Jacobson radical, nil radical

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Quiz 1: Elements of quotient rings, equivalence classes, etc.

$k[X]$, k a field, $I = (x-20)$
$= \{ f(x)(x-20) \mid f(x) \in k[X] \}$

$k[X]/I$ its elements are not elements of $k[X]$

$g(x) + f(x)(x-20) \in k[X]/I$ Doesn't make sense

$g(x)+I,\ f(x)+I \subseteq k[X]/I$ $\quad f(x) \sim g(x)$ if $f(x)-g(x) \in I$

$f(x), g(x) \in k[X]$

$k[X]/I \cong k$

$$k \subset \quad \varphi : k[X] \to k$$
$$f \mapsto f(20)$$
$$\ker(\varphi) = (x-20)k[X] = I$$

$n \in k[X]$
$n \pmod{x-20} \in k[X]/(x-20)$

$\bar{\varphi} : k[X]/I \to k$
$\quad \overset{\shortparallel}{k}$

$k[X]/(x-20) \neq k$
$\cong k$

$k[X]/(X) \neq k$

$$R = \frac{\mathbb{Z}[X]}{(x-1, x^2)} \qquad I = (x-1, x^2) = \mathbb{Z}[X]$$
$\varphi \downarrow$
$\mathbb{Z}$

$x(x-1) - x^2 \in I$
$-x \in I \Rightarrow -x + x - 1 \in I$
$\qquad\qquad\qquad\quad \overset{\shortparallel}{1}$

$0_R = x^2 + (x-1, x^2) = \overline{x^2} = q(x^2)$ where $q : \mathbb{Z}[X] \to \frac{\mathbb{Z}[X]}{(x-1, x^2)}$

$0 = \varphi(0_R) = \varphi(\overline{x^2}) = \varphi(\bar{x}\bar{x})$
$= \varphi(\bar{x})\varphi(\bar{x})$

$\bar{x} = q(x) = x + (x-1, x^2)$

$\Rightarrow \varphi(\bar{x}) = 0 \Rightarrow \varphi(1_R) = 0_R$

$1_R = 1 + (x-1, x^2)$
$= x + (x-1, x^2) = \bar{x}$

Last time: We talked about maximal ideals.

$\mathbb{Z}[X]$ $\quad (X)$ is prime but not maximal
$\qquad\quad (2)$ " " " " "

Comm ring with unity

* I, J ideals of R then $IJ = \{ a_1b_1 + \dots + a_nb_n \mid a_i \in I, b_i \in J, n \geq 1, 1 \leq i \leq n \}$

Q: $IJ \subseteq I \cap J$. Does equality hold?

Note: IJ is an ideal. $(IJ, +)$ is a group.
$r \in R$ & $\alpha \in IJ$ then $\alpha = a_1b_1 + \dots + a_nb_n$, $a_i \in I$ & $b_i \in J$
$\Rightarrow r\alpha = (ra_1)b_1 + \dots + (ra_n)b_n \in IJ$

* $IJ \subseteq I \cap J$ $\quad$ (ab where $a \in I$ & $b \in J$), then $ab \in I \cap J \Rightarrow IJ \subseteq I \cap J$.

$IJ \overset{?}{=} I \cap J$: Eg: $I = 2\mathbb{Z}$, $J = 2\mathbb{Z}$ $\quad I \cap J = 2\mathbb{Z}$, $IJ = 4\mathbb{Z}$
$\qquad I = 4\mathbb{Z}$, $J = 6\mathbb{Z}$ $\quad I \cap J = 12\mathbb{Z}$ $\quad IJ = 24\mathbb{Z}$

Jacobson radical: Let R be a nonzero comm ring with identity. The Jackson radical of R is defined to be intersection of all maximal ideals of R.

$$J(R) := \bigcap_{\substack{m \text{ maximal} \\ \text{ideal of } R}} m$$

Nil radical: $nil(R) := \{x \in R \mid x^n = 0\}$?
$=$ set of nilpotents of R

1) $nil(R)$ is an ideal of R.

2 $nil(R) \subseteq Jac(R)$

3 $x \in Jac(R) \iff 1 + ax$ is a unit for all $a \in R$

$x, y \in nil(R)$ & $r \in R$
$x^n = 0$ & $y^m = 0$ for some $n, m \geq 1$
$(rx)^n = 0$
$r^n x^n$

Binomial coeff

$$(x+y)^{n+m} = x^{n+m} + * x^{n+m-1} y + \dots + * x^n y^m + * x^{n-1} y^{m+1} + \dots$$
$$= 0$$

$\Rightarrow x + y \in nil(R)$.

2 $x \in nil(R) \Rightarrow x^n = 0 \in m$ for m any maximal ideal
$\Rightarrow x \in m$ ($\because m$ is a prime ideal
$x^n = x \cdot x^{n-1} \in m$

(In general P a prime ideal $a_1 \cdots a_n \in P$ then
$a_1 \in P$ or $a_2 \in P$ or $\dots$ or $a_n \in P$)

$$\Rightarrow x \in \bigcap_{\substack{m \text{ max ideal} \\ \text{of } R}} m \Rightarrow nil(R) \subseteq Jac(R)$$

In fact, $nil(R) \subseteq \bigcap_{P \text{ prime ideals of } R} P$.

⊗ $\text{nil}(R) = \bigcap_{P \text{ a prime ideals of } R} P$

Pf: $x \in \text{nil}(R) \Rightarrow x^n = 0$ for some n

$\Rightarrow x^n \in P$ for all prime ideal P.

$\Rightarrow x \in P \quad \forall P$ prime ideal of R

$\Rightarrow x \in \bigcap_{P \text{ a prime ideal of } R} P$

$\text{nil}(R) \subseteq \bigcap_{P \text{ prime ideal of } R} P$

$x \in \bigcap_{P \text{ prime ideal of } R} P$. WTS $x^n = 0$ for some n.

Suppose not

Let $S = \{1, x, x^2, x^3, \dots\}$ then $0 \notin S$

$\Omega = \{ I \subseteq R \mid I \text{ ideal s.t. } I \cap S = \phi \}$

Since $0 \notin S$ we have $\Omega \neq \phi$ $(\because (0) \in \Omega)$

Ω is a partially ordered set under inclusion

Let $\mathcal{C}$ be a chain in Ω.

Claim: Let $I_{\mathcal{C}} = \bigcup_{I \in \mathcal{C}} I$. Then $I_{\mathcal{C}}$ is an ideal.

Moreover $S \cap I_{\mathcal{C}} = \phi$.

$x, y \in I_{\mathcal{C}} \Rightarrow x \in I_1,\ y \in I_2,\ I_1, I_2 \in \mathcal{C}$

$\mathcal{C}$ a chain $\Rightarrow I_1 \subseteq I_2$ or $I_2 \subseteq I_1$

$\Rightarrow x + y \in I_{\mathcal{C}}$ & $rx \in I_{\mathcal{C}} \quad \forall r \in R$.

$\Rightarrow I_{\mathcal{C}}$ is an ideal.

Also $S \cap I = \phi \quad \forall I \in \mathcal{C}$

$\Rightarrow S \cap \left(\bigcup_{I \in \mathcal{C}} I \right) = \phi \Rightarrow S \cap I_{\mathcal{C}} = \phi$.

Hence by Zorn's lemma Ω has a maximal element m. caution $\not\Rightarrow (\because I \in \Omega \Rightarrow I \subseteq m)$

$\Rightarrow I \in \Omega$ & $m \subseteq I \Rightarrow m = I$

Claim: m is a prime ideal of R.

Claim $\Rightarrow$ $x \notin m_{A_i}$, contradicting $x \in \bigcap P$ (a prime ideal; P prime in R)

Pf of claim: Let

$ab \in m$ for $a, b \in R$. WTS $a \in m$ or $b \in m$

If $a \notin m$ & $b \notin m$ then

$aR + m \supsetneq m$ & $bR + m \supsetneq m$ $\Rightarrow$ $aR + m \in \Omega$ & $bR + m \in \Omega$

Hence $x^n \in aR + m$ for some n & $x^k \in bR + m$

for some k.

$\Rightarrow$ $x^n = r_1 a + y_1$ & $x^k = r_2 b + y_2$ for some $r_1, r_2 \in R$, $y_1, y_2 \in m$

$x^{n+k} = x^n x^k = (r_1 a + y_1)(r_2 b + y_2)$

$= \underbrace{r_1 r_2 ab}_{\in m} + \underbrace{y_1(r_2 b + y_2) + r_1 a y_2}_{\in m}$

Contradicting $m \cap S = \phi$.

Hence $a \in m$ or $b \in m$. Hence the claim.

$\blacksquare$

(*3) $x \in \mathrm{Jac}(R)$ iff $1 + ax$ is unit in R $\forall a \in R$.

($\Longrightarrow$): $ax \in \mathrm{Jac}(R)$

(*) Let R be a nonzero comm ring with unity. Let $I \subsetneq R$ ideal. Then $\exists$ a maximal ideal m of R containing I.

(Follows: Let $\overline{m}$ be a maximal ideal of R/I (this exist since R/I is a nonzero)

$m = q^{-1}(\overline{m})$ where $q : R \to R/I$

then $m \supseteq I$ & m is a max ideal of R ($\because$ ideals in R/I are in bijection with ideals of R containing I)

$$R/m \cong \frac{R/I}{m/I} \leftarrow \text{field}$$

Note $m/I = \overline{m}$)