general fact.

[Recall: if $f \in \mathcal{R}(\Omega)$, then $|f| \in \mathcal{R}(\Omega)$ & $\left|\int_\Omega f\right| \leq \int_\Omega |f|$.]

Here: $\mathcal{U}(|\tilde{f}|, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha \, v(B^2_\alpha)$

P, as above

$$= \sum_{\alpha \in \tilde{\Lambda}} M_\alpha \, v(B^2_\alpha)$$

[$\because$ $M_\alpha := \sup_{B^2_\alpha} |\tilde{f}| = 0 \ \forall \alpha \notin \tilde{\Lambda}$].

$$\leq M \times \underbrace{\sum_{\alpha \in \tilde{\Lambda}} v(B^2_\alpha)}_{< \varepsilon}.$$

$$< M \times \varepsilon.$$

$$\Rightarrow \inf \mathcal{U}(|\tilde{f}|, P) = 0 \quad \Rightarrow \int_{B^2} \tilde{f} = 0.$$

$$\Rightarrow \int_\Omega f = 0.$$ ☒

$\boxed{\text{Back}}$ to our thm: (Proof is similar).

if $\Omega = B^2$, nothing to prove.

Thm: $\underset{\text{bdd}}{\Omega} \supseteq \underset{\text{open}}{\mathcal{O}}$. Suppose $\bar{\Omega} \setminus \mathcal{O}$ is of ~~measure~~ content zero, $f \in \mathcal{B}(\Omega)$ & $f|_{\mathcal{O}}$ is continuous. Then $f \in \mathcal{R}(\Omega)$.

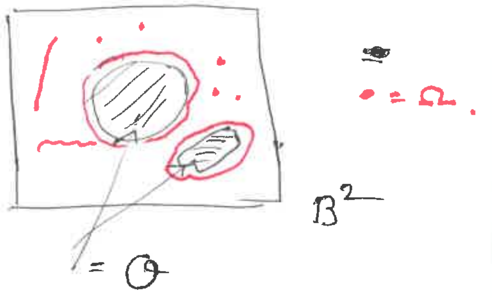
Proof: Let $\text{int}(B^2) \supseteq \bar{\Omega}$ & Consider $\tilde{f}$ on B^2 (extension of f).

Enough to prove that: $\mathcal{D}$, the set of points of discontinuity of $\tilde{f}$, is of measure zero.

Note that: (i) $\tilde{f}|_{\mathcal{O}}$ is cont. (ii) $\tilde{f}|_{\text{int}(B^2) \setminus \bar{\Omega}} \equiv 0$ is cont.

open set

& (iii) $\tilde{f}|_{\partial B^2} \equiv 0$ Cont.



$\Rightarrow \mathcal{D} \subseteq \bar{\Omega} \setminus \mathcal{O}$. ← Set of measure zero.

$\Rightarrow \mathcal{D}$ is a set of measure zero.

$\Rightarrow f \in \mathcal{R}(\Omega)$.

DANGER: Sets of ~~measure~~ content zero depends on the "dimension".

For instance: ① $[0,1] \subseteq \mathbb{R}$ is not of ~~zero measure~~ C·Z

but $[0,1] \times \{a\} \subseteq \mathbb{R}^2$ is of ~~measure zero~~ C·Z.

② ~~$\mathbb{Q} \cap [0,1]$ is of measure zero? Y/N:~~ ~~NO~~. C·Z

③ ~~$\mathbb{Q}\times\mathbb{Q} \cap ([0,1]\times[0,1])$ [illegible] ? Y/N:~~ ~~YES~~.

Fact: Let $f : B^2 \to \mathbb{R}$ be a cont. fn. Then

$$\boxed{\text{graph } f := \{(x, f(x)) : x \in B^2\}} \quad (\subseteq \mathbb{R}^3)$$

is a set of ~~measure~~ content zero.

Graphs have ~~measure~~ content zero.

works for $f : B^n \to \mathbb{R}$.

Proof: Let $\varepsilon > 0$. Note that: f is uniformly cont.

$\therefore \exists\, \delta > 0 \ni \quad |f(x) - f(y)| < \varepsilon \quad \forall\ |x-y| < \delta.$ $(x, y \in B^2)$

Next, on this $\delta > 0$, pick a partition ~~We can~~ P of B^2

s.t. the $\underbrace{\text{diameter}}_{=\text{diagonal}}$ of $B^2_\alpha < \delta \quad \forall\ \alpha \in \Lambda(P)$.

$$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall\ x, y \in B^2_\alpha, \quad \alpha \in \Lambda(P).$$

Set $I_\alpha := \{f(x) : x \in B^2_\alpha\}$.

The range set of $f|_{B^2_\alpha}$.

$\Rightarrow I_\alpha \subseteq \tilde{I}_\alpha$, for some interval of length at most ε. $\forall \alpha$.

$\therefore \{B^2_\alpha \times \tilde{I}_\alpha : \alpha \in \Lambda(P)\}$ is a cover of boxes of graph f. Also:

$\Lambda(P)$ is a finite set, &:

$$\sum_{\alpha\in\Lambda(P)} v(B^2_\alpha \times \tilde{I}_\alpha) = \sum_{\alpha\in\Lambda(P)} v(B^2_\alpha) \times v(\tilde{I}_\alpha)$$

$$\leq \sum_{\alpha\in\Lambda(P)} v(B^2_\alpha) \times \varepsilon.$$

$$= \underbrace{v(B^2)}_{\text{Constant}} \times \varepsilon.$$

$\Rightarrow$ measure of graph f is zero. $\square$

Infact, we have the following:

The proof is EVEN Better !!

Let $f \in \mathcal{R}([a,b])$. Then $G := \{(x, f(x)) : x \in [a,b]\} \subseteq \mathbb{R}^2$ is of ~~measure~~ Content zero.

Proof: We proceed along the same line:

Let $\varepsilon > 0$. $\exists$ $P \in \mathcal{P}([a,b])$ $\ni$

$$U(f,P) - L(f,P) < \varepsilon.$$

Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

& $B^2_i := [x_{i-1}, x_i] \times [m_i, M_i]$,

Here: $m_i = \inf_{[x_{i-1},x_i]} f$

& $M_i = \sup_{[x_{i-1},x_i]} f$.

$\therefore$ $G \subseteq \bigcup_{i=1}^n B^2_i$. Finally:

$$\sum_{i=1}^n v(B^2_i) = \sum_{i=1}^n v([x_{i-1}, x_i] \times [m_i, M_i])$$

$$= \sum_{i=1}^n (x_i - x_{i-1}) * (M_i - m_i).$$

$$= U(f,P) - L(f,P)$$

Smart proof? : Than P-42?

$$< \varepsilon. \quad \square$$

Back to Fubini's thm:

Recall: Let $f \in \mathcal{R}(B^2)$. Set $B_2 = [a,b] \times [c,d]$.

If $\int_a^b f(x,y)\,dx$ exists $\forall\, y \in [c,d]$, then

$$\int_{B^2} f = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{} \Big)\, dy. \quad \text{—— (1)}$$

Note: integrability of this fn. (in y) is guaranteed: by Fubini.

|||ly if, $\int_c^d f(x,y)\,dy$ exists for each $x \in [a,b]$, then

$$\int_{B^2} f = \int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\, dx. \quad \text{—— (2)}$$

If, in particular, $f \in C(B^2)$, then (1) = (2).

—— x ——.

Q: Fubini for $f \in \mathcal{R}(\Omega)$, $\Omega \subseteq \mathbb{B}^2$, bdd ??

How to think about it?

In fact: it is not easy to "evaluate" (COMPUTE) double integral over $\Omega \subseteq \mathbb{R}^2$ (bdd). However, with "some" control over Ω, one can do "something". It is as follows:

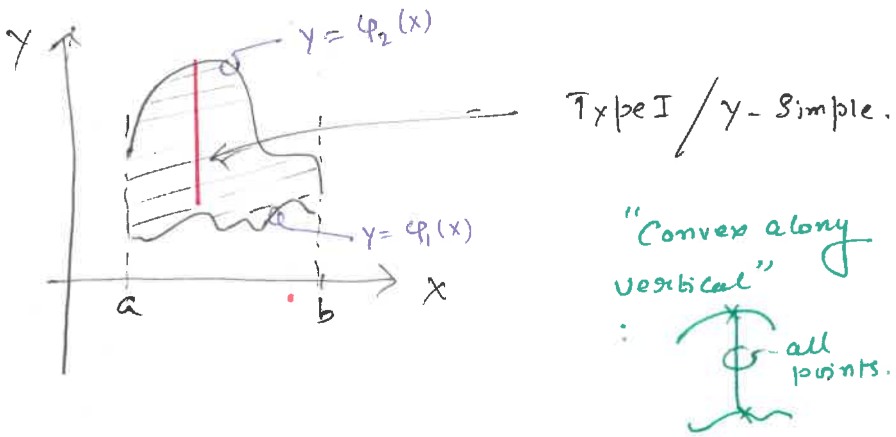
[Remark: Many/all of the results below works similarly in $\mathbb{R}^n$, $n \geqslant 3$. Atleast, think them in the setting of $\mathbb{R}^3$.]

Two Special domains (AKA: Elementary regions):

Def: A set $\Omega \subseteq \mathbb{R}^2$ is said to be y-Simple / Type I if $\exists\ \varphi_1, \varphi_2 \in \mathcal{R}([a,b])$ s.t.

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}.$$

Here:

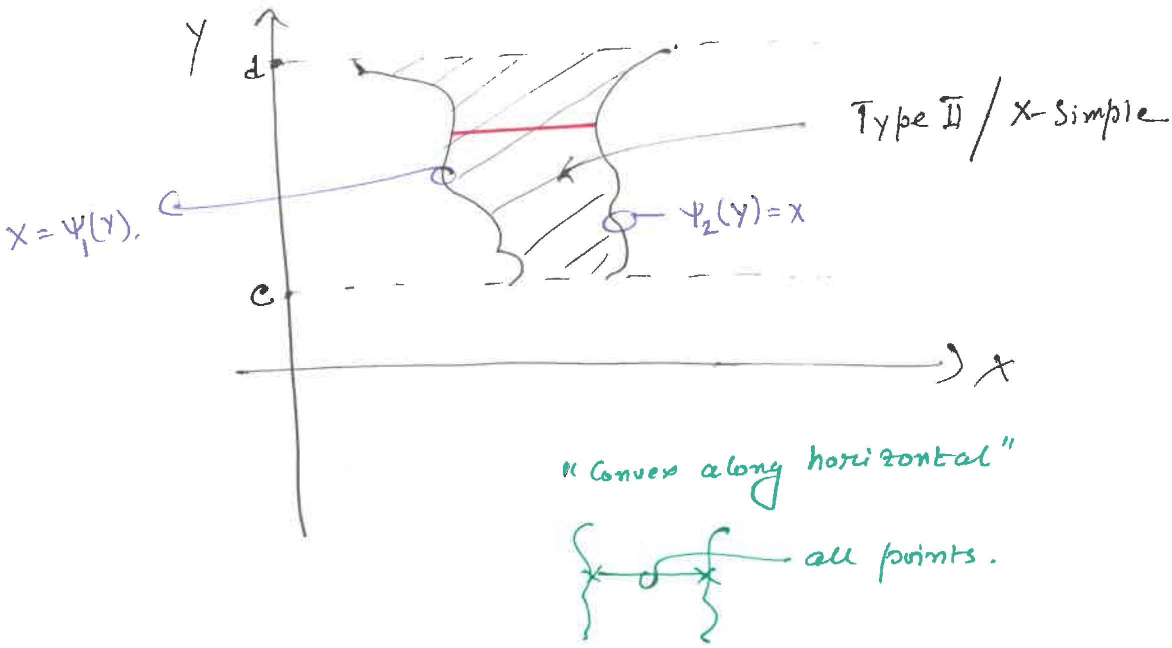


III^{ly} x-Simple / Type II regions are given by:

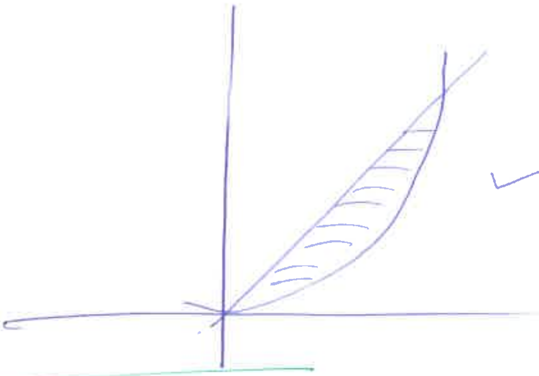
$$\Omega = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

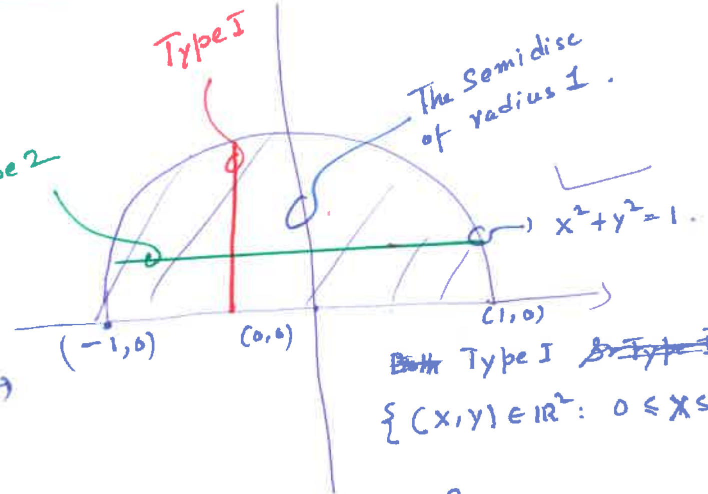
for some $\psi_1, \psi_2 \in \mathcal{R}[c,d]$.

Here:



eg:



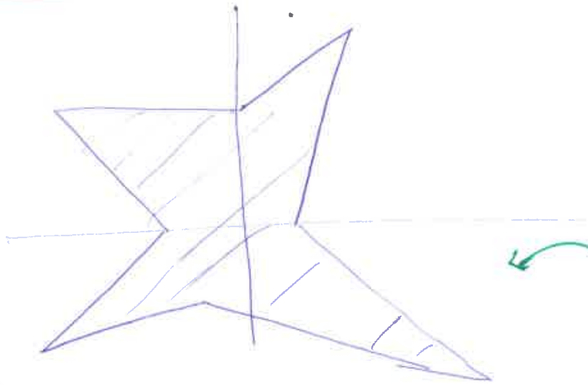


~~Both~~ Type I ~~& Type II~~.

$\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ 0\le y\le \sqrt{1-x^2}\}$.

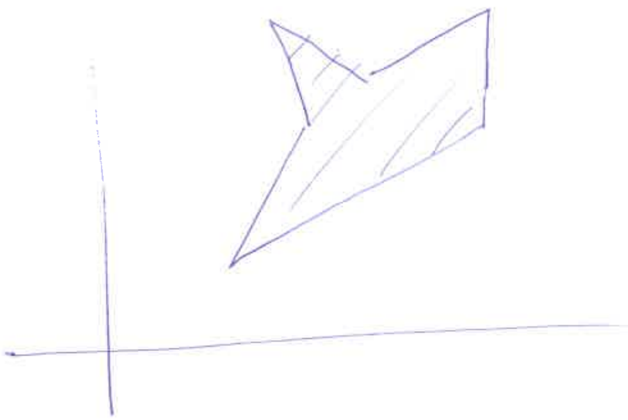
Also Type II:

$\{(x,y) : 0\le y\le 1\ \&\ -\sqrt{1-y^2}\le x\le\sqrt{1-y^2}\}$.



? ✗

BUT: Som of elementary regions!!



? ✗

Fubini

↓ ↓ ↓

Thm: Let $f \in \mathcal{R}(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ (an elementary region).

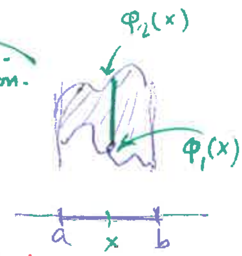
(I) If $\Omega = \{(x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$, for some $\varphi_1, \varphi_2 \in \mathcal{R}[a,b]$, and if $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$ exists $\forall x \in [a,b]$, then

$$\iint_\Omega f(x,y) = \int_a^b \Big(\underbrace{\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy}_{\text{THIS MUST EXISTS } \forall x} \Big)\,dx .$$

Just notation.

$\iint_\Omega f(x,y)\,dA = \int_\Omega f$

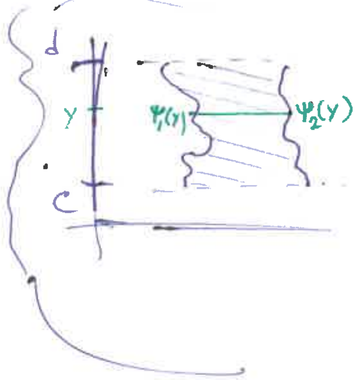
EASY interpretation.



: Then integrability is assured.

(II) If $\Omega = \{(x,y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$, for some $\psi_1, \psi_2 \in \mathcal{R}[c,d]$, and if $\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$ exists $\forall y \in [a,b]$, then

$$\iint_\Omega f = \int_c^d \Big(\underbrace{\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx}_{\text{THIS MUST EXISTS } \forall y \in [c,d]} \Big)\,dx .$$



Proof (Easy application of Fubini):

We ~~will~~ prove only (I), as (II) will be similar.

Get $c < d$ s.t. $\Omega \subseteq B^2 := [a,b] \times [c,d]$.

[In fact: $c = \inf_{[a,b]} \varphi_1$ & $d = \sup_{[a,b]} \varphi$ is one natural choice.]

Consider the extension fn. $\tilde{f} : B^2 \to \mathbb{R}$, where

$$\tilde{f}\big|_\Omega \equiv f \quad \& \quad \tilde{f}\big|_{B^2 \setminus \Omega} \equiv 0 .$$

We know $\tilde{f} \in \mathcal{R}(B^2)$. Now for each $x \in [a,b]$, $\int_c^d \tilde{f}(x,y)\,dy$ exists.

Indeed: (for fixed $x \in [a,b]$)

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \forall y \in [\varphi_1(x), \varphi_2(x)] \\ 0 & \forall y \in [c, \varphi_1(x)] \cup [\varphi_2(x), d] . \end{cases}$$

~~So~~ But $f(x,\cdot)\big|_{[\varphi_1(x), \varphi_2(x)]}$ & $\tilde{f}(x,\cdot)\big|_{[c,\varphi_1(x)] \cup [\varphi_2(x), d]}$ are integrable. So, by 1-variable result, $\tilde{f}(x,\cdot) \in \mathcal{R}[c,d]$.

Finally, again for fixed $x \in [a,b]$, by 1-variable additivity:

$$\int_c^d \tilde{f}(x,y)\,dy = \underbrace{\int_c^{\varphi_1(x)} \tilde{f}(x,y)\,dy}_{=0} + \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x,y)\,dy + \underbrace{\int_{\varphi_2(x)}^{d} \tilde{f}(x,y)\,dy}_{=0}.$$

$$= \int_{\varphi_1(x)}^{\varphi_2(x)} \tilde{f}(x,y)\,dy$$

$$= \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy. \quad \left[\because\ \tilde{f}(x,y) = f(x,y)\ \ \forall\ \varphi_1(x) \le y \le \varphi_2(x)\right].$$

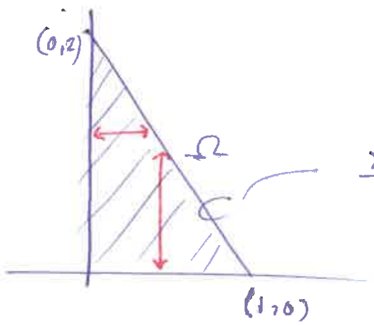
Then, by Fubini ($\because$ $\forall\ x \in [a,b]$, $\int_c^d \tilde{f}(x,y)\,dy$ exists):

$$\iint_\Omega f \ \underset{\text{DEF}}{=}\ \iint_{B^2} \tilde{f} \ \underset{\text{FUBINI}}{=}\ \int_a^b \left(\int_c^d \tilde{f}(x,y)\,dy \right) dx$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx.$$

~~Eg: Compute $\iint_\Omega f$, where $f \in R(\Omega)$ &~~ Ω.

Eg: Consider $f \in C(\Omega)$, where $\Omega =$

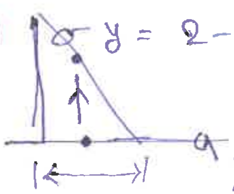


$\frac{x}{1} + \frac{y}{2} = 1$

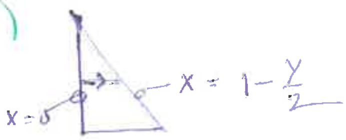
Clearly, Ω is both Type I & Type II.

Also, $f \in R(\Omega)$. Then

$$\int_\Omega f = \int_0^1 \left(\int_0^{2-2x} f(x,y)\,dy \right) dx$$



$$= \int_0^1 \left(\int_0^{1-y/2} f(x,y)\,dx \right) dy$$



Often, changing order of integration is useful. We will also see.