

general fact.

[Recall: if $f \in R(\Omega)$, then $|f| \in R(\Omega)$ & $\int_{\Omega} |f| \leq \int_{\Omega} f$.]

Here: $U(|\tilde{f}|, P) = \sum_{\alpha \in N(P)} M_{\alpha} v(B_{\alpha}^2)$

P , as above

$$= \sum_{\alpha \in \tilde{\Lambda}} M_{\alpha} v(B_{\alpha}^2) \quad \left[\because M_{\alpha} := \sup_{B_{\alpha}^2} |\tilde{f}| \right]$$

$$= 0 \quad \forall \alpha \notin \tilde{\Lambda}] .$$

$$\leq M \times \sum_{\alpha \in \tilde{\Lambda}} v(B_{\alpha}^2) .$$

$\underbrace{\quad}_{< \varepsilon}$

$$< M \times \varepsilon .$$

$$\Rightarrow \inf U(|\tilde{f}|, P) = 0 \quad \Rightarrow \int_{B^2} \tilde{f} = 0 .$$

$$\Rightarrow \int_{\Omega} \tilde{f} = 0 . \quad \square$$

Back to our thm: (Proof is similar).

Thm: $\bar{\Omega} \supseteq \Omega$. Suppose $\bar{\Omega} \setminus \Omega$ is of ~~measure~~ ^{content} zero,

\uparrow \uparrow
bdd. open

$f \in B(\bar{\Omega})$ & $f|_{\Omega}$ is continuous. Then $f \in R(\Omega)$.

Proof: Let $\text{int}(B^2) \supseteq \bar{\Omega}$ & consider \tilde{f} on B^2 (extension of f).

Enough to prove that: \mathcal{D} , the set of points of discontinuity of \tilde{f} , is of measure zero.

Note that: (i) $\tilde{f}|_{\Omega}$ is cont. (ii) $\tilde{f}|_{\text{int}(B^2) \setminus \bar{\Omega}}$ is cont.

& (iii) $\tilde{f}|_{\partial B^2} = 0$ cont.

$\Rightarrow \mathcal{D} \subseteq \bar{\Omega} \setminus \Omega \leftarrow$ set of measure zero.

$\Rightarrow \mathcal{D}$ is a set of measure zero.

$\Rightarrow f \in R(\Omega)$.



B^2

$= \emptyset$

DANGER: Sets of measure zero depends on the "dimension".

For instance: ① $[0,1] \subseteq \mathbb{R}$ is not of ~~measure zero~~ $\mathbb{C} \cdot \mathbb{Z}$

but $[0,1] \times \{a\} \subseteq \mathbb{R}^2$ is of measure zero.

② ~~Q n [0,1]~~ is of measure zero? $\mathbb{C} \cdot \mathbb{Z}$: NO.

③ ~~Q n ([0,1] x [0,1])~~ is of measure zero? $\mathbb{C} \cdot \mathbb{Z}$: YES.

Fact: Let $f: B^2 \rightarrow \mathbb{R}$ be a cont. fn. Then

$$\text{Graph } f := \{(x, f(x)) : x \in B^2\} \quad (\subseteq \mathbb{R}^3)$$

Graphs have
~~measure zero~~
Content zero.

is a set of measure zero.

works for
 $f: B^n \rightarrow \mathbb{R}$

Proof. Let $\varepsilon > 0$. Note that: f is uniformly cont.

$$\therefore \exists s > 0 \text{ s.t. } |f(x) - f(y)| < \varepsilon \quad \forall |x - y| < s. \quad (x, y \in B^2)$$

Next, on this $s > 0$, pick a partition P of B^2

s.t. the diameter of B_α^2 $< s$ $\forall \alpha \in \Lambda(P)$.

$$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in B_\alpha^2, \alpha \in \Lambda(P).$$

Set $I_\alpha := \{f(x) : x \in B_\alpha^2\}$

$\Rightarrow I_\alpha$ is an interval of length at most ε . $\forall \alpha$.

The range set of $f|_{B_\alpha^2}$.

$\therefore \{B_\alpha^2 \times I_\alpha : \alpha \in \Lambda(P)\}$ is a cover of boxes of

graph f . Also:

As $\{(x, f(x)) \in B_\alpha^2 \times I_\alpha : x \in B_\alpha^2, \alpha \in \Lambda(P)\}$

$\Lambda(P)$ is a finite set, \therefore

$$\sum_{\alpha \in \Lambda(P)} v(B_\alpha^2 \times I_\alpha) = \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times v(I_\alpha^2)$$

$$\leq \sum_{\alpha \in \Lambda(P)} v(B_\alpha^2) \times \varepsilon.$$

$$= \underbrace{v(B^2)}_{\text{constant}} \times \varepsilon.$$

\Rightarrow measure of graph is zero. \square

In fact, we have the following:

The proof is even better!!

Let $f \in R([a, b])$. Then $G_f := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ is of measure zero.

Proof: We proceed along the same line:

Let $\varepsilon > 0$. $\exists P \in \mathcal{P}([a, b]) \ni$

$$U(f, P) - L(f, P) < \varepsilon.$$

Set $P: a = x_0 < x_1 < \dots < x_n = b$.

$\forall B_i^2 := [x_{i-1}, x_i] \times [m_i, M_i]$,

Here: $m_i = \inf_{[x_{i-1}, x_i]} f$

$M_i = \sup_{[x_{i-1}, x_i]} f$.

$\therefore G_f \subseteq \bigcup_{i=1}^n B_i^2$. Finally:

$$\begin{aligned} \sum_{i=1}^n v(B_i^2) &= \sum_{i=1}^n v([x_{i-1}, x_i] \times [m_i, M_i]) \\ &= \sum_{i=1}^n (x_i - x_{i-1}) \times (M_i - m_i), \\ &= U(f, P) - L(f, P) \\ &< \varepsilon. \end{aligned}$$

Smart proof?
Then P-42?

Back to Fubini's thm:

Recall: Let $f \in \mathcal{R}(B^2)$. Set $B_2 = [a,b] \times [c,d]$.

If $\int_a^b f(x,y) dx$ exists $\forall y \in [c,d]$, then

$$\iint_{B^2} f = \int_c^d \left(\int_a^b f(x,y) dx \right) dy. \quad \text{--- (1)}$$

Note: integrability of this fn. (in y) is guaranteed by Fubini.

$\int_a^b (\int_c^d f(x,y) dy) dx$ if, $\int_c^d f(x,y) dy$ exists for each $x \in [a,b]$, then

$$= \int_a^b (\int_c^d f(x,y) dy) dx$$

$$\iint_{B^2} f = \int_a^b \left(\int_c^d f(x,y) dy \right) dx. \quad \text{--- (2)}$$

If $f \in C(B^2)$, then (1) = (2).

, in particular,

--- \times ---.

Q: Fubini for $f \in \mathcal{R}(\Omega)$, $\Omega \subseteq B^2$, bdd ??

How to think about it?

In fact: it is not easy to evaluate double integral over

$\Omega \subseteq \mathbb{R}^2$. However, with some control over Ω ,

one can do "something". It is as follows:

[Remarks: Many/all of the results below works similarly in \mathbb{R}^n , $n \geq 3$.

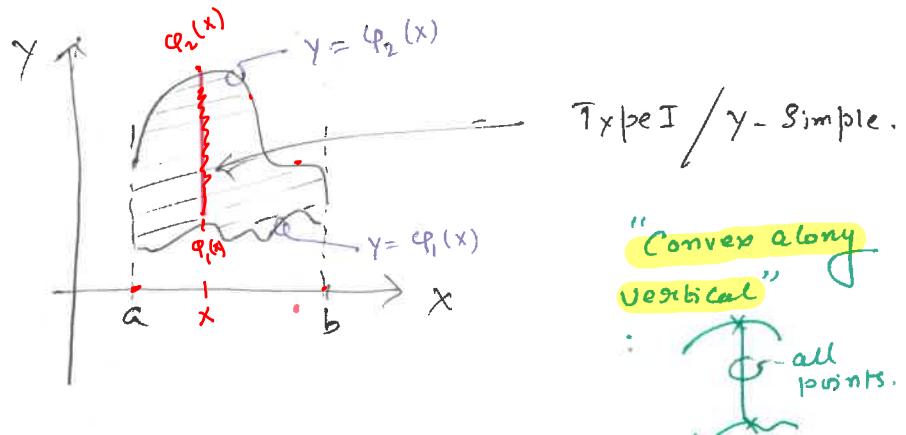
At least, think them in the setting of \mathbb{R}^3 .]

Two Special domains (AKA: Elementary regions) :

Def: A set $\Omega \subseteq \mathbb{R}^2$ is said to be y-simple / Type I if $\exists \varphi_1, \varphi_2 \in \mathcal{R}([a, b])$ s.t.

$$\Omega = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

Here:

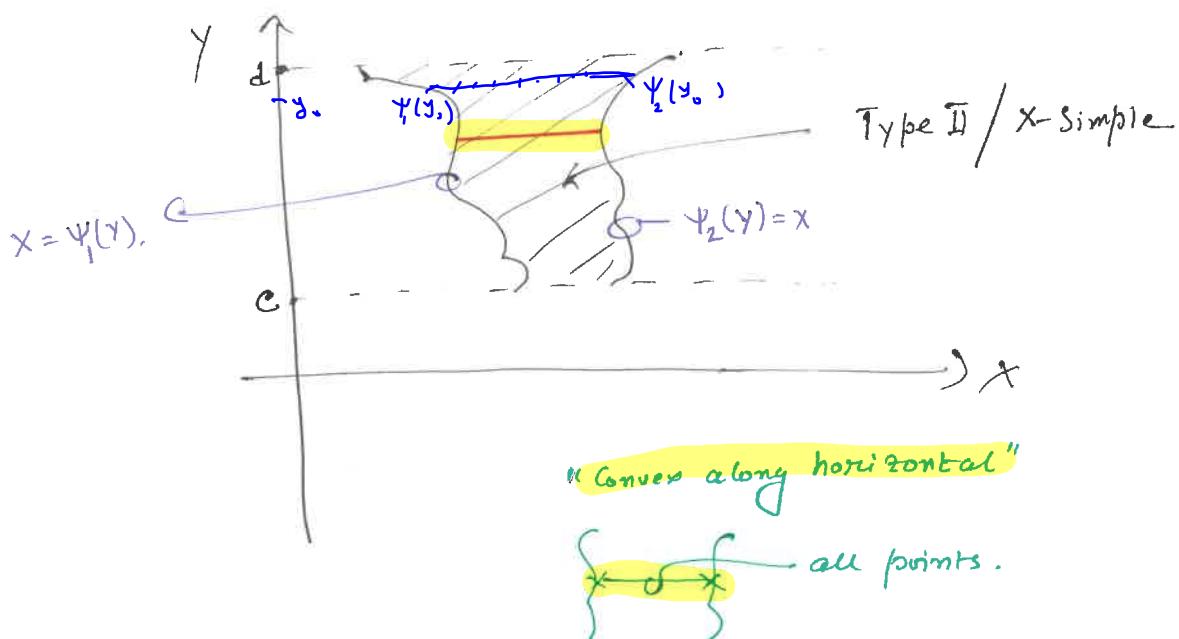


||| x-simple / Type II regions are given by:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$$

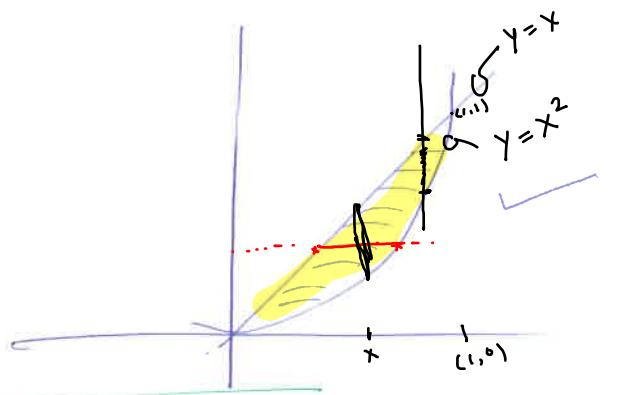
for some $\psi_1, \psi_2 \in \mathcal{R}[c, d]$.

Here:

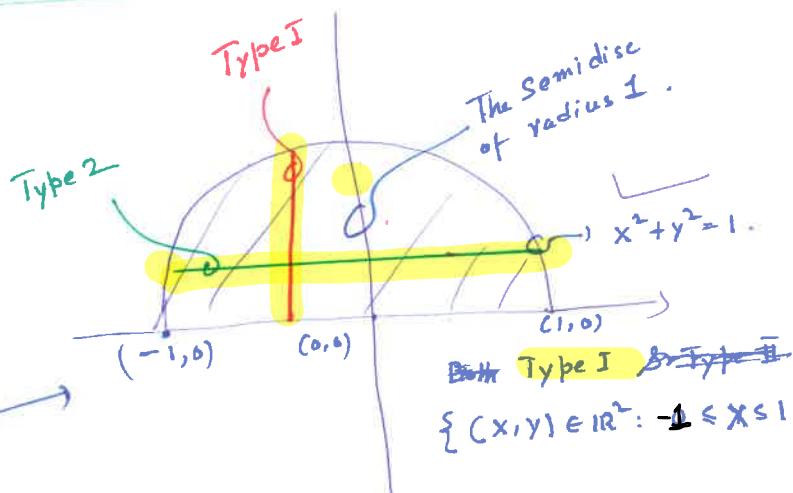


(46)

e.g:

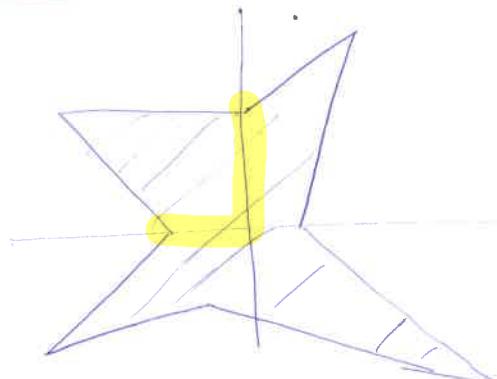


$$\{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq x\}$$



Also Type II:
 $\{(x, y) : 0 \leq y \leq 1 \text{ & } -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$.

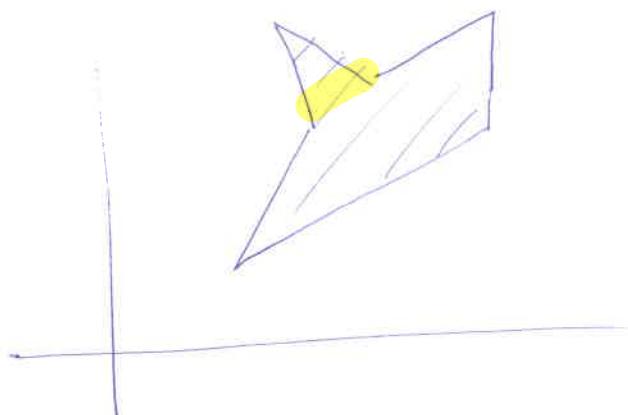
$$\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$



?

X

BUT: Sum of
elementary
regions!!



?

X

Fubini
 $\downarrow \downarrow \downarrow$

Thm: Let $f \in R(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ (an elementary region).

(I) If $\Omega = \{(x, y) : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$, for some $\varphi_1, \varphi_2 \in B[a, b]$, and if $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy$ exists $\forall x \in [a, b]$, then

$$\int_{\Omega} f(x, y) dx = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx.$$

THIS MUST EXIST $\forall x$: Then integrability is assured.

(II) If $\Omega = \{(x, y) : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$, for some $\psi_1, \psi_2 \in B[c, d]$, and if $\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx$ exists $\forall y \in [c, d]$, then

$$\int_{\Omega} f(x, y) dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right) dy.$$

THIS MUST EXIST $\forall y \in [c, d]$.

Proof (Easy application of Fubini):

We will prove only (I), as (II) will be similar.

Get $c < d$ s.t. $\Omega \subseteq B^2 := [a, b] \times [c, d]$.

[In fact: $c = \inf_{[a, b]} \varphi_1$ & $d = \sup_{[a, b]} \varphi_2$ is one natural choice.]

Consider the extension $\tilde{f} : B^2 \rightarrow \mathbb{R}$, where

$$\tilde{f}|_{\Omega} = f \quad \tilde{f}|_{B^2 \setminus \Omega} = 0.$$

We know $\tilde{f} \in R(B^2)$. Now for each $x \in [a, b]$, $\int_c^d \tilde{f}(x, y) dy$ exists.

Indeed: $\tilde{f}(x, y) = \begin{cases} f(x, y) & \forall y \in [\varphi_1(x), \varphi_2(x)] \\ 0 & \forall y \in [c, \varphi_1(x)] \cup [\varphi_2(x), d] \end{cases}$
 for fixed $x \in [a, b]$

So, But $f(x, \cdot)|_{[\varphi_1(x), \varphi_2(x)]}$ & $f(x, \cdot)|_{[c, \varphi_1(x)] \cup [\varphi_2(x), d]}$

are integrable. So, by 1-variable result, $\tilde{f}(x, \cdot) \in R[c, d]$.

Finally, again for fixed $x \in [a, b]$, by 1-variable additivity:

$$\int_c^d \tilde{f}(x, y) dy = \int_c^{\varphi_1(x)} \tilde{f}(x, y) dy + \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x, y) dy + \int_{\varphi_2(x)}^d \tilde{f}(x, y) dy.$$

$$= \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x, y) dy$$

$$= \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy. \quad \left[\because \tilde{f}(x, y) = f(x, y) \text{ and } \varphi_1(x) \leq y \leq \varphi_2(x) \right].$$

Then, by Fubini ($\because \forall x \in [a, b]$, $\int_c^d \tilde{f}(x, y) dy$ exists):

$$\iint_{\Omega} f \stackrel{\text{DEF}}{=} \iint_{B^2} \tilde{f} \stackrel{\text{FUBINI}}{=} \int_a^b \left(\int_c^d \tilde{f}(x, y) dy \right) dx$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \right) dx.$$

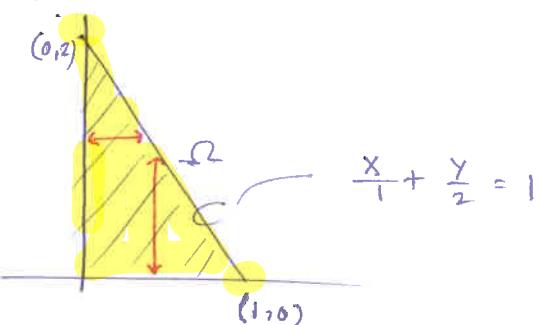
\square

~~Ex:~~ Compute $\int_{\Omega} f$, where $f \in R(\Omega)$

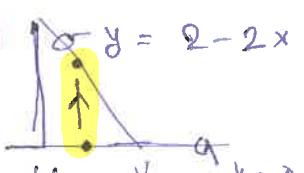
Eg: Consider $f \in C(\Omega)$, where $\Omega =$

Clearly, Ω is both Type I & Type II.

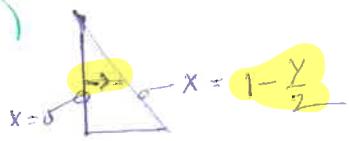
Also, $f \in R(\Omega)$. Then



$$\int_{\Omega} f = \int_0^{2-2x} \left(\int_0^{2-2x} f(x, y) dy \right) dx$$



$$= \int_0^{1-y/2} \left(\int_0^{1-y/2} f(x, y) dx \right) dy$$



Often, changing order of integration is useful.
We will also see.