Exercises – I solutions:

(1) Let $c \in [a,b]$ be a point where f attains global maximum value m.

Let $A = \{ x \in [a,c] : f(y) = m \ \forall\, y \in [x,c] \}$

Since f is continuous, Let $a' = \inf A$

$f(a') = m$ and so $a' \in A \Rightarrow A = [a', c]$

Since a' is a local minimum, $f(a') \leq f(x) \ \forall\, x \in I$

$\Rightarrow f(x) = m \ \forall\, x \in I$ = nbhd of a'.

Therefore $a' = a$, because otherwise I would contain a point of A less than a'. Thus $A = [a,c]$.

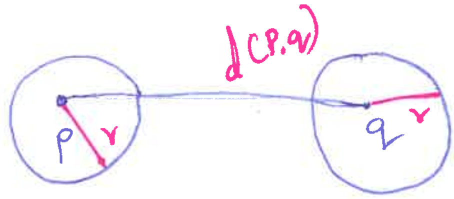
Analogously, ~~by~~ by considering

$B = \{ x \in [c,b] : f(y) = m \ \forall\, y \in [c,x] \}$

we get that $\sup B = b$ ie) $B = [c,b]$

$\Rightarrow$ f is constant on $[a,b]$.

(3) $r = \dfrac{d(p,q)}{4} > 0.$



(4) $(0,1) \in \mathbb{R} \setminus \mathbb{R}_f$ U^c = finite = closed in $\mathbb{R}$

$\Rightarrow \boxed{\mathbb{R}_f \subsetneq \mathbb{R}.}$

(5) • $(0,1) \in \mathbb{R} \setminus \mathbb{R}_c$

• $\mathbb{R} \setminus \{1, 1/2, 1/3, \cdots\} \in \mathbb{R}_c \setminus \mathbb{R}$, $0 \notin A^\circ$

There $\mathbb{R}$ & $\mathbb{R}_c$ are not comparable.

(6)

$\mathcal{B} = \{ (a,b) : a, b \in \mathbb{R} \}$.

$\mathcal{B}_1 = \{ (a,b) : a, b \in \mathbb{Q} \}$

$\mathcal{B}_2 = \{ (a,b) : a, b \in A \}$, $\bar{A} = \mathbb{R}$

[illegible], etc.

(7) Arithmetic progression topology:

$A_{a,b} = \{ a + nb : n \in \mathbb{Z} \} = \{ \cdots, a-2b, a-b, a, a+b, a+2b, \cdots \}$

$\mathcal{A} = \{ A_{a,b} : a, b \in \mathbb{Z} \}$ is a basis for a topology on $\mathbb{Z}$.

• $x \in A_{x,1} \quad \forall x \in \mathbb{Z}$.

• $x \in A_{a_1,b_1} \cap A_{a_2,b_2} \Rightarrow x \in A_{x,b_1} \cap A_{x,b_2}$

$\Rightarrow x \in A_{x, b_1 b_2} \subseteq A_{a_1,b_1} \cap A_{a_2,b_2}$.

(8)

$$\mathbb{Z} = \coprod_{k=0}^{p-1} (p\mathbb{Z} + k) \Rightarrow (p\mathbb{Z})^c = \coprod_{k=1}^{p-1} p\mathbb{Z} + k$$

$$= \bigcup_{k=1}^{p-1} A_{k,p} = \text{open}.$$

$\therefore$ $p\mathbb{Z}$ is closed

• $A_{a,b}$ is basis $\Rightarrow$ every open set is an infinite set.

Suppose there are only finitely many prime numbers. Lets call them $p_1, p_2, \cdots, p_N$.

$$\mathbb{Z} = \bigcup_{i=1}^{N} p_i\mathbb{Z} \cup \{\pm 1\}$$

~~open~~ ~~open~~ closed (finite union of closed sets)

$$\Rightarrow \{\pm 1\} = \left(\bigcup_{i=1}^{N} p_i\mathbb{Z}\right)^c = \text{open} \Rightarrow \Longleftarrow .$$

Hence # of primes is infinite.

9 $U \subseteq Y \subseteq X$.

U is open in Y, Y is open in X.

$\Updownarrow$

$U = A \cap Y$, $A =$ open in X

$\therefore U =$ open in X.

10 For $a \neq x$, choose $U_x \in \tau$ s.t. $a \notin U_x$, $x \in U_x$.

$$\{a\}^c = \bigcup_{x \in X \setminus \{a\}} U_x = \text{open}. \Rightarrow \{a\} \text{ is closed.}$$

11 $X = \prod X_\alpha$, $x \neq y \Rightarrow x_\beta \neq y_\beta$ for some $\beta \in J$.

$\Rightarrow x_\beta \in U_\beta \subseteq X_\beta$, $y_\beta \in V_\beta \subseteq X_\alpha$, $U_\beta \cap V_\beta = \phi$.

$U = \pi_\beta^{-1}(U_\beta)$, $V = \pi_\beta^{-1}(V_\beta)$, $x \in U$, $y \in V$

$U \cap V = \phi$.

$\therefore X$ is Hausdorff.

12 $\Delta = \{(x, x) : x \in X\}$ w.L.G $|X| > 1$.

suppose X is T_2. & $(a,b) \in X \times X \setminus \Delta$.

Since $a \neq b$, U_a, U_b is separation for a & b.

$\odot \; (U_a \times U_b) \cap \Delta = \phi$. (why?)

$\rightarrow \Delta^c$ is open ie.) Δ is closed.

Conversely suppose that Δ is closed. Let $a \neq b$

$(a,b) \in U \times V \subseteq \Delta^c \Rightarrow U \cap V = \phi$

$\therefore X$ is T_2 space.

(13) $Z = \bigcap_{i=1}^{K} f_i^{-1}\{0\}$ = closed. $(\because f_i$ is cts.$)$

(14) $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$

$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\} \rightarrow$ closed sets. $f|_{\{x\}}$ is cts $\forall x$.

but f is not continuous.

(15) $\mathbb{R}^n \setminus \{\bar{0}\}$ is connected $\Leftrightarrow n > 1$.

$\therefore f: \mathbb{R}^n \longrightarrow \mathbb{R}$ cannot be homeomorphic if $n > 1$.

(16) $B_1(0) = \{ x : \|x\|^2 = \sum_{i=1}^{n} |x_i|^2 < 1 \}$

$f: B_1(0) \rightarrow \mathbb{R}^n$ by $x \longmapsto \dfrac{x}{1 - \|x\|}$

$f^{-1}: \mathbb{R}^n \rightarrow B_1(0)$ by $y \longmapsto \dfrac{y}{1 + \|y\|}$

verify that f is homeomorphism b/w $B_1(0)$ & $\mathbb{R}^n$

(17) $S^n = \{ (x_1, x_2, \dots, x_{n+1}) : \sum x_i^2 = 1 \}$ $\quad P = (0, 0, 0, \dots, 1)$

(Stereographic projection)

$$f : S^n \setminus \{P\} \to \mathbb{R}^n$$

$$(x_1, x_2, \dots, x_{n+1}) \longmapsto \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right)$$

Verify that f is homeomorphism.

(18) $f : X \xrightarrow{\text{cts}} Y$.

$$X \xrightarrow{T} gr(f) = \{ (x, f(x)) : x \in X \}$$

$$x \overset{T}{\longmapsto} (x, f(x))$$

Then T is natural homeomorphism between X & $gr(f)$.

(19) $d(f, g) = \|f - g\|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|$, $f, g \in B[0,1]$

It is trivial to check that d is a metric on $B[0,1]$.

(20) $f : X \to \mathbb{R}$ (every fn is continuous.)

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$$

χ_A is cts $\iff$ A is open & closed.

$\Rightarrow$ X has discrete topology to get every fn $f : X \to \mathbb{R}$ is continuous.

(21) X is connected, locally constant. $f : X \to \mathbb{R}$ is cts. $\forall a \in \mathbb{R}$.

$\Rightarrow$ $f^{-1}\{a\}$ is open. & $f^{-1}(\mathbb{R} \setminus a)$ is open.

Thus f must be constant (why?)

Hint: $X = f^{-1}\{a\} \cup f^{-1}(\mathbb{R} \setminus \{a\})$.

22) a) $\boxed{\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}}$

Proof: $A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \overline{A}$
$A \cap B \subseteq B \Rightarrow \overline{A \cap B} \subseteq \overline{B}$ $\Big\} \Rightarrow \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

$\phi = \overline{\mathbb{Q} \cap \mathbb{Q}^c} \subsetneq \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R}$

b) $\boxed{\overline{A \setminus B} \supseteq \overline{A} \setminus \overline{B}}$

Proof:
$\overline{A - B} \setminus \overline{B} = \overline{A} \setminus \overline{B}$

$\phi = \overline{\mathbb{Q}} \setminus \overline{\mathbb{Q}^c} \subsetneq \overline{\mathbb{Q} \setminus \mathbb{Q}^c} = \overline{\mathbb{Q}} = \mathbb{R}$.

23) $x \in \text{Int}(A) \Rightarrow x \notin \overline{A^c} \Rightarrow x \notin bd(A)$

i.e) $\boxed{A^o \cap bd(A) = \phi}$

$A^o, bd(A) \subseteq \overline{A} \Rightarrow A^o \cup bd(A) \subseteq \overline{A}$.
by defn of $\overline{A}$, $\overline{A} \subseteq A^o \cup bd(A)$

$\boxed{\therefore \overline{A} = A^o \sqcup bd(A)}$

$bd(A) = \phi \Leftrightarrow \overline{A} = A^o \Leftrightarrow A^o = A = \overline{A}$
$\Leftrightarrow A$ is closed & open.

2) Thm Let $f : [0,\infty) \to [0,\infty)$ such that $f \uparrow$, $f(x) = 0 \Leftrightarrow x = 0$, $f(x+y) \leq f(x) + f(y)$ $\forall x, y$. Then $f \circ d$ is a metric whenever d is a metric on X.

Eg $f(x) = \frac{x}{1+x}$, $f(x) = \min\{1, x\}$, $f(x) = \sqrt{x}$, etc.