Thm: Let $\gamma:[a,b]\to\mathbb{R}^n$ be a C^1-curve. Then $\ell(\gamma)$ exists (i.e. γ is rectifiable), and

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\,dt .$$

$\therefore$ C^1 curve $\Rightarrow$ rectifiable.

Proof. Let $\varepsilon>0$. Set $I=\int_a^b\|\gamma'(t)\|\,dt$. $\leftarrow$ Recall: it exists.

Claim: $\exists\ \delta>0$ s.t. $|I-\underbrace{\ell(\gamma,P)}|<\varepsilon \quad \forall\, P\in\mathcal{P}[a,b]$ s.t. $\|P\|<\delta$.

Recall: $\left(\sum_{i=1}^{m}\|\gamma(t_i)-\gamma(t_{i-1})\| \text{ if } P: a=t_0<t_1<\cdots<t_m=b\right)$

[Back Calculation: $\ell(\gamma,P)=\sum_{i=1}^{m}\|\gamma(t_i)-\gamma(t_{i-1})\|$. $\leftarrow$ A given partition of $[a,b]$. $a=t_0<t_1<\cdots<t_m=b$.

$$=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}\left(\gamma_j(t_i)-\gamma_j(t_{i-1})\right)^2\right]^{\frac12},$$

where $\gamma(t)=(\gamma_1(t),\dots,\gamma_n(t))$; $\gamma_j:[a,b]\to\mathbb{R}$.

$\because$ γ is C^1 $\Rightarrow$ $\gamma_j:[a,b]\to\mathbb{R}$ is C^1, $j=1,\dots,n$.

$\therefore$ By MVT [!! BTW: there is no vector-valued MVT !!],

$\forall\, j\in\{1,\dots,n\}$ & $i\in\{1,\dots,m\}$, $\exists\ t_{ij}\in[t_{i-1},t_i]$

s.t. $\gamma_j(t_i)-\gamma_j(t_{i-1})=\gamma_j'(t_{ij})\,(t_i-t_{i-1})$.

$$\therefore\ \ell(\gamma,P)=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}\gamma_j'(t_{ij})^2\times(t_i-t_{i-1})^2\right]^{\frac12}$$

$$=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}\gamma_j'(t_{ij})^2\right]^{\frac12}\times(t_i-t_{i-1}).$$

$\leftarrow$ Trouble. —— 1

Recall: Riemann Sum for $f\in\mathcal{B}[a,b]$ is $S(f,P)=\sum_{i=1}^{n}f(\xi_i)\,|I_i|$ $\forall\, P\in\mathcal{P}([a,b])$ & $\xi_i\in I_i$ is i-th tag.

We know, $f\in\mathcal{R}[a,b]$ $\Rightarrow$ $\lim_{\|P\|\to0}S(f,P)$ exists. Then $\int_a^b f=\lim_{\|P\|\to0}S(f,P)$.

Finally: The tag set is just restricted on just $\xi_i\in[x_{i-1},x_i]$.

$\boxed{\text{Suppose,}}$ we have that: $t_{ij}=t_i^*\in[t_{i-1},t_i]\ \forall\, j=1,\dots,n$

j-free. $j=1,\dots,m$.

i.e., ~~the choice of~~ t_{ij} is independent of the choice of j, then:

$$\ell(\gamma,P)=\sum_{i=1}^{m}\left[\sum_{j=1}^{n}\gamma_j'(t_i^*)\right]^{\frac12}(t_i-t_{i-1})$$

$$=\sum_{i=1}^{m}\|\gamma'(t_i^*)\|\,(t_i-t_{i-1})=S(\|\gamma'\|,P)$$

The Riemann Sum of $\|\gamma'\|$ for P.

$\Rightarrow \ell(\gamma, P) = \mathcal{S}(\|\gamma'\|, P)$. ← This would finish the proof as $\|\gamma'\| \in \mathcal{R}[a,b]$.

So, we need to work on "t_{ij}" part.].

Define $B^n = [a,b] \times \cdots \times [a,b] = [a,b]^n$ ← a box in $\mathbb{R}^n$,

& $\Gamma : B^n \longrightarrow \mathbb{R}$ by

$$\Gamma(t_1, \ldots, t_n) = \left[\sum_{j=1}^{n} \gamma_j'(t_j)^2\right]^{\frac{1}{2}}.$$

Note

$\Rightarrow \Gamma(t, \ldots, t) = \|\gamma'(t)\|$, $\forall t \in [a,b]$.

$\because \gamma_j \in C^1$, $\Gamma : B^n \longrightarrow \mathbb{R}$ is continuous.

$\Rightarrow$ Γ is uniformly continuous [$\because B^n$ is compact.]

$\therefore$ For $\varepsilon > 0$, $\exists\, \delta_1 > 0$ $\ni$

$$|\Gamma(x) - \Gamma(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall\ \|x - y\| < \delta_1. \qquad (2)$$

Now By (1): $\ell(\gamma, P) = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \gamma_j'(t_{ij})^2\right]^{\frac{1}{2}} \times (t_i - t_{i-1})$.

↑Recall: This is the trouble. (i-th part).

$$= \sum_{i=1}^{m} \Gamma(t_{i1}, t_{i2}, \ldots, t_{in})\ (t_i - t_{i-1}).$$

Here $P : a = t_0 < t_1 < \cdots < t_m = b$ a partition

of $[a,b]$ $\ni$ $\|P\| = \max\limits_{1 \le i \le n} (t_i - t_{i-1}) < \delta_1$.

Moreover, ~~$\hat{\mathcal{R}}(\gamma, P) := \sum_{i=1}^{m} \|\gamma'(t_i)\| (t_i - t_{i-1})$~~

if $\mathcal{S}(\gamma, P) := \sum_{i=1}^{m} \|\gamma'(t_i)\|\ (t_i - t_{i-1})$, then

$t_i \in [t_{i-1}, t_i]$ is the tag point.

$$\mathcal{S}(\gamma, P) = \sum_{i=1}^{m} \Gamma(t_i, \ldots, t_i)\ (t_i - t_{i-1}).$$

The Riemann sum.

$$\therefore \quad \ell(\gamma, P) - \mathfrak{S}(\gamma, P) = \sum_{i=1}^{n} \Big(\Gamma(t_{i1}, \ldots, t_{in}) - \Gamma(t_i, \ldots, t_i) \Big) \times (t_i - t_{i-1}).$$

$$\Rightarrow \big|\ell(\gamma, P) - \mathfrak{S}(\gamma, P)\big| \leqslant \sum_{i=1}^{n} \big| \Gamma(t_{i1}, \ldots, t_{in}) - \Gamma(t_i, \ldots, t_i) \big| \, (t_i - t_{i-1}).$$

$$\underset{\text{by } (2)}{<} \frac{\varepsilon}{2(b-a)} \times \underbrace{\sum_{i=1} (t_i - t_{i-1})}_{= b-a}$$

$$= \frac{\varepsilon}{2}.$$

$$\therefore \quad \underbrace{\big|\ell(\gamma, P) - \mathfrak{S}(\gamma, P)\big| < \frac{\varepsilon}{2}}_{\text{The needed estimate.}} \qquad \forall \; P \in \mathcal{P}[a,b] \;\&\; \|P\| < \delta_1.$$

Also, as $\|\gamma'\| \in \mathcal{R}[a,b]$, $\exists \; \delta_2 > 0$ s.t.

$$\Big| \mathfrak{S}(\gamma, P) - \int_a^b \|\gamma'(t)\| \, dt \Big| < \frac{\varepsilon}{2}. \qquad \forall \|P\| < \delta_2.$$

$\therefore$ For $\delta := \min\{\delta_1, \delta_2\}$ & $\forall P$ $\ni$ $\|P\| < \delta$, we have:

$$\Big|\ell(\gamma, P) - \int \|\gamma'(t)\| dt\Big| \leqslant \big|\ell(\gamma, P) - \mathfrak{S}(\gamma, P)\big| + \Big| \mathfrak{S}(\gamma, P) - \int_a^b \|\gamma'(t)\| \, dt \Big|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\therefore \lim_{\|P\| \to 0} \ell(\gamma, P) = \int_a^b \|\gamma'\|.$$

$\square$

Cor: A piecewise smooth parametrized curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is rectifiable. Moreover $\ell(\gamma) = \int_a^b \|\gamma'\|$.

← However, rectifiable $\not\Rightarrow$ piecewise smooth
Consider: graph of the Cantor function
↳ Devil's staircase.

Proof: Let ~~[illegible]~~ $a = x_0 < x_1 < \cdots < x_m = b$ be a partition of $[a,b]$ s.t. $\gamma_i := \gamma\big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is a smooth parametrized curve, $\forall\, i = 1, \ldots, m$.

$$\Rightarrow \gamma = \bigcup_{i=1}^{m} \gamma_i \longleftarrow \text{Smooth } \forall i$$

Let $\varepsilon > 0$.

$\because$ γ is uniformly continuous ($\because$ Curve $\Rightarrow$ Cont.), $\exists\ \tilde{\delta} > 0$

$\ni$ $\|\gamma(s) - \gamma(t)\| < \dfrac{\varepsilon}{6m}$ $\forall\ |s-t| < \tilde{\delta}$.

Suppose $P \in \mathcal{P}[a,b]$ s.t. $\underbrace{\|P\|}_{\text{mesh}} < \tilde{\delta}$.

Let $\tilde{P} \supset P$ (↑ refinement), where ~~$\{x_i\}_{i=0}^m$ are also~~ the nodes of $\tilde{P}$ are $\{x_i\}_{i=0}^m \cup \{\text{nodes of } P\}$. [Clearly, $\|\tilde{P}\| < \tilde{\delta}$].

Then
$$|\ell(\gamma, \tilde{P}) - \ell(\gamma, P)| = \Big| \sum_{s_i \in \lambda(\tilde{P})\setminus\{a\}} \|\gamma(s_{i-1}) - \gamma(s_i)\| - \sum_{t_j \in \lambda(P)\setminus\{a\}} \|\gamma(t_{j-1}) - \gamma(t_j)\| \Big| \quad (\dagger)$$

s_i ~~index of nodes of $\tilde{P}$~~ t_j ~~index of nodes of P.~~

[Here $\lambda(P)$ = nodes of P &
$\lambda(\tilde{P})$ = — " — $\tilde{P}$.]

~~$\sum_{i\in\lambda(\tilde{P})} \|\gamma(s_{i-1}) - \gamma(s_i)\| + \sum_{j \in \lambda(P)} \|\gamma(t_{j-1}) - \gamma(t_j)\|$~~

Observe that $\lambda(\tilde{P}) = \lambda(P) \cup \{x_i\}_{i=0}^m$.

Apply ~~[illegible]~~ triangle inequality ~~in terms of~~ to $(\dagger)$ $\|\gamma(s_{i-1}) - \gamma(s_i)\|$, we get [Note: $\sum_{\lambda(P)}$ has m terms & $\sum_{\lambda(\tilde{P})}$ has at most 2m terms. Also each term can be dominated by $\varepsilon/6m$.] ~~if necessary, [illegible] we get~~

$$|\ell(\gamma, \tilde{P}) - \ell(\gamma, P)| \leq 3m \times \{\|\gamma(s) - \gamma(t)\| : s,t \in [a,b] \ \&\ |s-t| < \tilde{\delta}\}.$$

$$< 3m \times \frac{\varepsilon}{6m}$$

$$= \frac{\varepsilon}{2}.$$

$$\Rightarrow \left| \ell(\gamma, \tilde{P}) - \ell(\gamma, P) \right| < \frac{\varepsilon}{2}. \quad \text{Fact 1}$$

Now, $\forall\, i = 1, \ldots, m$, $\gamma_i := \gamma\big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is rectifiable.

$\therefore$ For $\frac{\varepsilon}{2m} > 0$, $\exists\ \delta_i > 0$ s.t. $\forall\ P_i \in \mathcal{P}([x_{i-1}, x_i])$ with $\|P_i\| < \delta_i$, we have

$$\left| \ell(\gamma_i, P_i) - \int_{x_{i-1}}^{x_i} \|\gamma_i'\| \right| < \frac{\varepsilon}{2m}. \quad \dagger$$

Set $\delta := \min\{\tilde{\delta}, \delta_1, \ldots, \delta_m\}$.

Let $P \in \mathcal{P}([a,b])$ $\ni$ $\|P\| < \delta$. (Like the previous construction of $\tilde{P}$. $\Rightarrow \|\tilde{P}\| < \delta$, where nodes of $\tilde{P}$ = nodes of $P \cup \{x_i\}_{i=1}^m$)

$$\therefore \left| \ell(\gamma, P) - \int_a^b \|\gamma'\| \right| \leq \left| \ell(\gamma, P) - \ell(\gamma, \tilde{P}) \right| + \left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right|.$$

$$\underset{\text{by Fact 1.}}{<} \frac{\varepsilon}{2} + \left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right|$$

Now $\left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right| = \left| \sum_{i=1}^m \ell(\gamma_i, \tilde{P}_i) - \sum_{i=1}^m \int_{x_{i-1}}^{x_i} \|\gamma'\| \right|$

$\tilde{P}_i \in \mathcal{P}([x_{i-1}, x_i])$ defined by $\tilde{P}_i := \tilde{P} \cap [x_{i-1}, x_i]$ $\forall\ i = 1, \ldots, m$.

$$\leq \sum_{i=1}^m \left| \ell(\gamma_i, \tilde{P}_i) - \int_{x_{i-1}}^{x_i} \|\gamma'\| \right|$$

$$< \frac{\varepsilon}{2m} \times m \quad (\text{by } \dagger) \quad \text{As } \|\tilde{P}_i\| \leq \|\tilde{P}\| < \delta \leq \delta_i\ \forall i.$$

$$= \frac{\varepsilon}{2}.$$

$$\therefore \left| \ell(\gamma, P) - \int_a^b \|\gamma'\| \right| < \varepsilon \quad \forall\ P \in \mathcal{P}([a,b]) \text{ s.t. } \|P\| < \delta.$$

$$\Rightarrow \ell(\gamma) = \int_a^b \|\gamma'\|. \quad \blacksquare$$

~~Theo:~~

Remark: Recall that a parametrized curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is smooth, if $\gamma'(t) \neq 0 \quad \forall\, t$.

$\because \gamma'(t) = (3t^2, 6t^5) \Rightarrow \gamma'(0) = (0,0)$.

(†) # Clearly $\gamma(t) = (t^3, t^6)$ is non-smooth at $t = 0$.

But $\tilde{\gamma}(t) = (t, t^2)$ is smooth at $t = 0$. [$\because \tilde{\gamma}'(t) = (1, 2t) \neq (0,0) \ \forall t$]

But the path of γ & the path of $\tilde{\gamma}$ are the same (i.e. the same trace/path).

$\Rightarrow$ Smoothness is NOT an intrinsic property of the curve as just path/subset/trace. Smoothness is a soul property of parametrization.

However:

Thm: Consider a smooth curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ & a parametrization $\varphi : [\tilde{a}, \tilde{b}] \longrightarrow [a,b]$, smooth, then $\tilde{\gamma} = \gamma \circ \varphi$ is also smooth.

A reparametrization of γ, but we are not assuming onto.

$$
\begin{tikzcd}
 & \mathbb{R}^n \\
\tilde{I} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\varphi"] & I \arrow[u, "\gamma"']
\end{tikzcd}
$$

Proof: Easy: $\tilde{\gamma}(s) = \gamma(\varphi(s))$

$\Rightarrow \tilde{\gamma}' = \gamma'(\varphi(s)) * \varphi'(s)$ [Chain rule]

$\because \gamma'$ & φ' non-vanishing ✓ $\Rightarrow \tilde{\gamma}'$ is non-vanishing ✓.

$\boxed{\text{TH}}$

eg: (From †) For $\gamma(t) = (t^3, t^6)$ & $\tilde{\gamma}(t) = (t, t^2)$, $\varphi(s) = s^{\frac{1}{3}}$. [$\Rightarrow \tilde{\gamma} = \gamma \circ \varphi$]

But φ is not even diff. at 0.

$\boxed{\text{TH}}$