TOPOLOGY

Contents

1. Introduction

Webster defines the word "Topology" to be the study of the geographical location of a place. The development of any subject has many facets : (not necessarily in this order) basic assumptions (axioms), pinning down the basic definitions, central questions in the subject and the development of ideas (technical tools) to find answers to the questions.

Topology pervades all of mathematics. The subject of topology has a long history and would probably have its roots in the Greek school of geometeers. In recent times the subject can probably be traced back to the work of Euler in the 18th century culiminating in the ideas of Riesz, Hausdorff (and several others) that led to the present day definition of a topological space in the early part of the 20th century[1]. A journey spanning two millenia.

Loosely, the subject of topology can be thought of as the study of properties of a geometric object that are unchanged under a *continuous deformation* of the object. In other words we allow change of shape of objects while maintaing the information about which points were close to each other so that after the object is deformed the points that were close still remain "close". This information is encoded by open sets. Thus, while we allow change of shape, we do not permit "tearing". This is in contrast with euclidean geometry where two triangles are isomorphic (or same) if the two overlap when one is placed over the other, no deformations of the sides being allowed. Thus allowing deformations is a weakening of the notion of sameness when compared with euclidean geometry. With this notion, one is forced to say that the triangle and the circle are the same. Topology explores the properties of the space that remain unchanged when one object is deformed into another.

[1]The article https://u.math.biu.ac.il/ megereli/TopHistory.pdf, for example, traces the recent history of the subject. See also the references therein.

Our discussion in this set of notes will be based on the book Topology by James R. Munkres. We assume familiarity with the notion of sets, various operations (like unions, intersection complementation) with sets and the notion of functions and their properties (composition, one-one, onto).

2. Sets : finite, infinite,...

As we have already mentioned we assume some familiarity with the notion of a set, operations on sets, and the notion of a function and some of the basic definitions concerning functions. In this section we discuss a basic construction with sets and a fundamental way in which sets can be compared. We begin with the following definition.

Definition 2.1. Let I be a set (called the indexing set). Suppose that for each $i \in I$ there is associated a set A_i, then the family

$$\mathcal{A} = \{A_i \,:\, i \in I\} = \{A_i\}_{i \in I}$$

is called an *indexed family* of sets indexed by the set I. It is possible that $A_i = A_j$ for some $i \neq j$. We refer to the set A_i as an element of the indexed family $\mathcal{A}$.

For example suppose that $I = \{1, 2, \ldots, n\}$, then an indexed family of sets indexed by I is denoted by

$$\mathcal{A} = \{A_1, A_2, \ldots, A_n\}.$$

If $I = \mathbb{N} = \{1, 2, \ldots\}$, then an indexed family

$$\mathcal{A} = \{A_1\}_{i \in \mathbb{N}}$$

is often denoted by $\mathcal{A} = \{A_1, A_2, \ldots\}$.

One defines the union, intersection of the elements of an indexed family of sets in an obvious way. Indeed, if $\mathcal{A} = \{A_i\}_{i \in I}$ is an indexed family of sets, then we define

$$\bigcup_{i \in I} A_i = \{x \,:\, x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \,:\, x \in A_i \text{ for all } i \in I\}.$$

If $I = \{1, 2, \ldots, n\}$, then the following notation

$$\bigcup_{i} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

is also used.

A construction of importance is the *cartesian product* of the elements of an indexed family of sets.

Definition 2.2. Let X be a set. A n-tuple of elements in X is a function

$$\mathbf{x} = \{1, 2, \dots, n\} \longrightarrow X$$

We denote by x_i, the element $\mathbf{x}(i) \in X$. Traditionally, the function $\mathbf{x}$ is denoted by the notation

$$\mathbf{x} = (x_1, x_2, \dots, x_n).$$

Definition 2.3. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be an indexed family of sets. Thus $\mathcal{A}$ is indexed by $I = \{1, 2, \dots, n\}$. The cartesian product of this indexed family is denoted by

$$\Pi_i A_i = A_1 \times \cdots \times A_n$$

and is by definition the set of n-tuples $\mathbf{x}$ in $X = \cup_i A_i$ such that $x_i \in A_i$.

In other words,

$$\Pi_i A_i = \{\mathbf{x} : \{1, 2, \dots, n\} \longrightarrow \cup_i A_i \, : \, \mathbf{x}(i) = x_i \in A_i\}.$$

By our agreed upon notation,

$$\Pi_i A_i = \{\mathbf{x} = (x_1, \dots, x_n) \, : \, x_i \in A_i\}.$$

Similar definition applies for an arbitrary family of indexed sets.

Definition 2.4. Let $\mathcal{A} = \{A_i\}_{i \in I}$ be an indexed family of sets indexed by the set I. The cartesian product of the family $\mathcal{A}$ is denoted by $\Pi_i A_i$ and is defined to be the set of all functions

$$\mathbf{x} : I \longrightarrow \bigcup_i A_i$$

such that $\mathbf{x}(i) = x_i \in A_i$. The function $\mathbf{x}$ is denoted by

$$\mathbf{x} = (x_i)_{i \in I}$$

We say that x_i is the i-th coordinate of $\mathbf{x}$.

Suppose $\{A_i\}_{i \in I}$ is an indexed family of sets indexed by I. Assume that $A_i = A$ for all $i \in I$, When $I = \{1, 2, \dots, n\}$ we use the notation

$$\Pi_i A_i = A^n$$

to denote the the cartesian product and the right hand side is called the n-fold product of A with itself. If $I = \{1, 2, \dots\}$, then we use the notation

$$\Pi_n A_i = A^\omega$$

to denote the cartesian product and the right hand side is called the countable product of A with itself.

The above definitions of the cartesian product are simple and it is important that we understand them well. We shall be using the cartesian product construction often.

Remark 2.5. Set theory is fundamental in the study of mathematics. There are many models of set theory with differing basic axioms. It is therefore important to fix a model of set theory with which we work. Although we will not spell this out explicitly, we will assume that in our setup the cartesian product of an arbitrary family of *nonempty* sets is *nonempty*. This seemingly self evident fact is an important tool in many situations. This assumption is equivalent to any of the following : axiom of choice, Zorn's lemma, maximum principle, the well-ordering theorem and many others[2].

Having discussed the notion of cartesian products we move on to discuss ways in which two sets can be compared. Towards this we make some definitions. First recall that two sets A, B are said to be in *bijective correspondence* if the exists a bijective function $f : A \longrightarrow B$.

Definition 2.6. A set A is said to be *finite* if it is in bijective correspondence with the set $\{1, 2, \ldots, n\}$ for some non-negative integer n. In this case we say that the set X has *cardinality* n. By convention the set $\{1, 2, \ldots, n\}$ is empty if $n = 0$. The set A is said to be infinite if it is not finite.

That the cardinality of a finite set is well defined needs a proof. This and some other consequences will following from the seemingly self evident facts that we prove below.

Proposition 2.7. Let n be a positive integer. Let A be a set with $a \in A$. Then there exists a bijection

$$f : A \longrightarrow \{1, 2, \ldots, n + 1\}$$

if and only if there exists a bijection

$$g : A - \{a\} \longrightarrow \{1, 2, \ldots, n\}.$$

Proof. If g is a bijection, then clearly f can be defined to be a bijection. Conversely assume the existen of f. Then we define g as follows. Let $f(a) = k$ with $1 \leq k \leq n+1$. We set

$$g(x) = \begin{cases} f(x) & \text{if } f(x) < k \\ f(x) - 1 & \text{if } f(x) > k \end{cases}$$

It is easy to check that g is a bijection. □

Proposition 2.8. Let $f : A \longrightarrow \{1, 2, \ldots, n\}$ be a bijection for some positive integer n. If B is a proper subset of A, then there does not exist any bijection $g : B \longrightarrow \{1, 2, \ldots, n\}$. If B is nonempty, then there exists a bijection $B \longrightarrow \{1, 2, \ldots, m\}$ for some positive integer $m < n$.

[2]Also look at https://plato.stanford.edu/entries/axiom-choice/ for an interesting discussion.

Proof. The proposition is true if B is the empty set. So we assume B is nonempty and induct. The proposition is clearly true when $n = 1$. Assume the truth of the proposition for n and let

$$f : A \longrightarrow \{1, 2, \ldots, n + 1\}$$

be a bijection and B a proper subset of A. Fix $b \in B$ (B is nonempty) and $a \in (A-B)$ (B is a proper subset). By Proposition 2.7 there exists a bijection

$$g : A - \{b\} \longrightarrow \{1, 2, \ldots, n\}.$$

As $B - \{b\} \subseteq A - \{b\}$ (and is a proper subset), the induction hypothesis now implies that

(1) there does not exist any bijection

$$B - \{b\} \longrightarrow \{1, 2, \ldots, n\}$$

and

(2) either $B - \{b\}$ is empty or there exists a bijection

$$B - \{b\} \longrightarrow \{1, 2, \ldots, p\}$$

for some $p < n$ is it is nonempty.

The point (1) along with Proposition 2.7 tells us that there does not exist any bijection

$$B \longrightarrow \{1, 2, \ldots, n\}.$$

To prove the last claim of the proposition we observe that if $B - \{b\} = \emptyset$, then there is a bijection $B \longrightarrow \{1\}$ and if it is nonempty, then again by Proposition 2.7 there is a bijection

$$B \longrightarrow \{1, 2, \ldots, p + 1\}.$$

But as $p < n$, we have $p + 1 < n + 1$ completing the inductive step and the proof of the proposition. □

Here are some consequences of the above observations.

Corollary 2.9. If A is a finite set, then there does not exist any bijection of A with a proper subset of of itself.

Proof. Let $B \subseteq A$ be a proper subset. Assume that there exists a bijection $f : B \longrightarrow A$ Since A is finite, there exists a bijection $g : A \longrightarrow \{1, 2, \ldots, n\}$ for some positive integer n. The composition

$$g^{-1} \circ f : B \longrightarrow \{1, 2, \ldots, n\}$$

is a bijection contradicting Proposition 2.8. □

Thus if a set A is in bijective correspondence with a proper subset of iteself, then it must be infinite. Using this fact it is easy to show that the integers $\mathbb{Z}$ is infinite.

Corollary 2.10. Cardinality of a finite set is well defined.

Proof. Exercise. □

Another self evident fact that needs a proof is the following.

Corollary 2.11. A subset of a finite set is finite.

Proof. This follows from Proposition 2.8. Notice that the cardinality of B is strictly less than that of A.

□

Corollary 2.12. Let B be a nonempty set. Then the following are equivalent.

(1) B is finite.
(2) There exists a surjective function $f : \{1, 2, \ldots, n\} \longrightarrow B$ for some positive integer n.
(3) There exists an injective function $g : B \longrightarrow \{1, 2, \ldots, n\}$ for some positove integer n.

Proof. Clearly, (1) implies (2). We now show that (2) implies (3). So assume that $f : \{1, 2, \ldots, n\} \longrightarrow B$ is a surjective function. Then we immediately notice that for any distinct $b, b' \in B$ we have

$$f^{-1}(b) \neq \emptyset; \quad f^{-1}(b) \cap f^{-1}(b') = \emptyset.$$

Now define $g : B \longrightarrow \{1, 2, \ldots, n\}$ by setting $g(b)$ to be the smallest element of $g^{-1}(b)$. This is evidently injective. Thus (2) implies (3). Finally we show that (3) implies (1) which will complete the proof. So assume that $g : B \longrightarrow \{1, 2, \ldots, n\}$ is injective. Thus B is in bijective correspondence with a subset of the finite set $\{1, 2, , \ldots, n\}$. By Corollary 2.11 B must be finite. Thus (3) implies (1). This completes the proof. □

Corollary 2.13. Finte union and products of finite sets is finite.

Proof. Exercise. □

Here are some examples.

Example 2.14. Let $X = \{0, 1\}$. We consider the countable cartesian product X^ω. By definition, X^ω the set of all functions

$$\mathbf{x} : I = \{1.2. \ldots\} \longrightarrow \{0, 1\}.$$

Recalling our notation, we have that

$$\{0, 1\}^\omega = \{\mathbf{x} = (x_i)_i \, : \, x_i = 0, 1\}.$$

We often write

$$\mathbf{x} = (x_1, x_2, x_3, \ldots)$$

and think of $\mathbf{x}$ as sequence in X. Let A be the subset of X^ω defined by

$$A = \{\mathbf{x} = (x_i) \,:\, x_1 = 0\}.$$

In other words A is the subset of all sequences with first term 0. Observe that A is a proper subset of X^ω. The function $f : X^\omega \longrightarrow A$ defined by

$$f(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

is clearly a bijection.

Example 2.15. Let $A = \{1, 2, \ldots, n\}$ and let $\mathcal{P}(A)$ be the set of all subsets of of A (also called the power set of A. Let $X = \{0, 1\}$. Define a function

$$f : \mathcal{P}(A) \longrightarrow X^n$$

as follows. Given $B \in \mathcal{P}(A)$ consider the function

$$f(B) : I \longrightarrow X$$

defined by

$$f(B)(i) = \begin{cases} 1 & \text{if } i \in B \\ 0 & \text{otherwise} \end{cases}$$

Then by definition $f(B) \in X^n$. Then it is easy to check that f is a bijection. It is an exercise to show that if A is finite, then so is $\mathcal{P}(A)$.

We now discuss infinite sets. We begin with a definition. We shall use the notation $\mathbb{Z}_+$ to denote the positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers.

Definition 2.16. A set A is said to be *countably infinite* if there exists a bijectiion

$$f : A \longrightarrow \mathbb{Z}_+$$

of A with the set of positive integers. A set A is said to be *countable* if it is either finite or countably infinite. A set that is not countable is said to be *uncountable.*

Here are some examples.

Example 2.17. The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}_+$ defined by

$$f(n) = \begin{cases} 2n & n > 0 \\ -2n + 1 & n \leq 0 \end{cases}$$

is a bijection showing that $\mathbb{Z}$ is countably infinite.

Example 2.18. The function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ defined by

$$f(m, n) = 2^{m-1}(2n - 1)$$

is a bijection showing that the cartesian product $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.

The following theorem tells us several interesting ways to decide when a set is countable.

Theorem 2.19. Let B be a nonempty set. Then the following are equivalent.

(1) B is countable.
(2) There is a surjective function $f : \mathbb{Z}_+ \longrightarrow B$.
(3) There exists an injective function $g : B \longrightarrow \mathbb{Z}_+$.

Proof. Assume that B is countable. If B is countably infinite, we are done. If B is finite then there exists a bijection $\{1, 2, \ldots, n\} \longrightarrow B$ for some positive integer n. This can be extended to a surjective function $\mathbb{Z}_+ \longrightarrow B$. Thus (1) implies (2).

Next assume that there exists a surjective function $f : \mathbb{Z}_+ \longrightarrow B$. Then an injective function $g : B \longrightarrow \mathbb{Z}_+$ can be constructed just as in Corollary 2.12. Thus (2) implies (3).

Finally assume that there is an injective function $g : B \longrightarrow \mathbb{Z}_+$. Thus B is in bijective correspondence with its image $g(B)$. If $g(B)$ is finite, then B is finite and hence countable. If $g(B)$ is infinite we proceed as follows.

We will use the following property of positive integers : Every nonempty subset of $\mathbb{Z}_+$ has a smallest element. This is called the well-ordering principle.

We now construct a bijection $h : \mathbb{Z}_+ \longrightarrow g(B)$ thereby completing the proof that $g(B)$ is countably infinite. First define $h(1)$ to be the smallest element of $g(B)$. Assume that we have defined $h(1), \ldots, h(n-1)$. We define $h(n)$ to be the smallest element of the set

$$g(B) - \{h(1), h(2), \ldots, h(n-1)\}.$$

The set $g(B) - \{h(1), h(2), \ldots, h(n-1)\}$ is nonempty. For if it were empty h would be a surjective map from

$$h : \{1, 2, \ldots, n-1\} \longrightarrow g(B)$$

which woul mean (by Corollary 2.12) that $g(B)$ is finite. A contradiction. Hence $h(n)$ is well defined. Thus the function

$$h : \mathbb{Z}_+ \longrightarrow g(B)$$

is well defined[3].

We first verify that h is one one. Suppose $m < n$. Then, by construction,

$$h(n) \in g(B) - \{h(1), \ldots, h(m), h(m+1), \ldots, h(n-1)\}$$

so in particular $h(m) \neq h(n)$.

Finally we verify that h is surjective. Let $p \in g(B)$. We shall show that there exists m with $h(m) = p$. Since h is injective, $h(\mathbb{Z}_+)$ is infinite and hence there exists $n \in \mathbb{Z}_+$ such that

$$h(n) > p$$

[3]Such a definition, where the function h is defined in terms of itself is called a recursive definition and we say that the formula that defines h is a recursion formula.

Thus the set
$$\{a \in \mathbb{Z}_+ \,:\, h(a) \geq p\}$$
is nonempty and therefore must have a smallest element say m. Then whenever $i < m$ we must have
$$h(i) < p.$$
Thus
$$p \notin \{h(1), \ldots, h(m-1)\}.$$
Since $h(m)$ is the smallest element of the set
$$g(B) - \{h(1), \ldots, h(m-1)\}$$
we must have $h(m) \leq p$. Whence $h(m) = p$. This completes the proof. $\square$

The proof actually shows that every infinite subset of $\mathbb{Z}_+$ is countably infinite. we note this below.

Theorem 2.20. Every infinite subset of $\mathbb{Z}_+$ is countable infinite. $\square$.

We immediately have the following consequence.

Corollary 2.21. $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.

Proof. You can choose your favorite way to prove this. We could also invoke Theorem 2.19. The proof is left as an exercise. $\square$

Theorem 2.22. A countable union of countable sets is countable. Finite products of countable sets is ccountable.

Proof. Let $\mathcal{A} = \{A_i\}_{i \in I}$ be an indexed family of countable sets indexed by $I = \{1, 2, \ldots n\}$ or $\{1, 2, 3, \ldots\}$.

First assume that $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ so that the collection is finite and indexed by $I = \{1, 2, \ldots, n\}$. To prove
$$\bigcup_{i=1}^{n} A_i$$
is countable it is enough to prove that the union of two countable sets (say A and B) is countable. Since A and B are countable we can "list" the sets as
$$A = \{x_1, x_2, \ldots\}$$
$$B = \{y_1, y_2, \ldots\}.$$
The function
$$h : A \cup B \longrightarrow \mathbb{Z}_+$$
defined by
$$h(z) = \begin{cases} 2n-1 & \text{if } z = x_n \\ 2n & \text{if } z = y_n \end{cases}$$

is a bijection. Thus $A \cup B$ is countable. By induction, a finite union of countable sets is countable.

Next assume that the collection $\mathcal{A}$ is countable infinite so that it is indexed by $I = \{1, 2, \ldots\}$. Thus

$$\mathcal{A} = \{A_1, A_2, \ldots\}.$$

We shall construct a surjective function

$$h : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \bigcup_i A_i$$

which will show (by Theorem 2.19) that $\cup_i A_i$ is countably infinite. Since each A_i is countable we may list each element of A_i in a sequence. The function h is defined by

$$h(m, n) = \text{the } n\text{-th element of } A_m.$$

Clearly, h is onto. Thus $\cup_i A_i$ is countably infinite.

Finally we check that a finite product of countable sets is countable. Again, it is enough to show that the product of two countable sets is countable. So suppose that A, B are countably infinite. Then writing

$$A = \{x_1, x_2, \ldots\}; \quad B = \{y_1, y_2, \ldots\}$$

the function

$$g : A \times B \longrightarrow \mathbb{Z}_+$$

defined by

$$f(x_i, y_j) = 2^i 3^j$$

is injective. Thus $A \times B$ is countable. $\square$

We end our discussion by two examples of uncountable sets.

Example 2.23. Let $X = \{0, 1\}$. We shall see that the countable product of X with itself, which we had agreed to denote by X^ω, is uncountable. Elements of X^ω are sequences where each term of the sequence is either 0 or 1. The following argument is due to Georg Cantor and is called Cantor's diagonal argument. Assume that there is a bijective function

$$g : \mathbb{Z}_+ \longrightarrow X^\omega = \{y_1, y_2, \ldots\}$$

where each y_i is a sequence of zeros and ones. This leads to a contradiction as follows. We shall construct a sequence $\{x_n\}$ of zeros and ones as follows. Set

$$x_n = \begin{cases} 0 & \text{if the } n\text{-th term of } y_n \text{ is } 1 \\ 1 & \text{if the } n\text{-th term of } y_n \text{ is } 0 \end{cases}$$

By construction, $\{x_n\} \in X^\omega$ but is not in the image of g as it differs from each y_i. This contradiction shows that X^ω must be uncountable.

Example 2.24. Let A be a set. Suppose that

$$f : A \longrightarrow \mathcal{P}(A)$$

is a function. We claim that f cannot be surjective. Consider the set

$$B = \{a \in A \,:\, a \notin f(a)\}.$$

Since f is assumed to be surjective, there exists an element $a' \in A$ such that

$$f(a') = B.$$

Now if $a' \in B$, then by the definition of B we have

$$a' \notin f(a') = B$$

a contradiction. On the other hand if $a' \notin B$, then by the definition of B we must have

$$a' \in f(a') = B$$

and we have a contradiction again. Thus f cannot be surjective. This shows that there is no injective function

$$g : \mathcal{P}(A) \longrightarrow A.$$

In particular, if we let $A = \mathbb{Z}_+$, then there is no injective function

$$\mathbb{Z}_+ \longrightarrow \mathcal{P}(\mathbb{Z}_+.$$

This shows that $\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

Here are some exercises.

Exercise 2.25. Exhibit a map

$$f : \mathcal{P}(\mathbb{Z}_+) \longrightarrow \{0,1\}^\omega$$

that is a bijection. This gives another proof that $\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

Exercise 2.26. A real number x is said to be *algebraic* (over rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients. Show that the set of algebraic numbers is countable.

Exercise 2.27. Determine whether the following sets are countable or not.

(1) The set of all functions $f : \{0,1\} \longrightarrow \mathbb{Z}_+$.
(2) The set B_n of all functions $\{1, 2, \ldots, n\} \longrightarrow \mathbb{Z}_+$.
(3) The set $C = \cup_n B_n$.
(4) The set of all functions $\mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$.

Exercise 2.28. We say that two sets A and B *have the same cardinality* if there exists a bijection $f : A \longrightarrow B$. Suppose that $B \subseteq A$. If there exists an injective map

$$f : A \longrightarrow B$$

then A and B have the same cardinality. Further show that (Schroeder-Bernstein theorem) if A and B are two sets and there exist injective maps $A \longrightarrow B$ and $B \longrightarrow A$, then A and B have the same cardinality.

Exercise 2.29. Construct a bijection $f : [a, b] \longrightarrow [c, d]$.

Exercise 2.30. Let $\mathcal{B}$ denote the set of countable subsets of $\{0, 1\}^\omega$. Show that $\mathcal{B}$ and $\{0, 1\}^\omega$ have the same cardinality.

Exercise 2.31. Show that every infinite set contains a subset that is countably infinite.

Exercise 2.32. Let A be an infinite set. Show that there is a function $f : A \longrightarrow A$ that is injective but not surjective.

Exercise 2.33. Show that a finitely generated group is countable. Is a countably generated group countable?

Exercise 2.34. Show that if A and B have the same cardinality, then so do $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Is the converse true?

3. Topological spaces

We begin the study of topological spaces. In this section we define the the notion of a topological space and look at some examples.

Definition 3.1. Let X be a set. A *topology* τ on X is a set consisting of subsets of X such that

(1) $\emptyset, X \in \tau$.
(2) τ is closed with respect to finite unions.
(3) τ is closed with respect to arbitrary unions.

A topological space is a pair(X, τ) where X is a set and τ a topology on X. A topology on a set X is therefore just a distinguised collection of subsets of X subject to satisfying the above three conditions. We shall often forget to mention τ and just say that X is a topological space when there is no danger of confusion. The subsets that belong to the topology have a name.

Definition 3.2. Let (X, τ) be a topological space. A subset $U \subseteq X$ is said to be *open* in X if $U \in \tau$.

In a topological space, therefore, the whole space is an open set. Here are some examples.

Example 3.3. Let X be a set. Then $\tau = \mathcal{P}(X)$ is a topology on X called the *discrete topology* on X. Note that in the discrete topology every subset of X is open in X.

Example 3.4. Given a set X, the collection

$$\mathcal{I} = \{\emptyset, \{X\}\}$$

is a topology on X called the *indiscrete topology* on X. It has just two open sets.

The two examples above exhibit two "extreme" topologies on a set. Every topology τ on a set X satisfies

$$\mathcal{I} \subseteq \tau \subseteq \mathcal{P}(X).$$

Example 3.5. Let X be a set. Define

$$\tau_f = \{U \subseteq X \,:\, X - U \text{ is either finite or equals } X\}.$$

It is easy to check that τ_f is a topology on X called the *finite complement* topology on X.

Observe that if the set X is finite, then the finite complement topology is the same as the discrete topology on X. The two topologies differ when the set is infinite. For example in the discrete topology singletons are always open (since every subset is open) but if the set X is infinite, then in the finite complement topology on X singletons are never open.

A variation of the above example is the following.

Example 3.6. Let X be a set. Define

$$\tau_c = \{U \subseteq X \,:\, X - U \text{ is either countable or equals } X\}.$$

It is easy to check that τ_c is a topology on X called the *cocountable* topology on X.

Again, if X is finite, the cocountable topology is the same as the discrete topology. Actually more is true. If X is countable, then the cocountable topology coincides with the discrete topology. The difference becomes apparent when the the set X is uncountable.

Often two topologies can be compared.

Definition 3.7. Let τ, τ' be two topologies on a set X. Assume that $\tau \subseteq \tau'$. We then say that τ' is *finer* than τ. We also say that τ is *coarser* than τ'

Let (X, τ) be a topological space. Instead of describing all the open sets in τ it is often convenient to specify a much smaller collection of open sets that determines the topology τ. This collection is called a basis for the topology. Here is the definition.

Definition 3.8. Let X be a set and $\mathcal{B}$ a collection of subsets of X. We say that $\mathcal{B}$ is a *basis* for a topology on X if

(1) $X = \bigcup_{B \in \mathcal{B}} B$,
(2) if $B, B' \in \mathcal{B}$ and $x \in B \cap B'$, then there exists $B'' \in \mathcal{B}$ with

$$x \in B'' \subseteq B \cap B'.$$

Let $\mathcal{B}$ be a basis for a topology on a set X. Then the topology τ generated (or determined) by $\mathcal{B}$ is defined in the following obvious manner.

Proposition 3.9. Let $\mathcal{B}$ be a basis for a topology on a set X. Let $\tau_{\mathcal{B}}$ be defined to be

$$\tau = \tau_{\mathcal{B}} = \{U \subseteq X \, : \text{ for each } x \in U, \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq U\}.$$

Then $\tau = \tau_{\mathcal{B}}$ is a topology on X called the topology generated by the basis $\mathcal{B}$.

Proof. This is a straightforward verification. It is clear that $\emptyset \in \tau$. That $X \in \tau$ follows from the fact that the union of the elements of $\mathcal{B}$ equals X.

Next let $\{U_i\}$ be an indexed family of sets in τ and let $x \in \cup_i U_i$. Thus $x \in U_j$ for some j. Since $U_j \in \tau$, there exists $B \in \mathcal{B}$ with

$$x \in B \subseteq U_j \subseteq \bigcup_i U_i.$$

This shows that $\cup_i U_i \in \tau$. Hence τ is closed with respect to arbitrary unions.

Finally suppose that $\{U_1, \dots, U_r\}$ is a finite collection of sets in τ and let

$$x \in U_1 \cap \cdots \cap U_r.$$

Then we can find $B_i \in \mathcal{B}$ with

$$x \in B_i \subseteq U_i$$

for $i = 1, 2, \dots, r$. Since $\mathcal{B}$ is a basis there exists $B \in \mathcal{B}$ with

$$x \in B \subseteq B_1 \cap B_2 \cdots \cap B_r \subseteq U_1 \cap \cdots \cap U_r.$$

Thus $\cap_i U_i \in \tau$. This shows that the collection τ is closed with respect to finite unions. Thus τ is a topology on X. $\square$

Remark 3.10. Here is an equivalent way of describing the topology generated by a basis. If $\mathcal{B}$ is a basis for a topology on X, then the topology determined by $\mathcal{B}$ is the collection of all possible unions of elements of $\mathcal{B}$. This verification is left as an exercise. Thus if we know a basis for the topology on X we know all open sets in X. Observe that if $\mathcal{B}$ is a basis for the topology τ, then

(1) every $B \in \mathcal{B}$ is an open set in X and
(2) every open set in X is the union of basic open sets, that is, the union of elements of $\mathcal{B}$.

Here are some examples

Example 3.11. Let X be a set and let $\mathcal{B}$ be the collection

$$\mathcal{B} = \{\{x\} \,:\, b \in X\}$$

of singleton subsets of X. Then $\mathcal{B}$ is a basis for a familiar topology on X.

Example 3.12. Consider the collection

$$\mathcal{B} = \{(a, b) \,:\, a < b \text{ and } a, b \in \mathbb{R}\}$$

of open intervals in the real line. This collection is easily seen to be basis for a topology on $\mathbb{R}$. The topology determined by this collection is called the *usual topology* on $\mathbb{R}$.

In view of Remark 3.10, in the usual topology on $\mathbb{R}$, a subset $U \subseteq \mathbb{R}$ is open if and only if it is the union of open intervals. Observe that in the usual topology on $\mathbb{R}$, the set $U = [0, 1)$ is not an open set. For we cannot find an interval (a, b) with

$$0 \in (a, b) \subseteq [0, 1).$$

Example 3.13. The collection of half open intervals of the form

$$\mathcal{B} = \{[a, b) \,:\, a < b\}$$

is a basis for a topology on $\mathbb{R}$ called the *lower limit* topology. We shall denote by $\mathbb{R}_\ell$ the real line with the lower limit topology.

Note that in the lower limit topology on $\mathbb{R}$, the subset

$$U = [0, 1)$$

is open by definition. Further, as

$$(0, 1) = \bigcup_n [1/n, 1)$$

we see that the open interval $U = (0, 1)$ (which is open in the usual topology on $\mathbb{R}$) is also open in the lower limit topology on $\mathbb{R}$. It is immediate now that

$$\tau \subseteq \tau_\ell$$

where τ is the usual topology on $\mathbb{R}$ and τ_ℓ is the lower limit topology on $\mathbb{R}$. Thus the lower limit topology on $\mathbb{R}$ is finer than the usual topology on $\mathbb{R}$.

Here is one more topology on the real line which along with the lower limit topology is useful to construct counter examples.

Definition 3.14. Let

$$K = \{1/n \,:\, n \in \mathbb{Z}_+\}.$$

Let $\mathcal{B}$ be the collection of open intervals of the form (a, b), $a < b$, and sets of the form $(a, b) - K$. Then one easily sees that $\mathcal{B}$ is a basis for a topology τ_K on $\mathbb{R}$ called the K-topology on $\mathbb{R}$. The real line with the K-topology is denoted by $\mathbb{R}_K$. It is immediate from the definition that the topology τ_K is finer than the usual topology on $\mathbb{R}$.

We note that the set $(-1,1)-K$ (which is open in the K-topology on $\mathbb{R}$) is not open in the usual topology on $\mathbb{R}$. For if $(-1,1)-K$ is open in the usual topology on $\mathbb{R}$, we should be able to find a basic open set (a,b) in the usual topology such that

$$0 \in (a,b) \subseteq (-1,1)-K.$$

But such an interval does not exist. Thus the K-topology is strictly finer than the usual topology on $\mathbb{R}$.

Given a topology on a set X, it is often useful to be able to identify a basis for the topology that we are working with. This can be done as below.

Lemma 3.15. Let (X,τ) be a topological space. Let $\mathcal{B}$ be the collection of open sets in X with the property that for each open set U in X and each $x \in U$, there exists $B \in \mathcal{B}$ with

$$x \in B \subseteq U.$$

Then $\mathcal{B}$ is a basis for the topology τ.

Proof. There are two points that need to be checked : (i) $\mathcal{B}$ is a basis for a topology on X and (ii) the topology determined by $\mathcal{B}$ equals τ.

We first check that $\mathcal{B}$ is basis for a topology on X. The definition of the collection $\mathcal{B}$ tells us that

$$X = \bigcup_{B \in \mathcal{B}} B.$$

Next, let $x \in B \cap B'$ with $B, B' \in \mathcal{B}$. As $B, B \in \mathcal{B} \subseteq \tau$ we have that B, B' are open in X and hence so is $B \cap B'$. Thus by definition of the collection $\mathcal{B}$ there exists $B'' \in \mathcal{B}$ with

$$x \in B'' \subseteq B \cap B'.$$

This completes the proof that $\mathcal{B}$ is a basis for a topology, say τ', on X.

Next we check (ii). Since $\mathcal{B} \subseteq \tau$ it is immediate that $\tau' \subseteq \tau$. Now let $U \in \tau$ so that U is an open subset of X (in the τ topology. We shall show that $U \in \tau'$. To prove this we need to check that U is the union of elements of $\mathcal{B}$. But this is precisely how the collection $\mathcal{B}$ was defined. Thus $\tau' = \tau$ completing the proof. $\square$

There is something more "basic" than a basis.

Definition 3.16. A collection $\mathcal{S}$ of subsets of X is called a *subbasis* for a topology on X if the union of elements of $\mathcal{S}$ equals X. The topology determined by $\mathcal{S}$ is the collection of all arbitrary unions of finite intersections of elements of X.

Given a topological space (X,τ) there always exists a basis and a subbasis for τ. Here are some exercises.

Exercise 3.17. Prove the claims made in Example 3.5 and Example 3.6.

Exercise 3.18. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ' respectively on X. Then show that the following are equivalent.

(1) τ' is finer than τ.
(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there exists a basis element $B' \in \mathcal{B}'$ with
$$x \in B' \subseteq B.$$

Exercise 3.19. Verify that the lower limit topology and the K-topology are indeed topologies on $\mathbb{R}$.

Exercise 3.20. Prove the claim in Example 3.16.

Exercise 3.21. Let A be a subset of the topological space X. Suppose that for each $x \in A$ there is an open set U such that
$$x \in U \subseteq A.$$
Show that A is open in X. This is a useful criteria to show that a set is open.

Exercise 3.22. Let $\{\tau_i\}$ be a family of topologies on X. Show that
$$\tau = \bigcap_i \tau_i$$
is a topology on X. Further show that there is a unique smallest topology on X containg all the τ_i and a unique largest topology on X contained in all the τ_i.

Exercise 3.23. Let $\mathcal{B}$ be a basis for a topology τ on X. Show that τ is the intersection of all topologies on X that contain $\mathcal{B}$. Is the same true if $\mathcal{B}$ is a subbasis?

Exercise 3.24. Show that the topologies $\mathbb{R}_K$ and $\mathbb{R}_\ell$ are not comparable.

Exercise 3.25. Give an example to show that a basis for a topology need not be unique.

Exercise 3.26. Exhibit a basis each for the discrete and indiscrete topologies on a set X.

Exercise 3.27. Exhibit a subbasis for the usual topology on $\mathbb{R}$.