Lecture 4: Group ring, nilpotents, zero divisors

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⊛ Let $R = \mathbb{Z}[X]$, $I_1 = 2\mathbb{Z}[X]$, $I_2 = X\,\mathbb{Z}[X]$

$\underline{I_1 + I_2} = \{ f(X) \in R \mid f(0) \text{ is even} \} \doteq T$

$f(X) = X^3 + 3X^2 + 2$

$I_1 \ni (2) + (X^3 + 3X^2) \in I_2$

$I_1 \ni (2 + 4X) + (X^3 + 3X^2 - 4X) \in I_2$.

$T \subseteq I_1 + I_2$ ✓

$f \in I_1 + I_2$ then $f = 2g + Xh$ for some $g, h \in R$

Let $g = b_m X^m + b_{m-1} X^{m-1} + \dots + b_1 X + b_0$, $b_i \in \mathbb{Z}$

then $f = \underbrace{2b_0}_{\text{even}} + Xh_1$ for some $h_1 \in R$

"or simply $f(0) = 2g(0)$ is even"

⊛ Def^n Group ring: Let $(R, +, \cdot)$ be a (comm) ring with unity and G be a group.

The group ring $R[G] = \{ \underbrace{r_1 g_1 + r_2 g_2 + \dots + r_n g_n}_{\text{formal sum}} \mid r_i \in R \;\&\; g_i \in G,\ n \geq 1 \}$

More precisely, $\underline{R[G]} = \{ f : G \to R \mid f(g) = 0$ for all but finitely many $g \in G \}$

$\boxed{f \leftrightarrow \sum_{g \in G} f(g) g}$

Given $a, b \in R[G]$, $(a \oplus b)(g) = a(g) + b(g)$

$(a \odot b)(g) = \sum_{h \in G} a(h)\, b(h^{-1}g) \in R$

(Explicitly $(r_1 g_1 + r_2 g_2 + r_3 g_3) \cdot (s_1 h_1 + s_2 h_2) = r_1 s_1 g_1 h_1 + r_1 s_2 g_1 h_2 + r_2 s_1 g_2 h_1 + r_2 s_2 g_2 h_2 + r_3 s_1 g_3 h_1 + r_3 s_2 g_3 h_2$ then combine same terms)

$g, g' \in G$

$1g,\ 1g' \in R[G]$

$1g \odot 1g'$ "

$1gg' \in R[G]$

Check $(R[G], \oplus, \odot)$ is a ring with unity. (Exc.)

$0_{R[G]}$ is the zero function.

$\underline{1_{R[G]}} = ?$ $1_R e$ where $e \in G$ is identity.

is the multiplicative identity of $R[G]$.

$(a \cdot 1_{R[G]})(g) = \sum_{h \in G} a(h)\, 1(h^{-1}g) = a(g)\, 1(g^{-1}g) + 0$
$= a(g)$

$\Rightarrow$ $1_{R[G]}$ is the unity of $R[G]$.

<u>Example</u>: $R = \mathbb{Z}$, 1) $G = \{e\}$ then $R[G] \cong R$

2) $G = \{e, g, g^2\}$ a group of order 3.

$\mathbb{Z}[G] = \{ae + bg + cg^2 \mid a, b, c \in \mathbb{Z}\} \overset{?}{\not\cong} \mathbb{Z}^3$ $(\pm 1, \pm 1, \pm 1)$

$1 \cdot g \in (\mathbb{Z}[G])^\times$ $\because (1 \cdot g)^3 = 1 \cdot e$

?

3) $G = S_3 = \{e, \underset{\sigma}{\underset{\shortparallel}{(1\,2\,3)}}, \underset{\tau}{\underset{\shortparallel}{(1\,2)}}, \underset{\sigma^2}{\underset{\shortparallel}{(3\,2\,1)}}, \underset{\sigma\tau}{\underset{\shortparallel}{(1\,3)}}, \underset{\tau\sigma}{\underset{\shortparallel}{(2\,3)}}\}$

$$\boxed{\mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^3 - 1)}$$

$\mathbb{Z}[S_3]$ is not commutative

$\checkmark$

Def^n: Let R be a ring with unity.
An element $a \in R$ is said to be a zero divisor if $\exists\, b \in R$, $b \neq 0$ s.t. $ab = 0$.
An element $c \in R$ is called nilpotent if $\exists\, n \geq 1$, $n \in \mathbb{N}$ s.t. $c^n = 0$.

Example: 1) $R = \mathbb{Z}$, zero divisor: 0
nilpotent: 0

2) $\mathbb{Z}/12\mathbb{Z}$, zero divisor: $2, 4, 6, 3, 9, 8, 10, 0$
nilpotent: $0, 6$

3) $\mathbb{Z}[G]$, $G = \{e, g, g^2\}$
zero divisor: 0, $(e-g)(e+g+g^2)$
$= e + \cancel{g} + \cancel{g^2} - \cancel{g} - \cancel{g^2} - g^3$
$= 0$

4) $M_{n\times n}(\mathbb{R})$, zero divisor: Any singular matrix A

$$A \cdot \operatorname{adj}(A) = \det(A)\, I = 0$$

$$A : \mathbb{R}^n \xrightarrow{A\cdot} \mathbb{R}^n$$

$$0 \neq v \in \ker(A) = \operatorname{Null}(A)$$

$$B = [\, v \mid 0 \mid 0 \mid \cdots \mid 0 \,] \neq 0$$

$$AB = 0$$

(*) If $a \in R$ is not a zero divisor then it is called a nonzero divisor in R.

* Let R be a nonzero ring with unity. Then units are nonzero divisors

Pf: Let $u \in R$ be a unit. If

$$ub = 0 \quad \text{in } R$$

$$\Rightarrow \quad u^{-1}ub = u^{-1}0$$

$$\Rightarrow \quad b = 0$$

$\boxed{}$

* Nilpotents are zero divisors.