We need one observation:

Lemma: Let $K \subseteq \mathbb{R}^n$ be a compact set of measure zero. Let $\varepsilon > 0$. Then $\exists$ open ~~rectangles~~ boxes $B_1, \dots, B_m$ (for some $m = m(\varepsilon)$) s.t.

$$\bigcup_{i=1}^{m} B_i \supset K \quad \& \quad \sum_{i=1}^{m} v(B_i) < \varepsilon.$$

Proof: Just compactness of K: Let $\varepsilon > 0$. Then $\exists$ boxes $\{B_i\}_{i=1}^{\infty}$ s.t. $\sum_{i=1}^{\infty} v(B_i) < \varepsilon$ & $\bigcup_{i=1}^{\infty} B_i \supseteq K$.

But K compact $\Rightarrow \exists\, m \in \mathbb{N}$ s.t. $\bigcup_{i=1}^{m} B_i \supseteq K$. Clearly, $\sum_{i=1}^{m} v(B_i) \leq \sum_{i=1}^{\infty} v(B_i) < \varepsilon$. $\boxtimes$

Remark: We can safely replace boxes by open/closed balls.

\# **Thm:** (Riemann - Lebesgue thm): Let $f \in \mathcal{B}(B^n)$. Then $f \in \mathcal{R}(B^n)$ $\Longleftrightarrow$ the set of discontinuity of f is of measure zero.

Proof: Set $\mathcal{D} := \{x \in B^n : f \text{ is not cont. at } x\}$.

$\therefore \mathcal{D} = \{x \in B^n : \text{osc}(f, x) > 0\}$.

"$\Leftarrow$" Claim: $\mathcal{D}$ be of measure zero. [Assumption: $f \in \mathcal{R}(B^n)$].

$\forall\, m \in \mathbb{N}$, Set $\underbrace{\mathcal{D}_m}_{\text{closed in } \mathbb{R}^n} = \{x \in B^n : \text{osc}(f, x) \geq \frac{1}{m}\}$.

$\therefore \mathcal{D}_m \uparrow$.

Note that: $\mathcal{D} = \bigcup_{m \geq 1} \mathcal{D}_m$.

So, enough to prove that $\mathcal{D}_m$ is of measure zero, $\forall m$.

(28)

Fix $m \in \mathbb{N}$. Goal: $\mathcal{D}_m = \left\{ x \in B^n : \operatorname{osc}(f, x) \geq \frac{1}{m} \right\}$ is of measure zero.

Let $\varepsilon > 0$. (fix it). [$\because$ both m & ε are fixed.]

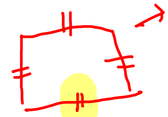
$\because f \in \mathcal{R}(B^n)$, $\exists\ P^m$ (~~a just P~~) a partition of B^n s.t.

$$\underbrace{U(f, P) - L(f, P)}_{\parallel} < \varepsilon.$$

i.e., $$\sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \varepsilon.$$

Note that: $\Lambda(P)$ is a finite set.

Let $\Lambda(P) := I \underset{\uparrow}{\sqcup} J$,
disjoint union.



Enough to prove $\mathbb{R}^2 \geq$ is of measure zero.

where $I = \{ \alpha \in \Lambda(P) : \operatorname{int}(B^n_\alpha) \cap \mathcal{D}_m \neq \varphi \}$.

& $J = \{ \alpha \in \Lambda(P) : \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m = \varphi \}$.

$$\therefore \mathcal{D}_m \subseteq \left[\bigcup_{\alpha \in I} \operatorname{int}(B^n_\alpha) \right] \cup \underbrace{\left[\bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha \right]}_{} \quad \text{— (28.1)}$$

~~Now for each $\alpha \in I$,~~ ~~as $\exists\, x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$~~

the set of all boundaries of the sub-boxes B^n_α, $\alpha \in \Lambda(P)$.

This should be negligible!!

Let $\alpha \in I$. So $\exists\, x \in \operatorname{int}(B^m_\alpha) \cap \mathcal{D}_m$.

Fix it for time being.

$\therefore \operatorname{osc}(f, x) \geq \frac{1}{m}$.

$$\parallel$$

$$\inf_{\delta > 0} \left[\sup_{z, y \in B_\delta(x)} [f(z) - f(y)] \right] \longleftarrow (*)$$

$\because x \in \operatorname{int}(B^n_\alpha)$, $\exists\ \delta > 0$ s.t. $B_\delta(x) \subseteq B^n_\alpha$.

Now since $M_\alpha - m_\alpha = \sup\limits_{z, y \in B^n_\alpha} [f(z) - f(y)]$, and $B_\delta(x) \subseteq B^n_\alpha$ we have:

$$\geq \sup_{z, y \in B_\delta(x)} [f(z) - f(y)].$$

$$M_\alpha - m_\alpha \geq \frac{1}{m}.$$

Hence:

$$\varepsilon > \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha)$$

$$\geq \sum_{\alpha \in I} (M_\alpha - m_\alpha)\, v(B^n_\alpha). \qquad [\because \Lambda(P) = I \amalg J]$$

$$\geq \sum_{\alpha \in I} \frac{1}{m} \times v(B^n_\alpha)$$

$\varepsilon \to \frac{\varepsilon}{m}$

$$= \frac{1}{m} \times \sum_{\alpha \in I} v(B^n_\alpha).$$

$\varepsilon.$

$$\Rightarrow \sum_{\alpha \in I} v(B^n_\alpha) < m\,\varepsilon. \qquad \text{———} \quad (\dagger)$$

Now look at (28.1):

$$\mathcal{D}_m \subseteq \underbrace{\left[\bigcup_{\alpha \in I} \operatorname{int}(B^n_\alpha)\right]}_{\substack{\text{is of measure} \\ \text{zero by } (\dagger)}} \cup \underbrace{\left[\bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha\right]}_{\substack{\text{finite Union of boundaries} \\ \text{of Sub-boxes.} \\ \Downarrow \\ \text{measure zero.} \\ \text{(HW)}}}$$

$\Rightarrow \mathcal{D}_m$ is of measure zero.

This proves $f \in \mathcal{R}(B^n) \Rightarrow \mathcal{D}$ is of measure zero.

"$\Leftarrow$" Suppose $\mathcal{D} := \{x \in B^n : \operatorname{osc}(f, x) > 0\}$ is of measure zero.

Claim: $f \in \mathcal{R}(B^n)$.

Let $\varepsilon > 0$. Set

$$\mathcal{D}_\varepsilon := \{x \in B^n : \operatorname{osc}(f, x) \geq \varepsilon\}$$

$\uparrow$ A closed set in $\mathbb{R}^n$.

$\Rightarrow$ $\mathcal{D}_\varepsilon$ is compact & of measure zero $[\because \mathcal{D}_\varepsilon \subseteq B^n$ & $\mathcal{D}_\varepsilon \subseteq \mathcal{D}]$.

Then for that $\varepsilon > 0$ itself, $\exists$ open boxes $\{B_i\}_{i=1}^m$ s.t. (finitely many)

"$\bigcup_{i=1}^m B_i \supseteq \mathcal{D}_\varepsilon$ ① & $\sum_{i=1}^m v(B_i) < \varepsilon$." (1a)

Then $B := B^n \setminus \left[\bigcup_{i=1}^m B_i\right]$ ② is again compact.

Now $x \in B \Rightarrow \operatorname{osc}(f, x) < \varepsilon$. ~~$\forall x \in B$~~

$\therefore$ $\exists$ a closed box C_x s.t. $x \in \operatorname{int} C_x$, ~~s.t.~~ ~~So that~~ In particular:

$$\sup_{y,z \in C_x} (f(y) - f(z)) < \varepsilon \quad (*)$$

$= \inf_{\delta > 0} \sup_{y,z \in B_\delta(x)} [f(z) - f(y)]$

$[\because \{y \in B^n : \operatorname{osc}(f, y) < \varepsilon\}$ is an open set.$]$

Again, by compactness of B, $\exists$ $x_1, \dots, x_p$ in B s.t.

$$\bigcup_{i=1}^p C_{x_i} \supseteq B. \quad ③$$

$[x_i \in \operatorname{int} C_{x_i},\ i = 1, \dots, p.]$

[Note that C_{x_i} may be considered as $\subseteq B^n$]

"Let P be a partition "out of $\{B_i, C_{x_j} : 1 \leq i \leq m,\ 1 \leq j \leq p\}$."

i.e: ~~So~~ B^n_α, $\alpha \in \Lambda(P)$, is either in $\overline{B_i}$, for some i, or, in C_{x_j}, for some j.

[See ①, ② & ③ & note that $i = 1, \dots, m$ & $j = 1, \dots, p$ } finite set.]

Get a partition: $\Lambda(P) = I \amalg J$ $\leftarrow$ B part.

I: those $\alpha \in \Lambda(P)$ s.t. $B^n_\alpha \subseteq \overline{B_i}$ for some $i = 1, \dots, m$. ($\mathcal{D}_\varepsilon$ part)

J: those $\alpha \in \Lambda(P)$ s.t. $B^n_\alpha \subseteq C_{x_j}$ for some $j = 1, \dots, p$.

$$\therefore\ U(f,P) - L(f,P) = \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B_\alpha^n)$$

$$= \sum_{\alpha \in I} \underbrace{(M_\alpha - m_\alpha)}_{\leq 2M} v(B_\alpha^n) + \sum_{\alpha \in J} \underbrace{(M_\alpha - m_\alpha)}_{\leq \varepsilon \text{ by } (*)} v(B_\alpha^n)$$

$M := \max_{B^n} |f|$.

$$\cancel{\leq \varepsilon \times \sum_{\alpha \in I} v(B_\alpha^n) + 2M \times \sum_{\alpha \in J} v(B_\alpha^n)}$$

$$\leq 2M \times \underbrace{\sum_{\alpha \in I} v(B_\alpha^n)}_{< \varepsilon \text{ by } (1a)} + \varepsilon \times \sum_{\alpha \in J} v(B_\alpha^n)$$

$$\leq \varepsilon \times \Big[2M + \sum_{\alpha \in J} v(B_\alpha^n)\Big]$$

$$\leq \varepsilon \times \Big[2M + \sum_{\alpha \in \Lambda(P)} v(B_\alpha^n)\Big]$$

$$= \varepsilon \times \big[2M + v(B^n)\big].$$

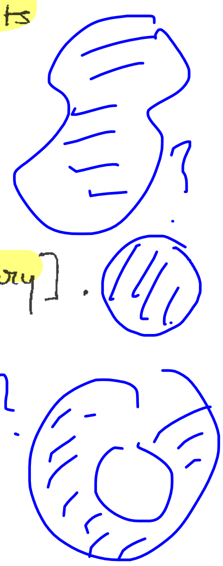
$\Rightarrow\ U(f,P) - L(f,P) \leq \varepsilon \times \tilde{M}$ for some $\tilde{M} > 0$.

$\Rightarrow\ f \in \mathcal{R}(B^n)$. ▨

~~Let $P^n \in \mathcal{P}(B^n)$~~.

Integration over bounded domains/sets

instead of boxes.



Let $\Omega \subseteq \mathbb{R}^n$ be a bdd set [Assume closed if necessary].

Assume $f \in \mathcal{B}(\Omega)$: A bdd fn.

Q: How to define $\int_\Omega f$?

Ans: We only know the answer for $\Omega = B^n$!

Also, recall, we need grids (i.e. B^n_α, $\alpha \in \Lambda(P)$, $P \in \mathcal{P}(B^n)$) to define integrations $\int_{B^n} f$!!

So, how to define $\int_\Omega f$?

One way : Get a box $B^n \supseteq \Omega$. Define

$$\tilde{f} : B^n \longrightarrow \mathbb{R} \text{ by}$$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega. \\ 0 & \text{if } x \in B^n \setminus \Omega. \end{cases}$$

Then define $\int_\Omega f = \int_{B^n} \tilde{f}$!!

Q: Looks ok, but: (1) $\int_\Omega f$ = independent of the choice of B^n?

(2) $\tilde{f} \in \mathcal{R}(B^n)$?

intution says these should do!!

We need to fix this first.

A Couple of observations:

(1) Let B^n_1 & B^n_2 be two boxes. Then either:

(i) $B^n_1 \cap B^n_2$ is a box, or

(ii) $B^n_1 \cap B^n_2 = \varphi$, or (iii) $B^n_1 \cap B^n_2$ is a $\subseteq$ of a face of B^n_1 & a face of B^n_2. — HW —