Graph Theory

Lecture 18

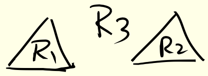
Planar Graphs - 2

Applications of Euler's formula.

Theorem (Euler 1758) If a connected planar graph has n vertices, e edges & f faces, then

$$n - e + f = 2.$$

Q. What happens if G is not connected?

$G =$ 

$\Rightarrow n - e + f = 3.$

$$\boxed{n - e + f = 2 + (r-1) \text{ where } r = \# \text{ connected comp.}}$$

Thm: ① If G is planar with $n \geq 3$ then $e \leq 3n - 6$

② If girth$(G) \geq 4$ & planar then $e \leq 2n - 4$

pf. ① If G is not connected, add an edge between two comp. of G ($G + e$ will still be planar (Exercise)). successively doing this, we can assume that G is connected. If n, e, f are as in Euler's theorem, then $\boxed{n - e + f = 2}$.

Want to use $\sum l(f_i) = 2e$.

f_i's are faces & $l(f_i)$ are the lengths of the boundary walk of f_i. Since every face is bounded by at least a triangle, $\sum l(f_i) \geq 3f \Rightarrow 3f \leq 2e$.

or $f \leq \frac{2}{3} e$

$$\Rightarrow n - e + \frac{2}{3} e \geq 2 \Rightarrow \boxed{e \leq 3n - 6}$$

② If G is triangle free then $l(f) \geq 4$

$$\Rightarrow \quad 2e = \sum l(f_i) \geq 4f. \text{ ie } \underline{f \leq \tfrac{1}{2}e}$$

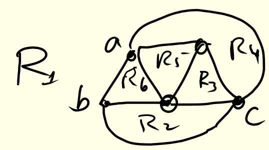
$$\Rightarrow \quad n - e + \tfrac{1}{2}e \geq 2 \quad \Rightarrow \quad \boxed{e \leq 2n-4}$$

<u>QED</u>.

① $\Rightarrow$ K_5 the complete graph on 5 vertices is <u>not</u> planar

② $\Rightarrow$ $K_{3,3}$ the complete bipartite graph is <u>not</u> planar.

If $e = 3n-6$ then every face must be enclosed by a triangle & G is connected.



5-vertices
9-edges
$\Rightarrow e = 3n-6$.

such a graph is called a triangulated graph. Maximal planar graph on n-vertices.

① $\Rightarrow$ Every planar graph must have a vertex of $\deg \leq 5$.

(If not; $2e = \sum_{v \in V} \deg v \geq 6n$.

$\Rightarrow e \geq 3n$ but $e \leq 3n-6$ contradiction.)

$\Rightarrow$ Every subgraph of a planar graph has vertex of $\deg \leq 5$.

<u>Colouring of Planar graphs</u>.

<u>Thm</u>: One can order vertices of a planar graph say $v_1, \ldots, v_n$ such that $\forall i$; $\exists$ at most 5 neighbours of v_i in $v_1, v_2, \ldots, v_{i-1}$.

<u>Remark</u>: Greedy algorithm $\Rightarrow$ any planar graph is 6-colourable.

<u>pf</u>. Put v_n = vertex of $\deg \leq 5$ in G.

v_{n-1} = vertex of deg ≤ 5 in $\langle G - v_n \rangle$

& so on. QED!

Thm: Every planar graph is 5-colourable.

Standard trick in colouring of graph

Colouring of graph is a function $f : V(G) \to \{1, \ldots, k\}$ s.t. $f(v_i) \neq f(v_j)$ if $v_i v_j \in E(G)$.

$\cancel{f^{-1}(i)} = V_i = \{ v \in V(G) \mid f(v) = i \}$. Then G is a subgraph of a k-partite graph. Let G_{ij} denote the induced bipartite graph on $V_i \cup V_j$. These subgraphs usually play a useful role in various proofs.

Proof: Induction on $|V(G)| = n$

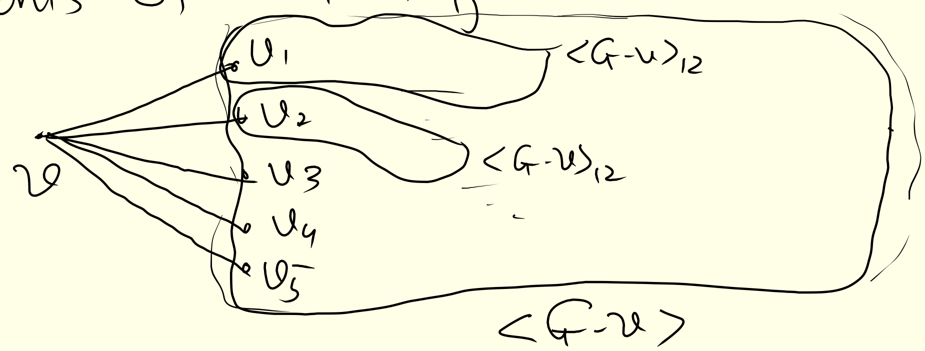
If $n \leq 5$ there is nothing to prove.

Assume that any planar graph with $n-1$ vertices is 5 colourable. Let G be a planar graph on n-vertices. Since G is planar, $\exists\, v \in V(G)$ s.t. $\deg(v) \leq 5$.

- If $\deg v < 5$ then colour $\langle G - v \rangle$ by 5-colours & there will be one colour missing amongst neighbours of v. Give that colour to v to get a 5-colouring of G.
- $\deg v = 5$. If $\exists$ a colouring of $G - v$ giving only at most colours to the neighbours of v, then we are done like before!

∴ Assume that every colouring $G-v$ gives 5-different colours to the neighbours of v. WLOG assume that $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ & v_i gets colour i in a colouring of $\langle G-v\rangle$

If v_i & v_j belong to different connected components of $\langle G-v\rangle_{ij}$ then we interchange colours i & j <u>only</u> on the conn. comp. of $\langle G-v\rangle_{ij}$ containing v_i



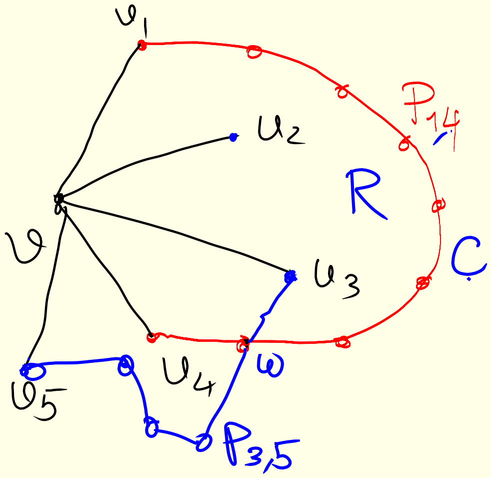
$\Rightarrow$ both v_i & v_j will get colour j & colour i is missing from $N(v)$. $\Rightarrow$ G has 5-colouring by giving colour i to v.

* Thus we can assume: G has a vertex v of deg 5. All five colours occur in $N(v)$. And $\forall\, i \neq j$ the pair $\{v_i, v_j\}$ lie in the same connected component of the subgraph $\langle G-v\rangle_{ij}$.

$\Rightarrow$ $\forall\, v_i \neq v_j$ $\exists$ a path in $\langle G-v\rangle_{ij}$ joining v_i & v_j. Let P_{ij} denote such paths $\forall\, 1 \le i \neq j \le 5$.

Now, look at the following diagram in G.

Jordan Curve Theorem



The path P_{14} together with edges vv_1 & vv_4 form a cycle C s.t. $u_3 \in$ interior of C, $v_5 \in$ exterior of C.

Then the path $P_{3,5}$ must intersect C. Since $P_{3,5} \subset \langle G-V \rangle_{3,5}$, $P_{3,5}$ can not contain v. $\therefore \exists\, w \in P_{1,4} \cap P_{3,5}$

contradiction!

$\therefore$ $*$ can not happen.

$\Rightarrow$ G is 5-colourable.

QED.