It is easy to show that $\nabla \times F = 0$.

However, $\int_C F \cdot d\mathbf{r} \neq 0$, where $C : \mathbf{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$. $0 \leq \theta \leq 2\pi$.

unit circle.

Indeed: $$\int_C F \cdot d\mathbf{r} = \int_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

$$= \int_0^{2\pi} \left(\frac{-\sin\theta}{\sin^2\theta + \cos^2\theta}\right) d(\cos\theta) + \left(\frac{\cos\theta}{\sin^2\theta + \cos^2\theta}\right) d(\sin\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = \int_0^{2\pi} d\theta$$

$$= 2\pi \neq 0. \qquad \blacksquare$$

Remark: What went wrong?

Well, perhaps, F is not C^1 (or not even defined/diff./cont) at $(0,0)$. So $F : \mathbb{R}^2 \to \mathbb{R}^2$ is NOT a good choice.

Okay: So, lets consider $F : \mathcal{O}_2 \to \mathbb{R}^2$, where $\mathcal{O}_2 = \mathbb{R}^2 \setminus \{(0,0)\}$

or $\{(x,y) : x^2+y^2 < 1\} \setminus \{(0,0)\}$.

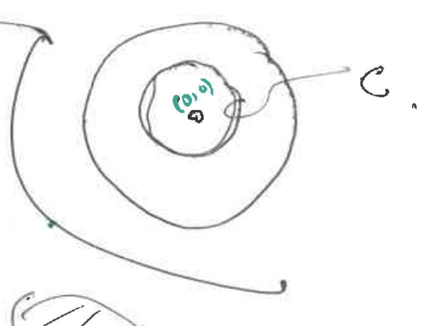
punctured disc

BUT, AGAIN, we can ~~will~~ ~~have~~ will consider a circle C & prove $\int_C F \cdot d\mathbf{r} \neq 0$.

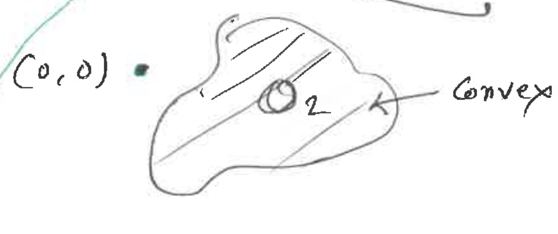
the same as above!

THEN?

The trouble is $(0,0)$, the singularity being in the interior of $\mathcal{O}_2$.

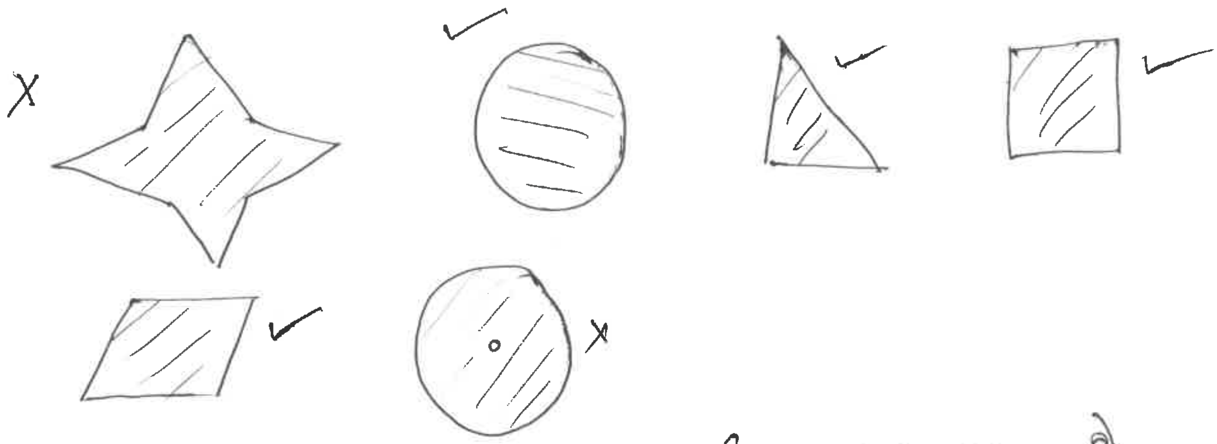


In fact: If $\mathcal{O}_2$ convex ~~(a.a)~~ & $(0,0) \notin \mathcal{O}_2$, then it will do!!



Remark: Now, suppose we have $F : O_2 \to \mathbb{R}^2$ s.t. any pair of points in O_2 can be connected via a line in O_2

(← We call it as CONVEX domain).



If we know F is conservative, then we know $\nabla F = F := (P, Q)$, & then we can simply follow the method of eg (2) in Page-60 to solve it for P & Q.

[See ✡ in page 60].

Q: But, how to determine F is conservative?

"Ans:" Green's thm.

"Curl:"

Recall curl of a v.f. F is $\nabla \times F$. : The measure of tendency of F to swirl/create whirlpool. Like

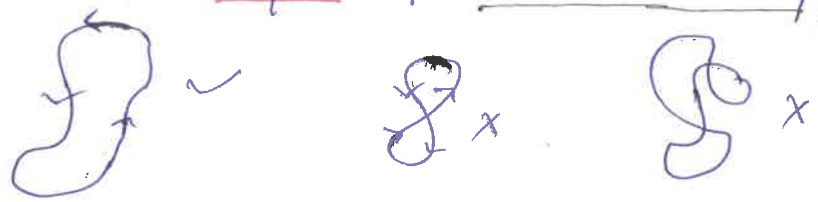
← $\neq 0$ curl i.e, there will be swirl/whirlpool.

But

X

Def: Let $\mathcal{D} \to$ open + connected subset of $\mathbb{R}^2$. We say that $\mathcal{D}$ is <u>simply connected</u> if, whenever $C \subseteq \mathcal{D}$ a simple closed curve, C can be shrunk continuosly / gradually to a point inside $\mathcal{D}$.

[# A curve C is <u>simple</u> if it has no self intersections.



CROSS ← Not allowed.

i.e.: If parametrizations of C are <u>injective</u> !!
↓ except initial & terminal points.]

$\mathcal{D} \to$ open + connected.

Then $\mathcal{D}$ is simply connected $\iff$ if $C \subseteq \mathcal{D}$ is a simple closed curve, then the interior of $C \subseteq \mathcal{D}$.

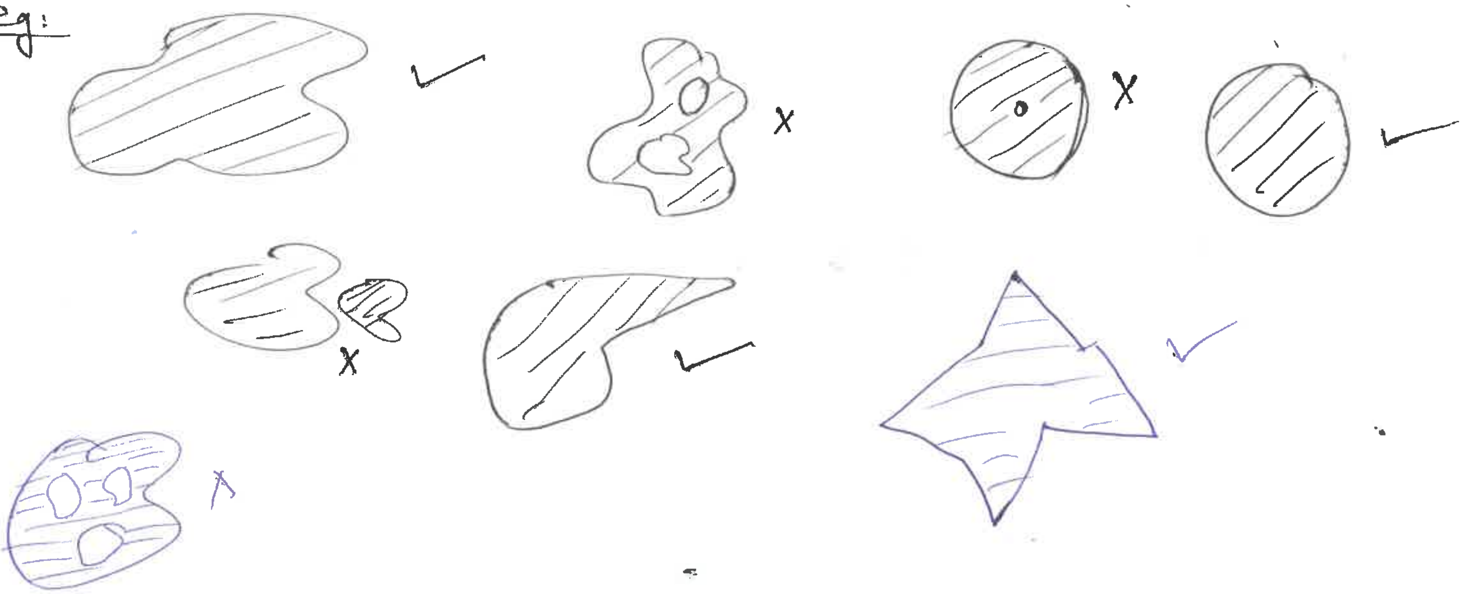
$\iff$ $\underbrace{\mathbb{R}^2_\infty}_{} \setminus \mathcal{D}$ is connected. [Ahlfor's : Complex Analysis].

$\mathbb{R}^2_\infty = \mathbb{R}^2 \cup \{\infty\} \cong S^2$ (sphere in $\mathbb{R}^2$: through stereographic projection).

NO PROOF!!

Remark: More precise / accurate defn needs the notion of fundamental groups / homotopy theory.

eg:

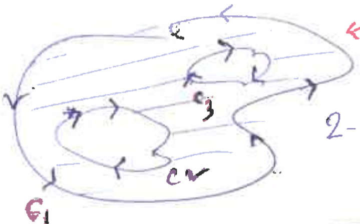


SEE AFTER Green's THEOREM

Green's thm for "n-holed domains".

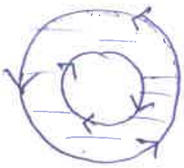
A domain (open + connected) bounded by finitely many piecewise simple C^1-curves.

eg:



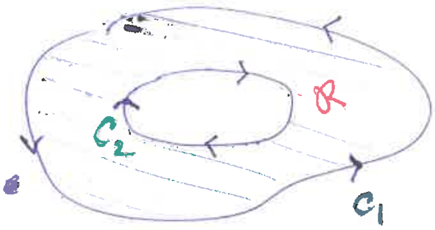
$\leftarrow C = C_1 \cup C_2 \cup C_3$

2-holed domain



$\leftarrow$ 1-hole (Annulus).

Consider:



$C = C_1 \cup C_2$.

Q: $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{?}{\underset{?}{=}} \int_C P\,dx + Q\,dy$??

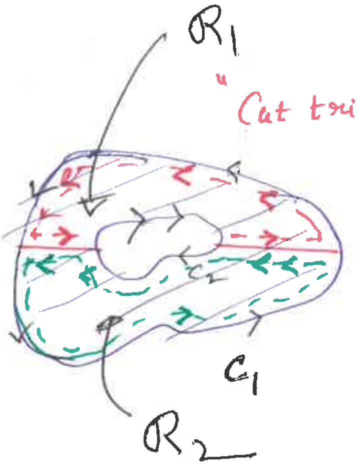
Ans: Yes.

$$\underbrace{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\parallel}$$

$$\iint_{R_1} + \iint_{R_2} \overset{\text{Green's thm}}{=} \int_{\partial R_1} P\,dx + Q\,dy + \int_{\partial R_2} P\,dx + Q\,dy$$

$$= \int_{C_1 \cup C_2} P\,dx + Q\,dy.$$

"Cut trick"



$R_1, R_2 \rightarrow$ Simply connected.

[∵ $C_1 \cup C_2 =$ ]

Illy n-holed domain.

Green's theorem: (in $\mathbb{R}^2$: Line vs. Area integrations)

Thm: Let $\mathcal{R} \subseteq \mathbb{R}^2$ be [a simply connected domain] ~~Region~~ (~~open + bdd + connected~~) with boundary curve C (parametrized such a way so that $\mathcal{R}$ is "to the left").
Let P, Q be a C^1-vector field on $\mathcal{R}$ (i.e. C^1 on an open set containing $\mathcal{R}$). Then

$$\underbrace{\int_C P\,dx + Q\,dy}_{\parallel\ (\dagger)} = \underbrace{\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\parallel} .$$

$$\int_C \vec{F} \cdot d\vec{\gamma} = \int \mathrm{Curl}(\vec{F})\, dA .$$

Where $\vec{F} = (P, Q) : \mathcal{R} \to \mathbb{R}^2$ & (recall that)

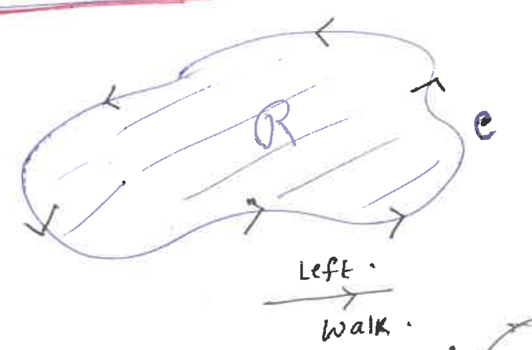
$$\mathrm{Curl}\,\vec{F} := \begin{vmatrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} := \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{Scalar field in 2 dim}} .$$

$(\dagger)$ [**Recall:** If $C = \mathrm{Ran}\,\gamma$ (Gamma), $\gamma = (x(t), y(t))$, $t \in [a,b]$, then for $\vec{F} = (P, Q)$, we have

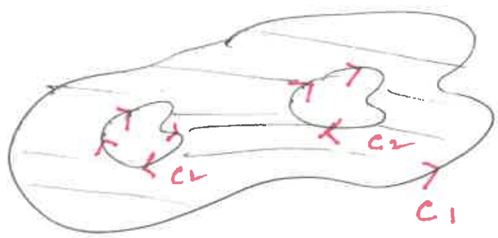
$$\int_C \vec{F} \cdot d\vec{\gamma} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$
$$= \int_a^b \big((P, Q) \cdot (x'(t), y'(t)) \big)\, dt$$
$$= \int_a^b \left(P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right) \times dt$$
$$= \int_C P\,dx + \underbrace{Q\,dy}$$]

Here: $dx = x'(t)dt$, $P = P(x(t), y(t))$.

"to the left" (orientation of C):



(Now see Page 64)



$C = C_1 \cup C_2 \cup C_3$.

Remark: Why $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Why "$-$"?

Think $\vec{F} = (P, Q)$ as $\vec{F} = (P, Q, 0)$.

Then $\underbrace{\text{Curl}(\vec{F})}$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$

This is the curl used in the statement but with a little care

$= \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$.

$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k$.

So $\left\{ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k \right\} \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

dot product

So the precise statement is:

$$\int_R \underbrace{\text{Curl}(\vec{F}) \cdot k \; dA} = \int_C \vec{F} \cdot d\vec{r}.$$

magnitude of Curl $(\vec{F})$

The normal vector to the plane.

Of course, Curl of planar v.f's is a vector pointing towards k, the normal to the plane.

Note $\int_C \vec{F} \cdot d\vec{r}$ = Circulation of $\vec{F}$ around C.
or Work done by $\vec{F}$ around C.

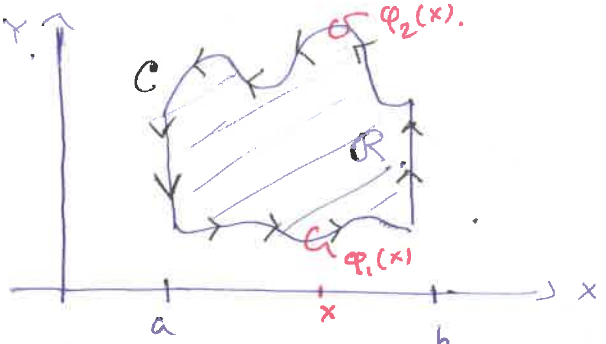
& $\int_R \text{Curl}(\vec{F}) \cdot k \; dA$ = Sum of all infinitely small Circulations in the region R.

Proof: Not in ~~the~~ our scope. (In fact: Green's thm $\Leftarrow$ Stokes thm (in $\mathbb{R}^3$).
AND: Stoke's thm fits / suits well in $\mathbb{R}^n$ but from exterior product + differential forms point of view).

However, here is a Simple ~~ver~~ version

Let $\mathcal{R} = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x) \}$ $\leftarrow$ elementary domain (closed).

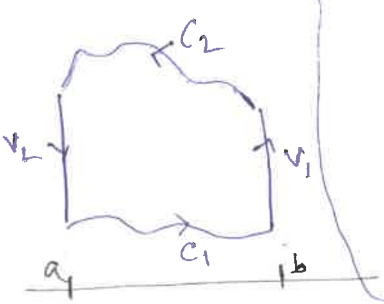
$P, Q \in C^1(\Omega)$, where $\underset{\text{open}}{\Omega} \supseteq \mathcal{R}$. Set $C = \partial \mathcal{R}$.



Claim: $\displaystyle\int_C P\,dx + Q\,dy = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$

We first prove: $\boxed{\displaystyle\int_{\mathcal{R}} -\frac{\partial P}{\partial y}\, dA = \int_C P\,dx}$. AND THEN $\boxed{\displaystyle\int_C Q\,dy = \int_{\mathcal{R}} \frac{\partial Q}{\partial x}}$

Indeed: $\displaystyle\int_{\mathcal{R}} -\frac{\partial P}{\partial y}\, dA = -\int_{\mathcal{R}} \frac{\partial P}{\partial y}\, dA = -\iint_{\mathcal{R}} \frac{\partial P}{\partial y}\, dx\, dy = -\int_a^b \int_{y=\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$



$$= -\int_a^b \Big[P(x,y) \Big]_{y=\varphi_1(x)}^{\varphi_2(x)} dx$$

$$= \int_a^b \Big(P(x, \varphi_1(x)) - P(x, \varphi_2(x)) \Big) dx$$

$$= \int_a^b P(t, \varphi_1(t))\, dt - \int_a^b P(t, \varphi_2(t))\, dt.$$

Now $C = C_1 \cup V_1 \cup C_2 \cup V_2$.

So, $\displaystyle\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{V_1} P\,dx + \int_{V_2} P\,dx$.

$\because$ on V_1: $x = a \Rightarrow \dfrac{dx}{dt} = 0$. So $\displaystyle\int_{V_1} P\,dx = \int_{\varphi_1(a)}^{\varphi_2(a)} P(a,y) \overbrace{\frac{dx}{dt}}^{=0} dt = \int 0\, dt = 0.$

[V_1 is given by $t \longmapsto (a, t) = (x(t), y(t))$, $\varphi_1(a) \leq t \leq \varphi_2(a)$]

$\therefore \displaystyle\int_{V_1} P\,dx = 0$. Illy $\displaystyle\int_{V_2} P\,dx = 0$.

Note that $C_1 : t \longmapsto (x(t), y(t)) := (t, \varphi_1(t))$, $t \in [a,b]$. ← parametrization of C_1.

$\therefore \frac{dx}{dt} = 1$ [$\because x(t) = t$].

$$\therefore \int_{C_1} P\,dx = \int_a^b P(t, \varphi(t)) \underbrace{\frac{dx}{dt}}_{=1} dt = \int_a^b P(t, \varphi(t))\,dt$$

Also $C_2 : t \longmapsto (x(t), y(t)) = (t, \varphi_2(t))$. $t \in [a,b]$.

~~the fact that~~ ~~$a \Rightarrow b \Rightarrow \varphi_1(b) \Rightarrow \varphi(a)$~~

$\therefore x(t) = t \Rightarrow \frac{dx}{dt} = 1$,

$$\therefore \int_{C_2} P\,dx = -\int_a^b P(t, \varphi_2(t))\,dt$$

"$-$": due to the opposite orientation of C_2.

Hence $\int_R -\frac{\partial P}{\partial y}\,dA = \int_C P\,dx$.

Illy $\int \frac{\partial Q}{\partial x}\,dA = \int Q\,dy$.

$\boxed{i)}$

Remark: Using the above, for boxes $B^2 \subseteq \mathbb{R}^2$, a way longer limiting approach will lead Green's theorem for domains. However, the natural way to ~~prove~~ get this as a Corollary of Stokes theorem.

Before we go to Stoke's thm, let's look at some examples:

eg: Compute $\int_C \langle x^2 - y^2, 2xy \rangle \cdot d\gamma$, where $C = \partial([0,1] \times [0,1])$

Sol: By Green's thm: $\int_C \langle \underbrace{x^2-y^2}_{P}, \underbrace{2xy}_{Q} \rangle \cdot d\gamma = \int_{[0,1]^2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$= \int_{[0,1]^2} (2y + 2y)\,dA = 4 \int_{[0,1]}\int_{[0,1]} y\,dA = 4 \int_0^1 \int_0^1 y\,dy\,dx$$

$$= 4 \times \frac{1}{2} \times \left[y^2 \right]_0^1 = 2. \quad \text{Ans.}$$

← Perhaps easy!!

Cor: Area formula.

1–1 i.e. no cross

Let $C = \partial R$, where C is truely oriented (Simple) closed curve.

i.e. R is on left.

Then $\underline{\text{Area}(R)} = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C x\,dy - y\,dx$.

Proof:. Simple idea: Choose $(P, Q) := F$ so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1. \Rightarrow \int_C P\,dx + Q\,dy = \int_R 1\,dA$$

$= \text{Area of } R$.

Here $(0, x)$, $(-y, 0)$, $(\frac{1}{2}x, -\frac{1}{2}y)$ does the job !!

eg: Area of ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\gamma(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$.

$\therefore$ Area $= \frac{1}{2}\int x\,dy - y\,dx$.

$$x = a\cos t,\quad y = b\sin t \Rightarrow \frac{dx}{dt} = -a\sin t,\quad \frac{dy}{dt} = b\cos t.$$

$$= \frac{1}{2}\int_0^{2\pi}\{a\cos t \times b\cos t - b\sin t(-a\sin t)\}dt$$

$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2 t + \sin^2 t)\,dt$$

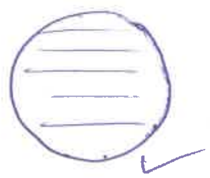
$$= \frac{1}{2}ab \times 2\pi = \pi ab.$$

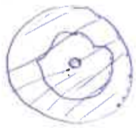
[$\therefore$ Green's Theorem Could be useful for line through double & double through line both !!]

Def: Let $\mathcal{D} \subseteq \mathbb{R}^n$ $(n = 2$ or $3)$ be a domain (open + connected). Then $\mathcal{D}$ is Simply Connected if each closed curve in $\mathcal{D}$ can be shrunk continuously / gradually to a point inside $\mathcal{D}$.

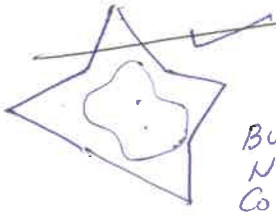
Covered in Page-63

$\mathbb{R}^2 \setminus \{(0,0)\}$ X.

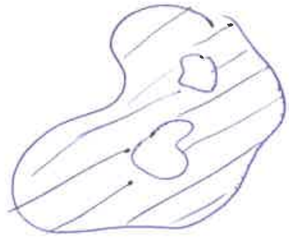




X



BUT NOT Convex.



X 2-holed domain.

lly n-holed domain X.

Thm: Let $\mathcal{D}$ be a Simply Connected domain in $\mathbb{R}^2$ & ~~[illegible]~~ F be a C^1-vector field on $\mathcal{D}$. Then F is Conservative $\iff$ $\underbrace{\nabla \times F}$ $= 0$ on $\mathcal{D}$.

Recall: in $\mathbb{R}^2$ $\nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ if $F = (P, Q)$.

Proof: "$\Rightarrow$" By defn. of $\nabla \times F$.

"$\Leftarrow$" Simply ~~Greens~~ Green's thm. ☑

[Recall: If $F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$, then $\nabla \times F = 0$ But F is not Conservative. Surely F is C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$ or open unit Ball $\setminus \{(0,0)\}$, BUT NONE OF THEM are simply Connected: So, Conservative has a lot to do with the nature of the domain of definitions.]

— ✲ —

Def: Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a v.f. Then $\text{Div}(F)$ (the divergence of F) is defined by:

$$\text{Div}(F) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

So, if $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$, then $\boxed{\text{Div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}$

lly 3-variables.