GT - Part VII

Complete Course on Algorithm for GATE - CS & IT

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ex

$$T(n) = T(n/5) + T\left(\frac{3n}{5}\right) + c$$

$$T(n) \le c\{2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{\log_{5/3} n}]$$

$$\le c\left[\frac{1\left(2^{\log_{5/3} n} - 1\right)}{2-1}\right]$$

$$\le c \cdot 2^{\log_{5/3} n} \Longrightarrow O\left(2^{\log_{5/3} n}\right) \Longrightarrow O\left(n^{\log_{5/3} 2}\right)$$

$$T(n) \ge c\{2^0 + 2^1 + 2^2 + \cdots + 2^{\log_5 n}]$$

$$\ge c \cdot \{2^{\log_5 n}]$$

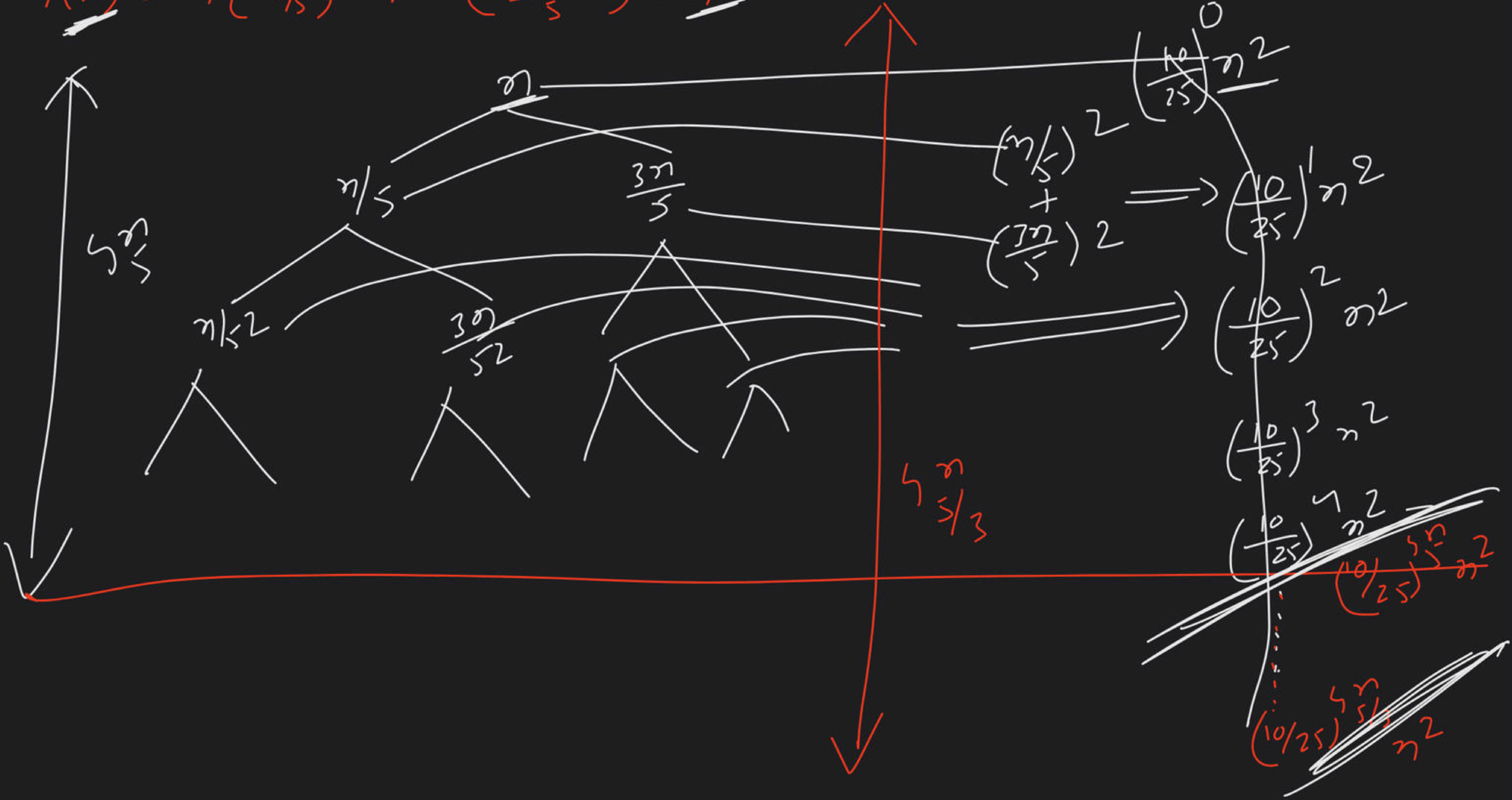
$$= \Omega\left(2^{\log_5 n}\right) = \Omega\left(n^{\log_5 2}\right)$$

$$\boxed{n^{\log_5 2} \le T(n) \le n^{\log_{5/3} n}}$$

$$T(n) = O\left(n^{\log_{5/3} 2}\right)$$

$$T(n) = \Omega\left(n^{\log_5 2}\right)$$

$$T(n) = T(n/5) + T\left(\frac{3n}{5}\right) + n^2$$



$$T(n) \le n^2 \left[\left(\frac{10}{25}\right)^0 + \left(\frac{10}{25}\right)^1 + \left(\frac{10}{25}\right)^2 + \cdots + \left(\frac{10}{25}\right)^{\log_{5/3} n} \right]$$

$$\le n^2 \cdot O(1) \implies O(n^2)$$

$$T(n) \ge n^2 \left[\left(\frac{10}{25}\right)^0 + \left(\frac{10}{25}\right)^1 + \left(\frac{10}{25}\right)^2 + \cdots + \left(\frac{10}{25}\right)^{\log_5 n} \right]$$

$$\ge n^2 \cdot O(1) = \Omega(n^2)$$

$$n^2 \le T(n) \le n^2 \implies \boxed{T(n) = \Theta(n^2)}$$

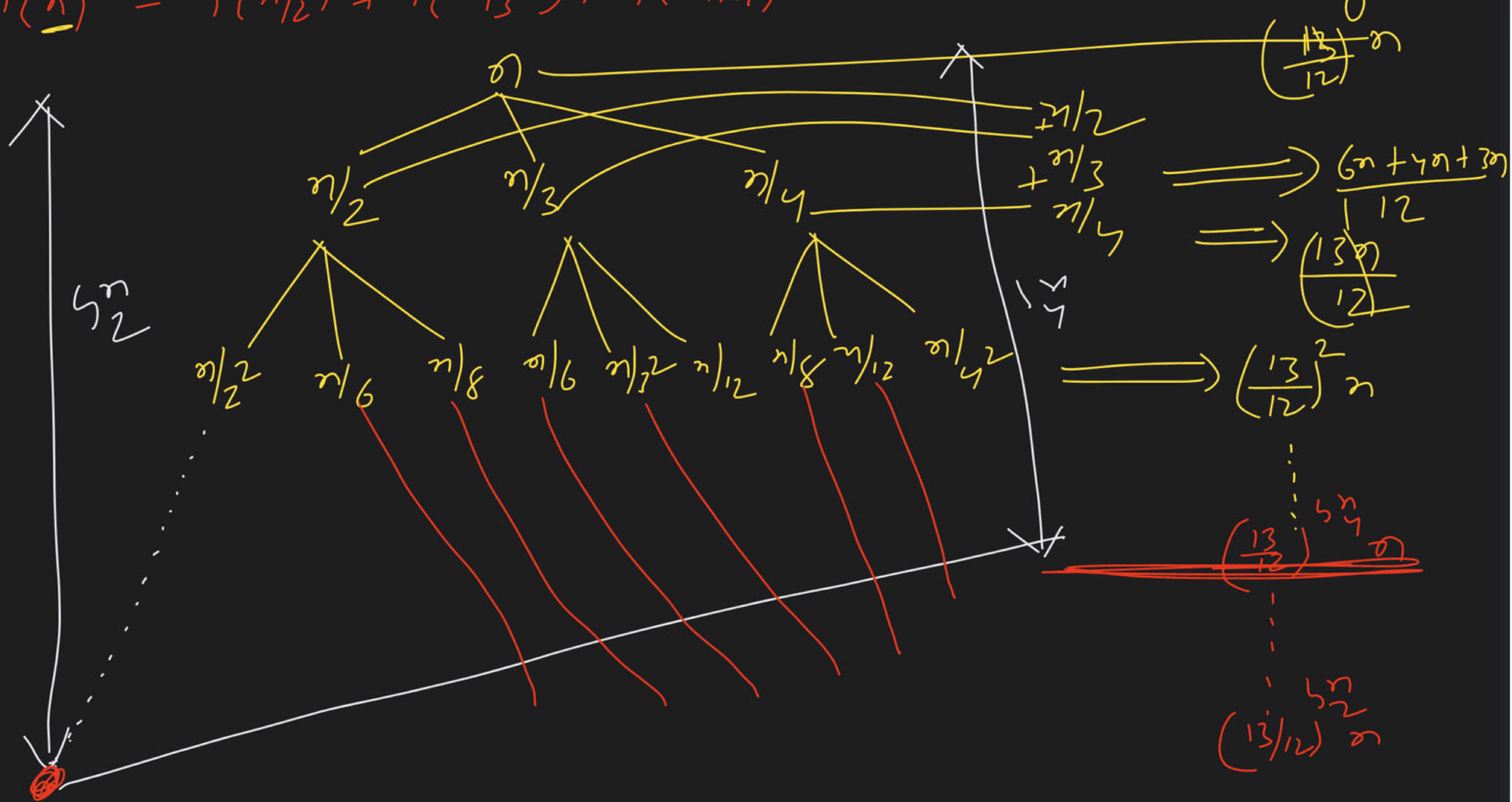
$$= \cancel{\Omega(1)}$$

$$= \cancel{\Omega(n)}$$

$$= \cancel{O(n^3)}$$

$$= \cancel{O(n^{10})}$$

$$T(n) = T(n/2) + T(n/3) + T(n/4) + n$$



$$T(n) \leq n\left[\left(\frac{13}{12}\right)^0 + \left(\frac{13}{12}\right)^1 + \cdots + \left(\frac{13}{12}\right)^{\log_2 n}\right]$$

$$\leq n \cdot \left(\frac{13}{12}\right)^{\log_2 n} \leq n \cdot n^{\log_2 \frac{13}{12}} = O\left(n \cdot n^{\log_2 \frac{13}{12}}\right)$$

$$T(n) \geq n\left[\left(\frac{13}{12}\right)^0 + \left(\frac{13}{12}\right)^1 + \cdots + \left(\frac{13}{12}\right)^{\log_4 n}\right]$$

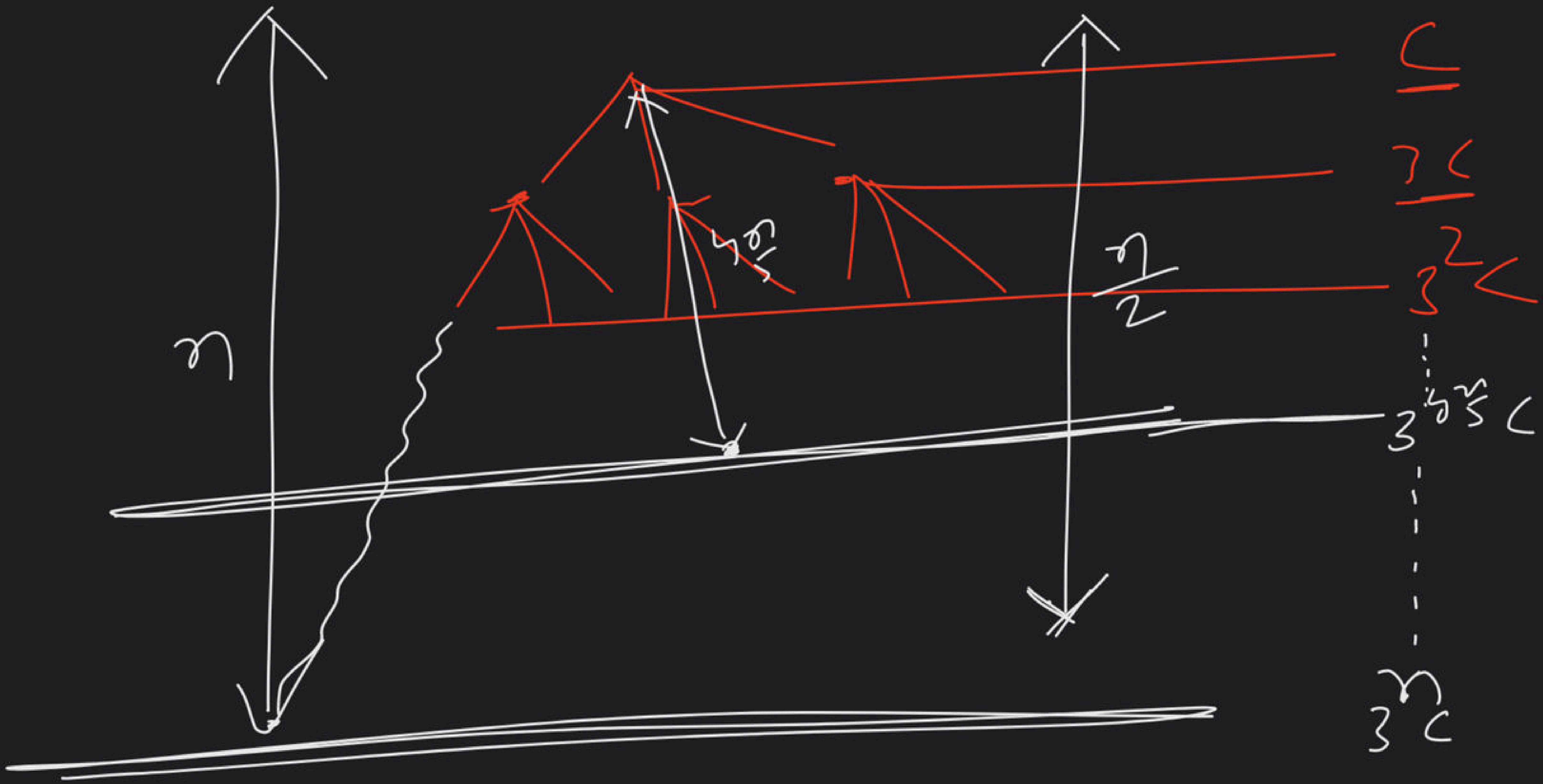
$$\geq n \cdot n^{\log_4 \frac{13}{12}} = \Omega\left(n \cdot n^{\log_4 \frac{13}{12}}\right)$$

$$n^{1+\log_4 \frac{13}{12}} \leq T(n) \leq n^{1+\log_2 \frac{13}{12}}$$

$$T(n) = O\left(n^{1+\log_2 \frac{13}{12}}\right)$$

$$T(n) = \Omega\left(n^{1+\log_4 \frac{13}{12}}\right)$$

$$T(n) = T(n-1) + T(n/5) + T(n-2) + c$$



$$T(n) \le c\left[3^0 + 3^1 + 3^2 + \cdots + 3^n\right]$$

$$\le c \cdot 3^n = O(3^n)$$

$$T(n) \ge c\left[3^0 + 3^1 + 3^2 + \cdots + 3^{\log_5 n}\right]$$

$$\ge c \cdot 3^{\log_5 n} = \Omega(3^{\log_5 n}) = \Omega(n^{\log_5 3})$$

$$= \Omega(n^{0.682})$$

$$n^{0.682} \le T(n) \le 3^n$$

$$T(n) = O(3^n)$$

$$= \Omega(n^{0.682})$$

Master theorem

$$\boxed{T(n) = aT(n/b) + f(n)}$$

where $a \geq 1$, $b > 1$ are constants & $f(n)$ is the function.

ex①

$$T(n) = 8T(n/2) + n^2$$

$$a = 8 \quad b = 2 \quad f(n) = n^2$$

$f(n)$	$n^{\log_b a}$ big
$\Downarrow$	right
n^2	$n^{\log_2 8} \Rightarrow n^3$

$n^3 \Rightarrow \frac{n^3}{n^1} \Rightarrow n^{3-1}$

bigger by Polynomial (n)

big

$\Theta(n^3)$

ex

$T(n) = 2T(n/2) + n^2$

$f(n)$

$\Downarrow$

n^2

$a=2,\ b=2 \qquad \log_b a = 1$

$n^{\log_b a}$

$\Downarrow$

n^1 small

right

$\Longrightarrow n^1 < n^1 \Longrightarrow n^{1+1}$

Small by polynomial (n)

big

$\Longrightarrow \Theta(n^2)$

$$T(n) = 7T(n/2) + n^{10}$$

$f(n)$ | $n^{\log_b a}$

$\Downarrow$

n^{10} | $n^{\log_2 7} \Rightarrow n^{2.81}$

Small

$n^{2.81} \Rightarrow n^{7.19} \cdot n^{2.81} \Rightarrow n^{7.19+2.81}$

Small by Polynomial

big

$\Theta(n^{10})$

$T(n) = 2T(n/2) + n\log n$

$f(n)$

$\Downarrow$

$n\log n$

$n^{\log_b a}$

$\Downarrow$

$n^{\log_2 2}$

$\Downarrow$

n

small

big

$(\log n)\, n$

Small by logarit but not polynom

$\cancel{\Theta(n\log n)}$

$$T(n) = 2T(n/2) + n$$

$f(n)$	$n^{\log_b a}$
$\Downarrow$	$\Downarrow$
n	$n\,\theta(\log n)^0 \Longrightarrow \theta\left(n(\log n)^{0+1}\right) \Longrightarrow \theta(n\log n)$

$$T(n) = 2T(n/2) + n\log n$$

$f(n)$	$n^{\log_b a}$
$\Downarrow$	$\Downarrow$
$n\log n$	$n \Longrightarrow n\,\theta(\log n)^1 \Longrightarrow \theta\left(n\cdot(\log n)^{1+1}\right)$

$$\theta\left(n\cdot(\log n)^2\right)$$

ex

$f(n) \Rightarrow (\log n)^{10}$

$n^{\log_b a} \Rightarrow (\log n)^2 \Longrightarrow \boxed{(\log n)^2 \cdot (\log n)^8} \Longrightarrow \Theta\left((\log n)^2 \cdot (\log n)^{8+1}\right)$

$\Longrightarrow \Theta\left((\log n)^{11}\right)$

case (2) — equal

RHS smaller by log n

case (2)

$f(n) = \Theta\left(n^{\log_b a} \cdot (\log n)^k\right)$ where k is constant $k > 0$

$T(n) = \Theta\left(n^{\log_b a} \cdot (\log n)^{k+1}\right)$

$f(n)$

$\Downarrow$

$(\log n)^{5}$

$n^{6} \leq 5$

$\Downarrow$

$(\log n)^{8}$

big

$\Rightarrow (\log n)^{8} * (\log n)^{-3}$ 6

$f(n)$ $\Downarrow$	n^2 5 $\Downarrow$
n	$2^n \cdot n$
n	big $n^2 \log n$

$2^n \cdot n \Rightarrow \Theta(2^n n)$

$n^2 \log n \Rightarrow n \cdot n \log n \Rightarrow \Theta(n^2 \log n)$

$n, n \log n, n^2, n^3$

$$T(n) = \boxed{2^n}\, T(n/2) + n$$

$$f(n) \Downarrow n$$

$$n^{\log_b a} \Downarrow n^{\log_2 2^n} \Longrightarrow n^n$$

big

$$\Longrightarrow n \cdot n^{n-1} \Longrightarrow \cancel{\Theta(n^n)}$$

n^c | $c > 0$

The master theorem

The master method depends on the following theorem.

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

$\frac{n^{\log_b}}{n^{\epsilon}}$

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

$n^{\log_b} \cdot n^{\epsilon}$