(i) $\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{SS_{xx}}\right)$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Note $\sum_{i=1}^{n} (x_i - \bar{x})\bar{y} = \sum_{i=1}^{n} (x_i \bar{y}) - \sum_{i=1}^{n} \bar{x}\bar{y}$

$$= n\bar{x}\bar{y} - n\bar{x}\bar{y} = 0$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2} - \frac{\sum_{i=1}^{n} (x_i - \bar{x})\bar{y}}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \quad (=0)$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Set $c_i = \frac{(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$

then $\hat{\beta}_1 = \sum_{i=1}^{n} c_i y_i \Rightarrow \hat{\beta}_1$ is linear combination

of y_i's. As done in probability theory

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linear combination of normals is normal. Hence

$\hat{\beta} \sim N(\mu, \sigma^2)$ for some μ, σ^2.

ie $\mathbb{E}(\hat{\beta}) = \mu$ $Var(\hat{\beta}) = \sigma^2$

Now it is sufficient to show

$\mathbb{E}(\hat{\beta}) = \beta_1$ $Var(\sigma \hat{\beta}) = \frac{\sigma^2}{SS_{xx}}$

Again

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\cancel{\beta_0} + \beta_1 x_i + \epsilon_i - \cancel{\beta_0} - \beta_1 \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\beta_1(x_i - \bar{x}) + \epsilon_i)}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

by linearity of Expectation & $\mathbb{E}(\epsilon_i) = 0$

we have

$\mathbb{E}[\hat{\beta}_1) = \mathbb{E}(\beta_1) = \beta_1.$

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Now

$$Var(\hat{\beta}_1) = Var\left(\beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)$$

($Var(a+x) = Var(x)$ where a is constant)

$$= Var\left(\frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}[x_i - \bar{x}]^2}\right)$$

($Var(ax) = a^2 Var(x)$)

$$= \left(\frac{1}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)^2 Var\left(\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i\right)$$

($Var(x+y) = Var(x) + Var(y)$ x & y indep)

$$= \left(\frac{1}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)^2 \left(\sum_{i=1}^{n}(x_i - \bar{x})^2 Var\,\epsilon_i\right)$$

($\epsilon_i \sim N(0, \sigma^2)$)

$$= \left(\frac{1}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)^2 \sum_{i=1}^{n}(x_i - \bar{x})^2 \sigma^2$$

$$= \sigma^2 \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2}$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sigma^2}{SS_{xx}}$$

Hence $\hat{\beta} \sim N\left(\hat{\beta}_1, \frac{\sigma^2}{SS_{xx}}\right)$

② for fixed x

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

then $\hat{y} \sim N\left(\beta_0 + \beta_1 x, \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{SS_{xx}}\right)\sigma^2\right)$

Note $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

$(\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x})$

$$= \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x$$

$$= \hat{\beta}_1 (x - \bar{x}) + \bar{y}$$

(Also $\hat{\beta}_1 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x}) y_i}{SS_{xx}}$ proved in question 1)

$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x}) y_i (x - \bar{x})}{SS_{xx}} + \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\hat{y} = \sum_{i=1}^{n}\left[\frac{1}{n} + \frac{(x_i - \bar{x})(x - \bar{x})}{SS_{xx}}\right] y_i \quad \text{—— } (*)$$

As $\hat{y}$ is linear combination of y_i

& y_i's are all normal Hence

$\hat{y}$ is normaly distributed.

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Now

$$\mathbb{E}\hat{y} = \mathbb{E}(\hat{\beta}_0 + \hat{\beta}_1 x)$$

$$= \mathbb{E}(\hat{\beta}_0) + x\,\mathbb{E}(\hat{\beta}_1)$$

$$\mathbb{E}\hat{y} = \beta_0 + x\beta_1 \quad \text{--- (1)}$$

Also from (*)

$$\mathrm{Var}(\hat{y}) = \mathrm{Var}\left[\left(\sum_{i=1}^{n} \frac{1}{n} + \frac{(x_i-\bar{x})(x-\bar{x})}{SS_{xx}}\right) y_i\right]$$

~~Since~~ y_i's ~~are~~ iid.

$$= \sum_{i=1}^{n}\left(\frac{1}{n} + \frac{(x_i-\bar{x})(x-\bar{x})}{SS_{xx}}\right)^2 \mathrm{var}(y_i)$$

$\mathrm{Var}(y_i) = \sigma^2$

$$= \sum_{i=1}^{n} \sigma^2\left(\frac{1}{n^2} + \frac{(x_i-\bar{x})^2(x-\bar{x})^2}{SS_{xx}^2} + \frac{2(x_i-\bar{x})(x-\bar{x})}{nS_{xx}}\right)$$

$$= \sigma^2\left(\sum_{i=1}^{n}\frac{1}{n^2} + \frac{(x-\bar{x})^2}{SS_{xx}^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{2(x-\bar{x})}{nS_{xx}}\sum_{i=1}^{n}(x_i-\bar{x})\right)$$

$$= \sigma^2\left(\frac{n}{n^2} + \frac{(x-\bar{x})^2}{SS_{xx}^2}\cdot SS_{xx} + \frac{2(x-\bar{x})}{nS_{xx}}\underbrace{(n\bar{x}-n\bar{x})}_{(=0)}\right)$$

$$\mathrm{Var}(\hat{Y}) = \sigma^2 \left(\frac{1}{n} + \frac{(X-\bar{X})^2}{SS_{xx}} \right) \longrightarrow ②$$

Since $\hat{Y}$ is normally distributed

& $\mathbb{E}\,\hat{Y} = \beta_0 + \beta_1 x$

& $\mathrm{Var}(\hat{Y}) = \sigma^2 \left(\frac{1}{n} + \frac{(X-\bar{X})^2}{SS_{xx}} \right)$

Hence

$$\hat{Y} \sim N\left(\beta_0 + \beta_1 x,\ \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{SS_{xx}}\right)\sigma^2 \right)$$

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