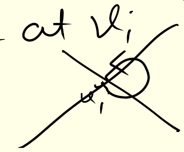
Graph Theory

Lecture 24 — Adjacency Matrix

Linear Algebra in Graph Theory.
- Adjacency matrix
- * oriented / non-oriented Incidence matrix
- * Laplacian matrix.

Proposition :- Let A denote the adjacency matrix of a graph G. The $(i,j)^{th}$ entry of A^k counts the number of $v_i - v_j$ walks of length k; for $k \geq 1$.

proof :- For $k=1$, G is simple $\therefore$ $A_{ii} = 0 \quad \forall i$
$=$ walks of length 1 at v_i



$A_{ij} = 1$ iff $v_iv_j \in E(G)$.

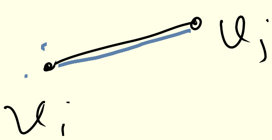
$\therefore$ statement is true for $k=1$.

Now use induction. $A^{k+1} = A \cdot A^k$ to prove the result.

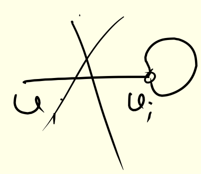
(Exercise)

for example

$A^2_{ii} = \deg v_i = $ walks of length 2 at v_i
$= \|R_i\|^2$.

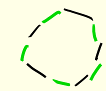


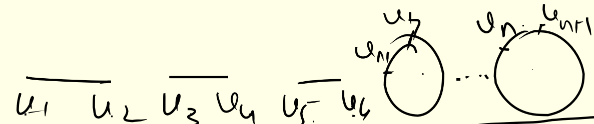
$A^2_{ij} \quad i \neq j$
$=$ \# paths of length 2 bet^n v_i & v_j.



Theorem (Harary 1962) Let G be a (simple) graph & ~~H~~ H be the set of spanning subgraphs in which every connected component is regular of deg 1 or 2 (ie $H \in \mathbb{H}$ iff $V(H) = V(G)$ & every conn. comp. of H is an edge or a cycle)

$\forall H \in \mathbb{H}$ let $k(H) = \#$ conn comp. of H & $\delta(H) = \#$ cycles in H.



{ex. if $H =$  $\Rightarrow$ $k(H) = 5$, $\delta(H) = 2$ }

Then,
$$\boxed{\det A(G) = \sum_{H \in \mathbb{H}} (-1)^{n-k(H)} \, 2^{\delta(H)}} \; ; \; n = |V(G)|$$

Remark :-

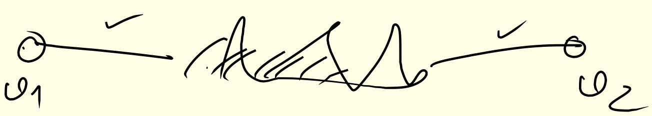
If $\mathbb{H} = \phi$; then $\det A(G) = 0$!

ex. Any tree on odd vertices. $\Rightarrow \det A(G) = 0$!!

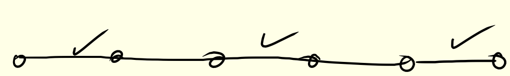
A tree with even vertex has at most one perfect matching. $\Rightarrow |\mathbb{H}| = 1$.

Any tree has two vertices of deg 1.

Any perfect matching must contain those two edges. Now $\langle T - \{u_1, u_2\} \rangle$



If $\exists$ a perfect matching $\det A(G) = (-1)^{n/2}$.

 $\det \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} = -1$.

Recall :-

pf :- $\det M_{n \times n} = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn} \sigma} \prod M_{i\sigma(i)}$

$\prod M_{i\sigma(i)} = 0$ if any one of $(i, \sigma(i))^{th}$ entry of M is 0.

In particular for $M = A(G)$ adj. matrix of a simple graph, $A_{ii} = 0 \ \forall i$. $\Rightarrow$ any σ which "fixes" a number gives summand 0.

$\rightarrow$ Derangement is a permutation on n letters that does not have any fixed point. $\leftarrow$

Any σ can be written as product of disjoint cycles $(1\ \sigma(1) \dots \sigma^i(1))(k\ \sigma(k)\ \sigma^2(k) \dots \sigma^j(k)) \dots$

If $\underline{\sigma = \underbrace{(a_{i_1}\ b_{i_1})}_{\tau_1} \dots \underbrace{(a_{i_r}\ b_{i_r})}_{\tau_r} C_1 \dots C_s}$ where each C_i is a cycle of length ≥ 3 then

sign : $S_n \rightarrow \{+1, -1\}$ is a gp. homo.

$\text{sign}(\sigma\tau) = \text{sign}\,\sigma . \text{sign}\,\tau$.

if $\text{sign}(\sigma) = (-1)^{\Sigma \underline{\text{sign}\,\tau_i} + \text{sign}\,C_j}$

$\therefore$ $\text{sign}\,\tau_i = (-1)^1$. $\text{sign}(C_i) = (-1)^{(\ell(C_i)-1)}$

$$\text{sign}\,\sigma = (-1)^t \quad \text{where } t = r + \sum_{i=1}^{s} (\ell(C_i) - 1)$$

$$= 2r + \underbrace{\sum_{i=1}^{s} \ell(C_i)}_{n} - \underbrace{(r+s)}_{k(H)}$$

$$\underline{n - (r+s)}.$$

$$\text{sign}\,\sigma = (-1)^{n-(r+s)}$$

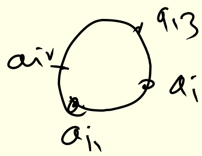
Since $\det A(G) = \sum_\sigma (-1)^{\text{sign}\,\sigma} \underline{\prod A_{i\sigma(i)}}$

$A_{i\sigma(i)} \neq 0$ iff $v_i v_{\sigma(i)} \in E(G)$

$\therefore$ Any derangement $\sigma = (a_{i_1} b_{i_1}) \dots (a_{i_r} b_{i_r}) C_1 \dots C_s$ that gives a non-zero in the RHS of $\det A(G)$ formula, must give us a collection of edges $v_i v_{\sigma(i)}$ such that $v_{\sigma(i)} v_{\sigma\sigma(i)}$

the graph consisting of edges $v_i v_{\sigma(i)}$ can have $1 \le \deg \le 2$. Since $v_{a_{i_1}}, \ldots, v_{a_{i_r}}, v_{b_{i_1}}, \ldots, v_{b_{i_r}}$ are the only vertices of deg 1, we see that this graph is a spanning subgraph whose conn. comp. are reg. of deg 1 or 2. Infact we know that $\exists$ s cycles & r edges in H.

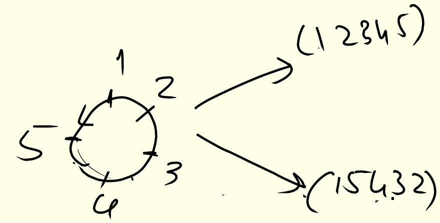
<u>Conclusion</u> :- Any derangement $\sigma \in S_n$ s.t. $\prod_{i=1}^{n} A_{i\sigma(i)} \neq 0$ gives $H \in \mathbb{H}$ s.t. # cycles in H (as graph) = # cycles of length ≥ 3 in σ as a perm.



$\longleftrightarrow (a_{i_1} \ldots a_{i_r})$

$\sigma = (1 \ldots 2 \; a_{i_1} \ldots a_{i_s})$

$\sum_{\sigma \in S_n} (-1)^{\text{sign}} \prod A_{i\sigma(i)}$

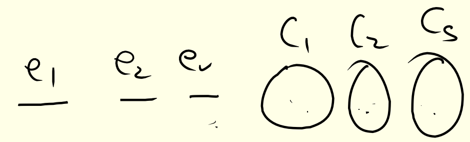


<u>Claim</u> :-

~~However~~ for every $H \in \mathbb{H}$ having $s(H)$ 2-regular components, $\exists$ $2^{s(H)}$ derangements of $\{1, 2, \ldots, n\}$ whose associated spanning subgraph is H.

+ All these $2^{s(H)}$ derangements have sign $(-1)^{n-k(H)}$.

<u>proof</u> of the claim :- Given H =



construct σ which has transposition $\tau_1 \ldots \tau_r$ corr. to edges $e_1 \ldots e_r$ & for each cycle $C_i = (a^i_1, \ldots, a^i_{n_i})$ construct cycles $(a^i_1 \ldots a^i_{n_i})$ or $(a^i_1, a^i_{n_i}, a^i_{n_i - 1}, \ldots, a^i_2)$

$\Rightarrow \exists$ 2^s σ's giving same set of edges.

$$\therefore \quad \det(A(G)) = \sum_{H \in \mathcal{H}} (-1)^{n-k(H)} \cdot 2^{c(H)}$$

$$\| \qquad \therefore$$

$$\sum_{\sigma \in S_n} (-1)^{\operatorname{sign}\sigma} \prod_{i=1}^{n} A_{i\sigma(i)}$$

QED. — x — x — x —

$$x^n - \boxed{\sum \lambda_i} x^{n-1} + \cdots \qquad + (-1)^n \prod \lambda_i^{*} \qquad = \prod_{i=1}^{n} (x - \lambda_i)$$

$\sum \lambda_i \to$ elementary symmetric functions

S_n acting $\mathbb{Z}[x_1 \ldots x_n]^{S_n}$

$x_1 + \cdots + x_n$, $\prod_{i<j} x_i x_j$, $\prod_{i<j<k} x_i x_j x_k$ $\prod_{i=1}^{n} x_i$

$$\mathbb{Z}[x_1 \ldots x_n]^{S_n} \subset \mathbb{Z}[x_1 \ldots x_n]$$

— x — x — x —