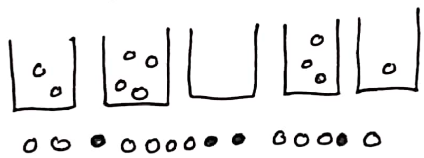
Solution Set HW 3

1. (i) If the quantum number of the i^{th} oscillator is denoted by n_i, the statement that the total energy of the system is equal to $\frac{1}{2}Nh\nu + Mh\nu$ implies that

$$n_1 + n_2 + \dots + n_N = M.$$

$\therefore$ ~~the thermodynami~~ the number of microstates Ω_M with total energy E
= # of ways of distributing M white balls among N labeled boxes. A box may be empty since $n_i = 0$ is possible.



As evident from the figure, this can be obtained by finding the # of permutations of placing all the white balls in a row with $N-1$ black balls that denote the dividing walls. If one labels all the balls with the running numbers, $1, 2, \dots M+N-1$, the # of permutations is $(M+N-1)!$

$$\Omega_M = \frac{(M+N-1)!}{M!\,(N-1)!}$$

$$S = k \ln \Omega_M.$$

for $N, M \gg 1$

$$S = k\{(M+N)\ln(M+N) - M\ln M - N\ln N\}.$$

We know

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial M}\frac{\partial M}{\partial E}$$

$$= k\ln\left(\frac{M+N}{M}\right)\cdot\frac{\partial M}{\partial E} = \frac{k}{h\nu}\ln\frac{M+\frac{1}{2}N+\frac{1}{2}N}{M+\frac{1}{2}N-\frac{1}{2}N}.$$

$$\frac{1}{T} = \ln\left(\frac{E/N + \frac{1}{2}h\nu}{E/N - \frac{1}{2}h\nu}\right).$$

or $$\frac{E + \frac{1}{2}Nh\nu}{E - \frac{1}{2}Nh\nu} = e^{h\nu/kT}$$

Solving for E

$$\boxed{E = N\left\{\frac{1}{2}h\nu + \frac{h\nu}{e^{h\nu/kT} - 1}\right\}}$$

2. If N_- particles are in state $-\varepsilon$ and N_+ in state $+\varepsilon$.

Total energy $E = M\varepsilon = -N_-\varepsilon + N_+\varepsilon$, $M = N_+ - N_-$

$\because$ $N = N_+ + N_-$

$$N_- = \frac{1}{2}(N-M), \quad N_+ = \frac{1}{2}(N+M)$$

Now there are $\frac{N!}{N_+!\,N_-!}$ ways of choosing N_- particles out of N to occupy the state $-\varepsilon$, each of which gives a different microscopic state with energy E. Hence

$$\Omega_M = \frac{N!}{[\frac{1}{2}(N-M)]!\,[\frac{1}{2}(N+M)]!}$$

$$S(E) = k\ln\Omega_M$$

$$\simeq k\left\{N\ln N - \frac{1}{2}(N-M)\ln\frac{1}{2}(N-M) - \frac{1}{2}(N+M)\ln\frac{1}{2}(N+M)\right\}$$

$$= k\left\{N_-\ln\left(\frac{N_-}{N}\right) + N_+\ln\left(\frac{N_+}{N}\right)\right\}$$

$$\boxed{\frac{1}{T} = \frac{1}{\varepsilon}\frac{\partial S}{\partial M} = \frac{1}{2}\frac{k}{\varepsilon}\ln\frac{N - E/\varepsilon}{N + E/\varepsilon}}$$

$\Rightarrow$ $T < 0$ for $E > 0$

$T > 0$ for $E < 0$

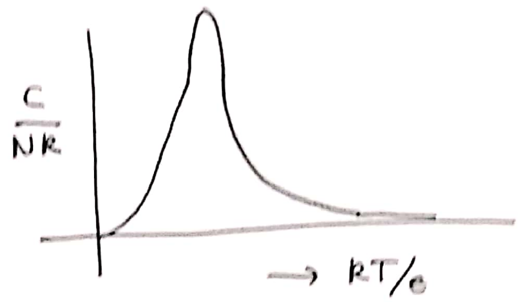
$$\frac{N_-}{N_+} = \frac{N-M}{N+M} = e^{2\varepsilon/kT}$$

$$\frac{N_-}{N} = \frac{e^{\varepsilon/kT}}{e^{\varepsilon/kT} + e^{-\varepsilon/kT}}$$

$$\frac{N_+}{N} = \frac{e^{-\varepsilon/kT}}{e^{\varepsilon/kT} + e^{-\varepsilon/kT}}$$

$$E = -(N_- - N_+)\varepsilon = -N\varepsilon \tanh(\varepsilon/kT).$$

$$C = \frac{dE}{dT} = \frac{Nk\left(\frac{\varepsilon}{kT}\right)^2}{\cosh^2 \varepsilon/kT}.$$



roughly.

→ specific heat → 0, at very low and very high temp. anomalous behavior!

3. The z-axis is used to measure the vertical position of the particles along the axis of the vessel.

the energy per particle is given by

$$\varepsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz \quad \Big\} \quad \sigma: \text{cross section of vessel.}$$

$$Z = \frac{1}{h_0^3} \int\int_\sigma dx\,dy \int_0^\infty dz \int_{-\infty}^{+\infty} dp_x\,dp_y\,dp_z\, e^{-\varepsilon/kT}.$$

$$= \frac{\sigma}{h_0^3}(2\pi m kT)^{3/2} \int_0^\infty e^{-mgz/kT}$$

$$Z = \frac{\sigma kT}{mg}\left(\frac{2\pi m kT}{h^2}\right)^{3/2}.$$

Partition fn. for N particles.

$$\tilde{Z} = \frac{1}{N!}(Z)^N = \frac{1}{N!}\left(\frac{\sigma kT}{mg}\right)^N \left(\frac{2\pi m kT}{h_0^2}\right)^{\frac{3N}{2}}.$$

$$F = -kT \ln Z = -NkT \ln\left[\frac{\sigma kTe}{Nmg}\left(\frac{2\pi m kT}{h_0^2}\right)^{3/2}\right]$$

$$E = kT^2 \frac{\partial \ln Z}{\partial T} = \frac{5}{2} NkT$$

$$C = \frac{5}{2} Nk.$$