

Thm: (Gauss/divergence thm) [Relates triple integrals \leftrightarrow surface integrals.]

Let $D \subseteq \mathbb{R}^3$ be a "solid" $\therefore \partial D$ (the boundary of D) is an (~~piecewise~~) oriented surface, $F = \langle P, Q, R \rangle$ be a C^1 v.f. on an open set containing $D \cup \partial D (= \bar{D})$. Then

$$\text{The Flux: } \int_{\partial D} \vec{F} \cdot d\vec{S} = \int_D \operatorname{div} \vec{F}$$

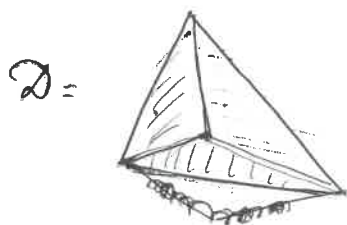
Note: (1) $\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

i.e. $\boxed{\operatorname{div} \vec{F} = \nabla \cdot \vec{F}}$
gradient dot product.

Also, note that $\int_D \operatorname{div} \vec{F} = \int_D \underbrace{\operatorname{div} \vec{F} \, dV}_{\text{a triple integral}} = \int_D \int \int \nabla \cdot \vec{F} \, dx \, dy \, dz$

$$\therefore \int_{\partial D} \vec{F} \cdot d\vec{S} = \int_D \int \int \operatorname{div} \vec{F} \, dV = \int_D \int \int \nabla \cdot \vec{F} \, dx \, dy \, dz$$

(2) ∂D could be piecewise parametrized, i.e., piecewise smooth. For instance:



$$\partial D = \bigcup_{i=1}^4 T_i, \text{ where } T_i \text{ are triangular face.}$$

(3) In the above case, just consider parametrizations for each T_i ($i=1, 2, 3, 4$)

Proof: Again, the general version is "beyond our scope".

Here we consider "elementary solid" like elementary region in Green's thm proof.

$$\mathcal{D} := \{ (x, y, z) : \varphi_1(x, y) \leq z \leq \varphi_2(x, y), x \in [a, b], y \in [c, d] \}$$

Vertically
convex.

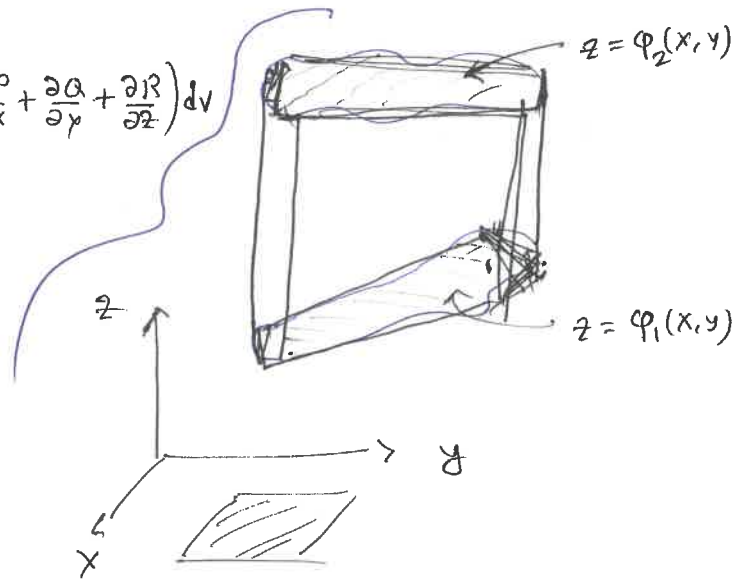
Goal: $\int_{\partial \mathcal{D}} \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv$

Like Green's thm, ~~we prove~~ split

$$\partial \mathcal{D} = \bigcup_{i=1}^6 S_i$$

S_i are faces of \mathcal{D} .

total 6



Suppose S_1 is top.

Also, again like Green's thm proof:

We prove $\int_{\partial \mathcal{D}} \langle 0, 0, R \rangle \cdot d\vec{S} = \int_{\mathcal{D}} \frac{\partial R}{\partial z} dv$ etc.

To this ~~end~~ end, we proceed as follows:

Note: $\int_{\mathcal{D}} \frac{\partial R}{\partial z} dv = \int_{y=c}^d \int_{x=a}^b \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} \frac{\partial R}{\partial z} dz dx dy = \int_c^d \int_a^b \left[R(x, y, \varphi_2(x, y)) - R(x, y, \varphi_1(x, y)) \right] dx dy$

Now $\int_{S_1} \langle 0, 0, R \rangle \cdot d\vec{S} = \iint_{S_1} \langle 0, 0, R \rangle \cdot \left\langle -\frac{\partial \varphi_2}{\partial x}, -\frac{\partial \varphi_2}{\partial y}, 1 \right\rangle dx dy$
 $= \iint_c^d \int_a^b R(x, y, \varphi_2(x, y)) dx dy$

The top.
 $z = \varphi_2(x, y)$.
Graph surface.

$\int_{S_2} \langle 0, 0, R \rangle \cdot d\vec{S} = - \iint_c^d \int_a^b R(x, y, \varphi_1(x, y)) dx dy$

$z = \varphi_1(x, y)$
bottom surface.

AND

$\int_{S_j} \langle 0, 0, R \rangle \cdot d\vec{S} = 0 \quad \forall S_j \text{ vertical wall}$ [Similar to Green's thm]

$j = 3, 4, 5, 6$

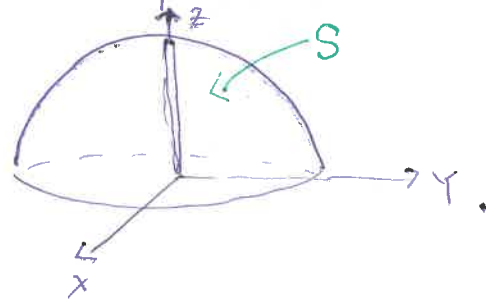
$\therefore \int_S \langle 0, 0, R \rangle \cdot d\vec{S} = \sum_{j=1}^4 \int_{S_j} \langle 0, 0, R \rangle \cdot d\vec{S} = \int_{S_1} + \int_{S_2}$
 $= \int_c^d \int_a^b \left[R(x, y, \varphi_2(x, y)) - R(x, y, \varphi_1(x, y)) \right] dx dy = \int_{\mathcal{D}} \frac{\partial R}{\partial z} dv$

□

eg: Compute $\int_S \vec{F} \cdot d\vec{S}$, where S is the hemisphere

$$x^2 + y^2 + z^2 = 1, \quad \underbrace{z > 0}_{\text{or } z \geq 0}.$$

$$\vec{F}(x, y, z) = \langle x+y, z^2, x^2 \rangle.$$



Sol: We wish to apply Divergence theorem:

$$\int_{S \cup S_1} \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \text{div } \vec{F} \, dV.$$

\mathcal{D} = Solid hemisphere.

$$S_1 = \text{bottom of the hemisphere} \\ = \{(x, y, 0) : x^2 + y^2 \leq 1\}.$$

$$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \text{div } \vec{F} \, dV - \int_{S_1} \vec{F} \cdot d\vec{S} \quad \text{--- (1)}$$

$$\vec{F} = \langle x+y, z^2, x^2 \rangle.$$

$$\therefore \text{div } \vec{F} = 1 + 0 + 0 = 1$$

$$\begin{aligned} \text{Now } \int_{\mathcal{D}} \text{div } \vec{F} \, dV &= \int_{\mathcal{D}} 1 \, dV \\ &= \text{Area}(\mathcal{D}) \\ &= \frac{2\pi}{3}. \end{aligned}$$

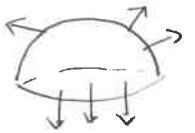
$$\int_{S_1} \vec{F} \cdot d\vec{S} = \int_{S_1} \vec{F} \cdot \vec{n} \, dS$$

$$\text{Here } \vec{n} = \langle 0, 0, 1 \rangle.$$

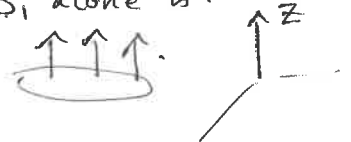
$$\begin{aligned} \therefore \int_{S_1} \vec{F} \cdot \vec{n} \, dS &= \int_{S_1} x^2 \, dS \\ &= \int_0^1 \int_0^{2\pi} u^2 \cos^2 v \, u \, du \, dv \\ &= \int_0^1 \int_0^{2\pi} u^3 \cos^2 v \, du \, dv \\ &= \dots = \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} \therefore (1) \Rightarrow \int_S \vec{F} \cdot d\vec{S} &= \frac{2\pi}{3} + \frac{\pi}{4} \\ &= \frac{11\pi}{4}. \end{aligned}$$

REMEMBER: orientation of $S \cup S_1$ is



However, orientation of S_1 alone is:



$$\text{Recall: } \int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n} \, dS.$$

S_1 is given by:

$$\vec{r}(u, v) = (u \cos v, u \sin v, 0) \\ \text{for } 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

$$\begin{aligned} \therefore \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} \\ &= \langle 0, 0, u \rangle. \end{aligned}$$

$$\Rightarrow \|\vec{r}_u \times \vec{r}_v\| = u$$

Thm: Stokes theorem

Let C be a piecewise C^1 curve enclosing an oriented surface S in \mathbb{R}^3 . Suppose $\vec{F} = \langle P, Q, R \rangle$ be a C^1 -v.f. on an open set containing S . Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Remark:

$$\# \nabla \times \vec{F} = \text{Curl } \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{So } \int_C \vec{F} \cdot d\vec{r} = \int_S \text{Curl } \vec{F} \cdot d\vec{S}$$

$R=0 \Rightarrow$ Green's thm. BUT: " \uparrow "

Do the proof for $S: z = f(x, y)$ (i.e. graph surface).
Here, we need to "use Green's thm" In true & full version:
it goes other way around though!

Proof: Suppose $S = \text{graph } f = \{(x, y, f(x, y)) : (x, y) \in R\}$. f is C^1 .

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz \quad \left[\begin{array}{l} \because z = f(x, y) \\ \Rightarrow dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \end{array} \right]$$

$$= \int_C \left(P + R \frac{\partial f}{\partial x} \right) dx + \left(Q + R \frac{\partial f}{\partial y} \right) dy$$

$(\tilde{C}) \rightarrow$

Green's thm

$$= \int_R \left(\frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) \right) dA$$

$[\tilde{C} = \{(x, y) : (x, y, z) \in C \text{ for some } z\}]$

i.e. \tilde{C} is projection of C onto XY -plane

$$\text{Now } \frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) = \frac{\partial Q}{\partial x} + \frac{\partial^2 f}{\partial x \partial y} R + \frac{\partial R}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = \frac{\partial P}{\partial y} + \frac{\partial^2 f}{\partial x \partial y} R + \frac{\partial R}{\partial y}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_R$$

Note P, Q, R are fns of (x, y, z) & $z = f(x, y)$.

[\therefore In particular: if $\omega = P(x, y, z)$, then

$$\frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} \cdot \frac{\partial f}{\partial x}$$

$$(*) \Rightarrow \frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial x}$$

$$\text{So } \frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) = Q_x + Q_z f_x + \left(R_x + R_z f_x \right) f_y + R \frac{\partial^2 f}{\partial x \partial y} \quad \text{--- (1)}$$

$$\frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = P_y + P_z f_y + \left(R_y + R_z f_y \right) f_x + R f_{xy} \quad \text{--- (2)}$$

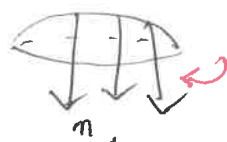
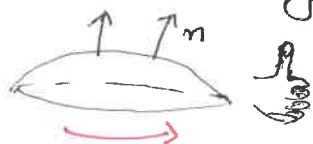
$$\therefore (1) - (2) \Rightarrow$$

$$\frac{\partial}{\partial x} \left(Q + R \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial f}{\partial x} \right) = f_x (Q_z - R_y) + f_y (R_x - P_z) + (Q_x - P_y)$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_R f_x (Q_z - R_y) + f_y (R_x - P_z) + (Q_x - P_y) \\ &= \int_R \langle -f_x, -f_y, 1 \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \int_S (\nabla \times \vec{F}) \cdot d\vec{S} \end{aligned}$$

BTW: How to figure out orientation of S vs. C ? RIGHT HAND RULE!!

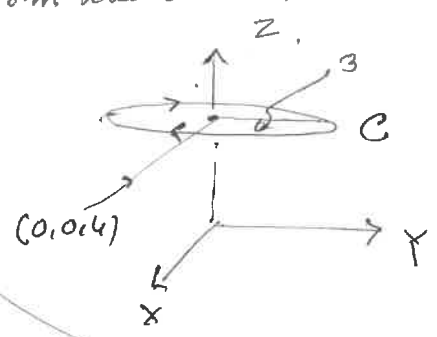
Place thumb along n of $S \Rightarrow$ The remaining 4 fingers will direct C .



eg: (A) $\int_C \vec{F} \cdot d\vec{r}$, where $C: x^2 + y^2 = 9, z = 4$, oriented clockwise when viewed from the above.

$\vec{F} = \langle -y, x, xyz \rangle$

Sol: $C: \vec{r}(t) = \langle 3\cos t, 3\sin t, 4 \rangle$
 $0 \leq t \leq 2\pi$



$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= - \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t \sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle dt \\ &= \dots = -18\pi \end{aligned}$$

∴ we need

(B) $S =$ the surface: $x^2 + y^2 + z^2 = 25$, below the plane $z = 4$, oriented s.t. the unit normal vector at $(0, 0, -5)$ is $\langle 0, 0, -1 \rangle$.

Compute $\int_S \underbrace{\text{Curl } \vec{F}}_{= \nabla \times \vec{F}} \cdot d\vec{S}$

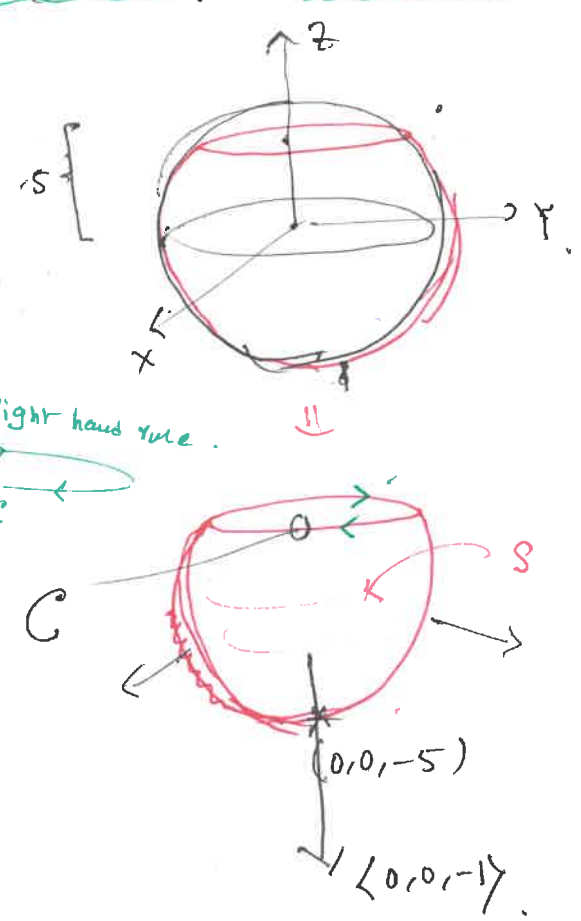
Sol: By Stokes' Thm.

$$\begin{aligned} \int_S \text{Curl } \vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{r} \\ &= -18\pi \end{aligned}$$

(By (A))

Stokes' Thm. Simplifies!!

A fairly simple



Inverse & implicit fn. theorems.

Recall the inverse fn. thm: let $O \subseteq \mathbb{R}$ open & $f: O \rightarrow \mathbb{R}$ be a C^1 fn.

If $f'(x_0) \neq 0$ for some $x_0 \in O$, then $\exists (a, b) \ni x_0$ s.t.

- (i) $(a, b) \subseteq O$,
- (ii) $f: (a, b) \rightarrow \underbrace{f(a, b)}_{\text{an open interval}}$ is bijection.
- (iii) $f^{-1}: f(a, b) \rightarrow (a, b)$ is diff. (C^1 fn.)
- (iv) $(f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \forall x \in (a, b).$

— The \mathbb{R}^n -version (No proof: see Spivak's book) —

Thm: (Inverse fn. thm).

Let $O_n \subseteq \mathbb{R}^n$ be open & $f: O_n \rightarrow \mathbb{R}^n$ be a C^1 -fn. If $Df(x_0)$ is invertible for some $x_0 \in O_n$, then \exists an open neighborhood $B_n \subseteq O_n$ s.t. $x_0 \in B_n$ &

- i) $f: B_n \rightarrow \underbrace{f(B_n)}_{\Rightarrow \text{open}}$ is bijection.
- ii) $f^{-1}: f(B_n) \rightarrow B_n$ is ~~diff~~ C^1 (\Rightarrow diff.)
- iii) $Df^{-1}(f(x)) = (Df(x))^{-1} \quad \forall x \in B_n.$

Moral of the story.

So, if $Df(x_0) \in M_n(\mathbb{R})$ inv. then locally f is invertible. ($\&$ diffeomorphism).

— x —

Implicit fn. thm.

Motivation:

We are mostly familiar with fns of the form

$$y = f(x), \quad z = f(x, y), \quad x_{n+1} = f(x_1, \dots, x_n) \text{ etc.}$$

Like: $\underbrace{y = \sin x}_{y - \sin x = 0}, \quad \underbrace{z = xy + y^2}_{z - xy - y^2 = 0} \text{ etc.}$

What about writing " $y = f(x)$ " if we have: $F(x, y) = 0$?

i.e.: if F is a fn. of (x, y) & if $F(x, y) = 0$

Can it implicit fn. or solution. \Rightarrow $y = f(x)$ for some f ? i.e. $F(x, f(x)) = 0$?
 [Also, we wish the solution f to be diff.].

Of course, this is a question of general interest.

eg1: Suppose $F(x, y) = ax + by + c$.

Now $ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$ ($b \neq 0$).

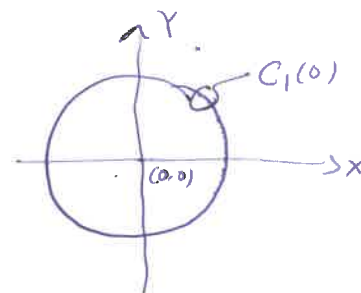
So, here $f(x) = -\frac{a}{b}x - \frac{c}{b} \quad \forall x$.

Note: $\frac{\partial F}{\partial y} = b$. So $\frac{\partial F}{\partial y} (=b) \neq 0 \Rightarrow F(x, f(x)) = 0$
 for some diff. f !!

Trivial but impressive observation. \rightarrow So $\frac{\partial F}{\partial y} \neq 0 \Rightarrow \exists$ diff. fn. f s.t. $F(x, f(x)) = 0$.

eg2: $F(x, y) = x^2 + y^2 - 1$.

We know $F(x, y) = 0 \Leftrightarrow (x, y) \in C_1(0)$.



Note that $y = \pm \sqrt{1-x^2}$.

$\Rightarrow y = f(x)$ for (even any fn.) f is NOT POSSIBLE!!

However: if $y > 0$, then $y = f(x) = \sqrt{1-x^2}$
 & if $y < 0$, then $y = f(x) = -\sqrt{1-x^2}$.

~~Now~~ \therefore If $y > 0$ (or $y < 0$), then $\exists f$ s.t.
 $F(x, f(x)) = 0$.

Note $\frac{\partial F}{\partial y} = 2y$. & so if $y > 0$ (or $y < 0$), then
 $\frac{\partial F}{\partial y} \neq 0$.

Eg 3: Suppose $F(x, y) = 0$ in a nbd of $(a, b) \in \mathbb{R}^2$.

Suppose $y = f(x)$ be a diff / C^1 solution to this:

$$F(x, f(x)) = 0.$$

implicit solution to F .

$$\Rightarrow \frac{\partial F}{\partial x} \frac{dx}{\frac{dx}{dx}=1} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0. \quad [\text{Chain rule}]$$

$$\Rightarrow \left[\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \right] \quad \left[\text{if } \frac{\partial F}{\partial y} \neq 0 \right] \quad \text{at } (a, b)$$

Again: $\frac{\partial F}{\partial y} \neq 0$ plays important role here.

This also says: if $y = f(x)$ is a C^1 solution to $F(x, y) = 0$ near (a, b) , then $\frac{\partial F}{\partial y} \neq 0$ at (a, b) is necessary to recover $\frac{dy}{dx}$!!

i.e.: derivative of F can be computed by differentiating

$$F(x, f(x)) = 0. \quad \text{~~Computing } \frac{dy}{dx} \text{ from}~~$$

~~Thm: Let $\mathcal{O} \subseteq \mathbb{R}^{n+m}$ be open, $f: \mathcal{O} \rightarrow \mathbb{R}^m$ a C^1 fn. Suppose $(a, b) \in \mathcal{O}$, $F(a, b) = 0$ & $\det \left(\frac{\partial f_i}{\partial y_j} \right) \neq 0$ at (a, b) .~~

Setting: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $n, m \geq 1$. $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. ~~Fixed~~ Let $(a, b) \in \mathcal{O} \subseteq \mathbb{R}^n \times \mathbb{R}^m$. $F: \mathcal{O} \rightarrow \mathbb{R}^m$ & $F = (f_1, \dots, f_m)$.

& $f_i: \mathcal{O} \rightarrow \mathbb{R} \quad \forall i = 1, \dots, m$. Finally, $f_i(x, y) = f_i(\underbrace{x_1, \dots, x_n}_{\in \mathbb{R}^n}, \underbrace{y_1, \dots, y_m}_{\in \mathbb{R}^m})$.

Thm: (Implicit fn. thm) : Suppose $F \in C^1(\mathcal{O})$ & $F(a, b) = 0$.

If $\boxed{\det \left(\frac{\partial f_i}{\partial y_j}(a, b) \right) \neq 0}_{1 \leq i, j \leq m}$, then \exists open sets $\underbrace{U \subseteq \mathbb{R}^n}_a, \underbrace{V \subseteq \mathbb{R}^m}_b$

& a C^1 -fn $f: U \rightarrow V$ s.t.

$$F(x, f(x)) = 0 \quad \forall x \in U.$$

Proof: Define $\tilde{F}: \Theta \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ by

$$\tilde{F}(x, y) = \cancel{(x, F(x))} \cdot (x, F(x, y)), \quad \forall (x, y) \in \Theta.$$

Now $J_{\tilde{F}}(a, b) = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & J_F(a, b) \end{bmatrix} \Rightarrow \det[J_{\tilde{F}}(a, b)] \neq 0.$

\therefore By Inverse fn thm: $\exists U_0 \subseteq \Theta, V_0 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ open sets s.t. $(a, b) \in U_0, (a, 0) \in V_0$ &
 $\tilde{F}(a, b)$

$\tilde{F}: U_0 \rightarrow V_0$ has a differentiable inverse.

Note: $\uparrow U_0 = A \times B$ for $\begin{matrix} A \subseteq \mathbb{R}^n \\ \downarrow a \end{matrix}, \begin{matrix} B \subseteq \mathbb{R}^m \\ \downarrow b \end{matrix}$ open.

We can consider

Call: $U = A$.

[\therefore Boxes forms basic open sets.]

Clearly, $\tilde{F}^{-1}(\underset{\substack{\uparrow \\ V_0}}{x, y}) = (x, g(x, y))$ for some diff. g .

$$[\therefore \tilde{F}(x, y) = (x, F(x, y))]$$

Now $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ $\left. \begin{matrix} (x, y) \mapsto y \end{matrix} \right\} \Rightarrow \pi \circ \tilde{F} = F.$

$$\begin{aligned} \therefore F(x, g(x, y)) &= \cancel{(F \circ g)(x, y)} F \circ \tilde{F}^{-1}(x, y) \\ &= \pi \circ \tilde{F} \circ \tilde{F}^{-1}(x, y) \end{aligned}$$

$$= \pi(x, y)$$

$$= y.$$

$$\forall (x, y) \in A \times B.$$

$$\Rightarrow F(x, g(x, 0)) = 0.$$

So, define $f(x) := g(x, 0)$. $\forall x \in A = U$. \square

————— X —————