

Thm: (Gauss/divergence thm)

[Relates triple integrals \leftrightarrow surface integrals.]

Let $D \subseteq \mathbb{R}^3$ be a "solid". $\exists \cdot \partial D$ (the boundary of D) is an ~~piecewise~~ oriented surface, $F = \langle P, Q, R \rangle$ be a C^1 v.f. on an open set containing $D \cup \partial D (= \bar{D})$. Then

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \int_D \operatorname{div} \vec{F}$$

The Flux.

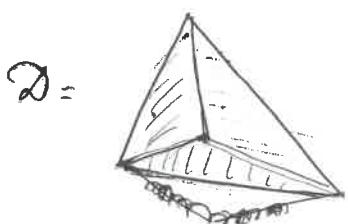
Note: (1) $\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

i.e. $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$ gradient dot product

Also, note that $\int_D \operatorname{div} \vec{F} = \int_D \operatorname{div} \vec{F} \underbrace{dV}_{\text{Volume.}} = \int \int \int_D \nabla \cdot \vec{F} dx dy dz$
a triple integral

$$\therefore \int_{\partial D} \vec{F} \cdot d\vec{s} = \int \int \int_D \operatorname{div} \vec{F} dV = \int \int \int_D \nabla \cdot \vec{F} dx dy dz$$

(2) ∂D could be piecewise ~~&~~ parametrized, i.e., piecewise smooth. For instance:



$$\partial D = \bigcup_{i=1}^4 T_i, \text{ where}$$

T_i are triangular face.

(3) In the above case, just consider parametrizations for each T_i ($i=1, 2, 3, 4$)

Proof:

Again, the general version is "beyond our scope".

Here we consider "elementary solid" like elementary region in Green's theorem proof.

(72)

$$\underline{\mathcal{D}} := \{(x, y, z) : \varphi_1(x, y) \leq z \leq \varphi_2(x, y), x \in [a, b], y \in [c, d]\}$$

vertically
convex.

$$\text{Goal: } \int_{\partial D} \vec{F} \cdot d\vec{S} = \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv$$

Like Green's thm, we prove: split

$$\partial D = \bigcup_{i=1}^6 S_i$$

S_i are faces of \mathcal{D} .
total 6

Suppose specified top.

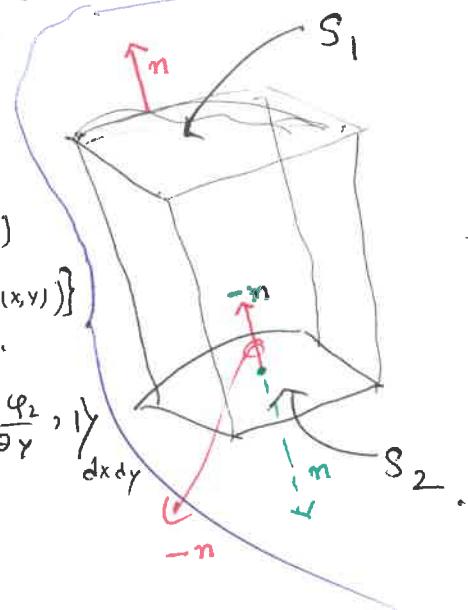
Also, again like Green's thm proof:

We prove $\int_{\partial D} \langle 0, 0, R \rangle \cdot d\vec{S} = \int_D \frac{\partial R}{\partial z} dv$ etc.

To this end, we proceed as follows:

Note: $\int_D \frac{\partial R}{\partial z} dv = \int_{y=c}^d \int_{x=a}^b \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R}{\partial z} dz dx dy = \int_c^d \int_a^b \left[R(x, y, \varphi_2(x, y)) - R(x, y, \varphi_1(x, y)) \right] dx dy$

Now $\int_{S_1} \langle 0, 0, R \rangle \cdot d\vec{S} = \int_{S_1} \int \langle 0, 0, R \rangle \cdot \left\langle -\frac{\partial \varphi_2}{\partial x}, -\frac{\partial \varphi_2}{\partial y}, 1 \right\rangle dx dy$
 The top.
 $z = \varphi_2(x, y)$:
 Graph surface. $= \int_c^d \int_a^b R(x, y, \varphi_2(x, y)) dx dy$.



likewise $\int_{S_2} \langle 0, 0, R \rangle \cdot d\vec{S} = - \int_{S_2} \int R(x, y, \varphi_1(x, y)) dx dy$.

AND $\int_{S_j} \langle 0, 0, R \rangle \cdot d\vec{S} = 0$ for S_j vertical wall Similar to Green's thm.

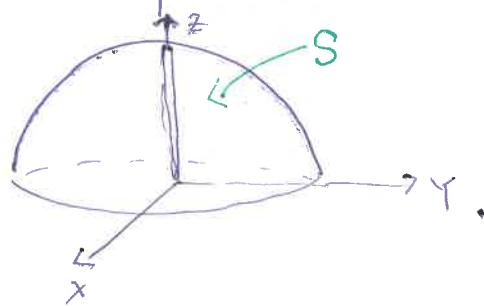
$$\begin{aligned} \therefore \int_S \langle 0, 0, R \rangle \cdot d\vec{S} &= \sum_{j=1}^4 \int_{S_j} \langle 0, 0, R \rangle \cdot d\vec{S} = \int_{S_1} + \int_{S_2} \\ &= \int_c^d \int_a^b \left\{ R(x, y, \varphi_2(x, y)) - R(x, y, \varphi_1(x, y)) \right\} dx dy = \int_D \frac{\partial R}{\partial z} dv. \end{aligned}$$

□

Eg: Compute $\int_S \vec{F} \cdot d\vec{S}$, where S is the hemisphere $x^2 + y^2 + z^2 = 1, z > 0$.

$$\vec{F}(x, y, z) = \langle x+y, z^2, xz \rangle.$$

Sol: We wish to apply Divergence Thm:



$$\int_{S \cup S_1} \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \operatorname{div} \vec{F} dV.$$

$$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \operatorname{div} \vec{F} dV - \int_{S_1} \vec{F} \cdot d\vec{S} \quad \text{①}$$

$$\begin{aligned} \text{Now } \int_{\mathcal{D}} \operatorname{div} \vec{F} dV &= \int_{\mathcal{D}} 1 \\ &= \text{Area } (\mathcal{D}) \\ &= \frac{2\pi}{3}. \end{aligned}$$

$$\therefore \int_{S_1} \vec{F} \cdot d\vec{S} = \int_{S_1} \vec{F} \cdot \vec{n} ds$$

$$\text{Here } \vec{n} = \langle 0, 0, 1 \rangle.$$

$$\begin{aligned} \therefore \int_{S_1} \vec{F} \cdot \vec{n} ds &= \int_{S_1} x^2 ds \\ &= \int_0^1 \int_0^{2\pi} u^2 \cos^2 v \, du \, dv \\ &= \int_0^1 \int_0^{2\pi} u^3 \cdot \cos^2 v \, du \, dv. \\ &\approx \dots = \underline{\frac{\pi}{4}}. \end{aligned}$$

$$\therefore \text{①} \Rightarrow \int_S \vec{F} \cdot d\vec{S} = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12}, \quad \text{②}$$

$$\begin{aligned} \mathcal{D} &= \text{Solid hemisphere,} \\ \&S_1 = \text{bottom of the } \mathcal{D} \\ &= \{(x, y, 0) : x^2 + y^2 \leq 1\}, \\ \vec{F} &= \langle x+y, z^2, xz \rangle. \end{aligned}$$

$$\begin{aligned} \because \operatorname{div} \vec{F} &= 1+0+0 \\ &= 1 \end{aligned}$$

* REMEMBER: orientation of $S \cup S_1$ is

However, orientation of S_1 alone is:

Recall: $\int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n} ds.$

S_1 is given by:

$$\begin{aligned} \vec{r}(u, v) &= (u \cos v, u \sin v, 0) \\ 0 \leq u \leq 1, & 0 \leq v \leq 2\pi. \end{aligned}$$

$$\begin{aligned} \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= \langle 0, 0, u \rangle. \end{aligned}$$

$$\Rightarrow \|\vec{r}_u \times \vec{r}_v\| = u$$

Theorem: Stokes theorem

Let C be a piecewise C^1 curve enclosing an oriented surface S in \mathbb{R}^3 . Suppose $\vec{F} = \langle P, Q, R \rangle$ be a C^1 -v.f. on an open set containing S . Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Remark: $\nabla \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$. So $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$

$R=0 \Rightarrow$ Green's thm. BUT: "↑" (i.e. graph surface).
 # Do the proof for $S: z = f(x, y)$ In true & full version:
 Here, we need to "use Green's thm"
 it goes other way around though!

Proof. Suppose $S = \text{graph } f = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}\}$, $f \in C^1$.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz . \quad \left[\because z = f(x, y) \Rightarrow dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right]$$

$$= \int_C \left(P + R \frac{\partial f}{\partial x} \right) dx + \left(Q + R \frac{\partial f}{\partial y} \right) dy . \quad \left[\tilde{C} = \{(x, y) : (x, y, z) \in C \text{ for some } z\} \right]$$

$$= \int_R \left(\frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) \right) dt . \quad \text{i.e.: } \tilde{C} \text{ is projection of } C \text{ onto XY-plane}$$

$$\text{Now } \frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) = \frac{\partial Q}{\partial x} + \frac{\partial^2 f}{\partial x \partial y} \quad \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = \frac{\partial P}{\partial y} - \frac{\partial^2 f}{\partial x \partial y} .$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_R$$

Note P, Q, R are fn's of $(x, y, z) \Rightarrow z = f(x, y)$.

\therefore In particular: if $\omega = P(x, y, z)$, then

$$\frac{\partial P}{\partial x} = \frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial P}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x} .$$

" " "

$$= \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} \cdot \frac{\partial f}{\partial x} .$$

* — $\boxed{\Rightarrow \frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} \frac{\partial f}{\partial x}}$

so $\frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) = Q_x + Q_z f_x + \cancel{R_x} \left(R_x + R_z f_x \right) f_y + R \frac{\partial^2 f}{\partial x \partial y} . \quad (1)$

By * $\cancel{Q} \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = P_y + P_z f_y + \left(R_y + R_z f_y \right) f_x + R f_{xy} . \quad (2)$

$\therefore (1) - (2) \Rightarrow$

$$\frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = f_x (Q_z - R_y) + f_y (R_x - P_z) + (Q_x - P_y)$$

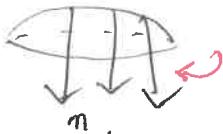
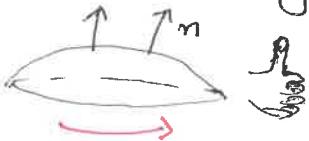
$\therefore \int_C \vec{F} \cdot d\vec{r} = \cancel{\oint_{C_R} \vec{F} \cdot d\vec{r}} \int_R f_x (Q_z - R_y) + f_y (R_x - P_z) + (Q_x - P_y)$

$$= \int_R \langle -f_x, -f_y, 1 \rangle \cdot \underbrace{\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle}_{\nabla \times \vec{F}}$$

$$= \int_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

BTW: How to figure out orientation of S vs. C ? RIGHT HAND RULE!!

place thumb along n of $S \Rightarrow$ The remaining 4 fingers will direct C .



$$\vec{F} = \langle -y, x, xy \rangle.$$

Eg: A) $\int_C \vec{F} \cdot d\vec{r}$, where $C: x^2 + y^2 = 9, z = 4$, oriented clockwise when viewed from the above.

Sol: $C: \vec{r}(t) = \langle 3\cos t, 3\sin t, 4 \rangle, 0 \leq t \leq 2\pi.$



$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

\because we need



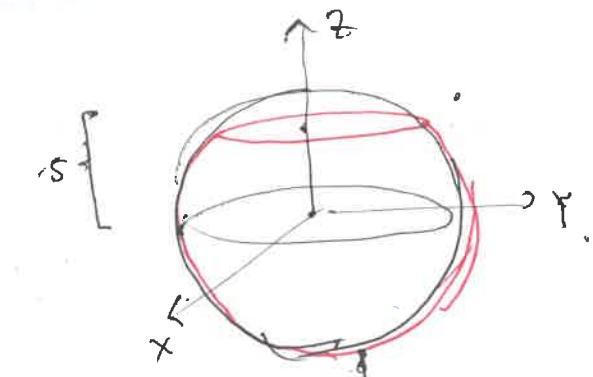
$$= \int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t \sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle dt$$

$$= \dots = -18\pi.$$

B)

$S =$ the surface: $x^2 + y^2 + z^2 = 25$, below the plane $z = 4$
oriented s.t. the unit normal vector at $(0, 0, -5)$ is $\langle 0, 0, -1 \rangle$.

Compute $\int_S \text{curl } \vec{F} \cdot d\vec{S}.$



Sol: By Stokes' Rul:

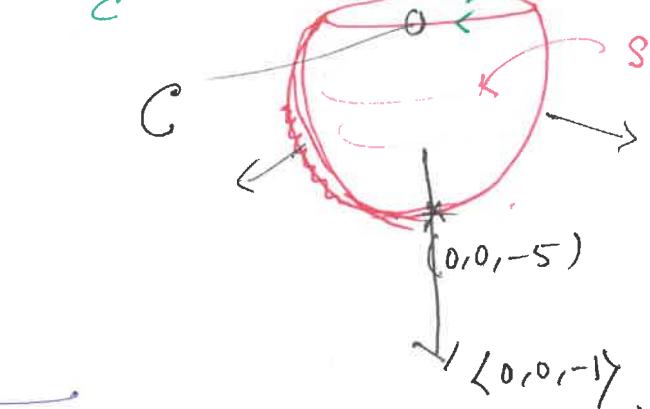
$$\int_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$$= -18\pi$$

(By A)

Stokes' thm. Simplifies!!

By Right hand rule.



A fairly simple

Inverse & implicit fn. theorems.

Recall the inverse fn. thm: Let $\Omega \subseteq \mathbb{R}$ open & $f: \Omega \rightarrow \mathbb{R}$ be a C^1 fn.

If $f'(x_0) \neq 0$ for some $x_0 \in \Omega$, then $\exists (a, b) \ni x_0$ s.t.

- (i) $(a, b) \subseteq \Omega$,
 - (ii) $f: (a, b) \rightarrow \underbrace{f(a, b)}$ is bijection.
an open interval.
 - (iii) $f^{-1}: f(a, b) \rightarrow (a, b)$ is diff. (C^1 fn.)
 - (iv) $(f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \forall x \in (a, b).$
-

— The \mathbb{R}^n -version (No proof: see Spivak's book) — .

Thm: (Inverse fn. thm).

Let $\Omega_n \subseteq \mathbb{R}^n$ be open & $f: \Omega_n \rightarrow \mathbb{R}^n$ be a C^1 -fn. If $Df(x_0)$ is invertible for some $x_0 \in \Omega_n$, then \exists an open neighbourhood $B_n \subseteq \Omega_n$ s.t. $x_0 \in B_n$ &

- i) $f: B_n \rightarrow \underbrace{f(B_n)}$ is bijection.
= open.
- ii) $f^{-1}: f(B_n) \rightarrow B_n$ is ~~diff~~ C^1 (\Rightarrow diff.).
- iii) $Df^{-1}(f(x)) = (Df(x))^{-1} \quad \forall x \in B_n.$

Moral of
the Story.

So, if $Df(x_0) \in M_n(\mathbb{R})$ inv. then locally f is invertible.

(\Rightarrow diffeomorphism).

— x —

Implicit fn. thm:

Motivation:

We are mostly familiar with fn's of the form

$$y = f(x), \quad z = f(x, y), \quad x_{n+1} = f(x_1, \dots, x_n) \text{ etc.}$$

Like: $y = \sin x$, $z = xy + y^2$ etc.

$$\underbrace{y - \sin x}_0 = 0 \quad \underbrace{z - xy - y^2}_0 = 0.$$

What about "writing" " $y = f(x)$ " if we have: $F(x, y) = 0$?

i.e.: if F is a fn. of (x, y) & if $F(x, y) = 0$

*Call it implicit fn.
or solution.* \iff $y = f(x)$ for some f ? i.e. $F(x, f(x)) = 0$?
[Also, we wish the solution f to be diff.].

Of Course, this is a question of general interest .

e.g.: Suppose $F(x, y) = ax + by + c$.

$$\text{Now } ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b} \quad (\text{if } b \neq 0).$$

$$\text{So, here } f(x) = -\frac{a}{b}x - \frac{c}{b} \quad \forall x.$$

Note: $\frac{\partial F}{\partial y} = b$. So $\frac{\partial F}{\partial y} (= b) \neq 0 \Rightarrow F(x, f(x)) = 0$

for some diff. f !!

Trivial but impressive observation. \rightarrow So $\frac{\partial F}{\partial y} \neq 0 \Rightarrow \exists$ diff. fn. f s.t. $F(x, f(x)) = 0$.

e.g.: $F(x, y) = x^2 + y^2 - 1$.

We know $F(x, y) = 0 \Leftrightarrow (x, y)$ in $C_1(0)$.

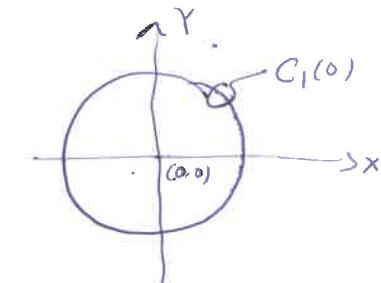
$$\text{Note that } y = \pm \sqrt{1-x^2}.$$

$\Rightarrow y = f(x)$ for (even any fn.) f is NOT POSSIBLE !!

However: if $y > 0$, then $y = f(x) = \sqrt{1-x^2}$
 \nexists if $y < 0$ — $y = f(x) = -\sqrt{1-x^2}$.

~~No such f~~ \therefore If $y > 0$ (or $y < 0$), then \exists f s.t.
 $F(x, f(x)) = 0$.

Note $\frac{\partial F}{\partial y} = 2y$. \nexists so if $y > 0$ (or $y < 0$), then
 $\frac{\partial F}{\partial y} \neq 0$.



Ex 3: Suppose $F(x, y) = 0$ in a nbh of $(a, b) \in \mathbb{R}^2$.

Suppose $y = f(x)$ be a diff / C^1 solution to this:

$$F(x, f(x)) = 0.$$

implicit solution to F .

$$\Rightarrow \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0. \quad [C^1 \text{ chain rule}]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}} \quad [\text{if } \frac{\partial F}{\partial y} \neq 0] \quad \underline{\text{at } (a, b)}$$

Again: $\frac{\partial F}{\partial y} \neq 0$ plays important role here.

This also says: if $y = f(x)$ is a C^1 solution to $F(x, y) = 0$ near (a, b) , then $\frac{\partial F}{\partial y} \neq 0$ at (a, b) is necessary to recover $\frac{dy}{dx}$!!.

i.e. derivative of F can be computed by differentiating

$$F(x, f(x)) = 0. \quad \cancel{\text{by implicit diff}}$$

~~Thm: Let $\Omega \subseteq \mathbb{R}^{n+m}$ be open, $F: \Omega \rightarrow \mathbb{R}^m$ a C^1 -fn. Suppose $(a, b) \in \Omega$, $F(a, b) = 0$ & $\det \left(\frac{\partial f_i}{\partial y_j} \right)_{i, j=1}^m \neq 0$ at (a, b) .~~

Setting: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $n, m \geq 1$. $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. ~~fixed~~ Let

$(a, b) \in \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$. $F: \Omega \rightarrow \mathbb{R}^m$ & $F = (f_1, \dots, f_m)$.

& $f_i: \Omega \rightarrow \mathbb{R}$ & $i = 1, \dots, m$. Finally, $f_i(x, y) = f_i(\underbrace{x_1, \dots, x_n}_{\in \mathbb{R}^n}, \underbrace{y_1, \dots, y_m}_{\in \mathbb{R}^m})$.

Thm: (Implicit fn. thm): Suppose $F \in C^1(\Omega)$ & $F(a, b) = 0$.

If $\boxed{\det \left(\frac{\partial f_i}{\partial y_j}(a, b) \right)_{i, j=1}^m \neq 0}$, then \exists open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ $\downarrow_a \uparrow_b$

& a C^1 -fn $f: U \rightarrow V$ s.t.

$$F(x, f(x)) = 0 \quad \forall x \in U.$$

Pruf: Define $\tilde{F}: \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ by

$$\tilde{F}(x, y) = (x, F(x, y)), \quad \forall (x, y) \in \Theta.$$

$$\text{Now } \tilde{J}_F(a, b) = \begin{bmatrix} I_{n \times n} & O \\ O & J_F(a, b) \end{bmatrix} \Rightarrow \det[\tilde{J}_F(a, b)] \neq 0.$$

$\therefore B_p$ Inverse fn. thm: $\exists U_0 \subseteq \Theta, V_0 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ open sets
 s.t. $(a, b) \in U_0, (a, 0) \in V_0 \quad \checkmark$
 $\widetilde{\widetilde{F}}(a, b)$

$\tilde{F}: U_0 \rightarrow V_0$. has a differentiable inverse.

Note: $\underline{U_0 = A \times B}$ for $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ open.

We can consider
 Call: $U = A$. [\because Boxes forms ~~at~~ basic open sets.]

Call: $U = A$.

Clearly, $\tilde{F}^{-1}\left(\frac{x}{y}, y\right) = (x, g(x, y))$ for some diff. g.
 $\therefore \tilde{F}\left(x, y\right) = (x, F(x, y))$.

$$\text{Now } \Pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \quad \left\{ \begin{array}{l} (x, y) \mapsto y \end{array} \right\} \Rightarrow \Pi \circ \tilde{F} = F.$$

$$\therefore F(x, g(x, y)) = (\text{for } g) \rightsquigarrow F \circ \tilde{F}^{-1}(x, y)$$

$$= \tilde{\pi}_0 \tilde{F}_0 \tilde{F}^{-1}(x, y)$$

$$= \pi(x, y)$$

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$\nexists (x, y) \in A \times B$.

$$\Rightarrow F(x, g(x_0)) = 0 .$$

So, define $f(x) := g(x_0)$. $\forall x \in A = U$. \square

\leftarrow 