Lecture 8: Maximal ideals.

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Let R be a comm ring with unity. Recall:

1) $I \subseteq R$ is a prime ideal if $I \neq R$, $ab \in I \Rightarrow a \in I$ or $b \in I$.

2) $I \subseteq R$ is a prime ideal $\iff$ R/I is an integral domain.

3) $m \subsetneq R$ is a maximal ideal if $\nexists$ any ideal $m \subsetneq I \subsetneq R$

Observation: $I \subseteq R$ an ideal. $I = R$ iff $1_R \in I$.

Prop: Let R be comm ring with unity and I be an R-ideal. Then I is a maximal ideal iff R/I is a field.

Lemma: Let R be a comm ring with unity. Then R is a field iff the only ideals in R are 0 and R.

Pf: $(\Rightarrow)$: Let $I \subseteq R$ be an ideal. $I \neq (0) \Rightarrow$
$\exists a \in I$ & $a \neq 0$ $\Rightarrow 1 = a^{-1}a \in I \Rightarrow I = R$.
Since R is a field $a^{-1} \in R$

$(\Leftarrow)$: Let $a \in R$ & $a \neq 0$ then $aR \neq 0 \Rightarrow$
$aR = R \Rightarrow \exists b \in R$ s.t. $ab = 1_R$.
Hence R is a field.

Proof of the proposition: $I \subseteq R$ is maximal ideal
$\iff$ the only ideals in R/I are the 0 ideal & R/I. ($\because$ ideals in R/I are in bijection with ideals of R containing I, and I being maximal the two such ideals are I & R whose images under $q_I : R \to R/I$ are 0 R/I-ideal and R/I.)

$\overset{\text{Lemma}}{\iff}$ R/I is a field.

Cor: R a comm ring with unity & $I \subseteq R$ a maximal ideal then I is a prime ideal of R.

Pf: I is a maximal ideal of $R \underset{\text{(*) Prop}}{\Rightarrow} R/I$ is a field $\overset{\text{obvious}}{\Rightarrow} R/I$ is an int domain $\underset{②}{\Rightarrow} I$ is a prime ideal.

Another proof of the cor: Let $I \subseteq R$ be a maximal ideal. Then $I \neq R$. Let $ab \in I$ for $a, b \in R$.

$Ra + I$ is an R-ideal containing I.

By maximality of I, $Ra + I = I \Rightarrow a \in I$

or $Ra + I = R$

$\Downarrow$

$\exists r \in R$ & $x \in I$ s.t.

$ra + x = 1$

$\Rightarrow \underbrace{rab}_{I} + \underbrace{xb}_{I} = b \quad \Rightarrow b \in I.$ $\blacksquare$

$(\because ab \in I, \; x \in I)$

Converse is not true: $(0) \subseteq \mathbb{Z}$ is a prime ideal but not maximal.

Question: Is every nonzero prime ideal of a ~~ring~~ integral dom R maximal?

$\mathbb{Z}[X]$ Ex: $(2), (X)$ in $\mathbb{Z}[X]$

$(2) = 2\mathbb{Z}[X] = I$

$\dfrac{\mathbb{Z}[X]}{(2)} \cong \mathbb{Z}/(2)\,[X]$

Last time

$I = X\mathbb{Z}[X]$, let $\underbrace{f(X)}_{\sum_{i=0}^{n} a_i X^i} \underbrace{g(X)}_{\sum_{i=0}^{m} b_i X^i} \in I \Rightarrow$

$f(X) g(X) = X h(X)$

$f(0) g(0) = 0 \Rightarrow f(0) = 0$ or $g(0) = 0$

$f(0)=0 \Downarrow a_0 = 0 \Downarrow f(X) = X\left(\sum_{i=1}^{n} a_i X^{i-1}\right) \in I$

$g(0)=0 \Downarrow b_0 = 0$, by $g(X) \in I$

$\{f \in \mathbb{Z}[X] \mid f(0) \text{ even}\} = (2, X)$

$I \subsetneq (2, X) \subsetneq \mathbb{Z}[X]$

Or check $\dfrac{\mathbb{Z}[X]}{(X)} \cong \mathbb{Z}$

$$\frac{\mathbb{Z}[X]}{(2, X)} \cong \mathbb{Z}/(2)$$

Thm: Every nonzero comm ring with unity R contains a maximal ideal.

Zorn's lemma: Let $(\Omega, \leq)$ be a nonempty partially ordered set. Assume that every chain in Ω has an upper bound in Ω then Ω has a maximal element.

partially ordered means $\leq$ relation is reflexive

anti-symmetric

$(a \leq b \ \& \ b \leq a \Rightarrow a = b)$

and transitive.

A chain C in Ω is a totally ordered subset

i.e. $\forall a, b \in C$ $\quad a \leq b$ or $b \leq a$.

C has an upper bound in Ω means $\exists$ $m \in \Omega$ s.t. $\forall a \in C$ $\quad a \leq m$.

m is a maximal element of Ω means

if $m \leq a$ for some $a \in \Omega$ then $a = m$.

Zorn's lemma is equivalent to Axiom of choice

AC: Let I be a set and $\{A_\alpha\}_{\alpha \in I}$ be a collection of set.

Then $\exists$ a set A s.t. A contains exactly one element from each A_α $\forall \alpha \in I$.

Pf of the thm: Let R be a nonzero comm ring with unity. Let $\Omega = \{ I \subseteq R \mid I \text{ is a proper } R\text{-ideal} \}$.
Then $\Omega \neq \phi$. Ω is a partially ordered by inclusion.

R is not the zero ring. $I \leq I'$ if $I \subseteq I'$

Let $C = \{ I_\alpha \}_{\alpha \in J}$ be a chain in Ω.
J is an indexing set

Let $I = \bigcup_{\alpha \in J} I_\alpha$. Claim: I is proper R-ideal.

Pf: Let $a, b \in I$ then $a \in I_{\alpha_0}$ & $b \in I_{\beta_0}$ for some $\alpha_0, \beta_0 \in J$. Since $\{ I_\alpha \}_{\alpha \in J}$ is totally ordered

$I_{\alpha_0} \subseteq I_{\beta_0}$ or $I_{\beta_0} \subseteq I_{\alpha_0} \Rightarrow a, b \in I_{\beta_0}$ or I_{α_0}

$\Rightarrow a+b \in I_\alpha$ or I_β for any $r \in R$
& ra

$\Rightarrow a+b$ & $ra \in I$ " " $r \in R$

$\Rightarrow I$ is an R-ideal

If $I = R \Rightarrow 1 \in I \Rightarrow 1 \in I_\alpha$ for some $\alpha \in J$
$\Rightarrow I_\alpha = R$, which contradicts $I_\alpha \in \Omega$.

Hence the claim. i.e. $I \in \Omega$ and I is an upper bound of $\{ I_\alpha \}_{\alpha \in J}$.

Hence by Zorn's lemma Ω has a maximal element say M. Then M is a maximal ideal of R by definition.

$\square$