Lecture 6: Isomorphism theorems

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- R an integral domain, $S \subseteq R$ subring (containing 1_R) then S is an int domain.
- R/I has a ring structure. $q: R \to R/I$ the quot. ring homo. surjective.
- In R/I, $a+I = q(a) = \bar{a}$, hence $\overline{a+b} = \bar{a} + \bar{b}$ & $\overline{ab} = \bar{a}\,\bar{b}$

Prop: Let R be an integral domain then $R[X]$ is an integral domain.
Hence $R[x_1, \dots, x_n]$ is also an " " .

Pf: Let $f(x), g(x) \in R[X]$ be nonzero elements then

$f(x) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_0$ for some $a_i \in R$, $a_n \neq 0$

$g(x) = b_m X^m + b_{m-1} X^{m-1} + \dots + b_0$ for some $b_i \in R$, $b_m \neq 0$

Then $f(x) g(x) = a_n b_m X^{n+m} + \dots + a_0 b_0$

R int domain $\Rightarrow a_n b_m \neq 0 \Rightarrow f(x) g(x) \neq 0$. ∎

⊛ $R[X]$ int domain $\Rightarrow R$ is an int domain.

⊛ Ideals in R/I.

Eg: $\mathbb{Z}$ $\quad \mathbb{Z}/n\mathbb{Z}$ $\quad a\mathbb{Z}/n\mathbb{Z}$ where $a|n$

$\{ b\mathbb{Z} \mid b|n \} \leftarrow$ ideals of $\mathbb{Z}$ containing $n\mathbb{Z}$

Prop: Ideals of R/I are in bijection with ideals of R containing I.
The bijection is given by: $J \subseteq R$ ideal containing I then
$q(J) = J/I \subseteq R/I$ is an ideal of R/I. Here $q: R \to R/I$ is the quotient map.
$W \subseteq R/I$ be an ideal then $q^{-1}(W)$ is an ideal of R containing I.

Pf: J/I is closed under addition ✓

Let $r+I \in R/I$ & $a+I \in J/I$ then $a \in J$

$\Rightarrow ra \in J \Rightarrow ra + I \in J/I \Rightarrow (r+I)(a+I) \in J/I$

Hence J/I is an ideal of R/I.

$I = q^{-1}(0)$

Also $q^{-1}(W)$ is an ideal of R for W an ideal of R/I and $I \subseteq q^{-1}(W)$ ✓ clear

Lemma: $\varphi: A \to B$ be a ring homo of comm rings with unity and $J \subseteq B$ be an ideal of B then $\varphi^{-1}(J)$ is an ideal of A.

Pf: $a_1, a_2 \in \varphi^{-1}(J) \Rightarrow \varphi(a_1), \varphi(a_2) \in J \Rightarrow \varphi(a_1 + a_2) \in J$

$\Rightarrow a_1 + a_2 \in \varphi^{-1}(J)$.

$\varphi(r) \in B$, J is an ideal of B

$r \in A$ & $a \in \varphi^{-1}(J) \Rightarrow \varphi(a) \in J \Rightarrow \varphi(r)\varphi(a) \in J$

$\xRightarrow{\varphi \text{ is a ring homo}} \varphi(ra) \in J \Rightarrow ra \in \varphi^{-1}(J) \Rightarrow \varphi^{-1}(J)$ is an ideal of A.

- $I \subseteq J \subseteq R$ ideal then $q^{-1}(J/I) = J$ $\quad \left(\begin{array}{l} a \in q^{-1}(J/I) \\ \Leftrightarrow a+I \in J/I \Leftrightarrow a \in J \end{array} \right)$
- $W \subseteq R/I$ be an ideal of R/I then $q^{-1}(W)/I = W$

$a+I \in W \Leftrightarrow a \in q^{-1}(W) \Leftrightarrow a+I \in q^{-1}(W)/I$

∎

(*) $\varphi : A \to B$ ring homo, $\psi : B \to C$ " then $\psi \circ \varphi : A \to C$ is a ring homo.

$$\psi \circ \varphi (a_1 a_2) = \psi(\varphi(a_1)\varphi(a_2)) = \psi(\varphi(a_1))\,\psi(\varphi(a_2))$$

Isomorphism theorems

First isom thm: Let $\varphi : A \to B$ be a surjective ring homo. Then the induced map

$$\bar{\varphi} : A/\ker\varphi \longrightarrow B \quad \text{is an isomorphism.}$$
$$\underset{a+\ker\varphi}{\overset{\|}{\bar{a}}} \longmapsto \varphi(a)$$

Prop: Let $\varphi : A \to B$ be a ring homo. and $K \subseteq \ker(\varphi)$ be an A-ideal. Then there exist a ring homo

$$\bar{\varphi} : A/K \longrightarrow B \quad \text{s.t. } \bar{\varphi} \circ q = \varphi$$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "q"'] & B \\ & A/K \arrow[u, dashed, "\bar{\varphi}"'] \end{tikzcd}$$

is a commutative diagram.

In particular, if $K = \ker(\varphi)$ then $\bar{\varphi}$ is injective

Pf: $\bar{\varphi} : A/K \longrightarrow B$ is well-defined
$$\bar{a} \longmapsto \varphi(a)$$

Let $\underset{}{\overset{a+K}{\bar{a}}} = \bar{b}$ for $a, b \in A \Rightarrow a - b \in K \subseteq \ker(\varphi)$

$\Rightarrow \varphi(a-b) = 0$ in B

$\Rightarrow \varphi(a) = \varphi(b)$ in B. Hence $\bar{\varphi}$ is well-defined

$\bar{\varphi}$ is a ring homo. :

$$\bar{\varphi}(\bar{a}+\bar{b}) = \bar{\varphi}(\overline{a+b}) = \varphi(a+b) = \varphi(a)+\varphi(b)$$

$\uparrow \varphi$ is a ring homo

$$\bar{\varphi}(\bar{a}) + \bar{\varphi}(\bar{b}) = \varphi(a) + \varphi(b)$$

III'ly $\bar{\varphi}(\bar{a}\bar{b}) = \bar{\varphi}(\overline{ab}) = \varphi(ab) = \varphi(a)\varphi(b) = \bar{\varphi}(\bar{a})\bar{\varphi}(\bar{b})$

For $a \in A$

$$\bar{\varphi} \circ q(a) = \bar{\varphi}(\bar{a}) = \varphi(a) \qquad (q : A \to A/K)$$

$\Rightarrow \bar{\varphi} \circ q = \varphi$

Now if $K = \ker(\varphi)$ and $\bar{a} \in A/K$ be s.t. $\bar{\varphi}(\bar{a}) = 0$ in B then $\varphi(a) = 0$. Hence $a \in K = \ker(\varphi)$

$\Rightarrow \bar{a} = 0$ in A/K.

Hence $\ker(\bar{\varphi}) = 0$; i.e. $\bar{\varphi}$ is injective. ☒

Pf of 1st isom thm:

By prop. $\bar{\varphi} : A/\ker\varphi \longrightarrow B$ is an injective ring homo. But φ is surjective & $\bar{\varphi} \circ q = \varphi \Rightarrow \bar{\varphi}$ is surjective

Hence $\bar{\varphi}$ is an isomorphism.

Second isom thm: Let R be a comm ring with unity. Let $S \subseteq R$ be a subring & I be an R-ideal. Then $S+I$ is subring of R, $S \cap I$ is an S-ideal and

$$S+I/_I \cong S/_{S\cap I} \text{ as rings.}$$

Pf: $\alpha, \alpha' \in S+I$

$\Rightarrow \alpha = r + a$ for some $r \in S$ & $a \in I$

$\alpha' = r' + a'$ " " $r' \in S$ & $a' \in I$

$\Rightarrow \alpha + \alpha' = \underbrace{(r+r')}_{\in S} + \underbrace{(a+a')}_{\in I} \in S+I$

& $\alpha \cdot \alpha' = (r+a)(r'+a') = \underbrace{rr'}_{\in S} + \underbrace{a(r'+a') + ra'}_{\in I} \in S+I$

$S \cap I = i^{-1}(I)$ where $i : S \hookrightarrow R$ is the inclusion map.

Hence $S \cap I$ is an S-ideal. (By Lemma)

Let $S \xhookrightarrow{i} S+I \xrightarrow{q} S+I/_I$

$\varphi = q \circ i$. Then φ is a ring homo. $\alpha \in S+I/_I \Rightarrow$

$(r+a)+I = \alpha = \overline{r+a}$ for some $r \in S$ & $a \in I$

$\Rightarrow \alpha = \bar{r}$ $(\because r+a-r = a \in I)$

$\Rightarrow \alpha = \varphi(r)$

Hence φ is surj

Claim: $\ker(\varphi) = S \cap I$

$\alpha \in \ker(\varphi) \Rightarrow \varphi(\alpha) = 0$ & $\alpha \in S$

$\Rightarrow \alpha + I = 0$ in $S+I/I$ & $\alpha \in S$

$\Rightarrow \alpha \in I \cap S$.

$\alpha \in S \cap I \Rightarrow \varphi(\alpha) = 0$ $(\because I = \ker(\varphi))$

Hence by 1^{st} isom thm

$$S/S\cap I \cong S+I/I$$

$\boxed{}$

Ex
$$\frac{\mathbb{Z}[x]}{(2, x^2-2)\mathbb{Z}[x]} \cong \frac{\mathbb{Z}/2\mathbb{Z}[Y]}{(Y^2)}$$

$I = (2, x^2-2) \subseteq \mathbb{Z}[x] = R$

$J = (x^2-2) \subseteq I$

$K = (2) \subseteq I$

R/K

$$R/I \cong \frac{R/K}{I/K}$$

$$\frac{\mathbb{Z}[x]}{(2, x^2-2)} \cong \frac{\frac{\mathbb{Z}[x]}{(2)}}{I/K}$$

$$\mathbb{Z}[x] \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z}[x] \qquad \cong \frac{\mathbb{Z}/2\mathbb{Z}[x]}{(x^2-\bar{2})} \cong \frac{\mathbb{Z}/2\mathbb{Z}[x]}{(x^2)}$$

$f \longmapsto f \pmod 2$

$\ker(\varphi) = (2) = 2\mathbb{Z}[x]$