Lecture 28: Noetherian modules

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19:00

Defⁿ/Prop: Let R be a ring and M be an R-module. TFAE

① Every R-submodule of M is finitely generated.

② Every increasing chain of R-submodules of M is eventually constant
i.e. $M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq \dots$ are R-submod of M
then $\exists N$ s.t. $M_n = M_N \ \forall n > N$.

③ Every nonempty collection Ω of R-submod of M has a maximal element.

An R-mod M satisfying the above equivalent conditions is called a noetherian module.

Pf: ① ⇒ ②: Take $M_s = \bigcup_{i \geq 0} M_i$, M_s is f.g. by $m_1, \dots, m_\ell$. Take N s.t. $m_1, \dots, m_\ell \in M_N$.

② ⇒ ③: $M_0 \in \Omega$ as $\Omega \neq \emptyset$. Lack of maximal element in Ω allows us to build a chain of strictly inc seq of R-submod of M.

③ ⇒ ①: Let M_0 be a submod of M. Take $m_1 \in M_0$, $M_1 = Rm_1$, $m_2 \in M_0 \setminus M_1$ if nonempty
$M_2 = Rm_1 + Rm_2$ and so on to construct $\Omega = \{M_i \mid i \geq 1\}$
a collection of R-submod of M without a maximal element
if M is not fin gen.

Examples: ① R a noetherian ring then R as an R-module is noetherian.
Since every submod of R is an R-ideal.
In fact R as an R-mod is noetherian iff R is a noeth ring.

② $R = \mathbb{Z}$ then every f.g. $\mathbb{Z}$-mod is noeth. By str. thm
$\mathbb{Z}^r \oplus M$ where M is a finite abelian grp.

③ R a ring M an R-mod s.t. M is a finite set then
M is noeth.

④ $R = \mathbb{Z}$ then $\mathbb{Q}$ is not noetherian $\mathbb{Z}$-mod.
$\mathbb{Z} \times \mathbb{Z} \times \dots$ infinite copies. are non noetherian R-mod

(*) M is a noeth R-mod then any R-submod of M is noeth.

④ M a f.g. R-mod & N an R-submod of M then M/N is a f.g. R-mod.

Pf: Let $M = Rm_1 + \dots + Rm_n$ then $M/N = R\bar{m}_1 + \dots + R\bar{m}_n$

④ M a noeth R-mod & N submod $\Rightarrow$ M/N is noeth

Pf: $\bar{K} \subseteq M/N$ is a submod then $\bar{K} = K/N$ for some R-submod K of M containing N.

Prop: R noeth ring and M a f.g. R-module then M is a noetherian R-module.

Pf: Let $m_1, \dots, m_n \in M$ s.t.

$M = Rm_1 + \dots + Rm_n$

Let
$$\varphi : R^n \longrightarrow M$$
$$e_i \longmapsto m_i$$
$$(a_1, \dots, a_n) \longmapsto \sum_{i=1}^n a_i m_i \quad \text{for } a_i \in R$$

φ is surjective ✓ and R-linear.

Let $x \in R$ & $\underset{\sim}{a}, \underset{\sim}{b} \in R^n$

$$\begin{aligned}
\varphi(x\underset{\sim}{a} + \underset{\sim}{b}) &= \varphi((xa_1 + b_1, xa_2 + b_2, \dots, xa_n + b_n)) \\
&= \sum_{i=1}^n (xa_i + b_i) m_i \\
&= \sum_{i=1}^n [x(a_i m_i) + b_i m_i] \\
&= x\sum_{i=1}^n a_i m_i + \sum_{i=1}^n b_i m_i \\
&= x\varphi(\underset{\sim}{a}) + \varphi(\underset{\sim}{b})
\end{aligned}$$

$\Rightarrow \quad M \cong R^n / \ker(\varphi)$ by 1st isom thm.

Hence enough to show R^n is noeth R-mod $\forall\ n \geq 1$.

Proof by induction. $n=1$ ✓ Since R is noeth ring.

Assume R^{n-1} is a noeth R-mod for some $n \geq 2$.

- $R^n = R^{n-1} \times R$. WTS every submod of R^n is f.g.

Let $M \subseteq R^n$ be a R-submod.

$R^{n-1} \subseteq R^n$ as $\{(a_1, \dots, a_{n-1}, 0) \mid a_i \in R\}$ is an R-submod of R^n.

$$\pi : R^n \longrightarrow R \quad \text{is R-lin map}$$
$$(a_1, \dots, a_n) \longmapsto a_n$$

$\ker(\pi) = R^{n-1} = \{(a_1, \dots, a_{n-1}, 0) \mid a_i \in R\}$

$M_1 = M \cap \ker(\pi) \subseteq R^{n-1}$ is a R-submod of R^{n-1}

M_1 is f.g. since R^{n-1} is noeth. say by $\{x_1, \dots, x_k\} \subseteq M$

$M_2 = \pi(M)$ is a submod of R. Hence M_2 is also f.g. R-mod say by $\{y_1, \dots, y_l\}$

Let $z_i \in M$ be s.t. $\pi(z_i) = y_i \quad 1 \leq i \leq l$.

<u>Claim</u>: $\{x_1, \dots, x_k, z_1, \dots, z_\ell\}$ gen M.

Let $x \in M$ then

$\pi(x) \in M_2$

$$\Rightarrow \pi(x) = \sum_{i=1}^{\ell} a_i y_i \text{ for some } a_i \in R$$

$$\Rightarrow \pi(x) = \sum_{i=1}^{\ell} a_i \pi(z_i)$$

$$\Rightarrow \pi\left(x - \sum_{i=1}^{\ell} a_i z_i\right) = 0$$

$$x - \sum_{i=1}^{\ell} a_i z_i \in \ker(\pi) \cap M = M_1$$

$$\Rightarrow x - \sum_{i=1}^{\ell} a_i z_i = \sum_{j=1}^{k} b_j x_j \text{ for } b_j \in R, \ 1 \le j \le k$$

$$\Rightarrow x = \sum_{i=1}^{\ell} a_i z_i + \sum_{j=1}^{k} b_j x_j$$