Curves and Surfaces. (in $\mathbb{R}^n$).

$$I \xrightarrow[\text{Cont.}]{\subseteq \mathbb{R}} \mathbb{R}^n$$

$$O_2 \xrightarrow[\text{Cont.}]{} \mathbb{R}^n \quad (n \geq 3).$$

$\subseteq \mathbb{R}^2$, open & ∂O_2 's of content zero.
i.e. O_2 has area.

(Goal)

Roughly:
Curve $\leftrightarrow$ 1 dimensional object.
Surface $\leftrightarrow$ 2 dim. object.

Differentiate, integrate, & then also relate them.

Notation: ① $I = [a,b]$, $a < b$. ② $C^1 = $ ~~Smooth~~ Continuously diff. fn's.
$(c,b], (-\infty, a), (-\infty,\infty)$.

Defo 1) A parametrized Curve/path is a Continuous fn.

$$\gamma : I \longrightarrow \mathbb{R}^n.$$

2) Given a parametrized Curve γ, $\{\gamma(t) : t \in I\}$ is Called ~~the~~ a path.

just a set / subset of $\mathbb{R}^n$
~~given as~~: the range of a parametrized Curve.

3) A parametrized Curve is ~~smooth~~ $\underset{C^1\text{-Curve}}{}$ if γ is a C^1- fn.

Recall: ① $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$.
Here $\gamma(t)$ is C^1 means $\gamma_i : I \to \mathbb{R}$
is ~~smooth~~ C^1, $\forall$ $1 \leq i \leq n$.
② If $I = [a,b]$, then $\gamma'(a)$ or $\gamma'(b)$
will be defined as one-sided
limits [or, γ has ~~a~~ a C^1-
extension to an open set $O \supseteq I$]

4) A C^1 Curve $\gamma : I \longrightarrow \mathbb{R}^n$
is said to be smooth
if $\gamma'(t) \neq 0 \quad \forall t \in I$.

you dont want
the direction
to change rapidly.

eg: ① $\gamma : [0,1] \longrightarrow \mathbb{R}^2$, defined by

$$\gamma(t) = \underbrace{(1,2)}_{\in \mathbb{R}^2} + t\underbrace{(-2,1)}_{\in \mathbb{R}^2} \qquad t \in [0,1].$$

$$= (1-2t, 2+t).$$

$\gamma'(t) = (-2, 1)$
$\|$
Smooth curve.

A line: $\begin{array}{l} x = 1-2t \\ y = 2+t \end{array}\Big\}$ parametric form. $\implies \frac{1-x}{2} = y-2$
$\implies x + 2y = 5$.

(-1, 3)
γ.
(1,2)

② $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$ defined by $\quad \gamma(t) = (r\cos t, r\sin t)$. $\quad r > 0$.

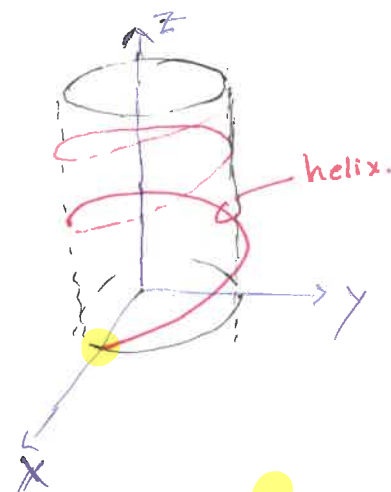Here $\gamma'(t) = (-r\sin t, r\cos t)$. $\quad \Rightarrow \gamma'(t) \neq 0 \quad \forall t$.

$\therefore \gamma$ represents a circle oriented counterclockwise.

③ $\gamma : [0, \overset{= 4\pi/n\pi}{a}] \longrightarrow \mathbb{R}^3, \quad \gamma(t) := (r\cos t, r\sin t, ct), \quad r > 0, c \neq 0.$

If $t = 0 : \quad \gamma(0) = (r, 0, 0).$

The ~~curve~~ path is known as Helix.

Smooth curve.

helix.

④ $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2$ defined by

$\gamma(t) = (|t|, t) \quad$ is not a

C^1 - curve $(\Rightarrow$ non-smooth$)$.

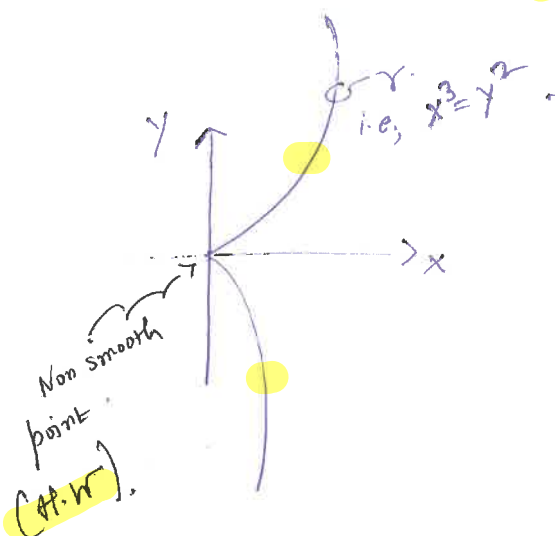Also, $\gamma(t) = (0, t^2)$ is C^1 - curve but not smooth:

$\quad \gamma'(t) = (0, 2t) \quad \Rightarrow \gamma'(0) = 0.$

$(0,0) \xrightarrow{} (0,1)$
$\gamma(t) = t(0,1)$
$\quad = (0, t)$

⑤ $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2, \quad \gamma(t) = (t^2, t^3) \quad \leftarrow$ ~~Cusps~~ Cuspidal Cubic.

C^1 but non-smooth.

i.e., $x^3 = y^2$.

y

Non smooth point $(H.W)$.

$> x$

Here $x = t^2, y = t^3.$
$\Rightarrow x^3 = y^2$
Cuspidal Cubic.

⑥ Recall : path is the range/trace of a prametrized curve.

Then for $\gamma(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$, the corresponding

$$\text{path} = \{ \gamma(t) : \cancel{r\cos t, r} \quad 0 \leq t \leq 2\pi \}$$

$$= \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2 \}$$

twice rotation

$$= \{ \tilde{\gamma}(t) : 0 \leq t \leq 2\pi \}, \quad \text{where} \quad \tilde{\gamma}(t) = (r\cos 2t, r\sin 2t)$$
$$t \in [0, 2\pi].$$

∴ γ & $\tilde{\gamma}$ are

∴ $\cancel{C\gamma \text{ is a path}}$ $\tilde{\gamma}, \gamma$ are different parameterizations

of $C_\gamma(0)$.

the path

⑦ Given a ~~smooth~~ fn. $f : I \to \mathbb{R}^n$, define $\gamma(t) = (t, f(t))$, $t \in I$.

∴ γ is a parameterization of the graph of f.

parameterizations of graphs.

If $f \in C^1$, then γ is smooth. $\in [0,1]$

$t \mapsto (t, t)$

$[0, 1] \ni t \mapsto (t^2, t^2)$

⑧ $\gamma(t) = (e^t \cos t, e^t \sin t)$ $\quad t \in I \leftarrow$ Any interval.

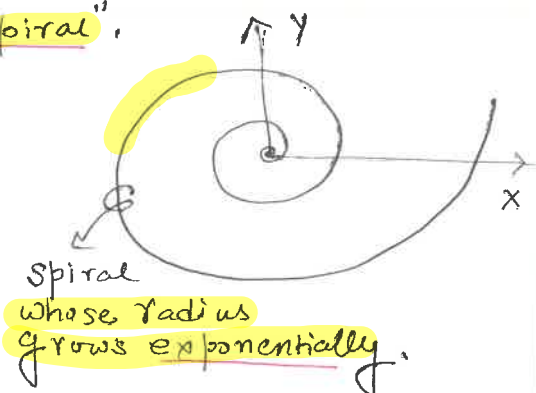$\downarrow = x \qquad \downarrow = y$

∴ $x^2 + y^2 = e^{2t}$ (Not a good representation)

But $\sqrt{x^2 + y^2} = e^t$.

∴ In polar coordinate: $r = e^t$. Also $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} = \cos t$

$\cancel{r\cos\theta = r e^t \cos t}$

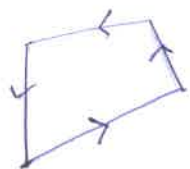$\Downarrow$

$\theta = t \quad (t = \theta + 2n\pi)$

∴ $r = e^\theta$ ← polar representation of γ.

Called "Logarithmic Spiral".

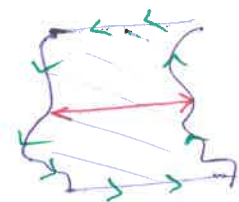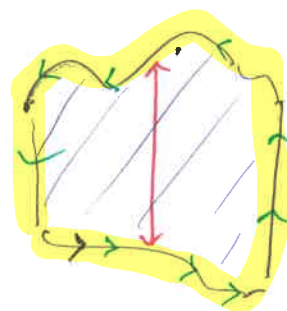spiral whose radius grows exponentially.

Def: A parametrized curve $\gamma : I \longrightarrow \mathbb{R}^n$ is called **piecewise smooth** if $\exists$ a partition of $I = [a,b]$, say, $a = x_0 < x_1 < \cdots < x_n = b$, s.t. $\gamma \Big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is a smooth parametrized curve, $i = 1, \ldots, n$.

eg:

boundary of Type I & Type II:

equivalent Curves:

Consider $\quad t \overset{\gamma}{\longmapsto} (r\cos t, r\sin t) \qquad t \in [0, 2\pi]$.

$\quad t \overset{\tilde{\gamma}}{\longmapsto} (r\cos 2t, r\sin 2t) \qquad t \in [0, \pi]$.

Clearly, $\quad \gamma(2t) = \tilde{\gamma}(t)$.

or $\quad \gamma(\varphi(t)) = \tilde{\gamma}(t)$, where $\varphi(t) = 2t$.

$\tilde{\gamma}$ is a reparametrization of γ.   a parametrization.

\# But often, we need φ to be a "good" parametrization!! $\Rightarrow$

Def: Two parametrized curves $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ & $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \longrightarrow \mathbb{R}^n$ are said to be **equivalent** if $\exists$ strictly increasing parametrized curve $\varphi : [\tilde{a}, \tilde{b}] \longrightarrow [a,b]$, onto & differentiable (often C^1) s.t. $\tilde{\gamma} = \gamma \circ \varphi$,
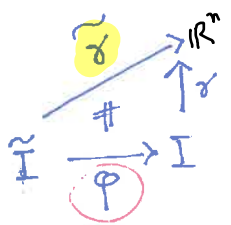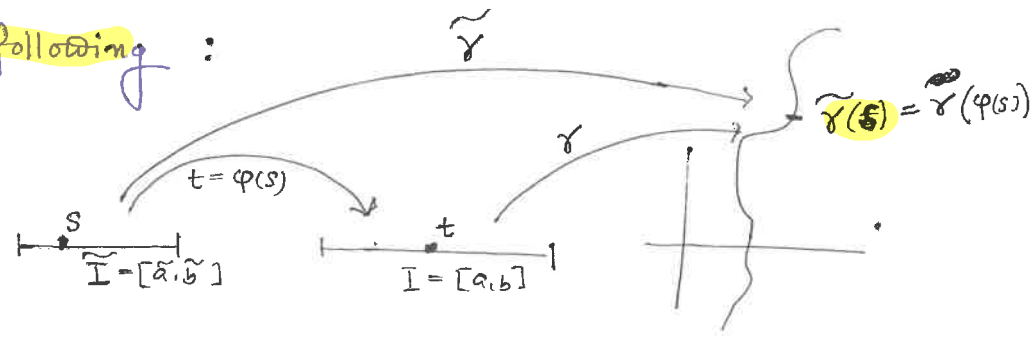
A reparametrization.   a "smooth"/C^1 parametrization.

So, we have the following :

$\tilde{\gamma}$

$$\tilde{I} \xrightarrow{\varphi} I$$

$\tilde{\gamma} \to \mathbb{R}^n$

$\#$

$\uparrow \gamma$

i.e.

$t = \varphi(s)$

$\tilde{I} = [\tilde{a}, \tilde{b}]$

$I = [a, b]$

$\tilde{\gamma}(s) = \gamma(\varphi(s))$

If $\gamma, \tilde{\gamma}$ are C^1, or smooth, then we also impose the same to φ.

$\therefore$ φ is change of time. Here $\tilde{\gamma}$, at time s, is at $\tilde{\gamma}(s)$, where γ arrives there at time $t = \varphi(s)$!!

$\#$ From now on: Curve $\longleftrightarrow$ parametrized curve.

Def: ~~Given~~ Let $\gamma : I \to \mathbb{R}^n$ be a $\overset{C^1}{\checkmark}$ curve.

(1) $\| \gamma'(t) \| :=$ "Speed" of γ at time $t \in I$.

(2) $\displaystyle\int_{t_1}^{t_2} \| \gamma'(t) \| \, dt :=$ Arc length of γ between times t_1 & t_2. $(t_1 < t_2)$.

Question:
$\#$ Speed is somewhat clear (practical point of view), as $\| \gamma'(t) \|$ is the magnitude of the velocity $\gamma'(t)$. But what is the interpretation of arc length ?? — WAIT.

Remark: 1) For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2} = d(x, \underline{0})$$

$\underline{0} = (0, \dots, 0)$

2) Let $\gamma(t) = (x_1(t), \dots, x_n(t))$.

$\therefore \gamma'(t) = (x_1'(t), \dots, x_n'(t))$.

$$\Rightarrow \| \gamma'(t) \| = \sqrt{\sum_{i=1}^{n} (x_i'(t))^2}.$$

$\therefore \gamma'$ is C^1 $\Rightarrow$ ~~each~~ x_i, $1 \le i \le n$, is C^1

$\Rightarrow t \mapsto \| \gamma'(t) \| \in C(I)$.

$\Rightarrow$ Arc length is well defined.

But, there is another way, (perhaps more natural) to introduce arc length of curves. We do it by ~~linear~~ polygonal approximation.

Consider a path $\gamma : [a,b] \longrightarrow \mathbb{R}^n$.

Let P be a partition of $[a,b]$, i.e.
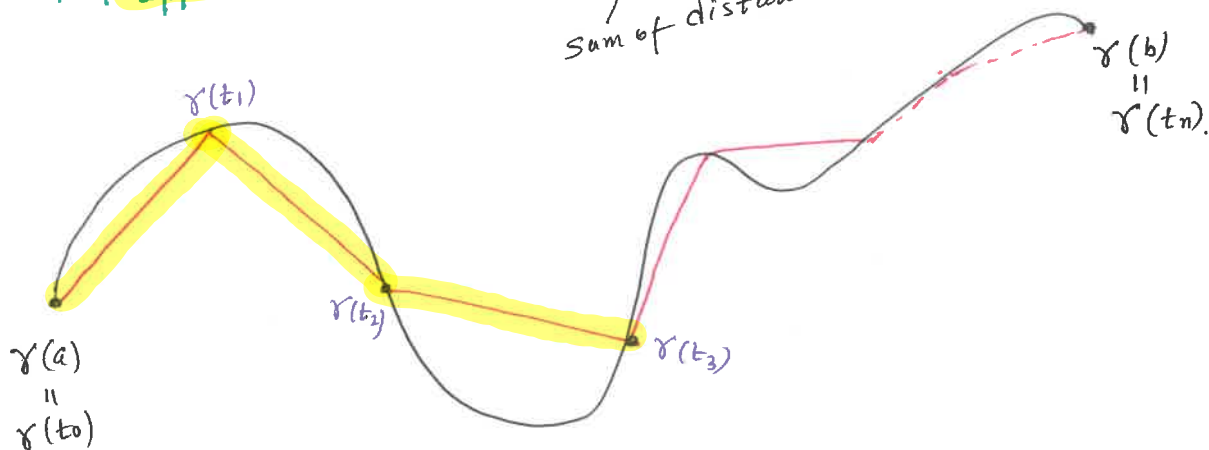
$$P : a = t_0 < t_1 < \cdots < t_n = b.$$

Now we consider $\{ \gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n) \}$, and then the distance between $\gamma(t_{i-1})$ & $\gamma(t_i)$ as: $\| \gamma(t_i) - \gamma(t_{i-1}) \|$,

$$i = 1, \cdots, n.$$

Finally, define:

$$\ell(\gamma, P) := \sum_{i=1}^{n} \| \gamma(t_i) - \gamma(t_{i-1}) \|.$$

polygonal approximation.

sum of distances between $\gamma(t_i)$.

$\gamma(t_1)$

$\gamma(t_2)$

$\gamma(t_3)$

$\gamma(a)$ = $\gamma(t_0)$

$\gamma(b)$ = $\gamma(t_n)$.

<u>Def</u>: A curve $\gamma : [a,b] \longrightarrow \mathbb{R}$ is said to have arc length, or to be rectifiable, if

if exists, it is !.

$$\lim_{\|P\| \to 0} \ell(\gamma, P) := \ell(\gamma) \quad \text{exists}.$$

length of γ.

For $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|\ell(\gamma, P) - \ell(\gamma)| < \varepsilon \quad \forall P \text{ s.t. } \|P\| < \delta.$$

mesh = max {sub interval of P}