Technische Universität Berlin

Chair of Geodesy and Adjustment Theory

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Selected Sections of Adjustment Calculation

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Exercise 1

Hypothesis Testing

- Type I and II errors, statistical tests -

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Table of Content

List of Figures

Task 1

Explain type I and type II error in your own words and find some application and describe them. Therefore, a detailed literature research is necessary. Keep in mind that you have to refer all the sources and list them in a bibliography at the end of the report.

Objective: The main objective of this task is to describe the Type I and Type II error along with its application.

Hypothesis testing is the use of statistics to determine the probability that a given hypothesis is true. It mainly consist of four steps:

1. Formulate Null hypothesis and alternative hypothesis
2. Identifying Test Statistics that is used to access the truth of null hypothesis
3. Compute P-value. Smaller P-value tends to stronger evidence against null hypothesis
4. Compare P value to an acceptable significance value alpha.

Now, when we do this hypothesis test, two types of errors are possible Type I and Type II

Type I: It is also known as "False Positive". As it rejects null hypothesis when it is actually true[8]. This is the error of accepting an alternative hypothesis when the result can be attributed to chance.

Application: A Jury sent an innocent guy to jail where the Null hypothesis is "He is innocent". In this case, the Jury is rejecting the Null hypothesis when it is true.

Type II: It is also known as "False Negative". As it accepts null hypothesis when the alternative hypothesis is actually false. Alternatively, this is the error of failing to accept an alternative hypothesis.

Application: A Jury set a criminal guy free where the Null hypothesis is "He is innocent". In this case, the Jury is accepting the Null hypothesis when it is actually false.

Task 2

- **Plot the χ^2-distribution for different degrees of freedom f $= [1, 2, 3, 4, 5, 7, 9]$ and interpret the resulting graph.**
- **Plot the t-distribution for different degrees of freedom f $= [1, 2, 3, 4, 5, 10, 50]$ as well as the normal distribution N(0,1) and interpret the resulting graphs.**

Objective: The main objective of this task is to plot χ^2-distribution and t-distribution for different degree of freedom and plot the normal distribution N(0,1) with t-distribution. Apart from that we need to interpret the resulting graphs.

Brief description about χ^2-Distribution and t-Distribution:

- *χ^2-Distribution:*

The χ^2 distribution with f degrees of freedom is the distribution of a sum of the squares of the f independent standard normal random variables[5]. It is often used to judge "how far away" some number is from some other members[6]. Mathematically,

X_j is independent, standard normal random variables

$X_j \sim N(0,1)$ with $j = 1,2 \ldots f$ where f is the degree of freedom

The sum of their squares

$$\chi^2 = \sum_{j=1}^{f} X_j^2 \qquad \ldots (2.1)$$

For simplification substituting $t = \chi^2$

$$f(t) = \begin{cases} 0 & for\; t \leq 0 \\ \dfrac{1}{2^{\frac{f}{2}}\Gamma(\frac{f}{2})} . t^{\frac{f}{2}-1} . \exp(-\dfrac{t}{2}) & for\; t > 0 \end{cases} \qquad \ldots (2.2)$$

with Gamma function

$$\Gamma\left(\frac{f}{2}\right) = \int_0^\infty t^{\frac{f}{2}-1} . \exp\left(-\frac{f}{2}\right) dt \qquad \ldots (2.3)$$

- *t-Distribution*:

The t-distribution is a continuous probability distribution that arise when estimating the mean of a normally distributed function in a situation where sample size is small and standard deviation is unknown[7] . Mathematically it can be defined as

L as realization of an arbitrary $N(\mu, \sigma^2)$ distributed random variable L, transformation to standard normal distribution $N(0,1)$

$$l \sim N(\mu, \sigma^2) \qquad \ldots (2.4)$$

$$\bar{\varepsilon} = \frac{l-\mu}{\sigma} = \frac{\varepsilon}{\sigma} \quad \text{... (2.5)}$$

$$\bar{\varepsilon} \sim N(0,1) \quad \text{... (2.6)}$$

Theoretical standard deviation is not normally known. So replacing it with empirical standard deviation s.

$$t_f = \frac{l-\mu}{s} = \frac{\varepsilon}{s} \sim t_f \quad \text{... (2.7)}$$

$$f \to \infty \Rightarrow \begin{cases} s^2 \to \sigma^2 \\ t_f \to \bar{\varepsilon} \end{cases} \Rightarrow t - dist \ \Rightarrow N(0,1) \quad \text{... (2.8)}$$

Probability density function of t-distribution

$$f(t) = \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{\pi.f}\,\Gamma\left(\frac{f}{2}\right)}.(1+\frac{t^2}{f})^{-\frac{f+1}{2}} \quad \text{... (2.9)}$$

for $-\infty \leq t \leq +\infty$ with Gamma Function

$$\Gamma(t) = \int_0^\infty t^{f-1} e^{-t} dt \quad \text{... (2.10)}$$

Plotting the Graphs:

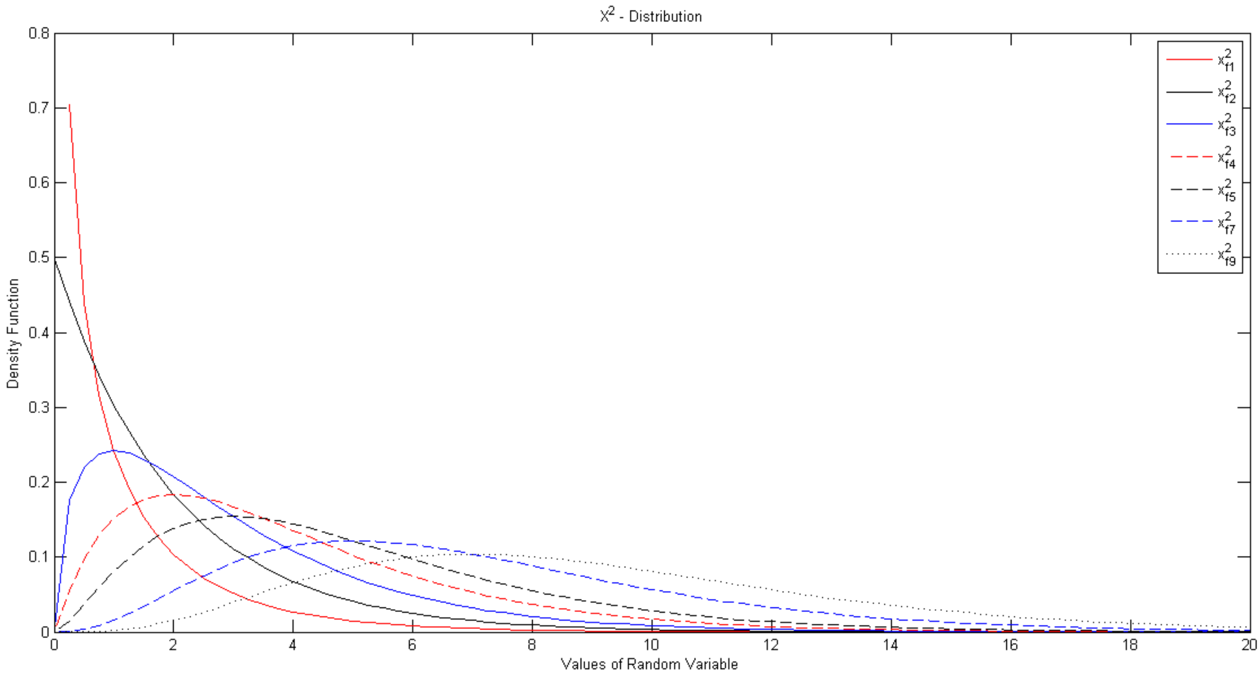

Fig. 1 χ^2-Distribution

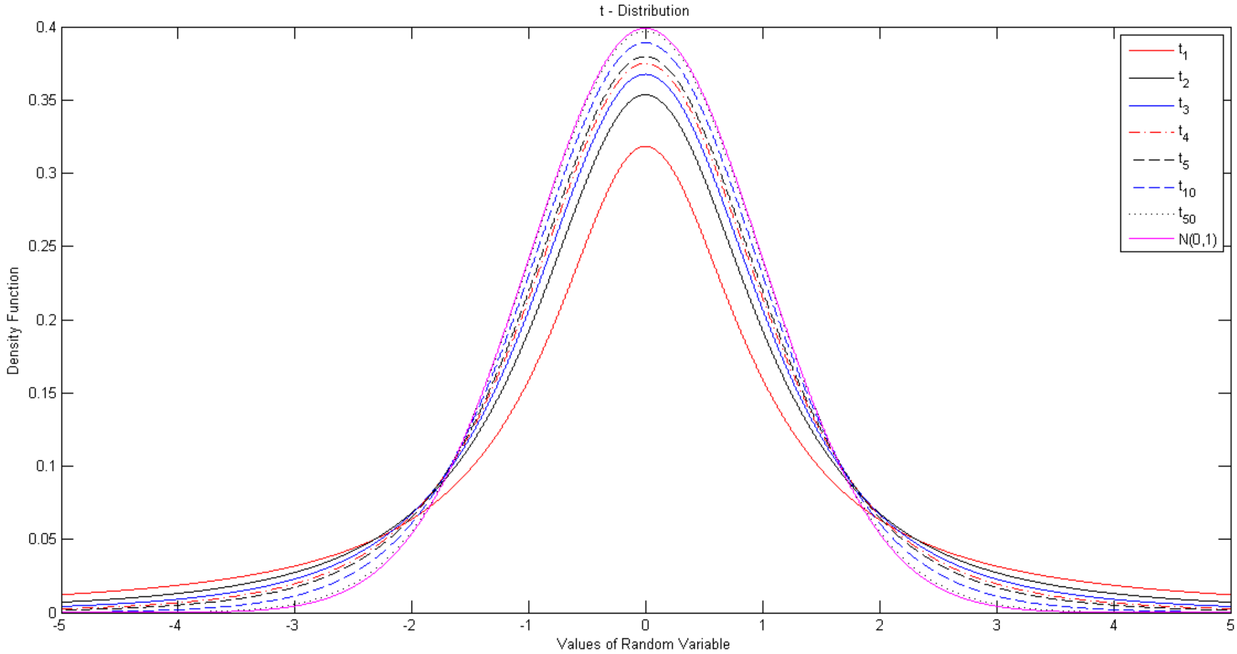

Fig. 2 t-Distribution

Brief Discussion:

χ^2*-Distribution* : When the degree of freedom increases the graph becomes more symmetric. Along with that, when the degree of freedom increases above 3 and tends to higher value then the pattern of the graph resembles.

t-Distribution : When the degree of freedom is 50 it is sufficiently close to normal distribution[2].

Task 3

During a previously carried out survey the students have observed the distance between two fixed points and the measurements can be found in "distances.txt".

- **Load the measurement from the "distance.txt"**
- **Calculate the mean value, the standard deviation of the mean value as well as the standard deviation of a single measurement.**
- **Calculate and plot the residuals.**
- **Calculate the confidence limits with $S = 95\%$ and $S = 99\%$ for the expectation value as well as for a single measurement.**
- **Comment and evaluate all results!**

Objective: The main objective of this task is to calculate the mean value, standard deviation of mean value as well as the standard deviation of a single measurement from the data sheet. Along with that we need to plot the residuals of the data given. Lastly, we need to find the confidence limits with different values of S for the expectation value and for a single measurement.

Given:

Data Set	Values
Data	distances.txt
S	95%
S	95%

Brief Explanation of Mean Value, Standard Deviation:

- *Mean Value*: Mean value can be defined as the average of the numbers. Precisely, it can be defined as the central value of a set of numbers. Mathematically, it can be defined as[1]

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \dots (3.1)$$

where, n is the number of observation.

- *Standard Deviation*: Standard deviation can be defined as the measure of the dispersion of a set of data from its mean value. It is the measure of the absolute variability of a distribution. The higher is the dispersion, the greater is the standard deviation and on the same side greater will be the magnitude of the deviation of the value from their mean, Mathematically, it can be defined as[1]

$$s_l = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \text{... (3.2)}$$

where, n is the number of observations. The above equation of standard deviation is when expectation of the random variable is known.

Now when the expectation of the random variable is unknown then,

$$s_l = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \text{... (3.3)}$$

where, $(n-1)$ is the degree of freedom.

Theoretical Standard Deviation of an arithmetic mean can be mathematically defined as

$$s_{\bar{l}}^2 = \frac{s_l^2}{n} \qquad \text{... (3.4)}$$

or,

$$s_{\bar{l}} = \frac{s_l}{\sqrt{n}} \qquad \text{... (3.5)}$$

3.2 Results of Mean Value and Standard Deviation:

Data Set	**Mathematical Expression**	**Values**
Mean Value	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	51.0021 m
Standard Deviation of Single Observation	$s_l = \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2}$	0.0158 m
Standard Deviation of Arithmetic Mean	$s_{\bar{l}} = \frac{s_l}{\sqrt{n}}$	0.0032 m

MATLAB Output:

```
Mean_Value =

   51.0021


Standard_Deviation_for_Single_Obervation =

    0.0158


Standard_Deviation_for_Arithmetic_Mean =

    0.0032
```

3.3 Residuals:

Residuals can be mathematically defined as[1]

$$v_j = (\bar{x} - x_j)\ \forall\, j \qquad \text{... (3.6)}$$

where, x_j is the data set, $\bar{x}$ is the mean value of the data set.
The vector of residuals can be defined as

$$\vec{v}_{n,1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \bar{x} - x_1 \\ \bar{x} - x_2 \\ \vdots \\ \bar{x} - x_n \end{bmatrix} \qquad \text{... (3.7)}$$

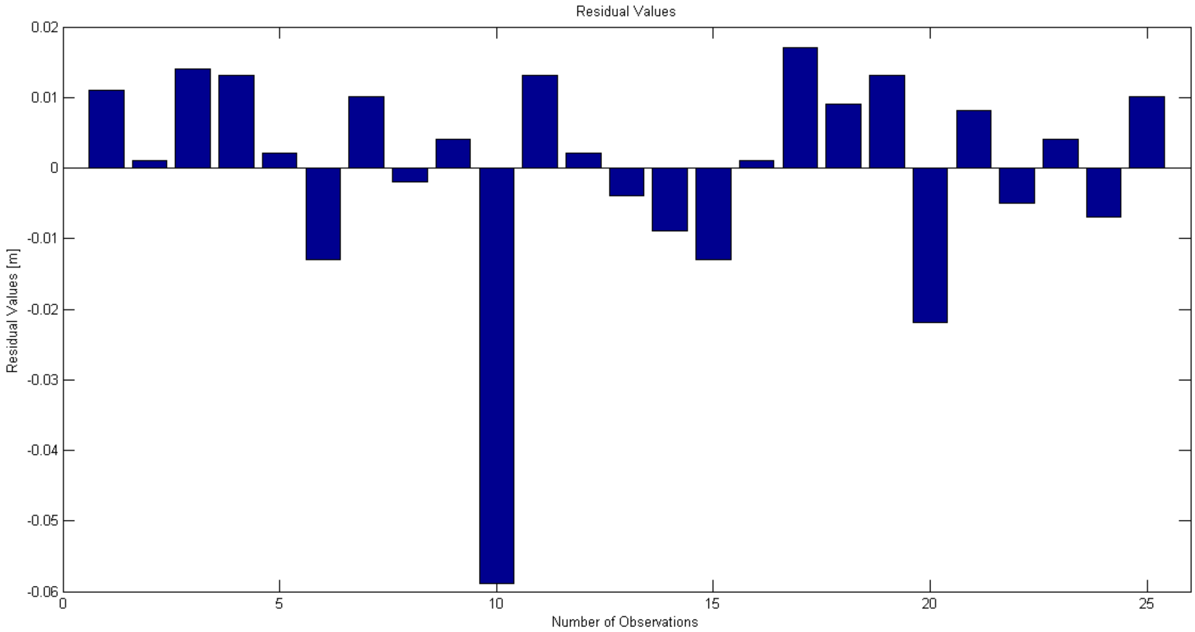

Fig. 3 Residual Values

3.4 Confidence Limits for the expectation value as well as for a Single Measurement:

In this case, we need to find out the boundaries of the confidence interval. The confidence interval for the expectation can be calculated from the following mathematical equations[2].

$$P\left(t_{f,1-\frac{\alpha}{2}} \leq t_f \leq t_{f,\frac{\alpha}{2}}\right) = 1 - \alpha \qquad \text{... (3.8)}$$

Considering symmetrical distribution,

$$t_{f,\frac{\alpha}{2}} = -t_{f,1-\frac{\alpha}{2}} \qquad \text{... (3.9)}$$

Inserting $t_f = \frac{l-\mu}{s}$

Now using mathematical operations and rearranging, we get

$$P\left(l + t_{f,\frac{\alpha}{2}}.s \geq \mu \geq l - t_{f,\frac{\alpha}{2}}.s\right) = 1 - \alpha \qquad \text{... (3.10)}$$

Finally,

$$P\left(l - t_{f,\frac{\alpha}{2}}.s \leq \mu \leq l + t_{f,\frac{\alpha}{2}}.s\right) = 1 - \alpha \qquad \text{... (3.11)}$$

$$a = l - t_{f,\frac{\alpha}{2}}.s\ [Lower\ Boundary] \qquad \text{... (3.12)}$$

$$b = l + t_{f,\frac{\alpha}{2}}.s\ [Upper\ Boundary] \qquad \text{... (3.13)}$$

Results

Confidence Limit for the Expectation Value:

$$\mathbf{50.9956 \leq \boldsymbol{\mu} \leq 51.0086} \qquad [\boldsymbol{S} = \mathbf{95\%}]$$

$$\mathbf{50.9933 \leq \boldsymbol{\mu} \leq 51.0109} \qquad [\boldsymbol{S} = \mathbf{99\%}]$$

Confidence Limit for Single Observation:

$$\mathbf{50.9695 \leq \boldsymbol{\mu} \leq 51.0347} \qquad [\boldsymbol{S} = \mathbf{95\%}]$$

$$\mathbf{50.9579 \leq \boldsymbol{\mu} \leq 51.0462} \qquad [\boldsymbol{S} = \mathbf{99\%}]$$

MATLAB Output

```
Lower_Boundary_for_Single_Observation_95 =

   50.9695


Upper_Boundary_for_Single_Observation_95 =

   51.0347


Lower_Boundary_for_Mean_Value_95 =

   50.9956


Upper_Boundary_for_Mean_Value_95 =

   51.0086


Lower_Boundary_for_Single_Observation_99 =

   50.9579


Upper_Boundary_for_Single_Observation_99 =

   51.0462


Lower_Boundary_for_Mean_Value_99 =

   50.9933


Upper_Boundary_for_Mean_Value_99 =

   51.0109
```

Brief Discussion:

- From the graph, we can say that the value corresponding to observation number 10 has the maximum deviation from the mean value. So, it can be considered as the blunder error.

- By calculating the confidence interval, we can conclude that the mean value in each case lies within the boundary limit.

Task 4

Describe the similarities between a two-tailed hypothesis and a confidence interval.

Objective: The objective of this task is to define the similarities between two-tailed hypothesis and a confidence interval.

In both the cases, two tailed hypothesis and confidence interval, both have lower boundary and upper boundary which is determined by the confidence value. When the confidence value increases the distance between the boundaries increases.

Task 5

Student claims he studied at least 6 hours a week. A sample of 16 weeks shows that the average number of hours he studied was 5.5, with a standard deviation of 1.0. Is his claim true?

Objective: The objective of this task is to check whether the claim by the student is true or not

Given:

Data Set	Values
Expected Value $E(x)$	6 hours
Mean Value $\bar{x}$	5.5 hours
Number of Samples n	16
Standard Deviation $S_{\bar{x}}$	1.0 hour

Based on the given data set, we can set the

Null Hypothesis (H_0) : $\bar{x} = E(x)$

Alternative Hypothesis (H_A) : $\bar{x} < E(x)$ [One Sided Problem]

Distribution: t-Distribution [2]

Test Statistic: $T_t = \frac{|\bar{x}-E(x)|}{S_{\bar{x}}} = \frac{|6.0-5.5|}{1.0} = 0.5$

Confidence Level: S = 95% , $\alpha = 5\% = 0.05$

Critical Value: $\alpha = 0.05, f = 16 - 1 = 15, \rightarrow\ t_{0.05,15} = 1.753$[2]

where, f = degree of freedom.

Test Decision: $T_t <\ t_{0.05,15}$

Fail to reject Null Hypothesis. So, the student's claim is true.

Task 6

A mean value $\bar{x} = 10.0$ with a variance of $s^2 = 0.07$ is obtained after 12 observations. Is the deviation between the obtained and given variance of $\sigma^2 = 0.10$ significant different, with an error level of 5%?

Objective: The objective of this task is to check whether the deviation between the obtained and given variance is significantly different or not.

Given:

Data Set	Values
Mean value $\bar{x}$	10.0
Theoretical Variance σ^2	0.10
Observed Variance S^2	0.07
Number of Samples n	12
Degree of freedom f	12-1 = 11

Based on the given data set, we can set the

Null Hypothesis (H_0) : $s_0^2 = \sigma_0^2$

Alternative Hypothesis (H_A) : $s_0^2 \neq \sigma_0^2$ [Two Sided Problem]

Distribution: χ^2-Distribution [2]

Test Statistic: $T_t = f\frac{s_0^2}{\sigma_0^2} = 7.7$

Confidence Level: S = 95% , $\alpha = 5\% = 0.05$

Critical Value: $\alpha = 0.05, f = 16 - 1 = 15, \rightarrow \chi_{0.05,11} = 21.92$[2]

where, f = degree of freedom.

Test Decision: $T_t < \chi_{0.05,11}$

Fail to reject Null Hypothesis. So, we obtained that the deviation between obtained and the given variance is equivalent.

Task 7

The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of size $n_1 = 15$ and $n_2 = 17$ are selected, and the sample means and sample variance are $\overline{\chi 1} = 8.73$, $s_1^2 = 0.35$, $\overline{\chi 2} = 8.68$ and $s_2^2 = 0.40$, respectively. Assume thst $\sigma_1^2 = \sigma_2^2$ and that the data are drawn from a normal distribution.

- **Is there evidence to support that claim that the two machines produce rods with different mean diameters with an error level of 5%.**

Objective: The objective of this task is to check whether the two machines produce rods with different diameters with an error level of 5%.

Given:

Data Set	Values
Sample Mean of First Machine $\overline{\chi 1}$	8.73
Sample Mean of Second Machine $\overline{\chi 2}$	8.68
Sample Variance of First Machine s_1^2	0.35
Sample Variance of Second Machine s_2^2	0.40
Number of Samples for First Machine n_1	15
Number of Samples for Second Machine n_2	17
Error Level	5%

Based on the given data set, we can set the

Null Hypothesis (H_0) : $s_1^2 = s_2^2 = \sigma^2$

Alternative Hypothesis (H_A) : $s_1^2 < s_2^2$ [Two Sided Problem]

Distribution: F-Distribution [2]

Test Statistic: $T_t = \frac{s_1^2}{s_2^2} = 1.1429$

Confidence Level: S = 95% , $\alpha = 5\% = 0.05$

Critical Value: $\alpha = 0.05, f_1 = 15 - 1 = 14,\ \ f_2 = 17 - 1 = 16 \ \rightarrow F_{14,16,0.025} = 2.9234$[2]

where, f = degree of freedom.

Test Decision: $T_t < F_{14,16,0.025}$

Critical value is greater than test value so we are accepting the Null Hypothesis. So, we obtained that two machines produce rods of same mean diameter.

Bibliography

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6. http://pj.freefaculty.org/guides/stat/Distributions/DistributionWriteups/ChiSquare/ChiSquare.pdf
7. https://en.wikipedia.org/wiki/Student%27s_t-distribution
8. http://www.stat.berkeley.edu/~hhuang/STAT141/Lecture-FDR.pdf

Source Code

Task 2

```
%------------------------------------------------------------------------
%
%   SELECTED SECTIONS OF ADJUSTMENT CALCULATION
%   Chi-Square Distribution
%
%   Task            : 2
%   Author          : Arghadeep Mazumder
%   Version         : May 23, 2017
%
%------------------------------------------------------------------------

clc;
clear all;
close all;

x=0:0.20:15;

A=chi2pdf(x,1);
figure;
plot(x,A,'r','DisplayName','x^2_f_1')
hold on

B=chi2pdf(x,2);
plot(x,B,'k','DisplayName','x^2_f_2')
hold on

C=chi2pdf(x,3);
plot(x,C,'b','DisplayName','x^2_f_3')
hold on

D=chi2pdf(x,4);
plot(x,D,'r--','DisplayName','x^2_f_4')
hold on

E=chi2pdf(x,5);
plot(x,E,'k--','DisplayName','x^2_f_5')
hold on

F=chi2pdf(x,7);
plot(x,F,'b--','DisplayName','x^2_f_7')
hold on

G=chi2pdf(x,9);
plot(x,G,'k:','DisplayName','x^2_f_9')
hold on

title('X^2 - Distribution')
xlabel('Values of Random Variable')
ylabel('Density Function')
legend('show')

%------------------------------------------------------------------------
%
%   SELECTED SECTIONS OF ADJUSTMENT CALCULATION
%   t-Square Distribution
%
%   Task            : 2
%   Author          : Arghadeep Mazumder
%   Version         : May 23, 2017
%
%------------------------------------------------------------------------

clc;
clear all;
close all;
```

```
x=-5:0.0001:5;

A=tpdf(x,1);
figure;
plot(x,A,'r','DisplayName','t_1')
hold on

B=tpdf(x,2);
plot(x,B,'k','DisplayName','t_2')
hold on

C=tpdf(x,3);
plot(x,C,'b','DisplayName','t_3')
hold on

D=tpdf(x,4);
plot(x,D,'r-.','DisplayName','t_4')
hold on

E=tpdf(x,5);
plot(x,E,'k--','DisplayName','t_5')
hold on

F=tpdf(x,10);
plot(x,F,'b--','DisplayName','t_1_0')
hold on

G=tpdf(x,50);
plot(x,G,'k:','DisplayName','t_5_0')
hold on

H= normpdf(x,0,1);
plot(x,H,'m','DisplayName','N(0,1)')
hold on

title('t - Distribution')
xlabel('Values of Random Variable')
ylabel('Density Function')
legend('show')
```

Task 3

```
clear;
clc;
close all;

data = load('distances.txt');
n = length(data); %Loading the file 'distances.txt'
dof = n-1; % Degree of Freedom
mv = mean(data); %Finding the Mean Value
Mean_Value= mv
std_dev = std(data); %Calculating the Standard Deviation for Sigle Observaton
Standard_Deviation_for_Single_Obervation=std_dev
std_mean = std_dev/sqrt(n); %Calculating Standard Deviation for Arithmetic Mean
Standard_Deviation_for_Arithmetic_Mean=std_mean
res = ones(n,1)*mv - data;  %Calculating Residual Values
figure;
b=bar(res); % Plotting Residual Values in bar format
xlim([0 26]) % Setting the limits for the X-axis
title('Residual Values')
xlabel('Number of Observations')
ylabel('Residual Values [m]')

%Calculating Confidence Limit with S=95% for the expectation value as
% well as for a single measurement

p_95 = .975;

a_95_so = mv - tinv(p_95,dof)*std_dev;
b_95_so = mv + tinv(p_95,dof)*std_dev;
```

```
Lower_Boundary_for_Single_Observation_95=a_95_so
Upper_Boundary_for_Single_Observation_95=b_95_so


a_95_mv = mv - tinv(p_95,dof)*std_mean;
b_95_mv = mv + tinv(p_95,dof)*std_mean;

Lower_Boundary_for_Mean_Value_95=a_95_mv
Upper_Boundary_for_Mean_Value_95=b_95_mv

%Calculating Confidence Limit with S=99% for the expectation value as
% well as for a single measurement

p_99 = 0.995;
a_99_so = mv - tinv(p_99,dof)*std_dev;
b_99_so = mv + tinv(p_99,dof)*std_dev;

Lower_Boundary_for_Single_Observation_99=a_99_so
Upper_Boundary_for_Single_Observation_99=b_99_so


a_99_mv = mv - tinv(p_99,dof)*std_mean;
b_99_mv = mv + tinv(p_99,dof)*std_mean;

Lower_Boundary_for_Mean_Value_99=a_99_mv
Upper_Boundary_for_Mean_Value_99=b_99_mv
```