Thm: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a C^1-curve. Then $\ell(\gamma)$ exists (i.e. γ is rectifiable), and

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt .$$

$\ell(\gamma) = \lim_{\|P\|\to 0} \ell(\gamma, P)$

$\therefore$ C^1 curve $\Rightarrow$ rectifiable.

Proof: Let $\varepsilon > 0$. Set $I = \int_a^b \|\gamma'(t)\|\,dt$. $\leftarrow$ Recall: it exists.

Claim: $\exists\ \delta > 0$ s.t. $|I - \ell(\gamma, P)| < \varepsilon \quad \forall P \in \mathcal{P}[a,b]$ s.t. $\|P\| < \delta$.

Recall: $\ell(\gamma,P) = \sum_{i=1}^m \|\gamma(t_i) - \gamma(t_{i-1})\|$ if $P: a = t_0 < t_1 < \cdots < t_m = b$

[~~Back Calculation:~~ $\ell(\gamma, P)$ $\leftarrow$ A given partition of $[a,b]$. $a = t_0 < t_1 < \cdots < t_m = b$.

$$\ell(\gamma,P) = \sum_{i=1}^m \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

$$= \sum_{i=1}^m \left[\sum_{j=1}^n (\gamma_j(t_i) - \gamma_j(t_{i-1}))^2\right]^{\frac{1}{2}},$$

where $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$; $\gamma_i : [a,b] \to \mathbb{R}$.

$\because$ γ is C^1 $\Rightarrow$ $\gamma_j : [a,b] \to \mathbb{R}$ is C^1, $j = 1, \ldots, n$.

$\therefore$ By MVT [!! BTW: there is no vector-valued MVT !!],

$\forall j \in \{1, \ldots, n\}$ & $i \in \{1, \ldots, m\}$, $\exists\ t_{ij} \in [t_{i-1}, t_i]$

s.t. $\gamma_j(t_i) - \gamma_j(t_{i-1}) = \gamma_j'(t_{ij})\,(t_i - t_{i-1})$.

$$\therefore\ \ell(\gamma, P) = \sum_{i=1}^m \left[\sum_{j=1}^n \gamma_j'(t_{ij})^2 \times (t_i - t_{i-1})^2\right]^{\frac{1}{2}}$$

$$= \sum_{i=1}^m \left[\sum_{j=1}^n \gamma_j'(t_{ij})^2\right]^{\frac{1}{2}} \times (t_i - t_{i-1}).$$

Trouble. —— ①

Recall: Riemann Sum for $f \in \mathcal{B}[a,b]$ is $S(f,P) = \sum_{i=1}^n f(\xi_i)\,|I_i|$ $\forall P \in \mathcal{P}([a,b])$ & $\xi_i \in I_i$ is i-th tag.

We know, $f \in \mathcal{R}[a,b]$ $\Rightarrow \lim_{\|P\|\to 0} S(f,P)$ exists. Then $\int_a^b f = \lim_{\|P\|\to 0} S(f,P)$.

Finally: The tag set is ~~just~~ restricted only on ~~just~~ the sub-intervals $\xi_i \in [x_{i-1}, x_i]$.

$\boxed{\text{Suppose,}}$ we have that: $t_{ij} = t_i^* \in [t_{i-1}, t_i]$ $\forall j = 1, \ldots, n$

t_i^*: j-free. $i = 1, \ldots, m$.

i.e., ~~the choice of~~ t_{ij} is independent of the choice of j, then:

$$\ell(\gamma, P) = \sum_{i=1}^m \left[\sum_{j=1}^n \gamma_j'(t_i^*)\right]^{\frac{1}{2}} (t_i - t_{i-1})$$

$$= \sum_{i=1}^m \|\gamma'(t_i^*)\|\,(t_i - t_{i-1}) = S(\|\gamma'\|, P)$$

The Riemann Sum of $\|\gamma'\|$ for P.

$\Rightarrow \ell(\gamma, P) = \mathfrak{S}(\|\gamma'\|, P)$. $\leftarrow$ This would finish the proof as $\|\gamma'\| \in \mathcal{R}[a,b]$.

So, we need to work on "t_{ij}" part.].

Define $B^n = [a,b] \times \cdots \times [a,b] = [a,b]^n \leftarrow$ a box in $\mathbb{R}^n$,

& $\Gamma : B^n \longrightarrow \mathbb{R}$ by

$$\Gamma(t_1, \ldots, t_n) = \left[\sum_{j=1}^{n} \gamma_j'(t_j)^2 \right]^{\frac{1}{2}}.$$

Note: $\Gamma(t, \ldots, t) = \|\gamma'(t)\|$, $\forall t \in [a,b]$.

$\because \gamma_j \in C^1$, $\Gamma : B^n \longrightarrow \mathbb{R}$ is continuous.

$\Rightarrow \Gamma$ is uniformly continuous [$\because B^n$ is compact.]

$\therefore$ For $\varepsilon > 0$, $\exists\ \delta_1 > 0$ $\ni$

$$|\Gamma(x) - \Gamma(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall\ \|x - y\| < \delta_1. \qquad (2)$$

Now By (1): $\ell(\gamma, P) = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \gamma_j'(t_{ij})^2 \right]^{\frac{1}{2}} \times (t_i - t_{i-1})$.

$\uparrow$ Recall: This is the trouble. (i-th part).

$$= \sum_{i=1}^{m} \Gamma(t_{i1}, t_{i2}, \ldots, t_{in})\,(t_i - t_{i-1}).$$

Here $P: a = t_0 < t_1 < \cdots < t_m = b$ a partition of $[a,b]$ $\ni$ $\|P\| = \max_{1 \le i \le n} (t_i - t_{i-1}) < \delta_1$.

Moreover, ~~$\hat{\mathcal{R}}(\gamma, P) := \sum_{i=1} \|\gamma'(t_i)\| (t_i - t_{i-1})$~~

if $\mathfrak{S}(\gamma, P) := \sum_{i=1}^{m} \|\gamma'(t_i)\|\,(t_i - t_{i-1})$, then

$t_i \in [t_{i-1}, t_i]$ is the tag point.

$$\mathfrak{S}(\gamma, P) = \sum_{i=1}^{m} \Gamma(t_i, \ldots, t_i)\,(t_i - t_{i-1}).$$

The Riemann Sum.

$$\therefore \quad \underset{\nearrow \text{ polygonal approx.}}{\ell(\gamma, P)} - \underset{\uparrow \text{ Riemann Sum of } \|\gamma'\|}{\mathbb{S}(\gamma, P)} = \sum_{i=1}^{n} \left(\Gamma(t_{i1}, \ldots, t_{in}) - \Gamma(t_i, \ldots, t_i) \right) \times (t_i - t_{i-1}).$$

$$\Rightarrow |\ell(\gamma,P) - \mathbb{S}(\gamma,P)| \leq \sum_{i=1}^{n} \left| \Gamma(t_{i1}, \ldots, t_{in}) - \Gamma(t_i, \ldots, t_i) \right| (t_i - t_{i-1}).$$

$$\underset{\text{by } (2)}{<} \frac{\varepsilon}{2(b-a)} \times \underbrace{\sum_{i=1} (t_i - t_{i-1})}_{= b-a}$$

$$= \frac{\varepsilon}{2}.$$

$$\therefore \underbrace{|\ell(\gamma,P) - \mathbb{S}(\gamma,P)| < \frac{\varepsilon}{2}}_{\text{The needed estimate.}} \quad \forall \; P \in \mathcal{P}[a,b] \;\&\; \|P\| < \delta_1.$$

Also, as $\|\gamma'\| \in \mathcal{R}[a,b]$, $\exists \; \delta_2 > 0$ s.t.

$$\Big| \underset{\swarrow \|\gamma'\|}{\mathbb{S}(\gamma, P)} - \int_a^b \|\gamma'(t)\| \, dt \Big| < \frac{\varepsilon}{2}. \quad \forall \|P\| < \delta_2.$$

$|\ell(\gamma,P) - \int \ldots|$

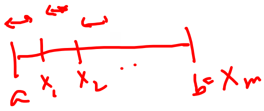
$\therefore$ For $\delta := \min\{\delta_1, \delta_2\}$ & $\forall P$ s.t. $\|P\| < \delta$, we have:

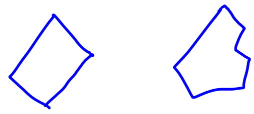
$$\left|\ell(\gamma,P) - \int \|\gamma'(t)\| dt\right| \leq \left| \ell(\gamma,P) - \mathbb{S}(\gamma,P) \right| + \left| \underset{\nwarrow (S)}{\mathbb{S}(\gamma,P)} - \int_a^b \|\gamma'(t)\| \, dt \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\therefore \lim_{\|P\| \to 0} \ell(\gamma, P) = \int_a^b \|\gamma'\|.$$

$\boxed{///}$

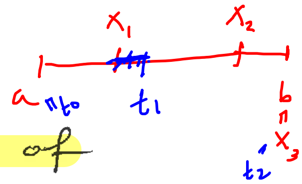




Cor: A piecewise smooth parametrized curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is rectifiable.
Moreover $\ell(\gamma) = \int_a^b \|\gamma'\|$. ⟵ However, rectifiable $\not\Rightarrow$ piecewise smooth
Consider: graph of the Cantor function
↳ Devil's staircase.

Proof: Let ~~[illegible]~~ $a = x_0 < x_1 < \cdots < x_m = b$ be a partition of $[a,b]$
s.t. $\gamma_i = \gamma\big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is a smooth parametrized curve, $\forall\, i = 1, \ldots, m$. $\Rightarrow \gamma = \bigcup_{i=1}^m \gamma_i$ ⟵ Smooth $\forall\, i$

Let $\varepsilon > 0$.
$\because$ γ is uniformly continuous ($\because$ Curve $\Rightarrow$ Cont.), $\exists\, \tilde{\delta} > 0$
s.t. $\|\gamma(s) - \gamma(t)\| < \dfrac{\varepsilon}{6m}$ $\quad \forall\, |s-t| < \tilde{\delta}$.



Suppose $P \in \mathcal{P}[a,b]$ s.t. $\underbrace{\|P\|}_{\text{mesh}} < \tilde{\delta}$.

Let $\tilde{P} \supset P$, where ~~$\{x_i\}_{i=0}^m$ are also~~ the nodes of
$\uparrow$ refinement

[Clearly, $\|\tilde{P}\| < \tilde{\delta}$]

$\tilde{P}$ are $\{x_i\}_{i=0}^m \cup \{\text{nodes of } P\}$.

nodes of $P = p$.
.... $\tilde{P} \le m + p$.

Then $\left|\ell(\gamma, \tilde{P}) - \ell(\gamma, P)\right| = \left| \sum \|\gamma(s_{i-1}) - \gamma(s_i)\| - \sum \|\gamma(t_{j-1}) - \gamma(t_j)\| \right|$

$s_i \in \Lambda(\tilde{P}) \setminus \{a\}$ ~~index of nodes of $\tilde{P}$~~ $\qquad$ # $m+p$

$t_j \in \Lambda(P) \setminus \{a\}$ ~~index of nodes of P.~~ $\qquad$ p ⟶ (+)

[Here $\Lambda(P)$ = nodes of P &
$\Lambda(\tilde{P})$ = —— " —— $\tilde{P}$.]

~~$= \sum_{i \in \Lambda(\tilde{P})} \|\gamma(s_{i-1}) - \gamma(s_i)\| + \sum_{j \in \Lambda(P)} \|\gamma(t_{j-1}) - \gamma(t_j)\|$~~

Observe that $\Lambda(\tilde{P}) = \Lambda(P) \cup \{x_i\}_{i=0}^m$. ok

Apply ~~[illegible]~~ triangle inequality ~~in terms of~~ & to (+) $\|\gamma(s_{i-1}) - \gamma(s_i)\|$,
~~if necessary~~, ~~[illegible]~~ ~~we get~~ we get [~~Note: $\sum_{\Lambda(P)}$ has m terms~~ & ~~$\sum_{\Lambda(\tilde{P})}$ has at most $2m$ terms~~ ~~Also each term can be dominated by $\varepsilon/6m$.~~]

$$\left|\ell(\gamma, \tilde{P}) - \ell(\gamma, P)\right| \le 3m \times \{\|\gamma(s) - \gamma(t)\| : s, t \in [a,b] \ \& \ |s-t| < \tilde{\delta}\}.$$

$$< 3m \times \frac{\varepsilon}{6m}$$

$$= \frac{\varepsilon}{2}.$$

$$\Rightarrow \left| \ell(\gamma, \tilde{P}) - \ell(\gamma, P) \right| < \frac{\varepsilon}{2}. \quad \text{—— Fact 1}$$

Now, $\forall\, i = 1, \ldots, m$, $\quad \gamma_i := \gamma\big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is rectifiable.

$\therefore$ For $\frac{\varepsilon}{2m} > 0$, $\exists\ \delta_i > 0$ s.t. $\forall\ P_i \in \mathcal{P}([x_{i-1}, x_i])$ with $\|P_i\| < \delta_i$, we have

$$\left| \ell(\gamma_i, P_i) - \int_{x_{i-1}}^{x_i} \|\gamma_i'\| \right| < \frac{\varepsilon}{2m}. \quad \text{—— } (\dagger)$$

Like the previous Construction of $\tilde{P}$.

Set $\delta := \min\{\tilde{\delta}, \delta_1, \ldots, \delta_m\}$.

Let $P \in \mathcal{P}([a,b]) \ni \|P\| < \delta$. $\left(\Rightarrow \|\tilde{P}\| < \delta\right.$, where nodes of $\tilde{P}$ = nodes of $P \cup \{x_i\}_{i=1}^m\left.\right)$

$$\therefore \left| \ell(\gamma, P) - \int_a^b \|\gamma'\| \right| \leq \left| \ell(\gamma, P) - \ell(\gamma, \tilde{P}) \right| + \left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right|.$$

$$\underset{\text{by Fact 1.}}{<} \frac{\varepsilon}{2} + \left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right|$$

Now $\left| \ell(\gamma, \tilde{P}) - \int_a^b \|\gamma'\| \right| = \left| \sum_{i=1}^m \ell(\gamma_i, \tilde{P}_i) - \sum_{i=1}^m \int_{x_{i-1}}^{x_i} \|\gamma'\| \right|$

$\tilde{P}_i \in \mathcal{P}([x_{i-1}, x_i])$ defined by $\tilde{P}_i := \tilde{P} \cap [x_{i-1}, x_i]$ $\forall\ i = 1, \ldots, m$.

$$\leq \sum_{i=1}^m \left| \ell(\gamma_i, \tilde{P}_i) - \int_{x_{i-1}}^{x_i} \|\gamma'\| \right|$$

$$< \frac{\varepsilon}{2m} \times m \quad (\text{by } (\dagger)) \quad \text{As } \|\tilde{P}_i\| \leq \|\tilde{P}\| < \delta \leq \delta_i \ \forall i.$$

$$= \frac{\varepsilon}{2}.$$

$\therefore \left| \ell(\gamma, P) - \int_a^b \|\gamma'\| \right| < \varepsilon \quad \forall\ P \in \mathcal{P}([a,b])$ s.t. $\|P\| < \delta$.

$$\Rightarrow \ell(\gamma) = \int_a^b \|\gamma'\|. \qquad ///$$

~~Thm:~~

Remark: Recall that a parametrized curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is smooth, if $\gamma'(t) \neq 0 \quad \forall\, t$.

$\because \gamma'(t) = (3t^2, 6t^5) \Rightarrow \gamma'(0) = (0,0)$.

(†) # Clearly $\gamma(t) = (t^3, t^6)$ is non-smooth at $t = 0$.

But $\tilde{\gamma}(t) = (t, t^2)$ is smooth at $t = 0$. $[\because \tilde{\gamma}'(t) = (1, 2t) \neq (0,0) \ \forall t]$

But the path of γ & the path of $\tilde{\gamma}$ are the same (i.e. the same trace/path).

$\Rightarrow$ Smoothness is NOT an intrinsic property of ~~the curve~~ curves as just a ~~path~~ / subset ~~/ trace~~ of $\mathbb{R}^n$. Smoothness is a soul property of parametrization.

However:

Thm: Consider a smooth curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ & a parametrization $\varphi : [\tilde{a}, \tilde{b}] \longrightarrow [a,b]$, smooth, then $\tilde{\gamma} = \gamma \circ \varphi$ is also smooth.

A reparametrization of γ, but we are not assuming onto.

$$\begin{tikzcd} & \mathbb{R}^n \\ \tilde{I} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\varphi"] & I \arrow[u, "\gamma"'] \end{tikzcd}$$

Proof: Easy: $\tilde{\gamma}(s) = \gamma(\varphi(s))$

$\Rightarrow \tilde{\gamma}' = \gamma'(\varphi(s)) * \varphi'(s)$ [Chain rule]

$\because \gamma'$ & φ' non-vanishing $\Rightarrow \tilde{\gamma}'$ is non-vanishing. ☒

eg: For $\gamma(t) = (t^3, t^6)$ & $\tilde{\gamma}(t) = (t, t^2)$, $\varphi(s) = s^{\frac{1}{3}}$. $[\Rightarrow \tilde{\gamma} = \gamma \circ \varphi]$

(From †) But φ is not even diff. at 0. ☒