CS409 — Algorithmic Game Theory
Notes for Lectures 10 to 13
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1 Mixed strategies and mixed Nash equilibria

Consider an n-player game in which each player $i = 1, 2, \dots, n$ has a finite set of *(pure) strategies* S_i. In a *(pure) strategy profile* $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$, for every $i = 1, 2, \dots, n$, we have a pure strategy $s_i \in S_i$ of player i; we write $\boldsymbol{S}$ for the set $S_1 \times S_2 \times \cdots \times S_n$ of pure strategy profiles. For every player i, we have the payoff function $p_i : \boldsymbol{S} \to \mathbb{R}$; the payoff of player i in a pure strategy profile $\boldsymbol{s}$ is $p_i(\boldsymbol{s})$.

A *mixed strategy* of player i is a probability distribution $\sigma_i : S_i \to \mathbb{R}$ over the set of pure strategies S_i of player i; in other words, we have that $\sigma_i(s) \geq 0$ for all $s \in S_i$, and $\sum_{s \in S_i} \sigma_i(s) = 1$. In a *(mixed) strategy profile* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$, for every $i = 1, 2, \dots, n$, we have a mixed strategy σ_i of player i. A mixed strategy profile induces a probability distribution

$$\boldsymbol{\sigma}(\boldsymbol{s}) \;=\; \sigma_1(s_1) \cdot \sigma_2(s_2) \cdot \cdots \cdot \sigma_n(s_n)$$

on the set $\boldsymbol{S}$ of pure strategy profiles. The *expected payoff* of player i in a strategy profile $\boldsymbol{\sigma}$ is defined by:

$$p_i(\boldsymbol{\sigma}) \;=\; \sum_{\boldsymbol{s} \in \boldsymbol{S}} \boldsymbol{\sigma}(\boldsymbol{s}) \cdot p_i(\boldsymbol{s}).$$

We sometimes write $\boldsymbol{\sigma}_{-i}$ for the $(n-1)$-strategy profile without the strategy σ_i of player i, and we write $(\boldsymbol{\sigma}_{-i}, \tau)$, where τ is a mixed strategy of player i, to denote the n-strategy profile, in which player i plays strategy τ and the other players play their strategies in the strategy profile $\boldsymbol{\sigma}$. It follows that $(\boldsymbol{\sigma}_{-i}, \sigma_i) = \boldsymbol{\sigma}$ and hence $p_i(\boldsymbol{\sigma}_{-i}, \sigma_i) = p_i(\boldsymbol{\sigma})$.

A *mixed Nash equilibrium* is a mixed strategy profile $\boldsymbol{\sigma}$, such that for every player i, and for every mixed strategy τ of player i, we have:

$$p_i(\boldsymbol{\sigma}_{-i}, \tau) \;\leq\; p_i(\boldsymbol{\sigma}_{-i}, \sigma_i) \;=\; p_i(\boldsymbol{\sigma}).$$

In other words, a mixed Nash equilibrium is a strategy profile in which no player can increase their payoff by unilaterally deviating from their strategy in the strategy profile.

*The notes are partially based on: Notes *Einführung in die Spieltheorie* by Simon Fischer (RWTH Aachen, 2007), slides *Game Theory Principles* by Wooyoung Lim (University of Pittsburgh, 2009), and the book *Thinking Strategically* by Dixit and Nalebuff (New York, 1991).

2 Two-player zero-sum games

Assume, without loss of generality, that players I and II have pure strategy sets $S_1 = \{1, 2, \dots, m_1\}$ and $S_2 = \{1, 2, \dots, m_2\}$. We represent mixed strategies of player i by m_i-dimentional vectors $\boldsymbol{\sigma}_i \in \Delta^{m_i}$, where

$$\Delta^m = \{\boldsymbol{x} \in \mathbb{R}^m : x_1 + x_2 + \cdots + x_m = 1, \text{ and } x_j \geq 0 \text{ for all } j = 1, 2, \dots, m\}.$$

- For any two pure strategies $s_1 \in S_1$ and $s_2 \in S_2$, we have $p_1(s_1, s_2) = -p_2(s_1, s_2)$.
- The payoff can be written as a $m_1 \times m_2$ matrix

$$\boldsymbol{A} = \begin{pmatrix} p_1(1,1) & \cdots & p_1(1, m_2) \\ \vdots & \ddots & \vdots \\ p_1(m_1, 1) & \cdots & p_1(m_1, m_2) \end{pmatrix}$$

  for player I and $\boldsymbol{B} = -\boldsymbol{A}$ for player II.
- Let $\boldsymbol{x}$ be a mixed strategy for player I and $\boldsymbol{y}$ be a mixed strategy for player II. Then, player I has expected payoff $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y}$ and player II has expected payoff $\boldsymbol{x}^T \boldsymbol{B} \boldsymbol{y} = -\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y}$.

2.1 An example

Consider the following payoff matrix:

$$\boldsymbol{A} = \begin{pmatrix} 28 & 1 & -38 & -11 \\ 4 & 3 & 2 & -3 \\ 5 & -3 & 4 & 3 \\ -19 & -9 & 29 & 1 \end{pmatrix} .$$

What does it mean for player II if player I chooses the mixed strategy $(0, 1/2, 1/2, 0)^T$? What is the expected payoff for player II if she chooses the first, second, third, or fourth column? This can be easily calculated by multiplying the transposed strategy vector of player I with the payoff matrix of player II.

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}^T \begin{pmatrix} -28 & -1 & 38 & 11 \\ -4 & -3 & -2 & 3 \\ -5 & 3 & -4 & -3 \\ 19 & 9 & -29 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} & 0 & -3 & 0 \end{pmatrix} .$$

The first column gives expected payoff -9/2, the second one 0, the third one -3, and the fourth one 0. Clearly, choosing the first or third column is not a good option since the second and fourth column give a higher expected payoff. Player II is indifferent between the second and fourth column. Any probability distribution on these two choices is a best response to player I's mixed strategy.
What does it mean for player I if player II chooses the mixed strategy $(0, 1/2, 0, 1/2)^T$? What is the expected payoff for player I if she chooses the first, second, third, or fourth row? Again, this can be easily calculated by multiplying the payoff matrix of player I with the strategy vector of player II.

$$\begin{pmatrix} 28 & 1 & -38 & -11 \\ 4 & 3 & 2 & -3 \\ 5 & -3 & 4 & 3 \\ -19 & -9 & 29 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 0 \\ -4 \end{pmatrix} .$$

The first row gives expected payoff -5, the second one 0, the third one 0, and the fourth one -4. Clearly, choosing the first or fourth row is not a good option since the second and third row give a higher expected payoff. Player I is indifferent between the second and third row. Any probability distribution on these two choices is a best response to player II's mixed strategy.

Thus, the investigated mixed strategies of player I and II are best responses to one another and form a Nash equilibrium. The expected payoff of player I in this equilibrium is

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}^T \begin{pmatrix} 28 & 1 & -38 & -11 \\ 4 & 3 & 2 & -3 \\ 5 & -3 & 4 & 3 \\ -19 & -9 & 29 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = 0 \ .$$

2.2 Linear programming refresher

Before we continue, we have to remind ourselves of some facts about linear programs. A linear program (LP) is an optimization problem in which we want to maximize (or minimize) a linear objective function subject to a set of linear constraints. That means we are given a matrix M and vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ and the goal is to find a vector $\boldsymbol{x}$ such that $\boldsymbol{c}^T\boldsymbol{x}$ is as large as possible but at the same time $\boldsymbol{M}\boldsymbol{x} \leq \boldsymbol{b}$ is satisfied and all entries in $\boldsymbol{x}$ are nonnegative. Here, the inequality $\boldsymbol{M}\boldsymbol{x} \leq \boldsymbol{b}$ should be read component-wise, that is, the i-th entry of $\boldsymbol{M}\boldsymbol{x}$ is at most as large as the $i - th$ entry of $\boldsymbol{b}$ for all i. Therefore, the standard form of a linear program is

$$\begin{array}{lrcl} \max_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} & & \\ \text{subject to} & \boldsymbol{M}\boldsymbol{x} & \leq & \boldsymbol{b} \\ & \boldsymbol{x} & \geq & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \ . \end{array}$$

A linear program in which the goal is to minimize the objective function has the standard form

$$\begin{array}{lrcl} \min_{\boldsymbol{y}} & \boldsymbol{b}^T\boldsymbol{y} & & \\ \text{subject to} & \boldsymbol{M}^T\boldsymbol{y} & \geq & \boldsymbol{c} \\ & \boldsymbol{y} & \geq & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \ . \end{array}$$

As it turns out, these two linear programs are intimately related to one another. One is called the *primal* and the other one is called the *dual* (of the primal). The dual of a linear program is obtained by switching min and max, switching $\boldsymbol{b}$ and $\boldsymbol{c}$, changing the direction of the first inequality and by transposing the matrix M. You will notice that, if we form the dual of the dual, we obtain the primal again.

In the following section, we make use of the following important theorem.

Theorem 2.1 (Strong duality theorem)**.** *If the primal or the dual has an optimal solution of finite value, then so does the other and the values of the optimal solutions are the same.*

It is also worth noting that optimal solutions of linear programs can be efficiently computed, both in theory and in practice.

2.3 Existence and structure of Nash equilibria

A possible approach of player I is to try to maximize her *guaranteed* expected payoff. For this player I chooses a mixed strategy $\boldsymbol{x}$ that gives the highest expected payoff under the assumption that player II chooses a response that is the worst for player I. This worst-case assumption is realistic for two-player zero-sum games since the players' interests are diametrically opposed.

Formally, player I wants to maximize the guaranteed expected payoff v, that is,

$$\begin{array}{lrcl} \max_{\boldsymbol{x}\in\Delta^{m_1},v} & & v & \\ \text{subject to} & \boldsymbol{x}^T\boldsymbol{A} & \geq & \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}^T \end{array}$$

The condition $\boldsymbol{x} \in \Delta^{m_1}$ just means that entries in $\boldsymbol{x}$ have to be nonnegative and have to sum up to 1. This can be rewritten as $\boldsymbol{x} \geq \mathbf{0}$ and $\mathbf{1}^T\boldsymbol{x} = 1$, where $\mathbf{1}$ and $\mathbf{0}$ denote m_1-dimensional vectors in which every entry is 1 and 0, respectively. Assume that all entries in $\boldsymbol{A}$ are nonnegative (if not, we can add a large number to every entry of $\boldsymbol{A}$, which does change the payoffs but not the structure of the game and its equilibria). Then, it is sufficient to require that $\mathbf{1}^T\boldsymbol{x} \leq 1$ instead of $\mathbf{1}^T\boldsymbol{x} = 1$. Combining all of this, we obtain the following linear program for player I:

$$\begin{array}{lrcl} \max & (0,\ldots,0,1)\begin{pmatrix} x_1 \\ \vdots \\ x_{m_1} \\ v \end{pmatrix} & & \\ \text{subject to} & \begin{pmatrix} & & & 1 \\ & -\boldsymbol{A}^T & & \vdots \\ & & & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_{m_1} \\ v \end{pmatrix} & \leq & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \\ & \begin{pmatrix} x_1 \\ \vdots \\ x_{m_1} \\ v \end{pmatrix} & \geq & \mathbf{0} \end{array}$$

Assume player II also wants to maximize her guaranteed expected payoff. For two-player zero-sum games, this is equivalent with the aim to minimize the expected payoff of player I. The same line of reasoning as before gives us the following linear program for player II.

$$\begin{array}{lrcl} \min & (0,\ldots,0,1)\begin{pmatrix} y_1 \\ \vdots \\ y_{m_2} \\ w \end{pmatrix} & & \\ \text{subject to} & \begin{pmatrix} & & & 1 \\ & -\boldsymbol{A} & & \vdots \\ & & & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_{m_2} \\ w \end{pmatrix} & \geq & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \\ & \begin{pmatrix} y_1 \\ \vdots \\ y_{m_2} \\ w \end{pmatrix} & \geq & \mathbf{0} \end{array}$$

These two linear programs are dual to one another and therefore have the same value. Hence, player I has a strategy that guarantees her an expected payoff of some value v and player II has a strategy that guarantees that player I is not able to get more than v in expectation. Clearly these two strategies form a Nash equilibrium of the game and they have a special name.

Definition 2.2 (Maximin and minimax strategies)**.** *A strategy $\boldsymbol{x} \in \Delta^{m_1}$ is called maximin-strategy if*

$$\boldsymbol{x} \in \arg \max_{\boldsymbol{x}' \in \Delta^{m_1}} \min_{j \in [m_2]} (\boldsymbol{A}^T \boldsymbol{x}')_j \ .$$

A strategy $\boldsymbol{y} \in \Delta^{m_2}$ is called minimax-strategy if

$$\boldsymbol{y} \in \arg \min_{\boldsymbol{y}' \in \Delta^{m_2}} \max_{i \in [m_1]} (\boldsymbol{A}\boldsymbol{y}')_i \ .$$

We call a pair of maximin- and minimax-strategies a minimax-equilibrium.

Theorem 2.3. *For two-player zero-sum games:*

- *A pair of maximin- and minimax-strategies forms a Nash equilibrium.*
- *All Nash equilibria result in the same expected payoff. We call the expected payoff of player I the value of the game.*
- *A Nash equilibrium can be computed in polynomial time (in the number of pure strategies).*

Example 2.4 (Rock, paper, scissors)**.** *Consider the following well-known game.*

I \ II	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

We may already see what the equilibrium in this game is, but how could we derive it if we didn't? Following the approach described above, we can formulate an optimization problem for player I, which, when solved, will give us a maximin-strategy.

$$\begin{array}{llllll}
\max_{v,\boldsymbol{x}} & v & & & & \\
\text{subject to} & & x_2 & -x_3 & \geq & v \\
& -x_1 & & +x_3 & \geq & v \\
& x_1 & -x_2 & & \geq & v \\
& x_1 & +x_2 & +x_3 & = & 1 \\
& x_1, & x_2, & x_3 & \geq & 0 \ .
\end{array}$$

We can do the same for player II, which will allow us to derive a minimax-strategy.

$$\begin{array}{llllll}
\min_{w,\boldsymbol{y}} & w & & & & \\
\text{subject to} & & -y_2 & +y_3 & \leq & w \\
& y_1 & & -y_3 & \leq & w \\
& -y_1 & +y_2 & & \leq & w \\
& y_1 & +y_2 & +y_3 & = & 1 \\
& y_1, & y_2, & y_3 & \geq & 0 \ .
\end{array}$$

To save us some work, we will not bring these linear program into its standard form and work directly with them as stated above. The first thing we might observe is that, if all constraints of the first LP are satisfied, we must have $v \leq 0$. This follows because summing up the first three constraints gives $0 \geq 3 \cdot v$. A similar observation for the second LP shows that $w \geq 0$. Now note that, due to strong duality, for the optimal solutions of the LPs v and w must be equal. But if v is always nonpositive and w is always nonnegative they can only be equal if they are in fact 0. This already tells us what the value of the game is, namely, 0. This should not be surprising. The game is completely symmetric for the two players and therefore it is quite intuitive that neither of them should have an advantage over the other.

The next goal is to find the actual mixed strategies. This got a little easier, now that we know what the optimal values of v and w are. Specifically, we are looking for $\boldsymbol{x}$ satisfying the following constraints

$$\begin{array}{rrrcl} & x_2 & -x_3 & \geq & 0 \\ -x_1 & & +x_3 & \geq & 0 \\ x_1 & -x_2 & & \geq & 0 \\ x_1 & +x_2 & +x_3 & = & 1 \\ x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

or, equivalently, $x_2 \geq x_3 \geq x_1 \geq x_2$, $x_1 + x_2 + x_3 = 1$, and $x_1, x_2, x_3 \geq 0$. Clearly the first change of inequalities can only be satisfied if the values of x_1, x_2, and x_3 are equal, i.e., $x_1 = x_2 = x_3$. Combined with $x_1 + x_2 + x_3 = 1$, this gives us the solution $x_1 = x_2 = x_3 = 1/3$. We can approach the second LP in the same way to determine that $y_1 = y_2 = y_3 = 1/3$.

Note that, in general, it does not have to be as easy as in this example. Deriving a solution like this either requires some creativity or the application of an algorithm for solving LPs (which can be complicated). For sufficiently simple and small examples the former should usually be possible.

To make sure that we made no mistake we can also test our solution. Player I can guarantee an expected payoff of 0 for herself by choosing each of the pure strategies with probability $1/3$. This is easy to verify. We have

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

and therefore the expected payoff in this case is always 0 no matter what player II chooses to do.

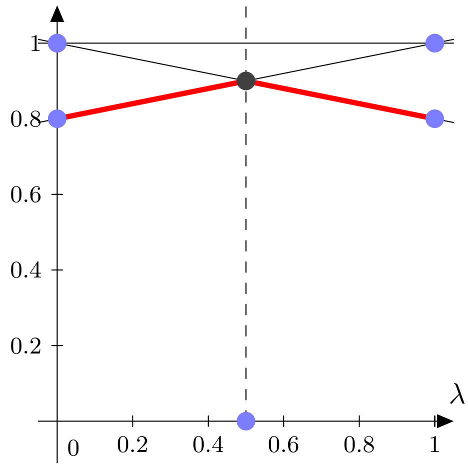
Player II on the other hand can choose each of her pure strategy with probability $1/3$ and thereby guarantee that the expected payoff of player I will never be larger than 0.

We found a pair of maximin- and minimax-strategies. They form a Nash equilibrium and the value of this game is 0.

Example 2.5. *Consider the following game*

I \ II	l	m	r
T	*0.8*	*1*	*1*
B	*1*	*1*	*0.8*

Let player I choose strategy T with probability λ and strategy B with probability $1-\lambda$. The expected payoff for player I is $1 - 0.2\lambda$, 1, and $0.8 + 0.2\lambda$ if player II choses the strategies l, m, and r, respectively. Since player II wants to minimize the expected payoff of player I, the best she can achieve is the lower envelope of these three functions.



Player I chooses λ such that her expected payoff is maximized which is the case for $\lambda = 1/2$. Hence, her strategy will be $\boldsymbol{x} = (1/2, 1/2)^T$ and the value of the game is 0.9.

Player II can guarantee that payoff by choosing the strategy $\boldsymbol{y} = (1/2, 0, 1/2)^T$.

It is easy to verify that this is correct. By choosing $\boldsymbol{x}$, player I can guarantee a payoff of 0.9 since

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0.8 & 1 & 1 \\ 1 & 1 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 1 & 0.9 \end{pmatrix} \quad .$$

Player II can choose strategy $\boldsymbol{y}$. This guarantees that player I will never get a payoff of more than 0.9 since

$$\begin{pmatrix} 0.8 & 1 & 1 \\ 1 & 1 & 0.8 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.9 \end{pmatrix} \quad .$$

We found a pair of maximin- and minimax-strategies. They form a Nash equilibrium and the value of this game is 0.9.