Lecture 18: Ideals in Localization, Local rings

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⊛ Let R be a comm ring with unity and S a mult. subset of R. Then $\varphi(s)$ is a unit in $S^{-1}R$ $\forall s \in S$. Here

$$\varphi : R \to S^{-1}R \quad \text{is the natural map.}$$
$$r \mapsto \frac{r}{1}$$

<u>Thm</u> (Universal property of Localization):
Let R be comm ring with unity. $S \subseteq R$ be a mult. subset of R. Let

$$f : R \longrightarrow A \quad \text{be a ring homomorphism}$$

where A is a comm ring with unity such that $\forall s \in S$, $f(s)$ is a unit in A. Then $\exists!$ ring homo.

$$\hat{f} : S^{-1}R \to A \quad \text{s.t.} \quad \hat{f} \circ \varphi = f.$$

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[dr, "\varphi"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & A \\
& S^{-1}R \arrow[u, dashed, "\exists! \hat{f}"']
\end{tikzcd}$$

⊛ $0 \in S \Rightarrow S^{-1}R = \{0\}$

⊛ $\forall s \in S$ nonzero divisor $\iff \varphi : R \to S^{-1}R$ is injective

⊛ $\ker(\varphi) = \{ r \in R \mid sr = 0 \text{ for some } s \in S \}$.

⊛ Let R be an integral domain, a field K is called the <u>field of fractions</u> of R if R is a subring of K and no proper subfield of K contains R.

⊛ R an int domain and $S = R \setminus \{0\}$ then $\varphi : R \to S^{-1}R$ is injective ring homo & $S^{-1}R$ is the field of fractions of R where we identify R with $\varphi(R)$.

(*) $\mathbb{Z}[\pi]$, $\mathbb{Z}[x]$, $\mathbb{Z}[e]$ are isom rings
$\mathbb{Q}(\pi)$, $\mathbb{Q}(x)$, $\mathbb{Q}(e)$ are their fraction fields

More formally, let R be an integral domain. The <u>field of fractions of R</u> is an injective ring homo. $i: R \hookrightarrow K$ s.t. K is a field and for any subfield K_0 of K containing $i(R)$, $K_0 = K$.

$$\boxed{\frac{\mathbb{Z}[x]}{(x^3-2)}}$$

(*) $\mathbb{Z}[\sqrt[3]{2}] \subseteq \mathbb{Q}(\sqrt[3]{2})$

$\mathbb{Z}[\sqrt[3]{2}] \cong \mathbb{Z}[\omega\sqrt[3]{2}]$ $\qquad$ $\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(\omega\sqrt[3]{2})$

(*) Field of fraction is unique upto unique isomorphism. i.e.

i.e.
$$\begin{tikzcd} R \arrow[r, hook, "i"] \arrow[dr, hook, "i'"'] & K \arrow[d, dashed, "\exists! f"] \\ & K' \end{tikzcd}$$
are field of fractions

$\exists! f: K \to K'$ s.t. f is an isom. & $f \circ i = i'$

(*) S consist of units then
$\varphi: R \to S^{-1}R$ is an isomorphism.
$r \mapsto \frac{r}{1}$

<u>Pf:</u> $id: R \to R$ is a ring homo.

$id(s) = s$ is a unit $\forall\, s \in S$.

Universal prop of localization $\Longrightarrow$ $\exists!\ \widehat{id}$ s.t.

$$\begin{tikzcd} R \arrow[r, "id"] \arrow[dr, "\varphi"'] & R \\ & S^{-1}R \arrow[u, "\exists!\widehat{id}"'] \end{tikzcd}$$

$\frac{r}{s} \mapsto s^{-1}r = \frac{r}{s}$

$\widehat{id} \circ \varphi = id$

φ is injective ($\because \widehat{id} \circ \varphi$ is injective)

Let $\frac{r}{s} \in S^{-1}R$ then

$$\frac{r}{s} = \frac{r}{1} \cdot \frac{1}{s} = \frac{r}{1} \cdot \frac{s^{-1}}{1} = \varphi(r)\varphi(s^{-1})$$

Note $s^{-1} \in R$

Ex: R a comm ring with unity, $x \in R$; $S = \{1, x, x^2, \dots\}$

Then $S^{-1}R = R[\frac{1}{x}] \cong \dfrac{R[Z]}{(xZ-1)}$

$\uparrow$ Notation

where $R[Z]$ is the polynomial ring over R.

<u>Pf:</u> $f: R \longrightarrow \dfrac{R[Z]}{(xZ-1)}$

$r \longmapsto \bar{r}$

$$R \xrightarrow{i} R[Z] \xrightarrow{q} \frac{R[Z]}{(xZ-1)}$$

$f = q \circ i$

Note: $f(x)\,\bar{Z} = \bar{x}\,\bar{Z} = \bar{1}$ $\left(\because xZ-1 \in (xZ-1)\right)$

$\Rightarrow f(x)$ is a unit in $\dfrac{R[Z]}{(xZ-1)}$

$\Rightarrow \exists \tilde{f}: S^{-1}R \longrightarrow \dfrac{R[Z]}{(xZ-1)}$ s.t.

$\tilde{f}\left(\frac{r}{s}\right) = f(s)^{-1} f(r)$ $\forall s \in S$

$= \bar{Z}^n \bar{r}$

Note $s = x^n$ for some n.

$$\alpha : R[Z] \longrightarrow S^{-1}R$$
$$Z \longmapsto \frac{1}{x}$$
$$a_n Z^n + a_{n-1}Z^{n-1} + \dots + a_0 \longmapsto \frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \dots + \frac{a_1}{x} + \frac{a_0}{1}$$
where $a_i \in R$

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, "\varphi"'] & R[Z] \arrow[r, "q"] \arrow[d, "\alpha"] & \frac{R[Z]}{(xZ-1)} \arrow[dl, bend right, "\tilde{\alpha}"'] \\ & S^{-1}R \arrow[ur, bend right, "\tilde{f}"'] & \end{tikzcd}$$

α is a ring homo. $\left(\because\ \alpha(p(Z)) = p\left(\frac{1}{x}\right)\right)$

$$\alpha(xZ-1) = \frac{x}{x} - \frac{1}{1} = 0$$
$$\Rightarrow (xZ-1) \subseteq \ker(\varphi)$$

$$\tilde{\alpha}\circ q = \alpha, \quad \alpha\circ i = \varphi$$
$$\tilde{f}\circ\varphi = q\circ i\ (= f)$$

1st isom. $\Rightarrow$ $$\tilde{\alpha} : \frac{R[Z]}{(xZ-1)} \longrightarrow S^{-1}R$$

$$\tilde{\alpha}\left(\overline{p(Z)}\right) \longmapsto \alpha(p(Z)) = p\left(\frac{1}{x}\right)$$

Check: $$\tilde{\alpha}\circ\tilde{f}\left(\frac{r}{s}\right) = \tilde{\alpha}\left(\overline{Z}^n\,\overline{r}\right) = r\,\frac{1}{x^n}$$
$$= \frac{r}{s}$$

($s \in S \Rightarrow s = x^n$ for some n)

$$\tilde{f}\circ\tilde{\alpha}\left(\overline{p(Z)}\right) = \tilde{f}\left(p\left(\frac{1}{x}\right)\right) = \tilde{f}\left(\frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \dots + \frac{a_0}{1}\right)$$

where $p(Z) = a_n Z^n + \dots + a_0$

$$= \tilde{f}\left(\frac{a_n + a_{n-1}x + \dots + a_0 x^n}{x^n}\right)$$
$$= \overline{Z}^n\left(\overline{a_n} + \overline{a_{n-1}x} + \dots + \overline{a_0 x^n}\right)$$
$$= \overline{a_n}\,\overline{Z}^n + \overline{a_{n-1}}\,\overline{Z}^{n-1} + \dots + \overline{a_1}\,\overline{Z} + \overline{a_0}$$
$$= \overline{p(Z)}$$

$\boxtimes$

Local rings: A comm ring with unity is called a local ring if it has exactly one maximal ideal.

Examples: 1) Fields
2) valuation rings
3) $\mathbb{Z}/4\mathbb{Z}$, More generally $\mathbb{Z}/p^n\mathbb{Z}$ $n \geq 1$ & p prime are local rings

$R = \mathbb{Z}$, $S = \{1, 2, 2^2, \ldots\}$

$S^{-1}R = \mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$

$\overset{\shortparallel}{\{\frac{a}{b} \mid b = 2^n \text{ for some } n\}}$

4) $R = \mathbb{Z}$, $S = \{\text{odd integers}\} = \mathbb{Z} \setminus 2\mathbb{Z}$

$S^{-1}R = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \text{ odd}\} = \mathbb{Z}_{(2)}$

$\mathbb{Z} \subseteq S^{-1}R \subseteq \mathbb{Q}$

(2)

$2S^{-1}R$ is the maximal ideal of $S^{-1}R$. $\overset{\shortparallel}{\mathbb{Z}_{(2)}}$

So $S^{-1}R$ is local ring