Lecture 5: Reduced ring, integral domains; quotient rings.

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Recall: $a \in R$ is a zero divisor if $ab = 0$ for some $b \underset{\neq 0}{\in} R$

$a \in R$ is nilpotent if $a^n = 0$ for some $n \geq 1$.

Def^n: Let R be a nonzero comm ring with unity. It is said to be **reduced** if it doesnot contain nonzero nilpotents.

Def^n: A ring R is said to be an **integral domain** if it is a nonzero comm ring with unity and it does not contain any nonzero zero divisors. i.e. every nonzero element of R is a nonzerodivisor.

⊛ Let R be a comm ring with unity, $a \in R$ be a nonzero divisor then

$ab = ac \Rightarrow b = c$ (concellation property holds)

In particular if R is an integral domain then

$ab = ac \Rightarrow a = 0$ or $b = c$.

Pf: $ab = ac \Rightarrow a(b-c) = 0$

$\Rightarrow (b-c) = 0$ ($\because$ a is a nonzerodivisor)

$\Rightarrow b = c$

If R is an int domain

$ab = ac$ & $a \neq 0 \Rightarrow b = c$. ▨

Examples: 1) $\mathbb{Z} \leftarrow$ Integral domain

⊛ Integral domains are reduced rings.

2) $\mathbb{Z}/n\mathbb{Z} \leftarrow$ • For n prime $\mathbb{Z}/n$ is an int. domain

• n not a prime then $\mathbb{Z}/n\mathbb{Z}$ is not an int domain.

• (Exc) $\mathbb{Z}/n\mathbb{Z}$ is reduced iff $p^2 \nmid n$ for any prime p.

3) $\mathbb{Z}[X]$

4) $\mathbb{Q}[X]$ ← Integral domains.

5) valuation ring ← Int domains

⊛ R an int domain and $R_1 \subseteq R$ subring. Then R_1 is an int domain.

6) $\mathbb{Z}^2 \leftarrow$ $(1,0)$ is a zerodivisor ($\because (1,0)(0,1) = (0,0)$)

$(a,b)^n = (a^n, b^n) \Rightarrow \{(a,b) \neq 0 \Rightarrow (a,b)^n \neq 0\}$

7) $\mathcal{C}([0,1]) \leftarrow$ $\mathbb{R}$-valued cont. functions on $[0,1]$.

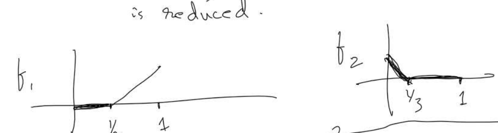
• Let $f \in \mathcal{C}([0,1])$ $f^n \equiv 0$

$\Rightarrow (f(x))^n = 0 \quad \forall x$

$\Rightarrow f(x) = 0 \quad \forall x$

$\Rightarrow f \equiv 0$

$\Rightarrow \mathcal{C}([0,1])$ is reduced.

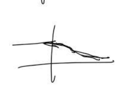
• 

$f_1 f_2 \equiv 0$

$f(x) - 1$

$f(x) = \begin{cases} e^{-x^2} & x > 0 \\ 1 & x \leq 0 \end{cases}$

$f'(x) = 2xe^{-x^2}$



Def^{n}/Prop: Let R be a comm ring with unity and I be an ideal of R. Then R/I with usual addition $((a+I)\oplus(b+I) = (a+b)+I)$ and the multiplication given by

$$(a+I)\odot(b+I) := (ab+I)$$

makes $(R/I, \oplus, \odot)$ into a ring. Moreover the map

$$q: R \longrightarrow R/I \qquad a \longmapsto a+I$$

is a ring surjective homomorphism.

The ring R/I is called the quotient of R by the ideal I and $q: R \to R/I$ is called the quotient map.

Pf:

Claim: $(a+I)\odot(b+I) := (ab+I)$ is well-defined.

Pf: $a+I = a'+I$ & $b+I = b'+I$

$\Rightarrow a-a' \in I$ & $b-b' \in I$

$(a-a')b + a'(b-b') \in I$

$\Rightarrow ab - \cancel{a'b} + \cancel{a'b} - a'b' \in I$

$\Rightarrow ab - a'b' \in I$

$\Rightarrow ab + I = a'b' + I$

Assoc (Easy exc.)

Distributive law: $(a+I)\odot(b+I \oplus c+I)$

$= (a+I)\odot((b+c)+I)$

$= a(b+c)+I$

$= (ab+ac)+I$ (by Dist axiom in R)

$= (ab+I)\oplus(ac+I)$

$= [(a+I)\odot(b+I)]\oplus[(a+I)\odot(c+I)]$

Check $1\oplus I$ is the multiplicative identity.

III'ly check other axioms.

$$q: R \longrightarrow R/I \qquad a \longmapsto a+I \ (= \bar{a} \text{ notation!})$$

q is a group homo (Group theory)

$q(ab) = ab+I$

$= (a+I)\odot(b+I) = q(a)\odot q(b)$ ▨

Example 1) $\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$, $n \longmapsto [n]$ ← ideal is $I = n\mathbb{Z}$, $R = \mathbb{Z}$

2) $q: \mathbb{Q}[x] \longrightarrow \dfrac{\mathbb{Q}[x]}{(x^2-2)\mathbb{Q}[x]}$ ← int domain $\cong \mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$

3) $q: \mathbb{R}[x] \longrightarrow \dfrac{\mathbb{R}[x]}{(x^2-2)\mathbb{R}[x]}$ ← $\overline{x-\sqrt{2}}$, $\overline{x+\sqrt{2}}$

4) $q: \mathbb{Z}[x] \longrightarrow \dfrac{\mathbb{Z}[x]}{(x^2-2)\mathbb{Z}[x]} \cong \mathbb{Z}[\sqrt{2}]$

5) $\dfrac{\mathbb{Z}[x]}{(2, x^2-2)\mathbb{Z}[x]} \cong \dfrac{\mathbb{Z}/2\mathbb{Z}[Y]}{Y^2\, \mathbb{Z}/2\mathbb{Z}[Y]}$ } Next class

6) $\dfrac{\mathbb{Z}[x]}{(5, x^2-2)\mathbb{Z}[x]} \cong \dfrac{\mathbb{Z}/5\mathbb{Z}[x]}{(x^2-2)}$ ← Int domain

$\varphi: \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$

$f(x) \longmapsto f(\sqrt{2})$

φ is a ring homo (Check!)

$\ker(\varphi) \ni x^2-2 = g(x)$

$g(\sqrt{2}) = 0$

$(x^2-2)\mathbb{Q}[x] \subseteq \ker(\varphi)$

Let $f(x) \in \mathbb{Q}[x]$ & $f(x) \in \ker(\varphi)$

$f(x) = g(x)q(x) + r(x)$ ← Remainder thm

$\deg(r(x)) \leq 1$

$0 = f(\sqrt{2}) \Rightarrow r(\sqrt{2}) = 0$
$\Rightarrow r(x) = 0$ } $\Rightarrow f(x) \in (x^2-2)\mathbb{Q}[x]$

$\Rightarrow \ker(\varphi) = (x^2-2)\mathbb{Q}[x]$

$\xRightarrow{1^{st}\text{ isom}} \mathbb{Q}[x]/(x^2-2)\mathbb{Q}[x] \cong \mathbb{Q}[\sqrt{2}]$

⊛ Every ideal is a kernel of a ring homo. This follows from the fact that $\ker(q) = I$.

Pf: Let $a \in I$

$q(a) = a + I = 0 + I$

$\Rightarrow a \in \ker(q)$

$a \in \ker(q) \Rightarrow q(a) = 0 + I$

$\Rightarrow a + I = 0 + I$

$\Rightarrow a \in I$.

⊛ $\dfrac{\mathbb{R}[x]}{(x^2-2)\mathbb{R}[x]}$ is not an integral domain ($(x^2-2)\mathbb{R}[x] = I$ say)

$x - \sqrt{2} \in \mathbb{R}[x]$

$q(x-\sqrt{2}) = \overline{x-\sqrt{2}} = (x-\sqrt{2}) + I \neq 0 + I$

$q(x+\sqrt{2}) = \overline{x+\sqrt{2}} = (x+\sqrt{2}) + I \neq 0 + I$

$((x-\sqrt{2}) + I) \cdot ((x+\sqrt{2}) + I) = ((x-\sqrt{2})(x+\sqrt{2}) + I)$
$= x^2 - 2 + I = 0 + I$