Algebra III (Notes16-19)

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- $S \subseteq R$ is said to be a multiplicative subset if $1 \in S$ and $\forall x, y \in S, xy \in S$
- $P \to$ prime ideal $\Rightarrow R \setminus P$ is a multiplicative subset of R
- $S \times R = \{(s, r) \mid s \in S, r \in R\}$, $(s_1, r_1) \sim (s_2, r_2)$ if $s(s_1 r_2 - s_2 r_1) = 0$ for some $s \in S$
- $\sim$ is an equivalence relation and the set of equivalence classes $(S \times R)/\sim$ is denoted by $S^{-1}R$. The equivalence class $[(s, r)]$ is denoted by $\frac{r}{s}$. $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$ and $\frac{r_1}{s_1} . \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$. $(S^{-1}R, +, .)$ is a commutative ring with unity.
- Natural map $\begin{aligned} \phi : R &\to S^{-1}R \\ r &\mapsto \frac{r}{1} \end{aligned}$ is a ring homomorphism and $ker(\phi) = \{r \in R \mid sr = 0, 0 \neq s \in S\}$.

  So $S \to$ integral domain $\Rightarrow ker\phi = \{0\} \Rightarrow \phi$ is injective
- $R \to$ integral domain, a field K is called the "field of fractions" of R if R is a subring of K and no proper subfield of K contains R. It is denoted by $frac(R)$ or $QF(R)$
- If K is a field containing R as a subring then K contains $frac(R)$
- $R \to$ integral domain and $S = R \setminus \{0\}$, then $S^{-1}R$ is the field of fractions of R . Also $\phi : R \hookrightarrow frac(R)$ is injective
- $S \to$ multiplicative subset of R, then $\phi(s)$ is a unit in $S^{-1}R$ $\forall s \in S$
- $0 \in S \Rightarrow S^{-1}R = \{0\}$
- Let $f : R \to A$ be a ring homomorphism ($A \to$ comm ring with unity) such that $f(s)$ is a unit in A $\forall s \in S$, then $\exists!$ ring homomorphism $\hat{f} : S^{-1}R \to A$ such that $\hat{f} \circ \phi = f$
- $R \to$ integral domain, then the field of fractions of R is an injective ring homo $i : R \hookrightarrow K$ such that K is a field and for any subfield K_0 of K containing $i(R)$, $K_0 = K$
- Field of fraction is unique upto isomorphism i.e. if $i : R \hookrightarrow K$ and $i' : R \hookrightarrow K'$ are field of fractions, then $\exists!$ $f : K \to K'$ such that f is an isomorphism and $f \circ i = i'$
- S consists of units $\Rightarrow \phi$ is an isomorphism
- $x \in R$, $S = \{1, x, x^2, ...\} \Rightarrow S^{-1}R = R\left[\frac{1}{x}\right] \cong \frac{R[z]}{(xz - 1)}$
- A commutative ring with unity R is called a "local ring" if it has exactly one maximal ideal.
- $P \to$ prime ideal, $S = R \setminus P \Rightarrow S^{-1}R$ (also denoted by R_P) is a local ring with the unique maximal ideal $PS^{-1}R = S^{-1}P = \left\{\frac{r}{s} \mid r \in P, s \notin P\right\}$
- $m \to$ maximal ideal of $R \Rightarrow$ TFAE:
  - ✔ R is a local ring
  - ✔ The set of nonunits of R form the ideal m
  - ✔ $1 + x$ is a unit $\forall x \in m$
- $I \subseteq S^{-1}R \to$ ideal $\Rightarrow J = \phi^{-1}(I)$ is an ideal of R if I is a proper ideal $J \cap S = \phi$. Also $(\phi(J)) = JS^{-1}R = I$
- Every ideal in $S^{-1}R$ is of the form $JS^{-1}R$ for some R-ideal J