Lecture 27: Caley-Hamilton theorem, Nakayama's lemma

13 November 2020
10:45

Note $\varphi : M \longrightarrow M$ is a R- lin map.
$m \longmapsto X \cdot m$

$\mathrm{Hom}_R(R[X], R)$

R-mod + $\varphi : R[X] \to R$
$\downarrow$
$R[X]$-mod
$\downarrow$
R-mod + lin map

$\mathrm{End}_R(M)$

$\{ R[X]\text{-modules} \} \xrightarrow{\theta_1} \{ R\text{-modules} + \text{an } R\text{-lin endo of the module} \}$, with θ_2 going back.

$X \cdot m := \varphi(m)$

<u>Caley-Hamilton theorem</u>: Let R be a ring and $A \in M_{n \times n}(R)$. Let $p_A(X) = \det(XI - A) \in R[X]$. Then $p_A(A) = 0$ in $\mathrm{End}_R(R^n) = M_{n\times n}(R)$.

<u>Thm</u>: Let M be a f.g. R-mod. and $\varphi \in \mathrm{End}_R(M)$ s.t. $\varphi(M) \subseteq IM$ where I is an R-ideal.

Then $\varphi^n + a_1 \varphi^{n-1} + \dots + a_{n-1}\varphi + a_n = 0$ for some $a_i \in I^i \subseteq I$, $1 \le i \le n$.

$I^i = \underbrace{I \cdot I \cdots I}_{i \text{ times}}$

<u>Cor</u> (Nakayama's): Let M be f.g. R-module s.t. $M = IM$ for some ideal $I \subseteq \mathrm{Jac}(R)$ then $M = 0$.

<u>Cor (Nakayama)</u>: Let (R, m) be a local ring and let M be a f.g. R-mod s.t. $M = mM$ then $M = 0$.

<u>Pf</u>: $M = IM$ where $I \subseteq \mathrm{Jac}(R)$ and M is f.g.
Take $\varphi = \mathrm{Id}$ and apply the thm to conclude that

$$I + a_1 I + a_2 I + \dots + a_n I = 0 \quad \text{in } \mathrm{End}_R(M)$$
where $a_i \in I$ $\forall\, 1 \le i \le n$

$\Rightarrow (1 + a) I = 0$ in $\mathrm{End}_R(M)$ where $a = a_1 + \dots + a_n \in I$

$\boxed{M \text{ is an } R\text{-mod} \implies \mu : R \longrightarrow \mathrm{End}_R(M)}$

So $(1+a) \cdot m = 0$ $\forall\, m \in M$
But $1+a$ is a unit in R as $a \in I \subseteq \mathrm{Jac}(R)$

$\implies m = 0$ $\forall\, m \in M$.
$\implies M = 0$

Pf of C-H thm:

$((a_{ij})) = A$ defines a linear map

$$\varphi_A : R^n \longrightarrow R^n \qquad e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k^{th} \text{ spot}$$

$$e_k \longmapsto \sum_{i=1}^{n} a_{ki} e_i$$

$$\varphi_A(e_k) = e_k A$$

So can make the R-mod R^n into an $R[X]$-mod via.

$$X \cdot e_k = \varphi_A(e_k) = \sum_{i=1}^{n} a_{ki} e_i \qquad 1 \le k \le n$$

$$X \cdot e_1 - a_{11} e_1 - a_{12} e_2 - \cdots - a_{1n} e_n = 0 \leftarrow \text{zero of } R^n$$

$$-a_{21} e_1 + X \cdot e_2 - a_{22} e_2 - a_{23} e_3 \cdots - a_{2n} e_n = 0$$

$$\vdots$$

$$-a_{n1} e_1 + \cdots \quad - a_{nn-1} e_{n-1} + X e_n - a_{nn} e_n = 0$$

$$(XI - A) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = 0$$

$$B = \mathrm{Adj}(XI - A) \in M_{n \times n}(R[X])$$

$$B(XI - A) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{zero of } R^n$$

$$\det(XI - A)\, I \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad \det(x - IA)\, e_k = 0 \quad \forall\, 1 \le k \le n$$

$$\Longrightarrow \quad p_A(x) \cdot e_k = 0 \quad \forall\, 1 \le k \le n$$

$$\Longrightarrow \quad p_A(\varphi_A)(e_k) = 0 \quad \forall\, 1 \le k \le n$$

$$\Longrightarrow \quad e_k\, p_A(A) = 0 \quad \left(\because \ \varphi_A(e_k) = e_k A \right) \quad \forall\, 1 \le k \le n$$

$$\Longrightarrow \quad p_A(A) = 0 \quad \left(\because \ \{e_1, \ldots, e_n\} \text{ gen } \mathbb{R}^n \right)$$

$\blacksquare$

Pf of thm

We know M is f.g. R-mod

Let $\{e_1, \dots, e_n\}$ be gen set of M.

$\varphi : M \longrightarrow M$ R-linear

So this gives an $R[X]$-mod str on M

$$X \cdot m = \varphi(m) \quad \forall m \in M$$

$$\varphi(e_k) = X \cdot e_k = \sum_{j=1}^{n} a_{kj} e_j \quad 1 \le k \le n$$

Moreover

$$\varphi(M) \subseteq I M$$

$\Rightarrow$ we can choose $a_{kj} \in I$

$$\left(\begin{array}{l} X \cdot e_k \in IM \\ X \cdot e_k = \sum b_j m_j \quad b_j \in I \\ m_j = \sum_{l=1}^{n} \alpha_{jl} e_l \end{array} \right)$$

$$(XI - A) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0_M \\ \vdots \\ 0_M \end{pmatrix}$$

$\vdots$

$$\Rightarrow \quad p_A(x) \cdot e_k = 0 \quad \forall\, 1 \le k \le n$$

$$\Rightarrow \quad p_A(\varphi)(e_k) = 0 \quad \forall\, 1 \le k \le n$$

$$\Rightarrow \quad p_A(\varphi) = 0 \quad \left(\because \{e_1, \dots, e_n\} \text{ gen } M \right)$$

$$p_A(X) = \det(XI - A)$$

$$= \det\begin{pmatrix} X - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & X - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & \cdots & -a_{n\,n-1} & X - a_{nn} \end{pmatrix}$$

$$= X^n + a_1 X^{n-1} + \cdots + a_n$$

$$a_1 \in I,\ a_2 \in I^2,\ \dots,\ a_n \in I^n$$