Lecture 31: Structure theorem for f.g. modules over PID

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Defn: Let R be an int domain and M be an R-mod.
Then $\text{rank}(M) = \text{v.s. dim}_{S^{-1}R}(S^{-1}M)$ where $S = R \setminus \{0\}$.
$= \dim_K(S^{-1}M)$ where $S^{-1}R = K = \text{frac}(R)$ as vector space

$\text{rank}(M) =$ size of the largest l.i. subset of M.

Defn: Let R be ring & M be an R-mod.
$T(M) = \{ m \in M : \exists r \in R, r \neq 0 \text{ \& } rm = 0 \} \iff \{ m \in M \mid \text{Ann}(m) \neq 0 \}$
$T(M)$ is a submod of M if R is an int domain.
M is called torsion free R-mod if $T(M) = 0$.

Let R be an int domain M an R-mod then
$M/T(M)$ is torsion free.

R an int dom M an R-mod s.t. $M = T(M)$ then $\text{rank}(M) = 0$.

such modules are called torsion modules

Pf: Let $\alpha \in S^{-1}M$ then $\alpha = \frac{m}{s}$ for $m \in M$ & $s \in R \setminus \{0\}$
$\exists r \in R$, $r \neq 0$ s.t. $rm = 0$ in M $\Rightarrow \alpha = \frac{m}{s} = \frac{rm}{rs} = \frac{0}{rs} = \frac{0}{1}$ $(\because r \in R \setminus \{0\} = S)$

Hence $S^{-1}M = 0 \Rightarrow \text{rank}(M) = 0$

Even converse holds. Because $\text{rank}(M) = 0 \Rightarrow S^{-1}M = 0$

$\Rightarrow \frac{m}{s} = \frac{0}{1}$ in $S^{-1}M$ $\forall m \in M$ & $s \in R \setminus \{0\}$

$\Rightarrow \frac{m}{1} = \frac{0}{1}$ in $S^{-1}M$

$\Rightarrow \exists r \in S = R \setminus \{0\}$ s.t. $r(m - 0) = 0 \Rightarrow rm = 0$

$\Rightarrow m \in T(M)$ $\forall m \in M$.

<u>Thm</u> (Str thm): Let R be a PID and M be a f.g. R-mod. Then

$$M \cong R^k \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

where $k = \text{rank}(M)$ and $a_1, \ldots, a_m \in R$ are nonzero nonunits s.t. $a_1 | a_2 | a_3 | \cdots | a_m$. Here k and m could be 0.

<u>Cor</u>: R a PID and M a f.g. torsion free R-mod then M is free R-mod.

Pf: $M \cong R^k \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$. But if $m \geq 1$ then $m \mapsto (0, \bar{1}, 0, \ldots, 0) \in RHS$ then $a_1 m = a_1(0, \bar{1}, 0, \ldots, 0) = (0, 0, \ldots, 0)$. Contradicting M is torsion free.

<u>Cor</u>: Let G be a f.g. abelian group then

$$G \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \mathbb{Z}/a_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$$

for some $n \geq 0$ and $a_1, \ldots, a_m \in \mathbb{Z}$ are s.t.

$$a_i | a_{i+1} \quad 1 \leq i \leq m-1.$$

Pf: $R = \mathbb{Z}$ then G is f.g. $\mathbb{Z}$-mod and apply str thm.

Cor: R a PID and M a submod of a f.g. free mod then M is free.

Pf: $M \subseteq R^n \Rightarrow M$ is also torsion free.

R noeth $\Rightarrow R^n$ is a noeth R-mod $\Rightarrow M$ is noeth R-mod.

Hence M is f.g. R-mod. $\blacksquare$

Example: 1) $R = \mathbb{Z}[X]$ or $\mathbb{Q}[X, Y]$.

$M = (2, X)$ or $(X, Y) = M$

M is f.g. torsion free R-mod.

Is M free? Note rank$(M) = 1$

But M is not a principal ideal.

$2, X$ are not l.i. as

$X \cdot 2 + (-2) \cdot X = 0$ in M.

2) R a PID. M torsion free $\overset{?}{\Longrightarrow} M$ is free

Example: $\mathbb{Q}$ as a $\mathbb{Z}$-module

Let $S \subseteq \mathbb{Q}$ s.t. $|S| \geq 2$

then S is lin dep.

$sp\left(\frac{r}{s}\right) - qr\left(p/q\right) = 0$ & $\mathbb{Q} \ncong \mathbb{Z}$.

<u>Prop:</u> Let R be a PID and F be a free R-module of rank n. Let N be a submodule of F. Then N is a free R-mod of rank $m \leq n$. Moreover there is a basis $x_1, \dots, x_n$ of F and $\exists\, a_1, \dots, a_m \in R^\times$ s.t. $a_1 | a_2 | \dots | a_m$ and $\{a_1x_1, a_2x_2, \dots, a_mx_m\}$ is a basis of N.

<u>Prop $\Rightarrow$ str thm:</u> Let M be a f.g. R-mod.

Let $m_1, \dots, m_n \in M$ s.t. $M = Rm_1 + \dots + Rm_n$.

$$\Rightarrow \quad \varphi : F \longrightarrow M \qquad e_i \longmapsto m_i$$

where $\underset{R^n}{F}$ is a free mod of rank n and e_i's are std. basis vectors.

Then φ extends to R-lin surj map.

$\varphi((b_1, \dots, b_n)) = \sum_{i=1}^{n} b_i m_i$. We know φ is R-lin and since $\{m_i \mid 1 \leq i \leq n\}$ gen M, φ is surj.

$N = \ker(\varphi) \subseteq F$. By prop., $\exists$ $m \leq n$

a basis $\{x_1, \dots, x_n\}$ of F and $a_1, \dots, a_m \in R^\times$ s.t. $a_1 | a_2 | \dots | a_m$ and $\{a_1x_1, a_2x_2, \dots, a_mx_m\}$ is a basis of N.

$$N \subseteq F = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$$

$$\| $$

$$Ra_1x_1 \oplus Ra_2x_2 \oplus \cdots \oplus Ra_mx_m$$

$$M \cong F/N \quad (\text{by } 1^{st} \text{ isom thm})$$

$$= \frac{Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n}{Ra_1x_1 \oplus Ra_2x_2 \oplus \cdots \oplus Ra_mx_m}$$

$$= Rx_1/Ra_1x_1 \oplus \frac{Rx_2}{Ra_2x_2} \oplus \cdots \oplus Rx_m/Ra_mx_m \oplus Rx_{m+1} \oplus \cdots \oplus Rx_n$$

($\because$ $M_1, M_2, \ldots, M_n$ are R-mod & $N_i \subseteq M_i$ are R-submod then

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n / N_1 \oplus N_2 \oplus \cdots \oplus N_n \cong M_1/N_1 \oplus M_2/N_2 \oplus \cdots \oplus M_n/N_n \,)$$

$$\cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m}$$

$$\left(\quad Rx_1 \overset{\alpha}{\cong} R \right.$$

$$x_1 \longleftrightarrow 1$$

$$x_1 \longmapsto 1$$

$$\left. \alpha(Ra_1x_1) = (a_1) \right)$$

$a_1 | a_2 | \cdots | a_m \qquad a_i \in R^\times$.

Let $a_1, \ldots, a_r$ be units $r \le m$.

then $M \cong R/(a_{r+1}) \oplus \cdots \oplus R/(a_m) \oplus R^{n-m}$

and we get the str thm by replacing m by $m-r$ and a_i by a_{r+i}.

Finally note that $\operatorname{rank}(M) = \dim_{S^{-1}R}(S^{-1}M)$

where $S = R \setminus \{0\}$

$\Rightarrow S^{-1}(RHS) = (S^{-1}R)^{n-m}$

$\Rightarrow \operatorname{rank}(M) = n-m = k$.