Graph Theory

Lecture 7

Connectivity.

Motivation. Typically graph may depict roadways network. It is of practical concern if we are able to go from point A to point B using the roads.

$\Rightarrow$ (connectedness).

Also it useful if this connectedness remains if we ~~remove~~ remove certain no. of roads or nodes.

$\Rightarrow$ (connectivity)

$\rightarrow$ robust connectedness is preferred.

Def^n ① Let G be a graph, a vertex cut of G is a set $S \subset V(G)$ s.t. the induced subgraph $\langle V(G) \setminus S \rangle$, has more than one connected components OR $|V(G) \setminus S| = 1$.

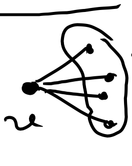
② A graph G is said to be k-connected if $|V(G)| > k$ and $\langle V(G) \setminus X \rangle$ is connected $\forall\ X \subset V(G)$ with $|X| < k$.

③ The connectivity $\kappa(G)$ of a graph G is the no. $\max\{k \mid G \text{ is } k\text{-connected}\}$

Examples. ① K_n is $n-1$ connected & in fact.

$$\kappa(K_n) = n-1.$$

② $\kappa(G) \le \delta(G)$; where $\delta(G) = \min_{u \in G} d(u)$



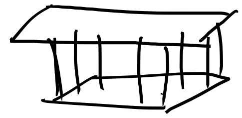
$N(u) = X$.

take X = neighbours of u with $d(u) = \delta(G)$.

Remark. If $\kappa(G) = k$ then every vertex must have deg. $\geq k$. Which means that no. of edges $\geq \lceil \frac{k \cdot n}{2} \rceil$.

Q. Can equality be acheived?

Exercise :- ① $V = \mathbb{F}_2^k$, $x \sim x + e_i$, $1 \leq i \leq k$ k-reg. graph.

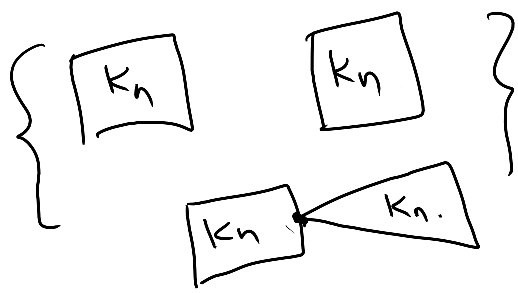
This =  Q_k - hypercube graph has connectivity $= k$.

$|V(Q_k)| = 2^k$.

② Harary graphs. $\forall\ k, n$ $\exists$ Graphs $G(k,n)$ s.t. $\delta(G_{k,n}) = k$, $\kappa(G_{k,n}) = k$ & $|E(G_{k,n})| = \lceil \frac{kn}{2} \rceil$

page 151 (D. West - Graph Theory).

Examples



connectivity $= 0$.
$\delta(G) = n-1$.

connectivity $= 1$
$\delta(G) = n-1$

$\Rightarrow$ The above bound $\kappa(G) \leq \delta(G)$ is very weak in general.

(Mader 1972) **Theorem** :- Every graph whose average degree is $4k$ contains a k-connected subgraph.

$\left(\text{Avg deg} = \dfrac{\sum d(v)}{|V(G)|} \right)$

In order to prove this theorem we prove following technical result.

Proposition:- Let G be a graph such that

① $n = |V(G)| \geq 2k-1$

② $|E(G)| \geq (2k-3)(n+1-k) + 1$

Then, G has a k-connected subgraph H.

Remark :- Since the induced graph on $V(H)$ can have more edges than H, its connectivity $\geq$ conn. of H. Hence we may assume H is an induced subgraph.

Pf. :- Induction on n.

Step 1 $\quad n = 2k-1 \; ; \Rightarrow (2k-3)(n+1-k)+1 = \binom{n}{2}$

$\Rightarrow \quad |E(G)| \geq |E(K_n)| \Rightarrow G = K_n$.

Take H = induced graph on any k-subset !

Step 2 $\quad$ Induction hypothesis. Stat. is true $\forall$ $2k-1 \leq n-1$

Step 3 $\quad$ Let $|V(G)| = m$. (a) Assume that $\exists$ a vertex $v \in V(G)$ s.t. $d(v) \leq 2k-3$.

i.e. $\boxed{\delta(G) \leq 2k-3}$

Remove this v to get new graph G_1 on $m-1$ vertices. Since $m \geq 2k$. $\boxed{m-1 \geq 2k-1}$

Since $|E(G)| \geq (2k-3)(m+1-k)+1$.

$\Rightarrow |E(G_1)| \geq |E(G)| - (2k-3)$

$\geq (2k-3)(m-k)+1$

$\Rightarrow G_1$ has k-conn. subgraph & hence G has !

Case b $\quad \boxed{\delta(G) \geq 2k-2}$

If G itself is k - connected, then take $H = G$!

$\therefore$ assume that $\exists \; X \subset V(G)$ s.t. $|X| \leq k-1$ and $\langle G - X \rangle$ is not connected.

Let G_1, G_2 be subgraphs of $\langle G - X \rangle$ s.t. $G_1 \cup G_2 = \langle G - X \rangle$ & $\nexists$ an edge with one end pt in $V(G_1)$ & other in $V(G_2)$.

$= \langle G - X \rangle$

Note that $G_1 \cup X$ & $G_2 \cup X$ have < no. of vertices than $V(G)$. if $|E(\langle G_i \cup X \rangle)| \geq (2k-3)(|V(G_i \cup X)|+1-k)+1$. Then apply ind. hyp.

$$|E(G)| \le |E(\langle G_1 \cup X\rangle)| + |E(\langle G_2 \cup X\rangle)|; \quad \left(\begin{array}{l} |E(G_i \cup X)| \\ \le (2k-3)(\ell_i+1-k). \end{array} \quad \ell_i = |V(G_i)| + |X| \right)$$

$$\le (2k-3)(\underbrace{|V(G_1) \cup X|}_{=\ell_1} + \underbrace{|V(G_2) \cup X|}_{=\ell_2} - 2k+2)$$

$$= (2k-3)(n + \underset{\le k-1}{|X|} - 2k+2)$$

$$\le (2k-3)(n-k+1) \qquad \text{contradiction !!!!}$$

$\Rightarrow$ We can't have

$$E(\langle G_1 \cup X\rangle) \le (2k-3)(|V(G_1 \cup X)| + 1 - k)$$

AND

$$E(\langle G_2 \cup X\rangle) \le (2k-3)(|V(G_2 \cup X)| + 1 - k)$$

$\Rightarrow$ Either G_1 or G_2 has a k-conn. subgraph by induction hypothesis.

QED.

Corollary (Mader's thm:) : Since avg. deg $\ge 4k$, $\exists$ at least one vertex of deg. $\ge 4k$.

$$\Rightarrow |V(G)| \ge 4k+1 > 2k-1. \quad \text{—(1)}.$$

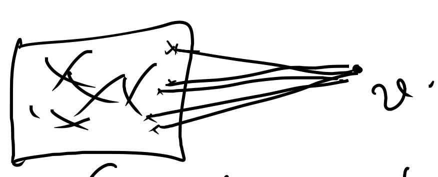
$$2|E(G)| \ge 4kn. \Rightarrow |E(G)| \ge 2kn.$$

Is it true? $|E(G)| \ge (2k-3)(n+\underbrace{1-k}_{\le 0})+1$? YES!!!

$\therefore$ Apply previous proposition.

QED.

Exercises :-



$d(v) \ge k$.

G - k-connected.

If G is k-connected & $G_1 = G \cup v$ with $\deg_{G_1}(v) \ge k$, then G_1 is also k-connected.