Thm: (Gauss / divergence thm) [Relates triple integrals $\longleftrightarrow$ Surface integrals.]

Let $\mathcal{D} \subseteq \mathbb{R}^3$ be a "solid" $\cdot \ni \cdot$ $\partial \mathcal{D}$ (the boundary of $\mathcal{D}$) is an (~~piecewise~~) oriented surface, $F = \langle P, Q, R \rangle$ be a C^1 v.f. on an open set containing $\mathcal{D} \cup \partial\mathcal{D} . (= \overline{\mathcal{D}})$. Then

$$\underbrace{\int_{\partial \mathcal{D}} \vec{F} \cdot d\vec{S}}_{\text{The Flux.}} = \int_{\mathcal{D}} \operatorname{div} \vec{F} \qquad \operatorname{div} \vec{F} = P_x + Q_y + R_z$$

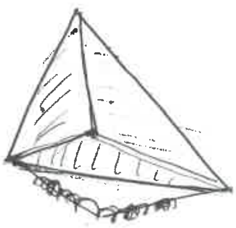
Note: (1) $\operatorname{div} \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.

i.e $\boxed{\operatorname{div} \vec{F} = \underbrace{\nabla}_{\text{gradient}} \cdot \vec{F}}$ dot product.

Also, note that $\displaystyle\int_{\mathcal{D}} \operatorname{div} \vec{F} = \underbrace{\int_{\mathcal{D}} \operatorname{div} \vec{F} \, \underbrace{dV}_{\text{volume.}}}_{\text{a triple integral}} = \iiint_{\mathcal{D}} \nabla \cdot \vec{F} \, dx\,dy\,dz$

$$\therefore \int_{\partial \mathcal{D}} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{D}} \operatorname{div} \vec{F} \, dV = \iiint_{\mathcal{D}} \nabla \cdot \vec{F} \, dx\,dy\,dz$$

(2) $\partial \mathcal{D}$ could be piecewise parametrized, i.e., piecewise smooth. For instance:

$\mathcal{D} =$  $\partial \mathcal{D} = \bigcup_{i=1}^{4} T_i$, where T_i are triangular face.

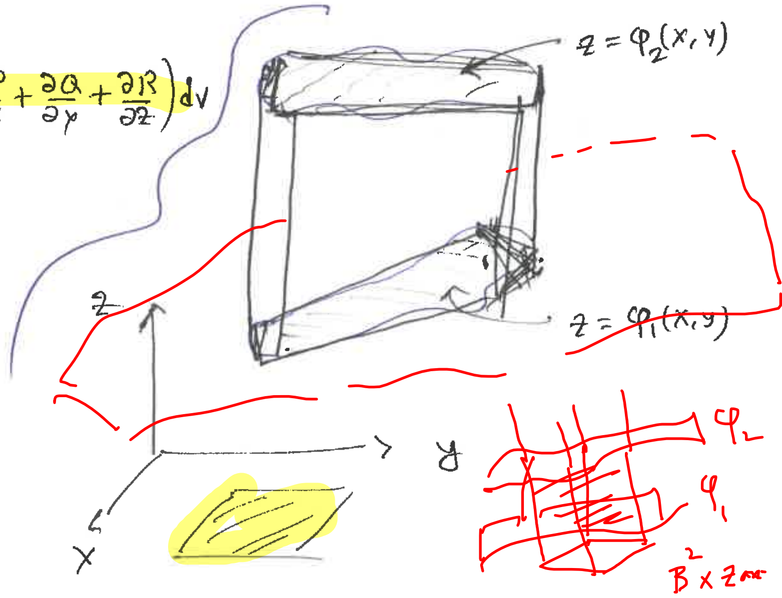
(3) In the above case, just consider parametrizations for each T_i $(i = 1, 2, 3, 4)$

Proof: Again, the general version is "beyond our scope". Here we consider "elementary solid" like elementary region in Green's thm proof.

$\mathcal{D} := \{ (x,y,z) : \varphi_1(x,y) \leq z \leq \varphi_2(x,y),\ x \in [a,b],\ y \in [c,d] \}$.

vertically convex.

Goal: $\displaystyle\int_{\partial\mathcal{D}} \vec{F} \cdot d\vec{S} = \int_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$



Like Green's thm, ~~we prove:~~ split

$$\partial\mathcal{D} = \bigcup_{i=1}^{6} S_i .$$

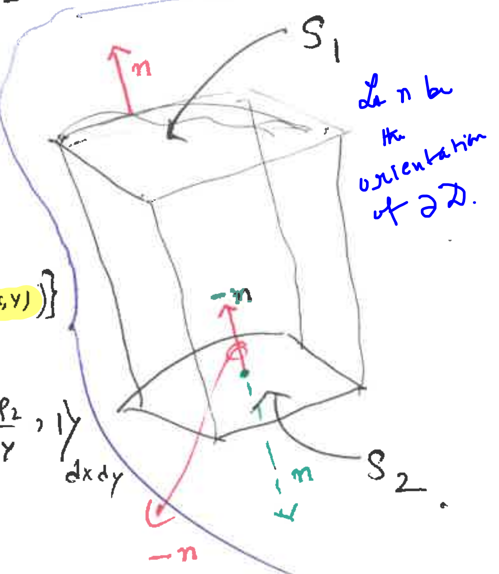
S_i are faces of $\mathcal{D}$.

total 6

~~Suppose~~

Also, again like Green's thm proof:

we prove $\displaystyle\int_{\partial\mathcal{D}} \langle 0,0,R \rangle \cdot d\vec{S} = \int_{\mathcal{D}} \frac{\partial R}{\partial z}\, dV$ etc.



To this ~~cause~~ end, we proceed as follows:

Note: $\displaystyle\int_{\mathcal{D}} \frac{\partial R}{\partial z}\, dV = \int_{y=c}^{d} \int_{x=a}^{b} \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R}{\partial z}\, dz\, dx\, dy = \int_c^d \int_a^b \{ R(x,y,\varphi_2(x,y)) - R(x,y,\varphi_1(x,y)) \}\, dx\, dy.$

Now $\displaystyle\int_{S_1} \langle 0,0,R \rangle \cdot d\vec{S} = \iint_{S_1} \langle 0,0,R \rangle \cdot \left\langle -\frac{\partial \varphi_2}{\partial x}, -\frac{\partial \varphi_2}{\partial y}, 1 \right\rangle dx\, dy$

The top. $z = \varphi_2(x,y)$: Graph surface.

$$= \int_c^d \int_a^b R(x,y,\varphi_2(x,y))\, dx\, dy .$$

Illy $\displaystyle\int_{S_2} \langle 0,0,R \rangle \cdot d\vec{S} = -\iint_{S_2} R(x,y,\varphi_1(x,y))\, dx\, dy .$

$z = \varphi_1(x,y)$ bottom surface.

AND $\displaystyle\int_{S_j} \langle 0,0,R \rangle \cdot d\vec{S} = 0 \quad \forall\ S_j$ vertical wall [Similar to Green's thm].

$j = 3, 4, 5, 6$

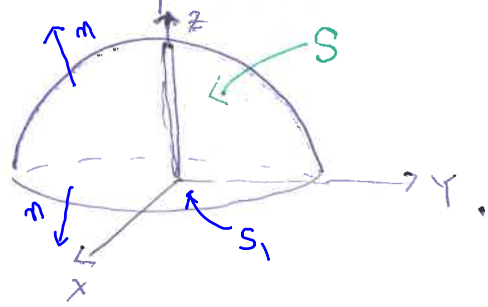
$$\therefore \int_S \langle 0,0,R \rangle \cdot d\vec{S} = \sum_{j=1}^{4} \int_{S_j} \langle 0,0,R \rangle \cdot d\vec{S} = \int_{S_1} + \int_{S_2}$$

$$= \int_c^d \int_a^b \{ R(x,y,\varphi_2(x,y)) - R(x,y,\varphi_1(x,y)) \}\, dx\, dy = \int_{\mathcal{D}} \frac{\partial R}{\partial z}\, dV . \quad \boxtimes$$

eg: Compute $\int_S \vec{F}\cdot d\vec{S}$, where S is the hemisphere

$x^2+y^2+z^2=1$, $\underset{\text{or } z\geq 0}{\underbrace{z>0}}$.

& $F(x,y,z) = \langle x+y, z^2, x^2\rangle$.



Sol: We wish to apply divergence thm:

$$\int_{S\cup S_1} \vec{F}\cdot d\vec{S} = \int_{\mathcal{D}} \operatorname{div}\vec{F}$$

$\mathcal{D}$ = Solid hemisphere.

& S_1 = bottom of the —I—

$= \{(x,y,0) : x^2+y^2\leq 1\}$.

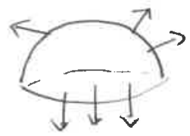
$$\Rightarrow \int_S \vec{F}\cdot d\vec{S} = \int_{\mathcal{D}} \operatorname{div}\vec{F} \;\overset{+\;(*)}{-}\; \int_{S_1} \vec{F}\cdot d\vec{S} \quad \text{—— (1)}$$

$\vec{F} = \langle x+y, z^2, x^2\rangle$.

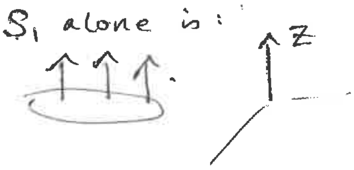
$\left[\because \operatorname{div}\vec{F} = 1+0+0 = 1\right]$

$$\text{Now } \int_{\mathcal{D}} \operatorname{div}\vec{F} = \int_{\mathcal{D}} 1 = \text{Area}(\mathcal{D}) = \frac{2\pi}{3}.$$

(*) REMEMBER: orientation of $S\cup S_1$ is



However, orientation of S_1 alone is:



$$\& \int_{S_1} \vec{F}\cdot d\vec{S} = \int_{S_1} \vec{F}\cdot n \, dS$$

Recall: $\int_S \vec{F}\cdot d\vec{S} = \int_S \vec{F}\cdot\vec{n}\,dS$.

Here $n = \langle 0,0,1\rangle$.

$$\therefore \int_{S_1} \vec{F}\cdot n\,dS = \int_{S_1} x^2\,dS$$

$$= \int_0^1\int_0^{2\pi} u^2\cos^2 v\; u\,du\,dv$$

$$= \int_0^1\int_0^{2\pi} u^3\cdot\cos^2 v\,du\,dv.$$

$$= \cdots = \pi/4.$$

S_1 is given by:

$r(u,v) = (u\cos v, u\sin v, 0)$

$0\leq u\leq 1$, $0\leq v\leq 2\pi$.

$$\therefore r_u\times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0,0,u\rangle.$$

$\Rightarrow \|r_u\times r_v\| = u$

$$\therefore (1) \Rightarrow \int_S \vec{F}\cdot d\vec{S} = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{4}.$$

Ans.

Thm: (Stokes theorem)

Let C be a piecewise C^1 curve enclosing an oriented surface S in $\mathbb{R}^3$. Suppose $\vec{F} = (P, Q, R)$ be a C^1- v.f. on an open set containing S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_S (\nabla\times\vec{F})\cdot d\vec{S}.$$

Remark:

\# $\nabla\times\vec{F} = \text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$. So $\int_C \vec{F}\cdot d\vec{r} = \int_S \text{Curl}\,\vec{F}\cdot d\vec{S}$.

\# $R = 0 \Rightarrow$ Green's thm. BUT: " $\uparrow$ "

\# Do the proof for S: $z = f(x,y)$ (i.e. graph surface). In trace of full version:
Here, we need to "use Green's thm"
it goes other way around though!

Proof: Suppose S = graph f $= \{(x,y,f(x,y)) : (x,y) \in \mathcal{R}\}$. f is C^1.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

$\left[\because z = f(x,y) \Rightarrow dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right]$

$$= \int_{\tilde{C}} \left(P + R\frac{\partial f}{\partial x}\right)dx + \left(Q + R\frac{\partial f}{\partial y}\right)dy.$$

$\left[\tilde{C} = \{(x,y) : (x,y,z) \in C \text{ for some } z\}\right]$

i.e.: $\tilde{C}$ is projection of C onto xy-plane

$$\overset{\text{Green's thm}}{=} \int_{\mathcal{R}} \left(\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right)\right)dA$$

Now $\frac{\partial}{\partial x}\left(Q + \frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x} + \frac{\partial^2 f}{\partial x\partial y}$ & $\frac{\partial}{\partial y}\left(P + \frac{\partial f}{\partial x}\right) = \frac{\partial P}{\partial y} - \frac{\partial^2 f}{\partial x\partial y}$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{\mathcal{R}}$$

Note P, Q, R are fn's of (x, y, z) & $z = f(x,y)$.

[∴ In particular: if $\omega = P(x,y,z)$, then

$$\frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x}\cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \cancel{\frac{\partial P}{\partial y}}\cdot \underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial P}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z}\cdot\frac{\partial f}{\partial x}. \quad]$$

(*) — $\boxed{\Rightarrow \frac{\partial \omega}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial x}}$

So

$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) = Q_x + Q_z f_x + (R_x + R_z f_x) f_y + R\frac{\partial^2 f}{\partial x \partial y}. \quad —(1)$$

By (*)

&

$$\frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right) = P_y + P_z f_y + (R_y + R_z f_y) f_x + R f_{xy} \quad —(2)$$

∴ (1) − (2) ⇒

$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right) = f_x(Q_z - R_y) + f_y(R_x - P_z) + (Q_x - P_y)$$

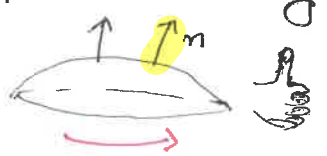
$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_R f_x(Q_z - R_y) + f_y(R_x - P_z) + (Q_x - P_y)$$

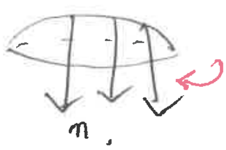
$$= \iint_R \langle -f_x, -f_y, 1\rangle \cdot \underbrace{\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle}_{\nabla \times \vec{F}}.$$

$$= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

BTW: How to figure out orientation of S vs. C? RIGHT HAND RULE!!

place thumb along n of S ⟹ The remaining 4 fingers will direct C.

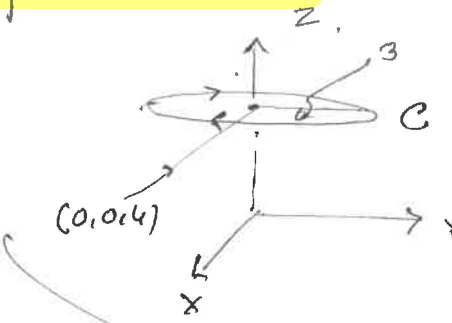


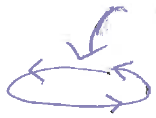


$\vec{F} = \langle -y, x, xyz \rangle$.

eg: (A) $\int_C \vec{F} \cdot d\vec{r}$, where $C: x^2+y^2=9,\ z=4$, oriented clockwise when viewed from the above.

Sol: $C: \vec{r}(t) = \langle 3\cos t, 3\sin t, 4 \rangle$.
$0 \le t \le 2\pi$.





$\therefore \int_C \vec{F} \cdot d\vec{r} = - \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

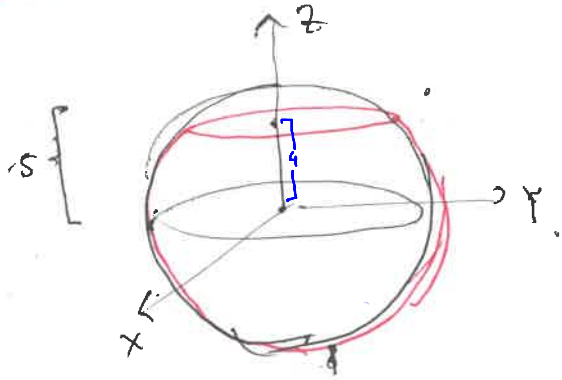
$\because$ we need

$= \int_0^{2\pi} \langle -3\sin t, 3\cos t, 36\cos t \sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle\, dt$

$= \cdots = -18\pi$.

(B) S = the surface: part of $x^2+y^2+z^2=25$, below the plane $z=4$ oriented s.t. the unit normal vector at $(0,0,-5)$ is $\langle 0,0,-1 \rangle$.

Compute $\int_S \underbrace{\text{Curl}\,\vec{F}}_{=\nabla \times \vec{F}} \cdot d\vec{S}$.



Sol: By Stoke's thm:

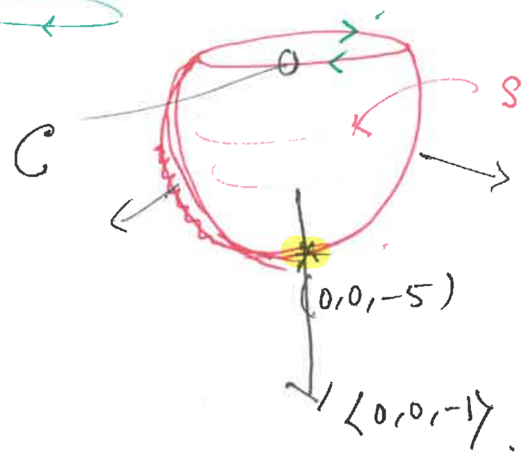
$\int_S \text{Curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$

$= -18\pi$

(By (A))

Stoke's thm. Simplifies!!

By Right hand rule.



~~A fairly simple~~

Inverse & implicit fn. theorems.

Recall the inverse fn. thm: Let $\Theta \subseteq \mathbb{R}$ open & $f : \Theta \to \mathbb{R}$ be a C' fn.

If $f'(x_0) \neq 0$ for some $x_0 \in \Theta$, then $\exists$ $(a,b) \ni x_0$ s.t.

(i) $(a,b) \subseteq \Theta$,

(ii) $f : (a,b) \to \underbrace{f(a,b)}_{\text{an open interval.}}$ is bijection.

(iii) $f^{-1} : f(a,b) \to (a,b)$ is diff. (C' fn.)

(iv) $$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \forall x \in (a,b).$$

— The $\mathbb{R}^n$ - version (No proof : see Spivak's book) —.

Thm: (Inverse fn. thm).

Let $\Theta_n \subseteq \mathbb{R}^n$ be open & $f : \Theta_n \to \mathbb{R}^n$ be a C'- fn. If $Df(x_0)$ is invertible for some $x_0 \in \Theta_n$, then $\exists$ an open neighborhood $B_n \subseteq \Theta_n$ s.t. $x_0 \in B_n$ &

i) $f : B_n \to \underbrace{f(B_n)}_{\Rightarrow \text{open}}$ is bijection.

ii) $f^{-1} : f(B_n) \to B_n$ is ~~diff~~ C^1 ($\Rightarrow$ diff.)

iii) $Df^{-1}(f(x)) = (Df(x))^{-1}$ $\forall x \in B_n$.

So, if $Df(x_0) \in M_n(\mathbb{R})$ inv. then locally f is invertible. (& diffeomorphism)

Moral of the story.

— x —

Implicit fn. thm:

Motivation:

We are mostly familiar with fn's of the form

$y = f(x)$, $z = f(x,y)$, $x_{n+1} = f(x_1, \ldots, x_n)$ etc

$F(x,y,z) = \sin(xyz)$ $\Downarrow$? $z = f(x,y)$ $\Downarrow$ $F(x,y,f(x,y)) = 0$.

Like: $\underbrace{y = \sin x}_{y - \sin x = 0}$, $\underbrace{z = xy + y^2}_{z - xy - y^2 = 0}$ etc.

$F(x,y) = 0$ $\quad$ $F(x,y,z) = 0 \rightsquigarrow \overset{?}{\Longrightarrow} z = f(x,y)$

What about "writing" "$y = f(x)$" if we have: $F(x,y)=0$?

i.e: if F is a fn. of (x,y) & if $F(x,y)=0$

Call it implicit fn. or Solution.

$\Longrightarrow$? $\boxed{y = f(x)}$ for some f ? i.e. $\boxed{F(x,f(x))=0}$?

[Also, we wish the solution f to be diff.]

Of Course, this is a question of general interest.

eg1: Suppose $F(x,y) = ax+by+c$.

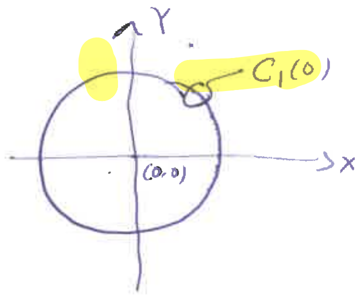
Now $ax+by+c=0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$ (if $b \neq 0$).

So, here $f(x) = -\frac{a}{b}x - \frac{c}{b}$ $\forall x$.

Note: $\frac{\partial F}{\partial y} = b$. So $\frac{\partial F}{\partial y} (=b) \neq 0 \Rightarrow F(x, f(x)) = 0$ for some diff. f!!

Trivial but impressive observation. $\longrightarrow$ So $\frac{\partial F}{\partial y} \neq 0 \Rightarrow \exists$ diff. fn. f $\ni$. $F(x,f(x))=0$.

eg2: $F(x,y) = x^2+y^2-1$.



We know $F(x,y)=0 \iff (x,y)$ in $C_1(0)$.

Note that $y = \pm\sqrt{1-x^2}$.

$\Rightarrow$ $y = f(x)$ for (even any fn) f is NOT POSSIBLE !!

However: if $y \geq 0$, then $y = f(x) = \sqrt{1-x^2}$

& if $y<0$ ——— $y = f(x) = -\sqrt{1-x^2}$.

$\therefore$ If $y \geq 0$ (or $y<0$), then $\exists$ f s.t.

$F(x, f(x)) = 0$.

Note $\frac{\partial F}{\partial y} = 2y$. & so if $y>0$ (or $y<0$), then

$\frac{\partial F}{\partial y} \neq 0$.

Eg 3: Suppose $F(x,y)=0$ in a nbd of $(a,b)\in\mathbb{R}^2$.

Suppose $y=f(x)$ be a diff / C^1 solution to this:

$F(x,f(x))=0$. implicit solution to F.

$$\Rightarrow \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0. \quad [\text{Chain rule}]$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}} \quad \left[\text{if } \frac{\partial F}{\partial y}\neq 0\right] \text{ at } (a,b)$$

Again: $\frac{\partial F}{\partial y}\neq 0$ plays important role here.

This also says: if $y=f(x)$ is a C^1 solution to $F(x,y)=0$ near (a,b), then $\frac{\partial F}{\partial y}\neq 0$ at (a,b) is necessary to recover $\frac{dy}{dx}$!!

i.e: derivative of f can be computed by differentiating $F(x,f(x))=0$. ~~& looking at the~~

~~Thm: Let $\mathcal{O}\subseteq\mathbb{R}^{n+m}$ be open, $F:\mathcal{O}\to\mathbb{R}^m$ $=(f_1,\dots,f_m)$ a C^1-fn. Suppose $(a,b)\in\mathcal{O}$, $F(a,b)=0$ & $\det\left(\frac{\partial f_i}{\partial y_j}\right)\neq 0$ at (a,b).~~ ($a\in\mathbb{R}^n$, $b\in\mathbb{R}^m$)

Setting: $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, $n,m\geq 1$. $a\in\mathbb{R}^n$, $b\in\mathbb{R}^m$. ~~Fixed~~ Let $(a,b)\in\underbrace{\mathcal{O}}_{\text{open}}\subseteq\mathbb{R}^n\times\mathbb{R}^m$. $F:\mathcal{O}\longrightarrow\mathbb{R}^m$ & $F=(f_1,\dots,f_m)$.

& $f_i:\mathcal{O}\longrightarrow\mathbb{R}$ $\forall\, i=1,\dots,m$. Finally, $f_i(x,y)=f_i(\underbrace{x_1,\dots,x_n}_{\in\mathbb{R}^n},\underbrace{y_1,\dots,y_m}_{\in\mathbb{R}^m})$.

Thm: (Implicit fn. thm): Suppose ~~f~~ $F\in C^1(\mathcal{O})$ & $F(a,b)=0$.

If $\boxed{\det\underbrace{\left(\frac{\partial f_i}{\partial y_j}(a,b)\right)}_{=M}{}_{1\leq i,j\leq m}\neq 0}$, then $\exists$ open sets $\mathcal{U}\subseteq\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$ ($a\in\mathcal{U}$, $b\in V$)

& a C^1-fn $f:\mathcal{U}\longrightarrow V$ s.t.

$$F(x,f(x))=0 \quad \forall\, x\in\mathcal{U}.$$

$(y_1,\dots,y_m)=f(x)$

Proof: Define $\tilde{F} : \Theta \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ by

$$\tilde{F}(x,y) = \cancel{(x, F(x))}\ (x, F(x,y)), \ \forall (x,y) \in \Theta .$$

Now $J_{\tilde{F}}(a,b) = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & \cancel{J_F(a,b)}\ M \end{bmatrix}$ $\Rightarrow \det[J_{\tilde{F}}(a,b)] \neq 0$.

$\therefore$ By Inverse fn thm: $\exists$ $U_0 \subseteq \Theta$, $V_0 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ open sets s.t. $(a,b) \in U_0$, $\underbrace{(a,0)}_{=\tilde{F}(a,b)} \in V_0$ &

$\tilde{F} : U_0 \longrightarrow V_0$. has a differentiable inverse.

Note: $U_0 = A \times B$ for $\underset{a}{\overset{\Psi}{A}} \subseteq \mathbb{R}^n$, $\underset{b}{\overset{\Psi}{B}} \subseteq \mathbb{R}^m$ open.
We can consider [$\because$ Boxes forms basic open sets.]

Call: $U = A$.

Clearly, $\tilde{F}^{-1}(\underset{V_0}{\underset{\uparrow}{x,y}}) = (x, g(x,y))$ for some diff. g.
[$\because \tilde{F}(x,y) = (x, F(x,y))$].

Now $\left.\begin{array}{l} \pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \\ (x,y) \longmapsto y \end{array}\right\}$ $\Rightarrow \pi \circ \tilde{F} = F$.

$$\begin{aligned} \therefore F(x, g(x,y)) &= F \circ \tilde{F}^{-1}(x,y) \\ &= \pi \circ \tilde{F} \circ \tilde{F}^{-1}(x,y) \\ &= \pi(x,y) \\ &= y. \end{aligned} \quad \forall (x,y) \in A \times B.$$

$\Rightarrow F(x, g(x,0)) = 0$.

So, define $f(x) := g(x,0)$. $\forall x \in A = U$. $\boxed{//}$

---- X ----