

$$\therefore \underline{f}(x) = \int_{B^n} \underline{f}(x, y) dV(y) \quad \text{and} \quad \bar{f}(x) = \int_{B^n} \bar{f}(x, y) dV(y) \quad \forall x \in B^m.$$

$\underline{f}(x, y)$

We now prove a BIG result :

$$f: B^{m+n} \rightarrow \mathbb{R}; B^{m+n} = B^m \times B^n$$

(x, y)

Thm: Suppose $f \in R(B^{m+n})$. Then $\underline{f}, \bar{f} \in R(B^m)$

$$\int_{B^m} \underline{f} dV(x) = \int_{B^m} \bar{f} dV(x) = \int_{B^{m+n}} f dV(x, y).$$

i.e.,

$$\int_{B^m} \left(\int_{B^n} f(x, y) dV(y) \right) dV(x) = \int_{B^m} \left(\int_{B^n} \bar{f}(x, y) dV(y) \right) dV(x)$$

$$= \int_{B^{m+n}} f(x, y) dV.$$

$\therefore \int_{B^{m+n}} f = \text{iterations of lower \& upper.}$
They always exist

Cor: [Note]: Similar statement holds if we consider slice f_x 's
w.r.t. y : If $f \in R(B^{m+n})$, then

$$\int f(x, y) dV_y = \int_{B^n} \left(\int_{B^m} f(x, y) dV(x) \right) dV(y) = \int_{B^n} \left(\underbrace{\int_{B^m} f(x, y) dV(x)}_{\in R(B^m)} \right) dV(y) \in R(B^n)$$

- the proof will be the same.

Cor: Suppose $f \in R(B^{m+n})$. If f_x is the fn. $\int f(x, y) dV(y)$, then,

$\int f(x, y) dV(y) \in R(B^m)$, [or] $\int_{B^m} f(x, y) dV(x) \in R(B^n)$

i.e., the iteration is integrable.

$f_x \in R(B^n) \quad \forall x \in B^m$

$f_x: B^n \rightarrow \mathbb{R}$

$f_x(y) = \int f(x, y) dV(y)$

$f_y \in R(B^m)$

$f_y(x) = \int f(x, y) dV(y)$

$$\int f dV = \int_{B^{m+n}} \int_{B^m} f(x, y) dV(y) dV(x).$$

$$\int f dV = \int_{B^{m+n}} \int_{B^n} f(x, y) dV(x) dV(y).$$

(20)

Remark: Suppose $f \in C(B^{m+n})$. Then for each $x \in B^m$,
the slice $f_x \in C(B^n)$. [Ily $f_y \in C(B^m) \forall y \in B^n$].

Trivial. $\xrightarrow{\text{Then}}$ $x \mapsto \int_{B^n} f(x, y) dV(y) \in \mathcal{R}(B^m)$ $\boxed{=}$
 $\xrightarrow{B^n}$ Then, in particular $\int_{B^n} f_x = \int_{B^n} f_x$. (Ily f_{xy}).
Thus:

Corr: If $f \in C(B^{m+n})$, then

$$\int_{B^{m+n}} f dV = \int_{B^m} \left(\int_{B^n} f dV(y) \right) dV(x) = \int_{B^m} \left(\int_{B^n} f dV(y) \right) dV(x).$$

Continuous
version of
Fubini

Proof of Fubini's thm: $\xrightarrow{\text{To prove: } \int_{B^m} f dx = \int_{B^m} \bar{f} dx = \int f dx}$ - We need to show that f, \bar{f} are integrable.

Let $P \in \mathcal{P}(B^{m+n})$. So $P = P^m \times P^n$ for some partitions $P^m \in \mathcal{P}(B^m)$ & $P^n \in \mathcal{P}(B^n)$.

We aim to prove f integrable. Now $L(f, P) = \sum_{\alpha \in \Lambda(P)} m_\alpha \times \nu(B_\alpha^{m+n})$,

$$= \sum_{\alpha(P^m) \in \Lambda(P^m)} \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} \nu(B_{(\alpha(P^m), \alpha(P^n))}^{m+n})$$

$\xrightarrow{\alpha \in \Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)}$ $\Rightarrow \alpha = \alpha(P^m) \times \alpha(P^n)$.

$$= \nu(B_{\alpha(P^m)}^m) \times \nu(B_{\alpha(P^n)}^n).$$

T.D: $\sum_{\alpha} \exists P \in \mathcal{P}(B^m)$
 $\epsilon \quad U(f, P) - L(f, P) < \epsilon$
 $\cap \quad U(f, P) - L(f, P) = 0$

$$\sum_{\alpha(P^m) \in \Lambda(P^m)} \left(\sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} \nu(B_{(\alpha(P^m), \alpha(P^n))}^n) \right) \times \nu(B_{\alpha(P^m)}^m).$$

Attack.

For each $x \in B^m$, set $m_{\alpha(P^m)}(x) = \inf_{y \in B^n} f(x, y)$ $\xrightarrow{\alpha = f_x(y)}$.

$$\therefore \forall x \in B_{\alpha(P^m)}^m, m_{\alpha(P^m)}(x) \geq m_{(\alpha(P^m), \alpha(P^n))}.$$

$$\text{If } U(f, P) \therefore \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} \nu(B_{\alpha(P^n)}^m)$$

$$\leq \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{\alpha(P^n)}(x) \times \nu(B_{\alpha(P^n)}^m) \quad \forall x \in B_{\alpha(P^n)}^m$$

~~$\vdash \vdash \vdash \vdash \vdash$~~

$$= L(f_x, P^n) \leq \int_{B^n} f_x \, d\nu(y) \quad \forall x \in B_{\alpha(P^n)}^m.$$

$$= \underline{\underline{f}}(x)$$

$$\leq \inf_{x \in B_{\alpha(P^m)}^m} \underline{\underline{f}} \quad . \quad (= \underline{\underline{m}}_{B_{\alpha(P^m)}^m})$$

$$\begin{aligned} \therefore L(f, P) &\leq \sum_{\alpha(P^m) \in \Lambda(P^m)} \underline{\underline{m}}_{B_{\alpha(P^m)}^m} \nu(B_{\alpha(P^m)}^m) \\ &= L(\underline{\underline{f}}, P^m). \quad (\because P = P^m \times P^n). \end{aligned}$$

$$\text{i.e., } \underline{\underline{L}(f, P)} \leq L(\underline{\underline{f}}, P^m).$$

If $U(f, P) \geq U(\underline{\underline{f}}, P^m)$. Now the rest is standard.
 [Recall: we aim at proving that $\underline{\underline{f}}$ is integrable.]

$$\therefore U(\underline{\underline{f}}, P^m) - L(\underline{\underline{f}}, P^m) \leq U(f, P) - L(f, P).$$

Also ~~each~~ $\underline{\underline{f}} \in R(B^{m+n})$.

$$\Rightarrow \underline{\underline{f}} \in R(B^m). \quad \text{If } \underline{\underline{f}} \in R(B^n)$$

$$\text{Finally, } L(f, P) \leq L(\underline{\underline{f}}, P^m) \leq U(\underline{\underline{f}}, P^m) \leq U(f, P)$$

$$\Rightarrow L(f, P) \leq L(\underline{\underline{f}}, P^m) \leq U(f, P). \Rightarrow \int_{B^m} \underline{\underline{f}} = \int_{B^{m+n}} f \quad \blacksquare$$

$\nexists P \neq P^m$

Cor: If $f \in C(B^n)$, then

$$\int_{B^n} f d\nu(x) = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Also denoted by

$$\int_{B^n} f(x_1, \dots, x_n) dx_1 \cdots dx_n$$

all commutes.

Moreover (and in particular: $n=2$):

$$\int_{[a_1, b_1] \times [a_2, b_2]} f d\nu = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) dx_2 dx_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) dx_1 dx_2$$

(i.e., Changing order of integration is okay).

Very useful.

Eg: 1) $\mathbb{R}[x_1, \dots, x_n] \subseteq \mathcal{R}(B^n)$. \forall box $B^n \subseteq \mathbb{R}^n$

polynomial ring (\because cont.)

$$\begin{aligned} 2) \int_{[0,1]^2} xy \, dx dy &= \int_0^1 \left(\int_0^1 xy \, dx \right) dy \\ &= \int_0^1 y \times \left[\frac{x^2}{2} \right]_0^1 dy = \frac{1}{2} \times \int_0^1 y \, dy \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} \text{OR. } \int_{[0,1]^2} xy \, dx dy &= \left(\int_0^1 x \, dx \right) \times \left(\int_0^1 y \, dy \right) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

$$3) f(x, y) = \begin{cases} -1 & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q}, \\ 1 & \text{if } (x, y) \in \mathbb{Q}^c \times \mathbb{Q}^c, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \notin \mathcal{R}([0,1]^2)$.
— HW —

Sad fellow.

4) Let $f(x) = c \quad \forall x \in B^n$. Then

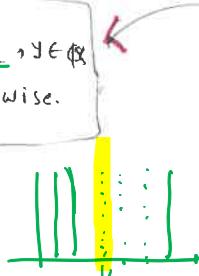
$$\int_{B^n} f \, dv = c \times v(B^n).$$

— We will do more soon —

Note: We have seen an example of $f \in \mathcal{B}(B^2)$ (or $f \in \mathcal{B}(B^n)$) s.t. $f \in R(B^2)$ BUT f is NOT continuous!!

Q: No. of points of discontinuity? Ans: 1 in that particular example.

Page 16: $f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$



We can confidently deal with finitely many points of discontinuity in that example too!!

BUT, the real question is: Can we integrate a bdd fn. with too many points of discontinuity?

Ans: Seems vague question: unless you spell out "too many" means "how many"!! i.e., What is "too many"?

"Vague answer": As long as those points of discontinuity fail to generate a "volume"? That's a good one.

Let's get into this (already it's too complicated):

Clearly, we need to talk about Small sets (Small in terms of Volume)

"small sets"

Def: A subset A of \mathbb{R}^n is of measure zero if for $\varepsilon > 0$

exists a countable boxes $\{B_m^n\}_{m=1}^{\infty}$ with "the total volume" $\sum m v(B_m^n) < \varepsilon$.

a cover of A by

i.e., $\bigcup_{m=1}^{\infty} B_m^n \supseteq A \quad \& \quad \sum_m v(B_m^n) < \varepsilon.$

$$\begin{aligned} & \text{C.R.} \\ & A = \{a\} \\ & \varepsilon > 0 \\ & \text{s.t. } (a - \varepsilon, a + \varepsilon) \subset (-\infty, \infty) = \mathbb{R}. \end{aligned}$$

[Note: If n is clear from the context, then we will simply write B instead of B^n .]

Note: 1) Let A be a finite set. Then A is of measure zero.

$\varepsilon > 0$ is given.

Suppose $A = \{a_1, \dots, a_n\}$, $\varepsilon > 0$. Consider $r = \frac{\varepsilon}{2n}$ & open boxes around a_m s.t. $v(B_m) < \frac{\varepsilon}{2n}, \dots$. \rightarrow compactness argument?

2) Suppose $A = \{a_n\}_{n=1}^{\infty}$ \rightarrow a countable set.

: Choose box B_n $\forall n \geq 1$ s.t.

$$v(B_n) < \frac{\varepsilon}{2^n} \quad \& \quad a_n \in B_n.$$

$$\Rightarrow \sum v(B_n) < \varepsilon. \Rightarrow A \text{ is of measure zero.}$$

3) Use the above to prove the following: Suppose $\{A_n\}$ be a seqn. of subsets of \mathbb{R}^n of measure 0. Then $A := \bigcup_{n=1}^{\infty} A_n$ is also of measure 0.

Proof: Let $\varepsilon > 0$. $\forall n$, get C_n = a countable collection of boxes s.t. $\bigcup_{c \in C_n} c \supseteq A_n \quad \& \quad \sum_{c \in C_n} v(c) < \frac{\varepsilon}{2^n}$.

Then $C = \bigcup_{n=1}^{\infty} C_n$ is a countable collection of boxes which covers $A = \bigcup_{n=1}^{\infty} A_n \quad \& \quad \sum_{c \in C} v(c) < \varepsilon$. \square

Countable union of
Sets of measure
0 is of measure
0.

Standard trick.

We need one more notion:

Let $f \in \mathcal{B}(\Omega_n)$. Let $x_0 \in \Omega_n$.

open in \mathbb{R}^n

Define $\text{osc}(f, x_0) := \lim_{\delta \rightarrow 0} \text{osc}_{\delta}(f, x_0)$ by:
 Oscillation of f at x_0

$$\text{osc}(f, x_0) := \lim_{\delta \rightarrow 0} \left[\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) \right].$$

$\therefore \text{osc}$ is a fn. : $\Omega_n \rightarrow \mathbb{R}$ defined by

$$\text{osc}(x) = \text{osc}(f, x), \quad \forall x \in \Omega_n.$$

Note: (1) $\forall \delta > 0, \sup_{B_\delta(x_0)} f > \inf_{B_\delta(x_0)} f \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f > 0$.

~~f is bdd~~ Moreover $\delta \downarrow \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \downarrow$

$\therefore \text{osc}(f, x_0)$ exists $\forall x_0 \in \Omega_n$.

(2) f is cont. at $x_0 \Leftrightarrow \text{osc}(f, x_0) = 0$. $\left[\because \lim_{\delta \rightarrow 0} [\cdot] = 0 \right]$

osc is a measure of discontinuity of f at points in Ω_n .

Proof: Let $\text{osc}(f, x_0) = 0$. Let $\varepsilon > 0$.

$\exists \delta > 0$ s.t. $\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) < \varepsilon$.

$$= \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \quad \begin{array}{l} \text{we know} \\ \text{this} \\ \text{useful.} \end{array}$$

$$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in B_\delta(x_0)$$

In particular: $|f(x) - f(x_0)| < \varepsilon \quad \forall x \in B_\delta(x_0)$.

\Leftarrow Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in B_\delta(x_0)$.

$$\Rightarrow |f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(y) - f(x_0)| < \varepsilon.$$

$$\Rightarrow \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \leq \varepsilon \quad \Rightarrow \text{osc}(f, x_0) = 0. \quad \begin{array}{l} \forall x, y \in B_\delta(x_0) \\ \square \end{array}$$

$$(3) \quad \text{osc}(f, x_0) = \inf_{\delta > 0} \left\{ \sup \left\{ |f(z) - f(y)| : z, y \in B_\delta(x_0) \right\} \right\}, \quad \text{just observed.}$$

(4) Let $\alpha > 0$. Then $\{x \in C_n : \text{osc}(f, x) \geq \alpha\}$ is closed. Let $f \in \mathcal{B}(C_n)$, $C_n \subseteq \mathbb{R}^n$ closed.

Then $\{x \in C_n : \text{osc}(f, x) \geq \alpha\}$ is closed like $C_n = \mathbb{R}^n$.
 $\{x \in C_n : \text{osc}(f, x) < \alpha\}$ is open in \mathbb{R}^n . But Assume $\text{int}(C_n) \neq \emptyset$.

Proof. Let $C := \{x \in C_n : \text{osc}(f, x) \geq \alpha\}$. Claim: C is closed.
 Let $x \in \mathbb{R}^n \setminus C$.

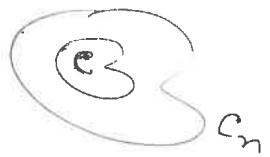
$\Rightarrow x \notin C_n$ or $x \in C_n$ but $x \notin C$.

case 1

case 2

$\exists \delta > 0$ s.t. $B_\delta(x) \subseteq C_n^c$ [$\because C_n$ is closed]

$\Rightarrow B_\delta(x) \subseteq C^c \Rightarrow C^c$ open. $\Rightarrow C$ closed.



Let $x \in C_n$ but $x \notin C$.

$\Rightarrow x \in C_n$ & $\text{osc}(f, x) < \alpha$

$\therefore \exists \delta > 0$ s.t. $\sup \left\{ |f(z) - f(y)| : z, y \in B_\delta(x) \right\} < \alpha$.

(maybe $\leq \alpha$
+ does not matter)

Consider an open box $B \subseteq B_\delta(x)$.
 Let $y \in B$.

$\therefore \forall y \in B$, open $\exists \delta_1 > 0$ s.t. $B_{\delta_1}(y) \subseteq B_\delta(x)$.

\therefore In particular: $\sup \left\{ |f(z) - f(w)| : z, w \in B_{\delta_1}(y) \right\} < \alpha$

$\Rightarrow \text{osc}(f, y) < \alpha$.

Thus, $\forall y \in B$, open $\text{osc}(f, y) < \alpha$. $\Rightarrow B \subseteq \mathbb{R}^n \setminus C$.

$\Rightarrow C$ is closed. \square