Thm: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a rectifiable parametrized curve, & let $\tilde{\gamma}$ be a parametrization of γ.

$$\begin{tikzcd} & \mathbb{R}^n \\ {[c,d]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\varphi"'] & {[a,b]} \arrow[u, "\gamma"'] \end{tikzcd}$$

φ: strictly increasing + onto + Cont. ~~diff~~.

Then $\tilde{\gamma}$ is rectifiable & $\ell(\gamma) = \ell(\tilde{\gamma})$. i.e: length is invariant.

$\lim_{\|P\| \to 0} \ell(\gamma, P)$

($\Rightarrow$ parametrizations, as defined above, makes good sense.).

Proof: Let $\varepsilon > 0$. $\because$ γ is rectifiable, $\exists\ \delta > 0$ $\ni$.

$$\left| \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| - \cancel{\int_a^b \|\gamma'\|}\ \ell(\gamma) \right| < \varepsilon. \quad \text{— (1)}$$

$\forall\ P := a = t_0 < t_1 < \cdots < t_n = b$ & $\|P\| < \delta$.

Also, $\exists\ \tilde{\delta} > 0$ $\ni$. $|\varphi(s) - \varphi(t)| < \delta$ $\forall\ |s-t| < \tilde{\delta}$. — (2)

By uniform Continuity of φ.

Note: $\tilde{\gamma} = \gamma \circ \varphi : [c,d] \to \mathbb{R}^n$.

Let $\tilde{P}$ be a partition of $[c,d]$.

Set $\tilde{P}$: $c = s_0 < s_1 < \cdots < s_m = b$.

$\because$ $\varphi \uparrow$ & onto: $P \in \mathcal{P}([a,b])$, where

$$P:\ a = \underset{\varphi(s_0)}{\underset{\parallel}{t_0}} < \underset{\varphi(s_1)}{\underset{\parallel}{t_1}} < \cdots < \underset{\varphi(s_m)}{\underset{\parallel}{s_m}} = b.$$

Also $$\sum_{i=1}^{m} \|\underbrace{\tilde{\gamma}(s_i)}_{\gamma(\varphi(s_i))} - \underbrace{\tilde{\gamma}(s_{i-1})}_{\gamma(\varphi(s_{i-1}))}\| = \sum_{i=1}^{m} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

& note that: if $\|\tilde{P}\| < \tilde{\delta} \Rightarrow \|P\| < \delta$ [by (2)]

$$\therefore (1) \Rightarrow \left| \sum \|\underbrace{\gamma(\varphi(s_i))}_{\tilde{\gamma}(s_i)} - \underbrace{\gamma(\varphi(s_{i-1}))}_{=\tilde{\gamma}(s_{i-1})}\| - \cancel{\int_a^b \|\gamma'\|}\ \ell(\gamma) \right| < \varepsilon.$$

$\Rightarrow$ $\tilde{\gamma}$ is rectifiable &

$$\ell(\tilde{\gamma}) = \cancel{\int_a^b \|\gamma'\|}\ \ell(\gamma).$$

QED

Remark: Usually, identification of rectifiable curve is not so easy.

See the notion of bounded variation. In fact: A curve $\gamma=(\gamma_1,\dots,\gamma_n):[a,b]\to\mathbb{R}^n$ is rectifiable $\Leftrightarrow$ $\gamma_i:[a,b]\to\mathbb{R}$ is of bounded variation $\forall i$.

A Cool fact of "parametrizations":

Thm: Suppose $\gamma:[a,b]\to\mathbb{R}^n$ be a smooth curve. Then $\exists$ a parametrization $\tilde{\gamma}$ of γ s.t.

$$\|\tilde{\gamma}'(s)\| = 1 \qquad \forall\ s\in[c,d].$$

The same route followed by the path with uniform (unit) speed!!

Here

$$\begin{tikzcd} & \mathbb{R}^n \\ {[c,d]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\varphi"'] & {[a,b]} \arrow[u, "\gamma"'] \end{tikzcd}$$

Proof: So, we want φ, strictly increasing + onto + diff. s.t.

$$\tilde{\gamma}=\gamma\circ\varphi \ \ \& \ \ \|\tilde{\gamma}'(s)\|=1.$$

If we have such φ, then

$$\tilde{\gamma}'(s)=\gamma'(\varphi(s))\cdot\varphi'(s).$$

$$\& \ \ 1=\|\tilde{\gamma}'(s)\| = \|\gamma'(\varphi(s))\|\ |\varphi'(s)|$$

$$\Rightarrow \ |\varphi'(s)| = \frac{1}{\|\gamma'(\underbrace{\varphi(s)}_{\varphi(s)=t})\|}.$$

$\leftarrow$ This is what is expected!!

~~So, if $\varphi'(s)>0$, then~~

But $\varphi\uparrow \Rightarrow \varphi'>0 \Rightarrow$ $\varphi'(s)=\dfrac{1}{\|\gamma'(t)\|}$

$\therefore$ ~~The choice of φ is~~ If $\tilde{\varphi}(t):=\int_a^t \|\gamma'\|$.

then $\tilde{\varphi}'(t)=\|\gamma'(t)\|$.

Set $\varphi(t)=\dfrac{1}{\tilde{\varphi}(t)}$. i.e. $\varphi(t)=\dfrac{1}{\int_a^t\|\gamma'\|}$.

$\therefore$ If we define $\varphi(t)=\dfrac{1}{\int_a^t\|\gamma'\|}$, then

$\tilde{\gamma}:=\gamma\circ\varphi$ will have unit speed. Also, φ is a parametrization. $\boxed{14}$

eg: $\gamma(t) := (t, 1-|t|)$. $t \in \mathbb{R}$.

$\therefore$ γ is not diff at 0 $\Rightarrow$ γ is not ~~diff~~ smooth at $t=0$.

Moreover, γ cannot be made smooth via any reparametrization: Consider a reparametrization:

$$\tilde{\gamma} = \gamma \circ \varphi$$

φ ↑, diff, onto.

Assume WLOG: $\varphi(0) = 0$.

Now $\tilde{\gamma}(s) = \gamma(\varphi(s))$

$$\Rightarrow \tilde{\gamma}'(s) = \underbrace{\gamma'(\varphi(s))}\,\varphi'(s).$$

Trouble at $s=0$. As $\gamma'(0)$ is not defined.

In this case, γ has a corner at 0. V.

|| i.e, a corner is a true non-smooth point, which cannot be reparametrized to a smooth curve.

— x —

eg: (1) Consider $\gamma(t) = (\alpha t, \beta t - 16t^2)$. $\alpha, \beta \in \mathbb{R} \setminus \{0\}$.

eqn. of the trajectory of a thrown ball:
$x = \alpha t$, $y = \beta t - 16t^2$.
$\Rightarrow \boxed{y = \frac{\beta}{\alpha}x - 16\frac{\beta}{\alpha^2}x^2.}$

$\therefore \gamma'(t) = (\alpha, \beta - 32t)$.

$$\therefore \ell(\gamma) = \int \|\gamma'\|$$

$$= \int \sqrt{\alpha^2 + (\beta - 32t)^2}\, dt.$$

(2) Circumference of the circle $x^2 + y^2 = r^2$.

Here: $\gamma(t) = (r\cos t, r\sin t)$. $0 \leq t \leq 2\pi$.

$\therefore\ \gamma'(t) = (-r\sin t, r\cos t)$.

$\therefore\ \|\gamma'(t)\|^2 = r^2. \Rightarrow \|\gamma'(t)\| = r\ \forall t.$

$\therefore\ L(\gamma) = \int_0^{2\pi} r\,dt = 2\pi r.$ ← Known fact: of course.

(3) Arc length of graphs: Let $f:[a,b]\to\mathbb{R}$ be a C^1-fn.

Define $\gamma(t) = (t, f(t))$. ← graph of f.

Then $\ell(\gamma) = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. $(y = f(x))$.

← The familiar formula.!!

Why?

$\left[\because\ \ell(\gamma) = \int_a^b \sqrt{1^2+\left(\frac{df}{dt}\right)^2}\,dt\right]$.

Remark: We know Smooth (or piecewise smooth) $\Rightarrow$ rectifiable $\Rightarrow$ Any reparametrization is also rectifiable & arc length remains invariant.

So, in particular (see p-13): If

$$\begin{array}{ccc} & & \mathbb{R}^n \\ & \overset{\tilde\gamma}{\nearrow} & \uparrow \gamma \\ [c,d] & \xrightarrow[\varphi]{} & [a,b] \end{array}$$

then $\ell(\gamma) = \ell(\tilde\gamma)$ i.e., $\int_a^b \|\gamma'\| = \int_c^d \|\tilde\gamma'\|$.

Useful & minimum requirement.

~~Now, we define line integration:~~

Defi: Let $f \in Cont(S)$ ($\mathbb{R}$-valued) & let $S = \operatorname{ran}\gamma$, where $\gamma:[a,b]\to\mathbb{R}^n$ a piecewise smooth parametrized curve. Then we define

$$\int_\gamma f = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt.$$

Line integrals (Some basic intro:).

Scalar field = Scalar-valued fn like $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.
Vector field = vector-valued fn like $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Often, Scalar/vector fields are atleast Cont: fn. i.e. all Components are Cont.

[A Scalar field ⟹ a special vector field. Like:
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Scalar field & diff.
Then ∇f is a vector field. (known as gradient field).].

Q: Given a Scalar field $f : \underset{\text{or } \mathcal{O}_n \subseteq \mathbb{R}^n, \text{ or } C}{\mathbb{R}^n} \longrightarrow \mathbb{R}$, a bounded $\underbrace{\text{Curve}}_{\text{w.r.to Some parametrization.}}$
$C \subseteq \mathbb{R}^n$, how to define

$$\int_C f \quad ?$$

[One way: $\because \underset{\text{bdd}}{C} \subseteq \mathbb{R}^n$, we appeal to our previous integration theory & define $\int_C f = \int_{B^n} \tilde{f}$, $B^n \supseteq C$ a box!
the extension fn.

BUT, Since $n > 1$, & C is a Curve, C is of Content zero.

$$\Rightarrow \int_{B^n} \tilde{f} = 0 \Rightarrow \int_C f = 0$$

— Not so desirable.

⟹ WRONG WAY. !!].

Right way:

Def: Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a Smooth Curve, $C = \operatorname{ran} \gamma$ (the path of γ) & let $f \in \mathcal{B}(C)$. [Note: $C \subseteq \mathbb{R}^n$].
Given $P : a = t_0 < t_1 \cdots < t_m = b$, a partition of $[a,b]$, define

$$U(f,P) = \sum_{i=1}^{m} M_i \delta_i \quad \& \quad L(f,P) = \sum_{i=1}^{m} m_i \delta_i, \text{ where}$$

$$\delta_i := \| \underbrace{\gamma([t_i, t_{i-1}])}_{:= C_i \subseteq \mathbb{R}^n} \|, \quad M_i := \sup_{C_i} f, \quad m_i := \inf_{C_i} f \quad \forall i = 1, \dots, m.$$

Length

We call $U(f, P) \longrightarrow$ upper sum of f w.r.to P (& γ)

& $L(f, P) \longrightarrow$ lower —— " ——.

Dont miss that: γ is here!!

$[\|\gamma([t_{i-1}, t_i])\| = \|\gamma(t_i) - \gamma(t_{i-1})\|.]$

Def: Given a rectifiable (/piecewise smooth) curve $\gamma : [a,b] \to \mathbb{R}^n$, set $C = \operatorname{ran} \gamma$ (the path). We say $f \in \mathcal{B}(C)$ is integrable if

$$\inf_{P \in \mathcal{P}[a,b]} f = \sup_{P \in \mathcal{P}[a,b]} f. \qquad \text{———} \circledast$$

In this case, the common value of the integration is will be denoted by:

$$\underbrace{\int_C f \, ds.}$$

Called: line (or contour) integral.

The following is convincing: but the proof is left to you (it will be slightly non-trivial but routine computation):

Thm: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a rectifiable curve, $C = \operatorname{ran} \gamma$ & $f \in \mathcal{B}(C)$. Then:

(1) $f \in \operatorname{Cont}(C) \Rightarrow f$ is integrable (in the sense of line integrals).

(2) f is line integrable $\Leftrightarrow \lim_{\|P\| \to 0} \sum_{i \in \Lambda(P), \eta_i \in C_i} f(\eta_i) s_i$ exists. (Tag set.)

In this case, $\int_C f = \lim_{\|P\| \to 0} \left[\sum_{i \in \Lambda(P)} f(\eta_i) s_i \right].$

For you $\Rightarrow$ (3)* If γ is C^1 & $f \in \mathcal{R}(C)$ [in the sense of $\circledast$], then

$$\boxed{\int_C f \, ds = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt.}$$