Sequence & Series of functions:

For $\{a_n\} \subseteq \mathbb{R}$ to converge (to a, say): We only need to know "if a_n's are close enough".

The Cauchy criterion: "for $\varepsilon > 0$ if $\exists N \in \mathbb{N}$ $(n \equiv n(\varepsilon))$ $\ni$ $|a_n - a_m| < \varepsilon$ $\forall n, m \geq N$"

$d(a_n, a_m) < \varepsilon$

usual metric of $\mathbb{R}_u$.

$\Leftrightarrow$ $\{a_n\}$ is convergent.

(*) metric structure, i.e., Completeness of $\mathbb{R}$ plays THE role. [i.e. $\mathbb{R}_u$ is complete.]

Aim: Consider $\{f_n\} \subseteq \underbrace{\mathcal{F}(\overset{\text{Subset of } \mathbb{R}.}{S})}_{\text{Set of fn's: } S \to \mathbb{R}}$: a seqn of fn's.

"Figure out" Convergency of $\{f_n\}$, i.e. explain $f_n \longrightarrow f$ for some $f \in \mathcal{F}(S)$.

[?? But what about metric structure like $\mathbb{R}_u$ in (*)?]
The real question!!

Obs: $\forall x \in S$, $\{f_n(x)\} \subseteq \mathbb{R}_u$. Here we can certainly talk about convergency of $\underbrace{\{f_n(x)\}}_{\subseteq \mathbb{R}_u.}$ $\forall x \in S$.

Known as pointwise convergency.

Def: $\{f_n\}$ Converges pointwise (on the set S) to f if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in S.$$

We write $f_n \xrightarrow{\text{pointwise}} f$ & say: f is the pointwise limit of $\{f_n\}$.

Obs: Given $\{f_n\}$ & $f \in \mathcal{F}(S)$, $f_n \xrightarrow{\text{point}} f$

$\Leftrightarrow$ For $\varepsilon > 0$, $x \in S$ $\exists N \equiv N(\varepsilon, x) \in \mathbb{N}$ $\ni$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

$\Leftrightarrow$ For $\varepsilon > 0$, $x \in S$, $\exists N \equiv N(\varepsilon, x) \in \mathbb{N}$ $\ni$ $|f_n(x) - f_m(x)| < \varepsilon$ $\forall n, m \geq N$.

Remark: ① If $\{f_n(x)\}$ is Cauchy, $\forall\, x \in S$, then we can safely define
$f(x) := \lim_{n\to\infty} f_n(x) \quad \forall x$.

② The notion of pointwise convergency is still not well done:
We must talk about $\boxed{\text{closedness}}$ of $\{f_n\}$!!
i.e. A $\underbrace{\text{METRIC}}_{\text{How to get ?}}$ flavour of / among functions !!

To the least: one may think about getting rid of x for $N(\varepsilon, x)$ in the pointwise defn.:

We say, f is the uniform limit of $\{f_n\}$.

Def. Let $\{f_n\} \cup \{f\} \subseteq \mathcal{F}(S)$. We say that $f_n \xrightarrow{\text{uni}} f$ (i.e. f_n converges to f) (on S) if for $\varepsilon > 0$ $\exists$ $N \equiv N(\varepsilon) \in \mathbb{N}$
$\ni$.
$$\underbrace{|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N.}_{\text{This indicates the following:}}$$

Def: For $f, g \in \mathcal{F}(S)$, define
$$d(f,g) := \sup_{x\in S} |f(x) - g(x)|.$$
Sup metric

& $$\|f\|_\infty := \sup_{x \in S} |f(x)|.$$
Sup norm.

\# Depending on S & f, d or $\|\cdot\|_\infty$ will carry good meaning !!
For instance: $d(\cdot,\cdot)$ is a metric (Complete) on $C[a,b]$ or $B[a,b]$.

Fact: $f_n \xrightarrow{\text{unif}} f \iff f_n \to f$ in $\|\cdot\|_\infty$
i.e., $d(f_n, f) \to 0$.

\# [$\because$ d is THE METRIC which can take the role of distance among functions !

Obs: Suppose $f_n \xrightarrow{\text{Unif}} f$. Then $f_n \xrightarrow{\text{point.}} f$. i.e.: unif $\Rightarrow$ pointwise.

Proof: Easy: as $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in S,\ n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N \ \&\ \forall x \in S.$

Q: "$\Leftarrow$" ? No. (A big NO.)

However: pointwise limit is the fn. to look for : for uniform convergence !!

i.e: pointwise is the 1st BUT NOT the last step.

eg: ① $f_n(x) = x^n$. $\forall n \geq 1$. $x \in [0,1] = S$.

Now $\forall x \in [0,1)$, $x^n \to 0$.

& if $x = 1$, then $f_n(x) = f_n(1) = 1^n \to 1$..

So $f_n \xrightarrow{p} f$, where $f(x) = \begin{cases} 0 & \forall x \in [0,1) \\ 1 & x = 1 \end{cases}$.

Obs: ⓐ Here $\{f_n\} \subseteq C[0,1]$. But the pointwise limit $f \notin C[0,1]$. $\Rightarrow$ point. is NOT good for $C[0,1]$!!

pointwise $\not\Rightarrow$ respecting cont.

$\not\Rightarrow$ —ıı— bdd

ⓑ $d(f_n, f) = \sup_{x \in [0,1]} |f_n(x) - f(x)| = 1 \quad \forall n$

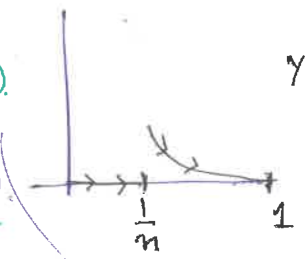
$[\because f \equiv 0$ on $[0,1)$ & $|x^n| \leq 1$

$\Rightarrow f_n \overset{u}{\not\to} f$. but $f_n \xrightarrow{p} f$.

② Let $f_n(x) = \begin{cases} 0 & 0 < x \leq \frac{1}{n} \\ \frac{1}{x} & \frac{1}{n} \leq x \leq 1 \end{cases}$ $\quad f_n \in \mathcal{F}((0,1])$.

Here $\lim_{n\to\infty} f_n(x) = \frac{1}{x} \quad \forall x \in (0,1]$. (fixed)

$y = f_n(x)$.



i.e. $f_n \xrightarrow{p} f$, when $f(x) = \frac{1}{x}$, $x \in (0,1]$

Clearly, $|f_n(x)| \leq n \quad \forall n$. & $x \in (0,1]$

$\Rightarrow \{f_n\}$ is a bdd seqn. but the pointwise limit f is NOT.

$\therefore$ Pointwise limit does not respect bddness.

uniform does !!

Remark: However, if $\{f_n\}$ be s.t. $f_n \xrightarrow{u} f$ & $\{f_n\}$ are bdd, then f is also bdd : Let $\varepsilon > 0$. $\therefore \exists N \ni |f_n(x) - f(x)| < \varepsilon \ \forall n \geq N, \forall x \in S$.

$$\therefore \ |f(x)| \leq |f_N(x) - f(x)| + |f_N(x)|.$$
$$< \varepsilon + |f_N(x)|$$
$$\leq \varepsilon + \underbrace{\sup_{x \in S} |f_N(x)|}_{< \infty}. \qquad \forall x \in S.$$

$\Rightarrow$ f is bdd.

$\therefore$ unif. Cnv. is good with bddness!!

Q: what about integration, diff. (etc) under uniform / pointwise ??

(3) Suppose $f_n \xrightarrow{p} f$, $f_n \in \mathcal{R}[a,b]$.

$$\overset{?}{\Rightarrow} f \in \mathcal{R}[a,b] \ ? \quad \text{or} \quad \underbrace{\lim \int f_n = \int \underbrace{\lim f_n}_{= f}}_{\text{interchanging limits ?}}.$$

Ans: point : ✗
unif : ✓.

Let $f_n(x) = nx(1-x^2)^n$. $x \in [0,1]$.

Clearly, $f_n \in C[0,1] \subseteq \mathcal{R}[0,1]$.

Now for $x \in (0,1)$,

$$0 \leq f_{n+1}(x) = \cancel{nx(1-x^2)^n} \ (n+1)x(1-x^2)^{n+1}.$$
$$= \cancel{\text{not}} \ \{nx(1-x^2)^n\} \times \left\{\frac{n+1}{n}(1-x^2)\right\}$$
$$= f_n(x) \left\{\frac{n+1}{n}(1-x^2)\right\}.$$

$$\Rightarrow 0 \leq \frac{f_{n+1}(x)}{f_n(x)} = \frac{n+1}{n}(1-x^2) \longrightarrow (1-x^2) < 1$$
as $x \in (0,1)$.

$$\Leftarrow \left[\lim \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_n \to 0\right]$$

$$\Rightarrow \lim_{n \to \infty} f_n(x) = 0 \quad \forall x \in [0,1].$$

[Note: if $x = 0, 1$, then $f_n(x) = 0 \ \forall n$]

$\Rightarrow$ $f_n \xrightarrow{p} f$ where $f \equiv 0$.

Now $\int_0^1 f_n(x)\,dx = n\int_0^1 x(1-x^2)^n\,dx$.

$1-x^2 \to t$.
$2x\,dx = -dt$
$[0,1] \to [1,0]$.

$= n\,\frac{1}{2}\int_0^1 t^n\,dt$.

$= \frac{1}{2}\frac{n}{n+1}$. $\longrightarrow \underline{\frac{1}{2}}$ $(\neq 0)$ as $n\to\infty$.

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \longrightarrow \frac{1}{2} \neq \int_0^1 \lim f_n(x) = 0.$

Q: What if $f_n \xrightarrow{u} f$? Ans: Yes: wait.

(4). Convergency vs derivatives:

derivative vs. pointwise ? $\to$ X
— " — uniform ? $\to$ X.

Set $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$.

Now $|f_n(x)| = \frac{1}{\sqrt{n}}|\sin(nx)| \leq \frac{1}{\sqrt{n}}$ $\forall\, x\in\mathbb{R}$, $n\in\mathbb{N}$.

$\frac{1}{\sqrt{n}} \to 0$ as $n\to\infty$.

RHS is x free $\Rightarrow$ unif. conv. ✓

$\Rightarrow |f_n(x) - 0| \leq \frac{1}{\sqrt{n}} \longrightarrow 0$ as $n\to\infty$. $\forall\, x\in\mathbb{R}$.

$\Rightarrow \left\{\frac{\sin(nx)}{\sqrt{n}}\right\}$ is uniformly convergent (& converges to the zero fn) on $\mathbb{R}$. Set $f\equiv 0$ on $\mathbb{R}$.

In particular $\therefore \lim_{n\to\infty} f_n = f$ (uniform)

& $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall\, x\in\mathbb{R}$. [$\because$ unif. $\Rightarrow$ point.]

Now $f_n'(x) = \sqrt{n}\cos(nx)$. $\Longrightarrow \{f_n'\}$ does not even converge pointwise [For instance: $f_n'(0) = \sqrt{n}\ \forall n$].

But f' exists & $f' \equiv 0$. So, in particular,

$f_n \xrightarrow{u} f$ but $f_n' \overset{p}{\not\to} f'$ ($\Rightarrow f_n' \overset{u}{\not\to} f'$).

Some theory : But before that, we note again: Let $\{f_n\} \cup \{f\} \subseteq \mathcal{F}(S)$, $S \subseteq \mathbb{R}$.

Then $f_n \xrightarrow{u} f \iff M_n \longrightarrow 0$, where $M_n = d(f_n, f) = \sup_{x \in S} |f_n(x) - f(x)|$

$\Downarrow \; \not\Uparrow$

$f_n \xrightarrow{p} f$

$\iff$ For $\varepsilon > 0$, $\exists N \in \mathbb{N}$ ∋ $\forall m, n \geq N$

Cauchy criterion

$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in S$.

Proof.

["⇒" let $\varepsilon > 0$. As $f_n \xrightarrow{u} f$ or $M_n \to 0$, $\exists N$ ∋

$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall x \in S$ & $n \geq N$.

$\Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in S$ & $m, n \geq N$.

"⇐" ∵ $\mathbb{R}$ is complete, $\lim_{n\to\infty} f_n(x) := f(x)$ defines a fn f on S.

Claim: $f_n \xrightarrow{u} f$. But, for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ ∋

$|f_m(x) - f_n(x)| < \varepsilon \quad \forall m, n \geq N$. $x \in S$.

∴ For any $n \geq N$, taking $m \to \infty$, we have:

$|f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N$, $x \in S$.]

eg: $f_n(x) = x^n$, $|x| \leq c$, $0 < c < 1$.

Then $f_n \xrightarrow{p} 0$. Also $M_n = \sup_{|x| \leq c} |x|^n \leq c^n \longrightarrow 0 \Rightarrow f_n \xrightarrow{u} 0$ on $[-c, c]$.

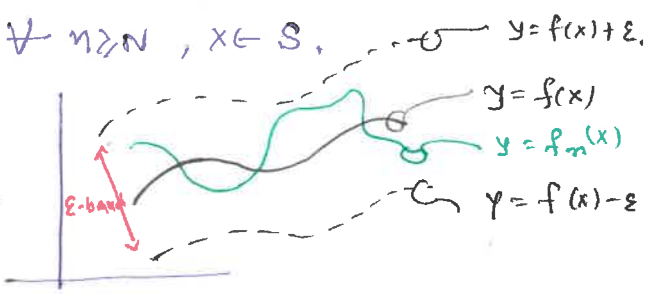
x-free.

~~Thm~~ (Continuity)

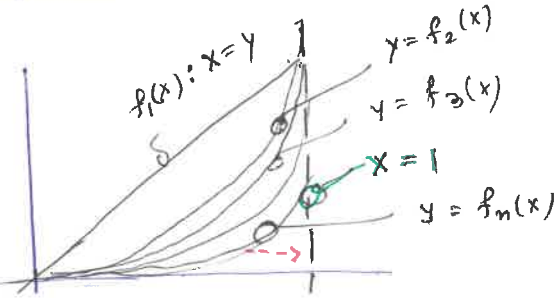
What's going on between pointwise $\overset{\not\longrightarrow}{\longleftarrow}$ uniform convergences?

For u.c: we need a fn $f : S \to \mathbb{R}$ s.t. : given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ ∋ $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall n \geq N$, $x \in S$,

i.e. $\{(x, f_n(x)) : x \in S\} \subseteq$ "the band ⁓⁓⁓" $\forall n \geq N$.



So Clearly, $f(x) = x^n$ fails this band: (at $x = 1$).



Continuity

Thm: Let $f_n, f : S \longrightarrow \mathbb{R}$, $n \geq 1$, and let $f_n \xrightarrow{u} f$. If $x_0 \in S$ & f_n is cont. at x_0 $\forall n$, then f is also cont. at x_0.

So, $\lim\limits_{x \to x_0} \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n(x)$.

Proof. ~~Let $x_0 \in S'$~~

Let $x_0 \in \text{int } S$

or $x_0 \in S \cap S'$.

or, just take $S = [a,b]$

[Limit point / or: just do it as in subspace metric. BTW: All results works as is for f_n's $f_n : \underbrace{(X, d)}_{\text{metric space}} \longrightarrow \mathbb{R}$]

Let $\varepsilon > 0$. $\exists$ $N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall x \in S$. —— (*)

$\because$ f_N ~~$\in$ Cont(S)~~ is cont. at x_0, $\exists$ $\delta > 0$ s.t. $|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}$ $\forall |x - x_0| < \delta$ & $x \in S$.

$\therefore$ $|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_N(x)|}_{< \varepsilon/3 \text{ by (*)}} + |f_N(x) - f_N(x_0)| + \underbrace{|f_N(x_0) - f(x_0)|}_{< \frac{\varepsilon}{3} \text{ by (*)}}$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ $\quad \forall |x - x_0| < \delta$.

$\Rightarrow$ f is cont. at x_0. ☒

\# Cor: If $f_n \in \text{Cont}[0,1] = C[0,1]$ & $f_n \xrightarrow{u} f \Rightarrow f \in \text{Cont}[0,1] = C[0,1]$.

Converse is not true: $S = (-1, 1)$; $f_n(x) = x^n$, $n \geq 1$. $f(x) := 0$ $\forall x \in S$.

Then $f_n \xrightarrow{p} f$. Here $f_n, f \in$ Cont(S).

However $f_n \not\xrightarrow{u} f$. (HW)

Integration

Thm: Let $f_n \in \mathcal{R}[a,b]$ $\forall n$ & $f_n \xrightarrow{u} f$. Then $f \in \mathcal{R}[a,b]$ &

$$\lim \int_a^b f_n = \int_a^b \lim f_n \left(= \int_a^b f \right). \longleftarrow \lim \int = \int \lim.$$

Proof: We know $f \in \mathcal{B}[a,b]$. Set $M_n = \sup\limits_{x \in [a,b]} |f_n(x) - f(x)|$. We know $M_n \to 0$.

Also: $f_n(x) - M_n \leq f(x) \leq f_n(x) + M_n$ $\forall x \in [a,b]$.

$$\Rightarrow \underbrace{\underline{\int} f_n - M_n(b-a) \leq \underline{\int} f}_{\text{get it separately}} \leq \underbrace{\overline{\int} f \leq \overline{\int} f_n + M_n(b-a)}. \quad —— (*)$$

But $f_n \in \mathcal{R}[a,b] \Rightarrow \underline{\int} f_n = \overline{\int} f_n \quad \forall n.$

$\therefore \quad 0 \leq \overline{\int} f - \underline{\int} f \leq 2 M_n (b-a) \longrightarrow 0$ as $n \to \infty$ $[\because M_n \to 0]$.

$\Rightarrow \overline{\int} f = \underline{\int} f \quad \Rightarrow \quad f \in \mathcal{R}[a,b].$

Again, $\circledast \Rightarrow \int_a^b f_n - M_n(b-a) \leq \int_a^b f \leq \int_a^b f_n + M_n(b-a).$

$\Rightarrow \left| \int_a^b f_n - \int_a^b f \right| \leq M_n (b-a).$

$\Rightarrow \lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \quad \boxtimes$

Derivative : But see eq4 in page 5.

So ~~many~~ many tailor ~~tailo~~ made assumptions : less useful compare to cont. & integ.

Thm: Suppose $f_n \in C^1([a,b])$, $n \geq 1$, and let

1) $\{f_n'\}$ is uniformly conv. &

2) $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a,b]$.

Then $\{f_n\}$ is u.c. & $\underbrace{\lim_{n\to\infty} f_n}_{\text{the uniform limit}} := f \in C^1([a,b])$. Moreover:

$$f_n' \xrightarrow{p} f'.$$

Proof: Define $g(x) := \lim f_n'(x)$. $x \in [a,b]$. $(\because f_n' \to g)$

$\because f_n' \in C[a,b] \Rightarrow g \in C[a,b]$. &

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n' = \int_{x_0}^{x} g \quad \forall x \in [a,b]. \longrightarrow \circledast$$

Set $\varphi(x) = \int_{x_0}^{x} g \quad \forall x \in [a,b]$.

$\therefore \varphi$ is diff. & $\varphi'(x) = g(x)$, $x \in [a,b]$. $\Rightarrow \varphi \in C^1[a,b]$

Now $\int_{x_0}^{x} f_n' = f_n(x) - f_n(x_0)$.

$\therefore \circledast \Rightarrow \lim_{n\to\infty} [f_n(x) - f_n(x_0)] = \int_{x_0}^{x} g = \varphi(x).$

$\Rightarrow f_n \xrightarrow{p} \varphi + c := f \quad [c := \lim_{n\to\infty} f_n(x_0)]$.

& $f_n' \xrightarrow{p} f' \quad [f' = \varphi' = g].$

$\boxtimes$

Most of the above results works in the setting of metric spaces!!

Not for exam.

Series of functions:

Consider $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}([a,b])$ — you can replace $[a,b]$ by $S \subseteq \mathbb{R}$ or even by a metric space (Complete is better).

Consider the formal sum 1) $\sum_1^\infty f_n$ ⟵ Series of f_n's.

2) $\sum_1^\infty f_n(x)$ ⟵ pointwise series of f_n's. $x \in [a,b]$.

Def: Given a series $\sum f_n$, set

$S_n \in \mathcal{F}([a,b])$ $\forall n \geq 1$ by

$$S_n(x) = \sum_{m=1}^{n} f_m(x) \quad \forall x \in [a,b].$$

n-th partial sum

We say that the series converges

1) uniformly if $\{S_n\}$ conv. unif. on $[a,b]$.

2) pointwise if ~~$\{S_n\}$ conv.~~ $\{S_n(x)\}$ conv. $\forall x \in [a,b]$.

3) absolutely if $\sum_{n=1}^{\infty} |f_n(x)|$ conv. pointwise on $[a,b]$.

useful in power series.

The following are easy:

Cont. ⟶ 1) If $f_n \in C[a,b]$ & $\sum f_n$ conv. unif. then $\sum f_n \in C[a,b]$.

Integ. ⟶ 2) If $f_n \in \mathcal{R}[a,b]$ & $\sum f_n$ conv. unif. then $\sum f_n \in \mathcal{R}[a,b]$

$$\& \quad \int_a^b \sum f_n = \sum \int_a^b f_n .$$

3) If $\sum f_n$ conv. unif, then $f_n \to 0$ unif.

[Suppose $f = \sum f_n$. Then $\forall x \in [a,b]$ & n large,

$$|f_n(x)| = |S_n(x) - S_{n-1}(x)| \leq |S_n(x) - f(x)| + |S_{n-1}(x) - f(x)|$$

$\because S_n \xrightarrow{u} f \Rightarrow |f_n(x)| < \varepsilon \quad \forall x \in [a,b]$ & $n \geq N(\varepsilon)$ where $\varepsilon > 0$ is given.

$$\Rightarrow f_n \xrightarrow{u} 0 . \ ]$$

~~Defn:~~ Just like series of real no.s, we have:

Thm: Suppose $|f_n(x)| \le M_n$ $\forall n$, $x \in [a,b]$. If $\sum M_n < \infty$

Weierstrass' M-test

then $\sum f_n$ is unif. conv as well as absolutely convergent.

Proof: Follow the real series case.

eg: 1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is u.c. on $\mathbb{R}$: $\left|\frac{\sin nx}{n^2}\right| \le \frac{1}{n^2}$ & $\sum \frac{1}{n^2} < \infty$.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ is u.c. on $[p, \infty)$, $\forall p > 0$.

$$\because \frac{x}{1+n^2x^2} \le \frac{x}{n^2x^2} \le \frac{1}{n^2 x} \le \frac{1}{n^2 p}$$

& $\sum \frac{1}{n^2 p} < \infty$.

~~Obs: If $\sum f_n$ is absolutely conv then $\sum f_n$ is u.c.~~

~~Proof: $\because |S_n(x)| \le \sum_{m=1}^{n} |f_m(x)|$~~

Remark: $\sum |f_n|$ is unif conv. $\Rightarrow$ $\sum f_n$ is unif. conv.

Imp. eg: $f_n(x) := x^n$. Then $\displaystyle\sum_{n=0}^{\infty} f_n = (1-x)^{-1}$ $x \in (-1,1)$.

$= \sum_{n=0}^{\infty} x^n$

geometric series.

Easy to prove

Take the above example & proceed to "power series".

— 0 —