Set 3

Thm: Let $f \in \mathcal{B}(B^n)$, $P, \tilde{P} \in \mathcal{P}(B^n)$ & $\tilde{P}$ a refinement of P.

Then: $L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$.

Proof: Of course, we only have to prove

$$L(f,P) \leq L(f,\tilde{P}).$$

The middle "$\leq$" is known. The remaining one $U(f,\tilde{P}) \leq U(f,P)$ will be llar.

$\because$ $\tilde{P}$ is a refinement of P,

$$\exists\ \eta : \Lambda(\tilde{P}) \longrightarrow \Lambda(P)$$

s.t. $B^n_{\alpha(\tilde{P})} \subseteq B^n_{\underbrace{\eta(\alpha(\tilde{P}))}_{\in \Lambda(P)}}$ $\forall\ \alpha(\tilde{P}) \in \Lambda(\tilde{P})$.

Use α instead of $\alpha(\tilde{P})$ if $\tilde{P}$ ~~meaning~~ is clear from the context.

Careful: $B^n_{\alpha(\tilde{P})}$, $B^n_{\eta(\alpha(\tilde{P}))}$ $B^n_{\alpha(P)}$ ~~etc~~ are all subsets / sub boxes of the box B^n.

Recall: $\Lambda(P) = \{ (\underbrace{j_1, \dots, j_n}_{=\alpha \text{ or } \alpha(P)}) : 1 \leq j_i \leq n_i,\ 1 \leq i \leq n \}$

Where $P_i : a_i = x_{i,0} < x_{i,1} < \dots < x_{i,n_i} = b_i$

$B^n_\alpha = [x_{1,j_1-1}, x_{1,j_1}] \times \dots \times [x_{n,j_n-1}, x_{n,j_n}]$
$(\because j_i = 1, 2, \dots, n_i)$.

Let $\beta(P) \in \Lambda(P)$.

Look at $\{ \alpha(\tilde{P}) \in \Lambda(\tilde{P}) : \eta(\alpha(\tilde{P})) = \beta(P) \}$.

$$:= \Lambda(\beta(P))$$

$$\therefore\ \Lambda(\beta(P)) := \eta^{-1}(\{\beta(P)\}) \subseteq \Lambda(\tilde{P}).$$

[$\because$ $\Lambda(\beta(P))$ contains those $\alpha \in \Lambda(\tilde{P})$ for which $B^n_{\eta(\alpha)} \subseteq B^n_{\beta(P)}$.]

$\eta^{-1}(\{\beta(P)\})$ = Those $\alpha(\tilde{P}) \in \Lambda(\tilde{P})$ $\ni$ $B^n_{\alpha(\tilde{P})}$ is a subbox of $B^n_{\beta(P)}$.

Recall: ~~Then~~

$$L(f,\tilde{P}) = \sum_{\alpha(\tilde{P})\in\Lambda(\tilde{P})} m_{\alpha(\tilde{P})} \times v\left(B^n_{\alpha(\tilde{P})}\right).$$

$$\left[\, m_{\alpha(\tilde{P})} := \inf_{B^n_{\alpha(\tilde{P})}} f \,\right]$$

& IIIly

$$L(f,P) = \sum_{\beta(P)\in\Lambda(P)} m_{\beta(P)} \times v\left(B^n_{\beta(P)}\right)$$

$$\left[\, M_{\beta(P)} = \sup_{B^n_{\beta(P)}} f \right.$$

~~implies that~~:
Then

$$L(f,\tilde{P}) = \sum_{\alpha(\tilde{P})\in\Lambda(\tilde{P})} m_{\alpha(\tilde{P})} \times v\left(B^n_{\alpha(\tilde{P})}\right).$$

$$\left[\ \because\ \Lambda(\tilde{P}) = \bigcup_{\beta(P)\in\Lambda(P)} \eta^{-1}(\{\beta(P)\})\right.$$

~~$m_{\alpha(\tilde{P})}$~~. ~~as $\alpha = \alpha(\tilde{P})$~~.

$$= \sum_{\beta(P)\in\Lambda(P)}\ \sum_{\alpha(\tilde{P})\in\eta^{-1}(\{\beta(P)\})} m_{\alpha(\tilde{P})} \times v\left(B^n_{\alpha(\tilde{P})}\right) \quad \text{———} \ (*)$$

~~Here~~ ~~$\alpha = \alpha(\tilde{P})$~~.

Note that

$$B^n_{\alpha(\tilde{P})} \subseteq B^n_{\eta(\alpha(\tilde{P}))} = B^n_{\beta(P)} \quad \text{if } \alpha(\tilde{P}) \in \eta^{-1}(\{\beta(P)\}).$$

$$\Rightarrow\ m_{\alpha(\tilde{P})} \geq m_{\beta(P)} \quad \text{if } \alpha(\tilde{P}) \in \eta^{-1}(\{\beta(P)\})$$

$$\therefore\ (*) \Rightarrow\ L(f,\tilde{P}) \geq \sum_{\beta(P)\in\Lambda(P)} m_{\beta(P)} \times \underbrace{\sum_{\alpha(\tilde{P})\in\eta^{-1}(\{\beta(P)\})} v\left(B^n_{\alpha(\tilde{P})}\right)}_{=\, v\left(B^n_{\beta(P)}\right)}.$$

$\because\ B^n_{\beta(P)} \supseteq B^n_{\alpha(\tilde{P})}$ $\forall\, \eta(\alpha(\tilde{P})) = \beta(P)$

$$= \sum_{\beta(P)\in\Lambda(P)} m_{\beta(P)}\, v\left(B^n_{\beta(P)}\right)$$

$$= L(f,P). \quad \blacksquare$$

Recall:

Lower Darboux int. $\displaystyle \underline{\int_{B^n}} f = \sup\{L(f,P) : P \in \mathcal{P}(B^n)\}.$

upper Darboux int. $\displaystyle \overline{\int_{B^n}} f = \inf\{U(f,P) : P \in \mathcal{P}(B^n)\}.$

So:

<u>Cor:</u> $\forall f \in \mathcal{B}(B^n)$, $\quad \displaystyle \underline{\int_{B^n}} f \leq \overline{\int_{B^n}} f.$

\# A Little more is true:

$$m \times v(B^n) \leq \underline{\int_{B^n}} f \leq \overline{\int_{B^n}} f \leq M \times v(B^n).$$

Always bounded.

<u>Recall:</u> f is (Riemann/Darboux) integrable if

$$\underbrace{\int_{B^n} f\,dx}_{\text{def: ``Volume'' (if } n>2 \text{, or ``Area'')}} := \underline{\int_{B^n}} f = \overline{\int_{B^n}} f.$$

<u>def:</u> ~~volum~~ "Volume" (if $n > 2$, or "Area")

<u>Also</u> denoted by

$$\int_{B^n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

<u>or</u> $\displaystyle \int_{B^n} f\, dV$ <u>or</u> $\displaystyle \int_{B^n} f(x)\, dV(x)$

etc.

Cor: $\forall\ P, Q \in \mathcal{P}(B^n)$, $f \in \mathcal{B}(B^n)$,

$$L(f, P) \leq U(f, Q).$$

[$\because$ $L \leq U$ Always.].

Proof: $\tilde{P} := P \cup Q$ $\longleftarrow$ the common refinement.

$$\therefore\ L(f, P) \leq L(f, \tilde{P})$$

$$\&\ \ U(f, \tilde{P}) \leq U(f, Q).$$

But $L(f, \tilde{P}) \leq U(f, \tilde{P})$.

$\therefore$ The result follows. $\square$

Notation: $\mathcal{R}(B^n) = \{ f \in \mathcal{B}(B^n) : f \text{ is integrable.} \}$.

(Little more than V.S : we will see.).

HW: $\mathcal{R}(B^n)$ is a vector space over $\mathbb{R}$:

i.e., $rf + g \in \mathcal{R}(B^n)$ $\forall\ r \in \mathbb{R}$, $f, g \in \mathcal{R}(B^n)$

& also
$$\int_{B^n} (rf + g)\, dv = r \int_{B^n} f\, dv + \int_{B^n} g\, dv.$$

HW: If $f, g \in \mathcal{R}(B^n)$ & $f(x) \leq g(x)\ \forall\ x \in B^n$,

then
$$\int_{B^n} f\, dv \leq \int_{B^n} g\, dv \quad \text{(Monotonic.)}$$

Thm: (Classification 1): Let $f \in \mathcal{B}(B^n)$. Then $f \in \mathcal{R}(B^n) \iff$ for $\varepsilon > 0$ $\exists$ $P\,(\equiv P(\varepsilon))$ in $\mathcal{P}(B^n)$ $\ni$ $U(f,P) - L(f,P) < \varepsilon$.

Proof: "$\Rightarrow$" If $f \in \mathcal{R}(B^n)$, then

$$\inf_P U(f,P) - \sup_P L(f,P) = 0.$$

$$\Rightarrow \inf_P \Big(U(f,P) - L(f,P) \Big) = 0$$

— done —

"$\Leftarrow$" Let $\varepsilon > 0$ & $P \in \mathcal{P}(B^n)$ be s.t.

$$\underbrace{U(f,P)}_{\substack{\vee\!\!\wr \\ \overline{\int} f}} - \underbrace{L(f,P)}_{\substack{\wedge\!\!\wr \\ \underline{\int} f}} < \varepsilon.$$

$$\Rightarrow \overline{\int_{B^n}} f - \underline{\int_{B^n}} f \leq U(f,P) - L(f,P) < \varepsilon.$$

$$\Rightarrow 0 \leq \overline{\int_{B^n}} f - \underline{\int_{B^n}} f < \varepsilon. \quad \forall \varepsilon > 0.$$

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Books:
1) Apostol : Calculus 2.
2) S. Lang : A 1st Course in Calculus.
3) Calculus on manifold: M. Spivak.

BIG HW: (Check $n = 1$ case)

Let $f, g \in \mathcal{R}(B^n)$. Then

1) $|f| \in \mathcal{R}(B^n)$ &

$$\left| \int_{B^n} f \, dv \right| \leq \int_{B^n} |f| \, dv.$$

(**Here:** $|f|(x) = |f(x)| \ \forall x \in B^n$).

2) $f \cdot g \in \mathcal{R}(B^n)$.

(Here: $(f \cdot g)(x) = f(x) \cdot g(x). \ \forall x \in B^n.$).

$\therefore$ **Cor:** $\mathcal{R}(B^n)$ is a ring with unit over $\mathbb{R}$. [**HW:** Zero fn. & 1 fn. $\in \mathcal{R}(B^n)$.]

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