Curves and Surfaces (in $\mathbb{R}^n$).

$I \subseteq \mathbb{R}^n \xrightarrow{\text{Cont.}} \mathbb{R}^n$

$O_2 \xrightarrow{\text{Cont.}} \mathbb{R}^n$

$\subseteq \mathbb{R}^2$, open & ∂O_2 is of Content zero. i.e: O_2 has area.

Roughly: Curve ⟺ 1 dimensional object.

Surface ⟺ 2 dim. object.

Goal

Differentiate, integrate, & then also relate them.

Notation: ① $I = [a,b]$, $a<b$. / $(a,b]$, $(-\infty, a)$, $(-\infty,\infty)$. ② $C^1 =$ ~~Smooth~~ Continuously diff. fn's.

Def: 1) A parametrized Curve/path is a Continuous fn.

$\gamma : I \longrightarrow \mathbb{R}^n$.

2) Given a parametrized Curve, $\underbrace{\{\gamma(t) : t \in I\}}_{\text{just a set / subset of } \mathbb{R}^n}$ is called the/a path.

~~given as~~: the range of a parametrized Curve.

3) A parametrized γ Curve is ~~Smooth~~ C^1-Curve if γ is a C^1- fn.

[Recall: ① $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$.

Here $\gamma(t)$ is C^1 means $\gamma_i : I \to \mathbb{R}$ is ~~Smooth~~ C^1, $\forall\ 1 \le i \le n$.

② If $I = [a,b]$, then $\gamma'(a)$ or $\gamma'(b)$ will be defined as one-sided limits [or, γ has a C^1- extension to an open set $O \supseteq I$]]

4) A C^1- Curve $\gamma : I \longrightarrow \mathbb{R}^n$ is said to be Smooth if $\underbrace{\gamma'(t) \neq 0}_{\text{you dont want the direction to change rapidly.}}$ $\forall t \in I$.

eg: ① $\gamma : [0,1] \longrightarrow \mathbb{R}^2$, defined by

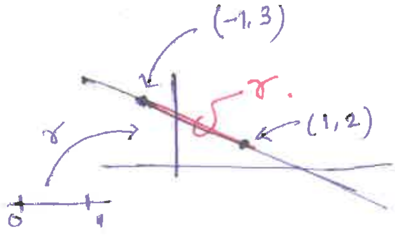
$\gamma(t) = (1,2) + t(-2,1)$ $t \in [0,1]$.

$= (1-2t,\ 2+t)$.

$\gamma'(t) = (-2,1)$
$\Vert$
Smooth curve.

A line: $\left.\begin{array}{l} x = 1-2t \\ y = 2+t \end{array}\right\}$ parametric form. $\Rightarrow \frac{1-x}{2} = y-2$

$\Rightarrow x + 2y = 5$.



(2) $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$ defined by $\gamma(t) = (r\cos t, r\sin t)$.

Here $\gamma'(t) = (-r\sin t, r\cos t)$. $\Rightarrow \gamma'(t) \neq 0 \quad \forall\, t$.

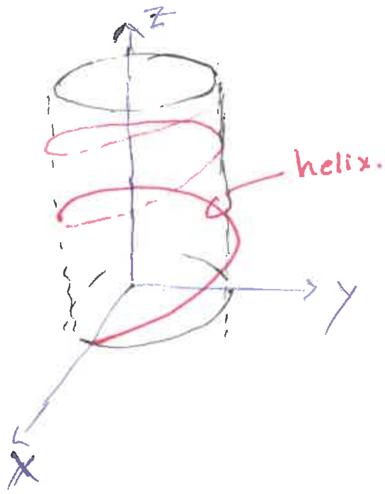
$\therefore$ γ represents a Circle oriented Counterclockwise.

(3) $\gamma : [0, \overset{= 4\pi/n\pi}{a}] \longrightarrow \mathbb{R}^3$, $\gamma(t) := (r\cos t, r\sin t, ct)$, $r > 0, c \neq 0$.

If $t = 0$: $\gamma(0) = (r, 0, 0)$.

The ~~Curve~~ path is known as Helix.

Smooth Curve.



(4) $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2$ defined by

$\gamma(t) = (|t|, t)$ is not a

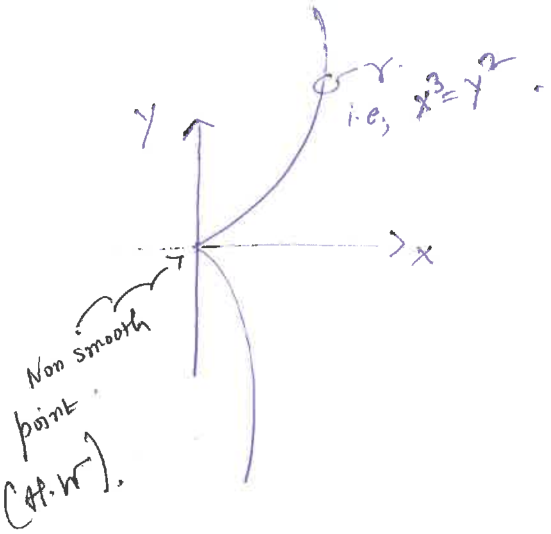
C^1 - Curve ($\Rightarrow$ non-smooth).

Also, $\gamma(t) = (0, t^2)$ is C^1 - curve but not smooth:

$\gamma'(t) = (0, 2t) \Rightarrow \gamma'(0) = 0$.

(5) $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2$, $\gamma(t) = (t^2, t^3)$ $\longleftarrow$ ~~Cusps~~ Cuspidal Cubic.

C^1 but non-smooth.



Here $x = t^2$, $y = t^3$.

$\Rightarrow \underbrace{x^3 = y^2}_{\text{Cuspidal Cubic}}$.

(6) Recall : path is the range / trace of a parametrized curve.

Then for $\gamma(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$, the corresponding

path $= \{ \gamma(t) : 0 \le t \le 2\pi \}$

$= \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2 \}$

twice rotation.

$= \{ \tilde{\gamma}(t) : 0 \le t \le 2\pi \}$, where $\tilde{\gamma}(t) = (r\sin 2t, r\cos 2t)$
$t \in [0, 2\pi]$.

$\therefore$ γ & $\tilde{\gamma}$ are different parametrizations of the path $C_r(0)$.

(7) Given a smooth fn $f : I \to \mathbb{R}$, define $\gamma(t) = (t, f(t))$, $t \in I$.

$\therefore$ γ is a parametrization of the graph of f.

parametrizations of graphs.

(8) $\gamma(t) = (\underbrace{e^t \cos t}_{=x}, \underbrace{e^t \sin t}_{=y})$ $t \in I$ ← Any interval.

$\therefore$ $x^2 + y^2 = e^{2t}$ (NOT a Good representation)

But $\sqrt{x^2+y^2} = e^t$.

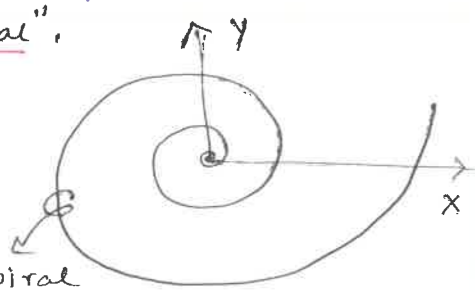
$\therefore$ In polar coordinate: $r = e^t$. Also $\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \cos t$.

$\Downarrow$

$\theta = t$ $(t = \theta + 2m\pi)$

$\therefore$ $\underbrace{r = e^{\theta}}$ ← polar representation of γ.

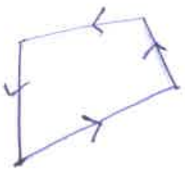
Called "Logarithmic spiral".



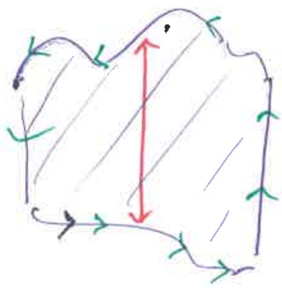
Spiral whose radius grows exponentially.

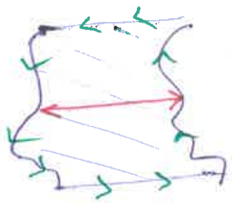
Def: A parametrized curve $\gamma : I \longrightarrow \mathbb{R}^n$ is called piecewise smooth if $\exists$ a partition of $I = [a,b]$, say, $a = x_0 < x_1 < \cdots < x_n = b$, s.t. $\gamma\Big|_{[x_{i-1}, x_i]} : [x_{i-1}, x_i] \longrightarrow \mathbb{R}^n$ is a smooth parametrized curve, $i = 1, \ldots, n$.

eg:



γ boundary of Type I & Type II:





equivalent curves:

Consider $t \xmapsto{\gamma} (r\cos t, r\sin t)$ $\quad t \in [0, 2\pi]$.

$t \xmapsto{\tilde{\gamma}} (r\cos 2t, r\sin 2t)$ $\quad t \in [0, \pi]$.

Clearly, $\gamma(2t) = \tilde{\gamma}(t)$.

or $\gamma(\varphi(t)) = \tilde{\gamma}(t)$, where $\varphi(t) = 2t$.

$\tilde{\gamma}$ is a reparametrization of γ. $\quad$ a parametrization.

\# But often, we need φ to be a "good" parametrization!! $\Rightarrow$

Def: Two parametrized curves $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ & $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \longrightarrow \mathbb{R}^n$ are said to be equivalent if $\exists$ strictly increasing parametrized curve $\varphi : [\tilde{a}, \tilde{b}] \longrightarrow [a,b]$, onto & differentiable (often C^1)

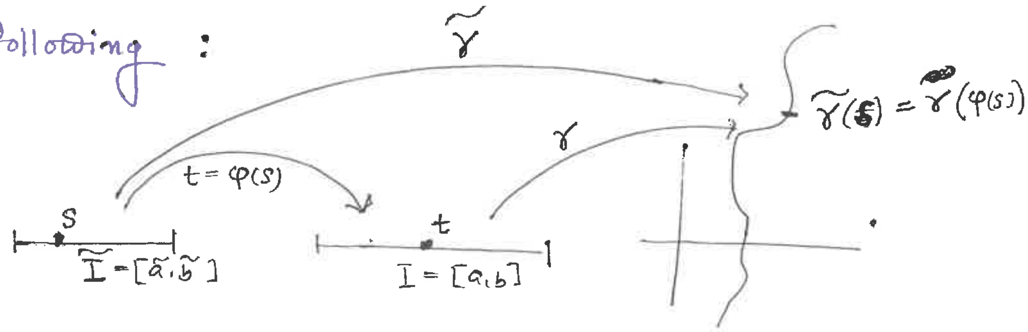
s.t. $\tilde{\gamma} = \gamma \circ \varphi$.

A reparametrization. $\quad$ a "smooth"/C^1 parametrization.

So, we have the following :

$$\begin{tikzcd} & \mathbb{R}^n \\ \tilde{I} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\varphi"'] & I \arrow[u, "\gamma"'] \arrow[ul, phantom, "\#"] \end{tikzcd}$$

i.e.



If $\gamma, \tilde{\gamma}$ are C^1, or smooth, then we also impose the same to φ.

$\therefore$ φ is change of time. Here $\tilde{\gamma}$, at time s, is at $\tilde{\gamma}(s)$, where γ arrives there at time $t=\varphi(s)$!!

\# From now on: Curve $\longleftrightarrow$ parametrized Curve.

Def: ~~Given~~ Let $\gamma : I \longrightarrow \mathbb{R}^n$ be a curve.

(1) $\|\gamma'(t)\| :=$ "speed" of γ at time $t \in I$.

(2) $\displaystyle\int_{t_1}^{t_2} \|\gamma'(t)\|\, dt :=$ Arc length of γ between times t_1 & t_2. $(t_1<t_2)$.

Question:
\# Speed is somewhat clear (practical point of view), as $\|\gamma'(t)\|$ is the magnitude of the velocity $\gamma'(t)$. But what is the interpretation of arc length ?? — WAIT.

Remark: 1) For $x=(x_1,\dots,x_n) \in \mathbb{R}^n$,

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} = d(x, \underline{0}) \quad \underline{0} = (0,\dots,0).$$

2) Let $\gamma(t) = (x_1(t), \dots, x_n(t))$.

$\therefore$ $\gamma'(t) = (x_1'(t), \dots, x_n'(t))$.

$$\Rightarrow \|\gamma'(t)\| = \sqrt{\sum_{i=1}^{n} (x_i'(t))^2}.$$

$\because$ γ is C^1 $\Rightarrow$ ~~each~~ x_i, $1\le i\le n$, is C^1

$\Rightarrow t \mapsto \|\gamma'(t)\| \in C(I)$.

$\Rightarrow$ Arc length is well defined.

But, there is another way (perhaps more natural) to introduce arc length of curves. We do it by ~~linear~~ polygonal approximation

Consider a path $\gamma : [a,b] \longrightarrow \mathbb{R}^n$.

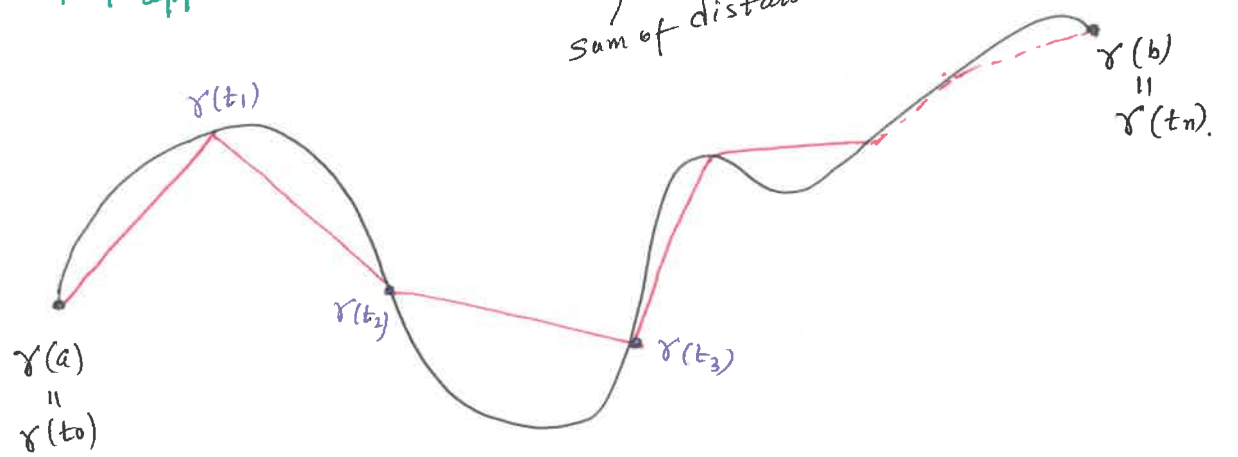
Let P be a partition of $[a,b]$, i.e.,

$$P: a = t_0 < t_1 < \cdots < t_n = b.$$

Now we consider $\{\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_n)\}$, and then the distance between $\gamma(t_{i-1})$ & $\gamma(t_i)$ as: $\|\gamma(t_i) - \gamma(t_{i-1})\|$, $i = 1, \ldots, n$.

Finally, define:

$$\underbrace{\ell(\gamma, P)}_{\text{polygonal approximation}} := \underbrace{\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|}_{\text{sum of distances between } \gamma(t_i)}.$$



Def: A curve $\gamma : [a,b] \longrightarrow \mathbb{R}$ is said to have arclength, or to be rectifiable, if

$$\lim_{\|P\| \to 0} \ell(\gamma, P) := \underbrace{\ell(\gamma)}_{\text{length of } \gamma} \text{ exists.}$$

if exists, it is !.

For $\varepsilon > 0$, $\exists\ \delta > 0$ s.t

$|\ell(\gamma, P) - \ell(\gamma)| < \varepsilon \quad \forall P$ s.t. $\underbrace{\|P\|}_{\text{mesh}} < \delta$.

mesh = max {sub interval of P}.