Recall: $\int_S f \, dS := \int_R f \circ r \, \| r_u \times r_v \| \, dA$.

Surface integral of the scalar field $f \in$ Cont(S) over the surface S.

Where $r : R \longrightarrow \mathbb{R}^3$ is a parametrization. (which is independent of the value of the integration).

eg: Evaluate $\int_S (x^2+y^2+z^2) \, dS$, where S is the portion of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

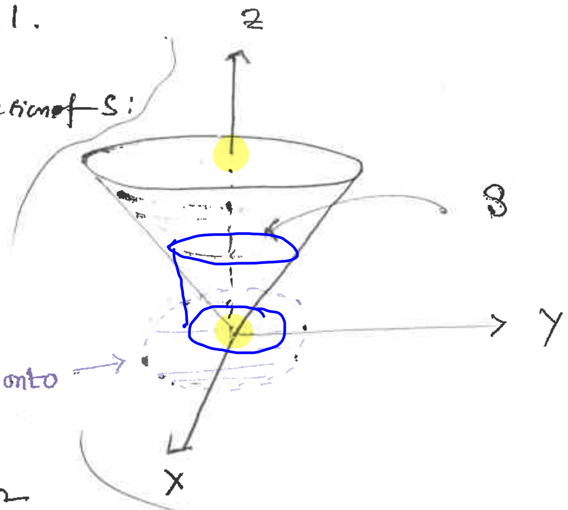
Sol: ~~Here,~~ We consider the following parametrization of S:

$r(x,y) = (x, y, \sqrt{x^2+y^2})$

$\forall (x,y) \in R = \{(x,y) : x^2+y^2 \leq 1\}$

The graph of $(x,y) \mapsto \sqrt{x^2+y^2}$.

The shadow of S onto xy-plane.



$\therefore \| r_x \times r_y \| = \sqrt{1 + f_x^2 + f_y^2}$,

where $f(x,y) = \sqrt{x^2+y^2}$ $(= z)$.

Known fact. or reprove it.

$\because \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ & $\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$.

We have: $\| r_x \times r_y \| = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}$.

$\therefore \int_S (x^2+y^2+z^2) \, dS = \int_R \underbrace{(x^2+y^2+(x^2+y^2))}_{= f \circ r} \sqrt{2} \, dA$.

$= 2\sqrt{2} \int_{x^2+y^2 \leq 1} (x^2+y^2) \, dA$.

$\overset{(*)}{=} 2\sqrt{2} \int_0^{2\pi} \int_0^1 \rho^2 \cdot \rho \, d\rho \, d\theta = \sqrt{2}\pi$. Ans.

(*) $x \longrightarrow \rho \cos\theta$

$y \longrightarrow \rho \sin\theta$.

$\Rightarrow x^2+y^2 = \rho^2$.

& $|J| = \rho$.

Jacobian

Surface integrals of $\boxed{\text{vector fields}}$:

Recall : vector fields are fn's of the form $F : \mathbb{R}^n \to \mathbb{R}^n$.

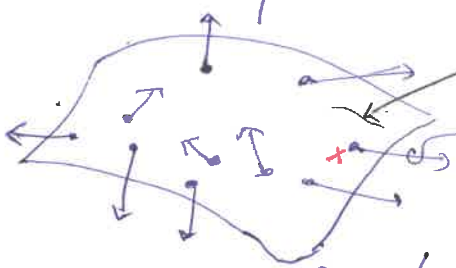
Here our interest is in vector fields $F : \mathcal{O}_3 \to \mathbb{R}^3 / \mathcal{O}_2 \to \mathbb{R}^2$.

eg: electric fields, magnetic fields, velocity field of a fluid/gas.

Let $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a velocity field of a fluid. Consider a surface $S \subseteq \mathbb{R}^3$.

or $\mathcal{O}_3 \subseteq \mathbb{R}^3$.

$\boxed{\text{OR Simply } F}$



Q: How much the vector field / amount of fluid (Also known as the $\boxed{\text{FLUX}}$ of the vector field $\vec{F}$) passes through the surface ?

Ans: Surface integral of $\vec{F}$ over S.

$\int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt$

Let's explain this.

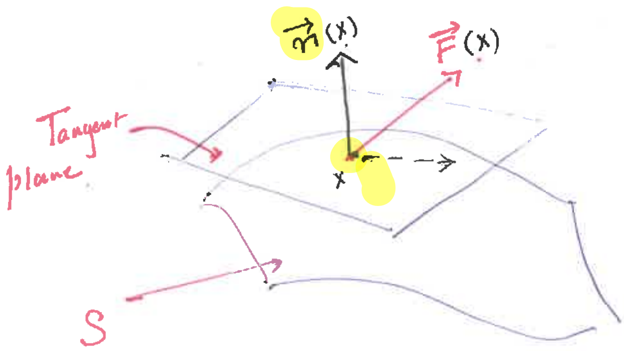
[Recall $\int_C \vec{F} \cdot d\vec{s}$ = work done by $\vec{F}$ along C.]

dr

Here we want to talk about / define

$$\int_S \vec{F} \cdot d\vec{s}.$$

So, we want to compute / measure the extent to which $\vec{F}$ is PUSHING ALONG the surface S. Let's consider "one point" situation:



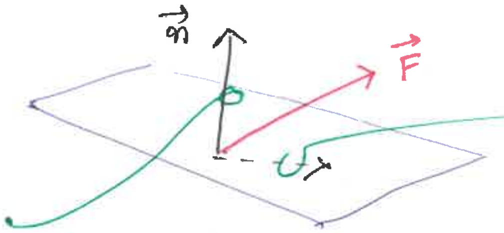
$\vec{n}$ normal vector.

$\vec{F}$ vector field.

Evidently here the components of $\vec{F}$ are: (i) one along $\vec{n}$, the vector $\perp$ to the tangent plane, & (ii) one along the tangent plane.

Clearly, (ii) (i.e. the component in the tangent plane) is NOT pushing through the surface !!

i.e: WALK ALONG NORMALS !!



No contribution to the flux.

$\vec{F}\cdot\vec{n}$ the only contribution to flux by $\vec{F}$ at this point.

Clearly, we must define

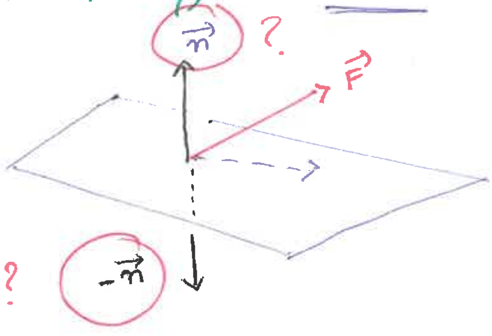
$$\int_S \vec{F}\cdot d\vec{S} \;:=\; \int_S \vec{F}\cdot\vec{n}\; dS$$

"Surface integral of the vector field $\vec{F}$ along S" = Flux / amount of fluid flowing through S.

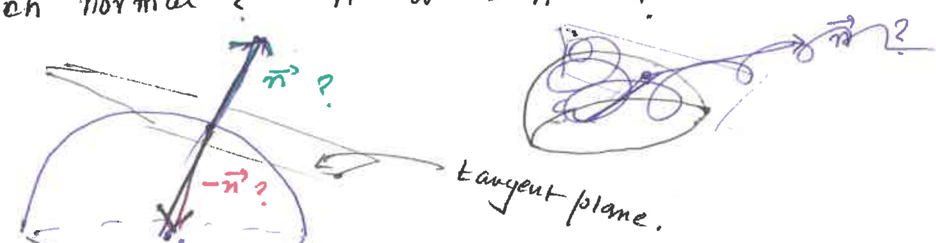
Known object. Surface integral of the scalar field $\vec{F}\cdot\vec{n} : S \longrightarrow \mathbb{R}$.

Remark: With the above "one point" view, in fact the above def of $\int \vec{F}\cdot d\vec{S}$ may be set forth as "partition–limit of Riemann integrable fn." as the way we did in previous cases.

BUT: there is atleast (/ or perhaps only) one trouble:



Which normal ? $\vec{n}$ or $-\vec{n}$?



Ans: Of course, $\vec{n}$: the direction along which the vector is pushing off !! In the sphere case.

Ans: Whatever it is, it should be "Consistent", i.e. Continuous !! We call it "orientation" of S (if $\exists$ such a choice/possibility).

Def: A surface $S \subseteq \mathbb{R}^3$ is said to be oriented if $\exists$ a continuous fn $\vec{n} : S \to \mathbb{R}^3$ (a vector field) $\ni$ $\vec{n}(x)$ is normal to S at x, $\forall x \in S$, &

$$\underbrace{\|\vec{n}(x)\| = 1}_{\text{unit vector}} \quad \forall x \in S.$$

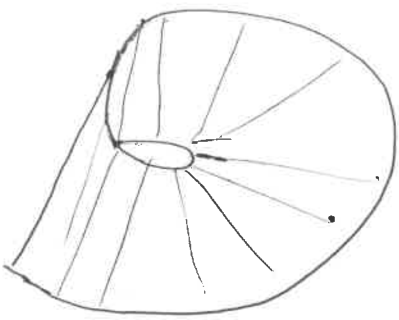
Often $\vec{n} \leftrightarrow n$.

Remark: The idea of orientation is clear: Consider just "one" "SIDE" of the surface & consider the choice of $\vec{n}$ along that SIDE.

So, any S is orientable (with two sides)?

No!! Möbius band/strip is not orientable: it has only one "side" ← whatever it means.

Another one: Klein bottle.



← Can you get a parametrization of Möbius strip?

— x —

So, along with "Oriented" care, we define:

Def: (Surface integral of a vector field $\vec{F}$ along an oriented surface S):

$$\int_S \vec{F} \cdot d\vec{S} := \int_S \vec{F} \cdot \underbrace{n}_{\substack{\uparrow \\ \text{The orientation of the surface } S}} \, dS.$$

The orientation $\vec{n}: S \to \mathbb{R}^3$ is known as the normal field.

eg: (1) $S = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ ← the sphere.

Here $\vec{n}(x) = \frac{1}{\|x\|} x \quad \forall x \in S$ (Clearly, cont.).

most useful.

(2) Consider the graph(f) : (the graph surface corresponding to $f \in C'(\mathcal{O}_2)$)

$$S := \text{graph}(f) = \{(x, y, f(x,y)) : (x,y) \in \mathcal{O}_2\}.$$

Recall: $r : \mathcal{O}_2 \longrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (x, y, f(x,y))$.

is a parametrization of S.

Here $r_x \times r_y = \langle -f_x, -f_y, 1 \rangle$.

$\because f \in C_1$, $(r_x \times r_y)(x,y) = \langle -f_x(x,y), -f_y(x,y), 1 \rangle$ is cont. on $\mathcal{O}_2$.

Set $\boxed{\vec{n} := \dfrac{r_x \times r_y}{\|r_x \times r_y\|}}$

$\therefore \vec{n}$ is an orientation of graph(f).

This will be our orientation for graph(f).

of $\int \vec{F} \cdot d\vec{s}$

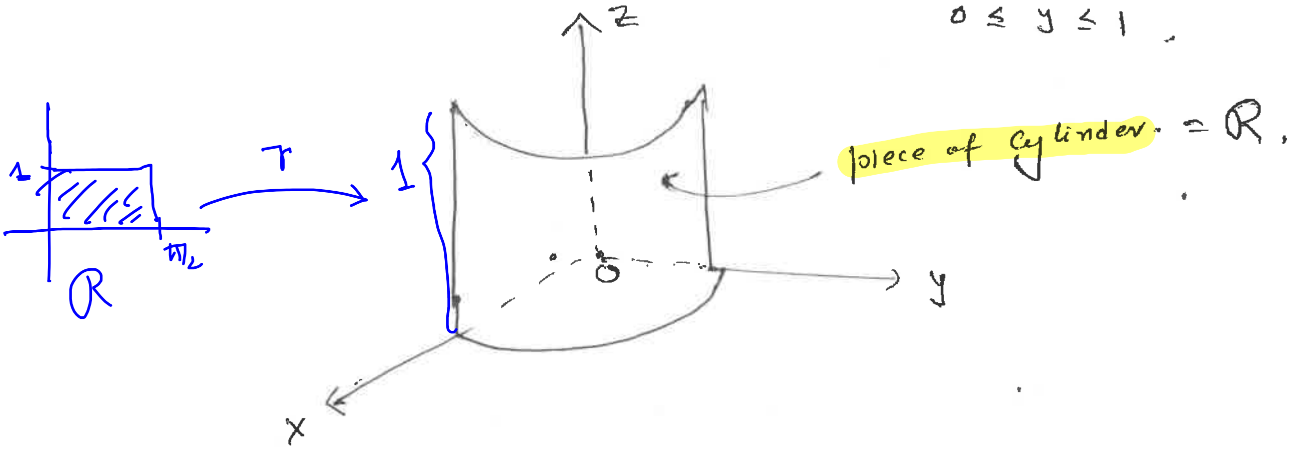
eg: We will do it : but you will soon realize, computation of $\int_S \vec{F} \cdot d\vec{s}$ is complicated, in general. There must be an easier way!! Still, lets fix some examples.

eg: $\vec{F}(x,y,z) = \langle x, y, z \rangle$ on S, where [See Page 47]

$S = \text{range}$; $\quad r(x,y) = (\cos x, \sin x, y)$

$0 \le x \le \pi/2$

$0 \le y \le 1$.



We want to compute $\displaystyle\int_S \vec{F} \cdot d\vec{S}$.

We know (See Page 47):

$r_x \times r_y = (\cos x, \sin x, 0)$. $\leftarrow$ Cont. right?

$$\therefore \vec{n} = \frac{r_x \times r_y}{\|r_x \times r_y\|} = r_x \times r_y$$

$$\therefore \int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n} \, dS$$ $\leftarrow$ Surface integral of the scalar field $\vec{F} \cdot \vec{n}$.

$$= \int_R \vec{F}(r(x,y)) \cdot \vec{n}(x,y) \underbrace{\|\vec{n}(x,y)\|}_{=1} \, dA$$ $\leftarrow$ & it will be 1 always!!

$$= \int_R \langle \cos x, \sin x, y \rangle \cdot \langle \cos x, \sin x, 0 \rangle \, dA$$

$$= \int_R (\cos^2 x + \sin^2 x) \, dA = \int_R 1 \, dA$$

$$\left(= \text{Area}(R) \right) \therefore \int_0^1 \int_0^{\pi/2} 1 \, dx \, dy = \pi/2.$$

Ans.

So, we have the following integrations:

Line integrals Vs. **Surface integrals.**

I

$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$

(length of $\gamma : [a,b] \to \mathbb{R}^n$).

$S \leftarrow$ a surface with ~~Surf. Area (R)~~ ~~(area of a region)~~ a parametrization: $r : R \to \mathbb{R}^3$, $R \subseteq \mathbb{R}^2$. Then

Surface area of S $= \underbrace{\iint_R \|r_x \times r_y\|\, dA}_{\text{Riemann double integration}}$.

II

$\int_C f = \int_a^b f(\gamma(t))\, \|\gamma'(t)\|\, dt$

f → Scalar field

C → C^1 piecewise smooth curve.

1-variable standard Riemann integ.

$\gamma : [a,b] \to \mathbb{R}^n$ a parametrization of C.

(Integration of Scalar field).

$\iint_S f\, dS = \iint_R f \circ r\, \|r_x \times r_y\|\, dA$

f → Scalar field

S → Surface.

$r : R \to \mathbb{R}^3$ a parametrization of S.

(Integration of scalar field over/along surface S)

III

$\int_C F \cdot dr = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$

F → vector field

$\gamma : [a,b] \to \mathbb{R}^3$ a parametrization of C.

(line integral of vector field) or Work done.

$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot \vec{n}\, dS$

F or $\vec{F}$ if it is clear from the context.

(Flux/surface integral of a vector field F along oriented surface S).

Back to FTC (in line integrals): Let $f : O_n \to \mathbb{R}$ be a C^1-scalar field, C be a piecewise C^1-curve (or Smooth) in O_n joining two points A & B. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A). \quad \text{———} \quad \text{✡}$$

line integrals of gradient field.

gradient field.

[See Page: 25]

Given a v.f. g if we know that $g = \nabla f$ for some $f \in C^1$ (??), then we know $\int g \cdot d\mathbf{r}$!!

$\therefore$ If C is closed, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

Motivation

~~Def: A Curve $\gamma : [a,b] \to \mathbb{R}^n$ is said to be a simple closed Curve if~~

Def: A vector field $\vec{F}$ (or F if it is clear) on O_n ($\subseteq \mathbb{R}^n$, open) is called **conservative** if $F = \nabla f$ for some C^1-scalar field f. In this case, f is called a **potential fn.** of F.

The R.H.S. of ✡ is path-independent (choice of C free, so long as C connects A & B).

Fact: Let F be a v.f. on $O_n \subseteq \mathbb{R}^n$. TFAE: ($n = 2, 3$).

1. F is conservative.
2. $\int_C F \cdot d\mathbf{r} = 0 \quad \forall$ piecewise smooth/C^1-curve $C \subseteq O_n$.
3. $\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}, \quad \forall\ C_1, C_2$ piecewise smooth curves in O_n with the same initial & end points.

$\therefore$ work done is independent of C but end points.

$C = C_1 \cup -C_2$

i.e: line integrals of F are path independent.

— HW (Easy) —

Given a v.f F on $\mathcal{O}_3$: Q1: F is Conservative?

Q2: If F is Conservative, then how to Compute a potential?

Q: A v.f. F is necessarily Conservative? Ans: No.

Lets see: Let $F = (P,Q,R)$ be a v.f. & f be a potential fn. of F (in $\mathbb{R}^3$).

$\Rightarrow \nabla f = F = (P, Q, R)$ ~~graph~~.

$\Rightarrow \frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad \frac{\partial f}{\partial z} = R.$

$\Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$ — (1) $\left(\because = \frac{\partial^2 f}{\partial x \partial y} \text{ as } f \in C^1 \right)$.

& Illy $\boxed{\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$ — (2) & $\boxed{\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}}$ — (3)

$\therefore$ ~~$\nabla f = F$~~ $\Rightarrow$ $\underbrace{(1) \text{ \& } (2) \text{ \& } (3) \text{ holds}}_{\text{Combining: } \boxed{\nabla \times F = 0}}$ (the necessary part).

F is Conservative. $= \langle P, Q, R \rangle$

Here $\nabla \times F := \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

A v.f

$\underset{\text{Formal}}{=} i\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - j\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$

$\therefore F = (P, Q, R)$ is Conservative $\Rightarrow \nabla \times F = 0.$

B.T.W: Curl of a v.f. F is defined by:

$\nabla \times F.$ ← Another v.f.

eg: (1) $F(x,y) = \langle xy, 1 - x^2 \rangle$ in $\mathbb{R}^2$.

$= \langle P, Q \rangle$ (Say). $\rightarrow \frac{\partial P}{\partial y} = x, \quad \frac{\partial Q}{\partial x} = -2x$

Now $\nabla \times F = \begin{vmatrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x - x = -3x \neq 0$ (if $x \neq 0$).

$\Rightarrow F$ is not Conservative!!

(2) Let $F(x,y) = \langle y-3,\ x+2 \rangle$. $\left(= \langle P, Q \rangle \text{ say} \right)$.

Here $\dfrac{\partial P}{\partial y} = 1 = \dfrac{\partial Q}{\partial x}$. $P = y-3$, $Q = x+2$.

If F is conservative, then $\exists f: \mathbb{R}^2 \to \mathbb{R}$. $\nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle = \langle P, Q \rangle$.

To find a potential of F:

$$\therefore \left. \begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \end{aligned} \right\} \longleftarrow \text{We got to Solve it.}$$

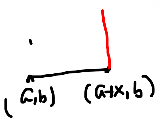
i.e, Solve $\dfrac{\partial f}{\partial x} = y-3$ — (1) $\dfrac{\partial f}{\partial y} = x+2$ — (2)

Solving PDE !!

(1) $\xRightarrow{\text{int w.r.to } x}$ $\int (y-3)\, dx = f + \varphi(y)$ $\longleftarrow$ Why ?? ✡

for some φ.

Need FTC & integration over lines.



$\Rightarrow f = xy - 3x - \varphi(y)$ ~~and~~ . —— (3)

$\xRightarrow{\text{Keeping in mind (2)}}$ $\dfrac{\partial f}{\partial y} = x - \varphi'(y)$.

$\therefore$ (2) $\Longrightarrow$ $x - \varphi'(y) = x + 2$

$\Rightarrow \varphi'(y) = -2$

$\underset{\text{FTC}}{\Rightarrow}$ $\varphi = -2y + \underbrace{k}_{\text{Constant.}}$

& we Consider $k=0$. $\longleftarrow$ No harm !!

Convexity of $\mathbb{R}^2$ $\longrightarrow$ $\Theta_2 \subseteq \mathbb{R}^2$

$\therefore$ (3) $\Rightarrow$ $f = xy - 3x + 2y$.

$\therefore$ $F(x,y) = xy - 3x + 2y$ is the potential function of F.

— x —

Q: Suppose $\nabla \times F = 0$. $\overset{?}{\Rightarrow}$ F is Conservative ?

No: eg: $F(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$. $\forall (x,y) \in \Theta_2$

$\Theta_2 \not\ni (0,0)$.