QUIZ

(1) The degree of a splitting field of $x^6 - 3$ over $\mathbb{Q}$ is
   (a) 3.
   (b) 6.
   (c) 12.
   (d) none of the above.

   Answer: (c). The required splitting field is $K = \mathbb{Q}(\sqrt[6]{3}, \frac{1+i\sqrt{3}}{2}) = \mathbb{Q}(\sqrt[6]{3}, i)$. Note that $\frac{1+i\sqrt{3}}{2}$ is the primitive 6th root of unity $\exp(2i\pi/6)$.

(2) The degree of a splitting field of $x^6 + 3$ over $\mathbb{Q}$ is
   (a) 3.
   (b) 6.
   (c) 12.
   (d) none of the above.

   Answer: (b). The required splitting field is $K = \mathbb{Q}(\sqrt[6]{-3}, \frac{1+i\sqrt{3}}{2}) = \mathbb{Q}(\sqrt[6]{-3})$, since $i\sqrt{3} = (\sqrt[6]{-3})^3$.

(3) The degree of a splitting field of $x^6 - 1$ over $\mathbb{Q}$ is
   (a) 2.
   (b) 3.
   (c) 5.
   (d) 6.

   Answer: (a). The required splitting field is $K = \mathbb{Q}(\xi)$, where $\xi = exp(2i\pi/6)$ is a primitive 6th root of unity. The minimal polynomial (over $\mathbb{Q}$) of ξ divides $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$, so it must be x^2+x+1 or x^2-x+1. Now, $x^2 + x + 1$ is the minimal polynomial (over $\mathbb{Q}$) of $\omega = \exp(2i\pi/3) = \xi^2$ and $\omega^2 = \exp(4i\pi/3) = \xi^4$. So the minimal polynomial of ξ (over $\mathbb{Q}$) is $x^2 - x + 1$. The other root of this irreducible polynomial over $\mathbb{Q}$ is $\xi^5 = \xi^{-1}$.

(4) The degree of a splitting field of $x^6 - 1$ over $\mathbb{F}_2$ is
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 6.

   Answer: (b). $x^6 - 1 = (x^3 - 1)^2$ over $\mathbb{F}_2$. Now, $x^3 - 1 = (x - 1)(x^2 + x + 1)$. So the required splitting field K is got by attaching a root - and hence both roots - of $x^2 + x + 1$ (which can easily be checked to be an irreducible polynomial over $\mathbb{F}_2$). Hence K has degree 2 over $\mathbb{F}_2$.

(5) The degree of a splitting field of $x^6 - 1$ over $\mathbb{F}_3$ is
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 6.

Answer: (a). $x^6 - 1 = (x^2 - 1)^3$ over $\mathbb{F}_3$, and $x^2 - 1 = (x-1)(x+1)$ over $\mathbb{F}_3$, so the required splitting field K is $\mathbb{F}_3$.

(6) The degree of a splitting field of $x^6 - 1$ over $\mathbb{F}_5$ is
 (a) 1.
 (b) 2.
 (c) 3.
 (d) 4.

Answer: (b). $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$ over $\mathbb{F}_5$, and it is easy to check that both the degree 2 polynomials x^2+x+1 and x^2-x+1 are irreducible over $\mathbb{F}_5$. Let α be root of x^2+x+1, attach α to $\mathbb{F}_5$ to get a degree 2 extension K. Now K also contains a root of x^2-x+1 and hence is the required splitting field. This is because $\alpha+1$ satisfies x^2-x+1, as $(\alpha+1)^2-(\alpha+1)+1 = \alpha^2+2\alpha+1-\alpha-1+1 = \alpha^2+\alpha+1 = 0$ (in K).