

Exercise If G is a (simple) non-Hamiltonian graph on n vertices, then prove that G has at most $\binom{n-1}{2} + 1$ edges.

Solution Induction on n .

① If $n=3$ then $\binom{n-1}{2} + 1 = 2$.

Only K_3 has more edges & it is Hamiltonian.

② Assume the result for $n-1$.

Note that this means that any graph with $n-1$ vertices & $\geq \binom{n-2}{2} + 2$ edges must be Hamiltonian.

(*) — { which in turn implies that any $n-1$ vertex graph with $\binom{n-2}{2} + 1$ number of edges must have a Hamilton path. (because adding an edge gives Hamilton cycle)

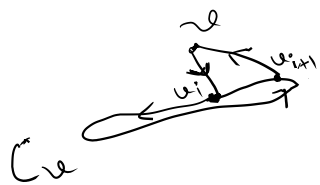
③ Now let G be a graph on n vertices which is not Hamiltonian.

Case a

$\forall v \in V(G)$, The induced graph on $V(G) - v$ is

Hamiltonian. Let C_v denote a Hamilton cycle of $G - v$. Then note that v cannot

be adjacent to two consecutive vertices of C_v . (If so, then $v \rightarrow v_i \rightarrow C_v \rightarrow v_{i+1} \rightarrow v$ gives Hamilton cycle of G !)



$$\text{Thus, } \deg_G(v) \leq \lceil \frac{n-1}{2} \rceil \leq \frac{n}{2}.$$

$$\Rightarrow \sum \deg_G(v) \leq n \cdot \frac{n}{2} = \frac{n^2}{2}.$$

Since $\sum \deg v = 2|E(G)|$,
we get $|E(G)| \leq \frac{n^2}{4}$

$$\text{But } \frac{(n-1)(n-2)}{2} + 1 - \frac{n^2}{4} \geq 0 \quad \forall n \geq 4.$$

Hence result is true in this case.

case b) $\exists v \in V(G)$; s.t. $G-v$ is not Hamiltonian.

$$\Rightarrow |E(G)| \leq \deg_G v + \binom{n-2}{2} + 1$$

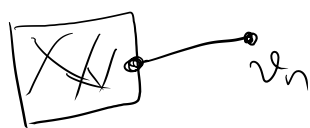
\hookrightarrow by induction hypothesis.

Now if $G-v$ has $\binom{n-2}{2} + 1$ edges, then by (*) above, it has a Hamiltonian path. $\Rightarrow \deg v \neq n-1$ (otherwise v is adjacent to the end points of this path & hence gives Hamilton cycle for G , a contradiction)

$$\Rightarrow \deg v + |E(G-v)| \leq (n-2) + \binom{n-2}{2} + 1 \quad \underline{\underline{\text{always}}}$$

\parallel
 $\binom{n-1}{2} + 1$

This completes the inductive step & hence the proof.

Remark :- The graph 

gives example in which the upper bound is attained.