Theorem. Let θ be a real parameter and let X have density $f(x|\theta)$ with MLR in $T(x)$. Then
(i) for testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$, there exists a UMP level α test given by

$$\phi(x) = \begin{cases} 1 & \text{when } T(x) > C; \\ \gamma & \text{when } T(x) = C; \\ 0 & \text{when } T(x) < C, \end{cases} \qquad (*)$$

where C and γ are determined by

$$E_{\theta_0}\phi(X) = \alpha. \qquad (**)$$

(ii) The power function $E_\theta \phi(X)$ of this test is strictly increasing for all points θ for which $E_\theta \phi(X) < 1$.
(iii) For all θ', the test given by (*) and (**) is UMP for testing $H_0 : \theta \le \theta'$ versus $H_1 : \theta > \theta'$ at the level $\alpha' = E_{\theta'}\phi(X)$.
(iv) For any $\theta < \theta_0$, the test (*) and (**) minimizes $E_\theta \phi(X)$ among all tests satisfying (**).

Proof. We first prove the existence of ϕ satisfying (*) and (**). Then we show that (ii) holds for this ϕ. Then we prove the UMP part of (i).

Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, $\theta_1 > \theta_0$. From N-P Lemma, MP test for this is of the form:

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} > k; \\ 0 & \text{if } \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} < k. \end{cases}$$

But $p_\theta(x) = f(x|\theta)$ has MLR in $T(x)$. Therefore,

$$\frac{f(x|\theta_1)}{f(x|\theta_0)} > k \text{ iff } T(x) > C \text{ for some } C\ .$$

Hence,

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > C; \\ 0 & \text{if } T(x) < C. \end{cases}$$

Now, from N-P Lemma (a), there exist C and γ satisfying (*) and (**). These do not depend on θ_1. (Note, due to MLR, dependence of MP on θ_1 has been eliminated.) From N-P Lemma (b) this test is MP for testing $H_0 : \theta = \theta'$ versus $H_1 : \theta = \theta''$ at level $\alpha' = E_{\theta'}\phi(X)$ if $\theta' < \theta''$. This is because, from MLR,

$$\frac{f(x|\theta'')}{f(x|\theta')} > k_1 \text{ iff } T(x) > C_1 \text{ for some } C_1$$

and the MP test is exactly of the same form as ϕ. Now from the corollary to N-P Lemma, we have $E_{\theta''}\phi(X) > E_{\theta'}\phi(X)$, whenever $\theta'' > \theta'$. This proves (ii). The fact that $E_\theta\phi(X)$ is strictly increasing implies that

$$E_\theta\phi(X) \le \alpha \;\forall\; \theta \le \theta_0. \qquad (***)$$

Now let us check what test is UMP for testing $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$. For this, consider the class $\mathcal{C}_1$ of all tests that satisfy (***). (We want to know what is best in $\mathcal{C}_1$.) This class $\mathcal{C}_1$ is contained in the class $\mathcal{C}_2$ of tests that satisfy $E_{\theta_0}\phi(X) \le \alpha$. In other words, we have,

$$\mathcal{C}_1 = \{\phi : E_\theta\phi(X) \le \alpha \;\forall\; \theta \le \theta_0\} \subset \mathcal{C}_2 = \{\phi : E_{\theta_0}\phi(X) \le \alpha\},$$

since $E_\theta\phi(X) \le \alpha \;\forall\; \theta \le \theta_0$ implies $E_{\theta_0}\phi(X) \le \alpha$. From N-P Lemma (b), MP level α test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ (for any $\theta_1 > \theta_0$) is given by (*) and (**). That means that ϕ satisfying (*) and (**) is best (MP) in $\mathcal{C}_2$ and this ϕ belongs to $\mathcal{C}_1$ also. Therefore it is best in $\mathcal{C}_1$. However, this test does not depend on θ_1. Therefore it is UMP for $H_1 : \theta > \theta_0$. i.e., for any $\theta > \theta_0$, $E_\theta\phi(X) \ge E_\theta\phi^*(X)$ for all $\phi^* \in \mathcal{C}_1$.

Example. $X \sim$ Binomial(n, p) and it is of interest to test $H_0 : p \le p_0$ versus $H_1 : p > p_0$. (Consider a clinical trial where the efficacy of a drug is being checked.) MLR exists in $T(x) = x$. Therefore, UMP exists and is given by

$$\phi(x) = \begin{cases} 1 & \text{if } x > x_0; \\ 0 & \text{if } x < x_0; \\ \gamma & \text{if } x = x_0. \end{cases}$$

Choose γ and x_0 to satisfy

$$E_{p_0}\phi(X) = P_{p_0}(X > x_0) + \gamma P_{p_0}(X = x_0) = \alpha.$$

Let $\alpha = 0.05$, $n = 10$ and $p_0 = 1/2$. We have

x	$f(x\|p = 1/2)$	$P_{p=1/2}(X \ge x)$
10	0.000977	0.00098
9	0.009766	0.01074
8	0.043945	0.05469

Thus, $x_0 = 8$ and

$$\gamma = \frac{\alpha - P(X > x_0|p = p_0)}{P(X = x_0|p = p_0)} = \frac{0.05 - 0.01074}{0.04395} = 0.8933.$$

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, σ^2 known. Test $H_0 : \mu \le \mu_0$ versus $H_1 : \mu > \mu_0$. $\bar{X}$ is sufficient for μ and $\bar{X} \sim N(\mu, \sigma^2/n)$. It was shown previously, directly, that the likelihood ratio is an increasing function of $T(\bar{x}) = \bar{x}$. This means MLR which can also be shown using the fact that $N(\mu, \sigma^2)$, σ^2 known is a one-parameter exponential family. Since

$$f(\bar{x}|\mu) = \sqrt{\frac{n}{2\pi}}\frac{1}{\sigma}\exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right) = \exp\left(\frac{n\mu}{\sigma^2}\bar{x} - \frac{n}{2\sigma^2}\mu^2 - \frac{n}{2\sigma^2}\bar{x}^2 + \cdots\right),$$

which establishes MLR in $T(\bar{x}) = \bar{x}$. Therefore UMP level α test is

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \bar{x} > C; \\ \gamma & \text{if } \bar{x} = C; \\ 0 & \text{if } \bar{x} < C. \end{cases}$$

Choice of γ is not needed since $P_\mu(\bar{X} = C) = 0$. We need

$$\alpha = P_{\mu_0}\left(\bar{X} > C\right) = P_{\mu_0}\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{C-\mu_0}{\sigma/\sqrt{n}}\right),$$

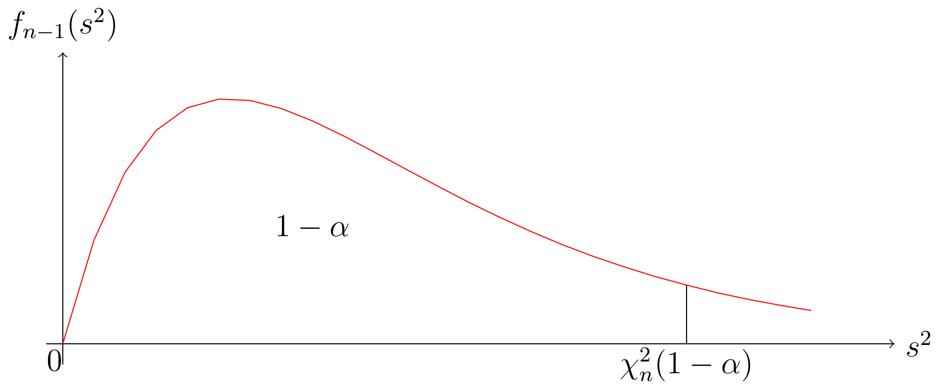
so that $\frac{C-\mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$ or $C = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$. Thus, the UMP test simply rejects H_0 if $\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}$. This is simply the standard Z-test.

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu_0, \sigma^2)$, μ_0 known. Test $H_0 : \sigma^2 \le \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$. $S^2 = \sum_{i=1}^n (X_i - \mu_0)^2$ is sufficient for σ^2 and
$S^2 = \sum_{i=1}^n (X_i - \mu_0)^2/\sigma^2 \sim \chi_n^2$.

$$\begin{aligned} f_{S^2}(s^2|\sigma^2) &= \exp\left(-\frac{s^2}{2\sigma^2}\right)\left(\frac{s^2}{2\sigma^2}\right)^{n/2-1} \times \text{ constant} \\ &= \exp\left(-\frac{1}{2\sigma^2}s^2 - \frac{n}{2}\log(\sigma^2) + (\frac{n}{2}-1)\log(s^2) + \cdots\right). \end{aligned}$$

Thus we have a one-parameter exponential family with MLR in $T(s^2) = s^2$. Therefore the UMP level α test is

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } s^2 > C; \\ 0 & \text{if } s^2 < C. \end{cases}$$



C is determined by

$$\alpha = P_{\sigma_0^2}\left(S^2 > C\right) = P_{\sigma_0^2}\left(\frac{S^2}{\sigma_0^2} > \frac{C}{\sigma_0^2}\right).$$

Since $S^2/\sigma_0^2 \sim \chi_n^2$, when $\sigma^2 = \sigma_0^2$, we have that $C/\sigma_0^2 = \chi_n^2(1-\alpha)$