

Lecture 33: Applications of structure theorem

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Prop: Let R be a PID and F be a free R -module of rank n . Let N be a submodule of F . Then N is a free R -mod of rank $m \leq n$. Moreover there is a basis x_1, \dots, x_n of F and $\exists a_1, \dots, a_m \in R^\times$ s.t. $a_1 | a_2 | \dots | a_m$ and $\{a_1 x_1, a_2 x_2, \dots, a_m x_m\}$ is a basis of N .

Thm (Str thm): Let R be a PID and M be a f.g. R -mod. Then

$$M \cong R^k \oplus R/(a_1) \oplus R/(a_2) \oplus \dots \oplus R/(a_m)$$

where $k = \text{rank}(M)$ and $a_1, \dots, a_m \in R$ are nonzero nonunits s.t. $a_1 | a_2 | a_3 | \dots | a_m$. Here k and m could be 0.

Thm: (Str thm version 2) Let R be a PID and M be a f.g. R -mod. Then

$$M \cong R^k \oplus R/p_1^{r_{11}} \oplus R/p_1^{r_{12}} \oplus R/p_1^{r_{13}} \oplus \dots \oplus R/p_1^{r_{1m_1}} \\ \oplus R/p_2^{r_{21}} \oplus R/p_2^{r_{22}} \oplus \dots \oplus R/p_2^{r_{2m_2}} \\ \vdots \\ \oplus R/p_m^{r_{m1}} \oplus R/p_m^{r_{m2}} \oplus \dots \oplus R/p_m^{r_{mm_m}}$$

$k = \text{rank}(M)$

where p_1, \dots, p_m are irreducible elements of R , $r_{ij} \geq r_{i,j-1} \forall 1 \leq i \leq m, 2 \leq j \leq n_i$ are positive integers.

Q. G is a f.g. abelian group then

$$G \cong \mathbb{Z}^k \oplus \mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_r^{n_r}$$

p_1, \dots, p_r are prime nos.

$|G| = 36$, G is abelian

$$36 = 3^2 \cdot 2^2$$

$$G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \text{ or } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

$$n_1 = 2, p_1 = 2, p_2 = 3$$

$$n_2 = 2$$

$$\& n_{11} = n_{12} = 1 = n_{21} = n_{22}$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \text{ or } \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$$

$$\mathbb{Z}/36\mathbb{Z}$$

Version 1 & 2 are equivalent -

$$\text{Let } a_m = p_1^{g_{11}} p_2^{g_{21}} \dots p_m^{g_{m1}}$$

$$g_{ij} \geq 1 \quad 1 \leq j \leq m$$

$$a_{m-1} = p_1^{g_{12}} p_2^{g_{22}} p_3^{g_{32}} \dots p_m^{g_{m2}}$$

(upto unit)

where $g_{ij} \geq 0$.

$$\vdots$$

$$a_2 = p_1^{g_{1,m-1}} p_2^{g_{2,m-1}} \dots p_m^{g_{m,m-1}}$$

$$a_1 = p_1^{g_{1m}} p_2^{g_{2m}} \dots p_m^{g_{mm}}$$

Since $a_j | a_{j+1}$ we have $g_{i,m-j+1} \leq g_{i,m-j}$
 $\forall 1 \leq i \leq m$
 $1 \leq j \leq m-1$

Note that some g_{ij} may be 0.

\swarrow CRT

$$\text{Now } R/(a_m) \cong R/(p_1^{g_{11}}) \oplus \dots \oplus R/(p_m^{g_{m1}})$$

$$(\because (p_i^{g_{1i}}, p_j^{g_{1j}}) = 1 \text{ if } i \neq j)$$

Now rearrange

$$M \cong R^k \oplus R/(a_1) \oplus \dots \oplus R/(a_m)$$

$$\cong R^k \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{11}}\right) \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{21}}\right) \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{m1}}\right)$$

$$\oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{12}}\right) \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{22}}\right) \oplus \dots \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{m2}}\right)$$

$$\oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{1m}}\right) \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{2m}}\right) \oplus \dots \oplus R/\left(\begin{pmatrix} p_1 & \dots & p_m \end{pmatrix}^{r_{mm}}\right)$$

Note that r_{1m}, r_{2m} may be 0.
 Write the transpose^{of above} and ignore
 the terms where $r_{ij} = 0$ to
 obtain version 2.

Version 2 \Rightarrow version 1

$$M \cong R^k \oplus R/p_1^{r_{11}} \oplus \dots \oplus R/p_1^{r_{1n_1}}$$

$$r_{ij} \geq r_{i,j-1}$$

$$R/p_m^{r_{m1}} \oplus \dots \oplus R/p_m^{r_{mn_m}}$$

$$\text{Let } l = \max(n_1, \dots, n_m)$$

$$M \cong R^k \oplus \underbrace{0 \oplus \dots \oplus R/p_1^{r_{11}} \dots \oplus R/p_1^{r_{1n_1}}}_{l} \oplus \underbrace{0 \oplus \dots \oplus R/p_2^{r_{21}} \oplus R/p_2^{r_{2n_2}}}_{l} \oplus \dots \oplus \underbrace{0 \oplus \dots \oplus R/p_m^{r_{m1}} \dots \oplus R/p_m^{r_{mn_m}}}_{l}$$

$$\text{Let } a_m = p_1^{r_{1n_m}} p_2^{r_{2n_m}} \dots p_m^{r_{mn_m}}$$

$$a_{m-1} = p_1^{r_{1,n_{m-1}}} p_2^{r_{2,n_{m-1}}} \dots p_m^{r_{m,n_{m-1}}}$$

$$a_1 = p_1^{r_{1,n_1-m+1}} p_2^{r_{2,n_2-m+1}} \dots p_m^{r_{m,n_m-m+1}}$$

$$M \cong R^k \oplus R/(a_1) \oplus \dots \oplus R/(a_m)$$

Also condition $r_{ij} \geq r_{i,j-1}$

$$\Rightarrow a_1 | a_2 | \dots | a_m$$

convention \leftarrow
is $r_{ij} = 0$ if $j \leq 0$.



Thm: (Rational form)
 $A \in M_{n \times n}(k)$ k a field.

where A is similar to
 $a_i = x^{n_i} + b_{n_i-1}x^{n_i-1} + \dots + b_1x + b_0$
 $1 \leq i \leq m$

$$\begin{bmatrix} R_{a_1} & & 0 \\ & \ddots & \\ 0 & & R_{a_m} \end{bmatrix}$$

$$\sum_{i=1}^m n_i = n$$

$$a_1 | a_2 | \dots | a_m$$

where $R_{a_i} = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & 0 & 1 \\ 0 & & & 0 \end{bmatrix}_{n_i \times n_i}$ matrix

$$\exists P \text{ nonsing s.t. } P^{-1}AP = R$$

Pf: $A : \underset{\substack{\uparrow \\ V}}{k^n} \rightarrow \underset{\substack{\uparrow \\ V}}{k^n}$ k -lin map.

This gives V as $k[x]$ -mod str.

via $v \in V$

$$X \cdot v = Av$$

$$f(X) = a_n X^n + \dots + a_1 X + a_0$$

In general $f(X) \cdot v = f(A)v$
 $= (a_n A^n + \dots + a_1 A + a_0 I)v$

$k[x]$ is a PID and V is f.g. $k[x]$ -mod

What is $\text{rank}(V)$ as a $k[x]$ -mod?

$m_A(x) \in k[x]$
 minimal poly

$$m_A(x) \cdot v = m_A(A)v = 0 \quad (\text{Caley-Hamilton})$$

$\Rightarrow V$ is torsion $k[x]$ -mod

So by str. thm

$$V \cong R/(a_1) \oplus \dots \oplus R/(a_m) \text{ as } R\text{-mod}$$

$$\text{where } R = k[X]$$

$$a_1 | a_2 | \dots | a_m \quad a_i \in k[X]$$

May assume a_i are monic

$$A.v = X.v$$

$$a_i(X) = X^{n_i} + b_{n_i-1}X^{n_i-1} + \dots + b_0$$

So choose the basis

$$R/(a_i) \cong k[X]/(a_i(X)) \cong k \oplus k\bar{X} \oplus k\bar{X}^2 \oplus \dots \oplus k\bar{X}^{n_i-1}$$

$$B = \{1, \bar{X}, \bar{X}^2, \bar{X}^3, \dots, \bar{X}^{n_i-1}, \dots\}$$

What is the matrix of A w.r.t B .

$$X \cdot 1 = 0 \cdot 1 + 1\bar{X} + 0\bar{X}^2 + \dots$$

$$X \cdot \bar{X} = \bar{X}^2 = 0 \cdot 1 + 0\bar{X} + 1\bar{X}^2 + \dots$$

$$X \cdot \bar{X}^{n-2} = \bar{X}^{n-1} = 0 \cdot 1 + 0\bar{X} + \dots + 0\bar{X}^{n-2} + 1\bar{X}^{n-1}$$

$$X \cdot \bar{X}^{n-1} = \bar{X}^n = -b_0 1 - b_1 \bar{X} - b_2 \bar{X}^2 - \dots$$

$$\begin{bmatrix} 0 & & & -b_{n-1} \\ 1 & 0 & & \vdots \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 - b_0 \end{bmatrix}$$