Lecture 16: Field of fractions, Localization

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⊛ Construction of $\mathbb{Z}$ to $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, \begin{array}{l} a, b \in \mathbb{Z} \\ b \neq 0 \end{array} \right\} = \underline{\{ (a,b) \mid a, b \in \mathbb{Z},\ b \neq 0 \}}/\sim \quad \boxed{(a,b) \sim (c,d) \text{ if } ad = bc}$$

So we are inverting all nonzero elements of $\mathbb{Z}$ to obtain $\mathbb{Q}$.

Lets generalize this to arbitrary rings.

Def^n: Let R be a comm ring with unity & $S \subseteq R$ be subset. Then S is said to be multiplicative subset if $1 \in S$ & $\forall\, x, y \in S$, $xy \in S$.

Example: 1) $S = \{1\}$ (R any ring) 2') $S = R$ ← Not interesting

2) S = set of units

3) R an integral domain like $\mathbb{Z}$; $S = R \setminus \{0\}$. ←

4) In $\mathbb{Z}$, $S = \mathbb{Z}_{>0}$

5) R any ring, & $x \in R$, $S = \{1, x, x^2, x^3, \dots\}$ is a multiplicative set.

Not interesting for localization 6) R any ring, $S = I \cup \{1\}$ is a multiplicative set.

7) Let R be any ring and $P \subseteq R$ a prime ideal. Then $S = R \setminus P$ is a multiplicative set.

$\supset$. $b \sim S \Rightarrow a\,b \notin P \overset{(\because P \text{ prime ideal})}{\Longrightarrow} ab \notin P \Rightarrow ab \in S$.

Define a relation on $S \times R$ where R is comm ring with unity and S is a mult set.

$$S \times R = \{ (s, r) \mid s \in S \ \& \ r \in R \}$$

$(s_1, r_1) \sim (s_2, r_2)$ if $s(s_2 r_1 - s_1 r_2) = 0$ for some $s \in S$.

<u>Prop:</u> $\sim$ is an equivalence relation

<u>Pf:</u> $(s_1, r_1) \sim (s_1, r_1)$ by taking $s = 1$
$1(s_1 r_1 - s_1 r_1) = 0$

$\sim$ is reflexive

$\sim$ is symmetric

$(s_1, r_1) \sim (s_2, r_2)$ then $\exists s \in S$

s.t. $s(s_2 r_1 - s_1 r_2) = 0$

$\Rightarrow s(s_1 r_2 - s_2 r_1) = 0$

$\Rightarrow (s_2, r_2) \sim (s_1, r_1)$

$\sim$ is transitive:

Let $(s_1, r_1) \sim (s_2, r_2)$ & $(s_2, r_2) \sim (s_3, r_3)$

$\exists s, s' \in S$ s.t.

$s(s_2 r_1 - s_1 r_2) = 0$ ... (1) & $s'(s_3 r_2 - s_2 r_3) = 0$ ... (2)

$s' s_3$ (1) $+ s s_1$ (2) gives

$$s' s s_2 s_3 r_1 - s' s \cancel{s_1 s_3} r_2 + s s' \cancel{s_3} s_1 r_2 - s s' s_2 s_1 r_3 = 0$$

$$s' s s_2 (s_3 r_1 - s_1 r_3) = 0$$

Also $s' s s_2 \in S$ ($\because S$ is multiplicative)

Hence $\sim$ is an equivalence relation.

Def^n/Prop: The set of equivalence classes $S \times R/\sim$ is denoted by $S^{-1}R$. The equivalence class $[(s,r)]$ will be denoted by $\frac{r}{s}$.

The binary operators
$$\frac{r_1}{s_1} \oplus \frac{r_2}{s_2} := \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$$
and
$$\frac{r_1}{s_1} \odot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$
are well defined. Moreover $(S^{-1}R, \oplus, \odot)$ is a commutative ring with unity. The map

$$\varphi : R \longrightarrow S^{-1}R \qquad r \longmapsto \frac{r}{1}$$

is a ring homo.

Pf: $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$ $\quad r_1, r_1', r_2, r_2' \in R \implies (s_1, r_1) \sim (s_1', r_1')$

$\frac{r_2}{s_2} = \frac{r_2'}{s_2'}$ $\quad s_1, s_1', s_2, s_2' \in S \implies (s_2, r_2) \sim (s_2', r_2')$

WTS: $$\frac{s_2 r_1 + s_1 r_2}{s_1 s_2} = \frac{s_2' r_1' + s_1' r_2'}{s_1' s_2'}$$

$\exists\, u, v \in S$ s.t.

$u(s_1' r_1 - s_1 r_1') = 0$ ← ① $\quad$ & $\quad v(s_2' r_2 - s_2 r_2') = 0$ ← ②

WTS: $\exists\, w \in S$ s.t.

$$w\left[s_1' s_2' (s_2 r_1 + s_1 r_2) - s_1 s_2 (s_2' r_1' + s_1' r_2')\right] = 0$$

$s_2 s_2' v$ ① $+ s_1 s_1' u$ ② gives

$$s_2 s_2' v u s_1' r_1 - s_2 s_2' v u s_1 r_1' + s_1 s_1' u v s_2' r_2 - s_1 s_1' u v s_2 r_2' = 0$$

$$uv\left[s_1' s_2' (s_2 r_1 + s_1 r_2) - s_1 s_2 (s_2' r_1' + s_1' r_2')\right] = 0$$

So take $w = uv \in S$ ($\because S$ is multiplicative)

|||ly $\odot$ is well-defined.

Claim: $(S^{-1}R, \oplus, \odot)$ is a comm ring with unity

1) $(S^{-1}R, \oplus)$ is an abelian group

- Check $\oplus$ is associative (check)
- $\frac{0}{1}$ is the additive identity ✓ $\frac{r_1}{s_1} \oplus \frac{0}{1} = \frac{1 r_1 + s_1 0}{s_1} = \frac{r_1}{s_1}$
- $\oplus$ is commutative ✓
- $\frac{r}{s} \oplus \frac{-r}{s} = 0$. $\frac{r}{s} \oplus \frac{-r}{s} = \frac{sr - sr}{ss} = \frac{0}{ss} = \frac{0}{1}$

But $\frac{0}{ss} = \frac{0}{1}$ $(\because 1(1 \cdot 0 - ss \cdot 0) = 0)$

2)
- $\odot$ is associative
  easily follows from $\cdot$ is assoc in R
- $\odot$ is commutative
  easily " " $\cdot$ " comm in R.
- $\frac{1}{1}$ is unity. (trivial)
- Distributive laws (check!)

$$\varphi(r + r') = \frac{r + r'}{1} = \frac{r}{1} \oplus \frac{r'}{1} = \varphi(r) \oplus \varphi(r')$$

$$\varphi(r r') = \frac{r r'}{1} = \frac{r}{1} \odot \frac{r'}{1} = \varphi(r)\, \varphi(r')$$

Hence φ is ring homo. Also $\varphi(1) = \frac{1}{1}$ is unity in $S^{-1}R$.