TECHNICAL UNIVERSITY BERLIN

GEODESY AND GEOINFORMATION SCIENCE

ADJUSTMENT CALCULATION

HOMEWORK III

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TASK I

Objective: In this task, our job is to plot the residual and the adjusted results of an inclinometer where the first column represents the time axis and second column represents the inclination. On the same side we also need to determine the unknown parameters of the polynomial of 4th degree via least square methods.

Functional Model: From the given question, the functional model can be written as

$$y_i = at_i^4 + bt_i^3 + ct_i^2 + dt_i^1 + e$$

where, y represents the inclination and t represents the time. The above equation represents the 4th degree polynomial equation with the unknown parameters a, b, c, d and e.

Observation Equation: The observation equation can be written as

$$y_i + v_i = at_i^4 + bt_i^3 + ct_i^2 + dt_i^1 + e$$

Parameters	Types
y	Observation parameter
t	Error free parameter
a	Unknown parameter
b	Unknown parameter
c	Unknown parameter
d	Unknown parameter
e	Unknown parameter

Now, before approaching further we need to check whether the problem is linear or non-linear. A fourth-degree polynomial equation is a non-linear equation but the unknown parameters in the given problem is linear. So, we can consider the problem as a **Linear Adjustment Problem**.

To plot the residuals, we need to proceed step by step. At the very first we need to setup our stochastic model with the condition that the measurements are uncorrelated.

Vector of observation: $\boldsymbol{L_{n,1}} = [L_1 \quad L_2 \quad \dots . \quad L_n]^T$

Variance covariance matrix of the observation:

$$\Sigma_{LL} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1 L_2} & \cdots & \sigma_{L_1 L_n} \\ \sigma_{2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2 L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_n L_1} & \sigma_{L_n L_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$

Co-factor matrix of the observations: $\boldsymbol{Q_{LL}} = \frac{1}{\sigma_0^2} \Sigma_{LL}$

Weight matrix of the observation: $\boldsymbol{P} = Q_{LL}^{-1}$

Vector of adjusted unknown: $\hat{\boldsymbol{X}} = [\widehat{X_1} \quad \widehat{X_2} \quad \dots \quad \widehat{X_u}]^T$

Design Matrix: $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1u} \\ a_{21} & a_{22} & \dots & a_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nu} \end{bmatrix}$

Vectors of residuals: $\boldsymbol{v_{n,1}} = [v_1 \quad v_2 \quad \dots . \quad v_n]^T$

Observation equation: $L + v = A\hat{X}$

Normal equation: $A^T PA\hat{X} = A^T PL$

Vector of absolute values: $n = A^T PL$

Normal equations: $N\hat{X} = n$

Inversion of normal matrix: $Q_{\hat{X}\hat{X}} = N^{-1}$

Solution for the unknowns: $\hat{X} = Q_{\hat{X}\hat{X}}\, n$

Vector of residuals: $v = A\hat{X} - L$

Vector of adjusted observations: $\hat{L} = L + v$

Final Check: $\hat{L} = \varphi(\hat{X})$

Empirical reference standard dev: $s_0 = \sqrt{\frac{v^T Pv}{n-u}}$

Co-factor matrix of adjusted unknown: $Q_{\hat{X}\hat{X}}$

VCM of adjusted unknown: $\Sigma_{\hat{X}\hat{X}} = s_0^2 Q_{\hat{X}\hat{X}}$

Cofactor matrix of adjusted obs: $Q_{\hat{L}\hat{L}} = AQ_{\hat{X}\hat{X}}A^T$

VCM of adjusted observation: $\Sigma_{\hat{L}\hat{L}} = s_0^2 Q_{\hat{L}\hat{L}}$

Co-factor matrix of the residuals: $Q_{VV} = Q_{LL} - Q_{\hat{L}\hat{L}}$

VCM of the residuals: $\Sigma_{VV} = s_0^2\, Q_{VV}$

Now, using the above steps in the program we obtain the following results.

Unknown Parameters	**Values**	**Standard Deviation**
a	0.00075	0.0341×10^{-3}
b	0.0022	0.2062×10^{-3}
c	-0.0087	0.4116×10^{-3}
d	0.0266	0.3034×10^{-3}
e	-0.00051	0.0656×10^{-3}

Plots:

Residual vs Observation

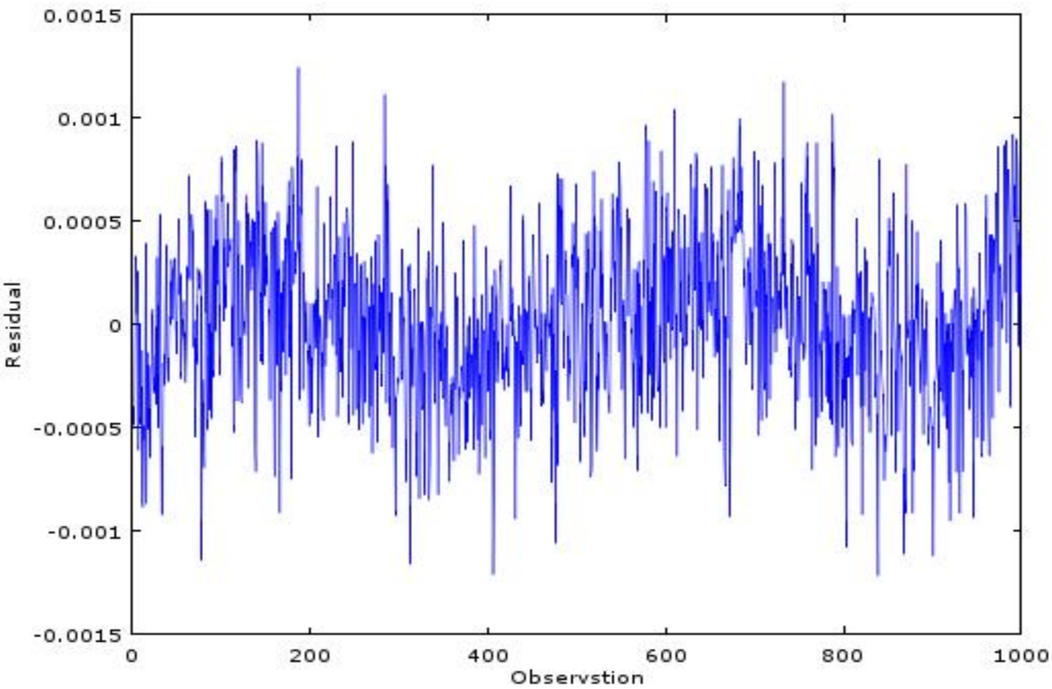

Fig. 1

Plot of Adjusted Unknown

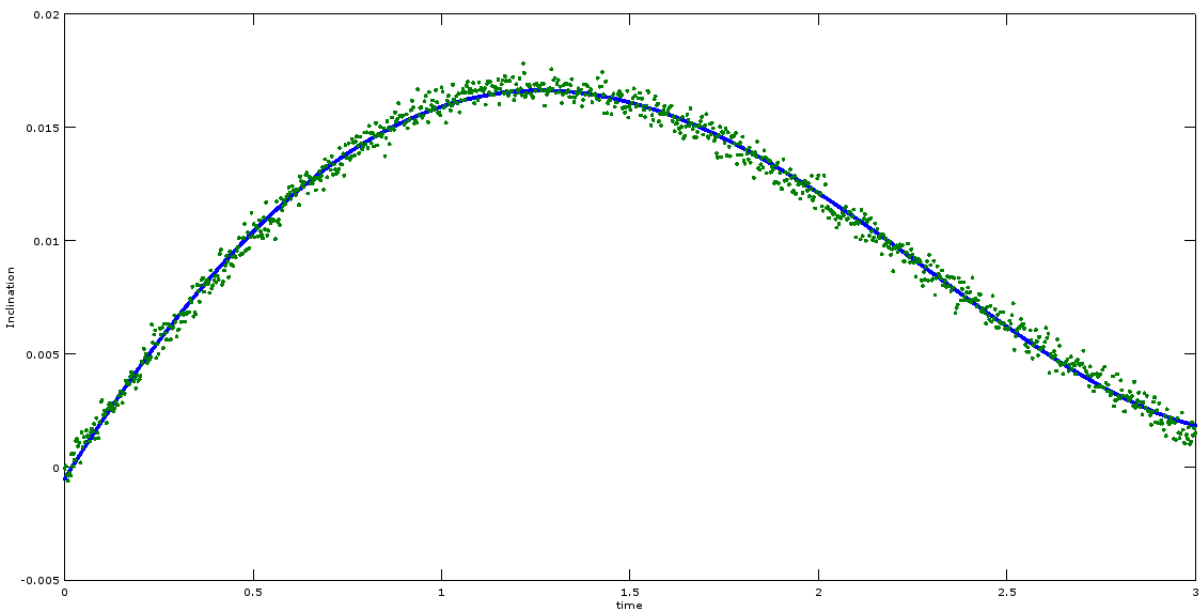

Fig. 2

TASK II

Objective: In this task, our job is to plot the residual and the adjusted results for the given series of measurement where the first column represents the x values and second column represents y(x). On the same side we also need to determine the unknown parameters of the polynomial via least square methods.

Functional Model: When we plot the given data set, we get a curve of exponentially increasing function.

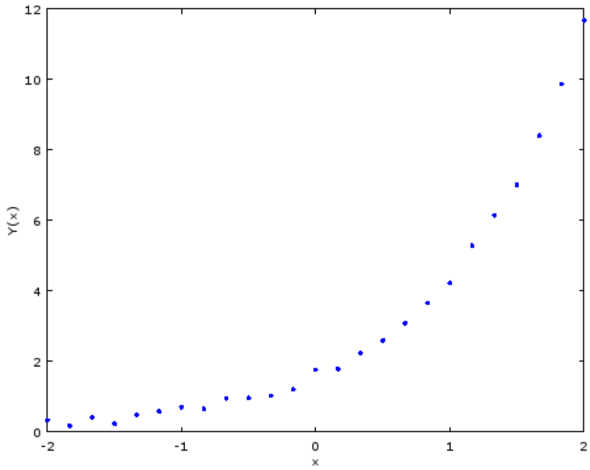

Fig. 3

So, considering the general exponential function.

$$y_i = ae^{bx_i}$$

Parameters	**Types**
y	Observation parameter
x	Error free parameter
a	Unknown parameter
b	Unknown parameter

Now, before approaching further we need to understand the type of the problem. Here the function is nonlinear. The parameter a is linear but the parameter b is nonlinear. So we must use the nonlinear adjustment functional model.

To plot the residuals, we need to proceed step by step. At the very first we need to setup our stochastic model with the condition that the measurements are uncorrelated.

Vector of observation: $\boldsymbol{L_{n,1}} = [L_1 \quad L_2 \quad \ldots. \quad L_n]^T$

Variance covariance matrix of the observation:

$$\Sigma_{LL} = \begin{bmatrix} \sigma_{L_1}^2 & \sigma_{L_1L_2} & \cdots & \sigma_{L_1L_n} \\ \sigma_{2L_1} & \sigma_{L_2}^2 & \cdots & \sigma_{L_2L_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{L_nL_1} & \sigma_{L_nL_2} & \cdots & \sigma_{L_n}^2 \end{bmatrix}$$

VCM of the reduced observation from the variance co-variance propagation with the functional model

$$\Sigma_{ll} = \Sigma_{LL}$$

Co-factor matrix of the observations:	$\boldsymbol{Q_{LL}} = \frac{1}{\sigma_0^2}\Sigma_{LL}$
Weight matrix of the observation:	$\boldsymbol{P} = Q_{LL}^{-1}$
Vector of reduced observation:	$l = L - L^0$
Vectors of residuals:	$\boldsymbol{v_{n,1}} = [v_1 \quad v_2 \quad \dots. \quad v_n]^T$
Observation equation:	$L + v = A\hat{x}$
Normal equation:	$A^T PA\hat{X} = A^T Pl$
Normal matrix:	$N = A^T PA$
Vector of absolute values:	$n = A^T Pl$
Normal equations:	$N\hat{x} = n$
Inversion of normal matrix:	$Q_{\hat{x}\hat{x}} = N^{-1}$
Solution for the unknowns:	$\hat{X} = \widehat{X^0} + \hat{x}$
Vector of residuals:	$v = A\hat{x} - L$
Vector of adjusted observations:	$\hat{L} = L + v$
Check 1:	$\max\lvert\hat{x}_i\rvert \leq \varepsilon \quad \forall\, i = 1 \dots. u$
Check 2:	$\max\left\lvert\hat{L}_i - \varphi_i(\hat{X})\right\rvert \leq \delta \quad \forall\, i = 1 \dots. n$
Empirical reference standard dev:	$s_0 = \sqrt{\frac{v^T Pv}{n-u}}$
Co-factor matrix of adjusted unknown:	$Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}}$
VCM of adjusted unknown:	$\Sigma_{\hat{X}\hat{X}} = s_0^2 Q_{\hat{X}\hat{X}}$
Cofactor matrix of adjusted obs:	$Q_{\hat{L}\hat{L}} = AQ_{\hat{X}\hat{X}}A^T$
VCM of adjusted observation:	$\Sigma_{\hat{L}\hat{L}} = s_0^2 Q_{\hat{L}\hat{L}}$
Co-factor matrix of the residuals:	$Q_{VV} = Q_{LL} - Q_{\hat{L}\hat{L}}$
VCM of the residuals:	$\Sigma_{VV} = s_0^2\, Q_{VV}$

Unknown Parameters	Values	Standard Deviation
a	0.99198	0.0241585
b	1.60478	0.0091106

Plots:

Residual vs Observation

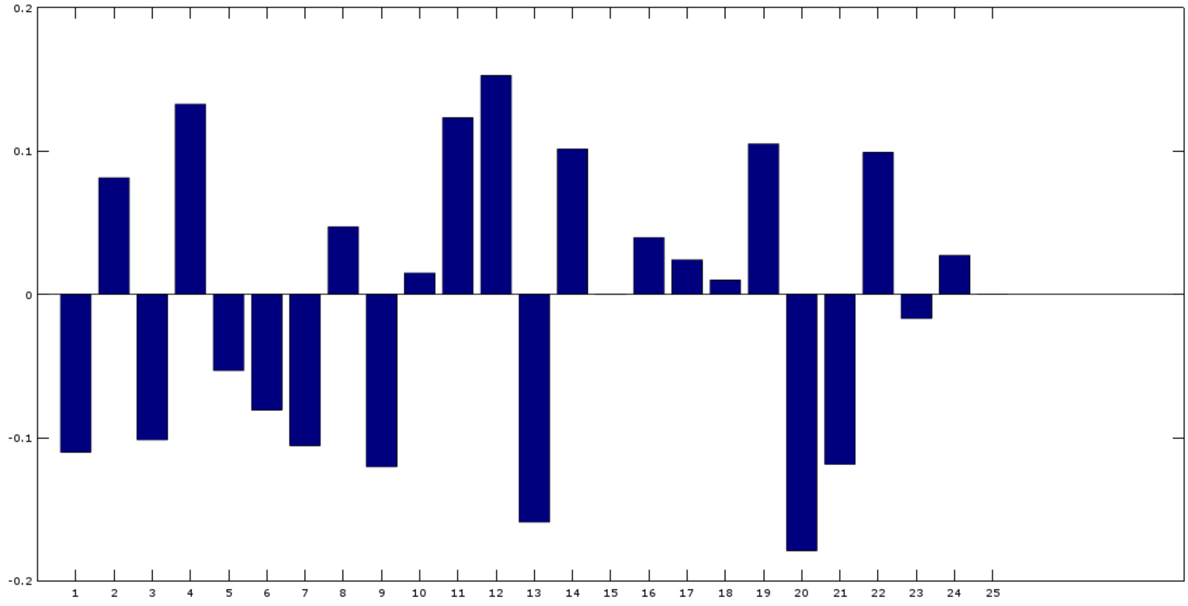

Fig. 4

Plot of Adjusted Unknown

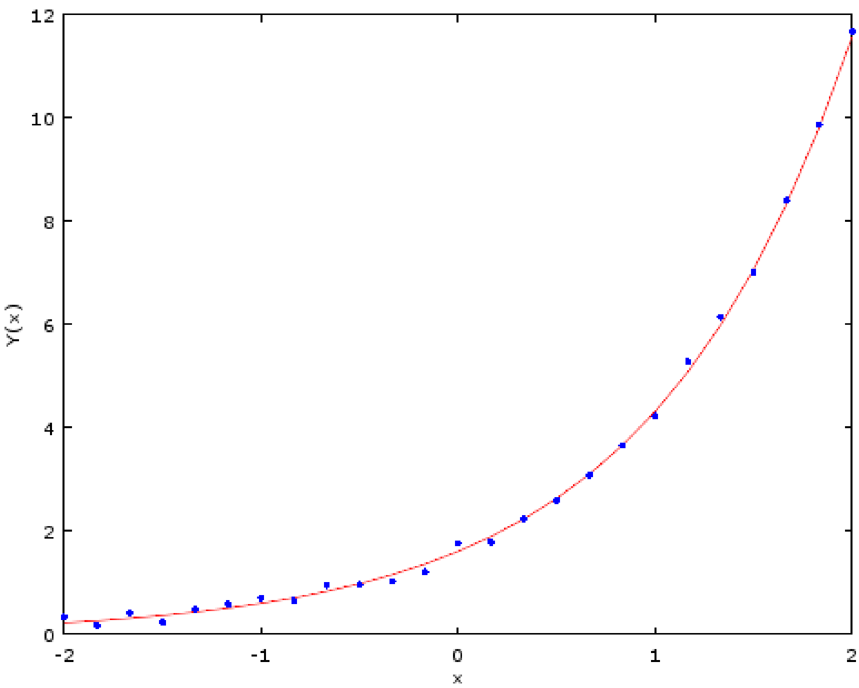

Fig. 5

TASK III

Objective: In this task, our job is to calculate the adjusted volume of the cube of side a and the mass of m. The mass and the length of the cube is given along with the standard deviation.

Functional Model: The volume of a cube is given by the following equation

$$V = a^3$$

So, the length of the side can be written as

$$a = \sqrt[3]{V}$$

The mass of the cube can be written as the product of its mass and density.

$$m = V \times \rho$$

Parameters	**Types**
a	Observation parameter
m	Observation parameter
V	Unknown parameter
ρ	Error free parameter

Now before approaching further we need to understand the type of the problem. As we need to calculate the adjusted volume of the cube and the function is non-linear, so we have to follow the non-linear adjustment functional model.

In this case, we have the standard deviation. So, the variance covariance matrix of observation is

$$\Sigma_{ll} = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_m^2 \end{bmatrix}$$

Now the co-factor matrix of observation is given by

$$\boldsymbol{Q_{LL}} = \frac{1}{\sigma_0^2} \Sigma_{LL}$$

The weight matrix of observation is given by

$$\boldsymbol{P} = Q_{LL}^{-1}$$

As the function is non-linear, we need to linearize the function in order to create the design matrix. So, using the Jacobian matrix is given by

$$J = A_{(1,2)} = \begin{bmatrix} \frac{1}{3} V^{-\frac{2}{3}} & \rho \end{bmatrix}$$

Now using the following steps we determine the required value.

Normal matrix: $N = A^T P A = 3.411 \times 10^4$

Vector of absolute values: $n = A^T P l = 2.571 \times 10^{-4}$

Normal equations: $\hat{x} = N^{-1} n = 7.540 \times 10^{-9}$

Solution for the unknowns: $\hat{X} = \widehat{X^0} + \hat{x} = 1.6875$

The residual and adjusted observation $\hat{L} = L + v$ using the following equations:

Vector of residuals: $v = A\hat{x} - L = \begin{bmatrix} 0.0306 \\ -0.0804 \end{bmatrix}$

Vector of adjusted observations: $\hat{L} = L + v = \begin{bmatrix} 1.190 \\ 15.069 \end{bmatrix}$

Now from the theory we used in the task 2, we use the check condition.

Check 1: $\max|\hat{x}_i| \leq \varepsilon \quad \forall\, i = 1 \ldots . u$

Check 2: $\max\left|\hat{L}_i - \varphi_i(\hat{X})\right| \leq \delta \quad \forall\, i = 1 \ldots . n$

Unknown Parameters	**Values**	**Standard Deviation**
V	1.6875 cm^3	0.034216

TASK IV

Objective: In this task, our job is to plot the residual and the adjusted results for the given ellipse. On the same side we also need to determine the unknown parameters of the ellipse via least square methods.

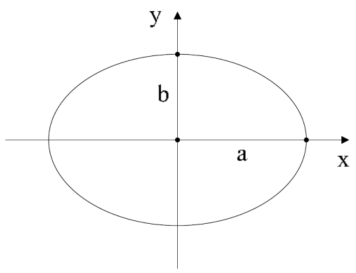

Fig. 6

Functional Model: The equation of the ellipse is given as

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

Now the functional model will be as,

$$y = b\sqrt{1 - (\frac{x}{a})^2}$$

Observation Equation: The observation equation can be written as

$$y + v = b\sqrt{1 - (\frac{x}{a})^2}$$

$$\hat{y}_i + v_i = b\sqrt{1 - (\frac{\hat{x}}{a})^2}$$

Now before approaching further we need to understand the type of the problem. As we need to calculate the unknown parameters of the ellipse and the function is non-linear, so we have to follow the **non-linear adjustment functional model**.

Parameters	**Types**
x	Observation parameter [Error free]
y	Observation parameter
a	Unknown parameter
b	Unknown parameter

To plot the residuals, we need to proceed step by step. At the very first we need to setup our stochastic model with the condition that the measurements are uncorrelated and of same standard deviation. So,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Vector of observation: $\boldsymbol{L_{n,1}} = [L_1 \quad L_2 \quad \dots. \quad L_n]^T$

$$\boldsymbol{L_{5,1}} = [0.673 \quad -2.080 \quad -2.200 \quad 2.088 \quad -0.669]^T$$

Vector of reduced observation: $l = L - L^0$

Design matrix: $J = A_{(5,2)} = \begin{bmatrix} \frac{\partial L_1}{\partial a} & \frac{\partial L_1}{\partial b} \end{bmatrix}^T = \begin{bmatrix} \frac{bx^2}{a^3\sqrt{1-(\frac{x}{a})^2}} & \sqrt{1-(\frac{x}{a})^2} \end{bmatrix}^T$

Vectors of residuals: $\boldsymbol{v_{n,1}} = [v_1 \quad v_2 \quad \dots. \quad v_n]^T$

Observation equation: $L + v = A\hat{x}$

Normal equation: $A^T PA\hat{X} = A^T Pl$

Normal matrix: $N = A^T PA$

Vector of absolute values: $n = A^T Pl$

Normal equations: $N\hat{x} = n$

Inversion of normal matrix: $Q_{\hat{x}\hat{x}} = N^{-1}$

Solution for the unknowns: $\hat{X} = \widehat{X^0} + \hat{x}$

Vector of residuals: $v = A\hat{x} - L$

Vector of adjusted observations: $\hat{L} = L + v$

Check 1: $\max|\hat{x}_i| \leq \varepsilon \quad \forall\, i = 1 \dots. u$

Check 2: $\max\left|\hat{L}_i - \varphi_i(\hat{X})\right| \leq \delta \quad \forall\, i = 1 \dots. n$

Empirical reference standard dev: $s_0 = \sqrt{\frac{v^T Pv}{n-u}}$

Co-factor matrix of adjusted unknown: $Q_{\hat{X}\hat{X}} = Q_{\hat{x}\hat{x}}$

VCM of adjusted unknown: $\Sigma_{\hat{X}\hat{X}} = s_0^2 Q_{\hat{X}\hat{X}}$

Cofactor matrix of adjusted obs: $Q_{\hat{L}\hat{L}} = AQ_{\hat{X}\hat{X}}A^T$

VCM of adjusted observation: $\Sigma_{\hat{L}\hat{L}} = s_0^2 Q_{\hat{L}\hat{L}}$

Co-factor matrix of the residuals: $Q_{VV} = Q_{LL} - Q_{\hat{L}\hat{L}}$

VCM of the residuals: $\Sigma_{VV} = s_0^2\, Q_{VV}$

Using the following equations in the program, we get the following results

Unknown Parameters	Values	Standard Deviation
a	3.1503	0.0013510
b	2.1985	0.0022947

Plots:

Residual vs Observation

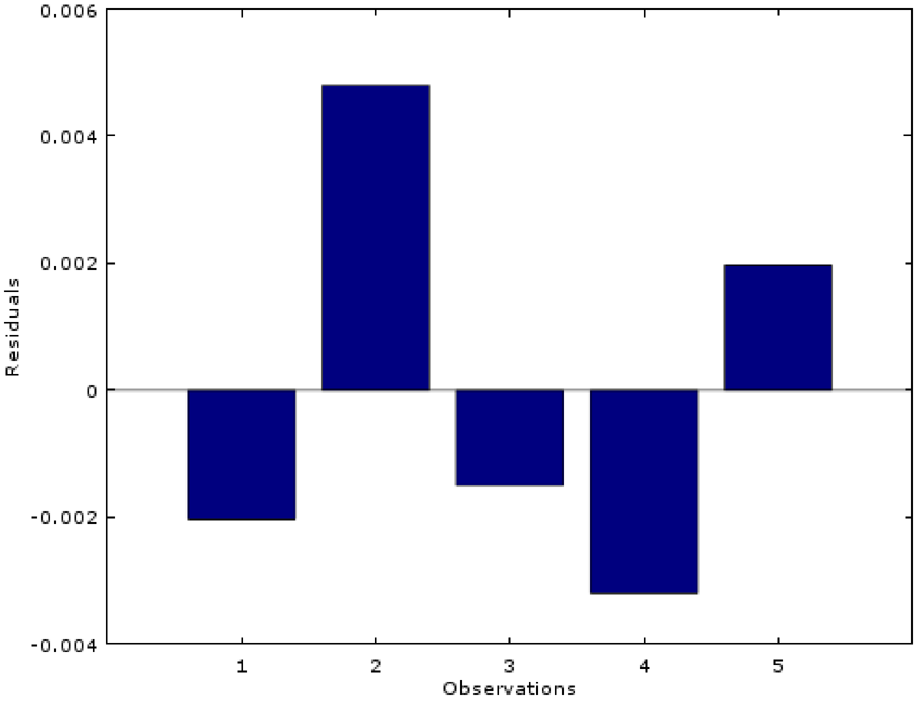

Fig. 7

Reference

Lecture notes, Prof. Dr. Frank Neitzel, 2016-17

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