Lecture 10: Product of rings

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Recall: R comm ring with unity.

$\mathrm{Jac}(R) = \bigcap_{m \text{ maximal ideals of } R} m$ (Jacobson radical)

$\cup\!\!\!\mid$

$\mathrm{nil}(R) = \sqrt{(0)} = \{x \in R \mid x^n = 0 \text{ for } n \geq 1\}$ is an ideal.

⊛ $\mathrm{nil}(R) = \bigcap_{P \text{ prime ideals in } R} P$ (Nil radical of R)

⊛ Let R be a comm ring with unity $I \subsetneq R$ be a proper ideal [i.e. $I \neq R$]. Then $\exists$ a maximal ideal m of R s.t. $I \subseteq m$.

⊛ $x \in \mathrm{Jac}(R) \iff 1 + ax$ is a unit in R $\forall a \in R$.

Pf: $(\Rightarrow)$: $x \in \mathrm{Jac}(R)$, suppose $1+ax$ is not a unit in R for some $a \in R$. Then $I = (1+ax)R \subsetneq R$. Hence $\exists$ m maximal ideal of R containing I. In particular $1+ax \in m$. But $ax \in m$ (as $x \in m$). Hence $1 \in m$ contradicting m is a maximal ideal.

$(\Leftarrow)$: $1+ax$ is a unit $\forall a \in R$. Let m be a maximal ideal of R. If $x \notin m$ then $Rx + m = R \Rightarrow$ $\exists y \in m$ and $a \in R$ s.t. $-ax + y = 1 \Rightarrow 1 + ax = y \in m$ contradicting $1+ax$ is a unit. Hence x belongs to every maximal ideal, i.e. $x \in \mathrm{Jac}(R)$.

Examples:

Ring R	$\mathbb{Z}$	$\mathbb{Q}$ or any field	$\mathbb{Q}[X]$	$\mathbb{Z}[X]$
$\mathrm{Jac}(R)$	0	0	0	0
$\mathrm{nil}(R)$	0	0	0	0

⊛ $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ if p is a prime

$n \in \mathbb{Z}$ $n = p_1^{a_1} \cdots p_n^{a_n}$ then $n \in p_i\mathbb{Z}$

⊛ Maximal ideal of $\mathbb{Q}[x]$, (x), $(x-a)$ $a \in \mathbb{Q}$.

$f(x) \in \mathbb{Q}[x]$, if $f(x) \in \mathrm{Jac}(\mathbb{Q}[x])$

$\Rightarrow f(a) = 0 \ \forall a \in \mathbb{Q} \Rightarrow f(x) = 0$

$2 \in \mathbb{Q}[x]$ then $2^{-1} = \frac{1}{2} \in \mathbb{Q}[x] \Rightarrow (2, ..) = \mathbb{Q}[x]$

⊛ Maximal ideal of $\mathbb{Z}[x]$, $(2, X)$, $(p, X-a)$ p prime & $a \in \mathbb{Z}$

Let R be a comm ring with unity and I be a proper R-ideal, i.e. $I \subsetneq R$.

Which ideal in R correspond to $nil(R/I)$?

i.e. $q: R \longrightarrow R/I$, What is $q^{-1}(nil(R/I))$?

$\sqrt{I} := q^{-1}(nil(R/I)) = \{ r \in R \mid r^n \in I \text{ for some } n \geq 1\}$

$rad(I)$

$nil(R/I) = \{ r+I \in R/I \mid r^n \in I \text{ for some } n \geq 1\}$

⊛ $\sqrt{I} = \bigcap_{\substack{P \text{ prime in } R \\ \& \ I \subseteq P}} P$

Exc

Example 1) $R = \mathbb{Z}$, $I = 12\mathbb{Z}$

$\sqrt{I} = 6\mathbb{Z}$, $Jac(R/I) = ?$ $6\mathbb{Z}/12\mathbb{Z}$

$nil(\mathbb{Z}/12\mathbb{Z}) = \{\bar{0}, \bar{6}\}$ $= \{\bar{0}, \bar{6}\}$

⊛ What is an example of a ring R s.t.

$Jac(R) \supsetneq nil(R)$? Valuation rings

Def^n / Prop:

Product of rings: Let $R_1, R_2, \ldots, R_n$ be comm rings with unity then $R_1 \times R_2 \times \cdots \times R_n$ with component wise addition and multiplication is also a comm ring with unity.

$(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in R = R_1 \times \cdots \times R_n$

$\Rightarrow a_i, b_i \in R_i$

$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) := (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$

$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$

Check all the rings axioms for R.

$1_R = (1_{R_1}, 1_{R_2}, \ldots, 1_{R_n})$ is unity of R.

Note: $p_i : R \longrightarrow R_i$ is a ring homo. (trivial)

$(a_1, \ldots, a_n) \longmapsto a_i$

Ideals in product of rings

⊗ Let $I \subseteq R = R_1 \times R_2 \times \cdots \times R_n$ be an ideal of R then

$I = I_1 \times \cdots \times I_n$.

{Note: $I_1 \times I_2 \times \cdots \times I_n$ is a R-ideal if I_j is an R_j-ideal $1 \le j \le n$.

Pf: $I_j = p_j(I)$ is an ideal. $a_j, b_j \in I_j$ & $r_j \in R_j$ $\exists r \in R$ s.t. $p_j(r) = r_j$ ($\because p_j$ is surj)

$\Rightarrow \exists a, b \in I$ s.t. $a_j = p_j(a)$ & $b_j = p_j(b)$

$\Rightarrow p_j(a+b) = a_j + b_j \in I_j$ ($\because a+b \in I$)

$p_j(ra) = p_j(r)\, p_j(a) = r_j a_j \in I_j$ ($\because ra \in I$)

So I_j are ideals. Claim: $I = I_1 \times \cdots \times I_n$. ($\supseteq$: $(0,0,0,\ldots,a_j,0,\ldots,0) \in I$ for $\forall a_j \in I_j \; \forall j$)

$\exists a \in I$ s.t. $p_j(a) = a_j$, $a = (r_1, r_2, \ldots, r_{j-1}, a_j, r_{j+1}, \ldots, r_n)$ for some $r_i \in R_i$ $i \ne j$

j^{th} place

$e_j = (0,0,\ldots,0,1,0,\ldots,0) \in R$ & $e_j a = (0,0,\ldots,0,a_j,0,0,\ldots,0) \in I$

$\supseteq$: ✓

$\subseteq$: $a \in I \Rightarrow a = (a_1, \ldots, a_n)$ & $a_j \in p_j(I) = I_j$

$\Rightarrow a \in I_1 \times \cdots \times I_n$

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