Indian Statistical Institute

Analysis III (HW - 4) Due date: Take your sweet time Instructor : Jaydeb Sarkar

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NOTE: (i) $B^n = \Pi_{i=1}^n [a_i, b_i]$. (ii) $R(B^n)$ = the set of all Riemann integrable functions on B^n. (iii) $v(\Omega)$ =volume of $\Omega(\subseteq \mathbb{R}^n)$, whenever $n \geq 3$. (iv) $A(\Omega)$ = area of $\Omega(\subseteq \mathbb{R}^2)$.

(1) Consider a closed box $B^n \subseteq \mathbb{R}^n$ and a matrix $A = (a_{ij})_{i,j=1}^n$. Prove that $A(B^n) = \{Ax : x \in R^n\} \subseteq \mathbb{R}^n$ has a volume, and

$$v(A(B^n)) = |\det A| v(B^n).$$

[Hint: Enough to prove for elementary matrices.]

(2) Prove that the volume of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi abc}{3}$.

(3) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

(4) Compute $\int_\Omega \frac{y}{x} dA$, where Ω is the region bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = \frac{1}{2}x$.

(5) Let $0 < \epsilon < 1$. Prove that $A(\Omega) \leq \epsilon^{\frac{1}{2}}$, where

$$\Omega = \{(x, y) : x, y \geq 0,\ 0 < x^2 + y^2 < 1,\ 0 \leq \frac{x^2}{x^2 + y^2} < 1\}.$$

(6) Compute $\int_\Omega \exp(\frac{x-y}{x+y}) dA$, where $\Omega = \{(x, y) : x, y \geq 0, x + y \leq 1\}$.
[Hint: Use the substitution: $u = x + y$ and $v = x - y$.]

(7) Find the volume generated by the cone $z = \sqrt{x^2 + y^2}$ and $0 \leq z \leq 3$.

(8) Compute $\int_0^1 \int_0^{\sqrt{x}} y \exp(\sqrt{x}) dy\, dx$. [Hint: Use $x \mapsto x^2$ and $y \mapsto y$.]

(9) Evaluate $\int_0^1 \int_0^z \int_0^y \exp\Big((1 - x)^3\Big) dx\, dy\, dz$.

(10) Evaluate

$$\int\limits_{x^2+y^2+z^2 \leq 1} \exp\left((x^2 + y^2 + z^2)^{\frac{3}{2}}\right).$$