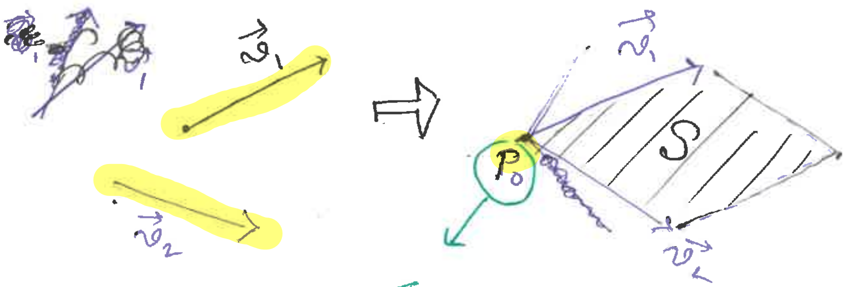
Goal: Compute area of a surface S.

i.e., Area of $S = \operatorname{ran} r$, where r is a parametrization of S.

Remark: ~~Let's first observe~~ If $\vec{OP} = \langle a_1, b_1, c_1 \rangle = \vec{v}_1$ & $\vec{OQ} = \langle a_2, b_2, c_2 \rangle = \vec{v}_2$, then for $\vec{OP_0} = \langle a_0, b_0, c_0 \rangle = \vec{v}_0$ fixed, we define

$$S := \{ \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2 \; : \; 0 < t_1, t_2 < 1 \}.$$

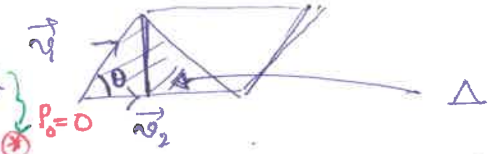
open parallelogram based at (corner) $\vec{v}_0$ (or P_0) bounded by the sides $\vec{v}_1$ & $\vec{v}_2$. (in $\mathbb{R}^3$).



Fact: $\boxed{\text{Area of } S = \| \vec{v}_1 \times \vec{v}_2 \|}$ (*) —— HW.

Hint: Just compute $\vec{v}_1 \times \vec{v}_2$ & then $\| \vec{v}_1 \times \vec{v}_2 \|$. $\Longleftarrow$ Method (1)

Method (2) $\Longrightarrow$



Area $S = 2 \times$ area of Δ

$= 2 \times \left[\frac{1}{2} \times \| v_2 \| \times \| v_1 \| \sin\theta \right]$

[(*) $\to$ Translate P_0 to O; Area will remain unchanged.]

Observe: S is a surface with $S = \operatorname{ran} r$, where

$r : \underbrace{(0,1) \times (0,1)}_{= \mathcal{R}} \longrightarrow \mathbb{R}^3$ given by

Don't worry about (0,1) instead of [0,1]. We want $\mathcal{R}$ to be open.

$$r(t_1, t_2) = \vec{v}_0 + t_1 \vec{v}_1 + t_2 \vec{v}_2.$$

$(t_1, t_2) \in (0,1) \times (0,1)$

$$\boxed{\therefore \text{ Area of } \operatorname{ran} r = \| \vec{v}_1 \times \vec{v}_2 \|.}$$

A basic & simple step:

Recall that $z = ax + by + c$ represts a plane. Moreover, if $(x,y) \in B^2 \leftarrow$ an open box in $\mathbb{R}^2$, then the graph of $f(x,y) = ax+by+c$ given by $\boxed{S = \{(x, y, ax+by+c) : (x,y) \in B^2\}}$ is a surface.

Here $\vec{r}(x,y) = (x, y, ax+by+c) \quad \forall (x,y) \in B^2$.

"plane segment (bounded)".

FACT: $\boxed{\text{Area } S = \sqrt{1+a^2+b^2} \times \text{Area of}(B^2)}$ $\leftarrow$ S as above

Proof: Recall that Area $\tilde{S} = \|\vec{v}_1 \times \vec{v}_2\|$, where

$$\tilde{S} = \{\underbrace{\vec{v}_0}_{\downarrow} + t_1\vec{v}_1 + t_2\vec{v}_2 : (t_1,t_2) \in (0,1)\times(0,1)\}.$$

Recall: As far area is concerned, $\vec{v}_0 = \vec{0}$.

(We represent S as $\tilde{S}$ as above).

Let $B^2 = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$.

Set $\vec{v}_0 = \langle \alpha_1, \alpha_2, a\alpha_1 + b\alpha_2 + c \rangle$.

Also, set $\vec{v}_1 = \langle 0, \beta_2 - \alpha_2, b(\beta_2 - \alpha_2) \rangle$

& $\vec{v}_2 = \langle \beta_1 - \alpha_1, 0, a(\beta_1 - \alpha_1) \rangle$.

Then $\{\vec{v}_0 + t_1\vec{v}_1 + t_2\vec{v}_2 : (t_1,t_2) \in (0,1)\times(0,1)\}$

$= S$.

$\therefore$ Area $S = \|\vec{v}_1 \times \vec{v}_2\|$.

Now $\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 0 & \beta_2-\alpha_2 & b(\beta_2-\alpha_2) \\ \beta_1-\alpha_1 & 0 & a(\beta_1-\alpha_1) \end{vmatrix}$

$= \langle a(\beta_2-\alpha_2)(\beta_1-\alpha_1), b(\beta_2-\alpha_2)(\beta_1-\alpha_1), -(\beta_2-\alpha_2)(\beta_1-\alpha_1) \rangle$

$\therefore \|\vec{v}_1 \times \vec{v}_2\| = \sqrt{a^2+b^2+1} \times (\beta_2-\alpha_2)(\beta_1-\alpha_1)$

$= \sqrt{a^2+b^2+1} \times \text{Area}(B^2)$. $\boxtimes$

Remark: The factor "$\sqrt{1+a^2+b^2}$" is interesting. Indeed, for $z = ax+by+c$ $(= f(x,y))$,

$$a = f_x \quad , \quad b = f_y .$$

$$\therefore \boxed{\sqrt{1+a^2+b^2} = \sqrt{1+f_x^2+f_y^2}} \; !!$$

$\square$

With the above area formula, we can now talk about the idea of defining / obtaining. Surface area :

Let S be the Surface parametrized by $r : R \to \mathbb{R}^3$, ~~where~~

i.e. $r(u,v) \in \mathbb{R}^3$, $\quad (u,v) \in R := [\alpha_1,\beta_1) \times (\alpha_2,\beta_2)$

& $\operatorname{ran} r = S$.

for Simplicity, lets take $R = B^2$.

~~Consider a partition of B^2 : Say P~~
~~So $P = \{ B^2_\alpha : \alpha \in \Lambda(P) \}$.~~

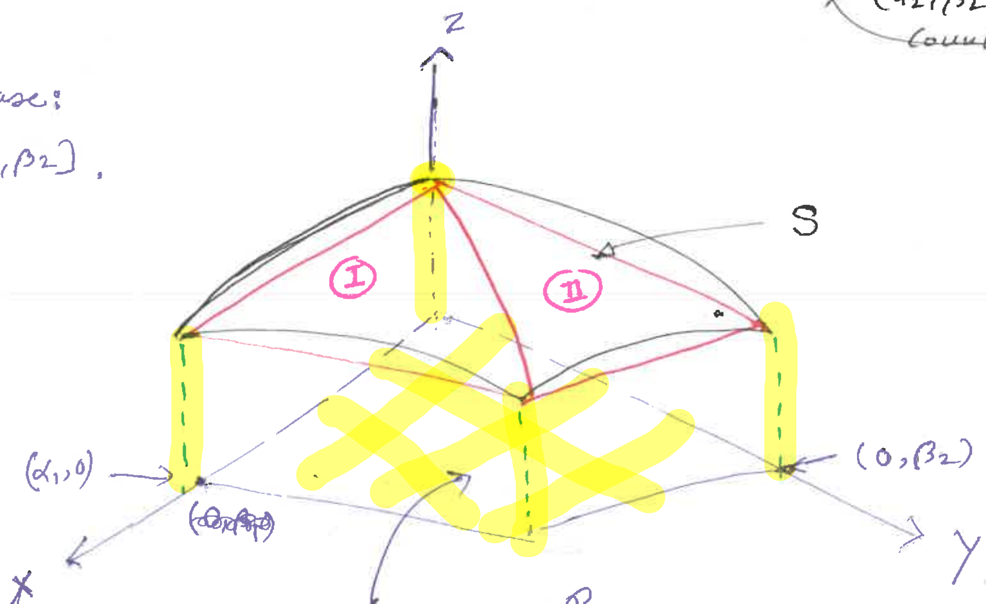
~~don't worry about "one" Side open (α_1,β_1) & (α_2,β_2). It doesn't count anything.~~

Pause

So

A Simplest Case:

$R = [0,\alpha_1] \times [0,\beta_2]$.



The bottom $= R$

$R = [0,\alpha_1] \times [0,\beta_2]$.

Then (BAD) area of $S \approx \underbrace{\triangle_{\mathrm{I}} + \triangle_{\mathrm{II}}}$

Sum of two polygonal surfaces.

& we know the area of this by our previous computation.

polygonal approximation.

THE IDEA IS to take this consideration & proceed to limit approach.

Known route!!

So, in the B^2-setting, pick a partition of B^2, say P.

$\therefore \; B^2 = \bigcup_{\alpha \in \Lambda(P)} B^2_\alpha$. ← May be, at this point, consider B^2 to be closed.

$\forall\, \alpha \in \Lambda(P)$, pick (arbitrary) $x_\alpha \, (= (u_\alpha, v_\alpha)) \in \text{int}(B^2_\alpha)$

Consider the tangent plane to the surface S at $r(x_\alpha)$: i.e, the linear / tangent plane approximation as follows:

$r^{(\alpha)} : B^2_\alpha \longrightarrow \mathbb{R}^3$ defined by:

$$r^{(\alpha)}(x) = \Big(\underbrace{(Dr)(x_\alpha)}_{\text{Total derivative}} \Big) (x - x_\alpha) + r(x_\alpha).$$

$$= [r_u \;\; r_v](x_\alpha) = [r_u(x_\alpha) \;\; r_v(x_\alpha)].$$

plane or parallelogram

$\therefore$ We have $z = r^{(\alpha)}(x) = r_u(x_\alpha)\, x + r_v(x_\alpha)\, y + c$ (for some $c \in \mathbb{R}$)

& area of the above parallelogram (By FACT in p-44)

$$= \| r_u(x_\alpha) \times r_v(x_\alpha) \| \, (\text{Area } B^2_\alpha)$$

$\therefore$ Area of S under the partition P

$$= \sum_{\alpha \in \Lambda(P)} \| r_u(x_\alpha) \times r_v(x_\alpha) \| \, \text{Vol}(B^2_\alpha).$$

This is the Riemann sum of the fn. $x \longmapsto \| r_u(x) \times r_v(x) \|$.

This is integrable as r is a C^1-fn.

So, $\lim_{\|P\| \to 0} (\text{RHS}) = \text{Area of } S$.

We KNOW how to compute this.

$\Longrightarrow$ We may define: Area of $S := \int_R \| r_u \times r_v \| \, dA$.

The Riemann integration in 2-variables w.r.to "area".

Def:

$$\text{Area}(S) := \int_{R} \|r_u \times r_v\| \, dA .$$

where $R \subseteq \mathbb{R}^2$ is a region & $S = \operatorname{ran} r$ for some parametrization $r : R \longrightarrow \mathbb{R}^3$ of S.

Fact: Suppose $f \in C^1(R)$, $R \subseteq \mathbb{R}^2$ a region. Recall $r : R \longrightarrow \mathbb{R}^3$, defined by $r(u,v) = (u, v, f(u,v))$ is a parametrization of graph(f). Also, recall that (see Page 33):

$$r_u \times r_v = (-f_u, -f_v, 1).$$

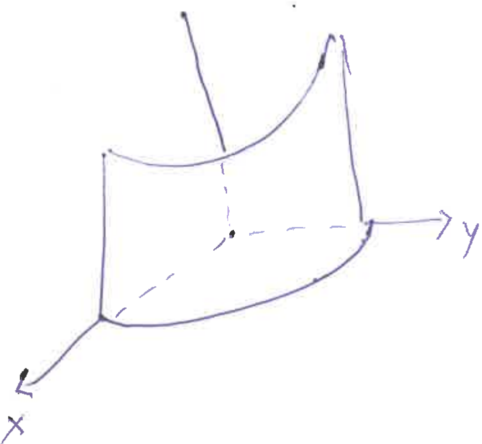
$$\Rightarrow \|r_u \times r_v\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2}.$$

$$\therefore \text{Area}(\text{graph}(f)) = \int_{R} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, \underbrace{dx\,dy}_{= dA}.$$

eg: Consider truncated cylinder: $r : R \longrightarrow \mathbb{R}^3$, where

$$R = \{(x,y) : 0 \leq x \leq \pi/2, \ 0 \leq y \leq 1\}$$

& $r(x,y) = (\cos x, \sin x, y)$.



Here $r_x = (-\sin x, \cos x, 0)$

$r_y = (0, 0, 1)$

$$\therefore r_x \times r_y = \begin{vmatrix} i & j & k \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos x, \sin x, 0)$$

$$\therefore \|r_x \times r_y\| = \sqrt{\cos^2 x + \sin^2 x} = 1.$$

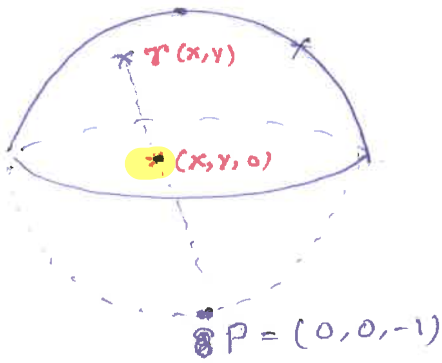
$$\therefore \text{Area} = \int_{R} 1 \, ds = \text{Area}(R) = \pi/2 .$$

eg: (Area of a hemisphere)

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1,\ z>0\}.$$

The upper hemisphere.

Apply stereographic projection:



i.e. $\mathcal{R} = B_1((0,0)) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

$r(x,y)$ = the point of intersection of the upper hemisphere & the line joining P & $(x,y,0)$.

Then $$r(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$$ $\forall (x,y) \in \mathcal{R}$. — HW.

NOT good for computation!!

Here, we apply spherical coordinate (for ease of computation):

We define $\mathcal{R} = \{(u,v) \in \mathbb{R}^2 : 0<u<\pi,\ 0<v<\pi\}$.

& $r(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$. $\forall (u,v) \in \mathcal{R}$.

Then $\| r_u \times r_v \| \overset{?\,(HW).}{=} |\sin u| = \sin u.$

HW: Check this from Surface of revolution point of view.

$\therefore$ Area(S) = $\displaystyle\int_{\mathcal{R}} \sin u \underbrace{du\, dv}_{dA}$

The upper hemisphere

$$= \int_0^{\pi}\int_0^{\pi} \sin u\, du\, dv$$

$$= \pi \times \int_0^{\pi} \sin u\, du$$

$$= \pi \times [-\cos u]_0^{\pi} = \pi \times 2$$

$$= 2\pi.$$

Surface integrals of scalar fields:

Let S be a surface in $\mathbb{R}^3$, $S \subset \underset{\text{open}}{\underline{O_3}} \subseteq \mathbb{R}^3$, & let $f \in Cont(O_3)$ or $\mathcal{R}(O_3)$. Then, as in the area computation, (see the bottom of Page 46).

We can prove:

$$\lim_{\|P\| \to 0} \sum_{\alpha \in \Lambda(P)} f(r(x_\alpha)) \, \|r_u(x_\alpha) \times r_v(x_\alpha)\| \, vol(B^2_\alpha).$$

$$= \int_{\mathcal{R}} f \circ r \, \|r_u \times r_v\| \, dA.$$

Where $r: \mathcal{R} \to \mathbb{R}^3$ is a parametrization of the surface S.

Hence, we define:

def: Given a surface S, $f \in Cont(O_3)$, where $O_3 \supseteq S$ is an open subset of $\mathbb{R}^3$, the surface integral of f over the surface S is defined by:

$$\int_S f \, ds := \int_{\mathcal{R}} f \circ r \, \|r_u \times r_v\| \, dA$$

Q: independant of the choice of r?

where $r: \mathcal{R} \to \mathbb{R}^3$ is a parametrization of S.

Remark: "~~The RHS of the above~~": Suppose a curved metal plate / thin film lies along the surface S with density fn $f: S \to \mathbb{R}$. Then ~~the RHS of the above definition~~ ~~is~~

$$\int_S f \, ds = \text{mass of the metal plate.}!!$$

Independence of parametrizations: (Sketch).

Suppose $\gamma : R \longrightarrow \mathbb{R}^3$ & $\tilde{\gamma} : \tilde{R} \longrightarrow \mathbb{R}^3$ be two parametrizations of S.

$\therefore$ $S = \operatorname{ran} \gamma = \operatorname{ran} \tilde{\gamma}$. So, we have the following.

$$\begin{tikzcd} & S & \\ R \arrow[ur, "\gamma"] \arrow[rr, dashed] & & \tilde{R} \arrow[ul, "\tilde{\gamma}"'] \end{tikzcd}$$

? Get φ s.t. #

i.e: $\gamma(u,v) = \tilde{\gamma}(\varphi(u,v))$.

For $(u,v) \in R$, define $\varphi(u,v) = (\tilde{u}, \tilde{v}) \in \tilde{R}$ as

$\varphi = \tilde{\gamma}^{-1} \circ \gamma$ $\longleftarrow$ Remember, γ & $\tilde{\gamma}$ are one-to-one maps. & $\operatorname{ran} \gamma = \operatorname{ran} \tilde{\gamma}$.

$\therefore$ $\varphi : R \longrightarrow \tilde{R}$ is a bijection!!

P.T. φ is C^1-fn. & $(J\varphi)(u,v)$ invertible $\forall (u,v) \in R$.

Then apply the change of variables theorem/ formula along with chain rule $\Longrightarrow$

$$\int_R f(\gamma(u,v)) \, \|\gamma_u \times \gamma_v\| = \int_{\tilde{R}} f(\tilde{\gamma}(\tilde{u},\tilde{v})) \, \|\tilde{\gamma}_{\tilde{u}} \times \tilde{\gamma}_{\tilde{v}}\|$$

$\boxed{///}$