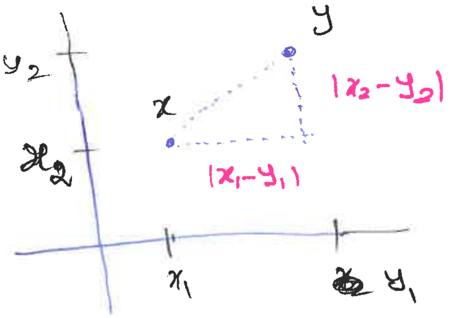
Metric space (Examples):

- $X = \mathbb{R}$ or $\mathbb{C}$ with usual metric $d(x,y) = |x-y|$

- $X = \mathbb{R}^2$ $\quad x = (x_1, x_2)$ $\quad y = (y_1, y_2)$

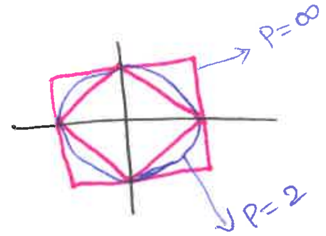


$$d_1(x,y) = |x_1-y_1| + |x_2-y_2|$$

$$d_2(x,y) = \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2}$$

$$d_\infty(x,y) = \operatorname{Max}\{|x_1-y_1|, |y_2-x_2|\}$$

$$d_p(x,y) = \left(|x_1-y_1|^p + |x_2-y_2|^p\right)^{1/p}$$



- $X = \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$ $\quad y = (y_1, y_2, \ldots y_n)$

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}, \quad 1 \le p < \infty$$

$$d_\infty(x,y) = \operatorname{Max}\{|x_1-y_1|, |x_2-y_2|, \ldots |x_n-y_n|\}$$

- $X = \mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \cdots$, $\quad x = (x_1, x_2, \ldots)$, $\quad y = (y_1, y_2, \ldots)$

$$d_p(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} \quad \text{is meaningless}$$

$\because$ $d_p\left((1/\sqrt{n}), (0)\right) = \left(\sum_{n=1}^{\infty} \frac{1}{n^{p/2}}\right)^{1/p}$ may not be convergent sequence.

For each $p \in [1, \infty)$, $\ell^p := \left\{ x \in \mathbb{R}^\omega : \left(\sum_i |x_n|^p\right)^{1/p} < \infty \right\}$

$\ell^\infty = \left\{ x \in \mathbb{R}^\omega : \sup\{|x_1|, |x_2|, \ldots\} < \infty \right\}$

$$d_p(x,y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}, \quad x, y \in \ell^p$$

$$d_\infty(x,y) := \sup\{|x_1-y_1|, |x_2-y_2|, \ldots\}, \quad x, y \in \ell^\infty.$$

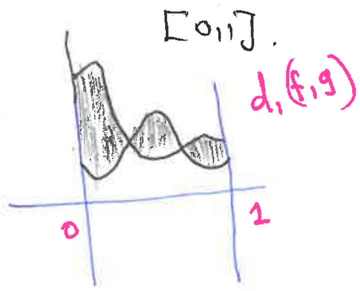
- $X = C[0,1]$ = The set of all (real valued) continuous functions on $[0,1]$.

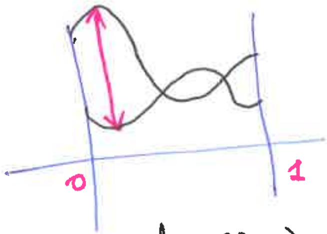
$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$

$$d_2(f,g) = \left(\int_0^1 |f(x) - g(x)|^2\, dx\right)^{1/2}$$

$$d_p(f,g) = \left(\int_0^1 |f(x) - g(x)|^p\, dx\right)^{1/p}$$

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$





$d_\infty(f,g)$.

- Hamming metric in computer science
- Helly metric in Game theory
- Edge counting metric in Graph theory
- Aeroplane distance in Heisenberg group
- Mahalanobis distance in Statistics
- Expectation in Probability
- Rank in Matrix theory
- stereographic projection in complex analysis.
- operator norm in Functional analysis.

=== x ===

Metric spaces:-

(1) Let (X, d) be a metric space such that

$$d(x,y) \le \max\{ d(x,z), d(z,y) \} \quad \forall\ x,y,z \in X.$$

(a) Prove that for every $x,y,z \in X$, atleast any two of the three quantities $d(x,y)$, $d(y,z)$, $d(z,x)$ will be same.

(b) Let $b \in B(a,r) := \{ x \in X : d(a,x) < r \}$.

Show that b is also a centre for $B(a,r)$

i.e) $B(b,r) = B(a,r)$

(c) Prove that $B(a,r)$ is closed.

Sol:-

(a) Suppose $d(x,z) = d(z,y)$, then we are through.

W.L.G $d(x,z) < d(y,z)$.

- $d(x,y) \le \max\{ d(x,z), d(y,z)\} = d(y,z)$

- $d(y,z) \le \max\{ d(x,y), d(x,z)\}$

$= d(x,y) \quad (\because d(x,z) < d(y,z))$

$$\boxed{\therefore d(x,y) = d(y,z)}$$

(b) Fix $b \in B(a,r)$. Let $x \in B(a,r)$.

$d(b,x) \le \max\{ d(a,b), d(a,x)\} < \max\{r,r\} = r$

$\therefore x \in B(b,r) \Rightarrow \boxed{B(a,r) \subseteq B(b,r)}$

Let $x \in B(b,r)$. Then $d(a,x) \le \max\{ d(a,b), d(b,x)\} < r$

$\therefore x \in B(a,r) \Rightarrow \boxed{B(b,r) \subseteq B(a,r)}$

(c) <u>Claim:</u> $B(a,r)$ is closed.

Let $x \notin B(a,r)$. <u>claim:</u> $B(x,r) \cap B(a,r) = \phi$

suppose $y \in B(x,r) \cap B(a,r)$.

Then $B(x,r) = B(y,r) = B(a,r) \Rightarrow x \in B(a,r)$.

which is contradiction. $\therefore B(x,r) \cap B(a,r) = \phi$.

$\Rightarrow x \in B(x,r) \subseteq B(a,r)^c$ $\quad \therefore B(a,r)^c$ is open

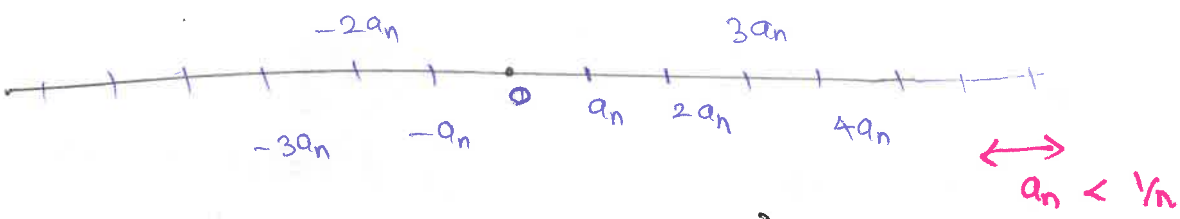
Hence $B(a,r)$ is closed.

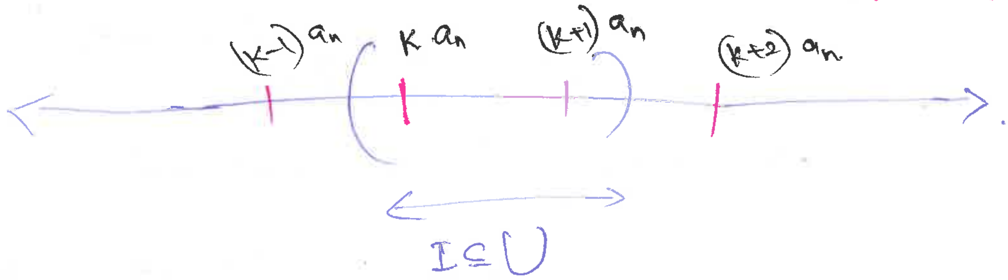
② Let A be a subgroup of $\mathbb{R}$ under addition.
Then prove that A is discrete or A is dense in $\mathbb{R}$.

<u>Sol:</u>

<u>Hint:</u> consider $b := \inf\{a > 0 : a \in A\}$

<u>case-I</u>
If $b = 0$ then $\forall n \in \mathbb{N}$, we can choose $a_n \in A \cap (0, 1/n)$





Let U be a nonempty open set in $\mathbb{R}$.

$\Rightarrow U$ contains some interval $(I \subseteq U)$ with $l(I) > 0$.

choose $n \in \mathbb{N}$ s.t $1/n < l(I)$.

Since $l(0,a_n) < 1/n < l(I)$, $\exists\, K \in \mathbb{N}$ s.t

$K \cdot a_n \in I \subseteq U$. $\therefore$ $K \cdot a_n \in A \cap U = \phi$.

i.e.) A intersects every open set of $\mathbb{R}$.

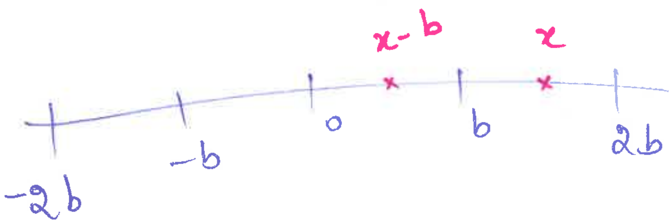
Hence A is dense in $\mathbb{R}$ ie) $\bar{A} = \mathbb{R}$.

Case-II $b = \inf\{a > 0 : a \in A\} > 0$.

- $b \in A$ (why?)
- $b\mathbb{Z} \subseteq A$.
- on $(0,b)$, $(-b,0)$, A has no element.

ie) $A \cap (0,b) = \phi$, $A \cap (-b,0) = \phi$

- $A \cap (-2b,-b) = \phi$ & $A \cap (b,2b) = \phi$.



- $A \cap (-nb, -(n-1)b) = \phi$ & $A \cap ((n-1)b, nb) = \phi$

$\forall n = 1,2, \dots$

- $\boxed{A = b\mathbb{Z}}$ $\therefore$ A is discrete subgroup of $\mathbb{R}$.

3 a) Give all continuous functions from $\mathbb{R}$ to $\mathbb{R}_l$.

b) Give all cts fns from $\mathbb{R}$ to $\mathbb{Q}$.

Soln: Let $f : \mathbb{R} \to \mathbb{R}_l$ be a constant map.

$f \equiv c$

$$f^{-1}(U) = \begin{cases} \phi & \text{if } c \notin U \\ \mathbb{R} & \text{if } c \in U \end{cases}$$

$\therefore$ f is continuous.

- Let $f : \mathbb{R} \to \mathbb{R}_\ell$ is not constant map.

$\Rightarrow$ choose $a_1 < a_2$ and $a_1, a_2 \in \text{Range}(f)$.

Take ~~such~~ $a \in \mathbb{R}$ such that $a_1 < a < a_2$

$(-\infty, a)$ & $[a, \infty)$ are open sets in $\mathbb{R}_\ell$

$\Rightarrow$ $U_1 = f^{-1}(-\infty, a)$ & $U_2 = f^{-1}[a, \infty)$ are open sets in $\mathbb{R}$.

- $U_1, U_2 \neq \phi$
- $U_1 \cap U_2 = \phi$

Note that $U_1 \cup U_2 = \mathbb{R}$ (why?)

$\Rightarrow$ $\mathbb{R}$ is union of two non-empty disjoint open sets. which gives contradiction.

$\therefore$ constants functions are only cts fns from $\mathbb{R}$ to $\mathbb{R}_\ell$.

(b) $f : \mathbb{R} \to \mathbb{Q}$

$f(\mathbb{R}) =$ connected ($\because$ f is cts)

$\Rightarrow$ $f(\mathbb{R})$ is singleton in $\mathbb{Q}$.

$\therefore$ only constant fns are cts maps from $\mathbb{R}$ to $\mathbb{Q}$.

(4) (i) Give a bijection between $(0,1)$ & $[0,1]$.

(ii) Give a continuous bijection b/w $(0,1)$ & $[0,1]$.

Sol:-

cmpt $\quad$ non- cmpt

- $f: [0,1] \to (0,1)$ $\quad$ X

  cts + bijective.

- $f: (0,1) \to [0,1]$ $\quad$ X

  cts + bijective. $\quad$ w.L.G $a < b$

  $a \mapsto 0$ $\qquad$ $[a,b] \longleftrightarrow [0,1]$

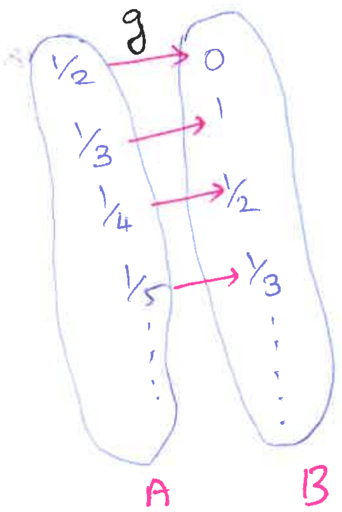
  $b \mapsto 1$ $\qquad$ (by Intermediate Value theorem)

  $\Rightarrow$ $f: (0,1) \to [0,1]$ is not bijective.

- $A = \{1/2, 1/3, 1/4, 1/5, \dots\}$

  $B = \{0, 1, 1/2, 1/3, 1/4, \dots\}$

  $C = (0,1) \setminus A = [0,1] \setminus B$



  $f: (0,1) \longrightarrow [0,1]$

  $f: A \cup C \longrightarrow B \cup C.$

  $$f(x) = \begin{cases} x & , x \in C \\ g(x) & , x \in A. \end{cases}$$

  Thus f is a bijective map b/w $(0,1)$ & $[0,1]$.

- $M_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\} \simeq \mathbb{R}^4$

  $GL_2(\mathbb{R}) = \{ A \in M_2(\mathbb{R}) : A \text{ is an invertible matrix} \}$

  $SL_2(\mathbb{R}) = \{ A \in M_2(\mathbb{R}) : \det(A) = 1 \}.$

<u>Qn:</u> (i) Is $GL_2(\mathbb{R})$ open? Is it closed?

(ii) what about $SL_2(\mathbb{R})$?

<u>soln</u>

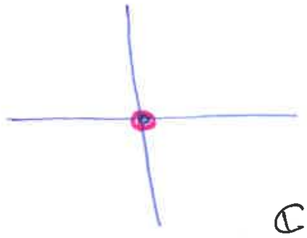
$T: M_2(\mathbb{R}) \longrightarrow \mathbb{R}$

$A \longrightarrow |A| = \det(A) = ad - bc$

cts fn.

- $T^{-1}(\mathbb{R}\setminus\{0\}) = GL_2(\mathbb{R})$ = open
- $T^{-1}\{0\}$ = set of singular matrices = closed
- $T^{-1}\{1\} = SL_2(\mathbb{R})$ = Closed.
- $GL_2(\mathbb{R}) \xrightarrow{T} \mathbb{R}\setminus\{0\}$ = not connected.

  $\hookrightarrow$ not connected.

-  

$\mathbb{R}\setminus\{0\}$ (not connected)

$\mathbb{C}\setminus\{0\}$ (connected !)

<u>Note:</u> $GL_2(\mathbb{C})$ is connected.

- As $\mathbb{R}^4 \approx M_2(\mathbb{R})$ is connected, there is no nontrivial proper set which is both open & closed.

$=\!=\!= \times =\!=\!=$