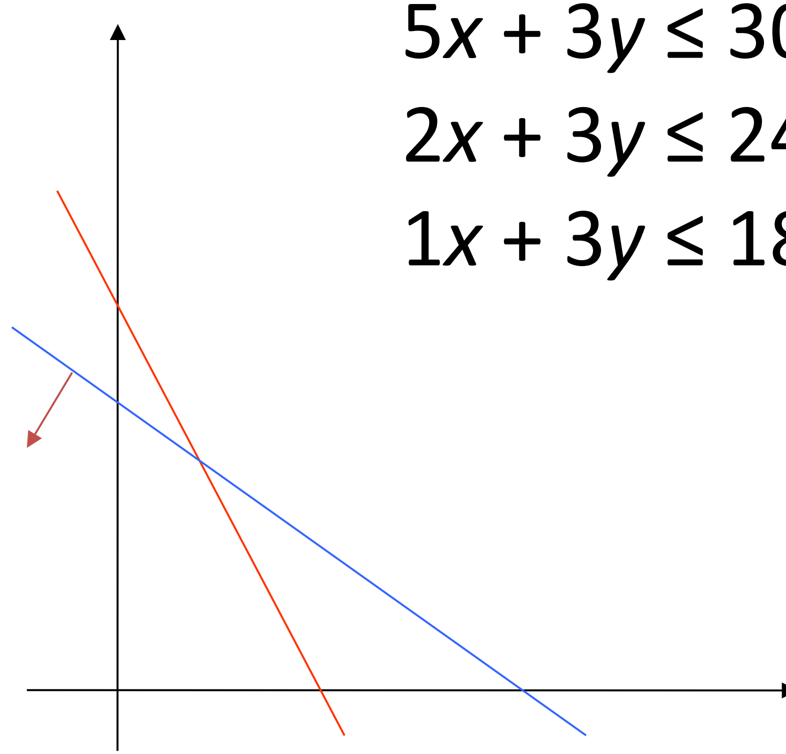
$$2x + 3y = 24$$

2. The set of points satisfying the constraint

$$2x + 3y \le 24$$

is under the line since the origin satisfies the constraint

$$5x + 3y \le 30$$
$$2x + 3y \le 24$$
$$1x + 3y \le 18$$



Feasible Domain (3/4)

1. Draw the line

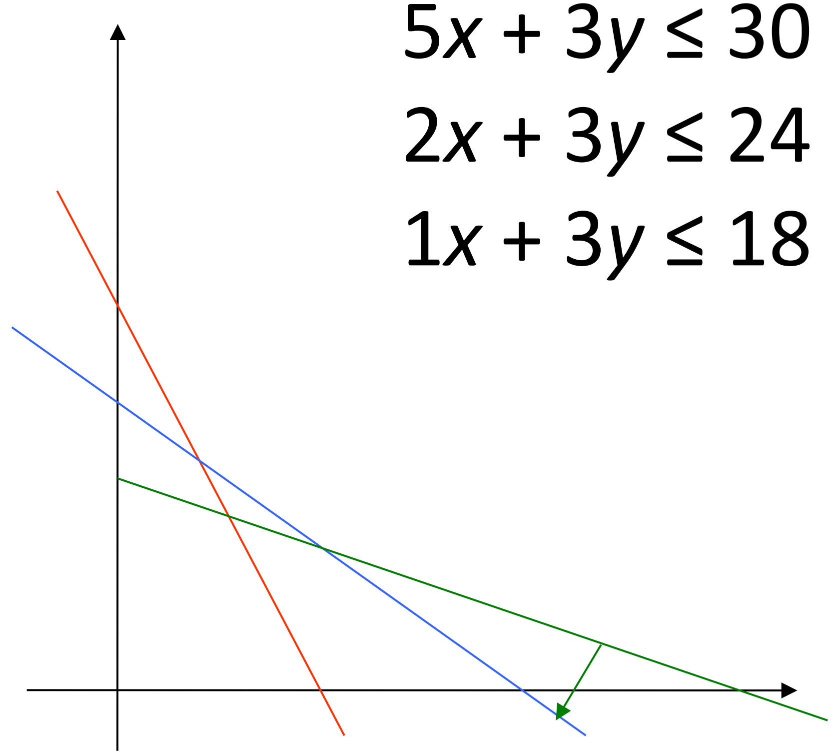
$$1x + 3y = 18$$

2. The set of points satisfying the constraint

$$1x + 3y \leq 18$$

is under the line since the origin satisfies the constraint

$$5x + 3y \leq 30$$
$$2x + 3y \leq 24$$
$$1x + 3y \leq 18$$



Feasible Domain (4/4)

The set of feasible points

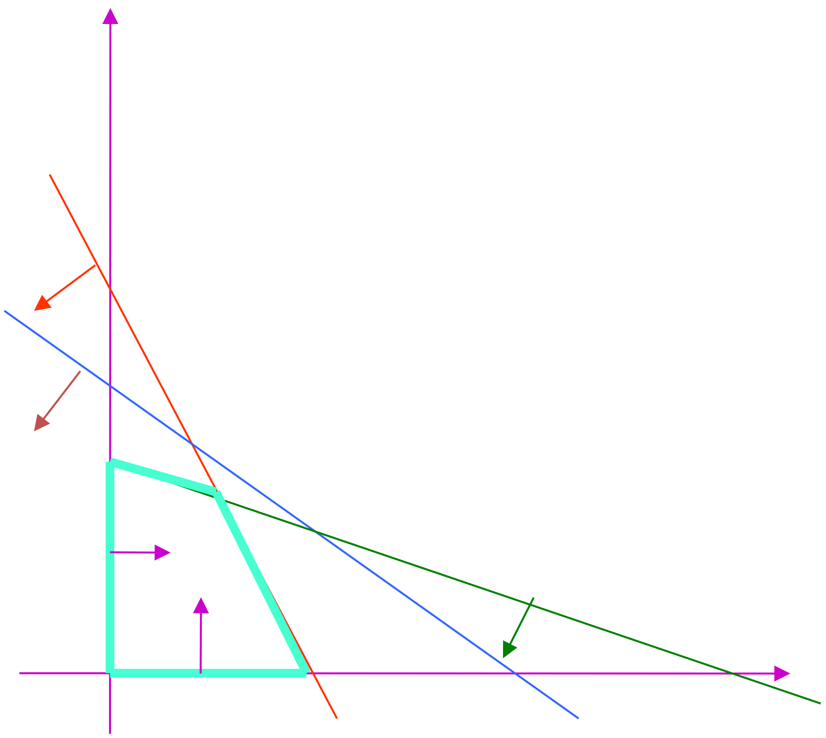
for the system

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$

$$x, y \geq 0$$



Solving the problem

1. Consider the economic function :

$$z = -8x - 6y.$$

2. The more we move away on the right of the origin, the more the objective function decreases:

$$x = 0 \text{ and } y = 0 \Rightarrow z = 0$$

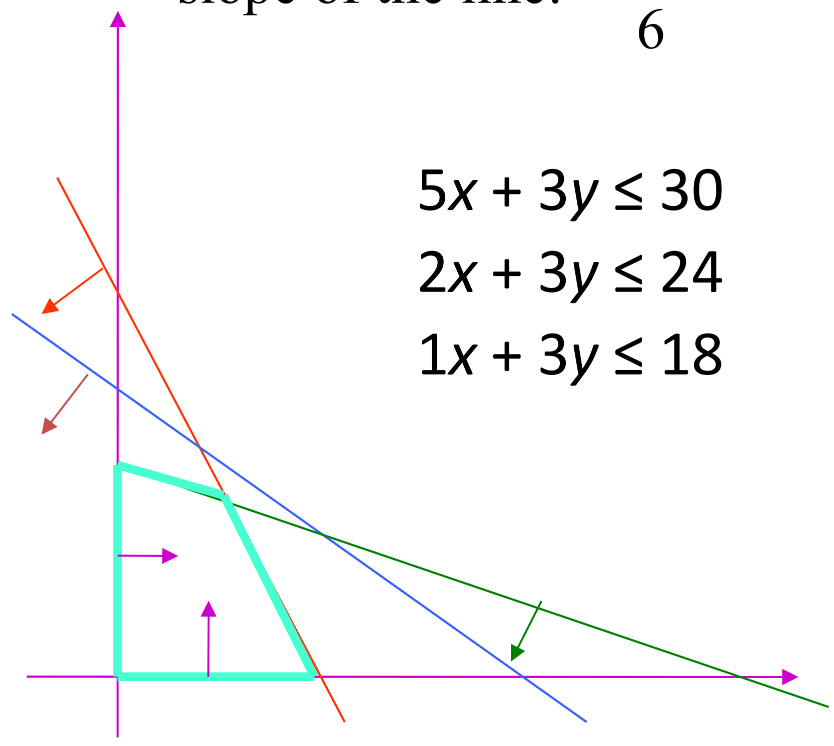
$$y = -\frac{8}{6}x - \frac{z}{6}$$

slope of the line: $-\frac{8}{6}$

$5x + 3y \le 30$

$2x + 3y \le 24$

$1x + 3y \le 18$



Solving the problem

Consider the economic function:

$$z = -8x - 6y.$$

The more we move away on the right of the origin, the more the objective function decreases:

$x = 0$ and $y = 0 \Rightarrow z = 0$

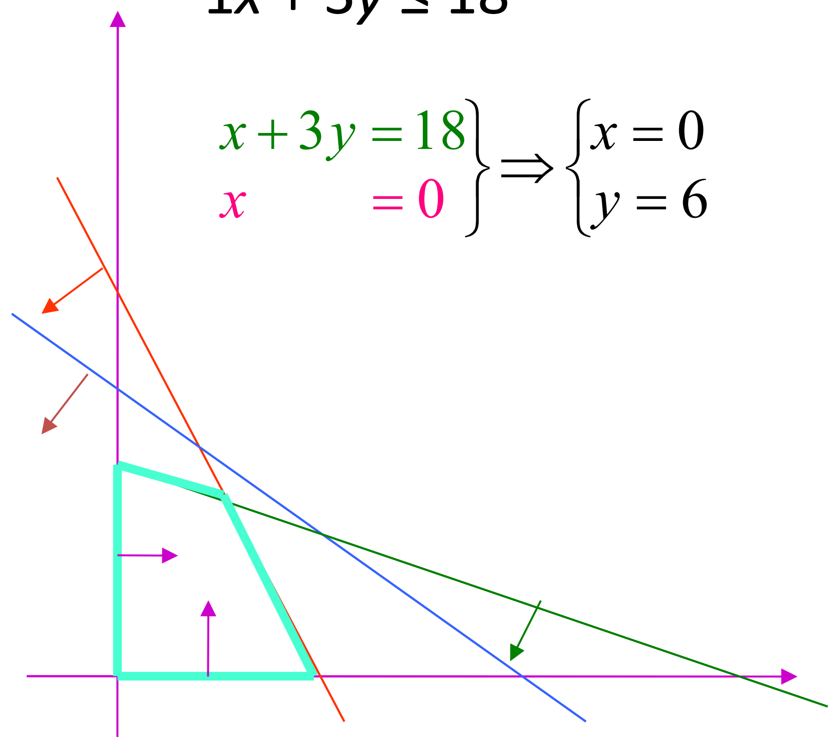
$x = 0$ and $y = 6 \Rightarrow z = -36$

$5x + 3y \leq 30$

$2x + 3y \leq 24$

$1x + 3y \leq 18$

$$\left.\begin{array}{l} x + 3y = 18 \\ x \qquad\;\; = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = 0 \\ y = 6 \end{array}\right.$$



Solving the problem

$$5x + 3y \leq 30$$
$$2x + 3y \leq 24$$
$$1x + 3y \leq 18$$

Consider the economic function :

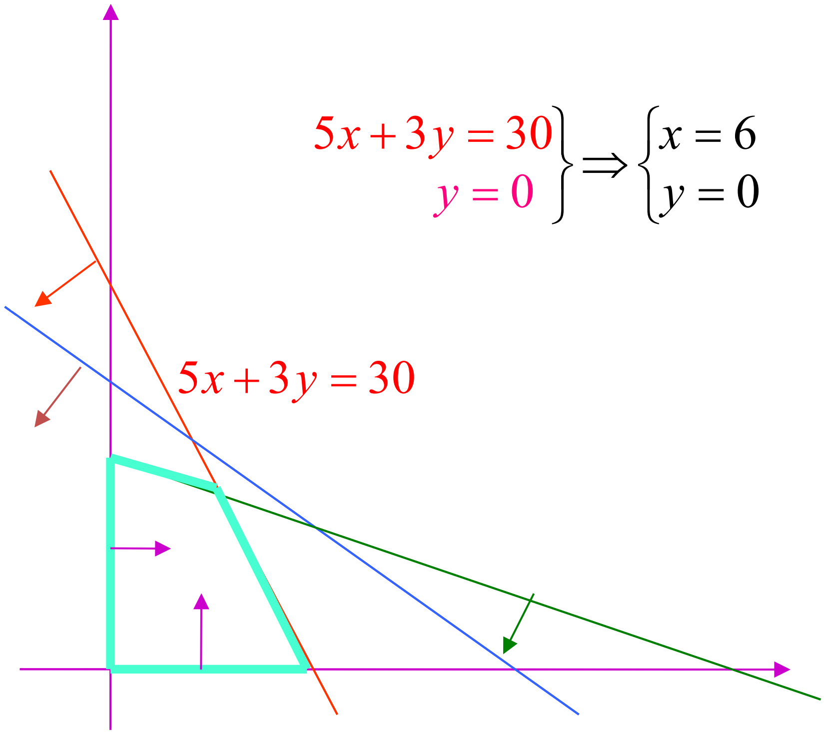
$$z = -8x - 6y.$$

The more we move away on the right of the origin, the more the objective function decreases:

$x = 0$ and $y = 0 \Rightarrow z = 0$

$x = 0$ and $y = 6 \Rightarrow z = -36$

$x = 6$ and $y = 0 \Rightarrow z = -48$



Solving the problem

1. Consider the economic function:

$$z = -8x - 6y.$$

2. The more we move away on the right of the origin, the more the objective function decreases:

   $x = 0$ and $y = 0 \Rightarrow z = 0$

   $x = 0$ and $y = 6 \Rightarrow z = -36$

   $x = 6$ and $y = 0 \Rightarrow z = -48$

   $x = 3$ and $y = 5 \Rightarrow z = -54$.

3. **Cannot move further on the right without going out of the feasible domain.**

$$5x + 3y \leq 30$$

$$2x + 3y \leq 24$$

$$1x + 3y \leq 18$$