Lecture 29: Localization of modules

20 November 2020
10:11

Recall: An R-module M is noetherian if all its submod are finitely generated.

(*) Submodules and quotient modules of noetherian modules are noetherian

⊙ Let M be an R-mod $N \subseteq M$ be noeth R-submod s.t. M/N is noeth then M is noeth.

(**) R a noetherian ring. An R-mod M is noetherian iff it is f.g.

Cor: R noeth ring. M a f.g. R-mod then any submod of M is f.g.

Example: $R = k[x_1, x_2, \dots]$ and $M = R$. Then M is generated by 1_R as an R-mod. But $I = (x_1, x_2, \dots) \subseteq M$ is not f.g. R-mod.

Localization of R-modules

Def^n/Prop: Let R be comm ring, $S \subseteq R$ be a mult set and M be an R-mod.

Define a relation on $S \times M = \{(s,m) \mid s \in S,\ m \in M\}$ as follows:

$(s_1, m_1) \sim (s_2, m_2)$ if $\exists s \in S$ s.t. $s \cdot (s_1 m_2 - s_2 m_1) = 0_M$. ① Then $\sim$ is an equivalence relation. Let $\frac{m}{s}$ denote the equivalence class $[(s,m)]$ for $(s,m) \in S \times M$ and $S^{-1}M = S \times M/\sim$. ② Then $\frac{m_1}{s_1} \oplus \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$ is a well-defined binary operator on $S^{-1}M$. ③ The map

$$S^{-1}R \times S^{-1}M \xrightarrow{\sigma} S^{-1}M$$
$$\left(\frac{r}{s}, \frac{m}{s'}\right) \longmapsto \frac{rm}{ss'}$$

is well-defined. ④ Moreover $S^{-1}M$ is a $S^{-1}R$-module via σ as the scalar multiplication. ⑤ In particular $S^{-1}M$ is an R-mod.

⑥ The map $\varphi : M \longrightarrow S^{-1}M$, $m \longmapsto \frac{m}{1}$ is an R-lin map.

Pf: $\sim$ is reflexive and symmetric follows trivially.

$(s_1, m_1) \sim (s_2, m_2)$ & $(s_2, m_2) \sim (s_3, m_3)$

$\exists u \in S$ & $v \in S$

$u \cdot (s_1 m_2 - s_2 m_1) = 0_M$ & $v \cdot (s_2 m_3 - s_3 m_2) = 0$

(1) (2)

$s_3 v (1) + s_1 u (2)$

$$s_3 u v \cancel{s_1} m_2 - s_3 u v s_2 m_1 + s_1 u v s_2 m_3 - s_1 u \cancel{v s_3} m_2 = 0$$

$$s_2 u v (s_1 m_3 - s_3 m_1) = 0 \qquad s_2 u v \in S$$

$\implies (s_1, m_1) \sim (s_3, m_3)$.

(2) & (3) same as the ring case

For (4): • $S^{-1}M$ is an abelian group:

Note $\frac{0}{1} \oplus \frac{m}{s} = \frac{s.0 + 1.m}{s} = \frac{m}{s}$

So $\frac{0}{1}$ is the additive identity.

Assoc. $\left(\frac{m_1}{s_1} \oplus \frac{m_2}{s_2}\right) \oplus \frac{m_3}{s_3} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2} \oplus \frac{m_3}{s_3}$

$$|| \qquad = \frac{s_3 s_2 m_1 + s_3 s_1 m_2 + s_1 s_2 m_3}{s_1 s_2 s_3}$$

$$\frac{m_1}{s_1} \oplus \left(\frac{m_2}{s_2} \oplus \frac{m_3}{s_3}\right) = \frac{m_1}{s_1} \oplus \frac{s_3 m_2 + s_2 m_3}{s_2 s_3}$$

$$= \frac{s_2 s_3 m_1 + s_1 s_3 m_2 + s_1 s_2 m_3}{s_1 s_2 s_3}$$

$$\frac{-m}{s} \oplus \frac{m}{s} = \frac{0}{s^2} = \frac{0}{1}$$

• $\frac{1}{1} \cdot \frac{m}{1} = \frac{1.m}{1.1} = \frac{m}{1}$

• $\left(\frac{r_1}{s_1} + \frac{r_2}{s_2}\right) \cdot \frac{m}{s} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} \cdot \frac{m}{s}$

$$|| \qquad = \frac{s_2 r_1 m + s_1 r_2 m}{s_1 s_2 s}$$

$$\frac{r_1}{s_1} \cdot \frac{m}{s} + \frac{r_2}{s_2} \cdot \frac{m}{s} = \frac{r_1 m}{s_1 s} + \frac{r_2 m}{s_2 s}$$

$$= \frac{s_2 s r_1 m + s_1 s r_2 m}{s_1 s_2 s^2} = \frac{s(s_2 r_1 m + s_1 r_2 m)}{s(s_1 s_2 s)}$$

$$= \frac{s_2 r_1 m + s_1 r_2 m}{s_1 s_2 s}$$

$|||^{ly}$ check

• $\frac{r}{s}\left(\frac{m_1}{s_1} + \frac{m_2}{s_2}\right) = \frac{r}{s} \cdot \frac{m_1}{s_1} + \frac{r}{s} \cdot \frac{m_2}{s_2}$

• $\frac{r_1}{s_1} \cdot \left(\frac{r_2}{s_2} \cdot \frac{m}{s}\right) = \frac{r_1 r_2 m}{s_1 s_2 s} = \left(\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}\right) \cdot \frac{m}{s}$

Recall $R \longrightarrow S^{-1}R$ is a ring homo.
$r \longmapsto \frac{r}{1}$

Hence $S^{-1}M$ is an R-mod.

In fact $r \cdot \frac{m}{s} = \frac{r}{1} \cdot \frac{m}{s} = \frac{rm}{s}$.

Finally the map $\varphi : M \longrightarrow S^{-1}M$
$m \longmapsto \frac{m}{1}$

is R-lin. $\forall m_1, m_2 \in M$

$$\varphi(m_1 + m_2) = \frac{m_1 + m_2}{1} = \frac{m_1}{1} \oplus \frac{m_2}{1}$$
$$= \varphi(m_1) \oplus \varphi(m_2)$$

So φ is a grp homo.

For $r \in R$ & $m \in M$ $\varphi(r \cdot m) = \frac{rm}{1} = r \cdot \frac{m}{1}$
$$= r\,\varphi(m)$$

So φ is R-mod homo.

Example: 1) $M = R$ then $S^{-1}M = S^{-1}R$ as an $S^{-1}R$-mod.

(2) $R = \mathbb{Z}$ and $M = \mathbb{Z} \times \mathbb{Z}/15$

(a) $S_3 = \{1, 3, 3^2, \dots\}$ $\leftarrow$ $S_3^{-1}M \cong \mathbb{Z}[\frac{1}{3}] \times \mathbb{Z}/5\mathbb{Z}$

(b) $S_{15} = \{1, 15, 15^2, \dots\}$ $S_{15}^{-1}M \cong \mathbb{Z}[\frac{1}{15}]$

(c) $S = \mathbb{Z} \setminus \{0\}$ $S^{-1}M \cong \mathbb{Q}$

$$S_3^{-1}(\mathbb{Z}/15\mathbb{Z}) = \left\{ \frac{[a]_{15}}{3^n} \,\middle|\, [a]_{15} \in \mathbb{Z}/15, \; n \geq 0 \right\}$$

Note $\frac{[1]}{1}$ has order 5 $\frac{[2]}{1}, \frac{[3]}{1}, \frac{[4]}{1}, \frac{[0]}{1}$
$\in S_3^{-1}(\mathbb{Z}/15\mathbb{Z})$

given 3^n

$\frac{[1]}{9} \overset{?}{=} \frac{[4]}{1}$ $3^n[1 - 36] = 0$ in $\mathbb{Z}/15$

Let a be s.t
$a3^n \equiv 1 \pmod 5$
then $\frac{[1]}{3^n} = \frac{[a]}{1}$ $\frac{[b]}{3^n} = \frac{[ab]}{1}$

$$\mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$$

(b) $\mathbb{Z}[\frac{1}{15}]$

(c) $\mathbb{Q}$