Thm: Let $f \in \underbrace{C(B^n)}_{\text{Set of Cont. fn's on } B^n}$. Then $f \in \mathcal{R}(B^n)$. i.e; $C(B^n) \subseteq \mathcal{R}(B^n)$.

Proof: B^n is compact. $\Rightarrow$ f is uniformly cont. on B^n.
$[\because B^n \subseteq \mathbb{R}^n_u]$.

Let $\varepsilon > 0$. So $\exists$ $\delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{2\underbrace{v(B^n)}_{\text{volume of the box}}} \quad \forall \ \|x-y\| < \delta. \qquad (*)$$

[Halt: Diameter of a box $\prod_{i=1}^{n}[a_i, b_i]$ = largest diagonal. Check.

= max { distance of v_1, v_2 : v_1, v_2 are vertices of the box }.

In the sense of metric space $\mathbb{R}^n_u$.

||ly, if P is a partition of B^n $\left(= \prod_{i=1}^{n}[a_i, b_i]\right)$,

then $\underbrace{\|P\|}_{\text{mesh of } P} = \max\{\text{diameter of } B^n_\alpha : \alpha \in \Lambda(P)\}$

Note: $\Lambda(P)$ is a finite set.]

Now for that $\delta > 0$, pick a partition P of B^n s.t.
$\|P\| < \delta$.

[Remark: This is always possible. Think $n = 1$ case.]
But possibly long computation for $n > 1$.

$\forall \alpha \in \Lambda(P)$, pick & then fix $a_\alpha \in B^n_\alpha$.

$\therefore \|a_\alpha - x\| < \delta \quad \forall x \in B^n_\alpha \qquad (\& \ \forall \alpha \in \Lambda(P))$.

Then $(*)$ $\Rightarrow$ $|f(x) - f(a_\alpha)| < \tilde{\varepsilon} \quad \forall \ x \in B^n_\alpha. \quad (\forall \alpha \in \Lambda(P))$.

$\left[\tilde{\varepsilon} := \varepsilon / 2v(B^n)\right]$

$$\Rightarrow \quad f(a_\alpha) - \tilde{\varepsilon} < f(x) < f(a_\alpha) + \tilde{\varepsilon} \quad \forall\, x \in B^n_\alpha .$$

Taking sup-inf: $\quad f(a_\alpha) - \tilde{\varepsilon} \leqslant m_\alpha \leqslant M_\alpha \leqslant f(a_\alpha) + \tilde{\varepsilon}.$

Recall [$m_\alpha = \inf\{f(x) : x \in B^n_\alpha\}$
Illy M_α.
Here $\alpha \equiv \alpha(P)$]

Thus: $\forall\, \alpha \in \Lambda(P)$ (or $\alpha(P) \in \Lambda(P)$), we have:

$$f(a_\alpha) - \tilde{\varepsilon} \leqslant m_\alpha \leqslant M_\alpha \leqslant f(a_\alpha) + \tilde{\varepsilon}$$

$$\Rightarrow \sum_{\alpha \in \Lambda(P)} \left(f(a_\alpha) - \tilde{\varepsilon}\right) \times v(B^n_\alpha) \leqslant L(f,P) \leqslant U(f,P)$$

$$\leqslant \sum_{\alpha \in \Lambda(P)} \left(f(a_\alpha) + \tilde{\varepsilon}\right) \times v(B^n_\alpha)$$

But the left-most term $= \underbrace{\left(\sum f(a_\alpha) \times v(B^n_\alpha)\right)}_{:= c} - \underbrace{\tilde{\varepsilon}}_{= \frac{\varepsilon}{2 v(B^n)}} \times v(B^n)$

$$:= c - \frac{\varepsilon}{2}.$$

& the right-most term $= c + \frac{\varepsilon}{2}$.

$$\therefore \quad c - \frac{\varepsilon}{2} \leqslant L(f,P) \leqslant U(f,P) \leqslant c + \frac{\varepsilon}{2}.$$

$$\therefore \quad 0 \leqslant \overline{\int_{B^n}} f - \underline{\int_{B^n}} f \quad \cancel{\leqslant \left(c + \frac{\varepsilon}{2}\right) - \left(c - \frac{\varepsilon}{2}\right)}$$

Always true.

$$\leqslant U(f,P) - L(f,P)$$

$$\leqslant \left(c + \frac{\varepsilon}{2}\right) - \left(c - \frac{\varepsilon}{2}\right) = \varepsilon$$

$$\Rightarrow \quad U(f,P) - L(f,P) < \varepsilon \quad \Rightarrow \quad f \in \mathcal{R}(B^n). \qquad \boxtimes$$

Now, again (like $n=1$ case) it is time to talk about Computing $\underbrace{\int_{B^n} f\,dv}$, $f \in R(B^n)$.

If $n=1$, then $\int_a^b \longleftrightarrow \sum_{n=1}^{\infty}$

i.e., $\int$ is a continuous analogue of infinite series.

So, if $n=2$, then $\underbrace{\int_{[a_1,b_1]\times[a_2,b_2]} f\,dv} \longleftrightarrow \sum_{m,n=1}^{\infty} a_{mn}$!!

i.e., a cont. analogue of "double series" ??

Ans: It should be.

So, "very briefly", lets talk about double sequences & series.
HW/upto you.

As usual, a double seqn (or even n-seqn) is a fn
$f : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ or $X \to$ a M.S.).
We write f as $\{f(m,n)\}$ or simply $\{a_{mn}\}_{m,n\geq 0}$

Often, it is helpful to assume $\mathbb{N} = \{0,1,2,\cdots\}$. or just usual $\mathbb{N}$.

Def: A double seqn $\{a_{mn}\}$ is said to be convergent if there is a real no. a so that: $\forall\ \varepsilon>0\ \exists\ N\in\mathbb{N}$
$\ni$ $|a_{mn}-a| < \varepsilon \quad \forall\ m,n \geq N$.

We write: $\boxed{\lim_{m,n\to\infty} a_{mn} = a.}$

HW: Limit is ! (if exists).

eg: 1) $a_{mn} := \frac{1}{m+n}$. $\forall\, m,n \geq 1$.

For $\varepsilon > 0$, choose $N \in \mathbb{N}$ ∍ $N > \frac{1}{2\varepsilon}$.

So, if $m,n \geq N \Rightarrow m,n > \frac{1}{2\varepsilon} \Rightarrow m+n > \frac{1}{\varepsilon}$

$\Rightarrow \frac{1}{m+n} < \varepsilon$.

$\Rightarrow |a_{mn} - 0| < \varepsilon \quad \forall\, m,n \geq N$.

$\therefore \lim\limits_{m,n\to\infty} a_{mn} = 0$.

2) $a_{mn} := (-1)^{m+n} \times \left(\frac{1}{m} + \frac{1}{n}\right)$.

$\therefore |a_{mn}| = \frac{1}{m} + \frac{1}{n}$

$\therefore$ if $\varepsilon > 0$, then choose $N \in \mathbb{N}$ ∍ $N > \frac{2}{\varepsilon}$.

$\therefore \forall\, m,n \geq N$, $m,n > \frac{2}{\varepsilon}$.

$\Rightarrow \frac{1}{m}, \frac{1}{n} < \frac{\varepsilon}{2}$. $\Rightarrow \frac{1}{m} + \frac{1}{n} < \varepsilon$.

$\Rightarrow \lim\limits_{m,n\to\infty} a_{mn} = 0$.

3) $a_{mn} = \frac{mn}{m^2+n^2}$. $\forall\, m,n \geq 1$.

Now (OLD TRICK) $m = n \Rightarrow a_{mn} = \frac{n^2}{2n^2} = \frac{1}{2}$.

Remember?

$\therefore \lim\limits_{\substack{m,n\to\infty \\ m=n}} a_{mn} = \frac{1}{2}$.

But $m = -n \Rightarrow a_{mn} = -\frac{1}{2}$.

$\therefore \lim\limits_{\substack{m,n\to\infty \\ m=-n}} a_{mn} = -\frac{1}{2}$.

$\therefore \lim\limits_{m,n\to\infty} a_{mn}$ DNE (why?).

??

∴ One sided exists But NOT both sided !!

Even worse !!

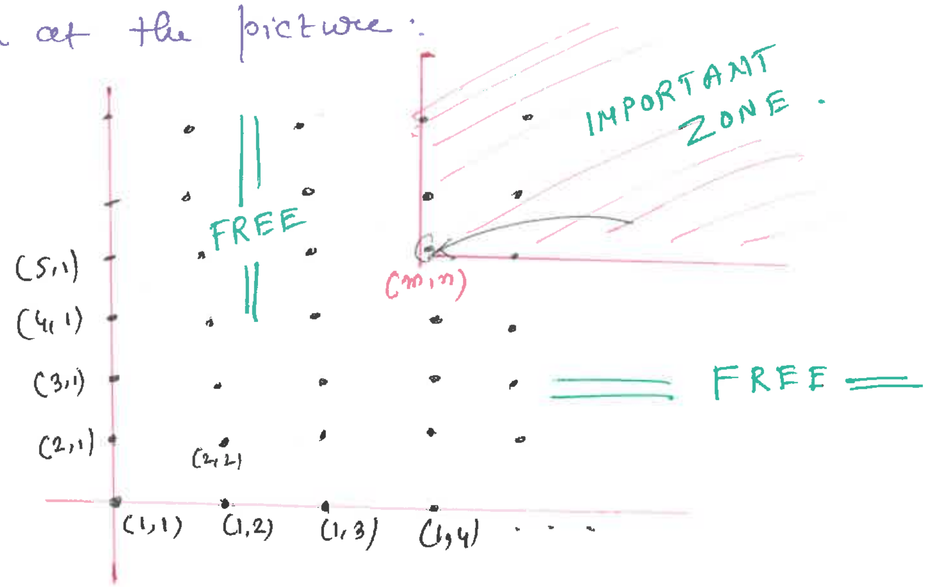
Lets look at the defn again: for $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|a_{mn} - a| < \varepsilon$. $\forall m, n \geq N$

So, let us consider eg 1 in page (10). Define

$$b_{mn} = \begin{cases} n & \text{if } m = 1 \\ \frac{1}{m+n} & \text{if } m > 1. \end{cases}$$

$\Rightarrow \lim_{m,n\to\infty} b_{mn} = 0$ (Agree ?) BUT $\{b_{mn}\}$ is NOT bounded.

Lets look at the picture:



Q: How to compute $\lim_{m,n\to\infty} a_{mn}$ (if exists) ?

[BTW: We must get back to R.I. along with similar type questions &c.]

Maybe: we compute $\lim_{n\to\infty} a_{mn}$ (treating m fixed) & then $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$? So, if all goes well, we say:

$$\lim_{m,n\to\infty} a_{mn} = \lim_{m\to\infty}\lim_{n\to\infty} a_{mn} \quad \left(= \lim_{n\to\infty}\lim_{m\to\infty} a_{mn}\right) \;!!$$

Let's give a name to it:

Def: $\{a_{mn}\}$ is said to have an ~~iter~~ <u>iterated limit</u> if

$\hat{a}_m := \lim_{n\to\infty} a_{mn}$ exists $\forall\, m \geq 1$ & $\hat{a}_m \to \hat{a}$ for some $\hat{a}$

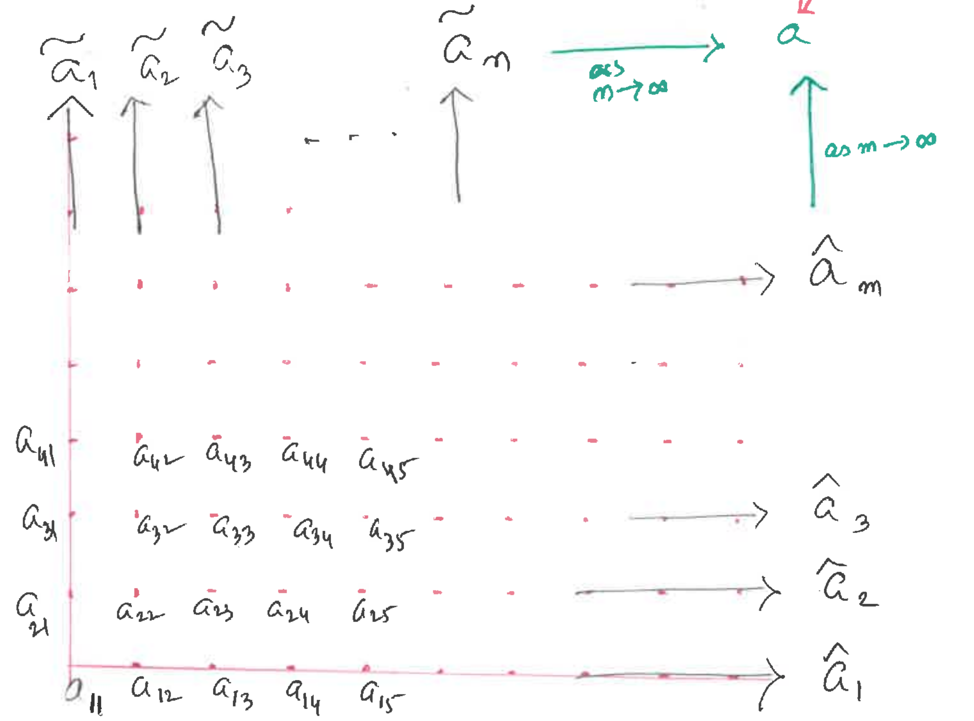
or $\tilde{a}_n := \lim_{m\to\infty} a_{mn}$ exists $\forall\, n \geq 1$ & $\tilde{a}_n \to \tilde{a}$ —— " —— $\tilde{a}$.

[We write: $\hat{a} = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}$ & $\tilde{a} = \lim_{n\to\infty} \lim_{m\to\infty} a_{mn}$.]

Q: How to relate $a, \hat{a}, \tilde{a}$ (if there is any)?

or relate

In "picture"



And in theorem

Thm: Let $\lim_{m,n\to\infty} a_{mn}$ exists & ~~the iterated~~ limit $\lim_{n\to\infty} a_{mn}$ exists <u>$\forall\, m$</u>. Then the iterated limit ~~exis~~

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{mn} \text{ exists } \&$$

$$\lim_{m,n\to\infty} a_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}.$$

Stress on this.

11(4) if $\lim_{m,n\to\infty} a_{mn}$ exists & $\lim_{m\to\infty} a_{mn}$ exists $\forall n$, then $\lim_{n\to\infty}\lim_{m\to\infty} a_{mn}$ exists & $= \lim_{m,n\to\infty} a_{mn}$.

Proof: We only prove the first one. Let $\lim_{m,n\to\infty} a_{mn} := a$.
$\forall\, m$, set $\hat{a}_m := \lim_{n\to\infty} a_{mn}$.

Let $\varepsilon > 0$. $\exists\, N \in \mathbb{N} \ni |a_{mn} - a| < \frac{\varepsilon}{2}$ $\forall\, m, n \geq N$.

Also, for each m, $\exists\, N_m \in \mathbb{N} \ni$

$$|a_{mn} - \hat{a}_m| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_m .$$

$$\therefore\ |\hat{a}_m - a| = |(\hat{a}_m - a_{mn}) + (a_{mn} - a)|$$

n IN

$$\leq |\hat{a}_m - a_{mn}| + |a_{mn} - a|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

n OUT

Here assume $n = \max\{N, N_m\}$

$$\forall\, m \geq N.$$

$$\Rightarrow \hat{a}_m \to a \text{ as } m \to \infty.$$

[OR: $|\hat{a}_m - a| = |\lim_{n\to\infty} a_{mn} - a|$

$$= \lim_{n\to\infty} |a_{mn} - a| \quad \leftarrow \text{by Continuity of } x \mapsto |x|.$$

$$\leq \varepsilon$$]

Cor: If $\lim_{m,n\to\infty} a_{mn} = a$ (i.e. the limit exists) & both $\lim_{m\to\infty} a_{mn}$ & $\lim_{n\to\infty} a_{mn}$ exists, then $\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} = \lim_{n\to\infty}\lim_{m\to\infty} a_{mn} = a$.

"We need them all"

eg: double limit $\not\Rightarrow$ single limit.

$$a_{mn} := (-1)^{m+n} \times \left(\frac{1}{m}+\frac{1}{n}\right).$$

We know $a_{mn} \to 0$ as $m,n\to\infty$.

However, for fixed m: $\quad a_{mn} = (-1)^m \times \underbrace{\left[\frac{(-1)^n}{m} + \frac{(-1)^n}{n}\right]}_{\text{DNE as } n\to\infty}$

$$\Rightarrow \lim_{n\to\infty} a_{mn} \text{ DNE } \forall\, m.$$

$$\text{IIIly } \lim_{m\to\infty} a_{mn} \text{ DNE } \forall\, n.$$

eg: Single limit $\not\Rightarrow$ double limit.

$a_{mn} = \dfrac{mn}{m^2+n^2}$. We have seen $\lim_{m,n\to\infty} a_{mn}$ DNE.

But, for fixed m, $\quad a_{mn} = m \times \dfrac{n}{m^2+n^2} \leq m \times \dfrac{n}{n^2} \to 0$ as $n\to\infty$.

IIIly — " — n, $\quad a_{mn} \to 0$ as $m\to\infty$.

All in all: if we know "Double Limit" exists & one or both single limit(s) exists, then we can compute the double limit.