Thm: Let $\underbrace{f \in C(B^n)}_{\text{Set of Cont. fn's on } B^n}$. Then $f \in \mathcal{R}(B^n)$. i.e; $C(B^n) \subseteq \mathcal{R}(B^n)$.

Proof. B^n is compact. $\Rightarrow$ f is uniformly cont. on B^n.
$[\because B^n \subseteq \mathbb{R}^n_u]$.

Let $\varepsilon > 0$. So $\exists\ \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{2\,\underbrace{v(B^n)}_{\text{volume of the box}}} \quad \forall\ \|x-y\| < \delta. \qquad (*)$$

[Halt: Diameter of a box $\prod_{i=1}^{n}[a_i, b_i]$ = largest diagonal. Check.

= max { distance of v_1, v_2 : v_1, v_2 are vertices of the box }.

In the sense of metric space $\mathbb{R}^n_u$.

I/l_y, if P is a partition of $B^n \left(= \prod_{i=1}^{n}[a_i, b_i]\right)$,

then $\underbrace{\|P\|}_{\text{mesh of } P} = \text{Max}\{\text{diameter of } B^n_\alpha : \alpha \in \Lambda(P)\}$

Note: $\Lambda(P)$ is a finite set.]

Now for that $\delta > 0$, pick a partition P of B^n s.t.
$\|P\| < \delta$.

[Remark: This is always possible. Think $n=1$ case.] But possibly long computation for $n > 1$. Do it for $n=2$.

$\forall\ \alpha \in \Lambda(P)$, pick & then fix $a_\alpha \in B^n_\alpha$.

$\therefore \|a_\alpha - x\| < \delta \quad \forall\ x \in B^n_\alpha \quad (\&\ \forall\ \alpha \in \Lambda(P))$.

Then $(*)$ $\Rightarrow$ $|f(x) - f(a_\alpha)| < \tilde{\varepsilon} \quad \forall\ x \in B^n_\alpha \quad (\forall \alpha \in \Lambda(P))$.

$[\tilde{\varepsilon} := \varepsilon / 2v(B^n)]$

$$\Rightarrow \quad f(a_\alpha) - \tilde{\varepsilon} < f(x) < f(a_\alpha) + \tilde{\varepsilon} \quad \forall\ x \in B^n_\alpha.$$

Taking sup-inf: $\quad f(a_\alpha) - \tilde{\varepsilon} \leqslant m_\alpha \leqslant M_\alpha \leqslant f(a_\alpha) + \tilde{\varepsilon}.$

Recall $\left[m_\alpha = \inf\{f(x) : x \in B^n_\alpha\} \right.$
$\quad$ Ily M_α.
$\quad$ # Here $\left. \alpha \equiv \alpha(P) \right]$

Thus: $\quad \forall\ \alpha \in \Lambda(P)$ (or $\alpha(P) \in \Lambda(P)$), we have:

$$f(a_\alpha) - \tilde{\varepsilon} \leqslant m_\alpha \leqslant M_\alpha \leqslant f(a_\alpha) + \tilde{\varepsilon}$$

$$\Rightarrow \sum_{\alpha \in \Lambda(P)} \left(f(a_\alpha) - \tilde{\varepsilon}\right) \times v(B^n_\alpha) \leqslant L(f, P) \leqslant U(f, P)$$

$$\leqslant \sum_{\alpha \in \Lambda(P)} \left(f(a_\alpha) + \tilde{\varepsilon}\right) \times v(B^n_\alpha).$$

But the ~~left~~ most term $= \underbrace{\left(\sum_{\alpha \in \Lambda(P)} f(a_\alpha) \times v(B^n_\alpha)\right)}_{:= c\ = c(P)} - \underbrace{\tilde{\varepsilon}}_{= \frac{\varepsilon}{2 v(B^n)}} \times v(B^n)$

$$:= c - \frac{\varepsilon}{2}.$$

& the right-most term $= c + \frac{\varepsilon}{2}.$

$$\therefore \quad c - \frac{\varepsilon}{2} \leqslant L(f, P) \leqslant U(f, P) \leqslant c + \frac{\varepsilon}{2}.$$

$$\therefore \quad 0 \leqslant \underbrace{\overline{\int_{B^n}} f}\ - \underbrace{\underline{\int_{B^n}} f} \quad \cancel{\leqslant \left(c + \frac{\varepsilon}{2}\right) - \left(c - \frac{\varepsilon}{2}\right)}$$

Always true.

$$\leqslant U(f, P) - L(f, P)$$

$$\leqslant \left(c + \frac{\varepsilon}{2}\right) - \left(c - \frac{\varepsilon}{2}\right) = \varepsilon$$

$$\Rightarrow U(f, P) - L(f, P) < \varepsilon \quad \Rightarrow f \in \mathcal{R}(B^n). \quad \boxtimes$$

Now, again (like $n=1$ case) it is time to talk about **Computing** $\int_{B^n} f\,dv$, $f \in \mathcal{R}(B^n)$.

If $n=1$, then $\int_a^b \longleftrightarrow \sum_{n=1}^{\infty}$

i.e., $\int$ is a continuous analogue of infinite series.

So, if $n=2$, then $\int_{[a_1,b_1]\times[a_2,b_2]} f\,dv \longleftrightarrow \sum_{m,n=1}^{\infty} a_{mn}$!!

i.e., a cont. analogue of "double series" ??

Ans: It should be.

So, "very briefly", lets talk about double sequences & series.

HW/upto you.

As usual, a double seqn (or even n-seqn) is a fn

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R} \quad (\text{or } \mathbb{C} \text{ or } X \to \text{a M.S.}).$$

We write f as $\{f(m,n)\}$ or simply $\{a_{mn}\}_{m,n \geq 0}$

Often, it is helpful to assume $\mathbb{N} = \{0,1,2,\ldots\}$. or just usual $\mathbb{N}$.

Def: A double seqn $\{a_{mn}\}$ is said to be convergent if there is a real no. a so that: $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$

$\ni$ $|a_{mn} - a| < \varepsilon \quad \forall\, m, n \geq N.$ $\Leftarrow$ $m \geq N_1$, $n \geq N_2$

limit of the seqn

We write: $\boxed{\lim_{m,n \to \infty} a_{mn} = a.}$

HW: Limit is ! (if exists).

eg: 1) $a_{mn} := \dfrac{1}{m+n}$ $\quad \forall\, m,n \geq 1$.

For $\varepsilon > 0$, choose $N \in \mathbb{N}$ $\ni$ $N > \frac{1}{2\varepsilon}$.

So, if $m,n \geq N \Rightarrow m,n > \frac{1}{2\varepsilon} \Rightarrow m+n > \frac{1}{\varepsilon}$

$\Rightarrow \dfrac{1}{m+n} < \varepsilon$.

$\Rightarrow |a_{mn} - 0| < \varepsilon \quad \forall\, m,n \geq N$.

$\therefore \lim\limits_{m,n\to\infty} a_{mn} = 0$.

2) $a_{mn} := (-1)^{m+n} \times \left(\frac{1}{m} + \frac{1}{n}\right)$.

$\therefore |a_{mn}| = \frac{1}{m} + \frac{1}{n}$

$\therefore$ if $\varepsilon > 0$, then choose $N \in \mathbb{N} \ni N > \frac{2}{\varepsilon}$.

$\therefore \forall\, m,n \geq N$, $\quad m,n > \frac{2}{\varepsilon}$.

$\Rightarrow \frac{1}{m}, \frac{1}{n} < \frac{\varepsilon}{2}$. $\Rightarrow \frac{1}{m} + \frac{1}{n} < \varepsilon$.

$\Rightarrow \lim\limits_{m,n\to\infty} a_{mn} = 0$.

3) $a_{mn} = \dfrac{mn}{m^2+n^2}$. $\quad \forall\, m,n \geq 1$.

Now (OLD TRICK) $m = n \Rightarrow a_{mn} = \dfrac{n^2}{2n^2} = \frac{1}{2}$.

Remember ?

$\therefore \lim\limits_{\substack{m,n\to\infty \\ m=n}} a_{mn} = \frac{1}{2}$.

But $m = 2n \Rightarrow a_{mn} = \frac{2}{5}$

$\therefore \lim\limits_{\substack{m,n\to\infty \\ m=-n}} a_{mn} = \frac{2}{5}$.

$\therefore \lim\limits_{m,n\to\infty} a_{mn}$ DNE (why ?).

??

$\therefore$ One sided exists But NOT both sided !!

Even worse !!

Lets look at the defn again: for $\varepsilon>0$ $\exists N \in \mathbb{N}$ s.t. $|a_{mn}-a|<\varepsilon$. $\forall m,n \geq N$

So, let us consider eg 1 in page 10. Define

$$b_{mn} = \begin{cases} n & \text{if } m=1 \\ \frac{1}{m+n} & \text{if } m>1. \end{cases}$$

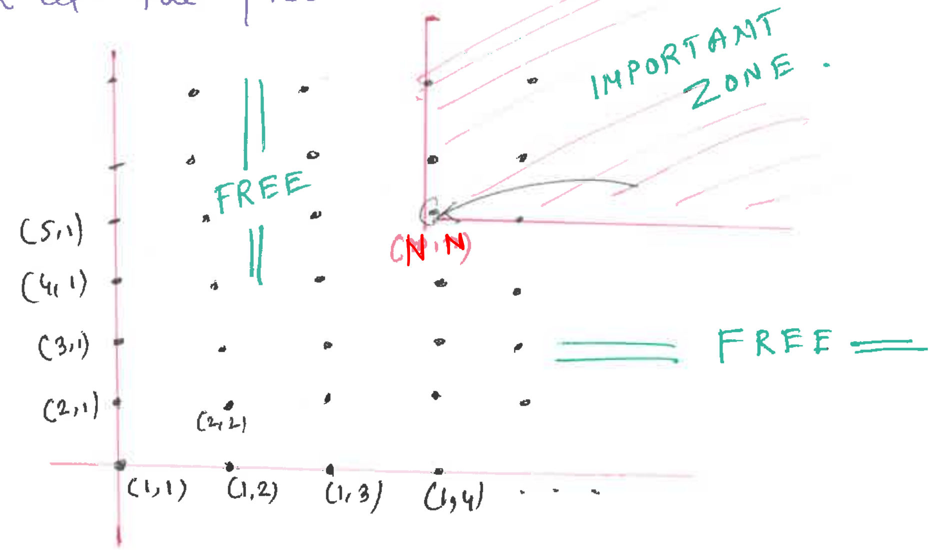
$N \neq 1$

$b_{1n} = n$

$m=1$. $b_{1n} \not\to$ Conv.

$$\Rightarrow \lim_{m,n\to\infty} b_{mn} = 0$$ (Agree?) BUT $\{b_{mn}\}$ is NOT bounded.

Lets look at the picture:



Q: How to compute $\lim_{m,n\to\infty} a_{mn}$ (if exists)?

[BTW: We must get back to R.I. along with similar type questions etc.]

Maybe: we compute $\lim_{n\to\infty} a_{mn}$ (treating m fixed) & then $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$? So, if all goes well, we say:

$$\lim_{m,n\to\infty} a_{mn} = \lim_{m\to\infty}\lim_{n\to\infty} a_{mn} \left(= \lim_{n\to\infty}\lim_{m\to\infty} a_{mn}\right)!!$$

Let's give a name to it:

Def: $\{a_{mn}\}$ is said to have an ~~iter~~ iterated limit if

$\hat{a}_m := \lim_{n\to\infty} a_{mn}$ exists $\forall\, m \geq 1$ & $\hat{a}_m \to \hat{a}$ for some $\hat{a}$

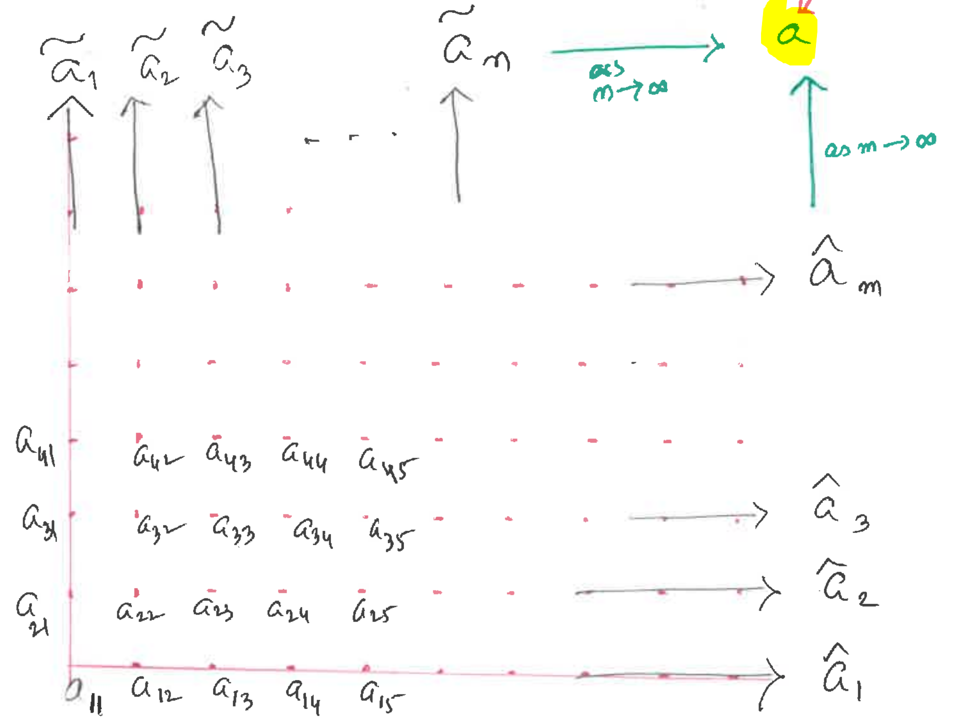
or $\tilde{a}_n := \lim_{m\to\infty} a_{mn}$ exists $\forall\, n \geq 1$ & $\tilde{a}_n \to \tilde{a}$ —||— $\tilde{a}$.

[We write: $\hat{a} = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}$ & $\tilde{a} = \lim_{n\to\infty} \lim_{m\to\infty} a_{mn}$.]

Q: How to relate, $a, \hat{a}, \tilde{a}$ (if there is any)?

/ or relate

In "picture"



And in theorem

$\Downarrow$

Thm: Let $\lim_{m,n\to\infty} a_{mn}$ exists & ~~the iterated~~ limit $\lim_{n\to\infty} a_{mn}$ exists

$\forall\, m$. Then the iterated limit ~~exis~~

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{mn} \text{ exists } \&$$

$$\lim_{m,n\to\infty} a_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} a_{mn}.$$

Stress on this.

11.4) if $\lim_{m,n\to\infty} a_{mn}$ exists & $\lim_{m\to\infty} a_{mn}$ exists $\forall n$, then

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{mn} \text{ exists } \& = \lim_{m,n\to\infty} a_{mn}.$$

Proof: We only prove the first one. Let $\lim_{m,n\to\infty} a_{mn} := a$.

For all $\forall m$, set $\hat{a}_m := \lim_{n\to\infty} a_{mn}$.

· Let $\varepsilon > 0$. $\exists\, N \in \mathbb{N} \ni \cdot \quad |a_{mn} - a| < \frac{\varepsilon}{2} \quad \forall\, m,n \geq N$.

Also, for each m, $\exists\, N_m \in \mathbb{N} \ni \cdot$

$$|a_{mn} - \hat{a}_m| < \frac{\varepsilon}{2} \quad \forall\, n \geq N_m.$$

$$\therefore\ |\hat{a}_m - a| = |(\hat{a}_m - a_{mn}) + (a_{mn} - a)|$$

n IN

$$\leq |\hat{a}_m - a_{mn}| + |a_{mn} - a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \forall\, m \geq N.$$

n OUT

Here assume $n = \max\{N, N_m\}$

$$\Rightarrow \hat{a}_m \longrightarrow a \text{ as } m \longrightarrow \infty.$$

[OR: $|\hat{a}_m - a| = |\lim_{n\to\infty} a_{mn} - a|$

$$= \lim_{n\to\infty} |a_{mn} - a| \quad \longleftarrow \text{by Continuity of } x \mapsto |x|.$$

$$\leq \varepsilon \quad]$$

Cor: If $\lim\limits_{m,n\to\infty} a_{mn} = a$ (i.e. the limit exists) & both $\lim\limits_{m\to\infty} a_{mn}$ & $\lim\limits_{n\to\infty} a_{mn}$ exists, then $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} a_{mn} = \lim\limits_{n\to\infty}\lim\limits_{m\to\infty} a_{mn} = a$.

"We need them all"

eg: double limit $\not\Rightarrow$ single limit.

$$a_{mn} := (-1)^{m+n} \times \left(\frac{1}{m}+\frac{1}{n}\right).$$

We know $a_{mn} \to 0$ as $m,n\to\infty$.

However, for fixed m: $a_{mn} = (-1)^m \times \underbrace{\left[\frac{(-1)^n}{m} + \frac{(-1)^n}{n}\right]}_{\text{DNE as } n\to\infty}$

$$\Rightarrow \lim_{n\to\infty} a_{mn} \text{ DNE } \forall\, m.$$

$$\text{Illy } \lim_{m\to\infty} a_{mn} \text{ DNE } \forall\, n.$$

eg: Single limit $\not\Rightarrow$ double limit.

$a_{mn} = \dfrac{mn}{m^2+n^2}$. We have seen $\lim\limits_{m,n\to\infty} a_{mn}$ DNE.

But, for fixed m, $a_{mn} = m \times \dfrac{n}{m^2+n^2} \le m \times \dfrac{n}{n^2} \to 0$ as $n\to\infty$.

Illy — " — n, $a_{mn} \to 0$ as $m\to\infty$.

All in all: if we know "Double Limit" exists & one or both single limit(s) exists, then we can compute the double limit.