

general fact.

[Recall: if $f \in R(\Omega)$, then $|f| \in R(\Omega)$ & $|\int_{\Omega} f| \leq \int_{\Omega} |f|$.]

Here: $U(\tilde{f}, P) = \sum_{\alpha \in \tilde{A}(P)} M_{\alpha} v(B_{\alpha}^2)$
P, as above
 $= \sum_{\alpha \in \tilde{A}} M_{\alpha} v(B_{\alpha}^2)$ $\because M_{\alpha} := \sup_{B_{\alpha}^2} |f| = 0 \forall \alpha \notin \tilde{A}$
 $\leq M \times \sum_{\alpha \in \tilde{A}} v(B_{\alpha}^2) < \varepsilon$
 $< M \times \varepsilon$

$\Rightarrow \inf U(\tilde{f}, P) = 0 \Rightarrow \int_{B^2} \tilde{f} = 0$

$\Rightarrow \int_{\Omega} f = 0$ \square

Back to our thm: (Proof is similar).

if $\Omega = B^2$, nothing to prove.

Thm: $\Omega \supseteq \emptyset$. Suppose $\bar{\Omega} \setminus \emptyset$ is of ~~measure~~ ^{Content} zero,
 \uparrow bdd. \uparrow open

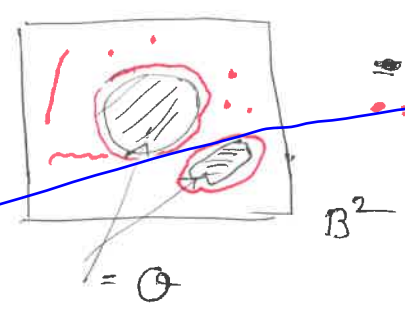
$f \in B(\Omega)$ & $f|_{\emptyset}$ is continuous. Then $f \in R(\Omega)$.

Proof: Let $\text{int}(B^2) \supseteq \bar{\Omega}$ & Consider \tilde{f} on B^2 (extension of f).

Enoug to prove that: \mathcal{D} , the set of points of discontinuity of \tilde{f} , is of measure zero.

Note that: (i) $\tilde{f}|_{\emptyset}$ is Cont. (ii) $\tilde{f}|_{\text{int}(B^2) \setminus \bar{\Omega}} \equiv 0$ is Cont.
 \nearrow open set.
(iii) $\tilde{f}|_{\partial B^2} \equiv 0$ Cont.

$\Rightarrow \mathcal{D} \subseteq \bar{\Omega} \setminus \emptyset \leftarrow$ Set of measure zero.
 $\Rightarrow \mathcal{D}$ is a set of measure zero.
 $\Rightarrow f \in R(\Omega)$.



DANGER: Sets of content zero depends on the "dimension".

For instance: $[0,1] \subseteq \mathbb{R}$ is not of measure zero ^{C.Z}

but $[0,1] \times \{a\} \subseteq \mathbb{R}^2$ is of measure zero ^{C.Z}

~~② $\mathbb{Q} \cap [0,1]$ is of measure zero? X/N: NO.~~

~~③ $\mathbb{Q} \times \mathbb{Q} \cap ([0,1] \times [0,1])$ is of measure zero? X/N: YES.~~

Fact: Let $f: B^2 \rightarrow \mathbb{R}$ be a Cont. fn. Then

$$\text{graph } f := \{(x, f(x)) : x \in B^2\} \quad (\subseteq \mathbb{R}^3)$$

is a set of Content zero.

Graphs have measure zero.
Content

works for $f: B^n \rightarrow \mathbb{R}$.

Proof: Let $\varepsilon > 0$. Note that: f is uniformly Cont.

$$\therefore \exists \delta > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \quad \forall |x - y| < \delta. \\ (x, y \in B^2)$$

Next, on this $\delta > 0$, pick a partition P of B^2

S.t. the diameter of $B_\alpha^2 < \delta \quad \forall \alpha \in \Lambda(P)$.
"diagonal"

$$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in B_\alpha^2, \alpha \in \Lambda(P).$$

Set $I_\alpha := \{f(x) : x \in B_\alpha^2\}$

The range set of $f|_{B_\alpha^2}$.

$\Rightarrow I_\alpha \subseteq$ interval of length at most ε ^{for some} $\forall \alpha$.

$\therefore \{B_\alpha^2 \times I_\alpha : \alpha \in \Lambda(P)\}$ is a cover of boxes of

graph f . Also:

As $(x, f(x)) \in B_\alpha^2 \times I_\alpha$
 $\forall x \in B_\alpha^2, (\alpha \in \Lambda(P))$

$\Lambda(P)$ is a finite set, $\mathcal{S}:-$

(43)

$$\sum_{\alpha \in \Lambda(P)} v(B_\alpha^2 \times \tilde{I}_\alpha) = \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times v(\tilde{I}_\alpha)$$

$$\leq \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times \varepsilon.$$

$$= v(B^2) \times \varepsilon.$$

Constant.

\Rightarrow ^{Content} ~~measure~~ of graph f is zero. \square

In fact, we have the following:

The proof is EVEN Better!!

Let $f \in \mathcal{R}([a, b])$. Then $G := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ is of ~~measure~~ ^{Content} zero.

Proof: We proceed along the same line:

Let $\varepsilon > 0$. $\exists P \in \mathcal{P}([a, b])$ s.t.

$$U(f, P) - L(f, P) < \varepsilon.$$

Set $P: a = x_0 < x_1 < \dots < x_n = b$.

$$\mathcal{S} \quad B_i^2 := [x_{i-1}, x_i] \times [m_i, M_i],$$

$$\text{Here: } m_i = \inf_{[x_{i-1}, x_i]} f$$

$$\mathcal{S} \quad M_i = \sup_{[x_{i-1}, x_i]} f.$$

$\therefore G \subseteq \bigcup_{i=1}^n B_i^2$. Finally:

$$\sum_{i=1}^n v(B_i^2) = \sum_{i=1}^n v([x_{i-1}, x_i] \times [m_i, M_i])$$

$$= \sum_{i=1}^n (x_i - x_{i-1}) \times (M_i - m_i).$$

$$= U(f, P) - L(f, P)$$

$$< \varepsilon.$$

\square

Smart proof? \rightarrow
: Then P-42?

Back to Fubini's thm:

Recall: Let $f \in \mathcal{R}(B^2)$. Set $B_2 = [a,b] \times [c,d]$.

If $f \in C(B^2)$.

If $\int_a^b f(x,y) dx$ exists $\forall y \in [c,d]$, then

$\iint_{[a,b] \times [c,d]} f$
 $= \int_c^d \left(\int_a^b f(x,y) dx \right) dy$
 $= \int_a^b \left(\int_c^d f(x,y) dy \right) dx$

$$\int_{B^2} f = \int_c^d \left(\int_a^b f(x,y) dx \right) dy \quad \text{--- (1)}$$

Note: integrability of this f_y (in x) is guaranteed by Fubini.

if, $\int_c^d f(x,y) dy$ exists for each $x \in [a,b]$, then

$$\int_{B^2} f = \int_a^b \left(\int_c^d f(x,y) dy \right) dx \quad \text{--- (2)}$$

If $f \in C(B^2)$, then (1) = (2).

in particular,

Q: Fubini for $f \in \mathcal{R}(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, bdd ??

How to think about it?

In fact: it is not easy to "evaluate" double integral over

COMPUTE

$\Omega \subseteq \mathbb{R}^2$
 \uparrow
bdd.

However, with "Some Control" over Ω ,

one can do "Something". It is as follows:

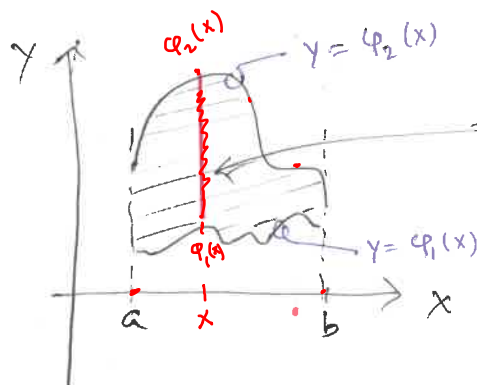
[Remark: Many/all of the results below works similarly in \mathbb{R}^n , $n \geq 3$.
At least, think them in the setting of \mathbb{R}^3 .]

Two special domains (AKA: Elementary regions):

Def: A set $\Omega \subseteq \mathbb{R}^2$ is said to be y-simple / Type I if $\exists \varphi_1, \varphi_2 \in \mathcal{R}([a, b])$ s.t.

$$\Omega = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

Here:



Type I / y-simple.

"Convex along vertical"



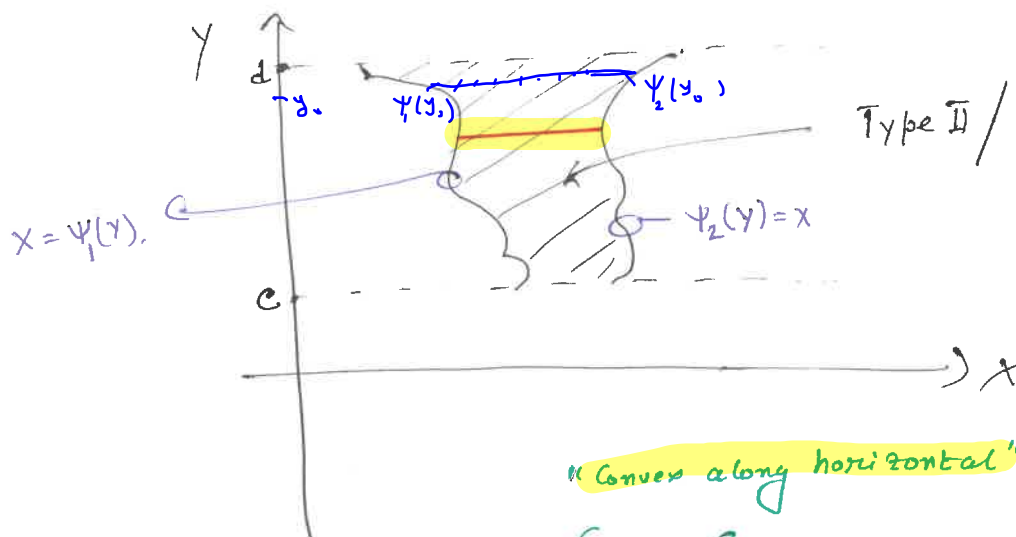
1/4

x-simple / Type II regions are given by:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$$

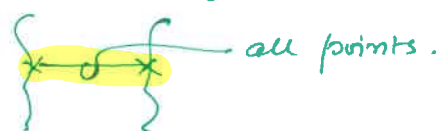
for some $\psi_1, \psi_2 \in \mathcal{R}[c, d]$.

Here:

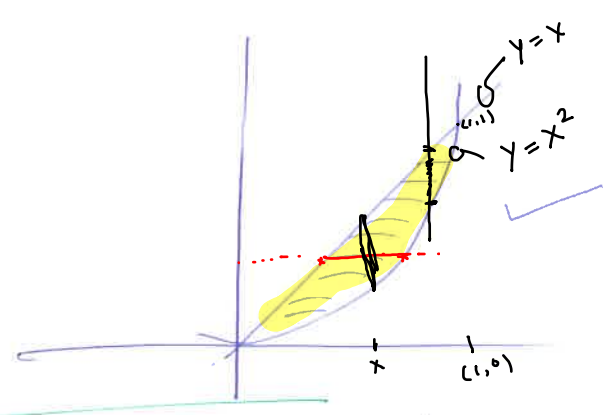


Type II / x-simple

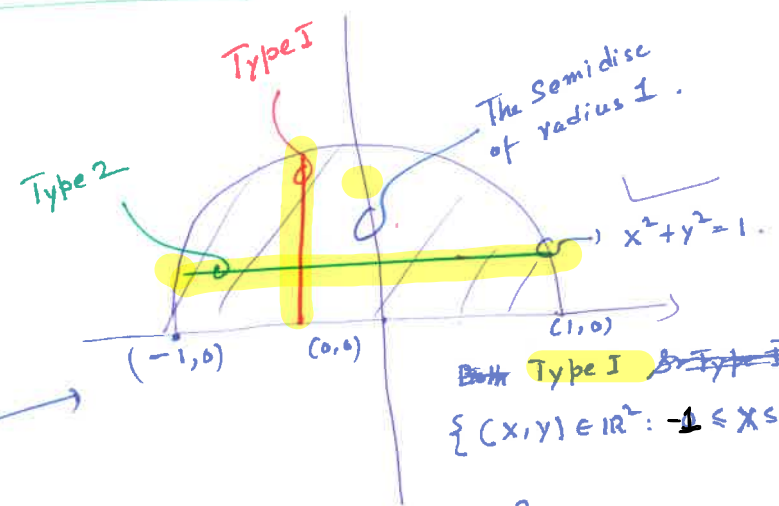
"Convex along horizontal"



eg:



$$\{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq x\}$$

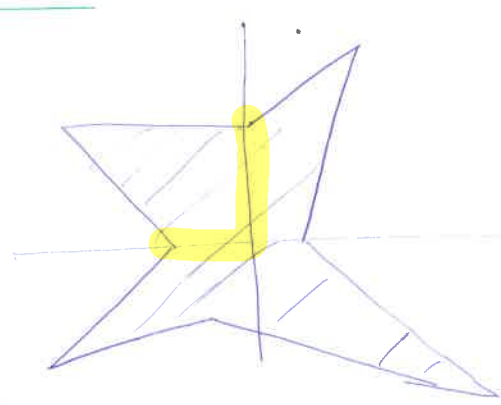
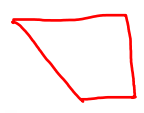


Both Type I & Type II.

$$\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$

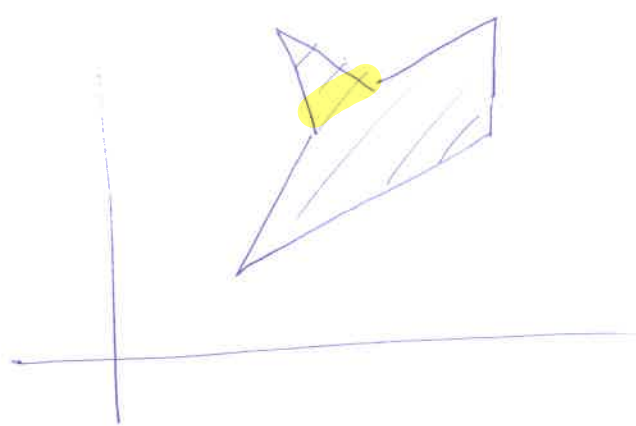
Also Type II:

$$\{(x, y) : 0 \leq y \leq 1, -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$$



? X

BUT : Sum of elementary regions!!



? X

Thm: Let $f \in \mathcal{R}(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ (an elementary region).

(I) If $\Omega = \{(x, y) : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, for some $\varphi_1, \varphi_2 \in \mathcal{B}[a, b]$, and if $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy$ exists $\forall x \in [a, b]$, then

$$\iint_{\Omega} f(x, y) dA = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx.$$

Just notation. $\int_{\Omega} f(x, y) dA = \int_{\Omega} f$

EASY interpretation. $\varphi_2(x)$ $\varphi_1(x)$

THIS MUST EXIST $\forall x$: Then integrability is assured.

(II) If $\Omega = \{(x, y) : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$, for some $\psi_1, \psi_2 \in \mathcal{B}[c, d]$, and if $\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx$ exists $\forall y \in [c, d]$, then

$$\iint_{\Omega} f = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right) dy$$

THIS MUST EXIST $\forall y \in [c, d]$.

Proof (Easy application of Fubini):

We ~~will~~ prove only (I), as (II) will be similar.

Get $c < d$ s.t. $\Omega \subseteq B^2 := [a, b] \times [c, d]$.

[In fact: $c = \inf_{[a, b]} \varphi_1$ & $d = \sup_{[a, b]} \varphi_2$ is one natural choice.]

Consider the extension $\tilde{f} : B^2 \rightarrow \mathbb{R}$, where $\tilde{f}|_{\Omega} \equiv f$ & $\tilde{f}|_{B^2 \setminus \Omega} \equiv 0$.

We know $\tilde{f} \in \mathcal{R}(B^2)$. Now for each $x \in [a, b]$, $\int_c^d \tilde{f}(x, y) dy$ exists.

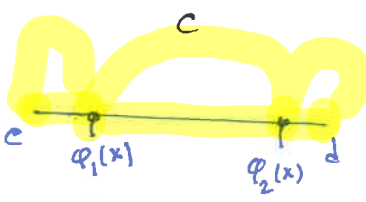
Indeed: $\tilde{f}(x, y) = \begin{cases} f(x, y) & \forall y \in [\varphi_1(x), \varphi_2(x)] \\ 0 & \forall y \in [c, \varphi_1(x)] \cup [\varphi_2(x), d] \end{cases}$
for fixed $x \in [a, b]$

So But $f(x, \cdot)|_{[\varphi_1(x), \varphi_2(x)]}$ & $f(x, \cdot)|_{[c, \varphi_1(x)] \cup [\varphi_2(x), d]}$

are integrable. So, by 1-variable result, $\tilde{f}(x, \cdot) \in \mathcal{R}[c, d]$.

Finally, again for fixed $x \in [a, b]$, by 1-variable additivity:

$$\begin{aligned} \int_c^d \tilde{f}(x, y) dy &= \int_c^{\varphi_1(x)} \tilde{f}(x, y) dy + \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x, y) dy + \int_{\varphi_2(x)}^d \tilde{f}(x, y) dy \\ &= \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x, y) dy \\ &= \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy. \end{aligned}$$




$\left[\because \tilde{f}(x, y) = f(x, y) \quad \forall \varphi_1(x) \leq y \leq \varphi_2(x) \right]$

Then, by Fubini ($\because \forall x \in [a, b], \int_c^d \tilde{f}(x, y) dy$ exists):

$$\begin{aligned} \iint_{\Omega} f &\stackrel{\text{DEF}}{=} \iint_{B^2} \tilde{f} \stackrel{\text{FUBINI}}{=} \int_a^b \left(\int_c^d \tilde{f}(x, y) dy \right) dx \\ &= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx. \end{aligned}$$



eg: Complete  f , where $f \in R(\Omega)$

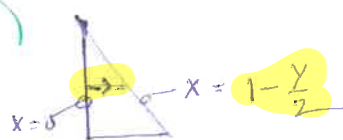
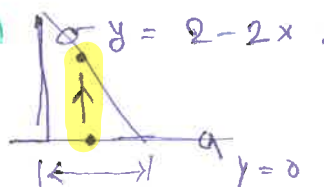
eg: Consider $f \in C(\Omega)$, where $\Omega =$

Clearly, Ω is both Type I & Type II.

Also, $f \in R(\Omega)$. Then

$$\iint_{\Omega} f = \int_0^1 \left(\int_0^{2-2x} f(x, y) dy \right) dx$$

$$= \int_0^2 \left(\int_0^{1-\frac{y}{2}} f(x, y) dx \right) dy$$



Often, changing order of integration is useful. We will also see.