TIME TO GET OUT OF $\underbrace{\text{"Sets of measure zero"}}_{\text{For Lebesgue measure}}$ :–

For us; in this class; we deal with the following (A different "Smallness" of Subsets of $\mathbb{R}^n$):

**Def:** Let $S \subseteq \mathbb{R}^n$. We say that $S$ has $\boxed{\text{Content zero}}$ (c.z) if for $\varepsilon > 0$, $\exists$ closed boxes $\{B_1^n, \cdots, B_m^n\}$, for some $m \in \mathbb{N}$, $\ni$

i) $S \subseteq \bigcup_{j=1}^{m} B_j^n$

FINITE

& ii) $\sum_{j=1}^{m} v(B_j^n) < \varepsilon$.

\# Replacing $B_j^n$ by open boxes $B_j^n$ : NO PROBLEM.

Remark : (1) Let $S \subseteq \mathbb{R}^n$. Then $S$ is of measure zero whenever $S$ has c.z

(2) If $f \in \mathcal{B}(B^n)$ & $\mathcal{D} := \{x \in B^n : f \text{ is NOT cont. at } x\}$,

then the following holds: $\mathcal{D}$ has c.z. $\Rightarrow f \in \mathcal{R}(B^n)$.
[Follows from (1).]

(3) "Unofficial": c.z $\xrightarrow{\text{goes to}}$ Riemann integration.
$\downarrow$ FINER $\qquad \downarrow$ FINER.
m.z $\xrightarrow{\text{goes to.}}$ $\underbrace{\text{Lebesgue integration}}$.

This covers more sets & fn's. Way more!!

eg: 1) Let $U \subseteq \mathbb{R}^n$ unbounded. Then $U$ cannot have c.z.

Proof: Easy: all boxes are bounded.

So if $U \subset \bigcup_{i=1}^{n} B_i^n$, for some boxes $\{B_i^n\}$, finite

then $U$ is bdd.

$\therefore$ $\boxed{S \subseteq \mathbb{R}^n \text{ is of c.z.} \Rightarrow S \text{ is bdd}}$

Contrast: $\mathbb{N}$, $\mathbb{Q}$ etc. are of m.z. !!

2) $\left.\begin{array}{l}\text{bdd set} \\ + \\ \text{m.z}\end{array}\right\} \not\Rightarrow$ c.z.

Proof: Trivial example: $S := [0,1] \cap \mathbb{Q}$.

$\Downarrow$

A countable set.

$\Downarrow$

m.z.

As $\overline{S} = [0,1]$, a finite cover of $S$ (by closed intervals) will also cover $[0,1]$.

But $v([0,1]) = 1$.

$\Rightarrow$ $[0,1]$ is not of c.z.

$\Rightarrow$ $S$ is not of c.z.

3) If $S \subseteq \mathbb{R}^n$ is of c.z, then $\partial S$ is also of c.z.

: Suppose $m \in \mathbb{N}$ ∋.

$$S \subseteq \bigcup_{j=1}^{m} B_j^n$$

$$\Rightarrow \partial S \subseteq \bigcup_{j=1}^{m} B_j^m$$ [∵ finite union of $B_j^m$ are all closed.]

∴ $S$ is of c.z. $\Rightarrow$ $\partial S$ is also so!!

Contrast: $\mathbb{Q} \cap [0,1]$ is of m.s.

But $\partial(\mathbb{Q} \cap [0,1])$ is not!

4) Finite subsets of $\mathbb{R}^n$ is of c.z.

5) Let $S \subseteq \mathbb{R}^n$ is of c.z. Then ~~s.t.~~ $int(S) \neq \varphi$. ~~Then S is of c.z.~~

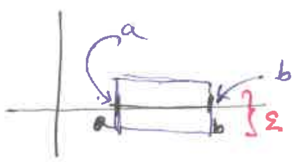
[HW: Do the easy case first: $n=1$; if $(a,b) \subseteq S$, then $S$ is not of c.z.]

6) Let $S$ be a line segment in $\mathbb{R}^n$ ($n>1$) (think $n=2$ to the least). Then $S$ is of c.z.

Proof: $n=2$: Consider $S = \{(x,0) : a \leq x \leq b\}$.

For $\varepsilon > 0$; $B_\varepsilon := \left\{[a,b] \times \left[\frac{-\varepsilon}{2(b-a)}, \frac{\varepsilon}{2(b-a)}\right]\right\}$

$\Rightarrow v(B_\varepsilon) = \varepsilon$. & $B_\varepsilon \supseteq S$.



7) If $S \subseteq \mathbb{R}^n$ is compact + measure zero, then $S$ is of content zero.

Proof: Easy (Also see : Page 27).

8) Let $S \subseteq \mathbb{R}^n$ is of c.z.

\# If $A \subseteq S$, then $A$ & $\overline{A}$ are of c.z.

↓ so $\overline{S}$ is of c.z

Contrast!! $\mathbb{Q}$ is of m.z, but $\overline{\mathbb{Q}}$ is NOT!!

This is where you see finite cover vs infinite cover

c.z   m.z.

9) Let $f : B^2 \to \mathbb{R}$ & $f \in \mathcal{R}(B^2)$.

$G_f := \{(x, f(x)) : x \in B^2\} \subseteq \mathbb{R}^3$.

graph of $f$.

Then $G_f$ is of c.z.

Proof: Wait.

—— x ——

10)

From now on: $\underline{n=2}$ will be our setting.

~~Note on measure zero~~; We continue with "Content zero":

**Thm:** Let $\underset{\text{bdd}}{\Omega} \supseteq \underset{\text{open}}{\mathcal{O}}$, $\Omega \subseteq \mathbb{R}^2$ & let $\bar{\Omega} \setminus \mathcal{O}$ is of ~~measure~~ content zero.

Suppose $f \in \mathcal{B}(\Omega)$ & $f|_{\mathcal{O}}$ is continuous. Then $f \in \mathcal{R}(\Omega)$.

Proof: Wait.

**Remark:** (1) Recall: Riemann-Lebesgue thm says: for $f \in \mathcal{B}(B^2)$, $f \in \mathcal{R}(B^2) \iff$ the set of discontinuity of $f$ is of measure zero.

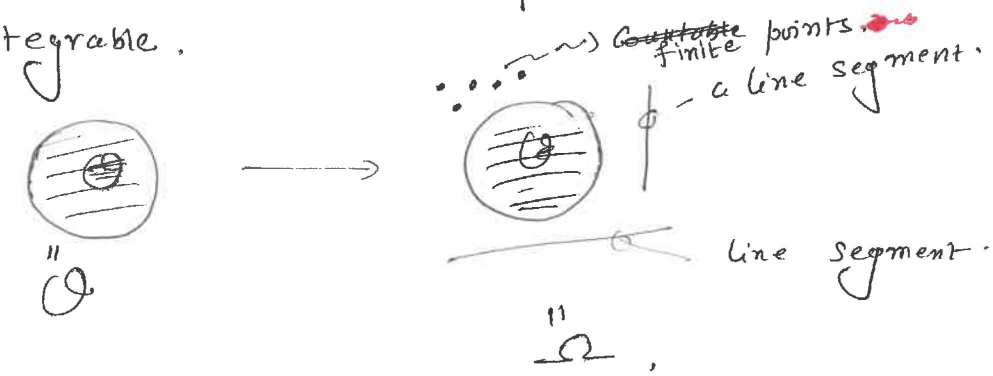
(2) From this perspective: the above thm is different:

ⓐ $\Omega$ is a bdd subset of $\mathbb{R}^2$.
ⓑ $\bar{\Omega} \setminus \mathcal{O}$ is of c.z.

(3) In particular: Consider a continuous fn. $f$ on $\underset{\text{open}}{\mathcal{O}} \subseteq \mathbb{R}^2$.

Any extension (but bdd) of $f$ to any bdd set $\Omega$ s.t. $\bar{\Omega} \setminus \mathcal{O}$ is of c.z. ~~measure zero~~ will be integrable.

eg:




(4) We are hoping the following:

$$\left[\begin{array}{l} \text{Let } f \in \mathcal{B}(\underset{\text{bdd}}{\Omega}) \text{ \& let } \Omega \text{ is of } \text{~~measure~~ content zero.} \\ \\ \text{Then } f \in \mathcal{R}(\Omega) \text{ \& } \displaystyle\int_\Omega f = 0. \end{array}\right.$$

# Should be useful.

Proof. Consider a box $B^2$ s.t. $\mathrm{int}(B^2) \supseteq \overline{\Omega}$. Recall $\tilde{f} \in \mathcal{B}(B^2)$ is the extension of $f$:

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \forall (x,y) \in \Omega \\ 0 & \forall (x,y) \in B^2 \setminus \Omega. \end{cases}$$

Note that $\tilde{f}\Big|_{\mathrm{int}(B^2)\setminus\overline{\Omega}} \equiv 0$. & $\mathrm{int}(B^2)\setminus\overline{\Omega}$ is an open set. Thus $\tilde{f}\Big|_{\mathrm{int}(B^2)\setminus\overline{\Omega}}$ is cont. fn. on $\mathrm{int}(B^2)\setminus\overline{\Omega}$.

Moreover, $\Omega$ is of ~~measure~~ content zero $\Rightarrow$ $\overline{\Omega}$ is of ~~measure~~ content zero, [~~HW~~ ~~&~~ Easy].

$\Rightarrow$ $\overline{\Omega}$ is of measure zero.

$\because$ the set of points of discontinuity of $\tilde{f} \subseteq \overline{\Omega}$, (~~namely~~ $\overline{\Omega}$) and $\overline{\Omega}$ is is of measure zero, it follows

Important

that $\tilde{f} \in \mathcal{R}(B^2)$. $\leftarrow$ By Riemann-Lebesgue classification.

To prove: $\int_{B^2} \tilde{f} \left( = \int_\Omega f \right) = 0$: Let $\varepsilon > 0$.

Set $M = \sup_\Omega |f|$.

Now for $\varepsilon > 0$, $\exists$ a partition $P$ of $B^2$ s.t.

$$\sum_{\alpha \in \tilde{\Lambda}} \mathrm{vol}(B^2_\alpha) < \varepsilon \quad \& \quad \bigcup_{\alpha \in \tilde{\Lambda}} (B^2_\alpha) \supseteq \Omega.$$

(for some $\tilde{\Lambda} \subseteq \Lambda(P)$).

Is: [Infact: get a finite cover of $\Omega$ with total area $< \varepsilon$ & then ~~...~~ add some more subboxes to cover the entire $B^2$: that will be the partition $P$.]

[Recall: if $f \in \mathcal{R}(\Omega)$, then $|f| \in \mathcal{R}(\Omega)$ & $\left|\int_\Omega f\right| \leq \int_\Omega |f|$.]

general fact.

Here: $U(|\tilde{f}|, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha \, v(B^2_\alpha)$

P, as above

$$= \sum_{\alpha \in \tilde{\Lambda}} M_\alpha \, v(B^2_\alpha) \qquad \left[\because M_\alpha := \sup_{B^2_\alpha} |f| = 0 \ \forall \alpha \notin \tilde{\Lambda}\right].$$

$$\leq M \times \underbrace{\sum_{\alpha \in \tilde{\Lambda}} v(B^2_\alpha)}_{< \varepsilon}.$$

$$< M \times \varepsilon.$$

$$\Rightarrow \inf U(|\tilde{f}|, P) = 0 \quad \Rightarrow \int_{B^2} \tilde{f} = 0.$$

$$\Rightarrow \int_\Omega f = 0. \qquad \boxtimes$$

$\boxed{\text{Back}}$ to our thm: (Proof is similar).

if $\Omega = B^2$, nothing to prove.

Thm: $\underset{\text{bdd}}{\Omega} \supseteq \underset{\text{open}}{\mathcal{O}}$. Suppose $\bar{\Omega} \setminus \mathcal{O}$ is of content ~~measure~~ zero, $f \in \mathcal{B}(\Omega)$ & $f|_{\mathcal{O}}$ is continuous. Then $f \in \mathcal{R}(\Omega)$.

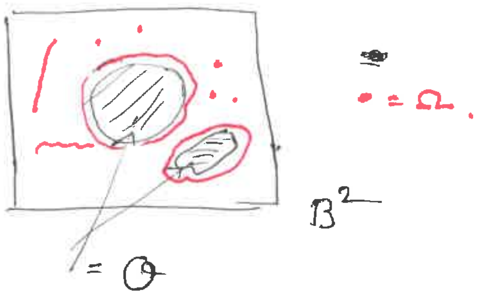
Proof: Let $\text{int}(B^2) \supseteq \bar{\Omega}$ & consider $\tilde{f}$ on $B^2$ (extension of $f$).

Enough to prove that: $\mathcal{D}$, the set of points of discontinuity of $\tilde{f}$, is of measure zero.

Note that: (i) $\tilde{f}|_{\mathcal{O}}$ is cont. (ii) $\tilde{f}|_{\text{int}(B^2) \setminus \bar{\Omega}} \equiv 0$ is cont.

open set.

& (iii) $\tilde{f}|_{\partial B^2} \equiv 0$ cont.




$\Rightarrow \mathcal{D} \subseteq \bar{\Omega} \setminus \mathcal{O} \leftarrow$ Set of measure zero.

$\Rightarrow \mathcal{D}$ is a set of measure zero.

$\Rightarrow f \in \mathcal{R}(\Omega)$.

DANGER: Sets of ~~measure~~ content zero depends on the "dimension".

For instance: ① $[0,1] \subseteq \mathbb{R}$ is not of ~~zero measure~~ C.Z

but $[0,1] \times \{a\} \subseteq \mathbb{R}^2$ is of ~~measure zero~~ C.Z.

~~② $\mathbb{Q} \cap [0,1]$ is of measure zero? Y/N: NO~~ C.Z

~~③ $\mathbb{Q} \times \mathbb{Q} \cap ([0,1] \times [0,1])$ — II — ? Y/N: YES.~~

Fact: Let $f: B^2 \to \mathbb{R}$ be a cont fn. Then

$$\boxed{\text{graph } f := \{(x, f(x)) : x \in B^2\}} \quad (\subseteq \mathbb{R}^3)$$

is a set of ~~measure~~ content zero.

Graphs have ~~measure~~ content zero.

Proof: Let $\varepsilon > 0$. Note that: $f$ is uniformly cont.

$\therefore \exists\, \delta > 0$ $\ni$ $|f(x) - f(y)| < \varepsilon \quad \forall\, |x-y| < \delta$.

$(x, y \in B^2)$

Next, on this $\delta > 0$, pick a partition ~~We are~~ $P$ of $B^2$

S.t. the $\underbrace{\text{diameter}}_{\substack{\parallel \\ \text{diagonal}}}$ of $B^2_\alpha < \delta \quad \forall\, \alpha \in \Lambda(P)$.

$$\Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall\, x, y \in B^2_\alpha, \quad \alpha \in \Lambda(P).$$

Set $I_\alpha := \{f(x) : x \in B^2_\alpha\}$.

The range set of $f|_{B^2_\alpha}$.

$\Rightarrow \quad I_\alpha \subseteq \tilde{I}_\alpha$, for some interval of length atmost $\varepsilon$. $\forall \alpha$.

$\therefore \{B^2_\alpha \times \tilde{I}_\alpha : \alpha \in N(P)\}$ is a cover of boxes of graph $f$. Also:

$\Lambda(P)$ is a finite set, &:

$$\sum_{\alpha \in \Lambda(P)} v(B_\alpha^2 \times \tilde{I}_\alpha) = \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times v(\tilde{I}_\alpha)$$

$$\leqslant \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times \varepsilon.$$

$$= \underbrace{v(B^2)}_{\text{Constant}} \times \varepsilon.$$

$\Rightarrow$ measure of graph $f$ is zero. $\boxed{//}$

Infact, we have the following:

Better !! Let $f \in \mathcal{R}([a,b])$. Then $G := \{(x, f(x)) : x \in [a,b]\} \subseteq \mathbb{R}^2$ is of ~~measure~~ Content zero.

Proof: We proceed along the same line:

Let $\varepsilon > 0$. $\exists\ P \in \mathcal{P}([a,b]) \ni$

$$U(f,P) - L(f,P) < \varepsilon.$$

Set $P : a = x_0 < x_1 < \cdots < x_n = b$.

& $B_i^2 := [x_{i-1}, x_i] \times [m_i, M_i]$,

Here: $m_i = \inf_{[x_{i-1}, x_i]} f$

& $M_i = \sup_{[x_{i-1}, x_i]} f$.

$\therefore\ G \subseteq \bigcup_{i=1}^{n} B_i^2$. Finally:

$$\sum_{i=1}^{n} v(B_i^2) = \sum_{i=1}^{n} v([x_{i-1}, x_i] \times [m_i, M_i])$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \times (M_i - m_i).$$

$$= U(f,P) - L(f,P)$$

$$< \varepsilon. \qquad \boxed{//}$$

Smart proof ? Than P-42 ?

## Back to Fubini's thm:

Recall: Let $f \in \mathcal{R}(B^2)$. Set $B_2 = [a,b] \times [c,d]$.

If $\int_a^b f(x,y)\,dx$ exists $\forall\, y \in [c,d]$, then

$$\int_{B^2} f = \int_c^d \left( \int_a^b f(x,y)\,dx \right) dy. \quad \text{—— (1)}$$

Illy if, $\int_c^d f(x,y)\,dy$ exists for each $x \in [a,b]$, then

$$\int_{B^2} f = \int_a^b \left( \int_c^d f(x,y)\,dy \right) dx \quad . \quad \text{—— (2)}$$

# If, in particular, $f \in C(B^2)$, then (1) = (2).

—— x ——.

Q: Fubini for $f \in \mathcal{R}(\Omega)$, $\Omega \subseteq \mathbb{B}^2$, bdd ??

How to think about it?

In fact: it is not easy to "evaluate" COMPUTE double integral over $\Omega \subseteq \mathbb{R}^2$. However, with "some" control over $\Omega$,
bdd

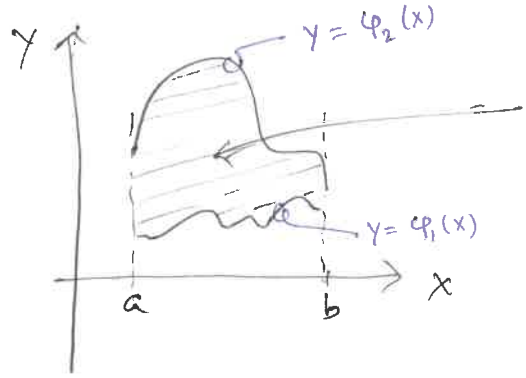
one can do "something". It is as follows:

Two special domains: ( AKA: Elementary regions ) :

**Def:** A set $\Omega \subseteq \mathbb{R}^2$ is said to be y-Simple / Type I if $\exists\ \varphi_1, \varphi_2 \in \mathcal{R}([a,b])$ s.t.

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

Here:



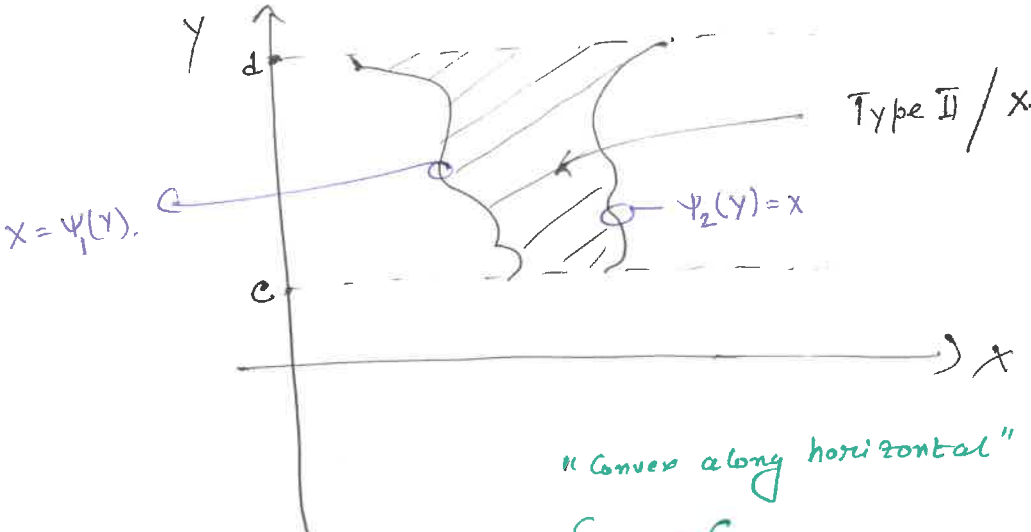

"Convex along vertical" :


||y X-Simple / Type II regions are given by:

$$\Omega = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}.$$

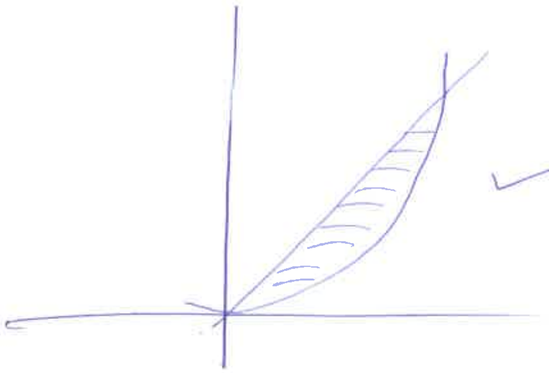
for some $\psi_1, \psi_2 \in \mathcal{R}[c,d]$.

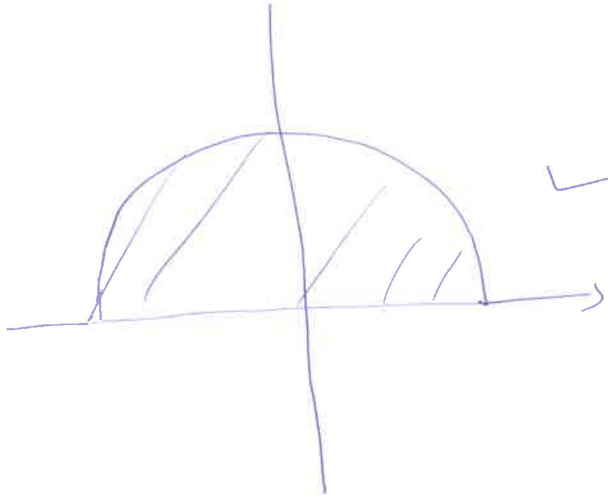
Here:

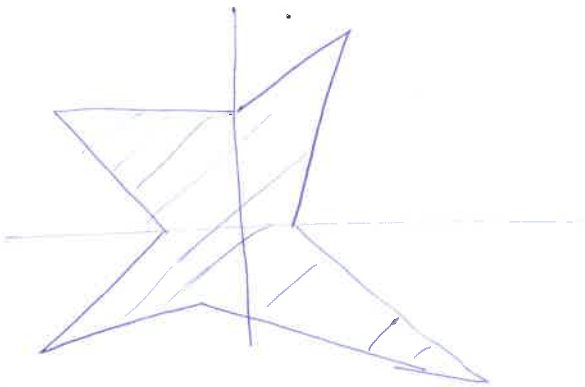



" Convex along horizontal "

eg:







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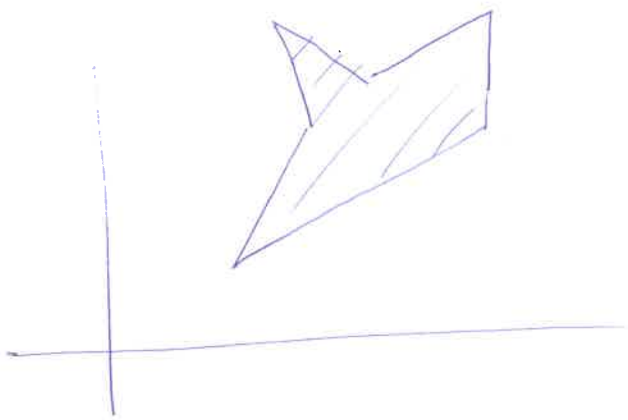

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