

Goal: Compute area of a surface S .

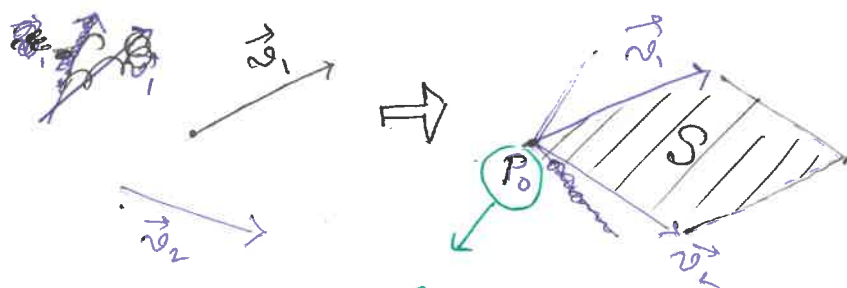
↓
i.e., Area of $S = \text{ran } \mathbf{r}$, where \mathbf{r} is a parametrization of S .

Remark:

~~Let's first observe~~ If $\vec{OP} = \langle a_1, b_1, c_1 \rangle$ & $\vec{OQ} = \langle a_2, b_2, c_2 \rangle$,
then for $\vec{OP}_0 = \langle a_0, b_0, c_0 \rangle$ fixed, we define

$$S := \{ \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 : 0 < t_1, t_2 < 1 \}.$$

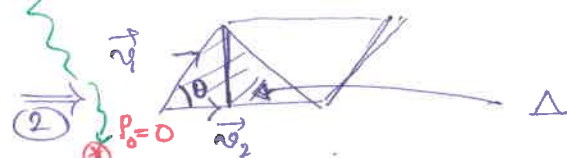
↓
open parallelogram based at \vec{v}_0 (or P_0) bounded by the
sides \vec{v}_1 & \vec{v}_2 . (in \mathbb{R}^3). ← corner



Fact: $\text{Area of } S = \| \vec{v}_1 \times \vec{v}_2 \|$ ⊗ H.W.

Hint: Just compute
 $\vec{v}_1 \times \vec{v}_2$
then $\| \vec{v}_1 \times \vec{v}_2 \|$.

Method ①



$$\begin{aligned} \text{Area } S &= 2 \times \text{area of } \Delta \\ &= 2 \times \left[\frac{1}{2} \times \| \vec{v}_2 \| \times \| \vec{v}_1 \| \sin \theta \right] \end{aligned}$$

⊗ → Translate P_0 to O ; Area will remain unchanged.

Observe: S is a surface with $S = \text{ran } \mathbf{r}$, where

$$\mathbf{r} : (0,1) \times (0,1) \rightarrow \mathbb{R}^3 \text{ given by}$$

$$\mathbf{r}(t_1, t_2) = \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2.$$

$$(t_1, t_2) \in (0,1) \times (0,1).$$

$\therefore \text{Area of } \text{ran } \mathbf{r} = \| \vec{v}_1 \times \vec{v}_2 \|$

Don't worry about
(0,1) instead of
[0,1]. We want R to be
open.

A basic step:

Recall that $z = ax + by + c$ represents a plane. Moreover, if $(x, y) \in B^2 \leftarrow$ an open box in \mathbb{R}^2 , then the graph of $f(x, y) = ax + by + c$ given by $S = \{(x, y, ax + by + c) : (x, y) \in B^2\}$ is a surface.
Here $\vec{r}(x, y) = (x, y, ax + by + c) \quad \forall (x, y) \in B^2$.

FACT:

$$\text{Area } S = \sqrt{1 + a^2 + b^2} \times \text{Area of } (B^2). \quad \leftarrow S \text{ as above}$$

Proof: Recall that $\text{Area } \tilde{S} = \|\vec{v}_1 \times \vec{v}_2\|$, where

$$\tilde{S} = \left\{ \underbrace{\vec{v}_0}_{\downarrow} + t_1 \vec{v}_1 + t_2 \vec{v}_2 : (t_1, t_2) \in (0, 1) \times (0, 1) \right\}$$

 Recall: As far as area is concerned, $\vec{v}_0 = \vec{0}$.
 (We represent S as \tilde{S} as above).
 $\downarrow \downarrow \downarrow \downarrow$

$$\text{Let } B^2 = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2).$$

$$\text{Set } \vec{v}_0 = \langle \alpha_1, \alpha_2, a\alpha_1 + b\alpha_2 + c \rangle.$$

$$\text{Also, set } \vec{v}_1 = \langle 0, \beta_2 - \alpha_2, b(\beta_2 - \alpha_2) \rangle$$

$$\vee \vec{v}_2 = \langle \beta_1 - \alpha_1, 0, a(\beta_1 - \alpha_1) \rangle.$$

$$\text{Then } \{ \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 : (t_1, t_2) \in (0, 1) \times (0, 1) \} = S.$$

$$\therefore \text{Area } S = \|\vec{v}_1 \times \vec{v}_2\|.$$

$$\text{Now } \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \beta_2 - \alpha_2 & b(\beta_2 - \alpha_2) \\ \beta_1 - \alpha_1 & 0 & a(\beta_1 - \alpha_1) \end{vmatrix}$$

$$= \langle a(\beta_2 - \alpha_2)(\beta_1 - \alpha_1), b(\beta_2 - \alpha_2)(\beta_1 - \alpha_1), -(\beta_2 - \alpha_2)(\beta_1 - \alpha_1) \rangle$$

$$\therefore \|\vec{v}_1 \times \vec{v}_2\| = \sqrt{a^2 + b^2 + 1} \times (\beta_2 - \alpha_2)(\beta_1 - \alpha_1)$$

$$= \sqrt{a^2 + b^2 + 1} \times \text{Area}(B^2).$$

□

Remark: The factor " $\sqrt{1+a^2+b^2}$ " is interesting. Indeed,
for $z = ax + by + c$ ($= f(x, y)$),
 $a = f_x$, $b = f_y$.

$$\therefore \boxed{\sqrt{1+a^2+b^2} = \sqrt{1+f_x^2+f_y^2}} \quad !! \quad \square$$

With the above area formula, we can now talk about the idea of defining/obtaining Surface area:

Let S be the surface parametrized by $r: R \rightarrow \mathbb{R}^3$, where
i.e. $r(u, v) \in \mathbb{R}^3$, $(u, v) \in R := [a_1, b_1] \times [a_2, b_2]$
 $\text{ran } r = S$.

for simplicity, let's
take $R = B^2$.

Consider a partition of B^2 : say P

$$\text{So } P = \{B_x^2 : x \in \Lambda(P)\}$$

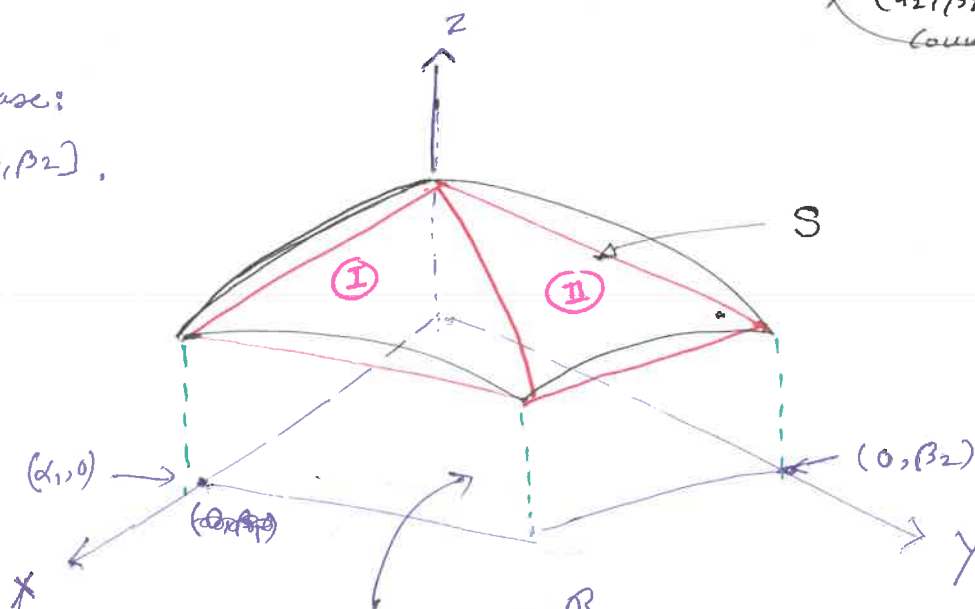
Don't worry about
"one" side open (a_1, b_1)
 (a_2, b_2) . It doesn't
count anything.

Pause

So

A simplest case:

$$R = [0, \alpha_1] \times [0, \beta_2]$$



The bottom = R
 ~~$[0, \alpha_1] \times [0, \beta_2]$~~

$$R = [0, \alpha_1] \times [0, \beta_2]$$

Then (BAD) area of $S \approx \triangle I + \triangle II$

polygonal
approximation.

Sum of two polygonal surfaces.

We know the area of this by our
previous computation.

THE IDEA IS to take this consideration & proceed to limit approach.

known route!!

So, in the B^2 -setting, pick a partition of B^2 , say P .

$$\therefore B^2 = \bigcup_{\alpha \in N(P)} B_\alpha^2.$$

← May be, at this point, consider B^2 to be closed.

$\forall \alpha \in N(P)$, pick (arbitrary) $x_\alpha = (u_\alpha, v_\alpha) \in \text{int}(B_\alpha^2)$

Consider the tangent plane to the surface S at $r(x_\alpha)$:
i.e., the linear/tangent plane approximation as follows:

~~$r(x)$~~ $r^{(x)}: B_\alpha^2 \rightarrow \mathbb{R}^3$ defined by:

$$r^{(x)}(x) = \underbrace{\left((D r)(x_\alpha) \right)}_{\text{Total derivative}} (x - x_\alpha) + r(x_\alpha).$$

plane or
parallelogram

$$= [r_u \ r_v](x_\alpha) = [r_u(x_\alpha) \ r_v(x_\alpha)].$$

\therefore We have $z = r^{(x)}(x) = r_u(x_\alpha)x + r_v(x_\alpha)y + c$ (for some $c \in \mathbb{R}$)

$\&$ area of the above parallelogram (By FACT in P-44)

$$= \|r_u(x_\alpha) \times r_v(x_\alpha)\| (\text{Area } B_\alpha^2)$$

\therefore Area of S under the partition P

$$= \sum_{\alpha \in N(P)} \|r_u(x_\alpha) \times r_v(x_\alpha)\| \text{Vol}(B_\alpha^2).$$

This is the Riemann sum of the fr.

$$x \mapsto \|r_u(x) \times r_v(x)\|.$$

This is integrable as r is a C^1 -fn.

So, $\lim_{\|P\| \rightarrow 0} (\text{RHS}) = \text{Area of } S.$

\Rightarrow we may define:

$$\text{Area of } S := \int_{\mathcal{R}} \|r_u \times r_v\| dA.$$

We KNOW how to compute this.

The Riemann integration in 2-variables w.r.to "area".

Def:

$$\text{Area}(S) := \int_R \|r_u \times r_v\| \, dA.$$

Where $R \subseteq \mathbb{R}^2$ is a region & $S = \text{ran } r$ for some parametrization $r: R \rightarrow \mathbb{R}^3$ of S .

Fact: Suppose $f \in C^1(R)$, $R \subseteq \mathbb{R}^2$ a region. Recall

$r: R \rightarrow \mathbb{R}^3$, defined by

$r(u, v) = (u, v, f(u, v))$ is a parametrization of $\text{graph}(f)$. Also, recall that (see Page 33):

$$r_u \times r_v = (-f_u, -f_v, 1).$$

$$\Rightarrow \|r_u \times r_v\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}.$$

$$\therefore \text{Area}(\text{graph}(f)) = \int_R \underbrace{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}_{\|r_u \times r_v\|} \, dx \, dy.$$

eg:

Consider truncated cylinder: $r: R \rightarrow \mathbb{R}^3$, where

$$R = \{(x, y) : 0 \leq x \leq \pi/2, 0 \leq y \leq 1\}$$

$$r(x, y) = (\cos x, \sin x, y).$$

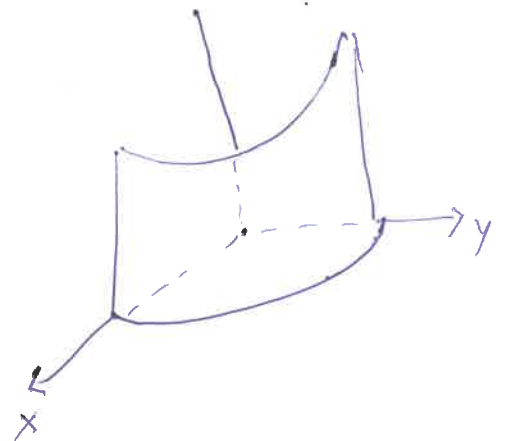
Here $r_x = (-\sin x, \cos x, 0)$

$$r_y = (0, 0, 1)$$

$$\therefore r_x \times r_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos x, \sin x, 0)$$

$$\therefore \|r_x \times r_y\| = \sqrt{\cos^2 x + \sin^2 x} = 1.$$

$$\therefore \text{Area} = \int_R 1 \, ds = \text{Area}(R) = \pi/2.$$



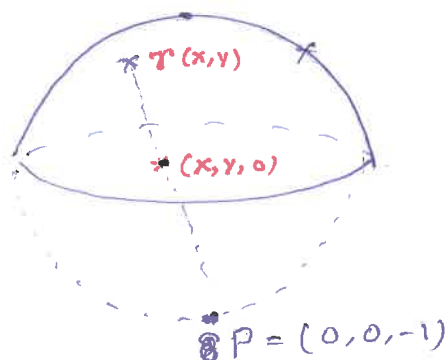
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eg: (Area of a hemisphere)

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

→ The upper hemisphere.

Apply stereographic projection:



i.e. $R = B_1(0, 0) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

$T(x, y)$ = the point of intersection of the upper hemisphere & the line joining P & $(x, y, 0)$.

$$\text{Then } T(x, y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right)$$

$$\forall (x, y) \in R.$$

← NOT good for computation!! HW.

Here, we apply spherical coordinates (for ease of computation):
We define $R = \{(u, v) \in \mathbb{R}^2 : 0 \leq u < \pi, 0 \leq v < 2\pi\}$

$$T(u, v) = (\sin u \cos v, \sin u \sin v, \cos u) \quad \forall (u, v) \in R.$$

→ HW: Check this from surface of revolution point of view.

$$\text{Then } \|T_u \times T_v\| \stackrel{?(HW)}{=} |\sin u| = \sin u.$$

$$\begin{aligned} \therefore \text{Area}(S) &= \int_R \sin u \, du \, dv \\ &= \int_0^\pi \int_0^{2\pi} \sin u \, du \, dv \\ &= 2\pi \int_0^\pi \sin u \, du \\ &= 2\pi \times [-\cos u]_0^\pi = 2\pi \times 2 = 4\pi \end{aligned}$$

The upper hemisphere

Surface integrals of scalar fields:

Let S be a surface in \mathbb{R}^3 , $S \subset \underset{\text{open}}{O_3} \subseteq \mathbb{R}^3$, & let $f \in \text{Cont}(O_3) \approx \mathbb{R}(O_3)$. Then, as in the area computation, (see the bottom of Page 46).

We can prove:

$$\lim_{\|P\| \rightarrow 0} \sum_{\alpha \in \Lambda(P)} f(r(x_\alpha)) \|r_u(x_\alpha) \times r_v(x_\alpha)\| \text{vol}(B_\alpha^2).$$

$$= \int_R f \circ r \|r_u \times r_v\| dA.$$

Where $r: R \rightarrow \mathbb{R}^3$ is a parametrization of the surface S .

Hence, we define:

Def: Given a surface S , $f \in \text{Cont}(O_3)$, where $O_3 \supseteq S$ is an open subset of \mathbb{R}^3 , the Surface integral of f over the surface S is defined by:

$$\int_S f ds := \int_R f \circ r \|r_u \times r_v\| dA$$

Q: independent of the choice of r ?

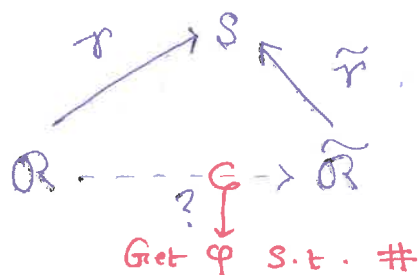
where $r: R \rightarrow \mathbb{R}^3$ is a parametrization of S .

Remark: "The RHS of the above": Suppose a curved metal plate/thin film lies along the surface S with density f_u . $f: S \rightarrow \mathbb{R}$. Then the RHS of the above definition, i.e., $\int_S f ds =$ mass of the metal plate.!!

Independence of parametrizations: (Sketch).

Suppose $r: R \rightarrow \mathbb{R}^3$ & $\tilde{r}: \tilde{R} \rightarrow \mathbb{R}^3$ be two parametrizations of S . ~~Both parametrizations.~~

$\therefore S = \text{ran } r = \text{ran } \tilde{r}$. So, we have the following.



i.e: $r(u,v) = \tilde{r}(\varphi(u,v))$.

For $(u,v) \in R$, define $\varphi(u,v) = (\tilde{u}, \tilde{v}) \in \tilde{R}$ as

$$\varphi = \tilde{r}^{-1} \circ r \quad \leftarrow \text{Remember, } r \text{ \& } \tilde{r} \text{ are one-to-one maps.}$$

$\& \text{ran } r = \text{ran } \tilde{r}$.

$\therefore \varphi: R \rightarrow \tilde{R}$ is a bijection !!

P.T. φ is C^1 -fn. & $(J\varphi)(u,v)$ invertible $\forall (u,v) \in R$.

Then apply the change of variables theorem/ formula along with chain rule \Rightarrow

$$\int_R f(r(u,v)) \|r_u \times r_v\| = \int_{\tilde{R}} f(\tilde{r}(\tilde{u}, \tilde{v})) \|\tilde{r}_{\tilde{u}} \times \tilde{r}_{\tilde{v}}\|$$

