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Exact solution of 3D Timoshenko beam problem using linked interpolation of arbitrary order

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Abstract For arbitrary polynomial loading and a sufficient finite number of nodal points N, the solution for the 3D Timoshenko beam differential equations is polynomial and given as $\boldsymbol{\theta} = \sum_{i=1}^{N} I_i \boldsymbol{\theta}_i$ for the rotation field and $\mathbf{u} = \sum_{i=1}^{N+1} J_i \mathbf{u}_i$ for the displacement field, where I_i and J_i are the Lagrangian polynomials of order $N-1$ and N, respectively. It has been demonstrated in this work that the exact solution for the displacement field may be also written in a number of alternative ways involving contributions of the nodal rotations including $\mathbf{u} = \sum_{i=1}^{N} I_i \left[\mathbf{u}_i + \frac{1}{N}(\boldsymbol{\theta} - \boldsymbol{\theta}_i) \times \mathbf{R}_i\right]$, where $\mathbf{R}_i$ are the beam nodal positions.

Keywords Linked interpolation · Timoshenko beam · Higher-order interpolation

1 Introduction

The Timoshenko beam problem is one of the classic problems in elasticity, which has attracted a widespread attention owing to its obvious usefulness in practical engineering design. In contrast to many other problems in computational mechanics and structural engineering, the Timoshenko beam problem is relatively simple, its behaviour well understood and its closed-form solution widely reported [1].

To use this solution in large-scale structural problems, possibly consisting of additional structural members for which a numerical approximation to the true solution is the most we can hope for, it becomes of interest to express it in terms of a set of basic unknown parameters appropriate for application of a particular numerical method. These are often the displacement and rotation vectors at a chosen number of nodal points, and we thus talk about interpolating the problem solution for the displacement and the rotation fields exactly.

Such exact interpolation has been also widely reported [2–6] and often praised for automatically eliminating the notorious shear-locking anomaly [4,7]. It turns out that there exist a number of forms which this interpolation can take, from a highly coupled one in which the displacement and the rotation field both depend on the nodal displacements and rotations as well as the problem geometric, material and loading data [2,3,5,8] to a completely uncoupled interpolation of the displacement and the rotation fields. The plurality of interpolation choices leading to the same answer has been studied in [4], where a family of so-called *virgin elements* is introduced, in which the above independent interpolation for the rotation and displacement fields is presumed at the outset. A family of *constrained elements* has been proposed next, in which a number of internal degrees of freedom are eliminated by reducing the shear strain in the element to a certain theoretically justified lower order, thus leading to what the authors call *the interdependent interpolation* of the rotation and the lateral displacement fields.

When faced with such a variety of interpolation choices giving the exact solution, a natural instinct in the implementation of a numerical procedure is to go for the one which involves the smallest number of

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unknown parameters. Sometimes, however, we may wish to have interpolation which is free from the problem parameters, e.g. in multi-purpose codes capable of handling materially non-linear problems, which requires interpolation with additional number of nodal unknowns. In geometric non-linearity, the economy in the number of nodal unknowns is not of prime concern either, as the extra degrees of freedom become increasingly useful in describing departure from the shape of the deformation in linear case. Also, combining beam structural elements with appropriate plate and shell elements in finite-element codes may not be feasible without additional internal nodes.

In this paper we investigate the family of exact interpolation functions for the 3D Timoshenko beam further and show that, for arbitrary polynomial loading, they follow a very structured pattern. We derive this result by consistently providing a sufficient finite number of nodal points for the displacement and the rotation vectors, which should depend on the order of the polynomial describing the applied loading. Expressing some of the internal degrees of freedom in terms of the remaining degrees of freedom leads to various types of the constrained interpolation including those of [4], and we particularly emphasise the interpolation for the displacement field using equal number of (the same) nodes for the rotation and the displacement vectors.

2 Timoshenko beam problem

The Timoshenko beam theory assumes a beam to be a 3D body with two of its dimensions considerably smaller than the third—the longitudinal dimension along which the beam has a length L. Intersection of the beam with a plane orthogonal to its longitudinal dimension defines a beam cross-section, which the theory defines as rigid. The line of centroids of the cross-sections, in contrast, is deformable. Specific to the Timoshenko beam theory is that, in the deformed state, the cross-sections are not necessarily orthogonal to the centroidal axis of the beam. Here, we consider a straight Timoshenko beam of a uniform cross-section made of a linearly elastic material. The vector differential equations for such a problem may be derived, e.g. from [1] as

$$\frac{d^3\boldsymbol{\theta}}{\mathrm{d}x^3} = \mathbf{C}_M^{-1}\left(\mathbf{G}_1 \times \mathbf{n} - \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}x}\right) \quad \text{and} \quad \frac{d^2\mathbf{r}}{\mathrm{d}x^2} = -\mathbf{C}_N^{-1}\mathbf{n} - \mathbf{G}_1 \times \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}x}, \tag{1}$$

where $0 \leq x \leq L$ is the arc-length co-ordinate of the beam centroidal axis along which a constant unit vector $\mathbf{G}_1$ is directed, $\mathbf{n}$ and $\mathbf{m}$ are the distributed force and moment load vectors, L is the beam length, and $\mathbf{C}_N = \mathrm{diag}(EA, GA_2, GA_3)$ and $\mathbf{C}_M = \mathrm{diag}(GI_t, EI_2, EI_3)$ are constant diagonal 3×3 matrices of geometric and material properties of the beam cross-section. Here, EA is the axial stiffness of the cross section, GA_2 and GA_3 are the shear stiffnesses in the principal directions, EI_2 and EI_3 are the bending stiffnesses with respect to the principal directions, and GI_t is the torsional stiffness of the cross section. Solving the above differential equations subject to particular boundary conditions results in the vector functions $\mathbf{r}(x)$ and $\boldsymbol{\theta}(x)$ describing the position of the deformed centroidal axis of the beam and the evolution of the rotation of the cross-sections along the length of the beam.

Introducing a linear transformation $x = \frac{L}{2}(1+\xi) \Leftrightarrow \xi = 2\frac{x}{L} - 1 \Rightarrow \frac{d}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{d}{\mathrm{d}\xi} = \frac{2}{L}\frac{d}{\mathrm{d}\xi}$ between the arc-length parameter x and the natural co-ordinate $-1 \leq \xi \leq 1$, as well as a pair of dimensionless loading functions

$$\boldsymbol{\mu} = \frac{L^3}{8}\mathbf{C}_M^{-1}\left(\mathbf{G}_1 \times \mathbf{n} - \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}X^1}\right) \quad \text{and} \quad \boldsymbol{\nu} = \frac{L}{2}\mathbf{C}_N^{-1}\mathbf{n}, \tag{2}$$

eq. (1) is re-written as

$$\boldsymbol{\theta}''' = \boldsymbol{\mu} \quad \text{and} \quad \mathbf{r}'' = -\frac{L}{2}(\boldsymbol{\nu} + \mathbf{G}_1 \times \boldsymbol{\theta}'),$$

where the prime ($'$) denotes a differentiation with respect to the natural co-ordinate ξ. These differential equations may now be solved for $\boldsymbol{\theta}$ and $\mathbf{r}$ to give

$$\boldsymbol{\theta} = \int\int\int \boldsymbol{\mu}\,\mathrm{d}\xi\,\mathrm{d}\xi\,\mathrm{d}\xi + \frac{1}{2}\mathbf{C}_1\xi^2 + \mathbf{C}_2\xi + \mathbf{C}_3, \tag{3}$$

$$\mathbf{r} = -\frac{L}{2}\left[\int\int \boldsymbol{\nu}\,\mathrm{d}\xi\,\mathrm{d}\xi + \mathbf{C}_4\xi + \mathbf{C}_5 + \mathbf{G}_1 \times \left(\int\int\int\int \boldsymbol{\mu}\,\mathrm{d}\xi\,\mathrm{d}\xi\,\mathrm{d}\xi\,\mathrm{d}\xi + \frac{1}{6}\mathbf{C}_1\xi^3 + \frac{1}{2}\mathbf{C}_2\xi^2\right)\right], \tag{4}$$

where $\mathbf{C}_1, \ldots, \mathbf{C}_5$ are vector constants to be determined from the boundary conditions.

3 Exact representation of the solution in terms of nodal positions and rotations

The integration constants in (3) and (4) are easily computed if a sufficient number of kinematic or static boundary conditions are known. These equations may thus be expressed in terms of the known boundary values for $\boldsymbol{\theta}$ and $\mathbf{r}$ leading to a finite element capable of providing the analytical solution. Specific to the interpolation obtained in this way is that it becomes dependent on the material and cross-sectional properties [2,3,5,6] as well as the loading functions $\boldsymbol{\mu}(\xi)$ and $\boldsymbol{\nu}(\xi)$ [8].

For the mechanical problem as defined (linear problem, straight beam, constant cross section, static loading) there is little motivation for re-casting this result in an alternative form with additional nodal degrees of freedom but, as discussed in the Introduction, such alternatives become interesting for the situations where complete adherence to the analytical solution cannot be claimed.

For polynomial loading functions of arbitrary order

$$\boldsymbol{\mu} = \sum_{i=0}^{m} \boldsymbol{\mu}_i \xi^i \quad \text{and} \quad \boldsymbol{\nu} = \sum_{i=0}^{n} \boldsymbol{\nu}_i \xi^i, \tag{5}$$

where $m, n \in \mathcal{N}$, the results for $\boldsymbol{\theta}$ and $\mathbf{r}$ in (3) and (4) can be written in terms of a finite number of parameters $\mathbf{C}_1, \ldots, \mathbf{C}_5, \boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_m, \boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_n$ as

$$\boldsymbol{\theta} = \sum_{i=0}^{m} \frac{1}{i+3}\frac{1}{i+2}\frac{1}{i+1}\boldsymbol{\mu}_i \xi^{i+3} + \frac{1}{2}\mathbf{C}_1\xi^2 + \mathbf{C}_2\xi + \mathbf{C}_3, \tag{6}$$

$$\begin{aligned}\mathbf{r} = -\frac{L}{2}\Bigg[&\sum_{i=0}^{n}\frac{1}{i+2}\frac{1}{i+1}\boldsymbol{\nu}_i\xi^{i+2} + \mathbf{C}_4\xi + \mathbf{C}_5 \\ &+ \mathbf{G}_1 \times \left(\sum_{i=0}^{m}\frac{1}{i+4}\frac{1}{i+3}\frac{1}{i+2}\frac{1}{i+1}\boldsymbol{\mu}_i\xi^{i+4} + \frac{1}{6}\mathbf{C}_1\xi^3 + \frac{1}{2}\mathbf{C}_2\xi^2\right)\Bigg].\end{aligned} \tag{7}$$

For a sufficient number of known kinematic conditions ($\boldsymbol{\theta}$ and $\mathbf{r}$ at a number of co-ordinates) the integration constants and the loading parameters may be obtained in terms of these values. Obviously, (6) and (7) may thus be re-expressed in terms of a finite number of known values for $\boldsymbol{\theta}$ and $\mathbf{r}$ (nodal positions and rotations). A so-called displacement-based finite element with exact polynomial interpolation may be constructed next using standard methodology [7,9].

3.1 Exact solution for the rotation field in terms of nodal rotations

Let us start by re-writing (6) as

$$\boldsymbol{\theta}(\xi) = \sum_{q=0}^{m+3} \boldsymbol{\alpha}_q \xi^q, \tag{8}$$

where $\boldsymbol{\alpha}_0 = \mathbf{C}_3$, $\boldsymbol{\alpha}_1 = \mathbf{C}_2$, $\boldsymbol{\alpha}_2 = \frac{1}{2}\mathbf{C}_1$, and, for $3 \le q \le m+3$, $\boldsymbol{\alpha}_q = \frac{1}{q}\frac{1}{q-1}\frac{1}{q-2}\boldsymbol{\mu}_{q-3}$. It is clear that, in order to provide the exact solution, the interpolation for the rotation field has to be of order $m+3$ or higher requiring $M \ge m+4$ ($M \ge 3$ if $\boldsymbol{\mu} = \mathbf{0}$) nodes at which

$$\boldsymbol{\theta}_1 = \boldsymbol{\theta}(\xi_1), \ldots, \boldsymbol{\theta}_M = \boldsymbol{\theta}(\xi_M),$$

for the chosen nodal co-ordinates $\xi_1, \ldots, \xi_M$. Obviously, evaluating (8) at these nodal points gives M conditions on constants $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_{m+3}$. In order to express them in terms of $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_M$, it is convenient to re-write (8) as

$$\boldsymbol{\theta}(\xi) = \sum_{q=0}^{m+3} \boldsymbol{\alpha}_q \xi^q + \sum_{q=m+4}^{M-1} \overbrace{\boldsymbol{\alpha}_q}^{=\mathbf{0}} \xi^q = \sum_{q=0}^{M-1} \boldsymbol{\alpha}_q \xi^q, \tag{9}$$

and evaluate it at the chosen nodal co-ordinates $\xi_1, \ldots, \xi_M$ to obtain

$$\sum_{q=0}^{M-1} \xi_i^q \boldsymbol{\alpha}_q = \boldsymbol{\theta}_i, \quad i = 1, \ldots, M. \tag{10}$$

The coefficients in the above system of linear equations constitute the so-called Vandermonde matrix, which can be explicitly inverted (e.g. [10]). It is known (e.g. [11]) that the columns of the inverse of the Vandermonde matrix are simply the Lagrangian polynomials expressed as vectors in the basis $(\xi^0, \xi^1, \xi^2, \ldots, \xi^{M-2}, \xi^{M-1})$. The proof is given in Appendix. For a Lagrangian polynomial

$$I_p(\xi) = \sum_{q=0}^{M-1} d_{p,q}\xi^q, \tag{11}$$

where $p = 1, \ldots, M$, the solution of (10) is $\boldsymbol{\alpha}_q = \sum_{p=1}^{M} d_{p,q}\boldsymbol{\theta}_p$ (also resulting in $\boldsymbol{\alpha}_q = \mathbf{0}$ for $q > m+3$). Substituting this result in (9) now gives

$$\boldsymbol{\theta}(\xi) = \sum_{q=0}^{M-1}\sum_{p=1}^{M} d_{p,q}\boldsymbol{\theta}_p\xi^q = \sum_{p=1}^{M}\sum_{q=0}^{M-1} d_{p,q}\xi^q\boldsymbol{\theta}_p = \sum_{p=1}^{M} I_p(\xi)\boldsymbol{\theta}_p, \tag{12}$$

where the last identity follows from (11). This result shows that the standard Lagrangian interpolation for rotations of order $m+3$ or higher provides the exact solution for rotations for the Timoshenko beam problem. If there is no distributed loading (point loads only), or if $\mathbf{G}_1 \times \mathbf{n} - \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}X^1} = \mathbf{0}$, a quadratic (three-noded) interpolation for rotations suffices for the exact solution.

3.2 Exact solution for the position field in terms of nodal positions and rotations

In a similar manner, we first re-write (7) as

$$\mathbf{r}(\xi) = -\frac{L}{2}\left(\sum_{j=0}^{n+2}\boldsymbol{\beta}_j\xi^j + \mathbf{G}_1 \times \int_{-1}^{\xi}\boldsymbol{\theta}\,\mathrm{d}\eta\right), \tag{13}$$

where $\boldsymbol{\beta}_0 = \mathbf{C}_5$, $\boldsymbol{\beta}_1 = \mathbf{C}_4$, and, for $2 \le j \le n+2$, $\boldsymbol{\beta}_j = \frac{1}{j}\frac{1}{j-1}\boldsymbol{\nu}_{j-2}$. Again, in order to provide the exact solution, interpolation for the part of the position field defined by the parameters $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n+2}$ has to be of order $n+2$ or higher for which $N \ge n+3$ ($N \ge 2$ if $\boldsymbol{\nu} = \mathbf{0}$) nodes are needed. At a chosen set of nodal co-ordinates $\bar{\xi}_1, \ldots, \bar{\xi}_N$, not necessarily coincident with the one used for interpolating rotations, these positions are

$$\mathbf{r}_1 = \mathbf{r}(\bar{\xi}_1), \ldots, \mathbf{r}_N = \mathbf{r}(\bar{\xi}_N). \tag{14}$$

Again, it is useful to recognise that the sum in (13) may be written as

$$\sum_{j=0}^{n+2}\boldsymbol{\beta}_j\xi^j = \sum_{j=0}^{n+2}\boldsymbol{\beta}_j\xi^j + \sum_{j=n+3}^{N-1}\overbrace{\boldsymbol{\beta}_j}^{=\mathbf{0}}\xi^j = \sum_{j=0}^{N-1}\boldsymbol{\beta}_j\xi^j$$

to express (13) as

$$\mathbf{r}(\xi) = -\frac{L}{2}\left(\sum_{j=0}^{N-1}\boldsymbol{\beta}_j\xi^j + \mathbf{G}_1 \times \int_{-1}^{\xi}\boldsymbol{\theta}\,\mathrm{d}\eta\right). \tag{15}$$

and evaluate it at the nodal points in (14):

$$\sum_{j=0}^{N-1} \bar{\xi}_i^j \boldsymbol{\beta}_j = -\left(\frac{2}{L}\mathbf{r}_i + \mathbf{G}_1 \times \int_{-1}^{\bar{\xi}_i} \boldsymbol{\theta}(\eta)\mathrm{d}\eta \right), \quad i = 1, \ldots, N,$$

from where $\boldsymbol{\beta}_0, \ldots, \boldsymbol{\beta}_{N-1}$ may be obtained in terms of the nodal positions and rotations (again, for $j > n+2$, $\boldsymbol{\beta}_j$ are known to vanish). It is obvious that each of the unknown parameters $\boldsymbol{\beta}_0, \ldots, \boldsymbol{\beta}_{N-1}$ is composed of two parts: one which is dependent on the nodal positions $\mathbf{r}_i$, $i = 1, \ldots, N$, and the other dependent on the nodal rotations $\boldsymbol{\theta}_p$, $p = 1, \ldots, M$, hidden in $\boldsymbol{\theta}(\eta)$ as per (12). Let us denote these parts, respectively, as $\boldsymbol{\beta}_{j,r}$ and $\boldsymbol{\beta}_{j,\theta}$ so that, for $j = 1, \ldots, N-1$,

$$\boldsymbol{\beta}_j = \boldsymbol{\beta}_{j,r} + \boldsymbol{\beta}_{j,\theta},$$

where $\boldsymbol{\beta}_{j,r}$ and $\boldsymbol{\beta}_{j,\theta}$ are accordingly obtained from the following two systems of linear equations:

$$\sum_{j=0}^{N-1} \bar{\xi}_i^j \boldsymbol{\beta}_{j,r} = -\frac{2}{L}\mathbf{r}_i \quad \text{and} \quad \sum_{j=0}^{N-1} \bar{\xi}_i^j \boldsymbol{\beta}_{j,\theta} = -\mathbf{G}_1 \times \int_{-1}^{\bar{\xi}_i} \boldsymbol{\theta}(\eta)\mathrm{d}\eta, \quad i = 1, \ldots, N. \tag{16}$$

Finally, this means that $\mathbf{r}(\xi)$ in (15) may be expressed as

$$\mathbf{r}(\xi) = \mathbf{r}_r(\xi) + \mathbf{r}_\theta(\xi), \tag{17}$$

with

$$\mathbf{r}_r(\xi) = -\frac{L}{2}\sum_{j=0}^{N-1} \boldsymbol{\beta}_{j,r}\xi^j \quad \text{and} \quad \mathbf{r}_\theta(\xi) = -\frac{L}{2}\left(\sum_{j=0}^{N-1} \boldsymbol{\beta}_{j,\theta}\xi^j + \mathbf{G}_1 \times \int_{-1}^{\xi} \boldsymbol{\theta}(\eta)\mathrm{d}\eta \right). \tag{18}$$

Note that from (18) and (16) it follows that, at the nodes $(\bar{\xi}_1, \ldots, \bar{\xi}_N)$, $\mathbf{r}_r(\bar{\xi}_i)$ and $\mathbf{r}_\theta(\bar{\xi}_i)$ turn out to be

$$\mathbf{r}_r(\bar{\xi}_i) = \mathbf{r}_i \quad \text{and} \quad \mathbf{r}_\theta(\bar{\xi}_i) = \mathbf{0}.$$

As a result, $\mathbf{r}_r(\xi)$ may be expressed in terms of the Lagrangian polynomials

$$J_k(\xi) = \sum_{j=0}^{N-1} \bar{d}_{k,j}\xi^j \tag{19}$$

as $\mathbf{r}_r(\xi) = \sum_{k=1}^{N} J_k(\xi)\mathbf{r}_k$ and $\mathbf{r}_\theta(\xi)$ is a polynomial with zeroes at the nodes $\bar{\xi}_1, \ldots, \bar{\xi}_N$, i.e. $\mathbf{r}_\theta(\xi)$ may be eventually written as a product of $\prod_{k=1}^{N+1}(\xi - \bar{\xi}_k)$ and a vector coefficient depending on the nodal rotations $\boldsymbol{\theta}_p = \boldsymbol{\theta}(\xi_p)$, $p = 1, \ldots, M$ and possibly on another function of ξ.

3.2.1 Part of the solution for the position field dependent on the nodal positions

Part of the solution for the position field dependent on the nodal positions $\mathbf{r}_r(\xi)$ is obtained by solving the system of linear equations in $(16)_1$ for $\boldsymbol{\beta}_{j,r}$, $j = 0, \ldots, N-1$, and substituting the result into $(18)_1$. As before, the matrix of coefficients to this system is a Vandermonde matrix and in Appendix the solution of the Vandermonde system is proven to be equal to what here turns out to be

$$\boldsymbol{\beta}_{j,r} = -\frac{2}{L}\sum_{k=1}^{N} \bar{d}_{k,j}\mathbf{r}_k, \quad j = 1, \ldots, N \tag{20}$$

with the parameters $\bar{d}_{k,j}$ being the coefficients of the Lagrangian polynomials (19). Substituting (20) in $(18)_1$ now gives

$$\mathbf{r}_r(\xi) = -\frac{L}{2}\sum_{j=0}^{N-1}\left(-\frac{2}{L}\sum_{k=1}^{N} \bar{d}_{k,j}\mathbf{r}_k \right)\xi^j = \sum_{k=1}^{N}\sum_{j=0}^{N-1} \bar{d}_{k,j}\xi^j\mathbf{r}_k = \sum_{k=1}^{N} J_k(\xi)\mathbf{r}_k, \tag{21}$$

where the last identity follows from (19). This result strengthens the conclusion expressed in the last sentence before 3.2.1 and shows not only that $\mathbf{r}_r(\xi)$ may be given in terms of the Lagrangian polynomials $J_k(\xi)$ in conjunction with the nodal values $\mathbf{r}_k$, but that in fact such a result is equivalent to $(18)_1$.

3.2.2 *Part of the solution for the position field dependent on the nodal rotations*

Part of the solution for the position field dependent on the nodal rotations $\mathbf{r}_\theta(\xi)$ is obtained by solving the system of linear equations in $(16)_2$ for $\boldsymbol{\beta}_{j,\theta}$, $j = 0, \ldots, N-1$, and substituting the result into $(18)_2$. The matrix of coefficients to this system is the same Vandermonde matrix as in 3.2.1 and accordingly the solution is

$$\boldsymbol{\beta}_{j,\theta} = -\sum_{k=1}^{N} \bar{d}_{k,j} \mathbf{G}_1 \times \int_{-1}^{\bar{\xi}_k} \boldsymbol{\theta} \mathrm{d}\eta, \quad j = 1, \ldots, N.$$

Substituting this result in $(18)_2$ now gives

$$\begin{aligned}
\mathbf{r}_\theta(\xi) &= \frac{L}{2}\left(\sum_{j=0}^{N-1}\sum_{k=1}^{N} \bar{d}_{k,j}\mathbf{G}_1 \times \int_{-1}^{\bar{\xi}_k} \boldsymbol{\theta}\mathrm{d}\eta \xi^j - \mathbf{G}_1 \times \int_{-1}^{\xi} \boldsymbol{\theta}\mathrm{d}\eta\right) \\
&= \frac{L}{2}\mathbf{G}_1 \times \left(\sum_{k=1}^{N}\sum_{j=0}^{N-1} \bar{d}_{k,j}\xi^j \int_{-1}^{\bar{\xi}_k} \boldsymbol{\theta}\mathrm{d}\eta - \int_{-1}^{\xi} \boldsymbol{\theta}\mathrm{d}\eta\right) = \frac{L}{2}\mathbf{G}_1 \times \left(\sum_{k=1}^{N} J_k \int_{-1}^{\bar{\xi}_k} \boldsymbol{\theta}\mathrm{d}\eta - \int_{-1}^{\xi} \boldsymbol{\theta}\mathrm{d}\eta\right).
\end{aligned}$$

where the last identity follows from (19). Owing to the completeness property of Lagrangian polynomials $\sum_{k=1}^{N} J_k(\xi) = 1$, this result can be further written as

$$\mathbf{r}_\theta(\xi) = \frac{L}{2}\mathbf{G}_1 \times \sum_{k=1}^{N} J_k \left(\int_{-1}^{\bar{\xi}_k} \boldsymbol{\theta}\mathrm{d}\eta - \int_{-1}^{\xi} \boldsymbol{\theta}\mathrm{d}\eta\right) = \frac{L}{2}\mathbf{G}_1 \times \sum_{k=1}^{N} J_k \int_{\xi}^{\bar{\xi}_k} \boldsymbol{\theta}\mathrm{d}\eta.$$

Substituting (11) and (12) this becomes

$$\begin{aligned}
\mathbf{r}_\theta(\xi) &= \frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{k=1}^{N} J_k \int_{\xi}^{\bar{\xi}_k} \sum_{q=0}^{M-1} d_{p,q}\xi^q \mathrm{d}\eta = \frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{k=1}^{N} J_k \sum_{q=0}^{M-1} \frac{\bar{\xi}_k^{q+1} - \xi^{q+1}}{q+1} d_q^p \\
&= \frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{q=0}^{M-1} \frac{\sum_{k=1}^{N} J_k \bar{\xi}_k^{q+1} - \sum_{k=1}^{N} J_k \xi^{q+1}}{q+1} d_{p,q}
\end{aligned}$$

with $\sum_{k=1}^{N} J_k = 1$. Substituting (19) this further becomes

$$\mathbf{r}_\theta(\xi) = \frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{q=0}^{M-1} \frac{\sum_{j=0}^{N-1} \xi^j \sum_{k=1}^{N} \bar{d}_{k,j}\bar{\xi}_k^{q+1} - \xi^{q+1}}{q+1} d_{p,q},$$

where, for $q+1 < N$, $\sum_{k=1}^{N} \bar{d}_{k,j}\bar{\xi}_k^{q+1} = \delta_{q+2,j+1}$, with $\delta_{s,r}$ being the Kronecker symbol equal to unity for $r = s$ and zero otherwise. The proof is given in Appendix. Consequently, the sum over the fraction in the above expression becomes

$$\sum_{q=0}^{N-2} \frac{\sum_{j=0}^{N-1} \xi^j \delta_{q+2,j+1} - \xi^{q+1}}{q+1} d_{p,q} + \sum_{q=N-1}^{M-1} \frac{\sum_{j=0}^{N-1} \xi^j \sum_{k=1}^{N} \bar{d}_{k,j}\bar{\xi}_k^{q+1} - \xi^{q+1}}{q+1} d_{p,q},$$

where the first term vanishes leaving

$$\begin{aligned}\mathbf{r}_\theta(\xi) &= \frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{q=N-1}^{M-1} \frac{\sum_{k=1}^{N}\left(\sum_{j=0}^{N-1} \bar{d}_{k,j}\xi^j\right)\bar{\xi}_k^{q+1} - \xi^{q+1}}{q+1} d_{p,q} \\ &= -\frac{L}{2}\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{q=N-1}^{M-1} \frac{\sum_{k=1}^{N} J_k\left(\xi^{q+1} - \bar{\xi}_k^{q+1}\right)}{q+1} d_{p,q},\end{aligned} \tag{22}$$

since $\sum_{j=0}^{N-1} \bar{d}_{k,j}\xi^j = J_k$ and $1 = \sum_{k=1}^{N} J_k$. The sum $\sum_{k=1}^{N} J_k\left(\xi^{q+1} - \bar{\xi}_k^{q+1}\right)$ in (22) may be further simplified in the following manner. We first factorise the difference of equal powers $\xi^{q+1} - \bar{\xi}_k^{q+1}$ as

$$\xi^{q+1} - \bar{\xi}_k^{q+1} = \left(\xi - \bar{\xi}_k\right)\left(\xi^q + \xi^{q-1}\bar{\xi}_k + \cdots + \bar{\xi}_k^q\right) = \left(\xi - \bar{\xi}_k\right)\sum_{r=1}^{q+1} \xi^{q+1-r}\bar{\xi}_k^{r-1}$$

and recall the results from Appendix to write the Lagrangian polynomials $J_k(\xi)$ as

$$J_k(\xi) = \prod_{i=1, i\neq k}^{N} \frac{\left(\xi - \bar{\xi}_i\right)}{\left(\bar{\xi}_k - \bar{\xi}_i\right)} = \bar{d}_{k,N-1} \prod_{i=1, i\neq k}^{N} \left(\xi - \bar{\xi}_i\right)$$

and obtain

$$\begin{aligned}\sum_{k=1}^{N} J_k\left(\xi^{q+1} - \bar{\xi}_k^{q+1}\right) &= \sum_{k=1}^{N} \bar{d}_{k,N-1} \prod_{i=1, i\neq k}^{N} \left(\xi - \bar{\xi}_i\right)\left(\xi - \bar{\xi}_k\right)\sum_{r=1}^{q+1} \xi^{q+1-r}\bar{\xi}_k^{r-1} \\ &= \prod_{i=1}^{N}\left(\xi - \bar{\xi}_i\right)\sum_{r=1}^{q+1}\left(\sum_{k=1}^{N} \bar{d}_{k,N-1}\bar{\xi}_k^{r-1}\right)\xi^{q+1-r} = \xi^{q+1-N}\prod_{i=1}^{N}\left(\xi - \bar{\xi}_i\right)\end{aligned} \tag{23}$$

since, as shown in Appendix, $\sum_{k=1}^{N} \bar{d}_{k,N-1}\bar{\xi}_k^{r-1} = \delta_{r,N}$. Substituting (23) in (22) thus finally gives

$$\mathbf{r}_\theta(\xi) = -\frac{L}{2}\prod_{i=1}^{N}\left(\xi - \bar{\xi}_i\right)\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{q=N-1}^{M-1} \frac{\xi^{q+1-N}}{q+1} d_{p,q}. \tag{24}$$

4 Discussion on the family of linked interpolation functions derived

For an arbitrary number $M \geq m+4$ of nodal co-ordinates for rotations $\xi_1, \ldots, \xi_M$ and an arbitrary number $N \geq n+3$ of nodal co-ordinates for positions $\bar{\xi}_1, \ldots, \bar{\xi}_N$, the exact solution for the rotation vector is given in (12) and the exact solution for the position vector is obtained by substituting (21) and (24) in (17) giving

$$\boldsymbol{\theta}(\xi) = \sum_{p=1}^{M} I_p(\xi)\boldsymbol{\theta}_p \tag{25}$$

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} J_k(\xi)\mathbf{r}_k - \frac{L}{2}\prod_{i=1}^{N}\left(\xi - \bar{\xi}_i\right)\mathbf{G}_1 \times \sum_{p=1}^{M} \boldsymbol{\theta}_p \sum_{j=0}^{M-N} \frac{d_{p,N-1+j}}{N+j}\xi^j, \tag{26}$$

where, as defined by (11), $d_{p,N-1+j}$ is a coefficient in pth Lagrangian polynomial of order $N-1$ multiplying ξ^{N-1+j}. As explained in (2) and (5), m and n here are the orders of the polynomial loading functions $\boldsymbol{\mu} = \frac{L^3}{8}\mathbf{C}_M^{-1}(\mathbf{G}_1 \times \mathbf{n} - \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}X^1})$ and $\boldsymbol{\nu} = \frac{L}{2}\mathbf{C}_N^{-1}\mathbf{n}$, respectively.

4.1 Solution in terms of a minimum number of nodal positions and rotations

With a minimum number of nodal points $M = m + 4$ and $N = n + 3$, the number of integration and loading constants in solution (6) and (7) of the differential equations is exactly matched by the number of nodal rotations and positions. Interpolation (26) therefore implicitly includes the one-to-one correspondence between the two sets of parameters.

A particularly interesting case is the one with $m = n$, i.e. the case where the distributed moment loading is of a degree at most by one larger than the degree of the distributed force loading, including the situation with no moment loading at all. The minimum number of nodes needed for the exact solution is then $M = n + 4$ and $N = n + 3$, i.e. $N = M - 1$ and $\mathbf{r}(\xi)$ in (26) turns into

$$\begin{aligned}\mathbf{r}(\xi) &= \sum_{k=1}^{M-1} J_k(\xi)\mathbf{r}_k - \frac{L}{2}\prod_{i=1}^{M-1}\left(\xi - \bar{\xi}_i\right)\mathbf{G}_1 \times \sum_{p=1}^{M}\boldsymbol{\theta}_p \sum_{j=0}^{1}\frac{d_{p,M-2+j}}{M-1+j}\xi^j \\ &= \sum_{k=1}^{M-1} J_k(\xi)\mathbf{r}_k - \frac{L}{2}\prod_{i=1}^{M-1}\left(\xi - \bar{\xi}_i\right)\mathbf{G}_1 \times \sum_{p=1}^{M}\boldsymbol{\theta}_p\left(\frac{d_{p,M-2}}{M-1} + \frac{d_{p,M-1}}{M}\xi\right), \\ &= \sum_{k=1}^{M-1} J_k(\xi)\mathbf{r}_k - \frac{L}{2}\prod_{i=1}^{M-1}\left(\xi - \bar{\xi}_i\right)\mathbf{G}_1 \times \sum_{p=1}^{M}\frac{\frac{\xi}{M} - \frac{\sum_{q=1,q\neq p}^{M}\xi_q}{M-1}}{\prod_{r=1,r\neq p}^{M}\left(\xi_p - \xi_r\right)}\boldsymbol{\theta}_p, \end{aligned} \tag{27}$$

since $d_{p,M-2} = -d_{p,M-1}\sum_{q=1,q\neq p}^{M}\xi_q$ and $d_{p,M-1} = 1/\prod_{r=1,r\neq p}^{M}\left(\xi_p - \xi_r\right)$ (see Appendix).

Remark 1 The above result can be understood as a generalisation of the result given in Chapter 10.5 of [7] for arbitrary order of interpolation. In this work the authors have considered a Timoshenko beam subject to point loads for which they have presented an exact interpolation, which is free from geometric and material characteristics, i.e. problem-independent. As already argued, this necessitates an internal datum needed to evaluate the fifth unknown constant in (3) and (4). For $M = 3$ and evenly spaced nodes (27) reads

$$\mathbf{r} = \frac{1-\xi}{2}\mathbf{r}_1 + \frac{1+\xi}{2}\mathbf{r}_2 + L(1-\xi^2)\mathbf{G}_1 \times \left[\left(\frac{\xi}{12} - \frac{1}{8}\right)\boldsymbol{\theta}_1 - \frac{\xi}{6}\boldsymbol{\theta}_2 + \left(\frac{\xi}{12} + \frac{1}{8}\right)\boldsymbol{\theta}_3\right],$$

which is equivalent to the 2D form given in equation (10.75) of [7].

4.2 Solution in terms of a redundant number of nodal positions or rotations

At times it may be convenient to use a larger set of mutually dependent nodal parameters, i.e. $M > m + 4$ with $N = n + 3$, or $N > n + 3$ with $M = m + 4$ or both. There exist two particularly interesting and computationally useful sets of such redundant degrees of freedom: the first with the number of the nodal points used for interpolating the position vector larger than the number of the nodal points used for interpolating the rotation vector by one ($N = M + 1$), and the second with the equal number of the (same) nodal points used for interpolating the position and the rotation vector ($M = N$ and $\bar{\xi}_i = \xi_i$ for $i = 1, \ldots, N$). We shall now analyse these two cases in more detail.

4.2.1 Number of nodal points for positions larger than number of nodal points for rotations by one ($N = M + 1$)

In this case (26) simply turns into

$$\mathbf{r}(\xi) = \sum_{k=1}^{M+1} J_k(\xi)\mathbf{r}_k,$$

showing that, for $M \geq m + 4$ and $N \geq n + 3$, $N = M + 1$ is the minimum number of the position nodal points needed to separate the interpolation for positions from the rotational degrees of freedom. At the same time, therefore, it is also the minimum number of the position nodal points needed to express the exact solution using independent interpolations for the positions and the rotations. In [4] the elements with such interpolation are called the *virgin elements*.

4.2.2 Same nodal points for positions and rotations ($\bar{\xi}_i = \xi_i$ for $i = 1, \ldots, N$)

In this case $J_k(\xi) = I_k(\xi)$ and (26) turns into

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} I_k(\xi)\mathbf{r}_k - \frac{L}{2}\prod_{i=1}^{N}(\xi - \xi_i)\,\mathbf{G}_1 \times \sum_{p=1}^{N}\boldsymbol{\theta}_p \frac{d_{p,N-1}}{N}. \tag{28}$$

Noting from (11) that $N-1$st derivative of $I_p(\xi)$ is $I_p^{(N-1)}(\xi) = d_{p,N-1}(N-1)!$ the above result becomes

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} I_k(\xi)\mathbf{r}_k - \frac{1}{N!}\frac{L}{2}\prod_{i=1}^{N}(\xi - \xi_i)\,\mathbf{G}_1 \times \boldsymbol{\theta}_p^{(N-1)}$$

showing that the standard Lagrangian interpolation using only the translational degrees of freedom suffices for the exact solution when $N-1$st derivative of the interpolation for rotations vanishes. This, of course, is precisely the last situation analysed in which this derivative would vanish owing to the fact that the rotation was interpolated using a polynomial of the order smaller than $N-1$. Equation (28) is worth analysing further. As, for $N = M$, $d_{p,N-1} = 1/\prod_{r=1,r\neq p}^{N}\left(\xi_p - \xi_r\right)$ (see Appendix), this result turns into

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} I_k(\xi)\mathbf{r}_k - \frac{1}{N}\frac{L}{2}\prod_{i=1}^{N}(\xi - \xi_i)\,\mathbf{G}_1 \times \sum_{p=1}^{N}\frac{\boldsymbol{\theta}_p}{\prod_{r=1,r\neq p}^{N}\left(\xi_p - \xi_r\right)} \tag{29}$$

and takes a particularly elegant form for the elements with equidistant spacing between the nodes. In this case, the nodal co-ordinates are given as $\xi_i = \frac{2i-(N+1)}{N-1}$; hence

$$\begin{aligned}\frac{1}{\prod_{r=1,r\neq p}^{N}\left(\xi_p - \xi_r\right)} &= \frac{\left(\frac{N-1}{2}\right)^{N-1}}{(p-1)(p-2)\ldots 1}\frac{1}{(-1)(-2)\ldots(p-N)} = \frac{\left(\frac{N-1}{2}\right)^{N-1}}{(p-1)!}\frac{(-1)^{N-p}}{(N-p)!}\\ &= \left(\frac{N-1}{2}\right)^{N-1}\frac{(-1)^{p-N}}{(N-1)!}\frac{\frac{(N-1)!}{(N-p)!}}{(p-1)!} = \left(-\frac{N-1}{2}\right)^{N-1}\frac{(-1)^{p-1}}{(N-1)!}\binom{N-1}{p-1}.\end{aligned}$$

As $\xi - \xi_i$ now turns out to be equal to $2\left(\frac{1+\xi}{2} - \frac{i-1}{N-1}\right)$, (29) becomes

$$\begin{aligned}\mathbf{r} &= \sum_{k=1}^{N} I_k\mathbf{r}_k - \frac{L}{N}\prod_{j=2}^{N}\left(-\frac{N-1}{j-1}\right)\prod_{i=1}^{N}\left(\frac{1+\xi}{2} - \frac{i-1}{N-1}\right)\mathbf{G}_1 \times \sum_{p=1}^{N}(-1)^{p-1}\binom{N-1}{p-1}\boldsymbol{\theta}_p,\\ &= \sum_{k=1}^{N} I_k\mathbf{r}_k - \frac{L}{N}\frac{1+\xi}{2}\prod_{i=2}^{N}\left(1 - \frac{N-1}{i-1}\frac{1+\xi}{2}\right)\mathbf{G}_1 \times \sum_{p=1}^{N}(-1)^{p-1}\binom{N-1}{p-1}\boldsymbol{\theta}_p,\end{aligned}$$

where the coefficients in the second sum are the binomial coefficients forming a Pascal triangle. Introducing a set of linear functions $N_i(\xi)$ having a zero value at the node ξ_i and a unit value at the node $\xi_1 = -1$ (apart from $N_1(\xi)$ which has a unit value at the node $\xi_N = 1$), correspondingly defined as $N_1(\xi) = \frac{1+\xi}{2}$ and $N_i(\xi) = 1 - \frac{N-1}{i-1}\frac{1+\xi}{2}$ for $i = 2, \ldots, N$ and shown in Fig. 1, the above result turns into

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} I_k(\xi)\mathbf{r}_k - \frac{L}{N}\prod_{i=1}^{N}N_i(\xi)\sum_{p=1}^{N}(-1)^{p-1}\binom{N-1}{p-1}\mathbf{G}_1 \times \boldsymbol{\theta}_p. \tag{30}$$

Remark 2 Some special cases of this result are well known. For a two-noded element (30) gives $\mathbf{r}(\xi) = \frac{1-\xi}{2}\mathbf{r}_1 + \frac{1+\xi}{2}\mathbf{r}_2 - \frac{L}{8}\left(1-\xi^2\right)\mathbf{G}_1 \times (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2)$, which has been often used as a basis for development of constant strain beam elements (see equation (10.84) in [7]) and also plate elements and their extensions (see [12] for an application to Mindlin plates). The three-node case of (30) reads $\mathbf{r}(\xi) = -\xi\frac{1-\xi}{2}\mathbf{r}_1 + \left(1-\xi^2\right)\mathbf{r}_2 + \xi\frac{1+\xi}{2}\mathbf{r}_3 + \frac{L}{12}\xi\left(1-\xi^2\right)\mathbf{G}_1 \times (\boldsymbol{\theta}_1 - 2\boldsymbol{\theta}_2 + \boldsymbol{\theta}_3)$ and has been also reported and used for development of higher-order Timoshenko beam elements [4] as well as triangular and rectangular Mindlin plate elements [13,14].

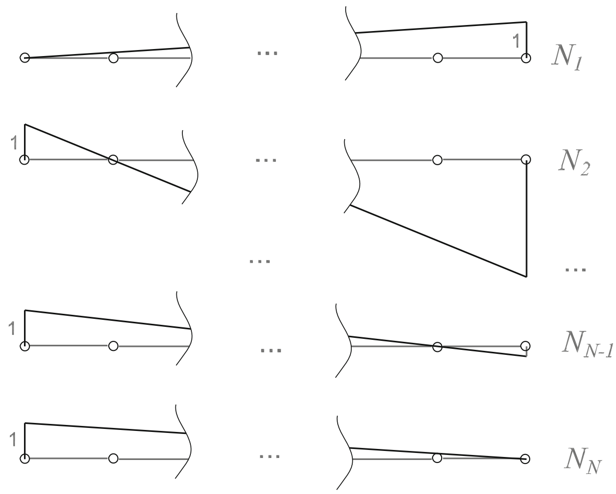

Fig. 1 Linear functions $N_i(\xi)$ used in interpolation (30)

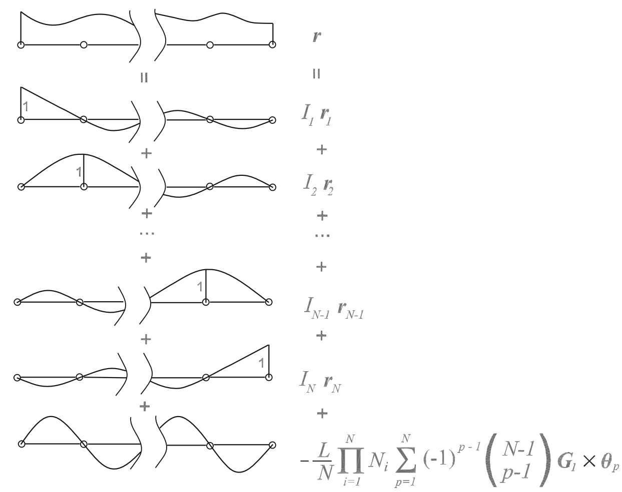

Fig. 2 Graphical representation of interpolation (30)

The meaning of interpolation given in (30) is illustrated in Fig. 2. Obviously, the exact solution is obtained by enhancing the standard Lagrangian interpolation of order $N-1$ with an extra polynomial of order N passing through all the nodal points scaled by a linear combination of the nodal rotations using the binomial coefficients proportional to the length of the beam and inversely proportional to the number of nodes.

Turning back to internal nodes with arbitrary spacing provides another illustrative view at (29). Since $I_k(\xi) = \prod_{i=1, i\neq k}^{N} \frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}$, this equation simplifies and becomes

$$\mathbf{r}(\xi) = \sum_{k=1}^{N} I_k(\xi) \left[\mathbf{r}_k - \frac{1}{N}\frac{L}{2}\mathbf{G}_1 \times (\xi - \xi_k)\,\boldsymbol{\theta}_k \right],$$

or, introducing the position vector of the undeformed beam centroidal line $\mathbf{R}(\xi) = \frac{1}{2}(1-\xi)\,\mathbf{R}_1 + \frac{1}{2}(1+\xi)\,\mathbf{R}_2$ with $\mathbf{R}_1$ and $\mathbf{R}_2$ as the position vectors of the boundary nodes in the undeformed state,

$$\begin{aligned} \mathbf{r} &= \sum_{k=1}^{N} I_k \left[\mathbf{r}_k - \frac{1}{N}(\mathbf{R} - \mathbf{R}_k) \times \boldsymbol{\theta}_k \right] = \sum_{k=1}^{N} I_k \mathbf{r}_k - \frac{1}{N}\left(\mathbf{R} \times \sum_{k=1}^{N} I_k \boldsymbol{\theta}_k - \sum_{k=1}^{N} I_k \mathbf{R}_k \times \boldsymbol{\theta}_k \right) \\ &= \sum_{k=1}^{N} I_k \mathbf{r}_k - \frac{1}{N}\left(\mathbf{R} \times \boldsymbol{\theta} - \sum_{k=1}^{N} I_k \mathbf{R}_k \times \boldsymbol{\theta}_k \right) = \sum_{k=1}^{N} I_k \left[\mathbf{r}_k + \frac{1}{N}(\boldsymbol{\theta} - \boldsymbol{\theta}_k) \times \mathbf{R}_k \right], \end{aligned} \quad (31)$$

since, for a straight beam, $\mathbf{R}(\xi) = \sum_{k=1}^{N} I_k(\xi)\mathbf{R}_k$. Obviously, since $\mathbf{r}(\xi) = \mathbf{R}(\xi) + \mathbf{u}(\xi)$ and $\mathbf{r}_k = \mathbf{R}_k + \mathbf{u}_k$, the same interpolation also applies to the displacement vector, i.e. $\mathbf{u} = \sum_{k=1}^{N} I_k \left[\mathbf{u}_k + \frac{1}{N}(\boldsymbol{\theta} - \boldsymbol{\theta}_k) \times \mathbf{R}_k\right]$. An exceedingly elegant alternative expression for the interpolation of the position vector follows from (31) as

$$\mathbf{r} = \sum_{k=1}^{N} I^k \left\{ \mathbf{r}_k + \frac{1}{N} \left[(\boldsymbol{\theta} - \boldsymbol{\theta}_k) \times \mathbf{r}_k - \overbrace{(\boldsymbol{\theta} - \boldsymbol{\theta}_k) \times \mathbf{u}_k}^{\text{S.O.T.}} \right] \right\}.$$

Bearing in mind that in the presently considered linear theory all second-order terms are assumed to vanish, we further obtain $\mathbf{r} = \sum_{k=1}^{N} I_k \left(\mathbf{I} + \frac{1}{N}\widehat{\boldsymbol{\theta}-\boldsymbol{\theta}_k}\right)\mathbf{r}_k$, with $\mathbf{I}$ a 3D unity tensor with the hat denoting the cross-product operator such that $\widehat{\mathbf{a}}\mathbf{b} = \mathbf{a} \times \mathbf{b}$ for any 3D vectors $\mathbf{a}$, $\mathbf{b}$. The position vector in this case can be therefore defined by a generalised interpolation

$$\mathbf{r} = \sum_{k=1}^{N} \tilde{\mathbf{I}}_k \mathbf{r}_k \quad \text{with} \quad \tilde{\mathbf{I}}_k = I_k \left(\mathbf{I} + \frac{1}{N} \widehat{\boldsymbol{\theta}-\boldsymbol{\theta}_k} \right),$$

where $\boldsymbol{\theta} - \boldsymbol{\theta}_k$ can be computed as $\boldsymbol{\theta} - \boldsymbol{\theta}_k = \sum_{j=1}^{N} \left(I_j - \delta_{jk}\right)\boldsymbol{\theta}_j$. Apparently, this result may be taken as a viable basis for the development of non-linear beam finite elements with a capability to provide the exact analytical result in linear analysis.

5 Conclusions

In this work a family of linked interpolation functions of arbitrary order for Timoshenko beam elements has been derived and thoroughly analysed. We have eliminated all the material, geometric and loading parameters from the interpolation in order to arrive at a *problem-independent* result. Also, we have limited our attention to polynomial loading of arbitrary order and shown that with a sufficient finite number of internal nodes it is always possible to obtain the exact result.

For a distributed force loading of arbitrary order, three situations have been analysed in more detail: (i) the linked interpolation with a minimum number of parameters in which the number of nodes with translational degrees of freedom is smaller than the number of nodes with rotational degrees of freedom by one, (ii) the linked interpolation with the same nodal points for the translational and the rotational degrees of freedom and (iii) the interpolation in which the number of nodes for the translational degrees of freedom is larger than the number of nodes for the rotational degrees of freedom by one resulting in the standard independent interpolation for the two fields using Lagrangian polynomials of different order. The second of these situations appears to be particularly elegant both mathematically and computationally and in this paper it has been presented in its general form of which some of the known linked interpolations reported in the literature have been shown to be the special cases.

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Appendix: Inverse of a Vandermonde matrix

Over a domain $-1 \leq \xi \leq 1$ a function $f(\xi)$ may be approximated by a simple polynomial expansion

$$f(\xi) = \sum_{i=0}^{n-1} a_i \xi^i, \tag{32}$$

where the coefficients a_i are obtained by solving the so-called Vandermonde problem

$$\mathbf{Va} = \mathbf{f} \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \dots & \xi_1^{j-1} & \dots & \xi_1^{n-1} \\ 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^{j-1} & \dots & \xi_2^{n-1} \\ 1 & \xi_3 & \xi_3^2 & \dots & \xi_3^{j-1} & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 1 & \xi_i & \xi_i^2 & \dots & \xi_i^{j-1} & \dots & \xi_i^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 1 & \xi_n & \xi_n^2 & \dots & \xi_n^{j-1} & \dots & \xi_n^{n-1} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{j-1} \\ \vdots \\ a_{n-1} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{Bmatrix} \tag{33}$$

with $f_i = f(\xi_i)$ and ξ_i the chosen nodal co-ordinates. Alternatively, the same function may be also approximated using the Lagrangian interpolation polynomials $I_j(\xi) = \prod_{k=1,k\neq j}^{n} \frac{\xi-\xi_k}{\xi_j-\xi_k}$ as

$$f(\xi) = \sum_{j=1}^{n} I_j(\xi) f_j. \tag{34}$$

This result may be expanded into a power series of the type

$$I_j(\xi) = \sum_{i=0}^{n-1} d_{j,i}\xi^i \tag{35}$$

with known coefficients $d_{j,i}$ e.g. $d_{j,n-1} = \frac{1}{\prod_{k=1,k\neq j}^{n}(\xi_j-\xi_k)}$, $d_{j,n-2} = -d_{j,n-1}\sum_{k=1,k\neq j}^{n} \xi_k$ and $d_{j,0} = \prod_{k=1,k\neq j}^{n} \frac{-\xi_k}{\xi_j-\xi_k}$. Substituting (35) in (34) now gives

$$f(\xi) = \sum_{j=1}^{n}\sum_{i=0}^{n-1} d_{j,i}\xi^i f_j = \sum_{i=0}^{n-1}\left(\sum_{j=1}^{n} d_{j,i} f_j\right)\xi^i,$$

which may be compared to (32) to provide the solution of the Vandermonde problem (33) as $a_i = \sum_{j=1}^{n} d_{j,i} f_j$.

As the Vandermonde problem implies $\mathbf{a} = \mathbf{Wf}$, where $\mathbf{W} = \mathbf{V}^{-1}$, it follows from the above equation that the element w_{ij} of the inverse of the Vandermonde matrix is

$$w_{ij} = d_{j,i-1},$$

where $d_{j,i}$ is the coefficient in the jth Lagrangian polynomial of the order $n-1$ multiplying ξ^i as shown in (35). In other words, the columns of the inverse of the nth order Vandermonde matrix are the Lagrangian polynomials of order $n-1$ written in the basis $\{\xi^0, \dots, \xi^{n-1}\}$. For example, for a quadratic Lagrangian interpolation with equidistant nodes $I_1(\xi) = -\frac{1}{2}\xi(1-\xi)$, $I_2(\xi) = (1-\xi)(1+\xi)$ and $I_3(\xi) = \frac{1}{2}\xi(1+\xi)$, the inverse of the Vandermonde matrix reads

$$\mathbf{W} = \mathbf{V}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

Since the elements of the Vandermonde matrix and its inverse are $v_{ij} = \xi_i^{j-1}$ and $w_{ij} = d_{j,i-1}$, the following results are obtained from $\mathbf{VW} = \mathbf{WV} = \mathbf{I}$:

$$\sum_{k=0}^{n-1} \xi_i^k d_{j,k} = \delta_{i,j} \quad \text{and} \quad \sum_{k=1}^{n} d_{k,i-1}\xi_k^{j-1} = \delta_{i,j}, \tag{36}$$

of which the first reproduces the well-known property $I_j(\xi_i) = \delta_{i,j}$ of the Lagrangian polynomials (35).

The second result in (36) is more interesting and perhaps not so well known: *multiplying a chosen power (between* 0 *and* $n-1$) *of a nodal co-ordinate* ξ_k *with the coefficient in the* k*th Lagrangian polynomial associated with a certain power of* ξ*, and summing over all the nodes* $k = 1, \ldots, n$*, gives a unity if the two powers are equal and zero otherwise*. A weaker but perhaps more illustrative conclusion follows by multiplying $(36)_2$ with ξ^{i-1}, summing the result over $i = 1, \ldots, n$, and substituting (35): $\sum_{k=1}^{n} \xi_k^j I_k(\xi) = \xi^j$, $j = 0, \ldots, n-1$. For $j = 0$ this of course turns into the standard completeness property of the Lagrangian polynomials $\sum_{k=1}^{n} I_k(\xi) = 1$.

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