$$\therefore \quad \underline{f}(x) = \underline{\int_{B^n}} f(x,y)\, dv(y) \quad \& \quad \overline{f}(x) = \overline{\int_{B^n}} f(x,y)\, dv(x) \quad \forall x \in B^m.$$

We now prove a BIG result :

Thm: [Fubini thm] Suppose $f \in \mathcal{R}(B^{m+n})$. Then $\underline{f}, \overline{f} \in \mathcal{R}(B^m)$ &

$$\int_{B^m} \underline{f}\, dv(x) = \int_{B^m} \overline{f}\, dv(x) = \int_{B^{m+n}} f\, dv(x,y).$$

i.e.,
$$\int_{B^m} \left(\underline{\int_{B^n}} f(x,y)\, dv(y) \right) dv(x) = \int_{B^m} \left(\overline{\int_{B^n}} f(x,y)\, dv(y) \right) dv(x)$$

$$= \int_{B^{m+n}} f(x,y)\, dv .$$

$\rightsquigarrow \therefore \int_{B^{m+n}} f$ = iterations of lower & upper.
They always exists

[**Note:** Similar statement holds if we consider slice fn's w.r.t. y : If $f \in \mathcal{R}(B^{m+n})$, then

$$\int f(x,y)\, dv = \int_{B^n} \underbrace{\left(\underline{\int_{B^m}} f(x,y)\, dv(x) \right)}_{\in \mathcal{R}(B^n)} dv(y) = \int_{B^n} \underbrace{\left(\overline{\int_{B^m}} f(x,y)\, dv(x) \right)}_{\in \mathcal{R}(B^n)} dv(y)$$

– the proof will be the same.]

#

Cor: Suppose $f \in \mathcal{R}(B^{m+n})$. If ~~f~~ the fn. ~~$y \mapsto \int f(x,y)\, dv(x)$~~

$x \longmapsto \int_{B^n} f(x,y)\, dv(y) \in \mathcal{R}(B^m)$, [or $y \longmapsto \int_{B^m} f(x,y)\, dv(x) \in \mathcal{R}(B^n)$]

then,
$$\int_{B^{m+n}} f\, dv = \int_{B^m} \int_{B^n} f(x,y)\, dv(y)\, dv(x).$$

i.e., the iteration is integrable.

$$\left[\text{or, } \int_{B^{m+n}} f\, dv = \int_{B^n} \int_{B^m} f(x,y)\, dv(x)\, dv(y). \right]$$

Remark: Suppose $f \in C(B^{m+n})$. Then for each $x \in B^m$, the slice fn $f_x \in C(B^n)$. [IIly $f_y \in C(B^m)$ $\forall y \in B^n$].

Trivial.

~~$x \mapsto \int_{B^n} f(x,y)\, dv(y) \in \mathcal{R}(B^m)$~~ Then, in particular $\underline{\int_{B^n}} f_x = \overline{\int_{B^n}} f_x$. (IIly for y).

Thus:

Cor: If $f \in C(B^{m+n})$, then

$$\int_{B^{m+n}} f\, dv = \int_{B^n} \left(\int_{B^m} f\, dv(x) \right) dv(y) = \int_{B^m} \left(\int_{B^n} f\, dv(y) \right) dv(x).$$

Continuous version of Fubini

Proof of Fubini's thm: [To prove: $\int_{B^m} \underline{f}\, dx = \int_{B^m} \overline{f}\, dx = \int f\, dy\, dx$]

– We need to show that $\underline{f}, \overline{f}$ are integrable.

Let $P \in \mathcal{P}(B^{m+n})$. So $P = P^m \times P^n$ for some partitions $P^m \in \mathcal{P}(B^m)$ & $P^n \in \mathcal{P}(B^n)$.

We aim to prove $\underline{f}$ integrable.

Now $L(f,P) = \sum_{\alpha \in \Lambda(P)} m_\alpha \times v(B^{m+n}_\alpha)$.

$$= \sum_{\alpha(P^m) \in \Lambda(P^m)} \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m),\alpha(P^n))} v(B^{m+n}_{(\alpha(P^m),\alpha(P^n))})$$

[$\because \alpha \in \Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)$ $\Rightarrow \alpha = \alpha(P^m) \times \alpha(P^n)$.

$= v(B^m_{\alpha(P^m)}) \times v(B^n_{\alpha(P^n)})$.

x-variable. y-variable

$$= \sum_{\alpha(P^m) \in \Lambda(P^m)} \left(\sum_{\alpha(P^n) \in \Lambda(P^n)} \underbrace{m_{(\alpha(P^m),\alpha(P^n))}}_{\text{Attack.}} v(B^n_{\alpha(P^n)}) \right) \times v(B^m_{\alpha(P^m)}).$$

For each $x \in B^m$, & $\alpha(P^n) \in \Lambda(P^n)$, set $m_{\alpha(P^n)}(x) = \inf_{y \in B^n_{\alpha(P^n)}} f(x,y)$ $= f_x(y)$.

$\therefore \forall x \in B^m_{\alpha(P^m)}$, $m_{\alpha(P^n)}(x) \geq m_{(\alpha(P^m),\alpha(P^n))}$

$$\therefore \sum_{\alpha(P^n)\in\Lambda(P^n)} m_{(\alpha(P^n),\alpha(P^m))}\, v\left(B^n_{\alpha(P^n)}\right)$$

$$\leq \sum_{\alpha(P^n)\in\Lambda(P^n)} m_{\alpha(P^n)}(x) \times v\left(B^n_{\alpha(P^n)}\right) \qquad \forall\, x\in B^m_{\alpha(P^m)}$$

$$= L(f_x, P^n) \leq \int_{B^n} f_x \, dv(y) \qquad \forall\, x\in B^m_{\alpha(P^m)}.$$

$$= \underline{f}(x) \qquad \text{— " —}.$$

$$\leq \inf_{x\in B^m_{\alpha(P^m)}} \underline{f} \quad . := m_{B^m_{\alpha(P^m)}}$$

$$\therefore \quad L(f,P) \leq \sum_{\alpha(P^m)\in\Lambda(P^m)} m_{B^m_{\alpha(P^m)}}\, v\left(B^m_{\alpha(P^m)}\right)$$

$$= L(\underline{f}, P^m). \qquad (\because P = P^m \times P^n).$$

i.e., $L(f,P) \leq L(\underline{f}, P^m)$.

$\|^{ly}$ $U(f,P) \geq U(\underline{f}, P^m)$. *Now the rest is standard.*

[Recall: we aim at proving that $\underline{f}$ is integrable.]

$$\therefore \quad U(\underline{f}, P^m) - L(\underline{f}, P^m) \leq U(f,P) - L(f,P).$$

Also ~~each~~ $f \in \mathcal{R}(B^{m+n})$.

$\Rightarrow$ $\underline{f} \in \mathcal{R}(B^m)$. $\|^{ly}$ $\overline{f} \in \mathcal{R}(B^n)$.

Finally, $L(f,P) \leq L(\underline{f}, P^m) \leq U(\underline{f}, P^m) \leq U(f,P)$

$\Rightarrow L(f,P) \leq L(\underline{f}, P^m) \leq U(f,P)$ $\forall P$ & $\forall P^m$. $\Rightarrow \int_{B^m} \underline{f} = \int_{B^{m+n}} f$. ◻

Cor: If $f \in C(B^n)$, then

$$\int_{B^n} f \, dv(x) = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n .$$

Also denoted by

$$\int_{B^n} f(x_1, \ldots, x_n) \, \underbrace{dx_1 \cdots dx_n}_{\text{all Commutes.}}$$

Moreover (& in particular: $n=2$):

$$\int_{[a_1,b_1]\times[a_2,b_2]} f \, dv = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) \, dx_2 \, dx_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) \, dx_1 \, dx_2$$

(i.e, Changing orders of integration is okay).

very useful.

eg: 1) ~~$\mathbb{R}[x_1, \ldots, x_n]$~~ $\mathbb{R}[x_1, \ldots, x_n] \subseteq \mathcal{R}(B^n)$, $\forall$ box $B^n \subseteq \mathbb{R}^n$

polynomial ring ($\because$ Cont.)

2) $$\int_{[0,1]^2} xy \, dx \, dy = \int_0^1 \left(\int_0^1 xy \, dx \right) dy$$

$$= \int_0^1 y \times \left[\frac{x^2}{2} \right]_0^1 dy = \frac{1}{2} \times \int_0^1 y \, dy$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \quad \text{Ans}$$

OR $$\int_{[0,1]^2} xy \, dx \, dy = \left(\int_0^1 x \, dx \right) \times \left(\int_0^1 y \, dy \right)$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

3) $$f(x,y) = \begin{cases} -1 & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q}. \\ 1 & \text{if } (x,y) \in \mathbb{Q}^c \times \mathbb{Q}^c. \\ 0 & \text{otherwise.} \end{cases}$$

Sad fellow.

Then $f \notin \mathcal{R}([0,1]^2)$.

— HW —

4) Let $f(x) = c \quad \forall x \in B^n$. Then

$$\int_{B^n} f\, dv = c \times v(B^n).$$

— We will do more soon — .

Note: We have seen an example of $f \in \mathcal{B}(B^2)$ (or $f \in \mathcal{B}(B^n)$) s.t. $f \in \mathcal{R}(B^2)$ BUT f is NOT Continuous !!

Q: No. of points of discontinuity ? Ans: 1 in that particular example.

Page 16: $f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \\ 0 & \text{Otherwise.} \end{cases}$

We can confidently deal with finitely many points of discontinuity in that example too!!

BUT, the real question is: Can we integrate a bdd fn. with too many points of discontinuity ?

Ans: Seems Vague question : unless you spell out "too many" means "how many" !! ??

i.e., What is "too many" ?

"Vague answer" : As long as those points of discontinuity fail to generate a "Volume" ?? That's a good one.

Lets get into this (already it's too complicated):

Clearly, we need to talk about Small sets (Small in terms of Volume)

"Small sets"

Def: A subset A of $\mathbb{R}^n$ is of measure zero if for $\varepsilon > 0$ $\exists$ a cover of A by countable boxes $\{B^n_m\}_{m=1}^{\infty}$ with "the total volume" $< \varepsilon$.

i.e., $\bigcup_{m=1}^{\infty} B^n_m \supseteq A$ & $\sum v(B^n_m) < \varepsilon$.

[Note: If n is clear from the context, then we will simply write B instead of B^n.].

Note: 1) Let $A \subseteq \mathbb{R}^n$ be a finite set. Then A is of measure zero.

$\varepsilon > 0$ is given.

[Suppose $A = \{a_1, \dots, a_n\}$, $\varepsilon > 0$. Consider $r = \frac{\varepsilon}{2n}$ & boxes B_m around a_m s.t. $v(B_m) < \frac{\varepsilon}{2n}$ $\forall m = 1, \dots, n$ .....]. $\longrightarrow$ Compactness argument?

2) Suppose $A = \{a_n\}_{n=1}^{\infty}$ $\longrightarrow$ a countable set.

: Choose box B_n $\forall n \geq 1$ s.t.

$v(B_n) < \frac{\varepsilon}{2^n}$ & $a_n \in B_n$.

$\Rightarrow$ $\sum v(B_n) < \varepsilon$. $\Rightarrow$ A is of measure zero.

3) Use the above to prove the following: Suppose $\{A_n\}$ be a seqn. of subsets of $\mathbb{R}^n$ of measure 0. Then $A := \bigcup_{n=1}^{\infty} A_n$ is also of measure 0.

Countable union of sets of measure 0 is of measure 0.

Proof: Let $\varepsilon > 0$. $\forall n$, get $\mathcal{C}_n$ = a countable collection of boxes s.t. $\bigcup_{C \in \mathcal{C}_n} C \supseteq A_n$ & $\sum_{C \in \mathcal{C}_n} v(C) < \frac{\varepsilon}{2^n}$.

Standard trick.

Then $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a countable collection of boxes which covers $A = \bigcup_{n=1}^{\infty} A_n$ & $\sum_{C \in \mathcal{C}} v(C) < \varepsilon$. ∎

We need one more notion:

Let $f \in \mathcal{B}(\underbrace{\mathcal{O}_n}_{\text{open in } \mathbb{R}^n})$. Let $x_0 \in \mathcal{O}_n$.

Define $\operatorname{osc}(f, x_0)$ ~~as~~ by:

Oscillation of f at x_0

$$\operatorname{osc}(f, x_0) := \lim_{\delta \to 0} \left[\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) \right].$$

$\therefore$ osc is a fn. : $\mathcal{O}_n \longrightarrow \mathbb{R}$ defined by

$$\operatorname{osc}(x) = \operatorname{osc}(f, x). \ \forall x \in \mathcal{O}_n.$$

Note: (1) $\forall \delta > 0$, $\sup_{B_\delta(x_0)} f \geq \inf_{B_\delta(x_0)} f \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \geq 0$.

~~f is bdd~~ Moreover $\delta \downarrow \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \downarrow$.

$\therefore$ $\operatorname{osc}(f, x_0)$ exists $\forall x_0 \in \mathcal{O}_n$.

(2) f is cont. at $x_0 \iff \operatorname{osc}(f, x_0) = 0$.

osc is a measure of discontinuity of f at points in $\mathcal{O}_n$.

Proof: "$\Rightarrow$" Let $\operatorname{osc}(f, x_0) = 0$. Let $\varepsilon > 0$.

[$\because \lim_{\delta \to 0} [\ ] = 0$.]

$\therefore \exists \delta > 0$ s.t. $\underbrace{\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x)} < \varepsilon$.

$$= \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)|$$

we know this. useful.

$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in B_\delta(x_0)$

In particular: $|f(x) - f(x_0)| < \varepsilon \quad \forall x \in B_\delta(x_0)$.

"$\Leftarrow$" Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in B_\delta(x_0)$.

$\Rightarrow |f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(y) - f(x_0)| < \varepsilon$. $\forall x, y \in B_\delta(x_0)$.

$\Rightarrow \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \leq \varepsilon \Rightarrow \operatorname{osc}(f, x_0) = 0$. ☐

(3) $\operatorname{osc}(f, x_0) = \inf\limits_{\delta>0} \left\{ \sup |f(x) - f(y)| : x, y \in B_\delta(x_0) \right\}$ ← just observed.

(4) Let $\alpha > 0$ ~~Then~~ & let $f \in \mathcal{B}(C_n)$, $C_n \subseteq \mathbb{R}^n$ closed. Like $C_n = B^n$. But Assume $\operatorname{int}(C_n) \neq \varphi$.

Then $\{x \in C_n : \operatorname{osc}(f, x) \geq \alpha\}$ is ~~open~~ closed

& $\{x \in C_n : \operatorname{osc}(f, x) < \alpha\}$ is open in $\mathbb{R}^n_u$.

Proof. Let $C := \{x \in C_n : \operatorname{osc}(f, x) \geq \alpha\}$ Claim: C is closed. We prove $\mathbb{R}^n \setminus C$ open.

Let $x \in \mathbb{R}^n \setminus C$.

$\Rightarrow \underbrace{x \notin C_n}_{\text{Case 1}}$ or $\underbrace{x \in C_n \text{ but } x \notin C}_{\text{Case 2}}$.

$\Downarrow$

$\exists\, \delta > 0$ s.t. $B_\delta(x) \subseteq C_n^c$ [$\because$ C_n is closed]

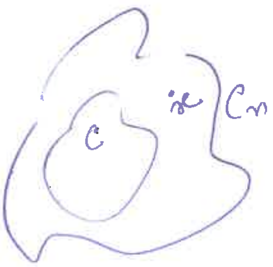
$\Rightarrow B_\delta(x) \subseteq C^c \Rightarrow C^c$ open $\Rightarrow C$ closed.

Let $x \in C_n$ but $x \notin C$.

$\Rightarrow x \in C_n$ & $\operatorname{osc}(f, x) < \alpha$

$\therefore \exists\, \delta > 0$ s.t. $\sup\left\{ |f(z) - f(y)| : z, y \in B_\delta(x_0) \right\} < \alpha$.

(maybe $\leq \alpha$ does not matter)



Consider an open box $B \subseteq B_\delta(x)$.

Let $y \in B$.

$\because$ $y \in \underbrace{B}_{\text{open}}$, ~~then~~ $\exists\, \delta_1 > 0$ s.t. $B_{\delta_1}(y) \subseteq B_\delta(x)$.

$\therefore$ In particular: $\sup\left\{ |f(z) - f(w)| : z, w \in B_{\delta_1}(y) \right\} < \alpha$

$f(w)$

$\Rightarrow \operatorname{osc}(f, y) < \alpha$.

Thus, $\forall\, y \in \underbrace{B}_{\text{open}}$, $\operatorname{osc}(f, y) < \alpha$. $\Rightarrow \underbrace{B}_{\text{open}} \subseteq \mathbb{R}^n \setminus C$.

$\Rightarrow C$ is closed. ☑