Lecture 30: Localization, rank, torsion

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Recall:

Localization of R-modules

Def^n/Prop: Let R be comm ring, $S \subseteq R$ be a mult set and M be an R-mod.

Define a relation on $S \times M = \{(s,m) \mid s \in S, m \in M\}$ as follows}

$(s_1, m_1) \sim (s_2, m_2)$ if $\exists s \in S$ s.t. $s(s_1 m_2 - s_2 m_1) = 0_M$. ① Then $\sim$ is an equivalence relation. Let $\frac{m}{s}$ denote the equivalence class $[(s,m)]$ for $(s,m) \in S \times M$ and $S^{-1}M = S \times M / \sim$. ② Then $\frac{m_1}{s_1} \oplus \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$ is a well-defined binary operator on $S^{-1}M$. ③ The map

$$S^{-1}R \times S^{-1}M \xrightarrow{\sigma} S^{-1}M$$
$$\left(\frac{r}{s}, \frac{m}{s'}\right) \longmapsto \frac{rm}{ss'}$$

is well-defined. ④ Moreover $S^{-1}M$ is a $S^{-1}R$-module via σ as the scalar multiplication. ⑤ In particular $S^{-1}M$ is an R-mod.

⑥ The map $\varphi : M \longrightarrow S^{-1}M$, $m \longmapsto \frac{m}{1}$ is an R-lin map.

Basic properties

⊛ R a ring, $S \subseteq R$ a mult subset. Let M, N be R-modules then $S^{-1}(M \times N) \cong S^{-1}M \times S^{-1}N$ as $S^{-1}R$-mod.

⊛ Note $\varphi\left(\frac{(m,n)}{s}\right) = \left(\frac{m}{s}, \frac{n}{s}\right)$ is well-defined $\varphi : S^{-1}(M \times N) \to S^{-1}M \times S^{-1}N$

$\frac{(m_1, n_1)}{s_1} = \frac{(m,n)}{s}$ in $S^{-1}(M \times N)$ $\Rightarrow \exists u \in S$ s.t. $u\left(s(m_1, n_1) - s_1(m,n)\right) = 0$ in $M \times N$

$\Rightarrow \left(u(sm_1 - s_1 m), u(sn_1 - s_1 n)\right) = 0$ in $M \times N$

$\Rightarrow u(sm_1 - s_1 m) = 0$ in M & $u(sn_1 - s_1 n) = 0$ in N

$\Rightarrow \frac{m_1}{s_1} = \frac{m}{s}$ in $S^{-1}M$ & $\frac{n_1}{s_1} = \frac{n}{s}$ in $S^{-1}N$

$\Rightarrow \left(\frac{m_1}{s_1}, \frac{n_1}{s_1}\right) = \left(\frac{m}{s}, \frac{n}{s}\right)$ in $S^{-1}M \times S^{-1}N$

φ is $S^{-1}R$-linear. (Check!) $\rightarrow$ $\varphi\left(\frac{r}{s'} \cdot \frac{(m,n)}{s}\right) = \varphi\left(\frac{(rm, rn)}{s's}\right)$ for $\frac{r}{s'} \in S^{-1}R$ & $\frac{(m,n)}{s} \in S^{-1}(M \times N)$

$= \left(\frac{rm}{s's}, \frac{rn}{s's}\right)$

$= \frac{r}{s'} \cdot \left(\frac{m}{s}, \frac{n}{s}\right)$

$= \frac{r}{s'} \varphi\left(\frac{(m,n)}{s}\right)$

$\psi : S^{-1}M \times S^{-1}N \longrightarrow S^{-1}(M \times N)$

$\left(\frac{m}{s}, \frac{n}{s'}\right) \longmapsto \frac{(s'm, sn)}{ss'}$

Check ψ is well-defined

Check φ & ψ are inverses to each other

$$\psi \circ \varphi\left(\frac{(m,n)}{s}\right) = \psi\left(\left(\frac{m}{s}, \frac{n}{s}\right)\right) = \frac{(sm, sn)}{s^2} = \frac{(m,n)}{s} \Rightarrow \psi \circ \varphi = id$$

$$\varphi \circ \psi\left(\frac{m}{s}, \frac{n}{s'}\right) = \varphi\left(\frac{(s'm, sn)}{s's}\right) = \left(\frac{s'm}{s's}, \frac{sn}{ss'}\right) = \left(\frac{m}{s}, \frac{n}{s'}\right) \Rightarrow \varphi \circ \psi = id.$$

(*) R, S as above, $I \subseteq R$ an ideal & $M = R/I$

$S^{-1}M \cong S^{-1}R / S^{-1}I$ as $S^{-1}R$-mod. $S^{-1}I = IS^{-1}R = \{\frac{x}{s} \mid x \in I, s \in S\}$

In particular if $I \cap S \neq \phi \Rightarrow S^{-1}(R/I) = 0$ $S = \{1, 3, 3^2, \dots\}$ $M = \mathbb{Z}/81\mathbb{Z}$

In fact more generally if N is a submod of an R-mod M. Then $\underline{S^{-1}(M/N) \cong S^{-1}M / S^{-1}N}$

Pf: $\varphi : S^{-1}M \longrightarrow S^{-1}(M/N)$

$$\frac{m}{s} \longmapsto \frac{\bar{m}}{s} \quad \text{where } \bar{m} = m + N \text{ in } M/N$$

φ is well-defined: $\frac{m}{s} = \frac{m'}{s'} \Rightarrow \exists u \in S$ s.t. $u(s'm - sm') = 0$ in M

$\Rightarrow u(s'\bar{m} - s\bar{m}') = 0$ in M/N

($\because q : M \to M/N$ is R-lin map)

$\Rightarrow \frac{\bar{m}}{s} = \frac{\bar{m}'}{s}$ in $S^{-1}(M/N)$

Check φ is $S^{-1}R$-lin (because q is)

and surjective by definition.

$$\ker(\varphi) = \{\tfrac{m}{s} \mid \tfrac{\bar{m}}{s} = \tfrac{0}{1} \text{ in } S^{-1}(M/N)\}$$

$$= \{\tfrac{m}{s} \mid \exists u \in S \ \ u\bar{m} = 0 \text{ in } M/N\}$$

$$= \{\tfrac{m}{s} \mid \exists u \in S \ \ um \in N\}$$

Claim: $S^{-1}N \xrightarrow{i} S^{-1}M$ $S \times M/\sim$ $S^{-1}R$-linear

$S \times N/\sim$ $\quad \frac{n}{s} \longmapsto \frac{n}{s}$ is injective, with image $\ker(\varphi)$.

Claim implies $S^{-1}N$ is an $S^{-1}R$-submod of $S^{-1}M$ and

By 1st isom thm $S^{-1}M / S^{-1}N \cong S^{-1}(M/N)$.

Pf of claim: $\frac{n}{s} = \frac{0}{1}$ in $S^{-1}M$ for $n \in N$ & $s \in S$

$\Rightarrow$ $\exists\, u \in S$ s.t. $un = 0$ in M.

$$\overset{\Downarrow}{un = 0} \text{ in } N$$

$\Rightarrow$ $\frac{n}{s} = \frac{0}{1}$ in $S^{-1}N$

Hence i is injective.

$i\left(\frac{n}{s}+\frac{n'}{s'}\right) = i\left(\frac{s'n+sn'}{ss'}\right) = \frac{s'n+sn'}{ss'} = \frac{n}{s}+\frac{n'}{s'}$

well-defined

i is well-defined R-linear is tautological.

For $n \in N$ and $s \in S$, $\frac{n}{s} \in \ker(\varphi)$ as $1 \cdot n \in N$

Take $u=1$

So $S^{-1}N = i(S^{-1}N) \subseteq \ker(\varphi)$.

Let $\frac{m}{s}$ be s.t. $um \in N$ for some $u \in S$ then

$\frac{m}{s} = \frac{um}{us} \in S^{-1}N$. Hence $\ker(\varphi) = S^{-1}N$ $\blacksquare$

(*) $\varphi: N \longrightarrow M$ an R-mod homo

then $S^{-1}\varphi: S^{-1}N \longrightarrow S^{-1}M$ is an $S^{-1}R$-mod homo

$$\frac{n}{s} \longmapsto \frac{\varphi(n)}{s}$$

φ inj $\Rightarrow$ $S^{-1}\varphi$ inj

φ surj $\Rightarrow$ $S^{-1}\varphi$ surj

Exc

Rank of an R-mod for R an integral domain.

Def^n: Let R be an int domain and M be an R-mod. Then $\operatorname{rank}(M) = \operatorname{vsdim}_{S^{-1}R}(S^{-1}M)$ where $S = R\setminus\{0\}$.

$= \dim_K(S^{-1}M)$ where $S^{-1}R = K = \operatorname{frac}(R)$ as vector space

Example: ① $R = \mathbb{Z}$, $M = \mathbb{Z}^2 \oplus \mathbb{Q} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$

$\operatorname{rank}(M) = 3$

$S = R\setminus\{0\}$

$$S^{-1}M \cong S^{-1}\mathbb{Z} \oplus S^{-1}\mathbb{Z} \oplus S^{-1}\mathbb{Q} \oplus S^{-1}\mathbb{Z}/15S^{-1}\mathbb{Z} \oplus \frac{S^{-1}\mathbb{Z}}{9S^{-1}\mathbb{Z}}$$

$$\cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} = \mathbb{Q}^3$$

$S^{-1}\mathbb{Q} \cong \mathbb{Q}$ check that this is an isom of $\mathbb{Q}$-vs.

$\alpha/s \longmapsto \frac{1}{s}\cdot\alpha$

$\dfrac{m/n}{s} = \dfrac{m}{ns}$ in $S^{-1}\mathbb{Q}$

② $R = \mathbb{Z}$, $M = 2\mathbb{Z}$, $\operatorname{rank}(M) = ?$

$S^{-1}M \cong \mathbb{Q}$ $\operatorname{rank}(M) = 1$

⊛ $I \subseteq R$ is a nonzero ideal of an int domain R then $\operatorname{rank}(I) = 1$

Universal propery of Localization

M an R-mod & $S \subseteq R$ mult subset

$\varphi : M \to S^{-1}M$ $\quad \varphi$ is R-linear $\quad S^{-1}M$ is $S^{-1}R$-mod

Let N be an $S^{-1}R$-mod then N has an R-mod via $r \cdot n = \frac{r}{1} \cdot n$

Let $\theta : M \to N$ which is R-linear then $\exists!$ $S^{-1}R$-lin map $\tilde{\theta} : S^{-1}M \to N$ s.t. $\tilde{\theta} \circ \varphi = \theta$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\theta"'] & S^{-1}M \arrow[d, dashed, "\tilde{\theta}"] \\ & N \end{tikzcd}$$

Pf: Claim: $\tilde{\theta}\left(\frac{m}{s}\right) := \frac{1}{s} \cdot \theta(m)$ is well-defined $S^{-1}R$-linear

$\frac{m}{s} = \frac{m'}{s'}$ in $S^{-1}M \Rightarrow \exists u \in S$ s.t. $u(s'm - sm') = 0$ in M

$\Rightarrow \theta(u(s'm - sm')) = 0$ in N

$\Rightarrow u(s'\theta(m) - s\theta(m')) = 0$ in N

$\overset{\frac{1}{u}}{\Rightarrow} s'\theta(m) - s\theta(m') = 0$ in N as u unit in $S^{-1}R$

$\overset{\cdot \frac{1}{ss'}}{\Rightarrow} \frac{1}{s}\theta(m) = \frac{1}{s'}\theta(m')$

check $\tilde{\theta}\left(\frac{m_1}{s_1} + \frac{r}{s}\frac{m_2}{s_2}\right) = \tilde{\theta}\left(\frac{m_1}{s_1}\right) + \frac{r}{s}\tilde{\theta}\left(\frac{m_2}{s_2}\right)$

check uniqueness $\alpha : S^{-1}M \to N$ s.t. $\alpha \circ \varphi = \theta$ then show $\alpha = \tilde{\theta}$ $\blacksquare$

Def^n: Let R be ring & M be an R-mod.

$$T(M) = \{ m \in M : \exists r \in R, r \neq 0 \text{ \& } rm = 0 \}$$

$T(M)$ is a submod of M if R is an int domain.

M is called torsion free R-mod if $T(M) = 0$.

(*) Let R be an int domain M an R-mod then $M/T(M)$ is torsion free.

Pf: $r \cdot \bar{m} = 0$ in $M/T(M)$ $\quad r \neq 0$

$\Rightarrow \overline{rm} = 0$ in $M/T(M)$

$\Rightarrow rm \in T(M)$

$\overset{\exists}{\Rightarrow} r' \in R, r' \neq 0$ s.t. $r'rm = 0$

$\Rightarrow m \in T(M$ $\quad$ ($\because r'r \neq 0$ as R int dom)

$\Rightarrow \bar{m} = 0$

(*) R an int dom M an R-mod s.t. $M = T(M)$ then $\text{rank}(M) = 0$.