# Lecture 23: Modules, submodules, linear maps

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<u>Def</u>: Let $R$ be a commutative ring with unity. An <u>R-module</u> is a $(M, +, s, R)$ where $+$ is a binary operator on $M$ and $s: R \times M \to M$ is a function satisfying the following axioms

1) $(M, +)$ is an abelian group with identity $0_M$.
2) $s(\alpha, x_1 + x_2) = s(\alpha, x_1) + s(\alpha, x_2)$ $\forall \alpha \in R$ & $x_1, x_2 \in M$
3) $s(\alpha_1 + \alpha_2, x) = s(\alpha_1, x) + s(\alpha_2, x)$ $\forall x \in M$, $\forall \alpha_1, \alpha_2 \in R$
4) $s(\alpha_1 \alpha_2, x) = s(\alpha_1, s(\alpha_2, x))$
5) $s(1, x) = x$ $\forall x \in M$.

$\alpha \in R$ & $x \in M$ then $\alpha x \in M$, $\alpha x = s(\alpha, x)$

$(\alpha_1 + \alpha_2) \cdot x = \alpha_1 x + \alpha_2 x$

$(\alpha_1 \alpha_2) \cdot x = \alpha_1 \cdot (\alpha_2 \cdot x)$

$1 \cdot x = x$

<u>Ex</u>: 1) $R$ a field then $R$-modules are $R$-vs

2) $R = \mathbb{Z}$ then $M = \mathbb{Z}$ then $s: R \times M \to M$ is the usual multi of integers then $M$ is an $R$-mod.

More generally $R$ a ring then $(R, +)$ is an $R$-mod. as well.

3) $R \subseteq R'$ then any $R'$-mod is an $R$-mod.

$M$ is an $R$-module where $R$ is a ring.

<u>Facts</u>: 1) $0_R \cdot m = 0_M$ $\forall m \in M$ $\quad (s(0_R, m) = 0_M)$

2) $r \cdot 0_M = 0_M$ $\forall r \in R$ $\quad (s(r, 0_M) = 0_M)$

$$0_R \cdot m = (0_R + 0_R) \cdot m = 0_R \cdot m + 0_R \cdot m$$

$$\Rightarrow 0_R \cdot m = 0_M \quad \text{Similarly 2)}$$

Example 3) $R$ a ring, $I$ an $R$-ideal. Then $I$ is an $R$-module w.r.t usual multiplication as scalar multiplication.

## Submodule and quotient module $(M, +, s)$

**Def$^n$**: Let $R$ be a ring and $M$ be an $R$-module. An $R$-submodule $N$ of $M$ is a subset $N \subseteq M$ s.t. $(N, +, s_{|R\times N})$ is an $R$-module.

i.e. $N$ is a subgroup of $M$ and $\forall\, r \in R$ & $n \in N$, $s(r,n) \in N$.

$$\Updownarrow$$

$$r\cdot n \in N$$

(*) $N$ is a $R$-submodule of $M$ iff $\forall\, n_1, n_2 \in N$, $n_1 + n_2 \in N$ and $\forall\, r \in R$ & $n \in N$, $r\cdot n \in N$

**Example**: 1) $R$ is an $R$-module & $I$ is an $R$-submodule of $R$.

2) $\{0_M\}$ and $M$ are $R$-submod of $M$.

3) $m \in M$ and $Rm = \{rm \mid r \in R\}$ is an $R$-submod of $M$.

(*) Let $M$ be a $R$-mod and $I$ an $R$-ideal then $IM = \left\{ x_1m_1 + x_2m_2 + \cdots + x_nm_n \;\middle|\; \begin{array}{l} n \geq 1,\ x_1, \ldots, x_n \in I \\ \&\ m_1, \ldots, m_n \in M \end{array} \right\}$ is an $R$-submodule of $M$.

**Pf**: Closed under addition is trivial.

Let $\alpha \in IM$ & $r \in R$ then $\alpha = x_1m_1 + \cdots + x_nm_n$ for some $x_i \in I$ & $m_i \in M$ $1 \leq i \leq n$

then
$$\begin{aligned} r\alpha &= r(x_1m_1 + \cdots + x_nm_n) \\ &= r(x_1m_1) + \cdots + r(x_nm_n) \\ &= (rx_1)m_1 + \cdots + (rx_n)m_n \\ &\in IM \quad \left(\because I \text{ is an ideal } rx_i \in I \ \forall\, 1 \leq i \leq n\right) \end{aligned}$$

# Quotient modules

Prop/Def$^n$: Let $M$ be an $R$-mod & $N$ be an $R$-submod. Then the abelian group $M/N$ has a natural $R$-mod structure given by $r\cdot(m+N) = rm + N$. The $R$-mod $M/N$ with this scalar multication is called the quotient of $M$ by $N$.

Pf: WTS: $r\cdot(m+N) = r.m + N$ is well-defined

Let $m, m' \in M$ be s.t. $m+N = m'+N$ & $r \in R$

$\Rightarrow m - m' \in N$

$\Rightarrow r(m-m') \in N$ ($\because$ $N$ is an $R$-submod)

$\Rightarrow rm - rm' \in N$

$\Rightarrow rm + N = rm' + N$

Hence scalar multiplication is well-defined

Note $M/N$ is an abelian grp

For $r_1, r_2 \in M$ & $m+N \in M/N$

$$\begin{aligned}(r_1+r_2)\cdot(m+N) &= (r_1+r_2)m + N \\ &= (r_1 m + r_2 m) + N \\ &= (r_1 m + N) + (r_2 m + N) \\ &= r_1\cdot(m+N) + r_2(m+N)\end{aligned}$$

lll'y check $r\cdot(m_1+N + m_2+N) = r(m_1+N) + r(m_2+N)$

$$(r_1 r_2)\cdot(m+N) = r_1(r_2(m+N))$$

$$1\cdot(m+N) = 1m + N = m + N$$

In particular, $M/IM$ is an $R$-mod $\forall$ $R$-mod $M$ and $R$-ideal $I$.

<u>Prop</u>: Let $M$ be an $R$-mod & $I$ an $R$-ideal then $M/IM$ is naturally an $R/I$-module where the scalar multiplication is given by.

$$R/I \times M/IM \xrightarrow{s} M/IM$$

$$(r+I,\ m+IM) \longmapsto rm+IM$$

$$(\bar{r},\ \bar{m}) \longmapsto \overline{rm} \qquad (\bar{r}\cdot\bar{m} = \overline{rm})$$

<u>Pf</u>: $s$ is well-defined

Note $r\cdot(m+IM) = rm+IM$ is well-define

$$\begin{cases} \text{Let } r+I = r'+I \text{ for } r, r' \in R \\ \& \quad m+IM = m'+IM \text{ for } m, m' \in M \end{cases}$$

$\Rightarrow x := r-r' \in I$ & $m-m' \in IM$

WTS: $rm+IM = r'm'+IM$

$$rm - r'm' = (r-r')m + r'(m-m')$$
$$= \underbrace{xm}_{\in IM} + \underbrace{r'(m-m')}_{\in IM} \quad (\because IM \text{ is a } R\text{-submod of } M)$$

$\Rightarrow rm+IM = r'm'+IM$

Check that this is a module structure.