Generalized Likelihood Ratio Tests (GLRT)

UMP tests do not exist in all but simple situations. UMPU tests also may not exist. How does one conduct a test then? The approach that seems reasonable is to derive tests heuristically, and then check for their optimality.

Let $X \sim P_\theta, \theta \in \Theta$ having density $f(x|\theta)$. Consider testing

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1.$$

Then the Generalized Likelihood Ratio statistic is defined to be

$$L(x) = \frac{\sup_{\theta \in \Theta_1} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)}.$$

Reject H_0 if L is too large. This is a reasonable approach because we saw earlier that $\frac{f(x|\theta_1)}{f(x|\theta_0)}$ can be looked upon as evidence against $H_0 : \theta = \theta_0$ and in favour of $H_1 : \theta = \theta_1$. Now, $\sup_{\theta \in \Theta_1} f(x|\theta)$ is the best evidence for $H_1 : \theta \in \Theta_1$ whereas $\sup_{\theta \in \Theta_0} f(x|\theta)$ is the best evidence for $H_0 : \theta \in \Theta_0$. Suppose $\Theta = \Theta_0 \cup \Theta_1$. Consider

$$\lambda(x) = \frac{\sup_{\theta \in \Theta} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)}.$$

Then $\lambda(x) = \max\{L(x), 1\}$ since

$$\lambda(x) = \begin{cases} 1 & \text{if } \sup_{\theta \in \Theta_0} f(x|\theta) \geq \sup_{\theta \in \Theta_1} f(x|\theta); \\ L(x) & \text{if } \sup_{\theta \in \Theta_0} f(x|\theta) < \sup_{\theta \in \Theta_1} f(x|\theta). \end{cases}$$

Note that

$$\lambda_n(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n|\hat{\theta})}{f(x_1, \ldots, x_n|\hat{\theta}_0)},$$

where
$\hat{\theta}$ = MLE of θ in Θ,
$\hat{\theta}_0$ = MLE of θ in Θ_0.
If an increasing function of $\lambda(\mathbf{X})$ has a standard distribution under H_0, then it can be used to construct the test.

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, both μ and σ^2 unknown. Test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. Then

$$\Theta_0 = \{(\mu = 0, \sigma^2), \sigma^2 > 0\}, \qquad \Theta_1 = \{(\mu, \sigma^2), -\infty < \mu < \infty, \mu \neq 0, \sigma^2 > 0\}.$$

MLE are needed to compute the GLR statistic: unrestricted and, restricted to Θ_0.

$$\hat{\theta} = (\hat{\mu} = \bar{X}, \qquad \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2),$$

$$\hat{\theta}_0 = (\hat{\mu}_0 = 0, \qquad \hat{\sigma^2}_0 = \frac{1}{n}\sum_{i=1}^{n} X_i^2).$$

Therefore,

$$\begin{aligned}
\lambda(\mathbf{x}) &= \frac{f(\mathbf{x}|\hat{\theta})}{f(\mathbf{x}|\hat{\theta}_0)} \\
&= \frac{(2\pi)^{-n/2}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-n/2}\exp\left(-\frac{1}{2\hat{\sigma}^2}\left\{\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\hat{\mu})^2\right\}\right)}{(2\pi)^{-n/2}\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)^{-n/2}\exp\left(-\frac{1}{2\hat{\sigma}_0^2}\left\{\sum_{i=1}^{n}x_i^2\right\}\right)} \\
&= \frac{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-n/2}}{\left(\sum_{i=1}^{n}x_i^2\right)^{-n/2}} = \left(\frac{\sum_{i=1}^{n}x_i^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{n/2} \\
&= \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2 + n\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{n/2} = \left(1 + \frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{n/2}.
\end{aligned}$$

Note that $\lambda(\mathbf{x})$ is an increasing function of

$$T^2 = \frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)},$$

and therefore of $|T|$, where

$$T = \frac{\sqrt{n}\bar{X}}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2/(n-1)}} \sim t_{n-1}, \text{ if } H_0 \text{ is true.}$$

Therefore the GLRT rejects H_0 if

$$\left|\frac{\sqrt{n}\bar{x}}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)}}\right| > t_{n-1}(1-\alpha/2).$$

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, σ^2 known. Test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Derive the GLR statistic and show that GLRT rejects H_0 when

$$\left|\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma}\right| > z_{1-\alpha/2}.$$

Note. Classical or Frequentist test procedure (which is what we have been discussing) is predetermined. $\mathbf{x}$ or data is used only to check whether it falls in the rejection region or not. Exact value of $\mathbf{x}$ is not relevant. What is reported is the level α and whether $\phi(\mathbf{x})$ is 1, γ or 0. If H_0 is true, then the test procedure will ensure that, if used over and over again, the long-run average rejection rate is α.

Confidence Sets and Hypothesis Tests

For a confidence set, we want $S(X) \subset \Theta$ such that

$$P_\theta\left(\theta \in S(X)\right) \geq 1 - \alpha \text{ for all } \theta \in \Theta.$$

Then $S(X)$ is said to be 100(1-α)% confidence set for θ. Suppose we have available to us a test procedure for testing $H_0 : \theta = \theta'$ versus $H_1 : \theta \neq \theta'$ for any $\theta' \in \Theta$. Let $A(\theta') \subset \mathcal{X}$ be the acceptance region of the level α test of $H_0 : \theta = \theta'$ versus $H_1 : \theta \neq \theta'$. Define

$$\begin{aligned} S(x) &= \{\theta' \in \Theta \text{ such that } x \in A(\theta')\} \\ &= \{ \text{ all } \theta' \text{ for which } H_0 : \theta = \theta' \text{ will be accepted if } x \text{ is observed. } \} \end{aligned}$$

Then $\theta \in S(x)$ iff $x \in A(\theta)$. Therefore,

$$P_\theta\left(\theta \in S(X)\right) = P_\theta\left(X \in A(\theta)\right) \geq 1 - \alpha.$$

Therefore, $S(X)$ is 100(1-α)% confidence set for θ.

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, σ^2 known. Then $\bar{X}$ is sufficient and $\bar{X} \sim N(\mu, \sigma^2/n)$. Recall that GLRT for testing $H_0 : \mu = \mu'$ versus $H_1 : \mu \neq \mu'$ rejects H_0 when

$$\left|\frac{\sqrt{n}(\bar{x} - \mu')}{\sigma}\right| > z_{1-\alpha/2}.$$

Therefore, its acceptance region is

$$A(\mu') = \left\{\bar{x} : \left|\frac{\bar{x} - \mu'}{\sigma/\sqrt{n}}\right| \leq z_{1-\alpha/2}\right\}.$$

Hence,

$$S(\bar{x}) = \left\{\mu' : \left|\frac{\bar{x} - \mu'}{\sigma/\sqrt{n}}\right| \leq z_{1-\alpha/2}\right\}.$$

Therefore the resulting confidence set (interval) is

$$S(\bar{X}) = \bar{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, both μ and σ^2 unknown. It is of interest to construct a confidence set for μ. $(\bar{X}, S^2 = \sum_{i=1}^n (X_i - \bar{X})^2)$ is sufficient. Let $s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2)$. Consider the GLRT for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Its acceptance region is

$$\begin{aligned} A(\mu_0) &= \left\{ (\bar{x}, s^2) : \left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \leq t_{n-1}(1 - \alpha/2) \right\}, \text{ so that} \\ S(\bar{x}, s^2) &= \left\{ \mu : \left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| \leq t_{n-1}(1 - \alpha/2) \right\}. \end{aligned}$$

This yields the confidence interval:

$$S(\bar{X}, s^2) = \bar{X} - t_{n-1}(1 - \alpha/2)\frac{s}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{n-1}(1 - \alpha/2)\frac{s}{\sqrt{n}}.$$