

Finiteness of Complete Intersection Dimensions of RHOM and EXT Modules

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ABSTRACT. In this paper, we explore the implications of the finiteness of complete intersection dimensions for RHom complexes and Ext modules. We prove various stability results and criteria for detecting finite complete intersection homological dimension of complexes and modules. In addition, we introduce and explore the concept of CI-perfect modules. We also study the vanishing of Ext when certain Hom module have finite complete intersection homological dimension. In this direction, we improve a result by Ghosh and Samanta, prove the Auslander-Reiten conjecture for finitely generated modules M over a Noetherian local ring R such that $\text{Hom}_R(M, R)$ or $\text{Hom}_R(M, M)$ has finite complete intersection injective dimension, and provide Gorenstein criteria.

1. INTRODUCTION

Throughout this paper, unless otherwise specified, we assume that R is a commutative Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . We will frequently regard M and N as complexes of (not necessarily finitely generated) R -modules; however, whenever we use M and N to denote R -modules, they will be assumed to be finitely generated.

Finiteness of homological dimensions of RHom complexes and their homologies is an active and long-standing research area in commutative algebra. We list some standard stability results previously established over the past few decades:

- (1) [11, Proposition 16.4.32] $\text{id}_R(\mathbf{RH}\text{om}_R(M, N)) < \infty \implies \text{pd}_R(M) < \infty$ and $\text{id}_R(N) < \infty$.
- (2) [10, Lemma 6.2.12] $\text{pd}_R(\mathbf{RH}\text{om}_R(M, N)) < \infty$ and $\text{id}_R(N) < \infty \implies \text{id}_R(M) < \infty$.
- (3) [11, Proposition 16.4.19] $\text{pd}_R(\mathbf{RH}\text{om}_R(M, N)) < \infty$ and $\text{pd}_R(M) < \infty \implies \text{pd}_R(N) < \infty$.
- (4) [21, Theorem 5.6] $\text{G-dim}_R(\mathbf{RH}\text{om}_R(M, N)) < \infty$ and $\text{pd}_R(M) < \infty \implies \text{G-dim}_R(N) < \infty$.
- (5) [29, Theorem 3.11] $\text{CI-dim}_R(\mathbf{RH}\text{om}_R(M, N)) < \infty$ and $\text{pd}_R(M) < \infty \implies \text{CI-dim}_R(N) < \infty$.

More stability results concerning the finiteness of homological dimensions of RHom can be found in [30, 31, 36, 12, 34]. In a similar vein, several papers have studied the consequences of Ext modules having finite projective, injective, Gorenstein projective, or Gorenstein injective dimensions [19, 26, 38, 14, 25, 15, 20, 23]. Recently, an initial result considering Ext modules with finite complete intersection dimension was established by Kimura [23, Proposition 4.6].

In this paper, we study the consequences of $\mathbf{RH}\text{om}_R(M, N)$ and its homologies having finite complete intersection homological dimension. Complete intersection dimension for finitely generated modules over Noetherian ring was first established by Avramov, Gasharov, and Peeva [5] as an homological invariant that measures how close a given ring is to being a complete intersection. As one might expect, a Noetherian local ring is a complete intersection if and only if the residue field has finite complete intersection dimension in a version of the classical Auslander-Buchsbaum-Serre theorem. The concept of complete intersection dimension was expanded to include non-finitely generated modules by Sahandi, Sharif, and Yassemi [28], who also first defined the complete intersection injective dimension (or CI-id) of a module. Sather-Wagstaff has extensively studied complete intersection dimension, and expanded its study to complexes in [29, 30].

Throughout our paper, we often prove in parallel results using multiple homological dimensions. In order to be concise, we generally refer to projective dimension and complete intersection dimension as H-dim, and injective and upper complete intersection injective dimension Hid.

We obtain several stability results concerning finite complete intersection homological dimension of RHom complexes and several consequences of having Ext modules with finite complete intersection homological

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dimension. After providing background for the derived category of a ring and complete intersection homological dimensions, we first establish some stability results concerning complete intersection dimension of $\mathbf{R}\mathrm{Hom}$ complexes. In particular, we prove our first main theorem:

Theorem 1.1. *Let M and N be non-acyclic complexes in $D^f_{\square}(R)$ such that $\mathbf{R}\mathrm{Hom}_R(M, N)$ is in $D^f_{\square}(R)$. Assume that $\mathrm{H-dim}_R(\mathbf{R}\mathrm{Hom}_R(M, N)) < \infty$.*

- (1) *If $\mathrm{G-dim}_R(N) < \infty$, then $\mathrm{H-dim}_R(M \oplus N) < \infty$.*
- (2) *If R is Cohen-Macaulay with a canonical module and $\mathrm{Gid}_R(M) < \infty$, then $\mathrm{Hid}_R(M \oplus N) < \infty$.*

We next present some consequences of having Ext modules with finite complete intersection homological dimension. We record our main results in the next two theorems:

Theorem 1.2. *Let M and N be non-zero R -modules such that $\mathrm{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Set $L = \bigoplus_{i=0}^{\infty} \mathrm{Ext}_R^i(M, N)$.*

- (1) *If $\mathrm{CI-dim}_R(M \oplus L) < \infty$, then $\mathrm{CI-dim}_R(N) < \infty$.*
- (2) *If $\mathrm{H-dim}_R(L) < \infty$ and $\mathrm{G-dim}_R(N) < \infty$, then $\mathrm{H-dim}_R(M \oplus N) < \infty$.*
- (3) *If $\mathrm{H-dim}_R(L) < \infty$ and $\mathrm{Gid}_R(M) < \infty$, then $\mathrm{Hid}_R(M \oplus N) < \infty$.*
- (4) *If $\mathrm{CI-id}_R(L) < \infty$, then there is a quasi-deformation for which $\mathrm{CI-dim}_R(M) < \infty$ and $\mathrm{CI-id}_R(N) < \infty$ are both finite.*
- (5) *If $\mathrm{CI}^*\text{-id}_R(L) < \infty$, then there is a quasi-deformation for which $\mathrm{CI-dim}_R(M) < \infty$ and $\mathrm{CI}^*\text{-id}_R(N) < \infty$ are both finite.*

Theorem 1.3. *Let R be a local ring with a normalized dualizing complex D , and let M and N be non-zero R -modules. Let $r \in \mathbb{N}$ be such that $r \geq \mathrm{depth}(R) - \mathrm{depth}(M)$. Assume that the following conditions hold:*

- (1) $\mathrm{H-dim}_R(\mathrm{Ext}_R^i(M, N)) < \infty$ for all $0 \leq i \leq r$.
- (2) $\mathrm{Ext}_R^i(M, N)$ has finite length for all $i > r$.
- (3) $\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(M, N \otimes_R^L D)) = 0$ for all $i < \mathrm{depth}(M)$.

The following statements then hold:

- (i) *If $\mathrm{G-dim}_R(N) < \infty$, then $\mathrm{H-dim}_R(M \oplus N) < \infty$.*
- (ii) *If R is Cohen-Macaulay and $\mathrm{Gid}_R(M) < \infty$, then $\mathrm{Hid}_R(M \oplus N) < \infty$.*

Theorem 1.3 is inspired by [23, Theorems 3.3 and 3.15] and their proofs. It strengthens these results in the sense that they can be recovered as special cases. In addition, while [23, Theorems 3.3 and 3.15] establishes finiteness of the homological dimension for a single module, our result yields such finiteness simultaneously for two modules.

Although there are some known examples of Ext modules having infinite projective dimension and finite CI-dim (see [20, Example 1.7]), in general it is difficult to construct non-simple examples of such Ext modules. To help produce Ext modules with finite CI-dim, we introduce and explore the concept of CI-perfect modules as a natural generalization of perfect modules. Our main result for CI-perfect modules is listed below:

Theorem 1.4. *Let M be an R -module. If M is CI-perfect of grade g , then $\mathrm{Ext}_R^g(M, R)$ is CI-perfect of grade g .*

In the final section of our paper, we study the interaction between the Auslander-Reiten conjecture and complete intersection dimensions. The Auslander-Reiten conjecture is a long-standing question in commutative algebra, and is written below for completeness.

Conjecture 1.5 (Auslander-Reiten conjecture). *Let M be a non-zero R -module. If $\mathrm{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$, then M is free.*

Ghosh and Samanta proved in [17, Theorem 3.9] that an R -module M is free under the assumption that $\mathrm{Ext}_R^i(M, R) = \mathrm{Ext}_R^j(M, M) = 0$ for all $1 \leq i \leq \mathrm{depth}(R)$ and $j > 0$, $\mathrm{CI-dim}_R(\mathrm{Hom}_R(M, M)) < \infty$, and either $\mathrm{G-dim}_R(M)$ or $\mathrm{G-dim}_R(\mathrm{Hom}_R(M, R))$ is finite. Using our results, we show that when $\mathrm{G-dim}_R(M) < \infty$, the vanishing of $\mathrm{Ext}_R^i(M, R)$ for all $1 \leq i \leq \mathrm{depth}(R)$ can be omitted. When $\mathrm{G-dim}_R(\mathrm{Hom}_R(M, R)) < \infty$, we do not know whether this vanishing condition can be removed; however, we prove that it can be dropped in the special case where $\mathrm{CI-dim}_R(\mathrm{Hom}_R(M, M)) = 0$.

Theorem 1.6. *Let M be a non-zero R -module such that $\mathrm{Ext}_R^i(M, M) = 0$ for all $i > 0$. Then the following conditions are equivalent:*

- (1) M is free.
- (2) $\text{G-dim}_R(\text{Hom}_R(M, R)) < \infty$ and $\text{CI-dim}_R(\text{Hom}_R(M, M)) = 0$.
- (3) $\text{G-dim}_R(M) < \infty$ and $\text{CI-dim}_R(\text{Hom}_R(M, M)) < \infty$.

Finally, we prove that the Auslander-Reiten conjecture holds provided that at least one of the modules $\text{Hom}_R(M, R)$ or $\text{Hom}_R(M, M)$ has finite complete intersection injective dimension, extending thus a result of Ghosh and Takahashi [19, Corollary 1.3].

Theorem 1.7. *Let M be a non-zero R -module. Suppose that one of the following conditions holds:*

- (1) $\text{CI-id}_R(\text{Hom}_R(M, M)) < \infty$ and $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$.
- (2) $\text{CI-id}_R(\text{Hom}_R(M, R)) < \infty$, $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$ and $\text{Ext}_R^{2j}(M, M) = 0$ for some $j > 0$.

Then M is free and R is Gorenstein.

However, most known cases in which the Auslander-Reiten conjecture holds require the vanishing of only finitely many Ext modules. We show in Theorem 6.8 that finitely many vanishing of $\text{Ext}(M, R)$ and $\text{Ext}(M, M)$ are enough to ensure that M is free. We also provide new characterizations of Gorenstein rings in terms of vanishing of certain Ext modules and finite upper complete intersection injective dimensions of Hom that are comparable to those of [19, 26].

2. BACKGROUND

2.1. The Derived Category of a Ring. Throughout this paper, we work in the derived category of the ring R , denoted $\mathbf{D}(R)$. This category is obtained from the category of chain complexes by formally inverting all quasi-isomorphisms. Unlike the category of chain complexes, the derived category is rarely abelian. Instead, the derived category is a triangulated category, where distinguished triangles take the place of short exact sequences.

We generally use the same notation and terminology as in this paper as in [11]. For convenience, we record some definitions and notations in this section.

A complex $M : \cdots \rightarrow M_{i+1} \rightarrow M_i \rightarrow M_{i-1} \rightarrow \cdots$ of R -modules is *bounded above* (resp. *below*) if $M_i = 0$ for all $i \gg 0$ (resp. $i \ll 0$). The complex M is *bounded* if M is bounded above and below; and it is *degreewise finitely generated* if M_i is finitely generated for each i .

For a complex M we use ∂_i^M to denote its i -th differential $M_i \rightarrow M_{i-1}$. The full subcategories $\mathbf{D}_{\square}(R)$, $\mathbf{D}_{\square}(R)$, $\mathbf{D}_{\square}(R)$ and $\mathbf{D}^f(R)$ of $\mathbf{D}(R)$ are defined by specifying their objects as follows:

$$\begin{aligned}\mathbf{D}_{\square}(R) &= \{M \in \mathbf{D}(R) \mid \text{H}(M) \text{ is bounded above}\} \\ \mathbf{D}_{\square}(R) &= \{M \in \mathbf{D}(R) \mid \text{H}(M) \text{ is bounded below}\} \\ \mathbf{D}_{\square}(R) &= \{M \in \mathbf{D}(R) \mid \text{H}(M) \text{ is bounded}\} \\ \mathbf{D}^f(R) &= \{M \in \mathbf{D}(R) \mid \text{H}(M) \text{ is degreewise finitely generated}\}.\end{aligned}$$

The full subcategory $\mathbf{D}^f(R) \cap \mathbf{D}_{\square}(R)$ is denoted by $\mathbf{D}_{\square}^f(R)$. Similarly, one defines the subcategories $\mathbf{D}_{\square}^f(R)$ and $\mathbf{D}_{\square}^f(R)$.

The classical functors $\text{Hom}_R(M, -)$, $\text{Hom}_R(-, M)$, and $- \otimes_R M$ defined in the category of R -complexes induce derived functors $\mathbf{R}\text{Hom}_R(M, -)$, $\mathbf{R}\text{Hom}_R(-, M)$, and $- \otimes_R^L M$ which operate on objects and morphisms in $\mathbf{D}(R)$. These derived functors are computed using semi-projective, semi-injective, and semi-flat replacements, which exist for any complex. Explicitly,

$$\mathbf{R}\text{Hom}_R(M, N) = \text{Hom}_R(P, N) \text{ (where } P \text{ is a semi-projective replacement of } M\text{),}$$

$$\mathbf{R}\text{Hom}_R(N, M) = \text{Hom}_R(N, I) \text{ (where } I \text{ is a semi-injective replacement of } M\text{),}$$

$$M \otimes_R^L N = F \otimes_R N \text{ (where } F \text{ is a semi-flat replacement of } M\text{).}$$

As for modules, one sets:

$$\begin{aligned}\text{Ext}_R^i(M, N) &= \text{H}_{-i}(\mathbf{R}\text{Hom}_R(M, N)) \\ \text{Tor}_i^R(M, N) &= \text{H}_i(M \otimes_R^L N).\end{aligned}$$

For more information about the derived category of a ring and its uses in commutative algebra, we recommend Christensen, Foxby, and Holm's excellent and thorough textbook [11].

2.2. Dualizing complex and deficiency modules. A *dualizing complex* D is a bounded complex of injective R -modules with finitely generated homologies such that the natural morphism $R \rightarrow \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism in $\mathrm{D}(R)$. Moreover, a dualizing complex D for R is called *normalized* if the equality $\sup D = \dim R$ holds.

If D is a dualizing complex for R , then the complex $\sum^{\dim R - \sup D} D$ is a normalized dualizing complex for R ([11, Lemma 18.2.23]). The ring R has a dualizing complex if and only if R is a homomorphic image of a Gorenstein local ring ([9, A.8.3.2] and [22, Corollary 1.4]). In particular, if R is complete, then R admits a dualizing complex.

For considering deficiency modules, we assume that R has a normalized dualizing complex D , i.e., R is a factor of a Gorenstein local ring S . Given an R -module M , set:

$$K^i(M) := \mathrm{Ext}_S^{\dim(S)-i}(M, S)$$

for all $i \in \mathbb{Z}$. For each i , the R -module $K^i(M)$ is called the i -th *deficiency module* of M ([33]). In particular, the R -module $K(M) := K^{\dim_R(M)}(M)$ is called the *canonical module* of M and generalizes the notion of a canonical module of a ring. In a certain sense, the deficiency modules of M measure the extent of the failure of M to be Cohen-Macaulay. The modules of deficiency satisfy the following isomorphism:

$$H_{\mathfrak{m}}^i(M) \cong \mathrm{Hom}_R(K^i(M), E_R) \text{ and } K^i(M) \cong H_i(\mathbf{R}\mathrm{Hom}_R(M, D)),$$

where E_R is the injective hull of k . The first isomorphism shows that $K^i(M) = 0$ for $i < \mathrm{depth}(M)$ or $i > \dim(M)$, and $K^i(M) \neq 0$ for $i = \mathrm{depth}(M)$ and $i = \dim(M)$.

2.3. Restricted flat dimension. Motivated by the results of [23], we consider the (large) restricted flat dimension to limit our hypotheses in Section 4 to finitely many Ext modules.

Definition 2.1. Let M be an R -module. The (large) restricted flat dimension is given by the following formula:

$$\mathrm{Rfd}_R(M) := \sup\{\mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

By [3, Theorem 1.1], $0 \leq \mathrm{Rfd}_R(M) < \infty$ if M is non-zero.

2.4. Gorenstein Dimensions. This section is a brief distillation of Chapter 9 in [11]. For more details, we recommend reading through this chapter.

A complex P of projective R -modules is called *totally acyclic* if it is acyclic and if $\mathrm{Hom}_R(P, L)$ is acyclic for any projective R -module L . A cokernel of a totally acyclic complex of projective modules is defined to be a *Gorenstein projective* module. Note that any projective module P is automatically Gorenstein projective, as it is a cokernel of the totally acyclic complex $0 \rightarrow P \rightarrow P \rightarrow 0$.

Definition 2.2. Let M be an complex in $\mathrm{D}(R)$. The *Gorenstein projective dimension* of M (denoted as $\mathrm{Gpd}_R(M)$) is

$$\mathrm{Gpd}_R(M) := \inf\{n \in \mathbb{Z} \mid \text{There exists a semi-projective replacement } P \text{ of } M \text{ such that } H_i(P) = 0 \text{ for all } i > n \text{ and } \mathrm{coker}(\partial_n^P) \text{ is Gorenstein projective.}\}.$$

A complex I of injective R -modules is called *totally acyclic* if it is acyclic and if $\mathrm{Hom}_R(E, I)$ is acyclic for any injective R -module E . A kernel of a totally acyclic complex of injective modules is defined to be a *Gorenstein injective* module. Note that any injective module I is automatically Gorenstein injective, as it is a kernel of the totally acyclic complex $0 \rightarrow I \rightarrow I \rightarrow 0$.

Definition 2.3. Let M be an complex in $\mathrm{D}(R)$. The *Gorenstein injective dimension* of M (denoted as $\mathrm{Gid}_R(M)$) is

$$\mathrm{Gid}_R(M) := \inf\{n \in \mathbb{Z} \mid \text{There exists a semi-injective replacement } I \text{ of } M \text{ such that } H_i(I) = 0 \text{ for all } i < n \text{ and } \mathrm{ker}(\partial_{-n}^I) \text{ is Gorenstein injective.}\}.$$

Finally, a complex F of flat R -modules is called *totally acyclic* if it is acyclic and if $F \otimes_R I$ is acyclic for any injective R -module I . A cokernel of a totally acyclic complex of flat modules is defined to be a *Gorenstein flat* module. Note that any flat module F is automatically Gorenstein flat, as it is a cokernel of the totally acyclic complex $0 \rightarrow F \rightarrow F \rightarrow 0$.

Definition 2.4. Let M be an complex in $\mathbf{D}(R)$. The *Gorenstein flat dimension* of M (denoted as $\mathrm{Gfd}_R(M)$) is

$$\mathrm{Gfd}_R(M) := \inf\{n \in \mathbb{Z} \mid \text{There exists a semi-flat replacement } F \text{ of } M \text{ such that } \mathrm{H}_i(F) = 0 \text{ for all } i > n \text{ and } \mathrm{coker}(\partial_n^F) \text{ is Gorenstein flat.}\}.$$

When working with complexes with finitely generated homology over Noetherian rings, the theory of Gorenstein dimensions simplifies considerably. In particular, the class of Gorenstein projective modules coincides exactly with the class of totally reflexive modules [11, Proposition 10.4.13], and the Gorenstein projective dimension of a complex M in $\mathbf{D}_{\square}^f(R)$ can be computed by the formula

$$\mathrm{Gpd}_R(M) = -\inf(\mathbf{R}\mathrm{Hom}_R(M, R))$$

provided that M is already known to have finite Gorenstein projective dimension [11, Corollary 10.4.16]. For historical reasons, the Gorenstein projective dimension of such a complex M is often simply called the Gorenstein dimension of M or $\mathrm{G-dim}_R(M)$.

2.5. Complete Intersection Dimensions. We first define a quasi-deformation which is essential to studying complete intersection dimension.

Definition 2.5. Let R be a local ring. A *quasi-deformation* of R is a diagram of local ring homomorphisms

$$R \longrightarrow R' \longleftarrow Q,$$

where R' is a flat R -module and where R' is a quotient of Q by a Q -regular sequence.

A local ring is a complete intersection if and only if there exists a quasi-deformation

$$R \longrightarrow \widehat{R} \longleftarrow Q,$$

where Q is a regular local ring.

Next, we define complete intersection dimensions in the context of complexes as in [30] but without restricting to homologically bounded complexes.

Definition 2.6. Let M be an object in $\mathbf{D}(R)$. Then the *complete intersection dimension*, the *complete intersection injective dimension* and *upper complete intersection injective dimension* are given by the following formulas:

$$\begin{aligned} \mathrm{CI-dim}_R(M) &:= \inf\{\mathrm{pd}_Q(R' \otimes_R^L M) - \mathrm{pd}_Q(R') \mid R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}, \\ \mathrm{CI-id}_R(M) &:= \inf\{\mathrm{id}_Q(R' \otimes_R^L M) - \mathrm{pd}_Q(R') \mid R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}. \\ \mathrm{CI}^*\mathrm{id}_R(M) &:= \inf\{\mathrm{id}_Q(R' \otimes_R^L M) - \mathrm{pd}_Q(R') \mid R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation with } R' \\ &\quad \text{having Gorenstein formal fibers and } R'/\mathfrak{m}R' \text{ Gorenstein}\}. \end{aligned}$$

When R is a complete intersection, then every bounded complex has finite complete intersection dimension and complete intersection injective dimension.

For any complex $M \in \mathbf{D}(R)$, we immediately have that $\mathrm{CI}^*\mathrm{id}_R(M) \leq \mathrm{CI-id}_R(M)$; it is unknown if the two quantities coincide in general.

For more information and results concerning complete intersection dimensions, we recommend the following resources [5, 29, 30].

Notation 2.7. Throughout this paper, we obtain parallel results using multiple homological dimensions. In this context, we use H -dim to denote projective dimension and complete intersection dimension, while H -id denotes their injective counterparts, injective dimension and upper complete intersection injective dimension.

Notation 2.8. Whenever we have a quasi-deformation $R \rightarrow R' \leftarrow Q$, for an R -module or complex M , we denote $M \otimes_R R'$ by M' .

Remark 2.9. Given R -modules M and N , it is unknown if $\mathrm{CI-dim}_R(M) < \infty$ and $\mathrm{CI-dim}_R(N) < \infty$ imply that $\mathrm{CI-dim}_R(M \oplus N) < \infty$. The difficulty of establishing this seemingly simple fact comes from the inability to establish a common quasi-deformation involving M and N . As a consequence, we often assume that the direct sum of a collection of modules has finite complete intersection dimension in order to establish a common quasi-deformation. The versions of this open problem with $\mathrm{CI-id}$ and $\mathrm{CI}^*\mathrm{id}$ are also unknown.

A result that will be used later in this paper is the following, which is a generalization of Ischebeck's formula involving finite complete intersection homological dimension.

Proposition 2.10. *Let M and N be non-zero R -modules. Suppose that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Then*

$$\text{depth}(R) - \text{depth}(M) = \sup\{i \geq 0 : \text{Ext}_R^i(M, N) \neq 0\},$$

if $\text{CI-dim}_R(M) < \infty$ or $\text{CI-id}_R(N) < \infty$.

Proof. When $\text{CI-dim}_R(M) < \infty$, this is proved in [1, Theorem 4.2]. Assume $\text{CI-id}_R(N) < \infty$. So there exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{id}_Q(N') < \infty$. Since $R \rightarrow R'$ is a local flat homomorphism, we have that $\sup\{i \geq 0 : \text{Ext}_R^i(M, N) \neq 0\} = \sup\{i \geq 0 : \text{Ext}_{R'}^i(M', N') \neq 0\}$ and $\text{depth}_R(R) - \text{depth}_R(M) = \text{depth}_{R'}(R') - \text{depth}_{R'}(M')$. In view of this, we can assume that $R = R'$ and that $R = Q/\mathbf{x}Q$ for some Q -regular sequence $\mathbf{x} = x_1, \dots, x_c$.

It then follows by Ischebeck's formula [8, 3.1.24] that $\sup\{i \geq 0 : \text{Ext}_Q^i(M, N) \neq 0\} = \text{depth}_Q(Q) - \text{depth}_Q(M)$. Moreover, by [1, Lemma 2.6], we have that $\sup\{i \geq 0 : \text{Ext}_Q^i(M, N) \neq 0\} = \sup\{i \geq 0 : \text{Ext}_R^i(M, N) \neq 0\} + c$. Therefore,

$$\begin{aligned} \sup\{i \geq 0 : \text{Ext}_R^i(M, N) \neq 0\} &= (\text{depth}_R(R) + c) - \text{depth}_Q(M) - c \\ &= \text{depth}_R(R) - \text{depth}_R(M), \end{aligned}$$

where the second equality follows using [8, 1.2.26(b)]. \square

3. FINITENESS OF CI-DIMENSIONS OF RHOM

We first work to establish some stability results of RHom complexes with respect to complete intersection dimension. We need a few necessary lemmas before we start in order to pass homological information between complexes over a ring and over a quotient of that ring. The two lemmas below were shared with us by Kaito Kimura.

Lemma 3.1. *Let $x \in \mathfrak{m}$ be a non-zero-divisor in R , and let M and N be complexes in $\mathsf{D}(R/xR)$. Then there is a distinguished triangle:*

$$\mathbf{R}\text{Hom}_R(M, N) \longrightarrow \mathbf{R}\text{Hom}_{R/xR}(M, N) \longrightarrow \mathbf{R}\text{Hom}_{R/xR}(M, N) \longrightarrow \Sigma \mathbf{R}\text{Hom}_R(M, N).$$

Proof. We first consider the short exact sequence given by multiplication by x , namely

$$0 \rightarrow R \xrightarrow{\cdot x} R \rightarrow R/xR \rightarrow 0$$

and then apply the functor $\mathbf{R}\text{Hom}_{R/xR}(M, \mathbf{R}\text{Hom}_R(-, N))$. By adjunction, this yields the distinguished triangle of the form

$$\mathbf{R}\text{Hom}_R(M, N) \longrightarrow \mathbf{R}\text{Hom}_{R/xR}(M, N) \longrightarrow \mathbf{R}\text{Hom}_{R/xR}(M, N) \longrightarrow \Sigma \mathbf{R}\text{Hom}_R(M, N)$$

which is the distinguished triangle we seek. \square

Lemma 3.2. *Let $x \in \mathfrak{m}$ be a non-zero-divisor in R , and M and N be complexes in $\mathsf{D}(R/xR)$. Suppose that S is a ring such that R is an S -algebra. If $\text{pd}_S(\mathbf{R}\text{Hom}_{R/xR}(M, N)) < \infty$, then $\text{pd}_S(\mathbf{R}\text{Hom}_R(M, N)) < \infty$, and if $\text{id}_S(\mathbf{R}\text{Hom}_{R/xR}(M, N)) < \infty$, then $\text{id}_S(\mathbf{R}\text{Hom}_R(M, N)) < \infty$.*

Proof. This follows from the previous lemma and from the two-out-of-three property for projective and injective dimension [11, Corollaries 8.1.9 and 8.2.9]. \square

We now obtain the following stability results for complete intersection dimensions.

Proposition 3.3. *Let M and N be non-acyclic complexes in $\mathsf{D}(R)$. For items (1) and (2) below, assume that M and N are in $\mathsf{D}_\square^f(R)$. For items (3) and (4) assume that M is in $\mathsf{D}_\square^f(R)$ and N is in $\mathsf{D}_\sqcup^f(R)$. For item (4), in addition, assume that $\mathbf{R}\text{Hom}_R(M, N)$ is in $\mathsf{D}_\square^f(R)$. The following results hold:*

- (1) *If $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, N) \oplus M) < \infty$, then $\text{CI-dim}_R(N) < \infty$.*
- (2) *If $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$ and $\text{id}_R(N) < \infty$, then $\text{CI}^*\text{id}_R(M) < \infty$.*
- (3) *If $\text{CI-id}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$, then there is a quasi-deformation for which $\text{CI-dim}_R(M)$ and $\text{CI-id}_R(N)$ are both finite.*

(4) If $\text{CI}^*\text{-id}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$, then there is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R'/\mathfrak{m}R'$ is artinian Gorenstein, Q is complete and $\text{pd}_Q(M \otimes_R R') < \infty$ and $\text{id}_Q(N \otimes_R R') < \infty$; and therefore $\text{CI-dim}_R(M)$ and $\text{CI}^*\text{-id}_R(N)$ are both finite.

Proof. (1) There is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{pd}_Q(\mathbf{R}\text{Hom}_{R'}(M', N')) < \infty$ and $\text{pd}_Q(M') < \infty$. By Lemma 3.2, $\text{pd}_Q(\mathbf{R}\text{Hom}_Q(M', N')) < \infty$. Now, by [11, Proposition 16.4.19], it follows that $\text{pd}_Q(N') < \infty$. Hence, $\text{CI-dim}_R(N) < \infty$.

(2) Note that $\mathbf{R}\text{Hom}_R(M, N)$ is homologically bounded since $\text{id}_R(N) < \infty$. By [30, Theorem F], there is a quasi-deformation $R \rightarrow R' \leftarrow Q$ with $R'/\mathfrak{m}R'$ artinian Gorenstein and Q complete having the property that $\text{pd}_Q(\mathbf{R}\text{Hom}_{R'}(M', N')) < \infty$. By Lemma 3.2, $\text{pd}_Q(\mathbf{R}\text{Hom}_Q(M', N')) < \infty$. On other hand, since $\text{id}_R(N) < \infty$, we have that by [11, Theorem 16.4.36], $\text{id}_{R'}(N') < \infty$. From [11, Proposition 15.4.10] we have that $\text{id}_Q(N') < \infty$ as R' has finite projective dimension over Q . Hence $\text{pd}_Q(M') < \infty$ by [10, Lemma 6.2.12], and so it follows that $\text{CI}^*\text{-id}_R(M) < \infty$ by the definition of upper CI-injective dimension.

(3) There is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{id}_Q(\mathbf{R}\text{Hom}_{R'}(M', N')) < \infty$. By Lemma 3.2, $\text{id}_Q(\mathbf{R}\text{Hom}_Q(M', N')) < \infty$. Then by [10, Lemma 6.2.10], $\text{pd}_Q(M') < \infty$ and $\text{id}_Q(N') < \infty$. Thus, there is a quasi-deformation that gives the finiteness of $\text{CI-dim}_R(M)$ and $\text{CI-id}_R(N)$.

(4) This is similar to the proof of the previous item using [30, Proposition 3.5]. □

To prepare our next stability results, we must prove that the derived tensor-evaluation morphism is an isomorphism under some non-standard conditions.

Theorem 3.4. *Let R be a Noetherian ring (not necessarily local) with finite Krull dimension, and let M be a complex in $D_{\square}^f(R)$, N a complex in $D(R)$ and I a complex in $D_{\square}(R)$. Assume that I has finite injective dimension, and that both $\mathbf{R}\text{Hom}_R(M, N)$ and N have finite Gorenstein flat dimension. Then*

$$\mathbf{R}\text{Hom}_R(M, N \otimes_R^L I) \simeq \mathbf{R}\text{Hom}_R(M, N) \otimes_R^L I.$$

Proof. Let F and G be semi-projective resolutions of M and N , respectively, and let J be a semi-injective replacement of I . By assumption, we may assume that F is a complex of finitely generated projective modules bounded below and that J is bounded. Set $n = \text{Gfd}_R(N)$. By [11, Proposition 9.3.19], $n \geq \sup G$ and therefore the natural morphism $\tau_{\subseteq n}^G : G \rightarrow G_{\subseteq n}$ is a quasi-isomorphism. Then it follows from [11, Proposition 9.3.29] and [9, A.2.10] that

$$\begin{aligned} \mathbf{R}\text{Hom}_R(M, N \otimes_R^L I) &= \mathbf{R}\text{Hom}_R(F, G \otimes_R J) \\ &\simeq \text{Hom}_R(F, G_{\subseteq n} \otimes_R J) \\ &\simeq \text{Hom}_R(F, G_{\subseteq n}) \otimes_R J. \end{aligned}$$

In addition, note that $\mathbf{R}\text{Hom}_R(M, N) \simeq \text{Hom}_R(F, G) \simeq \text{Hom}_R(F, G_{\subseteq n})$ and that $L := \text{Hom}_R(F, G_{\subseteq n})$ is a complex of Gorenstein flat modules bounded above.

Let $\Phi : P \rightarrow \text{Hom}_R(F, G_{\subseteq n})$ be a semi-projective resolution of L . Set $C = \text{cone}(\Phi)$. Then $C_i = L_i \oplus P_{i-1}$ for each i , and C is an acyclic complex of Gorenstein flat modules. Let $m = \text{Gfd}_R(\mathbf{R}\text{Hom}_R(M, N))$ and f_i denote the differential of C_i for each i . If $i > \sup L + 1$, note that f_i is the map $P_{i-1} \rightarrow P_{i-2}$. Since $m \geq \sup P$ by [11, Proposition 9.3.19], then $H_i(P) = 0$ for all $i > m$, and thus $\text{coker}(f_i)$ is Gorenstein flat for $i \gg 0$ by [11, Lemma 9.3.26].

We claim that $C \otimes_R J$ is acyclic. This follows from [11, A.11], since $C \otimes_R J_i$ is acyclic for each i , see Lemma A.2. Now, as $\text{cone}(\Phi \otimes_R J) \simeq C \otimes_R J$, it follows that $\Phi \otimes_R J$ is an isomorphism. Thus, $\mathbf{R}\text{Hom}_R(M, N) \otimes_R^L I \simeq P \otimes_R^L J \simeq P \otimes_R J \simeq \text{Hom}_R(F, G_{\subseteq n}) \otimes_R J$. Thus, this ensure the last isomorphism in the following:

$$\mathbf{R}\text{Hom}_R(M, N) \otimes_R^L I \simeq \mathbf{R}\text{Hom}_R(M, N \otimes_R^L I).$$

□

From now on, H-dim denotes projective dimension or complete intersection dimension. Also, we use Hid to denote injective dimension in the first case, and upper complete intersection injective dimension in the second case.

Theorem 3.5. *Let M and N be non-acyclic complexes in $D_{\square}^f(R)$ such that $\mathbf{R}\text{Hom}_R(M, N)$ is in $D_{\square}^f(R)$. Assume that $\text{G-dim}_R(N) < \infty$. If $\text{H-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$, then $\text{H-dim}_R(M \oplus N) < \infty$.*

Proof. We may assume that R is complete, and so has a dualizing complex D . In particular, $\text{G-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. Then by Proposition 3.4,

$$(3.1) \quad \mathbf{R}\text{Hom}_R(M, N \otimes_R^L D) \simeq \mathbf{R}\text{Hom}_R(M, N) \otimes_R^L D.$$

If $\text{H-dim}_R = \text{pd}_R$, then by (3.1) and Foxby-Sharp equivalence [11, Theorem 10.3.8], $\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D)$ is homologically bounded of finite injective dimension. Now, by [10, Lemma 6.2.10], $\text{pd}_R(M) < \infty$ and $\text{id}_R(N \otimes_R^L D) < \infty$. The last inequality implies that $\text{pd}_R(N) < \infty$ through Foxby-Sharp equivalence [11, Theorem 10.3.8].

If $\text{H-dim}_R = \text{CI-dim}_R$, then by (3.1) and [30, Theorem E(a)], we have that $\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D)$ is homologically bounded and $\text{CI}^*\text{-id}_R(\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D)) < \infty$. We then have, by Proposition 3.3(4), a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R'/\mathfrak{m}R'$ is artinian Gorenstein, $\text{pd}_Q(M \otimes_R R') < \infty$, and $\text{id}_Q((N \otimes_R^L D) \otimes_R^L R') < \infty$. Since $R'/\mathfrak{m}R'$ is Gorenstein, then $R' \otimes_R^L D$ is dualizing for R' . As Q is complete, it admits a dualizing complex D^Q . As $Q \rightarrow R$ is surjective with kernel generated by a Q -regular sequence, we have that $R' \otimes_Q^L D^Q$ is dualizing for R' . We then have that $R' \otimes_Q^L D^Q \simeq R' \otimes_R^L D$ (up to a shift). This yields the following isomorphisms (up to a shift)

$$\begin{aligned} (N \otimes_R^L D) \otimes_R^L R' &\simeq N \otimes_R^L (D \otimes_R^L R') \\ &\simeq N \otimes_R^L (R' \otimes_Q^L D^Q) \\ &\simeq (N \otimes_R R') \otimes_Q^L D^Q. \end{aligned}$$

Hence, $\text{id}_Q((N \otimes_R R') \otimes_Q^L D^Q) < \infty$, so $\text{pd}_Q(N \otimes_R R') < \infty$ through Foxby-Sharp equivalence [11, Theorem 10.3.8]. Consequently, $\text{CI-dim}_R(M \oplus N) < \infty$. \square

Corollary 3.6. *Let M and N be complexes in $\mathbf{D}_{\square}^f(R)$. Assume that $\text{pd}_R(\mathbf{R}\text{Hom}(M, N)) < \infty$ and that $\mathbf{R}\text{Hom}_R(M, N)$ is in $\mathbf{D}_{\square}^f(R)$. Then the following are simultaneously finite: $\text{pd}_R(M)$, $\text{CI-dim}_R(M)$, $\text{pd}_R(N)$, and $\text{CI-dim}_R(N)$.*

Proof. If $\text{pd}_R(M) < \infty$ (resp. $\text{pd}_R(N) < \infty$), then $\text{CI-dim}_R(M) < \infty$ (resp. $\text{CI-dim}_R(N) < \infty$). Thus, if we show the implications below, we will done.

$$\text{CI-dim}_R(M) < \infty \xrightarrow{(1)} \text{CI-dim}_R(N) < \infty \xrightarrow{(2)} \text{pd}_R(M \oplus N) < \infty$$

To prove (1), assume that $\text{CI-dim}_R(M) < \infty$. Then $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, N) \oplus M) < \infty$, by [29, Lemma 3.6]. Thus, by Proposition 3.3(1), $\text{CI-dim}_R(N) < \infty$.

Now, to prove (2), note that if $\text{CI-dim}_R(N) < \infty$, then by Theorem 3.5, $\text{pd}_R(M \oplus N) < \infty$. \square

Corollary 3.7. *Let M be a non-acyclic complex in $\mathbf{D}_{\square}^f(R)$ such that $\mathbf{R}\text{Hom}_R(M, M)$ is in $\mathbf{D}_{\square}^f(R)$. Assume that $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, M)) < \infty$. If $\text{G-dim}_R(M) < \infty$, then $\text{pd}_R(M) < \infty$.*

Proof. By Theorem 3.5, $\text{CI-dim}_R(M) < \infty$. Then, by [6, Proposition 5.7(1)], $\text{pd}_R(M) < \infty$. \square

We now seek to prove a dual statement to Theorem 3.5, but instead under the assumption that $\text{Gid}_R(M)$ is finite. We are only able to do this in the case that R is a Cohen-Macaulay local ring with a canonical module ω .

Lemma 3.8. *Let R be a Cohen-Macaulay local ring with a canonical module ω . Let I be a complex of injective modules bounded above. If $n = \text{Gid}_R(I) < \infty$, then the morphism $\tau_{\geq -n}^I : I_{\geq -n} \rightarrow I$ induces a quasi-isomorphism $\text{Hom}_R(\omega, \tau_{\geq -n}^I) : \text{Hom}_R(\omega, I_{\geq -n}) \rightarrow \text{Hom}_R(\omega, I)$.*

Proof. For simplicity, we may assume that $I_l = 0$ for all $i > 0$. By [11, 9.2.10], one has $-n \leq \inf I$, so the map $\tau_{\geq -n}^I : I_{\geq -n} \rightarrow I$ is an quasi-isomorphism.

We need to show that the cone of $\text{Hom}_R(\omega, \tau_{\geq -n}^I) : \text{Hom}_R(\omega, I_{\geq -n}) \rightarrow \text{Hom}_R(\omega, I)$ is acyclic. By [9, A.2.1.2], we have that

$$\text{cone}(\text{Hom}_R(\omega, \tau_{\geq -n}^I)) \simeq \text{Hom}_R(\omega, \text{cone}(\tau_{\geq -n}^I)).$$

Since $\tau_{\geq -n}^I$ is a quasi-isomorphism, then $Y := \text{cone}(\tau_{\geq -n}^I)$ is an acyclic complex bounded above. Moreover, Y is a complex by Gorenstein injective modules by [11, Lemma 9.2.11]. In addition, denoting by Z_l the kernel of $Y_l \rightarrow Y_{l-1}$, note that $Z_l^Y = 0$ for all $l \geq 0$.

Since each Y_j is Gorenstein injective, then Y_j is in the Bass class of R by [9, Theorem 6.1.7], and therefore $\text{Ext}_R^i(\omega, Y_j) = 0$ for all $i > 0$. Moreover, by [9, Lemma 4.1.6(c)],

$$\text{Ext}_R^1(\omega, Z_l^Y) \cong \text{Ext}_R^{1-l}(\omega, Z_0^Y) = 0$$

for $l \leq 0$. Thus, again by [9, Lemma 4.1.6(c)], $\text{Hom}_R(\omega, Y)$ is acyclic. \square

Theorem 3.9. *Let R be a Cohen-Macaulay local ring with a canonical module ω , and let M and N be non-acyclic complexes in $D_{\square}(R)$ and $D(R)$, respectively. Assume that $\text{Gid}_R(M) < \infty$ and $\text{Gfd}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. Then*

$$\mathbf{R}\text{Hom}(\mathbf{R}\text{Hom}_R(\omega, M), N) \simeq \omega \otimes_R^L \mathbf{R}\text{Hom}_R(M, N).$$

Proof. Let I and J be semi-injective replacement of M and N respectively. By assumption, we may assume that I is bounded above.

Set $n = \text{Gid}_R(M)$. Then the natural map $\tau_{\geq -n}^I : I_{\geq -n} \rightarrow I$ is a quasi-isomorphism. We have

$$\begin{aligned} \mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N) &\simeq \mathbf{R}\text{Hom}_R(\text{Hom}_R(\omega, I), N) \\ &\simeq \text{Hom}_R(\text{Hom}_R(\omega, I_{\geq -n}), J) \\ &\simeq \omega \otimes_R \text{Hom}_R(I_{\geq -n}, J). \end{aligned}$$

where the second isomorphism is due to Lemma 3.8, and the third is because of [11, Proposition 4.5.13(d)].

Set $L = \text{Hom}_R(I_{\geq -n}, J)$ and let $\Phi : P \rightarrow L$ be a projective resolution of L . We show that $\omega \otimes_R \Phi$ is a quasi-isomorphism. Set $C = \text{cone}(\Phi)$. Note that $C_i = L_i \oplus P_{i-1}$ for each i and that C is an acyclic complex of Gorenstein flat modules since L consist of Gorenstein flat modules, see [9, Corollary 6.3.6]. Let $m = \text{Gfd}_R(\mathbf{R}\text{Hom}_R(M, N))$ and f_i denote the differential of $C_i \rightarrow C_{i-1}$ for each i . If $i > \sup L^\sharp + 1$, note that f_i is the map $P_{i-1} \rightarrow P_{i-2}$. Since $m \geq \sup P$ by [11, Proposition 9.3.19], then $H_i(P) = 0$ for all $i > m$, and thus $\text{coker}(f_i)$ is Gorenstein flat for $i \gg 0$ by [11, Lemma 9.3.26].

If we denote by f_i the differential of C , note that for $i \gg 0$, $\text{coker}(f_i)$ has finite Gorenstein flat dimension. It follows from Lemma A.2(2) that $\text{cone}(\omega \otimes_R \Phi) \cong \omega \otimes_R C$ is acyclic.

Since $\omega \otimes_R \Phi$ is a quasi-isomorphism and $L \simeq \mathbf{R}\text{Hom}_R(M, N)$, we then have the following isomorphisms in $D(R)$:

$$\omega \otimes_R \text{Hom}_R(I_{\geq -n}, J) \simeq \omega \otimes_R P \simeq \omega \otimes_R^L \mathbf{R}\text{Hom}_R(M, N).$$

Thus, the required isomorphism follows. \square

Theorem 3.10. *Let R be a Cohen-Macaulay local ring. Let M and N be non-acyclic complexes in $D_{\square}^f(R)$ such that $\mathbf{R}\text{Hom}_R(M, N)$ is in $D_{\square}^f(R)$. Assume that $\text{Gid}_R(M) < \infty$. If $\text{H-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$, then $\text{Hid}_R(M \oplus N) < \infty$.*

Proof. We may assume that R is complete, and therefore R admits a canonical module ω . By assumption, we have $\text{G-dim}_R(\text{Hom}_R(M, N)) < \infty$. Therefore by Theorem 3.9,

$$(3.2) \quad \mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N) \simeq \omega \otimes_R^L \mathbf{R}\text{Hom}_R(M, N).$$

If $\text{H-dim}_R = \text{pd}_R$, then it follows from (3.2) and Foxby-Sharp equivalence that $\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N)$ is homologically bounded of finite injective dimension. By Bass series arguments [11, Proposition 16.4.32], this implies that $\text{pd}_R(\mathbf{R}\text{Hom}_R(\omega, M)) < \infty$ and $\text{id}_R(N) < \infty$. Again through Foxby-Sharp equivalence, we have that $\text{id}_R(M) < \infty$.

Suppose now that $\text{H-dim}_R = \text{CI-dim}_R$. Then by [30, Theorem E] and (3.2), we have that $\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N)$ is homologically bounded and $\text{CI}^*\text{-id}_R(\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N)) < \infty$. By Proposition 3.3(4), there is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R'/\mathfrak{m}R'$ is artinian Gorenstein, Q is complete, $\text{pd}_Q(\mathbf{R}\text{Hom}_R(\omega, M) \otimes_R R') < \infty$, and $\text{id}_Q(N \otimes_R R') < \infty$. We have that $\mathbf{R}\text{Hom}_R(\omega, M) \otimes_R R' \simeq \mathbf{R}\text{Hom}_{R'}(\omega \otimes_R R', M \otimes_R R')$ as R' is flat over R . Moreover, Q is Cohen-Macaulay since R' is, and therefore Q admits a canonical module ω_Q . Note that $\omega' = \omega \otimes_R R'$ is a canonical module for R' and that $\omega' \cong \omega_Q \otimes_Q R' \simeq \omega_Q \otimes_Q^L R'$.

Since

$$\mathbf{R}\text{Hom}_{R'}(\omega \otimes_R R', M \otimes_R R') \simeq \mathbf{R}\text{Hom}_{R'}(\omega_Q \otimes_Q^L R', M \otimes_R R') \simeq \mathbf{R}\text{Hom}_Q(\omega_Q, M \otimes_R R'),$$

then $\text{pd}_Q(\mathbf{R}\text{Hom}_Q(\omega_Q, M \otimes_R R')) < \infty$, whence it then follows that $\text{id}_Q(M \otimes_R R') < \infty$ using Foxby-Sharp equivalence. Thus, $\text{CI}^*\text{-id}_R(M \oplus N) < \infty$. \square

Corollary 3.11. *Let R be a Cohen-Macaulay local ring with canonical module ω , and let M be a non-acyclic complex in $D_{\square}^f(R)$ such that $\mathbf{R}\text{Hom}_R(M, M)$ is in $D_{\square}^f(R)$. Assume that $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, M)) < \infty$. If $\text{Gid}_R(M) < \infty$, then $\text{id}_R(M) < \infty$.*

Proof. By Theorem 3.10, $\text{CI}^*\text{-id}_R(M) < \infty$. Then, by [6, Proposition 5.7(2)], $\text{id}_R(M) < \infty$. \square

4. FINITENESS OF CI-DIMENSIONS OF EXT MODULES

In this section, we consider the consequences of assuming that certain Ext modules have finite complete intersection dimension or complete intersection injective dimension. In many theorems, we consider direct sums of Ext modules to force a uniform quasi-deformation that apply to each Ext module, see Remark 2.9. The following lemma will be used frequently.

Lemma 4.1. *Let X be a complex in $D_{\square}^f(R)$.*

- (1) *If $\text{CI-dim}_R(\bigoplus_{i \in \mathbb{Z}} H_i(X)) < \infty$, then $\text{CI-dim}_R(X) < \infty$.*
- (2) *If $\text{CI-id}_R(\bigoplus_{i \in \mathbb{Z}} H_i(X)) < \infty$, then $\text{CI-id}_R(X) < \infty$.*
- (3) *If $\text{CI}^*\text{-id}_R(\bigoplus_{i \in \mathbb{Z}} H_i(X)) < \infty$, then $\text{CI}^*\text{-id}_R(X) < \infty$.*

Proof. We prove item (1). There is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{pd}_Q(\bigoplus_{i \in \mathbb{Z}} H_i(X')) < \infty$. By [20, Lemma 3.2], we have that $\text{pd}_Q(X') < \infty$. Then, we have that $\text{CI-dim}_R(X) < \infty$.

The proof of items (2) and (3) are similar using [38, Lemma 4.1]. \square

4.1. Finite CI-dimension of Ext modules. The following theorem can be compared to [23, Theorem 1.1].

Theorem 4.2. *Let M and N be non-zero R -modules such that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Let $L = \bigoplus_{i=0}^{\text{Rfd}_R(M)} \text{Ext}_R^i(M, N)$. If $\text{CI-dim}_R(M \oplus L) < \infty$, then $\text{CI-dim}_R(N) < \infty$.*

Proof. Since we have that $\text{CI-dim}_R(M) < \infty$, we immediately have that $\text{G-dim}(M) < \infty$ [5, Theorem 1.4]. This implies that $\text{Rfd}_R(M) = \text{G-dim}_R(M)$. By Proposition 2.10, we have that $\text{Ext}_R^i(M, N) = 0$ for all $i > \text{depth}(R) - \text{depth}(M) = \text{Rfd}_R(M)$. Applying Lemma 4.1(1) to the complex $\mathbf{R}\text{Hom}_R(M, N) \oplus M$, we have that $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, N) \oplus M) < \infty$. Thus, by Proposition 3.3(1), $\text{CI-dim}_R(N) < \infty$. \square

Remark 4.3. Instead of requiring that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ in Theorem 4.2, it is sufficient to consider the weaker condition that $\text{Ext}_R^i(M, N)$ vanishes for $\text{cx}_R(M) + 1$ consecutive values of $i > \text{CI-dim}_R(M)$; see [4, Theorem 4.7].

Corollary 4.4. *Let M and N be non-zero R -modules. Assume that $\text{pd}_R(M) < \infty$ and set $L = \bigoplus_{i=0}^{\text{pd}_R(M)} \text{Ext}_R^i(M, N)$. If $\text{CI-dim}_R(L) < \infty$, then $\text{CI-dim}_R(N) < \infty$.*

Proof. This immediately follows from the previous theorem and [29, Lemma 3.6]. \square

Theorem 4.5. *Let M and N be non-zero R -modules such that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. If $\text{G-dim}_R(N) < \infty$ and $\text{H-dim}_R(\bigoplus_{i=0}^{\infty} \text{Ext}_R^i(M, N)) < \infty$, then $\text{H-dim}_R(M \oplus N) < \infty$.*

Proof. In each case, Lemma 4.1(1) or [20, Lemma 3.2] ensures that $\text{H-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. Therefore, the conclusion follows from Theorem 3.5. \square

Corollary 4.6. *Let M and N be non-zero R -modules, with M having finite Gorenstein dimension. Set $L = \bigoplus_{i=0}^{\text{G-dim}_R(M)} \text{Ext}_R^i(M, N)$. Suppose that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. If $\text{CI-dim}_R(N \oplus L) < \infty$, then $\text{CI-dim}_R(M) < \infty$.*

Proof. This follows from the previous theorem and [27, Theorem 2.6(ii)] \square

Theorem 4.7. *Let M and N be non-zero R -modules such that $\text{Gid}_R(M) < \infty$. Set $L = \bigoplus_{i=0}^{\infty} \text{Ext}_R^i(M, N)$, and assume that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. If $\text{H-dim}_R(L) < \infty$, then $\text{Hid}_R(M \oplus N) < \infty$.*

Proof. In each case, Lemma 4.1(1) or [20, Lemma 3.2] ensures that $\text{H-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. Therefore, the conclusion follows from Theorem 3.10. \square

The following theorem includes a version of [23, Theorem 1.2] for complete intersection dimension.

Theorem 4.8. *Let M and N be non-zero R -modules, and set $L = \bigoplus_{i=0}^{\text{Rfd}_R(M)} \text{Ext}_R^i(M, N)$. Assume that $\text{CI-dim}_R(L) < \infty$.*

- (1) *If $\text{id}_R(N) < \infty$, then $\text{CI}^*\text{-id}_R(M) < \infty$.*
- (2) *If $\text{id}_R(M) < \infty$, then $\text{CI}^*\text{-id}_R(N) < \infty$.*

Proof. In either case, we have that R is Cohen-Macaulay by Bass' conjecture, since it admits a non-zero finitely generated R -module of finite injective dimension. In particular, $\text{Rfd}_R(M) = \text{depth}(R) - \text{depth}(M)$ by [23, Lemma 3.9].

(1) By Lemma 4.1(1), we have $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. Then by Proposition 3.3(2), we must have that $\text{CI}^*\text{-id}_R(M) < \infty$.

(2) By [30, Corollary 3.7(a)], we may assume that R is complete. Moreover, applying [23, Corollary 3.17], we have $\text{Gid}_R(N) < \infty$. Therefore, we must then have that $\text{Ext}_R^i(M, N) = 0$ for all $i > \text{depth}(R) - \text{depth}(M)$ by [32, Theorem 2.10]. Now, the result follows from the previous theorem. \square

In the next theorem, we study the consequences of having deficiency modules with finite CI-dim, all them with the same quasi-deformation. It is a version of [20, Theorem 4.1(1)] for complete intersection dimension.

Theorem 4.9. *Let R have a dualizing complex D , and let M be a non-zero R -module such that $\text{CI-dim}_R \left(\bigoplus_{i=\text{depth}(M)}^{\dim(M)} K^i(M) \right) < \infty$. Then $\text{CI}^*\text{-id}_R(M) < \infty$ and R is Cohen-Macaulay.*

Proof. Lemma 4.1(1) implies that $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(M, D)) < \infty$, since the modules of deficiency are simply the homology modules of $\mathbf{R}\text{Hom}_R(M, D)$. We then have that $\text{CI}^*\text{-id}_R(M) < \infty$ by [30, Corollary 4.6]. This implies that R is Cohen-Macaulay by [24, Theorem 2.7]. \square

The following three results are inspired by the proofs of [23, Theorem 3.3 and Theorem 3.15]. As our proof of the next lemma is fairly long, we have placed it in an appendix at the end of this paper.

Lemma 4.10. *Let R be a local ring with $t = \text{depth}(R)$. Let D be a normalized dualizing complex of R and let*

$$L : \cdots \rightarrow L_{i+1} \rightarrow L_i \rightarrow L_{i-1} \rightarrow \cdots$$

be a complex of Gorenstein flat R -modules bounded above whose homologies are finitely generated. Assume in addition that C is a complex of finitely generated R -modules or that R is Cohen-Macaulay. Suppose that there is an integer r such that:

- (1) $\text{H-dim}_R(L_{\geq r}) < \infty$.
- (2) $H_i(L)$ has finite length for $i > r$.
- (3) $H_i(L \otimes_R D) = 0$ for all $i < t + r$.

Then $\text{Hid}_R(L \otimes_R D) < \infty$.

Theorem 4.11. *Let R be a local ring with a normalized dualizing complex D , and let M and N be non-zero R -modules. Let $r \in \mathbb{N}$ be such that $r \geq \text{depth}(R) - \text{depth}(M)$. Assume that the following conditions hold:*

- (1) $\text{G-dim}_R(N) < \infty$ and $\text{H-dim}_R(\text{Ext}_R^i(M, N)) < \infty$ for all $0 \leq i \leq r$.
- (2) $\text{Ext}_R^i(M, N)$ has finite length for all $i > r$.
- (3) $H_i(\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D)) = 0$ for all $i < \text{depth}(M)$.

Then $\text{H-dim}_R(M \oplus N) < \infty$.

Proof. Similarly as the beginning of the proof of Proposition 3.4, let $F : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ and $G : \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ be projective resolutions of finitely generated R -modules of M and N , and $n = \text{G-dim}_R(N) < \infty$, one can see that $\mathbf{R}\text{Hom}_R(M, N) \simeq \text{Hom}_R(F, G_{\leq n})$ and

$$\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D) \simeq \text{Hom}_R(F, G_{\leq n}) \otimes_R D.$$

Set $L = \text{Hom}_R(F, G_{\leq n})$. Note that L is a complex of totally reflexive modules. By (1), $\text{pd}_R(L_{\geq -r}) < \infty$, while by (2), $H_i(L)$ has finite length for $i > -r$. Moreover, since $r \geq \text{depth}(R) - \text{depth}(M)$, then by (3), $H_i(L \otimes_R D) = 0$ for $i < \text{depth}(R) - r$. Thus, Lemma 4.10 ensures that $\text{Hid}_R(L \otimes_R D) < \infty$. This implies that $\text{Hid}_R(\mathbf{R}\text{Hom}_R(M, N \otimes_R^L D)) < \infty$. Now, as in the proof of Theorem 3.5, it follows that $\text{H-dim}_R(M \oplus N) < \infty$. \square

Theorem 4.12. *Let R be a Cohen-Macaulay local ring with a canonical module ω , and let M and N be non-zero R -modules. Let $r \in \mathbb{N}$ such that $r \geq \text{depth}(R) - \text{depth}(M)$. Assume that the following conditions hold:*

- (1) $\text{Gid}_R(M) < \infty$ and $\text{H-dim}_R(\text{Ext}_R^i(M, N)) < \infty$ for all $0 \leq i \leq r$.
- (2) $\text{Ext}_R^i(M, N)$ has finite length for all $i > r$.
- (3) $\text{H}_i(\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N)) = 0$ for all $i < \text{depth}(M) - \dim(R)$.

Then $\text{Hid}_R(M \oplus N) < \infty$.

Proof. Let $I = (0 \rightarrow I_{-0} \rightarrow I_{-1} \rightarrow \dots)$, $J = (0 \rightarrow J_{-0} \rightarrow J_{-1} \rightarrow \dots)$ be minimal injective resolutions of M and N . Set $n = \text{Gid}_R(M)$. We have

$$\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N) \simeq \omega \otimes_R \text{Hom}_R(I_{\geq -n}, J) \simeq \Sigma^{-\dim(R)}(D \otimes_R \text{Hom}_R(I_{\geq -n}, J)).$$

The first isomorphism follows using the same reasoning as the proof of Theorem 3.9, while the second isomorphism follows by Lemma A.3 as $\Sigma^{\dim(R)}\omega \simeq D$ and $\text{Hom}_R(I_{\geq -n}, J)$ is a complex of Gorenstein flat modules.

Set $L = \text{Hom}_R(I_{\geq -n}, J) \simeq \mathbf{R}\text{Hom}_R(M, N)$. By item (1), $\text{H-dim}_R(L_{\geq -r}) < \infty$. Moreover, by item (2), $\text{H}_i(L)$ has finite length for all $i > -r$. By item (3) and the isomorphism above, $\text{H}_i(L \otimes_R D) = 0$ for all $i < \text{depth}(M)$. Since $r \geq \text{depth}(R) - \text{depth}(M)$, then $\text{H}_i(L \otimes_R D) = 0$ for all $i < \text{depth}(R) - r$. Then, by Lemma 4.10, $\text{Hid}_R(L \otimes_R D) < \infty$. Thus, $\text{Hid}_R(\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(\omega, M), N)) < \infty$. Now, as in the proof of Theorem 3.10, it follows that $\text{Hid}_R(M \oplus N) < \infty$. \square

4.2. Finite CI-injective dimensions of Ext modules. In this subsection, we investigate the consequences of assuming that $\bigoplus_{i=0}^{\infty} \text{Ext}_R^i(M, N)$ has finite complete intersection injective dimension. We show that this condition is sufficient to ensure that the complete intersection dimensions of M and N are finite.

Theorem 4.13. *Let M and N be non-zero R -modules such that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Let $L = \bigoplus_{i=0}^{\infty} \text{Ext}_R^i(M, N)$.*

- (1) *If $\text{CI-id}_R(L) < \infty$, then there is a quasi-deformation for which $\text{CI-dim}_R(M)$ and $\text{CI-id}_R(N)$ are both finite.*
- (2) *If $\text{CI}^*\text{-id}_R(L) < \infty$, then there is a quasi-deformation for which $\text{CI-dim}_R(M)$ and $\text{CI}^*\text{-id}_R(N)$ are both finite.*

Proof. It follows by Lemma 4.1(2) that $\text{CI-id}_R(\mathbf{R}\text{Hom}_R(M, N)) < \infty$. (1) then follows from Proposition 3.3(3).

The proof of (2) is similar using Lemma 4.1(3) and Proposition 3.3(4). \square

Corollary 4.14. *Let M and N be non-zero R -modules such that $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Set $L = \bigoplus_{i=0}^{\text{Rfd}_R(M)} \text{Ext}_R^i(M, N)$. If $\text{CI-id}_R(L \oplus N) < \infty$, then $\text{CI-dim}_R(M) < \infty$.*

Proof. Since $\text{CI-id}_R(N) < \infty$, then R is Cohen-Macaulay [24, Theorem 2.7]. Moreover, by Proposition 2.10, $\text{Ext}_R^i(M, N) = 0$ for all $i > \text{depth}(R) - \text{depth}(M)$. In addition, $\text{Rfd}_R(M) = \text{depth}(R) - \text{depth}(M)$ by [23, Lemma 3.9]. Thus, $\text{CI-id}_R(\bigoplus_{i=0}^{\infty} \text{Ext}_R^i(M, N)) < \infty$. The required conclusion now follows from Theorem 4.13. \square

Corollary 4.15. *Let M and N be non-zero R -modules. Set $L = \bigoplus_{i=0}^{\text{Rfd}_R(M)} \text{Ext}_R^i(M, N)$. If $\text{id}_R(N) < \infty$ and $\text{CI-id}_R(L) < \infty$, then $\text{CI-dim}_R(M) < \infty$.*

We finish this section studying the consequences of having deficiency modules with finite $\text{CI}^*\text{-id}$, all of them with the same quasi-deformation. The following proposition is a version of [20, Theorem 4.1(1)] for complete intersection dimension.

Theorem 4.16. *Let R have a dualizing complex D , and let M be a non-zero R -module such that $\text{CI}^*\text{-id}_R \left(\bigoplus_{i=\text{depth}(M)}^{\dim(M)} K^i(M) \right) < \infty$. Then $\text{CI-dim}_R(M) < \infty$ and R is Cohen-Macaulay.*

Proof. Lemma 4.1 immediately implies that $\text{CI}^*\text{-id}_R(\mathbf{R}\text{Hom}_R(M, D)) < \infty$, since the modules of deficiency are simply the homology modules of $\mathbf{R}\text{Hom}_R(M, D)$. We then have that $\text{CI-dim}_R(M) < \infty$ by [30, Corollary 4.6]. In addition, R is Cohen-Macaulay by [24, Theorem 2.7]. \square

5. CI-PERFECT MODULES

For the purpose of producing Ext modules with finite complete intersection dimension, in this section we introduce the notion of CI-perfect modules as a natural generalization of perfect modules.

For an R -module M , we recall that $\text{grade}(M) = \inf\{i \mid \text{Ext}_R^i(M, R) \neq 0\}$.

Definition 5.1. An R -module M is said to be *CI-perfect* if $\text{CI-dim}_R(M) = \text{grade}(M)$.

Remark 5.2. An R -module M is CI-perfect if and only if M is G-perfect and $\text{CI-dim}_R(M) < \infty$.

In order to prove the main result of this section, we first establish the following auxiliary lemma.

Lemma 5.3. Consider the following exact sequence of R -modules

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow 0$$

with $n \geq 2$. Suppose that there exist $i, l \in \mathbb{N}$ such that $\text{CI-dim}_R(X_i) < \infty$ and $\text{pd}_R(X_j) < \infty$ for all $j \in \{1, \dots, n\} - \{i, l\}$. Then $\text{CI-dim}_R(X_l) < \infty$.

Proof. There is a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{pd}_Q(X_i \otimes_R R') < \infty$. Since R' is flat over R , then

$$0 \rightarrow X_n \otimes_R R' \rightarrow X_{n-1} \otimes_R R' \rightarrow \cdots \rightarrow X_0 \otimes_R R' \rightarrow 0$$

is an exact sequence both of R' -modules and Q -modules. In addition, for each $j \in \{1, \dots, n\} - \{i, l\}$, one has $\text{pd}_Q(X_j \otimes_R R') < \infty$, see [5, Lemma 1.5]. Therefore, by using the exact sequence above, one concludes that $\text{pd}_Q(X_l \otimes_R R') < \infty$, and so $\text{CI-dim}_R(X_l) < \infty$. \square

We are now ready to prove the main result of this section, which extends [7, Lemma 3.5]. Moreover, it is a version of [37, Theorem 1.8] for complete intersection dimension.

Theorem 5.4. Let M be an R -module. If M is CI-perfect of grade g , then $\text{Ext}_R^g(M, R)$ is CI-perfect of grade g .

Proof. If $g = 0$, then $\text{CI-dim}_R(M) = 0$ and the result follows from [7, Lemma 3.5]. Suppose that $g > 0$. We set $M^* := \text{Hom}_R(M, R)$. Let

$$\cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a minimal free resolution of M . Since $\text{Ext}_R^i(M, R) = 0$ for all $1 \leq i \leq g-1$ and $M^* = 0$, dualizing the minimal free resolution gives an exact sequence

$$(5.1) \quad 0 \rightarrow F_0^* \rightarrow \cdots \rightarrow F_{g-1}^* \rightarrow (\Omega^g M)^* \rightarrow \text{Ext}_R^g(M, R) \rightarrow 0.$$

Since $\text{CI-dim}_R(\Omega^g M) = 0$, then $\text{CI-dim}_R((\Omega^g M)^*) = 0$ by [7, Lemma 3.5]. Now, applying Lemma 5.3 to the exact sequence (5.1), we have $\text{CI-dim}_R(\text{Ext}_R^g(M, R)) < \infty$. Thus, [37, Theorem 1.8] implies that $\text{Ext}_R^g(M, R)$ is CI-perfect. \square

This is analogous to [37, Corollary 2.4] for complete intersection dimension.

Proposition 5.5. Let M be an R -module with $\text{CI-dim}_R(M) < \infty$. If M is Cohen-Macaulay, then M is CI-perfect. Moreover, if R is Cohen-Macaulay, then the converse also holds, and we have that $\text{grade}(M) = \dim(R) - \text{depth}(M)$.

Proof. If M is Cohen-Macaulay, by [37, Corollary 2.4], M is G-perfect, so it follows from Remark 5.2 that M is CI-perfect. Conversely, assume that R is Cohen-Macaulay. Since M is CI-perfect, then M is G-perfect, and the conclusion follows from [37, Corollary 2.4]. The additional equality follows from [37, Theorem 2.1]. \square

Corollary 5.6. Let R be a Cohen-Macaulay ring of dimension d , and let M be a Cohen-Macaulay R -module of depth s . Then

$$\text{CI-dim}_R(M) < \infty \iff \text{CI-dim}_R(\text{Ext}_R^{d-s}(M, R)) < \infty.$$

Proof. The forward implication follows directly from Theorem 5.4 and Proposition 5.5.

To show the reverse implication, assume that $\text{CI-dim}_R(\text{Ext}_R^{d-s}(M, R)) < \infty$. It follows by [16, Theorem 2] that $\text{Ext}_R^{d-s}(M, R)$ is G-perfect of grade equal to $d-s$ and that $M \cong \text{Ext}_R^{d-s}(\text{Ext}_R^{d-s}(M, R), R)$. Thus $\text{Ext}_R^{d-s}(M, R)$ is CI-perfect and Theorem 5.4 implies that $\text{CI-dim}_R(M) < \infty$. \square

6. VANISHING OF EXT AND FINITE COMPLETE INTERSECTION HOMOLOGICAL DIMENSION OF HOM

In this section, we study vanishing of Ext, provided that that the self dual or algebraic dual of a module has finite complete intersection homological dimension. As an application, we prove the Auslander-Reiten conjecture in certain cases and provide Gorenstein criteria. The algebraic dual of a given module M , namely, $\text{Hom}_R(M, R)$, will be denoted by M^* . Moreover, $\text{edim}(R)$ will denote the embedding dimension of R .

In their partial solution to the Auslander-Reiten conjecture (Conjecture 1.5), Ghosh and Samanta [17], proved the following statement:

Theorem 6.1. [17, Theorem 3.9] *Let R be a local ring, and let M be a non-zero R -module such that $\text{Ext}_R^i(M, R) = 0$ for all $1 \leq i \leq \text{depth}(R)$, $\text{Ext}_R^j(M, M) = 0$ for all $j \geq 1$ and $\text{CI-dim}_R(\text{Hom}_R(M, M)) < \infty$. Suppose at least one of $\text{G-dim}_R(M)$ and $\text{G-dim}_R(M^*)$ is finite. Then M is free.*

As a consequence of our following results, we prove that the vanishing of $\text{Ext}_R^i(M, R)$ for all $1 \leq i \leq \text{depth}(R)$ can be dropped in the above statement. For the next proposition, given a non-negative integer k , we say that an R -module M satisfies (\tilde{S}_k) if $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{k, \text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})\}$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Proposition 6.2. *Let R be a local ring of dimension d , and let M and N be a non-zero R -modules with $\text{Supp}(N) \subseteq \text{Supp}(M)$. Suppose that $\text{G-dim}_R(N^*) < \infty$ and that $\text{Hom}_R(M, N)$ is totally reflexive. If $\text{Ext}_R^i(M, N) = 0$ for all $1 \leq i \leq d$, then N is totally reflexive.*

Proof. We claim that $\text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \text{Supp}(N)$. Indeed, let $\mathfrak{p} \in \text{Supp}(N)$. Since $\text{Ext}_R^i(M, N) = 0$ for all $1 \leq i \leq d$, we have that $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $1 \leq i \leq \text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$. By [26, Lemma 3.1(2)], we have that $\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$ and $\text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}))$. Since $\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is totally reflexive over $R_{\mathfrak{p}}$, its depth (as an $R_{\mathfrak{p}}$ -module) is $\text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Thus, $\text{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$, as desired.

Then, it is easy to see that N satisfies (\tilde{S}_t) , for $t = \text{depth}(R)$. Therefore, it follows from [25, Theorem 4.1] that N is totally reflexive. \square

Theorem 6.3. *Let M be a non-zero R -module such that $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$. Then the following conditions are equivalent:*

- (1) M is free.
- (2) $\text{G-dim}_R(M^*) < \infty$ and $\text{CI-dim}_R(\text{Hom}_R(M, M)) = 0$.
- (3) $\text{G-dim}_R(M) < \infty$ and $\text{CI-dim}_R(\text{Hom}_R(M, M)) < \infty$.

Proof. The implication (1) \Rightarrow (2) is trivial. On the other hand, (2) \Rightarrow (3) holds since M is totally reflexive by Proposition 6.2. For the implication (3) \Rightarrow (1), by Theorem 3.7 or 4.5, we have $\text{CI-dim}_R(M) < \infty$. Then, it follows from [1, Theorem 4.3], that M is free. \square

One can see in [18, Example 3.10] that the condition of self vanishing of Ext can not be omitted in Theorem 6.3.

Motivated by the results of [19] and [26], which provide partial solutions to the Auslander-Reiten conjecture under the assumption that certain Hom-modules have finite injective dimension or finite injective dimension with respect to a semidualizing module, we now establish several freeness criteria assuming that these Hom-modules have finite complete intersection injective dimension.

Theorem 6.4. *Let M be a non-zero R -module. Suppose that one of the following conditions holds:*

- (1) $\text{CI-id}_R(\text{Hom}_R(M, M)) < \infty$ and $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$.
- (2) $\text{CI-id}_R(M^*) < \infty$, $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$ and $\text{Ext}_R^{2j}(M, M) = 0$ for some $j > 0$.

Then M is free and R is Gorenstein.

Proof. (1) By Theorem 4.13, $\text{CI-dim}_R(M) < \infty$. Then by [1, Theorem 4.3], M is free. Thus, $\text{CI-id}_R(R) < \infty$ and hence R is Gorenstein by [24, Theorem 2.5].

(2) By Theorem 4.13, $\text{CI-dim}_R(M) < \infty$. Also, since $\text{Ext}_R^{2j}(M, M) = 0$ for some $j > 0$, then $\text{pd}_R(M) < \infty$ by [4, Theorem 4.2]. Now, as $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$, then M must be free. Thus, $\text{CI-id}_R(R) < \infty$, and then it follows from [24, Theorem 2.5] again that R is Gorenstein. \square

The following corollary extends [19, Corollary 2.16 and 2.12].

Corollary 6.5. *Let M be a non-zero R -module such that $\text{Hom}_R(M, M)$ or M^* has finite complete intersection injective dimension. Then the Auslander-Reiten conjecture holds true for M .*

For the rest of this section, we aim to weaken the vanishing conditions in Theorem 6.4. To this end, we consider the upper complete intersection injective dimension ($\text{CI}^*\text{-id}$) and establish a few auxiliary results.

Lemma 6.6. *Let Q be a complete local ring and x_1, \dots, x_c ($c \geq 0$) be a Q -regular sequence. Set $R = Q/(x_1, \dots, x_c)$, and let M be a non-zero R -module. If $\text{Ext}_R^2(M, M) = 0$, then there exists a finitely generated Q -module N such that x_1, \dots, x_c is an N -sequence and $M \cong N/(x_1, \dots, x_c)N$.*

Proof. This is an immediate consequence of [2, Proposition 1.7] and [13, Lemma 6.1]. \square

Lemma 6.7. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with canonical module ω , and let M be a non-zero R -module such that $\text{CI}^*\text{-id}_R(M) < \infty$. Then, there is a quasi-deformation $R \rightarrow R' \leftarrow Q$ of codimension $\text{cx}_R(\text{Hom}_R(\omega, M)) < \infty$ such that $R'/\mathfrak{m}R'$ is artinian Gorenstein, Q is complete and $\text{id}_Q(M \otimes_R R') < \infty$.*

Proof. By [30, Theorem E], $\mathbf{R}\text{Hom}_R(\omega, M)$ has finite complete intersection dimension. In addition, note that $\mathbf{R}\text{Hom}_R(\omega, M)$ is concentrated in a single degree (i.e., $\mathbf{R}\text{Hom}_R(\omega, M) \simeq \text{Hom}_R(\omega, M)$) as M is in the Bass class of R [9, Theorem 3.3.8].

By [4, 4.1.3], there is a quasi-deformation $R \rightarrow R' \leftarrow Q$ of codimension $\text{cx}_R(\mathbf{R}\text{Hom}_R(\omega, M)) < \infty$ such that $\text{pd}_Q(\mathbf{R}\text{Hom}_R(\omega, M) \otimes_R R') < \infty$.

By [30, Lemma 3.1] and its proof, there exists a quasi-deformation $R \rightarrow R'' \leftarrow Q'$ with the same codimension such that $R''/\mathfrak{m}R''$ is artinian Gorenstein, and Q' is complete. Following the proof of [30, Theorem F], we have that $\text{pd}_{Q'}(\mathbf{R}\text{Hom}_R(\omega, M) \otimes_R R'') < \infty$. Since R'' is flat over R , we have that $\text{pd}_{Q'}(\mathbf{R}\text{Hom}_{R''}(\omega \otimes_R R'', M \otimes_R R'')) < \infty$, and so $\text{pd}_{Q'}(\mathbf{R}\text{Hom}_{Q'}(\omega \otimes_R R'', M \otimes_R R'')) < \infty$ by Theorem 3.2. $\omega \otimes R''$ is the canonical module of R'' as $R''/\mathfrak{m}R''$ is Gorenstein. Q' is complete and so has a dualizing complex $D^{Q'}$ and as in the proof of Theorem 3.5, we may assume that $D^{Q'} \otimes_{Q'}^{\mathbf{L}} R'' \simeq \omega \otimes_R^{\mathbf{L}} R''$. We then have that

$$\begin{aligned} \mathbf{R}\text{Hom}_{Q'}(\omega \otimes_R R'', M \otimes_R R'') &\simeq \mathbf{R}\text{Hom}_{Q'}(D^{Q'} \otimes_{Q'}^{\mathbf{L}} R'', M \otimes_R R'') \\ &\simeq \mathbf{R}\text{Hom}_{Q'}(D^{Q'}, \mathbf{R}\text{Hom}_{Q'}(R'', M \otimes_R R'')) \\ &\simeq \mathbf{R}\text{Hom}_Q(D^{Q'}, M \otimes_R R''). \end{aligned}$$

It then follows that $\text{id}_{Q'}(M \otimes_R R'') < \infty$ using Foxby-Sharp equivalence. \square

Theorem 6.8. *Let M be a non-zero R -module. Suppose that one of the following conditions holds:*

- (1) $\text{CI}^*\text{-id}_R(\text{Hom}_R(M, M)) < \infty$ and $\text{Ext}_R^i(M, M) = \text{Ext}_R^j(M, R) = 0$ for all $1 \leq i \leq \max\{2, \text{edim}(R)\}$ and $1 \leq j \leq \text{edim}(R)$.
- (2) $\text{CI}^*\text{-id}_R(M^*) < \infty$, $\text{Ext}_R^2(M, M) = 0$, and $\text{Ext}_R^i(M, R) = 0$ for all $1 \leq i \leq \text{edim}(R)$.

Then M is free and R is Gorenstein.

Proof. Under either condition, we can assume that $R = \widehat{R}$, as the corresponding complete intersection injective dimension remains finite [30, Corollary 3.7], the Ext vanishings remain the same, and the freeness of \widehat{M} as an \widehat{R} -module implies the freeness of M as an R -module. In addition, we can assume that R is Cohen-Macaulay with canonical module ω , as there exists a finitely generated R -module of finite complete intersection injective dimension [24, Theorem 2.7].

(1) There exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that Q is complete and $\text{id}_Q(\text{Hom}_R(M, M) \otimes_R R') < \infty$ of codimension $c \leq \text{edim}(R) - \dim(R)$ by Lemma 6.7 and [5, Theorem 5.7]. Since $R \rightarrow R'$ is faithfully flat, we have that $\text{id}_Q(\text{Hom}_{R'}(M', M')) < \infty$, $\text{Ext}_{R'}^i(M', M') = \text{Ext}_{R'}^j(M', R') = 0$ for $1 \leq i \leq \max\{2, \text{edim}(R)\}$ and $1 \leq j \leq \text{edim}(R)$, and that M' is R' -free if and only if M is R -free. As a result, we can assume that $R = R'$ and that $R = Q/\mathbf{x}Q$ for some Q -regular sequence $\mathbf{x} = x_1, \dots, x_c$. Since Q is complete and $\text{Ext}_R^2(M, M) = 0$, by Lemma 6.6 there is a Q -module L such that \mathbf{x} is an L -sequence and $L/\mathbf{x}L \cong M$. Using [17, Lemma 3.8], we obtain that $\text{Ext}_Q^i(L, L) = \text{Ext}_Q^j(L, Q) = 0$ for $1 \leq i \leq \max\{2, \text{edim}(R)\}$ and $1 \leq j \leq \text{edim}(R)$. Moreover, by [14, Lemma 3.1] it follows that \mathbf{x} is also a $\text{Hom}_Q(L, L)$ -regular sequence and that

$$\text{Hom}_R(M, M) \cong \text{Hom}_Q(L, L)/\mathbf{x} \text{Hom}_Q(L, L).$$

Since $\text{id}_Q(\text{Hom}_R(M, M)) < \infty$, it follows from [35, Exercise 4.3.3] that $\text{id}_Q(\text{Hom}_Q(L, L)) < \infty$. Note that $\text{Ext}_Q^i(L, L) = \text{Ext}_Q^i(L, Q) = 0$ for all $1 \leq i \leq \dim(Q)$ since $\dim(Q) = \dim(R) + c$ and since $c \leq \text{edim}(R) - \dim(R)$. It then follows from [26, Theorem 3.2] that L is free as a Q -module, and so M must be free as an R -module as $M \cong L/\mathbf{x}L$.

(2) The proof is similar to the first case. There exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ of codimension $c \leq \text{edim}(R) - \dim(R)$ with Q complete such that $\text{id}_Q(\text{Hom}_R(M, R) \otimes_R R') < \infty$. As in the first case, we can assume that $R = R'$ and that $R = Q/\mathbf{x}Q$ for some Q -regular sequence $\mathbf{x} = x_1, \dots, x_c$.

Since Q is complete and $\text{Ext}_R^2(M, M) = 0$, by Lemma 6.6 there is a Q -module L such that \mathbf{x} is an L -sequence and $L/\mathbf{x}L \cong M$. By [17, Lemma 3.8], we have that $\text{Ext}_Q^i(L, Q) = 0$ for $1 \leq i \leq \text{edim}(R)$. Furthermore, it follows by [14, Lemma 3.1] that \mathbf{x} is also a $\text{Hom}_Q(L, Q)$ -regular sequence and that

$$\text{Hom}_R(M, R) \cong \text{Hom}_Q(L, Q)/\mathbf{x} \text{Hom}_Q(L, Q).$$

As $\text{id}_Q(\text{Hom}_R(M, R)) < \infty$, it follows again from [35, Exercise 4.3.3] that $\text{id}_Q(\text{Hom}_Q(L, Q)) < \infty$. Note that $\text{Ext}_Q^i(L, Q) = 0$ for all $1 \leq i \leq \dim(Q)$ since $\dim(Q) = \dim(R) + c$ as $R = Q/(x_1, \dots, x_c)$ and since $c \leq \text{edim}(R) - \dim(R)$. The freeness of L then follows from [19, Corollary 2.10].

Finally, we show that R is Gorenstein in both these cases. In both cases we have proved that M is free and then $\text{CI}^*\text{-id}_R(R) < \infty$. Hence, R is Gorenstein by [24, Theorem 2.5]. \square

The following proposition can be compared to [19, Proposition 3.2].

Proposition 6.9. *Let M be an R -module with $\text{Ext}_R^i(M, M) = 0$ for $1 \leq i \leq \max\{2, \text{edim}(R) - \dim(R)\}$. If $\text{depth}(\text{Hom}_R(M, M)) = \text{depth}(R)$ and $\text{CI}^*\text{-id}_R(\text{Hom}_R(M, M)) < \infty$, then R is Gorenstein.*

Proof. Replacing R with its completion, we can assume that R is complete (see [30, Corollary 3.7]). In addition, we can assume that R is Cohen-Macaulay with canonical module ω , as there exists a finitely generated R -module of finite complete intersection injective dimension [24, Theorem 2.7].

There exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R'/\mathfrak{m}R'$ is artinian Gorenstein, Q is complete and $\text{id}_Q(\text{Hom}_R(M, M) \otimes_R R') < \infty$ of codimension $c \leq \text{edim}(R) - \dim(R)$ by Lemma 6.7. Since $R \rightarrow R'$ is flat, we have that $\text{id}_Q(\text{Hom}_{R'}(M', M')) < \infty$ and that $\text{depth}_{R'}(\text{Hom}_{R'}(M', M')) = \text{depth}_{R'} R'$. Moreover, as we assume $R'/\mathfrak{m}R'$ to be Gorenstein, it follows that R is Gorenstein if and only R' is Gorenstein, by [11, Theorem 16.4.36]. In view of all this, we can assume that $R = R'$ and that $R = Q/\mathbf{x}Q$ for some Q -regular sequence $\mathbf{x} = x_1, \dots, x_c$. Since Q is complete, by Lemma 6.6 there is a Q -module L such that \mathbf{x} is an L -sequence and $L/\mathbf{x}L \cong M$. By [17, Lemma 3.8], we have that $\text{Ext}_Q^i(L, L) = 0$ for $1 \leq i \leq \max\{2, \text{edim}(R) - \dim(R)\}$. Therefore, it follows by [14, Lemma 3.1] that \mathbf{x} is also a $\text{Hom}_Q(L, L)$ -regular sequence and that

$$\text{Hom}_R(M, M) \cong \text{Hom}_Q(L, L)/\mathbf{x} \text{Hom}_Q(L, L).$$

Then

$$\text{depth}_Q(\text{Hom}_Q(L, L)) = \text{depth}_R(\text{Hom}_R(M, M)) + c = \text{depth}_R(R) + c = \text{depth}_Q(Q)$$

and $\text{id}_Q(\text{Hom}_Q(L, L)) < \infty$, by [35, Exercise 4.3.3]. Therefore, it follows by [19, Proposition 3.2] that Q is Gorenstein and so is R . \square

Inspired by [19, Theorem 3.6] and [26, Theorem 6.3], we obtain the following new characterizations of Gorenstein local rings in terms of vanishing of certain Ext and finite complete intersection injective dimensions of Hom.

Corollary 6.10. *The following statements are equivalent:*

- (1) R is Gorenstein.
- (2) R admits a non-zero module M such that $\text{CI-id}_R(\text{Hom}_R(M, M)) < \infty$ and $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$.
- (3) R admits a non-zero module M such that $\text{CI-id}_R(M^*) < \infty$, $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$ and $\text{Ext}_R^{2j}(M, M) = 0$ for some $j > 0$.
- (4) R admits a non-zero module M such that $\text{CI}^*\text{-id}_R(\text{Hom}_R(M, M)) < \infty$, and $\text{Ext}_R^i(M, M) = \text{Ext}_R^j(M, R) = 0$ for all $1 \leq i \leq \max\{2, \text{edim}(R)\}$ and $1 \leq j \leq \text{edim}(R)$.

- (5) R admits a non-zero module M such that $\text{CI}^*\text{-id}_R(M^*) < \infty$, $\text{Ext}_R^2(M, M) = 0$, and $\text{Ext}_R^i(M, R) = 0$ for all $1 \leq i \leq \text{edim}(R)$.
- (6) R admits a module M such that $\text{Ext}_R^i(M, M) = 0$ for $1 \leq i \leq \max\{2, \text{edim}(R) - \dim(R)\}$,
- $$\text{depth}(\text{Hom}_R(M, M)) = \text{depth}(R) \text{ and } \text{CI}^*\text{-id}_R(\text{Hom}_R(M, M)) < \infty.$$

Proof. Note that the implications (1) \Rightarrow (2), (3), (4), (5) hold when we take $M = R$. The reverse implications (2) \Rightarrow (1), (3) \Rightarrow (1), (4) \Rightarrow (1), (5) \Rightarrow (1) and (6) \Rightarrow (1) follow from Theorem 6.4((1) and (2)), Theorem 6.8((1) and (2)) and Proposition 6.9, respectively. \square

APPENDIX A. PROOF OF LEMMA 4.10

Let (R, \mathfrak{m}, k) be a local ring. For an R -module M , not necessarily finitely generated, its *depth* is defined as

$$\text{depth}_R(M) = \inf\{i \geq 0 : \text{Ext}_R^i(k, M) \neq 0\}.$$

Unlike the case of finitely generated modules, in general the depth of a module is not always finite. On the other hand, the depth lemma holds for all short exact sequences of modules, regardless of finite generation (see the proof of [8, Proposition 1.2.9]). Considering this and [11, Theorem 19.3.6], one can prove the following:

Lemma A.1. *Let R be a Noetherian ring (not necessarily local) with finite Krull dimension, and $t = \sup\{\text{depth}(R_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}$. Let n be a non-negative number such that $n \geq \max\{1, t\}$. If $0 \rightarrow K \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0$ is an exact sequence of modules where $\text{Gfd}_R(K) < \infty$ and each G_i is Gorenstein flat, then K is Gorenstein flat.*

We use Lemma A.1 in the proof of the following acyclicity result.

Lemma A.2. *Let R be a Noetherian ring (not necessarily local) with finite Krull dimension. Let (X, ∂) be an acyclic complex of Gorenstein flat R -modules. If $\text{coker}(\partial_n)$ has finite Gorenstein flat dimension for some n , then $\text{coker}(\partial_j)$ is a Gorenstein flat module for each j ; and if in addition:*

- (1) *If I is an injective R -module, then $X \otimes_R I$ is acyclic.*
- (2) *If R is Cohen-Macaulay local with a canonical module ω , then $X \otimes_R \omega$ is acyclic.*

Proof. Since X is acyclic, for any $i, j \in \mathbb{Z}$ such that $j \leq i$ we have an exact diagram

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & X_j & \xrightarrow{\partial_j} & X_{j-1} & \xrightarrow{\partial_{j-1}} & X_{j-2} & \rightarrow & \cdots & \rightarrow & X_{i-1} & \xrightarrow{\partial_i} & X_{i-2} & \rightarrow & \cdots \\ & & \searrow & & \swarrow & & \searrow & & & & \searrow & & \swarrow & & \searrow & & \cdots \\ & & & & \text{coker}(\partial_j) & & & & & & & \text{coker}(\partial_i) & & & & & \\ & & 0 & \nearrow & & 0 & \nearrow & & & & 0 & \nearrow & & 0 & \nearrow & & 0 \end{array}$$

From the two-out-of-three property for Gorenstein flat dimension ([11, Proposition 9.3.25]), note that if one of $\text{coker}(\partial_i)$ or $\text{coker}(\partial_j)$ has finite Gorenstein flat dimension, then the other also has finite Gorenstein flat dimension. Thus, $\text{coker}(\partial_j)$ has finite Gorenstein dimension for all j .

Set $t = \sup\{\text{depth}(R_{\mathfrak{p}}) : \mathfrak{p} \in \text{Spec}(R)\}$. Now, fix $j \in \mathbb{Z}$ and take i such that $j > t + i + 1$, in the diagram, we obtain an exact sequence

$$0 \rightarrow \text{coker}(\partial_j) \rightarrow X_{j-2} \rightarrow \cdots \rightarrow X_i.$$

It follows from Lemma A.1 that $\text{coker}(\partial_j)$ is Gorenstein flat. Thus, the first assertion is proved.

For proving item (1), by [11, Lemma 9.3.7(1)], we have that $\text{Tor}_i^R(I, \text{coker}(\partial_j)) = \text{Tor}_i^R(I, X_j) = 0$ for all i, j . Then it follows from [11, Proposition A.10] that $X \otimes_R I$ is acyclic.

The proof of item (2) is similar as that of item (1), using that fact that if L is a Gorenstein flat module, then $\text{Tor}_i^R(\omega, L) = 0$ for all $i > 0$, see [9, Theorems 3.4.6 and 5.2.5]. \square

Lemma A.3. *Let R be a Noetherian ring (not necessarily local). Let G be a complex of R -modules and $\Phi : I \rightarrow J$ a quasi-isomorphism of bounded complexes such that $\text{Tor}_R^m(I_i \oplus J_j, G_v) = 0$ for all $m > 0$ and all $i, j, v \in \mathbb{Z}$. Then $\Phi \otimes_R G$ is a quasi-isomorphism, and therefore $I \otimes_R G \cong J \otimes_R G$.*

Proof. We must prove that $\text{cone}(\Phi) \otimes_R G \cong \text{cone}(\Phi \otimes_R G)$ is acyclic. Set $X = \text{cone}(\Phi)$. Then X is a bounded acyclic complex. By [11, Proposition A.11], it is enough to show that $X \otimes_R G_v$ is acyclic for every $v \in \mathbb{Z}$.

Fix $v \in \mathbb{Z}$. By assumption, $\text{Tor}_i^R(G_v, X_l) = 0$ for all $l \in \mathbb{Z}$.

For each l , denote by C_l^X the cokernel of the differential $X_{l+1} \rightarrow X_l$. If $s = \inf X^\natural$, then $C_l^X = 0$ for all $l \leq s$. In addition, by [9, Lemma 4.1.7(c)], we have that $\text{Tor}_1^R(G_v, C_l^X) \cong \text{Tor}_{1+l-s}^R(G_v, C_s^X) = 0$ for all $l \geq s$. Thus, again by [9, Lemma 4.1.7(c)], $X \otimes_R G_v$ is acyclic. \square

Lemma A.4. *Let R be a Noetherian ring with a normalized dualizing complex D , and $t = \text{depth}(R)$. Let r be an integer. Let (C, ∂_i) be a complex of Gorenstein flat R -modules such that $H_i(C) = 0$ for all $i \geq r$. Assume that $\text{coker}(\partial_n)$ has finite Gorenstein flat dimension for some $n \geq r$ and that $C_{\mathfrak{p}}$ is an acyclic complex at each non-maximal prime ideal \mathfrak{p} . If each C_i is finitely generated or R is Cohen-Macaulay, then $H_i(C \otimes D) = 0$ for all $i \geq t + r$.*

Proof. Throughout this proof, we will use the normalized minimal semi-injective replacement for D as described in [11, Corollary 18.2.36]. Actually, in view of Lemma A.3, we may assume that

$$D_v = \bigoplus_{\dim(R/\mathfrak{p})=v} E(R/\mathfrak{p})$$

for each integer v .

Since $H_i(C) = 0$ for all $i \geq r$, then the complex

$$\cdots \rightarrow C_{j+1} \rightarrow C_j \rightarrow \cdots \rightarrow C_r \rightarrow C_{r-1}$$

is an exact sequence. Let \mathfrak{p} be a non-maximal prime ideal. Then $\text{coker}((\partial_n)_{\mathfrak{p}})$ has finite Gorenstein flat dimension over $R_{\mathfrak{p}}$. Since $C_{\mathfrak{p}}$ is a acyclic complex of Gorenstein flat $R_{\mathfrak{p}}$ -modules, then by Lemma A.2,

$$C \otimes_R E(R/\mathfrak{p}) \simeq C_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} E(R/\mathfrak{p}) \simeq 0.$$

Thus, $C \otimes_R D_i \simeq 0$ for all $i \neq 0$. Now, consider the natural short exact sequence

$$\begin{array}{ccc} D(i) : & (0 \longrightarrow 0 \longrightarrow D_i \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow 0) \\ & \downarrow & \parallel & \parallel & \parallel \\ D(i+1) : & (0 \longrightarrow D_{i+1} \longrightarrow D^i \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow 0) \\ & \parallel & \downarrow & \downarrow & \downarrow \\ & (0 \longrightarrow D_{i+1} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0). \end{array}$$

Then $C \otimes_R D(0) \simeq C \otimes_R D(1) \simeq \cdots \simeq C \otimes_R D(d) = C \otimes_R D$, and hence

$$H_i(C \otimes_R D) \cong H_i(C \otimes_R D_0).$$

Now, we prove that $\text{coker}(\partial_{r+t+2})$ is Gorenstein flat. If $t = 0$, this follows from [11, Theorem 19.3.6] or [11, Theorem 1.4.8].

Assume $t > 0$. Since the complex above is an exact sequence, we have two exact sequences

$$0 \rightarrow \text{coker}(\partial_{r+t+2}) \rightarrow C_{r+t} \rightarrow \cdots \rightarrow C_{r-1} \rightarrow \text{coker}(\partial_r) \rightarrow 0$$

and

$$\cdots \rightarrow C_{r+t+2} \rightarrow C_{r+t+1} \rightarrow \text{coker}(\partial_{r+t+2}) \rightarrow 0.$$

By the Lemma A.1 or depth Lemma, the first sequence shows that $\text{coker}(\partial_{r+t+2})$ is Gorenstein flat. Then considering the second exact sequence and applying Lemma A.2(1), we obtain that

$$\cdots \rightarrow C_{r+t+2} \otimes_R D_0 \rightarrow C_{r+t+1} \otimes_R D_0 \rightarrow \text{coker}(\partial_{r+t+2}) \otimes_R D_0 \rightarrow 0$$

is an exact sequence. Therefore, $H_i(C \otimes_R D) \cong H_i(C \otimes_R D_0) = 0$ for all $i \geq r + t$. \square

Proof of Lemma 4.10. By(1), there is a complex P of finitely generated projective modules

$$P : \cdots \rightarrow \cdots \rightarrow P_{r+1} \rightarrow P_r \rightarrow 0$$

and a quasi-isomorphism $P \xrightarrow{\sim} L_{\geq r}$. Let $\Phi : P \rightarrow L$ be the composition with the natural morphism $L_{\geq r} \rightarrow L$. Set $C = \text{cone}(\Phi)$. Then there is an exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow \Sigma P \rightarrow 0.$$

Tensoring with D , we obtain an exact sequence

$$0 \rightarrow L \otimes_R D \rightarrow C \otimes_R D \rightarrow \Sigma(P \otimes_R D) \rightarrow 0.$$

Since $H\text{-dim}_R(P) < \infty$, note that $P \otimes_R D \simeq P \otimes_R^L D$ has finite Hid dimension, see [11, Theorem 10.3.8] and [30, Theorem E]. Thus, if $C \otimes_R D$ is homologically trivial, then $L \otimes_R D \simeq (P \otimes_R D)$, and we will done. Since $H_i(P \otimes_R D) = 0$ for all $i < t + r$ by (3), we have that $H_i(C \otimes_R D) = 0$ for all $i < t + r$. Thus, it is enough to show that $H_i(C \otimes_R D) = 0$ for all $i \geq t + r$.

Note that C is a complex of Gorenstein flat modules. Moreover, by assumption, C is a complex of finitely generated R -modules or R is Cohen-Macaulay. We also have that $H_i(C) = 0$ for all $i \geq r$ because Φ induces isomorphism in the homologies of degree $\geq r$. Also, $H_i(C) \cong H_i(L)$ for $i < r$.

Let f_i the i th differential of C . Since L is bounded above, then for $i \ll 0$, f_i is the map $P_i \rightarrow P_{i-1}$, and thus $\text{coker}(f_i)$ has finite Gorenstein flat dimension. By item (2), note that $H_i(C_{\mathfrak{p}}) = 0$ for all i and for each non-maximal prime ideal \mathfrak{p} . Then the conclusion follows from Lemma A.4. \square

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