Compact Subspaces of $\mathbb{R}_l$ and $\mathbb{R}_K$

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In this brief note, we wish to categorise all possible compact subsets of $\mathbb{R}_l$ and $\mathbb{R}_K$. First of all, we state a few general theorems.

1 Some Generic Theorems

Let X be a topological space and X' be a finer topology on X.

Theorem 1.1. Any finite subset of X is compact

Proof. Let $A \subset X$ be finite. Then If $\mathcal{U}$ is an open cover of A, for each $a \in A$, $\exists U_a \in \mathcal{U}$ such that $a \in U_a$. Then $\cup_{a \in A} U_a$ is a finite subcover and we are done. $\square$

Theorem 1.2. $A \subset X'$ is compact, then $A \subset X$ is compact

Proof. X' is finer than X, so any open cover $\mathcal{U}$ of A in X is also an open cover of A in X'. But the latter has a finite subcover. But this is also a finite subcover in X. So we are done. $\square$

Theorem 1.3. Suppose a set $A \subset \mathbb{R}$ satisfies $\forall a \in A$, $\exists \epsilon_a > 0$ s.t. $I_a = (a - \epsilon_a, a) \subset \mathbb{R} \setminus A$, then A is countable.

Proof. Every I_a contains a rational number. Also I_a are disjoint for different $a \in A$. So we can form an injective function $f : A \to \mathbb{Q}$ by Axiom of Choice. Thus indeed, A is countable. $\square$

Subsets satisfying above condition will be called LG (left gap) subsets.

Now we analyse compact subsets of $\mathbb{R}_l$ and $\mathbb{R}_K$.

2 $\mathbb{R}_l$

Proposition 2.1. Any Compact subset of $\mathbb{R}_l$ is LG

Proof. Suppose A is a compact subset of $\mathbb{R}$. Let $a \in A$. Then define $\mathcal{U} = \{[a, \infty)\} \cup \{(-\infty, a - \frac{1}{n}) : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is an open cover. As A is compact, it has a finite subcover. But then this has only finitely many elements from $\{(-\infty, a - \frac{1}{n}) : n \in \mathbb{N}\}$. Say N is the largest natural number corresponding to which there is an interval $(-\infty, a - \frac{1}{N})$ in the subcover. Then $A \subset (-\infty, a - \frac{1}{N}) \cup [a, \infty)$. So setting $\epsilon_a = 1/N$ we get A is LG, as desired. $\square$

Now from whatever we have derived, we have the following strong result.

Theorem 2.2. If $A \subset \mathbb{R}_l$ is compact, then A is a compact LG subset of $\mathbb{R}$.

(Note that $LG - ness$ is not dependent on nature topology of $\mathbb{R}$ as it is defined only using the constructional properties, i.e., the usual topology)

Now we may naturally ask, is the converse true? In fact, it is true!! The proof cleverly relies on the following beautiful property of LG subsets.

Lemma 2.2.1. Suppose A is LG and $\mathcal{U}$ is a basic open cover of A in $\mathbb{R}_l$, then there is a basic open cover of A, say $\mathcal{U}'$, in $\mathbb{R}$ such that the collections in the subspace topology, both sets coincide.

Proof. Let $U \in \mathcal{U}$. Then $U = [a, b)$. If $a \notin A$, then consider $U' = (a, b)$ in $\mathcal{U}'$. Otherwise, consider $U' = (a - \epsilon_a, b)$. In either case, the subspace covers coincide and $\mathcal{U}'$ turns out to be an open cover of A in $\mathbb{R}$, as desired. $\square$

Theorem 2.3. If A is a compact subset of $\mathbb{R}$ which is LG, then A is compact in $\mathbb{R}_l$.

Proof. Let $\mathcal{U}$ be an basic open cover of A. We can simply consider the image of $\mathcal{U}$ in subspace topology, call that $\mathcal{U}_1$. Then by previous lemma, we derive $\mathcal{U}'$, an basic open cover in $\mathbb{R}$, such that if $\mathcal{U}_1'$ is the corresponding subspace open sets, then $\mathcal{U}_1' = \mathcal{U}_1$. But A is compact in $\mathbb{R}$ so $\mathcal{U}_1'$ hence $\mathcal{U}_1$ has a finite subcover, which can be drawn back to $\mathcal{U}$ giving us the desired result.

(Note : For compactness, enough to prove for basic open covers) □

So we get the following complete characerisation of compact subsets of $\mathbb{R}_l$

Theorem 2.4. $A \subset \mathbb{R}_l$ is compact iff A is compact in $\mathbb{R}$ and is LG

One more interesting point is the following theorem (A neat, better to visualise, formulation of LG sets)

Theorem 2.5. $A \subset \mathbb{R}$ is LG iff A is countable and if (a_i) is a convergent increasing sequence in A with limit in A, then (a_i) is eventually constant and equal to limit.

Proof. The forward direction is straightforward by Theorem 1.3, we get that A is countable. Additionally, suppose (a_i) is a convergent increasing sequence which is not eventually constant, then for each $\varepsilon > 0$, there is some $j \in \mathbb{N}$ s.t. $a_j \in (a - \epsilon, a)$, which contradicts A is LG.

For the converse, suppose A is countable with given convergence property. Let $a \in A$, then suppose for each $n \in \mathbb{N}$, there is an $a_n \in A$ such that $a - \frac{1}{n} < a_n < a$. Then clearly (a_n) is a convergent increasing sequence in A with limit in A but (a_i) is not eventually constant. This violates given claim. So there is a natural number n such that $(a - \frac{1}{n}, a) \cap A = \emptyset$

(An interesting point to realise is that "(a_i) is a convergent increasing sequence in A with limit in A, then (a_i) is eventually constant and equal to limit" implies A is countable. This is clear from the fact we never used countability in our proof of converse) □

Thus one can characterise as follows

Theorem 2.6. (The Complete Categorisation in numerous forms) $A \subset \mathbb{R}$ is compact in $\mathbb{R}$, then the following are equivalent

1. A is compact in $\mathbb{R}_l$
2. A is LG
3. A is countable and if (a_i) is a convergent increasing sequence in A with limit in A, then (a_i) is eventually constant and equal to limit.
4. A is countable and if (a_i) is an increasing sequence in A, then (a_i) is eventually constant and equal to limit.
5. $\forall B \subset A$, then $sup(B)$ exists and is in B.

Proof. Equivalence of the first 3 points has already been seen or is easy, we just note that (3) $\iff$ (4) $\iff$ (5) holds because compactness of A in $\mathbb{R}$ is equivalent to sequential compactness of A in $\mathbb{R}$ is equivalent to limit point compactness of A in $\mathbb{R}$ □

3 $\mathbb{R}_K$

Now we wish to determine all compact sets in $\mathbb{R}_K$. This is considerably easier.

Proposition 3.1. If $A \cap K$ is infinite, then A is not compact in $\mathbb{R}_K$

Proof. Let $\mathcal{U}$ be defined as the open cover $\{\mathbb{R}\setminus K\}\cup\{(\frac{1}{n}-\frac{1}{2n^2+2n}, \frac{1}{n}+\frac{1}{2n^2+2n}) : n \in \mathbb{N}\}$

Then each $k \in K$, there is a unique $U \in \mathcal{U}$ such that $k \in \mathcal{U}$. Thus, indeed $\mathcal{U}$ does not have a finite sub cover of A

□

Proposition 3.2. If $A \subset \mathbb{R}$ is compact and $A \cap K$ is finite, then A is compact in $\mathbb{R}_K$

Proof. Let $\mathcal{U}$ be any open cover of A. First of all, we observe that if we consider $A' = A \setminus K$, then there has to be a finite subcover of A' using sets of $\mathcal{U}' = \{U \setminus K : U \in \mathcal{U}\}$ by compactness of A in $\mathbb{R}$.

Moreover, for each $a \in A \cap K$, there is a $U_a \in \mathcal{U}$. Thus, we get finitely many elements of $\mathcal{U}$ covering A. Hence indeed, A is compact in $\mathbb{R}_K$ □

So we get the following theorem.

Theorem 3.3. A is compact in $\mathbb{R}$. Then A is compact in $\mathbb{R}_K$ iff $A \cap K$ is finite.

So indeed we have categorised all compact subsets of $\mathbb{R}_K$ and $\mathbb{R}_l$.

4 Note : The Basis of Our Considerations

We have this following beautiful theorem.

Theorem 4.1. X be a topological space and X' be the same set with some other topology. Also, say X and X' are comparable. Suppose both spaces are compact and Hausdorff. Then indeed $X = X'$

Proof. WLOG, let X' be finer than X. $Id : X' \to X$ is a bijective continuous map. Moreover, by Closed Map Lemma, Id is a closed map. Hence, indeed, Id is a homeomorphism, thereby proving our result. □

Using this, we realise the compact subsets of $\mathbb{R}$ and $\mathbb{R}_l$ or $\mathbb{R}_K$ cannot coincide. (As topologies are strictly finer for most non discrete compact subspaces). One immediate consequence (even without our analysis) is that $[0, 1]$ is not compact in either of these topologies - which is indeed surprising! If it were compact, then note that the subspace topology is strictly finer than the usual subspace topology, thereby contradicting the above theorem.

The things become trickier with $\mathbb{R}_K$ as K is a pretty local set, so taking intersections become necessary.

Finally, the theorem 1.2 tells us where to proceed and rest is some simple analysis.