

$$\underline{f}(x) = \int_{B^n} \underbrace{f(x,y)}_{f_x(y)} dV(y) \quad \& \quad \overline{f}(x) = \int_{B^n} f(x,y) dV(y) \quad \forall x \in B^m.$$

$$f: B^{m+n} \rightarrow \mathbb{R}; \quad B^{m+n} = B^m \times B^n$$

We now prove a BIG result:

Thm: Suppose $f \in \mathcal{R}(B^{m+n})$. Then $\underline{f}, \overline{f} \in \mathcal{R}(B^m)$ &
 Fubini thm

$$\int_{B^m} \underline{f} dV(x) = \int_{B^m} \overline{f} dV(x) = \int_{B^{m+n}} f dV(x,y).$$

$$\text{i.e.,} \quad \int_{B^m} \left(\int_{B^n} f(x,y) dV(y) \right) dV(x) = \int_{B^m} \left(\int_{B^n} \overline{f}(x,y) dV(y) \right) dV(x) \\ = \int_{B^{m+n}} f(x,y) dV$$

$\therefore \int_{B^{m+n}} f =$ iterations of lower & upper.
 They always exist.

Cor: [Note: Similar statement holds if we consider slice f_y 's w.r.t. y : If $f \in \mathcal{R}(B^{m+n})$, then

$$\int f(x,y) dV = \int_{B^n} \left(\int_{B^m} f(x,y) dV(x) \right) dV(y) = \int_{B^n} \underbrace{\left(\int_{B^m} f(x,y) dV(x) \right)}_{\in \mathcal{R}(B^n)} dV(y)$$

- the proof will be the same.

#

Cor: Suppose $f \in \mathcal{R}(B^{m+n})$. If $f_x \in \mathcal{R}(B^n) \quad \forall x \in B^m$
 $f_x(y) = \int_{B^n} f(x,y) dV(y)$

$$\text{then,} \quad \int_{B^m} \left(\int_{B^n} f(x,y) dV(y) \right) dV(x) \in \mathcal{R}(B^m) \quad \text{[or } \int_{B^m} \left(\int_{B^n} f(x,y) dV(y) \right) dV(x) \in \mathcal{R}(B^m) \text{]} \\ f_y \in \mathcal{R}(B^m) \quad \forall y$$

$$\int_{B^{m+n}} f dV = \int_{B^m} \int_{B^n} f(x,y) dV(y) dV(x)$$

$$\text{[or,} \quad \int_{B^{m+n}} f dV = \int_{B^n} \int_{B^m} f(x,y) dV(x) dV(y) \text{]}$$

i.e., the iteration is integrable.

Remark: Suppose $f \in C(B^{m+n})$. Then for each $x \in B^m$,
the slice fn $f_x \in C(B^n)$. [Illy $f_y \in C(B^m) \forall y \in B^n$].

Trivial. $x \mapsto \int_{B^n} f(x, y) dv(y) \in \mathcal{R}(B^m)$ [Trivial].
Then, in particular $\int_{B^n} f_x = \int_{B^n} f_y$. (Illy f_{xy}).

Thus:

Cor: If $f \in C(B^{m+n})$, then

$$\int_{B^{m+n}} f dv = \int_{B^n} \left(\int_{B^m} f dv(x) \right) dv(y) = \int_{B^m} \left(\int_{B^n} f dv(y) \right) dv(x).$$

Continuous
version of
Fubini

[To prove: $\int_{B^m} f dx = \int_{B^m} \bar{f} dx = \int f dx dx$]

- We need to show that f, \bar{f} are integrable.

Proof of Fubini's thm:

Let $P \in \mathcal{P}(B^{m+n})$. So $P = P^m \times P^n$ for some partitions $P^m \in \mathcal{P}(B^m)$ & $P^n \in \mathcal{P}(B^n)$.

We aim
to prove
 f integrable.

Now

$$L(f, P) = \sum_{\alpha \in \Lambda(P)} m_{\alpha} \times v(B_{\alpha}^{m+n}).$$

$$= \sum_{\alpha(P^m) \in \Lambda(P^m)} \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} v(B_{(\alpha(P^m), \alpha(P^n))}^{m+n})$$

x-variable y-variable

[$\because \alpha \in \Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)$
 $\Rightarrow \alpha = \alpha(P^m) \times \alpha(P^n)$.]

$$= v(B_{\alpha(P^m)}^m) \times v(B_{\alpha(P^n)}^n).$$

T.D:
 $\epsilon > 0, \exists \tilde{P} \in \mathcal{P}(B^n)$
s.t. $U(f, \tilde{P}) - L(f, \tilde{P}) < \epsilon$
w.l.o.g. $U(f, P) - L(f, P) < \epsilon$

$$\sum_{\alpha(P^m) \in \Lambda(P^m)} \left(\sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} v(B_{\alpha(P^n)}^n) \right) \times v(B_{\alpha(P^m)}^m).$$

Attack.

For each $x \in B^m$, set $m_{\alpha(P^n)}(x) = \inf_{y \in B^n} f(x, y) = f_x(y)$.

$$\therefore \forall x \in B_{\alpha(P^m)}^m, \quad m_{\alpha(P^n)}(x) \geq m_{(\alpha(P^m), \alpha(P^n))}$$

$$\therefore L(f, P) \leq \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{(\alpha(P^m), \alpha(P^n))} v(B_{\alpha(P^n)}^n)$$

$$\leq \sum_{\alpha(P^n) \in \Lambda(P^n)} m_{\alpha(P^n)}(x) \times v(B_{\alpha(P^n)}^n) \quad \forall x \in B_{\alpha(P^n)}^n$$

$$= L(f_x, P^n) \leq \int_{B^n} f_x dV(y) \quad \forall x \in B_{\alpha(P^n)}^n$$

$$= \underline{f}(x)$$

$$\leq \inf_{x \in B_{\alpha(P^n)}^m} \underline{f} = \underline{m}_{B_{\alpha(P^n)}^m}$$

$$\therefore L(f, P) \leq \sum_{\alpha(P^m) \in \Lambda(P^m)} m_{B_{\alpha(P^m)}^m} v(B_{\alpha(P^m)}^m) = L(\underline{f}, P^m) \quad (\because P = P^m \times P^n)$$

$$\text{i.e., } \underline{L}(f, P) \leq \underline{L}(f, P^m)$$

$$\underline{\underline{U}} \quad U(f, P) \geq U(\underline{f}, P^m) \quad \text{Now the rest is standard.}$$

[Recall: We aim at proving that \underline{f} is integrable.]

$$\therefore U(\underline{f}, P^m) - L(\underline{f}, P^m) \leq U(f, P) - L(f, P)$$

Also $\underline{f} \in \mathcal{R}(B^{m+n})$.

$$\Rightarrow \underline{f} \in \mathcal{R}(B^m) \quad \underline{\underline{U}} \quad \overline{f} \in \mathcal{R}(B^n)$$

$$\text{Finally, } L(f, P) \leq L(\underline{f}, P^m) \leq U(\underline{f}, P^m) \leq U(f, P)$$

$$\Rightarrow L(f, P) \leq L(\underline{f}, P^m) \leq U(f, P) \Rightarrow \int_{B^m} \underline{f} = \int_{B^{m+n}} f \quad \forall P \neq P^m$$

□

Cov: If $f \in C(B^n)$, then

$$\int_{B^n} f \, dv(x) = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n.$$

Also denoted by

$$\int_{B^n} f(x_1, \dots, x_n) \, dx_1 \cdots dx_n$$

all commutes.

Moreover (in particular: $n=2$):

$$\int_{[a_1, b_1] \times [a_2, b_2]} f \, dv = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) \, dx_2 \, dx_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2) \, dx_1 \, dx_2.$$

(i.e., Changing orders of integration is okay).

very useful.

eg: 1) ~~$\mathbb{R}[x_1, \dots, x_n]$~~ $\mathbb{R}[x_1, \dots, x_n] \subseteq \mathcal{R}(B^n)$, $\forall \text{ box } B^n \subseteq \mathbb{R}^n$

↖ polynomial ring (\because cont.)

$$\begin{aligned} 2) \quad \int_{[0,1]^2} xy \, dx \, dy &= \int_0^1 \left(\int_0^1 xy \, dx \right) dy \\ &= \int_0^1 y \times \left[\frac{x^2}{2} \right]_0^1 dy = \frac{1}{2} \times \int_0^1 y \, dy \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \quad \underline{\text{Ans}} \end{aligned}$$

$$\begin{aligned} \text{OR} \quad \int_{[0,1]^2} xy \, dx \, dy &= \left(\int_0^1 x \, dx \right) \times \left(\int_0^1 y \, dy \right) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

3) $f(x, y) = \begin{cases} -1 & \text{if } (x, y) \in \mathbb{Q} \times \mathbb{Q} \\ 1 & \text{if } (x, y) \in \mathbb{Q}^c \times \mathbb{Q}^c \\ 0 & \text{otherwise.} \end{cases}$

sad fellow.

Then $f \notin \mathcal{R}([0,1]^2)$.

— HW —

4) Let $f(x) = c \quad \forall x \in B^n$. Then

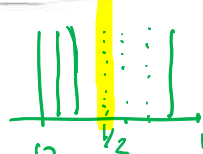
$$\int_{B^n} f \, dv = c \times v(B^n).$$

— We will do more soon —.

Note: We have seen an example of $f \in \mathcal{B}(B^2)$ (w $f \in \mathcal{B}(B^n)$)
s.t. $f \in \mathcal{R}(B^2)$ BUT f is NOT continuous!!

Q: No. of points of discontinuity? Ans: 1 in that particular example.

Page 16: $f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{R} \\ 0 & \text{otherwise.} \end{cases}$



We can confidently deal with finitely many points of discontinuity in that example too!!

BUT, the real question is: Can we integrate a bdd fn with too many points of discontinuity?

Ans: Seems Vague question: unless you spell out "too many" means "how many"!! 😐
i.e., What is "too many"?

"Vague answer": As long as those points of discontinuity fail to generate a "volume"??
↳ That's a good one.

Let's get into this (already it's too complicated):

Clearly, we need to talk about Small sets (Small in terms of Volume)

"Small sets"

Def: A subset A of \mathbb{R}^n is of **measure zero** if for $\varepsilon > 0$
 \exists Countable boxes $\{B_m^n\}_{m=1}^{\infty}$ with "the total volume" $< \varepsilon$.

a **cover of A by**

i.e., $\bigcup_{m=1}^{\infty} B_m^n \supseteq A$ & $\sum_m v(B_m^n) < \varepsilon$.

$\subseteq \mathbb{R}$

$A = \{\pi\}$

$\varepsilon > 0$
 $\exists \delta = (\varepsilon - \varepsilon_2, \varepsilon_2 + \varepsilon_1)$
 $v(-\delta) = \varepsilon$

[Note: If n is clear from the context, then we will simply write B instead of B^n .]

Note:

1) Let $A \subseteq \mathbb{R}^n$ be a finite set. Then A is of measure zero.

$\varepsilon > 0$ is given.

[Suppose $A = \{a_1, \dots, a_n\}$, $\varepsilon > 0$. Consider $r = \frac{\varepsilon}{2n}$ & ~~pick~~ boxes B_m around a_m s.t. $v(B_m) < \frac{\varepsilon}{2n}$ for $m=1, \dots, n$.]

Compactness argument?

2) Suppose $A = \{a_n\}_{n=1}^{\infty} \rightarrow$ a countable set.

: Choose box B_n $\forall n \geq 1$ s.t.

$v(B_n) < \frac{\varepsilon}{2^n}$ & $a_n \in B_n$.

$\Rightarrow \sum v(B_n) < \varepsilon. \Rightarrow A$ is of measure zero.

3) Use the above to prove the following: Suppose $\{A_n\}$ be a seqⁿ of subsets of \mathbb{R}^n of measure 0. Then $A := \bigcup_{n=1}^{\infty} A_n$ is also of measure 0.

Countable union of sets of measure 0 is of measure 0.

Proof: Let $\varepsilon > 0$. $\forall n$, get $\mathcal{C}_n =$ a countable collection of boxes s.t. $\bigcup_{C \in \mathcal{C}_n} C \supseteq A_n$ & $\sum_{C \in \mathcal{C}_n} v(C) < \frac{\varepsilon}{2^n}$.

Standard trick.

Then $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a countable collection of boxes which covers $A = \bigcup_{n=1}^{\infty} A_n$ & $\sum_{C \in \mathcal{C}} v(C) < \varepsilon$. \square

We need one more notion:

Let $f \in \mathcal{B}(\underbrace{U_n}_{\text{open in } \mathbb{R}^n})$. Let $x_0 \in U_n$.

Define $\text{osc}(f, x_0)$ ~~is~~ ~~the~~ ~~limit~~ ~~as~~ ~~$\delta \rightarrow 0$~~ by:

Oscillation
of f at x_0

$$\text{osc}(f, x_0) := \lim_{\delta \rightarrow 0} \left[\sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) \right].$$

$\therefore \text{osc}$ is a $f_n : U_n \rightarrow \mathbb{R}$ defined by

$$\text{osc}(x) = \text{osc}(f, x). \quad \forall x \in U_n.$$

Note: (1) $\forall \delta > 0, \sup_{B_\delta(x_0)} f \geq \inf_{B_\delta(x_0)} f \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \geq 0$.

~~f is not~~ Moreover $\delta \downarrow \Rightarrow \sup_{B_\delta(x_0)} f - \inf_{B_\delta(x_0)} f \downarrow$

$\therefore \text{osc}(f, x_0)$ exists $\forall x_0 \in U_n$.

(2) f is cont. at $x_0 \Leftrightarrow \text{osc}(f, x_0) = 0$. [$\because \lim_{\delta \rightarrow 0} [\] = 0$]

Proof: ~~"if"~~ Let $\text{osc}(f, x_0) = 0$. Let $\varepsilon > 0$.

$$\therefore \exists \delta > 0 \text{ s.t. } \sup_{x \in B_\delta(x_0)} f(x) - \inf_{x \in B_\delta(x_0)} f(x) < \varepsilon.$$

$$= \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \quad \leftarrow \text{we know this. useful.}$$

$$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in B_\delta(x_0)$$

$$\text{In particular: } |f(x) - f(x_0)| < \varepsilon \quad \forall x \in B_\delta(x_0).$$

" \Leftarrow " Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in B_\delta(x_0)$.

$$\Rightarrow |f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(y) - f(x_0)| < \varepsilon.$$

$$\Rightarrow \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| \leq \varepsilon \Rightarrow \text{osc}(f, x_0) = 0. \quad \forall x, y \in B_\delta(x_0)$$

(3) $\text{osc}(f, x_0) = \inf_{\delta > 0} \left\{ \sup_{x, y \in B_\delta(x_0)} |f(x) - f(y)| : x, y \in B_\delta(x_0) \right\}$. \leftarrow just observed.

(4) Let $\alpha > 0$. Then ~~there is~~ let $f \in B(C_n)$, $C_n \subseteq \mathbb{R}^n$ closed.
 Then $\{x \in C_n : \text{osc}(f, x) \geq \alpha\}$ is ~~open~~ ^{closed}
~~&~~ $\{x \in C_n : \text{osc}(f, x) < \alpha\}$ is open in \mathbb{R}^n_u .
 Like $C_n = \mathbb{B}^n$,
 But Assume $\text{int}(C_n) \neq \emptyset$.

Proof: Let $C := \{x \in C_n : \text{osc}(f, x) \geq \alpha\}$. claim: C is closed.
 We prove $\mathbb{R}^n \setminus C$ open.

Let $x \in \mathbb{R}^n \setminus C$.

$\Rightarrow x \notin C_n$ or $x \in C_n$ but $x \notin C$.

\Downarrow ~~Case 1~~
 Case 1

$\exists \delta > 0$ s.t. $B_\delta(x) \subseteq C_n^c$ [$\because C_n$ is closed].

$\Rightarrow B_\delta(x) \subseteq C^c \Rightarrow C^c$ open $\Rightarrow C$ closed.

Let $x \in C_n$ but $x \notin C$.

$\Rightarrow x \in C_n$ & $\text{osc}(f, x) < \alpha$

$\therefore \exists \delta > 0$ s.t. $\sup \left\{ |f(z) - f(w)| : z, w \in B_\delta(x) \right\} < \alpha$.

(maybe $\leq \alpha$
 & does not matter)

Consider an open box $B \subseteq B_\delta(x)$.

Let $y \in B$.

$\therefore \forall y \in B$, ~~then~~ $\exists \delta_1 > 0$ s.t. $B_{\delta_1}(y) \subseteq B_\delta(x)$.
_{open}

\therefore In particular: $\sup \left\{ |f(z) - f(w)| : z, w \in B_{\delta_1}(y) \right\} < \alpha$
 \uparrow
 $f(w)$

$\Rightarrow \text{osc}(f, y) < \alpha$.

Thus, $\forall y \in \underbrace{B}_{\text{open}}$, $\text{osc}(f, y) < \alpha$. $\Rightarrow \underbrace{B}_{\text{open}} \subseteq \mathbb{R}^n \setminus C$.

$\Rightarrow C$ is closed. \square

