

CRITICAL LEVEL-SET PERCOLATION ON FINITE GRAPHS AND SPECTRAL GAP

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ABSTRACT. We study the bond percolation on finite graphs induced by the level-sets of zero-average Gaussian free field on the associated metric graph above a given height (level) parameter $h \in \mathbb{R}$. We characterize the near- and off-critical phases of this model for any expanders family $\mathcal{G}_n = (V_n, E_n)$ with uniformly bounded degrees. In particular, we show that the volume of the largest open cluster at level h_n is of the order $|V_n|^{\frac{2}{3}}$ when h_n lies in the corresponding *critical window* which we identify as $|h_n| = O(|V_n|^{-\frac{1}{3}})$. Outside this window, the volume starts to deviate from $\Theta(|V_n|^{\frac{2}{3}})$ culminating into a linear order in the supercritical phase $h_n = h < 0$ (the giant component) and a logarithmic order in the subcritical phase $h_n = h > 0$. We deduce these from effective estimates on tail probabilities for the maximum volume of an open cluster at any level h for a generic base graph \mathcal{G} . The estimates depend on \mathcal{G} only through its size and upper and lower bounds on its degrees and spectral gap respectively. To the best of our knowledge, this is the first instance where a mean-field critical behavior is derived under such general setup for finite graphs. The generality of these estimates preclude any local approximation of \mathcal{G} by regular infinite trees — a standard approach in the area. Instead, our methods rely on exploiting the connection between spectral gap of the graph \mathcal{G} and its connection to the level-sets of zero-average Gaussian free field mediated via a set function we call the *zero-average capacity*.

1. INTRODUCTION

1.1. Background and motivation. The study of percolation on finite graphs has a long and rich history which has spanned several areas in mathematics. See the book [10] for a classical introduction to the topic and the survey article [66] along with the references therein for more recent developments in the context of *random graph models*. More general sequences of finite graphs were considered in [5, 11, 12]; see [45, 36, 43, 8, 9] for some recent works in this area. Beyond Bernoulli percolation, a broad class of models on infinite transient graphs with *long range correlations* has attracted extensive attention over the past two decades. Notable examples include the random interlacements and its vacant set (see, e.g., [63, 64, 49, 55, 22, 33, 34, 35, 40, 39]) and the level-set percolation of the Gaussian free field (see, for instance, [58, 65, 26, 25, 38, 32, 29, 37, 19, 70]) among others. It is therefore natural to study analogous models on finite graphs. Two such instances are the vacant sets of random walk [63, 69, 24, 68] and level-sets of the *zero-average Gaussian free field* (GFF) [1, 3, 67, 23]. In the current article, we are interested in the latter.

Introduced in the context of studying percolation of GFF level-sets on discrete tori in [1], the zero-average Gaussian free field is a “suitable” version of the free field that can be defined on any finite graph. See the recent work [41] regarding the extremal behavior of this process. In [3] (see also [2]), A. Abächerli and J. Černý looked into the level-set percolation of zero-average Gaussian free field on a certain class of finite *locally tree-like* regular graphs which includes the *large girth regular* expanders and typical realizations of *random regular* graphs. Subsequently,

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the supercritical phase of the model was analyzed for these families of graphs in [67] by J. Černý and in [23] by G. Conchon-Kerjan. In the current work, we initiate the study of the critical phase of (a variant of) this model on *any* sequence of finite graphs with growing sizes provided they have uniformly bounded degrees and uniformly positive spectral gaps.

Bernoulli percolation on such families of graphs, i.e., the uniform expanders, were studied by N. Alon, I. Benjamini and A. Stacey in [5] where they were primarily interested in the existence and uniqueness of giant clusters. Under the *additional* assumptions that these graphs are d -regular with diverging girth, they identified the critical probability for the emergence of a giant cluster as $\frac{1}{d-1}$ which is same as the critical probability for Bernoulli percolation on (infinite) d -regular trees. The *large girth* assumption is crucial as it allows to approximate neighborhoods of the graph of large radii with those of the regular tree. This was generalized in the very recent work [4]. Results similar to [5] were proved for the vacant sets of random walk in [69] (see also [24]) and the level-sets of the zero-average Gaussian free field in [67, 23].

The *mean-field* (near-)critical behavior for the Bernoulli percolation on sequences of *random regular* graphs was established by A. Nachmias and Y. Peres in [53] and by B. Pittel in [56]. J. Černý and A. Teixeira obtained similar results in [68] for the vacant set of random walk on random regular graphs. In [68], the authors crucially use an observation made by C. Cooper and A. Frieze from [24] whereby the vacant set of random walk on random regular graphs can be seen as an average (distributionally) over random graphs with prescribed (random) degree sequences whose critical regime is very well-understood (see, e.g. [42, 44]). See also [7] for more refined results on the geometry of critical clusters of the vacant set. The setting of random regular graphs thus plays an extremely important role in all these results.

One of the main novel features of the current article is that we derive mean-field bounds for a version of the level-set percolation of zero-average Gaussian free field on any finite graph $\mathcal{G} = (V, E)$ that only involve the size $|V|$ and upper and lower bounds on the degrees and the spectral gap of \mathcal{G} . To the best of our knowledge, this is the *first* instance where a mean-field characterization of the critical regime is established in such generality and precision for a percolation model (conjecturally) in the Erdős-Rényi universality class.

The version of the level-set percolation alluded to in the previous paragraph refers to the bond percolation model on a (finite) graph $\mathcal{G} = (V, E)$ defined as follows. Given a realization of the zero-average Gaussian free field $(\varphi(x) : x \in V)$ (see §1.2 below for definition) on the vertices of \mathcal{G} and $h \in \mathbb{R}$, we *open* each edge $\{x, y\} \in E$ independently (at level h) with conditional probability

$$(1.1) \quad 1 - \exp(-2(\varphi(x) - h)_+(\varphi(y) - h)_+) \text{ where } a_+ \stackrel{\text{def.}}{=} \max(a, 0) \text{ for } a \in \mathbb{R}$$

(cf. display (1.6) in [29] and also [13], Chap.IV, §26, p.67). This “rule” corresponds to including the edge (x, y) in the resulting random graph iff the value of the field φ , extended suitably to the metric graph $\tilde{\mathcal{G}}$ associated to \mathcal{G} where each edge is replaced by a unit interval connecting its two endpoints (see Section 2 for precise definitions), stays above level h .

The study of level-set percolation on metric graphs originated in the work [50] by T. Lupu for *transient graphs*. In recent years, there has been spectacular progress in this area for a class of transient graphs with *polynomial growth* which includes \mathbb{Z}^d , $d \geq 3$; see, e.g. [26, 25, 27, 29, 28, 37, 15, 30, 16, 14, 19, 70, 21, 17, 18, 20, 31]. See also the very recent articles [59, 60] where the authors consider level-set percolation on some finite subgraphs of \mathbb{Z}^2 and \mathbb{Z}^3 for the (metric graph) free field associated to the *transient* random walk killed at a rate depending on the size and geometry of the subgraphs. However, the zero-average Gaussian free field — which can be defined in a rather canonical manner on finite graphs — exhibit new aspects for the level-set percolation, not least due to the global zero-average constraint (see (2.30)) imposed by the *recurrence* of the underlying random walk. On the other hand, the existing works on

the level-set percolation of zero-average Gaussian free field (or, for that matter, the Bernoulli percolation, the vacant set of random walk etc.) on finite graphs require different degrees of control on the local geometry of the graph (see, e.g., [1, 3, 67, 53, 69, 68]). Unfortunately the setup under which we work in this paper does not allow any such control. Instead, we develop an approach that is based on “global” properties of the graph accessible only through its maximum degree and the spectral gap. We refer to §1.3 for an overview of our techniques.

1.2. Main results. For any finite (simple, connected) graph $\mathcal{G} = (V, E)$, the corresponding (discrete) *zero-average Gaussian free field* is a centered Gaussian process $\varphi_{\mathcal{G}} = (\varphi(x) : x \in V)$ with covariances given by the *zero-average Green function* $g_{\mathcal{G}}(\cdot, \cdot)$ on \mathcal{G} (see, e.g., displays (2.3)–(2.5) in [67, Section 2]). Now for any $h \in \mathbb{R}$, let $E^{\geq h} \subset E$ denote the subset of open edges at level h obtained as per the rule described around (1.1) above and $\mathcal{G}^{\geq h} = (V, E^{\geq h})$ denote the (random) subgraph induced by these edges. Also let $\mathcal{C}_x^{\geq h} = \mathcal{C}_x^{\geq h}(\mathcal{G})$ denote the *cluster* (i.e., the connected component) of $x \in V$ in the graph $\mathcal{G}^{\geq h}$ and $\mathcal{C}_{\max}^{\geq h} = \mathcal{C}_{\max}^{\geq h}(\mathcal{G})$ denote “a” cluster with *maximum volume* (chosen arbitrarily). Our first main result in this paper identifies the “critical window” around $h = 0$ in terms of the order of $|\mathcal{C}_{\max}^{\geq h}|$. In the sequel, we let $d_{\mathcal{G}}^*$ and $\lambda_{\mathcal{G}}^*$ denote the maximum degree and the spectral gap of \mathcal{G} (see (2.7) in Section 2 for definition) respectively and $\mathbb{P}_{\mathcal{G}}$ denote probability measure underlying everything, i.e., the field $\varphi_{\mathcal{G}}$ along with the variables $\{1_{\{e \in E^{\geq h}\}} : e \in E, h \in \mathbb{R}\}$.

Theorem 1.1 (Mean-field behavior inside the critical window). *Let $d > 1$, $\lambda > 0$ and $\mathcal{G} = (V, E)$ be any finite graph with $d_{\mathcal{G}}^* \leq d$ and $\lambda_{\mathcal{G}}^* \geq \lambda$. Then there exists $C = C(d, \lambda) \in (0, \infty)$ such that for any $h = A|V|^{-1/3}$ with $A \in \mathbb{R}$ and $\delta \in (0, 1)$ satisfying $|V| \geq \frac{(1+|A|)^8}{\delta^3}$, we have*

$$(1.2) \quad \mathbb{P}_{\mathcal{G}}[\delta|V|^{2/3} \leq |\mathcal{C}_{\max}^{\geq h}| \leq \frac{1}{\delta}|V|^{2/3}] \geq 1 - C(1 + |A|)\delta^{1/5}.$$

Our second main result concerns the order of $|\mathcal{C}_{\max}^{\geq h}|$ when h is farther away from 0.

Theorem 1.2 (Mean-field behavior outside the critical window). *Under the same assumptions as in Theorem 1.1, there exist $C = C(d, \lambda)$ and $c = c(d, \lambda)$ in $(0, \infty)$ such that for any $h = A|V|^{-1/3}$ with $A \in (1, c|V|^{1/3})$, we have*

$$(1.3) \quad \mathbb{P}_{\mathcal{G}}[|\mathcal{C}_{\max}^{\geq h}| \leq C \frac{\log(|V|h^3)}{h^2}] \geq 1 - \frac{C}{A}.$$

On the other hand, if $h = -A|V|^{-1/3}$ with $A \in (1, |V|^{1/3})$, we have

$$(1.4) \quad \mathbb{P}_{\mathcal{G}}[|\mathcal{C}_{\max}^{\geq h}| \geq c|h||V|] \geq 1 - \frac{C}{A^{1/5}}.$$

The typical order of $|\mathcal{C}_{\max}^{\geq h}|$ implied by these two theorems match those for the Erdős-Rényi random graphs and Bernoulli percolation on random regular graphs; cf., for instance, [53, 56] and see also the references therein.

As an immediate corollary of these results, we can identify the *critical parameter* for the emergence of a giant cluster in this percolation model on any uniform expanders sequence with uniformly bounded degrees as 0 and also characterize the nature of its phase transition as follows.

Theorem 1.3 (Nature of phase transition on expanders). *Let $d > 1$ and $\lambda > 0$. Also let $\mathcal{G}_n = (V_n, E_n)$ be a sequence of finite graphs such that $d_{\mathcal{G}_n}^* \leq d$ and $\lambda_{\mathcal{G}_n}^* \geq \lambda$ for all $n \geq 1$, and $|V_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then for any $h < 0$, there exists $c(h) = c(h, d, \lambda) \in (0, \infty)$ such that*

$$(1.5) \quad \lim_{n \rightarrow \infty} \mathbb{P}_{\mathcal{G}_n}[|\mathcal{C}_{\max}^{\geq h}| \geq c(h)|V_n|] = 1 \quad (\text{supercritical phase}).$$

When $h > 0$, there exists $C(h) = C(h, d, \lambda) \in (0, \infty)$ such that

$$(1.6) \quad \lim_{n \rightarrow \infty} \mathbb{P}_{\mathcal{G}_n}[|\mathcal{C}_{\max}^{\geq h}| \leq C(h) \log |V_n|] = 1 \quad (\text{subcritical phase}).$$

Finally for $h = 0$, we have

$$(1.7) \quad \lim_{n \rightarrow \infty} \mathbb{P}_{\mathcal{G}_n} [\delta |V|^{2/3} \leq |\mathcal{C}_{\max}^{\geq h}| \leq \frac{1}{\delta} |V|^{2/3}] = 1 \quad (\text{critical phase}).$$

Theorems 1.1 and 1.2 follow from more general bounds on the *tail probabilities* for $|\mathcal{C}_{\max}^{\geq h}|$, namely Theorems 7.1 and 7.2 which are stated and proved in Section 7.

1.3. Proof outline. We now present a brief account of the key ideas that go into the proof of our main results.

- **Dirichlet martingale for exploration of clusters.** It is often useful in the study of percolation clusters to “attach” a suitably defined martingale to a cluster exploration process and analyze its properties. See, e.g., [53, 54] for Bernoulli percolation on random graphs and [25] for the percolation of GFF level-sets on the metric graphs associated with \mathbb{Z}^d . In our situation, we identify the right martingale to be given by the functional

$$K \mapsto \tilde{\mathcal{E}}(\varphi, f_K)$$

where K ranges over “nice” compact subsets of $\tilde{\mathcal{G}}$ (the metric graph associated to \mathcal{G}), $\tilde{\mathcal{E}}(\cdot, \cdot)$ is the Dirichlet form on $\tilde{\mathcal{G}}$ (see (2.9)), φ is the corresponding zero-average GFF and f_K minimizes the form $\tilde{\mathcal{E}}(f_K, f_K)$ subject to the conditions

$$f_{|K} \equiv 1 \text{ and } \sum_{x \in V} d_x f(x) = 0$$

where d_x is the degree of the vertex x in \mathcal{G} . The constraint $\sum_{x \in V} d_x f_K(x) = 0$ manifests the zero-average condition. Indeed, as we explain in §2.4, φ is the Gaussian free field on $\tilde{\mathcal{G}}$ with free boundary condition *normalized* so that $\sum_{x \in V} d_x \varphi(x) = 0$. From an exploration of clusters at a level h , we obtain a continuous family $(K_t)_{t \geq 0}$ of random (nice) compact subsets of $\tilde{\mathcal{G}}$ such that the process

$$(\tilde{M}_t)_{t \geq 0} \stackrel{\text{def.}}{=} (\mathcal{E}(\varphi, f_{K_t}))_{t \geq 0} \text{ is a } \textit{continuous} \text{ martingale w.r.t. the filtration of } (K_t)_{t \geq 0}.$$

- **Linking \tilde{M}_t to cluster size via zero-average capacity.** The process $(\tilde{M}_t)_{t \geq 0}$ carries important information on the cluster volumes. When K_t lies inside a cluster (at level h), we have

$$\tilde{M}_t \geq \tilde{\mathcal{E}}(f_{K_t}, f_{K_t}) h.$$

On the other hand, when K_t is “close” to a cluster, we have

$$\tilde{M}_t = \tilde{\mathcal{E}}(f_{K_t}, f_{K_t}) h + o_{\mathbb{P}}(1)$$

where the $o_{\mathbb{P}}(1)$ term converges to 0 in \mathbb{P} *uniformly* over all compact sets that are not too large. We call the quantity $\tilde{\mathcal{E}}(f_K, f_K)$ as the *zero-average capacity* of the set K (see below Remark 3.3 in Section 3 for the reason behind such naming). The bound on the spectral gap gives yields the following linear isoperimetric condition for the zero-average capacity:

$\tilde{\mathcal{E}}(f_K, f_K)$ grows linearly in the volume of K when K is not large.

- **Long excursion events for \tilde{M}_t and computation of probabilities.** Carefully using the observations made in the previous item, we can translate events involving the existence of large percolation clusters into events involving the existence of long excursions for \tilde{M}_t away from some suitably chosen value after being reparametrized by the zero-average capacity of K_t — which also turns out to be the *quadratic variation* of M_t . At this point, we can compute the corresponding probabilities as probabilities of similar events for Brownian motion using Dubin-Schwarz (cf. [25]).

We now describe the organization of this article. In Section 2, we gather some necessary definitions and notions including the construction of the metric graph $\tilde{\mathcal{G}}$ and the corresponding zero-average Gaussian free field as well as some elements of potential theory on \mathcal{G} and $\tilde{\mathcal{G}}$. We formally introduce the zero-average capacity in Section 3 and prove some of its important properties. In Section 4, we establish the martingale property of the functional $K \mapsto \tilde{\mathcal{E}}(\varphi, f_K)$ (see the discussion above) w.r.t. set inclusion and record some of its useful features. Section 6 is concerned with the exploration process $(K_t)_{t \geq 0}$ (see above) which we formally describe in §6.1 and the exploration martingale $(\tilde{M}_t)_{t \geq 0}$ which we define in §6.2 using the construction in Section 6. Finally in Section 7, we state Theorems 7.1 and 7.2 regarding the tail probabilities of cluster size from which we derive our main results, i.e., Theorems 1.1 and 1.2 in a fairly straightforward manner. The remainder of the section is devoted to the proof of Theorems 7.1 and 7.2.

Notations and conventions. In the upcoming sections, we will assume the graph $\mathcal{G} = (V, E)$ to be fixed and drop it from the notations $\mathbb{P}_{\mathcal{G}}, d_{\mathcal{G}}^*$ etc. unless they need to be distinguished in some context. In view of the formulation of our main results, we may assume $|V| > 1$ without any loss of generality. We also assume that $d_{\mathcal{G}}^* \leq d$ and $\lambda_{\mathcal{G}}^* \geq \lambda$ for some $d > 1$ and $\lambda > 0$. We denote by c, c', C, C', \dots etc. finite, positive constants which are allowed to vary from place to place. All constants may implicitly depend on d and λ . Their dependence on other parameters will be made explicit. Numbered constants c_1, c_2, \dots and C_1, C_2, \dots stay fixed after their first appearance within the text.

2. PRELIMINARIES

In this section we collect several definitions, preliminary facts and results that will be used throughout the paper.

2.1. The metric graph $\tilde{\mathcal{G}}$. Let us start with the *metric graph* (or the *cable system*) $\tilde{\mathcal{G}}$ associated to \mathcal{G} which is topologically the 1-complex obtained by gluing a copy of the open unit interval $I_{\{x,y\}}$ at its endpoints $x, y \in V$ for each $\{x, y\} \in E$. We denote the (topological) closure and boundary of any $S \subset \tilde{\mathcal{G}}$ as \overline{S} and ∂S respectively. For any $x, y \in \overline{I_e}$ where $e \in E$, we can define the *intervals* (of $\tilde{\mathcal{G}}$) with endpoints at x and y that are open, closed, semi-closed at x and semi-closed at y in the obvious manner and denote them as (the ordering of x, y is irrelevant)

$$(2.1) \quad (x, y)_{\mathcal{G}}, [x, y]_{\mathcal{G}}, [x, y)_{\mathcal{G}} \text{ and } (x, y]_{\mathcal{G}} \text{ respectively.}$$

Equipping each $I_e, e \in E$ with the Lebesgue measure (of total mass 1), we get the push-forward measure ρ on $\tilde{\mathcal{G}}$ which assigns a unit length to each interval I_e . The space $\tilde{\mathcal{G}}$ is metrizable. In fact, the graph distance $d(\cdot, \cdot)$ on \mathcal{G} extends naturally to a distance on $\tilde{\mathcal{G}}$ which we also denote as $d(\cdot, \cdot)$. More precisely, for any $x, y \in \tilde{\mathcal{G}}$, we let

$$(2.2) \quad d(x, y) = \min_{x', y'} (\rho([x, x']_{\mathcal{G}}) + d(x', y') + \rho([y', y]_{\mathcal{G}})),$$

where the minimum is over all pairs of endpoints of (closed) intervals containing x and y respectively. In plain words, $d(x, y)$ is the minimum length of a continuous path between x, y . For any $x \in \tilde{\mathcal{G}}$ and $r \geq 0$, we denote by $B(x, r)$ the ball of radius r centered at x .

The metric space $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}, d)$ is compact and locally path-connected. Further, any connected subset of $\tilde{\mathcal{G}}$

(2.3) is path-connected with a finite boundary set and is a union of finitely many intervals.

For any $x \in \tilde{\mathcal{G}}$ and $S \subset \tilde{\mathcal{G}}$, we let $\mathcal{C}_x(S)$ denote the (*connected*) component of x inside S (as subspaces of $\tilde{\mathcal{G}}$). Notice that $\mathcal{C}_x(S) = \emptyset$ if $x \notin S$. In order to avoid confusion, we use the term

cluster only for the connected components of the random subgraph $\mathcal{G}^{\geq h}; h \in \mathbb{R}$ which we denote by $\mathcal{C}_x^{\geq h}$ (see §1.2) and reserve the term *component* for subsets of $\tilde{\mathcal{G}}$. More generally, for any $S' \subset \tilde{\mathcal{G}}$, we let $\mathcal{C}_{S'}(S) \stackrel{\text{def.}}{=} \cup_{x \in S'} \mathcal{C}_x(S)$.

The following two special collections of subsets of $\tilde{\mathcal{G}}$ will appear frequently in this article. We let

$$(2.4) \quad \mathcal{K} \text{ denote the collection of all non-empty compact subsets of } \tilde{\mathcal{G}}.$$

We further let

$$(2.5) \quad \begin{aligned} \mathcal{K}_{<\infty} &\text{ denote the sub-collection of } \mathcal{K} \text{ containing only sets } K \\ &\text{having finitely many components and satisfying } V \setminus K \neq \emptyset. \end{aligned}$$

2.2. Excerpts from the potential theory on $\tilde{\mathcal{G}}$. Next we gather some useful facts from the potential theory on \mathcal{G} and $\tilde{\mathcal{G}}$. For $f, g \in \mathbb{R}^V$, the *Dirichlet form* on \mathcal{G} is given by

$$(2.6) \quad \mathcal{E}(f, g) \stackrel{\text{def.}}{=} \sum_{\{x,y\} \in E} (f(x) - f(y))(g(x) - g(y)).$$

The *spectral gap* λ^* of \mathcal{G} can be defined in terms of the Dirichlet form as follows:

$$(2.7) \quad \lambda^* = \inf \left\{ \mathcal{E}(f, f) : \sum_{x \in V} d_x f^2(x) = 1, \sum_{x \in V} d_x f(x) = 0 \right\}.$$

For $S \subset V$, let $\partial_e S \stackrel{\text{def.}}{=} \{e \in E : e \text{ has an endpoint in both } S \text{ and } V \setminus S\}$ denote the *edge boundary* of S in the graph \mathcal{G} . The following is a consequence of the Cheeger's inequality (see, e.g., [51, Theorem 6.15]) and our standing assumption that $\lambda^* \geq \lambda$,

$$(2.8) \quad |\partial_e S| \geq c|S| \text{ for all } S \subset V \text{ with } |S| \leq |V|/2.$$

In particular, this implies that the graph \mathcal{G} is connected. Letting $\tilde{H}^1(\mathcal{G})$ denote the subspace of \mathbb{R}^V consisting of all $f \in \mathbb{R}^V$ satisfying $\sum_{x \in V} d_x f(x) = 0$, we see that $\tilde{H}^1(\mathcal{G})$ equipped with the inner product $\mathcal{E}(\cdot, \cdot)$ is a Hilbert space.

The corresponding Dirichlet form on $\tilde{\mathcal{G}}$ is defined as

$$(2.9) \quad \tilde{\mathcal{E}}(f, g) = \int_{\tilde{\mathcal{G}}} f' g' d\rho \text{ for all } f, g \in H^1(\tilde{\mathcal{G}})$$

where $H^1(\tilde{\mathcal{G}})$ denotes the space of all real-valued, *absolutely continuous* functions f on $\tilde{\mathcal{G}}$ (i.e., $f|_{I_e}$ is absolutely continuous on I_e w.r.t. the measure $\rho|_{I_e}$ for each $e \in E$) such that $f' \in L^2(\tilde{\mathcal{G}}, \rho)$. Like $\tilde{H}^1(\mathcal{G})$, we let $\tilde{H}^1(\tilde{\mathcal{G}})$ denote the subspace of $H^1(\tilde{\mathcal{G}})$ consisting of all $f \in H^1(\tilde{\mathcal{G}})$ satisfying $\sum_{x \in V} d_x f(x) = 0$ so that it is a Hilbert space when equipped with the form $\tilde{\mathcal{E}}(\cdot, \cdot)$. Owing to the simple structure of the connected subsets of $\tilde{\mathcal{G}}$ as elucidated in (2.3), we can similarly define a Dirichlet form $\tilde{\mathcal{E}}_K(\cdot, \cdot)$ and the corresponding Hilbert spaces $H^1(K)$ and $\tilde{H}^1(K)$ associated to any $K \subset \tilde{\mathcal{G}}$ with finitely many components (the constraint for $\tilde{H}^1(K)$ being $\sum_{x \in V \cap K} d_x f(x) = 0$). To simplify notations, we let

$$(2.10) \quad \tilde{\mathcal{E}}_{K^c}(f, g) \stackrel{\text{def.}}{=} \tilde{\mathcal{E}}_{\overline{K^c}}(f|_{\overline{K^c}}, g|_{\overline{K^c}}) \text{ for all } f, g : \tilde{\mathcal{G}} \rightarrow \mathbb{R} \text{ s.t. } f|_{\overline{K^c}}, g|_{\overline{K^c}} \in H^1(\overline{K^c}).$$

Of special interests to us are those functions which are “linear” on intervals of $\tilde{\mathcal{G}}$. More precisely, for any $K \subset \tilde{\mathcal{G}}$, interval $[x, y]_{\mathcal{G}} \subset K$ and function $f : K \rightarrow \mathbb{R}$, we say that f is *linear* on $[x, y]_{\mathcal{G}}$ if

$$(2.11) \quad f(z) = f(x) \frac{\rho([z, y]_{\mathcal{G}})}{\rho([x, y]_{\mathcal{G}})} + f(y) \frac{\rho([x, z]_{\mathcal{G}})}{\rho([x, y]_{\mathcal{G}})}$$

for all $z \in [x, y]_{\mathcal{G}}$. Clearly if f is linear on $[x, y]_{\mathcal{G}}$, then f is absolutely continuous with a constant derivative on the same interval. More generally, given any collection \mathcal{I} of (closed) intervals of $K (\subset \tilde{\mathcal{G}})$, we say that

$$(2.12) \quad f \text{ is linear on } \mathcal{I} \text{ provided } f \text{ is linear on each } I \subset \mathcal{I}.$$

We call f as *piecewise linear* (on its domain K) if f is linear on \mathcal{I} for some *finite* collection \mathcal{I} satisfying $\cup_{I \in \mathcal{I}} I = K$. Piecewise linear functions on $\tilde{\mathcal{G}}$ are elements of $H^1(\tilde{\mathcal{G}})$.

The vocabulary of piecewise linear functions allow us to obtain the appropriate Dirichlet forms on *discrete subgraphs* of $\tilde{\mathcal{G}}$ as *traces* of $\tilde{\mathcal{E}}(\cdot, \cdot)$ (see, e.g. [46, Chapter 1.3]). To this end, we recall the framework of *enhancements* from [27, Section 2.2]. Let $K \subset \tilde{\mathcal{G}}$ be compact with finitely many components so that the boundary ∂K is a finite set by (2.3). The *enhancement set* induced by K is given by

$$(2.13) \quad V^K \stackrel{\text{def.}}{=} V \cup \partial K.$$

$\tilde{\mathcal{G}}$ can be expressed as a union of closed intervals with disjoint interior whose endpoints are points in V^K . In the sequel,

$$(2.14) \quad \begin{aligned} &\text{we refer to this collection of intervals as } \mathcal{I}_K \text{ and denote by } \mathcal{I}_{-K} \text{ the} \\ &\text{sub-collection comprising only intervals that are } \textit{not} \text{ subsets of } K \end{aligned}$$

(recall the structure of connected subsets of $\tilde{\mathcal{G}}$ from (2.3)). Now let E^K denote the set of all (unordered) pairs $\{x, y\}$ of points in V^K that are the endpoints of some interval in \mathcal{I}_K . Then for any $f, g \in H^1(\tilde{\mathcal{G}})$ that are linear on \mathcal{I}_K (recall (2.12)), we have in view of (2.9) and (2.11):

$$(2.15) \quad \tilde{\mathcal{E}}(f, g) = \sum_{\{x, y\} \in E^K} (f(x) - f(y))(g(x) - g(y)) \frac{1}{\rho([x, y]_{\mathcal{G}})}$$

More so, it is not difficult to see that for any such f ,

$$(2.16) \quad \tilde{\mathcal{E}}(f, f) = \min_g \tilde{\mathcal{E}}(g, g),$$

where the minimum is over all $g \in H^1(\tilde{\mathcal{G}})$ satisfying $g|_{V^K} = f|_{V^K}$. Thus we get the following *trace form* on the *enhancement graph* $\mathcal{G}^K \stackrel{\text{def.}}{=} (V^K, E^K)$ (induced by K):

$$(2.17) \quad \mathcal{E}^K(f, g) \stackrel{\text{def.}}{=} \sum_{\{x, y\} \in E^K} (f(x) - f(y))(g(x) - g(y)) C_{\{x, y\}}^K$$

for $f, g \in \mathbb{R}^{V^K}$ where

$$(2.18) \quad C_{\{x, y\}}^K \stackrel{\text{def.}}{=} \frac{1}{\rho([x, y]_{\mathcal{G}})} \text{ for } \{x, y\} \in E^K \text{ and } 0 \text{ otherwise.}$$

Henceforth, we will assume the graph $\mathcal{G}^K = (V^K, E^K)$ to be equipped with the conductances $C_e^K, e \in E^K$. The corresponding *degree* of $x \in V^K$ is given by

$$(2.19) \quad C_x^K \stackrel{\text{def.}}{=} \sum_{\{x, y\} \in E^K} C_{\{x, y\}}^K.$$

The display (2.15) also gives us a way to extend the domain of the form $\tilde{\mathcal{E}}(\cdot, \cdot)$ as follows. For any function f and a piecewise linear function g on $\tilde{\mathcal{G}}$, let us define

$$(2.20) \quad \tilde{\mathcal{E}}(f, g) = \mathcal{E}^K(f^K, g)$$

where K is the set of endpoints of the intervals in \mathcal{I} on which g is linear (see below (2.12)) and f^K is the piecewise linear extension of $f|_{V^K}$ to $\tilde{\mathcal{G}}$ (cf. (2.11)).

2.3. The canonical diffusions on $\tilde{\mathcal{G}}$ and \mathcal{G} . Now one can define, for each $x \in \tilde{\mathcal{G}}$, a ρ -symmetric diffusion process $(\tilde{X}_t)_{t \geq 0}$ starting from x with state space $\tilde{\mathcal{G}}$ via its associated Dirichlet form $\tilde{\mathcal{E}}(\cdot, \cdot)$ which we can view as the $\tilde{\mathcal{G}}$ -valued Brownian motion. See [27, §2.1] and [50, Section 2] for a detailed account of this construction. We denote its law (and the corresponding expectation) by \mathbf{P}_x (and \mathbf{E}_x respectively). For $K \subset \tilde{\mathcal{G}}$, we let

$$(2.21) \quad H_K \stackrel{\text{def.}}{=} \inf\{t \geq 0 : \tilde{X}_t \in K\} \text{ and } T_K \stackrel{\text{def.}}{=} H_{K^c}$$

denote the hitting and exit time of the set K respectively. One can show (see [27, §2.1] and [50, Section 2]) that the process $\tilde{X} = (\tilde{X}_t)_{t \geq 0}$ admits of a continuous family of local times $(\ell_x(t))_{x \in \tilde{\mathcal{G}}, t \geq 0}$ under \mathbf{P}_x . Then for any open $U \subset \tilde{\mathcal{G}}$, we can define the corresponding *Green function* as

$$(2.22) \quad g_U(x, y) (= g_U(y, x)) = \mathbf{E}_x[\ell_y(T_U)] \text{ for all } x, y \in \tilde{\mathcal{G}}.$$

Let us make two useful observations on the Green functions which are standard (see, e.g., [50, display (2.1)]). For any $K \in \mathcal{K}$ (recall (2.4)) with finitely many components and $x \in \tilde{\mathcal{G}}$,

$$(2.23) \quad g_{K^c}(x, y) \text{ is linear on } \mathcal{I}_{-(K \cup \{x\})} \text{ (recall (2.14)) as a function in } y \in \tilde{\mathcal{G}}.$$

Further, with the supremum ranging over all $K \in \mathcal{K}$ with finitely many components,

$$(2.24) \quad G \stackrel{\text{def.}}{=} \sup_K g_{K^c}(x, y) < \infty$$

Now by taking *traces* of the process \tilde{X} on V^K (see, e.g., §2.1 and §2.2 in [27] for a formal treatment) we obtain, for each compact K with finitely many connected components, the canonical jump processes $(\tilde{X}_t^K)_{t \geq 0}$ on \mathcal{G}^K with jump rates $C_{\{x,y\}}^K, \{x, y\} \in E^K$ (see (2.18)) under the probability measures $\mathbf{P}_x, x \in V^K$. For any $K' \subset \tilde{\mathcal{G}}$, with a slight abuse of notation, we use $H_{K'}$ and $T_{K'}$ as in (2.21) to denote the hitting and exit time of the set $K' \cap V^K$ for \tilde{X}^K . As explained below display (2.4) in [27, §2.1], we have for any $K' \in \mathcal{K}$ with finitely many components such that $\partial K' \subset V^K$,

$$(2.25) \quad \begin{aligned} g_{(K')^c}(x, y) &= \mathbf{E}_x \left[\int_0^\infty 1_{\{\tilde{X}_t^K=y, t < H_{K'}\}} dt \right] \text{ and} \\ \mathbf{P}_x[\tilde{X}_{H_{K'}}^K = y] &= \mathbf{P}_x[\tilde{X}_{H_{K'}} = y] \text{ for } x, y \in V^K. \end{aligned}$$

Denoting the discrete skeleton of the continuous-time chain \tilde{X}^K by $X^K = (X_n^K)_{n \in \mathbb{N}}$, we have the following identities in view of (2.25) under the same setup:

$$(2.26) \quad \begin{aligned} g_{(K')^c}(x, y) &= \frac{g_{(K')^c}^K(x, y)}{C_y^K} \text{ where } g_{(K')^c}^K(x, y) \stackrel{\text{def.}}{=} \sum_{n \geq 0} \mathbf{P}_x[X_n^K = y, n < H_{K'}] \text{ and} \\ \mathbf{P}_x[\tilde{X}_{H_{K'}}^K = y] &= \mathbf{P}_x[X_{H_{K'}}^K = y] \text{ for } x, y \in V^K. \end{aligned}$$

Here, similarly as before, we use $H_{K'}$ to denote the hitting time of the set $K' \cap V^K$ by the discrete-time walk X^K to avoid clutter. The (discrete) *Laplacian* operator on \mathcal{G}^K corresponding to the walk X^K is given by

$$(2.27) \quad \Delta^K f(x) \stackrel{\text{def.}}{=} \sum_{\{x,y\} \in E^K} (f(y) - f(x)) C_{\{x,y\}}^K \text{ for all } f \in \mathbb{R}^{V^K} \text{ and } x \in V^K.$$

On the other hand, the Laplacian *killed* outside K is defined as

$$(2.28) \quad \Delta^K K^c f(x) = \sum_{\substack{\{x,y\} \in E^K, \\ y \in V^K \setminus K}} f(y) C_{\{x,y\}}^K - f(x) \left(\sum_{\{x,y\} \in E^K} C_{\{x,y\}}^K \right)$$

for all $f \in \mathbb{R}^{V^K \setminus K}$ and $x \in V^K \setminus K$. Often we need to work with a *common refinement* of enhancement sets induced by two sets K and K' in \mathcal{K} with finitely many connected components. To this end, we let

$$(2.29) \quad V^{K,K'}, E^{K,K'}, C_{\{\cdot,\cdot\}}^{K,K'}, C_{\{\cdot\}}^{K,K'} \text{ and } g^{K,K'}(\cdot, \cdot) \text{ denote the objects defined above corresponding to the compact set } K \cup \partial K' \text{ (which has finitely many components).}$$

2.4. The zero-average Gaussian free fields on \mathcal{G} and $\tilde{\mathcal{G}}$. Finally we come to the zero-average Gaussian free field on \mathcal{G} (rather the vertex set V) and its extension to the metric graph $\tilde{\mathcal{G}}$. Let us recall from the introduction that the *zero-average Gaussian free field* on the vertices of \mathcal{G} is a centered Gaussian process $(\varphi(x) : x \in V)$ with covariances given by the *zero-average Green function* $g_{\mathcal{G}}(\cdot, \cdot)$ on \mathcal{G} (see, for instance, displays (2.3)–(2.5) in [67, Section 2]). The field values $(\varphi(x) : x \in V)$ satisfy the following “zero-average” property (cf. [67, (1.2)]):

$$(2.30) \quad \sum_{x \in V} d_x \varphi(x) = 0, \quad \mathbb{P}\text{-a.s.}$$

where d_x is the degree of the vertex $x \in V$. Enlarging the underlying probability space if necessary (with the probability measure \mathbb{P}), we extend φ to the metric graph $\tilde{\mathcal{G}}$ (also denoted as φ) as follows. Conditionally on $(\varphi(x) : x \in V)$, the processes $(\varphi(x) : x \in \overline{I_{\{y,z\}}})$, $\{y, z\} \in E$ joining $\varphi(y)$ and $\varphi(z)$. See [50, Section 2] for further details. As already noted in the introduction, $e \in E^{\geq h}$ (see the beginning of §1.2) if and only if $\varphi(x) \geq h$ for all $x \in I_e$. Accordingly, for any $h \in \mathbb{R}$, we define the *level-set above height h* as

$$(2.31) \quad \tilde{\mathcal{G}}^{\geq h} = \{x \in \tilde{\mathcal{G}} : \varphi(x) \geq h\}.$$

The sets $\tilde{\mathcal{G}}^{\leq h}$, $\tilde{\mathcal{G}}^{< h}$ etc. are defined in a similar manner. Notice that φ is continuous as a function on $\tilde{\mathcal{G}}$ and hence $\tilde{\mathcal{G}}^{\geq h}$ is a compact subspace of $\tilde{\mathcal{G}}$. We denote the components in $\mathcal{G}^{\geq h}$, i.e., $\mathcal{C}_x(S \cap \tilde{\mathcal{G}}^{\geq h})$ and $\mathcal{C}_{S'}(S \cap \tilde{\mathcal{G}}^{\geq h})$ (see below (2.3)) as $\mathcal{C}_x^{\geq h}(S)$ and $\mathcal{C}_{S'}^{\geq h}(S)$ respectively.

The field φ is, in fact, the *canonical Gaussian free field* on the Hilbert space $\tilde{H}^1(\tilde{\mathcal{G}})$ equipped with the Dirichlet form $\tilde{\mathcal{E}}(\cdot, \cdot)$ (see (2.9) and the discussion around that). See, for instance, Section 2 in [62] and Section 1.7 of Chapter 1 in [6]. In particular,

$$(2.32) \quad \begin{aligned} \text{The process } (\tilde{\mathcal{E}}(\varphi, f) : f \in \tilde{H}^1(\tilde{\mathcal{G}})) \text{ (cf. (2.20)) is a } &\text{Gaussian Hilbert} \\ \text{space (see, e.g., [62, Definition 2.11]) of centered Gaussian variables with} \\ \text{covariances given by } \text{Cov}[\tilde{\mathcal{E}}(\varphi, f), \tilde{\mathcal{E}}(\varphi, g)] = \tilde{\mathcal{E}}(f, g) \text{ for all } f, g \in \tilde{H}^1(\tilde{\mathcal{G}}). \end{aligned}$$

Owing to the zero-average constraint (2.30), the field φ does *not* possess a *domain Markov* or *strong Markov* property in the traditional sense (see Section 1 in [3] or [67]). However, we can still characterize the conditional distributions of φ utilizing the Gaussian Hilbert space structure (see, e.g., [6, Theorem 1.52]). To this end, for any $U \subset \tilde{\mathcal{G}}$, let us consider the sigma-algebra $\mathcal{F}_U \stackrel{\text{def.}}{=} \sigma(\{\varphi(x) : x \in U\})$. A *random* set K taking values in $\mathcal{K}_{<\infty}$ (see (2.5)) satisfies

$$(2.33) \quad \{K \subset U\} \text{ is measurable for each } U \subset \tilde{\mathcal{G}}$$

and we define \mathcal{F}_K as the (sub-) σ -algebra generated by all events of the form $\{K \subset U\}$ with $U \subset \tilde{\mathcal{G}}$ open. Now for any $K \in \mathcal{K}_{<\infty}$ and function $\phi : K \rightarrow \mathbb{R}$, let

$$(2.34) \quad \begin{aligned} f_\phi : \mathcal{G} &\rightarrow \mathbb{R} \text{ denote the function minimizing } \tilde{\mathcal{E}}_{K^c}(f, f) \text{ (cf. (2.10)) over} \\ &\text{all } f : \tilde{\mathcal{G}} \rightarrow \mathbb{R} \text{ satisfying } f|_{\overline{K^c}} \in H^1(\overline{K^c}), f|_K = \phi \text{ and } \sum_{x \in V} d_x f(x) = 0. \end{aligned}$$

When $\phi \in H^1(K)$, we can simplify the above as follows.

$$(2.35) \quad f_\phi \text{ minimizes } \tilde{\mathcal{E}}(f, f) \text{ over all } f \in \tilde{H}^1(\tilde{\mathcal{G}}) \text{ subject to the constraint } f|_K = \phi.$$

We will give an explicit expression for f_ϕ in Lemma 3.1 in the next section. As in (2.16), we have that

$$(2.36) \quad f_\phi \text{ is linear on } \mathcal{I}_{-K} \text{ (see (2.14)).}$$

We are now ready to state the promised conditional law for φ ; cf. [3, Proposition 2.1] for the discrete version. For a formal proof, see [6, Theorem 1.52] or [62, Section 2.6] (cf. Lemma 3.7 below) and see Theorem 4 in Chapter 2, Section 2.4 of [61] for the version with random sets. Suppose that the random set K taking values in $\mathcal{K}_{<\infty}$ satisfies the property

$$(2.37) \quad \{K \subset U\} \subset \mathcal{F}_U \text{ for every (deterministic) open subsets } U \text{ of } \tilde{\mathcal{G}}.$$

Then, with $\tilde{H}_0^1(\overline{K^c})$ denoting the subspace of $\tilde{H}^1(\overline{K^c})$ (see below (2.9)) consisting of $f \in \tilde{H}_0^1(\overline{K^c})$ satisfying $f|_{\partial K} = 0$, we have:

$$(2.38) \quad \begin{aligned} &\text{conditionally on } \mathcal{F}_K, \varphi|_{\overline{K^c}} \text{ has the same law as } f_{\varphi|_K} + \varphi^{\overline{K^c}} \\ &\text{where } \varphi^{\overline{K^c}} \text{ is a Gaussian free field on the Hilbert space } \tilde{H}_0^1(\overline{K^c}). \end{aligned}$$

We end this section with a linear variance estimate for the sum of φ over subsets of V which will be useful later to control the ‘‘bulk’’ component of the exploration martingale introduced in Section 6. The bound follows from the expression of the zero-average green function on the graph \mathcal{G} (see [3, display (2.17)]) and the lower bound on the spectral gap λ_* (cf. 2.7) as per our standing assumption. We omit the proof.

Lemma 2.1. *For any $K \subset V$, we have*

$$(2.39) \quad \text{Var}\left[\sum_{x \in K \setminus \{v\}} d_x \varphi(x)\right] \leq C|K|.$$

3. ZERO-AVERAGE CAPACITY

In this section we will formally introduce the *zero-average capacity* and discuss some of its important properties. The *linear isoperimetry* inherent in the lower bound on the spectral gap of \mathcal{G} (see (2.8)), which is crucial for the probability estimates in Theorems 1.1 and 1.2, enter our analysis through this object.

There are two distinct ways to arrive at the zero-average capacity both of which will be important for us. The first of these is linked to a special expression for the function f_{1_K} from (2.34) where $K \in \mathcal{K}_{<\infty}$ (see (2.5)). We state it for general f_ϕ for applications in future sections.

Lemma 3.1 (An expression for f_ϕ). *Let $K \in \mathcal{K}_{<\infty}$ and $\phi : K \rightarrow \mathbb{R}$. Then, for $x \in \tilde{\mathcal{G}}$,*

$$(3.1) \quad f_\phi(x) = -\nu_\phi \sum_{y \in V} d_y g_{K^c}(y, x) + \mathbf{E}_x[\phi(\tilde{X}_{H_K})],$$

(see (2.21)–(2.22) for notations and compare with [3, Proposition 2.1]) where

$$(3.2) \quad \nu_\phi \stackrel{\text{def.}}{=} \frac{\sum_{x \in V} d_x \mathbf{E}_x[\phi(\tilde{X}_{H_K})]}{\sum_{x,y \in V} d_y d_x g_{K^c}(y,x)} (\in \mathbb{R}).$$

Proof. Since $K \in \mathcal{K}_{<\infty}$, i.e., in particular, $V \setminus K \neq \emptyset$ (recall (2.5)), the denominator in (3.2) is non-zero and hence $\nu_\phi \in \mathbb{R}$. Clearly both f_ϕ and the function defined on the right-hand side of (3.1) equal ϕ on K . In view of (2.23) and (2.36), both of these functions are also linear on \mathcal{I}_{-K} (recall (2.12) and (2.14)). Therefore it suffices to prove (3.1) *only* on $V^K \setminus K$ (see (2.13)). However the expression (3.1), when restricted to V^K , admits of the following equivalent formulation due to (2.26):

$$(3.3) \quad f_\phi(x) = \nu_\phi \sum_{y \in V} d_y \frac{g_{K^c}^K(y,x)}{C_x^K} + \mathbf{E}_x[\phi(X_{H_K}^K)] \text{ for } x \in V^K$$

(see (2.19) and (2.26) for notations).

In order to prove (3.3), first note that we obtain from the variational formulation in (2.34), the linearity of f_ϕ on \mathcal{I}_{-K} and the identity (2.17) that the vector $f_{\phi|V^K} = (f_\phi(x) : x \in V^K) \in \mathbb{R}^{V^K}$ minimizes the quadratic form

$$(3.4) \quad \begin{aligned} \mathcal{E}^K(f,f) &= \sum_{\{x,y\} \in E^K} (f(x) - f(y))^2 C_{\{x,y\}}^K \text{ over all } f \in \\ &\mathbb{R}^{V^K} \text{ satisfying } f|_{V^K \cap K} = \phi|_{V^K \cap K} \text{ and } \sum_{x \in V} d_x f(x) = 0. \end{aligned}$$

The space of *feasible* solutions is non-empty since $V \setminus K \neq \emptyset$. Using the method of Lagrange multipliers, we obtain that the optimal vector $f_{\phi|V^K \setminus K}$ is a solution to the system of equations

$$(3.5) \quad -\Delta^K f_{\phi|V^K}(x) + \nu \bar{d}_x = 0; x \in V^K \setminus K$$

(recall (2.27)) for some $\nu \in \mathbb{R}$ where $\bar{d}_x = d_x$ if $x \in V$ and 0 otherwise. Since $f_{\phi|V^K \cap K} = \phi|_{V^K \cap K}$, we can rewrite this system as

$$(3.6) \quad -\Delta_{K^c}^K f_{\phi|V^K \setminus K} = -\nu \bar{d}_{V^K \setminus K} + C_{V^K \setminus K, V^K \cap K}^K \phi|_{V^K \cap K},$$

(recall (2.28)) where $\bar{d}_{V^K \setminus K} = (\bar{d}_x : x \in V^K \setminus K) \in \mathbb{R}^{V \setminus K}$ and $C_{V^K \setminus K, V^K \cap K}^K$ is the linear map from $\mathbb{R}^{V^K \cap K}$ to $\mathbb{R}^{V^K \setminus K}$ given by $C_{V^K \setminus K, V^K \cap K}^K h(x) = \sum_{y \in V^K \cap K} C_{\{x,y\}}^K h(y)$ ($h \in \mathbb{R}^{V^K \cap K}, x \in V^K \setminus K$). Now we obtain the following expression for $f_\phi(x)$ in view of (3.6) when $x \in V^K \setminus K$:

$$f_\phi(x) = -\nu \sum_{y \in V \setminus K} d_y \frac{g_{K^c}^K(y,x)}{C_x^K} + \sum_{y \in V^K \setminus K, z \in V^K \cap K} g_{K^c}^K(x,y) \frac{C_{\{y,z\}}^K}{C_y^K} \phi(z)$$

(cf. (2.26) and (2.22)). Note that, by the *last-exit decomposition* for X^K applied to the set $V^K \setminus K$ (see, e.g., [48, Proposition 4.6.4]), we have

$$(3.7) \quad \sum_{y \in V^K \setminus K} g_{K^c}^K(x,y) \frac{C_{\{y,z\}}^K}{C_y^K} = \mathbf{P}_x[X_{H_K}^K = z],$$

for any $x \in V^K \setminus K$ and $z \in V^K \cap K$. Plugging this into the expression for f_ϕ above, we get

$$f_\phi(x) = -\nu \sum_{y \in V \setminus K} d_y \frac{g_{K^c}^K(y,x)}{C_x^K} + \mathbf{E}_x[\phi(X_{H_K}^K)] = -\nu \sum_{y \in V} d_y \frac{g_{K^c}^K(y,x)}{C_x^K} + \mathbf{E}_x[\phi(X_{H_K}^K)],$$

for $x \in V^K \setminus K$ where in the final step we used that $g_{K^c}^K(y, x) = 0$ for $y \in V^K \cap K (\supseteq V \cap K)$. This verifies (3.3) (both expressions equal ϕ on $V^K \cap K$) except for the equality between ν and ν_ϕ . To this end we will use the constraint $\sum_{x \in V} f_K(x) d_x = 0$ (recall (3.4)). In view of (3.1), with ν in place of ν_ϕ , this constraint leads to the equation

$$\sum_{x \in V} d_x \mathbf{E}_x[\phi(X_{H_K}^K)] - \nu \sum_{y \in V} d_y d_x \frac{g_{K^c}^K(y, x)}{C_x^K} = 0,$$

whence we get $\nu = \nu_\phi$ as in (3.2) by (2.26). \square

We are finally ready to define the so-called “zero-average capacity”.

Definition 3.2 (zero-average capacity). Let $K \in \mathcal{K}_{<\infty}$ and $1_K : K \rightarrow \mathbb{R}$ denote the constant function 1 on K . We define the *zero-average capacity* of K as the quantity $2|E|\nu_K$ where

$$(3.8) \quad \nu_K \stackrel{\text{def.}}{=} \nu_{1_K} \stackrel{(3.2)}{=} \frac{2|E|}{\sum_{x,y \in V} d_y d_x g_{K^c}(y, x)}.$$

Remark 3.3. Notice that, in view of (3.1), we can rewrite $f_K \stackrel{\text{def.}}{=} f_{1_K}$ as

$$(3.9) \quad f_K(x) = -\nu_K \sum_{y \in V} d_y g_{K^c}(y, x) + 1.$$

The rationale behind calling $2|E|\nu_K$ the *zero-average capacity* is given by the following identity. The reader can compare this with a corresponding variational formulation of the *capacity* of a set in transient graphs, see, e.g., [71, (2.10), p. 18] and also the discussion in the second item in §1.3.

Lemma 3.4 (Variational formula for ν_K). *For every $K \in \mathcal{K}_{<\infty}$, one has (cf. (2.35))*

$$(3.10) \quad \tilde{\mathcal{E}}(f_K, f_K) \stackrel{(2.17),(2.36)}{=} \mathcal{E}^K(f_{K|V^K}, f_{K|V^K}) = 2|E|\nu_K (\in (0, \infty)).$$

Proof. Applying the (discrete) Gauss–Green formula, we can write

$$(3.11) \quad \mathcal{E}^K(f_{K|V^K}, f_{K|V^K}) = \sum_{x \in V^K} -\Delta^K f_{K|V^K}(x) f_K(x).$$

Now by (3.5), we have $\Delta^K f_K(x) = \nu_K \bar{d}_x$ for all $x \in V^K \setminus K$ whereas by (3.9),

$$(3.12) \quad \Delta^K f_{K|V^K}(x) = -\nu_K \sum_{y \in V \setminus K, z \in V^K} d_y g_{K^c}^K(y, z) \frac{C_{\{z,x\}}^K}{C_z^K} = -\nu_K \sum_{y \in V \setminus K} d_y \mathbf{P}_y[X_{H_K}^K = x],$$

for $x \in V^K \cap K$. Note that we used the last-exit decomposition (cf. (3.7)) in the final step. Plugging these into the right-hand side of (3.11) we obtain

$$(3.13) \quad \begin{aligned} \mathcal{E}^K(f_{K|V^K}, f_{K|V^K}) &= -\nu_K \sum_{x \in V \setminus K} d_x f_K(x) + \nu_K \sum_{x \in V^K \cap K} f_K(x) \sum_{y \in V \setminus K} d_y \mathbf{P}_y[X_{H_K}^K = x] \\ &= -\nu_K \sum_{x \in V \setminus K} d_x f_K(x) + \nu_K \sum_{y \in V \setminus K} d_y \mathbf{P}_y[H_K < \infty] = \left(\sum_{x \in V} d_x\right) \nu_K, \end{aligned}$$

where we used $f_{K|V^K \cap K} = 1$ in the second step and then in the final step along with the fact that $\sum_{x \in V} d_x f_K(x) = 0$. \square

We now explore some useful properties of ν_K starting with monotonicity and volume order bounds in the following proposition. The lower bound on ν_K crucially relies on the lower bound on the *spectral gap* of the graph \mathcal{G} . In the sequel for any $K \subset \tilde{\mathcal{G}}$, we let $b_0(K)$ denote the number of components of K and $|K|_V \stackrel{\text{def.}}{=} |K \cap V|$.

Proposition 3.5 (Monotonicity and linear isoperimetry). *The mapping $K \mapsto \nu_K : \mathcal{K}_{<\infty} \rightarrow [0, \infty)$ is increasing w.r.t. set inclusion. Further for any $K \in \mathcal{K}_\infty$, we have*

$$(3.14) \quad c|K|_V \leq 2|E|\nu_K \text{ whereas, if } |K|_V + b_0(K) \leq c|V|, \text{ then } 2|E|\nu_K \leq C(|K|_V + b_0(K)).$$

Proof. The monotonicity of ν_K is clear in view of (2.34) and Lemma 3.4. For the lower bound in (3.14), consider the set $\lceil K \rceil \stackrel{\text{def.}}{=} K \cap V$. If $\lceil K \rceil \neq \emptyset$, i.e., $|K|_V > 0$ which we can assume without any loss of generality, then $\lceil K \rceil \in \mathcal{K}_\infty$. Hence using the monotonicity of ν_K , we get $\nu_K \geq \nu_{\lceil K \rceil}$. On the other hand, from Lemma 3.4 (see also (2.17)) and (2.7) we have $2|E|\nu_{\lceil K \rceil} \geq c|\lceil K \rceil|_V$. Combining these two we obtain $c|K|_V \leq 2|E|\nu_K$.

For the upper bound in (3.14), we will first construct a set $\lceil K \rceil \supset K$ in $\mathcal{K}_{<\infty}$ such that $\partial[\lceil K \rceil] \subset V$ and then derive the required upper bound on $\mathcal{E}(f^*, f^*)$ where $f^* \in \mathbb{R}^V$ satisfies $f^*_{|V \cap \lceil K \rceil} = 1_{V \cap \lceil K \rceil}$ as well as $\sum_{x \in V} d_x f^*(x) = 0$. This would yield the bound on $2|E|\nu_K$ in view of the monotonicity of ν_K , Lemma 3.4 (see (2.17)) and (3.4).

The set $\lceil K \rceil$ is formed by taking the union of K with any $\overline{I_e}$ such that $K \cap I_e \neq \emptyset$. Clearly, $b_0(\lceil K \rceil) \leq b_0(K)$ and $\partial[\lceil K \rceil] \subset V$. Before we can define the vector f^* , we need to obtain a suitable upper bound on $|\lceil K \rceil|_V$. To this end, let us first suppose that K is connected, i.e., $b_0(K) = 1$. It follows that in this case, $|\lceil K \rceil|_V \leq 2|K|_V + 1$. More generally, this implies that

$$(3.15) \quad |\lceil K \rceil|_V \leq 2|K|_V + b_0(K) \text{ for any } K \in \mathcal{K}_{<\infty}.$$

Now consider $f^* \in \mathbb{R}^V$ defined as $f^*(x) = 1$ for $x \in V \cap \lceil K \rceil$ and

$$(3.16) \quad f^*(x) = -\frac{\sum_{x \in V \cap \lceil K \rceil} d_x}{\sum_{x \in V \setminus \lceil K \rceil} d_x} \text{ for } x \in V \setminus \lceil K \rceil.$$

f^* is well-defined if $\lceil K \rceil \in \mathcal{K}_{<\infty}$, i.e., $V \setminus \lceil K \rceil \neq \emptyset$ which happens as soon as $2|K|_V + b_0(K) < |V|$ due to (3.15). Also in this case, $\sum_{x \in V} d_x f(x) = 0$. Further if $|K|_V + b_0(K) \leq c|V|$, we have $(0 \geq) f^*(x) \geq -C$ for $x \in \lceil K \rceil \cap V$. Thus

$$(3.17) \quad \mathcal{E}(f^*, f^*) \leq (1 + C)^2 |\lceil K \rceil|_V \leq C'(|K|_V + b_0(K))$$

whenever $|K|_V + b_0(K) \leq c|V|$ as desired. \square

Although the zero-average capacity is monotone like the usual capacity, it lacks an important property possessed by the latter, namely the *subadditivity* (see, e.g., [47, Proposition 2.2.1(b)]). This is evident from the simple observation that while $2|E|\nu_{\{x\}} \leq C$ for all $x \in V$ in view of Proposition 3.5, $\nu_V = \infty$ by Definition 3.2 (or the variational formula in Lemma 3.4). Notwithstanding, there is a weaker version which is sufficient for the purpose of this work.

Lemma 3.6 (Coarse Lipschitz property). *Let $K \in \mathcal{K}_{<\infty}$ and $\mathcal{C} \subset \tilde{\mathcal{G}}$ be such that $K \cup \mathcal{C} \in \mathcal{K}_{<\infty}$. Then*

$$(3.18) \quad (0 \leq) 2|E|(\nu_{K \cup \mathcal{C}} - \nu_K) \leq C(|\mathcal{C}|_V + b_0(\mathcal{C}))$$

for $|K|_V + |\mathcal{C}|_V + b_0(K) + b_0(\mathcal{C}) \leq c|V|$.

Later in Section 5, we will prove the (non-quantitative) continuity of the function $K \mapsto \nu_K$ w.r.t. the Hausdorff distance on $\mathcal{K}_{<\infty}$.

(3.18) follows from a refined version of the argument used for the proof of upper bound in Proposition 3.5 combined with a special case of the following result. We will use the full strength of this result in Section 4 below.

Lemma 3.7 (Orthogonality). *Let $K \in \mathcal{K}_{<\infty}$ and $\phi : K \rightarrow \mathbb{R}$. Then for any piecewise linear $g : \tilde{\mathcal{G}} \rightarrow \mathbb{R}$ (see below (2.12)) satisfying $g|_K = 0$ and $\sum_{x \in V} d_x g(x) = 0$, we have $\tilde{\mathcal{E}}(f_\phi, g) = 0$. On the other hand, if ϕ is also piecewise linear on K , then we have $\tilde{\mathcal{E}}(\psi, f_\phi) = 0$ for any $\psi : \tilde{\mathcal{G}} \rightarrow \mathbb{R}$ such that $\psi|_K = 0$ and $\sum_{x \in V} d_x \psi(x) = 0$.*

We will prove Lemma 3.7 shortly and finish the

Proof of Lemma 3.6. It suffices to find a piecewise linear function g satisfying (i) $g|_K = 0$ and $g|_{(K \cup \mathcal{C}) \setminus K} = 1$, (ii) $\sum_{x \in V} d_x g(x) = 0$ and (iii) $\tilde{\mathcal{E}}(g, g) \leq C(|\mathcal{C}|_V + b_0(\mathcal{C}))$. Indeed, it then follows similarly as in the proof of the upper bound on $2|E|\nu_K$ in Proposition 3.5 that

$$\nu_{K \cup \mathcal{C}} \stackrel{(3.10)}{\leq} \tilde{\mathcal{E}}(f_K + g, f_K + g) \stackrel{\text{Lem. 3.7}}{=} \tilde{\mathcal{E}}(f_K, f_K) + \tilde{\mathcal{E}}(g, g) \stackrel{(3.10), (iii)}{\leq} 2|E|\nu_K + C(|\mathcal{C}|_V + b_0(\mathcal{C})).$$

In order to construct the function g , we will use the sets $[K]$ and $[\mathcal{C}]$ introduced in the proof of Proposition 3.5. To this end consider the vector $g^* \in \mathbb{R}^V$ defined as $g^*|_{V \cap [K]} = 0$, $g^*|_{V \cap ([\mathcal{C}] \setminus [K])} = 1$ and

$$g^*(x) = -\frac{\sum_{x \in V \cap ([\mathcal{C}] \setminus [K])} d_x}{\sum_{x \in V \setminus ([K] \cup [\mathcal{C}])} d_x} \quad \text{for } x \in V \setminus ([K] \cup [\mathcal{C}])$$

(cf. (3.16)). It follows from (3.15) that g^* is well-defined for $|K|_V + |\mathcal{C}|_V + b_0(K) + b_0(\mathcal{C}) \leq c|V|$. Now let g be the piecewise linear function obtained by interpolating g^* linearly on each interval \overline{I}_e , $e \in E$. It is straightforward to check that g satisfies the properties (i) and (ii) whereas property (iii) follows from an analysis similar to the one leading to (3.17). \square

It remains to give the

Proof of Lemma 3.7. Since g is piecewise linear, we can write

$$\tilde{\mathcal{E}}_{K^c}(f_\phi + \lambda g, f_\phi + \lambda g) \stackrel{(2.10)}{=} \tilde{\mathcal{E}}_{K^c}(f_\phi, f_\phi) + 2\lambda \tilde{\mathcal{E}}_{K^c}(f_\phi, g) + \lambda^2 \tilde{\mathcal{E}}_{K^c}(f_\phi, f_\phi).$$

Therefore, if $\tilde{\mathcal{E}}_{K^c}(f_\phi, g) = \tilde{\mathcal{E}}(f_\phi, g) \neq 0$ (recall (2.10) and that $g|_K = 0$), we can choose λ with small enough absolute value so that $\tilde{\mathcal{E}}_{K^c}(f_\phi + \lambda g, f_\phi + \lambda g) < \tilde{\mathcal{E}}_{K^c}(f_\phi, f_\phi)$. On the other hand, from the properties of g we know that $f_\phi + \lambda g$ satisfies the constraints in (2.34) and hence $\tilde{\mathcal{E}}_{K^c}(f_\phi + \lambda g, f_\phi + \lambda g) \geq \tilde{\mathcal{E}}_{K^c}(f_\phi, f_\phi)$ which leads a contradiction. Thus $\tilde{\mathcal{E}}(f_\phi, g) = 0$.

For the second part, notice that in view of (2.20) and (2.12) as well as the condition $\psi|_K = 0$, we can define a piecewise linear function $\tilde{\psi}$ on $\tilde{\mathcal{G}}$ satisfying $\tilde{\psi}|_{K \cup V} = \psi|_{K \cup V}$ and $\tilde{\mathcal{E}}(\psi, f_\phi) = \tilde{\mathcal{E}}(\tilde{\psi}, f_\phi)$. Now we can follow a similar argument as above to deduce $\tilde{\mathcal{E}}(\tilde{\psi}, f_\phi) = 0$. \square

4. MARTINGALE PROPERTY OF DIRICHLET FORMS

In this rather short section, we will introduce a process indexed by the sets in $\mathcal{K}_{<\infty}$ (recall (2.5)). This process turns out to be a martingale, even when sampled at suitable random sets (see Proposition 4.4 below). Later in Section 6, we will use this to define a (continuous) $[0, \infty)$ indexed martingale “tied to” the exploration of components in $\tilde{\mathcal{G}}^{\geq h}$ (cf. (2.31)).

Definition 4.1. For any $K \in \mathcal{K}_{<\infty}$, we define the random variable M_K as (see (2.9) and (2.17) for notations)

$$(4.1) \quad M_K = \tilde{\mathcal{E}}(\varphi, f_K) \stackrel{(2.20)}{=} \mathcal{E}^K(\varphi|_{V^K}, f_{K|V^K}).$$

Lemma 4.2. *For any $K \in \mathcal{K}_{<\infty}$, we have*

$$(4.2) \quad M_K = \nu_K \sum_{x \in V^K \cap K} \varphi(x) \sum_{y \in V} d_y \mathbf{P}_y[\tilde{X}_{H_K} = x],$$

where ν_K is as in (3.8) (see also (2.13) and (2.21)). Further, we can write $M_K = M_K^{\text{blk}} + M_K^{\text{bdr}}$ with

$$(4.3) \quad M_K^{\text{blk}} \stackrel{\text{def.}}{=} \nu_K \sum_{x \in V^K \cap K} d_x \varphi(x) \text{ and } M_K^{\text{bdr}} \stackrel{\text{def.}}{=} \nu_K \sum_{x \in \partial K} \varphi(x) \left(\sum_{y \in V \setminus K} d_y \mathbf{P}_y[\tilde{X}_{H_K} = x] \right).$$

We refer to M_K^{blk} and M_K^{bdr} as the ‘‘bulk’’ and ‘‘boundary’’ part of M_K respectively.

Proof. From the (discrete) Gauss-Green formula we have (cf. (3.11))

$$M_K = \sum_{x \in V^K} -\Delta^K f_{K|V^K}(x) \varphi(x).$$

Plugging the expressions (3.5) and (3.12) into right-hand side of the above display, we obtain

$$(4.4) \quad \begin{aligned} M_K &= -\nu_K \sum_{x \in V \setminus K} d_x \varphi(x) + \nu_K \sum_{x \in V^K \cap K} \varphi(x) \sum_{y \in V \setminus K} d_y \mathbf{P}_y[\tilde{X}_{H_K} = x] \\ &= \nu_K \sum_{x \in V \cap K} d_x \varphi(x) + \nu_K \sum_{x \in V^K \cap K} \varphi(x) \sum_{y \in V \setminus K} d_y \mathbf{P}_y[\tilde{X}_{H_K} = x], \end{aligned}$$

where we wrote \tilde{X}_{H_K} instead of $X_{H_K}^K$ inside the probabilities owing to (2.26) and in the last step we used $\sum_{x \in V} d_x \varphi(x) = 0$ (cf. (2.30)). Since $\mathbf{P}_y[\tilde{X}_{H_K} = x] = 1$ for some $y \in V \cap K$ and $x \in V^K \cap K$ if $x = y$ and 0 otherwise, we obtain (4.2) from (4.4).

Alternatively, observe that in the second term on the right-hand side of (4.4), the only non-zero contributions come from the points $x \in \partial K$ whence we obtain (4.3). \square

As a direct consequence of the expression (4.2), we record below the ‘‘first’’ important property of M_K which will be useful later to link large component sizes of $\tilde{\mathcal{G}}^{\geq h}$ (recall (2.31)) to large values of M_K for subsets K of those components.

Lemma 4.3. *Let $K \in \mathcal{K}_{<\infty}$ and $h \in \mathbb{R}$. If $K \subset \tilde{\mathcal{G}}^{\geq h}$, then we have $M_K \geq 2h|E|\nu_K$.*

Proof. Note that when $K \subset \tilde{\mathcal{G}}^{\geq h}$,

$$\begin{aligned} M_K &\stackrel{(4.2)}{=} \nu_K \sum_{x \in V^K \cap K} \sum_{y \in V} d_y \mathbf{P}_y[X_{H_K}^K = x] \varphi(x) \geq h\nu_K \sum_{x \in V^K \cap K} \sum_{y \in V} d_y \mathbf{P}_y[X_{H_K}^K = x] \\ &= h\nu_K \sum_{y \in V} d_y \sum_{x \in V^K \cap K} \mathbf{P}_y[X_{H_K}^K = x] = h\nu_K \sum_{y \in V} d_y = 2h|E|\nu_K. \end{aligned} \quad \square$$

We now come to the most important result of this section.

Proposition 4.4 (Martingale property). *Let K be a random set (recall (2.33)) taking values in $\mathcal{K}_{<\infty}$ that satisfies property (2.38). Also let $K' \supset K$ be a (deterministic) set in $\mathcal{K}_{<\infty}$. Then,*

$$(4.5) \quad \mathbb{E}[M_{K'} | \mathcal{F}_K] = M_K, \mathbb{P}\text{-almost surely.}$$

Proof. As K' (hence $f_{K'}$) is deterministic, we can write in view of (2.20) and (4.1) that

$$\mathbb{E}[M_{K'} | \mathcal{F}_K] = \tilde{\mathcal{E}}(\mathbb{E}[\varphi | \mathcal{F}_K], f_{K'}), \mathbb{P}\text{-a.s.}$$

Since K satisfies property (2.38), we further have $\mathbb{E}[\varphi | \mathcal{F}_K] = f_{\varphi|K}$ \mathbb{P} -a.s. Combining these two, we get

$$\mathbb{E}[M_{K'} | \mathcal{F}_K] = \tilde{\mathcal{E}}(f_{\varphi|K}, f_{K'}).$$

Now noting that $f_{K'} = f_K = 1$ on K (recall that $K \subset K'$) and $\sum_{x \in V} d_x f_K(x) = \sum_{x \in V} d_x f_{K'}(x) = 0$ (recall (2.34)), we can deduce from Lemma 3.7 (the first part) and (4.1) that

$$\tilde{\mathcal{E}}(f_{\varphi|K}, f_{K'}) = \tilde{\mathcal{E}}(f_{\varphi|K}, f_{K'} - f_K) + \tilde{\mathcal{E}}(f_{\varphi|K}, f_K) = \tilde{\mathcal{E}}(f_{\varphi|K}, f_K).$$

Similarly since $\varphi = f_{\varphi|K}$ on K and $\sum_{x \in V} d_x f_{\varphi|K}(x) = \sum_{x \in V} d_x \varphi(x) = 0$ (2.30), we get

$$\tilde{\mathcal{E}}(f_{\varphi|K}, f_K) = \tilde{\mathcal{E}}(f_{\varphi|K} - \varphi, f_K) + \tilde{\mathcal{E}}(\varphi, f_K) = \tilde{\mathcal{E}}(\varphi, f_K) \stackrel{(4.1)}{=} M_K,$$

using the second part of Lemma 3.7. Together, the last three displays imply (4.5). \square

If we look at the conditional variance instead, we get:

Lemma 4.5. *In the same setup as in Proposition 4.4, we have*

$$(4.6) \quad \text{Var}[M_{K'} | \mathcal{F}_K] = 2|E|(\nu_{K'} - \nu_K) \text{ } \mathbb{P}\text{-almost surely.}$$

Proof. First of all, let us note that $M_{K'}$ is square integrable owing to its definition in (4.1), the variance formula (2.32) and also (3.10). So we can use the following standard decomposition:

$$(4.7) \quad \text{Var}[M_{K'}] = \text{Var}[\mathbb{E}[M_{K'} | \mathcal{F}_K]] + \mathbb{E}[\text{Var}[M_{K'} | \mathcal{F}_K]] \stackrel{(4.5)}{=} \text{Var}[M_K] + \mathbb{E}[\text{Var}[M_{K'} | \mathcal{F}_K]].$$

Now suppose that K is deterministic. Then from (4.1), (2.32) and (3.10), we get

$$(4.8) \quad \text{Var}[M_S] = 2|E|\nu_S \text{ for any } S \in \mathcal{K}_{<\infty}.$$

On the other hand, in view of (2.38), we obtain that the conditional variance $\text{Var}[M_{K'} | \mathcal{F}_K]$ is constant almost surely. Plugging these two observations into (4.7) gives us

$$\text{Var}[M_{K'} | \mathcal{F}_K] = 2|E|(\nu_{K'} - \nu_K) \text{ a.s.}$$

for any deterministic K (cf. (4.6)). Hence the same expression also holds when $K \subset K'$ is random but satisfies property (2.38), thus yielding the lemma. \square

5. REGULARITY OF THE PROCESS $(M_K)_{K \in \mathcal{K}_{<\infty}}$

In this section, we will establish the continuity of the functions $K \mapsto M_K$ and $K \mapsto \nu_K$ as K ranges over the family $\mathcal{K}_{<\infty}$ (see Definitions 4.1 and 3.2 and also (2.5)). These will be used to ensure the continuity of a $[0, \infty)$ -indexed ‘‘exploration martingale’’ and its quadratic variation process which we introduce in §6.2.

The relevant topology on $\mathcal{K}_{<\infty}$ is given by the *Hausdorff distance* which we briefly recall below. For any $S, S' \subset \tilde{\mathcal{G}}$, the Hausdorff distance (corresponding to the metric $d(\cdot, \cdot)$ on $\tilde{\mathcal{G}}$, recall (2.2)) between them is defined as

$$(5.1) \quad d_H(S, S') = \inf\{\varepsilon \geq 0 : S \subset B(S', \varepsilon) \text{ and } S' \subset B(S, \varepsilon)\},$$

where $B(S, \varepsilon) \stackrel{\text{def.}}{=} \cup_{x \in S} \{y \in \tilde{\mathcal{G}} : d(y, x) < \varepsilon\}$ denotes the ε -neighborhood of S . d_H is a metric on \mathcal{K} (see, e.g., [52, pp. 280-281]). In the sequel, we use \mathcal{K} (or $\mathcal{K}_{<\infty}$) to refer to both the collection and the associated metric space depending on the context. To keep the exposition simple, and because it is sufficient for our purpose, we only prove a restricted version of continuity as follows (see also Proposition 5.2 below).

Proposition 5.1 (On continuity of $K \mapsto M_K$). *Let $K, K' \in \mathcal{K}_{<\infty}$ be such that either $K \subset K'$ or $K' \subset K$. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, K, \mathcal{G}, \varphi) > 0$ satisfying*

$$|M_K - M_{K'}| < \varepsilon \text{ whenever } d_H(K, K') < \delta.$$

The proof needs a similar result for $K \mapsto \nu_K$ as an ingredient which is important on its own.

Proposition 5.2 (On continuity of $K \mapsto \nu_K$). *Let $K, K' \in \mathcal{K}_{<\infty}$ be as in Proposition 5.1. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, K, \mathcal{G}) > 0$ satisfying*

$$|\nu_K - \nu_{K'}| < \varepsilon \text{ whenever } d_H(K, K') < \delta.$$

Assuming Proposition 5.2, let us finish the

Proof of Proposition 5.1. Let us recall the expression of the martingale from (4.2) in Lemma 4.2:

$$M_K = \nu_K \sum_{z \in V} \sum_{x \in V^K \cap K} \varphi(x) \mathbf{P}_z[X_{H_K} = x], \text{ and } M_{K'} = \nu_{K'} \sum_{z \in V} \sum_{x \in V^{K'} \cap K'} \varphi(x) \mathbf{P}_z[X_{H_{K'}} = x],$$

where X is the (discrete-time) walk $X^{K, K'}$ (cf. (2.29) and (2.13) for relevant notations) and we invoked (2.26) in replacing \tilde{X} with X inside the probabilities. In view of Proposition 5.2 and since φ is a continuous function on $\tilde{\mathcal{G}}$, it therefore suffices to prove the following statement for any continuous function f on $\tilde{\mathcal{G}}$, $z \in V$ and $\varepsilon \in (0, 1)$. Letting

$$(5.2) \quad F(z, K) \stackrel{\text{def.}}{=} \sum_{x \in V^K \cap K} f(x) \mathbf{P}_z[X_{H_K} = x], \text{ and } F(z, K') \stackrel{\text{def.}}{=} \sum_{x \in V^{K'} \cap K'} f(x) \mathbf{P}_z[X_{H_{K'}} = x],$$

there exists $\delta = \delta(\varepsilon, K, \mathcal{G}, f) > 0$ such that

$$(5.3) \quad |F(z, K) - F(z, K')| < \varepsilon$$

whenever $d_H(K, K') < \delta$. To this end, let $M_f \stackrel{\text{def.}}{=} \sup_{x \in \tilde{\mathcal{G}}} |f(x)| \vee 1$. Also let

$$\delta_{f, \varepsilon} \stackrel{\text{def.}}{=} \sup\{\delta \in (0, 1) : |f(x) - f(y)| < \frac{\varepsilon}{8} \text{ whenever } d(x, y) < \delta, x, y \in \tilde{\mathcal{G}}\}.$$

Note that $M_f < \infty$ and $\delta_{f, \varepsilon} > 0$ since f is continuous on $\tilde{\mathcal{G}}$ which is compact. Now let

$$(5.4) \quad \delta \stackrel{\text{def.}}{=} \frac{\inf_{x \neq y \in V^K} d(x, y) \wedge \delta_{f, \varepsilon}}{100} \cdot \frac{\varepsilon}{M_f |V^K \cap K|}.$$

We call $x \in V^{K'} \cap K'$ as *visible* from V if either $x \in V$ or $x \in (y_1, y_2)_\mathcal{G}$ for some $y_1, y_2 \in V$ such that $[y_1, x]_\mathcal{G} \cap K' = \emptyset$ (cf. (2.1) for notation). Let $D_v^{K'}$ denote the set of all visible points in $V^{K'} \cap K'$. Then we can write

$$(5.5) \quad F(z, K') = \sum_{x \in D_v^{K'}} f(x) \mathbf{P}_z[X_{H_{K'}} = x],$$

as $\mathbf{P}_z[X_{H_{K'}} = x] = 0$ when $x \notin D_v^{K'}$ (see (5.2)). We claim that

$$(5.6) \quad D_v^{K'} \subset B(V^K \cap K, 2\delta).$$

(5.6) follows in fairly straightforward manner from our definition of δ in (5.4). So we omit further details and proceed with the proof of (5.3) assuming this.

From the choice of δ in (5.4), it is clear that $d(x, y) > 4\delta$ for any two distinct $x, y \in V^K \cap K$ and as a result, $B(x, 2\delta) \cap B(y, 2\delta) = \emptyset$ for such x and y . On the other hand, from (5.6) we have $D_v^{K'} \subset \cup_{x \in V^K \cap K} B(x, 2\delta)$. In view of (5.2) and (5.5), these two observations give us the expression:

$$|F(z, K) - F(z, K')| = \left| \sum_{x \in V^K \cap K} f(x) \mathbf{P}_z[X_{H_K} = x] - \sum_{x \in V^K \cap K} \sum_{y \in B(x, 2\delta) \cap D_v^{K'}} f(y) \mathbf{P}_z[X_{H_{K'}} = y] \right|.$$

Further since $2\delta < \delta_{f,\varepsilon}$ (see (5.4) and the display above it) and $\sup_{x \in \tilde{G}} |f(x)| \leq M_f$, we can bound the above as

$$(5.7) \quad |F(z, K) - F(z, K')| \leq M_f \sum_{x \in V^K \cap K} |\mathbf{P}_z[X_{H_K} = x] - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}]| + \frac{\varepsilon}{8}.$$

At this point, we will split our analysis into two separate cases, namely $K \subset K'$ and $K' \subset K$ starting with the former.

The case $K \subset K'$. Using the strong Markov property of X in the second line below, we obtain for any $x \in V^K \cap K$:

$$(5.8) \quad \begin{aligned} & \mathbf{P}_z[X_{H_K} = x] - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}] \\ &= \sum_{y \in D_v^{K'}} \mathbf{E}_z[1\{X_{H_{K'}} = y\} \mathbf{P}_y[X_{H_K} = x]] - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}] \\ &= P_x^+ + P_x^- - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}], \end{aligned}$$

where

$$(5.9) \quad \begin{aligned} P_x^+ &= \sum_{y \in B(x, 2\delta) \cap D_v^{K'}} \mathbf{E}_z[1\{X_{H_{K'}} = y\} \mathbf{P}_y[X_{H_K} = x]], \text{ and} \\ P_x^- &= \sum_{y \in D_v^{K'} \setminus B(x, 2\delta)} \mathbf{E}_z[1\{X_{H_{K'}} = y\} \mathbf{P}_y[X_{H_K} = x]]. \end{aligned}$$

Now for any $y \in B(x, 2\delta) \cap D_v^{K'}$, we want to obtain a lower bound on $\mathbf{P}_y[X_{H_K} = x]$ where we may assume $y \neq x$ without any loss of generality. Since $2\delta < \inf_{x \neq y \in V^K} d(x, y)$ by (5.4) and $d(x, y) < 2\delta$ with $x \in V^K \cap K$, there exist $y_1, y_2 \in V$ such that $x, y \in [y_1, y_2]_{\mathcal{G}}$ and $y \in (y_1, x)_{\mathcal{G}}$. Also from (5.4) we know that $d(x, x') \geq \frac{100M_f|V^K \cap K|}{\varepsilon} \delta$ for any $x' (\neq x) \in D^K$. Hence there exists $y_3 \in [y_1, x]_{\mathcal{G}} \cap V^{K, K'}$ such that $\rho([y_1, x]_{\mathcal{G}}) \geq \frac{100M_f|V^K \cap K|^2}{\varepsilon} \delta$ and

$$\{X_{H_K} = x, X_0 = y\} \supset \{X_{T_{(y_3, x)_{\mathcal{G}}}} = x, X_0 = y\},$$

where $T_{(y_3, x)_{\mathcal{G}}}$ is the exit time of X from the interval $(y_3, x)_{\mathcal{G}}$ (cf. (2.21)). Then using similar arguments as used in the proof of Lemma 5.3 below (see around (5.19) in particular), we get

$$(5.10) \quad \mathbf{P}_y[X_{H_K} = x] \geq 1 - \frac{\varepsilon}{100M_f|V^K \cap K|}.$$

Now from the definition of P_x^+ in (5.9), we have

$$P_x^+ - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}] \leq \sum_{y \in B(x, 2\delta) \cap D_v^{K'}} \mathbf{E}_z[1\{X_{H_{K'}} = y\}] - \mathbf{P}_z[X_{H_{K'}} \in B(x, \delta) \cap D_v^{K'}] = 0.$$

On the other hand, from (5.10) we get,

$$P_x^+ - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}] \geq -\frac{\varepsilon}{100M_f|V^K \cap K|}.$$

Together, these two bounds imply

$$(5.11) \quad |P_x^+ - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}]| \leq \frac{\varepsilon}{100M_f|V^K \cap K|}.$$

Next we want to bound the term P_x^- from above. Since $D_v^{K'} \subset \cup_{x \in V^K \cap K} B(x, 2\delta)$, any $y \in D_v^{K'} \setminus B(x, 2\delta)$ belongs to $B(x', 2\delta)$ for some $x' (\neq x) \in V^K \cap K$. Then (5.10) applied to the

point x' gives us that

$$\mathbf{P}_y[X_{H_K} = x'] \leq \frac{\varepsilon}{100M_f |V^K \cap K|}.$$

Plugging this into the expression of P_x^- in (5.9), we obtain

$$(5.12) \quad P_x^- \leq \frac{\varepsilon}{100M_f |V^K \cap K|}.$$

Combining (5.11) and (5.12), we get from (5.8):

$$(5.13) \quad |\mathbf{P}_z[X_{H_K} = x] - \mathbf{P}_z[X_{H_{K'}} \in B(x, 2\delta) \cap D_v^{K'}]| \leq \frac{\varepsilon}{100M_f |V^K \cap K|}.$$

Plugging (5.13) into (5.7), we obtain

$$(5.14) \quad |F(z, K) - F(z, K')| \leq M_f \frac{\varepsilon}{100M_f |V^K \cap K|} |V^K \cap K| + \frac{\varepsilon}{8} < \varepsilon,$$

which is precisely the bound (5.3) (in this case).

The case $K' \subset K$. This case can be dealt with in a similar fashion by applying the strong Markov property at time $H_{K'}$ instead of H_K . \square

We now prepare for the proof of Proposition 5.2 which requires the following lemma.

Lemma 5.3. *Let $K, K' \in \mathcal{K}_{<\infty}$ be as in Proposition 5.1. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, K, \mathcal{G}) > 0$ such that (see (2.22) for notation)*

$$\max_{x \in K, y \in V} g_{(K')^c}(x, y) \vee \max_{x \in K', y \in V} g_{K^c}(x, y) < \varepsilon \text{ whenever } d_H(K, K') < \delta.$$

We prove Lemma 5.3 at the end of this section and proceed with the

Proof of Proposition 5.2. In view of the expression of ν_K from (3.8), it is enough to prove the statement with

$$g_{K^c}(x, y) \stackrel{(2.25), (2.26)}{=} \frac{g_{K^c}^{K, K'}(x, y)}{C_y^{K, K'}},$$

in place of ν_K for all $x, y \in V$ (see (2.26) and (2.29) for notations). To this end, given $\varepsilon \in (0, 1)$, let $\delta = \delta(\varepsilon, K, \mathcal{G}) > 0$ be as in Lemma 5.3. We only give the proof when $K' \subset K$ and the proof for the other case is similar.

So let $K' (\in \mathcal{K}_{<\infty}) \subset K$ be such that $d_H(K, K') < \delta$. By the strong Markov property of the walk $X = X^{K, K'}$, we can write

$$\frac{g_{(K')^c}^{K, K'}(x, y)}{C_y^{K, K'}} - \frac{g_{K^c}^{K, K'}(x, y)}{C_y^{K, K'}} = \sum_{z \in V^K} \mathbf{P}_x[X_{H_K} = z] \frac{g_{(K')^c}^{K, K'}(z, y)}{C_y^{K, K'}} \stackrel{\text{Lem. 5.3, (2.26)}}{\leq} \varepsilon \mathbf{P}_x[H_K < \infty] = \varepsilon,$$

which is precisely the required bound. \square

It remains to give the

Proof of Lemma 5.3. The proof is split into two parts depending on whether $K' \subset K$ or $K \subset K'$.
The case $K' \subset K$. Notice that we only need to check the bound on $g_{(K')^c}(x, y)$ in this case with $x \in K \setminus K'$. Also due to (2.23), it suffices to consider points $x \in V^K \setminus K'$. Now given $\varepsilon \in (0, 1)$, let

$$(5.15) \quad \delta \stackrel{\text{def.}}{=} \frac{\inf_{x \neq y \in V^K} d(x, y)}{(G \vee 1)d} \cdot \frac{\varepsilon}{10},$$

with G from (2.24) and $K' (\in \mathcal{K}_{<\infty}) \subset K$ such that $d_H(K, K') < \delta$. In view of (2.26), we have $g_{(K')^c}(x, y) = \frac{g_{(K')^c}^{K, K'}(x, y)}{C_y^{K, K'}}$. In the sequel, we substitute $\tilde{g}_{(K')^c}(\cdot, \cdot)$, $\tilde{g}_{K^c}(\cdot, \cdot)$, X and $C_{(\cdot, \cdot)}$ for $g_{(K')^c}^{K, K'}(\cdot, \cdot)$, $g_{K^c}^{K, K'}(\cdot, \cdot)$, $X^{K, K'}$ and $C_{\{\cdot, \cdot\}}^{K, K'}$ respectively to ease notation. Using the Markov property of the (discrete-time) walk X , we can write

$$(5.16) \quad \frac{\tilde{g}_{(K')^c}(x, y)}{C_y} = \frac{1_{x=y} + \mathbf{E}_x[\tilde{g}_{(K')^c}(X_1, y)1_{\{X_1 \notin K'\}}]}{C_y} \stackrel{(2.24), (2.26)}{\leq} \frac{1_{x=y}}{C_y} + G\mathbf{P}_x[X_1 \notin K'],$$

where $x \in V^K$ and $y \in V$. We will derive suitable bounds on each of these terms. Let us start by noting the following expression:

$$(5.17) \quad \mathbf{P}_x[X_1 \in K'] = \frac{\sum_{\{x, z\} \in E^{K, K'}, z \in K'} C_{\{x, z\}}}{\sum_{\{x, z\} \in E^{K, K'}} C_{\{x, z\}}} = \frac{1}{1 + \frac{\sum_{\{x, z\} \in E^{K, K'}, z \in K \setminus K'} C_{\{x, z\}}}{\sum_{\{x, z\} \in E^{K, K'}, z \in K'} C_{\{x, z\}}}}.$$

For $z \in K$ such that $\{x, z\} \in E^{K, K'}$, we have

$$C_{\{x, z\}} \stackrel{(2.18), (2.2)}{=} \frac{1}{d(x, z)} \stackrel{(5.15)}{\leq} \frac{\varepsilon}{6d^2G\delta}.$$

On the other hand, since $d_H(K, K') < \delta$ and $x \in V^K \setminus K'$, there exists $z \in K'$ such that $d(x, z) < \delta$ (recall (5.1)). In fact, as $\delta < \inf_{x \neq y \in V^K} d(x, y)$ by our choice in (5.15), we have $(x, z) \in E^{K, K'}$ and thus $C_{\{x, z\}} = \frac{1}{d(x, z)} \geq \frac{1}{\delta}$. Plugging these two bounds into the right-hand side of (5.17), we get

$$\mathbf{P}_x[X_1 \in K'] \geq 1 - \frac{\varepsilon}{10G}.$$

A similar reasoning also tells us that $C_x \geq \frac{1}{\delta}$. Combined with (5.16), these estimates give us

$$\frac{\tilde{g}_{(K')^c}(x, y)}{C_y} \leq \delta + G \frac{\varepsilon}{10G} \stackrel{(5.15)}{\leq} \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

The case $K \subset K'$. In this case we only need to check the bound on $g_{K^c}(x, y)$ for $x \in V^{K'} \setminus K$. Let $K' (\in \mathcal{K}_{<\infty}) \supset K$ be such that $d_H(K, K') < \delta$ where δ is still as in (5.15).

Now let \hat{x} be a point in $V^{K, K'} \cap K$ achieving the minimum distance from x . Using similar arguments as in the previous case, we obtain that $\{x, \hat{x}\} \in \overline{I_{\{z_0, z_1\}}}$ for a unique $\{z_0, z_1\} \in E$. Without loss of generality, suppose that $x \in [z_0, \hat{x}]_G$. Then by the strong Markov property of X as well as (2.24) and (2.25), we get the following analogue of (5.16):

$$(5.18) \quad g_{K^c}(x, y) = \frac{\tilde{g}_{K^c}(x, y)}{C_y} \leq \frac{1_{x=y}}{C_y} + G\mathbf{P}_x[X_{T_{(z_0, \hat{x})_G}} \notin K],$$

where, as defined in (2.21), $T_{(z_0, \hat{x})_G}$ is the exit time of the walk from the set $[z_0, \hat{x}]_G$. From standard one-dimensional gambler's ruin type computation (cf. (2.18)), it follows that

$$(5.19) \quad \mathbf{P}_x[X_{T_{(z_0, \hat{x})_G}} \in K] \geq \frac{1}{1 + \frac{d(x, \hat{x})}{d(x, x_0)}}.$$

Remainder of the proof now follows similarly as in the previous case with (5.19) and (5.18) replacing (5.17) and (5.16) respectively. \square

6. EXPLORATION OF LEVEL-SET COMPONENTS

So far, we did not talk so much about percolation. In this section, we relate the percolation of $\tilde{\mathcal{G}}^{\geq h}$ (recall (2.31)) to the process $(M_K)_{K \in \mathcal{K}_{<\infty}}$ that we introduced and analyzed in the previous two sections. In order to do this, we first construct a continuous family of random sets $(K_t)_{t \geq 0}$ in §6.1 which arise from a certain exploration scheme revealing the components of V in $\tilde{\mathcal{G}}^{\geq h}$ successively. Then in §6.2, we define the exploration martingale $(\tilde{M}_t)_{t \geq 0}$ as $(M_{K_t})_{t \geq 0}$ and discuss a few of its useful properties.

6.1. Exploration process. Let $v \in V$ and $N > 1$ be an integer. By exploring the components of $(V \setminus \{v\}) \cup \partial B(v, \frac{1}{N})$ inside $\tilde{\mathcal{G}}^{\geq h} \setminus B(v, \frac{1}{N})$ (recall that $|V| > 1$) in a certain order, starting with those of $\partial B(v, \frac{1}{N})$, and “interpolating” between them, one can obtain a *continuous* family (w.r.t. Hausdorff distance) of non-decreasing random compact sets $(K_t)_{t \geq 0}$. We restrict our exploration *away* from the vertex v in order to prevent K_t ’s from consuming the entire vertex set V . This would enable us to define a martingale $(\tilde{M}_t)_{t \geq 0}$ attached to $(K_t)_{t \geq 0}$ in the next subsection using the results from Section 4. N is essentially a spurious parameter (see Lemma 6.11 below) and will be sent to ∞ at the end in all applications. We also want to design the exploration in a way so that the sets K_t satisfy some favorable properties, mainly in relation to the process $(\tilde{M}_t)_{t \geq 0}$. This is the subject of Proposition 6.1. See below (2.31) in Section 2 to recall our notations regarding the components in $\tilde{\mathcal{G}}^{\geq h}$.

Proposition 6.1 (Exploration process). *There exists a family of (random) compact subsets $(K_t)_{t \geq 0} = (K_t(v, h, N))_{t \geq 0}$ of $\tilde{\mathcal{G}} \setminus B(v, \frac{1}{N})$, with $K_0 = K_0(v, N) \stackrel{\text{def.}}{=} \partial B(v, \frac{1}{N})$, satisfying the following properties:*

1. *$(K_t)_{t \geq 0}$ is increasing in t with respect to set inclusion and the map $t \mapsto K_t : [0, \infty) \rightarrow \mathcal{K}$ is continuous (see below (5.1)).*
2. *For all $t \geq 0$, each component of K_t intersects K_0 and $\{K_t \subset U\} \in \mathcal{F}_U$ for any open $U \subset \tilde{\mathcal{G}}$ (cf. (2.38)).*
3. *There exist $n \in \mathbb{N}$ and (random) timepoints $0 \leq \tau_{1,1} \leq \tau_{1,2} \leq \dots \leq \tau_{n,1} \leq \tau_{n,2} < \infty$ such that the following hold \mathbb{P} -a.s.*
 - i. $K_{\tau_{1,1}} = \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup K_0$ whereas $V \setminus \{v\} \subset K_{\tau_{n,2}}$ and $K_t = K_{\tau_{n,2}}$ for all $t \geq \tau_{n,2}$.
 - ii. *If $t \in [\tau_{k,1}, \tau_{k,2}]$, then $K_t = K_{\tau_{k,1}} \cup I_t$ where I_t is an interval and $K_{\tau_{k,1}}$ further satisfies $\partial K_{\tau_{k,1}} \setminus \tilde{\mathcal{G}}^{\leq h} \subset K_0$ and $\mathbf{P}_x[\tilde{X}_{H_{K_{\tau_{k,1}}}} \in K_0 \setminus \tilde{\mathcal{G}}^{\leq h}] = 0$ for all $x \in V \setminus \{v\}$.*
 - iii. *On the other hand if $t \in (\tau_{k,2}, \tau_{k+1,1}]$, then $K_t = K_{\tau_{k,2}} \cup \mathcal{C}_t$ where $\mathcal{C}_t \in \mathcal{K}$ is a connected subset of $\mathcal{C}_{x_k}^{\geq h}(B^c(v, \frac{1}{N}))$ with $x_k \in V$.*

Proof. We will do a certain type of *breadth-first exploration*. However, since we are exploring the components in the metric graph $\tilde{\mathcal{G}}$ and we need to exercise delicate control on the nature of the sets K_t ’s as well as the value of φ on these sets (see properties 2 and 3), we need to take additional care.

Let us start by describing in detail the timepoints $\tau_{1,1}$ and $\tau_{1,2}$ along with the sets $(K_t)_{0 \leq t \leq \tau_{1,2}}$. The timepoints $\tau_{k,1}, \tau_{k,2}$ and the sets $(K_t)_{\tau_{k-1,2} \leq t \leq \tau_{k,2}}$ for $k \geq 2$ will then be constructed inductively. We will verify properties 1–3 alongside the construction.

As already hinted, we obtain the sets $(K_t)_{t \geq 0}$ as well as the timepoints $\tau_{k,i}$ ’s from a certain *breadth-first algorithm* exploring the components. To this end, we introduce yet another sequence of non-decreasing (random) time points $\{\tau_1^j : j \in \mathbb{N}\}$ which correspond to “discrete rounds” of the underlying algorithm. Let us start from $\tau_1^0 = 0$ by inspecting the set $\partial B(v, \frac{1}{N}) \cap \tilde{\mathcal{G}}^{\geq h} = \{x'_1, \dots, x'_{d'}\}$ where $x'_1, \dots, x'_{d'}$ are arranged according to some arbitrary but deterministic ordering on V (which we assume to be fixed from now on). Letting $x_1, \dots, x_{d'}$

denote the (distinct) neighbors of v such that $x'_i \in [v, x_i]_{\mathcal{G}}$ (recall (2.1)), consider the points

$$(6.1) \quad x''_i \stackrel{\text{def.}}{=} \sup\{x \in [x'_i, x_i]_{\mathcal{G}} : [x'_i, x]_{\mathcal{G}} \subset \tilde{\mathcal{G}}^{\geq h} \setminus B(v, \frac{1}{N})\},$$

for $1 \leq i \leq d' (\leq d)$ where we take the supremum to be x'_i if this set is empty. Next we set the time point τ_1^1 as $\sum_{i=1}^{d'} \rho([x'_i, x''_i]_{\mathcal{G}}) = \tau_1^0 + \sum_{i=1}^{d'} \rho([x'_i, x''_i]_{\mathcal{G}})$ (an empty summation is 0 by convention) and define the sets K_t for $t \in (\tau_1^0, \tau_1^1]$ as follows. Let $d_t \in \{0, \dots, d' - 1\}$ denote the unique index such that

$$\sum_{i=1}^{d_t-1} \rho([x'_i, x''_i]_{\mathcal{G}}) < t (= t - \tau_1^0) \leq \sum_{i=1}^{d_t} \rho([x'_i, x''_i]_{\mathcal{G}}).$$

Now let

$$(6.2) \quad K_t = K_0 \cup \bigcup_{1 \leq i < d_t} [x'_i, x''_i]_{\mathcal{G}} \cup [x'_{d_t}, x'''_{d_t}]_{\mathcal{G}},$$

where $x'''_{d_t} \in [x'_{d_t}, x''_{d_t}]_{\mathcal{G}}$ is such that

$$\rho([x'_{d_t}, x]_{\mathcal{G}}) = t - \rho(\bigcup_{i=1}^{d_t-1} [x'_i, x''_i]_{\mathcal{G}}) = t - \tau_1^0 - \rho(\bigcup_{i=1}^{d_t-1} [x'_i, x''_i]_{\mathcal{G}}).$$

This concludes the first round in the construction of $(K_t)_{0 \leq t \leq \tau_1}$. The following properties are direct consequences of this definition and the continuity of φ .

- (a) $(K_t)_{0 \leq t \leq \tau_1^1}$ is increasing in t and the map $t \mapsto K_t : [0, \tau_1^1] \rightarrow \mathcal{K}$ is continuous (cf. property 1); (b) $\tau_1^1 = 0$ if $\partial B(v, \frac{1}{N}) \cap \tilde{\mathcal{G}}^{\geq h} = \emptyset$ and for all $0 \leq t \leq \tau_1^1$, $K_t = \cup_{x \in K_0} \mathcal{C}_{x,t}$ where each $(x \in) \mathcal{C}_{x,t} \in \mathcal{K}$ is a connected subset of $\mathcal{C}_x^{\geq h}(B^c(v, \frac{1}{N}))$ if $x \in \tilde{\mathcal{G}}^{\geq h}$ and $\{x\}$ otherwise (cf. properties 2 and 3-iii);
- (c) $\partial K_{\tau_1^1} \setminus \partial(\mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup K_0) \subset V$; and, (d) $\partial K_{\tau_1^1} \setminus \tilde{\mathcal{G}}^{\leq h} \subset V \cup K_0$ and $\mathbf{P}_x[\tilde{X}_{H_{K_{\tau_1^1}}} \in K_0 \setminus \tilde{\mathcal{G}}^{\leq h}] = 0$ for all $x \in V \setminus \{v\}$ (cf. property 3-ii).

Also for any $t \in [0, \infty)$, we have (cf. property 2)

$$(6.4) \quad \{K_{t \wedge \tau_1^1} \subset U\} \in \mathcal{F}_U \text{ for any open } U \subset \tilde{\mathcal{G}}.$$

To see this, let us consider the timepoints $\tau_1^{1,U}$ and the sets $(K_t^U)_{0 \leq t \leq \tau_1^{1,U}}$ which are defined analogously to τ_1^1 and $(K_t)_{0 \leq t \leq \tau_1^1}$ respectively but with $\tilde{\mathcal{G}}$ replaced by U (see (6.1) above). Clearly, $\tau_1^{1,U}$ and $(K_{t \wedge \tau_1^{1,U}}^U)_{t \geq 0}$ are measurable relative to \mathcal{F}_U . Furthermore since U is open, for any $t \geq 0$,

$$(6.5) \quad K_{t \wedge \tau_1^1} \subset U \text{ if and only if } K_{t \wedge \tau_1^{1,U}}^U \subset U \text{ in which case } K_{t \wedge \tau_1^1} = K_{t \wedge \tau_1^{1,U}}^U.$$

Together these observations imply (6.4).

Now consider the set $S_1^1 \stackrel{\text{def.}}{=} \{x \in \partial K_{\tau_1^1} \cap V : \varphi(x) > h\}$. If $S_1^1 \neq \emptyset$, we continue almost in a similar way as before. More precisely, suppose that we have completed some round $j \geq 1$ with

$$(6.6) \quad S_1^j \stackrel{\text{def.}}{=} \{x \in \partial K_{\tau_1^j} \cap V : \varphi(x) > h\} \neq \emptyset.$$

Let $x_{S_1^j} \in V$ denote the minimum vertex in S_1^j . We construct the timepoint τ_1^{j+1} and the sets $(K_t)_{\tau_1^j < t \leq \tau_1^{j+1}}$ similarly as τ_1^1 and $(K_t)_{\tau_1^0 < t \leq \tau_1^1}$ above with $\{x_{S_1^j}\}$ and τ_1^j and $K_{\tau_1^j}$ in place of $\partial B(v, \frac{1}{N})$, $\tau_1^0 (= 0)$ and K_0 (in (6.2)) respectively. If, on the other hand, $S_1^j = \emptyset$, we simply let $\tau_1^{j+1} = \tau_1^j$. One can verify the properties (a)–(d) in (6.3) as well as (6.4) (and (6.5)) for τ_1^{j+1} instead of τ_1^1 using the same arguments inductively.

Since φ is continuous on $\tilde{\mathcal{G}}$, it follows from this construction and (6.6) that if $S_1^j \neq \emptyset$ for some $j \geq 1$, then the points $x_{S_1^\ell} \in V; 1 \leq \ell \leq j$ are distinct. Consequently $S_1^J = \emptyset$ for some $J \geq 1$. In the rest of this paragraph we shall argue that

$$(6.7) \quad K_{\tau_1^J} = \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup K_0 \text{ and } \partial K_{\tau_1^J} \setminus \tilde{\mathcal{G}}^{\leq h} \subset K_0 \text{ with probability 1 (under } \mathbb{P})$$

(cf. property 3-i). The inclusion above is an immediate consequence of property (d) in (6.3) and (6.6). For the equality of sets, let \mathcal{C}_x denote the component of $K_{\tau_1^J}$ containing $x \in K_0 (\subset K_{\tau_1^J})$. By property (b) of $K_{\tau_1^J}$ in (6.3), we have $K_{\tau_1^J} = \cup_{x \in K_0} \mathcal{C}_x$. Therefore it suffices to show that, \mathbb{P} -a.s., $\mathcal{C}_x = \mathcal{C}'_x$ for all $x \in K_0$ where \mathcal{C}'_x is the component of $\mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup K_0$ containing x . So let us suppose that $\mathcal{C}_x \neq \mathcal{C}'_x$ for some $x \in K_0$ with positive probability. In view of property (b) of $K_{\tau_1^J}$, we have $\mathcal{C}_x \subset \mathcal{C}'_x$. As \mathcal{C}_x and \mathcal{C}'_x are compact, connected subsets of $\tilde{\mathcal{G}}$ (recall that φ is continuous on $\tilde{\mathcal{G}}$) and connected subsets of $\tilde{\mathcal{G}}$ are path-connected (see (2.3)), it follows from this inclusion (and $\mathcal{C}_x \neq \mathcal{C}'_x$) that there exists $y \in \partial \mathcal{C}_x$ and a continuous path γ_y starting at y such that $\text{image}(\gamma_y) \subset K_0 \cup \mathcal{C}_x^{\geq h}(B^c(v, \frac{1}{N}))$ whereas $\text{image}(\gamma_y) \cap \mathcal{C}_x = \{y\}$. Consequently if $y \notin V$, then $y \in \partial \mathcal{C}_x \setminus \partial(\mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N}))$. But the latter set is empty owing to property (c) of $K_{\tau_1^J}$ and hence $y \in V$. However, since $\varphi(x) \neq h$ for any $x \in V$ almost surely (note that $\text{Var}[\varphi(x)] > 0$ in view of (4.2), (4.8) and (3.14)), we obtain from the definition of S_1^J in (6.6) and the inclusion in property (d) of $K_{\tau_1^J}$ in (6.3) that $\partial K_{\tau_1^J} \cap V = \emptyset$ with probability 1. This implies that $\partial \mathcal{C}_x \cap V = \emptyset$, and thus $y \notin V$, almost surely as $\partial \mathcal{C}_x \subset \partial K_{\tau_1^J}$ which holds because the space $\tilde{\mathcal{G}}$ is locally (path-)connected. Therefore we have arrived at a contradiction which shows that our starting assumption $\mathcal{C}_x \neq \mathcal{C}'_x$ is false and completes the verification of (6.7).

We now have a clear choice for $\tau_{1,1}$, namely

$$\tau_{1,1} \stackrel{\text{def.}}{=} \tau_1^J.$$

If $(V \setminus \{v\}) \setminus K_{\tau_{1,1}} \neq \emptyset$, there exists a minimum vertex $x \in (V \setminus \{v\}) \setminus K_{\tau_{1,1}}$ (say) such that $K_{\tau_{1,1}} \cap \overline{I_{\{x,x'\}}} \neq \emptyset$ for some $\{x, x'\} \in E$. Let $x_{\tau_{1,1}}$ denote the point in $K_{\tau_{1,1}} \cap \overline{I_{\{x,x'\}}}$ that is nearest to x (so, in particular, $x_{\tau_{1,1}} \in \partial K_{\tau_{1,1}}$). Also let

$$\tau_{1,2} \stackrel{\text{def.}}{=} \tau_{1,1} + \rho([x_{\tau_{1,1}}, x]_{\mathcal{G}}) \text{ and } K_t \stackrel{\text{def.}}{=} K_{\tau_{1,1}} \cup [x_{\tau_{1,1}}, x'_t]_{\mathcal{G}} \text{ for } \tau_{1,1} \leq t \leq \tau_{1,2},$$

where $x'_t \in [x_{\tau_{1,1}}, x]_{\mathcal{G}}$ is such that $\rho([x_{\tau_{1,1}}, x'_t]_{\mathcal{G}}) = t - \tau_{1,1}$. We call the point x as $x_{\tau_{1,2}}$ ($\in V \cap K_{\tau_{1,2}}$) in the sequel. If, on the other hand, $(V \setminus \{v\}) \setminus K_{\tau_{1,1}} = \emptyset$, we simply let $\tau_{1,2} = \tau_{1,1}$. Since $\tau_{1,1} = \tau_1^J$ satisfies (6.3) and (6.7), the following properties follow directly from the above definitions (we interpret $\{x_{\tau_{1,2}}\}$ as \emptyset when $(V \setminus \{v\}) \setminus K_{\tau_{1,1}} = \emptyset$).

- (a') $(K_t)_{0 \leq t \leq \tau_{1,2}}$ is increasing in t and the map $t \mapsto K_t : [0, \tau_{1,2}] \rightarrow \mathcal{K}$ is continuous;
- (b') for $0 \leq t \leq \tau_{1,2}$, each component of K_t intersects K_0 ;
- (c') for $\tau_{0,2}(=0) < t \leq \tau_{1,1}$, K_t has a component contained in $\mathcal{C}_x^{\geq h}(B^c(v, \frac{1}{N}))$ for some $x \in K_0$;
- (d') for $\tau_{1,1} < t \leq \tau_{1,2}$, $K_t = K_{\tau_{1,1}} \cup I_t$ for some interval I_t ; and,
- (e') $\partial K_{\tau_{1,2}} \setminus \tilde{\mathcal{G}}^{\leq h} \subset \{x_{\tau_{1,2}}\} \cup K_0$ a.s. whereas $\mathbf{P}_x[X_{H_{K_{\tau_{1,2}}}} \in K_0 \setminus \tilde{\mathcal{G}}^{\leq h}] = 0$ for all $x \in V \setminus \{v\}$.

Restricting this construction so as to only “look inside” an open set $U \subset \tilde{\mathcal{G}}$, as in (6.5), we also obtain property 6.4 (and 6.5) for $\tau_{1,2}$.

Now suppose that we have constructed timepoints $(\tau_{1,1} < \tau_{1,2} \leq) \tau_{2,1} \dots \leq \tau_{k,1} < \tau_{k,2}$ (notice the strict inequalities between $\tau_{\ell,1}$ and $\tau_{\ell,2}$'s) and the corresponding family of sets $(K_t)_{0 \leq t \leq \tau_{k,2}}$ for some $k \geq 1$ such that — in addition to the aforementioned properties of $(K_t)_{0 \leq t \leq \tau_{1,2}}, \tau_{1,1}$

and $\tau_{1,2}$ — one also has the following properties for each $1 < \ell \leq k$. (6.4) holds with $\tau_{\ell,2}$ in place of τ_1^1 , (6.7) holds with $\tau_{\ell-1,2}$ in place of τ_1^J , and properties (a'), (b') and (c')–(e') in (6.8) hold with $\tau_{\ell-1,2}$, $\tau_{\ell,1}$ and $\tau_{\ell,2}$ instead of $\tau_{0,2}$, $\tau_{1,1}$ and $\tau_{1,2}$ respectively for some $x_{\tau_{\ell,2}} \in V \cap K_{\tau_{\ell,2}}$. Finally, (6.8)-(b') is replaced by:

$$K_t = K_{\tau_{\ell,2}} \cup \mathcal{C}_t \text{ for some connected } C_t \subset \mathcal{C}_{x_{\tau_{\ell,2}}}^{\geq h}(B^c(v, \frac{1}{N})) \text{ for all } \tau_{\ell,1} < t \leq \tau_{\ell,2}.$$

If $(V \setminus \{v\}) \setminus K_{\tau_{k,1}} = \emptyset$, we stop and let $K_t = K_{\tau_{k,2}}$ for all $t \geq \tau_{k,2}$ and $\tau_{m,1} = \tau_{m,2} = \tau_{k,2}$ for all $m > k$. Properties 1–3 in Proposition 6.1 with $n = k$ are immediate consequences of our hypothesis.

Otherwise if $(V \setminus \{v\}) \setminus K_{\tau_{k,1}} \neq \emptyset$, we can repeat the same sequence of steps with $\{x_{K_{\tau_{k,2}}}\}, \tau_{k,2}$ and $\{x_{K_{\tau_{k,2}}}\}$ instead of $\partial B(v, \frac{1}{N}), \tau_{0,2}(=0)$ and K_0 (in (6.2)) respectively to obtain timepoints $0 \leq \tilde{\tau}_{k+1,1} \leq \tilde{\tau}_{k+1,2} < \infty$ and sets $(\tilde{K}_t)_{\tau_{k+1,1} \leq t \leq \tilde{\tau}_{k+1,2}}$. Thus letting $\tau_{k+1,i} = \tau_{k,2} + \tilde{\tau}_{k+1,i}$ ($i = 1, 2$) and $K_t \stackrel{\text{def.}}{=} K_{\tau_{k,1}} \cup \tilde{K}_{t-\tau_{k,2}}$ for $\tau_{k,1} < t \leq \tau_{k+1,2}$, we see that the hypothesis in the previous paragraph is satisfied with $k+1$ instead of k . However, since $x_{K_{\tau_{k,2}}} \in (K_{\tau_{k,2}} \setminus K_{\tau_{k,1}}) \cap V$ in this case, after some $K \leq |V|$ iterations of this sequence we will meet the stopping condition $(V \setminus \{v\}) \setminus K_{\tau_{K,1}} = \emptyset$ leading to our desired family of timepoints and the corresponding sets. \square

Next we present a result which shows that $\mathcal{C}_v^{\geq h}$ is not “too far” and can be easily reconstructed from $\mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N}))$ (see property 3-i above) on an event whose probability tends to 1 as $N \rightarrow \infty$. To this end, fix $v \in V, N \geq 1$ and $h \in \mathbb{R}$ and consider the following event:

$$(6.9) \quad F_v = F_v(h, N) \stackrel{\text{def.}}{=} \left\{ \overline{B(v, \frac{1}{N})} \subset \mathcal{G}^{\geq h} \text{ if } \varphi(v) \geq h \text{ and } \overline{B(v, \frac{1}{N})} \subset \mathcal{G}^{< h} \text{ otherwise} \right\}.$$

Lemma 6.2 (Asymptotic irrelevance of N). *On the event F_v we have, with $K_0 = \partial B(v, \frac{1}{N})$,*

$$(6.10) \quad \mathcal{C}_v^{\geq h} = \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup B(v, \frac{1}{N}),$$

if $\varphi(v) \geq h$ whereas

$$(6.11) \quad \mathcal{C}_v^{\geq h} = \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) = \emptyset,$$

if $\varphi(v) < h$. Furthermore,

$$(6.12) \quad \mathbb{P}[F_v^c] \leq \frac{1}{N^c}.$$

Proof. (6.11) is immediate since $\overline{B(v, \frac{1}{N})} \subset \mathcal{G}^{< h}$ on the event $F_v \cap \{\varphi(v) < h\}$ (cf. (6.9)). So we give the argument for (6.10). In this case we have

$$(6.13) \quad \overline{B(v, \frac{1}{N})} \subset \mathcal{G}^{\geq h}.$$

Let $y \in \mathcal{C}_v^{\geq h}$ and we will argue that $y \in \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup B(v, \frac{1}{N})$. This is clear if $y \in B(v, \frac{1}{N})$. Otherwise $y \in \mathcal{C}_v^{\geq h} \setminus B(v, \frac{1}{N})$ and hence must be connected to $\partial B(v, \frac{1}{N})(= K_0)$ in $B^c(x, \frac{1}{N}) \cap \mathcal{G}^{\geq h}$. But in that case $y \in \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N}))$. Thus we get

$$\mathcal{C}_v^{\geq h} \subset \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup B(v, \frac{1}{N}).$$

For the opposite inclusion, consider $y \in \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup B(v, \frac{1}{N})$. If $y \in B(v, \frac{1}{N})$, we see that $y \in \mathcal{C}_v^{\geq h}$ because of (6.13). Else $y \in \mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N}))$ and hence is connected to K_0 in $\mathcal{G}^{\geq h}$. Since $K_0 = \partial B(v, \frac{1}{N})$ is itself connected to v in $\mathcal{G}^{\geq h}$ due to (6.13), we therefore obtain

$$\mathcal{C}_{K_0}^{\geq h}(B^c(v, \frac{1}{N})) \cup B(v, \frac{1}{N}) \subset \mathcal{C}_v^{\geq h}.$$

Combined with the previous inclusion (6.10) follows.

Next we prove (6.12). Letting $\varepsilon > 0$ (choice will be specified later), we introduce the following events.

$$D_{v,1} = \{\varphi(v) \geq h + \varepsilon, \overline{B(v, \frac{1}{N})} \cap \tilde{\mathcal{G}}^{<h} \neq \emptyset\}, D_{v,2} = \{\varphi(v) \leq h - \varepsilon, \overline{B(v, \frac{1}{N})} \cap \tilde{\mathcal{G}}^{\geq h} \neq \emptyset\} \text{ and}$$

$$D_{v,3} = \{h - \varepsilon < \varphi(v) < h + \varepsilon\}.$$

Clearly $F_v^c \subset D_{v,1} \cup D_{v,2} \cup D_{v,3}$ (see (6.9)) and hence

$$(6.14) \quad \mathbb{P}[F_v^c] \leq \mathbb{P}[D_{v,1} \cup D_{v,2}] + \mathbb{P}[D_{v,3}].$$

We will bound each of these three probabilities separately starting with $\mathbb{P}[D_{v,3}]$. Since $\varphi(v) \sim N(0, \sigma_v^2)$ with $\sigma_v \geq c$ (follows from (4.2), (4.8) and (3.14)), we have

$$(6.15) \quad \mathbb{P}[D_{v,3}] = \mathbb{P}[h - \varepsilon < \varphi(v) < h + \varepsilon] \leq C\varepsilon.$$

Next we bound $\mathbb{P}[D_{v,1}]$ and $\mathbb{P}[D_{v,2}]$. Notice that, in view of the definitions of $D_{v,1}$ and $D_{v,2}$,

$$(6.16) \quad \begin{aligned} \mathbb{P}[D_{v,1} \cup D_{v,2}] &\leq \mathbb{P}\left[\sup_{x \in \overline{B(v, \frac{1}{N})}} |\varphi(v) - \varphi(x)| > \varepsilon/2\right] \\ &\leq d \sup_{e \sim v} \mathbb{P}\left[\sup_{x \in \overline{I_e}, d(v, x) \leq \frac{1}{N}} |\varphi(v) - \varphi(x)| > \varepsilon/2\right] \end{aligned}$$

where $e \sim v$ if v is an endpoint of e . Since the process $(\varphi(x))_{x \in \overline{I_e}}$, where $e = (v, y)$, is distributed as a standard Brownian bridge between $\varphi(v)$ and $\varphi(y)$ conditionally on $(\varphi(v), \varphi(y))$ (see below display (2.30) in Section 2), we can write

$$\mathbb{P}\left[\sup_{x \in \overline{I_e}, d(v, x) \leq \frac{1}{N}} |\varphi(v) - \varphi(x)| > \varepsilon/2\right] \leq \mathbb{P}[|\varphi(v) - \varphi(y)| > N\varepsilon/4] + \mathbf{P}^{\text{br}}[\sup_{t \in [0, \frac{1}{N}]} |B_t| > \varepsilon/4],$$

where B_t is a standard Brownian bridge of length 1 under \mathbf{P}^{br} . The first probability is bounded by $e^{-cN^2\varepsilon^2}$ as $\varphi(v) - \varphi(y) \sim N(0, \sigma_{vy}^2)$ for some $\sigma_{vy} \leq C$ whereas the second probability is bounded by $Ce^{-cN\varepsilon^2}$ which we obtain using standard results on Brownian bridges (see, e.g., [13, Chapter IV.26]). Plugging these two bounds into the previous display and the resulting bound into (6.16) we get

$$\mathbb{P}[D_{v,1} \cup D_{v,2}] \leq Ce^{-CN\varepsilon^2}.$$

Together with (6.15) this yields, in view of (6.14),

$$\mathbb{P}[F_v^c] \leq C(\exp(-cN\varepsilon^2) + \varepsilon).$$

Now setting $\varepsilon = N^{-1/4}$ we obtain (6.11). \square

6.2. Exploration martingale. For any $K \in \mathcal{K}_{<\infty}$, recall the random variable $M_K = \tilde{\mathcal{E}}(\varphi, f_K)$ from Section 4. Now given any $v \in V$, $h \in \mathbb{R}$ and integer $N > 1$, let us consider the family of sets $(K_t)_{t \geq 0} = (K_t(v, h, N))_{t \geq 0}$ given by Proposition 6.1. Since $K_t \in \mathcal{K}_{<\infty}$ by property 2 of $(K_t)_{t \geq 0}$, we may define

$$(6.17) \quad \tilde{M}_t = \tilde{M}_t(v, h, N) \stackrel{\text{def.}}{=} M_{K_t}$$

for all $t \geq 0$. In the following lemma, we record some important properties of the process $(\tilde{M}_t)_{t \geq 0}$.

Lemma 6.3 (Quadratic variation of $(\tilde{M}_t)_t$). *The process $(\tilde{M}_t)_{t \geq 0}$ is a continuous (\mathcal{F}_{K_t}) -martingale with quadratic variation given by*

$$(6.18) \quad \langle \tilde{M} \rangle_t = 2|E|(\nu_{K_t} - \nu_{K_0}), \quad t \geq 0.$$

Proof. The continuity of $\tilde{M}_t = M_{K_t}$ as a function of t follows from property 1 of K_t 's and Lemma 5.1. Owing to property 2, K_t satisfies property (2.38) in view of (2.37) for all $t \geq 0$. Also each K_t is a subset of the compact set $K_\infty \stackrel{\text{def.}}{=} \tilde{\mathcal{G}} \setminus B(v, \frac{1}{N}) \in \mathcal{K}_{<\infty}$. Consequently, by Proposition 4.4,

$$\tilde{M}_t = \mathbb{E}[M_{K_\infty} \mid \mathcal{F}_{K_t}]$$

and thus $(\tilde{M}_t)_{t \geq 0}$ is a (uniformly integrable) (\mathcal{F}_{K_t}) -martingale. Finally, since $\text{Var}[M_{K_\infty} \mid \mathcal{F}_{K_t}] = 2|E|(\nu_{K_\infty} - \nu_{K_t})$ by Lemma 4.5 which is \mathbb{P} -almost surely continuous as a function of t due to Lemma 5.2, it follows from [25, Corollary 10] that the quadratic variation of $(\tilde{M}_t)_{t \geq 0}$ is given by the expression

$$\langle \tilde{M} \rangle_t = \text{Var}[M_{K_\infty} \mid \mathcal{F}_{K_0}] - \text{Var}[M_{K_\infty} \mid \mathcal{F}_{K_t}] = 2|E|(\nu_{K_t} - \nu_{K_0})$$

for all $t \geq 0$. \square

The following result follows from application of the classical Dubins–Schwarz theorem (see, e.g., [57, Chapter V, Theorem 1.7] for the particular version used here and also [57, Chapter IV, Proposition 1.13]) to the continuous martingale $(\tilde{M}_t)_{t \geq 0}$. In Section 7, this will enable us to calculate *tail probabilities* for the clusters of $\mathcal{G}^{\geq h}$ after being translated into suitable events measurable relative to \tilde{M}_t .

Theorem 6.4. *For any $t \geq 0$, let $T_t \stackrel{\text{def.}}{=} \inf\{s \geq 0 : \langle \tilde{M} \rangle_s > t\}$. Then the time-changed process $(\tilde{M}_{T_t})_{t \geq 0}$ is distributed as $(B_{t_0 + t \wedge \langle \tilde{M} \rangle_\infty})_{t \geq 0}$ where $(B_t)_{t \geq 0}$ is a standard Brownian motion and $t_0 = 2|E|\nu_{K_0}$. Furthermore, \tilde{M}_s is constant on each interval $[T_{t-}, T_t]$ where $T_{t-} \stackrel{\text{def.}}{=} \lim_{u \uparrow t} T_u$ for $t \geq 0$.*

7. TAIL PROBABILITIES FOR CLUSTER VOLUME

This section is devoted to the proof our main results, namely, Theorems 1.1 and 1.2. The corresponding upper tail probabilities derive from the following result.

Theorem 7.1 (Upper bounds on cluster volume). *For any $\delta \in (0, 1)$ and $h = -A|V|^{-1/3}$ with $A \geq 0$, we have*

$$(7.1) \quad \mathbb{P}[|\mathcal{C}_{\max}^{\geq h}| \geq \frac{1}{\delta}|V|^{2/3}] \leq C(\delta^{3/2} + \delta A).$$

On the other hand, for $h = A|V|^{-1/3}$ and $\delta|V|^{2/3} \geq C$ we have

$$(7.2) \quad \mathbb{P}[|\mathcal{C}_{\max}^{\geq h}| \geq \delta|V|^{2/3}] \leq \frac{C}{\delta^{3/2}} e^{-cA^2\delta}.$$

For the lower tail probabilities we need:

Theorem 7.2 (Upper bounds on cluster volume). *For $h = A|V|^{-1/3}$ with $A \geq 0$ and any $\delta \in (0, 1)$ such that $|V| \geq \frac{(1+A)^8}{\delta^3}$, we have*

$$(7.3) \quad \mathbb{P}[|\mathcal{C}_{\max}^{\geq h}| \leq \delta|V|^{2/3}] \leq C(1+A)^{1/2}\delta^{1/5}.$$

On the other hand, there exists $c \in (0, \infty)$ such that for $h = -A|V|^{-1/3}$ and any $\delta \in (0, 1)$ satisfying $\frac{1}{\delta} \leq c \min(|V|^{1/3}, A)$, we have

$$(7.4) \quad \mathbb{P}[|\mathcal{C}_{\max}^{\geq h}| \leq \frac{1}{\delta}|V|^{2/3}] \leq C(\tilde{\delta}^{1/5} + e^{-cA^2\tilde{\delta}})$$

for all $|V|^{-1/3} \leq \tilde{\delta} \leq 1$.

We can now deduce our main results assuming these bounds.

Proofs of Theorems 1.1 and 1.2. The bound (1.2) in Theorem 1.1 follows by combining (7.1) in Theorem 7.1 with (7.4) in Theorem 7.2 above. For the bound (1.3) in Theorem 1.2, we use (7.2) in Theorem 7.1 with $\delta = C \frac{\log eA}{A^2}$ for a suitable $C \in (0, \infty)$. For (1.4), we apply (7.4) in Theorem 7.2 with the choices $\delta = cA$ and $\tilde{\delta}$ for a suitable $c \in (0, \infty)$. \square

In §7.1 and §7.2 below, we give the proofs of Theorems 7.1 and 7.2 respectively. In the sequel, for any two events E_1 and E_2 defined on the probability space underlying \mathbb{P} , we say that $E_1 \subset E_2$ on an event with (\mathbb{P} -) probability 1 if $E_1 \cap E \subset E_2 \cap E$ for some event E such that $\mathbb{P}[E] = 1$.

7.1. Upper tail probabilities for cluster volume. We prove Theorem 7.1 in this subsection. The following bound on the size of a generic cluster is crucial (see, e.g., [54] for similar upper bounds in the context of Erdős-Rényi graphs).

Lemma 7.3. *For $T \geq 1$ and $h = -A|V|^{-1/3}$ with $A \geq 0$, one has*

$$(7.5) \quad \max_{x \in V} \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \leq C \left(\frac{1}{\sqrt{T}} + A|V|^{-1/3} \right),$$

whereas for $h = A|V|^{1/3}$ and $T \geq C$, we have

$$(7.6) \quad \max_{x \in V} \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \leq \frac{C}{\sqrt{T}} e^{-ch^2 T}.$$

Before we prove Lemma 7.3, let us deduce Theorem 7.1 using it.

Proof of Theorem 7.1. For $T \geq 1$, consider the (random) set $N_T^{\geq h} \stackrel{\text{def.}}{=} \{x \in V : |\mathcal{C}_x^{\geq h}| \geq T\}$. By definition of $N_T^{\geq h}$ and the Markov inequality, we have

$$(7.7) \quad \mathbb{P}[|\mathcal{C}_{\max}^{\geq h}| \geq T] = \mathbb{P}[|N_T^{\geq h}| \geq T] \leq \frac{\mathbb{E}[N_T^{\geq h}]}{T}.$$

When $h = -A|V|^{-1/3}$, the expectation on the right-hand side can be bounded as:

$$\frac{\mathbb{E}[N_T^{\geq h}]}{T} = \frac{1}{T} \sum_{x \in V} \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \leq \frac{|V|}{T} \max_{x \in V} \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \stackrel{(7.5)}{\leq} C \left(\frac{|V|}{T^{3/2}} + \frac{A|V|^{2/3}}{T} \right).$$

Now setting $T = \frac{1}{\delta}|V|^{2/3}$, we can deduce (7.1) from these two bounds.

On the other hand, when $h = A|V|^{-1/3}$, we can recompute (7.7) as follows:

$$\frac{\mathbb{E}[N_T^{\geq h}]}{T} \leq \frac{|V|}{T} \max_{x \in V} \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \stackrel{(7.6)}{\leq} C \frac{|V|}{T^{3/2}} e^{-\frac{ch^2 T}{2}}$$

(provided $T \geq C$). Now plugging $T = \delta|V|^{2/3}$ we obtain (7.2). \square

Our next result is an important step in the pending proof of Lemma 7.3.

Lemma 7.4 (Large cluster to positivity of Brownian motion). *There exist $C_1, c_1 \in (0, \infty)$ such that for any $x \in V$, $h \in \mathbb{R}$ and $T \geq 2$ we have*

$$(7.8) \quad \mathbb{P}[|\mathcal{C}_x^{\geq h}| \geq T] \leq \mathbf{P}_0^B \left[\inf_{C_1 \leq t \leq c_1 T} (B_t - ht) \geq 0 \right],$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion under \mathbf{P}_0^B .

Proof. For $x \in V$ and $N \geq 1$ an integer, let $K_0 = K_0(x, N) = \partial B(x, \frac{1}{N})$ (cf. Proposition 6.1). Recall the event $F_x = F_x(h, N)$ from (6.9). It follows from (6.11) in Lemma 6.2 that

$$\{|\mathcal{C}_x^{\geq h}| \geq T\} \subset \{|\mathcal{C}_{K_0}^{\geq h}(B^c(x, \frac{1}{N}))|_V \geq T - 1\} \text{ on the event } F_x$$

where, as in Section 3, $|S|_V = |S \cap V|$ (note that $\{|\mathcal{C}_x^{\geq h}| \geq T\} \subset \{\varphi(x) \geq h\}$ as $T > 0$). Hence for $T \geq 2$, we have

$$\{|\mathcal{C}_x^{\geq h}| \geq T\} \subset \{|\mathcal{C}_{K_0}^{\geq h}(B^c(x, \frac{1}{N}))|_V \geq cT\} \text{ on the event } F_x.$$

Now consider the family of sets $(K_t)_{t \geq 0}$ given by Proposition 6.1 with $v = x$, h as in the statement of the lemma and N as above. From properties 1 and 3-i (the first part) of $(K_t)_{t \geq 0}$, we get the inclusion

$$\begin{aligned} \{|\mathcal{C}_{K_0}^{\geq h}(B^c(x, \frac{1}{N}))|_V \geq cT\} &\subset \{|K_{\tau_{1,1}}|_V \geq cT\} \\ &\subset \{\exists T' \geq 0 \text{ such that } K_t \subset \mathcal{G}^{\geq h} \text{ for } t \in [0, T'] \text{ with } |K_{T'}|_V \geq cT\} \end{aligned}$$

on an event with probability 1 (this will be henceforth assumed implicitly for all inclusions in the proof). Next consider the continuous (\mathcal{F}_{K_t}) -martingale $(M_{K_t})_{t \geq 0} = (\tilde{M}_t)_{t \geq 0}$ attached to the random sets $(K_t)_{t \geq 0}$ (see (6.17) and Lemma 6.3). Using Lemma 4.3, we can write

$$\begin{aligned} &\{\exists T' \geq 0 \text{ such that } K_t \subset \mathcal{G}^{\geq h} \text{ for } t \in [0, T'] \text{ with } |K_{T'}|_V \geq cT\} \\ &\subset \{\exists T' \geq 0 \text{ such that } \tilde{M}_t \geq 2h|E|\nu_{K_t} \text{ for } t \in [0, T'] \text{ with } |K_{T'}|_V \geq cT\} \end{aligned}$$

Recalling the definition of T_t from Theorem 6.4, we further have

$$\begin{aligned} &\{\exists T' \geq 0 \text{ such that } \tilde{M}_t \geq 2h|E|\nu_{K_t} \text{ for } t \in [0, T']\} \\ &\subset \{\exists T' \geq 0 \text{ such that } \tilde{M}_{T_r} \geq 2h|E|\nu_{K_{T_r}} \text{ for } r \in [0, \langle \tilde{M} \rangle_{T'}]\}. \end{aligned}$$

Combining the chain of inclusions comprising the last four displays, we obtain
(7.9)

$$\{|\mathcal{C}_x^{\geq h}| \geq T\} \subset \{\exists T' \geq 0 \text{ such that } \tilde{M}_{T_r} \geq 2h|E|\nu_{K_{T_r}} \text{ for } r \in [0, \langle \tilde{M} \rangle_{T'}] \text{ and } |K_{T'}|_V \geq cT\}$$

on the event F_x .

Theorem 6.4 also tells us that the process $(\tilde{M}_{T_r})_{r \geq 0}$ has the same law as the stopped (standard) Brownian motion $(B_{r_0+r \wedge \langle \tilde{M} \rangle_\infty})_{r \geq 0}$ with $r_0 = 2|E|\nu_{K_0}$. Since $\langle \tilde{M} \rangle_r = 2|E|\nu_{K_r} - r_0 = r \wedge \langle \tilde{M} \rangle_\infty - r_0$ by Lemma 6.3 and the continuity of quadratic variation, together with (7.9) and the lower bound on $2|E|\nu_K$ from Proposition 3.5 this yields:

$$\mathbb{P}[|\mathcal{C}_{K_0}^{\geq h}(B^c(x, \frac{1}{N}))|_V \geq cT] \leq \mathbf{P}_0^B \left[\inf_{r_0 \leq r \leq cT} (B_r - hr) \geq 0 \right] + \mathbb{P}[F_x^c].$$

From this and the bound (6.12) in Lemma 6.2, we can deduce (7.8) by sending $N \rightarrow \infty$. \square

Proof of Lemma 7.3. We first give the proof of (7.5). We can assume, without any loss of generality, that $T \geq C$ and $|h| = A|V|^{-1/3} \leq 1$. In view of Lemma 7.4, it suffices to prove the upper bound for $\mathbf{P}_0^B \left[\inf_{C_1 \leq t \leq c_1 T} (B_t - ht) \geq 0 \right]$. Notice that

$$\mathbf{P}_0^B \left[\inf_{C_1 \leq t \leq c_1 T} (B_t - ht) \geq 0 \right] \leq \bar{\mathbf{P}}_0^B \left[\inf_{0 \leq t \leq c_2 T} (B_t - ht) \geq X \right],$$

for some $c_2 > 0$ where $X \sim N(C_1 h, C_1)$ is independent of the standard Brownian motion $(B_t)_{t \geq 0}$ under $\bar{\mathbf{P}}_0^B$. Using existing results (see, e.g., [13, Equation 2.0.2 in Part II]), we can write,

$$\bar{\mathbf{P}}_0^B \left[\inf_{0 \leq t \leq c_2 T} (B_t - ht) \geq X \right] = \int_{x>0} \Phi \left(\frac{x - c_2 h T}{\sqrt{c_2 T}} \right) \phi_h(x) dx - \int_{x>0} e^{2xh} \bar{\Phi} \left(\frac{x + c_2 h T}{\sqrt{c_2 T}} \right) \phi_h(x) dx,$$

where $\phi_h(x) = \frac{1}{\sqrt{2\pi C_1}} \exp(-\frac{(x+C_1h)^2}{2C_1})$ is the density function of $N(-C_1h, C_1)$. Since $\bar{\Phi}(b) = \Phi(-b)$ and $e^{2ah} \geq 1 + 2ah$, we get the bound

$$(7.10) \quad \begin{aligned} & \bar{\mathbf{P}}_0^B \left[\inf_{0 \leq t \leq c_2 T} (B_t - ht) \geq X \right] \\ & \leq \int_{x>0} \left(\Phi\left(\frac{x - c_2 h T}{\sqrt{c_2 T}}\right) - \Phi\left(\frac{-x - c_2 h T}{\sqrt{c_2 T}}\right) \right) \phi_h(x) dx - \int_{x>0} 2xh\phi_h(x) dx = I_1 + I_2. \end{aligned}$$

Using the fact that $|\Phi(x) - \Phi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$, we obtain (recall that $h < 0$ for the final step)

$$(7.11) \quad \begin{aligned} I_1 & \leq \frac{C}{\sqrt{T}} \int_{x>0} x\phi_h(x) dx = \frac{C}{\sqrt{T}} \int_{x>0} (x + C_1 h)\phi_h(x) dx - \frac{C}{\sqrt{T}} C_1 h \int_{x>0} \phi_h(x) dx \\ & \leq \frac{C}{\sqrt{T}} (e^{-ch^2} - Ch). \end{aligned}$$

On the other hand, by same computations,

$$I_2 \leq -Ch e^{-ch^2} + Ch^2.$$

Since $-h = A|V|^{-1/3} \leq 1$ by our assumption, plugging these into (7.10) we get

$$\bar{\mathbf{P}}_0^B \left[\inf_{0 \leq t \leq c_2 T} (B_t - ht) \geq X \right] \leq C \left(\frac{1}{\sqrt{T}} + A|V|^{-1/3} \right)$$

which concludes the proof of (7.5).

We now give the proof of (7.6). We proceed similarly until (7.10) (recall that $T \geq C$) and note that $I_2 \leq 0$ as $h > 0$. So we only need to bound I_1 which we need to do more carefully. To this end, let us decompose

$$I_1 = \int_{\frac{c_2}{2} h T \geq x > 0} + \int_{x > \frac{c_2}{2} h T}$$

where the two terms correspond to the integral in I_1 (see (7.10)) restricted to the respective ranges. To bound the first term, we use the improved estimate $|\Phi(x) - \Phi(y)| \leq e^{-ch^2 T} |x - y|$ for $x, y \leq -\frac{c_2}{2} h T$. This gives us (cf. (7.11))

$$\int_{\frac{c_2}{2} h T \geq x > 0} \leq \frac{Ce^{-ch^2 T}}{\sqrt{T}} \int_{x>0} x\phi_h(x) dx \leq \frac{Ce^{-ch^2 T}}{\sqrt{T}}.$$

For the other part, we bound as in (7.11) and obtain a similar bound, i.e.,

$$\int_{x > \frac{c_2}{2} h T} \leq \frac{C}{\sqrt{T}} \int_{x > \frac{c_2}{2} h T} x\phi_h(x) dx \leq \frac{Ce^{-ch^2 T}}{\sqrt{T}}.$$

Adding these two bounds we get (7.6). \square

7.2. Lower tail probabilities for cluster volume. In this subsection we give the proof of Theorem 7.2. We will only give the proof of (7.3) as the proof of (7.4) follows by adapting parts of the argument in a relatively straightforward manner. Let $v \in V$ be arbitrary (but fixed) and consider the family of sets $(K_t)_{t \geq 0}$ obtained from Proposition 6.1 with v, h as above (7.3) and some integer $N > 1$. In the sequel v, h, A and $(K_t)_{t \geq 0}$ always refer to these particular choices.

Let us recall the sequence of (possibly random) timepoints $0 \leq \tau_{1,1} \leq \tau_{1,2} \leq \dots \leq \tau_{n,1} \leq \tau_{n,2} < \infty$ from property 3 of $(K_t)_{t \geq 0}$ in Proposition 6.1. We augment this sequence by letting $\tau_{0,1} = \tau_{0,2} = 0$. The following two events play crucial roles in the proof of (7.3). Attached to the family $(K_t)_{t \geq 0}$ is the continuous (\mathcal{F}_{K_t}) -martingale $(M_{K_t})_{t \geq 0} = (\tilde{M}_t)_{t \geq 0}$ ((6.17) and Lemma 6.3).

Throughout this section, $\delta', \delta'' \in (0, 1)$ and $\beta, b \in (0, \infty)$ represent generic parameters for our events whereas their dependence on N is kept implicit. Let us start with

$$(7.12) \quad F_1(\delta', \beta, b) \stackrel{\text{def.}}{=} \left\{ \begin{array}{l} \tilde{M}_t \leq (\delta' + \beta A \delta'^{1/2}) |V|^{1/3} \text{ for any } \tau_{k,1} \leq t \leq \tau_{k,2} \\ \text{and } 0 \leq k \leq n \text{ such that } |K_{\tau_{k,1}}|_V \leq b \delta'^{1/2} |V|^{2/3} \end{array} \right\}.$$

Next let us consider

$$(7.13) \quad F_2(\delta', \delta'', \beta, b) \stackrel{\text{def.}}{=} \left\{ \begin{array}{l} \exists 0 \leq t_1 \leq t_2 \text{ s.t. } 2|E|(\nu_{K_{t_2}} - \nu_{K_{t_1}}) \geq \beta \delta'' |V|^{2/3}, |K_{t_2}|_V \leq \\ b \delta'^{1/2} |V|^{2/3} \text{ and } \tilde{M}_t > (\delta' + \beta A \delta'^{1/2}) |V|^{1/3} \text{ for all } t_1 \leq t \leq t_2 \end{array} \right\}.$$

Lemma 7.5. *There exists $C_2 \in (0, \infty)$ such that for any $\delta, \delta' \in (0, 1)$, $\beta \geq C_2$, $b > 0$ and $|V| \geq C((1/\delta)^{3/2} \vee b^3)$, we have*

$$(7.14) \quad F_1(\delta', \beta, b) \cap F_2(\delta', \delta, \beta, b) \subset \{|\mathcal{C}_{\max}^{\geq h}| \geq \delta |V|^{2/3}\}$$

on an event with probability 1.

Proof. Since K_t 's are increasing by property 1 and $K_t = K_{\tau_{n,2}}$ for all $t \geq \tau_{n,2}$ by property 3-i (in Proposition 6.1), it follows from the definitions in (7.12) and (7.13) that

$$(7.15) \quad F_1(\delta', \beta, b) \cap F_2(\delta', \delta, \beta, b) \subset \left\{ \begin{array}{l} \exists 0 \leq k < n \text{ and } \tau_{k,2} < t < \tau_{k+1,1} \text{ s.t. } |K_t|_V \leq \\ b \delta'^{1/2} |V|^{2/3} \text{ and } 2|E|(\nu_{K_t} - \nu_{K_{\tau_{k,2}}}) \geq \beta \delta |V|^{2/3} \end{array} \right\}.$$

on an event with \mathbb{P} -probability 1 (we will assume this implicitly for all relevant inclusions in the remainder of the section). Now from property 3-iii and (the first part of) property 2, we know that, a.s., $K_t = K_{\tau_{k,2}} \cup \mathcal{C}_t$ for any $t \in (\tau_{k,2}, \tau_{k+1,1})$ and $0 \leq k < n$ where either $\mathcal{C}_t \in \mathcal{K}$ is a connected subset of $\mathcal{C}_{x_k}^{\geq h}(B^c(v, \frac{1}{N}))$ for some $x_k \in V$ or each component of \mathcal{C}_t is a connected subset of $\{x\} \cup \mathcal{C}_x^{\geq h}$ for some $x \in K_0 = \partial B(v, \frac{1}{N})$. By Lemma 3.6, we thus have that

$$2|E|(\nu_{K_t} - \nu_{K_{\tau_{k,2}}}) \leq C(|\mathcal{C}_t|_V + b_0(K_{\tau_{k,2}})),$$

provided $|\mathcal{C}_t|_V + |K_{\tau_{k,2}}|_V + b_0(K_{\tau_{k,2}}) \leq c|V|$. Since $b_0(K_{\tau_{k,2}}) \leq d$ by property 2, we can combine this display with (2.3) and (7.15) to deduce

$$F_1(\delta', \beta, b) \cap F_2(\delta', \delta, \beta, b) \subset \{|\mathcal{C}_{\max}^{\geq h}| \geq c \beta \delta |V|^{2/3}\} \text{ when } |V| \geq C((1/\beta \delta)^{3/2} \vee b^3).$$

From this (7.14) follows by letting $C_2 = \frac{1}{c} \vee 1$. \square

In our next result, we give lower bounds on the probabilities of these two events.

Lemma 7.6. *Let $\delta \in (0, 1)$. There exist $C \in [C_2, \infty)$, $c, c' \in (0, \infty)$ and $\delta' = \delta'(\delta, A) \in (0, 1)$ such that for any $|V| \geq (1+A)^8/\delta^3$ we have*

$$(7.16) \quad \mathbb{P}[(F_1(\delta', C, c))^c] \leq \frac{C}{|V|} \text{ and } \mathbb{P}[(F_2(\delta', \delta, C, c))^c] \leq C(1+A)^{1/2} \delta^{1/5} + 1_{\delta \geq c'(1+A)^{-2}}.$$

Proof of Theorem 7.2, (7.3). (7.3) is an immediate consequence of Lemmas 7.5 and 7.6. \square

Proceeding to the proof of Lemma 7.6, let us start with the probability bound for the event $F_2(\dots)$.

Lemma 7.7. *Let $\delta \in (0, 1)$, $\beta, b > 0$ and $\delta' = (1+4\beta A)^{-2} \delta^{4/5}$. Then there exist $C = C(b, \beta)$ and $c = c(b, \beta) \in (0, \infty)$ such that,*

$$(7.17) \quad \mathbb{P}[(F_2(\delta', \delta, \beta, b))^c] \leq C(1+A)^{1/2} \delta^{1/5} + 1_{\delta \geq c(1+A)^{-2}},$$

whenever $|V| \geq (1/\delta)^{3/2}$.

For the other part of (7.16), we have the following result.

Lemma 7.8. *There exist $C, c \in (0, \infty)$ such that for any $\delta' \in (0, 1)$ and $|V| \geq (1/\delta')^{15/4}$,*

$$(7.18) \quad \mathbb{P}[(F_1(\delta', C, c))^c] \leq \frac{C}{|V|}.$$

Proof of Lemma 7.6. Lemma 7.6 follows directly from Lemmas 7.7 and 7.8. \square

We now proceed to proving the last two lemmas starting with Lemma 7.7.

Proof of Lemma 7.7. Let us recall the stopping times T_t from Theorem 6.4 and the expression for the (continuous) quadratic variation $\langle M \rangle_s$ from Lemma 6.3. Also recall that by Theorem 6.4, the process \tilde{M}_s remains constant on each interval $[T_{t-}, T_t]$. In view of the definition (7.13), we can therefore write

$$F_2(\delta', \delta, \beta, b) = \left\{ \begin{array}{l} \exists 0 \leq r_1 \leq r_2 \text{ with } r_2 - r_1 \geq \beta\delta|V|^{2/3} \text{ s.t. } |K_{T_{r_2}}|_V \leq b\delta'^{1/2}|V|^{2/3}, \langle \tilde{M} \rangle_{T_{r_2}} = \\ 2|E|(\nu_{K_{T_{r_2}}} - \nu_{K_0}) = r_2 \text{ and } \tilde{M}_{T_r} > (\delta' + \beta A \delta'^{1/2})|V|^{1/3} \text{ for all } r_1 \leq r \leq r_2 \end{array} \right\}.$$

Further from Proposition 3.5 we have that $2|E|\nu_{K_{T_{r_2}}} \geq c|K_{T_{r_2}}|_V$. Hence, letting $r_0 = 2|E|\nu_{K_0}$, we obtain the inclusion

$$(7.19) \quad F_2(\delta', \delta, \beta, b) \supset \left\{ \begin{array}{l} \exists 0 \leq r_1 \leq r_2 \leq cb\delta'^{1/2}|V|^{2/3} - r_0 \text{ with } r_2 - r_1 \geq \beta\delta|V|^{2/3} \text{ s.t.} \\ \langle \tilde{M} \rangle_\infty \geq r_2 \text{ and } \tilde{M}_{T_r} > (\delta' + \beta A \delta'^{1/2})|V|^{1/3} \text{ for all } r_1 \leq r \leq r_2 \end{array} \right\}.$$

Next let us note, by property 3-i of $(K_t)_{t \geq 0}$, that $|K_{\tau_{n,2}}|_V = |V| - 1$ \mathbb{P} -a.s. and thus

$$\langle \tilde{M} \rangle_\infty \geq \langle \tilde{M} \rangle_{\tau_{n,2}} = 2|E|\nu_{K_{\tau_{n,2}}} - r_0 \geq c|V| - r_0$$

in view of Proposition 3.5. Consequently, the lower bound on $\langle \tilde{M} \rangle_\infty$ on the right-hand side in (7.19) can be dropped as soon as $|V| \geq Cb^3$. Since the process $(\tilde{M}_{T_r})_{r \geq 0}$ is distributed as the stopped (standard) Brownian motion $(B_{r_0+r \wedge \langle \tilde{M} \rangle_\infty})_{r \geq 0}$ by Theorem 6.4, the above observation leads to the following restatement of (7.19) in terms of the corresponding probabilities when $|V| \geq Cb^3$. For

$$F_{2,0}(\delta', \delta, \beta, b) \stackrel{\text{def.}}{=} \left\{ \begin{array}{l} \exists r_0 \leq r_1 \leq r_2 \leq c_3 b \delta'^{1/2} |V|^{2/3} \text{ with } r_2 - r_1 \geq \beta\delta|V|^{2/3} \text{ s.t.} \\ B_r > (\delta' + \beta A \delta'^{1/2})|V|^{1/3} \text{ for all } r_1 \leq r \leq r_2 \end{array} \right\},$$

where $c_3 \in (0, \infty)$, we have

$$(7.20) \quad \mathbf{P}_0^B[(F_{2,0}(\delta', \delta, \beta, b))^c] \geq \mathbb{P}[(F_2(\delta', \delta, \beta, b))^c],$$

where \mathbf{P}_0^B is the law of the standard Brownian motion $(B_r)_{r \geq 0}$. So it suffices to prove the bound (7.18) with $F_{2,0}(\delta', \delta, \beta, b)$ instead of $F_2(\delta', \delta, \beta, b)$. To this end,

$$(7.21) \quad \text{let } \delta'_1 = \frac{c_3}{2} b \delta'^{1/2} \text{ and } \delta_1 = \frac{2(1+4\beta A)}{c_3 b} \delta'_1 \text{ with } \delta' = \frac{1}{(1+4\beta A)^2} \delta^{4/5}$$

(recall the statement of Lemma 7.7). Now consider the stopping time

$$(7.22) \quad \tau_{\delta_1} \stackrel{\text{def.}}{=} \inf\{r \geq 0 : B_r \geq \delta_1 |V|^{1/3}\},$$

and the event

$$(7.23) \quad F_{2,1}(\delta', \delta, \beta, b) \stackrel{\text{def.}}{=} \left\{ r_0 \leq \tau_{\delta_1} \leq \delta'_1 |V|^{2/3}, \inf_{\tau_{\delta_1} \leq r \leq \tau_{\delta_1} + \beta \delta |V|^{2/3}} B_r > (\delta' + \beta A \delta'^{1/2}) |V|^{1/3} \right\}.$$

Clearly, for $\delta'_1 + \beta \delta \leq c_3 b \delta'^{1/2}$ which holds when $\delta \leq c(b, \beta)(1+A)^{-2}$ in view of our choices in (7.21), we have the inclusion

$$(7.24) \quad F_{2,1}(\delta', \delta, \beta, b) \subset F_{2,0}(\delta', \delta, \beta, b).$$

Shifting the focus to $F_{2,1}(\delta', \delta, \beta, b)$, note that (see (7.23)),

$$(7.25) \quad \begin{aligned} & (F_{2,1}(\delta', \delta, \beta, b))^c \\ & \subset \{\tau_{\delta_1} < r_0\} \cup \{\tau_{\delta_1} > \delta'_1 |V|^{2/3}\} \cup \left\{ \inf_{\tau_{\delta_1} \leq r \leq \tau_{\delta_1} + \beta \delta |V|^{2/3}} B_r \leq (\delta' + \beta A \delta'^{1/2}) |V|^{1/3} \right\}. \end{aligned}$$

Since $r_0 = 2|E|\nu_{K_0} \leq C$ (recall Lemma 3.4 and also that $K_0 = \partial B(v, \frac{1}{N})$ with $N \geq 2$) and the maximum of a standard Brownian motion up to a time t is distributed as the absolute value of a Gaussian variable with variance t , we get the following bounds:

$$\mathbf{P}_0^B[\tau_{\delta_1} < r_0] \leq e^{-c\delta_1^2 |V|^{\frac{2}{3}}} \stackrel{(7.21)}{\leq} e^{-c\delta^{\frac{4}{5}} |V|^{\frac{2}{3}}} \text{ and } \mathbf{P}_0^B[\tau_{\delta_1} > \delta'_1 |V|^{\frac{2}{3}}] \leq \frac{\delta_1}{\sqrt{\delta'_1}} \stackrel{(7.21)}{\leq} C(b, \beta)(1 + A)^{\frac{1}{2}} \delta^{\frac{1}{5}}.$$

Using similar estimates along with the strong Markov property of Brownian motion, we obtain

$$\mathbf{P}_0^B \left[\inf_{\tau_{\delta_1} \leq r \leq \tau_{\delta_1} + \beta \delta |V|^{\frac{2}{3}}} B_r \leq (\delta' + \beta A \delta'^{\frac{1}{2}}) |V|^{\frac{1}{3}} \right] \leq \exp \left(-\frac{(\delta_1 - \delta' - \beta A \delta'^{\frac{1}{2}})^2}{2\beta\delta} \right) \stackrel{(7.21)}{\leq} \exp(-c(\beta)(1/\delta)^{\frac{1}{5}})$$

for any $\delta \in (0, 1)$. Putting together these estimates we get from the inclusion (7.25) that

$$\mathbf{P}_0^B[(F_{2,1}(\delta', \delta, \beta, b))^c] \leq C(b, \beta)(1 + A)^{1/2} \delta^{1/5}$$

for all $\delta \in (0, 1)$ and $|V| \geq \frac{1}{\delta^{3/2}}$. This yields the bound (7.18) in view of (7.24) and (7.20). \square

Proof of Lemma 7.8 goes through a number of intermediate events. To this end, for any $\delta' \in (0, 1)$ and $\beta > 0$, consider the event

$$(7.26) \quad F_3(\delta', b) \stackrel{\text{def. }}{=} \left\{ \tilde{M}_t^{\text{blk}} \leq \delta' |V|^{1/3} \text{ for any } t \geq 0 \text{ such that } |K_t|_V \leq b \delta'^{1/2} |V|^{2/3} \right\},$$

where $\tilde{M}_t^{\text{blk}} = M_{K_t}^{\text{blk}}$ (see (4.3)).

Lemma 7.9. *There exists $c_4 > 0$ such that for any $\delta' \in (0, 1)$,*

$$(7.27) \quad \mathbb{P}[(F_3(\delta', c_4))^c] \leq Ce^{-c\delta'^{1/2}|V|^{2/3}}.$$

Proof. Note that $K_t \cup \{v\}$ is connected by property 2 of the family $(K_t)_{t \geq 0}$. Consequently, the set $(K_t \cup \{v\}) \cap V$ is connected in the graph \mathcal{G} . Therefore, it suffices to show in view of the definition of M_K^{blk} from (4.3) and the upper bound on ν_K from Proposition 3.5 that

$$(7.28) \quad \mathbb{P} \left[\bigcup_{K \in \mathcal{K}(c')} \left\{ \sum_{x \in K \setminus \{v\}} d_x \varphi(x) \geq c\delta' |V|^{1/3} \frac{|V|}{|K|} \right\} \right] \leq Ce^{-c\delta'^{1/2}|V|^{2/3}}$$

for some $c' \in (0, \infty)$ where

$$\mathcal{K}(c') \stackrel{\text{def. }}{=} \{K \subset V : v \in K, K \text{ is connected in } \mathcal{G} \text{ and } |K| \leq c'\delta'^{1/2} |V|^{2/3}\}.$$

Now, from Lemma 2.1 we know that $\sum_{x \in K \setminus \{v\}} d_x \varphi(x)$ is a centered Gaussian variable with variance bounded by $C|K|$ for all $K \in \mathcal{K}(c')$. Using Gaussian tail bounds, we thus obtain

$$\mathbb{P} \left[\sum_{x \in K \setminus \{v\}} d_x \varphi(x) \geq c\delta' |V|^{1/3} \frac{|V|}{|K|} \right] \leq e^{-c\delta'^2 \frac{|V|^{2+2/3}}{|K|^3}}.$$

Combined with the standard upper bound $|\mathcal{K}(c')| \leq e^{Cc'\delta'^{1/2}|V|^{2/3}}$ (recall that the maximum degree of any vertex in \mathcal{G} is d), this yields (7.28) for a suitable choice of c' via a union bound. \square

Now for any $\delta' \in (0, 1)$ and $\beta > 0$, let

$$(7.29) \quad F_4(\delta', \beta) \stackrel{\text{def. }}{=} \left\{ \begin{array}{l} \tilde{M}_{\tau_{k,1}} \leq (\delta' + \beta A \delta'^{1/2}) |V|^{1/3} + \beta \sqrt{\log |V|} \text{ for any } \\ 0 \leq k \leq n \text{ such that } |K_{\tau_{k,1}}|_V \leq c_4 \delta'^{1/2} |V|^{2/3} \end{array} \right\}.$$

Lemma 7.10. *There exists $C_3 > 0$ such that for any $\delta' \in (0, 1)$ and $|V| \geq (1/\delta')^{3/4}$,*

$$(7.30) \quad \mathbb{P}[F_3(\delta', c_4) \setminus F_4(\delta', C_3)] \leq \frac{C}{|V|}.$$

Proof. In view of Lemma 4.2, we can write

$$\tilde{M}_t = \tilde{M}_t^{\text{blk}} + \tilde{M}_t^{\text{bdr}},$$

where $\tilde{M}_t^{\text{blk}} = M_{K_t}^{\text{blk}}$ and $\tilde{M}_t^{\text{bdr}} = M_{K_t}^{\text{bdr}}$ for all $t \geq 0$. By (7.26) we know that, on the event $F_3(\delta')$,

$$\tilde{M}_t^{\text{blk}} \leq \delta' |V|^{1/3} \text{ whenever } |K_t|_V \leq c_4 \delta'^{1/2} |V|^{2/3}.$$

As to the term \tilde{M}_t^{bdr} , we have the following expression from (4.3).

$$\tilde{M}_t^{\text{bdr}} = \nu_{K_t} \sum_{x \in \partial K_t} \varphi(x) \left(\sum_{y \in V \setminus K_t} d_y \mathbf{P}_y [\tilde{X}_{H_{K_t}} = x] \right) = \nu_{K_t} \left(\sum_{\partial K_t \cap \tilde{\mathcal{G}}^{\leq h}} + \sum_{\partial K_t \setminus \tilde{\mathcal{G}}^{\leq h}} \right),$$

where the first and the second summations are over $x \in \partial K_t \cap \tilde{\mathcal{G}}^{\leq h}$ and $x \in \partial K_t \setminus \tilde{\mathcal{G}}^{\leq h}$ respectively. We can bound the first summation as follows (recall that $h = A|V|^{-1/3} \geq 0$):

$$\begin{aligned} \sum_{\partial K_t \cap \tilde{\mathcal{G}}^{\leq h}} &\leq h \sum_{x \in \partial K_t \setminus \partial B(v, \frac{1}{N})} \sum_{y \in V \setminus K_t} d_y \mathbf{P}_y [\tilde{X}_{H_{K_t}} = x] \leq h \sum_{y \in V \setminus K_t} d_y \sum_{x \in \partial K_t} \mathbf{P}_y [\tilde{X}_{H_{K_t}} = x] \\ &= h \sum_{y \in V \setminus K_t} d_y \leq 2|E|A|V|^{-1/3}. \end{aligned}$$

In order to bound the second term, we will use property 3-ii of $(K_t)_{t \geq 0}$ from Proposition 6.1 which tells us that $\partial K_t \setminus \tilde{\mathcal{G}}^{\leq h} \subset \partial B(v, \frac{1}{N})$ and $\mathbf{P}_y [\tilde{X}_{H_{K_t}} \in \partial B(v, \frac{1}{N}) \setminus \tilde{\mathcal{G}}^{\leq h}] = 0$ for all $y \in V \setminus \{v\}$ when $t = \tau_{k,1}$ ($k \geq 1$). Hence in this case,

$$\sum_{\partial K_t \setminus \tilde{\mathcal{G}}^{\leq h}} \leq \frac{1}{d_x} \sum_{x \in \partial B(v, \frac{1}{N})} |\varphi(x)|.$$

Combining all the previous displays we obtain the inclusion,

$$(7.31) \quad F_3(\delta', c_4) \setminus F_{4,0}(\delta', \alpha) \subset \left\{ \sum_{x \in \partial B(v, \frac{1}{N})} |\varphi(x)| \geq \alpha d \right\} \cup \{\tilde{M}_0 \geq \alpha\}$$

for any $\alpha \in (0, \infty)$ where

$$F_{4,0}(\delta', \alpha) \stackrel{\text{def.}}{=} \left\{ \begin{array}{l} \tilde{M}_{\tau_{k,1}} \leq (\delta' + 2|E|\nu_{K_{\tau_{k,1}}} A|V|^{-2/3})|V|^{1/3} + \alpha \text{ for any } \\ 0 \leq k \leq n \text{ such that } |K_{\tau_{k,1}}|_V \leq c_4 \delta'^{1/2} |V|^{2/3} \end{array} \right\}$$

(cf. (7.29)). Now from Proposition 3.5 we have $2|E|\nu_K \leq C(|K|_V + b_0(K))$ for any $K \in \mathcal{K}_{<\infty}$ such that $|K|_V + b_0(K) \leq c|V|$. Since each component of K_t intersects $K_0 = \partial B(v, \frac{1}{N})$ by property 2 of $(K_t)_{t \geq 0}$, i.e., $b_0(K_t) \leq d$, we get the upper bound

$$(7.32) \quad 2|E|\nu_{K_t} \leq C\delta'^{1/2}|V|^{2/3} \quad \forall t \geq 0 \text{ s.t. } |K_t|_V \leq c_4 \delta'^{1/2} |V|^{2/3} \text{ when } |V| \geq C \vee (1/\delta')^{3/4}$$

(recall our standing assumptions on δ and $|V|$). Plugging (7.32) into the definition of $F_{4,0}(\delta', \alpha)$, we obtain from (7.31) that

$$\begin{aligned} F_4(\delta') \setminus \left\{ \tilde{M}_{\tau_{k,1}} \leq (\delta' + CA\delta'^{1/2})|V|^{1/3} + \alpha \text{ for any } 0 \leq k \leq n \text{ s.t. } |K_{\tau_{k,1}}|_V \leq c_4 \delta'^{1/2} |V|^{2/3} \right\} \\ \subset \left\{ \sum_{x \in \partial B(v, \frac{1}{N})} |\varphi(x)| \geq \alpha d \right\} \cup \{\tilde{M}_0 \geq \alpha\}. \end{aligned}$$

From this we can now conclude the proof of (7.30) by setting $\alpha = C\sqrt{\log |V|}$ and using Gaussian tail bounds for $\varphi(x); x \in \partial B(v, \frac{1}{N})$ and \tilde{M}_0 , each of which is a centered Gaussian variable with variance bounded by some $C > 0$ (follows from (4.2), (4.8) and (3.14)). \square

Next for any $\delta' \in (0, 1)$ and $\beta > 0$, let us introduce the event

$$(7.33) \quad \begin{aligned} F_5(\delta', \beta) &= F_5(\delta', \beta, N) \\ &\stackrel{\text{def.}}{=} \left\{ \tilde{M}_t \leq \tilde{M}_{\tau_{k,1}} + \beta \sqrt{\log |V|} \text{ for } \tau_{k,1} \leq t \leq \tau_{k,2}, 0 \leq k \leq n \text{ s.t. } |K_{\tau_{k,1}}|_V \leq c_4 \delta'^{\frac{1}{2}} |V|^{\frac{2}{3}} \right\}. \end{aligned}$$

Lemma 7.11. *There exists $C_4 > 0$ such that for any $\delta' \in (0, 1)$,*

$$(7.34) \quad \mathbb{P}[(F_5(\delta', C_4))^c] \leq \frac{C}{|V|}.$$

Proof. Recall the definition of T_t from Theorem 6.4 and since \tilde{M}_s is constant on each interval $[T_{t-}, T_t]$ by the same result, we have the inclusion

$$(7.35) \quad \begin{aligned} F_5(\delta', \beta) \\ \supset \left\{ \tilde{M}_{T_r} - \tilde{M}_{T_{q_{k,1}}} \leq \beta \sqrt{\log |V|} \forall q_{k,1} \leq r \leq q_{k,2}, 0 \leq k \leq n \text{ s.t. } |K_{\tau_{k,1}}|_V \leq c_4 \delta'^{\frac{1}{2}} |V|^{\frac{2}{3}} \right\}. \end{aligned}$$

where $q_{k,i} = \langle \tilde{M} \rangle_{\tau_{k,i}}$. From (6.18) in Lemma 6.3 we know that $q_{k,i} = 2|E|(\nu_{K_{\tau_{k,i}}} - \nu_{K_0})$ ($i = 1, 2$). Also from property 3-ii of Proposition 6.1 we know that $K_{\tau_{k,2}} = K_{\tau_{k,1}} \cup I$ for some interval I . Thus in view of Lemma 3.6, there exists $C > 0$ such that $q_{k,2} - q_{k,1} \leq C b_0(K_{\tau_{k,1}})$ provided $|K_{\tau_{k,1}}|_V + b_0(K_{\tau_{k,1}}) + C \leq c|V|$. Since $b_0(K_{\tau_{k,1}}) \leq d$ by property 2 of Proposition 6.1, this gives us

$$(7.36) \quad q_{k,2} - q_{k,1} \leq C \text{ for } |V| \geq C.$$

Further, we know from Theorem 6.4 that the process $(\tilde{M}_{T_r})_{r \geq 0}$ has the same law as the stopped Brownian motion $(B_{r_0+r \wedge \langle \tilde{M} \rangle_\infty})_{r \geq 0}$ with $r_0 = 2|E|\nu_{K_0}$. Combining this with (7.36), (7.35) and (7.32) we obtain, for $|V| \geq C$,

$$\mathbb{P}[(F_5(\delta', \beta))^c] \leq \mathbf{P}_0^B \left[\sup_{0 \leq r \leq |V|, 0 \leq s \leq 1} (B_{r+s} - B_r) > \beta \sqrt{\log |V|} \right].$$

Covering the interval $[0, |V|]$ with intervals of length 2 having both endpoints in \mathbb{Z} and using (Gaussian) tail estimates for the maximum and minimum of a Brownian motion, we get via a union bound,

$$\mathbb{P}[(F_5(\delta', C))^c] = \mathbf{P}_0^B \left[\sup_{0 \leq r \leq |V|, 0 \leq s \leq 1} (B_{r+s} - B_r) > C \sqrt{\log |V|} \right] \leq \frac{C}{|V|}$$

for some $C > 0$. □

We are now ready to prove Lemma 7.8.

Proof of Lemma 7.8. It follows from the definitions of the events $F_3(\delta', c_4)$, $F_4(\delta', C_3)$ and $F_5(\delta', C_4)$ in (7.26), (7.29) and (7.33) respectively that

$$F_3\left(\frac{\delta'}{2}, \frac{c_4}{\sqrt{2}}\right) \cap F_4\left(\frac{\delta'}{2}, C_3\right) \cap F_5(\delta', C_4) \subset F_1\left(\delta', C_3, \frac{c_4}{\sqrt{2}}\right)$$

whenever $|V| \geq C$. (7.18) now follows from this inclusion via a union bound with the corresponding probability bounds provided by Lemmas 7.9, 7.10 and 7.11 respectively. □

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