Surface integrals

Recall: Curve C is a Cont (or C^1 or smooth or piecewise smooth etc) fn. (parametrization) $\gamma : [a,b] \longrightarrow \mathbb{R}^n$.

[Then we went on talking about $\int_C$ or $\int_\gamma$.]

Similarly, we want the notion of surfaces S ($\gamma : \mathcal{O}_2 \rightarrow \mathbb{R}^3$ Cont/smooth) & then want to talk about $\int_S$!!

Def: A bdd subset $\mathcal{R} \subseteq \mathbb{R}^n$ is said to be a region if $\mathcal{R}$ is open & $\mathcal{R}$ has an area ($\Leftrightarrow \partial\mathcal{R}$ is of content zero).

Mostly, ~~for us~~ we will deal with $\mathcal{R} \subseteq \mathbb{R}^2$.

Def: Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a region. A C^1 fn $\gamma : \mathcal{R} \longrightarrow \mathbb{R}^2$ is called a parametrized surface (with parameter space $\mathcal{R}$) if :

(i) the component fn's of γ have bounded partial derivatives.

(ii) $\gamma : \mathcal{R} \longrightarrow \mathbb{R}^2$ is one-to-one.

(iii) $\forall\ (u,v) \in \mathcal{R}$,

$$\gamma_u \underset{\substack{\uparrow \\ \text{Cross product}}}{\times} \gamma_v \Big|_{(u,v)} := \gamma_u(u,v) \times \gamma_v(u,v) \neq 0.$$

Def: A subset $S \subseteq \mathbb{R}^3$ is called a surface if $S = \operatorname{ran}\gamma$ for some parametrized surface $\gamma : \mathcal{R} \longrightarrow \mathbb{R}^3$.

Try to find similarity between surfaces & smooth curves.

Review on planes & normals :

~~Same explanation~~

Given two vectors $\vec{P} = \langle a_1, a_2, a_3 \rangle$, $\vec{Q} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$,

$\qquad\qquad \overset{\shortparallel}{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}} \qquad \overset{\shortparallel}{b_1\vec{i} + b_2\vec{j} + b_3\vec{k}}$

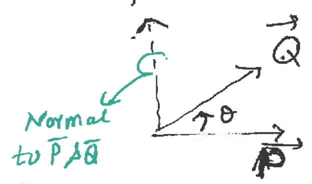
we define the cross product $\vec{P}\times\vec{Q}$ by :

$$\vec{P}\times\vec{Q} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\vec{P}$ & $\vec{Q}$ are linearly independent $\iff$ $\vec{P}\times\vec{Q} \neq \vec{0}$.

$\|\vec{P}\times\vec{Q}\| = \|\vec{P}\| \, \|\vec{Q}\| \sin\theta$.

The length/magnitude.



$\leftarrow$ in $\mathbb{R}^3$.

~~# A subset $P \subseteq \mathbb{R}^n$ is a plane if $\exists\, \gamma : S$ ...~~

~~# Eqn of planes (in $\mathbb{R}^3$)~~

~~Given A plane is determined by a point P_0 in the plane & a vector N orthogonal to the plane.~~

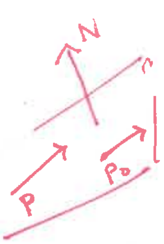
N $\to$ call it normal vector.] DO NOT CALL IT orthogonal anymore.

&

Def: For a point/vector $\vec{P_0} = \langle x_0, y_0, z_0 \rangle$ & a non-zero vector $\vec{N} = \langle a, b, c \rangle \neq \vec{0}$ in $\mathbb{R}^3$, the plane through $\vec{P_0}$ that is "normal" to $\vec{N}$ is the set $\mathcal{P} = \{ \vec{P_0} + \vec{P} \;:\; \vec{P}\cdot\vec{N} = 0,\ \vec{P} \text{ in } \mathbb{R}^3 \}$.

dot product.

($\vec{P}\cdot\vec{N} = 0 \iff \vec{P}$ is orthogonal to $\vec{N}$).



Note: Let $\vec{P_0} + \vec{Q_1}$, $\vec{P_0} + \vec{Q_2} \in \mathcal{P}$. ($\mathcal{P}$ a plane as above).

~~$\vec{Q_1}\cdot\vec{N} = \vec{Q_2}\cdot\vec{N} = 0$.~~

~~$\Rightarrow \vec{Q_1} = \vec{Q}$~~

$\hookrightarrow$ Suppose $\vec{Q_1}$ & $\vec{Q_2}$ are linearly independent.

$\Rightarrow \vec{Q_1}\times\vec{Q_2} \neq 0$. Also $(\vec{Q_1}\times\vec{Q_2})\cdot\vec{Q_i} = 0 \quad (i=1,2)$.

$\Rightarrow$ $\vec{Q}_1 \times \vec{Q}_2$ is orthogonal to both $\vec{Q}_1$ & $\vec{Q}_2$. [A general fact.]

~~Recall further,~~ $\Rightarrow$ $\vec{Q}_1 \times \vec{Q}_2 = \underset{\text{a scalar}}{c}\ \vec{N}$

[$\because$ $\dim \mathbb{R}^3 = 3$, & $\vec{Q}_1, \vec{Q}_2$ are lin. indep. $\Rightarrow$ $\vec{N}$ & $\vec{Q}_1 \times \vec{Q}_2$ must be linearly dep.]

i.e., $\vec{N} = \underset{\text{a scalar.}}{c} \left(\vec{Q}_1 \times \vec{Q}_2 \right)$

[OR, simply, $\{\vec{Q}_1, \vec{Q}_2, \vec{Q}_1 \times \vec{Q}_2\}$ will be a basis of $\mathbb{R}^3$.]

$\because$ $\{\vec{Q}_1, \vec{Q}_2$ & $\vec{Q}_1 \times \vec{Q}_2\}$ is a basis of $\mathbb{R}^3$, &

$\{ \vec{P}_0 + s_1 \vec{Q}_1 + s_2 \vec{Q}_2 \ : \ s_1, s_2 \in \mathbb{R} \} \subseteq \mathcal{P}$, AND

$N \notin$ LHS of the above, it follows that:

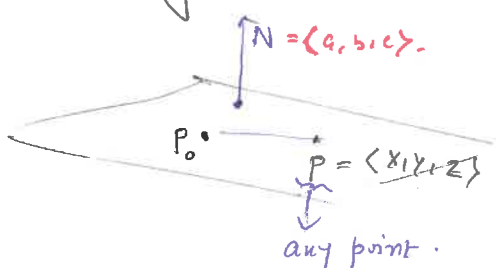
$$\boxed{\mathcal{P} = \{ \vec{P}_0 + s_1 \vec{Q}_1 + s_2 \vec{Q}_2 : \ s_1, s_2 \in \mathbb{R} \}.} \quad \text{——— (1)}$$

Representation of a plane. where $\vec{Q}_1, \vec{Q}_2 \perp \vec{N}$, & linearly independent.

As far as eqn. of a plane is concerned; we do as follows:

Given a (normal vector) $\vec{N} = \langle a, b, c \rangle$ & point $P_0 = \langle x_0, y_0, z_0 \rangle$, the eqn of the plane through P_0 & with $\vec{N}$ as a normal vector is given by:

$$\vec{N} \cdot \vec{P_0 P} = 0$$



$$\Rightarrow \ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0. \quad \text{——— (2)}$$

Cartesian form of a plane.

Also, (1) can be expressed as:

$$\mathbf{r}(s_1, s_2) = \vec{P}_0 + s_1 \vec{Q}_1 + s_2 \vec{Q}_2 . \quad \text{——— (3)}$$

Any $\vec{Q}_1 \not\parallel \vec{Q}_2$ will do.

In fact: ③ Can be used to define a plane!

\# Let $S \subseteq \mathbb{R}^n$ be a line or a plane (trace).

A vector $\vec{N} \in \mathbb{R}^n$ is said to be normal to S if

"Normal" defn. or Classification. $\longrightarrow$ $$\vec{N} \cdot (x-y) = 0 \quad \forall \; x, y \in S.$$

Remark: (1) let $S \subseteq \mathbb{R}^n$ be a line/plane. Then

S is a subspace (vector) of $\mathbb{R}^n \Leftrightarrow 0 \in S$.

(2) $S^{\perp} := \{ \vec{N} \in \mathbb{R}^n : \vec{N} \text{ is normal to } S \}$

is a subspace of $\mathbb{R}^n$.

Note: usually, we assume $\vec{N} \neq 0$ to avoid triviality.

(3) If S is a line, then $S^{\perp}$ is a plane.
($\subseteq \mathbb{R}^3$)

— HW —.

(4) If S is a plane in $\mathbb{R}^3$, then $S^{\perp}$ is a line.

— HW —.

Return to Surface:

Again, recall that given a region $\mathcal{R} \subseteq \mathbb{R}^2$ (open + $\partial \mathcal{R}$ is of zero content) a C^1 fn $\mathbf{r} : \mathcal{R} \longrightarrow \mathbb{R}^3$ is a parametrized surface with parameter space $\mathcal{R}$ if:

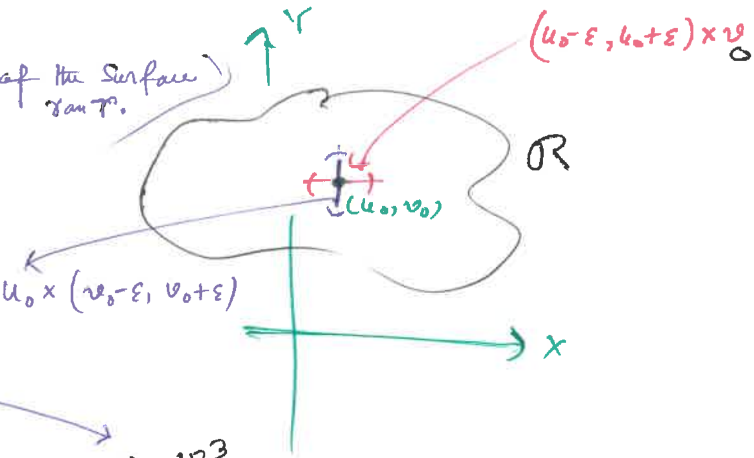
Often, this or $\mathbf{r}$ is also known as "regular param."

i) Components of $\mathbf{r}$ have bdd 1^{st} order partial derivatives. ← Often, we won't need/use this.

ii) $\mathbf{r}$ is injective. ← We will evaluate fn over ran $\mathbf{r}$.

iii) $\mathbf{r}_u \times \mathbf{r}_v \Big|_{(u,v)} \neq 0 \quad \forall (u,v) \in \mathcal{R}$.

(Def: $\mathbf{r}$ is said to be a prametrization of the surface ran $\mathbf{r}$.)



Note: on (iii): Let $(u_0, v_0) \in \mathcal{R}$.

$\because \mathcal{R}$ is open, $\exists\, \varepsilon > 0$ s.t.

$(u_0-\varepsilon, u_0+\varepsilon) \times v_0 \subseteq \mathcal{R}$

& $u_0 \times (v_0-\varepsilon, v_0+\varepsilon) \subseteq \mathcal{R}$.

So,

$$\begin{array}{ccccc} (-\varepsilon, \varepsilon) & \longrightarrow & \mathcal{R} & \longrightarrow & \mathbb{R}^3 \\ t & \longmapsto & (u_0+t, v_0) & \longmapsto & \mathbf{r}(u_0+t, v_0) \end{array}$$

defines a smooth curve in the surface $S = \operatorname{ran} \mathbf{r}$. Call it γ.

$\therefore \gamma(t) = \mathbf{r}(u_0+t, v_0) \quad \forall\, t \in (-\varepsilon, \varepsilon)$.

Clearly, by the chain rule:

$$\gamma'(t) = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial v}{\partial t}.$$

$$= \frac{\partial \mathbf{r}}{\partial u} \cdot 1 + \frac{\partial \mathbf{r}}{\partial v} \cdot 0$$

$$= \frac{\partial \mathbf{r}}{\partial u}.$$

$\Rightarrow \gamma'(0) = \frac{\partial \mathbf{r}}{\partial u}\Big|_{(u_0,v_0)}$ ← A tangent vector of a curve in S at $\mathbf{r}(u_0,v_0)$.

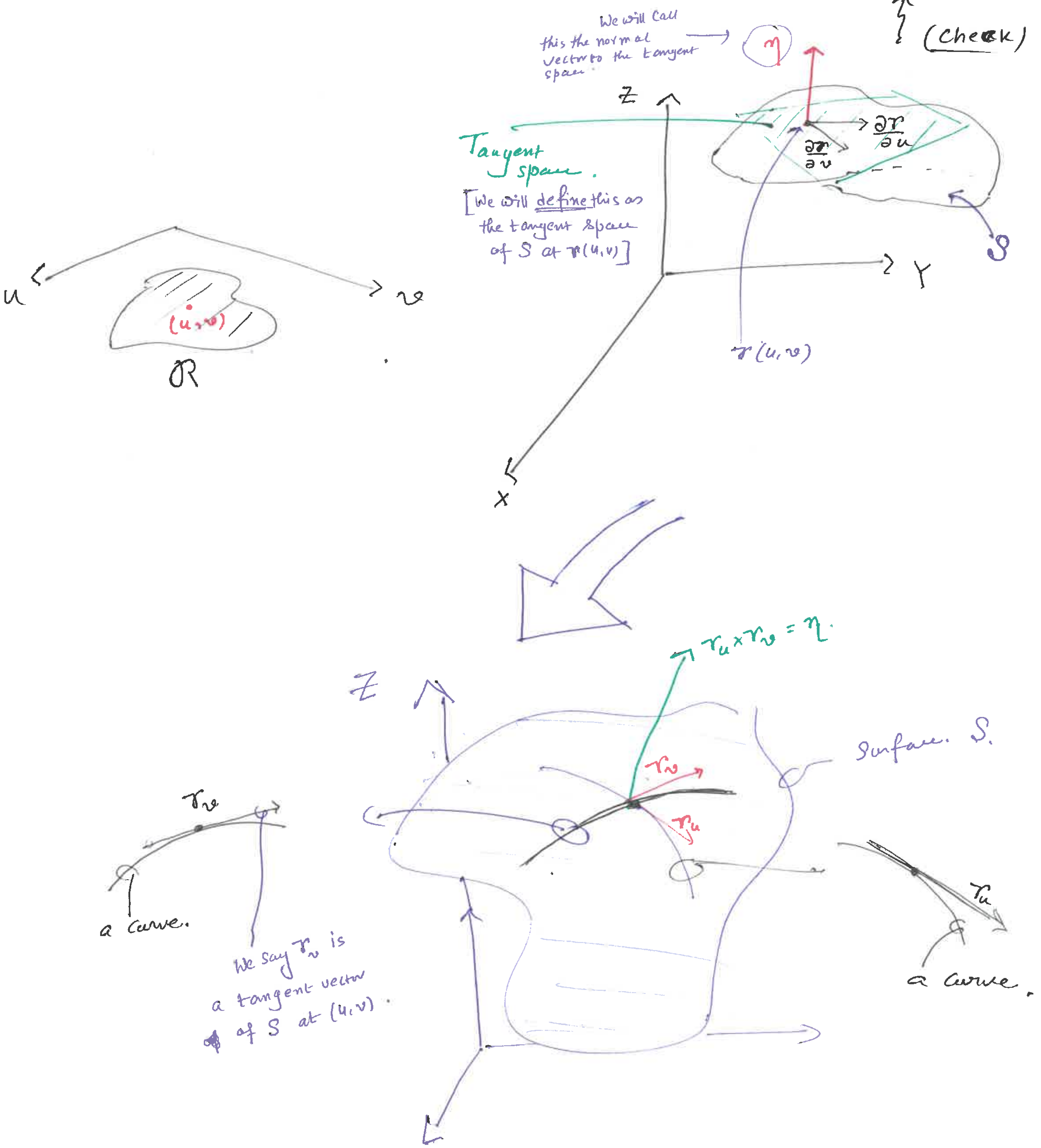
$\text{lly.} \quad \frac{\partial \mathbf{r}}{\partial v}\Big|_{(u_0,v_0)}$ ← A tangent vector of S at $\mathbf{r}(u_0,v_0)$

So, $\eta(u_0,v_0) := \mathbf{r}_u \times \mathbf{r}_v \Big|_{(u_0,v_0)} \neq 0$ is a normal vector to the pair of curves in S at $\mathbf{r}(u_0,v_0)$.

$\vec{\eta}(u_0,v_0)$

$\because \eta \neq 0$, (iii) $\Rightarrow$ the pair of curves in S have a normal vector $\eta(r(u,v))$ $\forall$ $r(u,v) \in S$ $\forall$ $(u,v) \in R$ (or just simply, $\forall$ $(u,v) \in R$).

$\because$ r is in $C'(R)$, it also follows that η is CONTINUOUS!! (check)



e.g.:

1) $\mathcal{O}_2 \subseteq \mathbb{R}$ be open, $f : \mathcal{O}_2 \to \mathbb{R}$ be C^1 fn. Consider the graph fn. $z = f(x,y)$, $(x,y) \in \mathcal{O}_2$.

i.e., graph of f = graph(f) = $\{(x, y, f(x,y)) : (x,y) \in \mathcal{O}_2\}$.

Then graph(f) is a parametrized surface. Indeed:

Consider the parametrization:

$$\mathbf{r}(u,v) := (u, v, f(u,v)), \quad (u,v) \in \mathcal{O}_2.$$

or simply $(x,y) \in \mathcal{O}_2$. But for the sake of computation / notation, we use (u,v).

Clearly, # $\mathbf{r}$ is C^1 ($\because$ $(u,v) \mapsto u, v, f(u,v)$ are C^1 fn.)

bdd 1^{st} order derivative must be assumed for f.

$\mathbf{r}$ is injective: trivial.

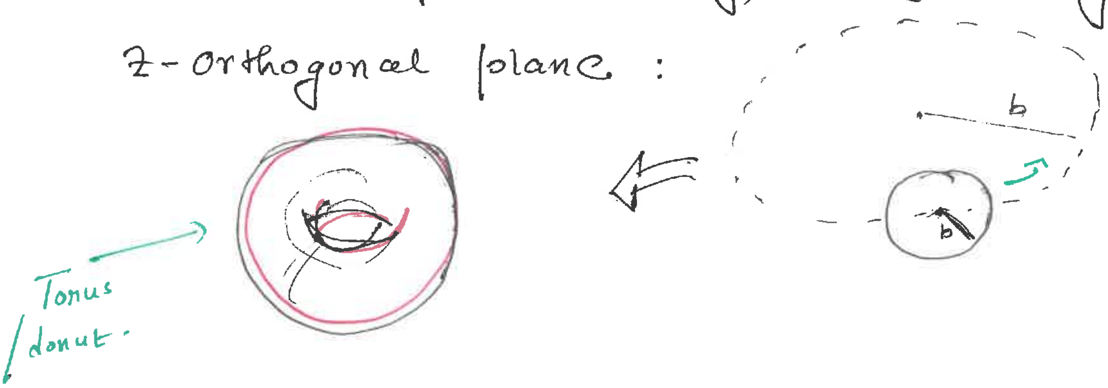
$\mathbf{r}_u = \left(1, 0, \frac{\partial f}{\partial u}\right)$, $\mathbf{r}_v = \left(0, 1, \frac{\partial f}{\partial v}\right)$.

$$\therefore \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1).$$

$$\Rightarrow \mathbf{r}_u \wedge \mathbf{r}_v \Big|_{(u,v)} \neq 0 \quad \forall (u,v) \in \mathcal{O}_2.$$

$\Rightarrow$ $\mathbf{r}$ is a parametrization of graph(f).

2) The torus: rotating a circle of radius (say) b about a circle of radius (say) a ($> b$) lying in an z-orthogonal plane:



We parametrize the above torus as follows:

$$r(u,v) = \big((a+b\cos u)\cos v,\ (a+b\cos u)\sin v,\ b\sin u \big)$$

$0 \le u, v \le 2\pi$

$\updownarrow$

Clearly: ~~this~~ is given by $\boxed{(x-b)^2 + z^2 = a^2}$.

$(0 < a < b)$

Also, it may be seen from: In xz-plane, a Circle of radius "a" Centered at "$(b,0)$" is given by

$$\left.\begin{aligned} x &= a\cos\theta + b \\ z &= a\sin\theta \end{aligned}\right\} \qquad \theta \in [0, 2\pi).$$

Then rotate the xz-plane around z-axis by

$$\left.\begin{aligned} x &= (a\cos\theta + b)\cos\varphi \\ y &= (a\cos\theta + b)\sin\varphi \\ z &= a\sin\theta \end{aligned}\right\} \qquad 0 \le \theta, \varphi \le 2\pi.$$

Anyway: r is injective.

$$r_u = \big(-b\sin u \cdot \cos v,\ -b\sin u\sin v,\ b\cos u \big)$$

$$r_v = \big(-(a+b\cos u)\sin v,\ (a+b\cos u)\cos v,\ 0 \big)$$

Then $r_u \times r_v = \{-b(a+b\cos u)\} \cdot \big(\cos u\cos v,\ \sin u\sin v,\ \cos u \big)$

$0 < u < \pi$
$0 \le v < 2\pi$

$\neq 0$ —— (HW).

$\therefore$ r is a parametrization of the torus.