From now on: $n=2$ will be our setting.

More on measure zero:

Thm: Let $\underset{\text{bdd}}{\Omega} \supseteq \underset{\text{open}}{\mathcal{O}}$, $\Omega \subseteq \mathbb{R}^2$ & let $\overline{\Omega} \setminus \mathcal{O}$ is of measure zero.

Suppose $f \in \mathcal{B}(\Omega)$ & $f|_{\mathcal{O}}$ is continuous. Then $f \in \mathcal{R}(\Omega)$.

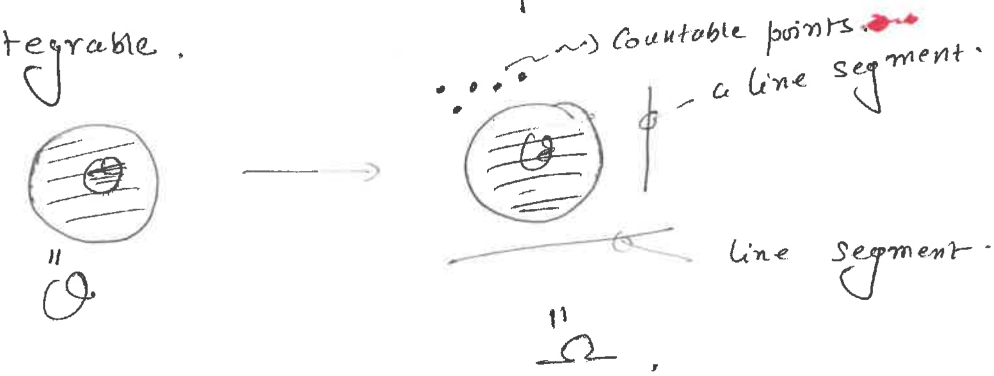
Remark: (1) Recall: Riemann-Lebesgue thm says: for $f \in \mathcal{B}(B^2)$, $f \in \mathcal{R}(B^2) \iff$ the set of discontinuity of f is of measure zero.

(2) From this perspective: the above thm is different: Ω is a bdd subset of $\mathbb{R}^2$.

(3) In particular: Consider a continuous fn. f on $\underset{\text{open}}{\mathcal{O}} \subseteq \mathbb{R}^2$.

Any extension (but bdd) of f to any bdd set Ω s.t. $\overline{\Omega} \setminus \mathcal{O}$ is of measure zero will be integrable.

eg:



(4) We are hoping the following:

Let $f \in \mathcal{B}(\underset{\text{bdd}}{\Omega})$ & let Ω is of measure zero.

Then $f \in \mathcal{R}(\Omega)$ & $\int_{\Omega} f = 0$.

Should be useful.

Proof. Consider a box B^2 s.t. $\mathrm{int}(B^2) \supseteq \overline{\Omega}$. Recall $\tilde{f} \in \mathcal{B}(B^2)$ is the extension of f:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \forall (x,y) \in \Omega \\ 0 & \forall (x,y) \in B^2 \setminus \Omega. \end{cases}$$

Note that $\tilde{f}\Big|_{\mathrm{int}(B^2)\setminus\overline{\Omega}} \equiv 0$. & $\mathrm{int}(B^2)\setminus\overline{\Omega}$ is an open set. Thus $\tilde{f}\Big|_{\mathrm{int}(B^2)\setminus\overline{\Omega}}$ is cont. fn. on $\mathrm{int}(B^2)\setminus\overline{\Omega}$.

Moreover, Ω is of measure zero $\Rightarrow$ $\overline{\Omega}$ is of measure zero. [HW] → Easy.

$\because$ the set of points of discontinuity of $\tilde{f}$ (namely $\overline{\Omega}$) is of measure zero, it follows that $\tilde{f} \in \mathcal{R}(B^2)$.

atmost

To prove: $\int_{B^2} \tilde{f} \left(= \int_{\Omega} f \right) = 0$: Let $\varepsilon > 0$.

Set $M = \sup_{\Omega} |f|$.

Now for $\varepsilon > 0$, $\exists$ a partition P of B^2 s.t.

$$\sum_{\alpha \in \tilde{\Lambda}} \mathrm{vol}(B^2_\alpha) < \varepsilon \quad \& \quad \bigcup_{\alpha \in \tilde{\Lambda}} (B^2_\alpha) \supseteq \Omega.$$

(for some $\tilde{\Lambda} \subseteq \Lambda(P)$).

Is: [Infact: get a finite cover of Ω with total area $< \varepsilon$ & then ~~extend~~ add some more subboxes to cover the entire B^2 : that will be the partition P.]

[Recall: if $f \in \mathcal{R}(\Omega)$, then $|f| \in \mathcal{R}(\Omega)$ & $\left|\int_\Omega f\right| \leq \int_\Omega |f|$.]

general fact.

Here: $\mathcal{U}(|\tilde{f}|, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha \, v(B^2_\alpha)$

P, as above

$$= \sum_{\alpha \in \tilde{\Lambda}} M_\alpha \, v(B^2_\alpha) \qquad \left[\because M_\alpha := \sup_{B^2_\alpha} |\tilde{f}| = 0 \ \forall \alpha \notin \tilde{\Lambda}\right].$$

$$\leq M \times \sum_{\alpha \in \tilde{\Lambda}} v(B^2_\alpha).$$

$$< M \times \varepsilon.$$

$$\Rightarrow \inf \mathcal{U}(|\tilde{f}|, P) = 0 \quad \Rightarrow \int_{B^2} \tilde{f} = 0.$$

$$\Rightarrow \int_\Omega f = 0. \qquad \square$$

Back to our thm:

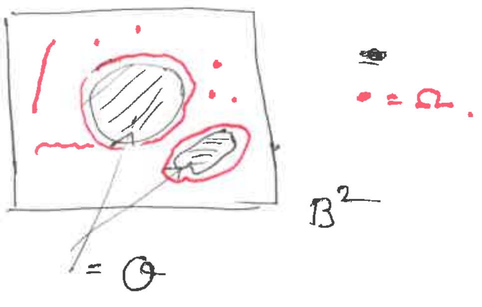
Thm: $\Omega \supseteq \mathcal{O}$ (Ω bdd, $\mathcal{O}$ open). Suppose $\bar{\Omega} \setminus \mathcal{O}$ is of measure zero, $f \in \mathcal{B}(\Omega)$ & $f|_{\mathcal{O}}$ is continuous. Then $f \in \mathcal{R}(\Omega)$.

Proof: Let $\mathrm{int}(B^2) \supseteq \bar{\Omega}$ & Consider $\tilde{f}$ on B^2 (extension of f).

Enough to prove that: $\mathcal{D}$, the set of points of discontinuity of $\tilde{f}$, is of measure zero.

Note that: (i) $\tilde{f}|_{\mathcal{O}}$ is cont. (ii) $\tilde{f}|_{\mathrm{int}(B^2) \setminus \bar{\Omega}} \equiv 0$ is cont.

& (iii) $\tilde{f}|_{\partial B^2} \equiv 0$ cont.



$\Rightarrow \mathcal{D} \subseteq \bar{\Omega} \setminus \mathcal{O} \leftarrow$ set of measure zero.

$\Rightarrow \mathcal{D}$ is a set of measure zero.

$\Rightarrow f \in \mathcal{R}(\Omega)$.

DANGER: Sets of measure zero depends on the "dimension".

For instance: ① $[0,1] \subseteq \mathbb{R}$ is not of zero measure

but $[0,1] \times \{a\} \subseteq \mathbb{R}^2$ is of measure zero

② $\mathbb{Q} \cap [0,1]$ is of measure zero ? Y/N :

③ $\mathbb{Q} \times \mathbb{Q} \cap ([0,1] \times [0,1])$ — " — ? Y/N :

Fact: Let $f : B^2 \to \mathbb{R}$ be a cont. fn. Then

Graphs have measure zero.

$$\operatorname{graph} f := \{(x, f(x)) : x \in B^2\} \quad (\subseteq \mathbb{R}^3)$$

is a set of measure zero.

Proof: Let $\varepsilon > 0$. Note that: f is uniformly cont.

$\therefore \exists\ \delta > 0 \ni \quad |f(x) - f(y)| < \varepsilon \quad \forall\ |x - y| < \delta.$
$(x, y \in B^2)$

Next, on this $\delta > 0$, pick a partition ~~we are in~~ P of B^2

s.t. the diameter of $B^2_\alpha < \delta \quad \forall\ \alpha \in \Lambda(P)$.

$$\Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall\ x, y \in B^2_\alpha, \quad \alpha \in \Lambda(P).$$

Set $I_\alpha := \{f(x) : x \in B^2_\alpha\}$.

The range set of $f|_{B^2_\alpha}$.

$\Rightarrow \quad I_\alpha \subseteq \tilde{I}_\alpha$, an interval of length atmost ε. $\forall \alpha$.

$\therefore \{B^2_\alpha \times \tilde{I}_\alpha : \alpha \in \Lambda(P)\}$ is a cover of boxes of graph f. Also:

$$\sum_{\alpha\in\Lambda(P)} v\left(B^2_\alpha \times \tilde{I}_\alpha\right) = \sum_{\alpha\in\Lambda(P)} v(B^2_\alpha) \times v(\tilde{I}_\alpha)$$

$$\leq \sum_{\alpha\in\Lambda(P)} v(B^2_\alpha) \times \varepsilon.$$

$$= \underbrace{v(B^2)}_{\text{Constant.}} \times \varepsilon.$$

$\Rightarrow$ measure of graph f is zero. $\boxed{///}$

Infact, we have the following:

Better !! Let $f\in\mathcal{R}([a,b])$. Then $G := \{(x,f(x)) : x\in[a,b]\} \subseteq \mathbb{R}^2$ is of measure zero.

Proof. We proceed along the same line:

Let $\varepsilon>0$. $\exists$ $P\in\mathcal{P}([a,b])$ $\ni$

$$U(f,P) - L(f,P) < \varepsilon.$$

Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

& $B^2_i := [x_{i-1}, x_i] \times [m_i, M_i]$,

Here: $m_i = \inf_{[x_{i-1},x_i]} f$

& $M_i = \sup_{[x_{i-1},x_i]} f$.

$\therefore$ $G \subseteq \bigcup_{i=1}^n B^2_i$. Finally:

$$\sum_{i=1}^n v(B^2_i) = \sum_{i=1}^n v\left([x_{i-1},x_i]\times[m_i,M_i]\right)$$

$$= \sum_{i=1}^n (x_i - x_{i-1}) \times (M_i - m_i).$$

Smart proof ? : Than P-42 ? $\rightarrow$

$$= U(f,P) - L(f,P)$$

$$< \varepsilon. \quad \boxed{///}$$

Back to Fubini's thm:

Recall: Let $f \in \mathcal{R}(B^2)$. Set $B_2 = [a,b] \times [c,d]$.

If $\int_a^b f(x,y)\,dx$ exists $\forall\, y \in [c,d]$, then

$$\int_{B^2} f = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy. \quad \text{—— } (1)$$

IIIly if, $\int_c^d f(x,y)\,dy$ exists for each $x \in [a,b]$, then

$$\int_{B^2} f = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \quad . \quad \text{—— } (2)$$

If $f \in C(B^2)$, then $(1) = (2)$.

—— x ——.

$\boxed{\text{Q: Fubini for } f \in \mathcal{R}(\Omega), \quad \Omega \subseteq \mathbb{B}^2, \text{ bdd ??}}$

How to think about it ?

In fact: it is not easy to "evaluate" double integral over
COMPUTE

$\Omega \subseteq \mathbb{R}^2$. However, with "some" control over Ω,
bdd

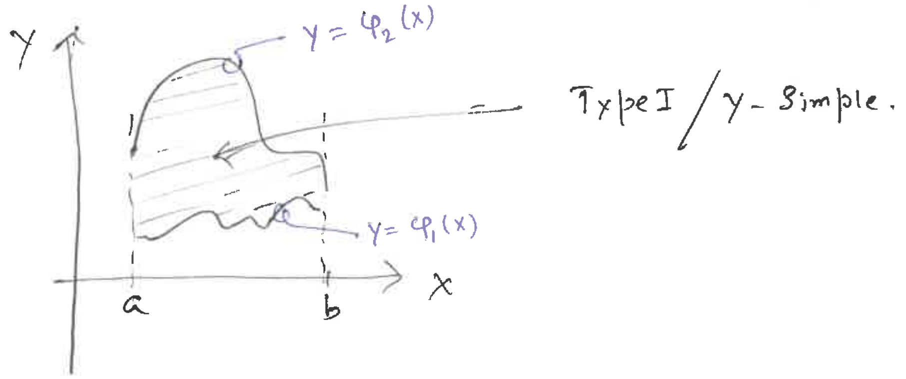
one can do "something". It is as follows:

Two special domains (AKA: Elementary regions):

Def: A set $\Omega \subseteq \mathbb{R}^2$ is said to be <u>y-Simple / Type I</u> if $\exists\ \varphi_1, \varphi_2 \in \mathcal{R}([a,b])$ s.t.

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}.$$

Here:

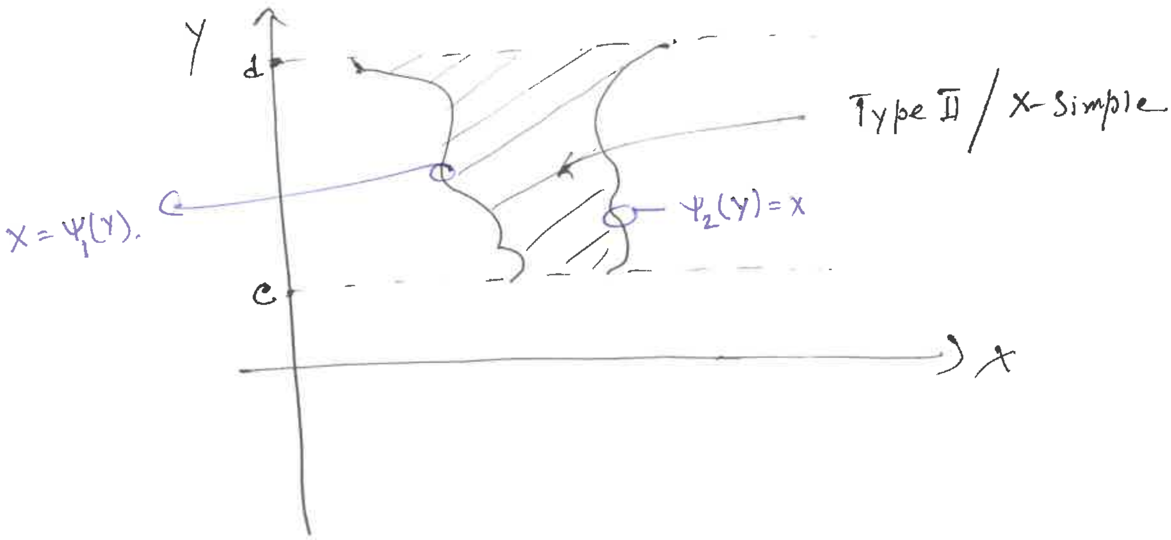


$\parallel_y$ <u>x-Simple / Type II</u> regions are given by:

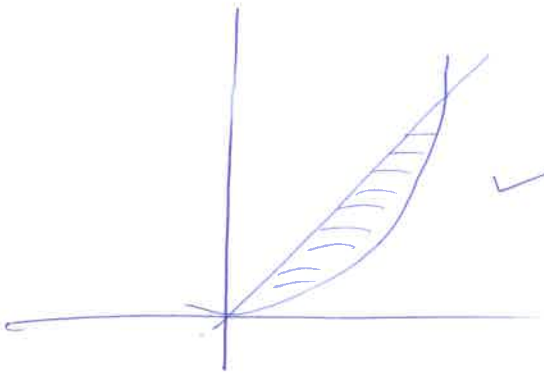
$$\Omega = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}.$$

for some $\psi_1, \psi_2 \in \mathcal{R}[c,d]$.

Here:



eg:



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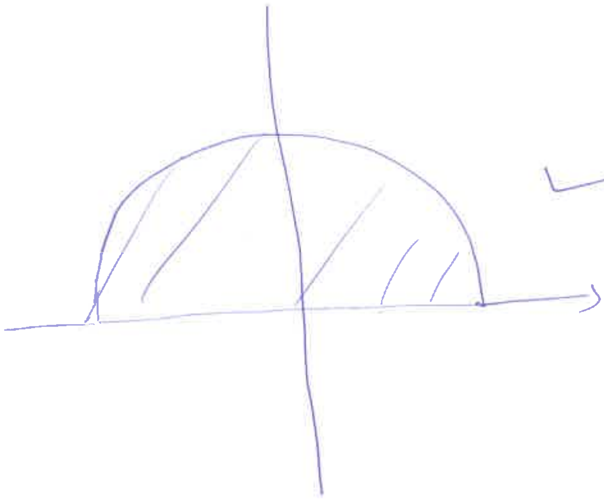

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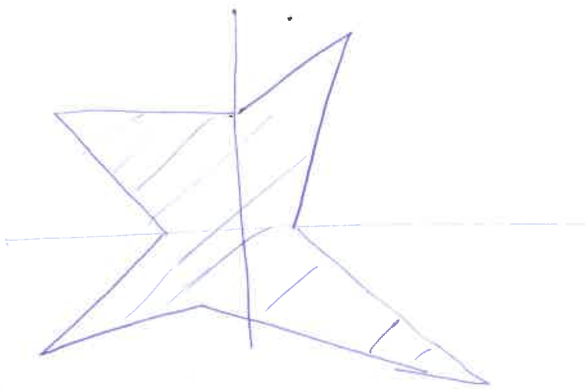

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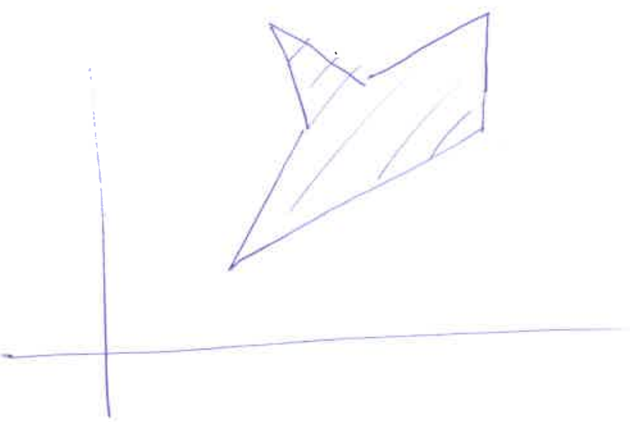

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