Introduction to AI: Propositional Logic I

François Jacquenet

Laboratoire Hubert Curien
Université de Saint-Etienne
Francois.Jacquenet@univ-st-etienne.fr

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Outline

Two types of sentences

In Propositional Logic, there are two types of sentences

- *simple sentences* that express simple facts about the world
- *compound sentences* that express logical relationships between the simpler sentences of which they are composed

Simple sentences in Propositional Logic are often called *proposition constants* or, sometimes, *logical constants*

Way of writing propositional constants: [a-z][a-zA-Z0-9_]*

Logical constants

Examples of logical constants

- p, q, r
- cloudy, cLoUdY
- john_loves_mary
- john_loves_mary_everyday_and_he_hopes_it_will_continue_until_his_death
- bjqybYgvjmOhzSfyvy42UyGbvFdc5se77gfReZ

Examples that are not logical constants

- 42
- Mary
- _mary

Compound sentences

Compound sentences are formed from simpler sentences and express relationships among the constituent sentences.

There are five types of compound sentences

- negations
- conjunctions
- disjunctions
- implications
- biconditionals

Negation

A negation consists of the negation operator $\neg$ and an arbitrary sentence (simple or compound), called the target.

Example: given the simple sentence p, we can form the negation of p by writing $(\neg p)$

Conjunction

A conjunction is a sequence of sentences (simple or compound) separated by occurrences of the $\wedge$ operator and enclosed in parentheses

The constituent sentences are called conjuncts

Example: we can form the conjunction of p and q by writing $(p \wedge q)$

Disjunction

A disjunction is a sequence of sentences (simple or compound) separated by occurrences of the $\vee$ operator and enclosed in parentheses

The constituent sentences are called disjuncts

Example: we can form the disjunction of p and q by writing $(p \vee q)$

Implication

An implication consists of a pair of sentences (simple or compound) separated by the $\Rightarrow$ operator and enclosed in parentheses

The sentence to the left of the operator is called the antecedent, and the sentence to the right is called the consequent

Example: we can form the implication of p and q by writing $(p \Rightarrow q)$

Biconditional

A biconditional consists of a pair of sentences (simple or compound) separated by the $\Leftrightarrow$ operator and enclosed in parentheses

A biconditional is a combination of an implication and a reverse implication.

Example: we can express the biconditional of p and q by writing $(p \Leftrightarrow q)$

Compound sentences

Compound sentence can be made up of either simple sentences or compound sentences or a mixture of the two

Examples of compound sentences:

- $(p \wedge q)$
- $((p \Rightarrow q) \vee r)$
- $((p \Rightarrow q) \vee (p \Rightarrow q))$
- $(((p \Rightarrow q) \vee (p \Rightarrow q)) \wedge (\neg((p \vee (\neg q)) \wedge r) \Rightarrow (r \wedge (\neg q))))$

Compound sentences

It's sometimes not very pleasant to use so many parentheses
We can omit them:

- $(\neg p) \rightsquigarrow \neg p$
- $(p \wedge q) \rightsquigarrow p \wedge q$

But there may be ambiguities

- consider $p \Rightarrow q \vee r$. Does it mean $(p \Rightarrow (q \vee r))$ or $((p \Rightarrow q) \vee r)$?
- consider $p \wedge q \vee r$. Does it mean $(p \wedge (q \vee r))$ or $((p \wedge q) \vee r)$?

Coumpound sentences

This problem can be solved considering precedence of operators

We have: $prec(\neg) > prec(\wedge) > prec(\vee) > prec(\Rightarrow) = prec(\Leftrightarrow)$

In case of sentences with operators having the same precedence $\rightarrow$ right associativity

$p \Rightarrow q \Rightarrow r \rightsquigarrow (p \Rightarrow (q \Rightarrow r))$

Propositional vocabulary and language

A propositional vocabulary is a set of proposition constants

A propositional language is the set of all propositional sentences that can be formed from a propositional vocabulary

Exercise

Say whether each of the following expressions is a syntactically legal sentence of Propositional Logic

1. $p \wedge \neg p$
2. $\neg p \vee \neg p$
3. $\neg(q \vee r)\neg q \Rightarrow \neg\neg p$
4. $(p \wedge q) \vee (p\neg \wedge q)$
5. $p \vee \neg q \wedge \neg p \vee \neg q \Rightarrow p \vee q$

Exercise

For each compound sentence below, give its fully parenthesized version

1. $p \vee q$
2. $p \vee \neg q \Rightarrow r$
3. $p \Rightarrow q \Rightarrow r$
4. $p \vee \neg(p \wedge q \Rightarrow r) \Leftrightarrow p$
5. $p \wedge q \vee r \Rightarrow q \Leftrightarrow p \wedge r \vee q$

Truth assignment

Logic is unconcerned with the real world significance of proposition constants

What is interesting is the relationship among the truth values of simple sentences and the truth values of compound sentences within which the simple sentences are contained

A truth assignment for a propositional vocabulary is a function assigning a truth value to each of the proposition constants of the vocabulary.

Truth assignment

- We use the digit 1 as a synonym for true and 0 as a synonym for false
- We refer to the value of a constant or expression under a truth assignment i by superscripting the constant or expression with i as the superscript
- Example:
  - Suppose we have the propositional vocabulary: p, q, r
  - a truth assignment i:

$$p^i = 1$$
$$q^i = 0$$
$$r^i = 1$$

  - another truth assignment j:

$$p^j = 0$$
$$q^j = 0$$
$$r^j = 1$$

Truth table

A truth table for a propositional language is a table showing all of the possible truth assignments for the proposition constants in the language

- The columns of the table correspond to the proposition constants of the language and some compound sentences
- The rows correspond to different truth assignments for those constants

Examples are given in the next slides

Truth table

If the truth value of a sentence is true, the truth value of its negation is false. If the truth value of a sentence is false, the truth value of its negation is true.

The truth table of the $\neg$ is the following:

Φ	$\neg\Phi$
1	0
0	1

Truth table

The truth value of a conjunction is true if and only if the truth values of its conjuncts are both true; otherwise, the truth value is false

The truth table of the $\wedge$ is the following:

Φ	Ψ	$\Phi \wedge \Psi$
1	1	1
1	0	0
0	1	0
0	0	0

Truth table

The truth value of a disjunction is true if and only if the truth value of at least one its disjuncts is true; otherwise, the truth value is false

The truth table of the $\vee$ is the following:

Φ	Ψ	$\Phi \vee \Psi$
1	1	1
1	0	1
0	1	1
0	0	0

Truth table

The truth value of an implication is false if and only if its antecedent is true and its consequent is false; otherwise, the truth value is true

The truth table of the $\Rightarrow$ is the following:

Φ	Ψ	$\Phi \Rightarrow \Psi$
1	1	1
1	0	0
0	1	1
0	0	1

Truth table

To better understand the concept of implication and it's truth table consider the example below
Let p means *"It's raining"*. Let q means *"the ground is wet"* Now let's look at the implication $p \Rightarrow q$

p	q	$p \Rightarrow q$	Comment
T	T	T	Yes, this always happens, when it rains the ground is wet
T	F	F	We never saw rain and the ground was dry, that's impossible
F	T	T	Yes, it may be possible that it doesn't rain and the ground is wet (for example you are watering it)
F	F	T	Yes, in most situations when it is not raining, the ground is dry

Truth table

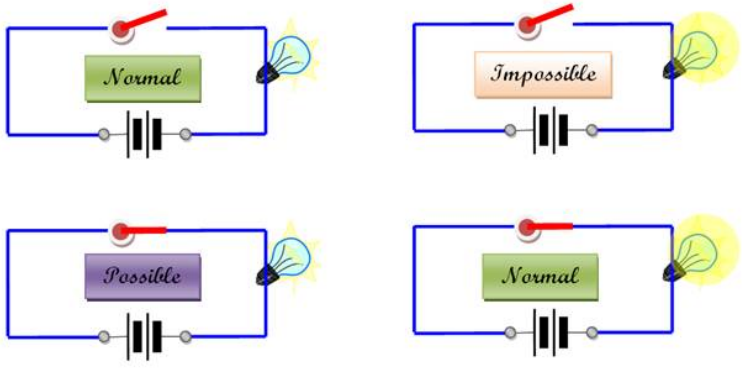
Consider the second example below

Let p means *"The number ends with a 0"*. Let q means *"The number may be a multiple of 5"* Now let's look at the implication $p \Rightarrow q$

p	q	$p \Rightarrow q$	Comment
T	T	T	Yes, by definition of a multiple of 5, a number that ends by 0 is a multiple of 5
T	F	F	It's impossible that a number that ends by a 0 is not a multiple of 5
F	T	T	Yes, if a number doesn't end by a 0 it may be a multiple of 5 (ending with a 5)
F	F	T	Yes, if a number doesn't end by a 0 it may not be a multiple of 5 (ending with a 1, 2, 3, 4, 6, 7, 8 or 9)

Truth table

Finally, consider the third example below
Let p means *"The switch is open"*. Let q means *"The bulb is off"*
Look at the diagram that summarizes the four possibilities



Truth table

Let p means *"The switch is open"*. Let q means *"The bulb is off"*
Now let's look at the truth table

p	q	$p \Rightarrow q$	Comment
T	T	T	Yes, if the switch is open, no electricity comes in the bulb and it's off
T	F	F	It's impossible that the switch is open and the bulb is on because no electricity circulates in the bulb
F	T	T	Yes, if the switch is closed it may happen that the bulb is off if it is burned out
F	F	T	Yes, it's the normal situation that if the switch is closed the bulb is on because electricity circulates in it

Truth table

A biconditional is true if and only if the truth values of its constituents agree, i.e. they are either both true or both false

The truth table of the $\Leftrightarrow$ is the following:

Φ	Ψ	$\Phi \Leftrightarrow \Psi$
1	1	1
1	0	0
0	1	0
0	0	1

Important definitions

- A truth assignment satisfies a sentence if and only if the sentence is true under that truth assignment
- A truth assignment falsifies a sentence if and only if the sentence is false under that truth assignment
- A truth assignment satisfies a set of sentences if and only if it satisfies every sentence in the set
- A truth assignment falsifies a set of sentences if and only if it falsifies at least one sentence in the set.

How to evaluate compound sentences?

Evaluation is the process of determining the truth values of compound sentences given a truth assignment for the truth values of proposition constants

How to proceed?

- Substitute true and false values for the proposition constants in our sentence
- Use operator semantics to evaluate subexpressions with these truth values as arguments
- Repeat, working from the inside out, until we have a truth value for the sentence as a whole

Example of evaluation

Consider the following assignment $i : p^i = 1; q^i = 0; r^i = 1$
We can prove that i satisfies $(p \vee q) \wedge (\neg q \vee r)$
Indeed:

$$\begin{gathered}(p \vee q) \wedge (\neg q \vee r)\\ (1 \vee 0) \wedge (\neg 0 \vee 1)\\ 1 \wedge (\neg 0 \vee 1)\\ 1 \wedge (1 \vee 1)\\ 1 \wedge 1\\ 1\end{gathered}$$

Example of evaluation

Consider now the assignment $j : p^j = 0; q^j = 1; r^j = 0$
We can prove that j does not satisfy $(p \vee q) \wedge (\neg q \vee r)$
Indeed:

$$
\begin{gathered}
(p \vee q) \wedge (\neg q \vee r) \\
(0 \vee 1) \wedge (\neg 1 \vee 0) \\
1 \wedge (\neg 1 \vee 0) \\
1 \wedge (0 \vee 0) \\
1 \wedge 0 \\
0
\end{gathered}
$$

Exercise

Consider a truth assignment in which p is true, q is false, r is true
Use this truth assignment to evaluate the following sentences

1. $p \Rightarrow q \wedge r$
2. $p \Rightarrow q \vee r$
3. $p \wedge q \Rightarrow r$
4. $p \wedge q \Rightarrow \neg r$
5. $p \wedge q \Leftrightarrow q \wedge r$

(take care to operator precedence)

Satisfaction vs Evaluation

Satisfaction is the opposite of evaluation.

- **Evaluation:** given a truth assignment we want to prove if some given sentences are true or false
- **Satisfaction:** given one or more compound sentences we want to figure out which truth assignments satisfy those sentences

There are effective procedures for finding truth assignments that satisfy Propositional Logic sentences

Example: method based on truth tables *(in the next parts of the course we will study other techniques to prove satisfaction)*

Example

Truth table for the compound sentence: $p \vee q \Rightarrow q \wedge r$

p	q	r	$p \vee q$	$q \wedge r$	$p \vee q \Rightarrow q \wedge r$
1	1	1	1	1	1
1	1	0	1	0	0
1	0	1	1	0	0
1	0	0	1	0	0
0	1	1	1	1	1
0	1	0	1	0	0
0	0	1	0	0	1
0	0	0	0	0	1

Assignments that satisfy this sentence: 1, 5, 7, 8

Complexity

- The disadvantage of the truth table method is computational complexity
- The size of a truth table for a language grows exponentially with the number of proposition constants in the language.
  - n constants $\Rightarrow 2^n$ rows
  - works well for a small number of constants
  - impractical when the number is large

Example: Chess team world (cf course 1)

- 4 girls
- 16 proposition constants (abby_likes_coddy, coddy_likes_dana, ...)
- $\Rightarrow 2^{16} = 65,536$ truth assignments...

$\Rightarrow$ Use symbolic manipulation (i.e. logical reasoning and proofs)

Natural Language and Propositional Logic

We focus on three properties of people: to be cool, to be funny, to be popular.
$\Rightarrow$ 3 proposition constants: c to mean that a person is *cool*, f to mean that a person is *funny*, p to mean that a person is *popular*
Note that we could have chosen:

- cool, funny, popular
- iqsdnfwxcv, azerty, mlkvbn
- ...

Natural Language and Propositional Logic

We want to translate four natural language sentences into PL:

- *If a person is cool or funny, then he is popular*
- *A person is popular only if he is either cool or funny*
- *A person is popular if and only if he is either cool or funny*
- *There is no one who is both cool and funny*

Consider the sentence: *If a person is cool or funny, then he is popular*

- the use of the words *"if"* and *"then"* suggests an *implication*
- the *condition* (cool or funny) is clearly a *disjunction*
- the conclusion (popular) is just a *simple fact*

$$c \vee f \Rightarrow p$$

Consider the sentence: *A person is popular only if he is either cool or funny*

- similar to the previous sentence
- but the presence of the phrase *"only if"* suggests that the conditionality goes the other way
- It is equivalent to the sentence: *If a person is popular, then he is either cool or funny*

$$p \Rightarrow c \vee f$$

Consider the sentence: *A person is popular if and only if he is either cool or funny*

- Use of *"if and only"* if suggests a biconditional
- Equivalent to the conjunction of the two implications shown above.
- The biconditional captures this conjunction in a more compact form

$$p \Leftrightarrow c \vee f$$

Consider the sentence: *There is no one who is both cool and funny*

- The word *"no"* here suggests a negation.
- We can rephrase this as: *It is not the case that there is a person who is both cool and funny*

$$\neg(c \wedge f)$$

TAKE CARE: just because we can translate sentences into the language of Propositional Logic does not mean that they are true.

BUT: we can use the evaluation procedure to determine which sentences are true and which are false

For example, suppose we were to imagine a person who is cool and funny and popular (c and f and p are all true)

First sentence (if a person is cool or funny, then he is popular)

$$c \vee f \Rightarrow p$$

$$(1 \vee 1) \Rightarrow 1$$

$$1 \Rightarrow 1$$

$$1$$

Second sentence (a person is popular only if he is either cool or funny)

$$p \Rightarrow c \vee f$$

$$1 \Rightarrow (1 \vee 1)$$

$$1 \Rightarrow 1$$

$$1$$

Third sentence (a person is popular if and only if he is either cool or funny)

$$p \Leftrightarrow c \vee f$$

$$1 \Leftrightarrow (1 \vee 1)$$

$$1 \Leftrightarrow 1$$

$$1$$

Fourth sentence (there is no one who is both cool and funny)

$$\neg(c \wedge f)$$

$$\neg(1 \wedge 1)$$

$$\neg 1$$

$$0$$

- Three of the sentences are true, while one is false.
- $\Rightarrow$ there is no person who is cool and funny and popular (assuming that the theory expressed in our sentences is correct).
- Note that there are cases where all four sentences are true
  - a person who is cool and popular but not funny
  - a person who is funny and popular but not cool.

Exercise: What about a person who is neither cool nor funny nor popular?

A small company makes widgets in a variety of constituent materials (aluminum, copper, iron), colors (red, green, blue, grey), and finishes (matte, textured, coated). Here is an order form for some widgets.

Order Form	
Material	☐ Aluminum ☑ Copper ☐ Iron
Color	☐ Red ☑ Green ☑ Blue ☐ Grey
Finish	☑ Matte ☐ Textured ☑ Coated

The company markets only a subset of all the possible combinations. The following sentences are constraints that characterize the possibilities. Your job is to determine which constraints are satisfied and which are violated by this particular order.

1. $aluminum \lor copper \lor iron$
2. $aluminum \Rightarrow grey$
3. $copper \land \neg coated \Rightarrow red$
4. $coated \land \neg copper \Rightarrow green$
5. $green \lor blue \Leftrightarrow \neg textured \land \neg iron$

Consider the sentences shown below. There are three proposition constants here, meaning that there are eight possible truth assignments. How many of these assignments satisfy all of these sentences?

- $p \vee q \vee r$
- $p \Rightarrow q \wedge r$
- $q \Rightarrow \neg r$

A small company makes widgets in a variety of constituent materials (aluminum, copper, iron), colors (red, green, blue, grey), and finishes (matte, textured, coated). Although there are more than one thousand possible combinations of widget features, the company markets only a subset of the possible combinations. The sentences below are some constraints that characterize the possibilities. Your job here is to select materials, colors, and finishes in such a way that all of the product constraints are satisfied. Note that there are multiple ways this can be done.

- $aluminum \vee copper \vee iron$
- $red \vee green \vee blue \vee grey$
- $aluminum \Rightarrow grey$
- $copper \wedge \neg coated \Rightarrow red$
- $iron \Rightarrow coated$