$\underline{\text{Recall}}$: • R-linear maps $M \xrightarrow{\varphi} N$ are maps which send $rm_1 + m_2$ to $r\varphi(m_1) + \varphi(m_2)$
$\forall r \in R, \ m_1, m_2 \in M.$ -->

• $\ker(\varphi)$ & $\text{Im}(\varphi)$ are submod of $M$ and $N$ resp.

• $N \subseteq M$ R-modules then $q: M \to M/N$ is R-lin

• Intersection of R-submod of an R-mod is an R-submod

• $N_1, N_2$ R-submod of $M$ then $N_1 + N_2 = \{n_1 + n_2 \mid n_1 \in N_1 \ \& \ n_2 \in N_2\}$ is an R-submod of $M$.

• First isom thm
$M \xrightarrow{\varphi} N \ \varphi$ R-lin then $M/\ker\varphi \cong \varphi(M)$

• Second isom thm $N_1, N_2 \subseteq M$ R-submod of $M$ then
$N_1 + N_2 / N_2 \cong N_1 / N_1 \cap N_2$

• Third isom thm: $K \subseteq N \subseteq M$ be R-modules then

$$(M/K)/(N/K) \cong M/N$$

⊗ Basics of linear maps
1) Let $M, N$ be R-modules & $\varphi, \psi: M \to N$ be R-lin maps
then (a) $\varphi + \psi: M \to N$ is an R-lin map.
$(\varphi + \psi)(m) = \varphi(m) + \psi(m)$

(b) For $r \in R$ $r\varphi: M \to N$ is an R-lin map
$m \mapsto r\varphi(m)$

2) Let $M, N, K$ be R-mod $\varphi: M \to N$ & $\psi: N \to K$ be
R-lin map then $M \xrightarrow{\psi \circ \varphi} K$ is R-lin.

Pf: Let $r \in R$ & $m_1, m_2 \in I$

a) $(\varphi + \psi)(r m_1 + m_2) = \varphi(r m_1 + m_2) + \psi(r m_1 + m_2)$

$$= r \varphi(m_1) + \varphi(m_2) + r \psi(m_1) + \psi(m_2)$$

$$= r(\varphi(m_1) + \psi(m_1)) + \varphi(m_2) + \psi(m_2)$$

$$= r(\varphi + \psi)(m_1) + (\varphi + \psi)(m_2)$$

Hence $\varphi + \psi$ is R-lin

$r' \in R$, $r \in R$

b) $(r \varphi)(r' m_1 + m_2) = r \cdot \varphi(r' m_1 + m_2)$

$$= r r' \varphi(m_1) + r \varphi(m_2)$$

$$= r'(r \varphi)(m_1) + (r \varphi)(m_2)$$

$\Rightarrow r \varphi$ is R-linear.

(2) Let $m_1, m_2 \in M$ & $r \in R$

$$\psi \circ \varphi(r m_1 + m_2) = \psi(r \varphi(m_1) + \varphi(m_2))$$

$$= r \psi(\varphi(m_1)) + \psi(\varphi(m_2))$$

$$= r \psi \circ \varphi(m_1) + \psi \circ \varphi(m_2)$$

So $\psi \circ \varphi$ is R-linear.

Cor: $M, N$ $R$-modules. Let $\text{Hom}_R(M, N)$ be the set $R$-lin maps from $M$ to $N$. Then $\text{Hom}_R(M, N)$ with addition of lin maps and scalar mult defined in ① is an $R$-mod.

Pf: Additive identity of $\text{Hom}_R(M, N)$
$$0 : M \longrightarrow N \quad \forall m \in M \text{ the zero map}$$
$$m \longmapsto 0$$

$\varphi$ $R$-lin then $-\varphi = -1 \cdot \varphi$ is also and $\varphi + \bar\varphi = 0$. So $\text{Hom}_R(M, N)$ is a abelian group.

• $r_1, r_2 \in R$ & $\varphi \in \text{Hom}_R(M, N)$

check: $(r_1 + r_2) \varphi \overset{?}{=} r_1 \varphi + r_2 \varphi$

Let $m \in M$
$$((r_1 + r_2) \varphi)(m) = (r_1 + r_2) \cdot \varphi(m)$$
$$= r_1 \varphi(m) + r_2 \varphi(m)$$
$$= (r_1 \varphi)(m) + (r_2 \varphi)(m)$$

• $(r_1 r_2) \varphi = r_1 (r_2 \varphi)$ for $r_1, r_2 \in R$ & $\varphi \in \text{Hom}_R(M, N)$

• $r(\varphi_1 + \varphi_2) = r \varphi_1 + r \varphi_2$ $\forall r \in R$ & $\varphi_1, \varphi_2 \in \text{Hom}_R(M, N)$

**Cor:** $\text{End}_R(M)$ is a ring where $M$ is an $R$-module.

Here $\text{End}_R(M) \doteq \text{Hom}_R(M,M)$ and addition is as in ① & multiplication is composition.

**Pf:** Exc

⊛ Free modules

Example: $R = \mathbb{Z}$ & $M = \mathbb{Z}/5\mathbb{Z}$ is an $R$-module.

$\{[1]_5\} \subseteq M$ is a gen set of $M$.

$5[1]_5 = 0_M = [0]_5$, but $5 \in R$. $5 \neq 0$

**Def$^n$:** • Let $M$ be an $R$-mod. Let $S \subseteq M$ be a subset of $M$. The $R$-submod of $M$ gen by $S$ is the smallest $R$-submod of $M$ containing $S$. This is denoted by $\langle S \rangle$.

We say $S$ is a gen set of $M$ if $\langle S \rangle = M$.

• An $R$-mod $M$ is said to be f.g. if $M$ is gen by a finite subset.

• An $R$-mod is called cyclic if it is gen by one element.

**Prop:** Let $M$ be an $R$-mod and $S$ be a subset of $M$. Then

$$\langle S \rangle = \left\{ r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \ \middle| \ \begin{array}{l} \text{where } n \geq 1, \\ m_i \in S \ \& \\ r_i \in R \ 1 \leq i \leq n \end{array} \right\}$$

$$\underbrace{\phantom{r_1 m_1 + r_2 m_2 + \cdots + r_n m_n}}_{N}$$

**Pf:** Let $N$ be the RHS.

Note that $S \subseteq N$.

Let $x, y \in N$ then

$$x = r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \quad \text{for some } r_i \in R \ \& \ m_i \in S$$

$$y = r_1' m_1' + r_2' m_2' + \cdots + r_{n'}' m_{n'}' \quad \text{for some } r_i' \in R \ \& \ m_i' \in S$$

$$\& \ r \in R$$

$$rx + y = r r_1 m_1 + r r_2 m_2 + \cdots + r r_n m_n$$
$$+ r_1' m_1' + r_2' m_2' + \cdots + r_{n'}' m_{n'}'$$

$$\in N$$

Hence $\langle S \rangle \subseteq N$.

Also if $m_1, \ldots, m_n \in S$ then $r_1 m_1 + \cdots + r_n m_n \in \langle S \rangle$ $\forall \ r_1, \ldots, r_n \in R$ since $\langle S \rangle$ is an $R$-submod of $M$. Hence $N \subseteq \langle S \rangle$.

Free modules: A subset $S$ of an $R$-module $M$ is called a basis

of $M$ if $S$ is a linearly independent gen set.

i.e. if $\sum_{i=1}^{n} r_i m_i = 0$ for some $n \geq 1$, $m_i \in S$ $1 \leq i \leq n$

$\qquad\qquad\qquad\qquad$ & $r_i \in R$ $1 \leq i \leq n$

$\qquad$ then $r_i = 0$ $\forall 1 \leq i \leq n$

$\qquad$ An $R$-module $M$ is called a free module

if it has a basis.

$\qquad$ Eg:1) $R = k[x,Y]$ $k$ is a field. Give an example

of a free $R$-module. Here $R$ is a free.

$\qquad$ $R^n$ where scalar multiplication is component-wise is

a free $R$-module $\forall n \geq 1$.

$\qquad$ 2) $R$ any ring then $R^n$ is a free $R$-mod.

Ⓜ Every f.g. free $R$-mod is isomorphic to $R^n$ for some $n$.

$\qquad$ Pf: Let $M$ be a f.g. free $R$-module. Then $M$

has basis $S$ containing $n$ element for some $n \geq 1$.

$\qquad\qquad \{x_1, \dots, x_n\}$

$\qquad \varphi : R^n \longrightarrow M$ $\qquad$ $\varphi$ is a well-defined.

$\qquad (r_1, \dots, r_n) \longmapsto \sum_{i=1}^{n} r_i x_i$

$\qquad \varphi\left(\underline{r} + \underline{r}'\right) = \sum_{i=1}^{n} (r_i + r_i') x_i$ $\qquad$ where $\underline{r} = (r_1, \dots, r_n)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{r}' = (r_1', \dots, r_n')$

$\qquad\qquad\qquad = \sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} r_i' x_i$

$\qquad\qquad\qquad = \varphi(\underline{r}) + \varphi(\underline{r}')$

$\qquad$ Check for $a \in R$ & $\underline{r} \in R^n$

$\qquad\qquad \varphi(a\underline{r}) = a\varphi(\underline{r})$

$\qquad$ So $\varphi$ is $R$-lin $\qquad$ and $\ker(\varphi) = \{\underline{r} \mid \sum_{i=1}^{n} r_i x_i = 0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \{\underline{0}\}$ $(\because \{x_i\}$ is a basis$\}$

$\qquad$ & $\varphi$ is surj, since $S$ is a gen set.