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Eigen Values and Eigen Vectors

Comprehensive Course on Engineering Mathematics

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LINEAR ALGEBRA

(EC/CE/MECH/EE/CSE/IT/IN/CHE)

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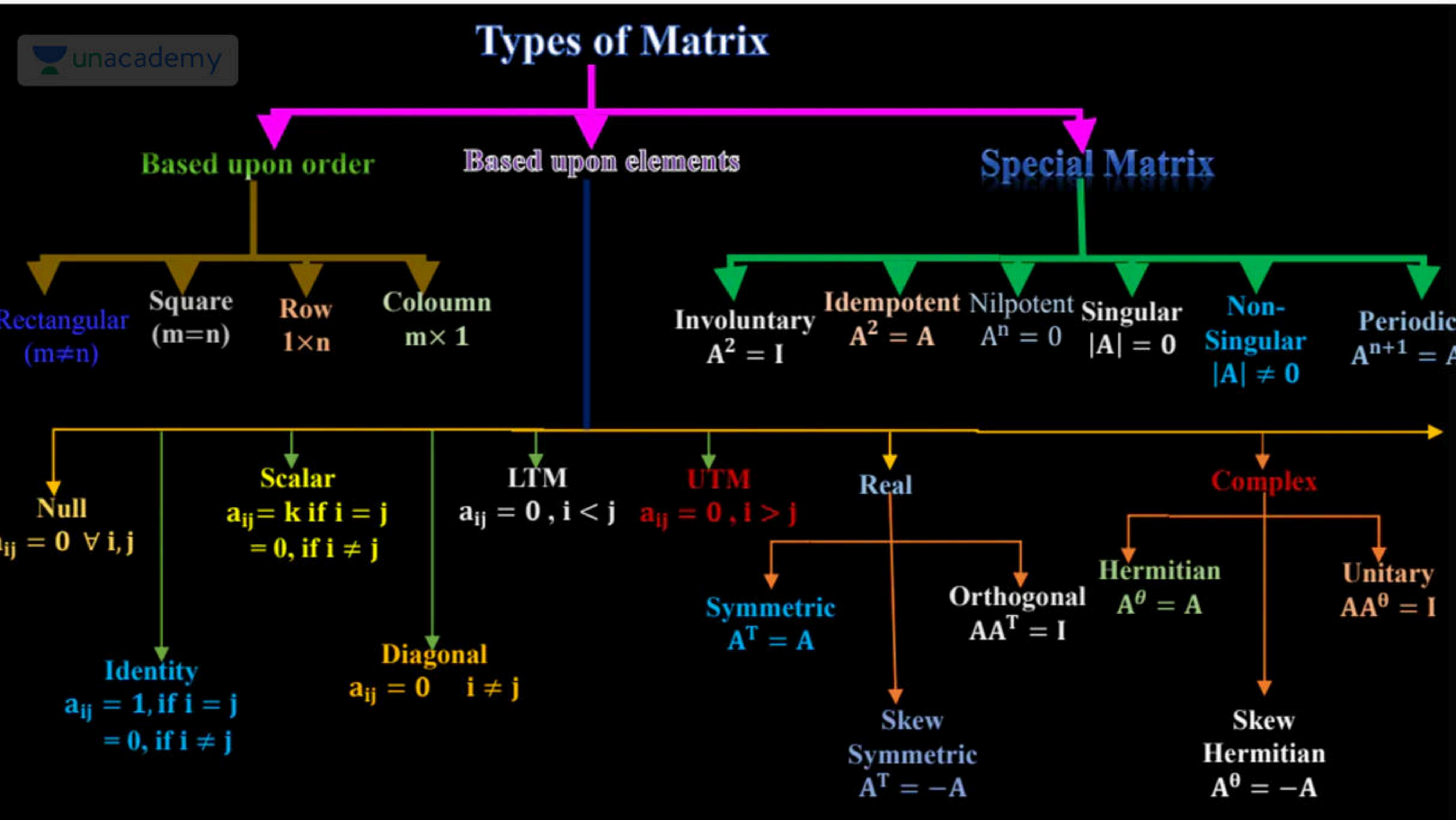
1. Types of matrices
2. Algebra of matrices
3. Determinant of matrix
4. Inverse of matrix
5. Rank of a matrix
6. Eigen values and Eigen vectors
7. System of linear equations

Matrix

A rectangular arrangement of numbers (which may be real or complex) in rows and columns is called as matrix.

Order of Matrix

A matrix having " m " rows and "n" columns is called a matrix of order m x n .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Types of Matrix
Based upon order
Based upon elements
Special Matrix
Rectangular (m≠n)
Square (m=n)
Row 1×n
Coloumn m× 1
Involuntary $A^2 = I$
Idempotent $A^2 = A$
Nilpotent $A^n = 0$
Singular $|A| = 0$
Non-Singular $|A| \neq 0$
Periodic $A^{n+1} = A$
Null $a_{ij} = 0 \ \forall$ i, j
Identity $a_{ij} = 1$, if i = j = 0, if i ≠ j
Scalar $a_{ij} = k$ if i = j = 0, if i ≠ j
Diagonal $a_{ij} = 0 \quad i \neq j$
LTM $a_{ij} = 0$, i < j
UTM $a_{ij} = 0$, i > j
Real
Symmetric $A^T = A$
Orthogonal $AA^T = I$
Skew Symmetric $A^T = -A$
Complex
Hermitian $A^\theta = A$
Unitary $AA^\theta = I$
Skew Hermitian $A^\theta = -A$

Rectangular matrix

If the number of rows is not equal to the number of columns in a matrix then the matrix is called a rectangular matrix.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 6 & 7 \\ 4 & 7 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 0 & 7 \end{bmatrix}$$

Square matrix:

If the number of rows is equal to the number of columns in a matrix then the matrix is called square matrix.

$$[2]$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

Row matrix:

If a matrix A has only one row and any number of columns then the matrix A is called a row matrix (or) row vector.

$$A = [1 - \;\; 4 \quad 7]$$

Column matrix:

If a matrix A has only one column and any number of rows then the matrix A is called a column matrix (or) column vector.

$$A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Null matrix or zero matrix:

If every element of a matrix is zero then the matrix is called null matrix .

$O_{2\times 3} =$

Identity matrix (Unit matrix)

A square matrix in which principal diagonal elements are ' 1' and rest all are zeros is called as identity matrix .

Properties

1. $AI =$

2. $I^{-1} =$

3. $I^{T} =$

4. $I^{n} =$

5. $|I| =$

6. $Adj\ (I) =$

Diagonal Matrix

In a square matrix if all the elements outside the principal diagonal are zeros , and at least one of the principal diagonal elements is non zero , then the matrix is called as diagonal matrix .

Scalar matrix

If all the principal diagonal elements of a diagonal matrix are same or equal then the matrix is called a scalar matrix.

Lower triangular matrix

If all the elements above the principal diagonal are zero in a square matrix then the matrix is called lower triangular matrix.

Upper triangular matrix:

If all the elements below the principal diagonal are zero in a square matrix then the matrix is called an upper triangular matrix.

Real matrix:

If all the elements of a matrix A are real numbers then the matrix A is called a real matrix,

$$[2]$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Complex matrix:

If at least one of the elements of a matrix A is purely imaginary (or) complex then the matrix A is called complex matrix. For example, the matrices

$[2i]$

$$\begin{bmatrix} 0 & 2 \\ 3i & 4+3i \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3i & 0 \\ 0 & 9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3+2i & 4 \\ 2 & 0 \end{bmatrix}$$

Principal diagonal of a square matrix

In a square matrix the diagonal from first element of the first row to the last element of the last row is called the principal diagonal of the square matrix .

Trace of a matrix

The sum of the principal diagonal elements of a square matrix A is called as trace of A and it is denoted by trace (A) or tr(A).

Transpose of a matrix

If a matrix $B_{n\times m}$ is obtained from a matrix $A_{m\times n}$, by changing its rows into columns and its columns into rows then the matrix $B_{n\times m}$ is called transpose of A and is denoted by A^T or A^1.

Properties

1. $(A^T)^T =$

2. $(KA)^T =$

3. $(A - B)^T =$

4. $(A\ B\ C\ )^T =$

5. $(A\ B\ C\ DEFGH)^T =$

6. $(A^T)^n =$

Symmetric matrix

a square matrix is said to be symmetric matrix , if .

Skew-symmetric matrix

a square matrix is said to be skew symmetric matrix , if

NOTE :

1. The diagonal elements of a skew- symmetric matrix are all zero.
2. The necessary and sufficient condition for square matrix A to be skew symmetric matrix is that $\boldsymbol{A}^T = -A$.
3. Sum of all the elements of skew symmetric matrix is zero . But vice versa may not be true.
4. Null matrix is both symmetric and skew symmetric

Properties of Symmetric and Skew Symmetric matrix

1. Every square matrix A can be uniquely expressed as a sum of symmetric & Skew- symmetric matrices.

2. If A is symmetric matrix , then kA is also symmetric matrix

3. If A is skew symmetric matrix , then kA is also skew symmetric matrix

4. If A and B are Symmetric matrix then $(K_1A + K_2B)$ is also symmetric matrix

5. If A and B are skew Symmetric matrix then $(K_1A + K_2B)$ is also skew symmetric matrix

6. If A is any real square matrix , then AA^T, A^TA are symmetric.

7. If A and B are square symmetric matrices of same order then AB is symmetric if and only if AB =BA.

8. If A and B are skew symmetric matrices of same order then AB is skew symmetric if and only if $AB = -BA$.

9. If A and B are symmetric

then

(AB + BA) -----> symmetric

(AB - BA) -----> skew symmetric

10. If A and B are skew symmetric then

(AB + BA) -----> symmetric

(AB - BA) -----> skew symmetric

Orthogonal matrix:

A square matrix $A_{n\times n}$ is said to be an orthogonal matrix if

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

$$\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$$

Note:

1. Every orthogonal matrix is non singular
2. If A is orthogonal then A^T *and* A^{-1} *is also orthogonal*
3. Product of orthogonal matrices is also orthogonal
4. A symmetric , orthogonal matrix is self inversed
5. If A and B are orthogonal matrices then AB and BA are also orthogonal matrices.
6. The determinant of any orthogonal matrix is $|A| = \pm 1$

Conjugate matrices

the matrix obtained from a given matrix A by replacing all the elements by their corresponding conjugate complex numbers is called conjugate of matrix A and denoted by $\overline{A}$

For example,

If $A = \begin{bmatrix} 1+i & 2i & 4 \\ 7 & 3-i & 2-3i \end{bmatrix}$

Properties of conjugate matrix

1. $\overline{(\overline{A})} = A$
2. $\overline{(A+B)} = \overline{A} + \overline{B}$
3. $\overline{(AB)} = \overline{A}\,\overline{B}$
4. $\overline{(kA)} = \overline{k}\,\overline{A}$,k is any complex number.
5. $\overline{A} = A$, then A is real matrix.
6. $\overline{A} = -A$, then A is purely imaginary matrix.

Transposed conjugate of a matrix

If $\overline{A}$ is a conjugate matrix of a complex matrix A then $(\overline{A})^T$ is called transposed conjugate of A and it is denoted by A^θ

Properties:

(i) $\left(A^{\theta}\right)^{\theta} = A$

(ii) $(A + B)^{\theta} = A^{\theta} + B^{\theta}$

(iii) $(kA)^{\theta} = \overline{K}A^{\theta}$

(iv) $(AB)^{\theta} = B^{\theta}A^{\theta}$

Hermitian matrix

A square matrix A is said to be Hermitian if

Note:

(i) The necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$.

(ii) The diagonal element of a Hermitian matrix are purely real.

Skew- Hermitian matrix

A square matrix A is said to be skew Hermitian if.

Note:

(i) The necessary & sufficient condition for a matrix A to be Skew-Hermitian is that

$$A^{\theta} = -A.$$

(ii) The diagonal elements of a skew-Hermitian matrix are either zero (or) purely imaginary.

(iii) Every complex square matrix A can be uniquely written as a sum of Hermitian and Skew-Hermitian matrices.

Unitary matrix

a square matrix A is said to be unitary if

Properties of unitary matrix

1. If A is unitary then A^T is also unitary
2. If A and B are unitary matrices then AB is also unitary.

Idempotent matrix

A square matrix A is said to be idempotent matrix , if

Involuntary matrix

A square matrix A is said to be involuntary matrix , if

NOTE:

1. Every involuntary matrix is non singular matrix
2. If A is involuntary matrix then $A = A^{-1}$
3. A is involuntary matrix then $(I+A)(I-A) = O$

Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exits at least one n , such that $A^n = 0$, then A is called as nilpotent matrix. The least value of n is called index of the nilpotent matrix A.

Note :

The trace of Nilpotent matrix is always zero.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

Non –Singular Matrix

A square matrix A is said to be non singular matrix , if

Singular Matrix

A square matrix A is said to be singular matrix , if

Properties

1. Product of non singular matrix is non singular matrix
2. Every nilpotent matrix is always singular
3. If A is idempotent matrix and if $A \neq I$, *then A is singular*
4. Involuntary matrix is always non singular
5. Orthogonal matrix is always non singular .

Periodic matrix

A square matrix A is called periodic if $A^{n+1} = A$ where k is a positive integer. The least value of k is called as period of A .

Equality of matrices

Two matrices $A_{m \times n}$ and $B_{p \times q}$ are said to be equal if

(i) they are of same order, i.e., $m = p$, $n = q$ and

(ii) their corresponding elements are equal.

Algebra of Matrix

Addition operation

If A and B are two matrices of same order then the matrix obtained by adding the corresponding elements of A and B respectively is called the sum of A and B and it is denoted by A+B.

1. $A+B = B+A$ (Commutative law)

2. $(A+B)+C = A+(B+C)$ (Associative law)

3. $A+(-A) = 0$(null matrix)

4. $A+B = A+C$ then $B = C$ (Left Cancellation law)

5. $B+A = C+A$, then $B = C$ (Right cancellation law)

Subtraction of matrices

If A & B are two matrices of same order then the matrix obtained by subtracting each elements of A is called from the corresponding elements of A is called the difference of A & B and it is denoted by A-B.

Scalar multiplication

If $A = (a_{ij})$ is an $m \times n$ matrix and k is any scalar (real number or complex number) then the matrix kA is obtained by multiplying every element of matrix A by a scalar k

If A= $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ are two matrices and k, l are two scalars then

(i) $k(A+B) =$

(ii) $(k+l)A=$

(iii) $(k\,l)A =$

(iv) $(-k)A =$

Multiplication of Matrices

If A & B are any two matrices such that the number of columns in A is equal to the number of rows in B then the product of A and B is denoted by AB .

1. If the product AB exists ,the product BA may or may not exist.

2. Matrix multiplication is not commutative

3.Multiplication of matrices is associative

4.Matrix multiplication is distributive over matrix addition

5. $A^m A^n = A^{m+n}$

6. $(A^m)^n = A^{mn}$

, then A and B may not be null matrix

8. If $AB = 0$, then BA may not be zero .

9. If $A^2 = B^2$, then A and B may not be equal

10. If A and B are two square matrix of same order then

$$(A+B)^2 = A^2 + B^2 + AB + BA$$

$$(A- B)^2 = A^2 + B^2 - AB - BA$$

Properties of Trace of a Matrix

If A & B are square matrices of order n, then

i. tr(A+B) = tr(A)+tr(B)

ii. tr(A-B) = tr(A)- tr(B)

iii. tr(AB) $\neq$ tr(A) tr(B)

iv. tr(BA) $\neq$tr(B) tr(A)

v. tr(AB) = tr(BA) , provided product AB and BA exits

vi. tr(kA) = k tr(A) where k is a scalar.

vii. tr$(A^T) = tr(A)$

viii. tr $(I_n) = n$

1. A real square matrix A is called skew-symmetric if

(GATE-16-ME)

(a) $A^T = A$

(b) $A^T = A^{-1}$

(c) $A^T = -A$

(d) $A^T = A + A^{-1}$

2. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$ then AB^T is equal to

(GATE-17-CE)

(a) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$

(d) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

3. Which one of the following matrices is singular?

(GATE-18-CE)

(a) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

4. If $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $Q^T P^T$ is

(GATE-21-CE)

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

5. If A is a square matrix then orthogonality property mandates

(GATE-21-CE)

(a) $AA^T = I$ (b) $AA^T = 0$

(c) $AA^T = A^{-2}$ (d) $AA^T = A^2$

6. If A and B are real symmetric matrices of order n then which of the following is true.

(GATE – 94[CS])

(a) $A A^T = I$

(b) $A = A - 1$

(c) $AB = BA$

(d) $(AB)^T = B^T A^T$

7. Given matrix $L = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$ and $M = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

then $L \times M$ is (GATE – 95[PI])

(a) $\begin{bmatrix} 8 & 1 \\ 13 & 2 \\ 22 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 8 \\ 2 & 13 \\ 5 & 22 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 2 \\ 9 & 4 \\ 0 & 5 \end{bmatrix}$

8. The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication.

(GATE – 96[CS])

(a) If $a = b$ (or) $\theta = n\pi$, n is an integer

(b) always

(c) never

(d) If $a\cos\theta \neq b\sin\theta$

9. If A and B are two matrices if both AB and BA exists

a) Only if A has as many rows as B has columns

b) Only if the order of A and B are same

c) Only if A and B are skew symmetric

d) Only if both A and B are symmetric

10. If A and B are two matrices if both AB and BA exists

a) Only if A has as many rows as B has columns

b) Only if the order of A and B are same

c) Only if A and B are skew symmetric

d) Only if both A and B are symmetric

11. An $n \times n$ array V is defined as follows $v[i,j] = i - j$ for all i, j, $1 \le i, j \le n$ then the sum of the elements of the array V is

(GATE–2000[CS])

(a) 0 (b) $n - 1$

(c) $n^2 - 3n + 2$ (d) $n(n + 1)$

12. The product $[P]\,[Q]^T$ of the following two matrices $[P]$ and $[Q]$

where $[P] = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $[Q] = \begin{bmatrix} 4 & 8 \\ 9 & 2 \end{bmatrix}$ is

(GATE–01[CE])

(a) $\begin{bmatrix} 32 & 24 \\ 56 & 46 \end{bmatrix}$ (b) $\begin{bmatrix} 46 & 56 \\ 24 & 32 \end{bmatrix}$

(c) $\begin{bmatrix} 35 & 22 \\ 61 & 42 \end{bmatrix}$ (d) $\begin{bmatrix} 32 & 56 \\ 24 & 46 \end{bmatrix}$

13. Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$, $[F]_{5\times1}$ are given. Matrices [B] and [E] are symmetric. Following statements are made with respect to their matrices.

(I) Matrix product $[F]^T\ [C]^T\ [B]\ [C]\ [F]$ is a scalar.

(II) Matrix product $[D]^T\ [F]\ [D]$ is always symmetric.

With reference to above statements which of the following applies?

(GATE–04[CE])

(a) statement (I) is true but (II) is false

(b) statement (I) is false but (II) is true

(c) both the statements are true

(d) both the statements are false

14. If A_{mxn} and B_{nxp} are $matrices,then$ the number of multiplication and additions in computing AB are

a)mnp , mp(n-1)

b)(m-1)np , mp(n-1)

c)mnp , (m-1)p(n-1)

d)mn , mp(n-1)

15. A square matrix B is symmetric if __

(GATE–09[CE])

(a) $B^T = -B$

(b) $B^T = B$

(c) $B^{-1} = B$

(d) $B^{-1} = B^T$

16. If a matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$ the transpose of product of these two matrices i.e., $(AB)^T$ is equal to

(GATE-11 [PI])

(a) $\begin{bmatrix} 28 & 19 \\ 34 & 47 \end{bmatrix}$

(b) $\begin{bmatrix} 19 & 34 \\ 47 & 28 \end{bmatrix}$

(c) $\begin{bmatrix} 48 & 33 \\ 28 & 19 \end{bmatrix}$

(d) $\begin{bmatrix} 28 & 19 \\ 48 & 33 \end{bmatrix}$

17. Consider a matrix 3×3 A whose $(i, j)^{th}$ element $= (i - j)^3$, then the matrix A will be

(GATE-2022-EEE)

(a) Symmetric
(b) Skew symmetric
(c) Unitary
(d) Null

18. [A] is a square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and differences of these matrices are defined as

$[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$ respectively. Which of the following statements is true? **(GATE–11[CS])**

(a) Both [S] and [D] are symmetric

(b) Both [S] and [D] are skew-symmetric

(c) [S] is skew-symmetric and [D] is symmetric

(d) [S] is symmetric and [D] is skew symmetric

19. What is the minimum number of multiplications involved in computing the matrix product PQR? Matrix P has 4 rows and 2 columns, matrix Q has 2 rows and 4 columns and matrix R has 4 rows and 1 column ________.

(GATE – 2013[CE])

20. If $A = \begin{bmatrix} 10 & 2k+5 \\ 3k-5 & k+5 \end{bmatrix}$ is a symmetric matrix, the value of k is________________

(GATE-2022-ME)

(a) 8

(b) 10

(c) -0.4

(d) $\frac{1+\sqrt{1561}}{12}$

Determinant of a square matrix of order two

1 • Asked by Rohit

Sir ye kaise true hai ? Because - minus to kisi bhi row se common ho skta hai na ?

Subtraction of matrices

If A & B are two matrices of $\boxed{\text{same order}}$ then the matrix obtained by subtracting each elements of A is called from the corresponding elements of A is called the difference of A & B and it is denoted by A-B.

$$A - B = A + (-B) =$$

$$B = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$-B = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$A + (-B) = \begin{bmatrix} \; \end{bmatrix}$$

$$2 - 3 = \underset{1\times1}{(2)} + \underset{1\times1}{(-3)}$$

Minor of an element

If $A = (a_{ij})$ is a square matrix of order 'n' then the minor of an element a_{ij} in A is the determinant of a square matrix that remains after deleting corresponding the i^{th} row and j^{th} column of A. It is denoted by M_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= a_{21}a_{33} - a_{31}a_{23}.$$

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 3 & -1 & 0 \\ 6 & 5 & 8 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} 2 & 4 \\ 6 & 5 \end{vmatrix}$$

$$= 10 - 24$$

$$= -14$$

Cofactor of an element

If $A = (a_{ij})$ is a square matrix of order 'n' then the cofactor of an element a_{ij} is denoted by A_{ij} and defined as $(-1)^{i+j} M_{ij}$ where M_{ij} is a minor of a_{ij}.

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 7 & 6 \\ 5 & 0 & 8 \end{bmatrix}$$

$$A_{22} = (-1)^{2+2} M_{22}$$

$$= (-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 8 \end{vmatrix} = (-1)^4 [16 - 15] = \underline{+1}$$

$$A = \begin{bmatrix} 2 & -3 & -1 & 7 \\ 1 & 8 & 4 & 9 \\ 0 & 2 & 6 & 5 \\ -5 & -8 & 1 & -7 \end{bmatrix}$$

$\underline{4 \times 4}$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 & 7 \\ 1 & 4 & 9 \\ -5 & 1 & -7 \end{vmatrix}$$

Determinant of 3rd and Higher order matrices

The determinant of a square matrix is equal to the sum of the products of the elements of a row with their corresponding cofactors. (or)

The determinant of a square matrix is equal to the sum of the products of the elements of a column with their corresponding cofactors.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 6 & -2 \\ -3 & 5 & 8 \end{bmatrix} \begin{array}{l} \\ \longrightarrow R_2 \\ \\ \end{array}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix} = +4$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 4 \\ -3 & 8 \end{vmatrix} = 20$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -3 & 5 \end{vmatrix} = -11$$

$$|A| = 7\,A_{21} + 6\,A_{22} - 2\,A_{23}.$$

$$= 7(4) + 6(20) - 2(-11)$$

$$= 170.$$

$$A = \begin{array}{c} \begin{array}{ccc} + & - & +C_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 4 \\ 7 & 6 & -2 \\ -3 & 5 & 8 \end{bmatrix} \end{array}$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 7 & 6 \\ -3 & 5 \end{vmatrix} = 53.$$

$$|A| = 4A_{13} - 2A_{23} + 8A_{33}.$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -3 & 5 \end{vmatrix} = -11$$

$$= 4(53) - 2(-11) + 8(-8)$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} = -8$$

$$= \underline{170}$$

Properties of the Determinants

1. Determinant of a matrix , expanding by any row (or column) is unique.

2. The sum of the products of the elements of any row(or column)by the co factors of another row (or column) is zero.

$$A = \begin{array}{cc} \begin{bmatrix} 2 & -4 & 3 \\ 1 & 6 & 7 \\ 8 & 5 & 9 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

R_2

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -4 & 3 \\ 5 & 9 \end{vmatrix} = +51$$

$$A_{22} = + \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6$$

$$A_{23} = - \begin{vmatrix} 2 & -4 \\ 8 & 5 \end{vmatrix} = -42.$$

$$2(51) - 4(-6) + 3(-42)$$

$$= 0$$

$$8(51) + 5(-6) + 9(-42) = 0$$

3. If A and B are two square matrices of same order then $|AB| = |A||B|$.

$$\boxed{\det(AB) = \det(A)\det(B)}$$

4. $|A^m| = |A|^m$

If $|A_{n\times n}| \neq 0$ then $|A^{-1}| = \frac{1}{|A|}$

$$|A^m| = |A|^m$$

$$|A^{-1}| = |A|^{-1} = \frac{1}{|A|}.$$

$$(|A| \neq 0)$$

5. If every element of a row (column) of a determinant of A is zero then $|A| = 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = 0(A_{21}) + 0(A_{22}) + 0(A_{23}) = 0$$

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

$$|A| = 0$$

6. If A is a square matrix of order n then $|A| = |A^T|$

$$|A^T| = |A|$$

7. If any two rows (or columns) of a determinant are identical then the value of determinant is zero.

$$A = \begin{array}{cc} \begin{bmatrix} 2 & 7 & 9 \\ 4 & 14 & 5 \\ 2 & 7 & 9 \end{bmatrix} & \begin{array}{l} \to R_1 \\ \\ \to R_3. \end{array} \end{array}$$

$$|A| = 0$$

$$A = \begin{array}{cc} \begin{bmatrix} 2 & 7 & 9 \\ 4 & 14 & 18 \\ 3 & -7 & 2 \end{bmatrix} & \begin{array}{l} \to R_1 \\ \to R_2 \\ \\ \end{array} \end{array}$$

$$A = 2\begin{bmatrix} 2 & 7 & 9 \\ 2 & 7 & 9 \\ 3 & -7 & 2 \end{bmatrix}$$

$$|A| = 0$$

8. If the elements of any two rows (or columns) of a determinant are proportional (or in the same ratio) then the value of the determinant is zero.

$$A = \begin{bmatrix} 2 & 7 & 9 \\ 4 & 14 & 18 \\ 3 & -7 & 2 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \end{array}$$

$$\boxed{R_2 = 2R_1}$$

$$A = 2\begin{bmatrix} 2 & 7 & 9 \\ 2 & 7 & 9 \\ 3 & -7 & 2 \end{bmatrix}$$

$$R_2 \propto R_1$$

$$|A| = 0$$

9. If any two rows (or columns) of a determinant are interchanged then the sign of determinant changes

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 8 \\ 2 & 4 \end{bmatrix}$$

$$|A| = -4 \qquad |B| = 20 - 16 = +4$$

10 .If each element of a row (or a column) of a determinant is multiplied by a constant ‘k’ then the value of the determinant will be multiplied by ‘k’

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$$

$$|A| = -4$$

$$B = \begin{bmatrix} 4 & 8 \\ 5 & 8 \end{bmatrix}$$

$$|B| = 32 - 40 = -8 = \underline{2}(-4)$$

11. $|kA_{n\times n}| = k^n|A_{n\times n}|$

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}_{2\times 2}$$

$$|A| = -4$$

$$B = kA$$

$$B = \begin{bmatrix} 2k & 4k \\ 5k & 8k \end{bmatrix}$$

$$|B| = 16k^2 - 20k^2 = -4k^2$$

$$|B| = (k)^2(-4)$$

$$\boxed{|B| = k^2|A|}$$

12. The determinant of upper triangular, lower triangular, diagonal, scalar (or) unit matrix is equal to product of its diagonal elements.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 7 & 6 \\ 0 & 0 & -2 \end{bmatrix} \qquad |A| = -14 \text{ (UTM)}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 2 & 0 \\ -1 & 6 & 5 \end{bmatrix} \qquad |B| = 10 \text{ (LTM)}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad |C| = -8.$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|D| = -8.$$

13. If sum of all elements of each row (or column) is zero , then the value of determinant is zero.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ 7 & -8 & 1 \end{bmatrix} \qquad |A| = 0$$

$$B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 2 & -8 \\ -3 & -6 & 1 \end{bmatrix} \qquad |B| = 0$$

14. If all elements of matrix are consecutive, then Det (A) = 0

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A| = 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = 4 - 6 = -2 \neq 0$$

$$B = \begin{bmatrix} 9 & 10 & 11 \\ 12 & 13 & 14 \\ 15 & 16 & 17 \end{bmatrix}$$

$$|B| = 0$$

Not valid 2×2 matrix

$n \geq 3$

15. If every row is starting with 2^{nd} element of previous row and the elements are consecutive in nature, then Det(A) = 0

$(n > 2)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \qquad |A| = 0$$

$$B = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 \end{bmatrix} \qquad |B| = 0$$

16. The number of terms in the expansion of determinant of A of nxn are $-\underline{n!}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$$|A| = a_{11}\left(\overset{①}{a_{22}a_{33}} - \overset{②}{a_{32}a_{23}}\right) - a_{12}\left(\overset{③}{a_{21}a_{33}} - \overset{④}{a_{31}a_{23}}\right)$$
$$+ a_{13}\left(\overset{⑤}{a_{21}a_{32}} - \overset{⑥}{a_{31}a_{22}}\right)$$

$6 = 3!$

$\underline{n!}$

$$A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}_{2\times 2}$$

$$|A| = 1 - a^2 = (1-a^2)^{\frac{2}{2}}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & a & 0 \\ 0 & a & 1 & 0 \\ a & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$|A| = (1-a^2)^2 = (1-a^2)^{\frac{4}{2}}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & a & 0 & 0 & 1 & 0 \\ a & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}.$$

$$|A| = (1-a^2)^{\frac{6}{2}} = (1-a^2)^3.$$

Q. The determinant of the matrix M shown below is ____________.

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(GATE – 2021 – IN)

$$|M| = 1\begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1(4)(4-6) - 2(3)(4-6)$$

$$= 4(-2) - 2(3)(-2) = -8+12 = +4.$$

Q. The value of the following determinant

$$\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$ is (GATE – 94[PI])

(a) 8 (b) 12

(c) –12 (d) –8

$$|A| = 1\left[9(25) - 256\right] - 4\left[100 - 144\right] + 9\left[64 - 81\right]$$

$$= -8$$

Q. If the determinant of the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -6 \\ 2 & 7 & 8 \end{bmatrix}$ is 26, then the determinant of the matrix $\begin{bmatrix} 2 & 7 & 8 \\ 0 & 5 & -6 \\ 1 & 3 & 2 \end{bmatrix}$ is

(GATE – 97[CE])

(a) – 26 (b) 26

(c) 0 (d) 52

Q. The determinant of the matrix

$$\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is (GATE – 97[CS])

(a) 11 (b) – 48

(c) 0 (d) – 24

Q. If A is any nxn matrix and k is a scalar then $|kA| = \alpha |A|$ where α is

(GATE–99[CE])

(a) kn

(b) n^k

(c) k^n

(d) $\frac{k}{n}$

Q. The number of terms in the expansion of general determinant of order n is

(GATE – 99[CE])

(a) n^2
(b) $n!$
(c) n
(d) $(n+1)^2$

Q. The determinant of the matrix $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{bmatrix}$ is (GATE–2000[CS])

(a) 4 (b) 0 (c) 15 (d) 20

Q. The determinant of the following matrix

$$\begin{bmatrix} 5 & 3 & 2 \\ 1 & 2 & 6 \\ 3 & 5 & 10 \end{bmatrix}$$

(GATE– 01[CE])

(a) – 76
(b) – 28
(c) 28
(d) 72

85. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$ is **(GATE – 02[EE])**

(a) 100

(b) 200

(c) 1

(d) 300

Q. The determinant of the matrix given below

is $\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ (GATE–05)

(a) -1

(b) 0

(c) 1

(d) 2

Q. The value of the determinant $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$ is

(GATE–09[PI])

(a) – 28
(b) – 24
(c) 32
(d) 36

Q. Consider the matrix

$$J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where α is a non – negative real number. The value of α for which det $(P) = 0$ is ______.

(GATE–14–EC–SET1)

Q. The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is ______.

(GATE–14–EC–SET2)

Q. Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is -12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is **(GATE-14-ME-SET1)**

(a) -96 (b) -24

(c) 24 (d) 96

Q. If any two columns of a determinant

$$P = \begin{vmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{vmatrix}$$ are interchanged, which one of the following statements regarding the value of the determinant is CORRECT?

(GATE – 15 – ME – Set 1)

(a) Absolute value remains unchanged but sign will change.

(b) Both absolute value and sign will change.

(c) Absolute value will change but sign will not change.

(d) Both absolute value and sign will remain unchanged.

Q. Matrix A as product of two other matrices is given by

$$A = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix}$$

The value of det(A) is______________ [round off to nearest integer]

(GATE-2022-PI)

Q. The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A) = 100 and trace(A) = 14. The value of $|a - b|$ is ________.

(GATE-16-EC)

Q. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ then det (A^{-1}) is __________ (correct to two decimal places).

(GATE-18-ME)

Q. For what value of x will the matrix given below become singular? $\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$

(GATE–04[ME])

a) -4 b) 4 c) 2 d)-2

Q. The determinant of the matrix given below

is $\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ (GATE–05)

(a) –1
(b) 0
(c) 1
(d) 2

Adjoint of a square matrix

If the elements of a matrix are replaced by corresponding co factors then the transpose of the resulting matrix is called the adjoint of the given matrix .

Inverse (or) reciprocal of a square matrix

If for a non-singular matrix A of order 'n' there exists another non-singular matrix B of order 'n' such that $AB = BA = I_n$ then B is called the inverse of A . it is denoted by A^{-1}.

- If the inverse of a square matrices A exists then the matrix is called invertible matrix.
- If A is a non-singular matrix of order n then $A^{-1} = \frac{1}{|A|}\, adj(A)$.

Q. The inverse of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is

(GATE-19-CE)

(a) $\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -\frac{4}{5} & -\frac{9}{5} \\ -3 & \frac{4}{5} & \frac{14}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -\frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

Q. The inverse of the matrix $S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is (GATE – 95[EE])

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & -2 \\ -2 & 2 & -2 \\ 0 & 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Q The matrix $\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$ is an inverse of the matrix $\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}$ **(GATE – 94[PI])**

(a) True (b) False

Q. If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$ then $a + b =$

(GATE–05[EE])

(a) $\frac{7}{20}$

(b) $\frac{3}{20}$

(c) $\frac{19}{60}$

(d) $\frac{11}{20}$

Q. Inverse of matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is

(GATE – 97[CE])

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Q. If $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ then $A^{-1} =$

(GATE – 98[EE])

(a) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 0 & 2 \\ 0 & -1/3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1/5 & 0 & 1/2 \\ 0 & 1/3 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/5 & 0 & -1/2 \\ 0 & 1/3 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Q. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then the top row of R^{-1} is (GATE–05[EE])

(a) [5 6 4]
(b) [5 −3 1]
(c) [2 0 −1]
(d) [2 −1 0]

Q. If $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ and

$\text{adj}(A) = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix}$ then $k =$

(GATE – 99)

(a) –5 (b) 3

(c) –3 (d) 5

Q. Consider the matrices X_{4x3}, Y_{4x3} and P_{2x3}. The order of $[P\,(X^T Y)^{-1} P^T]^T$ will be

(GATE–05[CE])

(a) 2×2

(b) 3×3

(c) 4×3

(d) 3×4

Q. Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \ (AA^T)^{-1} \text{ is}$$

(GATE–05[EC])

(a) $\frac{1}{4}I_4$

(b) $\frac{1}{2}I_4$

(c) I

(d) $\frac{1}{3}I_4$

Q. The inverse of matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(GATE–08[PI])

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

Properties of Inverse of a Matrix

Q. If A, B, C are square matrices of the same order then $(ABC)^{-1}$ is equal be

(GATE–2000[CE])

(a) $C^{-1} A^{-1} B^{-1}$ (b) $C^{-1} B^{-1} A^{-1}$

(c) $A^{-1} B^{-1} C^{-1}$ (d) $A^{-1} C^{-1} B^{-1}$

Q. Let A, B, C, D be n×n matrices, each with non-zero determinant. $ABCD = I$ then $B^{-1} =$

(GATE–04[CS])

(a) $D^{-1}C^{-1}A^{-1}$ (b) CDA

(c) ABC (d) does not exist

Q. The product of marices $(PQ)^{-1} P$ is

(GATE–08[CE])

(a) P^{-1}

(b) Q^{-1}

(c) $P^{-1} Q^{-1} P$

(d) $P Q P^{-1}$

Q. For a given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$,

where $i = \sqrt{-1}$, the inverse of matrix P is

(GATE – 15 – ME – Set 3)

(a) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$

(b) $\frac{1}{25}\begin{bmatrix} i & 4-3i \\ 4+3i & -i \end{bmatrix}$

(c) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

(d) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

Q. Let $M^4 = I$ (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:

(GATE – 16 – EC – Set 1)

(a) M^{4k+1}

(b) M^{4k+2}

(c) M^{4k+3}

(d) M^{4k}

Q. P and Q are two square matrices of the same order. Which of the following statement(s) is/are correct?

(GATE-2022-CE)

(a) If P and Q are invertible, then $[PQ]^{-1} = Q^{-1}P^{-1}$
(b) If P and Q are invertible, then $[QP]^{-1} = P^{-1}Q^{-1}$
(c) If P and Q are invertible, then $[PQ]^{-1} = P^{-1}Q^{-1}$
(d) If P and Q are not invertible, then $[PQ]^{-1} = Q^{-1}P^{-1}$

Elementary Transformations

1. Interchanging any two rows (or columns) this transformation is indicated as

2. Multiplication of the elements of any row (or column)by a scalar quantity , this can be indicated as

3. Addition of a constant multiple of the elements of any row (or column) to the corresponding elements of any other row (or column)

Q. If $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ then which of the following is a factor of Δ.

(GATE – 98[CS])

(a) $a + b$
(b) $a - b$
(c) abc
(d) $a + b + c$

Q. The determinant $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$ equals to

(GATE–07[PI])

(a) 0

(b) $2b(b-1)$

(c) $2(1-b)(1+2b)$

(d) $3b(1+b)$

Rank of a Matrix

A non-negative integer ‘r’ is said to be the rank of matrix A, if

(i) There exists at least one non-zero minor of order ‘r’.

(ii) all minors of order (r+1) if they exist, are zeros.

Then we rite Rank of A = $\rho(A)$ = r

(or)

Rank of matrix A is the number of linearly independent rows (or columns)of A

(or)

The number of non zero rows in the Row Echelon form .

Echelon form

A matrix A of order $m \times n$ is said to be in row echelon form if

1. The number of zeros before the first non-zero element in each row is less than the number of such zeros in the next non zero row.
2. If there are any Zero rows , they must be below the non-zero rows.

Q. A 5×7 matrix has all its entries equal to 1. Then the rank of a matrix is

(GATE – 94[EE])

(a) 7
(b) 5
(c) 1
(d) Zero

Q. The rank of the matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is

(GATE – 94[CS])

(a) 0

(b) 1

(c) 2

(d) 3

Q. Rank of the matrix $\begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$ is 3

(GATE – 94[ME])

(a) True (b) False

Q. The rank of the following (n+1) x (n+1) matrix, where 'a' is a real number is

$$\begin{bmatrix} 1 & a & a^2 & . & . & . & a^n \\ 1 & a & a^2 & . & . & . & a^n \\ . & & & & & & \\ . & & & & & & \\ 1 & a & a^2 & . & . & . & a^n \end{bmatrix}$$

(GATE – 95[EE])

(a) 1 (b) 2

(c) n (d) depends on the value of a

Q. The rank of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is

(GATE – 02[CS])

(a) 4 (b) 2 (c) 1 (d) 0

Q. Given matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is (GATE – 03[CE])

(a) 4 (b) 3

(c) 2 (d) 1

Q. Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i.j$. Then the rank of A is

(GATE–07[IN])

(a) 0 (b) 1

(c) $n - 1$ (d) n

Q. A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix.

Let matrix $A^{+} = (A^{T}A)^{-1}A^{T}$. Then which one of the following statement is false?

(GATE–08[EE])

(a) $AA^{+}A = A$

(b) $(AA^{+})^{2} = AA^{+}$

(c) $A^{+}A = I$

(d) $AA^{+}A = A^{+}$

Q. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is

(GATE–14–CE–SET2)

Q. Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i.j$. The rank of A is :

(GATE – 15 – CE – Set 2)

(a) 0
(b) 1
(c) $n - 1$
(d) n

Properties

1. Rank of a matrix will not be affected by the elementary transformations.
2. Rank of a null matrix is zero
3. $\rho(A_{m\times n}) \leq min\{m, n\}$.
4. If A is non singular matrix, then $\rho(A_{n\times n}) = n$.
5. If A is singular matrix ,then $\rho(A_{n\times n}) < n$.
6. $\rho(I_n) = n$.

7. $\rho\left(A^{T}\right) = \rho(A)$.

$\rho\left(A^{-1}\right) = \rho(A)$.

$\rho\left(AA^{T}\right) = \rho(A)$.

$\rho\left(A^{\theta}\right) = \rho(A)$.

$\rho\left(AA^{\theta}\right) = \rho(A)$.

8. The rank of a non zero row matrix of order 1x n is ………

The rank of a non zero column matrix of order m x1 is ………

If A is a non zero row matrix and B is non zero column matrix then rank of AB is…..

rank of BA is …..

9. $\rho(AB) \leq min\{\rho(A), \rho(B)\}$.

10. If A and B are square matrix of order n , then

$$\rho(AB) \geq \rho(A) + \rho(B) - n$$

10.If A and B are square matrix of order n , then

$$\text{Max}\{0,\ \rho(A) + \rho(B) - n\} \leq \rho(AB) \leq min\{\rho(A),\ \rho(B)\}$$

11.If A and B are square matrix of order n , then

$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq min\{\rho(A) + \rho(B), m, n\}$$

12. $\rho(A+B) \leq \{\rho(A)+\rho(B)\}$.

13. $\rho(A-B) \geq \{\rho(A)-\rho(B)\}$.

14. The rank of a diagonal matrix is equal to the number of non-zero diagonal elements.

15. The rank of a UTM is number of non zeros rows in the UTM .

The rank of a LTM is number of non zeros rows in the LTM .

16. If $\rho(A_{\mathrm{n}\times n}) = \mathrm{n}$, then $\rho(adj\ A) = n.$

17. If $\rho(A_{\mathrm{n}\times n}) = \mathrm{n} - 1$, then $\rho(adj\ A) = 1$.

18. If $\rho(A_{\mathrm{n}\times n}) < \mathrm{n} - 1$, then $\rho(adj\ A) = 0$.

Q. The rank of $(m \times n)$ matrix $(m < n)$ cannot be more than

(GATE – 94[EC])

(a) m
(b) n
(c) mn
(d) None

Q. $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$; $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$

If the rank of matrix A is N, then the rank of matrix B is

(GATE–14–EE–SET3)

(a) N/2 (b) N – 1

(c) N (d) 2 N

Q. Let A be a 4×3 real matrix which rank 2. Which one of the following statement is **TRUE**?

(GATE-16-EE)

(a) Rank of A^T is less than 2

(b) Rank of A^TA is equal to 2

(c) Rank of A^TA is greater than 2

(d) Rank of A^TA can be any number between 1 and 3

Q. If V is a non-zero vector of dimension 3×1, then the matrix $A = VV^T$ has a rank = ________

(GATE-17-IN)

Q. The rank of the matrix $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ is ________

(GATE-17-EC)

Q. The rank of the matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is

(a) 0 (b) 1 **(GATE-17-EC)**

(c) 2 (d) 3

Q. The rank of the following matrix is $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

(a) 1
(b) 2
(c) 3
(d) 4

Q. The rank of the matrix $\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$ is

(GATE-18-ME)

(a) 1 (b) 2

(c) 3 (d) 4

Q. The rank of the matrix, $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, is ________.

(GATE-19-EE)

Q. Let A and B be two $n \times n$ matrices over real numbers. Let rank(M) and det(M) denote the rank and determinant of a matrix M, respectively. Consider the following statements:

1. $\text{rank}(AB) = \text{rank}(A)\,\text{rank}(B)$

II. $\det(AB) = \det(A)\det(B)$

III. $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$

IV. $\det(A + B) \leq \det(A) + \det(B)$

Which of the above statements are TRUE?

(GATE-2020 (CS))

(a) III and IV only (b) II and III only

(c) I and IV only (d) I and II only

Q. The rank of the matrix $\begin{bmatrix} 1 & 2 & 2 & 3 \\ 3 & 4 & 2 & 5 \\ 5 & 6 & 2 & 7 \\ 7 & 8 & 2 & 9 \end{bmatrix}$ is

(GATE-21-CE)

(a) 4 (b) 2 (c) 1 (d) 3

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Q. The rank of the matrix $\begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 2 & 0 & 1 \\ -5 & 0 & 5 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ is

(GATE-21-CE)

(a) 1 (b) 2 (c) 3 (d) 4

Q. Given $M = \begin{bmatrix} 2 & 3 & 7 \\ 6 & 4 & 7 \\ 4 & 6 & 14 \end{bmatrix}$, which of the following statement(s) is/are correct?

(GATE-2022-IN)

(a) The rank of M is 2

(b) The rank of M is 3

(c) The rows of M are linearly independent

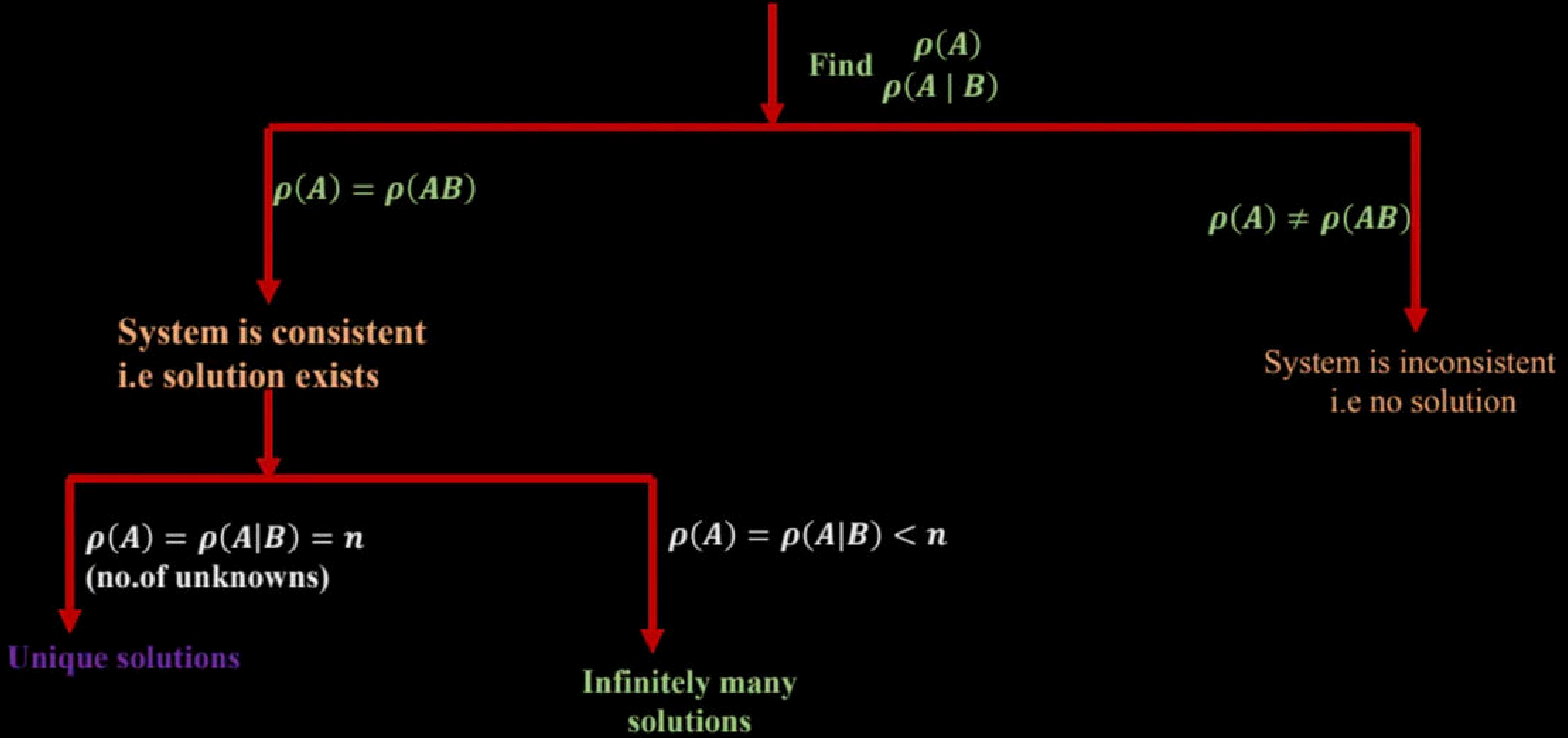
(d) The determinant of M is 0

System of Linear Equations

There two types of linear equations

1. Non Homogeneous system of linear equations
2. Homogeneous system of linear equations

Non Homogeneous equations
(AX= B)



Homogeneous equations
(AX= 0)

Always has a solution

$\rho(A) = n$
(no.of unknowns)

Unique solutions
(or)
Zero as a solution
(or)
Trivial solution

$\rho(A) < n$

Infinitely many solutions
(or)
Non trivial solution

Q. The following system of equations

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 1 \text{ has}$$

(GATE – 94[EC]

(a) Unique solution

(b) No solution

(c) Infinite number of solutions

(d) Only one solution

Q. Let $AX = B$ be a system of linear equations where A is an $m \times n$ matrix B is an $n \times 1$ column matrix which of the following is false?

(GATE – 96[CS])

(a) The system has a solution, if $\rho(A) = \rho(A/B)$

(b) If $m = n$ and B is a non – zero vector then the system has a unique solution.

(c) If $m < n$ and B is a zero vector then the system has infinitely many solutions.

(d) The system will have a trivial solution when $m = n$, B is the zero vector and rank of A is n.

Q. For the following set of simultaneous equations (GATE – 97[ME])

$$1.5x - 0.5y + z = 2$$
$$4x + 2y + 3z = 9$$
$$7x + y + 5z = 10$$

(a) the solution is unique

(b) infinitely many solutions exist

(c) the equations are incompatible

(d) finite many solutions exist

Q. Consider the following set of equations

$x + 2y = 5,$

$4x + 8y = 12,$

$3x + 6y + 3z = 15.$ This set

(GATE - 98[CS])

(a) has unique solution

(b) has no solution

(c) has infinite number of solutions

(d) has 3 solutions

Q. Consider the following system of linear equations (GATE – 03[CS])

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the 2^{nd} and 3^{rd} columns of the coefficient matrix are linearly dependent. For how many value of α, does systems of equations have infinitely many solutions.

(a) 0 (b) 1

(c) 2 (d) infinitely many

Q. A system of equations represented by $AX = 0$ where X is a column vector of unknown and A is a square matrix containing coefficients has a non-trivial solution when A is. **(GATE – 03)**

(a) non-singular
(b) singular
(c) symmetric
(d) Hermitian

Q. What values of x, y, z satisfy the following system of linear equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

(GATE – 04)

(a) $x = 6, y = 3, z = 2$

(b) $x = 12, y = 3, z = -4$

(c) $x = 6, y = 6, z = -4$

(d) $x = 12, y = -3, z = 4$

Q. How many solutions does the following system of linear equations have

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

(GATE–04[CS])

(a) infinitely many

(b) two distinct solutions

(c) unique

(d) none

Q. Consider the following system of equations in three real variable x_1, x_2 and x_3:

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + 5x_3 = 2$$
$$-x_1 + 4x_2 + x_3 = 3$$

This system of equations has

(GATE–05[CE])

(a) no solution

(b) a unique solution

(c) more than one but a finite number of solutions.

(d) an infinite number of solutions.

Solution of system of non-Homogeneous linear equations of 2-variables

Solution of system of Homogeneous linear equations of 2-variables

Q. Test whether the following vectors are linearly independent or not. Also find Basis , Dimension and Nullity

$$(1,2,3) \quad (1,1,3) \quad (2,4,9)$$

Q. Test whether the following vectors are linearly independent or not. Also find Basis , Dimension and Nullity

$$(1,1,-1,0)\quad(4,4,-3,1)\quad(-6,2,2,2)\qquad(9,9,-6,3)$$

Q. If nullity of matrix $A = \begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}$ is 1, then the value of K is------

Eigen values and Eigen Vectors

Q. Find the Eigen values and Eigen Vectors of the matrix $A=\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Q. Consider 3×3 matrix with every element being equal to 1. Its only non-zero eigenvalue is __________

(GATE-16-EE)

Properties of Eigen Values

1. Sum of the eigen values of a matrix is equal to the trace of the matrix
2. Product of the eigen values of a matrix is equal to the determinant of the matrix .
3. For lower triangular matrix (upper triangular matrix or diagonal matrix or scalar matrix or identity matrix), the Eigen values are same as diagonal elements of the matrix.

4. If λ is an Eigen value of a matrix A and k is a scalar then

(i) λ^m is Eigen value of matrix A^m

(ii) $\frac{1}{\lambda}$ is an eigen value of A^{-1}

(iii) $\bar{\lambda}$ is an eigen value of A^{θ} (where $\bar{\lambda}$ is the complex conjugate of λ)

(iv) $K\lambda$ is an Eigen value of matrix kA.

(v) $A \pm k$ is an Eigen value of matrix $A \pm kI$

(vi) $(\lambda \pm k)^n$ is an Eigen value of matrix $(A \pm kI)^n$

(vii) $\frac{|A|}{\lambda}$ is the eigen value of adjA

(viii) $a_0 + a_1\lambda + a_2\lambda^2$ is an Eigen value of matrix $a_0I + a_1A + a_2A^2$

Q. If A= $\begin{bmatrix} 4 & -1 & 0 \\ 0 & 6 & 5 \\ 0 & 0 & 5 \end{bmatrix}$, then find eigen values of

i. A

ii. 2A

iii. A^2

iv. A^θ

v. A^{-1}

vi. $A + 2I$

vii. $A - 3I$

viii. $(A - 2I)^2$

ix. $(A + I)^3$

x. $(A + 3I)^{-1}$

xi. $Adj\ A$

xii. $2A^2 + 3A + 4I$

5. For a matrix if a+ib is an eigen value of matrix A , then a-ib is also an eigen value of matrix A.
For a matrix if a+$\sqrt{b}$ is an eigen value of matrix A , then a- $\sqrt{b}$ is also an eigen value of matrix A.

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Type of Matrix	Eigen values
1. Symmetric matrix and Hermitian matrix	Real values
2.Skew symmetric matrix and skew Hermitian matrix	Zero (or) purely imaginary
3.Orthogonal matrix and Unitary matrix	Magnitude of eigen value is one $\lvert\lambda\rvert = 1$
4.Idempotent matrix	0 (or) 1
5.Involuntary matrix	-1 (or) +1
6. Nilpotent matrix	0

Q. The value of x for which the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$$

has zero as an eigen value is __________.

(GATE-16-EC)

Q. Consider a 2×2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

Where x is unknown. If the eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(GATE-16-EC)

(a) $+j\omega$ (b) $-j\omega$

(c) $+\omega$ (d) $-\omega$

Q. The condition for which the eigen values of matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive, is

(a) $k > \frac{1}{2}$ (b) $k > -2$ **(GATE-16-ME)**

(c) $k > 0$ (d) $k < -\frac{1}{2}$

Q. The eigen values of the matrix are $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(GATE-16-PI)

(a) i and -i
(b) 1 and -1
(c) 0 and 1
(d) 0 and -1

Q. Two eigen values of a 3×3 real matrix P are $\left(2+\sqrt{-1}\right)$ and 3. The determinant of P is ________.

(GATE-16-CSE)

Q. Suppose that the eigen values of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is _____.

(GATE-16-CSE)

Q. The eigen values of the matrix

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix} \text{ are}$$

(GATE-17-IN)

(a) $-1, 5, 6$

(b) $1, -5 \pm j6$

(c) $1, 5 \pm j6$

(d) $1, 5, 5$

1)

Q. Consider the 5×5 matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$

It is given that A has only one real eigen value. Then the real eigen value of A is

(a) -2.5 (b) 0

(c) 15 (d) 25

(GATE-17-EC)

Q. The eigen values of the matrix given below

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \text{ are}$$

(GATE-17-EE)

(a) (0, -1, -3)

(b) (0, -2, -3)

(c) (0, 2, 3)

(d) (0, 1, 3)

Q. The product of eigen values of the matrix P

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix} \text{ is}$$

(a) -6 (b) 2 **(GATE-17-ME)**

(c) 6 (d) -2

Q. The determinant of a 2×2 matrix is 50. If one eigen value of the matrix is 10, the other eigen value is ________.

(GATE-17-ME)

Q.Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigen values λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$, respectively. The value of $x_1^T x_2$ is _____

(GATE-17-ME)

Q. Consider the following simultaneous equations (with c_1 and c_2 being constants):

$$3x_1 + 2x_2 = c_1$$
$$4x_1 + x_2 = c_2$$

The characteristic equation for these simultaneous equations is

(GATE-17-CE)

(a) $\lambda^2 - 4\lambda - 5 = 0$ (b) $\lambda^2 - 4\lambda + 5 = 0$

(c) $\lambda^2 + 4\lambda - 5 = 0$ (d) $\lambda^2 + 4\lambda + 5 = 0$

Q. If the characteristic polynomial of a 3×3 matrix M over R (the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$. $a \epsilon R$, and one eigenvalue of M is 2, then the largest among the absolute values of the eigenvalues of M is __________.

(GATE-17-CSIT)

Q. Let N be a 3 by 3 matrix with real number entries. The matrix N is such that $N^2 = 0$. The eigen values of N are

(a) 0, 0, 0

(b) 0, 0, 1

(GATE-18-IN)

(c) 0, 1, 1

(c) 1, 1, 1

Q. Consider the following matrix: $R = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$

The absolute value of the product of Eigen values of R is ________.

(GATE-19-CSIT)

Q. A 4×4 matrix [P] is given below

$$[P] = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

The eigen values of [P] are

(GATE-2020(CE))

(a) 0, 3, 6, 6 (b) 1, 2, 3, 4

(c) 1, 2, 5, 7 (d) 3, 4, 5, 7

Q. If the sum and product of eigen values of a 2×2 matrix $\begin{bmatrix} 3 & p \\ p & q \end{bmatrix}$ are 4 and -1 respectively, then $|p|$ is__________ (in integer).

(GATE-2022-ME)

Q. Let p and q be real numbers such that $p^2 + q^2 = 1$. The eigenvalues of the matrix $\begin{bmatrix} p & q \\ q & -p \end{bmatrix}$ are

(GATE-21-EE)

(a) pq and -pq

(b) 1 and 1

(c) j and -j

(d) 1 and -1

Q. Which of the following is/are the eigenvector(s) for the matrix given below?

$$\begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix}$$

(GATE-2022-CSE)

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$

Q. Consider the following matrix :

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The largest eigenvalue of the above matrix is_____________.

(GATE-2021-cs

Q. A real 2×2 non-singular matrix A with repeated eigenvalue is given as

$$A = \begin{bmatrix} x & -3.0 \\ 3.0 & 4.0 \end{bmatrix}$$

where x is a real positive number . The value of x (rounded off to one decimal place) is_________.

(GATE – 2021 – EC)

Q. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

(GATE–08[ME])

(a) 0
(b) $\frac{1}{2}$
(c) 1
(d) 2

Q. The eigen vector pair of the matrix $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is (GATE–08[PI])

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q. The eigen values of a 2×2 matrix X are -2 and -3. The eigen values of matrix $(X+I)^{-1}(X+5I)$ are **(GATE–09[IN])**

(a) -3, -4

(b) -1, -2

(c) -1, -3

(d) -2, -4

Q. The eigen values of a skew-symmetric matrix are **(GATE–10[EC])**

(a) always zero

(b) always pure imaginary

(c) either zero (or) pure imaginary

(d) always real

Q. A real n×n matrix $A = [a_{ij}]$ is defined as follows $\begin{cases} a_{ij} = i, & \forall\, i = j \\ \quad = 0, & \text{otherwise} \end{cases}$.

The sum of all n eigen values of A is

(GATE–10[IN])

(a) $\frac{n(n+1)}{2}$

(b) $\frac{n(n-1)}{2}$

(c) $\frac{n(n+1)(2n+1)}{2}$

(d) n^2

Q. The product of the non-zero eigen values of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ is ______.

(GATE–14–CS–SET2)

Properties of Eigen vectors

1. For each Eigen value of a matrix there are infinitely many Eigen vectors.

   If X is an Eigen vector of matrix A corresponding to the Eigen value 'then kX (for every non-zero scalar k), is also an Eigen vector of A corresponding to the same Eigen value 'λ'.

2. If λ_1, λ_2, ..., λ_n are distinct Eigen values of a square matrix A of order 'n' then the corresponding Eigen vectors X_1, X_2, ...,X_n of matrix A are linearly independent.

3. If some Eigen values of matrix A are repeated then Eigen vectors of A may or may not be linearly independent.

4. The number of linearly independent eigen vectors of an eigen value 'λ' is $n - \rho(A - \lambda I)$

5. The eigen vectors of A, $A^{-1}, A^n, kA \; and \; f(A)$ are same .

Q. The matrix $A = \begin{bmatrix} 4 & 3 \\ 9 & -2 \end{bmatrix}$ has eigen values - 5 and 7.

The eigenvector(s) is/are____________

(GATE-2022-IN)

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 13 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

Q. If a matrix is squared, then **(GATE-2022-PI)**

(a) both eigenvalues and eigenvectors are retained.

(b) eigenvalues get squared but eigenvectors are retained.

(c) both eigenvalues and eigenvectors must change.

(d) eigenvalues are retained but eigenvectors change.

Q. The matrix $M = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$ has eigen values -3, -3, 5. An eigen vector corresponding to the eigen value 5 is $[1\ 2\ -1]^T$. Onc of the eigen vector of the matrix M^3 is

(GATE–11[IN])

(a) $[1\ \ 8\ \ -1]^T$
(b) $[1\ \ 2\ \ -1]^T$
(c) $[1\ \ \sqrt[3]{2}\ \ -1]^T$
(d) $[1\ \ 1\ \ -1]^T$

Cayley – Hamilton theorem

Every square matrix satisfies its own characteristic equation.

Applications of CH Theorem

1. To find higher powers of a matrix

2. To find Inverse of a matrix

Q. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then A^9 equals

(GATE–07[EE])

(a) 511 A + 510 I
(b) 309 A + 104 I
(c) 154 A + 155 I
(d) e^{9A}

Q. The characteristic equation of a 3×3 matrix P is defined as

$$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + 2\lambda + \lambda^2 + 1 = 0.$$

If I denotes identity matrix then the inverse of P will be

(GATE–08[EE])

(a) $P^2 + P + 2I$

(b) $P^2 + P + I$

(c) $-(P^2 + P + I)$

(d) $-(P^2 + P + 2I)$

Q. If matrix $X = \begin{bmatrix} a & 1 \\ -a^2 + a - 1 & 1 - a \end{bmatrix}$ and

$X^2 - X + I = O$ then the inverse of X is

(GATE – 04)

(a) $\begin{bmatrix} 1-a & -1 \\ a^2 & a \end{bmatrix}$

(b) $\begin{bmatrix} 1-a & -1 \\ a^2 - a + 1 & a \end{bmatrix}$

(c) $\begin{bmatrix} -a & 1 \\ -a^2 + a - 1 & a - 1 \end{bmatrix}$

(d) $\begin{bmatrix} a^2 - a + 1 & a \\ 1 & 1 - a \end{bmatrix}$

Q. If A= $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ then find the value of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 7A^2 - 2A + I$

Diagonalization of a Matrix

Applications

1. Finding the matrix from Eigen Vectors

2. Finding Higher powers of a matrix

Q. For a given 2x2 matrix A, it is observed that

$$A\begin{bmatrix}1\\-1\end{bmatrix}=-1\begin{bmatrix}1\\-1\end{bmatrix} \text{ and } A\begin{bmatrix}1\\-2\end{bmatrix}=-2\begin{bmatrix}1\\-2\end{bmatrix}$$

then the matrix A is (GATE-06[IN])

(a) $A=\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$

(b) $A=\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(c) $A=\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(d) $A=\begin{bmatrix}0 & -2\\1 & -3\end{bmatrix}$

Q. Given that $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the value of A^3 is

(GATE–12[EC, EE, IN])

(a) $15A + 12I$ (b) $19A + 30I$

(c) $17A + 15I$ (d) $17A + 21I$

Q. A matrix has eigen values -1 and -2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

(GATE – 13[EE])

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Q. For the matrix A satisfying the equation given below, the eigen values are

$$[A]\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(GATE–14–IN–SET1)

(a) (1, -j, j)
(b) (1, 1, 0)
(c) (1, 1, -1)
(d) (1, 0, 0)

Q. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$, where I is the 3×3 identity matrix.

The determinant of B is __________ (up to 1 decimal place)

(GATE-18-EC)

Q. Let the eigen values of a 2×2 matrix A be 1, -2 with eigen vectors x_1 and x_2 respectively. Then the eigen values and eigen vectors of the matrix $A^2 - 3A + 4I$ would respectively, be

(GATE-16-EE)

(a) 2, 14; x_1, x_2

(b) 2, 14; $x_1, x_2: x_1 - x_2$

(c) 2, 0; x_1, x_2

(d) 2, 0; $x_1 + x_2, x_1 - x_2$

Q. Consider an $n \times n$ matrix A and a non-zero $n \times 1$ vector p. Their product $Ap = \alpha^2 p$, where $\alpha \in \Re$ and $\alpha \notin \{-1, 0, 1\}$. Based on the given information, the eigen value of A^2 is:

(GATE-21-ME)

(a) α (b) α^2 (c) $\sqrt{\alpha}$ (d) α^4

Q. The matrix M is defined as

$$M = \left[\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\right]$$

and has eigenvalues 5 and - 2. The matrix Q is formed as

$$Q = M^3 - 4M^2 - 2M$$

Which of the following is/are the eigenvalue(s) of matrix Q (MSQ)

(GATE-2022-CE)

(a) 15 (b) 25

(c)-20 (d) -30

Q. Consider matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 1 & 1 \end{bmatrix}$, the matrix A is satisfy the equation $6A^{-1} = A^2 + cA + dI$ where c and d are scalars and I is the identity matrix, the (c + d) is equal to

(GATE-2022-EEE)

(a) 5
(b) 17
(c) −6
(d) 11

Q. A 3×3 matrix P is such that, $P^3 = P$. Then the eigen values of P are

(GATE-16-EE)

(a) 1, 1, -1

(b) 1, $0.5 + j0.866$, $0.5 - j0.866$

(c) 1, $-0.5 + j0.866$, $-0.5 - j0.866$

(d) 0, 1, -1

Q. The eigen values and the corresponding eigen vectors of a 2x2 matrix are given by

Eigen value	**Eigen vector**
$\lambda_1 = 8$	$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is **(GATE–06[EC])**

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

Q. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. The eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(GATE–06[EC]

(a) 2

(b) 4

(c) 6

(d) 8

191. In matrix equation $[A]\{X\} = \{R\}$.

$$[A] = \begin{bmatrix} 4 & 8 & 4 \\ 8 & 16 & -4 \\ 4 & -4 & 15 \end{bmatrix}, \{X\} = \begin{Bmatrix} 2 \\ 1 \\ 4 \end{Bmatrix} \text{ and } \{R\} = \begin{Bmatrix} 32 \\ 16 \\ 64 \end{Bmatrix}$$

One of the eigen values of matrix [A] is

(a) 16 (b) 15 **(GATE-19-ME)**

(c) 4 (d) 8

Q. Which of the following is/are the eigenvector(s) for the matrix given below?

$$\begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix}$$

(GATE-2022-CSE)

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$

Algebraic Multiplicity of an Eigen value

The number of times an eigen value is repeated is called as algebraic multiplicity (A.M) of that eigen value .

Geometric Multiplicity of an Eigen value

The number of linearly independent eigen vectors corresponding to an eigen value λ is called as geometric multiplicity of that eigen value λ.

The number of linearly independent eigen vectors of an eigen value 'λ' = $\boldsymbol{n - \rho(A - \lambda I)}$

1. Algebraic multiplicity of an eigen value $\geq$ Geometric multiplicity
2. Geometric multiplicity of every eigen value of a matrix is ≥ 1
3. If all the eigen values of a matrix are distinct then the matrix can be diagonalizable but converse need not be true .
4. A matrix is diagonalizable iff for every eigen value , geometric multiplicity is equal to algebraic multiplicity.
5. Every idempotent matrix , involuntary matrix , symmetric matrix, unitary matrix can be diagonalizable .
6. Nilpotent matrix can never be diagonalizable .
7. If all the elements are equal then A is diagonalizable .

Q. Consider a matrix P whose only eigenvectors are the multiples of $\begin{bmatrix}1\\4\end{bmatrix}$

Consider the following statements.

(I) P does not have an inverse

(II) P has a repeated eigenvalue

(III) P cannot be diagonalized

Which one of the following options is correct?

(a) Only I and III are necessarily true

(b) Only II is necessarily true

(c) Only I and II are necessarily true

(d) Only II and III are necessarily true

(GATE-18-CSIT)

LU Decomposition Method

Q. The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of lower triangular matrix [L] and an upper triangular [U]. The properly decomposed [L] and [U] matrices respectively are

(GATE–11[EE])

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

Q. Consider solving the following system of simultaneous equations using LU decomposition.

$x_1 + x_2 - 2x_2 = 4$

$x_2 + 3x_2 - x_2 = 7$

$2x_1 + x_2 - 5x_2 = 7$

Where L and U are denoted as

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & L_{33} \end{bmatrix}$$

Which one of the following is the correct combination of values for L_{32}, U_{33}, and x_1?

(GATE-2022-CSE)

(a) $L_{32} = 2$, $U_{33} = -\frac{1}{2}$, $x_1 = -1$

(b) $L_{32} = 2$, $U_{33} = 2$, $x_1 = -1$

(c) $L_{32} = -\frac{1}{2}$, $U_{33} = 2$, $x_1 = 0$

(d) $L_{32} = -\frac{1}{2}$, $U_{33} = -\frac{1}{2}$, $x_1 = 0$

Vectors

Q. Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i \mathrm{a}_i = 0$ where a_i are column vectors in R^n. Consider the set of linear equations $Ax = b$

Where $A = [\mathrm{a}_1, \ldots\ldots \mathrm{a}_n]$ and $b = \sum_{i=1}^{n} \mathrm{a}_i$. The set of equations has

(GATE-17-CSIT)

(a) A unique solution at $x = j_n$ where j_n denotes a n-dimensional vector of all 1

(b) no solution

(c) infinitely many solutions

(d) finitely many solutions