Remark: ① Given $\Omega \subseteq \mathbb{R}^2$, Ω has an area $\Leftrightarrow$ $\chi_\Omega \in \mathcal{R}(B^2)$ for some box $B^2 \supseteq \Omega$. In this case,

$$\text{Area}(\Omega) = \int_{B^2} \chi_\Omega .$$

[Def. $\chi_\Omega : B^2 \to \{0,1\}$ where $\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega . \end{cases}$ indicator/Characteristic fn.]

Proof: $\tilde{1}_\Omega = \chi_\Omega$.

② $\|^{ly}$ you may define/deduce, for $\Omega \subseteq \mathbb{R}^n$,

$$\underbrace{\text{Vol}(\Omega)}_{\text{volume of } \Omega} = \int_{B^n} \chi_\Omega . \qquad B^n \supseteq \Omega .$$

Thm: Let $\Omega \subseteq \mathbb{R}^n$ be bdd. Then Ω has a $\boxed{\text{volume}}$ $\Leftrightarrow$ $\partial\Omega$ is of content zero. $\Longleftarrow$ Let's do it for general $n \geq 2$.

Proof: "$\Leftarrow$" Suppose $\partial\Omega$ has content zero. Set $f := \underbrace{\tilde{1}_\Omega}_{\text{Corresponding to } B^n \supseteq \Omega} = \chi_\Omega$.

Clearly, f is cont. on $\text{int}(\Omega)$ ($\because f|_\Omega \equiv 1$).

Arguing along the same line of proof of thm in P-41:

$\mathcal{D} := \{ x \in B^n : f \text{ is not cont. at } x \}$. $B^n \supseteq \Omega$

Enough to prove that $\mathcal{D}$ is of measure zero.

So: (i) $f|_{\text{int}(\Omega)}$ is cont. on $\text{int}(\Omega)$.

(ii) $f|_{B^n \setminus \bar{\Omega}} \equiv 0$ is cont. on $B^n \setminus \bar{\Omega}$.

[Recall: $\partial\Omega = \bar{\Omega} \setminus \text{int}(\Omega)$.]

$\Rightarrow \mathcal{D} \subseteq \partial\Omega$ —(*)

$\because \partial\Omega$ is of content zero. $\Rightarrow \mathcal{D}$ is of content zero.

$\Rightarrow f \in \mathcal{R}(B^n)$ i.e., $\chi_\Omega \in \mathcal{R}(B^n)$.

i.e., Ω has a volume.

(*) if $B^n \supsetneq \Omega$, then $\mathcal{D} = \partial\Omega$.

"$\Rightarrow$" Let $B^n \supseteq \Omega$ & $\chi_\Omega = \tilde{1}_\Omega \in \mathcal{R}(B^n)$. Again: $f := \chi_\Omega$.

Claim: $\partial\Omega$ is of content zero.

Fix $\varepsilon > 0$. $\because f \in \mathcal{R}(B^n)$, $\exists\, P \in \mathcal{P}(B^n)$ s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2} . \qquad \longleftarrow \text{By integrability of } f.$$

$$\Rightarrow \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \frac{\varepsilon}{2} .$$

$\tilde{\Lambda} := \{ \alpha \in \Lambda(P) : B^n_\alpha \cap \Omega \neq \varphi$ & $B^n_\alpha \not\subseteq \Omega \}$

Not contained in.

Hint: if $x \in \partial\Omega$, then any open set $O \ni x$, $O \cap \Omega \neq \varphi$ & $O \not\subseteq \Omega$.

The point is: $M_\alpha = 1, \; m_\alpha = 0 \quad \forall \alpha \in \tilde{\Lambda}$.

$$\therefore \underbrace{\sum_{\alpha \in \tilde{\Lambda}} (M_\alpha - m_\alpha)\, v(B^n_\alpha)}_{= \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha)} \leq \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \frac{\varepsilon}{2}.$$

$$\Rightarrow \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha) < \frac{\varepsilon}{2}. \qquad \text{--- } (1)$$

On the other hand, ∂B^n_α is of content zero $\forall \alpha \in \Lambda(P)$.

[Known fact: Finite union of faces.]

$\Rightarrow \bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha$ is of content zero. ($\because$ finite union of c.z. set is of c.z.)

$\therefore \exists$ boxes ~~$\{P^n_\beta : \beta \in \Gamma'\}$~~ $\{B^n_1, \ldots, B^n_p\}$ $\ni \bigcup_{j=1}^{p} B^n_j \supseteq \bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha$

& $\sum_{j=1}^{p} v(B^n_j) < \frac{\varepsilon}{2}$. --- (2)

Nothing to do with B^n_α, $\alpha \in \Lambda(P)$.

Claim: $$\partial\Omega \subseteq \underbrace{\left(\bigcup_{\alpha \in \tilde{\Lambda}} B^n_\alpha \right)}_{\text{I}} \cup \underbrace{\left(\bigcup_{j=1}^{p} B^n_j \right)}_{\text{II}}.$$

Content zero

$\Downarrow$

$\partial\Omega$ is of content zero. by (1) & (2).

[$\because \sum_{j=1}^{p} v(B^n_j) + \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha) < \varepsilon$.]

AND we are done!!

Proof of the Claim:

Pick $x \in \partial\Omega \subseteq B^n$.

($\because B^n$ is closed)

$\therefore x \in B^n_\alpha$ for some $\alpha \in \Lambda(P)$. $\Rightarrow x \in \text{int}(B^n_\alpha)$ OR $x \in \partial B^n_\alpha$.

If $x \in \text{int}(B^n_\alpha)$: As $x \in \partial\Omega$ ~~& int(B^n_α) open~~ any open set containing x will hit Ω & Ω^c. int(B^n_α) also contains elements not in Ω [By the def. of bd points].

$\Rightarrow B^n_\alpha \cap \Omega \neq \varphi$ & $B^n_\alpha \not\subseteq \Omega$.

$\Rightarrow \alpha \in \tilde{\Lambda}$. $\Rightarrow x \in$ I.

If $x \in \partial B^n_\alpha$: Then $\partial B^n_\alpha \subseteq \dot{\bigcup}_{j=1}^{p} B^n_j$. $\Rightarrow x \in$ II .

$\therefore$ The claim holds good. ▨

Recall

Fact: Suppose $\Omega \subseteq \mathbb{R}^n$ is of content zero & $f \in \mathcal{B}(\Omega)$. Then $f \in \mathcal{R}(\Omega)$ & $\int_\Omega f = 0$. [Already done : P-39.]

Thm: Suppose $\Omega \subseteq \mathbb{R}^n$ bdd. Then :

Ω has an Volume ~~area~~ & ~~Ar~~ $\text{vol}(\Omega) = 0 \iff \Omega$ is of content zero.

Proof: "$\Rightarrow$" So, $\int_{B^n} \chi_\Omega = 0$. Let $\varepsilon > 0$.

$\chi_\Omega \in \mathcal{R}(\Omega)$

$\therefore$ Volume zero $=$ Content zero.

$$\Downarrow$$

$$0 = \overline{\int_{B^n}} \chi_\Omega = \inf\{U(\chi_\Omega, P) : P \in \mathcal{P}(B^n)\}$$

$\therefore \exists P \in \mathcal{P}(B^n) \ni U(\chi_\Omega, P) < \varepsilon$.

Set $\tilde{\Lambda} := \{\alpha \in \Lambda(P) : B^n_\alpha \cap \Omega \neq \varphi\}$.

Clearly, for $\alpha \in \Lambda(P)$, $\alpha \in \tilde{\Lambda} \iff M_\alpha = 1$. [$\because M_\alpha = \sup_{B^\alpha_n} \chi_\Omega = \sup_{B^\alpha_n} \chi_{B^\alpha_n \cap \Omega}$]

Also, $M_\alpha = 0 \ \forall \alpha \notin \tilde{\Lambda}$.

So
$$\varepsilon > U(\chi_\Omega, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha v(B^n_\alpha) = \sum_{\alpha \in \tilde{\Lambda}} M_\alpha v(B^n_\alpha)$$
$$= \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha).$$

Also, Since $\{B^n_\alpha : \alpha \in \Lambda(P)\}$ a partition of $B^n \supseteq \Omega$,

So $\{B^n_\alpha : \alpha \in \tilde{\Lambda}\}$ is a finite Cover ~~partition~~ of Ω & $\sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha) < \varepsilon$. $\Rightarrow \Omega$ is of Content zero.

"$\Leftarrow$" Let Ω is of Content zero. Then the above fact $\Rightarrow \chi_\Omega \in \mathcal{R}(\Omega)$ & $v(\Omega) = \int \chi_\Omega = 0$. ▨

Note: Let $\Omega_1 \subseteq \Omega$. Let $f \in \mathcal{R}(\Omega)$. We know $f|_{\Omega_1}$ need not be in $\mathcal{R}(\Omega_1)$.

[Simple example: $\Omega = [0,1]\times[0,1]$; $\Omega_1 = (\mathbb{Q}\times\mathbb{Q}) \cap \Omega$.
$f \equiv 1$. As we know Ω_1 ~~is not~~ does not have area.].

However, the following is impressive:

Thm: Let $\Omega_1 \subseteq \underbrace{\Omega}_{\text{bdd}} \subseteq \mathbb{R}^n$, and let $\partial\Omega_1$ is of content zero.

Then $f|_{\Omega_1} \in \mathcal{R}(\Omega_1) \quad \forall\ f \in \mathcal{R}(\Omega)$.

Proof: Consider $B^n \supseteq \Omega$. $\quad \therefore\ B^n \supseteq \Omega_1$.

Let $f \in \mathcal{R}(\Omega)$.

$\because\ \partial\Omega_1$ is of content zero, $\chi_{\Omega_1} \in \mathcal{R}(B^n)$.

Observe : $\widetilde{f|_{\Omega_1}} = \tilde{f}\,\chi_{\Omega_1}$ $\qquad$ both are : $B^n \to \mathbb{R}$.

$\circledast$

The extension of $f|_{\Omega_1} : \Omega_1 \to \mathbb{R}$ to $\widetilde{f|_{\Omega_1}} : B^n \to \mathbb{R}$

by $\left(f|_{\Omega_1}\right)\big|_{\Omega_1} = f|_{\Omega_1}$

& $\left(f|_{\Omega_1}\right)_{B^n \setminus \Omega_1} \equiv 0$.

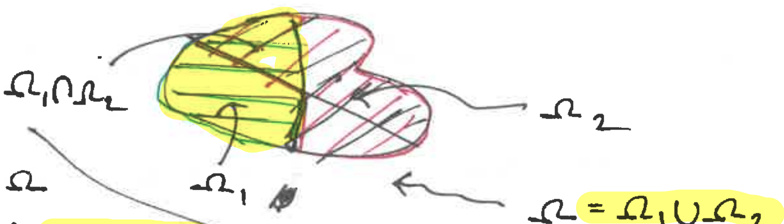
$\because\ \tilde{f},\ \chi_{\Omega_1} \in \mathcal{R}(B^n)$, by product formula,

$$\widetilde{f|_{\Omega_1}} \in \mathcal{R}(B^n).$$

i.e., $f|_{\Omega_1} \in \mathcal{R}(\Omega_1)$.

Remark: By $\circledast$, $\displaystyle\int_{\Omega_1} f|_{\Omega_1} = \int_{\Omega} \tilde{f}\,\chi_{\Omega_1}$. $\qquad \chi_{\Omega_1} : \Omega \to \mathbb{R}$. #.

Th: (Additivity of Sets):



Let $\Omega_1, \Omega_2 \subseteq \Omega$, $f \in \mathcal{B}(\Omega)$ & Ω be a bdd subset of $\mathbb{R}^n$. Suppose $\Omega=\Omega_1\cup\Omega_2$, $f|_X \in \mathcal{R}(X)$, where $X = \Omega_1, \Omega_2$ & $\Omega_1\cap\Omega_2$.

Then $f \in \mathcal{R}(\Omega)$ &
$$\int_\Omega f = \int_{\Omega_1} f|_{\Omega_1} + \int_{\Omega_2} f|_{\Omega_2} - \int_{\Omega_1\cap\Omega_2} f|_{\Omega_1\cap\Omega_2} \quad \text{———} \quad (*)$$

On the other hand, if $f \in \mathcal{R}(\Omega)$ & both $\partial\Omega_1$ & $\partial\Omega_2$ are of content zero, then $f|_X \in \mathcal{R}(X)$, $\forall\, X = \Omega_1, \Omega_2$ & $\Omega_1\cap\Omega_2$.

[$\Rightarrow$ $(*)$ also holds.].

Proof: Set $f_i = f|_{\Omega_i} : \Omega_i \longrightarrow \mathbb{R}$, $i=1,2$, & set $g = f| : \Omega_1\cap\Omega_2 \longrightarrow \mathbb{R}$.

Choose $B^n \supseteq \Omega$. $\Rightarrow$ $B^n \supseteq \Omega_1, \Omega_2, \Omega_1\cap\Omega_2$.

Then $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 - \tilde{g}$ ($\sim$: the extensions of these fn.'s to all B^n.).

$\because$ $f|_X \in \mathcal{R}(X)$ $\forall\, X = \Omega_1, \Omega_1, \Omega_1\cap\Omega_2$, by defn, it follows that $\tilde{f}_i, \tilde{g} \in \mathcal{R}(B^n)$.

$$\Rightarrow \tilde{f} \in \mathcal{R}(B^n) \;\&\; \int_\Omega f = \int_{B^n}\tilde{f} = \int_{B^n}\tilde{f}_1 + \int_{B^n}\tilde{f}_2 - \int_{B^n}\tilde{g}.$$
$$= \int_{\Omega_1} f_1 + \int_{\Omega_2} f_2 - \int_{\Omega_1\cap\Omega_2} g$$

For the 2nd part: Observe that

$$\partial\Omega_1, \partial\Omega_2, \partial(\Omega_1\cap\Omega_2) \subseteq \underbrace{\partial\Omega_1 \cup \partial\Omega_2}_{\text{Content zero.}}$$

$\Rightarrow$ $\overbrace{\partial\Omega_1, \partial\Omega_2 \;\&}$ $\partial(\Omega_1\cap\Omega_2)$ is of content zero.

$\Rightarrow$ $f|_{\Omega_1\cap\Omega_2} \in \mathcal{R}(\Omega_1\cap\Omega_2)$ & $f|_{\Omega_i} \in \mathcal{R}(\Omega_i)$ $i=1,2$.

$\boxtimes$

Cor: If $\underset{\text{bdd}}{\Omega_1, \Omega_2 \subseteq \Omega} \subseteq \mathbb{R}^n$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2$ is of content zero, and if $f \in \mathcal{B}(\Omega)$ s.t. $f|_{\Omega_1} \in \mathcal{R}(\Omega_1)$ & $f|_{\Omega_2} \in \mathcal{R}(\Omega_2)$, then $f \in \mathcal{R}(\Omega)$ &

$$\int_\Omega f = \int_{\Omega_1} f|_{\Omega_1} + \int_{\Omega_2} f|_{\Omega_2} .$$

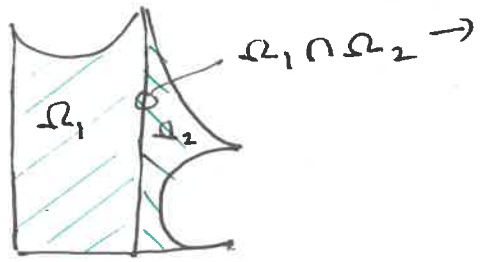
Useful. And also has been used.

Proof: We know, if $\underset{\text{bdd}}{X \subseteq \mathbb{R}^n}$ is of content zero & $g \in \mathcal{B}(X)$, then $g \in \mathcal{R}(X)$ & $\int_X g = 0$.

With this, the result follows from previous thm.

▨

Remark: So, if $\Omega = \Omega_1 \cup \Omega_2$



Then $\forall f \in \mathcal{B}(\Omega)$ s.t. $f|_{\Omega_i} \in \mathcal{R}(\Omega_i)$, $i = 1, 2$,

we have that $f \in \mathcal{R}(\Omega)$ &

$$\int_\Omega f = \int_{\Omega_1} f|_{\Omega_1} + \int_{\Omega_2} f|_{\Omega_2} .$$

This has been used & also will be very useful in integration of fn. over general bdd. sets.

Cor (Identity thm). Let $\underset{\text{bdd}}{\Omega \subseteq \mathbb{R}^n}$, $f \in \mathcal{R}(\Omega)$, $g \in \mathcal{B}(\Omega)$ & let $\mathcal{D} := \{x \in \Omega : f(x) \neq g(x)\}$ is of Content zero.

Then $g \in \mathcal{R}(\Omega)$ & $\int_\Omega f = \int_\Omega g$.

Proof: Set $F(X) = f(x) - g(x) \quad \forall\, x \in \Omega$.

$\therefore$ $\cancel{F}$ $F|_{\Omega \setminus \mathcal{D}} \equiv 0 \implies F \in \mathcal{R}(\Omega \setminus \mathcal{D})$ & $\int_{\Omega \setminus \mathcal{D}} F = 0$.

Also, $F|_{\mathcal{D}} \in \mathcal{R}(\mathcal{D})$ & $\int_{\mathcal{D}} F = 0$ [$\because$ $F \in \mathcal{B}(\Omega)$ & $\mathcal{D}$ is of Content zero.]

$\because$ $(\Omega \setminus \mathcal{D}) \cap \mathcal{D} = \varphi$ is of Content zero, it follows that $F \in \mathcal{R}(\Omega)$ &

$$\int_\Omega F = \underbrace{\int_{\mathcal{D}} F|_{\mathcal{D}}}_{=0} + \underbrace{\int_{\Omega\setminus\mathcal{D}} F|_{\Omega\setminus\mathcal{D}}}_{=0} = 0.$$

$\therefore$ $g = f - F \in \mathcal{R}(\Omega)$ & $\int_\Omega g = \int_\Omega (f - F) = \int_\Omega f - \int_\Omega F$

$= \int_\Omega f.$ $\blacksquare$

$g \in \mathcal{R}(\Omega)$.

$g(x) = 0$ a.e. Ω.

$\{\xi_i\}_{i=1}^m \subseteq [0,1]$.

$g(x) = 0 \quad \forall x \neq \xi_i$

$g(\xi_i) = r_i$

$\Rightarrow g \in \mathcal{R}([0,1])$ & $\int_0^1 g = 0$

\# If $f \in \mathcal{B}(\Omega)$ & $f(x) = 0 \quad \forall\, x$ but a Subset of Ω of Content zero, then $f \in \mathcal{R}(\Omega)$ & $\int_\Omega f = 0$.

$\mathcal{D} = \{x \in \Omega : f(x) \neq g(x)\}$ of Content zero.

\# So, if you change a Riemann integrable fn. to a new fn. by redefining it at a subset of Content zero, then the ~~integral~~ redefined fn. again will be integrable & will have the same integral value. ((

Note: $\Omega = [0,1]$.
$\mathcal{D} = \{\frac{1}{n} : n \in \mathbb{N}\}$
$\to$ Content zero.

\# $\boxed{\text{Think } n = 1 \text{ Case too!!}}$

Change of variables:

One of the most powerful tools.

$n=1$: Let $\varphi : \mathcal{O} \longrightarrow \mathbb{R}$ be a C^1-fn (i.e., Cont. diff.).
$\subseteq \mathbb{R}$ open

Assume $\varphi'(x) \neq 0$ $\forall x \in \mathcal{O}$. Also assume $\mathcal{O} \supseteq [a,b]$.

Then $\forall f \in C(\varphi[a,b])$, or $\mathcal{R}$

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \int_a^b (f\circ\varphi)(x)\ \varphi'(x)\,dx.$$

We did it for Cont. fn. Proof was simple application of FTC. What about FTC in $\mathbb{R}^n$?

Now, if $\varphi'(x) > 0$, $\varphi \uparrow \Rightarrow \varphi([a,b]) = [\varphi(a), \varphi(b)]$

Hly —— < 0, $\varphi \downarrow \Rightarrow \varphi([a,b]) = [\varphi(b), \varphi(a)]$.

$\therefore$ The above one is given by:

$$\int_{\varphi([a,b])} f = \int_{[a,b]} f\circ\varphi\ |\varphi'|.$$

i.e., $$\int_{\varphi([a,b])} f(x)\,dx = \int_{[a,b]} \underbrace{f(\varphi(x))}\ \underbrace{|\varphi'(x)|\,dx}.$$

NEW integrand $f\circ\varphi$

New? dx??

The 1-variable version of change of variable formula.

Q: What about $\mathcal{O}_n \subseteq \mathbb{R}^n$ version?

Pretty much same. But the proof is very involved!!