Analysis III (HW - 3) Due date: Take your sweet time Instructor : Jaydeb Sarkar

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NOTE: (i) $B^n = \Pi_{i=1}^n [a_i, b_i]$. (ii) $R(B^n)$ = the set of all Riemann integrable functions on B^n. (iii) $v(\Omega)$ =volume of $\Omega(\subseteq \mathbb{R}^n)$, whenever $n \geq 3$. (iv) $A(\Omega)$ = area of $\Omega(\subseteq \mathbb{R}^2)$.

(1) Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Suppose

$$\mathcal{D} = \{x \in [a, b] : f \text{ is not continuous at } x\}.$$

Prove that $\mathcal{D}$ is of measure zero.

(2) Let $f \in R(B^n)$. Suppose $f \geq 0$ and $\int_{B^n} f = 0$. If $\epsilon > 0$, then prove that $\{x \in B^n : f(x) > \epsilon\}$ is of measure zero.

(3) Show that the set $\{(x, y) : x^2 + 4y^2 = 16\}$ has content zero.

(4) Suppose $f : [0, 1] \times [0, 1] \to \mathbb{R}$ has continuous second partial derivatives. Suppose $f(0, 0) = 1, f(0, 1) = 2, f(1, 0) = 3$ and $f(1, 1) = 5$. Find

$$\iint_{[0,1]\times[0,1]} \frac{\partial^2 f}{\partial x \partial y}.$$

(5) Evaluate

$$(i) \iint_{[0,1]\times[1,2]} \frac{1}{2x + y}, \ (ii) \iint_{[1,2]\times[1,2]} \ln(x + y), \ (iii) \iint_{[0,1]\times[0,1]} x \exp(yx).$$

(6) Evaluate

$$(i) \int_0^{\frac{\pi}{2}} \int_0^{\cos y} x \sin y \, dx \, dy, \ (ii) \int_{-1}^{1} \int_0^{|x|} dy \, dx, \ (iii) \int_0^{2} \int_1^{3} |x - 2| \sin y \, dx \, dy.$$

(7) Let $f \in C([0, 1])$. Prove that

$$\Big(\int_0^1 f(x)\, dx\Big)^2 \leq \int_0^1 \Big(f(x)\Big)^2 dx.$$

[Hint: Consider $\int_0^1 \int_0^1 \Big(f(x) - f(y)\Big)^2 dy \, dx$.]

(8) Evaluate $\int\int_{\Omega} \sin(y^2)$, where Ω is the triangle bounded by the lines $x = 0$, $y = x$, and $y = \sqrt{\pi}$.

(9) Reverse the order of integration $I = \int_1^2 \int_{2-x}^{\sqrt{2x-x^2}} f(x, y) dy \, dx$, that is, express I as $\int_?^? \int_?^? f(x, y) dx \, dy$.

(10) Evaluate $\int\int_{\Omega} |xy|$, where Ω is the disk of radius 1 centered at the origin.

(11) Evaluate $\int\int_{\Omega} x^3 \exp(y^3)$, where $\Omega = \{(x, y) : 0 \leq x \leq 3, x^2 \leq y \leq 9\}$.

(12) Evaluate $\int_{\Omega} 3x^2 + 2y + z$, where

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : |x - y|, |y - z|, |x - z| \leq 1\}.$$

(13) Evaluate the volume of the solid bounded by the planes $x = 0$, $y = 0$, and $z = 0$, and $x + y + z = 1$.

(14) Find the area of the region bounded by $y = x$ and $y = x^2$.

(15) Find the volume of the region in $\mathbb{R}^3$ lying above the triangle with vertices $(-1, 0), (0, 1)$, and $(1, 0)$ and under the graph of the function $f(x, y) = x^2 y$.

(16) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set. Suppose Ω has a volume, and $v(\Omega) \neq 0$. Prove that Ω has an interior point.

(17) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set. If Ω has volume, then show that its closure $\overline{\Omega}$ must also have volume and that $v(\Omega) = v(\overline{\Omega})$.

(18) If $\Omega = \overline{B_1((0,0))}$, then prove that

$$\frac{\pi}{3} \leq \int_{\Omega} \frac{1}{\sqrt{x^2 + (y-2)^2}} \leq \pi.$$