Introduction to AI: Propositional Analysis

François Jacquenet

Laboratoire Hubert Curien
Université de Saint-Etienne
Francois.Jacquenet@univ-st-etienne.fr

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Outline

Validity, unsatisfiability, contingency

Validity

A sentence is valid if and only if it is satisfied by every truth assignment

Example: $(p \vee \neg p)$

Unsatisfiability

A sentence is unsatisfiable if and only if it is not satisfied by any truth assignment

Example: $(p \wedge \neg p)$

Contingency

A sentence is contingent if and only if there is some truth assignment that satisfies it and some truth assignment that falsifies it

Example: $(p \wedge q)$

Satisfiability, falsifiabillity

For many purposes, it is useful to group validity, contingency, and unsatisfiability into two groups

- A sentence is satisfiable if and only if it is valid or contingent (the sentence is satisfied by at least one truth assignment)
- A sentence is falsifiable if and only if it is unsatisfiable or contingent (the sentence is falsified by at least one truth assignment)

Exercise

Say whether each of the following sentences is *valid*, *contingent*, or *unsatisfiable*

- $(p \Rightarrow q) \vee (q \Rightarrow p)$
- $p \wedge (p \Rightarrow \neg q) \wedge q$
- $(p \Rightarrow (q \wedge r)) \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow r)$
- $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \wedge q) \Rightarrow r)$
- $(p \Rightarrow q) \wedge (p \Rightarrow \neg q)$
- $(\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$
- $((\neg p \Rightarrow q) \Rightarrow (\neg q \Rightarrow p)) \wedge (p \vee q)$
- $(\neg p \vee q) \Rightarrow (q \wedge (p \Leftrightarrow q))$
- $((\neg r \Rightarrow \neg p \wedge \neg q) \vee s) \Leftrightarrow (p \vee q \Rightarrow r \vee s)$
- $(p \wedge (q \Rightarrow r)) \Leftrightarrow ((\neg p \vee q) \Rightarrow (p \wedge r))$

Definition

Intuitively: two sentences are equivalent if they say the same thing (they are true in exactly the same worlds)

More formally: a sentence Φ is logically equivalent to a sentence Ψ if and only if every truth assignment that satisfies Φ satisfies Ψ and every truth assignment that satisfies Ψ satisfies Φ

Example

The sentence $\neg(p \vee q)$ is logically equivalent to the sentence $(\neg p \wedge \neg q)$
Indeed:

- If p and q are both true, then both sentences are false
- If p and q are both false, then both sentences are true
- If either p is true or q is true, then the disjunction in the first sentence is true and the sentence as a whole false
- If either p is true or q is true, then one of the conjuncts in the second sentence is false and so the sentence as a whole is false
- Since both sentences are satisfied by the same truth assignments, they are logically equivalent

Example

The sentences $(p \wedge q)$ and $(p \vee q)$ are not logically equivalent
Indeed:

- The first is false when p is true and q is false
- The second is true in the same situation
- Hence, they are not logically equivalent

How can we test equivalence?

Build a truth table containing a column for each sentence
Example for $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$

p	q	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
1	1	0	0
1	0	0	0
0	1	0	0
0	0	1	1

Both sentences are true for the same truth assignment

How can we test equivalence?

Example for $(p \wedge q)$ and $(p \vee q)$

p	q	$(p \wedge q)$	$(p \vee q)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Both sentences are not true for the same truth assignment

Substitutability

If a sentence Φ is logically equivalent to a sentence Ψ, then we can substitute Φ for Ψ in any Propositional Logic sentence and the result will be logically equivalent to the original sentence

Exercise

For each of the following pairs of sentences, determine whether or not the sentences are logically equivalent.

- $(p \Rightarrow q \vee r)$ and $(p \wedge q \Rightarrow r)$
- $(p \Rightarrow (q \Rightarrow r))$ and $(p \wedge q \Rightarrow r)$
- $(p \wedge q \Rightarrow r)$ and $(p \wedge r \Rightarrow q)$
- $((p \Rightarrow q \vee r) \wedge (p \Rightarrow r))$ and $(q \Rightarrow r)$
- $((p \Rightarrow q) \vee (q \Rightarrow r))$ and $(p \vee \neg p)$

Definition

We say that a sentence Φ logically entails a sentence Ψ (written $\Phi \models \Psi$) if and only if *every* truth assignment that satisfies Φ also satisfies Ψ

More generally, we say that a set of sentences Δ logically entails a sentence Ψ (written $\Delta \models \Psi$) if and only if *every* truth assignment that satisfies all of the sentences in Δ also satisfies Ψ.

Example

The sentence p logically entails the sentence $(p \vee q)$

Indeed, since a disjunction is true whenever one of its disjuncts is true, then $(p \vee q)$ must be true whenever p is true

On the other hand, the sentence p does not logically entail $(p \wedge q)$

Indeed, a conjunction is true if and only if both of its conjuncts are true, and q may be false (of course, any set of sentences containing both p and q does logically entail $(p \wedge q)$)

How can we test entailment?

Build a truth table containing a column for each sentence

If every row that satisfies the premises (contains a 1) also satisfies the conclusion (also contains a 1), then the premises logically entail the conclusion

Example: show that p entails $(p \vee q)$

p	q	p	$(p \vee q)$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	0	0

How can we test entailment?

Example: show that p doesn't entail $(p \wedge q)$

p	q	p	$(p \wedge q)$
1	1	1	1
1	0	1	0
0	1	0	0
0	0	0	0

How can we test entailment?

Now consider again the (simplified) problem: *If Mary loves Pat, then Mary loves Quincy. If it is Monday, then Mary loves Pat or Quincy. If it is Monday, does Mary love Quincy?*
We can formalized this in that way:

- Let p represent the possibility that Mary loves Pat
- let q represent the possibility that Mary loves Quincy
- let m represent the possibility that it is Monday

Then we can represent the information of this problem with the following logical sentences:

- $p \Rightarrow q$
- $m \Rightarrow p \vee q$

How can we test entailment?

So we wonder if the sentences $p \Rightarrow q$ and $m \Rightarrow p \vee q$ entails $m \Rightarrow q$
Let's build the truth table

m	p	q	$p \Rightarrow q$	$m \Rightarrow p \vee q$	$m \Rightarrow q$
1	1	1	1	1	1
1	1	0	0	1	0
1	0	1	1	1	1
1	0	0	1	0	0
0	1	1	1	1	1
0	1	0	0	1	1
0	0	1	1	1	1
0	0	0	1	1	1

YES, the entailment holds.

Exercise

Use the truth table method to answer the following questions about logical entailment.

- $\{p \Rightarrow q \vee r\} \models (p \Rightarrow r)$
- $\{p \Rightarrow r\} \models (p \Rightarrow q \vee r)$
- $\{q \Rightarrow r\} \models (p \Rightarrow q \vee r)$
- $\{p \Rightarrow q \vee r, p \Rightarrow r\} \models (q \Rightarrow r)$
- $\{p \Rightarrow q \vee r, q \Rightarrow r\} \models (p \Rightarrow r)$

Definition

A sentence Φ is consistent with a sentence Ψ if and only if there is a truth assignment that satisfies both Φ and Ψ

A sentence Φ is consistent with a set of sentences Δ if and only if there is a truth assignment that satisfies both Δ and Φ

Example

The sentence $(p \vee q)$ is consistent with the sentence $(p \wedge q)$

The sentence $(p \vee q)$ is not consistent with $(\neg p \wedge \neg q)$

How can we test consistency?

Using a truth table

p	q	$p \vee q$	$p \wedge q$	$\neg p \wedge \neg q$
1	1	1	1	0
1	0	1	0	0
0	1	1	0	0
0	0	0	0	1

We see that $p \vee q$ is consistent with $p \wedge q$
and that $p \vee q$ is not consistent with $\neg p \wedge \neg q$

TAKE CARE: just because two sentences are consistent does not mean that they are logically equivalent or that either sentence logically entails the other.

p	q	$p \vee q$	$\neg p \vee \neg q$
1	1	1	0
1	0	1	1
0	1	1	1
0	0	0	1

$p \vee q$ and $\neg p \vee \neg q$ are logically consistent, but they are not logically equivalent and neither sentence logically entails the other

TAKE CARE: if one sentence logically entails another this does not necessarily mean that the sentences are consistent

This situation occurs when one of the sentences is **unsatisfiable**

If a sentence is unsatisfiable, there are no truth assignments that satisfy it. So, by definition, every truth assignment that satisfies the sentence (there are none) trivially satisfies the other sentence.

Exercise

In each of the following cases, determine whether the given individual sentence is consistent with the given set of sentences.

- $\{p \vee q, p \vee \neg q, \neg p \vee q\}$ and $(\neg p \vee \neg q)$
- $\{p \Rightarrow r, q \Rightarrow r, p \vee q\}$ and r
- $\{p \Rightarrow r, q \Rightarrow r, p \vee q\}$ and $\neg r$
- $\{p \Rightarrow q \vee r, q \Rightarrow r\}$ and $p \wedge q$
- $\{p \Rightarrow q \vee r, q \Rightarrow r\}$ and $q \wedge r$

Equivalence Theorem

Equivalence Theorem

A sentence Φ and a sentence Ψ are logically equivalent if and only if the sentence $(\Phi \Leftrightarrow \Psi)$ is valid

Indeed: two sentences are logically equivalent if and only if they are satisfied by the same set of truth assignments
We know that a biconditional is true if and only if the truth values of the conditional sentences are the same
So, if two sentences are logically equivalent, they are satisfied by the same truth assignments, and so the corresponding biconditional must be valid
Conversely, if a biconditional is valid, the two component sentences must be satisfied by the same truth assignments and so they are logically equivalent

Deduction Theorem

Deduction Theorem

A sentence Φ logically entails a sentence Ψ if and only if $(\Phi \Rightarrow \Psi)$ is valid

More generally, a finite set of sentences $\{\Phi_1, \ldots, \Phi_n\}$ logically entails Φ if and only if the compound sentence $(\Phi_1 \wedge \cdots \wedge \Phi_n \Rightarrow \Phi)$ is valid

Indeed: If a sentence Φ logically entails a sentence Ψ, it means that any truth assignment that satisfies Φ also satisfies Ψ
Looking at the semantics of implications, we see that an implication is true if and only if every truth assignment that makes the antecedent true also makes the consequent true
Consequently, logical entailment holds exactly when the corresponding implication is valid

Unsatisfiability Theorem

Unsatisfiability Theorem

A set Δ of sentences logically entails a sentence Φ if and only if the set of sentences $\Delta \cup \{\neg\Phi\}$ is unsatisfiable

Suppose that Δ logically entails Φ. If a truth assignment satisfies Δ, then it must also satisfy Φ. But then it cannot satisfy $\neg\Phi$ Therefore, $\Delta \cup \{\neg\Phi\}$ is unsatisfiable

Suppose that $\Delta \cup \{\neg\Phi\}$ is unsatisfiable. Then every truth assignment that satisfies Δ must fail to satisfy $\neg\Phi$, i.e. it must satisfy Φ. Therefore, Δ must logically entail Φ

Consistency Theorem

Consistency Theorem

A sentence Φ is logically consistent with a sentence Ψ if and only if the sentence $(\Phi \wedge \Psi)$ is satisfiable

More generally, a sentence Φ is logically consistent with a finite set of sentences $\{\Phi_1, \ldots, \Phi_n\}$ if and only if the compound sentence $(\Phi_1 \wedge \cdots \wedge \Phi_n \wedge \Phi)$ is satisfiable

Indeed: A sentence Φ is logically consistent with a sentence Ψ if and only if there is a truth assignment that satisfies both Φ and Ψ. This is equivalent to saying that the sentence $(\Phi \wedge \Psi)$ is satisfiable

Exercise

For each of the statement below say wether it is true or false

- If Φ is equivalent to Ψ, then Φ entails Ψ
- If Φ is equivalent to Ψ, then Φ is consistent with Ψ
- If Φ entails Ψ, then Φ is equivalent to Ψ
- If Φ entails Ψ, then Φ is consistent with Ψ
- If Φ is consistent with Ψ, then Φ is equivalent to Ψ
- If Φ is consistent with Ψ, then Φ entails Ψ