(3) **Sphere of radius $a > 0$:**

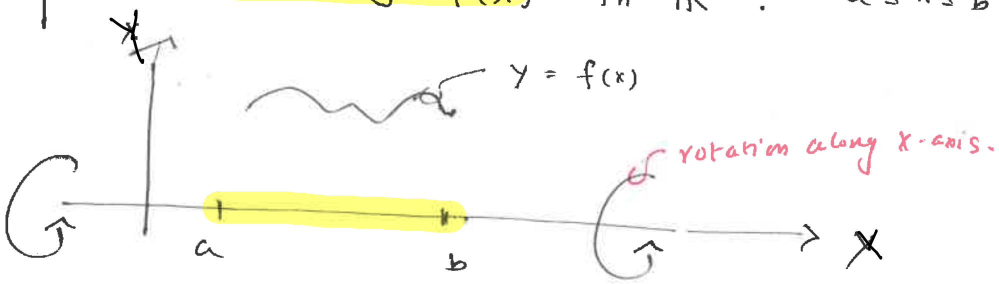
$$\mathbf{r}(u,v) = (a\sin u \cos v,\ a\sin u \sin v,\ a\cos u) \qquad \begin{array}{l} 0 < u < \pi \\ 0 \le v < 2\pi \end{array}$$

just the spherical coordinate.

$$\mathbf{r}_u \times \mathbf{r}_v \Big|_{(u,v)} = \sin u \ \mathbf{r}(u,v). \neq 0.$$

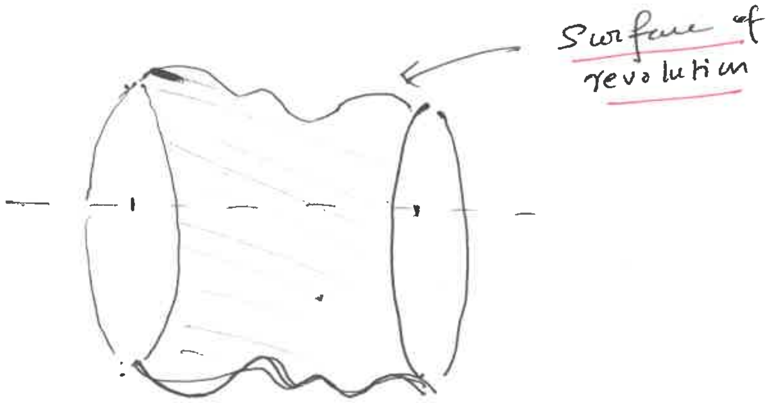
(4) **Surface of revolution:**

Torus & sphere are examples of "surface of revolution". More ~~Defn~~ specifically: Consider $y = f(x)$ in $\mathbb{R}^2$. $a \le x \le b$

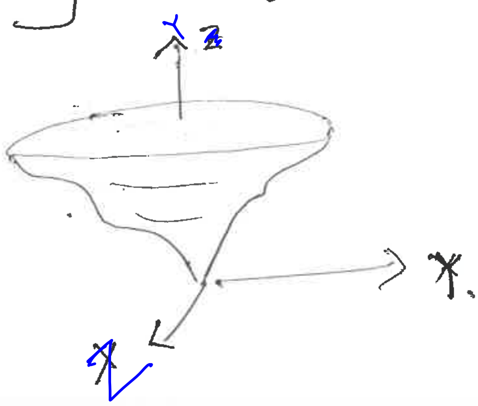


$\Downarrow$ revolve around y-axis.

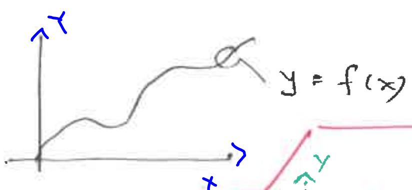
do it for any curve
around any line.



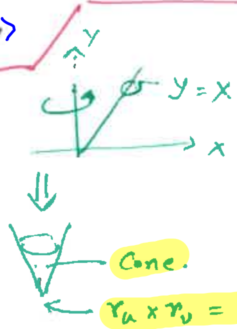
~~In general:~~ Also, think:



$\Leftarrow$



Remark:



Cone.

$r_u \times r_v = 0$??

In general: Consider $f, g \in C^1(\mathbb{R})$. $\searrow [a,b]$ $\quad f(t) \neq 0 \ \forall t$.

We define it as follows:

Consider a C^1-curve $\quad t \longmapsto (0, f(t), g(t)) \in \mathbb{R}^3$. $\quad t \in [a,b]$.

The surface of revolution generated by above around the z-axis is:

$$\gamma(u,v) = \big(f(u)\cos v,\ f(u)\sin v,\ g(u)\big) \qquad u \in [a,b].\ \ 0 \le v < 2\pi.$$

Then $$\gamma_u \times \gamma_v = f(u)\big(-g'(u)\cos v,\ -g'(u)\sin v,\ f'(u)\big)$$

$$(\neq 0).$$

~~(3) Cylindrical Coordinates~~

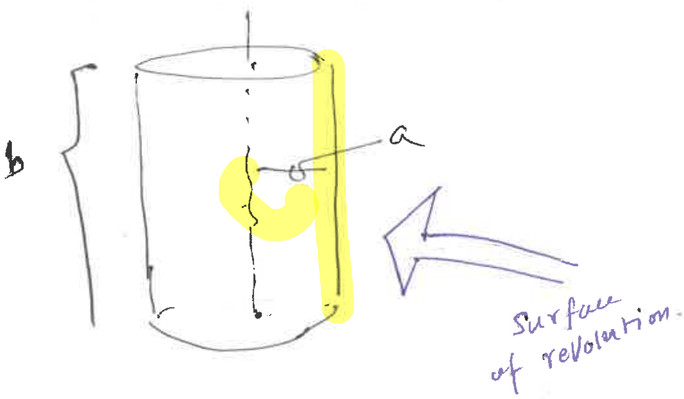
~~(4) Cylinder:~~

$$\cancel{\gamma(u,v) = (a\sin u\cos v,\ a\sin u\sin v,\ a\cos u)} \qquad \cancel{a>0.}\ \ \cancel{\text{fixed}}.$$

$$(u,v) \in \cancel{[0,\pi) \times [0,2\pi)}$$

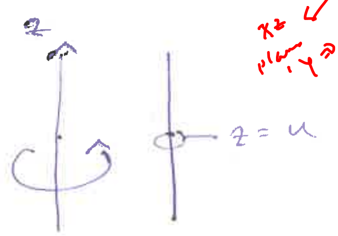
~~Surface of revolution.~~

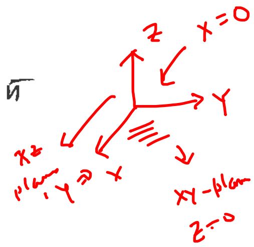
(4) Cylinder: $\quad \gamma(u,v) = (a\cos v,\ a\sin v,\ u)$. $\quad (a>0)$.



$a \le u \le b$

$0 \le v \le 2\pi$





Tangent plane & Normal vectors: (of Surfaces).

Let $\gamma : R \to \mathbb{R}^3$ be a parametrized surface.

$S = \operatorname{ran} \gamma$ (the Surface). Fix $P = \gamma(u_0, v_0)$, for some $(u_0, v_0) \in R$.

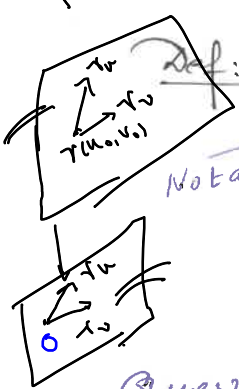
Then $\gamma_u(u_0, v_0)$ is tangent to $t \longmapsto \gamma(t, v_0)$ at $\gamma(u_0, v_0)$

& lly $\gamma_v(u_0, v_0)$ —— " —— $t \longmapsto \gamma(u_0, t)$ at $\gamma(u_0, v_0)$.

Curves lying on S.

[See Page 31].

$T_p(S)$



Def: T_pS := the tangent plane of S at P

Notation

:= the subspace spanned by the vectors $\gamma_u(u_0, v_0)$ & $\gamma_v(u_0, v_0)$.

defn

affine Subspace

A 2-dimensional

Tangent vectors of S at P are: $a\,\gamma_u(u_0,v_0) + b\,\gamma_v(u_0,v_0)$, $a, b \in \mathbb{R}$ ✡

Question: T_pS depends on the parametrization γ?

Ans: No.

Hw: Suppos $\tilde{\gamma} : \tilde{R} \to \mathbb{R}^3$ be a parametrization of S

(i.e, $S = \operatorname{ran} \tilde{\gamma}$) & let $\gamma(u_0, v_0) = \tilde{\gamma}(\tilde{u}_0, \tilde{v}_0) = P$.

Prove that $\operatorname{span}\{\gamma_u(u_0,v_0), \gamma_v(u_0,v_0)\} = \operatorname{span}\{\tilde{\gamma}_{\tilde{u}}(\tilde{u}_0,\tilde{v}_0), \tilde{\gamma}_{\tilde{v}}(\tilde{u}_0,\tilde{v}_0)\}$.

[Hint: $\gamma^{-1} \circ \tilde{\gamma}$ is a C^1-map from an open set around $(\tilde{u}_0, \tilde{v}_0)$ to an open set around (u_0, v_0). Apply Chain rule.]

Remark: The assumption that $\gamma_u \times \gamma_v \big|_{(u_0, v_0)} \neq 0$ assures that γ_u & γ_v are linearly independent at (u_0, v_0).

Def: Elements of T_pS are called tangent vectors of S at P.

(See ✡ above.)

Now consider ~~the~~ a graph fn. $z = f(x,y)$. $(x,y) \in \mathcal{O}_2$; f is C^1.

Then $r(u,v) = (u, v, f(u,v))$ is a parametrization of the surface:

$S := \operatorname{graph} f = \{(x, y, f(x,y)) : (x,y) \in \mathcal{O}_2\}$. ← See Page 33.

Also $r_u \times r_v = (-f_u, -f_v, 1)$ $\forall (u,v) \in \mathcal{O}_2$.

Let $P = (a, b, f(a,b)) \in S$.

∴ The eqn of the tangent plane $T_P S$ is:

$$(-f_u|_{(a,b)})(x-a) + (-f_v|_{(a,b)})(y-b) + (1)\cdot(z - f(a,b)) = 0,$$

i.e., $$\boxed{\underbrace{\frac{\partial f}{\partial u}\Big|_{(a,b)}}_{f_u(a,b)}(x-a) + \underbrace{\frac{\partial f}{\partial v}\Big|_{(a,b)}}_{f_v(a,b)}(y-b) - (z - f(a,b)) = 0.} \quad \text{— (T)}$$

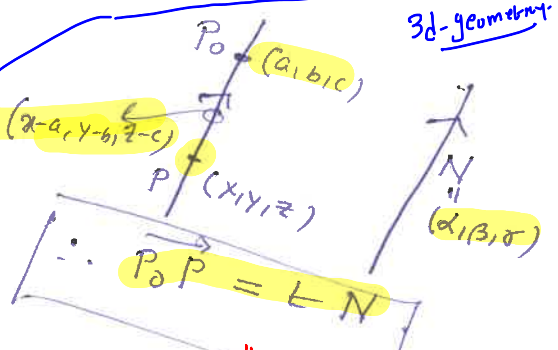
eqn. of a plane through P & ⊥ N.

Also eqn of the normal line N through P is given by:

$$\left.\begin{aligned} x - a &= \big(\underbrace{-f_u(a,b)}_{= -f_u(a,b)}\big)\, t \\ y - b &= \big(\underbrace{-f_v(a,b)}_{= -f_v(a,b)}\big)\, t \\ z - f(a,b) &= t \end{aligned}\right\} \quad \text{— (N)}$$

parametric eqn of a line through P in the direction of N.

$t \in \mathbb{R}$ is the parameter.



⇓

$(x-a, y-b, z-c) = (t\alpha, t\beta, t\gamma)$

⇓

$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$ eqn line.

OR (N) ⟺

$$\frac{x-a}{-f_u|_{(a,b)}} = \frac{y-b}{-f_v|_{(a,b)}} = \frac{z-f(a,b)}{1}.$$

↑

Symmetric eqn of the normal.

eg: Eqn. of tangent plane & normal line to $z = \frac{2x}{y} - x^2$ at $(1,1,1)$:

We straightaway compute $\frac{\partial z}{\partial x} = \frac{2}{y^2} - 2x$.

$$\frac{\partial z}{\partial y} = -\frac{2x}{y^2}.$$

$\therefore$ The normal vector $N = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \Big|_{(x=1,\, y=1)}$

$$= \left\langle -\frac{2}{y^2} + 2x, \frac{2x}{y^2}, 1 \right\rangle_{(1,1)}$$

$$= (0, 2, 1).$$

or $\langle 0,2,1 \rangle$ or $0i + 2j + k$.

$\therefore$ eqn of N is:

$$\left.\begin{aligned} x - 1 &= 0 \\ y - 1 &= 2 \\ z - 1 &= t \end{aligned}\right\}$$

eqn of tangent plane: $0(x-1) + 2(y-1) + 1\cdot(z-1) = 0$.

$$\Rightarrow 2y + z = 3.$$

eg: Consider the parametrized surface

$$r(u,v) = (u^2 - v^2, uv, u^2 + v^2).$$

$\therefore r_u = (2u, v, 2u)$, $r_v = (-2v, u, 2v)$.

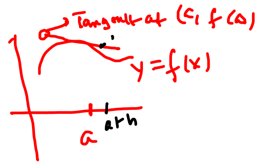
If $(u,v) = (2,1)$, then $r(u,v) = (3, 2, 5) := P$

& $r_u \times r_v \Big|_{(u,v)=(2,1)} = \begin{vmatrix} i & j & k \\ 4 & 1 & 4 \\ -2 & 2 & 2 \end{vmatrix} = \underbrace{(-6, -16, 10)}_{\vec{N}}$.

$\vec{N} \cdot \vec{PX}_{(x,y,z)}$

$\therefore$ At $(3,2,5)$, eqn. of tangent plane: $-6(x-3) - 16(y-2) + 10(z-5) = 0$.

& —— " —— Normal line: $\left. \begin{aligned} &x - 3 = -6t,\ y - 2 = -16t \\ &z - 5 = 10t. \end{aligned} \right\}$.



Approximation:

\# Let's get back to derivatives of fn's in $\mathbb{R}^2$.

Let $f \in C^1(\mathcal{O}_2)$. $\because f : \mathcal{O}_2 \to \mathbb{R}$ is diff. we have, for a fixed $(a,b) \in \mathcal{O}_2$:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \big(Df(a,b)\big)\,\underbrace{[x-a \quad y-b]^t}_{\begin{bmatrix} x-a \\ y-b \end{bmatrix} \in \mathbb{R}^2}}{\|(x,y) - (a,b)\|} = 0 \quad \text{——} (1)$$

$$Df(a,b) = \text{Total derivative of } f \text{ at } (a,b)$$
$$= [f_x(a,b) \quad f_y(a,b)].$$

So $\quad Df(a,b)\,[x-a \quad y-b]^t = f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b).$

So (1) $\Rightarrow$
$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Now recall (eqn (T), Page 38): The eqn of the tangent plane at $P(a, b, f(a,b))$ on the surface

$$S = \text{graph } f = \{ (x, y, f(x,y)) : (x,y) \in \mathcal{O}_2 \}$$ is

\# given by: $\quad z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$

Following 1-variable Calculus,

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{——} (2)$$

is Called the linear or the tangent $\boxed{\text{plane}}$ approximation of f (NEAR) at (a,b).

\# In particular, if $L(x,y) = $ R.H.S. of (2), then

$$(1) \Rightarrow \lim_{(x,y)\to(a,b)} \frac{f(x,y) - \text{"}L(x,y)\text{"}}{\|(x,y) - (a,b)\|} = 0.$$

"An affine plane"

A fn $L : \mathbb{R}^n \to \mathbb{R}$ is said to be affine if $\exists\, a_1, \dots, a_n \in \mathbb{R}$ & $a \in \mathbb{R}$ s.t. $L(x) = a + \sum_{i=1}^{n} a_i x_i$.

Of course, an affine map is linear $\Leftrightarrow a = 0$.

eg: Consider $f(x,y) = x e^{xy}$. ~~& (a,b) = (1,0)~~.

We want to find approximate value of $f(1\cdot 1, -0\cdot 1)$.

Sol: We do it by linear approximation of f near to $(1,0)$.

Here ~~def(expgen)~~ $\left.\begin{aligned} f_x &= (1+xy)\, e^{xy} \\ f_y &= x^2 e^{xy} \end{aligned}\right\}$.

$\therefore \left.\begin{aligned} f_x(1,0) &= e^0 = 1 \\ f_y(1,0) &= 1 \cdot e^0 = 1 \end{aligned}\right\}$.

$\therefore$ The linear approximation of f near $(1,0)$ is given by:

$$L(x,y) = f(1,0) + 1 \times (x-1) + 1 \times (y-0)$$
$$= 1 + x - 1 + y$$

$$\Rightarrow L(x,y) = x + y$$

$\therefore$ near $(1,0)$, $\quad x e^{xy} \approx x + y$.

So $f(1\cdot 1, -0\cdot 1) \approx L(1\cdot 1, -0\cdot 1)$
$$= 1\cdot 1 + (-0\cdot 1)$$
$$= 1.$$

Ans.

Here: linear approximation of $f(x,y)$ at (a,b) is simply the following:

Compute the normal vector $N = (-f_x(a,b), -f_y(a,b), 1)$. Then the tangent plane: $-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0$.

Then

$$\boxed{z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$ #

approximated value of f near (a,b).

eg: Use tangent plane to approximate $(1.99)^2 - \frac{1.99}{1.01}$.

Sol: First figure out a fn. Here

$$f(x,y) = x^2 - \frac{x}{y}. \quad \cancel{y} \quad (a,b) = (2,1).$$

$[\because (1.99, 1.01) \approx (2,1)]$

~~Now the normal of S = graph f at $(2,1,f(2,1))$~~ $= f(2,1)$

Then $f_x = 2x - \frac{1}{y}$, $f_y = \frac{x}{y^2}$.

So $f_x(2,1) = 3$, $f_y(2,1) = 2$.

$\therefore$ Eqn. of tangent plane at $(1,2,f(2,1)) = (1,2,2)$ is:

$$f(x,y) \approx z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$= 2 + 3(x-2) + 2(y-1)$$

$$= 3x + 2y - 6.$$

$$\therefore f(1.99, 1.01) \approx 3 \times (1.99) + 2 \times (1.01) - 6$$

$$= 5.97 + 2.02 - 6$$

$$= 7.99 - 6$$

$$= 1.99$$

i.e. $f(1.99, 1.01) \approx 1.99$.

Ans.