Leaving the theory of double series to your intution, let me return to integration:

So far, we know $\mathcal{C}(B^n) \subseteq \mathcal{R}(B^n)$.

Computing $\int_{B^n} f\, dv$ is HARD !! We need tools !! to Compute. Thankfully, there is one: "interated integration". Let's do it !!

Warm up Visualization:

Suppose $f : B^2 \longrightarrow \mathbb{R}$. $B^2 = [a_1, b_1] \times [a_2, b_2]$.

We set, for each fixed $x \in [a_1, b_1]$, $f_x : [a_2, b_2] \longrightarrow \mathbb{R}$ by

$$\underbrace{f_x}_{\substack{\uparrow \\ \text{Called slice fn.}}} (y) = f(x, y).$$

Ily $f_y : [a_1, b_1] \longrightarrow \mathbb{R}$ $\forall$ fixed $y \in [a_2, b_2]$.

BTW: $B^2 = \underbrace{B^1}_{\substack{\uparrow \\ \text{box in } \mathbb{R} \\ = [a_1, b_1]}} \times \underbrace{B^1}_{\substack{\text{box in } \mathbb{R} \\ = [a_2, b_2]}} \quad \left(= B_1^1 \times B_2^1 \text{ say} \right)$

Also, if P is a partition of $B^2 = [a_1, b_1] \times [a_2, b_2]$
$= B_1^1 \times B_2^1$ (say),

then $P = P_1 \times P_2$, where P_i is a partition of B_i^1.

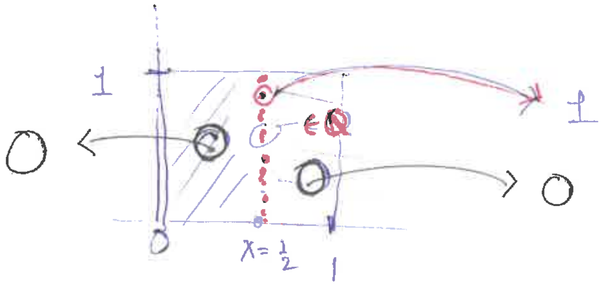
Also, $\Lambda(P) = \Lambda(P_1) \times \Lambda(P_2)$.

[In fact: $P_i = \pi_i(P)$. $\pi_i : P \longrightarrow P_i$

$$(x, y) \longmapsto \begin{cases} x & \text{if } i = 1 \\ y & \text{if } i = 2 \end{cases}$$]

We will adopt the above notation in the following generalization: $\longrightarrow$

An example: Consider $B^2 = [0,1] \times [0,1]$. And $f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \text{ \& } y \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$



Consider the slice ~~fn.~~ f_x at $x = \frac{1}{2}$. i.e, $f_{\frac{1}{2}} : B^1 \longrightarrow \mathbb{R}$.

Clearly $f_{\frac{1}{2}} \in \mathcal{B}[0,1]$ BUT $f_{\frac{1}{2}} \notin \mathcal{R}[0,1]$. $\longleftarrow$ Known to us.

Of course $f_x \in \mathcal{R}[0,1]$ $\forall\ x \neq \frac{1}{2}$. $(\because f_x \equiv 0)$.

On the other hand, $\forall\ y \in [0,1]$ (~~fixed~~), the slice fn. is given by:

$$f_y(x) \equiv \begin{cases} g(x) & \text{if } y \in \mathbb{Q} \\ 0 & \text{Otherwise.} \end{cases}$$

$\longleftarrow$ just atmost at one point discontinuity.

Here $g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$

(*) Continuous except $x = \frac{1}{2}$.

$$\therefore f_y \in \mathcal{R}[0,1] \text{ \& } \underbrace{\int_0^1 f_y\, dx}_{\displaystyle \int_0^1 f(x,y)\, dx} = 0 \quad \forall y$$

$$\int_0^1 f(x,y)\, dx = 0 \quad \forall y \in [0,1].$$

Hence
$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = 0$$

But
$$\int_0^1 f(x,y)\, dy \text{ DNE} \Rightarrow \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx$$

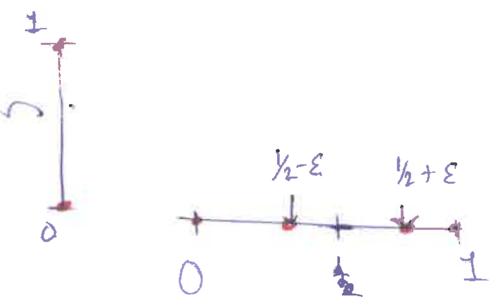
does not make sense.

In particular: $\displaystyle \int_0^1 \left(\int_0^1 f\, dx \right) dy \neq \int_0^1 \left(\int_0^1 f\, dy \right) dx.$

However $f \in R(B^2)$ & $\int_{B^2} f\, dv = 0$. ⟵ known analogy: double seqn.

<u>Proof</u>: Let $\varepsilon > 0$. Consider the following partition (the way we did it in $n=1$ case):

$P_2: 0 < 1$



$\rightsquigarrow P_1 = 0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1$.

$\therefore P_\varepsilon := P_1 \times P_2$. With $\Lambda(P) = \left\{ \left(\tfrac{1}{2}-\varepsilon, 1\right), \left(\tfrac{1}{2}+\varepsilon, 1\right), (1,1) \right\}$ — 3 nodes.

$\therefore \left\{ B^2_\alpha : \alpha \in \Lambda(P) \right\} = \left\{ \underbrace{[0, \tfrac{1}{2}-\varepsilon] \times I}_{:= B_{\alpha_1}}, \underbrace{[\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon] \times I}_{:= B_{\alpha_2}}, \underbrace{[\tfrac{1}{2}+\varepsilon, 1] \times I}_{:= B_{\alpha_3}} \right\}$

(Here $I = [0,1]$)

($\because f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$)

<u>Then</u>, $M_{\alpha_1} = M_{\alpha_3} = 0$; $M_{\alpha_2} = 1$

& $m_{\alpha_1} = m_{\alpha_2} = m_{\alpha_3} = 0$.

$\therefore L(f, P_\varepsilon) = 0$; $U(f, P_\varepsilon) = \underset{M_{\alpha_1}}{0} \times \underbrace{\left(\tfrac{1}{2} - \varepsilon\right)}_{v(B_{\alpha_1})} + \underset{M_{\alpha_2}}{1} \times \underbrace{2\varepsilon}_{v(B_{\alpha_2})}$

$+ \underset{M_{\alpha_3}}{0} \times \underbrace{\left(\tfrac{1}{2} - \varepsilon\right)}_{v(B_{\alpha_3})}$.

$= 2\varepsilon$.

$\therefore U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon$

$\Rightarrow f$ is integrable. <u>Also</u> $\overline{\int_{B^2}} f = 0$, $\underline{\int_{B^2}} f = 0 \Rightarrow \int_{B^2} f\, dv = 0$.

All in all : $f \in \mathcal{R}(B^2)$, $\int_{B^2} f \, dv = 0$, but $\int_0^1 \left(\int_0^1 f \, dy \right) dx$ DNE.

But $\int_{B^2} f \, dv = \int_0^1 \left(\int_0^1 f \, dx \right) dy = 0$.

$\square$

Setting: $B^{m+n} \underset{\text{box}}{\subseteq} \mathbb{R}^m \times \mathbb{R}^n$.

Clearly $B^{m+n} = B^m \times B^n$, for boxes $B^m \subseteq \mathbb{R}^m$ & $B^n \subseteq \mathbb{R}^n$.

Now let $P \in \mathcal{P}(B^{m+n})$. Then

$P = P^m \times P^n$, for some partitions P^m of B^m & P^n of B^n.

Also $\Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)$.

& $\forall \alpha(P) \in \Lambda(P)$, $B^{m+n}_{\alpha(P)} = B^m_{\alpha(P^m)} \times B^n_{\alpha(P^n)}$

for some $\alpha(P^m) \in \Lambda(P^m)$ & $\alpha(P^n) \in \Lambda(P^n)$.

And $\alpha(P^m) \times \alpha(P^n) = \alpha(P)$.

Finally, if $f \in \mathcal{B}(B^{m+n})$, we write $(x, y) \in B^{m+n}$ where $x \in B^m$, $y \in B^n$, and define $\underline{f}, \overline{f} : B^m \to \mathbb{R}$

by $\underline{f}(x) = \underline{\int_{B^n}} f_x \, dv(y)$, $\overline{f}(x) = \overline{\int_{B^n}} f_x \, dv(y)$.

$\therefore \underline{f}$ = lower Int. of f_x
$\overline{f}$ = upper —"—

Here, for each $x \in B^m$, $f_x : B^n \to \mathbb{R}$ is the slice fn. $f_x(y) = f(x, y)$, $\forall y \in B^n$.

Also $\int_{B^n} f_x \, dv(y) := \int_{B^n} f(x, y) \, dv(y)$.