Topologies on $\mathbb{R}$

- $\mathbb{R}_f = \{ A \subseteq \mathbb{R} : A^c = \mathbb{R} \text{ or finite} \}$ co-finite
- $\mathbb{R}_c = \{ A \subseteq \mathbb{R} : A^c = \mathbb{R} \text{ or countable} \}$ co-countable
- $\mathbb{R}_d = \langle \mathcal{B}_d \rangle$, $\mathcal{B}_d = \{ \{x\} : x \in \mathbb{R} \}$ discrete
- $\mathbb{R} = \langle \mathcal{B} \rangle$, $\mathcal{B} = \{ (a,b) : a<b, a,b \in \mathbb{R} \}$ usual
- $\mathbb{R}_\ell = \langle \mathcal{B}_\ell \rangle$, $\mathcal{B}_\ell = \{ [a,b) : a<b \}$ lower limit
- $\mathbb{R}_u = \langle \mathcal{B}_u \rangle$, $\mathcal{B}_u = \{ (a,b] : a<b \}$ upper limit
- $\mathbb{R}_K = \langle \mathcal{B}_K \rangle$, $\mathcal{B}_K = \{ (a,b) \text{ or } (a,b) \setminus K : a<b \}$ $\mathbb{R}_K$

  $K = \{ 1, 1/2, 1/3, \cdots \}$
- $\mathbb{R}_\infty = \langle \mathcal{B}_\infty \rangle$, $\mathcal{B}_\infty = \{ (a,\infty) : a \in \mathbb{R} \}$ $\mathbb{R}_\infty$
- $\mathbb{R}_{-\infty} = \langle \mathcal{B}_{-\infty} \rangle$, $\mathcal{B}_{-\infty} = \{ (-\infty,a) : a \in \mathbb{R} \}$ $\mathbb{R}_{-\infty}$

Problems:-

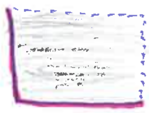
- $B_1 = \{ (a,b) : a<b, a,b \in \mathbb{Q} \}$ Is $\langle B_1 \rangle = \mathbb{R}$?
- $B_2 = \{ [a,b) : a<b, a,b \in \mathbb{Q} \}$. Is $\langle B_2 \rangle = \mathbb{R}_\ell$?

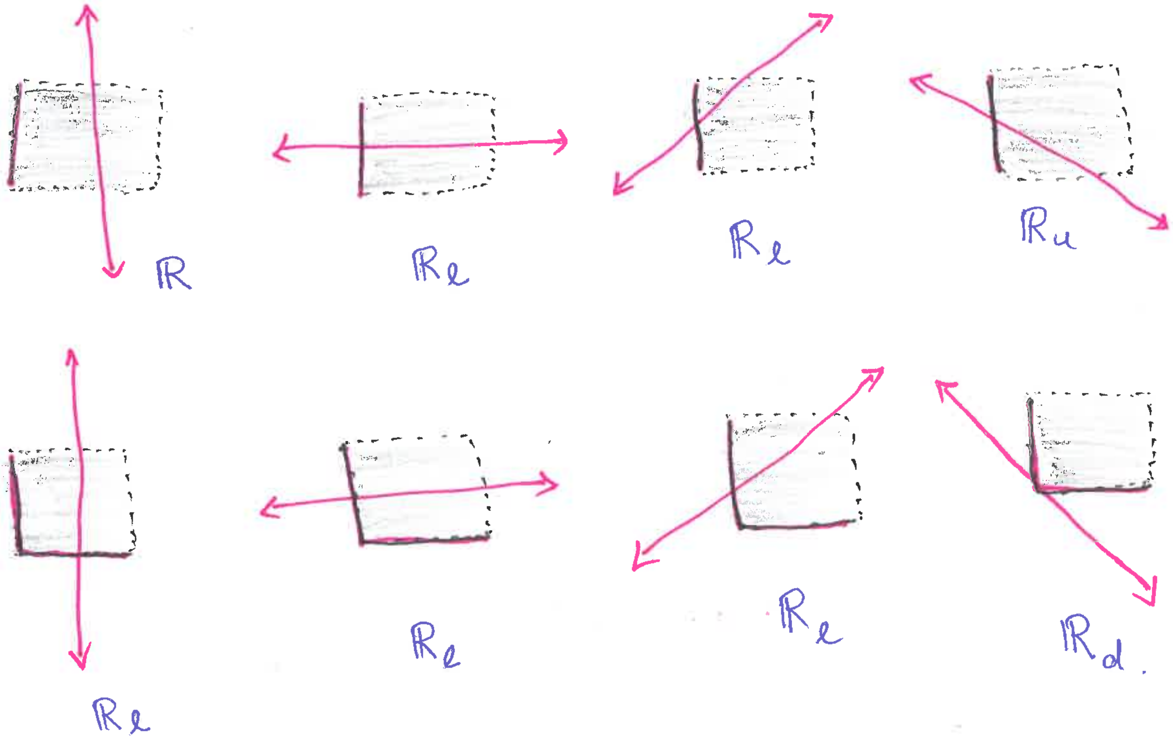
Sol:: $\langle B_1 \rangle = \mathbb{R}$, $\langle B_2 \rangle \subseteq \mathbb{R}_\ell$, $[\sqrt{2}, 2) \in \mathbb{R}_\ell \setminus \langle B_2 \rangle$.

- L is a straight line in $\mathbb{R}^2$. Describe the topology on L inherits
  - (i) as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$
  - (ii) as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$

Sol:

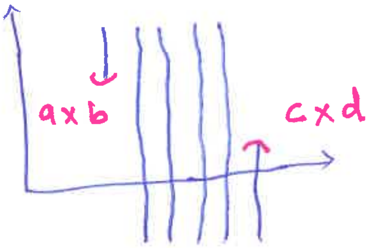
$\mathbb{R}_\ell \times \mathbb{R}$ 

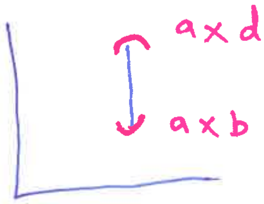
$\mathbb{R}_\ell \times \mathbb{R}_\ell$ 



- Prove that dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is same as the product topology $\mathbb{R}_d \times \mathbb{R}$.

Dictionary order:-





$$\mathcal{B}_1 = \{ (a\times b,\ c\times d) \ : \ a<c \ \text{(or)} \ b<d \ \text{if} \ a=c \}$$

$\mathcal{B}_2 = \{\{a\} \times (b,d)\}$ is also basis for dictionary topology.

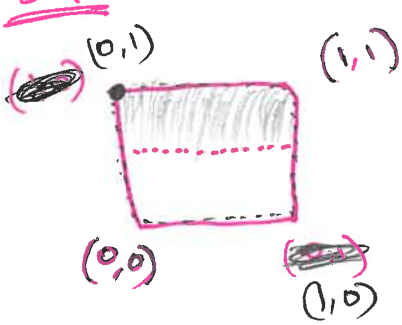
ie) Dictionary top $= \mathbb{R}_d \times \mathbb{R}$.

Problem:. Let $I = [0,1]$. consider the following Topologies.

- Product topology on $I \times I$
- Dictionary order top. on $I \times I$
- subspace top of $\mathbb{R} \times \mathbb{R}$ with dictionary order.

Compare the topologies.

Sol:.



open in first but not in ~~other~~ second.

$\{0\} \times (0,1)$ open in second but not in first.

Third one is finer than other two!

Problem: consider various Topologies on $\mathbb{R}$. To what point or points does the sequence $1/n$ converges?

Solution:

$L = \{x \in \mathbb{R} : 1/n \to x\}$

1. $\mathbb{R}$ $\qquad L = \{0\}$
2. $\mathbb{R}_d$ $\qquad L = \phi$
3. $\mathbb{R}_f$ $\qquad L = \mathbb{R}$
4. $\mathbb{R}_c$ $\qquad L = \phi$
5. $\mathbb{R}_\ell$ $\qquad L = \{0\}$
6. $\mathbb{R}_u$ $\qquad L = \phi$
7. $\mathbb{R}_\infty$ $\qquad L = (-\infty, 0]$
8. $\mathbb{R}_{-\infty}$ $\qquad L = [0, \infty)$
9. $\mathbb{R}_K$ $\qquad L = \phi$

x