Statistics C7Q2

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Let's define

$$I_1 := \left(\bar{x} - z_{0.025} \frac{s}{\sqrt{n}} , \bar{x} + z_{0.025} \frac{s}{\sqrt{n}} \right)$$

$$I_2 := \left(\bar{x} - z_{0.01} \frac{s}{\sqrt{n}} , \bar{x} + z_{0.04} \frac{s}{\sqrt{n}} \right)$$

1> To prove : I_2 is a 95% CI. (ie $\mathbb{P}(\mu \in I_2) = 0.95$)

Note $\{ \mu \in I_2 \}$ is same as the event -

$\{ -z_{0.01} < z < z_{0.04} \}$, where z is the z-score of $\bar{x}$.

Thus, $\mathbb{P}(\mu \in I_2)$

$$= \mathbb{P}(\{ -z_{0.01} < z < z_{0.04} \})$$

$$= 1 - \mathbb{P}(\{ z \geq z_{0.04} \}) - \mathbb{P}(\{ z \leq -z_{0.01} \}) \quad \cdots (*)$$

Now by definition,

$$z_\alpha = -\text{qnorm}(1-\alpha)$$

$$\Rightarrow \mathbb{P}(z > z_\alpha) = \alpha$$

So, $\mathbb{P}(\{ z \geq z_{0.04} \}) = 0.04$

$\mathbb{P}(\{ z \leq -z_{0.01} \}) = 0.01$

Thus, From $(*)$

$$\mathbb{P}(\mu \in I_2) = 1 - 0.04 - 0.01$$
$$= 0.95.$$

$\Rightarrow I_2$ is a 95% CI.

2> As both I_1 and I_2 are 95% CI's, we shall prefer the one with smaller length.

length of I_1 : $2 \times z_{0.025} \times \frac{s}{\sqrt{n}} \approx 3.91 \times \frac{s}{\sqrt{n}}$

length of I_2 : $z_{0.04} + z_{0.01} \times \frac{s}{\sqrt{n}} \approx 4.07 \times \frac{s}{\sqrt{n}}$

(Refer to the R-code attached)

Thus, $|I_1| < |I_2|$.

So, we shall prefer I_1.

Note : Among all the 95% CI, I_1 is the interval with shortest length. For an interval to be 95% CI it should swipe out 0.95 area under the graph of pdf of normal.

As the hight of the graph is decreasing from center towards the periphery to hold the same area the interval length will increase.

(this can proved mathematically).