We need one more notion:

Let $f \in \mathcal{B}(\mathcal{O}_n)$ ($\mathcal{O}_n$ open in $\mathbb{R}^n$). Let $x_0 \in \mathcal{O}_n$.

Define $osc(f, x_0)$ ~~$\lim$~~ ~~$\lim_{\delta \to 0}$~~ by:

Oscillation of f at x_0

Remark: $\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) = \sup_{x, y \in B_\delta(x_0)} (f(x) - f(y))$. — ($\ast\ast$)

$$osc(f, x_0) := \lim_{\delta \to 0} \left[\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) \right]. \quad \text{— } (\ast)$$

$\therefore$ osc is a fn. : $\mathcal{O}_n \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$osc(x) = osc(f, x)$. $\forall\, x \in \mathcal{O}_n$.

\# **Remark:** If $f \in \mathcal{B}(X)$, $X \subseteq \mathbb{R}^n$ (just a subset) & $x_0 \in X$, then define $osc(f, x_0) =$ $\ast$, BUT W.R.T (†) Subspace metric.

Note: (1) $\forall\, \delta > 0$, $\sup_{B_\delta(x_0)} f \geq \inf_{B_\delta(x_0)} f \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \geq 0$.

(†) So in ($\ast$), replace $B_\delta(x_0)$ by $X \cap B_\delta(x_0)$. So, $osc(f, x_0)$ is w.r.to subspace metric.

(2) Let $\delta_1 < \delta_2$. Then ($\ast\ast$) $\Rightarrow$ ~~f is bdd~~ ~~Moreover~~ ~~$\delta \downarrow \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \downarrow$~~

$$\sup_{x \in B_{\delta_1}(x_0)} f(x) - \inf_{x \in B_{\delta_1}(x_0)} f(x) \leq \sup_{x \in B_{\delta_2}(x_0)} f(x) - \inf_{x \in B_{\delta_2}(x_0)} f(x).$$

$\Rightarrow$ $osc(f, x_0)$ exists $\forall\, x_0 \in \mathcal{O}_n$ & $osc(f, x_0) \geq 0$. [$\because \delta \downarrow 0 \Rightarrow [\ ]$ in ($\ast$) $\downarrow$]

(3) f is cont. at $x_0 \iff osc(f, x_0) = 0$.

osc is a measure of discontinuity of f at points in $\mathcal{O}_n$.

[$\because \lim_{\delta \to 0} [\ ] = 0$]

Proof: "$\Rightarrow$" Let $osc(f, x_0) = 0$. Let $\varepsilon > 0$.

$\therefore \exists\, \delta > 0$ s.t. $\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) < \varepsilon$.

$$= \sup_{x, y \in B_\delta(x_0)} [f(x) - f(y)]$$

we know this. useful.

[Remark: Same proof w.r.to Subspace metric.]

$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall\, x, y \in B_\delta(x_0)$

In particular: $|f(x) - f(x_0)| < \varepsilon \quad \forall\, x \in B_\delta(x_0)$.

"$\Leftarrow$" Let $\varepsilon > 0$. $\exists\, \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{2} \quad \forall\, x \in B_\delta(x_0)$.

$\Rightarrow |f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(y) - f(x_0)| < \varepsilon$, $\forall\, x, y \in B_\delta(x_0)$.

$\Rightarrow \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \leq \varepsilon \Rightarrow osc(f, x_0) = 0$. ☒

(4) $osc(f, x_0) = \inf_{\delta>0} \left\{\sup |f(x) - f(y)| : x, y \in B_\delta(x_0)\right\}$. ← just observed.

(5) Let $\alpha > 0$ ~~Then~~ & let $f \in \mathcal{B}(C_n)$, $C_n \subseteq \mathbb{R}^n$ closed. Like $C_n = B^n$. But Assume $\operatorname{int}(C_n) \neq \varphi$ or invoke subspace metric in the proof.

Then $\{x \in C_n : osc(f, x) \geq \alpha\}$ is ~~open~~ closed

& $\{x \in C_n : osc(f, x) < \alpha\}$ is open in $\mathbb{R}^n_u$.

Proof. Let $C := \{x \in C_n : osc(f, x) \geq \alpha\}$. Claim: C is closed. We prove $\mathbb{R}^n \setminus C$ open.

Let $x \in \mathbb{R}^n \setminus C$.

$\Rightarrow \underbrace{x \notin C_n}_{\text{case 1}}$ or $\underbrace{x \in C_n \text{ but } x \notin C}_{\text{Case 2}}$.

$\Downarrow$ case 1

$\exists \delta > 0$ s.t. $B_\delta(x) \subseteq C_n^c$ [$\because C_n$ is closed].

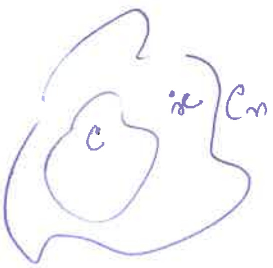
$\Rightarrow B_\delta(x) \subseteq C^c \Rightarrow$ ~~C^c open $\Rightarrow C$ closed~~.

Let $x \in C_n$ but $x \notin C$.

$\Rightarrow x \in C_n$ & $osc(f, x) < \alpha$

$\therefore \exists \delta > 0$ s.t. $\sup\left\{|f(z) - f(y)| : z, y \in B_\delta(x_0)\right\} < \alpha$.

~~(may be $\leq \alpha$ does not matter)~~



Consider an open box $B \subseteq B_\delta(x)$.

Let $y \in B$.

$\because y \in \underbrace{B}_{\text{open}}$, ~~then~~ $\exists \delta_1 > 0$ s.t. $B_{\delta_1}(y) \subseteq B_\delta(x)$.

$\therefore$ In particular: $\sup\left\{|f(z) - f(w)| : z, w \in B_{\delta_1}(y)\right\} < \alpha$

$f(w)$

$\Rightarrow osc(f, y) < \alpha$.

Thus, $\forall y \in \underbrace{B}_{\text{open}}$, $osc(f, y) < \alpha$. $\Rightarrow \underbrace{B}_{\text{open}} \subseteq \mathbb{R}^n \setminus C$.

$\Rightarrow C$ is closed. ☑

We need one observation:

Lemma: Let $K \subseteq \mathbb{R}^n$ be a compact set of measure zero. Let $\varepsilon > 0$. Then $\exists$ open boxes $B_1, \ldots, B_m$ (for some $m = m(\varepsilon)$) s.t.

$$\bigcup_{i=1}^{m} B_i \supset K \quad \& \quad \sum_{i=1}^{m} v(B_i) < \varepsilon.$$

Proof: Just compactness of K: Let $\varepsilon > 0$. Then $\exists$ boxes $\{B_i\}_{i=1}^{\infty}$ s.t. $\sum_{i=1}^{\infty} v(B_i) < \varepsilon$ & $\bigcup_{i=1}^{\infty} B_i \supseteq K$.

But K compact. $\Rightarrow \exists\, m \in \mathbb{N}$ s.t. $\bigcup_{i=1}^{m} B_i \supseteq K$. Clearly, $\sum_{i=1}^{m} v(B_i) \leq \sum_{i=1}^{\infty} v(B_i) < \varepsilon$. ☒

Remark: We can safely replace boxes by open/closed balls.

\# **Thm:** (Riemann - Lebesgue thm): Let $f \in \mathcal{B}(B^n)$. Then $f \in \mathcal{R}(B^n)$ $\Longleftrightarrow$ the set of discontinuity of f is of measure zero.

Proof: Set $\mathcal{D} := \{x \in B^n : f \text{ is not cont. at } x\}$.

$\therefore \mathcal{D} = \{x \in B^n : \operatorname{osc}(f, x) > 0\}$.

"$\Leftarrow$" Claim: $\mathcal{D}$ be of measure zero. [Assumption: $f \in \mathcal{R}(B^n)$].

$\forall m \in \mathbb{N}$, Set $\mathcal{D}_m = \{x \in B^n : \operatorname{osc}(f, x) \geq \frac{1}{m}\}$.

$\mathcal{D}_m$ closed in $\mathbb{R}^n$

$\therefore \mathcal{D}_m \downarrow$.

Note that: $\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}_m$.

So, enough to prove that $\mathcal{D}_m$ is of measure zero, $\forall m$.

$\boxed{\text{Fix } m \in \mathbb{N}.}$ Goal: $\mathcal{D}_m = \{x \in B^n : \operatorname{osc}(f, x) \geq \frac{1}{m}\}$ is of measure zero.

Let $\varepsilon > 0$. (fix it).

$\because f \in \mathcal{R}(B^n)$, $\exists$ P (~~a just P~~) a partition of B^n s.t.

$$\underbrace{U(f, P) - L(f, P)}_{\parallel} < \varepsilon.$$

i.e.,
$$\sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \varepsilon.$$

Note that: $\Lambda(P)$ is a finite set.

Let $\Lambda(P) := I \underset{\text{disjoint union}}{\sqcup} J$,

where $I = \{\alpha \in \Lambda(P) : \operatorname{int}(B^n_\alpha) \cap \mathcal{D}_m \neq \varphi\}$.

& $J = \{\alpha \in \Lambda(P) : \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m = \varphi\}$.

$\therefore \mathcal{D}_m \subseteq \left[\bigcup_{\alpha \in I} \operatorname{int}(B^n_\alpha)\right] \cup \underbrace{\left[\bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha\right]}$ —— (28.1)

the set of all boundaries of the sub-boxes B^n_α, $\alpha \in \Lambda(P)$.

~~Now for each $\alpha \in I$, as $\exists x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$~~

Let $\alpha \in I$. So $\exists x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$.

Fix it for time being.

$\therefore \operatorname{osc}(f, x) \geq \frac{1}{m}$.

$$\parallel$$

$$\inf_{\delta > 0} \left[\sup_{z, y \in B_\delta(x)} [f(z) - f(y)] \right] \longleftarrow (*)$$

$\because x \in \operatorname{int}(B^n_\alpha)$, $\exists$ $\delta > 0$ s.t. $B_\delta(x) \subseteq B^n_\alpha$.

$\because M_\alpha - m_\alpha = \sup_{z, y \in B^n_\alpha} [f(z) - f(y)]$, we have:

$$M_\alpha - m_\alpha \geq \frac{1}{m}.$$

Hence:

$$\varepsilon > \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B_\alpha^n)$$

$$\geq \sum_{\alpha \in I} (M_\alpha - m_\alpha)\, v(B_\alpha^n). \qquad [\because \Lambda(P) = I \amalg J]$$

$$\geq \sum_{\alpha \in I} \frac{1}{m} \times v(B_\alpha^n)$$

$$= \frac{1}{m} \times \sum_{\alpha \in I} v(B_\alpha^n).$$

$$\Rightarrow \sum_{\alpha \in I} v(B_\alpha^n) < m\,\varepsilon. \quad \text{———} \quad \text{(T)}$$

Now look at (28.1):

$$\mathcal{D}_m \subseteq \underbrace{\left[\bigcup_{\alpha \in I} \operatorname{int}(B_\alpha^n)\right]}_{\text{is of measure zero by (T)}} \cup \underbrace{\left[\bigcup_{\alpha \in \Lambda(P)} \partial B_\alpha^n\right]}_{\text{finite Union of boundaries of Sub-boxes.}}$$

finite Union of boundaries of Sub-boxes.
$\Downarrow$
measure zero. (HW).

$\Rightarrow \mathcal{D}_m$ is of measure zero.

This proves $f \in \mathcal{R}(B^n) \Rightarrow \mathcal{D}$ is of measure zero.