\# Leaving the theory of double sexies to your intution, let me return to integration:

So far, we know $C(B^n) \subsetneq R(B^n)$.

\# Computing $\int_{B^n} f \, dv$ is HARD!! We need (tools) to compute.

Thankfully, there is one: "interated integration". Let's do it!!

$$\# \{ \alpha_{m_1 m_2 \cdots m_n} \}$$

Warm up Visualization:

Suppose $f : B^2 \longrightarrow \mathbb{R}$. $\qquad B^2 = [a_1, b_1] \times [a_2, b_2]$.

We set, for each fixed $x \in [a_1, b_1]$, $f_x : [a_2, b_2] \longrightarrow \mathbb{R}$ by

$$f_x(y) = f(x, y).$$

$\underset{\uparrow}{\underline{\qquad}}$

Called slice f_x.

$\Vert_y$ $f_y : [a_1, b_1] \longrightarrow \mathbb{R}$ $\quad \forall$ fixed $y \in [a_2, b_2]$.

BTW: $B^2 = \underset{\uparrow}{B'} \times B'$ $\left(= B'_1 \times B'_2 \text{ say} \right)$

box in $\mathbb{R}$ $\qquad$ box in $\mathbb{R}$
$= [a_1, b_1]$ $\qquad = [a_2, b_2]$

Also, if P is a partition of $B^2 = [a_1, b_1] \times [a_2, b_2]$

$$= B'_1 \times B'_2 \text{ (say)},$$

then $\underline{P = P_1 \times P_2}$, where P_i is a partition of

Also, $\Lambda(P) = \Lambda(P_1) \times \Lambda(P_2)$. $\qquad\qquad B'_i$.

$\left[\text{In fact:} \quad P_i = \Pi_i(P). \qquad \Pi_i : P \longrightarrow P_i \right.$

$$\left. (x, y) \longmapsto \begin{cases} x & \text{if } i = 1 \\ y & \text{if } i = 2 \end{cases} \right]$$

We will adopt the above notation in the following generalization : $\longrightarrow$

An example: Consider $B^2 = [0,1] \times [0,1]$. And $f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ & \& y \in \mathbb{Q} \\ 0 & \text{Otherwise} \end{cases}$
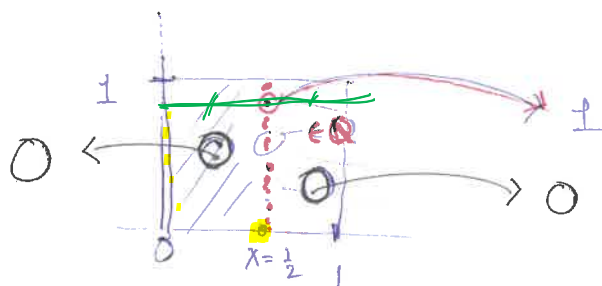
Consider the slice fn. f_x at $x = \frac{1}{2}$. i.e, $f_{\frac{1}{2}} : B^1 \to \mathbb{R}$.

Clearly $f_{\frac{1}{2}} \in B[0,1]$ BUT $f_{\frac{1}{2}} \notin \mathcal{R}[0,1]$. $\longleftarrow$ Known to us.

Of course $f_x \in \mathcal{R}[0,1] \quad \forall \ x \neq \frac{1}{2}$. $\left(\because f_x \equiv 0 \right)$.

On the other hand, $\forall \ y \in [0,1] \ (\text{fixed})$, the slice fn. is given by:

$$f_y(x) \equiv \begin{cases} g(x) & \text{if } y \in \mathbb{Q} \\ 0 & \text{Otherwise.} \end{cases} \longleftarrow \begin{array}{l} \text{just atmost} \\ \text{at one point} \\ \text{\circledast discontinuity.} \end{array}$$

Here $g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{Otherwise.} \end{cases}$

$\circledast$ Continuous except $x = \frac{1}{2}$.

$\therefore \ f_x \in \mathcal{R}[0,1]$ & $\displaystyle\int_0^1 f_y \, dx = 0 \quad \forall y$

$$\underbrace{}_{\parallel}$$

$$y \longmapsto \int_0^1 f(x,y) \, dx = 0 \quad \forall y \in [0,1].$$

Hence $\displaystyle\int_0^1 \left(\int_0^1 f(x,y) \, dx \right) dy = 0$

But $\displaystyle\int_0^1 f(x,y) \, dy$ DNE $\Rightarrow \int_0^1 \left(\int_0^1 f(x,y) \, dy \right) dx$
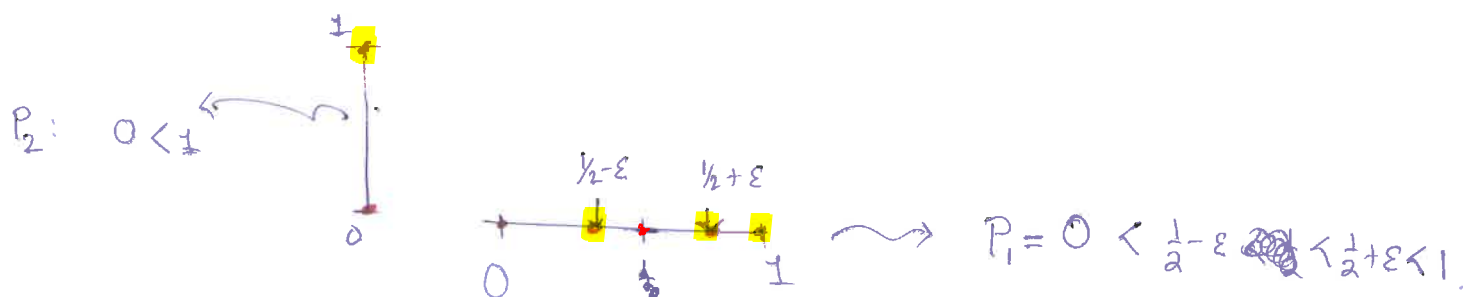
does not make sense.

In particular: $\displaystyle\int_0^1 \left(\int_0^1 f \, dx \right) dy \neq \int_0^1 \left(\int_0^1 f \, dy \right) dx$.

However $f \in R(B^2)$ & $\int_{B^2} f \, dv = 0$.

Proof: Let $\varepsilon > 0$. Consider the following partition (the way we did it in the $n=1$ case):

$P_2: \quad 0 < 1$

$\frac{1}{2} - \varepsilon \quad \frac{1}{2} + \varepsilon$

$\rightsquigarrow \quad P_1 = 0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1$

$\therefore \quad P_\varepsilon := P_1 \times P_2$. With $\Lambda(P) = \left\{ \left(\frac{1}{2}-\varepsilon, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}+\varepsilon, 0\right), (1,0) \right\}$

$= \left\{ \left(\frac{1}{2}-\varepsilon, 1\right), \left(\frac{1}{2}+\varepsilon, 1\right), (1,1) \right\}$

$\underbrace{}_{3 \text{ nodes.}}$

$\therefore \left\{ B^2_\alpha : \alpha \in \Lambda(P) \right\} = \left\{ [0, \frac{1}{2}-\varepsilon] \times I, \; [\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon] \times I, \; [\frac{1}{2}+\varepsilon, 1] \times I \right\}$

$\left(\text{Here } I = [0,1] \right)$

$\left(\because f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \right)$

$:= B_{\alpha_1} \qquad := B_{\alpha_2} \qquad := B_{\alpha_3}$.

Then, $M_{\alpha_1} = M_{\alpha_3} = 0; \quad M_{\alpha_2} = 1$

& $m_{\alpha_1} = m_{\alpha_2} = m_{\alpha_3} = 0$.

$\therefore L(f, P_\varepsilon) = 0; \qquad U(f, P_\varepsilon) = \underset{\substack{M_{\alpha_1} \\ \downarrow}}{0} \times \underset{v(B_{\alpha_1})}{\left(\frac{1}{2}-\varepsilon\right)} + \underset{\substack{M_{\alpha_2} \\ \downarrow}}{1} \times \underset{v(B_{\alpha_2})}{2\varepsilon}$

$+ \underset{\substack{\uparrow \\ M_{\alpha_3}}}{0} \times \underset{v(B_{\alpha_3})}{\left(\frac{1}{2}-\varepsilon\right)}$

$= 2\varepsilon$.

$\therefore U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon$

$\Rightarrow f$ is integrable. Also $\overline{\int_{B^2}} f = 0, \; \underline{\int_{B^2}} f = 0 \Rightarrow \int_{B^2} f \, dv = 0$.

All in all: $f \in R(B^2)$, $\displaystyle\int_{B^2} f \, dv = 0$, but $\displaystyle\int_0^1 \left(\int_0^1 f \, dy \right) dx$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad DNE.$

But $\displaystyle\int_{B^2} f \, dv = \int_0^1 \left(\int_0^1 f \, dx \right) dy = 0.$

$\boxed{\diagup\!\diagup}$

Setting:

$B^{m*n} \underset{box}{\subseteq} \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}.$

Clearly $B^{m+n} = B^m \times B^n$, for boxes $B^m \subseteq \mathbb{R}^m$ $\& B^n \subseteq \mathbb{R}^n.$

Now let $P \in \mathcal{P}(B^{m+n})$. Then

$\underline{P = P^m \times P^n}$, for some partitions P^m of B^m $\& P^n$ of B^n.

Also $\Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)$.

$\& \; \forall \alpha(P) \in \Lambda(P), \quad B^{m+n}_{\alpha(P)} = B^m_{\alpha(P^m)} \times B^n_{\alpha(P^n)}$

for some $\alpha(P^m) \in \Lambda(P^m)$ $\& \; \alpha(P^n) \in \Lambda(P^n).$

And $\alpha(P^m) \times \alpha(P^n)$ $= \alpha(P^{m+n}).$

Finally, if $f \in B(B^{m+n})$, we write $(x, y) \in B^{m+n}$

where $x \in B^m$, $y \in B^n$, and define $\underline{f}$, $\overline{f} : B^m \to \mathbb{R}$

by $\underline{f}(x) = \displaystyle\int_{B^n} f_x \, dv(y)$, $\overline{f}(x) = \displaystyle\overline{\int_{B^n} f_x \, dv(y)}$

$\therefore \underline{f} = $ lower int. of f_x
$\overline{f} = $ upper $-\!\!\!\!\;''\!\!\!\!\;-$.

Here, for each $x \in B^m$, $f_x : B^n \to \mathbb{R}$ is the

slice fn. $f_x(y) = f(x, y)$, $\forall y \in B^n$.

Also $\displaystyle\int_{B^n} f_x \, dv(y) := \int_{B^n} f(x, y) \, dv(y)$,