Lecture 3: More on polynomial rings and ideals

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(*) R a ring with unity then $R[x]$ consisting of polynomials with coefficient in R with addition and multiplication is a ring. We will denote elements of $R[x]$ as $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ instead of a function $a: \mathbb{Z}_{\geq 0} \to R$ with $a(i) = \begin{cases} a_i & i \leq n \\ 0 & \text{o.w.} \end{cases}$

Also $R \to R[x]$ is an injective ring homo.
$a \mapsto a$

So R is a subring of $R[x]$. Also if R is comm then $R[x]$ is comm ring.

<u>Ex</u> $\mathbb{Z}[\pi] \subseteq \mathbb{R}$ subring.

<u>Claim:</u> $\mathbb{Z}[x]$ the poly ring is isom to $\mathbb{Z}[\pi]$

Pf: $\varphi: \mathbb{Z}[x] \to \mathbb{Z}[\pi]$

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \mapsto a_n \pi^n + a_{n-1} \pi^{n-1} + \dots + a_0$$

$$\varphi(f(x)) = f(\pi)$$

$$\varphi(f+g) = (f+g)(\pi) = f(\pi) + g(\pi) = \varphi(f) + \varphi(g)$$

$\varphi(n) = n \quad \forall n \in \mathbb{Z}.$

lll'ly: $\varphi(fg) = \varphi(f)\varphi(g)$. So φ is a homomorphism

φ is clearly surjective.

"φ is injective" is a consequence of a deep result which says that "π is not a root of any nonzero polynomial with integer coefficient."

$$\varphi(f) = \varphi(g) \Rightarrow \varphi(f-g) = 0$$
$$\Rightarrow (f-g)(\pi) = 0$$
$$\Rightarrow f-g = 0$$
$$\Rightarrow f = g.$$

Hen φ is injective. So φ is an isom.

So now we have more ways of constructing new rings.

$(\mathbb{Z}[X])[Y]\ (\doteq \mathbb{Z}[X,Y])$

More generally, R a ring then the poly ring in n-variable $R[X_1,\dots,X_n] := (((R[X_1])[X_2])\cdots)[X_n]$

$R_0 := R$, $R_{i+1} = R_i[X]$ $i \geq 0$ then

$R[X_1,\dots,X_n] := R_n$

⊛ Ideals in a ring.

⊛ R a comm ring with unity. Let $\{I_\alpha \mid \alpha \in \Omega\}$ be a collection of ideals in R. Then

$I = \bigcap_{\alpha\in\Omega} I_\alpha$ is an ideal.

Pf: Same as subring

Let $a \in I$ & $r \in R$

$a \in I_\alpha\ \forall \alpha \in \Omega \Rightarrow ra \in I_\alpha\ \forall \alpha \in \Omega$

$\Rightarrow ra \in \bigcap_{\alpha\in\Omega} I_\alpha = I$

⊛ What about $I_1 \cup I_2$ if I_1, I_2 are ideals in R? Is it an ideal?

Ex In $\mathbb{Z}$ $2\mathbb{Z}$, $3\mathbb{Z}$ ideals in $\mathbb{Z}$

$2\mathbb{Z}$ " {even integers}, $3\mathbb{Z}$ " {integers which are multiple of 3}

$S = 2\mathbb{Z} \cup 3\mathbb{Z}$ $1 \notin S$ but $3, -2 \in S$

$\Rightarrow 3 + (-2) \notin S$

⊛ Let I_1, I_2 be ideals of a ring R then
$I_1 + I_2 :=$ the ideal generated by I_1 and I_2
i.e. the smallest ideal containing
I_1 & I_2.

⊛ Let $S \subseteq R$ be a subset of a ring R comm ring with unity
then $\langle S \rangle$ denotes the ideal generated by S, i.e. smallest ideal containing S

$$\langle S \rangle := \bigcap \{ I \mid I \text{ ideal of } R, S \subseteq I \}$$

Prop: Let $S \subseteq R$ be a subset of a ring R then

$$\langle S \rangle = \{ r_1 a_1 + r_2 a_2 + \dots + r_n a_n \mid n \geq 1,\ a_1, \dots, a_n \in S \ \& \ r_1, \dots, r_n \in R \}$$

Pf: Let $T = RHS$
Clearly $T \subseteq \langle S \rangle$ ($\because r_i a_i \in \langle S \rangle$ & $\sum_{i=1}^{n} r_i a_i \in \langle S \rangle$)

$S \subseteq T$. So enough to show
T is an ideal.
T is clearly closed under addition
Let $\sum_{i=1}^{n} r_i a_i \in T$ & $r \in R$ then

$$r\left(\sum_{i=1}^{n} r_i a_i\right) = \sum_{i=1}^{n} \underbrace{(r r_i)}_{\text{are in } R} \underbrace{a_i}_{\text{these are in } S} \in T$$

So T is an ideal. Hence $\langle S \rangle \subseteq T$. ☒

⊛ Let I_1, I_2 be ideals of R. Then

$$I_1 + I_2 = \{ a + b \mid a \in I_1 \ \& \ b \in I_2 \}$$

$$\begin{array}{c} \shortparallel \\ T \end{array}$$

Pf: RHS $\subseteq I_1 + I_2$ i.e. $T \subseteq I_1 + I_2$

Note $T \supseteq I_1 \cup I_2$ ✓

Let $a+b \in T$ & $a'+b' \in T$

i.e. $a, a' \in I_1$ & $b, b' \in I_2$

$$a+b+a'+b' = \underbrace{(a+a')}_{\in I_1} + \underbrace{(b+b')}_{\in I_2} \in T$$

Let $r \in R$, $a+b \in T$ $a \in I_1$ & $b \in I_2$

$$\Rightarrow r(a+b) = \underbrace{ra}_{\in I_1} + \underbrace{rb}_{\in I_2} \in T$$

Hence T is an ideal $\Rightarrow T = I_1 + I_2$.

Ex: In $\mathbb{Z}$, Compute $2\mathbb{Z} + 3\mathbb{Z}$? $\mathbb{Z}$

In $\mathbb{Z}[x]$ " $2\mathbb{Z}[x] + x\mathbb{Z}[x]$?

$= \{ f(x) \in \mathbb{Z}[x] \mid f(0) \text{ is even} \} = T$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in T$$

a_0 is even

$$f(x) = x(a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1) + a_0$$

$f(x) \in 2\mathbb{Z}[x] + x\mathbb{Z}[x] \Rightarrow T \subseteq 2\mathbb{Z}[x] + x\mathbb{Z}[x]$

$f(x) = 2g(x) + xh(x)$ for some $g(x), h(x) \in \mathbb{Z}[x]$

$g(x) = b_m x^m + \cdots + b_1 x + b_0$

$= 2b_0 + xh_1(x)$ where $h_1(x) = h(x) + b_m x^{m-1} + \cdots + b_2 x + b_1 \in \mathbb{Z}[x]$

$\Rightarrow f(x) \in T$.

$\Rightarrow 2\mathbb{Z}[x] + x\mathbb{Z}[x] \subseteq T$. Hence equality