Statistics

Chapter 6: Point Estimation

Where We're Going

- Estimate a population parameter with a sample statistic
- Method of Moments Estimator
- Bias and Variance
- Consistency and Asymptotic Normality

6.1: Point Estimation

- The unknown population parameter that we are interested in estimating is called the **target parameter**.

Parameter	Key Word or Phrase	Type of Data
μ	Mean, average	Quantitative
p	Proportion, percentage, fraction, rate	Qualitative
σ	Standard Deviation	Quantitative

6.1: Point Estimation

- A **point estimator** of a population parameter is a rule or formula that tells us how to use the sample data to calculate a *single* number that can be used to *estimate* the population parameter.

6.1: Point Estimation

- We often use the corresponding sample quantity to estimate the population quantity.

Parameter	Statistic	Type of Data
μ	$\bar{X}$	Quantitative
p	$\hat{p}$	Qualitative
σ	s	Quantitative

6.2 Method of moments(MoM)

- A general procedure is to equate population moments to sample moments and then solve for the parameter of interest.
- It can be seen directly that $\bar{X}$ and $\hat{p}$ are MoM estimators.

6.2 Method of moments(MoM)

- **MoM estimator of σ^2**
  - Note that $\sigma^2 = E(X-\mu)^2 = EX^2 - \mu^2$
  - The method of moments estimation of σ^2 would go as follows:

Equate EX^2 to $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ and EX to $\bar{X}$

Thus the equations are

$\sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ and μ= $\bar{X}$

6.2 Method of moments(MoM)

- Solve

$$\sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \text{ and } \mu = \bar{X}$$

to get the the MoM estimator of σ^2 as

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2$$

6.2 Method of moments(MoM)

- **MoM estimator of (n,p)**
- $X_1, X_2, \ldots X_k$~Bin(n,p) where both n and p are unknown.
- $EX = np$ and $\mathrm{E}X^2 = np + n(n-1)p^2$
- Equate these to $\mathrm{m}_1 = \bar{X}$ and $\mathrm{m}_2 = \overline{X^2}$

and solve to get the MoM estimators.

6.3 Bias and variance

If $\hat{\theta}$ is an estimator of θ, then the bias is defined as $\mathrm{E}(\hat{\theta}) - \theta$ and the variance is defined as $\mathrm{E}(\hat{\theta} - E\,\hat{\theta})^2$.

The probability distribution considered in the expectation is the sampling distribution of $\hat{\theta}$.

6.3 Bias and variance

- Example: Suppose $\theta = \mu$ and $\hat{\theta} = \bar{X}$.
$$E\left(\hat{\theta}\right) = E(\bar{X})$$
$$= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} EX_i = \frac{1}{n} n\mu = \mu = \theta$$
Hence Bias=0.

Such estimators are called unbiased.

6.3 Bias and variance

- Example: Suppose $\theta = \mu$ and $\hat{\theta} = \bar{X}$.

$$\mathrm{E}\left(\hat{\theta} - \theta\right)^2 = E(\bar{X} - \mu)^2 = E\left(\frac{1}{n}\sum X_i - \mu\right)^2 =$$

$$E\left(\frac{1}{n}\sum(X_i - \mu)\right)^2 = \frac{1}{n^2}\sum E(X_i - \mu)^2$$

The cross terms are zero by independence.

Finally the variance is $\frac{\sigma^2}{n}$.

6.4 Consistency and asymptotic normality

- An estimator $\hat{\theta}$ is consistent for a parameter θ if $\hat{\theta}$ converges to θ in probability (weak) or almost surely (strong) as the sample size goes to infinity.

  By law of large numbers, MoM estimators are consistent.

6.4 Consistency and Asymptotic normality

- An estimator $\hat{\theta}$ is asymptotically normal for a parameter θ if

$$\mathrm{T_n} = \sqrt{n}(\hat{\theta} - \theta)$$

converges in distribution to normal.

By Central Limit Theorem, MoM estimators are asymptotically normal.