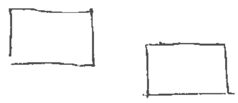
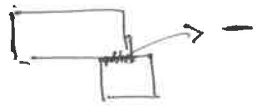
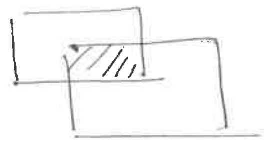
In figure $\left.\begin{matrix} \\ \underline{n=2} \end{matrix}\right\}$

(ii) $\rightarrow$ 

(iii) 

(i) 

So: prove it for $n=2$. It will go as is for general n.

(2) Suppose $f, g \in \mathcal{B}(B^n)$. If $f(x) = g(x)$ $\forall\, x \in \operatorname{int}(B^n)$,

then $f \in \mathcal{R}(B^n) \iff g \in \mathcal{R}(B^n)$.

In this case: $\displaystyle\int_{B^n} f = \int_{B^n} g$. ~~HW~~

(HW) Note: ① First, prove that $\displaystyle\underline{\int_{B^n}} f = \underline{\int_{B^n}} g$ & $\underbrace{\displaystyle\overline{\int_{B^n}} f = \overline{\int_{B^n}} g}_{\text{Similar proof.}}$

② [And Recall: $n=1$ case.]

③ $f \in \mathcal{R}(B^n) \iff$ ~~set~~ the set of points of discontinuity is of measure zero.

Will it be useful?

—— Will get back soon ——.

— x —

~~Lemma~~ Thm: let $\Omega \subseteq \mathbb{R}^n$ be bdd, $f \in \mathcal{B}(\Omega)$. Suppose $\underbrace{B^n_1, B^n_2}_{\text{boxes}} \supseteq \Omega$.

Define $f_i(x) = \begin{cases} f(x) & \forall\, x \in \Omega \\ 0 & \forall\, x \in B^n_i \setminus \Omega. \end{cases}$

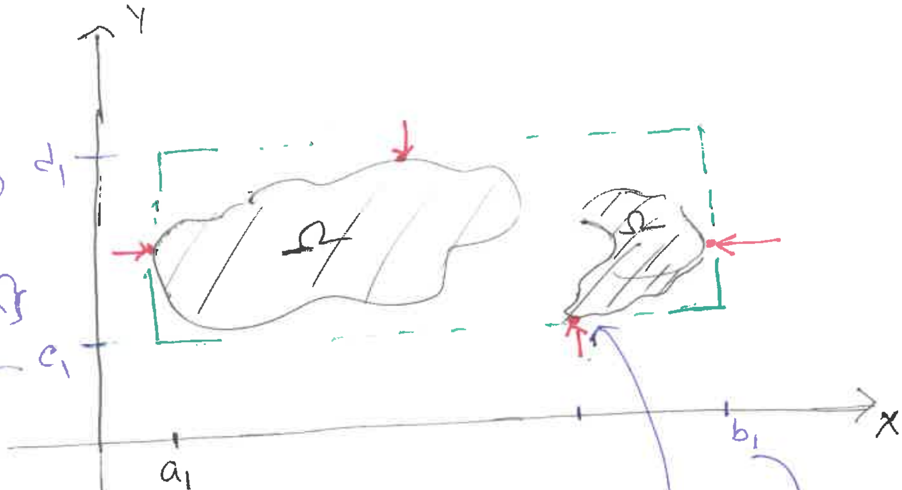
Then $f_1 \in \mathcal{R}(B^n_1) \iff f_2 \in \mathcal{R}(B^n_2)$. In this case:

$$\int_{B^n_1} f_1 = \int_{B^n_2} f_2 .$$

Proof:

Proof: Let's do it for $n = 2$. [general n: HW].

Let $B_1^2 = [a, b] \times [c, d]$.



$d_1 = \sup\{y \in \mathbb{R} : (x, y) \in \Omega \text{ for some } x \in \mathbb{R}\}$

$a_1 = \inf\{x \in \mathbb{R} : (x, y) \in \Omega \text{ for some } y \in \mathbb{R}\}$.

$c_1 = \inf\{y \in \mathbb{R} : (x, y) \in \Omega \text{ for some } x \in \mathbb{R}\}$

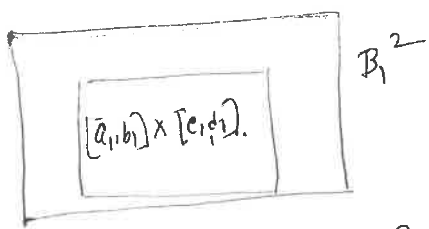
$b_1 = \sup\{x \in \mathbb{R} : (x, y) \in \Omega \text{ for some } y \in \mathbb{R}\}$

Note that $a_1, b_1, c_1, d_1 \in \mathbb{R}$ as Ω is a bdd subset of $\mathbb{R}^2$

Then $\Omega \subseteq [a_1, b_1] \times [c_1, d_1]$, & since

Since a_1, b_1, c_1 & d_1 are ~~de~~ uniquely determined by Ω, it follows that:

$B_1^2 = [a, b] \times [c, d] \supset \Omega$,



$a \leq a_1 \leq b_1 \leq b$ & $c \leq c_1 \leq d_1 \leq d$.

or, equivalently: $[a_1, b_1] \times [c_1, d_1] \subseteq B_1^2$.

Set $\tilde{f} \in \mathcal{B}([a_1, b_1] \times [c_1, d_1])$ by the restriction of f_1 to $[a_1, b_1] \times [c_1, d_1]$. i.e;

$$\tilde{f} = f_1 \Big|_{[a_1,b_1]\times[c_1,d_1]}$$

or
$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y)\in\Omega. \\ 0 & \text{if } (x,y)\in[a_1,b_1]\times[c_1,d_1]\setminus\Omega. \end{cases}$$

Enough to prove that: $\tilde{f}_1 \in \mathcal{R}([a_1,b_1]\times[c_1,d_1])$

$\iff f_1 \in \mathcal{R}(B_1^2)$.

[$\because$ We may just forget f_2.]

And: $\displaystyle\int_{[a_1,b_1]\times[c_1,d_1]} \tilde{f}_1 = \int_{B_1^2} f_1$.

[$\because$ the LHS is !ly determined by f & Ω.]

If $a_1 = b_1$, then $f_1(x,y)\equiv 0$ $\forall (x,y)\in B_1^2$ [or even $\mathbb{R}^2$]

except possibly at $x = a_1$.

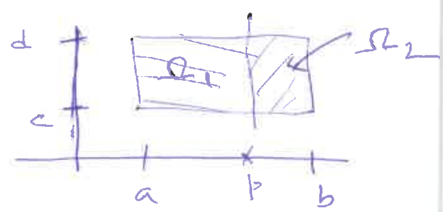
$\Rightarrow$ $f_1 \in \mathcal{R}(B_1^2)$ & $\displaystyle\int_{B_1^2} f_1 = 0$.

& $\tilde{f}\in\mathcal{R}([a_1,b_1]\times[c_1,d_1])$ & $\displaystyle\int_{[a_1,b_1]\times[c_1,d_1]} f = 0$.

HW: Let $f\in\mathcal{B}([a,b]\times[c,d])$ & $f(x,y)=0$ $\forall$ $(x,y)\in(a,b)\times(c,d)$.

① P.T. $f\in\mathcal{R}([a,b]\times[c,d])$ & $\displaystyle\int_{[a,b]\times[c,d]} f = 0$.

② Suppose $a<p<b$.



Then $\displaystyle\int_{[a,b]\times[c,d]} f = \int_{\Omega_1} f + \int_{\Omega_2} f$.

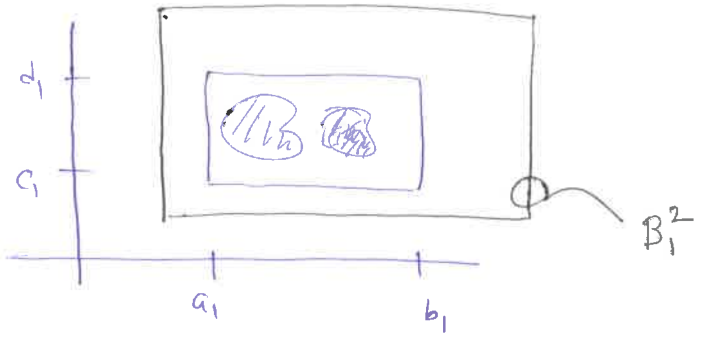
BOTH ARE INTEGRABLE.

$\underline{\text{Illy}}$ if $c_1 = d_1$, then $\int_{B_1^2} f_1 = 0$. & $\int_{[a_1,b_1]\times[c_1,d_1]} \tilde{f}_1 = 0$.

$\therefore$ Assume $a_1 < b_1$ & $c_1 < d_1$.

WLOG: assume $[a_1, b_1] \times [c_1, d_1] \subsetneq B_1^2$.

[Otherwise, nothing to prove.].

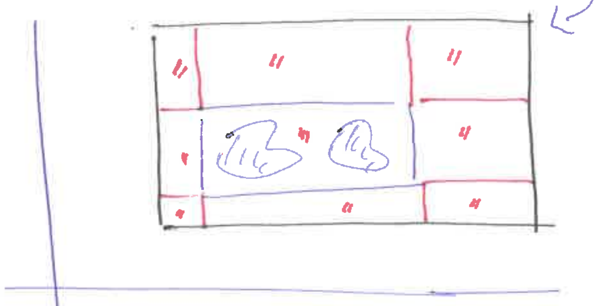


$/// = \Omega$.

Divide $B_1^2 = \coprod_i \tilde{B}_i^2$, where $\tilde{B}_i^2$ is a Sub-box & $i = 1, \dots, p$ for some $p \in \{2, 3, \dots, 9\}$.

In this case. You don't need "9" in the proof. $\therefore$ Lift this idea for general n.

AND $\tilde{B}_i^2 = [a_1, b_1] \times [c_1, d_1]$ for some $i \in \{1, \dots, p\}$.



$\boxed{\text{Apply the HW:}}$

~~Thus~~

If $f \in \mathcal{R}(B_1^2) \Rightarrow$

$$\int_{B_1^2} f = \sum_{i=1}^{p} \int_{\tilde{B}_i^2} f = \int_{[a_1,b_1]\times[c_1,d_1]} f.$$

all are integ. over $\tilde{B}_i^2$.

& Conversely (Again by HW).

$\boxtimes$

So, we have the following definition:

Def: Let $\Omega \subseteq \mathbb{R}^n$ be a bdd subset of $\mathbb{R}^n$. Suppose $f \in \mathcal{B}(\Omega)$. Let $B^n \subseteq \mathbb{R}^n$ be a box & suppose $\Omega \subseteq B^n$. Define $\tilde{f} \in \mathcal{B}(B^n)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \forall\, x \in \Omega. \\ 0 & \forall\, x \in B^n \setminus \Omega. \end{cases}$$

We say that $f \in \mathcal{R}(\Omega)$ (i.e, f is Riemann integrable) if $\tilde{f} \in \mathcal{R}(B^n)$. Also, we define

$$\int_\Omega f = \int_{B^n} \tilde{f}.$$

The Riemann integration of f over Ω.

The following properties are immediate:

Fact: Let $\Omega \subseteq \mathbb{R}^n$ be bounded, $f, g \in \mathcal{R}(\Omega)$, $\alpha \in \mathbb{R}$. Then:

1. $f + r g \in \mathcal{R}(\Omega)$ & $\displaystyle\int_\Omega f + rg = \int_\Omega f + r \int_\Omega g$.

2. $fg \in \mathcal{R}(\Omega)$, where $(fg)(x) = f(x)\, g(x)$ $\forall\, x \in \Omega$.

3. $|f| \in \mathcal{R}(\Omega)$ & $\displaystyle\left|\int_\Omega f\right| \leq \int_\Omega |f|$.

4. If $f(x) \geq g(x)$ $\forall x \in \Omega$, then $\displaystyle\int_\Omega f \geq \int_\Omega g$.

— x —

Def: Suppose $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$, $f \in \mathcal{B}(\Omega)$. The volume of the region generated by the "Surface" $z = f(x)$ & Ω is defined by

$$\int_\Omega f.$$