Lecture 17: Universal property of localization

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Define a relation on $S \times R$ where R is comm ring with unity and S is a mult set.

$S \times R = \{(s, r) \mid s \in S \ \& \ r \in R\}$

$(s_1, r_1) \sim (s_2, r_2)$ if $s(s_2 r_1 - s_1 r_2) = 0$ for some $s \in S$.

(*) $\sim$ is an equivalence relation. $[(s, r)] = \frac{r}{s}$

Def^n/Prop: The set of equivalence classes $S \times R/\sim$ is denoted by $S^{-1}R$. The equivalence class $[(s, r)]$ will be denoted by $\frac{r}{s}$.

The binary operators $\frac{r_1}{s_1} \oplus \frac{r_2}{s_2} := \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$ and

$\frac{r_1}{s_1} \odot \frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}$ are well

defined. Moreover $(S^{-1}R, \oplus, \odot)$ is a commutative ring with unity. The map

$$\varphi : R \longrightarrow S^{-1}R \qquad r \longmapsto \frac{r}{1}$$

is a ring homo.

$0_{S^{-1}R} = \frac{0}{1} = \frac{0}{s} \quad \forall s \in S$

$1_{S^{-1}R} = \frac{1}{1} = \frac{s}{s} \quad \forall s \in S$

Ex: ① $R = \mathbb{Z}$ & $S = \mathbb{Z} \setminus \{0\}$.

$S \times R = \{(s, r) \mid s \neq 0,\ s, r \in R\}$

$(s_1, r_1) \sim (s_2, r_2)$ if $\exists s \in \mathbb{Z} \setminus \{0\}$ s.t

$$s(s_2 r_1 - s_1 r_2) = 0$$
$$\Updownarrow$$
$$s_2 r_1 - s_1 r_2 = 0$$
$$\Updownarrow$$
$$s_2 r_1 = s_1 r_2$$

$\frac{r_1}{s_1} = \frac{r_2}{s_2}$ iff $s_2 r_1 = s_1 r_2$

So $S^{-1}R = \mathbb{Q}$.

Def w/ Prop: More generally if R is an integral domain & $S = R \setminus \{0\}$ then the ring $S^{-1}R$ is a field. This field is denoted by $\mathrm{frac}(R)$ or $QF(R)$ and is called field of fractions of R. Moreover $\varphi : R \hookrightarrow \mathrm{frac}(R)$ is injective and if K is a field containing R as a subring then K contains $\mathrm{frac}(R')$.

Pf: Let $\alpha \in S^{-1}R = \text{frac}(R)$, $\alpha \neq 0$ in $S^{-1}R$

$\alpha = \frac{r}{s}$ $r \in R$ & $s \in S = R \setminus \{0\}$

Since $\alpha \neq 0$ is $S^{-1}R \Rightarrow r \neq 0 \Rightarrow r \in S$

$\Rightarrow \frac{s}{r} \in S^{-1}R$. Then

$$\frac{s}{r} \odot \frac{r}{s} = \frac{sr}{sr} = \frac{1}{1} = 1_{S^{-1}R}$$

Hence $S^{-1}R$ is a field.

$$\varphi : R \longrightarrow S^{-1}R$$
$$r \longmapsto \frac{r}{1}$$

$$\ker(\varphi) = \left\{ r \;\middle|\; \frac{r}{1} = \frac{0}{1} \quad r \in R \right\}$$

$$= \left\{ r \in R \;\middle|\; \exists s \in R \setminus \{0\} \text{ s.t. } s(1 \cdot r - 1 \cdot 0) = 0_R \right\}$$

$$= \left\{ r \in R \;\middle|\; sr = 0,\ s \neq 0 \text{ some } s \in R \right\}$$

$$= \{ r \in R \mid r = 0 \} = \{0\}$$

$\Rightarrow \varphi$ is injective.

Let K be a field & $R \subseteq K$ be a subring. Let $\frac{r}{s} \in S^{-1}R$ then $s \neq 0$ in R & $r \in R \Rightarrow \frac{r}{s} \in K$.

So $S^{-1}R \subseteq K$. $\blacksquare$

⊛ Let R be a comm ring with unity and S a mult. subset of R. Then $\varphi(s)$ is a unit in $S^{-1}R$ $\forall\, s \in S$. Here

$$\varphi : R \to S^{-1}R \quad \text{is the natural map.}$$
$$r \mapsto \frac{r}{1}$$

Pf: $\varphi(s) = \frac{s}{1} \in S^{-1}R$, So $s \in S$.

so $\frac{1}{s} \in S^{-1}R$ and $\frac{s}{1} \odot \frac{1}{s} = \frac{s}{s} = \frac{1}{1} = 1_{S^{-1}R}$

Hence $\varphi(s)$ is a unit.

⊛ If $0 \in S$ then $S^{-1}R$ is the zero ring.

Pf: Claim $\frac{r}{s} = \frac{0}{1}$ $\forall\, r \in R$ & $s \in S$

the above equality holds if $u(1r - s0) = 0$ for some $u \in S$

Take $u = 0 \in S$. Hence the claim.

Hence $S^{-1}R = \{0\}$.

⊛ In general, S mult subset of a comm ring R & $\varphi : R \to S^{-1}R$ then
$$r \mapsto \frac{r}{1}$$

$$\ker(\varphi) = \{ r \in R \mid sr = 0 \text{ for some } s \in S \}.$$

In particular if S consist of nonzero divisors in R then φ is injective. Converse also holds.

Thm (Universal property of Localization):

Let R be comm ring with unity. $S \subseteq R$ be a mult. subset of R. Let

$$f : R \to A \text{ be a ring homomorphism}$$

where A is a comm ring with unity such that $\forall s \in S$ $f(s)$ is a unit in A. Then $\exists!$ ring homo.

$$\hat{f} : S^{-1}R \to A \quad \text{s.t.} \quad \hat{f} \circ \varphi = f.$$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\ S^{-1}\varphi \arrow[ur, dashed, "\exists! \hat{f}"'] & \end{tikzcd}$$

Pf: $\hat{f} : S^{-1}R \to A$

Let $\frac{r}{s} \in S^{-1}R$ for some $r \in R$ & $s \in S$

$$\hat{f}\left(\frac{r}{s}\right) = f(s)^{-1} f(r) \quad \left(\begin{array}{l}\text{Note } f(s) \text{ is a unit in } A. \\ \text{Hence } f(s)^{-1} \text{ make sense}\end{array}\right)$$

$\hat{f}$ is well-defined:

Let $\frac{r'}{s'} = \frac{r}{s}$

$\Rightarrow \exists u \in S$ s.t. $u(sr' - s'r) = 0$ in R

$\xRightarrow{\because f \text{ is a ring homo}}$ $f(u)\big(f(s)f(r') - f(s')f(r)\big) = 0$ in A

$\xRightarrow{f(u) \text{ is a unit}}$ $f(s)f(r') = f(s')f(r)$ in A

$\xRightarrow{\cdot f(s)^{-1} f(s')^{-1}}$ $f(s')^{-1}f(r') = f(s)^{-1}f(r)$

Hence $\underbrace{f(s')^{-1}f(r')}_{\hat{f}\left(\frac{r'}{s'}\right)} = \underbrace{f(s)^{-1}f(r)}_{\hat{f}\left(\frac{r}{s}\right)}$

$\hat{f}$ is well-defined.

For $\frac{r}{s}, \frac{r'}{s'} \in S^{-1}R$

$$\tilde{f}\left(\frac{r}{s}+\frac{r'}{s'}\right) = \tilde{f}\left(\frac{s'r+sr'}{ss'}\right)$$
$$= f(ss')^{-1} f(s'r+sr')$$
$$= f(s)^{-1} f(s')^{-1} \left[f(s') f(r) + f(s) f(r') \right]$$
$$= f(s)^{-1} f(r) + f(s')^{-1} f(r')$$
$$= \tilde{f}\left(\frac{r}{s}\right) + \tilde{f}\left(\frac{r'}{s'}\right)$$

lll^{ly} $\tilde{f}\left(\frac{r}{s}\cdot\frac{r'}{s'}\right) = \tilde{f}\left(\frac{r}{s}\right)\tilde{f}\left(\frac{r'}{s'}\right)$

For $r \in R$ $\quad \tilde{f}\circ\varphi(r) = \tilde{f}\left(\frac{r}{1}\right) = f(1)^{-1} f(r) = f(r)$

$\Longrightarrow \tilde{f}\circ\varphi = f$

Finally $\tilde{f}$ is unique: Let $h: S^{-1}R \to A$ be another ring homo. s.t. $h\circ\varphi = f$.

Let $\frac{r}{s} \in S^{-1}R$ $\quad h\left(\frac{r}{s}\right) = h\left(\frac{r}{1}\cdot\frac{1}{s}\right) = h\left(\frac{r}{1}\right) h\left(\frac{1}{s}\right)$
$$= h\circ\varphi(r)\cdot h\left(\frac{s}{1}\right)^{-1}$$
$$= f(r)\cdot\left(h\circ\varphi(s)\right)^{-1}$$
$$= f(r)\, f(s)^{-1}$$
$$= \tilde{f}\left(\frac{r}{s}\right)$$

$\Longrightarrow h = \tilde{f}$ $\quad\blacksquare$