# SUBSET SUM problem (ControleM1Complexity2013)

	$x_1$	$x_2$	$x_3$	$C_1$	$C_2$	$C_3$	$C_4$
$v_1$	1	0	0	1	0	0	1
$v_1'$	1	0	0	0	1	1	0
$v_2$	0	1	0	0	0	0	1
$v_2'$	0	1	0	1	0	1	0
$v_3$	0	0	1	0	0	1	1
$v_3'$	0	0	1	1	1	0	0
$s_1$	0	0	0	1	0	0	0
$s_1'$	0	0	0	2	0	0	0
$s_2$	0	0	0	0	1	0	0
$s_2'$	0	0	0	0	2	0	0
$s_3$	0	0	0	0	0	1	0
$s_3'$	0	0	0	0	0	2	0
$s_4$	0	0	0	0	0	0	1
$s_4'$	0	0	0	0	0	0	2
t	1	1	1	4	4	4	4

First check the polynomiality of this construction : Table has (2n + 2k + 1) rows and (n + k) columns. This is polynomial (parameters n AND k).

### 1 SAT $\Rightarrow$ SUBSET SUM

If  $\phi$  is True

Then at least one literal  $(x_i \text{ or } \neg x_i)$  per clause  $C_j$  is True

According to the rule specified in Question 4,  $v_i$  (resp.  $v'_i$ ) is then picked in A' (because True)

In the column indexed by  $C_i$ :

- if 2 others  $v_l$  (resp.  $v'_l$ ) corresponding to  $x_l$  (resp.  $\neg x_l$ ) in  $C_j$  have also been picked into A' then choose  $s_j$
- if only one  $v_l$  (resp.  $v'_l$ ) corresponding to  $x_l$  (resp.  $\neg x_l$ ) in  $C_j$  have also been picked into A' then choose  $s'_j$
- if no other  $v_l$  (resp.  $v'_l$ ) corresponding to  $x_l$  (resp.  $\neg x_l$ ) in  $C_j$  have also been picked into A' then choose  $s_j$  and  $s'_j$

This ensures that the numbers in column indexed by  $C_j$  sum to 4 Concerning the n first columns:

In the column indexed by  $x_i$ , only one of the numbers  $v_i$  and  $v'_i$  has been picked into A'.

This ensures that the numbers in column indexed by  $x_i$  sum to 1

#### 2 SUBSET SUM $\Rightarrow$ SAT

Once again: column by column

n first columns : sum to 1

Thus, **only one** of the numbers  $v_i$  and  $v'_i$  has been picked into A'

If  $v_i$  has been picked, let us set  $x_i$  to True If  $v_i'$  has been picked, then set  $\neg x_i$  to True

This ensures that we have built a truth assignment

k last columns : sum to 4

in column  $C_j$ ,  $s_j$  and  $s'_j$  are not sufficient to obtain a 4.

Thus at least one of the value 1 in the 2n first rows is required.

It is on a row  $v_i$  (resp.  $\neg x_i$ ) and encodes the occurrence of  $x_i$  (resp.  $\neg x_i$ ) in  $C_j$ .

As  $v_i$  (resp.  $v_i$ ) is in A' then  $x_i$  (resp.  $\neg x_i$ ) is set to True (see above) and thus  $C_i$  is True

#### 3 NP-hardness

3-SAT is known to be NP-complete, thus NP-hard (Definition 20 in the Slides)

From section 1. and 2., we have : 3-SAT  $\leq_P$  SUBSET SUM.

Thus (Prop. 6) SUBSET SUM is NP-hard.

## 4 NP-complete

SUBSET SUM is in NP: given  $A' \subseteq A$  a certificate, it suffices to check if  $\sum_{a \in A'} a = t$ . This takes O(|A|) operations (as A' is a subset of A).

Together with the fact that SUBSET SUM is NP-hard, we conclude that SUBSET SUM is NP-complete.