Some examples:

1) $\displaystyle \underbrace{\iint_{B^2}}_{\int_{B^2}\ \text{Convention.}} \underbrace{\sin(x+y)}_{f(x,y)} \underbrace{dA}_{\text{Convention: refers to area.}} = ?$; where $B^2 = [0,\pi] \times [-\pi/2, \pi/2]$.

i.e; Compute $\displaystyle\int_{B^2} \sin(x+y)$.

Sol:

$\sin(x+y) = \sin x \cos y + \cos x \sin y$. $\in \text{Cont}(B^2)$.

$$\therefore \int_{B^2} \sin(x+y) = \cancel{\int_0^{\pi} \left(\underbrace{\int_{-\pi/2}^{\pi/2} \left(\sin x \cos y + \cos x \sin y\right) dy}_{\pi} \right) dx}$$

$$\cancel{\sin x \int \cos y + \cos x \int \sin y}$$

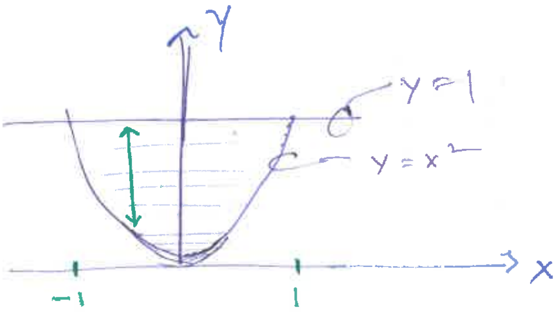
$$= \int_{B^2} \sin x \cos y + \int_{B^2} \cos x \sin y.$$

$$= \int_0^{\pi} \sin x\, dx \times \int_{-\pi/2}^{\pi/2} \cos y\, dy + \underbrace{\int_0^{\pi} \cos x\, dx}_{=0} \times \int_{-\pi/2}^{\pi/2} \sin y\, dy.$$

$$= 2 \times 2 + 0 = 4.$$

2) Ω = region bounded by $y = 1$ & $y = x^2$. Compute $\displaystyle\int_{\Omega} x^2 y$.

Here $\Omega = \left\{ (x,y) : \begin{array}{l} -1 \le x \le 1 \\ x^2 \le y \le 1 \end{array} \right\}$.



So $\displaystyle\int_{\Omega} f = -\int_{x=-1}^{1} \int_{y=1}^{y=x^2} x^2 y \, \underbrace{dA}_{\text{Area}}$

$\displaystyle = -\int_{-1}^{1} x^2 \left(\int_{y=1}^{y=x^2} y \, dy \right) dx. = -\int_{-1}^{1} x^2 \times \frac{1}{2}\left(x^4 - 1\right) dx.$

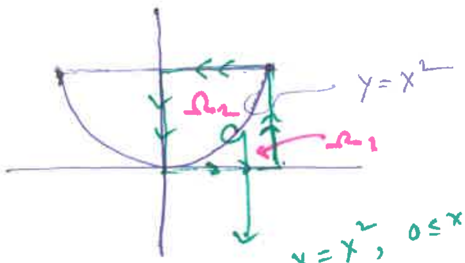
$\displaystyle = -\frac{1}{2} \int_{-1}^{1} \left(x^6 - x^2\right) dx. = -\frac{1}{2} \times 2 \times \left(\frac{1}{7} - \frac{1}{3}\right)$

$\displaystyle = \frac{4}{21}$. Ans

(3) Let $f(x,y) = \begin{cases} x & \text{if } y \le x^2 \\ y & \text{if } y > x^2 . \end{cases}$ $\qquad [0,1] \times [0,1] = B^2$.

Compute $\displaystyle\iint_{B^2} f$.

Sol: Write $B^2 = \Omega_1 \amalg \Omega_2$.



Along $y = x^2$, $0 \le x \le 1$, f is NOT Cont. But it is of Content zero.

Where $\Omega_1 = \{ (x,y) : 0 \le x \le 1, \ 0 \le y \le x^2 \}$

& $\Omega_2 = \{ (x,y) : 0 \le x \le 1, \ x^2 \le y \le 1 \}$.

Now $f|_{\Omega_1}$ is Cont. & $(f|_{\Omega_1})(x,y) = x$. So $f|_{\Omega_1} \in R(\Omega_1)$.

But $\left(f|_{\Omega_2}\right)(x,y) = \begin{cases} x & \text{if } y = x^2 \\ y & \text{if } x^2 < y \end{cases}$. ⟵ f along the curve.

$\Rightarrow f|_{\Omega_2}$ is not cont. along $y = x^2$, $0 \le x \le 1$.

But $f|_{\Omega_2}$ is cont. in the int(Ω_2).

$\Rightarrow f|_{\Omega_2} \in \mathcal{R}(\Omega_2)$.

$$\therefore \int_{B^2} f = \underbrace{\int_{\Omega_1} f}_{\text{Easy}} + \int_{\Omega_2} f .$$

⟵ How? Why? It is true but need a proof. – wait – ?

$$\int_0^1 x\left(\int_0^{x^2} dy\right)dx = \int_0^1 x^3\,dx = \frac{1}{4}.$$

Where: $$\int_{\Omega_2} f = \int_0^1\left(\int_{x^2}^1 y\,dy\right)dx = \frac{1}{2}\int_0^1 (1-x^4)\,dx = \cdots = \frac{2}{5}.$$

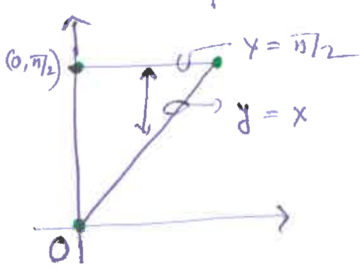
$$\therefore \int_{B^2} f = \frac{1}{4} + \frac{2}{5} = \frac{13}{20}.$$

(4) $\displaystyle\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\,dy\,dx = ?$ ⟵ Wow! Integrating $\frac{\sin y}{\cancel{x}}$??

$\in \mathcal{R}[?]$. At least, it is bdd (Right?).

We check the Ω first: $0 \le x \le \pi/2$, $x \le y \le \pi/2$.

So



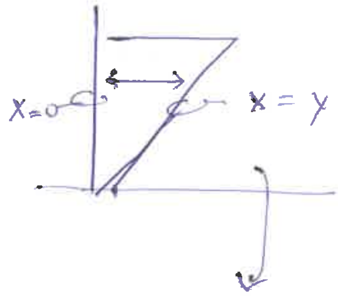
Clearly $f(x,y) = \frac{\sin y}{y} \in \mathcal{R}(\Omega)$.

Where $\Omega = \{(x,y) : 0 \le x \le \pi/2$ & $x \le y \le \pi/2\}$.

But how to evaluate?

By changing the order of integration (if we are lucky):

So we apply Fubini:



$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx = \int_\Omega \frac{\sin y}{y}\, dA$$

$$= \int_0^{\pi/2} \left(\int_0^{y} \frac{\sin y}{y}\, dx \right) dy$$

$$= \int_0^{\pi/2} \left(\frac{\sin y}{y} \underbrace{\int_0^{y} dx}_{= y} \right) dy$$

$$= \int_0^{\pi/2} \frac{\sin y}{y} \times y \cdot dy \qquad \longleftarrow \text{one point trouble.}$$

$$= \int_0^{\pi/2} \sin y\, dy = 1. \qquad \square$$

So changing the order of integration is a good deal (if you are lucky).

———— x ————

Q: But $\int_0^{\alpha} \frac{\sin x}{x}\, dx$, $\alpha > 0$, make sense?

Ans: Well, $x \longmapsto \frac{\sin x}{x}$ is cont. $\forall x \neq 0$.

Now $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

$\therefore$ We assign $\left(\frac{\sin x}{x}\right)(0) = 1$. $\rightsquigarrow$ turns $\frac{\sin x}{x} \in \mathcal{B}([0,\alpha])$.

Finally, it is cont. $\forall x$, possibly except $\underbrace{x = 0}_{\text{measure zero}}$.

$\therefore \frac{\sin x}{x} \in \mathcal{R}([0,\alpha])$, $\forall \alpha > 0$.

need a def.

Q: Let $\Omega \subseteq \mathbb{R}^2$. What is the "area" of Ω? Or, Can we Compute or talk about area of $\boxed{\text{any}}$ $\Omega \subseteq \mathbb{R}^2$?

||ly Volume of $\Omega \subseteq \mathbb{R}^3$?

· # Trouble is: how to define area!! Of course,

$$\text{Area}([a,b] \times [c,d]) \underset{\text{By defn.}}{:=} (b-a) \times (d-c).$$

Remark: One (good way) to get the R.H.S. as follows:

$$\int_{B^2} 1 \underset{\substack{1 \in C(B^2).\ \text{So Fubini}}}{=} \int_a^b \left(\int_c^d 1\, dy\right) dx \qquad \left(B^2 = [b-a] \times [d-c],\ 1 \equiv \text{Constant fn } 1\right).$$

$$= \left(\int_a^b 1\, dx\right) \times \left(\int_c^d 1\, dy\right)$$

$$= (b-a) \times (d-c).$$

We adopt the above as the defn. of "Area".

Def. ① Let $\Omega \subseteq \mathbb{R}^2$ be a bounded subset. Define

$$1_\Omega : \Omega \to \mathbb{R} \text{ by } 1_\Omega(x) = 1 \quad \forall x \in \Omega.$$

Write 1 instead of 1_Ω if Ω is clear from the context.

② We say that Ω has an area if $1_\Omega \in \mathcal{R}(\Omega)$.

And, in this case, we define

$$\text{Area}(\Omega) := A(\Omega) := \int_\Omega 1.$$

Of course, $\int_\Omega 1 = \int_{B^2} \tilde{1}$, where $\tilde{1}|_\Omega \equiv 1$ & $\tilde{1}|_{B^2 \setminus \Omega} \equiv 0$. [By the def. of $\int_\Omega$.]

Note: Set $\Omega := \{(x,y) : 0 \le x, y \le 1,\ x, y \in \mathbb{Q}\} = [0,1]\times[0,1] \cap (\mathbb{Q}\times\mathbb{Q})$.

Then $1 \notin \mathcal{R}(\Omega)$.

Proof: Recall $1 \in \mathcal{R}(\Omega)$ if $\tilde{1} \in \mathcal{R}(B^2)$, $B^2 = [0,1]\times[0,1]$ $(\because B^2 \supseteq \Omega)$.

& in that case $\int_\Omega 1 = \int_{B^2} \tilde{1}$.

But we prove that $1 \notin \mathcal{R}(\Omega)$. ← Same proof as that of the Dirichlet fn. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}^c \end{cases}$.

i.e; we prove that $\tilde{1} \notin \mathcal{R}(B^2)$. Enough. Right?

So, take any partition $P \in \mathcal{P}(B^2)$.

Use that $B^2 \cap (\mathbb{Q}\times\mathbb{Q})$ is dense in B^2 & get:

$$L(\tilde{1}, P) = 0 \quad \& \quad U(\tilde{1}, P) = \sum_{\alpha \in \Lambda(P)} v(B^2_\alpha) = v(B^2). \quad (\neq 0).$$

$$\Rightarrow \underline{\int_{B^2}} \tilde{1} = 0 \neq v(B^2) = \overline{\int_{B^2}} \tilde{1}.$$

$\therefore \tilde{f} \notin \mathcal{R}(B^2) \Rightarrow f \notin \mathcal{R}(\Omega)$.

Remark: From the above, & the fact that $1 \in \mathcal{R}(B^2)$, it is evident that integrability of a fn. also closely related with the nature of Ω.

And of course, nature of Ω should tell us if it has any area!! ← So it is not free.

BUT WAIT: My intution says that $\Omega = (\mathbb{Q}\times\mathbb{Q}) \cap ([0,1]\times[0,1])$ should be of "area = 0". In reality, we are saying the area DNE!!

THIS IS BAD!! — But, the trouble is the fact that Content of $\Omega \neq 0$. AND The measure of $\Omega = 0$!! So, measuring sets (i.e. Area) like Ω is GOOD in Lebesgue integration!

Remark: (1) Given $\Omega \subseteq \mathbb{R}^2$, Ω has an area $\Leftrightarrow$ $\chi_\Omega \in \mathcal{R}(B^2)$ for some box $B^2 \supseteq \Omega$. In this case,

$$\text{Area}(\Omega) = \int_{B^2} \chi_\Omega .$$

[Def. $\chi_\Omega : B^2 \to \{0,1\}$ where $\chi_\Omega(x) = \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. indicator/Characteristic fn.]

Proof: $\tilde{1}_\Omega = \chi_\Omega$.

(2) Ily you may define/deduce, for $\Omega \subseteq \mathbb{R}^n$,

$$\underbrace{\text{Vol}(\Omega)}_{\text{Volume of } \Omega} = \int_{B^n} \chi_\Omega . \qquad B^n \supseteq \Omega.$$

Thm: Let $\Omega \subseteq \mathbb{R}^n$ be bdd. Then Ω has a $\boxed{\text{volume}}$ $\Leftrightarrow$ $\partial\Omega$ is of content zero. ⟸ Lets do it for general $n \geq 2$.

Proof: "$\Leftarrow$" Suppose $\partial\Omega$ has content zero. Set $f := \underbrace{\tilde{1}_\Omega}_{\text{Corresponding to } B^n \supseteq \Omega} = \chi_\Omega$.

Clearly, f is cont. on Ω ($\because f|_\Omega \equiv 1$).

Arguing along the same line of proof of thm in P-41:

$$\mathcal{D} := \{x \in B^n : f \text{ is not cont. at } x\}.$$

Enough to prove that $\mathcal{D}$ is of measure zero.

So: (i) $f|_\Omega$ is cont. on Ω.

(ii) $f|_{B^n \setminus \bar{\Omega}} \equiv 0$ is cont. on $B^n \setminus \bar{\Omega}$.

$\Rightarrow \mathcal{D} \subseteq \partial\Omega$

$\because \partial\Omega$ is of content zero. $\Rightarrow \mathcal{D}$ is of content zero.

$\Rightarrow f \in \mathcal{R}(B^n)$ i.e., $\chi_\Omega \in \mathcal{R}(B^n)$.

i.e., Ω has a volume.

"$\Rightarrow$" Let $B^n \supseteq \Omega$ & $\chi_\Omega = \tilde{1}_\Omega \in \mathcal{R}(B^n)$. Again: $f := \chi_\Omega$.

Claim: $\partial\Omega$ is of content zero.

Fix $\varepsilon > 0$. $\because f \in \mathcal{R}(B^n)$, $\exists\, P \in \mathcal{P}(B^n)$ s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2}. \quad \longleftarrow \text{By integrability of } f.$$

$$\Rightarrow \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \frac{\varepsilon}{2}.$$

$$\tilde{\Lambda} := \{\alpha \in \Lambda(P) : B^n_\alpha \cap \Omega \neq \varphi \ \& \ B^n_\alpha \underset{\text{Not contained in.}}{\not\subseteq} \Omega\}$$

Hint: if $x \in \partial\Omega$, then any open set $O \ni x$, $O \cap \Omega \neq \varphi$ & $O \not\subseteq \Omega$.

The point is: $M_\alpha = 1$, $m_\alpha = 0$ $\forall \alpha \in \tilde{\Lambda}$.

$$\therefore \underbrace{\sum_{\alpha \in \tilde{\Lambda}} (M_\alpha - m_\alpha)\, v(B^n_\alpha)}_{= \sum_{\alpha\in\tilde{\Lambda}} v(B^n_\alpha)} \leq \sum_{\alpha \in \Lambda(P)} (M_\alpha - m_\alpha)\, v(B^n_\alpha) < \frac{\varepsilon}{2}.$$

$$\Rightarrow \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha) < \frac{\varepsilon}{2}. \quad \text{——} \quad (1)$$

On the other hand, ∂B^n_α is of content zero $\forall \alpha \in \Lambda(P)$.
[Known fact.].

$\Rightarrow \bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha$ is of content zero. ($\because$ finite union of c.z. set is of c.z.)

$\therefore \exists$ boxes ~~$\{P^n_\beta : \beta \in \Gamma\}$~~ $\{B^n_1, \dots, B^n_p\}$ $\ni$ $\bigcup_{j=1}^{p} B^n_j \supseteq \bigcup_{\alpha \in \Lambda(P)} \partial B^n_\alpha$

$$\& \sum_{j=1}^{p} v(B^n_j) < \frac{\varepsilon}{2}. \quad \text{——} \quad (2)$$

Claim: $$\partial\Omega \subseteq \underbrace{\left(\bigcup_{\alpha\in\tilde{\Lambda}} B^n_\alpha\right)}_{\text{I}} \cup \underbrace{\left(\bigcup_{j=1}^{p} B^n_j\right)}_{\text{II}}.$$

Content zero

$\Downarrow$

$\partial\Omega$ is of content zero. by (1) & (2).

AND we are done!!

[$\because \sum_{j=1}^{p} v(B^n_j) + \sum_{\alpha\in\tilde{\Lambda}} v(B^n_\alpha) < \varepsilon$.]

Proof of the Claim:

Pick $x \in \partial\Omega \subseteq B^n$.

$\therefore x \in B^n_\alpha$ for some $\alpha \in \Lambda(P)$. $\Rightarrow$ $x \in \text{int}(B^n_\alpha)$ OR $x \in \partial B^n_\alpha$.

If $x \in \text{int}(B^n_\alpha)$: As $x \in \partial\Omega$ ~~& $\text{int}(B^n_\alpha)$ open~~ $\text{int}(B^n_\alpha)$ also contains elements not in Ω [By the def. of bd points.].

$\Rightarrow B^n_\alpha \cap \Omega \neq \varphi$ & $B^n_\alpha \not\subseteq \Omega$.

$\Rightarrow \alpha \in \tilde{\Lambda}$. $\Rightarrow x \in$ (I).

If $x \in \partial B^n_\alpha$: Then $\partial B^n_\alpha \subseteq \dot{\bigcup}_{j=1}^{p} B^n_j$. $\Rightarrow x \in$ II.

$\therefore$ The claim holds good. ▨

Fact: Suppose $\Omega \subseteq \mathbb{R}^n$ is of Content zero & $f \in \mathcal{B}(\Omega)$. Then $f \in \mathcal{R}(\Omega)$ & $\int_\Omega f = 0$. [Already done : P-39.]

Thm: Suppose $\Omega \subseteq \mathbb{R}^n$ bdd. Then :

Ω has an ~~area~~ Volume & ~~Ar~~ $\text{vol}(\Omega) = 0 \Longleftrightarrow \Omega$ is of Content zero.

Proof: "$\Rightarrow$" So, $\int_{B^n} \chi_\Omega = 0$. Let $\varepsilon > 0$.

$$\Downarrow$$

$$0 = \overline{\int_{B^n}} \chi_\Omega = \inf\{ U(\chi_\Omega, P) : P \in \mathcal{P}(B^n)\}$$

$\therefore \exists\, P \in \mathcal{P}(B^n) \ni U(\chi_\Omega, P) < \varepsilon$.

Set $\tilde{\Lambda} := \{\alpha \in \Lambda(P) : B^n_\alpha \cap \Omega \neq \varphi\}$.

Clearly, for $\alpha \in \Lambda(P)$, $\alpha \in \tilde{\Lambda} \Longleftrightarrow M_\alpha = 1$.

Also, $M_\alpha = 0 \ \forall\, \alpha \notin \tilde{\Lambda}$.

So $$\varepsilon > U(\chi_\Omega, P) = \sum_{\alpha \in \Lambda(P)} M_\alpha\, v(B^n_\alpha) = \sum_{\alpha \in \tilde{\Lambda}} M_\alpha\, v(B^n_\alpha)$$

$$= \sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha).$$

Also, Since $\{B^n_\alpha : \alpha \in \Lambda(P)\}$ a partition of $B^n \supseteq \Omega$,

So $\{B^n_\alpha : \alpha \in \tilde{\Lambda}\}$ is a finite ~~partition~~ Cover of Ω &

$\sum_{\alpha \in \tilde{\Lambda}} v(B^n_\alpha) < \varepsilon$. $\Rightarrow \Omega$ is of Content zero.

"$\Leftarrow$" Let Ω is of Content zero. Then the above fact

$\Rightarrow \chi_\Omega \in \mathcal{R}(\Omega)$ & $v(\Omega) = \int \chi_\Omega = 0$. ▨

<u>Note:</u> Let $\Omega_1 \subseteq \Omega$. Let $f \in \mathcal{R}(\Omega)$. We know $f|_{\Omega_1}$ need not be in $\mathcal{R}(\Omega_1)$.

[Simple example: $\Omega = [0,1]\times[0,1]$; $\Omega_1 = (\mathbb{Q}\times\mathbb{Q}) \cap \Omega$. $f \equiv 1$. As we know Ω_1 ~~is not~~ does not have area.].

However, the following is impressive:

<u>Thm:</u> Let $\Omega_1 \subseteq \underbrace{\Omega}_{\text{bdd}} \subseteq \mathbb{R}^n$, and let $\partial\Omega_1$ is of content zero.

Then $f|_{\Omega_1} \in \mathcal{R}(\Omega_1) \quad \forall\, f \in \mathcal{R}(\Omega)$.

<u>Proof:</u> Consider $B^n \supseteq \Omega$. $\therefore\ B^n \supseteq \Omega_1$.

Let $f \in \mathcal{R}(\Omega)$.

$\because\ \partial\Omega_1$ is of content zero, $\chi_{\Omega_1} \in \mathcal{R}(B^n)$.

Observe : $\widetilde{f|_{\Omega_1}} = \tilde{f}\,\chi_{\Omega_1}$ — ⊛

both are : $B^n \to \mathbb{R}$.

The extension of $f|_{\Omega_1} : \Omega_1 \to \mathbb{R}$ to $\widetilde{f|_{\Omega_1}} : B^n \to \mathbb{R}$ by $(\widetilde{f|_{\Omega_1}})|_{\Omega_1} = f|_{\Omega_1}$ & $(\widetilde{f|_{\Omega_1}})|_{B^n \setminus \Omega_1} \equiv 0$.

$\because\ \tilde{f}, \chi_{\Omega_1} \in \mathcal{R}(B^n)$, by product formula,

$$\widetilde{f|_{\Omega_1}} \in \mathcal{R}(B^n).$$

i.e., $f|_{\Omega_1} \in \mathcal{R}(\Omega_1)$.

<u>Remark:</u> By ⊛, $\displaystyle\int_{\Omega_1} f|_{\Omega_1} = \int_{\Omega} f\,\chi_{\Omega_1}$. #