(3) Sphere of radius $a > 0$:

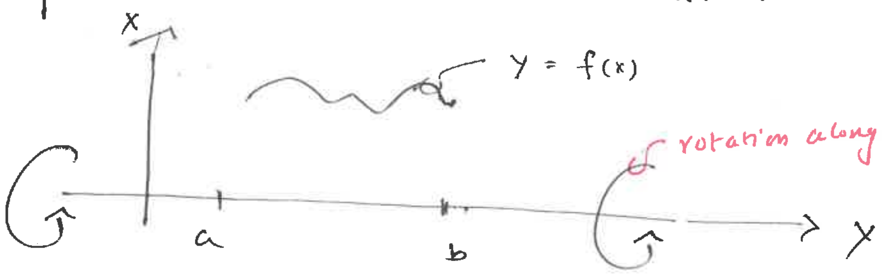
$$r(u,v) = (\sin u \cos v, \sin u \sin v, \cos u) \qquad \begin{array}{l} 0 < u < \pi \\ 0 \le v < 2\pi \end{array}$$

just the spherical coordinate.

$$r_u \times r_v \Big|_{(u,v)} = \sin u \; r(u,v). \neq 0 .$$

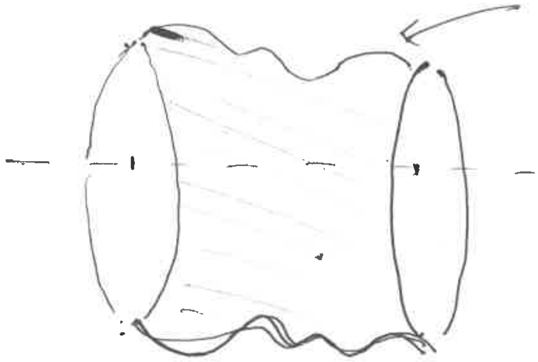
(4) Surface of revolution:

Torus & sphere are examples of "surface of revolution". More ~~Defn~~ specifically: Consider $y = f(x)$ in $\mathbb{R}^2$. $a \le x \le b$



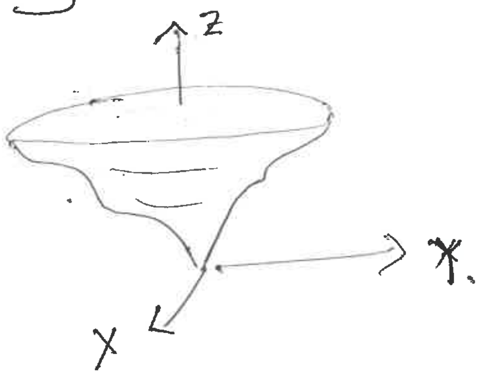
$\Downarrow$ revolve around y-axis.

do it for any curve around any line.



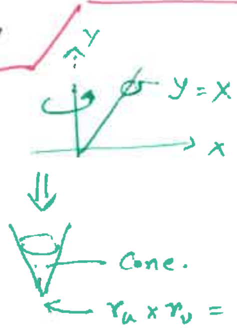
Surface of revolution

~~In general~~: Also, think:



$\Leftarrow$

Remark:



$\Downarrow$

Cone.

$r_u \times r_v = 0$??

We define it as follows:

Consider a C^1-curve $t \longmapsto (0, f(t), g(t)) \in \mathbb{R}^3$.
$t \in [a,b]$.

The surface of revolution generated by above around the z-axis is:

$$\gamma(u,v) = \big(f(u)\cos v,\ f(u)\sin v,\ g(u) \big) \qquad u \in [a,b],\ 0 \le v < 2\pi.$$

Then $\gamma_u \times \gamma_v = f(u)\big(-g'(u)\cos v,\ -g'(u)\sin v,\ f'(u) \big)$

$(\neq 0)$.

~~(5) Cylindrical Coordinates~~

~~(4) Cylinder:~~

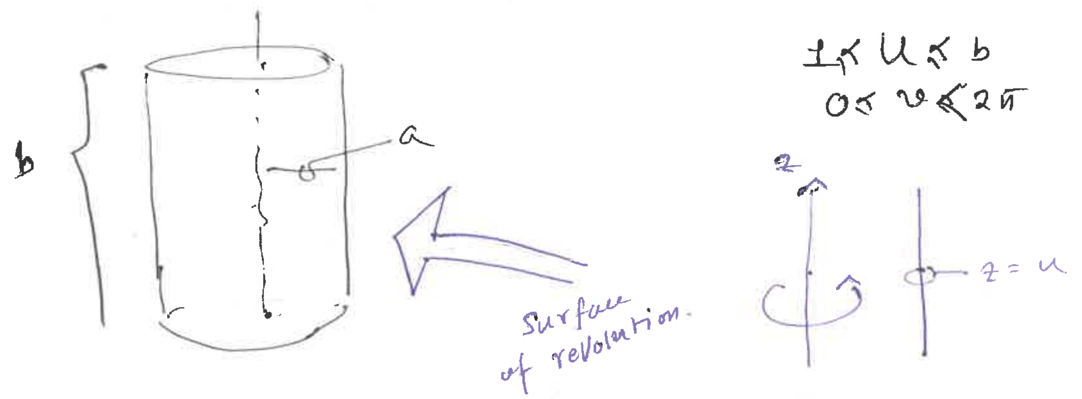
$\cancel{\gamma(u,v) = \big(a\sin u \cos v,\ a \sin u \sin v,\ a\cos u \big)}$ $a > 0$. ~~fixed~~.

$\cancel{(u,v) \in [0,\pi] \times [0, 2\pi]}$

~~Surface of revolution~~.

(4) Cylinder: $\gamma(u,v) = (a\cos v,\ a\sin v,\ u)$. $(a > 0)$.

$1 \le u \le b$
$0 \le v \le 2\pi$



Tangent plane & Normal vectors : (of Surfaces).

Let $r : R \to \mathbb{R}^3$ be a parametrized surface.
$S = \text{ran}\, r$ (the Surface). Fix $P = r(u_0, v_0)$, for some $(u_0, v_0) \in R$.

Then $r_u(u_0, v_0)$ is tangent to $t \longmapsto r(u, v_0)$ at $r(u_0, v_0)$
& Illy $r_v(u_0, v_0)$ —— " —— $t \longmapsto r(u_0, v)$ at $r(u_0, v_0)$.

Curves lying on S.

[See Page 31].

Def: T_pS = the tangent plane of S at P

Notation

:= the subspace spanned by the vectors $r_u(u_0, v_0)$ & $r_v(u_0, v_0)$. ← A 2-dimensional

defn

Tangent vectors of S at P are: $a\, r_u(u_0, v_0) + b\, r_v(u_0, v_0)$, $a, b \in \mathbb{R}$.

Question: T_pS depends on the parametrization r?

Ans: No.

Hw: Suppos $\tilde{r} : \tilde{R} \to \mathbb{R}^3$ be a parametrization of S (i.e, $S = \text{ran}\, \tilde{r}$) & let $r(u_0, v_0) = \tilde{r}(\tilde{u}_0, \tilde{v}_0) = P$.

Prove that $\text{span}\{ r_u(u_0, v_0), r_v(u_0, v_0) \} = \text{span}\{ \tilde{r}_{\tilde{u}}(\tilde{u}_0, \tilde{v}_0), \tilde{r}_{\tilde{v}}(\tilde{u}_0, \tilde{v}_0) \}$.

[Hint: $r^{-1} \circ \tilde{r}$ is a C^1-map from an open set around $(\tilde{u}_0, \tilde{v}_0)$ to an open set around (u_0, v_0). Apply Chain rule]

|| Remark: The assumption that $r_u \times r_v \big|_{(u_0, v_0)} \neq 0$ assures that r_u & r_v are linearly independent at (u_0, v_0).

Def: Elements of T_pS are called tangent vectors of S at P. (See ✡ above.)

Now consider ~~the~~ a graph fn. $z = f(x,y)$. $(x,y) \in \mathcal{O}_2$; f is C^1.

Then $r(u,v) = (u, v, f(u,v))$ is a parametrization of the surface:

$S := \text{graph } f = \{(x, y, f(x,y)) : (x,y) \in \mathcal{O}_2\}$. ← See Page 33.

Also $r_u \times r_v = (-f_u, -f_v, 1)$ $\forall (u,v) \in \mathcal{O}_2$.

Let $P = (a, b, f(a,b)) \in S$.

$\therefore$ The eqn of the tangent plane $\underbrace{\quad}_{T_P S}$ is:

$$\left(-f_u\big|_P\right)(x-a) + \left(-f_v\big|_P\right)(y-b) + (1)\cdot\left(z - f(a,b)\right) = 0,$$

i.e.,
$$\boxed{\underbrace{\frac{\partial f}{\partial u}\Big|_{(a,b)}}_{f_u(a,b)}(x-a) + \underbrace{\frac{\partial f}{\partial v}\Big|_{(a,b)}}_{f_v(a,b)}(y-b) - \left(z - f(a,b)\right) = 0.} \quad (T)$$

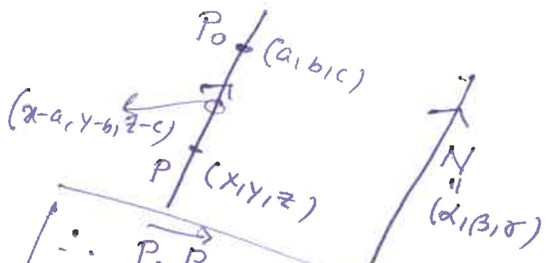
eqn. of a plane through P & $\perp$ N.

Also eqn of the normal line N is given by:

$$\left.\begin{aligned} x - a &= \big(\underbrace{-f_u(a,b)}_{= -f_u(a,b)}\big)\, t \\ y - b &= \big(\underbrace{-f_v(a,b)}_{= -f_v(a,b)}\big)\, t \\ z - f(a,b) &= t \end{aligned}\right\} \quad (N)$$

parametric eqn. of a line through P in the direction of N.

$t \in \mathbb{R}$ is the parameter.



$\therefore\ \overrightarrow{P_0 P} = tN$

OR (N) $\iff$

$$\frac{x-a}{-f_u\big|_{(a,b)}} = \frac{y-b}{-f_v\big|_{(a,b)}} = \frac{z - f(a,b)}{1}.$$

↑

Symmetric eqn. of the normal.

eg: Eqn. of tangent plane & normal line to $z = \frac{2x}{y} - x^2$ at $(1,1,1)$:

We straightaway compute $\frac{\partial z}{\partial x} = \frac{2}{y^2} - 2x$.

$$\frac{\partial z}{\partial y} = -\frac{2x}{y^2}.$$

$\therefore$ The normal vector $N = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \Big|_{(x=1,\,y=1)}$

$$= \left\langle -\frac{2}{y^2} + 2x, \frac{2x}{y^2}, 1 \right\rangle_{(1,1)}$$

$$= (0, 2, 1).$$

or $\langle 0,2,1\rangle$ or $0i + 2j + k$.

$\therefore$ eqn of N is:

$$\left.\begin{aligned} x-1 &= 0 \\ y-1 &= 2 \\ z-1 &= t \end{aligned}\right\}$$

eqn of tangent plane: $0(x-1) + 2(y-1) + 1\cdot(z-1) = 0$.

$$\Rightarrow 2y + z = 3.$$

eg: Consider the parametrized surface

$$r(u,v) = (u^2 - v^2, uv, u^2 + v^2).$$

$\therefore r_u = (2u, v, 2u)$, $r_v = (-2v, u, 2v)$.

If $(u,v) = (2,1)$, then $r(u,v) = (3,2,5)$.

& $r_u \times r_v \Big|_{(u,v)=(2,1)} = \begin{vmatrix} i & j & k \\ 4 & 1 & 4 \\ -2 & 2 & 2 \end{vmatrix} = (-6, -16, 10)$.

$\therefore$ At $(3,2,5)$, eqn. of tangent plane: $-6(x-3) - 16(y-2) + 10(z-5) = 0$.

& — " — Normal line: $\left.\begin{aligned} x-3 = -6t,\ y-2 = -16t \\ z-5 = 10t. \end{aligned}\right\}$.

Approximation:

Let's get back to derivatives of fn's in $\mathbb{R}^2$.

Let $f \in C^1(O_2)$. $\because f: O_2 \to \mathbb{R}$ is diff. we have, for a fixed $(a,b) \in O_2$:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \left(Df(a,b)\right)\left[x-a \quad y-b\right]^t}{\|(x,y)-(a,b)\|} = 0 \qquad (1)$$

$\begin{bmatrix} x-a \\ y-b \end{bmatrix} \in \mathbb{R}^2$

$Df(a,b)$ = Total derivative of f at (a,b)
$= [f_x(a,b) \quad f_y(a,b)]$.

So $Df(a,b)\,[x-a \quad y-b]^t = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

So (1) $\Rightarrow$ $$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

Now recall (eqn (T), Page 38): The eqn of the tangent plane at $P(a,b,f(a,b))$ on the surface

S = graph $f = \{(x,y,f(x,y)) : (x,y) \in O_2\}$. is

\# given by: $z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

Following 1-variable Calculus,

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad (2)$$

is called the linear or the tangent $\boxed{\text{plane}}$ approximation of f (NEAR) at (a,b).

\# In particular, if $L(x,y)$ = R.H.S. of (2), then

"An affine plane"

(1) $\Rightarrow$ $$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

A fn. $L: \mathbb{R}^n \to \mathbb{R}$ is said to be affine if $\exists\, a_1, \dots, a_n \in \mathbb{R}$ & $a \in \mathbb{R}$ s.t. $L(x) = a + \sum_{i=1}^{n} a_i x_i$.

Of course, an affine map is linear $\Leftrightarrow a = 0$.

eg: Consider $f(x,y) = x e^{xy}$. ~~$\&$ $(a,b) = (1,0)$~~.

We want to find approximate value of $f(1.1, -0.1)$.

Sol: We do it by linear approximation of f near to $(1,0)$.

Here ~~$\det(\text{expand})$~~
$$\left.\begin{aligned} f_x &= (1+xy)e^{xy} \\ f_y &= x^2 e^{xy} \end{aligned}\right\}.$$

$$\therefore \left.\begin{aligned} f_x(1,0) &= e^0 = 1 \\ f_y(1,0) &= 1 \cdot e^0 = 1 \end{aligned}\right\}.$$

$\therefore$ The linear approximation of f near $(1,0)$ is given by:

$$\begin{aligned} L(x,y) &= f(1,0) + 1 \times (x-1) + 1 \times (y-0) \\ &= 1 + x - 1 + y \end{aligned}$$

$$\Rightarrow L(x,y) = x + y$$

$\therefore$ near $(1,0)$, $\quad x e^{xy} \approx x + y$.

$$\begin{aligned} \text{So } f(1.1, -0.1) &\approx L(1.1, -0.1) \\ &= 1.1 + (-0.1) \\ &= 1. \end{aligned}$$

Ans.

Here: linear approximation of $f(x,y)$ at (a,b) is simply the following:

Compute the normal vector $N = (-f_x(a,b), -f_y(a,b), 1)$. Then the tangent plane:
$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0.$$

Then

$$\boxed{z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$$

approximated value of f near (a,b).

#

eg: Use tangent plane to approximate $(1.99)^2 - \frac{1.99}{1.01}$.

Sol: First figure out a fn. Here

$$f(x,y) = x^2 - \frac{x}{y}. \quad (a,b) = (2,1).$$

$[\because (1.99, 1.01) \approx (2,1)]$

~~Now the normal of S = graph f at $(2,1,f(2,1))$~~

$= f(2,1)$

Then $f_x = 2x - \frac{1}{y}$, $f_y = \frac{x}{y^2}$.

So $f_x(2,1) = 3$, $f_y(2,1) = 2$.

$\therefore$ Eqn. of tangent plane at $(1,2,f(2,1)) = (1,2,2)$ is:

$$f(x,y) \approx z = f(2,1) + f_x(2,1)\,(x-2) + f_y(2,1)\,(y-1)$$

$$= 2 + 3(x-2) + 2(y-1)$$

$$= 3x + 2y - 6.$$

$$\therefore f(1.99, 1.01) \approx 3\times(1.99) + 2\times(1.01) - 6$$

$$= 5.97 + 2.02 - 6$$

$$= 7.99 - 6$$

$$= 1.99.$$

i.e. $f(1.99, 1.01) \approx 1.99$.

Ans.