

## Lecture 33: Applications of structure theorem

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Prop: Let  $R$  be a PID and  $F$  be a free  $R$ -module of rank  $n$ . Let  $N$  be a submodule of  $F$ . Then  $N$  is a free  $R$ -mod of rank  $m \leq n$ . Moreover there is a basis  $x_1, \dots, x_n$  of  $F$  and  $\exists a_1, \dots, a_m \in R^\times$  s.t.  $a_1 | a_2 | \dots | a_m$  and  $\{a_1 x_1, a_2 x_2, \dots, a_m x_m\}$  is a basis of  $N$ .

Thm (Sta thm): Let  $R$  be a PID and  $M$  be a f.g.  $R$ -mod. Then

$$M \cong R^k \oplus R/(a_1) \oplus R/(a_2) \oplus \dots \oplus R/(a_m)$$

where  $k = \text{rank}(M)$  and  $a_1, \dots, a_m \in R$  are nonzero nonunits s.t.  $a_1 | a_2 | a_3 | \dots | a_m$ . Here  $k$  and  $m$  could be 0.

Thm: (Sta thm version 2) Let  $R$  be a PID and  $M$  be a f.g.  $R$ -mod. Then

$$\begin{aligned} M \cong & R^k \oplus R/\frac{r_{11}}{p_1} \oplus R/\frac{r_{12}}{p_1} \oplus R/\frac{r_{13}}{p_1} \oplus \dots \oplus R/\frac{r_{1n}}{p_1} \\ & \oplus R/\frac{r_{21}}{p_2} \oplus R/\frac{r_{22}}{p_2} \oplus \dots \oplus R/\frac{r_{2n}}{p_2} \\ & \vdots \\ & \oplus R/\frac{r_{m1}}{p_m} \oplus R/\frac{r_{m2}}{p_m} \oplus \dots \oplus R/\frac{r_{mn}}{p_m} \end{aligned} \quad k = \text{rank}(M)$$

where  $p_1, \dots, p_m$  are irreducible elements of  $R$ ,  $r_{ij} \geq r_{ij+1} \quad \forall 1 \leq i \leq m$ ,  $2 \leq j \leq n_i$  are positive integers.

$\text{Ex} \quad G$  is a f.g. abelian group then

$$G \cong \mathbb{Z}^k \oplus \mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_r^{n_r}$$

$p_1, \dots, p_r$  are prime nos.

$|G| = 36$ ,  $G$  is abelian

$$\mathbb{Z}/36\mathbb{Z}$$

$$36 = 3^2 \times 2^2$$

$$G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \quad \text{or}$$

$$n_1=2, p_1=2, p_2=3$$

$$\& n_2=2$$

$$\& n_{11}=n_{12}=1=n_{21}=n_{22}$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$$

Version 1 & 2 are equivalent

Let  $a_m = p_1^{g_{11}} p_2^{g_{21}} \cdots p_m^{g_{m1}}$   $g_{ij} \geq 1 \forall j \leq m$

$a_{m-1} = p_1^{g_{12}} p_2^{g_{22}} p_3^{g_{32}} \cdots p_m^{g_{m2}}$  (upto unit)

where  $g_{ij} \geq 0$ .

$\vdots$   
 $a_2 = p_1^{g_{1,m-1}} p_2^{g_{2,m-1}} \cdots p_m^{g_{m,m-1}}$

$a_1 = p_1^{g_{1m}} p_2^{g_{2m}} \cdots p_m^{g_{mm}}$

Since  $a_j | a_{j+1}$  we have  $g_{ij, m-j+1} \leq g_{i, m-j}$   
 $\forall 1 \leq i \leq m$   
 $1 \leq j \leq m-1$

Note that some  $g_{ij}$  may be 0.

CRT

( $\because (p_i^{g_{ii}}, p_j^{g_{jj}})$ )

Now  $R/(a_m) \cong R/(p_1^{g_{11}}) \oplus \cdots \oplus R/(p_m^{g_{m1}})$   $= 1$   
if  $i \neq j$

Now rearrange

$$M \cong R^k \oplus R/\langle a_1 \rangle \oplus \dots \oplus R/\langle a_m \rangle$$

$$\cong R^k \oplus R/\left(p_1^{r_{11}}\right) \oplus R/\left(p_2^{r_{21}}\right) \oplus R/\left(p_m^{r_{m1}}\right)$$

$$\oplus R/\left(p_1^{r_{12}}\right) \oplus R/\left(p_2^{r_{22}}\right) \oplus \dots \oplus R/\left(p_m^{r_{m2}}\right)$$

$$\vdots \oplus R/\left(p_1^{r_{1m}}\right) \oplus R/\left(p_2^{r_{2m}}\right) \oplus \dots \oplus R/\left(p_m^{r_{mm}}\right)$$

Note that  $r_{1m}, r_{2m}$  may be 0.

Write the transpose<sup>of above</sup> and ignore  
the terms where  $r_{ij} = 0$  to

Obtain version 2.

Version 2  $\Rightarrow$  version 1

$$M \cong R^k \oplus R/\mathfrak{p}_1^{g_{1,n_1}} \oplus \dots \oplus R/\mathfrak{p}_m^{g_{m,n_m}}$$

$g_{ij} \geq g_{ij-1}$

$$R/\mathfrak{p}_m^{g_{m,n_m}} \oplus \dots \oplus R/\mathfrak{p}_m^{g_{m,n_m}}$$

Let  $l = \max(n_1, \dots, n_m)$

$$M \cong R^k \oplus \underbrace{0 \oplus \dots \oplus 0}_{l-k} \oplus R/\mathfrak{p}_1^{g_{1,n_1}} \oplus \dots \oplus R/\mathfrak{p}_m^{g_{m,n_m}}$$

$$0 \oplus \dots \oplus 0 \oplus R/\mathfrak{p}_2^{g_{2,n_2}} \oplus R/\mathfrak{p}_3^{g_{3,n_3}} \oplus \dots \oplus R/\mathfrak{p}_m^{g_{m,n_m}}$$

$$0 \oplus \dots \oplus 0 \oplus R/\mathfrak{p}_m^{g_{m,n_m}}$$

Let  $a_m = \mathfrak{p}_1^{g_{1,n_1}} \mathfrak{p}_2^{g_{2,n_2}} \dots \mathfrak{p}_m^{g_{m,n_m}}$

$$a_{m-1} = \mathfrak{p}_1^{g_{1,n_1-1}} \mathfrak{p}_2^{g_{2,n_2-1}} \dots \mathfrak{p}_m^{g_{m,n_m-1}}$$

$$a_1 = \mathfrak{p}_1^{g_{1,n_1-m+1}} \mathfrak{p}_2^{g_{2,n_2-m+1}} \dots \mathfrak{p}_m^{g_{m,n_m-m+1}}$$

CRT  
 $M \cong R^k \oplus R/(a_1) \oplus \dots \oplus R/(a_m)$

Also condition  $g_{ij} \geq g_{ij-1}$

$$\Rightarrow a_1 | a_2 | \dots | a_m$$

Convention  
is  $g_{ij} = 0$  if  $j \leq 0$



Thm: (Rational form)  
 $A \in M_{n \times n}(k)$      $k$  a field.

where  $A$  is similar to  
 $a_i = x^{n_i} + b_{n_i-1}x^{n_i-1} + \dots + b_1x + b_0$   
 $1 \leq i \leq m$

$$\begin{bmatrix} R_{a_1} & & & \\ & \ddots & & \\ & & R_{a_m} & \\ & 0 & \ddots & \\ & & & 0 \end{bmatrix}$$

$$\sum_{i=1}^m n_i = n$$

$$a_1 | a_2 | \dots | a_m$$

where  $R_{a_i} = \begin{bmatrix} 0 & & & & -b_{n_i-1} \\ 0 & \ddots & & & \\ 0 & & \ddots & & -b_1 \\ 0 & & & \ddots & -b_0 \end{bmatrix}$   $n_i \times n_i$  matrix

$\exists P$  nonsing s.t.  $P^{-1}AP = R$

Pf:  $A : k^n \rightarrow k^n$   $k$ -lin map.  
 $\downarrow$        $\downarrow$

This gives  $V$  a  $k[x]$ -mod str.

via  $v \in V$   
 $x \cdot v = Av$        $f(x) = a_n x^n + \dots + a_1 x + a_0$

In general  $f(x) \cdot v = f(A) v$   
 $= (a_n A^n + \dots + a_1 A + a_0 I) v$

$k[x]$  is a PID and  $V$  is f.g.  $k[x]$ -mod

What is rank ( $V$ ) as a  $k[x]$ -mod?

$m_A(x) \in k[x]$  minimal poly       $m_A(x) \cdot v = m_A(A) v$   
 $= 0$  (Caley-Hamilton)

$\Rightarrow V$  is torsion  $k[x]$ -mod

So by Ste. Thm

$$V \cong R/(a_1) \oplus \cdots \oplus R/(a_m) \quad \text{as } R\text{-mod}$$

$$\text{where } R = k[x]$$

$$a_1 | a_2 | \cdots | a_m \quad a_i \in k[x]$$

May assume  $a_i$  are monic

$$A \cdot v = X \cdot v$$

$$a_i(x) = x^{n_i} + b_{n_i-1}x^{n_i-1} + \cdots + b_0$$

So choose the basis

$$R/(a_i) \cong k[x]/(a_i(x)) \cong k \oplus k\bar{x} \oplus k\bar{x}^2 \oplus \cdots \oplus k\bar{x}^{n_i-1}$$

$$B = \{1, \bar{x}, \bar{x}^2, \bar{x}^3, \dots, \bar{x}^{n_i-1}, \dots\}$$

What is the matrix of  $A$  w.r.t  $B$

$$X \cdot 1 = 0 \cdot 1 + 1 \bar{x} + 0 \bar{x}^2 + \cdots$$

$$X \cdot \bar{x} = \bar{x}^2 = 0 \cdot 1 + 0 \bar{x} + 1 \bar{x}^2 + \cdots$$

$$X \cdot \bar{x}^{n_i-1} = \bar{x}^n = 0 \cdot 1 + 0 \bar{x} + \cdots + 0 \bar{x}^{n_i-2} + 1 \bar{x}^{n_i-1}$$

$$X \cdot \bar{x}^n = \bar{x}^0 = -b_0 1 - b_1 \bar{x} + -b_n \bar{x}^n$$

$$\begin{bmatrix} 0 & & & & -b_{n_i-1} \\ 1 & 0 & & & \vdots \\ 0 & 1 & & & \\ \vdots & 0 & 1 & 0 & -b_1 \\ 0 & 0 & 0 & 1 & -b_0 \end{bmatrix}$$