Lecture 19: Local rings and Ideals in Localization

15 October 2020
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⊛ R a comm ring with unity. S a mult. set. We constructed a ring homo $\varphi: R \to S^{-1}R := \{\frac{r}{s} \mid r \in R, s \in S\}$ where $\frac{r}{s} = \frac{r'}{s'}$ if $\exists u \in S$ s.t. $u(s'r - sr') = 0$

$r \longmapsto \frac{r}{1}$

⊛ R int domain then $frac(R) = S^{-1}R$ where $S = R \setminus \{0\}$.

⊛ $R[\frac{1}{x}] := S^{-1}R \cong \frac{R[Z]}{(xZ-1)}$ where $S = \{1, x, x^2, \dots\}$

For $x \in R$

Eg: $S_1 = \{1, 30, 30^2, \dots\}$, $S_2 = \{1, 2, 3, 5, 2^{n_2}3^{n_3}5^{n_5} ; n_2, n_3, n_5 \geq 0\}$
$\subseteq \mathbb{Z}$. $\quad S_1^{-1}\mathbb{Z}$, $S_2^{-1}\mathbb{Z}$

$\mathbb{Z}[\frac{1}{30}] \subseteq \mathbb{Q}$

$S_1^{-1}\mathbb{Z} = S_2^{-1}\mathbb{Z}$ as subsets of $\mathbb{Q}$

$\subseteq$ ✓

$\frac{a}{2^{n_2}3^{n_3}5^{n_5}} = \frac{a2^{n-n_2}3^{n-n_3}5^{n-n_5}}{30^n}$

$n > n_2, n_3, n_5$

⊛ A comm ring with unity is called a <u>local ring</u> if it has exactly one maximal ideal.

<u>Example</u>: R a ring. $P \subseteq R$ a prime ideal. Then $S = R \setminus P$ is a mult. set. Then $S^{-1}R = \{\frac{r}{s} \mid r, s \in R, s \notin P\}$ is a local ring with $PS^{-1}R = \{\frac{r}{s} \mid r \in P, s \notin P\}$ the unique maximal ideal of $S^{-1}R$.

$\underset{S^{-1}P}{\shortparallel}$

<u>Pf</u>: $\frac{x_1}{s_1}, \frac{x_2}{s_2} \in PS^{-1}R$ then $\frac{x_1}{s_1} + \frac{x_2}{s_2} = \frac{s_2x_1 + s_1x_2}{s_1s_2} \in PS^{-1}R$ & $\frac{r}{s} \cdot \frac{x_1}{s_1} = \frac{rx_1}{ss_1} \in PS^{-1}R$.

$\frac{r}{s} \notin PS^{-1}R$ then $r \notin P \Rightarrow \frac{s}{r} \in S^{-1}R \Rightarrow \frac{r}{s} \cdot \frac{s}{r} = 1_{S^{-1}R} \Rightarrow \frac{r}{s}$ is a unit in $S^{-1}R$

Hence $PS^{-1}R$ is the maximal ideal of $S^{-1}R$. ∎

$S^{-1}R$ is also denoted by R_P.

<u>Prop</u>: Let R be a comm ring with unity and m a maximal ideal of R. TFAE

1) R is a local ring.
2) The set of nonunits of R form the ideal m.
3) $1 + x$ is a unit $\forall x \in m$.

Pf: (1) $\Rightarrow$ (2): Let J = set of non units in R.

Let $x \in J$, $(x) = I \subsetneq R$ ($\because x$ is nonunit)

So $\exists$ a max ideal of R containing I (and hence x). But (1) $\Rightarrow I \subseteq m$.

Hence $x \in m$. i.e. $J \subseteq m$

$m \subseteq J$ (trivial since m is a proper ideal)

$m = J$.

(2) $\Rightarrow$ (3): If $1+x$ is not a unit then by (2) $1+x \in m$ ($\because m$ contains all non units)

And $x \in m \Rightarrow 1 \in m$ a contradiction

$\Rightarrow 1+x$ is a unit.

(3) $\Rightarrow$ (1): Let $m' \subseteq R$ be another maximal ideal. Let $x \in m$

$\Rightarrow ax \in m \Rightarrow a \in R$

$\Rightarrow 1+ax$ is a unit $\forall a \in R$

$\Rightarrow x \in Jac(R)$

$\Rightarrow x \in m'$

$\Rightarrow m \subseteq m'$

$\Rightarrow m = m'$

Ideals of $S^{-1}R$: $\varphi : R \to S^{-1}R$; $\varphi(P)$ need
$r \mapsto \frac{r}{1}$

not be an ideal.

Ex: $\mathbb{Z} \hookrightarrow \mathbb{Q}$

$2\mathbb{Z}$ $\quad 2\mathbb{Z} \subseteq \mathbb{Q}$ not a $\mathbb{Q}$-ideal

$$\varphi(P)S^{-1}R = S^{-1}P := PS^{-1}R := \left\{ \frac{r}{s} \mid r \in P \;\&\; s \in S \right\}$$

$LHS = \left(\frac{r}{1} \mid r \in P \right)$ $\quad \frac{r}{1} \in S^{-1}P \quad \forall r \in P$

$$\Rightarrow \varphi(P)S^{-1}R \subseteq S^{-1}P$$

$$\frac{r}{s} \in S^{-1}P ; \; \frac{r}{s} = \frac{r}{1} \cdot \frac{1}{s}$$

$$\Rightarrow \frac{r}{s} \in \varphi(P)S^{-1}R$$

Let R be comm ring with unity & S a mult. subset. $\varphi : R \to S^{-1}R$ the nat' map.

Let $I \subseteq S^{-1}R$ be an ideal then

$J = \varphi^{-1}(I)$ is an ideal of R.

1) If I is a proper ideal $J \cap S = \phi$.

2) $(\varphi(J)) = JS^{-1}R = I$

Caution! $J \subseteq R$ ideal then $\varphi^{-1}(JS^{-1}R) \neq J$.

Pf: (1) Let $x \in J \cap S$ then

$\varphi(x) \in I$ but $x \in S \Rightarrow \varphi(x)$ is a

unit $\Rightarrow I = S^{-1}R$.

(2) Let $x \in \varphi(J) \Rightarrow \exists y \in J$ s.t.

$\varphi^{-1}(I)$

$x = \varphi(y) \Rightarrow x \in I$.

$\Rightarrow (\varphi(J)) \subseteq I$.

Let $x \in I \Rightarrow x = \frac{r}{s}$ $r \in R$ & $s \in S$

$\Rightarrow \frac{s}{1} \cdot \frac{r}{s} \in I$

$\Rightarrow \frac{r}{1} \in I$

$\Rightarrow r \in J$

$\Rightarrow \frac{r}{1} \in \varphi(J) \Rightarrow x = \frac{r}{s} = \frac{1}{s} \cdot \frac{r}{1} \in (\varphi(J))$

$\Rightarrow I \subseteq (\varphi(J))$.

<u>Example</u>: Every ideal in $S^{-1}R$ is of the form $JS^{-1}R$ for some R-ideal J.

1) $R = \mathbb{Q}[X, Y]$, $S = \{1, X, X^2, \ldots\}$

$S^{-1}R = \mathbb{Q}[X, Y, \frac{1}{X}]$

$J_1 = (X)$ then $J_1 S^{-1}R = S^{-1}R$

$J_2 = (X, Y)$ then $J_2 S^{-1}R = S^{-1}R$

$J_3 = (X+1)$ $\quad J_3 S^{-1}R = \left\{ \frac{(X+1) f(X,Y)}{X^n} \;\middle|\; n \geq 0,\ f(X,Y) \in R \right\}$

$J_4 = (XY)$, $J_5 = (X^2 Y)$

$J_4 S^{-1}R = (Y)$, $J_5 S^{-1}R = (Y)$

$\varphi^{-1}(J_4 S^{-1}R) = (Y)\,\mathbb{Q}[X, Y]$