Sequence & Series of functions:

For $\{a_n\} \subseteq \mathbb{R}$ to converge (to a, say): We only need to know "if a_n's are close enough".

The Cauchy Criterion: "for $\varepsilon > 0$ if $\exists\, N \in \mathbb{N}$ ($N = N(\varepsilon)$) $\ni$ $|a_n - a_m| < \varepsilon$ $\forall\, n, m \geq N$"

$d(a_n, a_m) < \varepsilon$

usual metric of $\mathbb{R}_u$.

$\Leftrightarrow$ $\{a_n\}$ is Convergent.

(*) metric structure, i.e., Completeness of $\mathbb{R}$ plays THE role. [i.e. $\mathbb{R}_u$ is complete.]

Aim: Consider $\underbrace{\{f_n\} \subseteq \mathcal{F}(S)}_{\text{Set of fn's: } S \to \mathbb{R}}$: a Seqn of fn's.

Subset of $\mathbb{R}$.

"Figure out" Convergency of $\{f_n\}$, i.e. explain $f_n \longrightarrow f$ for some $f \in \mathcal{F}(S)$.

[?? But what about metric structure like $\mathbb{R}_u$ in (*)?]

The real question!!

Obs: $\forall\, x \in S$, $\{f_n(x)\} \subseteq \mathbb{R}_u$. Here we can certainly talk about convergency of $\underset{\subseteq \mathbb{R}_u.}{\{f_n(x)\}}$ $\forall\, x \in S$.

Known as pointwise convergency.

Def: $\{f_n\}$ Converges pointwise (on the set S) to f if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall\, x \in S.$$

We write $f_n \xrightarrow{\text{pointwise}} f$ & Say: f is the pointwise limit of $\{f_n\}$.

Obs: Given $\{f_n\}$ & $f \in \mathcal{F}(S)$, $f_n \xrightarrow{\text{point}} f$

$\Longleftrightarrow$ For $\varepsilon > 0$, $x \in S$ $\exists\, N \equiv N(\varepsilon, x) \in \mathbb{N}$ $\ni$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq N.$$

$\Longleftrightarrow$ For $\varepsilon > 0$, $x \in S$, $\exists\, N \equiv N(\varepsilon, x) \in \mathbb{N}$ $\ni$ $|f_n(x) - f_m(x)| < \varepsilon$ $\forall\, n, m \geq N$.

Remark: ① If $\{f_n(x)\}$ is Cauchy, $\forall x \in S$, then we can safely define

$$f(x) := \lim_{n\to\infty} f_n(x) \quad \forall x.$$

② The notion of pointwise convergency is still not well done:

We must talk about $\boxed{\text{closedness}}$ of $\{f_n\}$!!

i.e. A $\underbrace{\text{METRIC}}_{\text{How to get ?}}$ flavour of / among functions !!

To the least : one may think about getting rid of x for $N(\varepsilon, x)$ in the pointwise defn. :

We say, f is the uniform limit of $\{f_n\}$.

Def. Let $\{f_n\} \cup \{f\} \subseteq \mathcal{F}(S)$. We say that $f_n \xrightarrow{uni} f$ (ie f_n converges to f) (on S) if for $\varepsilon > 0$ $\exists$ $N \equiv N(\varepsilon) \in \mathbb{N}$

$$\ni \cdot \quad \underbrace{|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N \;\&\; \forall x \in S.}_{\text{This indicates the following:}}$$

Def: For $f, g \in \mathcal{F}(S)$, define

Sup metric $\quad d(f,g) := \sup_{x\in S} |f(x) - g(x)|.$

& $\|f\|_\infty := \sup_{x \in S} |f(x)|.$

Sup norm.

\# Depending on S & f, d or $\|\cdot\|_\infty$ will carry good meaning !!

For instance: $d(\cdot,\cdot)$ is a metric (complete) on $C[a,b]$ or $B[a,b]$.

Fact: $f_n \xrightarrow{unif} f \iff f_n \to f$ in $\|\cdot\|_\infty$

i.e., $d(f_n, f) \to 0$.

\# [∵ d is THE METRIC which can take the role of distance among functions !

~~eg:~~

Obs: Suppose $f_n \xrightarrow{\text{Unif.}} f$. Then $f_n \xrightarrow{\text{point.}} f$. i.e.: unif $\Rightarrow$ pointwise.

Proof: Easy: as $|f_n(x) - f(x)| < \varepsilon \quad \forall x \in S,\ n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N \ \& \ \forall x \in S.$

Q: "$\Leftarrow$" ? **No.** (A big NO.)

However: pointwise limit is the fn. to look for: for uniform convergence !!

i.e: pointwise is the 1st BUT NOT the last step.

eg: ① $f_n(x) = x^n. \quad \forall n \geq 0 \qquad x \in [0,1] = S.$

$\{1, x, x^2, \ldots\}$.

Now $\forall x \in [0,1)$, $x^n \to 0$.

& if $x = 1$, then $f_n(x) = f_n(1) = 1^n \to 1$..

So $f_n \xrightarrow{p} f$, where $f(x) = \begin{cases} 0 & \forall x \in [0,1) \\ 1 & x = 1. \end{cases}$

Obs: (a) Here $\{f_n\} \subseteq C[0,1]$. But the pointwise limit $f \notin C[0,1]$. $\Rightarrow$ point. is NOT good for $C[0,1]$!! $\Big\} \Rightarrow f_n \overset{u}{\not\to} f$

pointwise $\not\Rightarrow$ respecting cont.

$\not\Rightarrow$ —"— bdd

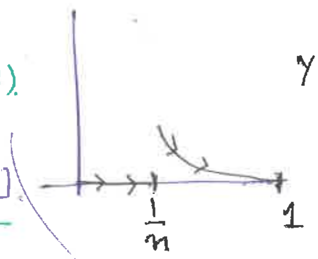
(b) $d(f_n, f) = \sup_{x \in [0,1]} |f_n(x) - f(x)| = 1 \quad \forall n$

$[\because f \equiv 0$ on $[0,1)$ & $|x^n| \leq 1$

$\Rightarrow f_n \overset{u}{\not\to} f$ but $f_n \xrightarrow{p} f$.

② For $\forall n \in \mathbb{N}$, define ~~Let~~ $f_n(x) = \begin{cases} 0 & 0 < x \leq \frac{1}{n} \\ \frac{1}{x} & \frac{1}{n} \leq x \leq 1 \end{cases} \qquad f_n \in \mathcal{F}((0,1]).$

Here $\lim_{n \to \infty} f_n(x) = \frac{1}{x} \quad \forall x \in (0,1]$. (fixed)



i.e. $f_n \xrightarrow{p} f$, when $f(x) = \frac{1}{x}$, $x \in (0,1]$

Clearly, $|f_n(x)| \leq n \quad \forall n.\ \& \ x \in (0,1]$

$\Rightarrow \{f_n\}$ is a bdd seq^n but the pointwise limit f is NOT.

$\therefore$ Pointwise limit does not respect bdd ness.

uniform does!!

Remark: However, if $\{f_n\}$ be s.t. $f_n \xrightarrow{u} f$ & $\{f_n\}$ are bdd, then f is also bdd: Let $\varepsilon > 0$. $\therefore \exists N \ni |f_n(x) - f(x)| < \varepsilon \;\; \forall n \geq N, \forall x \in S$.

$$\therefore |f(x)| \leq |f_N(x) - f(x)| + |f_N(x)|.$$
$$< \varepsilon + |f_N(x)|$$
$$\leq \varepsilon + \underbrace{\sup_{x\in S} |f_N(x)|}_{<\infty}. \qquad \forall x \in S.$$

$\Rightarrow$ f is bdd.

$\therefore$ unif. Conv. is good with bdd ness!!

Q: what about integration, diff (etc) under uniform/pointwise ??

(3) Suppose $f_n \xrightarrow{p} f$, $f_n \in \mathcal{R}[a,b]$. NO.

$\overset{?}{\Rightarrow} f \in \mathcal{R}[a,b]$? or $\underbrace{\lim \int f_n = \int \underbrace{\lim f_n}_{=f}}_{\text{interchanging limits ?}}$.

NO.

Ans: point: ✗
unif: ✓.

Let $f_n(x) = nx(1-x^2)^n$. $x \in [0,1]$.

Clearly, $f_n \in C[0,1] \subseteq \mathcal{R}[0,1]$.

Now for $x \in (0,1)$,

$$0 \leq f_{n+1}(x) = \cancel{n x(1-x^2)^n}\, (n+1)\, x\, (1-x^2)^{n+1}.$$
$$= \{nx(1-x^2)^n\} \times \left\{\frac{n+1}{n}(1-x^2)\right\}$$
$$= f_n(x) \left\{\frac{n+1}{n}(1-x^2)\right\}.$$

$\Rightarrow 0 \leq \dfrac{f_{n+1}(x)}{f_n(x)} = \dfrac{n+1}{n}(1-x^2) \longrightarrow (1-x^2) < 1$ as $x \in (0,1)$.

$\Leftarrow$ $\left[\lim \dfrac{a_{n+1}}{a_n} < 1 \Rightarrow a_n \to 0\right]$

$\Rightarrow \lim\limits_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1]$. [Note: if $x = 0, 1$, then $f_n(x) = 0 \;\forall n$]

$\Rightarrow f_n \xrightarrow{p} f$ where $f \equiv 0$.

Now $\int_0^1 f_n(x)\,dx = n\int_0^1 x(1-x^2)^n\,dx.$

$1-x^2 \to t.$
$2x\,dx = -dt$
$[0,1] \to [1,0].$

$= n\frac{1}{2}\int_0^1 t^n\,dt.$

$= \frac{1}{2}\frac{n}{n+1}. \longrightarrow \frac{1}{2}$ $(\neq 0)$ as $n\to\infty.$

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \longrightarrow \frac{1}{2} \neq \int_0^1 \lim f_n(x) = 0.$

Q: What if $f_n \xrightarrow{u} f$? Ans. Yes: wait.

(4). Convergency vs derivatives:

derivative vs. pointwise ? $\to$ X
— " — uniform ? $\to$ X.

Set $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$

Now $|f_n(x)| = \frac{1}{\sqrt{n}}|\sin(nx)| \leq \frac{1}{\sqrt{n}}$ $\forall\, x\in\mathbb{R}.$ $n\in\mathbb{N}.$

$\downarrow$
0 as $n\to\infty.$

$\forall x\in\mathbb{R}.$

$\Rightarrow |f_n(x) - 0| \leq \frac{1}{\sqrt{n}} \longrightarrow 0$ as $n\to\infty.$ $\forall\, x\in\mathbb{R}.$

RHS is x free
$\Rightarrow$ unif. conv. ✓

$\Rightarrow \left\{\frac{\sin(nx)}{\sqrt{n}}\right\}$ is uniformly convergent (& converges to the zero fn) on $\mathbb{R}$. Set $f\equiv 0$ on $\mathbb{R}.$

In particular $\therefore \lim_{n\to\infty} f_n = 0 f$ (uniform)

& $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall\, x\in\mathbb{R}.$ [$\because$ unif. $\Rightarrow$ point.]
$\| $
0 $\forall x\in\mathbb{R}.$ ✓

Now $f_n'(x) = \sqrt{n}\cos(nx).$ $\Longrightarrow \{f_n'\}$ does not even converge pointwise [For instance: $f_n'(0) = \sqrt{n}\ \forall n$].

But f' exists & $f' \equiv 0.$ So, in particular:

$f_n \xrightarrow{u} f$ but $f_n' \overset{p}{\not\longrightarrow} f'$ $(\Rightarrow f_n' \overset{u}{\not\longrightarrow} f').$

Some theory: But before that, we note again: Let $\{f_n\} \cup \{f\} \subseteq \mathcal{F}(S)$, $S \subseteq \mathbb{R}$.

Then $f_n \xrightarrow{u} f \iff M_n \to 0$, where $M_n = d(f_n, f) = \sup_{x \in S} |f_n(x) - f(x)|$

$\Downarrow$, $\not\Uparrow$

$f_n \xrightarrow{p} f$

$\iff$" For $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, m, n \geq N$

Cauchy criterion: $|f_n(x) - f_m(x)| < \varepsilon \quad \forall\, x \in S$.

Proof.

"$\Rightarrow$" let $\varepsilon > 0$. As $f_n \xrightarrow{u} f$ or $M_n \to 0$, $\exists\, N$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall\, x \in S$ & $n \geq N$.

$\Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall\, x \in S$ & $m, n \geq N$.

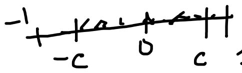
"$\Leftarrow$" $\because$ $\mathbb{R}_u$ Complete, $\lim_{n \to \infty} f_n(x) := f(x)$ defines a fn f on S.

Claim: $f_n \xrightarrow{u} f$. But, for $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$|f_m(x) - f_n(x)| < \varepsilon \quad \forall\, m, n \geq N$, $x \in S$.

$\therefore$ For any $n \geq N$, taking $m \to \infty$, we have:

$|f(x) - f_n(x)| < \varepsilon \quad \forall\, n \geq N$, $x \in S$.



eg: $f_n(x) = x^n$, $|x| \leq c$, $0 < c < 1$.

Then $f_n \xrightarrow{p} 0$. Also $M_n = \sup_{|x| \leq c} |x|^n \leq c^n \to 0 \Rightarrow f_n \xrightarrow{u} 0$ on $[-c, c]$.

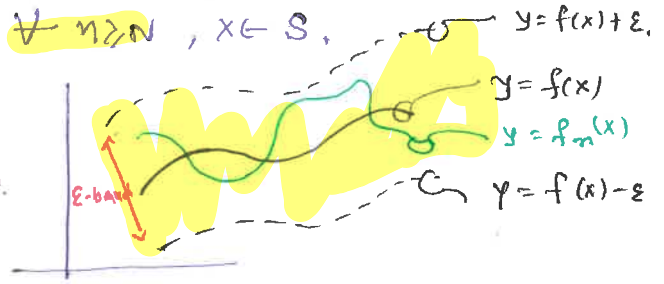
x-free.

~~Thm~~ (Continuity)

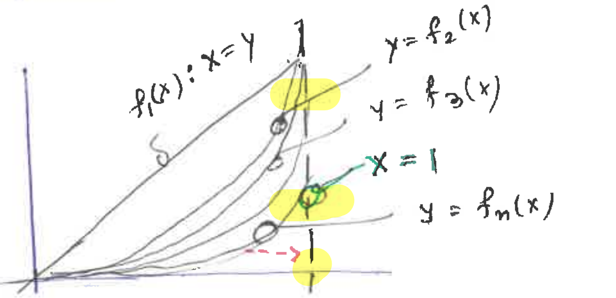
What's going on between pointwise $\overset{\not\Rightarrow}{\Leftarrow}$ uniform Convergences?

For u.c: we need a fn $f: S \to \mathbb{R}$ s.t.: given $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall\, n \geq N$, $x \in S$,

i.e. $\{(x, f_n(x)) : x \in S\} \subseteq$ "the band" $\forall\, n \geq N$.



So Clearly, $f(x) = x^n$ fails this band: (at $x = 1$).



Continuity

Thm: Let $f_n, f : S \longrightarrow \mathbb{R}$, $n \geq 1$, and let $f_n \xrightarrow{u} f$. If $x_0 \in S$ & f_n is Cont. at x_0 $\forall n$, then f is also Cont. at x_0.

So, $\lim\limits_{x \to x_0} \lim\limits_{n \to \infty} f_n(x) = \lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n(x)$.

Proof. ~~Let $x_0 \in S$~~

Let $x_0 \in \operatorname{int} S$

or $x_0 \in S \cap S'$.

or, just take $S = [a,b]$

[Limit point / or: just do it as in subspace metric. BTW: All results works as is for f_n's $f_n : \underbrace{(X, d)}_{\text{metric space}} \longrightarrow \mathbb{R}$]

Let $\varepsilon > 0$. $\exists\ N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall\ x \in S$. — (*)

$\because f_N$ ~~$\in \mathrm{Cont}(S)$~~ $\boxed{\text{is Cont. at } x_0}$, $\exists\ \delta > 0$ s.t. $|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}$

$\delta = \delta(x_0, \varepsilon, N)$ $\quad \forall\ |x - x_0| < \delta$ & $x \in S$.

$\therefore |f(x) - f(x_0)| \leq \underbrace{|f(x) - f_N(x)|}_{< \varepsilon/3 \text{ by } (*)} + |f_N(x) - f_N(x_0)| + \underbrace{|f_N(x_0) - f(x_0)|}_{< \frac{\varepsilon}{3} \text{ by } (*)}$

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \forall\ |x - x_0| < \delta$.

$\Rightarrow f$ is Cont. at x_0. ☒

\# Cor: If $f_n \in \mathrm{Cont}[0,1] = C[0,1]$ & $f_n \xrightarrow{u} f \Rightarrow f \in \mathrm{Cont}[0,1] = C[0,1]$.

Converse is not true: $S = (-1, 1)$; $f_n(x) = x^n$, $n \geq 1$. $f(x) := 0$ $\forall x \in S$.

Then $f_n \xrightarrow{p} f$. Here $f_n, f \in \mathrm{Cont}(S)$.

However $f_n \overset{u}{\not\longrightarrow} f$. (HW)

Integration

Thm: Let $f_n \in \mathcal{R}[a,b]$ $\forall n$ & $f_n \xrightarrow{u} f$. Then $f \in \mathcal{R}[a,b]$ &

$$\lim \int_a^b f_n = \int_a^b \lim f_n \left(= \int_a^b f \right). \longleftarrow \lim \int = \int \lim.$$

Proof: We know $f \in \mathcal{B}[a,b]$. Set $M_n = \sup\limits_{x \in [a,b]} |f_n(x) - f(x)|$. We know $M_n \to 0$.

Also: $f_n(x) - M_n \leq f(x) \leq f_n(x) + M_n$ $\forall\ x \in [a,b]$.

$|f_n(x) - f(x)| \leq M_n \Longrightarrow$
$\forall x \in [a,b]$

$\Rightarrow \underline{\int} f_n - M_n(b-a) \leq \underbrace{\underline{\int} f \leq \overline{\int} f}_{\text{get it separately}} \leq \overline{\int} f_n + M_n(b-a)$. — (*)

But $f_n \in \mathcal{R}[a,b] \Rightarrow \underline{\int} f_n = \overline{\int} f_n \quad \forall n.$

$\therefore \; 0 \leqslant \overline{\int} f - \underline{\int} f \leqslant 2 M_n (b-a) \longrightarrow 0$ as $n \to \infty$ $[\because M_n \to 0]$.

$\Rightarrow \overline{\int} f = \underline{\int} f \Rightarrow f \in \mathcal{R}[a,b].$

Again, $(*) \Rightarrow \displaystyle\int_a^b f_n - M_n(b-a) \leqslant \int_a^b f \leqslant \int_a^b f_n + M_n(b-a).$

$$\Rightarrow \left| \int_a^b f_n - \int_a^b f \right| \leqslant M_n (b-a).$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n = \int_a^b f. \qquad \boxtimes$$

Derivative: But see eq 4 in page 5.

So ~~many~~ many tailor ~~tailo~~ made assumptions: less useful compare to cont. & integ.

Thm: Suppose $f_n \in C^1([a,b])$, $n \geq 1$, and let

1) $\{f_n'\}$ is uniformly conv. &

2) $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a,b]$.

Then $\{f_n\}$ is u.c. & $\underbrace{\lim_{n\to\infty} f_n}_{\text{the uniform limit}} := f \in C^1([a,b])$. Moreover:

C-variable this works better.

$$f_n' \xrightarrow{p} f'.$$

Proof: Define $g(x) := \lim f_n'(x)$. $x \in [a,b]$. $(\because f_n' \to g)$

$\because f_n' \in C[a,b] \Rightarrow g \in C[a,b]$. &

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n' = \int_{x_0}^{x} g \quad \forall x \in [a,b]. \longrightarrow (*)$$

Set $\varphi(x) = \displaystyle\int_{x_0}^{x} g. \quad \forall x \in [a,b].$

$\therefore \varphi$ is diff. & $\varphi'(x) = g(x)$, $x \in [a,b]$. $\Rightarrow \varphi \in C^1[a,b]$

Now $\displaystyle\int_{x_0}^{x} f_n' = f_n(x) - f_n(x_0)$. $\forall x$ $\Rightarrow f_n(x) = \displaystyle\int_{x_0}^{x} f_n' + f_n(x_0)$

$\therefore (*) \Rightarrow$ $\forall x$ $\displaystyle\lim_{n\to\infty} [f_n(x) - f_n(x_0)] = \int_{x_0}^{x} g = \varphi(x).$ $\because f_n(x_0) \to c$ (say)

$\Rightarrow \forall x \; f_n(x) \xrightarrow{p} \varphi(x) + c := f$ ✓ $[c := \lim_{n\to\infty} f_n(x_0)]$.

& $f_n' \xrightarrow{p} f'$. $[f' = \varphi' = g]$. $\boxtimes$

\# Most of the above results works in the setting of metric spaces!!

Not for exam.

Series of functions:

Consider $\{f_n\}_{n\geq 1} \subseteq \mathcal{F}([a,b])$ — you can replace $[a,b]$ by $S \subseteq \mathbb{R}$ or even by a metric space (Complete is better).

Consider the formal sum 1) $\sum_1^{\infty} f_n$ ⟵ Series of f_n's.

2) $\sum_1^{\infty} f_n(x)$ ⟵ pointwise series of f_n's. $x \in [a,b]$.

Def: Given a series $\sum f_n$, set

$S_n \in \mathcal{F}([a,b])$ $\forall n \geq 1$ by

$$S_n(x) = \sum_{m=1}^{n} f_m(x) \quad \forall x \in [a,b].$$

n-th partial sum

We say that the series $\sum f_n$ converges

1) uniformly if $\{S_n\}$ conv. unif. on $[a,b]$.

2) pointwise if ~~$\{S_n\}$ conv.~~ $\{S_n(x)\}$ conv. $\forall x \in [a,b]$.

3) absolutely if $\sum_{n=1}^{\infty} |f_n(x)|$ conv. pointwise on $[a,b]$.

useful in power series.

The following are easy:

Cont. ⟶ 1) If $f_n \in C[a,b]$ & $\sum f_n$ conv. unif. then $\sum f_n \in C[a,b]$.

Integ. ⟶ 2) If $f_n \in \mathcal{R}[a,b]$ & $\sum f_n$ conv. unif. then $\sum f_n \in \mathcal{R}[a,b]$

& $$\int_a^b \sum f_n = \sum \int_a^b f_n .$$

3) If $\sum f_n$ conv. unif, then $f_n \longrightarrow 0$ unif.

[Suppose $f = \sum f_n$. Then $\forall x \in [a,b]$ & n large,

$|f_n(x)| = |S_n(x) - S_{n-1}(x)| \leq |S_n(x) - f(x)| + |S_{n-1}(x) - f(x)|$

$\because S_n \xrightarrow{u} f \Rightarrow |f_n(x)| < \varepsilon$ $\forall x \in [a,b]$ & $n \geq N(\varepsilon)$ where $\varepsilon > 0$ is given.

$\Rightarrow f_n \xrightarrow{u} 0$.]

~~Defn:~~ Just like series of real nos, we have:

$\sum M_n < \infty$

Thm: Suppose $|f_n(x)| \leq M_n \ \forall n, \ x \in [a,b]$. If ~~$\sum M_n < \infty$~~

Weierstrass' M-test

then $\sum f_n$ is unif. conv as well as absolutely convergent.

Proof: Follow the real series case.

eg: 1) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is u.c. on $\mathbb{R}$: $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$. & $\sum \frac{1}{n^2} < \infty$.

2) $\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ is u.c. on $[p,\infty)$, $\forall p > 0$.

$\because \frac{x}{1+n^2x^2} \leq \frac{x}{n^2x^2} \leq \frac{1}{n^2x} \leq \frac{1}{n^2p}$

& $\sum \frac{1}{n^2p} < \infty$.

~~Obs: If $\sum f_n$ is absolutely conv then $\sum f_n$ is u.c~~

~~Proof: $\because |S_n(x)| \leq \sum_{m=1}^{n} |f_m(x)|$~~

Remark: $\sum |f_n|$ is unif conv. $\Rightarrow$ $\sum f_n$ is unif. conv.

Imp. eg: $f_n(x) := x^n$. Then $\sum_{n=0}^{\infty} f_n = (1-x)^{-1}$ $x \in (-1,1)$.

$= \sum_{n=0}^{\infty} x^n$

geometric series.

Easy to prove

Take the above example & proceed to "power series".

—0—