Remark: For (3), you need a natural result:

HW

$f \in \mathcal{R}(C)$ [in the sense of $\circledast$ in page 18] $\Longleftrightarrow$

$$\lim_{\|P\| \to 0} \sum_{i \in \Lambda(P)} f(\eta_i) \, \| \gamma(t_i) - \gamma(t_{i-1}) \| \quad \text{exists.} \qquad \eta_i \in C_i, \text{ a tag.}$$

Moreover, in this case:

$$\int_C f \, ds = \lim_{\|P\| \to 0} \left[\sum_{i \in \Lambda(P)} f(\eta_i) \, \| \gamma(t_i) - \gamma(t_{i-1}) \| \right]$$

— x —

Remark: Often, ~~we~~ the definition of line integral appears as:

$$\int_C f = \int_a^b \underbrace{f(\gamma(t))}_{\text{Cont. fn}} \, \| \gamma'(t) \| \, dt. \longrightarrow \textcircled{T}$$

path/trace

$\int_C f$ or $\int_\gamma f$

γ : a piecewise smooth ~~f~~ curve.

Remark: To keep things in order: we must prove that the RHS of $\textcircled{T}$ is independent of choice of γ (depends only on C):

i.e. parametrizations.

Consider the following re-parametrization:

$$\begin{array}{ccc} & \overset{\tilde{\gamma}}{\nearrow} & \mathbb{R}^n \\ & & \uparrow \gamma \\ [c,d] & \xrightarrow{\varphi} & [a,b] \end{array}$$

φ: $\uparrow$, diff, onto.

$\therefore \tilde{\gamma} = \gamma \circ \varphi \Rightarrow \tilde{\gamma}'(t) = \gamma'(\varphi(t)) \, \varphi'(t)$.

$$\therefore \int_c^d f(\tilde{\gamma}(s)) \, \| \tilde{\gamma}'(s) \| \, ds = \int_c^d f(\gamma(\varphi(s))) \, \| \gamma'(\varphi(s)) \| \, \underbrace{\varphi'(s)}_{\neq 0.\ \text{why?}} \, ds.$$

$$= \int_{\varphi(c) = a}^{\varphi(d) = b} f(\gamma(t)) \, \| \gamma'(t) \| \, dt. \qquad [\varphi(s) \to t].$$

$\therefore \textcircled{T}$ is independent of choice of γ. $\boxtimes$

Due to the above observation, we write

$$\int_C f \quad \text{instead of} \quad \int_\gamma .$$

line integral of f over C. (via trace of a piecewise smooth curve).

But we often write $\int_\gamma$ with the same meaning.

Facts: Let C be a curve (with some parametrization of piecewise smooth curve $\gamma : [a,b] \to \mathbb{R}^n$, $\operatorname{ran}\gamma = C$), $f, g \in \operatorname{Cont}(C)$. If $r \in \mathbb{R}$, then we have:

works even for rectifiable γ.

(1) $$\int_\gamma f + rg = \int_C f + r\int_C g .$$

(2) if $f \geq g \Rightarrow \int_C f \geq \int_C g$.

(3) If $a < c < b$, $\gamma : [a,b] \to \mathbb{R}^n$ be s.t. $\operatorname{ran}\gamma = C$ & $\gamma_1 = \gamma|_{[a,c]}$, $\gamma_2 = \gamma|_{[c,b]}$, then

$$\int_C f = \int_{\underbrace{C_1}_{= \operatorname{ran}\gamma_1}} f + \int_{\underbrace{C_2}_{= \operatorname{ran}\gamma_2}} f .$$

(4) $$\left| \int_C f \right| \leq \int_C |f| .$$

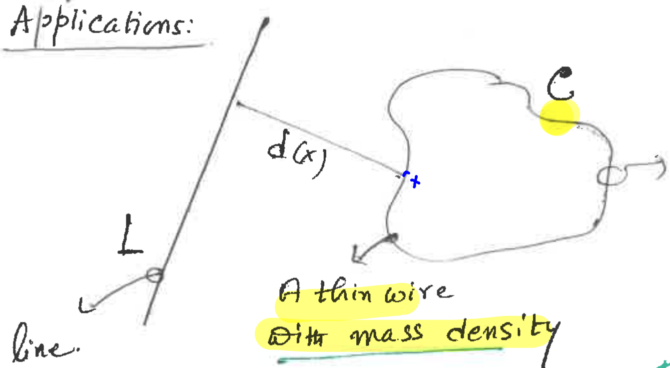
~~HW (Easy)~~.

(5) (Continuity) Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, \tilde{t} - t < \delta$,

$$\int_{C = \gamma[t, \tilde{t}]} |f| < \varepsilon .$$

— HW —.

Physics Applications:



C^1 curve, parametrized by one-to-one path $\gamma : [a,b] \to \mathbb{R}^n$ $(n \geq 1)$.

A thin wire with mass density at $x \in C$ as $g(x)$. ← Cont/integrable.

Then the total mass of the wire $:= \int_C g\, ds$. $(:= M)$.

The center of mass $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is given by:

$$\bar{x}_j = \frac{1}{M} \int_C x_j\, g\, ds, \qquad \forall j = 1, \ldots, n.$$

Also, if $L \subseteq \mathbb{R}^n$ is a line, & $d(x)$ = distance from $x \in C$ to L,

then the moment of inertia of C about L is:

$$I_L := \int_C d^2 g\, ds.$$

— x —

Unit speed

Thm: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a smooth curve. Then $\exists$ a reparametrization $\tilde{\gamma}$ of γ s.t. $\|\tilde{\gamma}'(s)\| = 1$ $\forall s$.

Proof: Fix $t_0 \in [a,b]$ & define $s : [a,b] \to \mathbb{R}$ by

$$s(t) = \int_{t_0}^{t} \|\gamma'\|. \qquad \forall t \in [a,b].$$

diff & in particular, cont. fn.

Set $\tilde{I} = \operatorname{ran} s$. $(\Rightarrow \tilde{I} = [\tilde{a}, \tilde{b}])$.

Now FTC $\Rightarrow$ $s'(t) = \|\gamma'(t)\| \neq 0$ $\forall t \in [a,b]$.

$\therefore$ $s : [a,b] \to \tilde{I} = [\tilde{a}, \tilde{b}]$ is smooth, bijective.

$\Rightarrow$ $s^{-1} : \tilde{I} \to [a,b]$ diff + bijective + smooth. [by inverse fn. thm].

Call $\varphi := S^{-1} : \tilde{I} \longrightarrow [a,b]$.

So, we have:

$$\begin{tikzcd} & \mathbb{R}^n \\ \tilde{I} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\varphi"'] & {[a,b]} \arrow[u, "\gamma"'] \end{tikzcd}$$

i.e. we consider $\tilde{\gamma} := \gamma \circ \varphi$.

$\therefore \; \tilde{\gamma}'(t) = \gamma'(\varphi(t))\, \varphi'(t).$

$$= \gamma'(\varphi(t)) \; \frac{1}{\underbrace{S'(\varphi(t))}_{= \|\gamma'(\varphi(t))\|}}.$$

$\because \; \varphi'(t) = \frac{d}{dt}\left(S^{-1}(t)\right)$

$$= \frac{1}{S'(S^{-1}(t))}$$

$$= \frac{1}{S'(\varphi(t))}.$$

$\therefore \; \|\tilde{\gamma}'(t)\| = 1$.

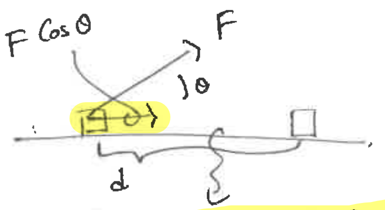
///

This is useful result, But, the solution $\tilde{\gamma}$ is not so explicit from computational point of views.

— x —

Applications:

$\boxed{\text{Work done:}}$



Linear or movement in $\mathbb{R}$.

Work done $:= \underbrace{(|F| \cos\theta)}_{\text{force}} \; \underbrace{|d|}_{\text{displacement}} \; (= \vec{F} \cdot \vec{d})$.

The classical result.

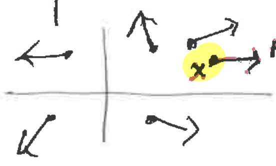
How to make it work ~~in~~ for movements (displacements) in $\underbrace{\mathbb{R}^2}_{\text{plane}}$ or $\underbrace{\mathbb{R}^3}_{\text{space}}$?

Ans: Consider a vector field (call it force field)

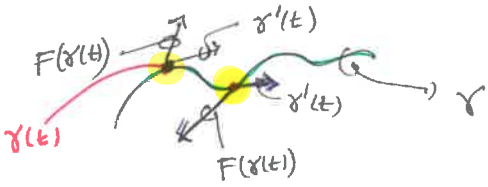
$$F : O_n \longrightarrow \mathbb{R}^n \quad (n = 2 \text{ or } 3).$$
$$O_n \subseteq \mathbb{R}^n$$

///

$\therefore \; \forall \, x \in O_n$, $F(x)$ represent a vector (force) at x. eg:



Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a curve.



In physics:
$\gamma = \mathbf{r}$.
$\therefore \gamma' = d\mathbf{r}$

Naturally, W = work done by the force on the particle moving along γ $(= \gamma)$ is:

$$W := \int_C F \cdot d\mathbf{r}$$

$d\mathbf{r}$: $\gamma'(t)\,dt$.

$\cdot$: dot product.

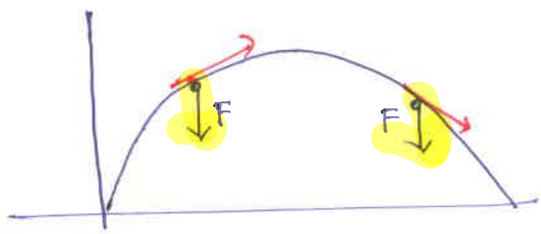
C: $C = \text{ran}\,\gamma$.

F: $F(\gamma(t))$

or,
$$\boxed{W = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt}$$

Take it as defn. or a fact. More later/soon!!

Think: A mass "m" projectile near the earth surface:



$F = \langle 0, -mg \rangle$

eg: Find the work done by the force field $F(x,y,z) = \langle xz, xy, zy \rangle$ along the curve C: $x = t^2$, $y = -t^3$, $z = t^4$, $0 \le t \le 1$.

Ans: Here $\gamma(t) = \langle t^2, -t^3, t^4 \rangle \Rightarrow \gamma'(t) = \langle 2t, -3t^2, 4t^3 \rangle$.

$$\therefore W = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^1 \langle t^6, -t^5, -t^7 \rangle \cdot \langle 2t, -3t^2, 4t^3 \rangle dt$$

$$= \int_0^1 (2t^7 + 3t^7 - 4t^{10})\,dt = \cdots = \frac{23}{88}.$$

Remark: Similar consideration applies to flow of a fluid along a curve.

Check with your physics lectures.

Now FTC for line integrals:

Recall: $\int_a^b f' = f(b) - f(a)$

$f \in C^1(I)$, $I \supseteq [a,b]$ open. (or even little general: f' exists & $f' \in \mathcal{R}[a,b]$).

We use the above for a similar result for line integrals:

A scalar field $f : \mathcal{O}_n \longrightarrow \mathbb{R}$, $\mathcal{O}_n \subseteq \mathbb{R}^n$ is given, Assume f is diff.

Look at ∇f, the gradient vector field & assume that ∇f is cont. (i.e., $\frac{\partial f}{\partial x_i} \in C(\mathcal{O}_n)$ $\forall$ $1 \le i \le n$).

i.e., we assume $f \in C^1(\mathcal{O}_n)$.

Assume that $P, Q \in \mathcal{O}_n$ & $\mathbf{r} = \gamma$ be a C^1- ~~path~~ curve s.t. ran $\gamma \subseteq \mathcal{O}_n$ & γ joins P & Q. ~~Define~~ Then

$$\int_C \nabla f \cdot d\mathbf{r} := \int f(\gamma(t)) \, \|\gamma'(t)\| \, dt .$$

Wait for the proof.

Line integrals of vector fields.

[Definition / Explanation: (Similar to "work done" part):

Let $F : \mathcal{O}_n \longrightarrow \mathbb{R}^n$ be a vector field. Let $\mathbf{r} : [a,b] \longrightarrow \mathcal{O}_n$ be a curve [Note: we have a different notation: $\mathbf{r}$ instead of γ]

Consider a partition $P : a = t_0 < t_1 < \cdots < t_m = b$.

$\mathbf{r}_i := \mathbf{r}(t_i)$ $\quad 1 \le i \le m$.

$\Delta \mathbf{r}_i := \mathbf{r}_{i+1} - \mathbf{r}_i \in \mathbb{R}^n$, ← kind of arc length.

Define $R(F, P) := \sum_{i=1}^{m} F(r_i) \cdot \Delta r_i$. $C = \operatorname{ran} r$.

vectors.

dot product.

Finally, define $\int_C F \cdot dr := \lim_{\|P\| \to 0} R(F, P)$.

(if exists).

Fact: Just like scalar fields, in this case as well:

$$\boxed{\underbrace{\int_C F \cdot dr}_{=W} = \int_a^b F(r(t)) \cdot r'(t)\, dt .}$$

line integral of vector fields.

— × —

Back to the gradient vector (FT of line integrals)

Thm: Let $f: O_n \to \mathbb{R}$ be a C^1-scalar field, r be a piecewise C^1-curve in O_n joining A & B. Then

$$\int_C \nabla f \cdot dr = f(B) - f(A). \qquad [C = \text{path of } r].$$

Proof: Here $\int_C \nabla f \cdot dr = \int_a^b \nabla f(r(t)) \cdot r'(t)\, dt$

[$r: [a,b] \to O_n$ a parametrization of the path C.] .

[Observe: $\dfrac{d}{dt}\big(\underbrace{f(r(t))}_{\text{A fn. of } t \text{: right?}}\big) \overset{\text{chain rule.}}{=} \dfrac{\partial f}{\partial x_1}\dfrac{dx_1}{dt} + \cdots + \dfrac{\partial f}{\partial x_n}\dfrac{dx_n}{dt}$

$f(r(t)) := f(x_1(t), x_2(t), \ldots, x_n(t))$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}.$$

$$\Rightarrow \frac{d}{dt}\left(f(\gamma(t))\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}$$

$$= \nabla f \cdot \underbrace{\left\langle \frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right\rangle}_{=\gamma'(t).} \Big].$$

The key point.

$$\therefore \int_C \nabla f \cdot d\gamma = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

$$= \int_a^b \frac{d}{dt}\left(f(\gamma(t))\right) dt.$$

$$= f(\gamma(b)) - f(\gamma(a)).$$

$$= f(B) - f(A). \qquad \left[\because \gamma(b) = B,\ \gamma(a) = A.\right]$$

$\boxed{//}$

Of course: $f(B) = f(A)$ if $A = B$ ($\Leftrightarrow \gamma$ is closed curve.) $\rightarrow \int_C \nabla f \cdot d\gamma = 0$

Cor: In the setting of above theorem, for any piecewise smooth curve C connecting to A itself (i.e. $\gamma(a) = \gamma(b) = A$),

$$\int_C \nabla f \cdot d\gamma = 0.$$

— x —

Hence, so far we have the following line integrals:

Let $f : O_n \to \mathbb{R}$, $F : O_n \to \mathbb{R}^n$ be scalar field & vector field, respectively. Assume that f & F are continuous. Let $\gamma\,(=\gamma) : [a,b] \to \mathbb{R}^n$ be a piecewise smooth curve s.t. $\operatorname{ran}\gamma = C \subseteq O_n$. Then

Line integral of a Scalar field → 1) $\displaystyle\int_C f\, ds = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\, dt.$

Line integral of a vector field. → 2) $\displaystyle\int_C F \cdot d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt.$