## Bousfield-Friedlander theorem

#### Arghan Dutta

November 2023

### 1 Proper Model Category

In a model category, fibrations are preserved under pullbacks, and cofibrations are preserved under pushouts. But weak equivalences, in general does not have such closure property. In a *proper model category*, weak equivalences are preserved under certain pullbacks and pushouts.

**Definition 1.1.** (Left proper model category) A model category is called *left proper* if weak equivalences are preserved under pushouts along cofibrations, i.e, for every  $f: B \to X$  in  $we(\mathcal{C})$  and for every  $i: B \to A$  in  $cof(\mathcal{C})$ , the pushout morphism  $i_*f: A \to A \coprod_B X$  is in  $we(\mathcal{C})$ .

$$\begin{array}{ccc}
A \coprod_{B} X & \longleftarrow & X \\
\downarrow_{i,f} & & \uparrow_{f} \\
A & \longleftarrow & B
\end{array}$$

**Definition 1.2.** (Right proper model category) A model category is called *right proper* if weak equivalences are preserved under pullbacks along fibrations, i.e, for every  $f: A \to Y$  in  $we(\mathcal{C})$  and for every  $p: X \to Y$  in  $fib(\mathcal{C})$ , the pullback morphism  $p^*f: X \times_Y A \to X$  is in  $we(\mathcal{C})$ .

$$\begin{array}{ccc}
X \times_Y A & \longrightarrow & A \\
\downarrow^{p^* f} & & \downarrow^f \\
X & \xrightarrow{p} & Y
\end{array}$$

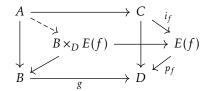
**Definition 1.3.** (Proper model category) A model category is called *proper* if it is both *left proper* and *right proper*.

## 2 Homotopy fiber squares

**Definition 2.1.** Let C be a proper model category. A commutative square in C,

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow f \\
B & \longrightarrow & D
\end{array}$$

is called a *homotopy fiber square* if for some factorization  $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$  of f where  $i_f$  is in we(C) and  $p_f$  is in fib(C),



the induced morphism  $A \to B \times_D E(f)$  is in  $we(\mathcal{C})$ .

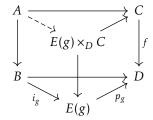
**Proposition 2.1.** Let  $\mathcal{C}$  be a proper model category. Then the following statements are equivalent:-

1. The commutative square in C,

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & & \downarrow f \\
B & \xrightarrow{g} & D
\end{array}$$

is a homotopy fiber square.

2. For some factorization  $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$  of g where  $i_g$  is in  $we(\mathcal{C})$  and  $p_g$  is in  $fib(\mathcal{C})$ ,



the induced morphism  $A \to E(g) \times_D C$  is in we(C).

*Proof.* (1  $\Longrightarrow$  2) Suppose the commutative square in (1) is a homotopy fiber square and take any factorization  $B \xrightarrow{i_g} E(g) \xrightarrow{p_g} D$  of g where  $i_g$  is in  $we(\mathcal{C})$  and

 $p_g$  is in fib(C). Then consider the commutative diagram,

$$A \xrightarrow{\hat{i_g}} E(g) \times_D C \xrightarrow{\hat{p_g}} C$$

$$\downarrow \hat{i_f} \qquad \downarrow \hat{i_f} \qquad \downarrow \hat{i_f}$$

$$B \times_D E(f) \xrightarrow{\hat{i_g}} E(g) \times_D E(f) \xrightarrow{\hat{p_g}} E(f)$$

$$\downarrow p_f \qquad \downarrow p_f$$

$$B \xrightarrow{\hat{i_g}} E(g) \xrightarrow{p_g} D$$

where  $C \xrightarrow{i_f} E(f) \xrightarrow{p_f} D$  is one such factorization of f,  $i_f$  in  $we(\mathcal{C})$  and  $p_f$  in  $fib(\mathcal{C})$ , such that  $\hat{i_f}$  is in  $we(\mathcal{C})$ . Now the bottom right square being a pullback,  $\tilde{p_g}$  and  $\tilde{p_f}$  is in  $fib(\mathcal{C})$ . Again  $\mathcal{C}$  being proper, and the bottom left and top right squares being pullbacks,  $\tilde{i_f}$ ,  $\tilde{i_g}$  is in  $we(\mathcal{C})$ . Finally by the 2-out-of-3 property,  $\hat{i_g}$  is in  $we(\mathcal{C})$ .

 $(2 \implies 1)$  Similar argument as above, just in this case we assume that  $\hat{i_g}$  is in  $we(\mathcal{C})$ , and conclude that  $\hat{i_f}$  is in  $we(\mathcal{C})$ .

Therefore, we conclude that (1) and (2) holds for every such factorization of f and g.

**Proposition 2.2.** Let C be a proper model category, and consider the commutative square.

$$\begin{array}{ccc}
A & \stackrel{\tilde{f}}{\longrightarrow} & C \\
f \downarrow & & \downarrow g \\
B & \stackrel{f}{\longrightarrow} & D
\end{array}$$

If f is in we(C), then  $\tilde{f}$  is in we(C) iff the above square is a homotopy fiber square.

*Proof.* Let  $g = p_g i_g$  be a factorization of p, where  $i_g$  is in  $we(\mathcal{C})$  and  $p_g$  is in  $fib(\mathcal{C})$ . Consider the commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{\tilde{f}} & C \\
\downarrow^{\tilde{i}} & & \downarrow^{i_g} \\
B \times_D E(g) & \xrightarrow{p_g^* f} & E(g) \\
f^* p_g \downarrow & & \downarrow p_g \\
B & \xrightarrow{f} & D
\end{array}$$

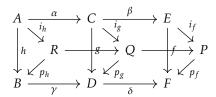
where  $p_g^*f$  is in  $we(\mathcal{C})$ , since  $\mathcal{C}$  is proper. Now, if  $\tilde{f}$  is in  $we(\mathcal{C})$ , then by the 2-out-of-3 property,  $\tilde{i}$  is in  $we(\mathcal{C})$ . Conversely, if  $\tilde{i}$  is in  $we(\mathcal{C})$ , then again by the 2-out-of-3 property,  $\tilde{f}$  is in  $we(\mathcal{C})$ .

**Proposition 2.3.** Let C be a proper model category and consider the commutative square,

$$\begin{array}{ccc}
A & \longrightarrow & C & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & D & \longrightarrow & F
\end{array}$$

where the right square is a homotopy fiber square. Then the left square is also a homotopy fiber square iff the total rectangle is a homotopy fiber square.

Proof. Consider the commutative diagram,



where we first factorize  $f=p_fi_f$ , where  $i_f$  is in  $we(\mathcal{C})$ , and  $p_f$  is in  $fib(\mathcal{C})$ . Then take the pullback of  $p_f$  along  $\delta$  in order to factorize  $g=p_gi_g$ . Since  $p_f$  is in  $fib(\mathcal{C})$ ,  $p_g$  is in  $fib(\mathcal{C})$ . Also, since the right square is a homotopy fiber square,  $i_g$  is in  $we(\mathcal{C})$ . Again take the pullback of  $p_g$  along  $\gamma$ , to factorize  $h=p_hi_h$ . Now suppose the left square is a homotopy fiber square. Then  $i_h$  is in  $we(\mathcal{C})$ . But  $p_h$  is a pullback of  $p_f$  along  $\delta\gamma$ , which implies that the total rectangle is a homotopy fiber square.

Conversely, let the total rectangle be a homotopy fiber square. Then  $i_h$  is in  $we(\mathcal{C})$ , which implies that that the left square in a homotopy fiber square.

**Proposition 2.4.** Every retract of a homotopy fiber square in  $\mathcal{C}^{\square}$  is a homotopy fiber square.

# 3 $\mathcal{Q}$ -structures on proper model categories

Let  $\mathcal{C}$  be a proper model category and  $\mathcal{Q}: \mathcal{C} \to \mathcal{C}$  be an endofunctor. A morphism  $f: X \to Y$  in  $\mathcal{C}$  is called

- a *Q*-equivalence if  $Q(f): Q(X) \to Q(Y)$  is a weak equivalence in C.
- a *Q-cofibration* if f is a cofibration in C.
- a *Q-fibration* if *f* has the *right lifting property* with respect to *Q*-trivial cofibrations.

**Definition 3.1.** (Quillen idempotent monad) Let C be a proper model category. A *Quillen idempotent monad* on C is

• an endofunctor  $Q: \mathcal{C} \to \mathcal{C}$ 

• a natural transformation  $\eta: \mathbf{1}_{\mathcal{C}} \to \mathcal{Q}$ 

such that

- 1. Q is homotopical, i.e., Q preserves weak equivalences.
- 2. For every object X in C, the morphisms,  $Q(\eta_X)$ ,  $\eta_{Q(X)}: Q(X) \to Q(Q(X))$  are weak equivalences.
- 3. For a pullback square in C,

$$\begin{array}{ccc}
X \times_Y A & \longrightarrow & A \\
\downarrow^{p^*f} & & & \downarrow^f \\
X & \xrightarrow{p} & Y
\end{array}$$

if p is a Q-fibration and f is a Q-equivalence, then  $p^*f$  is a Q-equivalence.

4. For a pushout square in C,

$$\begin{array}{ccc}
A \coprod_{B} X & \longleftarrow & X \\
\downarrow^{i_{*}f} & & \uparrow^{f} \\
A & \longleftarrow & B
\end{array}$$

if i is a Q-cofibration and f is a Q-equivalence, then  $i_*f$  is a Q-equivalence.

For a Quillen idempotent monad  $\mathcal{Q}$  on  $\mathcal{C}$ , let  $\mathcal{C}^{\mathcal{Q}}$  denote the category  $\mathcal{C}$  equipped with  $\mathcal{Q}$ -equivalences,  $\mathcal{Q}$ -fibrations and  $\mathcal{Q}$ -cofibrations.

**Lemma 3.1.** A morphism  $f: X \to Y$  in C is in  $we(C) \cap fib(C)$  iff f is a Q-trivial fibration.

*Proof.* Let  $f: X \to Y$  be in  $we(\mathcal{C}) \cap fib(\mathcal{C})$ . Then by (1), f is a  $\mathcal{Q}$ -equivalence. Now consider the commutative square,

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow f \\
B & \longrightarrow & Y
\end{array}$$

where i is a Q-trivial cofibration. Since i is a Q-cofibration, i is in  $cof(\mathcal{C})$  and f being in  $we(\mathcal{C}) \cap fib(\mathcal{C})$ , f has the right lifting property with respect to i. Hence, f is a Q-trivial fibration.

Conversely, let  $f: X \to Y$  in  $\mathcal C$  be a  $\mathcal Q$ -trivial fibration. Then  $f = p_f i_f$  where  $i_f$  is in  $cof(\mathcal C)$  and  $p_f$  is in  $we(\mathcal C) \cap fib(\mathcal C)$ . Now  $\mathcal Q(f) = \mathcal Q(p_f)\mathcal Q(i_f)$ , where  $\mathcal Q(f)$  is in  $we(\mathcal C)$ . Also by (1),  $\mathcal Q(p_f)$  is in  $we(\mathcal C)$ . By the 2-out-of-3 property,  $i_f$  is a  $\mathcal Q$ -equivalence. So,  $i_f$  is a  $\mathcal Q$ -trivial cofibration, hence has the left lifting property with respect to f. By the retract argument, f is a retract of  $p_f$ , therefore, f is in  $we(\mathcal C) \cap fib(\mathcal C)$ .

**Lemma 3.2.** If a morphism  $f: X \to Y$  is in fib(C), and  $\eta_X: X \to Q(X)$ ,  $\eta_Y: Y \to Q(Y)$  are in we(C), then f is a Q-fibration.

Proof. For any commutative square,

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & X \\
\downarrow i & & \downarrow f \\
B & \xrightarrow{\beta} & Y
\end{array}$$

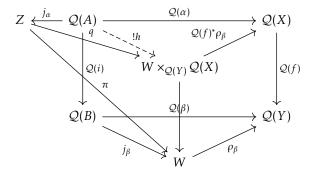
where i is a Q-trivial cofibration, it suffices to show that there exists a lift. First, we factorize the functorial image of the above commutative square,

$$Q(A) \xrightarrow{j_{\alpha}} Z \xrightarrow{\rho_{\alpha}} Q(X)$$

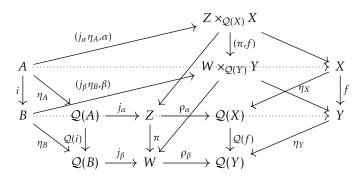
$$Q(i) \downarrow \qquad \qquad \downarrow \pi \qquad \qquad \downarrow Q(f)$$

$$Q(B) \xrightarrow{j_{\beta}} W \xrightarrow{\rho_{\beta}} Q(Y)$$

where,  $j_{\alpha}, j_{\beta}$  are in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and  $\rho_{\alpha}, \rho_{\beta}$  are in  $fib(\mathcal{C})$ . We obtain this factorization in the following way: first factorize  $\mathcal{Q}(\beta) = \rho_{\beta}j_{\beta}$ , where  $j_{\beta}$  is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and  $\rho_{\beta}$  is in  $fib(\mathcal{C})$ . Then take the pullback  $(\mathcal{Q}(f))^*(\rho_{\beta}) : W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X) \to \mathcal{Q}(X)$  of  $\rho_{\beta}$  along  $\mathcal{Q}(f)$ , which is in  $fib(\mathcal{C})$ , since  $\rho_{\beta}$  is. By the universal property, there exists an unique morphism  $h: \mathcal{Q}(A) \to W \times_{\mathcal{Q}(Y)} \mathcal{Q}(X)$ , such that  $\mathcal{Q}(\alpha) = (\mathcal{Q}(f))^*(\rho_{\beta})h$ . Again factorize  $h = qj_{\alpha}$ , where,  $j_{\alpha}$  is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and q is in  $fib(\mathcal{C})$ . Finally, set  $\rho_{\alpha} = (\mathcal{Q}(f))^*(\rho_{\beta})q$  and  $\pi = (\rho_{\beta})^*(\mathcal{Q}(f))q$ .



Now consider the pullback of the right square along the  $\eta$ -naturality square on f ,



we obtain the commutative diagram,

$$A \xrightarrow{(j_{\alpha}\eta_{A},\alpha)} Z \times_{\mathcal{Q}(X)} X \longrightarrow X$$

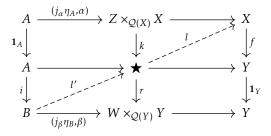
$$\downarrow \downarrow \downarrow (\pi,f) \qquad \qquad \downarrow f$$

$$B \xrightarrow{(j_{\beta}\eta_{B},\beta)} W \times_{\mathcal{Q}(Y)} Y \longrightarrow Y$$

where the left square is due to the universal property of the pullback. Now we show that  $(\pi, f)$  is in we(C). For that, consider the diagram,

$$\begin{array}{cccc} \mathcal{Q}(A) & \stackrel{j_{\alpha}}{\longrightarrow} Z & \stackrel{(\rho_{\alpha})^{*}\eta_{X}}{\longleftarrow} Z \times_{\mathcal{Q}(X)} X \\ \downarrow^{Q(i)} & & \downarrow^{\pi} & \downarrow^{(\pi,f)} \\ \mathcal{Q}(B) & \stackrel{j_{\beta}}{\longrightarrow} W & \stackrel{(\rho_{\beta})^{*}\eta_{Y}}{\longleftarrow} W \times_{\mathcal{Q}(Y)} Y \end{array}$$

Since  $\mathcal{C}$  is proper,  $(\rho_{\alpha})^*\eta_X$  and  $(\rho_{\beta})^*\eta_Y$  are in  $we(\mathcal{C})$ . By the 2-out-of-3 property,  $\pi$  is in  $we(\mathcal{C})$ , and which further implies that  $(\pi, f)$  is in  $we(\mathcal{C})$ . Finally factorize  $(\pi, f) = rk$ , where k is in  $cof(\mathcal{C})$  and r is in  $we(\mathcal{C}) \cap fib(\mathcal{C})$ . By the 2-out-of-3 property, k is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$ . Then in the following commutative diagram,



*i* being in cof(C) and *r* in  $we(C) \cap fib(C)$ , l' exists. Similarly, *k* being in  $we(C) \cap cof(C)$  and *f* in fib(C), l exists. Hence, ll' is our desired lift.

**Theorem 3.3.** (Bousfield-Friedlander theorem)  $C^{\mathbb{Q}}$  is a proper model catgeory, where  $we(C^{\mathbb{Q}})$ ,  $cof(C^{\mathbb{Q}})$  and  $fib(C^{\mathbb{Q}})$  are  $\mathbb{Q}$ -equivalences,  $\mathbb{Q}$ -fibrations and  $\mathbb{Q}$ -cofibrations respectively.

*Proof.* Since  $\mathcal{C}$  is a model category,  $\mathcal{C}^{\mathcal{Q}}$  has limits and colimits. Suppose h = gf, and two of the three morphisms f, g and h are in  $we(\mathcal{C}^{\mathcal{Q}})$ . Then  $\mathcal{Q}(h) = \mathcal{Q}(g)\mathcal{Q}(f)$ , and by the 2-out-of-3 property of  $\mathcal{C}$ , the third morphism is also in  $we(\mathcal{C}^{\mathcal{Q}})$ . This proves the 2-out-of-3 property of  $\mathcal{C}^{\mathcal{Q}}$ . Now, since  $cof(\mathcal{C}^{\mathcal{Q}}) = cof(\mathcal{C})$ , and by Lemma 2.1,  $we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}) = we(\mathcal{C}) \cap fib(\mathcal{C})$ , we have that  $(cof(\mathcal{C}^{\mathcal{Q}}), we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}}))$  is a weak factorization system. On the other hand, by the definition of  $fib(\mathcal{C}^{\mathcal{Q}})$ , we have  $fib(\mathcal{C}^{\mathcal{Q}}) = (we(\mathcal{C}^{\mathcal{Q}}) \cap cof(\mathcal{C}^{\mathcal{Q}}))^{|\mathcal{D}|}$ .

Now we consider a morphism  $f: X \to Y$  in  $\mathcal{C}^{\mathbb{Q}}$ . Then we factorize  $\mathcal{Q}(f)$ ,

$$Q(X) \xrightarrow{i} Z \xrightarrow{p} Q(Y)$$

*i* in  $we(\mathcal{C}) \cap cof(\mathcal{C})$ , hence in  $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$  and p in  $fib(\mathcal{C})$ . In the  $\eta$ -naturality square,

$$\begin{array}{ccc} \mathcal{Q}(X) & \xrightarrow{i} & Z & \xrightarrow{p} & \mathcal{Q}(Y) \\ \eta_{\mathcal{Q}(X)} \downarrow & & & \downarrow \eta_{Z} & & \downarrow \eta_{\mathcal{Q}(Y)} \\ \mathcal{Q}(\mathcal{Q}(X)) & \xrightarrow{\mathcal{Q}(i)} & \mathcal{Q}(Z) & \xrightarrow{\mathcal{Q}(p)} & \mathcal{Q}(\mathcal{Q}(Y)) \end{array}$$

since  $\eta_{\mathcal{Q}(X)}$ , i and  $\mathcal{Q}(i)$  are in  $we(\mathcal{C})$ , by 2-out-of-3 property,  $\eta_Z$  is in  $we(\mathcal{C})$ . So by Lemma 2.2, p is in  $fib(\mathcal{C}^{\mathcal{Q}})$ . Now, we factorize the  $\eta$ -naturality square on f, as the pullback corner morphism  $\tilde{i}$  followed by the pullback  $\tilde{p}$  of p along  $\eta_Y$ ,

$$X \xrightarrow{\tilde{i}} Z \times_{\mathcal{Q}(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\downarrow^{\eta_X} \qquad \qquad \downarrow^{\eta_Y} \qquad \qquad \downarrow^{\eta_Y}$$

$$\mathcal{Q}(X) \xrightarrow{i} Z \xrightarrow{p} \mathcal{Q}(Y)$$

By (3),  $\tilde{\eta}$  is in  $we(\mathcal{C}^{\mathbb{Q}})$ , since  $\eta_Y$  is in  $we(\mathcal{C}^{\mathbb{Q}})$  and p is in  $fib(\mathcal{C}^{\mathbb{Q}})$ . By the 2-out-of-3 property,  $\tilde{i}$  is in  $we(\mathcal{C}^{\mathbb{Q}})$ . Also, since p is in  $fib(\mathcal{C}^{\mathbb{Q}})$  and  $fib(\mathcal{C}^{\mathbb{Q}}) = (we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}}))^{\mathbb{Z}}$ ,  $\tilde{p}$  is in  $fib(\mathcal{C}^{\mathbb{Q}})$ . Finally, factorize  $\tilde{i} = \tilde{q}\tilde{j}$ , where  $\tilde{j}$  is in  $cof(\mathcal{C}^{\mathbb{Q}})$  and  $\tilde{q}$  is in  $we(\mathcal{C}^{\mathbb{Q}}) \cap fib(\mathcal{C}^{\mathbb{Q}})$ . By the 2-out-of-3 property of  $\mathcal{C}^{\mathbb{Q}}$ ,  $\tilde{j}$  is in  $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$  and  $\tilde{p}\tilde{q}$  is our required morphism in  $fib(\mathcal{C}^{\mathbb{Q}})$ . Let j be a morphism in  $\mathbb{Z}^{\mathbb{Q}}$ , and factorize j = rk, k in  $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$  and r in  $fib(\mathcal{C}^{\mathbb{Q}})$ . By the retract argument, j is a retract of k. But,  $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$  is closed under retracts, since  $cof(\mathcal{C}^{\mathbb{Q}}) = cof(\mathcal{C})$  and  $we(\mathcal{C}) \subseteq we(\mathcal{C}^{\mathbb{Q}})$  are closed under retracts. Therefore,  $we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}}) = \mathbb{Z}^{\mathbb{Q}}$  fib $(\mathcal{C}^{\mathbb{Q}})$  and  $(we(\mathcal{C}^{\mathbb{Q}}) \cap cof(\mathcal{C}^{\mathbb{Q}})$ ,  $fib(\mathcal{C}^{\mathbb{Q}})$ ) is a weak factorization system. The properness of  $\mathcal{C}^{\mathbb{Q}}$  follows from (3) and (4).

**Proposition 3.4.** A morphism  $f: X \to Y$  in C is a Q-fibration iff f is in fib(C)

and the  $\eta$ -naturality square on f,

$$X \xrightarrow{\eta_X} \mathcal{Q}(X)$$

$$f \downarrow \qquad \qquad \downarrow \mathcal{Q}(f)$$

$$Y \xrightarrow{\eta_Y} \mathcal{Q}(Y)$$

is a homotopy fiber square in C.

*Proof.* Let  $f: X \to Y$  in  $\mathcal{C}$  be a  $\mathcal{Q}$ -fibration. Then by the definition of a  $\mathcal{Q}$ -fibration, f is in  $fib(\mathcal{C})$ . We factorize  $\mathcal{Q}(f) = pi$ , where i is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and p is in  $fib(\mathcal{C})$ , and then consider the commutative diagram,

by the 2-out-of-3 property,  $\eta_Z$  is in  $we(\mathcal{C})$ . So by Lemma 3.2, p is in  $fib(\mathcal{C}^Q)$ . Now, since  $\mathcal{C}^Q$  is proper,  $p^*\eta_Y$  is in  $we(\mathcal{C}^Q)$ . Again by the 2-out-of-3 property of  $\mathcal{C}^Q$ ,  $\tilde{i}$  is in  $we(\mathcal{C}^Q)$ . In particular, by Proposition 2.2, the bottom right square is a homotopy fiber square in  $\mathcal{C}$ , and since p is in  $fib(\mathcal{C})$ , the top right square is also a homotopy fiber square in  $\mathcal{C}$ . Hence, by Proposition 2.3, the total right rectangle is a homotopy fiber square in  $\mathcal{C}$ . By the naturality of  $\eta$ , the total right rectangle is same as the commutative rectangle,

$$Z \times_{\mathcal{Q}(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\eta_{Z \times_{\mathcal{Q}(Y)} Y} \downarrow \qquad \qquad \downarrow \eta_{Y}$$

$$\mathcal{Q}(Z \times_{\mathcal{Q}(Y)} Y) \xrightarrow{\mathcal{Q}(\tilde{p})} \mathcal{Q}(Y)$$

$$\mathcal{Q}(p^{*}\eta_{Y}) \downarrow \qquad \qquad \downarrow \eta_{\mathcal{Q}(Y)}$$

$$\mathcal{Q}(Z) \xrightarrow{\mathcal{Q}(p)} \mathcal{Q}(\mathcal{Q}(Y))$$

Since  $p^*\eta_Y$  is in  $we(\mathcal{C}^\mathbb{Q})$ ,  $\mathcal{Q}(p^*\eta_Y)$  is in  $we(\mathcal{C})$ . Again by Proposition 2.2, the bottom square is a homotopy fiber square in  $\mathcal{C}$ . Since the total rectangle is a homotopy fiber square, by Proposition 2.3, the top square is a homotopy fiber square in  $\mathcal{C}$  as well. By the 2-out-of-3 property of  $\mathcal{C}^\mathbb{Q}$ , we factorize  $\tilde{i} = rk$ , k in  $we(\mathcal{C}^\mathbb{Q}) \cap cof(\mathcal{C}^\mathbb{Q})$  and r in  $we(\mathcal{C}^\mathbb{Q}) \cap fib(\mathcal{C}^\mathbb{Q})$ , which implies f is a retract of  $\tilde{p}r$ .

Now, the  $\eta$ -naturality square on r,

$$E(\tilde{i}) \xrightarrow{r} Y \times_{Q(Y)} Z$$

$$\downarrow^{\eta_{E(\tilde{i})}} \qquad \qquad \downarrow^{\eta_{Y} \times_{Q(Y)} Z}$$

$$Q(E(\tilde{i})) \xrightarrow{Q(r)} Q(Y \times_{Q(Y)} Z)$$

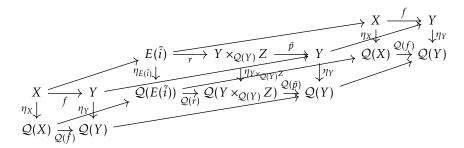
is a homotopy fiber square in  $\mathcal C$  since r is in  $we(\mathcal C^{\mathcal Q}) \cap fib(\mathcal C^{\mathcal Q}) = we(\mathcal C) \cap fib(\mathcal C)$  and  $\mathcal Q(r)$  is in  $we(\mathcal C)$ . It follows that the  $\eta$ -naturality square of  $\tilde pr$ ,

$$E(\tilde{i}) \xrightarrow{r} Y \times_{\mathcal{Q}(Y)} Z \xrightarrow{\tilde{p}} Y$$

$$\downarrow^{\eta_{E(\tilde{i})}} \qquad \qquad \downarrow^{\eta_{Y} \times_{\mathcal{Q}(Y)} Z} \qquad \downarrow^{\eta_{Y}}$$

$$\mathcal{Q}(E(\tilde{i})) \xrightarrow{\mathcal{Q}(r)} \mathcal{Q}(Y \times_{\mathcal{Q}(Y)} Z) \xrightarrow{\mathcal{Q}(\tilde{p})} \mathcal{Q}(Y)$$

is a homotopy fiber square in C. Now, f being a retract of  $\tilde{p}r$  in C,



it implies that the  $\eta$ -naturality square on f is a retract of the  $\eta$ -naturality square on  $\tilde{p}r$  in  $\mathcal{C}^{\square}$ . Therefore, by Proposition 2.4, the  $\eta$ -naturality square on f is a homotopy fiber square in  $\mathcal{C}$ .

Conversely, let f be in  $fib(\mathcal{C})$  and that the  $\eta$ -naturality square on f is a homotopy fiber square in  $\mathcal{C}$ . First we factor  $\mathcal{Q}(f) = pi$ , where i is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and p is in  $fib(\mathcal{C})$ . By the proof of Theorem 3.3,

$$X \xrightarrow{\tilde{i}} Z \times_{Q(Y)} Y \xrightarrow{\tilde{p}} Y$$

$$\uparrow_{\eta_X} \downarrow \qquad \qquad \downarrow_{\eta_{\tilde{i}}} \qquad \qquad \downarrow_{\eta_{Y}}$$

$$Q(X) \xrightarrow{i} Z \xrightarrow{p} Q(Y)$$

p is in  $fib(\mathcal{C}^{\mathcal{Q}})$ , which implies  $\tilde{p}$  is in  $fib(\mathcal{C}^{\mathcal{Q}})$  since  $fib(\mathcal{C}^{\mathcal{Q}})$  is closed under taking pullbacks. Now since the  $\eta$ -naturality square on f is a homotopy fiber square in  $\mathcal{C}$ ,  $\tilde{i}$  is in  $we(\mathcal{C})$ . Hence, by the 2-out-of-3 property,  $\tilde{i}$  admits a factorization  $\tilde{i}=qj$ , where j is in  $we(\mathcal{C}) \cap cof(\mathcal{C})$  and q is in  $we(\mathcal{C}) \cap fib(\mathcal{C}) = we(\mathcal{C}^{\mathcal{Q}}) \cap fib(\mathcal{C}^{\mathcal{Q}})$ . So we get,  $f = \tilde{p}\tilde{i} = (\tilde{p}q)j$ , where  $(\tilde{p}q)$  is in  $fib(\mathcal{C}^{\mathcal{Q}})$ . Therefore, by the retract argument, f is a retract of  $(\tilde{p}q)$ , i.e., in  $fib(\mathcal{C}^{\mathcal{Q}})$ .

#### References

- [BF78] Aldridge Knight Bousfield and Eric M. Friedlander. Homotopy theory of  $\gamma$ -spaces, spectra, and bisimplicial sets. 1978.
- [Bou00] Aldridge Knight Bousfield. On the telescopic homotopy theory of spaces. *Transactions of the American Mathematical Society*, 353:2391–2426, 2000.
- [Hir03] Philip S. Hirschhorn. Model categories and their localizations. 2003.
- [Hov07] M. Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, 2007.
- [Rie20] Emily Riehl. Homotopical categories: from model categories to  $(\infty,1)$ -categories, 2020.