(i) $X_1, \ldots, X_k$ be a sample of $X \sim Bin(n, p)$

Thus, we have,

$\mu = \mathbb{E}[X] = np$ & $\sigma^2 = Var(X) = np(1-p)$

Sample moments are given by,

$m_1 = \bar{X} = \sum_{i=1}^{k} X_i / k$ &

$m_2 = \overline{X^2} = \dfrac{\sum_{i=1}^{k} X_i^2}{k}$.

Now, equating the sample moments with population moments,

we get,

$m_1 = \mathbb{E}[X] = np$ & $m_2 = \mathbb{E}[X^2]$

$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$\Rightarrow Var(X) = m_2 - m_1^2$

$\Rightarrow m_2 - m_1^2 = np(1-p) = np - np^2$

$\Rightarrow m_2 - m_1^2 = m_1 - m_1 p$. (as $m_1 = np$)

Hence, $p = (m_1^2 + m_1 - m_2) / m_1$

& as $m_1 = np \Rightarrow n = m_1 / p$

Thus, $n = \dfrac{m_1^2}{m_1^2 + m_1 - m_2}$

(2) Here, we have $\theta = \sigma^2$ & $\hat{\theta} = s^2$

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}[s^2]$$

$$= \mathbb{E}\left[\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}\right]$$

which equals,

$$\mathbb{E}\left[\sum_{i=1}^{n} (X_i^2 - 2X_i\bar{X} + \bar{X}^2)\right] / (n-1)$$

$$= \frac{1}{n-1} \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2\right]$$

$$= \frac{1}{n-1} \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 - 2\bar{X}(n-\bar{X}) + n\bar{X}^2\right]$$

$$= \frac{1}{n-1} \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} \mathbb{E}[X_i^2] - \mathbb{E}[n\bar{X}^2]\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} \mathbb{E}[X_i^2] - n\,\mathbb{E}[\bar{X}^2]\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} \mathbb{E}[X_i^2] - n\,\mathbb{E}[\bar{X}]^2\right)$$

Now for each $i \in [n]$, $Var(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2$

which gives,

$$\mathbb{E}[X_i^2] = Var(X_i) + \mathbb{E}[X_i]^2$$

$$= \sigma^2 + \mu^2.$$

& $Var(\bar{X}) = \mathbb{E}[\bar{X}^2] - [\mathbb{E}(\bar{X})]^2$

which gives,

$$\mathbb{E}[\bar{X}^2] = Var(\bar{X}) + (\mathbb{E}[\bar{X}])^2$$

$$= \left(\frac{\sigma}{\sqrt{n}}\right)^2 + \mu^2 = \sigma^2/n + \mu^2$$

Also, X_i's are identically distributed.

$$\therefore \frac{1}{n-1}\left(\sum_{i=1}^{n}(\sigma^2+\mu^2) - n\left(\sigma^2/n + \mu^2\right)\right)$$

$$= \frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right)$$

$$= \sigma^2$$

Hence Bias $= \mathbb{E}[s^2] - \sigma^2 = \sigma^2 - \sigma^2 = 0$

Thus, s^2 is unbiased for σ^2

proved.

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