**Exercises**

1. The canonical form of a linear program is $\min \mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Show that a general linear program can be coverted to one in canonical form and vice versa.

2. Consider the problem: minimize $2x_1 + 3|x_2 - 10|$ subject to $|x_1 + 2| + |x_2| \leq 5$. Reformulate this as a linear programming problem.

3. Consider a linear optimization problem, with absolute values of the following form: minimize $\mathbf{c}'\mathbf{x} + \mathbf{d}'\mathbf{y}$ subject to $\mathbf{Ax} + \mathbf{By} \leq \mathbf{b}$, $y_i = |x_i|$ for all $i$. Assume that the entries of $\mathbf{B}$ and $\mathbf{d}$ are nonnegative.
   - Provide two different linear programming formulations of the above problem.
   - Show that the original problem and the two reformulations are equivalent in the sense that either all three are infeasible, or all three have the same optimal cost.
   - Provide an example to show that if $\mathbf{B}$ has negative entries, the problem may have a local minimum that is not a global minimum. (we will see later that this is never the case in linear programming problems. Hence, in the presence of such negative entries, a linear programming reformulation is implausible.)

4. Consider a school district with $I$ neighbourhoods, $J$ schools, and $G$ grades at each school. Each school $j$ has a capacity of $C_{jg}$ for grade $g$. In each neighbourhood $i$, the student population of grade $g$ is $S_{ig}$. Finally the distance of school $j$ from neighbourhood $i$ is $d_{ij}$. Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance travelled by all students. (You may ignore the fact that numbers of students must be integer)

5. Consider a set $P$ described by linear inequality constraints, that is, $P = \{x \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \leq b_i,\ i = 1, 2, \cdots, m\}$. A ball with center $\mathbf{y}$ and radius $r$ is defined as the set of all points within Euclidean distance $r$ from $\mathbf{y}$. We are interested in finding a ball with the largest possible radius, which is entirely contained within the set $P$. (the center of such a ball is called the Chebyshev center of $P$). Provide a linear programming formulation of this problem.

6. Suppose there are $N$ available currencies, and assume that one unit of currency $i$ can be exchanged for $r_{ij}$ units of currency $j$. (Naturally, we assume that $r_{ij} > 0$.) There are also certain regulations that impose a limit $u_i$ on the total amount of currency $i$ that can be exchanged on any given day. Suppose that we start with $B$ units of currency 1 and that we would like to maximize the number of units of currency $N$ that we end up with at the end of the day, through a sequence of currency transactions. Provide a linear programming formulation of this problem. Assume that for any sequence $i_1, \ldots, i_k$ of currencies, we have $r_{i_! i_2} r_{i_2 i_3} \cdots r_{i_{k-1} i_k} r_{i_k i_1} \leq 1$, which means that wealth cannot be multiplied by going through a cycle of currencies.

7. Consider a polyhedra $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ in standard form. Show that when $P$ is nonempty, we can assume full row rank of $\mathbf{A}$ without loss of generality.

8. [HW 1, due Sep 19] Suppose that $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, 2 \cdots m\}$ and $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{g}_i^T\mathbf{x} \geq h_i,\ i = 1, 2 \cdots, k\}$ are two representations of the same nonempty polyhedron. Suppose that the vectors $\mathbf{a}_1, \cdots \mathbf{a}_m$ span $\mathbb{R}^n$. Show that the same must be true for the vectors $\mathbf{g}_1, \cdots, \mathbf{g}_k$.

9. Consider the standard form polyhedron $\{\mathbf{x} : \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. Let $\mathbf{x}$ be a basic solution, and let $J = \{i : x_i \neq 0\}$. Show that a basis is associated with the basic solution $\mathbf{x}$ if and only if every column $\mathbf{A}_i$, $i \in J$, is in the basis.

10. [HW 2, due Sep 27] Consider the standard form polyhedron $\{\mathbf{x} : \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, and assume that the $m$ rows of the matrix $\mathbf{A}_{m\times n}$ are linearly independent.
    - Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate

- Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample.
- Suppose that a basic solution $\mathbf{x}$ is degenerate. Is it true that there exists an adjacent basic solution $\mathbf{y}$ which is degenerate? Prove or give a counterexample.

11. [HW 2, due Sep 27] Consider the standard form polyhedron $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$ has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else provide a counterexample.
    - If $n = m + 1$, then $P$ has at most two basic feasible solutions.
    - The set of all optimal solutions is bounded.
    - At every optimal solution, no more than $m$ variables can be positive.
    - If there is more than one optimal solution, then there are uncountably many optimal solutions.
    - If there are several optimal soulutions, then there exist at least two basic feasible solutions that are optimal.
    - Consider the problem of minimizing $\max\{\mathbf{c}^T\mathbf{x},\ \mathbf{d}^T\mathbf{x}\}$ over the set $P$. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of $P$.
12. Consider the standard form polyhedron $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$, of dimensions $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let $\mathbf{x}$ be an element of $P$ that has exactly $m$ positive components.
    - Show that $\mathbf{x}$ is a basic feasible solution.
    - Show that the above result is false if the nondegeneracy assumption is removed.
13. Let $P$ be a bounded polyhedron in $\mathbb{R}^n$, and let $\mathbf{a}$ be a vector in $\mathbb{R}^n$, and let $b$ be some scalar. We define
$$Q = \{\mathbf{x} \in P : \mathbf{a}^T\mathbf{x} = b\}.$$
Show that every extreme point of $Q$ is either an extreme point of $P$ or a convex combination fo two adjacent extreme points of $P$.
14. (Homework 3, due October 5) Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, 2, \cdots m\}$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are distinct basic feasible solutions that satisfy $\mathbf{a}_i^T\mathbf{u} = \mathbf{a}_i^T\mathbf{v} = b_i,\ i = 1, 2, \cdots n-1$, and that the vectors $\mathbf{a}_1, \cdots, \mathbf{a}_{n-1}$ are linearly independent. (In particular, $\mathbf{u}$ and $\mathbf{v}$ are adjacent). Let $L = \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v} | 0 \leq \lambda \leq 1\}$ be the segment that joins $\mathbf{u}$ and $\mathbf{v}$. Prove that $L = \{\mathbf{z} \in P | \mathbf{a}_i^T\mathbf{z} = b_i,\ i = 1, 2, \cdots n-1\}$.
15. Consider a polyhedron defined by the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$. Assume that the matrix $\mathbf{A}$ has linearly independent rows and that $u_i > 0$ for all $i$. Describe the basic solutions of this polyhedron.
16. Let $\mathbf{A}_1, \cdots \mathbf{A}_n$ be a collection of vectors in $\mathbb{R}^m$.
    (a) Let
$$C = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{A}_i : \lambda_1, \cdots, \lambda_n \geq 0 \right\}.$$
Show that any element of $C$ can be expressed as $\sum_{i=1}^{n} \lambda_i \mathbf{A}_i$, with each $\lambda_i \geq 0$ and at most $m$ of the coefficients $\lambda_i$ being nonzero. *Hint:* Consider the polyhedron
$$\Lambda = \left\{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i \mathbf{A}_i = \mathbf{y},\ \lambda_1, \cdots, \lambda_n \geq 0 \right\}.$$

(b) Let $P$ be the convex hull of the vectors $\mathbf{A}_i$. Show that any element of $P$ can be expressed in the form $\sum_{i=1}^n \lambda_i \mathbf{A}_i$, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i$, with at most $m+1$ of the coefficients $\lambda_i$ being nonzero.

17. (Homework 3, due October 5) Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$. Given any $\epsilon > 0$, show that there exists some $\bar{\mathbf{b}}$ with the following two properties:

    (a) The absolute value of every component of $\mathbf{b} - \bar{\mathbf{b}}$ is bounded by $\epsilon$.

    (b) Every basic feasible solution in the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \bar{\mathbf{b}}\}$ is nondegenerate.

18. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a convex function and let $S \subset \mathbb{R}^n$ be a convex set. Let $\mathbf{x}^*$ be an element of $S$. Suppose that $\mathbf{x}^*$ is a local optimum for the problem of minimizing $f(\mathbf{x})$ over $S$; that is, there exists some $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ for which $\|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon$. Prove that $\mathbf{x}^*$ is globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

19. **(Optimality conditions)** Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron $P$. Prove the following

    - A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}^T\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.
    - A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c}^T\mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

20. Let $\mathbf{x} \in P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Characterize the feasible directions at $\mathbf{x}$.

21. **(Conditions for a unique optimum)** Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove the following.

    - If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.
    - If $\mathbf{x}$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

22. **(Optimality conditions)** Consider a feasible solution $\mathbf{x}$ to a standard form problem, and let $Z = \{i : x_i = 0\}$. Show that $\mathbf{x}$ is an optimal solution if and only if the linear programming problem

    $$\text{minimize } \mathbf{c}^T\mathbf{d} \text{ subject to } \mathbf{A}\mathbf{d} = \mathbf{0},\ d_i \geq 0 \text{ for } i \in Z,$$

    has an optimal cost of zero.

23. **(Necessary and sufficient conditions for a unique optimum)** Consider a linear programming problem in standard form and suppose that $\mathbf{x}^*$ is an optimal basic feasible solution. Consider an optimal basis associated with $\mathbf{x}^*$. Let $B$ and $N$ be the set of basic and nonbasic indices, respectively. Let $I$ be the set of nonbasic indices $i$ for which the corresponding reduced costs $\bar{c}_i$ are zero.

    - Show that if $I$ is empty, then $\mathbf{x}^*$ is the only optimal solution.
    - Show that $\mathbf{x}^*$ is the unique optimal solution if and only if the following linear programming problem has an optimal value of zero: maximize $\sum_{i \in I} x_i$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, $x_i = 0$ for $i \in N \backslash I$, and $x_i \geq 0$ for $i \in B \cup I$.

24. Consider the problem: minimize $-2x_1 - x_2$ subject to $x_1 - x_2 \leq 2$, $x_1 + x_2 \leq 6$, $x_1, x_2 \geq 0$.

    - Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.
    - Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).
    - Draw a graphical representation of the problem in terms of the original variables $x_1, x_2$ and indicate the path taken by the simplex algorithm.

25. [Homework 4, due October 24] Solve completely (both Phase 1 and Phase 2) via the simplex method of the following problem: minimize $2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5$ subject to

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + x_5 &= 2 \\ x_1 + 2x_2 - 3x_4 + x_5 &= 2 \\ -x_1 - 4x_2 + 3x_3 &= 1 \\ x_1,\, x_2 \cdots x_5 &\geq 0. \end{aligned}$$

26. Consider the simplex method applied to a standard form problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. For each of the statements that follow, give either a proof or a counterexample.

- An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
- A variable that has just left the basis cannot reenter in the very next iteration.
- A variable that just entered the basis cannot leave in the very next iteration.
- If there is a nondegenerate optimal basis, then there exists a unique optimal basis.
- If $\mathbf{x}$ is an optimal solution found by the simplex method, no more than $m$ of its components can be positive, where $m$ is the number of equality constraints.

27. While solving a standard form problem, we arrive at the following tableau, with $x_3, x_4$, and $x_5$ being the basic variables:

| | | | | | |
|---|---|---|---|---|---|
| $-10$ | $\delta$ | $-2$ | 0 | 0 | 0 |
| 4 | $-1$ | $\eta$ | 1 | 0 | 0 |
| 1 | $\alpha$ | $-4$ | 0 | 1 | 0 |
| $\beta$ | $\gamma$ | 3 | 0 | 0 | 1 |

The entries $\alpha$, $\beta$, $\gamma, \delta, \eta$ in the tabeau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

- The current solution is optimal and there are multiple optimal solutions.
- The optimal cost is $-\infty$.
- The current solution is feasible but not optimal.

28. Consider a linear programming problem in standard form, desribed in terms of the following initial tableau:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\delta$ | 3 | $\gamma$ | $\xi$ |
| $\beta$ | 0 | 1 | 0 | $\alpha$ | 1 | 0 | 3 |
| 2 | 0 | 0 | 1 | $-2$ | 2 | $\eta$ | $-1$ |
| 3 | 1 | 0 | 0 | 0 | $-1$ | 2 | 1 |

The Greek letters in the tableau are unknown parameters. For each one of the following statements, find the ranges of values of the various parameters tath will make the statement to be true.

- Phase II of the simplex method can be applied using this as an initial tableau.
- The first row in the present tableau (below the row with the reduced costs) indicates that the problem is infeasible
- The corresponding basic solution is feasible, but we do not have an optimal basis.
- The corresponding basic solution is feasible and the first simplex iteration indicates that the optimal cost is $-\infty$.
- The corresponding basic solution is feasible, $x_6$ is a candidate for entering the basis, and when $x_6$ is the entering variable, $x_3$ leaves the basis.

- The corresponding basic soluiton is feasible, $x_7$ is a candidate for entering the basis, but if it does, the solution and the objective value remain unchanged.

29. [HW 5, due Nov 12] (**The simplex method with upper bound constraints**) Consider a problem of the form: $\min \mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$, where $\mathbf{A}$ has linearly independent rows and dimensions $m \times n$. Assume that $u_i > 0$ for all $i$.

   - Let $\mathbf{A}_{B(1)}, \cdots \mathbf{A}_{B(m)}$ be $m$ linearly independent columns of $\mathbf{A}$. We partition the set of all $i \neq B(1), B(2) \cdots B(m)$ into two disjoint subsets $L$ and $U$. We set $x_i = 0$ for all $i \in L$, and $x_i = u_i$ for all $i \in U$. We then solve the equation $\mathbf{Ax} = \mathbf{b}$ for the basic variables $x_{B(1)}, \cdots x_{B(m)}$. Show that the resulting vector $\mathbf{x}$ is a basic solution. Also, show that it is nondegenerate if and only if $x_i \neq 0$ and $x_i \neq u_i$ for every basic variable $x_i$.
   - For this part and the next, assume that the basic solution constructed above is feasible. We form the simplex tableau and compute the reduced costs as usual. Let $x_j$ be some nonbasic variable such that $x_j = 0$ and $\bar{c}_j < 0$. Increase $x_j$ by $\theta$, and adjust the basic variables from $\mathbf{x}_B$ to $\mathbf{x}_B - \theta\mathbf{B}^{-1}\mathbf{A}_j$. Given that we wish to preseve feasiblility, what is the largest possible value of $\theta$? How are the new basic columns determined?
   - Let $x_j$ be some nonbasic variable such that $x_j = u_j$ and $\bar{c}_j > 0$. We decrease $x_j$ by $\theta$, and adjust the basic variables from $\mathbf{x}_B$ to $\mathbf{x}_B + \theta\mathbf{B}^{-1}\mathbf{A}_j$. Given that we wish to preseve feasiblility, what is the largest possible value of $\theta$? How are the new basic columns determined?
   - Assuming that every basic feasible solution is nondegenerate, show that the cost strictly decreases with each iteration and the method terminates.

30. Suppose that the system of linear inequalities $\mathbf{Ax} \leq \mathbf{b}$ has at least one solution, and let $d$ be some scalar. Then show that the following are equivalent.

   - Every feasible solution to the system $\mathbf{Ax} \leq \mathbf{b}$ satisfies $\mathbf{c}^T\mathbf{x} \leq d$.
   - There exists some $\mathbf{p} \geq \mathbf{0}$ such that $\mathbf{p}^T\mathbf{A} = \mathbf{c}^T$ and $\mathbf{p}^T\mathbf{b} \leq d$.

31. Let $\mathbf{A}$ be a symmetric square matrix. Consider the linear programming problem: minimize $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} \geq \mathbf{c}$, $\mathbf{x} \geq \mathbf{0}$. Prove that if $\mathbf{x}^*$ satisfies $\mathbf{Ax}^* = \mathbf{c}$ and $\mathbf{x}^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is an optimal solution.

32. Consider a linear programming problem in standard form and assume that the rows of $\mathbf{A}$ are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

   - Let $\mathbf{x}^*$ be a basic feasible solution. Suppose that for every basis corresponding to $\mathbf{x}^*$, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less than $\mathbf{c}^T\mathbf{x}^*$.
   - The dual of the auxillary primal problem considered in Phase 1 of the simplex method is always feasible.
   - Let $p_i$ be the dual variable associated with the $i$th equality constraint in the primal. Eliminating the $i$th primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.
   - If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible.

33. **(Duality in piecewise linear convex optimization)** Consider the problem of minimizing $\max_{i=1,2\cdots m}(\mathbf{a}_i^T\mathbf{x} - b_i)$ over all $\mathbf{x} \in \mathbb{R}^n$. Let $v$ be the value of the optimal cost, assumed finite. Let $\mathbf{A}$ be the matrix with rows $\mathbf{a}_1^T, \mathbf{a}_2^T \cdots \mathbf{a}_m^T$, and let $\mathbf{b}$ be the vector with components $b_1, b_2 \cdots b_m$.

   - Consider any vector $\mathbf{p} \in \mathbb{R}^m$ that satisfies $\mathbf{p}^T\mathbf{A} = \mathbf{0}^T$, $\mathbf{p} \geq \mathbf{0}$ and $\sum_{i=1}^m p_i = 1$. Show that $-\mathbf{p}^T\mathbf{b} \leq v$.

- In order to obtain the best possible lower bound of the form considered in the above part, we form the linear programming problem

$$\text{maximize } -\mathbf{p}^T\mathbf{b} \text{ subject to } \mathbf{p}^T\mathbf{A} = \mathbf{0}^T,\ \mathbf{p}^T\mathbf{e} = 1,\ \mathbf{p} \geq \mathbf{0},$$

  where $\mathbf{e}$ is the vector with all components equal to 1. Show that the optimal cost in this problem is equal to $v$.

34. Consider the linear programming problem of minimizing $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Let $\mathbf{x}^*$ be an optimal solution, assumed to exist, and let $\mathbf{p}^*$ be an optimal solution to the dual.

    - Let $\bar{\mathbf{x}}$ be an optimal solution to the primal, when $\mathbf{c}$ is replaced by some $\bar{\mathbf{c}}$. Show that $(\bar{\mathbf{c}} - \mathbf{c})^T(\bar{\mathbf{x}} - \mathbf{x}^*) \leq 0$.
    - Let the cost vector be fixed at $\mathbf{c}$, but suppose that we now change $\mathbf{b}$ to $\bar{\mathbf{b}}$, and let $\bar{\mathbf{x}}$ be a corresponding optimal solution to the primal. Prove that $(\mathbf{p}^*)^T(\bar{\mathbf{b}} - \mathbf{b}) \leq \mathbf{c}^T(\bar{\mathbf{x}} - \mathbf{x}^*)$.

35. [HW 5, due Nov 12] **(Saddle points of the Lagrangean)** Consider the standard form problem of minimizing $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. We define the *Lagrangean* by

$$L(\mathbf{x}, \mathbf{p}) = \mathbf{c}^T\mathbf{x} + \mathbf{p}^T(\mathbf{b} - \mathbf{A}\mathbf{x}).$$

    Consider the following "game": player 1 chooses some $\mathbf{x} \geq \mathbf{0}$, and player 2 chooses some $\mathbf{p}$; then, player 1 pays to player 2 the amount $L(\mathbf{x}, \mathbf{p})$. Player 1 would like to minimize $L(\mathbf{x}, \mathbf{p})$, while player 2 would like to maximize it.

    A pair $(\mathbf{x}^*, \mathbf{p}^*)$, with $x^* \geq \mathbf{0}$, is call an *equilbrium* point (or a *saddle point*, or a *Nash equilibrium*) if

$$L(\mathbf{x}^*, \mathbf{p}) \leq L(\mathbf{x}^*, \mathbf{p}^*) \leq L(\mathbf{x}, \mathbf{p}^*), \quad \forall \mathbf{x} \geq \mathbf{0}, \forall \mathbf{p}.$$

    (Thus we have an equilibrium if no player is able to improve her performance by unilaterally modifying her choice).

    Show that a pair $(\mathbf{x}^*, \mathbf{p}^*)$ is an equilibrium point if and only if $\mathbf{x}^*$ and $\mathbf{p}^*$ are optimal solutions to the standard form problem under consideration and its dual, respectively.

36. Consider a linear programming problem in standard form which is infeasible, but which becomes feasible and has finite optimal cost when the last equality constraint is omitted. Show that the dual of the original (infeasible) problem is feasible and the optimal cost is infinite.

37. Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.