Lecture 34: Rational canonical form and Jordan form

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<u>Thm</u> (Rat'l canonical form): Let V be a n-dimensional vector space over a field k and $\varphi : V \to V$ be a k-linear map. Then $\exists$ a basis $\mathcal{B}$ of V s.t. that the matrix of φ w.r.t $\mathcal{B}$ is of the form.

$$R_\varphi = \begin{bmatrix} R_{a_1} & & & 0 \\ & R_{a_2} & & \\ & & \ddots & \\ 0 & & & R_{a_m} \end{bmatrix}$$

where for a monic poly $a(X) = X^l + b_{l-1}X^{l-1} + \dots + b_0$ of deg l, R_a is the $l \times l$ matrix

$$\begin{bmatrix} 0 & & & & -b_0 \\ 1 & 0 & & & -b_1 \\ & 1 & \ddots & & \vdots \\ & & & 0 & \vdots \\ & & & 1 & -b_{l-1} \end{bmatrix}$$

$a_1(X), \dots, a_m(X) \in k[X]$ are nonconstant monic poly s.t. $a_1 | a_2 | \cdots | a_m$.

Equivalently, $A \in M_{n \times n}(k)$ then $\exists$ a nonsingular matrix P s.t. $P^{-1}AP = R_\varphi$ for some $a_1, \dots, a_m \in k[x]$ nonconst. monic poly with $a_1 | a_2 | \cdots | a_m$.

<u>Pf:</u> $\varphi : V \to V$ is a k-lin map

$\Rightarrow$ V is a $k[X]$-mod s.t. $X \cdot v = \varphi(v)$ $\forall v \in V$. $f(X)\cdot v = f(\varphi)(v)$ Note $f(\varphi) \in \mathrm{End}_k(V)$

By str thm for f.g. mod over PID.

$V \overset{\theta}{\cong} R/(a_1) \oplus \dots \oplus R/(a_m)$ where $R = k[X]$ and $a_1, \dots, a_m$ are monic non const. poly and $a_1 | a_2 | \cdots | a_m$ as R-modules

(since rank$(V) = 0$ as $k[X]$-mod as $m_\varphi(x)$ annihilates V)
minimal poly of φ.

Note

$$\begin{tikzcd} V \arrow[r, "\varphi"] & V \arrow[d, "\theta^{-1}", "\wr"'] \\ \bigoplus_{i=1}^m R/(a_i) \arrow[u, "\theta", "\wr"'] \arrow[r, dashed] & \bigoplus_{i=1}^m R/(a_i) \end{tikzcd}$$

<u>Claim</u>: $\theta^{-1} \circ \varphi \circ \theta = \mu_X$ ← mult by X

Let $\alpha \in \bigoplus_{i=1}^m R/(a_i)$

$\theta^{-1} \circ \varphi \circ \theta(\alpha) = \theta^{-1}(X \cdot \theta(\alpha)) = X \cdot \theta^{-1}(\theta(\alpha)) = X \cdot \alpha = \mu_X(\alpha)$

θ^{-1} is $k[X]$-lineal

So enough to show μ_X that the matrix of μ_X w.r.t some basis is R_φ.

Note that $R/(a_i)$ is invariant under μ_X as

$$X \cdot (\alpha_1, \dots, \alpha_m) = (X \cdot \alpha_1, \dots, X \cdot \alpha_m) \quad \text{for } \alpha_i \in R/(a_i)$$

$1 \le i \le m$.

$[X^j]_i = X^j + (a_i)$

Also $B_i = \{[1]_i, [X]_i, [X]_i^2, \dots, [X]_i^{n_i - 1}\}$ is a basis of $R/(a_i)$ where a_i is a poly of deg n_i.

Let B be the ordered basis $B_1 \cup B_2 \cup \dots \cup B_m$. Then the matrix of μ_X w.r.t. B is

$$\begin{bmatrix} R_1 & & O \\ & \ddots & \\ O & & R_m \end{bmatrix}$$

where

R_i is the matrix of $\mu_X|_{R/(a_i)}$ w.r.t. the ordered basis B_i. (Since $R/(a_i)$ is invariant under μ_X.)

So enough to show $R_i = R_{a_i}$

matrix of $\mu_X|_{R/(a_i)}$ w.r.t $\mathcal{B}_i = \{[1]_i, [X]_i, \dots, [X^{n_i-1}]_i\}$.

Recall matrix of $\psi \in \mathrm{End}(V)$ w.r.t a ordered basis $\{v_1, \dots, v_n\}$ is $((c_{ij})) \in M_{n\times n}(k)$ where $\psi(v_i) = \sum_{j=1}^{n} c_{ji} v_j \quad 1 \le i \le n$.

$$\mu_X([1]_i) = X\cdot[1]_i = [X]_i = 0[1]_i + 1[X]_i + 0[X^2]_i + \dots + 0[X^{n_i-1}]_i$$

$$\mu_X([X]_i) = X\cdot[X]_i = [X^2]_i = 0[1]_i + 0[X]_i + 1[X^2]_i + \dots + 0[X^{n_i-1}]_i$$

$$\vdots$$

$$\mu_X([X^{n_i-2}]_i) = [X^{n_i-1}]_i = 0[1]_i + 0[X]_i + \dots + 0[X^{n_i-2}]_i + 1[X^{n_i-1}]_i$$

$$\mu_X([X^{n_i-1}]_i) = [X^{n_i}]_i = -b_0[1]_i + -b_1[X]_i + \dots + -b_{n_i-1}[X^{n_i-1}]_i$$

if $a_i(X) = X^{n_i} + b_{n_i-1}X^{n_i-1} + b_{n_i-2}X^{n_i-2} + \dots + b_1X + b_0$

Hence $R_i = R_{a_i} = \begin{pmatrix} 0 & 0 & & & & -b_0 \\ 1 & 0 & & 0 & & -b_1 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 & -b_{n_i-2} \\ 0 & 0 & 0 & \cdots & 1 & -b_{n_i-1} \end{pmatrix}$

Hence the matrix of μ_X w.r.t the ordered basis $\mathcal{B}$ is R_φ.

$\boxed{}$

Thm: (Jordan form) Let V be a n-dim'l vs over $\mathbb{C}$ (or an alg closed field). Let $\varphi : V \longrightarrow V$ be a $\mathbb{C}$-linear map. Then there exist a basis of $\mathcal{B}$ of V s.t. the matrix of φ w.r.t. $\mathcal{B}$ is of the form.

$$J_\varphi = \begin{bmatrix} J_{\lambda_1}^{r_{11}} & & & & & & & & \\ & J_{\lambda_1}^{r_{12}} & & & & & & \text{\Large 0} & \\ & & \ddots \; J_{\lambda_1}^{r_{1n_1}} & & & & & & \\ & & & J_{\lambda_2}^{r_{21}} & & & & & \\ & & & & \ddots \; J_{\lambda_2}^{r_{2n_2}} & & & & \\ & & & & & \ddots & & & \\ & \text{\Large 0} & & & & & J_{\lambda_m}^{r_{m1}} & & \\ & & & & & & & \ddots & \\ & & & & & & & & J_{\lambda_m}^{r_{mn_m}} \end{bmatrix}$$

where $\lambda_i \in \mathbb{C}$, $1 \le i \le m$

r_{ij} are positive integers.

$$J_\lambda^r = \begin{bmatrix} \lambda & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{bmatrix}$$ is a $r \times r$ matrix $\lambda \in \mathbb{C}$.

Equivalently, $A \in M_{n\times n}(\mathbb{C})$ then A is similar to J_φ for some $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ & r_{ij} positive integers.

str than as $\mathbb{C}[x]$-mod

$$V \cong R/p_1^{r_{11}} \oplus R/p_1^{r_{12}} \oplus \cdots \oplus R/p_1^{r_{1n_1}}$$

$$\oplus R/p_2^{r_{21}} \quad \cdots \quad \oplus R/p_2^{r_{2n_2}}$$

$$\vdots$$

$$\oplus R/p_m^{r_{m1}} \oplus \cdots \oplus R/p_m^{r_{mr_m}}$$

$p_i(x)$ are irred in $\mathbb{C}[x]$

(*) Every non const poly in $\mathbb{C}[x]$ is prod of linear factors (FTA).

$\Rightarrow p_i(x) = (x-\lambda_i)$ for some $\lambda_i \in \mathbb{C}$.

So $R/(p_i^{r_{ij}}) = \dfrac{\mathbb{C}[x]}{(x-\lambda_i)^{r_{ij}}}$

So we will choose a basis B_{ij} of $\dfrac{\mathbb{C}[x]}{(x-\lambda_i)^{r_{ij}}}$ s.t. the matrix of μ_x on $R/p_i^{r_{ij}}$ is $J_{\lambda_i}^{r_{ij}}$.

And this will complete the proof.

$$\mathcal{B}_{ij} = \{1,\ x-\lambda_i,\ (x-\lambda_i)^2, \dots, (x-\lambda_i)^{r_{ij}-1}\}$$

$$\mu_x(1) = x \cdot 1 = \lambda_i 1 + 1(x-\lambda_i) + 0\cdot(x-\lambda_i)^2 + \dots + 0(x-\lambda_i)^{r_{ij}-1}$$

$$\mu_x(x-\lambda_i) = x^2 - \lambda_i x = 0 1 + \lambda_i(x-\lambda_i) + 1(x-\lambda_i)^2 + 0 \dots$$

$$\vdots$$

$$\mu_x((x-\lambda_i)^{r_{ij}-2}) = x(x-\lambda_i)^{r_{ij}-2} = (x-\lambda_i)^{r_{ij}-1} + \lambda_i(x-\lambda_i)^{r_{ij}-2}$$

$$= 0 \cdot 1 + 0\cdot(x-\lambda_i) \dots + \lambda_i(x-\lambda_i)^{r_{ij}-2} + 1(x-\lambda_i)^{r_{ij}-1}$$

$$\mu_x((x-\lambda_i)^{r_{ij}-1}) = x(x-\lambda_i)^{r_{ij}-1} = \underbrace{(x-\lambda_i)^{r_{ij}}}_{0} + \lambda_i(x-\lambda_i)^{r_{ij}-1}$$

$$= 0\cdot 1 + 0\cdot(x-\lambda_i) + \dots + 0(x-\lambda_i)^{r_{ij}-2} + \lambda_i(x-\lambda_i)^{r_{ij}-1}$$

$$J_\lambda^{r_{ij}} = \begin{pmatrix} \lambda_i & 0 & & 0 & 0 \\ 1 & \lambda_i & 0 & \vdots & \vdots \\ 0 & 1 & \lambda_i & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & 1 & \lambda_i \end{pmatrix}$$

Hence the matrix of μ_x w.r.t. $\mathcal{B} = \mathcal{B}_1^{r_{11}} \cup \mathcal{B}_1^{r_{12}} \cup \dots \cup \mathcal{B}_1^{r_{1n_1}} \cup \mathcal{B}_2^{r_{21}} \cup \mathcal{B}_2^{r_{22}} \dots \cup \mathcal{B}_m^{r_{mn_m}}$ is

J_φ .