Change of variables thm

Thm: Let $\mathcal{O}_n \subseteq \mathbb{R}^n$ be open, $\varphi : \mathcal{O}_n \to \mathbb{R}^n$ be an injective & C^1-fn & let $J_\varphi(x)$ $\left(= \left[\frac{\partial \varphi_i}{\partial x_j}(x) \right] \right)$ is invertible $\forall x \in \mathcal{O}_n$.

($\Leftrightarrow \det J_\varphi(x) \neq 0$ $\forall x \in \mathcal{O}_n$.)

Let $\Omega \subseteq \mathcal{O}_n$ be s.t. $\Omega \cup \partial\Omega \subseteq \mathcal{O}_n$ & Ω has an area. ($\Leftrightarrow \partial\Omega$ is c.z).

If $f \in \mathcal{R}(\varphi(\Omega))$, then

$$\int_{\varphi(\Omega)} f = \int_{\Omega} \underbrace{f \circ \varphi \times \left| \det J_\varphi \right|}_{\in \mathcal{R}(\Omega) \text{ is assured.}}.$$

$$\left[\text{i.e., } \int_{\varphi(\Omega)} f(x)\, dx = \int_{\Omega} f(\varphi(x)) \left| \det J_\varphi(x) \right| dx . \right]$$

Proof: See Spivak, Page-67.

Remark:

1) There are many finer (or non-finer) variants of the above thm.

2) "$\varphi : \mathcal{O}_n \to \mathbb{R}^n$, a C^1-fn, with $J_\varphi(a)$ is invertible at $\boxed{\text{some point}}$ $a \in \mathcal{O}_n$" $\longrightarrow$ Also a strong statement. This is related with the inverse function theorem. We will talk about it soon.

3) Think $n=1$ case: $\varphi : (a,b) \to \mathbb{R}$ is diff. & $\varphi'(x) \neq 0$ $\forall x \in (a,b)$.
$\Rightarrow \varphi$ is injective.

[(*) actually, $(a-\varepsilon, b+\varepsilon)$, so that $[a,b] \subseteq (a-\varepsilon, b+\varepsilon)$.]

4) Think $n=1$ case: $\varphi : (a,b) \to \mathbb{R}$ be C^1-fn. ($\Rightarrow \varphi(a,b)$ is also an interval) & let $f \in \mathcal{R}(\varphi(a,b))$. Then ~~and in (a,b~~
$\forall$ $c,d \in (a,b)$ & $c < d$,

$$\int_c^d f(\varphi(x))\, \varphi'(x)\, dx = \int_{\varphi(c)}^{\varphi(d)} f(x)\, dx .$$

[(*) $c = a$, $d = b$ is okay.]

More Simply: if $\varphi : [a,b] \to \mathbb{R}$ is C^1-fn & $f \in \mathcal{R}(\varphi[a,b])$, then

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi)(x)\, \varphi'(x)\, dx .$$

If, in addition φ is injective, then:

$$\int_{\varphi([a,b])} f = \int_{[a,b]} (f\circ\varphi)(x)\,|\varphi'(x)|\,dx .$$

(5) "Above "injectivity" takes care of $\int_a^b$ vs $\int_b^a$, AS, IN $\mathbb{R}^n$, we do NOT HAVE $\int_a^b$ or $\int_b^a$. We just have

$$\int_{\Omega} \quad !!$$

$\therefore$ Injectivity of φ in the thm. is justified.

(6) "φ is 1-1" vs "J_φ invertible".

The latter $\Longrightarrow$ φ is locally 1-1. But NOT as a whole / globally.

—— Will see in inverse fn. thm.

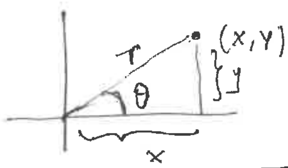
— x —

Examples:

(1) To Consider polar coordinate:

Let $x = r\cos\theta$, $y = r\sin\theta$. $r, \theta \in \mathbb{R}$.

So $\varphi : \underset{(r,\theta)}{\mathbb{R}^2} \longrightarrow \underset{(x,y)}{\mathbb{R}^2}$ defined by



$$\varphi(x,y) = \big(x(r,\theta),\ y(r,\theta)\big)$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x}. \end{cases}$$

So $J_\varphi = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}$

The Jacobian of φ at (x,y)
$= (r\cos\theta, r\sin\theta)$.

$$\Rightarrow \quad J_\varphi = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$$

$$\Rightarrow \quad \det J_\varphi = r\cos^2\theta + r\sin^2\theta = r.$$

i.e, $\det\left(J_\varphi(x,y)\right) = r \neq 0 \quad \forall (x,y) \neq (0,0)$.
or $r \neq 0$.

But, of course, φ is NOT injective (even if $(x,y) \neq (0,0)$).
($\because \theta \to \theta + 2n\pi$ will lead non-inj.)

We do the following (redefine φ as follows):

Given $(x,y) \neq (0,0)$, define $r = \sqrt{x^2+y^2}$. Pick
$! \ \theta \in [0, 2\pi) \ \ni \ (x,y) = (r\cos\theta, r\sin\theta)$.

Set $O_2 = \{(r,\theta) : r > 0, \ 0 < \theta < 2\pi\}$.
$= (0,\infty) \times (0, 2\pi)$
$\subseteq \mathbb{R}^2$.

Define $\varphi : O_2 \longrightarrow \mathbb{R}^2$ by
$\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$

$\therefore \varphi$ is C^1 & $J_\varphi(r,\theta) = r \neq 0 \quad \forall (r,\theta) \in O_2$.

Now clearly, φ is also 1-1.

So given $0 < r_1 < r_2$ & $0 < \theta_1 < \theta_2 < 2\pi$,
Set $\Omega = [r_1, r_2] \times [\theta_1, \theta_2]$. (or take open intervals)
If $f \in R(\varphi(\Omega))$, then

$$\int_{\varphi(\Omega)} f = \int_{\Omega} f\circ\varphi \times |\det J_\varphi|$$

i.e., $$\int_{\varphi(\Omega)} f(x,y)\, \underbrace{dx\,dy}_{\shortparallel\; dV(x,y)} = \int_{\Omega} f(\varphi(r,\theta))\; r\, \underbrace{dr\,d\theta}_{dV(r,\theta)}$$

$$= \int_{\Omega} f(r\cos\theta, r\sin\theta)\; r\; dr\,d\theta.$$

$$\overset{\text{If } f\in C(\varphi(\Omega)) \text{ \& then by Fubini}}{=} \int_{\theta_1}^{\theta_2}\left(\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\; r\, dr\right) d\theta.$$

(2) $$\int_{x^2+y^2<1} e^{-(x^2+y^2)} = \ ?$$

Sol: $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$. $\qquad \{(x,y) : x^2+y^2<1\} = \varphi\Big(\underbrace{[0,1]\times[0,2\pi]}\Big)$

0 ?
2π ?

$$\therefore\quad \underline{I} = \int_{[0,1]\times[0,2\pi]} e^{-r^2} \times r \times dr\, d\theta.$$

$$= \int_0^{2\pi}\left(\int_0^1 \left(e^{-r^2} r\, dr\right)\right) d\theta$$

? ?

$$= \int_0^{2\pi} d\theta \times \int_0^1 e^{-r^2} \times \frac{1}{2} \times d(r^2)$$

$$= 2\pi \times \frac{-1}{2} \times \left[e^{-r^2}\right]_0^1$$

$$= \pi\left(1 - e^{-1}\right).$$ ////

Why $r=0$?
$\theta = 2\pi$?

Remark: Note that $\varphi : (0,\infty) \times \cancel{(0, 2\pi)} \longrightarrow \mathbb{R}^2$, defined by $\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$ has a continuous extension

$$\tilde{\varphi} : [0,\infty) \times [0, 2\pi] \longrightarrow \mathbb{R}^2.$$

Then use the same limiting argument, as in $n=1$ case, one can make sense of the above $\int_0^{2\pi} \int_0^1$.

[Also, use, for instance, $\{(x,y) : x \in [0,1], y = 0\}$ is of content zero.].

3) Compute the area of $\Omega = \{(x,y) \in \mathbb{R}^2 : x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1\}$.

Sol: Define $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $\varphi(s,t) = (s\cos^3 t, s\sin^3 t)$.

$$\therefore \Omega = \varphi([0,1] \times [0, 2\pi])$$

So $\varphi\big|_{[0,1] \times [0,2\pi]}$ is cont. & 1-1 in the interior of $[0,1] \times [0, 2\pi]$.

Also,

$$J_\varphi = \begin{bmatrix} \cos^3 t & -3s \cancel{\cos t \sin t} \cos^2 t \sin t \\ \sin^3 t & 3s\sin^2 t \cdot \cos t \end{bmatrix}$$

$$\therefore \det(J_\varphi(s,t)) = 3s \times [\sin^2 t \cdot \cos^4 t + \sin^4 t \cdot \cos^2 t]$$

$$= 3s \cdot \sin^2 t \cdot \cos^2 t.$$

$$\neq 0 \quad \forall \; s \in (0,1], \; t \in (0, 2\pi).$$

$$\therefore \text{Area}(\Omega) = \text{Area}(\varphi([0,1] \times [0,2\pi]))$$

$$= \int_{\varphi([0,1] \times [0,2\pi])} 1.$$

$$= \int_{[0,1]\times[0,2\pi]} |J_\varphi(s,t)|\, ds\, dt$$

$$= \int_0^{2\pi}\int_0^1 3s\ \sin^2 t \times \cos^2 t\ dt\ ds.$$

$$= 3\int_0^{2\pi}\left(\int_0^1 s\,ds\right)\sin^2 t\cdot\cos^2 t\cdot dt$$

$$= \frac{3}{2}\times 1\times\frac{1}{4}\int_0^{2\pi}\sin^2(2t)\,dt.$$

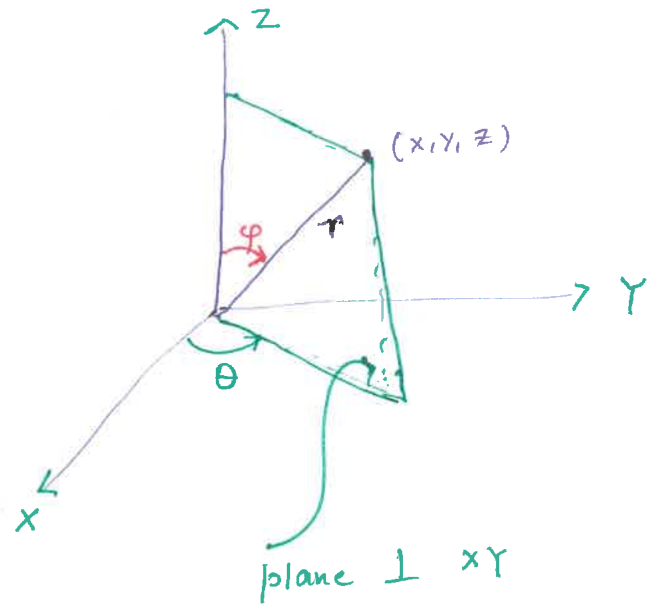
$$= \frac{3}{8}\times\int_0^{2\pi}\frac{1-\cos 4t}{2}\,dt = \frac{3}{8}\times\pi. \quad \text{Ans}$$

(4) Spherical Coordinate:

$\forall\ (x,y,z)\in\mathbb{R}^3\setminus\underbrace{\{(0,0,\alpha):\alpha\in\mathbb{R}\}}_{z\text{-axis}}$, Consider the triple (r,φ,θ) as:

$r=\sqrt{x^2+y^2+z^2}$; $\quad 0\le\theta<2\pi$ & $0<\varphi<\pi$

s.t. $(x,y,z)=(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$.



$(r,\varphi,\theta)\longrightarrow$ spherical Coordinate of (x,y,z).

Define $O_3 := \{(r, \varphi, \theta) : r > 0,\ 0 < \varphi < \pi,\ 0 < \theta < 2\pi\}$.

& $\Phi : O_3 \longrightarrow \mathbb{R}^3$ by

$$\Phi(r, \varphi, \theta) = (r \sin\varphi \cos\theta,\ r \sin\varphi \sin\theta,\ r \cos\varphi)$$
$$\forall\ (r, \varphi, \theta) \in O_3 .$$

$\therefore\ \Phi$ is C^1 & 1-1 on O_3. [& cont. extension to the boundary.]

Now,
$$J_\Phi = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$
$$= \begin{bmatrix} \sin\varphi \cos\theta & r\cos\varphi \cos\theta & -r\sin\varphi \sin\theta \\ \sin\varphi \sin\theta & r\cos\varphi \sin\theta & r\sin\varphi \cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{bmatrix}$$

$$\Rightarrow \det\left(J_\Phi(r, \varphi, \theta)\right) = r^2 \sin\varphi \neq 0$$
$\because\ r > 0$ & $\varphi \neq 0, \pi$.

$\therefore$ For $0 < r_1 < r_2$, $0 < \varphi_1 < \varphi_2 < \pi$ & $0 < \theta_1 < \theta_2 < 2\pi$,

& for $f \in \mathrm{Cont}\left(\Phi\left(\underbrace{[r_1, r_2] \times [\varphi_1, \varphi_2] \times [\theta_1, \theta_2]}_{:= \Omega}\right)\right)$,

We have: $\displaystyle\int_{\Phi(\Omega)} f(x, y, z)\, dx\, dy\, dz$

$$= \int_{\theta_1}^{\theta_2} \left\{ \int_{\varphi_1}^{\varphi_2} \left(\int_{r_1}^{r_2} f(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi) \times r^2 \sin\varphi\, dr \right) d\varphi \right\} d\theta .$$

Change of variables & Fubini as f is Cont.

PTO.

5 In particular:

If $\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2\}$

sphere of radius a centered at (0, 0, 0).

Then $\text{vol}(\Omega) = \int_\Omega 1 \underbrace{dx\,dy\,dz}_{dv}$.

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin\varphi \underbrace{dr \times d\varphi \times d\theta}_{\text{mind the ordering.}}$$

$= \ldots$

$\ldots$

$= \frac{4}{3}\pi a^3.$ ⟵ Your known & favourite formula.