It is easy to show that $\nabla \times F = 0$.

However, $\int_C F \cdot dr \neq 0$, where $C : r(\theta) = \langle \cos\theta, \sin\theta \rangle$. $0 \leq \theta \leq 2\pi$.

C ↑ unit circle.

Indeed: $\int_C F \cdot dr = \int_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$.

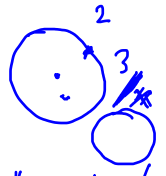
$$= \int_0^{2\pi} \left(\frac{-\sin\theta}{\sin^2\theta + \cos^2\theta}\right) d(\cos\theta) + \left(\frac{\cos\theta}{\sin^2\theta+\cos^2\theta}\right) d(\sin\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = \int_0^{2\pi} d\theta$$

$$= 2\pi \neq 0.$$

$\int_C \langle P, Q, R\rangle \cdot d\vec{r} = \int_C \underbrace{P\,dx + Q\,dy + R\,dz}_{\text{differential form}}$.

$x = X(t)$



$\frac{1}{2\pi} \int_C F \cdot dr = 1 / 2 / 3 \ldots$

Remark: What went wrong?

Well, perhaps, F is not C^1 (or not even defined / diff. / cont) at $(0,0)$. So $F : \mathbb{R}^2 \to \mathbb{R}^2$ is NOT a good choice.

Okay: So, lets consider $F : \mathcal{O}_2 \to \mathbb{R}^2$, where $\mathcal{O}_2 = \mathbb{R}^2 \setminus \{(0,0)\}$

or $\{(x,y) : x^2+y^2 < 1\} \setminus \{(0,0)\}$.

punctured disc

BUT, AGAIN, we can ~~will~~ ~~take~~ will

Consider a circle C & prove $\int_C F \cdot dr \neq 0$.

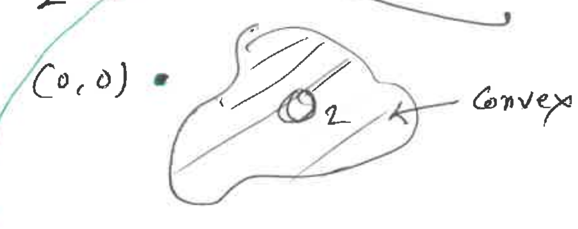
the same as above!

THEN?

The trouble is $(0,0)$, the singularity being in the interior of $\mathcal{O}_2$.

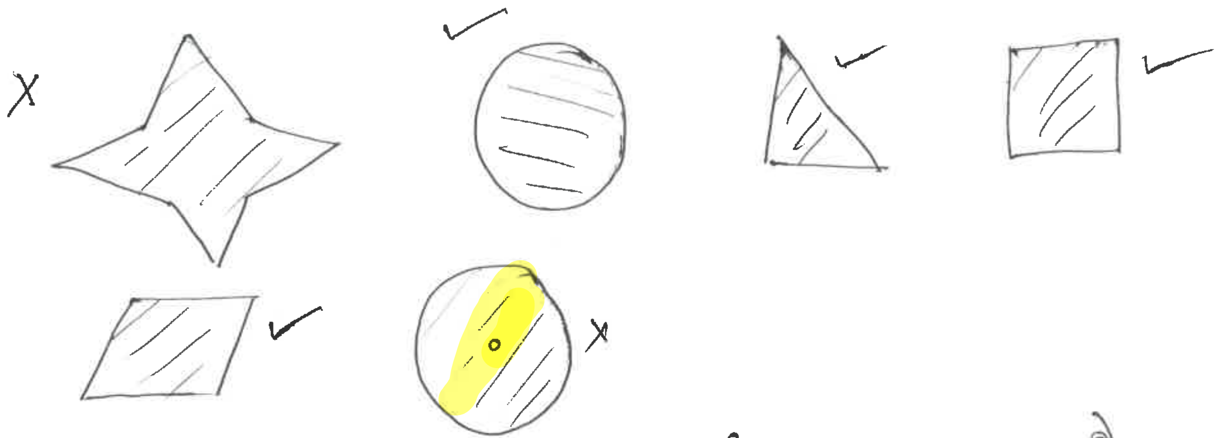
In fact: If $\mathcal{O}_2$ ~~(a.a)~~ ~~...~~ $\mathcal{O}_2$ convex

& $(0,0) \notin \mathcal{O}_2$, then it will do!!



Remark: Now, suppose we have $F : O_2 \to \mathbb{R}^2$ s.t. any pair of points in O_2 can be connected via a line in O_2

(← We call it as CONVEX domain).



If we know F is conservative,

then we know $\nabla F = F := (P, Q)$,

& then we can simply follow the method of eg (2) in Page-60 to solve it for P & Q.

[See ✡ in page 60].

Q: But, how to determine F is conservative?

"Ans:" Green's thm.

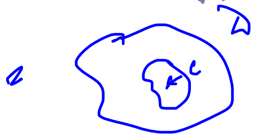
"Curl:"

Recall curl of a v.f. F is $\nabla \times F$: The measure of tendency of F to swirl/create whirlpool. Like

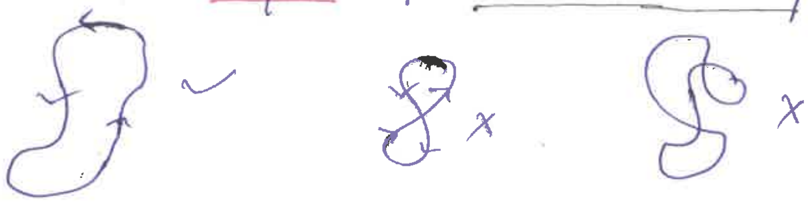
← $\neq 0$ curl i.e. there will be swirl/whirlpool.

But

Def: Let $\mathcal{D} \to$ open + connected subset of $\mathbb{R}^2$. We say that $\mathcal{D}$ is Simply Connected if, whenever $C \subseteq \mathcal{D}$ a simple & closed curve, C can be shrunk continuosly / gradually to a point inside $\mathcal{D}$.



[# A curve C is Simple if it has no self intersections.



CROSS ← Not allowed.

i.e.: If parametrizations of C are injective !!

except initial & terminal points.]

No PROOF!!

$\mathcal{D} \to$ open + connected. (in $\mathbb{R}^2$).

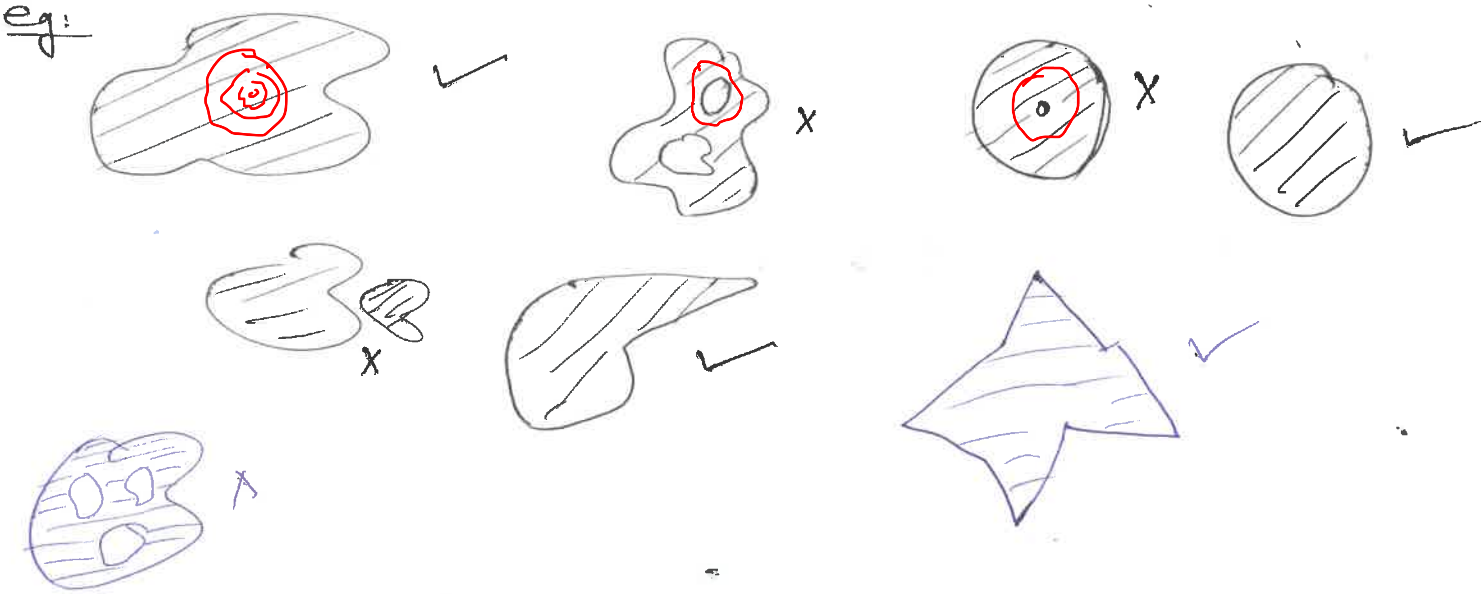
Then $\mathcal{D}$ is simply connected $\Longleftrightarrow$ if $C \subseteq \mathcal{D}$ is a simple closed curve, then the interior of $C \subseteq \mathcal{D}$.

$\Longleftrightarrow$ $\underbrace{\mathbb{R}^2_\infty}_{\substack{\| \\ \mathbb{R}^2 \cup \{\infty\}}} \setminus \mathcal{D}$ is connected. [Ahlfor's: Complex Analysis].

$\mathbb{R}^2 \cup \{\infty\} \cong S^2$ (Sphere in $\mathbb{R}^3$: through stereographic projection).

Remark: More precise / accurate defn needs the notion of fundamental groups / homotopy Theory.

eg:

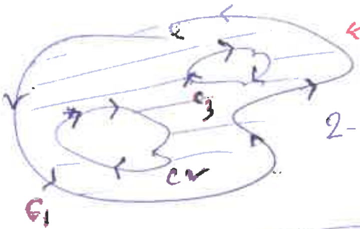


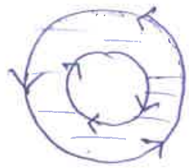
SEE AFTER Green's THEOREM

Green's thm for "n- holed domains":

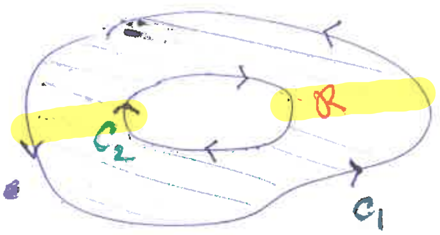
A domain (open + connected) bounded by finitely many piecewise simple C^1-curves.

eg:

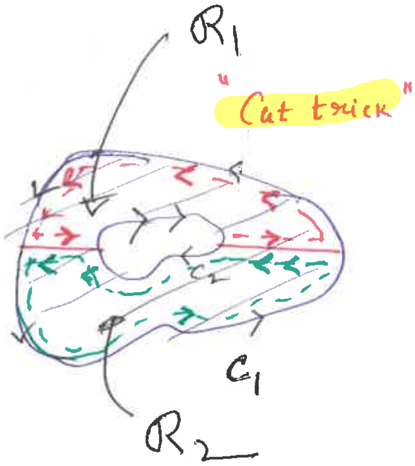




Consider:



$C = C_1 \cup C_2$.



Q: ~~Is~~ $\displaystyle\int_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{?}{=} \int_C P\,dx + Q\,dy$??

Ans: Yes.

$$\underbrace{\int_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\displaystyle \int_{\mathcal{R}_1} + \int_{\mathcal{R}_2}} \overset{\text{Green's thm}}{=} \int_{\partial \mathcal{R}_1} P\,dx + Q\,dy + \int_{\partial\mathcal{R}_2} P\,dx + Q\,dy.$$

$$= \int_{C_1 \cup C_2} P\,dx + Q\,dy.$$

$\mathcal{R}_1, \mathcal{R}_2 \to$ Simply connected.

[$C_1 \cup C_2 =$ ]

Illy n-holed domain.

Green's theorem: (in $\mathbb{R}^2$: Line vs. Area integrations)

Thm: Let $\mathcal{R} \subseteq \mathbb{R}^2$ be ~~a region~~ [a simply connected domain] (~~open + connected~~) with boundary curve C (parametrized such a way so that $\mathcal{R}$ is "to the left").
Let $\langle P, Q \rangle$ be a C^1 - vector field on $\mathcal{R}$ (i.e. C^1 on an open set containing $\mathcal{R}$). Then

$$\underbrace{\int_C P\,dx + Q\,dy}_{\parallel} = \underbrace{\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\parallel} .$$

$$\int_C \vec{F} \cdot d\vec{\gamma} \quad (\dagger) \quad = \int \operatorname{Curl}(\vec{F})\, dA .$$

Where $\vec{F} = (P, Q) : \mathcal{R} \to \mathbb{R}^2$ & (recall that)

$$\operatorname{Curl} \vec{F} := \begin{vmatrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} := \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{Scalar field in 2 dim}} .$$

$(\dagger)$ [Recall: If $C = \operatorname{ran} \gamma$ (Gamma), $\gamma = (x(t), y(t))$, $t \in [a,b]$, then for $\vec{F} = (P, Q)$, we have

$$\int_C \vec{F} \cdot d\vec{\gamma} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

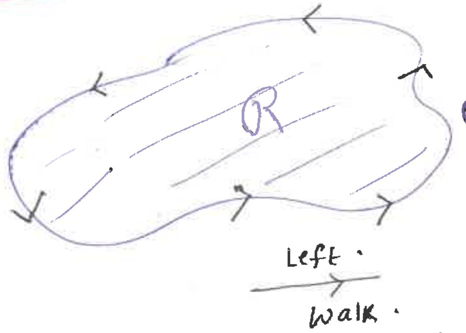
$$= \int_a^b \big((P, Q) \cdot (x'(t), y'(t)) \big) dt$$

$$= \int_a^b \left(P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right) \times dt$$

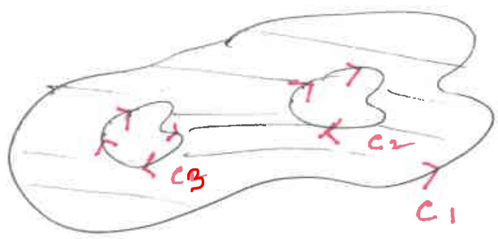
$$= \int_C P\,dx + \underbrace{Q\,dy}$$.]

Here: $dx = x'(t)dt$, $P = P(x(t), y(t))$.

"to the left" (orientation of C):



(Now see Page 64)



$C = C_1 \cup C_2 \cup C_3$.

Remark: Why $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\boxed{\text{Why "}-\text{" ?}}$

Think $\vec{F} = (P, Q)$ as $\vec{F} = (P, Q, 0)$.

Then $\underbrace{\text{Curl}(\vec{F})}$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$

This is the curl used in the statement but with a little care

$$= \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k.$$

So $\left\{ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k \right\} \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$

dot product

So the precise statement is:

$$\int_R \underbrace{\text{Curl}(\vec{F}) \cdot k}\ dA = \int_C \vec{F} \cdot d\vec{r}.$$

magnitude of Curl$(\vec{F})$

The normal vector to the plane.

Of course, Curl of planar v.f's is a vector pointing towards k, the normal to the plane.

Note $\int_C \vec{F} \cdot d\vec{r}$ = Circulation of $\vec{F}$ around C.
or Work done by $\vec{F}$ around C.

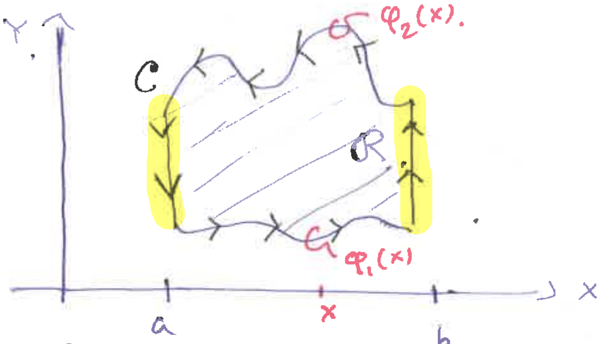
& $\int_R \text{Curl}(\vec{F}) \cdot k\ dA$ = Sum of all infinitely small Circulations in the region R.

Proof: Not in ~~the~~ our scope. (In fact: Green's thm $\Leftarrow$ Stokes thm (in $\mathbb{R}^3$).
AND: Stoke's thm fits/suits well in $\mathbb{R}^n$ but from exterior product + differential forms point of view).

However, here is a simple ~~ver~~ version

Let $\mathcal{R} = \{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$ ← elementary domain (closed).

$P, Q \in C^1(\Omega)$, where $\underset{\text{open}}{\Omega} \supseteq \mathcal{R}$. Set $C = \partial\mathcal{R}$.



Claim: $\displaystyle\int_C P\,dx + Q\,dy = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$

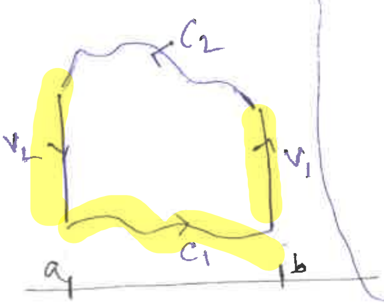
We first prove: $\boxed{\displaystyle\int_{\mathcal{R}} -\frac{\partial P}{\partial y}\,dA = \int_C P\,dx}$. AND THEN $\boxed{\displaystyle\int_C Q\,dy = \int_{\mathcal{R}} \frac{\partial Q}{\partial x}}$

Indeed: $\displaystyle\int_{\mathcal{R}} -\frac{\partial P}{\partial y}\,dA = -\int_{\mathcal{R}} \frac{\partial P}{\partial y}\,dA = -\iint_{\mathcal{R}} \frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b \int_{y=\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$

$$= -\int_a^b \Big[P(x,y)\Big]_{y=\varphi_1(x)}^{\varphi_2(x)} dx$$

$$= \int_a^b \Big(P(x,\varphi_1(x)) - P(x,\varphi_2(x))\Big)\,dx$$

$$= \int_a^b P(t,\varphi_1(t))\,dt - \int_a^b P(t,\varphi_2(t))\,dt.$$



Now $C = C_1 \cup V_1 \cup C_2 \cup V_2$.

So, $\displaystyle\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{V_1} P\,dx + \int_{V_2} P\,dx$.

$\because$ on V_1: $x = a \Rightarrow \dfrac{dx}{dt} = 0$. So $\displaystyle\int_{V_1} P\,dx = \int_{\varphi_1(a)}^{\varphi_2(a)} P(a,y)\underbrace{\frac{dx}{dt}}_{=0}\,dt = \int 0\,dt = 0.$

[V_1 is given by $t \longmapsto (a,t) = (x(t),y(t))$
$\varphi_1(a) \le t \le \varphi_2(a)$]

$\therefore \displaystyle\int_{V_1} P\,dx = 0.$ Illy $\displaystyle\int_{V_2} P\,dx = 0.$

Note that C_1 : $t \longmapsto (x(t), y(t)) := (t, \varphi_1(t))$ (68)

$t \in [a,b]$. ← parametrization of C_1.

$\therefore \frac{dx}{dt} = 1$ $[\because x(t) = 1]$.

$$\therefore \int_{C_1} P\,dx = \int_a^b P(t, \varphi(t)) \underbrace{\frac{dx}{dt}}_{=1} \partial t = \int_a^b P(t, \varphi(t))\, dt$$

Also C_2 : $t \longmapsto (x(t), y(t)) = (t, \varphi_2(t))$.

$t \in [a,b]$.

~~the fact that~~

a ~~⇒ b ⇒ $\varphi_1(b) \Rightarrow \varphi(a)$~~

$\therefore x(t) = t \Rightarrow \frac{dx}{dt} = 1$.

$$\therefore \int_{C_2} P\,dx = (-)\int_a^b P(t, \varphi_2(t))\, dt$$

"$-$": due to the opposite orientation of C_2.

Hence $\int_R -\frac{\partial P}{\partial y}\, dA = \int_C P\,dx$.

Hly $\int \frac{\partial Q}{\partial x} dA = \int Q\,dy$.

[6]

Remark: Using the above, for boxes $B^2 \subseteq \mathbb{R}^2$, a way longer limiting approach will lead Green's theorem for domains. However, the natural way to get ~~prove~~ this as a Corollary of Stokes theorem.

Before we go to Stoke's thm, let's look at some examples:

eg: Compute $\int_C \langle x^2 - y^2, 2xy \rangle \cdot d\mathbf{r}$, where $C = \partial([0,1] \times [0,1])$

Sol: By Green's thm: $\int_C \langle \underbrace{x^2-y^2}_{P}, \underbrace{2xy}_{Q} \rangle \cdot d\mathbf{r} = \int_{[0,1]^2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \int_{[0,1]^2} (2y + 2y)\, dA = 4 \int_{[0,1]}\int_{[0,1]} y\, dA = 4 \int_0^1 \int_0^1 y\, dy\, dx$$

$$= 4 \times \frac{1}{2} \times [y^2]_0^1 = 2.$$ Ans.

← Perhaps easy!!

Area formula.

1-1 i.e. no cross

Let $C = \partial R$, where C is +vely oriented simple closed curve.

i.e. R is on left.

Then $\text{Area}(R) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx$.

Proof: . Simple idea: Choose $(P, Q) := F$ so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1. \Rightarrow \int_C P\,dx + Q\,dy = \int_R 1\,dA = \text{Area of } R.$$

Here $(0, x)$, $(-y, 0)$, $(\frac{1}{2}x, -\frac{1}{2}y)$ does the job !!

eg: Area of ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\gamma(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$.

$\therefore$ Area $= \frac{1}{2}\int x\,dy - y\,dx$.

$x = a\cos t$, $y = b\sin t$ $\Rightarrow \frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t$.

$$= \frac{1}{2}\int_0^{2\pi}\{a\cos t \times b\cos t - b\sin t(-a\sin t)\}dt$$

$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2 t + \sin^2 t)\,dt$$

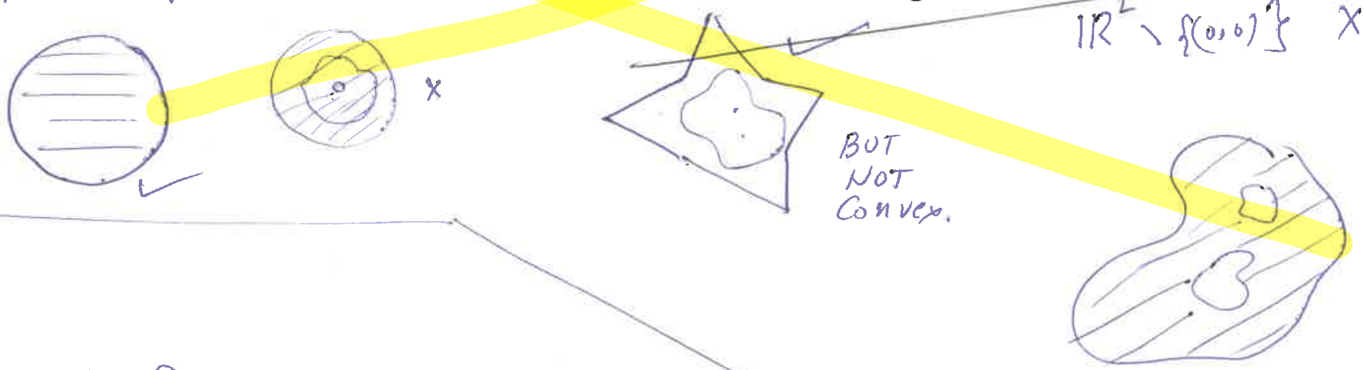
$$= \frac{1}{2}ab \times 2\pi = \pi ab.$$

[$\therefore$ Green's theorem could be useful for line through double as well as double through line both !!]

~~Def: Let $\mathcal{D} \subseteq \mathbb{R}^n$ ($n = 2$ or 3) be a domain (open + connected). Then $\mathcal{D}$ is Simply Connected if each closed curve in $\mathcal{D}$ can be shrunk continuously / gradually to a point inside $\mathcal{D}$.~~

Covered in Page-63

$\mathbb{R}^2 \setminus \{(0,0)\}$ X.



X 2-holed domain.

Illy n-holed domain X.

Thm: Let $\mathcal{D}$ be a Simply Connected domain in $\mathbb{R}^2$ & ~~Let~~ F be a C^1-vector field on $\mathcal{D}$. Then F is Conservative $\iff$ $\nabla \times F = 0$ on $\mathcal{D}$.

Recall: in $\mathbb{R}^2$ $\nabla \times F = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ if $F = (P, Q)$.

Proof: "$\Rightarrow$" By defn. of $\nabla \times F$.

"$\Leftarrow$" Simply ~~Green~~'s Green's thm. ∎

[Recall: If $F = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$, then $\nabla \times F = 0$ But F is not Conservative. Surely F is C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$ or open unit Ball $\setminus \{(0,0)\}$, BUT NONE OF THEM are simply connected: So, Conservative has a lot to do with the nature of the domain of definitions.]

— * —

Def: Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a v.f.
Then $\mathrm{Div}(F)$ (the divergence of F) is defined by:

$$\mathrm{Div}(F) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

So, if $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$, then $\boxed{\mathrm{Div}(F) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}}$

Illy 3-variables.