Indian Statistical Institute

Analysis III (HW - 2) Due date: Take your sweet time Instructor : Jaydeb Sarkar

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NOTE: (i) $B_r(x_0) = \{x \in \mathbb{R}^n : d_u(x, x_0) < r\}$. (ii) $D_r(x_0) = B_r(x_0) \setminus \{x\}$. (iii) $B^n = \Pi_{i=1}^n [a_i, b_i]$. (iv) $R(B^n)$ = the set of all Riemann integrable functions on B^n. (v) $C(B^n)$ = the set of all continuous functions on B^n. (vi) $v(B^n)$ =volume of B^n, whenever $n \geq 3$. (vii) $A(B^2)$ = area of B^2.

(1) Compute $\int_0^1 \int_{-1}^1 xe^{xy} dxdy$.

(2) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{q} & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise.} \end{cases}$$

(i) Prove that $f \in R([0,1] \times [0,1])$. (ii) Compute $\overline{\int_0^1} f(x,y)dx$ and $\underline{\int_0^1} f(x,y)dx$ for all $y \in [0,1]$. Prove that they are unequal for all $y \in \mathbb{Q}$. (iii) Prove that $\int_0^1 \int_0^1 f(x,y)dydx$ exists, but $\int_0^1 \int_0^1 f(x,y)dxdy$ does not.

(3) Fix $j \in \{1, \ldots, n\}$ and a scalar $r \in \mathbb{R}$. Prove that $X = \{x \in \mathbb{R}^n : x_j = r\}$ is of measure zero.

(4) Compute the oscillation $osc(f, x)$ for all $x \in [0,1]$, where

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap [0,1] \text{ and } y = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

(5) Compute the oscillation $osc(f, (x,y))$ for all $(x,y) \in \mathbb{R}^2$, where

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(6) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } \frac{1}{2^n} \leq x, y \leq \frac{1}{2^{n-1}} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $f \in R([0,1] \times [0,1])$. Also compute $\int f$.