Some examples:

(1) $$\iint_{B^2} \underbrace{\sin(x+y)}_{f(x,y)} \, dA = ? \quad ; \quad \text{where } B^2 = [0,\pi] \times [-\pi/2, \pi/2].$$

dA — Convention: refers to area.

$\int_{B^2}$ Convention.

$dx\,dy \,/\, dy\,dx$.

i.e. Compute $\int_{B^2} \sin(x+y)$

Sol:

$\sin(x+y) = \sin x \cos y + \cos x \sin y$. $\leftarrow$ Cont (B^2).

$$\therefore \int_{B^2} \sin(x+y) = \cancel{\int_0^{\pi} \left(\underbrace{\int_{-\pi/2}^{\pi/2} (\sin x \cos y + \cos x \sin y)\, dy}_{\pi} \right) dx}$$

$$\cancel{\sin x \int \cos y + \cos x \int \sin y}$$

$$= \int_{B^2} \sin x \cos y + \int_{B^2} \cos x \sin y.$$

$$= \int_0^{\pi} \sin x \, dx \times \int_{-\pi/2}^{\pi/2} \cos y \, dy + \underbrace{\int_0^{\pi} \cos x \, dx}_{=0} \times \int_{-\pi/2}^{\pi/2} \sin y \, dy.$$

$$= 2 \times 2 + 0 = 4.$$

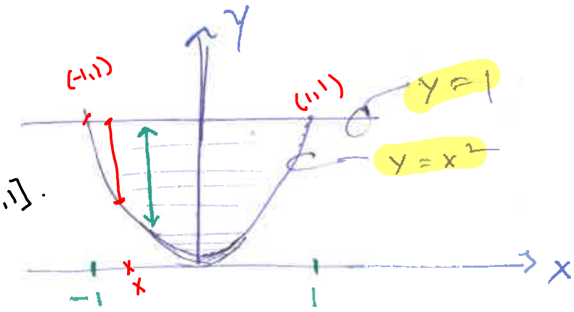
$B^2 = B_1 \times B_2$

$$\int_{B^2} f(x)\, g(y) = \int_{B_1} f \times \int_{B_2} g$$

(2) Ω = region bounded by $y = 1$ & $y = x^2$. Compute

$$\int_{\Omega} x^2 y.$$

Here $\Omega = \left\{ (x,y) : \begin{array}{l} -1 \le x \le 1 \\ x^2 \le y \le 1 \end{array} \right\}$



So $\displaystyle\int_\Omega f = -\int_{x=-1}^{1} \boxed{\int_{y=1}^{y=x^2} x^2 y} \underset{\text{Area.}}{dA}$ → should exists $\forall x \in [-1,1]$.

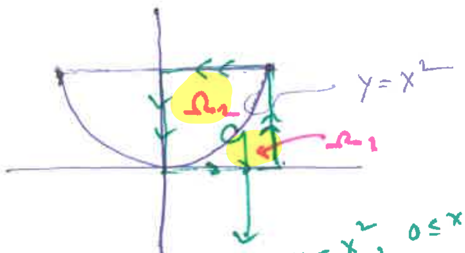
$$= -\int_{-1}^{1} x^2 \left(\int_{y=1}^{y=x^2} y\, dy \right) dx. \quad = -\int_{-1}^{1} x^2 \times \frac{1}{2}\left(x^4 - 1\right) dx.$$

$$= -\frac{1}{2}\int_{-1}^{1} (x^6 - x^2)\, dx. \quad = -\frac{1}{2} \times 2 \times \left(\frac{1}{7} - \frac{1}{3}\right)$$

$$= \frac{4}{21}. \quad \text{Ans.}$$

(3) Let $f(x,y) = \begin{cases} x & \text{if } y \le x^2 \\ y & \text{if } y > x^2. \end{cases}$ $\qquad [0,1] \times [0,1] = B^2.$

Compute $\displaystyle\iint_{B^2} f$.



Sol: Write $B^2 = \Omega_1 \cup \Omega_2$.

\# Along $y = x^2$, $0 \le x \le 1$, f is NOT cont. But it is of content zero.

Where $\Omega_1 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$

& $\Omega_2 = \{(x,y) : 0 \le x \le 1,\ x^2 < y \le 1\}$.

Now $f|_{\Omega_1}$ is cont. & $(f|_{\Omega_1})(x,y) = x$. So $f|_{\Omega_1} \in R(\Omega_1)$.

But $\left(f|_{\Omega_2}\right)(x,y) = \begin{cases} x & \text{if } y = x^2 \\ y & \text{if } x^2 < y \end{cases}$. ⟵ f along the curve.

$\Rightarrow f|_{\Omega_2}$ is not cont. along $y = x^2$, $0 \le x \le 1$.

But $f|_{\Omega_2}$ is cont. in the int (Ω_2).

$\Rightarrow f|_{\Omega_2} \in \mathcal{R}(\Omega_2)$.

$$\therefore \int_{B^2} f = \underbrace{\int_{\Omega_1} f}_{\text{Easy}} + \int_{\Omega_2} f .$$

⟵ How? Why? It is true but need a proof. – wait – ?

$$\int_{x=0}^{1} x\left(\int_{y=0}^{x^2} dy\right)dx = \int_0^1 x^3\,dx = \frac{1}{4}.$$

Where: $\displaystyle\int_{\Omega_2} f = \int_0^1\left(\int_{x^2}^1 y\,dy\right)dx = \frac{1}{2}\int_0^1 (1-x^4)\,dx$

$= \cdots = \frac{2}{5}$.

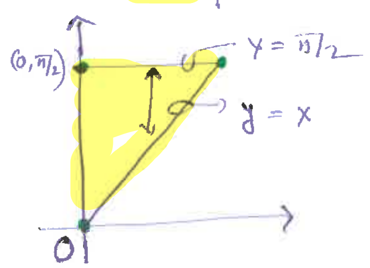
$$\therefore \int_{B^2} f = \frac{1}{4} + \frac{2}{5} = \frac{13}{20}.$$

(4) $\displaystyle\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\,dy\,dx = ?$ ⟵ Wow! Integrating $\frac{\sin y}{\cancel{y}}$??

$\in \mathcal{R}[\,?\,]$. At least, it is bdd (Right?).

We check the Ω first: $0 \le x \le \pi/2$, $x \le y \le \pi/2$.

So



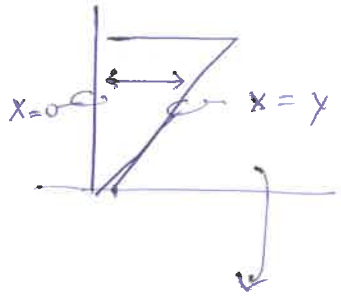
Clearly $f(x,y) = \frac{\sin y}{y} \in \mathcal{R}(\Omega)$.

Where $\Omega = \{(x,y) : 0 \le x \le \pi/2 \;\&\; x \le y \le \pi/2\}$.

But how to evaluate?

By changing the order of integration (if we are lucky):

So we apply Fubini:



$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx = \int_{\Omega} \frac{\sin y}{y}\, dA$$

$$= \int_0^{\pi/2} \left(\int_0^{y} \frac{\sin y}{y}\, dx \right) dy$$

$$= \int_0^{\pi/2} \left(\frac{\sin y}{y} \underbrace{\int_0^{y} dx}_{= y} \right) dy$$

$$= \int_0^{\pi/2} \frac{\sin y}{y} \times y \cdot dy \qquad \longleftarrow \text{one point trouble.}$$

$$= \int_0^{\pi/2} \sin y\, dy = 1. \qquad \boxed{//}$$

So changing the order of integration is a good deal (if you are lucky).

— x —

Q: But $\int_0^{\alpha} \frac{\sin x}{x}\, dx$, $\alpha > 0$, make sense?

$\frac{1}{x} \times [x + \cdots x^3 + \cdots x^5 + \cdots x^7 + \cdots]$
$= 1 + \cdots x^2 + \cdots x^4 + \cdots$

Ans: Well, $x \longmapsto \frac{\sin x}{x}$ is cont. $\forall x \neq 0$.

Now $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

$\therefore$ We assign $\left(\frac{\sin x}{x}\right)(0) = 1$. $\rightsquigarrow$ turns $\frac{\sin x}{x} \in C[0,\alpha]$.

Finally, it is cont. $\forall x$, possibly except $\underbrace{x = 0}$. measure zero.

$\therefore \frac{\sin x}{x} \in \mathcal{R}([0,\alpha])$, $\forall \alpha > 0$.

need a def.

Q: Let $\Omega \subseteq \mathbb{R}^2$. What is the "area" of Ω? Or, Can we Compute or talk about area of $\boxed{\text{any}}$ $\Omega \subseteq \mathbb{R}^2$?

$\|^{ly}$ Volume of $\Omega \subseteq \mathbb{R}^3$?

· # Trouble is: how to define area !! Of course,

$$\text{Area}\left([a,b] \times [c,d]\right) := (b-a) \times (d-c).$$

By defn.

Remark: One (good way) to get the R.H.S. as follows:

$$\int_{B^2} 1 = \int_a^b \left(\int_c^d 1 \, dy \right) dx \qquad \left(\begin{array}{l} B^2 = [b-a] \times [d-c] \\ 1 \equiv \text{Constant fn } 1 \end{array} \right).$$

$1 \in C(B^2)$. So Fubini

$$= \left(\int_a^b 1 \, dx \right) \times \left(\int_c^d 1 \, dy \right)$$

$$= (b-a) \times (d-c).$$

We adopt the above as the defn. of "Area".

Def. ① Let $\Omega \subseteq \mathbb{R}^2$ be a bounded subset. Define

$$1_\Omega : \Omega \to \mathbb{R} \text{ by } 1_\Omega(x) = 1 \quad \forall x \in \Omega.$$

Write 1 instead of 1_Ω if Ω is clear from the context.

For $f \in \mathcal{B}(\Omega)$, & $B^2 \supseteq \Omega$,
$$\tilde{f}(x) = \begin{cases} f(x) & \forall x \in \Omega \\ 0 & \forall x \in B^2 \setminus \Omega. \end{cases}$$

② We say that Ω has an area if $1_\Omega \in \mathcal{R}(\Omega)$.

And, in this case, we define

$$\text{Area}(\Omega) := A(\Omega) := \int_\Omega 1_\Omega.$$

Of course, $\int_\Omega 1 = \int_{B^2} \tilde{1}$, where $\tilde{1}|_\Omega = 1$ & $\tilde{1}|_{B^2 \setminus \Omega} \equiv 0$. [By the def. of $\int_\Omega$.]

$\tilde{1}$: r_f

Note: Set $\Omega := \{(x,y) : 0 \le x, y \le 1,\ x, y \in \mathbb{Q}\} = [0,1]\times[0,1] \cap (\mathbb{Q}\times\mathbb{Q})$

Then $\mathbb{1} \notin \mathcal{R}(\Omega)$. $\Omega \to$ X area.

Proof: Recall $\mathbb{1} \in \mathcal{R}(\Omega)$ if $\tilde{\mathbb{1}} \in \mathcal{R}(B^2)$, $B^2 = [0,1]\times[0,1]$ $(\because B^2 \supseteq \Omega)$.

& in that case $\int_\Omega \mathbb{1} = \int_{B^2} \tilde{\mathbb{1}}$.

~~But we prove that $\mathbb{1} \notin \mathcal{R}(\Omega)$.~~ ← Same proof as that of the Dirichlet fn. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}^c \end{cases}$.

So i.e; we prove that $\tilde{\mathbb{1}} \notin \mathcal{R}(B^2)$. Enough. Right?

$\tilde{\mathbb{1}}(x,y) = \begin{cases} 1 & (x,y) \in \mathbb{Q}\times\mathbb{Q} \\ 0 & \text{if } (x,y) \notin \mathbb{Q}\times\mathbb{Q} \end{cases}$

So, take any partition $P \in \mathcal{P}(B^2)$.

Use that $B^2 \cap (\mathbb{Q}\times\mathbb{Q})$ is dense in B^2 & get:

$$L(\tilde{\mathbb{1}}, P) = 0 \quad \& \quad U(\tilde{\mathbb{1}}, P) = \sum_{\alpha \in \Lambda(P)} v(B^2_\alpha) = v(B^2) \quad (\neq 0).$$

$$\Rightarrow \underline{\int_{B^2}} \tilde{\mathbb{1}} = 0 \neq v(B^2) = \overline{\int_{B^2}} \tilde{\mathbb{1}}.$$

$$\therefore \tilde{\mathbb{1}} \notin \mathcal{R}(B^2) \Rightarrow \mathbb{1} \notin \mathcal{R}(\Omega).$$

Remark: From the above, & the fact that $\mathbb{1} \in \mathcal{R}(B^2)$, it is evident that integrability of a fn. also closely related with the nature of Ω.

And of course, nature of Ω should tell us if it has any area!! ← So it is not free.

BUT WAIT: My intution says that $\Omega = (\mathbb{Q}\times\mathbb{Q}) \cap ([0,1]\times[0,1])$ should be of "area $= 0$". In reality, we are saying the area DNE!!

THIS IS BAD!! — But, the trouble is the fact that Content of $\Omega \neq 0$. AND The measure of $\Omega = 0$!! So, measuring sets (i.e. Area) like Ω is GOOD in Lebesgue integration!

Remark: ① Given $\Omega \subseteq \mathbb{R}^2$, Ω has an area $\Leftrightarrow$ $\chi_\Omega \in \mathcal{R}(B^2)$ for some box $B^2 \supseteq \Omega$. In this case,

$$\text{Area}(\Omega) = \int_{B^2} \chi_\Omega .$$

[Def. $\chi_\Omega : B^2 \to \{0,1\}$ where
$$\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega . \end{cases}$$
indicator / Characteristic fn.]

Proof: $\tilde{1}_\Omega = \chi_\Omega$.

② $\text{I}\!\text{I}^{ly}$ you may define / deduce, for $\Omega \subseteq \mathbb{R}^n$,

$$\underset{\text{Volume of } \Omega.}{\text{Vol}(\Omega)} = \int_{B^n} \chi_\Omega . \qquad B^n \supseteq \Omega.$$

Thm: Let $\Omega \subseteq \mathbb{R}^n$ be bdd. Then Ω has a $\boxed{\text{volume}}$ $\Leftrightarrow$ $\partial\Omega$ is of content zero.

$\Leftarrow$ Lets do it for general $n \geq 2$.

Proof: "$\Leftarrow$" Suppose $\partial\Omega$ has content zero. Set $f := \underbrace{\tilde{1}_\Omega}_{\text{Corresponding to } B^n \supseteq \Omega.} = \chi_\Omega$.

Clearly, f is Cont. on Ω ($\because f|_\Omega \equiv 1$).

Arguing along the same line of proof of thm in P-41:

$$\mathcal{D} := \{ x \in B^n : f \text{ is not cont. at } x \}.$$

Enough to prove that $\mathcal{D}$ is of measure zero.

So: (i) $f|_\Omega$ is Cont. on Ω.

(ii) $f|_{B^n \setminus \bar{\Omega}} \equiv 0$ is Cont. on $B^n \setminus \bar{\Omega}$.

$\Rightarrow \mathcal{D} \subseteq \partial\Omega$

$\because \partial\Omega$ is of content zero. $\Rightarrow \mathcal{D}$ is of Content zero.

$\Rightarrow f \in \mathcal{R}(B^n)$ i.e., $\chi_\Omega \in \mathcal{R}(B^n)$.

i.e., Ω has a volume.

"$\Rightarrow$" Let $B^n \supseteq \Omega$ & $\chi_\Omega = \tilde{1}_\Omega \in \mathcal{R}(B^n)$. Again: $f := \chi_\Omega$.

Claim: $\partial\Omega$ is of content zero.

Fix $\varepsilon > 0$. $\because f \in \mathcal{R}(B^n)$, $\exists\, P \in \mathcal{P}(B^n)$ s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2}. \qquad \leftarrow \text{By integrability of } f.$$

$$\Rightarrow \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \frac{\varepsilon}{2}.$$

$\tilde{\Lambda} := \{ \alpha \in \Lambda(P) : B_\alpha^n \cap \Omega \neq \varphi \text{ \& } B_\alpha^n \not\subseteq \Omega \}$

Not contained in.

Hint: if $x \in \partial\Omega$, then any open set $O \ni x$, $O \cap \Omega \neq \varphi$ & $O \not\subseteq \Omega$.

The point is: $M_\alpha = 1$, $m_\alpha = 0$ $\forall \alpha \in \tilde{\Lambda}$.

$$\therefore \underbrace{\sum_{\alpha \in \tilde{\Lambda}} (M_\alpha - m_\alpha)\, v(B_\alpha^n)}_{= \sum_{\alpha \in \tilde{\Lambda}} v(B_\alpha^n)} \leq \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B_\alpha^n) < \frac{\varepsilon}{2}.$$

$$\Rightarrow \sum_{\alpha \in \tilde{\Lambda}} v(B_\alpha^n) < \frac{\varepsilon}{2}. \quad \text{——— (1)}$$

On the other hand, ∂B_α^n is of content zero $\forall \alpha \in \Lambda(P)$. [Known fact.]

$\Rightarrow \bigcup_{\alpha \in \Lambda(P)} \partial B_\alpha^n$ is of content zero. ($\because$ finite union of c.z. set is of c.z.)

$\therefore \exists$ boxes $\cancel{\{P_\beta^n : \beta \in \Gamma\}}$ $\{B_1^n, \ldots, B_p^n\}$ $\ni$ $\bigcup_{j=1}^{p} B_j^n \supseteq \bigcup_{\alpha \in \Lambda(P)} \partial B_\alpha^n$

& $\sum_{j=1}^{p} v(B_j^n) < \frac{\varepsilon}{2}$. ——— (2)

Claim: $$\partial\Omega \subseteq \underbrace{\left(\bigcup_{\alpha \in \tilde{\Lambda}} B_\alpha^n \right)}_{\mathrm{I}} \cup \underbrace{\left(\bigcup_{j=1}^{p} B_j^n \right)}_{\mathrm{II}}.$$

Content zero

$\Downarrow$

$\partial\Omega$ is of Content zero. by (1) & (2).

[$\because \sum_{j=1}^{p} v(B_j^n) + \sum_{\alpha \in \tilde{\Lambda}} v(B_\alpha^n) < \varepsilon$.]

AND we are done!!

Proof of the Claim:

Pick $x \in \partial\Omega \subseteq B^n$.

$\therefore x \in B_\alpha^n$ for some $\alpha \in \Lambda(P)$. $\Rightarrow$ $x \in \operatorname{int}(B_\alpha^n)$ OR $x \in \partial B_\alpha^n$.

If $x \in \operatorname{int}(B_\alpha^n)$: As $x \in \partial\Omega$ ~~& $\operatorname{int}(B_\alpha^n)$ open~~ $\operatorname{int}(B_\alpha^n)$ also contains elements not in Ω [By the def. of bd points].

$\Rightarrow B_\alpha^n \cap \Omega \neq \varphi$ & $B_\alpha^n \not\subseteq \Omega$.

$\Rightarrow \alpha \in \tilde{\Lambda}$. $\Rightarrow x \in \mathrm{I}$.

If $x \in \partial B_\alpha^n$: Then $\partial B_\alpha^n \subseteq \bigcup_{j=1}^{p} B_j^n$. $\Rightarrow x \in$ (II).

$\therefore$ The claim holds good. ▨

Fact: Suppose $\Omega \subseteq \mathbb{R}^n$ is of content zero & $f \in \mathcal{B}(\Omega)$. Then $f \in \mathcal{R}(\Omega)$ & $\int_\Omega f = 0$. [Already done : P-39.]

Thm: Suppose $\Omega \subseteq \mathbb{R}^n$ bdd. Then :

Ω has an Volume ~~area~~ & ~~Ar~~ $\mathrm{vol}(\Omega) = 0 \Leftrightarrow \Omega$ is of content zero.

Proof: "$\Rightarrow$" So, $\int_{B^n} \chi_\Omega = 0$. Let $\varepsilon > 0$.

$$\Downarrow$$

$$0 = \overline{\int_{B^n}} \chi_\Omega = \inf \{ U(\chi_\Omega, P) : P \in \mathcal{P}(B^n) \}$$

$\therefore \exists P \in \mathcal{P}(B^n) \ni U(\chi_\Omega, P) < \varepsilon$.

Set $\tilde{\Lambda} := \{ \alpha \in \Lambda(P) : B_\alpha^n \cap \Omega \neq \varphi \}$.

Clearly, for $\alpha \in \Lambda(P)$, $\alpha \in \tilde{\Lambda} \Leftrightarrow M_\alpha = 1$.

Also, $M_\alpha = 0 \ \forall \alpha \notin \tilde{\Lambda}$.

So
$$\varepsilon > U(\chi_\Omega, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha v(B_\alpha^n) = \sum_{\alpha \in \tilde{\Lambda}} M_\alpha v(B_\alpha^n)$$
$$= \sum_{\alpha \in \tilde{\Lambda}} v(B_\alpha^n).$$

Also, Since $\{B_\alpha^n : \alpha \in \Lambda(P)\}$ a partition of $B^n \supseteq \Omega$,

So $\{B_\alpha^n : \alpha \in \tilde{\Lambda}\}$ is a finite Cover ~~partition~~ of Ω & $\sum_{\alpha \in \tilde{\Lambda}} v(B_\alpha^n) < \varepsilon$. $\Rightarrow \Omega$ is of Content zero.

"$\Leftarrow$" Let Ω is of Content zero. Then the above fact $\Rightarrow \chi_\Omega \in \mathcal{R}(\Omega)$ & $v(\Omega) = \int \chi_\Omega = 0$. ▨

Note: Let $\Omega_1 \subseteq \Omega$. Let $f \in \mathcal{R}(\Omega)$. We know $f|_{\Omega_1}$ need not be in $\mathcal{R}(\Omega_1)$.

[Simple example: $\Omega = [0,1] \times [0,1]$; $\Omega_1 = (\mathbb{Q} \times \mathbb{Q}) \cap \Omega$. $f \equiv 1$. As we know Ω_1 ~~is not~~ does not have area.].

However, the following is impressive:

Thm: Let $\Omega_1 \subseteq \underbrace{\Omega}_{\text{bdd}} \subseteq \mathbb{R}^n$, and let $\partial\Omega_1$ is of content zero.

Then $f|_{\Omega_1} \in \mathcal{R}(\Omega_1)$ $\forall\, f \in \mathcal{R}(\Omega)$.

Proof: Consider $B^n \supseteq \Omega$. $\therefore$ $B^n \supseteq \Omega_1$.

Let $f \in \mathcal{R}(\Omega)$.

$\because$ $\partial\Omega_1$ is of content zero, $\chi_{\Omega_1} \in \mathcal{R}(B^n)$.

Observe : $\widetilde{f|_{\Omega_1}} = \tilde{f}\,\chi_{\Omega_1}$ — ⊛

both are : $B^n \to \mathbb{R}$.

The extension of $f|_{\Omega_1} : \Omega_1 \to \mathbb{R}$ to $\widetilde{f|_{\Omega_1}} : B^n \to \mathbb{R}$

by $(\widetilde{f|_{\Omega_1}})|_{\Omega_1} = f|_{\Omega_1}$

& $(\widetilde{f|_{\Omega_1}})_{B^n \setminus \Omega_1} \equiv 0$.

$\because$ $\tilde{f}, \chi_{\Omega_1} \in \mathcal{R}(B^n)$, by product formula,

$$\widetilde{f|_{\Omega_1}} \in \mathcal{R}(B^n).$$

i.e., $f|_{\Omega_1} \in \mathcal{R}(\Omega_1)$.

Remark: By ⊛, $\displaystyle\int_{\Omega_1} f|_{\Omega_1} = \int_{\Omega} f\,\chi_{\Omega_1}$. #