1. Show that the column operations C_{jk}, $C_j(\alpha)$ and $C_{jk}(\alpha)$ on a matrix A is equivalent to postmultiplying A by elementary matrices $\mathbf{E}_{jk}$, $\mathbf{E}_j(\alpha)$ and $\mathbf{E}_{kj}(\beta)$ (note interchange of k and j) respectively.

2. Compute the inverses of $\mathbf{E}_{ik}$, $\mathbf{E}_i(\alpha)$ and $\mathbf{E}_{ik}(\beta)$.

3. Show that if $\mathbf{A}$ is obtained from $\mathbf{B}$ from elementary row operations, then $\mathbf{B}$ can be obtained from $\mathbf{A}$ by elementary row operations.

4. Let $\mathbf{P}$ be a permutation matrix, that is one obtained by permuting the columns of $\mathbf{I}$. What is the inverse of $\mathbf{P}$?

5. Do elementary row operations alter the column space of a matix?

6. Use Gaussian elimination to find the inverse of the matrix $\mathbf{A}$ and solve $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 6 \\ 11 \\ 3 \end{pmatrix}$$

7. Let $\mathbf{A}$ be an $m \times n$ matrix of rank r. Show that there exists non-singular matrices $\mathbf{P}$ of order m and $\mathbf{Q}$ of order n such that

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

(hint: Use elementary row and column operations).
Show next that there is a matrix $\mathbf{R}$ of order $m \times r$ and a matrix $\mathbf{S}$ of order $r \times n$ such that $\mathbf{A} = \mathbf{RS}$ (this is called a *rank-factorization* of $\mathbf{A}$). Show that the columns of $\mathbf{R}$ form a basis of the column space of $\mathbf{A}$ and the rows of $\mathbf{S}$ form a basis of the row space of $\mathbf{A}$.

8. A square matrix $\mathbf{H}$ of order n is said to be in *Hermite canonical form* (HCF) if: (i) $\mathbf{H}$ is upper triangular. (ii) each diagonal entry of $\mathbf{H}$ is 0 or 1. (iii) the i-th row of $\mathbf{H}$ is null if $h_{ii} = 0$, and (iv) the i-th column of $\mathbf{H}$ is $\mathbf{e_i}$ if $h_{ii} = 1$.

   - Given a HCF matrix $\mathbf{H}$, how does one obtain the rank and column space of $\mathbf{H}$?
   - Argue that any square matrix can be reduced to a matrix in HCF by elementary row operations.
   - Argue that for any HCF matrix $\mathbf{H}$ there exists a permutation matrix $\mathbf{P}$ such that

   $$\mathbf{P^T H P} = \begin{bmatrix} \mathbf{I}_r & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

   - Show that any HCF matrix $\mathbf{H}$ is idempotent, that is $\mathbf{H}^2 = \mathbf{H}$.
   - Show that the HCF matrix $\mathbf{H}$ obtained from a square matrix by elementary row operations is unique.
   (hint: suppose $\mathbf{G}$ and $\mathbf{H}$ are two possibilities. First argue that there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{G} = \mathbf{PH}$, from this obtain $\mathbf{GH} = \mathbf{G}$ and $\mathbf{HG} = \mathbf{H}$)

9. Solve the system $\mathbf{A}x = b$ (first check whether it is consistent and then find the general solution). Also find for $\mathbf{A}$, a g-inverse, the rank, a rank factorisation and a basis of the null space.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -4 \\ 0 & 1 & -3 \\ -1 & 0 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

10. As in the above exercise with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

11. Let $\mathbf{A}$ be an $m \times n$ matrix of rank $r < m$. Let $\mathbf{B}$ be a matrix in echelon form obtained from $\mathbf{A}$ by elementary row operations and let $\mathbf{P}$ be the transforming matrix. Show that the last $m - r$ rows of $\mathbf{P}$ form a basis of the solution space of $\mathbf{x^T A} = 0$.

12. For any two matrices $\mathbf{A}$ and $\mathbf{G}$ show that the following are equivalent.

    - $\mathbf{G}$ is a g-inverse of $\mathbf{A}$.
    - $\mathbf{AGA} = \mathbf{A}$.
    - $\mathbf{AG}$ is idempotent and $\rho(\mathbf{AG}) = \rho(\mathbf{A})$.
    - $\mathbf{GA}$ is idempotent and $\rho(\mathbf{GA}) = \rho(\mathbf{A})$.

13. Show that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are two distinct g-inverses of $\mathbf{A}$, show that $\alpha\mathbf{G}_1 + (1 - \alpha)\mathbf{G}_2$ is also a g-inverse for any $\alpha \in \mathbb{R}$.

14. Let $\mathbf{G}$ be a g-inverse of $\mathbf{A}$. Prove that

$$\{\mathbf{G} + (\mathbf{I} - \mathbf{GA})\mathbf{U} + \mathbf{V}(\mathbf{I} - \mathbf{AG}) : \mathbf{U}, \mathbf{V} \text{ arbitrary}\}$$

    is the class of all g-inverses of $\mathbf{A}$.

15. Find the $\mathbf{LU}$ decomposition of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

16. (a) Show that $< \mathbf{A}, \mathbf{B} >= \text{trace}(\mathbf{B}^*\mathbf{A})$ is an inner product on $\mathbb{C}^{m\times n}$.

    (b) Show that $< \mathbf{A}, \mathbf{B} >= \sum_{i=1}^n a_{ii}\bar{b}_{ii}$ is not an inner product on $\mathbb{C}^{n\times n}$.

    (c) Levt V be the vector space of all real valued continuous functions on an interval $[a, b]$. Fix $h \in V$ with $h(t) > 0$ for all $t \in [a, b]$. Show that $< f, g >= \int_a^b h(t)f(t)g(t)dt$ is an inner product on V.

    (d) Prove that $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ is not a norm on $\mathbb{R}^n$ for $0 < p < 1$ while it is a norm for $p \geq 1$ (called ℓ_p norm). Show also that $\|\mathbf{x}\|_\infty := \max_{1\leq i\leq n} |x_i|$ is a norm on $\mathbb{R}^n$ and for each fixed $\mathbf{x}$, $\|\mathbf{x}\|_p \to \|\mathbf{x}\|_\infty$ as $p \to \infty$.

17. Let $N(\cdot)$ be a norm on $\mathbb{R}^n$. For an $n \times n$ matrix, define

$$\|A\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{N(A\mathbf{x})}{N(\mathbf{x})}.$$

    (a) Show that $\|\cdot\|$ is a norm on $\mathbb{R}^{n\times n}$. This is the *matrix norm induced by the vector norm* N.

    (b) Show that $\|\mathbf{I}\| = 1$.

    (c) Show that $N(\mathbf{Ax}) \leq \|\mathbf{A}\|N(\mathbf{x})$.

    (d) Show that $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$.

    (e) Show that the matrix norm induced by the ℓ_1 norm on $\mathbb{R}^n$ is $\|\mathbf{A}\| = \max_j \sum_{i=1}^n |a_{ij}|$.

    (f) Show that the matrix norm induced by the ℓ_∞ norm on $\mathbb{R}^n$ is $\|\mathbf{A}\| = \max_i \sum_{j=1}^n |a_{ij}|$.

    (g) Let $\|\mathbf{A}\| < 1$, where the matrix norm is induced by a vector norm $N(\cdot)$. Then show that $\mathbf{I} - \mathbf{A}$ and $\mathbf{I} + \mathbf{A}$ are nonsingular. Show that

$$\frac{1}{1 + \|\mathbf{A}\|} \leq \|(\mathbf{I} - \mathbf{A})^{-1}\| \leq \frac{1}{1 - \|\mathbf{A}\|}.$$

    What are the bounds on $\|(\mathbf{I} + \mathbf{A})^{-1}\|$?

(h) Show that $\|\mathbf{A}\|_E = \sqrt{\sum_{i,j} |a_{i,j}|^2}$ is a norm on $\mathbb{R}^{n\times n}$.

(i) Let $\|\mathbf{A}\|_2$ be the matrix norm induced by ℓ_2 norm on $\mathbb{R}^n$. Show that $\|\mathbf{A}\|_2 \le \|\mathbf{A}\|_E \le \sqrt{n}\|\mathbf{A}\|_2$.

18. Show that any orthogonal set $\mathbf{A}$ (i.e. any two vectors in $\mathbf{A}$ are orthogonal) not containing the null vector is linearly independent.

19. Let $\mathbf{B} = \{\mathbf{x}_1, \cdots \mathbf{x}_n\}$ be an orthonormal basis of an inner product space V. Then show for any $\mathbf{x} \in V$

$$\mathbf{x} = \sum_{j=1}^{n} < \mathbf{x}, \mathbf{x}_j > \mathbf{x}_j$$

20. Recall the Gram-Schmidt orthogonalization process: Starting with a set $\{\mathbf{x}_1, \cdots \mathbf{x}_n\}$ recursively define for $k = 1, 2 \cdots n$

$$\mathbf{z}_k = \mathbf{x}_k - \sum_{j=1}^{k-1} < \mathbf{x}_k, \mathbf{y}_j > \mathbf{y}_j, \qquad \mathbf{y}_k = \begin{cases} \frac{\mathbf{z}_k}{\|\mathbf{z}_k\|} & \text{if } \mathbf{z}_k \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{z}_k = \mathbf{0} \end{cases}$$

The nonzero vectors among $\{\mathbf{y}_1, \cdots \mathbf{y}_n\}$ form an orthonormal set whose span is the subspace generated by $\{\mathbf{x}_1, \cdots \mathbf{x}_n\}$. Show this.

21. Show $< \mathbf{x}, \mathbf{y} >= \mathbf{y}^T\mathbf{A}\mathbf{x}$ on $\mathbb{R}^3$ is an inner product where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

Find an orthonormal basis of $S = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ and then extend to an orthonormal basis of $\mathbb{R}^3$.

22. (**QR**-decomposition) Let $\mathbf{A}$ be an $n \times s$ matrix with rank p. Let $\mathbf{y}_1, \mathbf{y}_2 \cdots \mathbf{y}_s$ be the vectors obtianed when the Gram-Schmidt orthogonalization process is applied to the columns of $\mathbf{A}$. Let $\mathbf{P} = [\mathbf{y}_1 : \mathbf{y}_2 : \cdots \mathbf{y}_s]$ and let $\mathbf{U}$ be the $s \times s$ upper triangular matrix $((u_{ik}))$ where

$$u_{ik} = \begin{cases} < \mathbf{A}_{*k}, \mathbf{y}_i > & \text{if } i < k \\ \|\mathbf{z}_k\| & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Then show that $\mathbf{A} = \mathbf{PU}$. Also show that if $\mathbf{Q}$ is the submatrix of $\mathbf{P}$ formed by the non-null columns and $\mathbf{R}$ is the submatrix of $\mathbf{U}$ formed by the corresponding rows, then $(\mathbf{Q}, \mathbf{R})$ is a rank-factorization of $\mathbf{A}$ and $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}_p$. When $\mathbf{A}$ is of full column rank, then $(\mathbf{Q}, \mathbf{R}) = (\mathbf{P}, \mathbf{U})$ is known as a **QR**-decomposition of $\mathbf{A}$.

Find the **QR**-decomposition of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 4 \end{bmatrix}$$

Show that the **QR** decomposition is unique if we insist that the diagonal elements of $\mathbf{R}$ are real and positive, i.e. if $\mathbf{A}$ is of full column rank, show that there exist unique matrices $\mathbf{Q}$ and $\mathbf{R}$ such that $\mathbf{A} = \mathbf{QR}$, $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$, $\mathbf{R}$ is upper triangular and $r_{ii} > 0$ for all i.

23. For subspaces S and T of a vector space, show that the following are equivalent: (i) $S + T$ is direct (ii) $S \cap T = \{\mathbf{0}\}$ (iii) $\mathbf{x} \in S - \{\mathbf{0}\}$, $\mathbf{y} \in T - \{0\}$ then $\mathbf{x}, \mathbf{y}$ are linearly independent. (iv) if $\mathbf{x} + \mathbf{y} = \mathbf{0}$, $\mathbf{x} \in S, \mathbf{y} \in T$ then $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$. (v) $\dim(S + T) = \dim(S) + \dim(T)$.

24. Consider the vector space $\mathbb{R}^5$ and the subspace

$$S = \{(\xi_1, \cdots \xi_5) : \xi_1 + \xi_4 = 0,\ 2\xi_1 + \xi_3 + \xi_5 = 0\}.$$

Find two complements of S. Then find the projection of the vector $(1,1,1,1,1)$ into S along each complement. Also find the projector matrices of S along each complement.

25. Let the underlying vector space be F^n where $F = \mathbb{R}$ or $\mathbb{C}$. Show that the following are equivalent: (i) $\mathbf{A}$ is a projector (ii) $\mathbf{A}^2 = \mathbf{A}$ (iii) $\mathcal{N}(\mathbf{A}) = \mathcal{C}(\mathbf{I} - \mathbf{A})$ (iv) $\rho(\mathbf{A}) + \rho(\mathbf{I} - \mathbf{A}) = n$ (v) $\mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{I} - \mathbf{A})$ is direct.

26. For a subspace S of an inner product space V denote the orthogonal complement of S

$$S^{\perp} = \{\mathbf{y} \in V :< \mathbf{y}, \mathbf{x} >= 0 \text{ for every } \mathbf{x} \in S\}$$

Consider the subspaces $S = \{(\xi_1, \cdots \xi_4) : \xi_1 = \xi_2 = \xi_3\}$ and $T = \{(\xi_1, \cdots \xi_4) : \xi_1 = \xi_2 \text{ and } \xi_4 = 0\}$ of $\mathbb{R}^4$. Find $S + T$, $S^{\perp}, T^{\perp}$

Let $B = \{\mathbf{x}_1, \cdots \mathbf{x}_k\}$ be an orthonormal basis of S and extend it to an orthonormal basis $\{\mathbf{x}_1 \cdots \mathbf{x}_n\}$ of V. Show that $S^{\perp}$ is the span of $\{\mathbf{x}_{k+1} \cdots \mathbf{x}_n\}$.

Show for subspaces S and T, $(S+T)^{\perp} = S^{\perp} \cap T^{\perp}$, $(S \cap T)^{\perp} = S^{\perp} + T^{\perp}$ and $(S^{\perp})^{\perp} = S$

What is the orthogonal projection $\mathbf{y}$ of a vector $\mathbf{x}$ into S along $S^{\perp}$? What can we say about $\mathbf{x} - \mathbf{y}$?

27. Show that the inverse (if it exists) of a lower triangular matrix is lower triangular. Similarly, show that the inverse of an upper triangular matrix is upper triangular.

28. Let $\mathbf{A}$ be an $n \times n$ real nonsingular matrix. All the matrices in the exercise have entries in $\mathbb{R}$.

   (a) Argue that there are permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ such that all the leading principal submatrices of $\mathbf{B} = \mathbf{PAQ}$ are nonsingular.

   (b) Show that the matrix $\mathbf{B}$ can be written uniquely as $\mathbf{B} = \mathbf{LDU}$ where $\mathbf{D}$ is a diagonal matrix, $\mathbf{L}$ is a lower triangular matrix with diagonal entries equal to 1, and $\mathbf{U}$ is an upper triangular matrix with diagonal entries equal to 1.

   (c) Show that any 2×2 diagonal matrix $\mathbf{D}$ of determinant 1 can be written as $\mathbf{D} = \mathbf{M}_1\mathbf{M}_2 \cdots \mathbf{M}_k$, where each of the 2×2 matrices $\mathbf{M}_i$ are either upper or lower triangular with all diagonal entries equal to 1. (*hint: think of appropriate elementary row and/or column operations which will convert the diagonal matrix to a lower or upper triangular matrix with* 1*'s on the diagonal*)

   (d) Show that any $n \times n$ diagonal matrix $\mathbf{D}$ of determinant 1 can be written as $\mathbf{D} = \mathbf{M}_1\mathbf{M}_2 \cdots \mathbf{M}_k$, where each of the $n \times n$ matrices $\mathbf{M}_i$ are either upper or lower triangular with all diagonal entries equal to 1. (*hint: induction*)

   (e) Assume $\det \mathbf{B} > 0$. Conclude that we can write $\mathbf{B} = (\det \mathbf{B})^{1/n} \cdot \mathbf{M}_1\mathbf{M}_2 \cdots \mathbf{M}_k$, where each of the $n \times n$ matrices $\mathbf{M}_i$ are either upper or lower triangular with all diagonal entries equal to 1.

   (f) Assume $\det \mathbf{B} > 0$. Consider the set $\mathcal{B}_n(\mathbb{R}) = \{\mathbf{M} \in \mathbb{R}^{n \times n} : \det \mathbf{M} = \det \mathbf{B}\}$. Construct a continuous map $f : [0,1] \to \mathcal{B}_n(\mathbb{R})$ such that $f(0) = \mathbf{B}$ and $f(1) = (\det \mathbf{B})^{1/n}\mathbf{I}$.

Remark: The conclusions of this exercise will continue to hold even if $\det \mathbf{B} < 0$, but then one has to consider matrices over $\mathbb{C}$, simply because $(\det \mathbf{B})^{1/n}$ would be a complex number.

29. Find the orthogonal projector $\mathbf{P}_\mathbf{A}$ into the column space of $\mathbf{A}$ where

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & -2 \\ -2 & 1 & -3 \end{bmatrix}$$

30. Denote by $< \cdot, \cdot >$ the canonical inner product on $\mathbb{C}^n$. Let $\mathbf{A}$ be an $n \times n$ complex matrix. Then show that the following are equivalent: (i) $\mathbf{A}$ is unitary. (ii) $< \mathbf{Ax}, \mathbf{Ay} >=< \mathbf{x}, \mathbf{y} >$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. (iii) $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$. (iv) $\|\mathbf{Ax}\| = 1$ whenever $\|\mathbf{x}\| = 1$ and $\mathbf{x} \in \mathbb{C}^n$. (v) $\|\mathbf{Ax} - \mathbf{Ay}\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. (vi) $\{\mathbf{Ax}_1, \cdots \mathbf{Ax}_n\}$ is an orthonormal basis of $\mathbb{C}^n$ whenever $\{\mathbf{x}_1, \cdots \mathbf{x}_n\}$ is.
    If $\mathbf{A}$ is real then show that the above continues to hold if unitary is replaced by orthogonal, and $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$.

31. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, that is $\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then show that there is an orthogonal matrix $\mathbf{A}$ and a vector $\mathbf{c}$ such that $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{c}$ for all $x \in \mathbb{R}^n$. (*hint: consider the function* $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$.)

32. Let $\chi = \{\mathbf{x}_1 \cdots \mathbf{x}_n\}$ be an orthonormal basis of an inner product space V, and let $\chi' = \{\mathbf{x}'_1 \cdots \mathbf{x}'_n\}$ be another orthonormal basis of V. Let $\mathbf{P}$ be the $n \times n$ transition matrix given by $\mathbf{x}'_k = \sum_{i=1}^n p_{ik}\mathbf{x}_i$. Show that $\mathbf{P}$ is unitary.