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Change of variables thm

Thm: Let $\mathcal{O}_n \subseteq \mathbb{R}^n$ be open, $\varphi : \mathcal{O}_n \to \mathbb{R}^n$ $\left(= (\varphi_1, \dots, \varphi_n)\right.$, $\left.\varphi_i : \mathcal{O}_n \to \mathbb{R}\right)$ be an injective & C^1-fn & let $J_\varphi(x)$ $\left(= \left[\frac{\partial \varphi_i}{\partial x_j}(x) \right] \right)$ is invertible $\forall x \in \mathcal{O}_n$. ($\Leftrightarrow \det J_\varphi(x) \neq 0 \ \forall x \in \mathcal{O}_n$.)

Let $\Omega \subseteq \mathcal{O}_n$ be s.t. $\Omega \cup \partial\Omega \subseteq \mathcal{O}_n$ & Ω has an area. ($\Leftrightarrow \partial\Omega$ is c.z).

If $f \in \mathcal{R}(\varphi(\Omega))$, then

$$\int_{\varphi(\Omega)} f = \int_{\Omega} f \circ \varphi \times |\det J_\varphi| .$$

$\in \mathcal{R}(\Omega)$ is assured.

$$\left[\text{i.e., } \int_{\varphi(\Omega)} f(u)\, du = \int_{\Omega} f(\varphi(x)) \, |\det J_\varphi(x)| \, dx . \right]$$

$dv(u)$ $\qquad$ $dv(x)$

$e_1 = (1,0)$

$\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$?this is nu me ti cal value

(x_1, y_1), (x_2, y_2)

$\det \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \text{Area}$.

Proof: See Spivak, Page-67.

Remark:

1) There are many finer (or non-finer) variants of the above thm.

2) "$\varphi : \mathcal{O}_n \to \mathbb{R}^n$, a C^1-fn, with $J_\varphi(a)$ is invertible at $\boxed{\text{some point}}$ $a \in \mathcal{O}_n$" $\to$ Also a strong statement. This is related with the inverse function theorem. We will talk about it soon.

3) Think $n=1$ Case: $\varphi : (a,b) \to \mathbb{R}$ is diff. & $\varphi'(x) \neq 0$ $\forall x \in (a,b)$.
$\Rightarrow \varphi$ is injective.

[(*) actually, $(a-\varepsilon, b+\varepsilon)$, so that $[a,b] \subseteq (a-\varepsilon, b+\varepsilon)$.]

4) Think $n=1$ Case: $\varphi : (a,b) \to \mathbb{R}$ be C^1-fn. ($\Rightarrow \varphi(a,b)$ is also an interval) & let $f \in \mathcal{R}(\varphi(a,b))$. Then ~~and in (a,b~~
$\forall\, c, d \in (a,b)$ & $c < d$,

$$\int_c^d f(\varphi(x))\, \varphi'(x)\, dx = \int_{\varphi(c)}^{\varphi(d)} f(x)\, dx .$$

[(*) $c = a$, $d = b$ is okay.]

More Simply: if $\varphi : [a,b] \to \mathbb{R}$ is C^1-fn & $f \in \mathcal{R}(\varphi[a,b])$, then

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi)(x)\, \varphi'(x)\, dx \leftarrow \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b-\varepsilon} f(\varphi(x))\varphi'(x)\, dx$$

If, in addition φ is injective, then:

$$\int_{\varphi([a,b])} f = \int_{[a,b]} (f\circ\varphi)(x)\,|\varphi'(x)|\,dx .$$

(5) "Above" "injectivity" takes care of $\int_a^b =$ vs $\int_b^a =$, AS, IN $\mathbb{R}^n$, we do NOT HAVE $\int_a^b$ or $\int_b^a$. We just have

$$\int_\Omega \quad !!$$

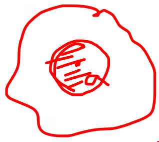
$a \rightleftarrows b$

$\therefore$ Injectivity of φ in the thm. is justified. $\square$

(6) "φ is 1-1" vs "J_φ invertible".

The latter $\Rightarrow$ φ is locally 1-1. But NOT as a whole / globally.

—— Will see in inverse fn. thm.



φ

$J_\varphi(a)$ is inv.

— ∗ —

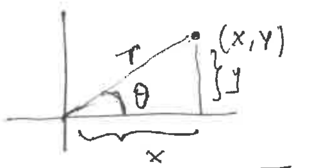
Examples:

(1) To consider polar coordinate:

Let $x = r\cos\theta$, $y = r\sin\theta$. $r,\theta\in\mathbb{R}$.

So $\varphi : \mathbb{R}^2_{(r,\theta)} \longrightarrow \mathbb{R}^2_{(x,y)}$ defined by

$$\varphi(r,\theta) = \big(x(r,\theta),\, y(r,\theta)\big)$$



$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$.

So $J_\varphi = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$

The Jacobian of φ at $(x,y) = (r\cos\theta, r\sin\theta)$.

$$\Rightarrow \quad J_\varphi = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

$$\Rightarrow \quad \det J_\varphi = r\cos^2\theta + r\sin^2\theta = r.$$

i.e, $\det\left(J_\varphi(\overset{r,\theta}{\cancel{x,y}})\right) = r \neq 0 \quad \forall (x,y) \neq (0,0)$.
or $r \neq 0$.

But, of course, φ is NOT injective (even if $(x,y) \neq (0,0)$).
($\because$ $\theta \to \theta + 2n\pi$ will lead non-inj.)

We do the following (redefine φ as follows):

Given $(x,y) \neq (0,0)$, define $r = \sqrt{x^2+y^2}$. Pick
$!\ \theta \in [0, 2\pi)$ $\ni$ $(x,y) = (r\cos\theta, r\sin\theta)$.

Set $O_2 = \{(r,\theta) : r > 0,\ 0 < \theta < 2\pi\}$.
$= (0,\infty) \times (0, 2\pi)$
$\subseteq \mathbb{R}^2$.

Define $\varphi : O_2 \longrightarrow \mathbb{R}^2$ by
$\varphi(r,\theta) = (\underbrace{r\cos\theta}_{=x}, \underbrace{r\sin\theta}_{=y})$

$\therefore$ φ is C^1 & $|J_\varphi(r,\theta)| = r \neq 0 \quad \forall (r,\theta) \in O_2$.

Now clearly, φ is also 1-1.

So given $0 < r_1 < r_2$ & $0 < \theta_1 < \theta_2 < 2\pi$,
Set $\Omega = [r_1, r_2] \times [\theta_1, \theta_2]$. (or take open Intervals)
If $f \in R(\varphi(\Omega))$, then

$$\int_{\varphi(\Omega)} f = \int_{\Omega} f \circ \varphi \times |\det J_\varphi|$$

$(r,\theta) \longrightarrow (x,y)$

$\theta_2 \qquad \varphi(\theta_2)$

i.e.,
$$\int_{\varphi(\Omega)} f(x,y) \underbrace{dx\,dy}_{dV(x,y)} = \int_{\Omega} f(\varphi(r,\theta))\, r \underbrace{dr\,d\theta}_{dV(r,\theta)}$$

$$= \int_{\Omega} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

$$= \int_{\theta_1}^{\theta_2} \left(\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr \right) d\theta.$$

If $f \in C(\varphi(\Omega))$ & then by Fubini

(2) $$\int_{x^2+y^2<1} e^{-(x^2+y^2)} = ?$$

Sol: $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$, $\quad \{(x,y) : x^2+y^2<1\} = \varphi\left(\underbrace{[0,1] \times [0,2\pi]}\right)$

1 ?
2π ?

$$\therefore \quad I = \int_{[0,1]\times[0,2\pi]} e^{-r^2} \times r \times dr\, d\theta.$$

$$= \int_0^{2\pi} ? \left(\int_0^1 ? \left(e^{-r^2} r\, dr \right) \right) d\theta$$

$$= \int_0^{2\pi} d\theta \times \int_0^1 e^{-r^2} \times \frac{1}{2} \times d(r^2)$$

$$= 2\pi \times \frac{-1}{2} \times \left[e^{-r^2} \right]_0^1$$

$$= \pi(1 - e^{-1}).$$

Why $r = 0$?
$\theta = 2\pi$?

Remark: Note that $\varphi : (0,\infty) \times (0, 2\pi) \longrightarrow \mathbb{R}^2$, defined by $\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$ has a continuous extension $\tilde{\varphi} : [0,\infty) \times [0, 2\pi] \longrightarrow \mathbb{R}^2$.

Then use the same limiting argument, as in $n=1$ case, one can make sense of the above $\int_0^{2\pi}\int_0^1$.

[Also, use, for instance, $\{(x,y) : x \in [0,1], y = 0\}$ is of content zero.].

3) Compute the area of $\Omega = \{(x,y) \in \mathbb{R}^2 : x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1\}$.

Sol: Define $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $\varphi(s,t) = (s\cos^3 t, s\sin^3 t)$.

$$\therefore \Omega = \varphi\left([0,1] \times [0, 2\pi]\right)$$

So $\varphi\big|_{[0,1]\times[0,2\pi]}$ is cont. & 1-1 in the interior of $[0,1] \times [0, 2\pi]$.

Also,

$$J_\varphi = \begin{bmatrix} \cos^3 t & -3s\cos^2 t\sin t \\ \sin^3 t & 3s\sin^2 t \cdot \cos t \end{bmatrix}$$

$$\therefore \det\left(J_\varphi(s,t)\right) = 3s \times \left[\sin^2 t \cdot \cos^4 t + \sin^4 t \cdot \cos^2 t\right]$$

$$= 3s \cdot \sin^2 t \cdot \cos^2 t.$$

$$\neq 0 \quad \forall\ s \in (0,1],\ t \in (0, 2\pi).$$

$$\therefore \text{Area}(\Omega) = \text{Area}\left(\varphi\left([0,1]\times[0,2\pi]\right)\right)$$

$$= \int_{\varphi([0,1]\times[0,2\pi])} 1.$$

$$= \int_{[0,1]\times[0,2\pi]} |J_\varphi(s,t)| \, ds \cdot dt$$

$$= \int_0^{2\pi} \int_0^1 3s \, \sin^2 t \times \cos^2 t \, dt \, ds.$$

$$= 3\int_0^{2\pi} \left(\int_0^1 s\, ds \right) \sin^2 t \cdot \cos^2 t \cdot dt$$

$$= \frac{3}{2} \times 1 \times \frac{1}{4} \int_0^{2\pi} \sin^2(2t)\, dt.$$

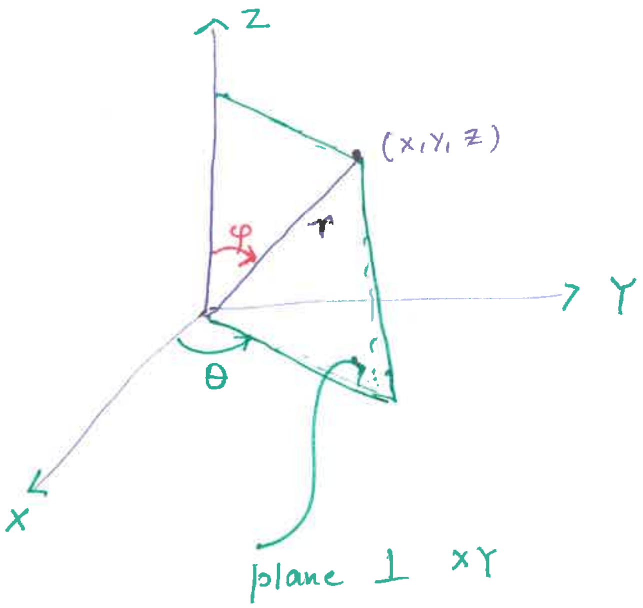
$$= \frac{3}{8} \times \int_0^{2\pi} \frac{1-\cos 4t}{2}\, dt = \frac{3}{8} \times \pi.$$ Ans

(4) Spherical Coordinate:

$\forall\ (x,y,z) \in \mathbb{R}^3 \setminus \underbrace{\{(0,0,\alpha) : \alpha \in \mathbb{R}\}}_{z\text{-axis}}$, Consider the triple (r, φ, θ) as:

$$r = \sqrt{x^2+y^2+z^2}\ ; \quad 0 \leq \theta < 2\pi \ \ \& \ \ 0 < \varphi < \pi$$

s.t. $(x,y,z) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$.



$(r, \varphi, \theta) \longrightarrow$ Spherical Coordinate of (x,y,z).

Define $\mathcal{O}_3 := \{(r, \varphi, \theta) : r > 0,\ 0 < \varphi < \pi,\ 0 < \theta < 2\pi\}$.

& $\Phi : \mathcal{O}_3 \longrightarrow \mathbb{R}^3$ by

$$\Phi(r, \varphi, \theta) = (r \sin\varphi \cos\theta,\ r \sin\varphi \sin\theta,\ r\cos\varphi)$$

$$\forall\ (r, \varphi, \theta) \in \mathcal{O}_3 .$$

$\therefore$ Φ is C^1 & 1-1 on $\mathcal{O}_3$. [& Cont. extension to the boundary.].

Now,

$$J_\Phi = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{bmatrix}$$

$$\Rightarrow \det\left(J_\Phi(r, \varphi, \theta)\right) = r^2 \sin\varphi \neq 0$$

$\because$ $r > 0$ & $\varphi \neq 0, \pi$.

$\therefore$ For $0 < r_1 < r_2$, $0 < \varphi_1 < \varphi_2 < \pi$ & $0 < \theta_1 < \theta_2 < 2\pi$,

& for $f \in \mathrm{Cont}\Big(\Phi\big(\underbrace{[r_1, r_2] \times [\varphi_1, \varphi_2] \times [\theta_1, \theta_2]}_{:= \Omega}\big)\Big)$,

We have:

$$\int_{\Phi(\Omega)} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_{\theta_1}^{\theta_2} \left\{ \int_{\varphi_1}^{\varphi_2} \left(\int_{r_1}^{r_2} f(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi) \times r^2 \sin\varphi\, dr \right) d\varphi \right\} d\theta .$$

Change of variables & Fubini as f is cont.

PTO.

5 In particular:

If $\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$

sphere of radius a centered at (0, 0, 0).

Then $\text{vol}(\Omega) = \int_{\Omega} 1 \underbrace{dx\, dy\, dz}_{dv}$.

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin\varphi \; \underbrace{dr \times d\varphi \times d\theta}_{\text{mind the ordering}}.$$

mind the ordering.

$= \ldots$

$\ldots$

$= \frac{4}{3} \pi a^3$.

Your known & favourite formula.