Compactness :

⊙ what are all the compact subsets of $\mathbb{R}$ under discrete ~~topo~~ topology ?

Ans: only finite sets. (why?)

⊙ $\mathbb{R}$ with indiscrete topology :-

Ans. Every subset is cmpt. (why?)

⊙ $\mathbb{R}$ with usual topology:-

Ans: $K \subseteq \mathbb{R}$ is cmpt $\Leftrightarrow$ K is closed + bounded.

⊙ $\mathbb{R}$ with co-finite topology:-

Ans: Every subset of $\mathbb{R}$ is cmpt.

Hint: Let $A \subseteq \mathbb{R}$, w.L.G $A \neq \phi$ choose $a \in A$.

Let $\{U_\alpha\}_{\alpha \in J}$ be open cover for A.

choose U_β s.t $a \in U_\beta$, $\beta \in J$. Then $(U_\beta)^c = \mathbb{R} \setminus U_\beta$ is finite. Thus finite open sets will cover remaining finitely many points, if any. Hence A is compact.

⊙ $\mathbb{R}$ with Co-countable topology :-

Ans: only finite subsets are cmpt.

Proof:. Finite subsets are clearly cmpt. suppose $A \subseteq \mathbb{R}$ is not finite. Thus $\{a_1, a_2, \dots\} \subseteq A$.

$U_n = \mathbb{R} \setminus \{a_n, a_{n+1}, a_{n+2}, \dots\}$ $\forall n \in \mathbb{N}$.

$U_1 \subseteq U_2 \subseteq \cdots\cdots$, $A \subseteq \bigcup_{n=1}^{\infty} U_n$.

But A does not have any finite subcover.

Therefore an infinite subset cannot be compact.

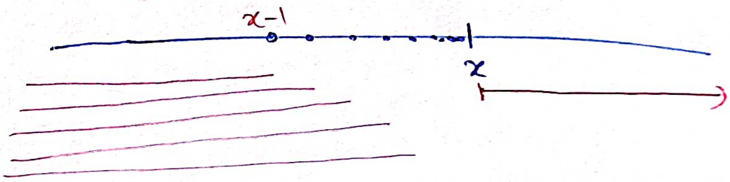
⊙ $\mathbb{R}$ with lower limit topology :.

▣ $f : \mathbb{R}_\ell \longrightarrow \mathbb{R}$, f is continuous as
$x \longmapsto x$. $\mathbb{R} \subseteq \mathbb{R}_\ell$.

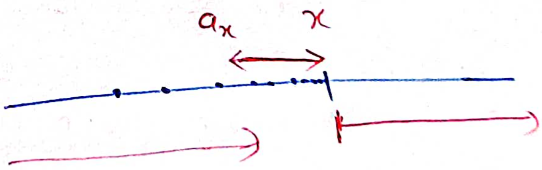
$\therefore$ If $A \subseteq \mathbb{R}_\ell$ is compact, then A is compact in $\mathbb{R}$ ie) A is closed & bounded. in $\mathbb{R}$

▣ consider a non-empty compact subset $K \subseteq \mathbb{R}_\ell$.
Fix an $x \in K$. consider the following open cover :

$\{ [x, +\infty) , (-\infty, x - 1/n) : n \in \mathbb{N} \}$



As K is compact, there is a finite subcover.



$\exists\, a_x \in \mathbb{R}$ s.t

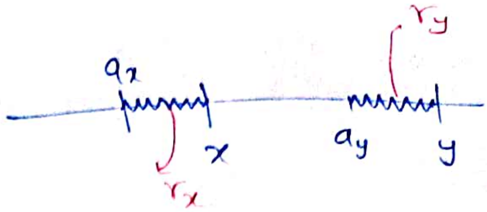
$[a_x, x) \cap K = \phi$.

$\therefore$ For every ~~limit~~ point l of K, there exists an ε such that there are no points of K with in $(l-\varepsilon, l)$.

~~ie)~~ ~~at~~ In particular, all of the limit points must be approached above.

For each $x \in K$, choose a rational number r_x s.t

$r_x \in (a_x, x)$



$h : K \to \mathbb{Q}$ = rationals.

$x \mapsto r_x$.

For $x \neq y$, $r_x \neq r_y$ $\therefore$ h is injective.

Hence K is countable.

Therefore $K \subseteq \mathbb{R}_l$ is cmpt $\Rightarrow$

(or) K is cmpt in $\mathbb{R}$.

(or) K is countable.

(&) every limit point of K is approached from above.

$\mathbb{R}$ with K- topology:

$K = \{1/n : n \in \mathbb{N}\}$ is an infinite set but without limit point (s). Hence, any subset of $\mathbb{R}$ which contains K cannot be compact in K- topology.

As $\mathbb{R} \subseteq \mathbb{R}_K$, Every compact subset of $\mathbb{R}_K$ must be compact in $\mathbb{R}$ with usual topology.

Cor: (i) $[0,1]$ is not compact in $\mathbb{R}_l$ topology.

(ii) $[0,1]$ is not compact in $\mathbb{R}_K$ topology.

Qn:

what are all the compact subsets in $\mathbb{R}_l$ topology & $\mathbb{R}_K$ topology?

- Let τ and τ' are two topologies on X such that X is cmpt + T_2 (Hausdorff) in both topologies. Prove that $\tau = \tau'$ or τ & τ' are not comparable.

Proof:..

Suppose τ & τ' are comparable. W.L.G $\tau \subseteq \tau'$.

Let A be an open set in τ' i.e.) $A \in \tau'$

$\Rightarrow A^c$ is closed in (X, τ')

$\Rightarrow A^c$ is cmpt in (X, τ')

$I : (X, \tau') \to (X, \tau)$
$x \mapsto x$
continuous.

$\Rightarrow A^c$ is cmpt in (X, τ)

$\Rightarrow A^c$ is closed in (X, τ)

$\Rightarrow A$ is open in (X, τ). ie) $A \in \tau$. $\Rightarrow \tau' \subseteq \tau$.

Hence $\boxed{\tau = \tau'}$

- Let P be polynomial in one variable.

$Z(P) = \{ x \in \mathbb{C} : P(x) = 0 \}$ = finite set $\Rightarrow$ compact.

- Let P be polynomial in two complex variables.

$Z(P) = \{ (z, w) \in \mathbb{C}^2 : P(z, w) = 0 \}$

Is $Z(P)$ compact?

$P(z, 1)$ is a poly of one variable with zero's $a_{11}, a_{12}, \cdots a_{1n_1}$

$\therefore \{ (a_{11}, 1), (a_{12}, 1), \ldots (a_{1n_1}, 1) \} \subseteq Z(P)$

Illy for each fixed n (second variable), we have some roots. $\therefore Z(P)$ is unbounded.

Hence $Z(P)$ is not compact.

- Prove that isometry on a compact metric space is hemeomorphism.

Proof:

Let (X, d) be cmpt metric space & $f : X \to X$ be an isometry.

ie.) $d(f(x), f(y)) = d(x, y) \quad \forall x, y \in X$.

- f is continuous (why?)

- For $x \neq y$, $d(f(x), f(y)) = d(x, y) > 0 \Rightarrow f(x) \neq f(y)$

$\therefore$ f is 1-1.

- Suppose f is not onto. choose $a \in X \setminus f(X)$

$x_1 = a$, $x_{n+1} = f(x_n) \; \forall n \geq 1$

$a \notin f(X) = $ cmpt.

$\exists \; \varepsilon > 0$ s.t $B(a, \varepsilon) \cap f(X) = \phi$

$d(x_n, x_1) \geq \varepsilon \quad \forall \; n > 1$

For $n > m > 1$,

$d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) = d(x_{n-1}, x_{m-1}) = \cdots$

$= d(x_{n-m+1}, x_1) \geq \varepsilon$.

$\therefore \; \forall \; n, m \in \mathbb{N}$ we have $d(x_n, x_m) \geq \varepsilon$.

Hence $\{x_n\}$ cannot have any convergent subsequence in a compact space X.

This leads a contradiction.

$\therefore$ f is onto.

$$\begin{array}{c} \text{cmpt} \\ f : X \xrightarrow{\text{cts}} X \rightarrow T_2 \\ \text{bijective} \end{array}$$

Hence f is homeomorphism.

Note: $f(n) = n+1$ is an isometry on $\mathbb{N}$ but not a homeomorphism.

- compact subset of a metric space is closed & bounded.

what about converse ?

$$\ell^2 = \{ x = (x_1, x_2, \dots) : \Sigma |x_n|^2 < \infty \}$$

Let B be the closed unit ball of ℓ^2.

i.e) $B = \{ x \in \ell^2 : \|x\| = (\Sigma |x_n|^2)^{1/2} \leq 1 \}$

$e_n = (0, 0, \dots 0, 1, 0, 0 \dots)$ $\qquad \|e_n\| = 1 \quad \forall n.$

$d(x, y) = \|x - y\|$, $\qquad d(e_n, e_m) = \|e_n - e_m\| = \sqrt{2} \quad \forall n, m.$

$\therefore \{e_n\}$ does not have any convergent subsequence.

This gives B is not compact but it is trivial to see that B is closed & bounded.

- $d(x, A) = \inf \{ d(x, y) : y \in A \}$, $x \in X$.

If A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$

$f : X \rightarrow \mathbb{R}$ $\quad$ Fix $x \in X$.

$y \mapsto d(x, y)$ is a continuous map.

since A is compact, $f|_A$ attains minimum.

$\therefore d(x, A) = d(x, a)$ for some $a \in A$.