

Goal: Compute area of a surface S .

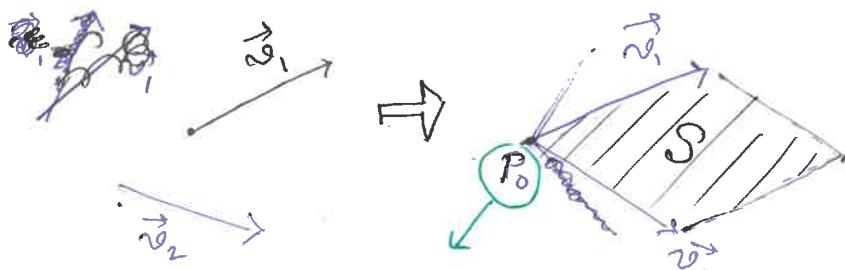
↓
i.e., Area of $S = \text{Vol } T$, where
 T is a parametrization of S .

Remark:

~~lets first observe~~ If $\vec{OP} = \langle a_1, b_1, c_1 \rangle$ & $\vec{OQ} = \langle a_2, b_2, c_2 \rangle$,
then for $\vec{OP}_0 = \langle a_0, b_0, c_0 \rangle$ fixed, we define

$$S := \left\{ \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 : 0 \leq t_1, t_2 \leq 1 \right\}.$$

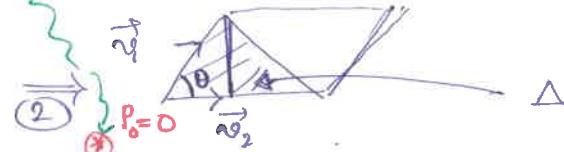
↓
open parallelogram based at \vec{v}_0 (or P_0) bounded by the
sides \vec{v}_1 & \vec{v}_2 . (in \mathbb{R}^3).



Fact: Area of $S = \|\vec{v}_1 \times \vec{v}_2\|$ — Hw.

Hint: Just compute $\vec{v}_1 \times \vec{v}_2$ \iff Method ①

then $\|\vec{v}_1 \times \vec{v}_2\|$.



Area $S = 2 \times \text{area of } \Delta$.

$$= 2 \times \left[\frac{1}{2} \times \|\vec{v}_2\| \times \|\vec{v}_1\| \sin \theta \right]$$

[*] → Translating P_0 to 0; Area will remain unchanged.

Observe: S is a surface with $S = \text{Vol } T$, where

$$T : (0, 1) \times (0, 1) \rightarrow \mathbb{R}^3 \text{ given by}$$

Dont worry about $(0,1)$ instead of $[0,1]$. We want R to be open.

$$T(t_1, t_2) = \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2.$$

$$(t_1, t_2) \in (0, 1) \times (0, 1)$$

∴ Area of $\text{Vol } T = \|\vec{v}_1 \times \vec{v}_2\|$.

A basic step:

Recall that $z = ax + by + c$ represents a plane. Moreover,

if $(x, y) \in B^2 \leftarrow$ an open box in \mathbb{R}^2 , then the graph of $f(x, y) = ax + by + c$

given by $S = \{(x, y, ax + by + c) : (x, y) \in B^2\}$ is a surface.

Here $\vec{\psi}(x, y) = (x, y, ax + by + c) \quad \forall (x, y) \in B^2$.

FACT:

$$\text{Area } S = \sqrt{1+a^2+b^2} \times \text{Area of } (B^2) \quad \leftarrow S \text{ as above}$$

"plane segment (bounded)".

Proof: Recall that $\text{Area } \tilde{S} = \|\vec{v}_1 \times \vec{v}_2\|$, where

$$\tilde{S} = \left\{ \underbrace{\vec{v}_0}_{\downarrow} + t_1 \vec{v}_1 + t_2 \vec{v}_2 : (t_1, t_2) \in (0, 1) \times (0, 1) \right\}.$$

Recall: As far area is concerned, $\vec{\psi}_0 = \vec{0}$.

(We represent S as \tilde{S} as above).

Let $B^2 = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$.

Set $\vec{v}_0 = \langle \alpha_1, \alpha_2, a\alpha_1 + b\alpha_2 + c \rangle$.

Also, set $\vec{v}_1 = \langle 0, \beta_2 - \alpha_2, b(\beta_2 - \alpha_2) \rangle$

$\vec{v}_2 = \langle \beta_1 - \alpha_1, 0, a(\beta_1 - \alpha_1) \rangle$.

Then $\{ \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 : (t_1, t_2) \in (0, 1) \times (0, 1) \}$

$= S$.

$\therefore \text{Area } S = \|\vec{v}_1 \times \vec{v}_2\|$.

$$\text{Now } \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 0 & \beta_2 - \alpha_2 & b(\beta_2 - \alpha_2) \\ \beta_1 - \alpha_1 & 0 & a(\beta_1 - \alpha_1) \end{vmatrix}$$

$$= \langle a(\beta_2 - \alpha_2)(\beta_1 - \alpha_1), b(\beta_2 - \alpha_2)(\beta_1 - \alpha_1), -(b\beta_2 - b\alpha_2)(\beta_1 - \alpha_1) \rangle$$

$$\therefore \|\vec{v}_1 \times \vec{v}_2\| = \sqrt{a^2 + b^2 + 1} \times (\beta_2 - \alpha_2)(\beta_1 - \alpha_1)$$

$$= \sqrt{a^2 + b^2 + 1} \times \text{Area}(B^2).$$

Remark: The factor " $\sqrt{1+a^2+b^2}$ " is interesting. Indeed, for $z = ax + by + c$ ($= f(x, y)$),
 $a = f_x$, $b = f_y$.
 $\therefore \sqrt{1+a^2+b^2} = \sqrt{1+f_x^2+f_y^2}$!!

With the above area formula, we can now talk about the idea of defining/obtaining surface area:

Let S be the surface parameterized by $\tau: R \rightarrow \mathbb{R}^3$, where i.e. $\tau(u, v) \in \mathbb{R}^3$, $(u, v) \in R := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$
 $\& \text{ran } \tau = S$.

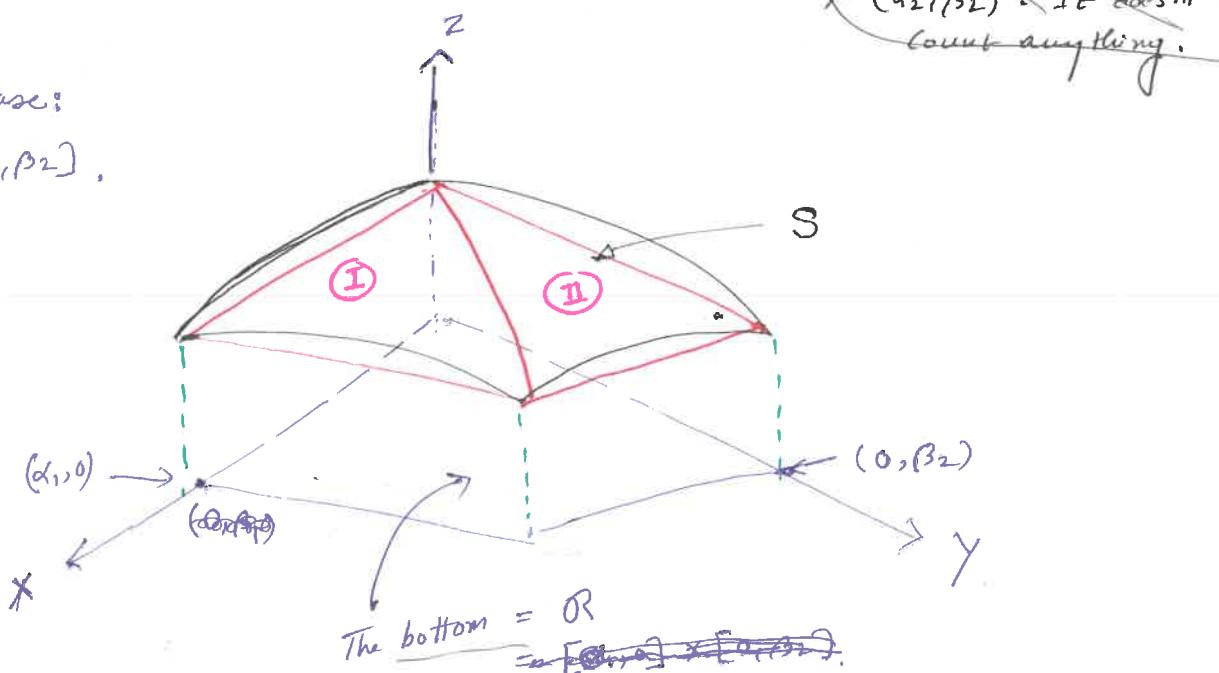
for simplicity, let's take $R = \mathbb{B}^2$.

Consider a partition of \mathbb{B}^2 : say P
 $\therefore P = \{B_\alpha : \alpha \in \Lambda(P)\}$.

Pause

A Simplest Case:

$$R = [0, \alpha_1] \times [0, \beta_2].$$



$$R = [0, \alpha_1] \times [0, \beta_2].$$

Then (BAD) area of $S \approx \triangle 1 + \triangle n$

polygona approximation.

Sum of two polygonal surfaces.
 We know the area of this by our previous computation.

THE IDEA IS to take this consideration & proceed to limit approach.

Known route!!

So, in the B^2 -setting, pick a partition of B^2 , say P .

$$\therefore B^2 = \bigcup_{\alpha \in N(P)} B_\alpha^2. \quad \leftarrow \text{May be, at this point, consider } B^2 \text{ to be closed.}$$

$\forall \alpha \in N(P)$, pick (arbitrary) $x_\alpha (= (u_\alpha, v_\alpha)) \in \text{int}(B_\alpha^2)$

Consider the tangent plane to the surface S at $r(x_\alpha)$: i.e., the linear/tangent plane approximation as follows:

~~Tangent~~ $r^{(\alpha)}: B_\alpha^2 \rightarrow \mathbb{R}^3$ defined by:

$$r^{(\alpha)}(x) = \underbrace{\left((Dr)(x_\alpha) \right)}_{\text{Total derivative}} (x - x_\alpha) + r(x_\alpha).$$

Plane or parallelogram

$$= [r_u \ r_v](x_\alpha) = [r_u(x_\alpha) \ r_v(x_\alpha)].$$

∴ We have $z = r^{(\alpha)}(x) = r_u(x_\alpha)x + r_v(x_\alpha)y + C$ (for some $C \in \mathbb{R}$)

∴ Area of the above parallelogram (By FACT in P-44)

$$= \|r_u(x_\alpha) \times r_v(x_\alpha)\| \quad (\text{Area } B_\alpha^2)$$

∴ Area of S under the partition P

$$= \sum_{\alpha \in N(P)} \|r_u(x_\alpha) \times r_v(x_\alpha)\|$$

This is the Riemann sum of the fn.

$$x \mapsto \|r_u(x) \times r_v(x)\|.$$

This is integrable as r is a C^1 -fn.

So, $\lim_{\|P\| \rightarrow 0} (\text{RHS}) = \text{Area of } S$.

⇒ We may define:

$$\int \text{Area of } S := \int \int \|r_u \times r_v\| dA.$$

We KNOW how to compute this.

The Riemann integration in 2-variables w.r.t "Area".

Def:

$$\text{Area}(S) := \int_{\mathcal{R}} \|T_u \times T_v\| dA.$$

Where $\mathcal{R} \subseteq \mathbb{R}^2$ is a region & $S = \text{range } \tau$ for some parameterization $\tau: \mathcal{R} \rightarrow \mathbb{R}^3$ of S .

Fact: Suppose $f \in C^1(\mathcal{R})$, $\mathcal{R} \subseteq \mathbb{R}^2$ a region. Recall

$\tau: \mathcal{R} \rightarrow \mathbb{R}^3$, defined by

$\tau(u, v) = (u, v, f(u, v))$ is a parameterization of $\text{graph}(f)$. Also, recall that (see page 33):

$$T_u \times T_v = (-f_u, -f_v, 1).$$

$$\Rightarrow \|T_u \times T_v\| = \sqrt{1 + (\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2}.$$

$$\therefore \text{Area}(\text{graph}(f)) = \int_{\mathcal{R}} \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dxdy. \quad \underbrace{dA}_{\text{dA}}$$

e.g.:

Consider truncated cylinder: $\tau: \mathcal{R} \rightarrow \mathbb{R}^3$, where

$$\mathcal{R} = \{(x, y) : 0 \leq x \leq \pi/2, 0 \leq y \leq 1\}$$

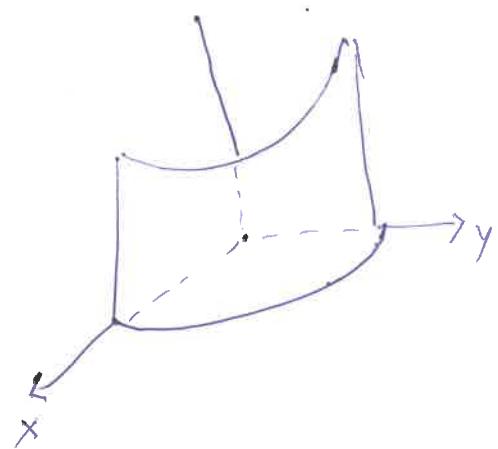
$$\therefore \tau(x, y) = (\cos x, \sin x, y).$$

$$\text{Here } T_x = (-\sin x, \cos x, 0)$$

$$T_y = (0, 0, 1)$$

$$\therefore T_x \times T_y = \begin{vmatrix} i & j & k \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos x, \sin x, 0)$$

$$\therefore \|T_x \times T_y\| = \sqrt{\cos^2 x + \sin^2 x} = 1.$$



$$\therefore \text{Area} = \int_{\mathcal{R}} 1 dA = \text{Area}(\mathcal{R}) = \frac{\pi}{2}.$$

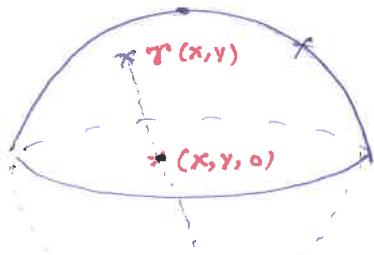
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e.g: (Area of a hemisphere)

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

The upper hemisphere.

Apply stereographic projection:



$$\text{i.e. } R = B_1(0,0) \\ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

$T(x, y)$ = the point of intersection of the upper hemisphere & the line joining P & $(x, y, 0)$.

$$\text{Then } T(x, y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right)$$

$\forall (x, y) \in R$. \leftarrow { - HW.
NOT good for computation!! }

Here, we apply spherical coordinate (for ease of computation):

We define $R = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \pi, 0 \leq v \leq \pi\}$.

$$T(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad \forall (u, v) \in R.$$

How: Check this from surface of revolution point of view.

$$\text{Then } \|T_u \times T_v\| = |\sin u|. \quad ?(\text{HW}).$$

$$\therefore \text{Area}(S) = \int_R \sin u \underbrace{\int du dv}_{dA}$$

$$= \int_0^{\pi} \int_0^{\pi} \sin u du dv$$

$$= \pi \int_0^{\pi} \int_0^{\pi} \sin u du dv$$

$$= \pi \int_0^{\pi} \left[-\cos u \right]_0^{\pi} du$$

$$= \pi \int_0^{\pi} 2 du$$

$$= \pi \times 2$$

$$= 2\pi$$

$$\frac{2\pi}{2}$$

Surface integrals of scalar fields:

Let S be a surface in \mathbb{R}^3 , $S \subset \overset{\text{open}}{\Omega_3} \subseteq \mathbb{R}^3$, & let $f \in \text{Cont}(\Omega_3)$ or $\mathcal{R}(\Omega_3)$. Then, as in the area computation, (see the bottom of Page 46).

We can prove:

$$\lim_{\|P\| \rightarrow 0} \sum_{\alpha \in \Lambda(P)} f(\tau(x_\alpha)) \|T_u(x_\alpha) \times T_v(x_\alpha)\| \text{Vol}(B_\alpha^2).$$

$$= \int_R f \circ \tau \|T_u \times T_v\| dA.$$

Where $\tau: R \rightarrow \mathbb{R}^3$ is a parametrization of the surface S .

Hence, we define:

Def: Given a surface S , $f \in \text{Cont}(\Omega_3)$, where $\Omega_3 \supseteq S$ is an open subset of \mathbb{R}^3 , the surface integral of f over the surface S is defined by:

$$\int_S f ds := \int_R f \circ \tau \|T_u \times T_v\| dA$$

Q: independent of the choice of τ ?

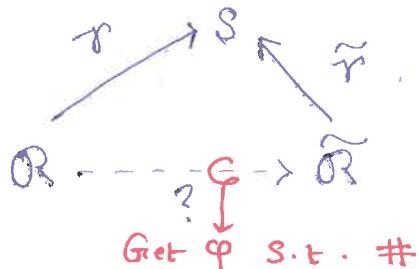
where $\tau: R \rightarrow \mathbb{R}^3$ is a parametrization of S .

Remark: "The RHS of the above": Suppose a curved metal plate/thin film lies along the surface S with density f_u . $f: S \rightarrow \mathbb{R}$. Then the RHS of the above definition, i.e., $\int_S f ds =$ mass of the metal plate !!

Independence of parametrizations: (Sketch).

Suppose $\tau: \mathbb{R} \rightarrow \mathbb{R}^3$ & $\tilde{\tau}: \tilde{\mathbb{R}} \rightarrow \mathbb{R}^3$ be two parameterizations of S . ~~So~~ ~~parametrize~~.

$\therefore S = \text{ran } \tau = \text{ran } \tilde{\tau}$. So, we have the following.



$$\text{i.e.: } \tau(u, v) = \tilde{\tau}(\varphi(u, v)).$$

For $(u, v) \in \mathbb{R}$, define $\varphi(u, v) = (\tilde{u}, \tilde{v}) \in \tilde{\mathbb{R}}$ as

$$\varphi = \tilde{\tau}^{-1} \circ \tau \quad \leftarrow \begin{array}{l} \text{Remember, } \tau \text{ & } \tilde{\tau} \text{ are} \\ \text{one-to-one maps.} \\ \& \text{ran } \tau = \text{ran } \tilde{\tau}. \end{array}$$

$\therefore \varphi: \mathbb{R} \rightarrow \tilde{\mathbb{R}}$ is a bijection !!.

P.T. φ is C^1 -fn. & $(J\varphi)(u, v)$ invertible $\forall (u, v) \in \mathbb{R}$.

Then apply the change of variables theorem/ formula along with chain rule \Rightarrow

$$\int_R f(\tau(u, v)) \| \tau_u \times \tau_v \| = \int_{\tilde{\mathbb{R}}} f(\tilde{\tau}(\tilde{u}, \tilde{v})) \| \tilde{\tau}_{\tilde{u}} \times \tilde{\tau}_{\tilde{v}} \|.$$

■