Graph Theory

Lecture 13

Tutte's 1-factor thm. (contd.)

G - graph; $q(G) = \#$ odd comp. of G. (conn. comp. with odd no. vertices)

<u>Thm.</u> :- G has perfect matching iff $q(G-S) \le |S| \ \forall S \subseteq V(G)$

<u>pf.</u> :- ② Assumed the condition. Assumed that G has no 1-factor.

We added edges to G to reach a stage where we can assume that $G+e$ has 1-factor, G has no 1-factor & G satisfies $q(G-S) \le |S|$.

If $U = \{v \mid \deg_G(v) = n-1 \ ; n = |V(G)|\}$

→ ⓐ $\langle G-U \rangle$ = disjoint union of complete graphs. ✓

ⓑ $\exists$ a conn. comp. of $\langle G \setminus U \rangle$ which is not a complete graph.

$\Rightarrow \exists$ vertices x, y, z, w s.t.

$xy, yz \in E(G)$ $\quad xz \notin E(G)$

i.e. $d_G(x,z) = 2$.

<u>AND</u> $\exists\, w \in V(G)$.

s.t. $yw \notin E(G)$.

By assumption on G, we know $G + xz$ has a perfect matching say M_1. (clearly $xz \in M_1$)

Ily $G + yw$ has 1-factor say M_2 ($yw \in M_2$).

Let H be the graph formed by $M_1 \triangle M_2$.

i.e. $V(H) = V(G)$; & $E(H) = E(M_1) \cup E(M_2) \setminus E(M_1 \cap M_2)$

Conn. comp. of H are either isolated vertices

or 2-regular.

Since xz & yw remain in $M_1 \triangle M_2$,

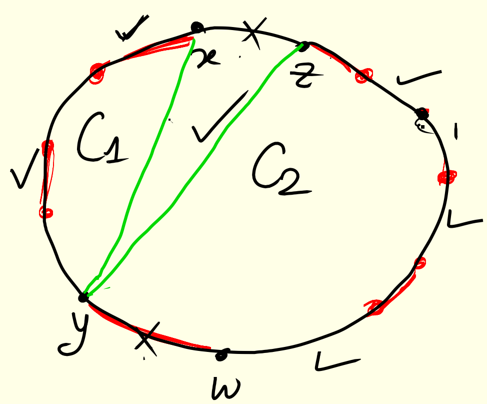
$$\deg_H(x) = \deg_H z = \deg_H y = \deg_H w = 2.$$

This is where we stopped last time.

$\Rightarrow$ the edge $e_1 = xz \in$ 2-regular conn. comp. of H

$\quad$ lly $e_2 = yw \in$ 2-regular conn. comp. of H

(a) e_1, e_2 are in same connected comp. of H. say C.

$C =$ 

red edges are from M_2

black edges are from M_1.

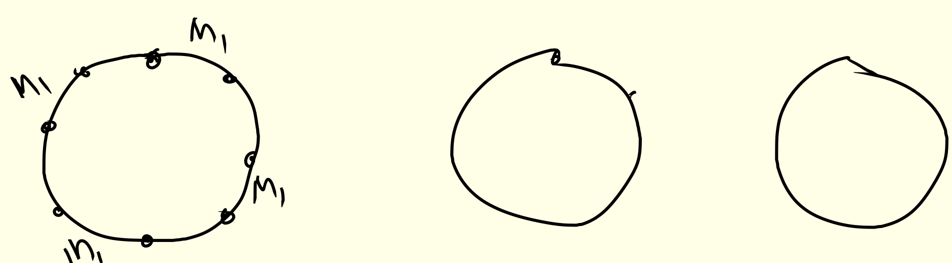
green edges are not in H but they exist in $E(G)$.

Form M by taking edges from M_2 in $C_1 + yz$.

& edges from M_1 in C_2

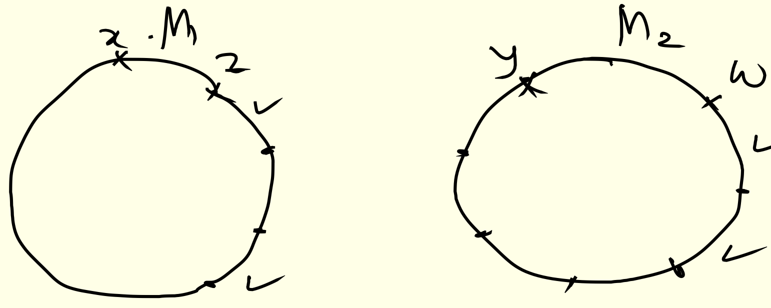
This M constitute a 1-factor of C.

which can be extended to a 1-factor of G by adding all common edges in $M_1 \cap M_2$, & choosing edges from M_1 from all other 2-reg. conn. comp. of H.



This M is a 1-factor of G. a contradiction.

(b) e_1, e_2 belong to distinct connected components of H.



Construct a 1-factor for G by taking all M_2-edges from C_1, all M_1-edges from C_2, all common edges from $M_1 \cap M_2$ and in all other 2-reg. conn. comp. of H take edges from M_1.

which again contradicts the construction of G being a graph without 1-factor.

QED.

As a consequence we reprove an old theorem of Petersen (1891)

Theorem (Petersen) Every 3-regular graph without any cut-edge has perfect matching.

pf. → Since $\boxed{\sum_{u \in V(G)} d(v) = 2|E(G)|}$, any $2k+1$-reg. graph must have even no. of vertices.

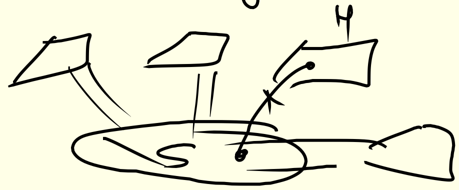
We will show that $\forall S \subseteq V(G)$, $q(G \setminus S) \leq |S|$.

Let $S \subseteq V(G)$. Let H be an odd comp. of $G \setminus S$.

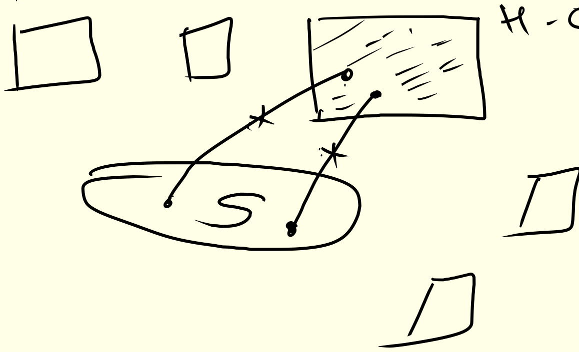
(First note that if we prove that every conn. comp. of G has a 1-factor, then G will have a 1-factor. $\therefore$ assume G is connected)

Since G is connected, $\exists$ at least one edge from H to S. (i.e. $\exists\, e \in E(G)$ s.t. one end pt of $e \in H$, other end pt. $\in S$)

If $\exists$ an H s.t. $\exists!\, e$ betn H & S then e will be an cut-edge of G. $\Rightarrow$ every H must have at least 2-edges joining H & S.



If H is odd comp. of $G \setminus S$ then we can't have exactly 2-edges betn H & S.



H - odd comp. of G-S.

then $\sum_{u \in \langle H \rangle} d(u) = \underline{3|V(H)|-2}$
$= \text{odd}$.
contradiction!

$\Rightarrow$ every odd comp. of $G \setminus S$ must have at least $\underline{\underline{3}}$-edges "joining it" to S.

$\#$ edges with exactly one end pt. in $S \le \sum \deg(S) \le 3|S|$.

But $\exists$ atleast $3 \cdot q(G$-$S)$ such edges

$\Rightarrow 3 \cdot q(G\text{-}S) \le 3 \cdot |S| \Rightarrow q(G\text{-}S) \le |S|$
$\forall\, S \subseteq V(G)$.

Now use Tutte's 1-factor theorem!

QED.

Defect version of Tutte's theorem.

$q(G\text{-}S) \le |S|$ but for some S,

$q(G-S) > |S|$. ; let $d(S) = \max_{S \subseteq V(G)} \{ q(G-S) - |S| \}$.

Note. $d(S)$ & $V(G)$ have same parity.

i.e. $d(S) \equiv V(G) \pmod 2$

Pf. (a) $|V(G)|$ - odd $\to$ $|S|$ - even $\Rightarrow$ $|G-S| = $ odd $\Rightarrow q(G-S)$ is odd $\Rightarrow d(S)$ odd.

$\to$ $|S|$ - odd $\Rightarrow$ $|G-S| = $ even $\Rightarrow q(G-S)$ even $\Rightarrow d(S)$ odd

(b) $|V(G)|$ - even $\to$ $|S|$ - even

$\to$ $|S|$ - odd

prove that $d(S) \equiv V(G) \bmod 2$.

(exercise)

Defect version of Tutte's thm.

(Berge 1958) The largest matching in an n-vertex graph G covers at least $n - \max_{S \subseteq V(G)} \{d(S)\}$ vertices.

$(q(G-S) - |S|)$

Pf :- Clearly $\max_{S \subseteq V(G)} d(S) \geq 0$ take $S = \phi$.

Let $d = \max d(S)$. Define G' by

$V(G') = V(G) \cup \{x_1, \dots, x_d\}$

& $E(G') = E(G) \cup \{x_i x_j\} \cup \{x_i v \mid v \in V(G)\}$.

We prove that Tutte's condition is satisfied by G'. (Exercise) $\Rightarrow$ G' has 1-factor

Then removing at most d edges gives a matching of required size for G.

QED.