<u>Thm</u> (Rat'l canonical form): Let $V$ be a $n$-dimensional vector space over a field $k$ and $\varphi: V \to V$ be a $k$-linear map. Then $\exists$ a basis $B$ of $V$ s.t. that the matrix of $\varphi$ w.r.t $B$ is of the form

$$R_\varphi = \begin{bmatrix} R_{a_1} & & & O \\ & R_{a_2} & & \\ & & \ddots & \\ O & & & R_{a_m} \end{bmatrix}$$

where for a monic poly $a(x) = x^\ell + b_{\ell-1} x^{\ell-1} + \cdots + b_0$ of

deg $\ell$, $R_a$ is the $\ell \times \ell$ matrix $\begin{bmatrix} 0 & 0 & & -b_0 \\ 1 & 0 & \ddots & -b_1 \\ & 1 & & \vdots \\ & & 0 & \vdots \\ & & 1 & -b_{\ell-1} \end{bmatrix}$

$a_1(x), \ldots, a_m(x) \in k[x]$ are nonconstant monic poly s.t. $a_1 | a_2 | \cdots | a_m$.

Equivalently, $A \in M_{n \times n}(k)$ then $\exists$ a nonsingular matrix $P$ s.t.
$$P^{-1} A P = R_\varphi \quad \text{for some} \quad a_1, \ldots, a_m \in k[x] \text{ nonconst, monic poly}$$
with $a_1 | a_2 | \cdots | a_m$.

<u>Thm</u>: (Jordan form) Let $V$ be a $n$-dim'l vs over $\mathbb{C}$ (or an alg closed field). Let $\varphi: V \to V$ be a $\mathbb{C}$-linear map. Then there exist a basis of $B$ of $V$ s.t. The matrix of $\varphi$ w.r.t. $B$ is of the form.

$$J_\varphi = \begin{bmatrix} J_{\lambda_1}^{n_{11}} & & & & & & & \\ & J_{\lambda_1}^{n_{12}} & & & & & O & \\ & & \ddots & J_{\lambda_1}^{n_{1s_1}} & & & & \\ & & & & J_{\lambda_2}^{n_{21}} & & & \\ & & & & & \ddots J_{\lambda_2}^{n_{2s_2}} & & \\ & O & & & & & J_{\lambda_m}^{n_{m1}} & \\ & & & & & & \ddots J_{\lambda_m}^{n_{m s_m}} & \end{bmatrix}$$

where $\lambda_i \in \mathbb{C}$
$1 \le i \le m$
$r_{ij}$ are positive integers.

$$J_\lambda^n = \begin{bmatrix} \lambda & 1 & & O \\ & \ddots & \ddots & \\ & & & 1 \\ O & & & \lambda \end{bmatrix} \quad \text{is a } r \times r \text{ matrix } \lambda \in \mathbb{C}.$$

$\longleftarrow \quad \mathbb{C}[x] / (x-\lambda)^r$

Equivalently, $A \in M_{n \times n}(\mathbb{C})$ then $A$ is similar to $J_\varphi$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ & $r_{ij}$ positive integers.

① What is the minimal poly of $\varphi$? Char poly of $\varphi$?

② What are eigen values of $\varphi$?

③ Note that $V \cong k[X]/(a_1) \oplus \cdots \oplus k[X]/(a_m)$

$a_1 | a_2 | \cdots | a_m$    $a_i \in k[X]$   non const monic poly.

minimal poly of $\varphi$ is the least deg monic poly $m_\varphi(X) \in k[X]$, s.t. $m_\varphi(\varphi)$ is the zero endo of $V$.    i.e. $m_\varphi(X) \cdot v = 0 \ \forall v \in V$

$\textcolor{red}{m_A(A) = 0}$

i.e. $(m_\varphi(X)) = Ann(V)$   $\textcolor{red}{r(\bar{1}, \bar{1}, \bar{1}, \ldots, \bar{1}) = \underset{\sim}{0}}$

But $Ann(V) = (a_m(X))$. Hence $a_m(X)$ is the minimal poly of $\varphi$. $\textcolor{red}{\nwarrow \text{as a } k[X]\text{-module}}$

char poly of $R_a = \det(XI - R_a)$

$$= \begin{pmatrix} X & 0 & & & b_0 \\ -1 & X & & \textcolor{red}{O} & \vdots \\ & -1 & \ddots & & \vdots \\ \textcolor{red}{O} & & -1 & X & b_{n-2} \\ & & & -1 & X+b_{n-1} \end{pmatrix}$$

$$= (X + b_{n-1}) X^{n-1} - b_{n-2} \det \begin{pmatrix} X & 0 & & 1 \\ -1 & \ddots & & O \\ & \ddots & X & 0 \\ O & & -1 & X & 0 \\ & & & 0 & -1 \end{pmatrix}$$

$$+ b_{n-3} \det \begin{pmatrix} X & 0 & & O \\ -1 & & & \\ & -1 & X & -1 & X \\ O & & & 0 & -1 \end{pmatrix} \cdots$$

$$= X^n + b_{n-1} X^{n-1} + b_{n-2} X^{n-2} + \cdots + b_0$$
$$= a(X)$$

So char of $R_\varphi$   $ch_\varphi(X) = a_1(X) a_2(X) \cdots a_m(X)$

<span style="color:red">* Minimal poly of $R_a$ is $a(X)$</span>

<span style="color:red">* minimal poly, char poly $J_\lambda^r$ is $(X-\lambda)^r$</span>

Eigen value of $\varphi$ are $\lambda_1, \ldots, \lambda_m$ of Jordan form.

Example: $V = \mathbb{C}^3$

$\quad\quad A: \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \quad\quad\quad\quad$ $V$ is a $\mathbb{C}[x]$-mod

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$V \overset{\theta}{\cong} \mathbb{C}[x]/(a_1) \oplus \mathbb{C}[x]/(a_2)$ or $\boxed{\mathbb{C}[x]/(a_1(x))}$

or $\boxed{\mathbb{C}[x]/(a_1) \oplus \mathbb{C}[x]/(a_2) \oplus \mathbb{C}[x]/(a_3)}$ $\cancel{X}$

$\quad\quad\quad\quad\quad \Downarrow \quad a_1 = a_2 = a_3$

$\text{if } m_A(x) = (x - \lambda) \implies A = \lambda I \text{ (Contra!)}$

$$ch_A(x) = (x-2)[(x-1)(x-4)-6] = \det \begin{pmatrix} x-1 & -2 & 0 \\ -3 & x-4 & 0 \\ 0 & 0 & x-2 \end{pmatrix}$$

$\quad\quad = (x-2)(x^2 - 5x + 4 - 6)$

$\quad\quad = (x-2)(x^2 - 5x - 2) = x^3 - 5x^2 - 2x$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad -2x^2 + 10x + 4$

$\quad\quad \dfrac{5 \pm \sqrt{33}}{2} \quad = x^3 - 7x^2 + 8x + 4$

$m_A(x) = ch_A(x)$

$\implies a_3(x) = ch_A(x) = m_A(x)$

$$\begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & -8 \\ 0 & 1 & 7 \end{pmatrix} \text{ rat'l form } A$$

Jordan form

$$\begin{pmatrix} 2 & & 0 \\ & \dfrac{5+\sqrt{33}}{2} & \\ 0 & & \dfrac{5-\sqrt{33}}{2} \end{pmatrix}$$