Surface integrals

Recall: Curve C is a Cont (or C' or smooth or piecewise smooth etc) fn. (parametrization) $r : [a,b] \longrightarrow \mathbb{R}^n$.

[Then we went on talking about $\int_C$ or $\int_r$.]

Similarly, we want the notion of Surfaces S $(r : \mathcal{O}_2 \to \mathbb{R}^3 \text{ Cont/smooth})$ & then want to talk about $\int_S$!!

Def: A Subset $R \subseteq \mathbb{R}^n$ (bdd) is said to be a region if R is open & R has an area ($\Longleftrightarrow \partial R$ is of content zero).

\# Mostly, ~~for this~~ we will deal with $R \subseteq \mathbb{R}^2$.

eg: $B^2 = [a_1, b_1] \times (a_2, b_2)$.

Def: Let $R \subseteq \mathbb{R}^2$ be a region. A C' fn. $r : R \longrightarrow \mathbb{R}^3$ is called a parametrized surface (with parameter space R) if:

  (i) the component fn's of r have bounded partial derivatives.

  (ii) $r : R \longrightarrow \mathbb{R}^3$ is one-to-one.

$$R \subseteq \mathbb{R}^2$$
$$\overrightarrow{(u,v)}$$

$r = (r_1, r_2, r_3)$ (u,v)

$r_u = (\frac{\partial r_1}{\partial u}, \frac{\partial r_2}{\partial u}, \frac{\partial r_3}{\partial u})$

  (iii) $\forall \ (u,v) \in R$,

$$r_u \times r_v \Big|_{(u,v)} := r_u(u,v) \times r_v(u,v) \neq 0.$$

$\uparrow$ Cross product.

Def: A Subset $S \subseteq \mathbb{R}^3$ is called a surface if $S = \operatorname{ran} r$ for some parametrized surface $r : R \longrightarrow \mathbb{R}^3$.

\# Try to find similarity between Surfaces & Smooth curves.

Review on planes & normals:

~~Some explanation~~

Given two vectors $\vec{P} = \langle a_1, a_2, a_3 \rangle$, $\vec{Q} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$,
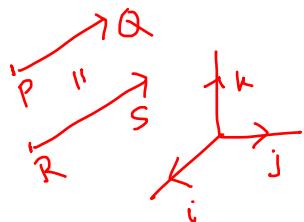
$$a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \qquad b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

we define the cross product $\vec{P} \times \vec{Q}$ by:

$$\vec{P} \times \vec{Q} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
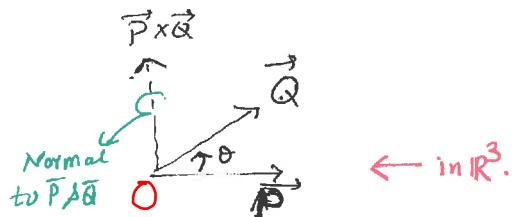
\# $\vec{P}$ & $\vec{Q}$ are linearly independent $\iff$ $\vec{P} \times \vec{Q} \neq \vec{0}$.

\# $\| \vec{P} \times \vec{Q} \| = \| \vec{P} \| \, \| \vec{Q} \| \sin\theta$.

The length/magnitude.

$\leftarrow$ in $\mathbb{R}^3$.

\# A subset $P \subseteq \mathbb{R}^n$ is a plane if $\exists ?: S$

\# Equ of planes (in $\mathbb{R}^3$)

~~Given~~ A plane is determined by a point P_0 in the plane & a vector N orthogonal to the plane. $N \to$ call it normal vector.] DO NOT CALL IT orthogonal anymore.

a non-zero vector

Def: For a point/vector $\vec{P_0} = \langle x_0, y_0, z_0 \rangle$ & $\vec{N} = \langle a, b, c \rangle \neq \vec{0}$ in $\mathbb{R}^3$, the plane through $\vec{P_0}$ that is "normal" to $\vec{N}$ is

the set $P = \{ \vec{P_0} + \vec{P} : \vec{P} \cdot \vec{N} = 0, \vec{P} \text{ in } \mathbb{R}^3 \}$.

dot product.

($\ast$ $\vec{P} \cdot \vec{N} = 0 \iff \vec{P}$ is orthogonal to $\vec{N}$).

Note: Let $\vec{P_0} + \vec{Q_1}, \vec{P_0} + \vec{Q_2} \in P$. ($P$ a plane as above).

$$\vec{Q_1} \cdot \vec{N} = \vec{Q_2} \cdot \vec{N} = 0$$

$\Rightarrow$ Suppose $\vec{Q_1}$ & $\vec{Q_2}$ are linearly independent.

$\Rightarrow$ $\vec{Q_1} \times \vec{Q_2} \neq 0$. Also $(\vec{Q_1} \times \vec{Q_2}) \cdot \vec{Q_i} = 0$ ($i = 1, 2$).

$\Rightarrow \quad \vec{Q_1} \times \vec{Q_2}$ is orthogonal to both $\vec{Q_1}$ & $\vec{Q_2}$. [A general fact.]

Recall further, $\Rightarrow \quad \vec{Q_1} \times \vec{Q_2} = \underset{\text{a scalar}}{c} \vec{N}$

$\left[\because \dim \mathbb{R}^3 = 3, \text{ so} \right.$
& $\vec{Q_1}, \vec{Q_2}$ are lin. indep.
$\Rightarrow \vec{N}$ & $\vec{Q_1} \times \vec{Q_2}$ must
be linearly dep. $\left. \right]$

i.e, $\quad \vec{N} = \underset{\text{a scalar.}}{c} (\vec{Q_1} \times \vec{Q_2})$

[OR, simply, $\{\vec{Q_1}, \vec{Q_2}, \vec{Q_1} \times \vec{Q_2}\}$
will be a basis of $\mathbb{R}^3$.]

$\because \{\vec{Q_1}, \vec{Q_2}$ & $\vec{Q_1} \times \vec{Q_2}\}$ is a basis of $\mathbb{R}^3$, &

$\{\vec{P_0} + s_1 \vec{Q_1} + s_2 \vec{Q_2} : s_1, s_2 \in \mathbb{R}\} \subseteq P$, AND

$N \notin$ LHS of the above, it follows that:

Not a subspace of $\mathbb{R}^3$. It is $\Leftrightarrow \vec{P_0} = 0$.

$$P = \{\vec{P_0} + s_1 \vec{Q_1} + s_2 \vec{Q_2} : s_1, s_2 \in \mathbb{R}\} \quad ——① $$

$\mathbb{R} \leftrightarrow B^2 \overset{(0,0)}{\supset}$
$(s_1, s_2) \in B^2$
$\underset{=}{\mathbb{R}}$

Representation of a plane.
Where $\vec{Q_1}, \vec{Q_2} \perp \vec{N}$,
& linearly independent.
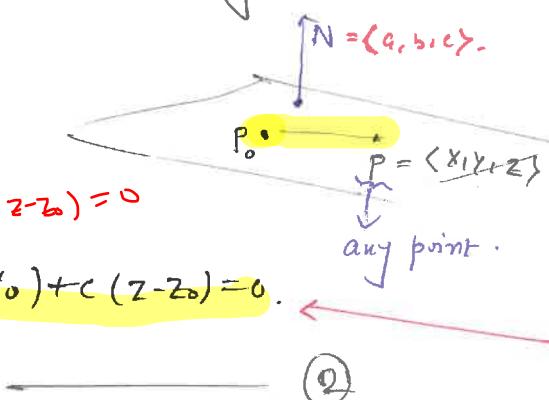
As far as eqn. of a plane is concered; we do as follows:

Given a (normal vector) $\vec{N} = \langle a, b, c \rangle$ & point $P_0 = \langle x_0, y_0, z_0 \rangle$,
the eqn of the plane through P_0 & with $\vec{N}$ as a normal
vector is given by:

$\vec{N} \cdot \vec{P_0 P} = 0$

$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$N = \langle a, b, c \rangle.$
P_0
$P = \langle x, y, z \rangle$
any point.

$\Rightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \quad ——② $

Cartesian form of a plane.

Also, ① can be expressed as:

$$r(\overset{u}{s_1}, \overset{v}{s_2}) = \vec{P_0} + \overset{u}{s_1} \vec{Q_1} + \overset{v}{s_2} \vec{Q_2}. \quad ——③ $$

$r_u = \vec{Q_1}$
$r_v = \vec{Q_2}$

$r_u \times r_v = \vec{Q_1} \times \vec{Q_2}$
$\neq 0$

$\Rightarrow \quad r|_{B_2}$ is a parametric surf.

Any $\vec{Q_1} \not\parallel \vec{Q_2}$ will do.

In fact: ③ Can be used to define a plane!

\# Let $S \subseteq \mathbb{R}^n$ be a line or a plane (trace).

A vector $\vec{N} \in \mathbb{R}^n$ is said to be normal to S if

$$\vec{N} \cdot (x-y) = 0 \qquad \forall \ x, y \in S.$$

"Normal" defn. or Classification.

Remark: (1) let $S \subseteq \mathbb{R}^3$ be a line / plane. Then
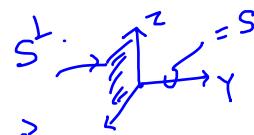
S is a subspace (vector) of $\mathbb{R}^3 \Leftrightarrow 0 \in S.$

(2) $S^\perp := \{ \vec{N} \in \mathbb{R}^3 : \vec{N} \text{ is normal to } S \}$

is a subspace of $\mathbb{R}^3$.

Note: usually, we assume $\vec{N} \neq 0$ to avoid triviality.

(3) If S is a line, then $S^\perp$ is a plane.
$(\subseteq \mathbb{R}^3)$ — HW — .

(4) If S is a plane in $\mathbb{R}^3$, then $S^\perp$ is a line. — HW — .

Return to Surface:

Again, recall that given a <mark>region</mark> $R \subseteq \mathbb{R}^2$ (open + ∂R is of zero content)

a C^1 fn. $r: R \to \mathbb{R}^3$ is a <mark>parametrized surface</mark> with <mark>parameter space</mark> R if:

Often, this is also known as "regular param".

i) components of r have $\boxed{bdd}$ 1st order /partial derivatives.

Often, we wont need/use this.

ii) r is <mark>injective</mark>. ← *We will evaluate fn over ran r.*

iii) $\left. T_u \times T_v \right|_{(u,v)} \neq 0 \quad \forall (u,v) \in R$.

Def: <mark>r is said to be a parametrization of the surface</mark> ran r.
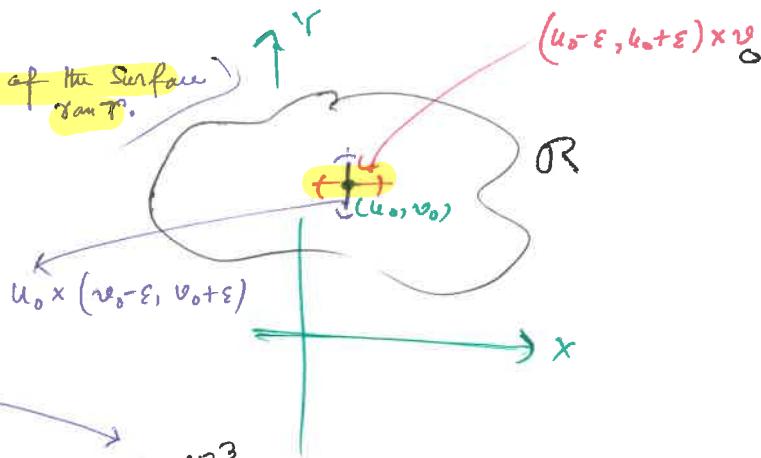
Note: on (iii): Let $(u_0, v_0) \in R$.

$\because R$ is open, $\exists \varepsilon > 0$ s.t.

$(u_0 - \varepsilon, u_0 + \varepsilon) \times v_0 \subseteq R$

& $u_0 \times (v_0 - \varepsilon, v_0 + \varepsilon) \subseteq R$.

$(u_0 - \varepsilon, u_0 + \varepsilon) \times v_0$

$u_0 \times (v_0 - \varepsilon, v_0 + \varepsilon)$

So,

$$(-\varepsilon, \varepsilon) \longrightarrow R \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (u_0 + t, v_0) \longmapsto r(u_0 + t, v_0)$$

defines a <mark>smooth curve</mark> in the surface $S = $ ran r. Call it γ.

$\therefore$ <mark>$\gamma(t) = r(u_0 + t, v_0)$</mark> $\forall t \in (-\varepsilon, \varepsilon)$.

Clearly, by the chain rule:

$$\gamma'(t) = \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$= \frac{\partial r}{\partial u} \cdot 1 + \frac{\partial r}{\partial v} \cdot 0$$

$$= \frac{\partial r}{\partial u} = T_u$$

$\therefore \ T_u \times T_v \neq 0 \Rightarrow \gamma'(t) \neq 0$.

$$\Rightarrow \gamma'(0) = \left. \frac{\partial r}{\partial u} \right|_{(u_0, v_0)}$$

← are A tangent vector of S at $r(u_0, v_0)$. *a curve in*
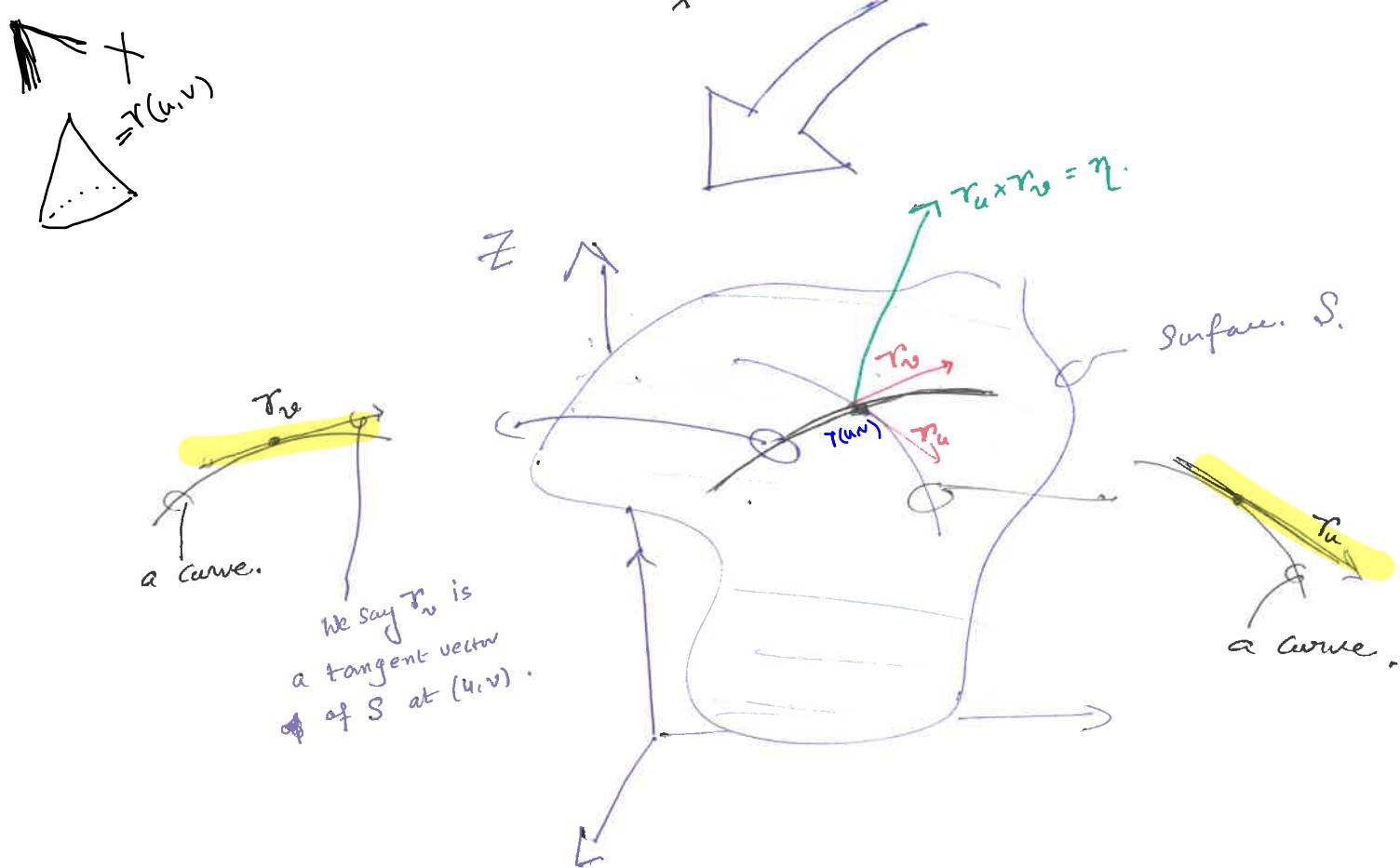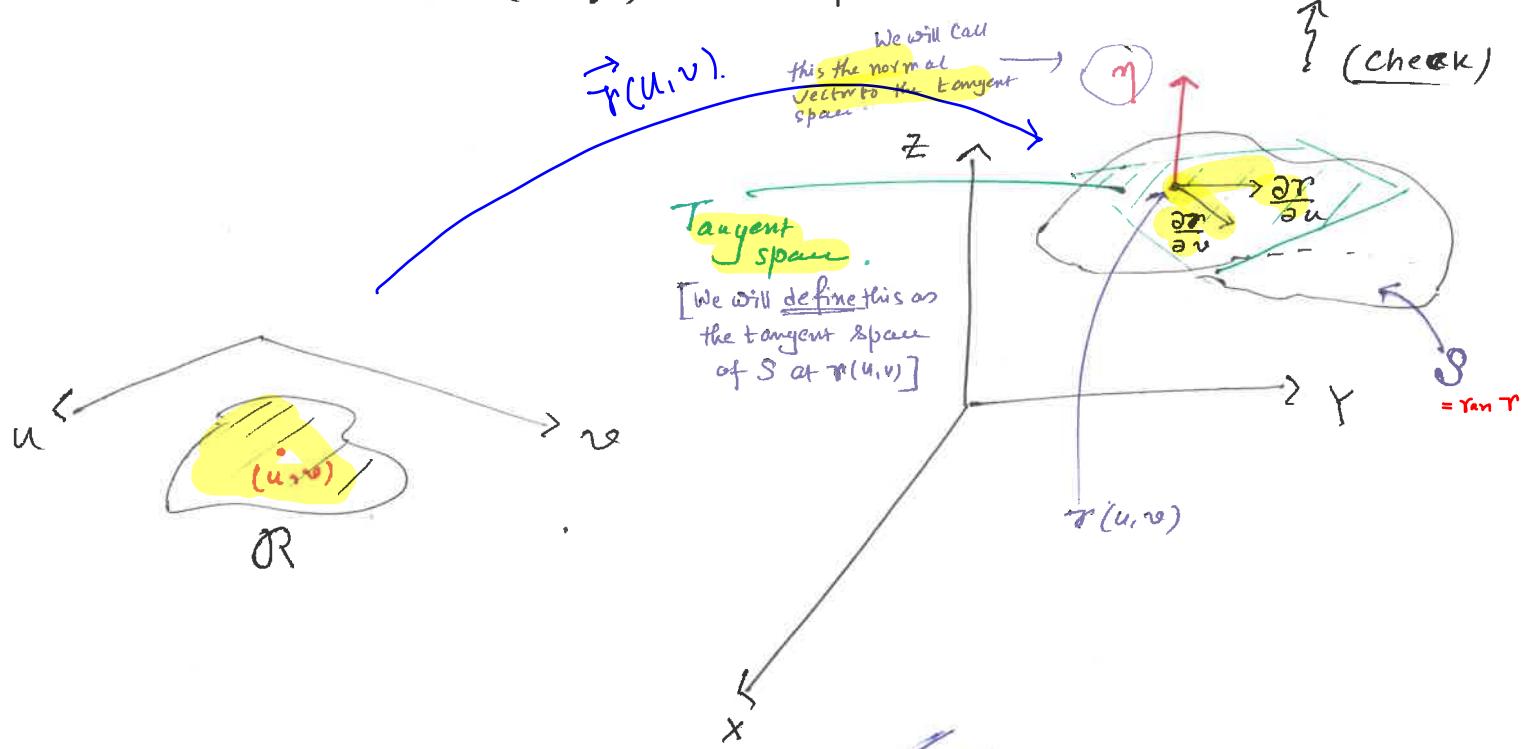
$\left. \frac{\partial r}{\partial v} \right|_{(u_0, v_0)}$ ← A tangent vector of S at $r(u_0, v_0)$

So, $\eta(u_0,v_0) := \left. T_u \times T_v \right|_{(u_0, v_0)} \neq 0$ is a normal vector to the pair of curves in S at $r(u_0, v_0)$.

$\because \eta \neq 0$, (iii) $\Longrightarrow$ [the pair of curves in] S have a normal vector $\eta(r(u,v))$ $\forall$.

$r(u,v) \in S$ $\forall$ $(u,v) \in R$ (or just simply, $\forall$ $(u,v) \in R$).

$\because$ r is in $C^1(R)$, it also follows that η is CONTINUOUS!! (check)

$\vec{r}(u,v)$.

We will call this the normal vector to the tangent space

η

Tangent space.

[We will define this as the tangent space of S at $r(u,v)$]

$\frac{\partial r}{\partial u}$ $\frac{\partial r}{\partial v}$

S = Tan T

z

y

x

$r(u,v)$

R

u (u,v) v

$= r(u,v)$

$r_u \times r_v = \eta$.

r_v r_u

$r(u,v)$

Surface S.

z

r_v a curve.

We say r_v is a tangent vector of S at (u,v).

r_u a curve.

eg:

1) $O_2 \subseteq \mathbb{R}$ be open., $f : O_2 \to \mathbb{R}$ be C^1 fn.. Consider the <u>graph fn.</u> $z = f(x,y)$, $(x,y) \in O_2$.

   i.e., graph of f = graph(f) $= \left\{ (x, y, f(x,y)) : (x,y) \in O_2 \right\}$.

Then graph(f) is a parametrized surface. Indeed :

Consider the parametrization :

$$ r(u,v) := \left(u, v, f(u,v) \right), \qquad (u,v) \in O_2. $$

Clearly, # r is C^1 $\left(\because (u,v) \mapsto u, v, f(u,v) \text{ are } C^1 \text{ fn.} \right)$

or simply $(x,y) \in O_2$. But for the sake of computation / notation, we use (u,v).

    # <u>bdd 1st order derivative must be assumed</u> for f .

    # r is injective : trivial.

    # $r_u = \left(1, 0, \frac{\partial f}{\partial u} \right)$, $r_v = \left(0, 1, \frac{\partial f}{\partial v} \right)$.
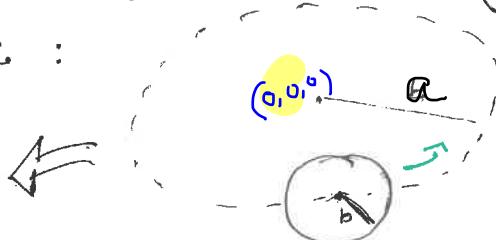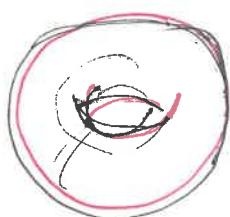
$$ \therefore \; r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1). $$

$$ \Rightarrow \; r_u \times r_v \Big|_{(u,v)} \neq 0 \qquad \forall \; (u,v) \in O_2. $$

$$ \Rightarrow \; r \text{ is a parametrization of } graph(f). $$

2) <u>The torus</u> : rotating a circle of radius (say) b about a circle of radius (say) a $(>b)$ lying in an z-orthogonal plane :

$(0,0,b)$ a

Torus / donut.

We parameterize the above torus as follows:

$$r(u,v) = \left((a+b\cos u)\cos v, \; (a+b\cos u)\sin v, \; b\sin u \right)$$

$$0 \leq u, v < 2\pi$$

$\updownarrow$

Clearly: this is given by $\boxed{(x-b)^2 + z^2 = a^2}$.

$(0 < a < b)$

Also, it may be seen from: In xz-plane, a circle of radius "a" centered at "$(b,0)$" is given by

$$x = a\cos\theta + b$$
$$z = a\sin\theta$$

$\theta \in [0, 2\pi]$.

Then rotate the xz-plane around z-axis by

$$x = (a\cos\theta + b)\cos\varphi$$
$$y = (a\cos\theta + b)\sin\varphi$$
$$z = a\sin\theta$$

$0 \leq \theta, \varphi < 2\pi$.

Anyway: r is injective.

$$r_u = \left(-b\sin u \cdot \cos v, \; -b\sin u \sin v, \; b\cos u \right)$$

$$r_v = \left(-(a+b\cos u)\sin v, \; (a+b\cos u)\cos v, \; 0 \right)$$

Then $\quad r_u \times r_v = \{-b(a+b\cos u)\} \cdot \left(\cos u \cos v, \; \sin u \sin v, \; \cos u \right)$

$$0 < u < 2\pi$$
$$0 \leq v < 2\pi$$

$$\neq 0 \qquad \text{(Hw)}.$$

$\therefore$ r is a parameterization of the torus.