Are there UMP tests for all fairly simple problems? Not in most cases.

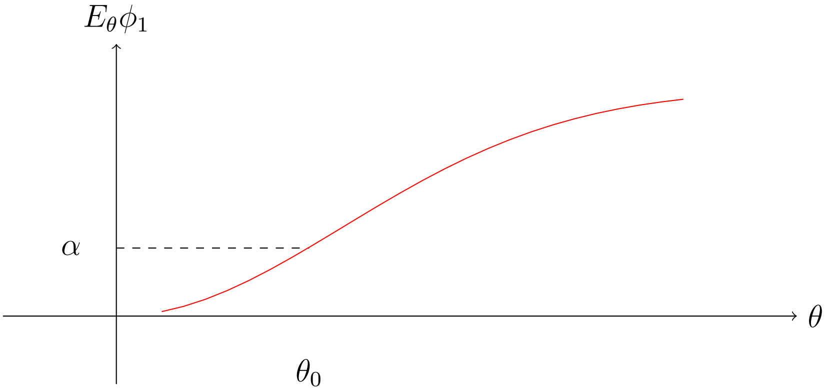
Example. $X_1, \dots, X_n$ i.i.d $N(\mu, \sigma^2)$, σ^2 unknown. Test $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$. This seems to be a simple problem, but N-P Lemma does not apply since the hypotheses are not simple:

$$\Theta_0 = \{(\mu_0, \sigma^2), \sigma^2 > 0\}, \qquad \Theta_1 = \{(\mu_1, \sigma^2), \sigma^2 > 0\}.$$

What about problems where $\Theta \subset \mathcal{R}^1$ and MLR exists? No, not in most cases, even then. Suppose $X \sim P_\theta, \theta \in \Theta \subset \mathcal{R}^1$ and MLR exists in $T(x)$. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. First consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. Then UMP test exists and is given by

$$\phi_1(x) = \begin{cases} 1 & \text{if } T(x) > C_1; \\ \gamma_1 & \text{if } T(x) = C_1; \\ 0 & \text{if } T(x) < C_1, \end{cases}$$

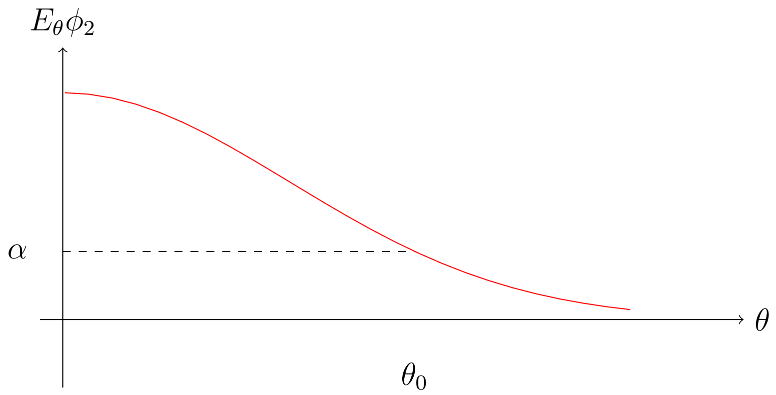
where C_1 and γ_1 are determined by $E_{\theta_0}\phi_1(X) = \alpha$. Observe the power function, $E_\theta \phi_1(X)$ of this test.



For all $\theta > \theta_0$, ϕ_1 maximizes the power among all level α tests. Does it maximize the power for any $\theta < \theta_0$? No. To see this, consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$. Then the UMP test is given by

$$\phi_2(x) = \begin{cases} 1 & \text{if } T(x) < C_2; \\ \gamma_2 & \text{if } T(x) = C_2; \\ 0 & \text{if } T(x) > C_2, \end{cases}$$

where C_2 and γ_2 are determined by $E_{\theta_0}\phi_2(X) = \alpha$. This test, by definition, maximizes the power for all $\theta < \theta_0$.



Therefore, no single test ϕ uniformly maximizes the power for all $\theta \neq \theta_0$ subject to $E_{\theta_0}\phi(X) = \alpha$.

One may argue that ϕ_1 and ϕ_2 both are clearly not reasonable in this case, since the power falls below the level for certain alternatives. Eliminate these by putting the condition

$$E_\theta\phi(X) \geq \alpha \text{ for all } \theta \neq \theta_0.$$

In general, for testing

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1$$

the test ϕ that maximizes the power $E_\theta\phi$ for $\theta \in \Theta_1$ subject to

$$\sup_{\theta\in\Theta_0} E_\theta\phi(X) \leq \alpha \text{ and } E_\theta\phi(X) \geq \alpha \text{ for all } \theta \neq \theta_0$$

is called the Uniformly Most Powerful Unbiased (UMPU) test. They exist under some very stringent conditions on the model density. These situations are rare. (See Lehmann, *TSH*.)

Generalized Likelihood Ratio Tests (GLRT)

UMP tests do not exist in all but simple situations. UMPU tests also may not exist. How does one conduct a test then? The approach that seems reasonable is to derive tests heuristically, and then check for their optimality.

Let $X \sim P_\theta, \theta \in \Theta$ having density $f(x|\theta)$. Consider testing

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1.$$

Then the Generalized Likelihood Ratio statistic is defined to be

$$L(x) = \frac{\sup_{\theta \in \Theta_1} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)}.$$

Reject H_0 if L is too large. This is a reasonable approach because we saw earlier that $\frac{f(x|\theta_1)}{f(x|\theta_0)}$ can be looked upon as evidence against $H_0 : \theta = \theta_0$ and in favour of $H_1 : \theta = \theta_1$. Now, $\sup_{\theta \in \Theta_1} f(x|\theta)$ is the best evidence for $H_1 : \theta \in \Theta_1$ whereas $\sup_{\theta \in \Theta_0} f(x|\theta)$ is the best evidence for $H_0 : \theta \in \Theta_0$. Suppose $\Theta = \Theta_0 \cup \Theta_1$. Consider

$$\lambda(x) = \frac{\sup_{\theta \in \Theta} f(x|\theta)}{\sup_{\theta \in \Theta_0} f(x|\theta)}.$$

Then $\lambda(x) = \max\{L(x), 1\}$ since

$$\lambda(x) = \begin{cases} 1 & \text{if } \sup_{\theta \in \Theta_0} f(x|\theta) \geq \sup_{\theta \in \Theta_1} f(x|\theta); \\ L(x) & \text{if } \sup_{\theta \in \Theta_0} f(x|\theta) < \sup_{\theta \in \Theta_1} f(x|\theta). \end{cases}$$

Note that

$$\lambda_n(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n|\hat{\theta})}{f(x_1, \ldots, x_n|\hat{\theta}_0)},$$

where
$\hat{\theta}$ = MLE of θ in Θ,
$\hat{\theta}_0$ = MLE of θ in Θ_0.
If an increasing function of $\lambda(\mathbf{X})$ has a standard distribution under H_0, then it can be used to construct the test.