Graph Theory

Lecture 9

Menger's Theorem.

2-connectedness in a graph. Whitney's theorems. Ear decomposition of a graph that is 2-connected.

Theorem :- If $x, y \in V(G)$ s.t. $xy \notin E(G)$, Then the min. size of an x-y cut equals the max. size of internally disjoint x-y paths.

pf. → x-y cut is a set $S \subseteq V(G) \setminus \{x, y\}$ s.t. in $\langle V(G) \setminus S \rangle$ x & y lie in diff. connected comp.

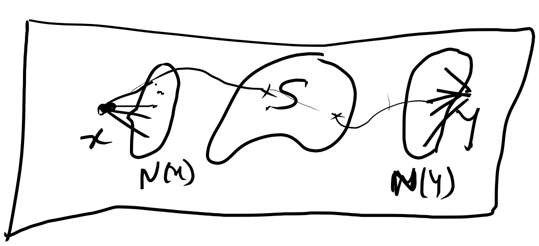
If x & y have k-internally disjoint paths, then any x-y cut must have at least one vertex from each of them.

$\Rightarrow$ $|x\text{-}y \text{ cut}| \geq |\text{int-disjt. } x\text{-}y \text{ paths}|$

$\Rightarrow$ min L.H.S. $\geq$ max. R.H.S.

Thus it remains to prove that if k = max no. int. disjt. x-y paths then $\exists$ a cut of size k. OR given a cut of size k we exhibit k internally disjt. x-y paths.

Let S be a minimum x-y cut.

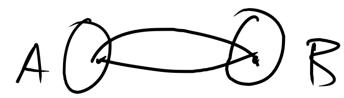


$\boxed{\text{case 1}}$ $\exists$ such an S which is different from $N(x)$ or $N(y)$.

Let $|S| = k$.

Idea :- We try to ~~get~~ construct k x-S paths & k S-y paths ~~and~~ and patch them up to get k int. disjt. x-y paths.

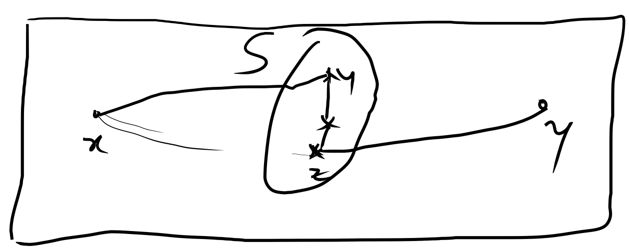
Def^n :- $A, B \subset V(G)$, an A-B path is a path whose starting pt is in A, end pt is in B & no other vertex is in $A \cup B$.



$\forall v \in A \cap B$, $\{v\}$ is trivial A-B path.

Let V_1 be the set of all vertices that belong to some x-S path. & V_2 be vertices in S-$\{y\}$ paths.

claim :- $V_1 \cap V_2 = S$.



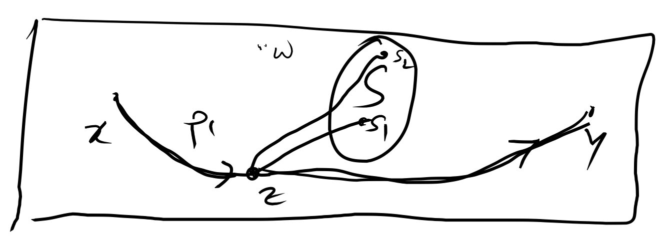
$\Rightarrow$ $y \notin V_1$ & $z \in V_2$.

$\forall z \in S$, $\exists$ a x-z path that is also a x-S path.

If not, then ever x-z path must contain another pt. of S.
In that case, $S \setminus \{z\}$ is also an x-y cut !! which contradicts the minimumness of $|S|$ as a x-y cut $\Rightarrow S \subseteq V_1$

$\Rightarrow$ $S \subseteq V_1 \cap V_2$ (Repeat above arg. for V_2)

If $V_1 \cap V_2 \ni z$ which is not in S then we can construct a x-y path avoiding S as follows



by joining x to z using an x-S path & followed by a z-y path given by a S-y path.

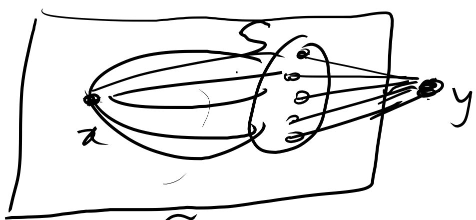
$\Rightarrow$ $S = V_1 \cap V_2$.

Use induction on $|V(G)|$

If $|V(G)|=2$, then $G = \dot{x} \quad \dot{y}$ $\Rightarrow$ min cut has 0 vertices & $\exists$ 0 x-y paths.

$\therefore$ Thm is true.

Now assume by way of induction that the thm is true $\forall$ G with $|V(G)| < n$.



$\tilde{H}_1 = \langle V_1 \rangle$

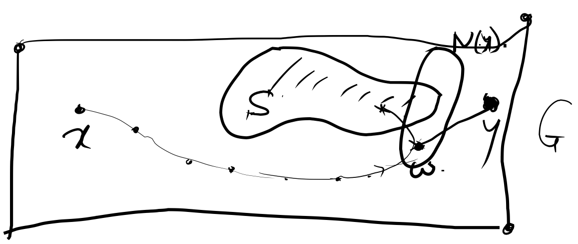
$H_1 = \langle V_1 \rangle \cup \{y'\}$ & edges are all edges in V_1 together with all edges of the type $\{sy' \mid s \in S\}$

Claims ① $|V(H_1)| < |V(G)|$. ✓

② S is min x-y' cut. $\rightarrow$

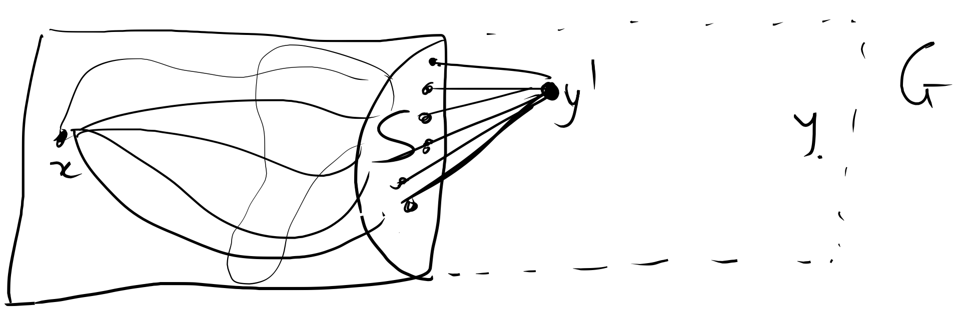
We need to prove that $|V(G) \setminus V_1| \geq 2$. to prove 1.

since $N(y) \setminus S$ is not in H_1



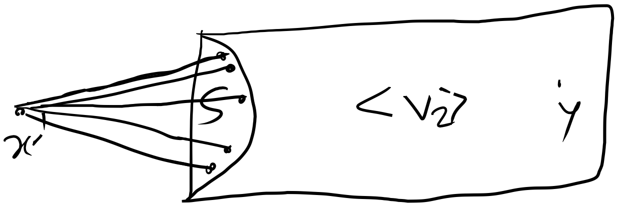
Clearly S is x-y' cut ($\because S = N(y')$)

Further S is also a minimum x-y' cut other we get a T an x-y' cut with $|T| < |S|$. Then T is also an x-S cut. $\Rightarrow$ T is x-y cut in G. contradiction.



H_1.

$\therefore$ By induction hypothesis, $\exists$ k internally disjoint x-y' paths in H_1.

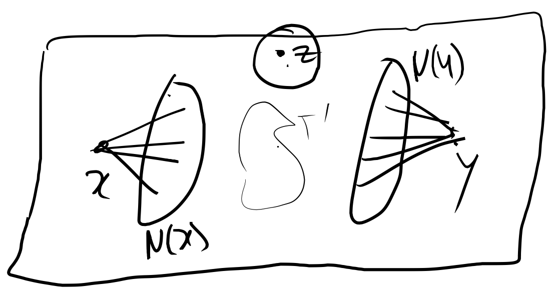


H_2

Illy $\exists$ k internally disjoint y-x' paths in H_2.

<u>Exercise</u> :- Using this data construct k-internally disjoint x-y paths.

$\boxed{\text{case 2}}$ There does not exist any min. x-y cut that is different from $N(x)$ or $N(y)$.



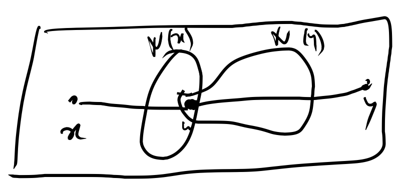
(case 2 a) $\exists$ z not in $\{x\} \cup N(x) \cup N(y) \cup \{y\}$

If $z \in$ some min x-y cut then assumption of case 2 will be contradicted. $\Rightarrow$ If an x-y cut T contains z then $|T| \geq k+1$. $\Rightarrow$ the min x-y cut of $\langle V(G) - z \rangle$ must have cardinality k.

$\Rightarrow$ can use ind. hyp. on $H = \langle V(G) - z \rangle$ to get k-int. disjt. x-y paths in $\langle V(G) - z \rangle$. & hence in G !!

(case 2b) If $N(x) \cap N(y) \neq \phi$. Say $w \in N(x) \cap N(y)$. Then <u>every</u> x-y cut must contain w.

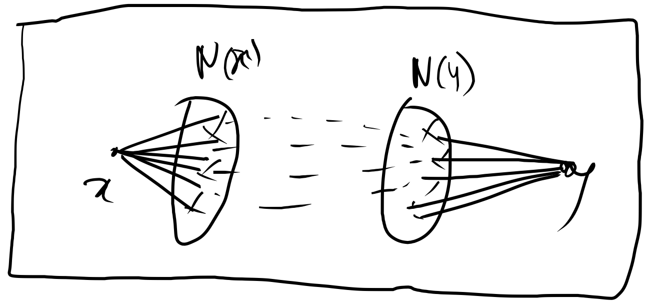
G



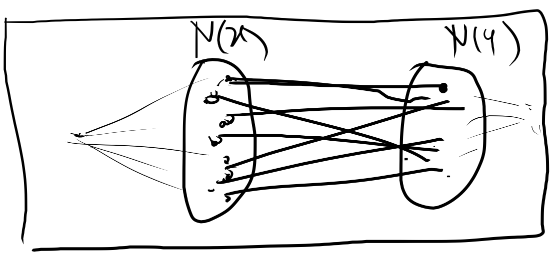
$\Rightarrow$ $\langle G - w \rangle$ has min. card. of x-y cut equal to $k-1$.

$\Rightarrow$ $\exists$ $k-1$ int. disj. path betn x & y in $\langle G-w \rangle$.
those + xwy are the k - int. disjt. paths betn x & y.

(case 2c)

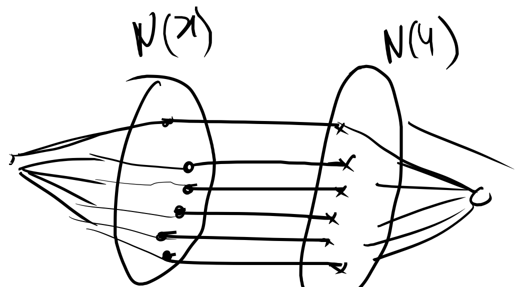


In this case remove all the edges whose both end pts are in $N(x)$ or $N(y)$ to get a bipartite subgraph on $N(x) \cup N(y)$



WLOG assume that
$|N(x)| \leq |N(y)|$

max # int. disjt x-y paths equals the max # of "parallel" edges. ie edges that do not meet.



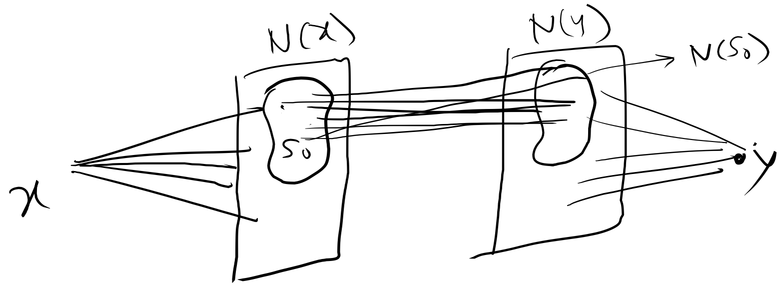
$\therefore$ the question reduces to showing that $\exists$ a set of $\parallel$ edges whose end pts contain $N(x)$.

Hall's condition In a bipartite graph such a set of parallel edges (matching) exists

(*) —— if $\forall \tilde{S} \subseteq N(x)$; $|N(\tilde{S})| \geq |\tilde{S}|$.

This condition is satisfied here due to the minimality of $|S|$.

If not $\exists\ S_0 \subset N(x)$ s.t. $|N(S_0)| < |S_0|$



then $\underline{N(x) \setminus S_0 \cup N(S_0)}$

has cardinality $< N(x)$

& it is still an x-y cut.

→ It remains to show that $(*) \Rightarrow$ required no. of parallel edges.

QED ?