We need one observation:

Lemma: Let $K \subseteq \mathbb{R}^n$ be a compact set of measure zero. Let $\varepsilon > 0$. Then $\exists$ open boxes $B_1, \ldots, B_m$ (for some $m = m(\varepsilon)$) s.t.

$$\bigcup_{i=1}^{m} B_i \supset K \quad \& \quad \sum_{i=1}^{m} v(B_i) < \varepsilon.$$

Proof: Just compactness of K: Let $\varepsilon > 0$. Then $\exists$ boxes $\{B_i\}_{i=1}^{\infty}$ s.t. $\sum_{i=1}^{\infty} v(B_i) < \varepsilon$ & $\bigcup_{i=1}^{\infty} B_i \supseteq K$.

But K compact $\Rightarrow \exists\, m \in \mathbb{N}$ s.t. $\bigcup_{i=1}^{m} B_i \supseteq K$. Clearly, $\sum_{i=1}^{m} v(B_i) \leq \sum_{i=1}^{\infty} v(B_i) < \varepsilon$. $\square$

Remark: We can safely replace boxes by open/closed balls.

\# **Thm:** (Riemann-Lebesgue thm): Let $f \in \mathcal{B}(B^n)$. Then $f \in \mathcal{R}(B^n)$ $\Longleftrightarrow$ the set of discontinuity of f is of measure zero.

Proof: Set $\mathcal{D} := \{x \in B^n : f \text{ is not cont. at } x\}$.

$\therefore \mathcal{D} = \{x \in B^n : \operatorname{osc}(f, x) > 0\}$.

"$\Rightarrow$" Claim: $\mathcal{D}$ be of measure zero. [Assumption: $f \in \mathcal{R}(B^n)$].

$\forall m \in \mathbb{N}$, Set $\underbrace{\mathcal{D}_m}_{\text{closed in } \mathbb{R}^n} = \{x \in B^n : \operatorname{osc}(f, x) \geq \frac{1}{m}\}$.

$\therefore \mathcal{D}_m \downarrow$.

Note that: $\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}_m$.

So, enough to prove that $\mathcal{D}_m$ is of measure zero, $\forall m$.

$\boxed{\text{Fix } m \in \mathbb{N}.}$ Goal: $\mathcal{D}_m = \left\{ x \in B^n : \operatorname{osc}(f, x) \geq \frac{1}{m} \right\}$ is of measure zero.

Let $\varepsilon > 0$. (fix it). [$\because$ both m & ε are fixed.]

$\because f \in \mathcal{R}(B^n)$, $\exists$ P^m (~~a just P~~) a partition of B^n s.t.

$$\underbrace{U(f, P) - L(f, P)}_{\shortparallel} < \varepsilon.$$

i.e., $$\sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \varepsilon.$$

Note that: $\Lambda(P)$ is a finite set.

Let $\Lambda(P) := I \sqcup J$,
disjoint union.

where $I = \left\{ \alpha \in \Lambda(P) : \operatorname{int}(B^n_\alpha) \cap \mathcal{D}_m \neq \varphi \right\}$.

& $J = \left\{ \alpha \in \Lambda(P) : \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m = \varphi \right\}$.

$$\therefore \mathcal{D}_m \subseteq \left[\bigcup_{\alpha \in I} \operatorname{int}(B^n_\alpha) \right] \cup \underbrace{\left[\bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha \right]} \quad \text{— (28.1)}$$

the set of all boundaries of the sub-boxes B^n_α, $\alpha \in \Lambda(P)$.

This should be negligible!!

~~Now for each $\alpha \in I$, as $\exists x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$~~

Let $\alpha \in I$. So $\exists x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$.

Fix it for time being.

$$\therefore \operatorname{osc}(f, x) \geq \frac{1}{m}.$$

$$\shortparallel$$

$$\inf_{\delta > 0} \left[\sup_{z, y \in B_\delta(x)} [f(z) - f(y)] \right] \quad \longleftarrow \circledast$$

$\because x \in \operatorname{int}(B^n_\alpha)$, $\exists$ $\delta > 0$ s.t. $B_\delta(x) \subseteq B^n_\alpha$.

Now since $M_\alpha - m_\alpha = \sup_{z, y \in B^n_\alpha} [f(z) - f(y)]$, and $B_\delta(x) \subseteq B^n_\alpha$ we have:

$$M_\alpha - m_\alpha \geq \frac{1}{m}.$$

Hence:

$$\varepsilon > \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha)$$

$$\geq \sum_{\alpha \in I} (M_\alpha - m_\alpha)\, v(B^n_\alpha). \qquad [\because \Lambda(P) = I \amalg J]$$

$$\geq \sum_{\alpha \in I} \frac{1}{m} \times v(B^n_\alpha)$$

$$= \frac{1}{m} \times \sum_{\alpha \in I} v(B^n_\alpha).$$

$$\Rightarrow \sum_{\alpha \in I} v(B^n_\alpha) < m\,\varepsilon. \qquad \text{——— } (\dagger)$$

Now look at (28.1):

$$\mathcal{D}_m \subseteq \underbrace{\Big[\bigcup_{\alpha \in I} \operatorname{int}(B^n_\alpha)\Big]}_{\text{is of measure zero by } (\dagger)} \cup \underbrace{\Big[\bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha\Big]}_{\text{finite Union of boundaries of Sub-boxes.}}$$

$\Downarrow$

measure zero. (HW).

$\Rightarrow \mathcal{D}_m$ is of measure zero.

This proves $f \in \mathcal{R}(B^n) \Rightarrow \mathcal{D}$ is of measure zero.

"$\Leftarrow$" Suppose $\mathcal{D} := \{x \in B^n : \operatorname{osc}(f, x) > 0\}$ is of measure zero.

Claim: $f \in \mathcal{R}(B^n)$.

Let $\varepsilon > 0$. Set

$$\mathcal{D}_\varepsilon := \{x \in B^n : \operatorname{osc}(f, x) \geq \varepsilon\}$$

$\uparrow$ A closed set in $\mathbb{R}^n$.

$\Rightarrow$ $\mathcal{D}_\varepsilon$ is compact & of measure zero [$\because$ $\mathcal{D}_\varepsilon \subseteq B^n$ & $\mathcal{D}_\varepsilon \subseteq \mathcal{D}$].

Then for that $\varepsilon > 0$ itself, $\exists$ open boxes $\{B_i\}_{i=1}^m$ s.t. (finitely many)

$$\bigcup_{i=1}^m B_i \supseteq \mathcal{D}_\varepsilon \quad (1) \qquad \& \qquad \sum_{i=1}^m v(B_i) < \varepsilon. \quad (1a)$$

Then $B := B^n \setminus \left[\bigcup_{i=1}^m B_i\right]$ (2) is again compact.

Now $x \in B \Rightarrow \operatorname{osc}(f, x) < \varepsilon$.

$\therefore$ $\exists$ a closed box C_x s.t. $x \in \operatorname{int} C_x$ s.t.

$$\sup_{y,z \in C_x} (f(y) - f(z)) < \varepsilon. \quad (*)$$

[$\because$ $\{y \in B^n : \operatorname{osc}(f, y) < \varepsilon\}$ is an open set.]

Again, by compactness of B, $\exists$ $x_1, \dots, x_p$ s.t.

$$\bigcup_{i=1}^p C_{x_i} \supseteq B. \quad (3)$$

[Note that C_{x_i} may be considered as $\subseteq B^n$]

Let P be a partition "out of $\{B_i, C_{x_j} : 1 \leq i \leq m, 1 \leq j \leq p\}$".

i.e: B^n_α, $\alpha \in \Lambda(P)$, is either in $\overline{B_i}$, for some i, or, in C_{x_j}, for some j.

[See (1), (2) & (3) & note that $i = 1, \dots, m$ & $j = 1, \dots, p$ } finite set.]

Get a partition: $\Lambda(P) = I \coprod J$

I: those $\alpha \in \Lambda(P)$ s.t. $B^n_\alpha \subseteq \overline{B_i}$ for some $i = 1, \dots, m$.

J: those $\alpha \in \Lambda(P)$ s.t. $B^n_\alpha \subseteq C_{x_j}$ for some $j = 1, \dots, p$.

$$\therefore \quad U(f,P) - L(f,P) = \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha)$$

$$= \sum_{\alpha \in I} \underbrace{(M_\alpha - m_\alpha)}_{\leq 2M} v(B^n_\alpha) + \sum_{\alpha \in J} \underbrace{(M_\alpha - m_\alpha)}_{\leq \varepsilon \text{ by } (*)} v(B^n_\alpha)$$

$M := \max_{B^n} f$.

$$\cancel{\leq \varepsilon \times \sum_{\alpha \in I} v(B^n_\alpha) + 2M \times \sum_{\alpha \in J} v(B^n_\alpha)}$$

$$\leq 2M \times \underbrace{\sum_{\alpha \in I} v(B^n_\alpha)}_{< \varepsilon \text{ by } (1a)} + \varepsilon \times \sum_{\alpha \in J} v(B^n_\alpha)$$

$$= \varepsilon \times \left[2M + \sum_{\alpha \in J} v(B^n_\alpha) \right]$$

$$\leq \varepsilon \times \left[2M + \sum_{\alpha \in \Lambda(P)} v(B^n_\alpha) \right]$$

$$= \varepsilon \times \left[2M + v(B^n) \right].$$

$\Rightarrow \quad U(f,P) - L(f,P) \leq \varepsilon \times \tilde{M}$ for some $\tilde{M} \geq 0$.

$\Rightarrow \quad f \in \mathcal{R}(B^n)$. $\blacksquare$

~~Let $P^n \in \mathcal{P}(B^n)$~~. Integration over $\underbrace{\text{bounded domains/sets}}_{\uparrow \text{ instead of boxes.}}$

Let $\Omega \subseteq \mathbb{R}^n$ be a bdd set [Assume closed if necessary].
Assume $f \in \mathcal{B}(\Omega)$: A bdd fn.

Q: How to define $\int_\Omega f$?

Ans: We only know the answer for $\Omega = B^n$!
Also, recall, we need grids (i.e., B^n_α, $\alpha \in \Lambda(P)$, $P \in \mathcal{P}(B^n)$) to define integrations $\int_{B^n} f$!!

So, how to define $\int_\Omega f$?

One way: Get a box $B^n \supseteq \Omega$. Define
$\tilde{f} : B^n \longrightarrow \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega. \\ 0 & \text{if } x \in B^n \setminus \Omega. \end{cases}$$

Then define $\int_\Omega f = \int_{B^n} \tilde{f}$!!

Q: Looks ok, but: ① $\int_\Omega f$ = independent of the choice of B^n?
② $\tilde{f} \in \mathcal{R}(B^n)$?

intution says these should do!!

We need to fix this first.

A Couple of observations:

(1) Let B^n_1 & B^n_2 be two boxes. Then either:
(i) $B^n_1 \cap B^n_2$ is a box, or
(ii) $B^n_1 \cap B^n_2 = \varphi$, or (iii) $B^n_1 \cap B^n_2$ is a $\subseteq$ of a face of B^n_1 & a face of B^n_2. — HW —