Lecture 24: Linear maps, Isomorphism theorems

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Recall: Modules, Submodules, Quotient modules over a ring R.

* $(M, +, s)$

Ex: $R = \mathbb{Z}$, $M = (\mathbb{Z}, +, \cdot)$ then every ideal is a submod of M, e.g. $n\mathbb{Z}$ $n \in \mathbb{Z}$. Then $\mathbb{Z}/n\mathbb{Z}$ is also a $\mathbb{Z}$-mod

$r \cdot [m]_n = [rm]_n$ $\quad r \cdot \overline{m} = \overline{rm}$

$r \cdot (m + nN) = rm + nN$

* The notion $\mathbb{Z}$-module is same as abelian groups.
  Let A be an abelian group the A has a natural $\mathbb{Z}$-mod str which is $n \cdot a = a + \dots + a$ if $n \geq 0$ o.w. $-a + \dots + -a$ if $n < 0$.

Def^n: Let R be a ring and M, N be R-modules. A map $\varphi: M \longrightarrow N$ is called an <u>R-linear map</u> or an <u>R-mod homo</u> if φ is a group homo and $\varphi(r \cdot m) = r \cdot \varphi(m)$ $\forall r \in R$ & $m \in M$.

More precisely, $(M, +, s_1)$ & $(N, +, s_2)$ are R-mod then

$\varphi(s_1(r, m)) = s_2(r, \varphi(m))$.

* $\varphi: M \longrightarrow N$ is an R-lin map iff $\varphi(rm_1 + m_2) = r\varphi(m_1) + \varphi(m_2)$ $\forall r \in R$ & $\forall m_1, m_2 \in M$

Example 1) M an R-mod & N a submod then $i: N \hookrightarrow M$ is an R-lin map also called R-mod monomorphism.

$q: M \longrightarrow M/N$ is a R-lin map also an R-lin epimorphism. For $r \in R$ & $m \in M$

$$q(r \cdot m) = rm + N$$
$$= r \cdot (m+N) \quad (R\text{-mod str on } M/N)$$
$$= r \cdot q(m)$$

2) $R = \mathbb{Z}/6\mathbb{Z}$, Does $\mathbb{Z}$ have $\mathbb{Z}/6\mathbb{Z}$-mod str.?

$n \in \mathbb{Z}$ & $[1]$

$[1] \cdot n = n$

$(\underbrace{[1] + [1] + \dots + [1]}_{6 \text{ times}}) \cdot n = n + n + \dots + n$

$0 = [0] \cdot n = 6n$ contradiction.

$M = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ $\quad ([a]_6, [b]_3) \in M$ then

$[r]_6 \in R$ then $[r]_6 \cdot ([a]_6, [b]_3) = ([ra]_6, [rb]_3)$

$N = \langle ([1]_6, [0]_3) \rangle$ then N is a submod

$M/N \cong \mathbb{Z}/3\mathbb{Z}$

* An R-lin map of R-mod is an <u>isom</u> if it is bijective.

$N_1 = \langle ([2]_6, [1]_3) \rangle$ then $M/N_1 \cong \mathbb{Z}/6\mathbb{Z} = \langle a \rangle$

$= \{ ([2]_6, [1]_3), ([4], [2]), ([0], [0]) \}$ $\quad M/N_1 = \langle (1, 0) \rangle$

$[1]_6 \cdot a = a \qquad [2] \cdot a = 2a$

Prop: Let $\varphi : M \to N$ be a R-lin map of R-mod
$\ker(\varphi)$ is an R-submod of M & $im(\varphi)$ is an R-submod of N.

Pf:

Let $x, y \in \ker(\varphi)$

Enough to show: $rx + y \in \ker\varphi \quad \forall r \in R$

$$\varphi(rx+y) = r\varphi(x) + \varphi(y) = r0_N + 0_N = 0_N$$

So $\ker\varphi$ is an R-submod of M.

$|||^{ly}$ $x, y \in im(\varphi)$ & $r \in R$ then

$$rx + y = r\varphi(x_1) + \varphi(y_1) \quad \text{for some } x_1, y_1 \in M$$
$$= \varphi(rx_1 + y_1)$$

$\Rightarrow$ $Im(\varphi)$ is an R-submod of N

$\boxtimes$

Isomorphism theorems

First Isom thm

<u>version 1</u>: Let R be a ring. Let $\varphi: M \to N$ be an R-mod homo. Let $K = \ker(\varphi)$ and let $K_1 \subseteq K$ be an R-submod of K. Then K_1 is also an R-submod of M. There exist a unique R-lin map $\tilde{\varphi}: M/K_1 \to N$ s.t. $\tilde{\varphi} \circ q_1 = \varphi$ where $q_1: M \to M/K_1$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "q_1"'] & N \\ & M/K_1 \arrow[u, dashed, "\tilde{\varphi}"'] \end{tikzcd}$$

Pf: For $m \in M$ $\tilde{\varphi}(m + K_1) = \varphi(m)$ (i.e. $\tilde{\varphi}(q_1(m)) = \varphi(m)$)

So $\tilde{\varphi}$ is well-defined group homo follows from the 1^{st} isom thm for groups. So we only need to check that $\tilde{\varphi}$ is R-lin.

For $r \in R$

$$\tilde{\varphi}(r \cdot (m + K_1)) = \tilde{\varphi}(rm + K_1) = \varphi(rm)$$
$$= r\varphi(m)$$
$$= r\tilde{\varphi}(m + K_1)$$

So $\tilde{\varphi}$ is R-lin.

Cor: Let $\varphi: M \to N$ is an R-mod epimorphism then $\bar{\varphi}: M/\ker\varphi \to N$ is an isomorphism

2^{nd} isomorphism thm

Let M be an R-mod.
Let N_1 and N_2
R-submod of M.

$$\frac{N_1 + N_2}{N_2} \cong N_1 / N_1 \cap N_2$$

Here $N_1 + N_2$ is the smallest submod of M containing N_1 & N_2. And $N_1 \cap N_2$ is also an R-submod of M.

(*) $N_1 + N_2 := \{ n_1 + n_2 \mid n_1 \in N_1 \text{ \& } n_2 \in N_2 \}$
is the smallest R-submod of M containing N_1 & N_2.

Easy to see $N_1 + N_2$ is closed under addition & scalar multi.

Let N_α be R-submod of M
$\alpha \in \Omega$ indexing set. then
$\bigcap_{\alpha \in \Omega} N_\alpha$ is an R-submod of M.

Third isom thm: Let M be an R-mod, N be an R-submod of M and K be an R-submod of N then

$$M/N \cong \frac{M/K}{N/K}$$

One checks that N/K is a submod of M/K.