Algebra III (Notes12-15)

- Arghyadip Chakraborty (bmat1907)

- A "Euclidean Domain(ED)" is an integral domain R such that there exists a function (called norm) $N : R^{\times} \to \mathbb{Z}_{\geq 0}$ satisfying the condition: for $a, b \in R^{\times}$ $\exists q, r \in R$ such that $a = bq + r$ with $N(r) < N(b)$ or r $= 0$
- An integral domain R is called "Principal Ideal Domain(PID)" if every R-ideal is principal i.e. generated by one element in R
- A nonzero nonunit $x \in R$ is called "irreducible" if $x = yz$ $(y, z \in R) \Rightarrow y$ or z is a unit.
- $x \in R$ is called a "prime element" if $x|ab$ $(a, b \in R) \Rightarrow x|a$ or $x|b$
- x is prime iff (x) is a prime ideal
- $R \to$ integral domain $\Rightarrow$ {prime elements in R} $\subseteq$ {irreducible elements in R}
- $R \to$ PID $\Rightarrow$ {irreducible elements in R} $=$ {prime elements in R}
- $R \to$ PID $\Rightarrow$ {nonzero prime ideals in R} $\subseteq$ {maximal ideals in R}
- $\{R \to \text{ED}\} \subseteq \{R \to \text{PID}\}$
- $d \in R$ is said to be a "gcd" of $a, b \in R$ if $d|a$, $d|b$ and if $d' \in R$ is such that $d'|a$ & $d'|b$ then $d'|d$
- $a, b \in R$ such that $(a, b) = (d)$ for some $d \in R$ then $d = gcd(a, b)$ and $d = ax + by$ for some $x, y \in R$. So $R \to$ PID $\Rightarrow (a, b) = (d)$ where $d = gcd(a, b)$
- $R[x] \to$ PID $\Rightarrow R \to$ field
- A nonzero nonunit $u \in R$ is called a "universal side divisor" if $\forall x \in R$ $\exists q \in R$ such that $x - qu$ is either zero or a unit.
- $R \to$ ED but $R \not\to$ field $\Rightarrow R$ contains a universal side divisor
- $N : R^{\times} \to \mathbb{Z}_{\geq 0}$ is a Dedekind-Hasse norm if $\forall a, b \in R^{\times}$ either $b|a$ or $\exists r \in (a, b)$ such that $N(r) < N(b)$ [i.e. $\exists x, y \in R$ such that $N(ax + by) < N(b)$]
- $R \to$ PID $\Leftrightarrow R$ has a Dedekind-Hasse norm
- An integral domain R is called "Unique Factorization Domain(UFD)" if any nonzero nonunit $x \in R$ can be uniquely written as product of irreducibles, where uniqueness means: if $x = p_1 p_2 ... p_n = q_1 q_2 ... q_m$ where p_i's and q_i's are irreducibles, then $n = m$ and after a reordering p_i & q_i are associates $\forall 1 \leq i \leq n$
- $x, y \in R$ are said to be "associates" if there exists some unit $u \in R$ such that $x = uy$. The relation of associates is an equivalence relation
- $\{R \to \text{ED}\} \subseteq \{R \to \text{PID}\} \subseteq \{R \to \text{UFD}\}$
- $R \to$ UFD $\Rightarrow$ {irreducible elements in R} $=$ {prime elements in R}
- $R \to$ UFD and $a = up_1^{e_1} p_2^{e_2} ... p_r^{e_r}$, $b = vq_1^{f_1} q_2^{f_2} ... q_r^{f_r}$ for some $r \geq 0$, $u, v \in R$, p_i's & q_i's irred in R, e_i's, f_i's $\in \mathbb{Z}_{\geq 0} \Rightarrow gcd(a, b) = p_1^{min(e_1, f_1)} p_2^{min(e_2, f_2)} ... p_r^{min(e_r, f_r)}$
- $R \to$ integral domain. $R \to$ UFD iff:
  - ✔ {irreducible elements in R} $\subseteq$ {prime elements in R}
  - ✔ Every strictly increasing chain of principal ideals is of finite length