TOPOLOGY

Contents

1. Introduction

Webster defines the word "Topology" to be the study of the geographical location of a place. The development of any subject has many facets : (not necessarily in this order) basic assumptions (axioms), pinning down the basic definitions, central questions in the subject and the development of ideas (technical tools) to find answers to the questions.

Topology pervades all of mathematics. The subject of topology has a long history and would probably have its roots in the Greek school of geometeers. In recent times the subject can probably be traced back to the work of Euler in the 18th century culiminating in the ideas of Riesz, Hausdorff (and several others) that led to the present day definition of a topological space in the early part of the 20th century[1]. A journey spanning two millenia.

Loosely, the subject of topology can be thought of as the study of properties of a geometric object that are unchanged under a *continuous deformation* of the object. In other words we allow change of shape of objects while maintaing the information about which points were close to each other so that after the object is deformed the points that were close still remain "close". This information is encoded by open sets. Thus, while we allow change of shape, we do not permit "tearing". This is in contrast with euclidean geometry where two triangles are isomorphic (or same) if the two overlap when one is placed over the other, no deformations of the sides being allowed. Thus allowing deformations is a weakening of the notion of sameness when compared with euclidean geometry. With this notion, one is forced to say that the triangle and the circle are the same. Topology explores the properties of the space that remain unchanged when one object is deformed into another.

Our discussion in this set of notes will be based on the book Topology by James R. Munkres. We assume familiarity with the notion of sets, various operations (like unions, intersection complementation) with sets and the notion of functions and their properties (composition, one-one, onto).

2. Sets : finite, infinite,...

As we have already mentioned we assume some familiarity with the notion of a set, operations on sets, and the notion of a function and some of the basic definitions concerning functions. In this section we discuss a basic construction with sets and a fundamental way in which sets can be compared. We begin with the following definition.

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[1]The article https://u.math.biu.ac.il/ megereli/TopHistory.pdf, for example, traces the recent history of the subject. See also the references therein.

Definition 2.1. Let I be a set (called the indexing set). Suppose that for each $i \in I$ there is associated a set A_i, then the family

$$\mathcal{A} = \{A_i \,:\, i \in I\} = \{A_i\}_{i\in I}$$

is called an *indexed family* of sets indexed by the set I. It is possible that $A_i = A_j$ for some $i \neq j$. We refer to the set A_i as an element of the indexed family $\mathcal{A}$.

For example suppose that $I = \{1, 2, \dots, n\}$, then an indexed family of sets indexed by I is denoted by

$$\mathcal{A} = \{A_1, A_2, \dots, A_n\}.$$

If $I = \mathbb{N} = \{1, 2, \dots\}$, then an indexed family

$$\mathcal{A} = \{A_1\}_{i\in\mathbb{N}}$$

is often denoted by $\mathcal{A} = \{A_1, A_2, \dots\}$.

One defines the union, intersection of the elements of an indexed family of sets in an obvious way. Indeed, if $\mathcal{A} = \{A_i\}_{i\in I}$ is an indexed family of sets, then we define

$$\bigcup_{i\in I} A_i = \{x \,:\, x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i\in I} A_i = \{x \,:\, x \in A_i \text{ for all } i \in I\}.$$

If $I = \{1, 2, \dots, n\}$, then the following notation

$$\bigcup_{i} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

is also used.

A construction of importance is the *cartesian product* of the elements of an indexed family of sets.

Definition 2.2. Let X be a set. A n-tuple of elements in X is a function

$$\mathbf{x} = \{1, 2, \dots, n\} \longrightarrow X$$

We denote by x_i, the element $\mathbf{x}(i) \in X$. Traditionally, the function $\mathbf{x}$ is denoted by the notation

$$\mathbf{x} = (x_1, x_2, \dots, x_n).$$

Definition 2.3. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be an indexed family of sets. Thus $\mathcal{A}$ is indexed by $I = \{1, 2, \dots, n\}$. The cartesian product of this indexed family is denoted by

$$\Pi_i A_i = A_1 \times \cdots \times A_n$$

and is by definition the set of n-tuples $\mathbf{x}$ in $X = \cup_i A_i$ such that $x_i \in A_i$.

In other words,

$$\Pi_i A_i = \{\mathbf{x} : \{1, 2, \dots, n\} \longrightarrow \cup_i A_i \,:\, \mathbf{x}(i) = x_i \in A_i\}.$$

By our agreed upon notation,

$$\Pi_i A_i = \{\mathbf{x} = (x_1, \dots, x_n) \,:\, x_i \in A_i\}.$$

Similar definition applies for an arbitrary family of indexed sets.

Definition 2.4. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an indexed family of sets indexed by the set I. The cartesian product of the family $\mathcal{A}$ is denoted by $\Pi_i A_i$ and is defined to be the set of all functions

$$\mathbf{x} : I \longrightarrow \bigcup_i A_i$$

such that $\mathbf{x}(i) = x_i \in A_i$. The function $\mathbf{x}$ is denoted by

$$\mathbf{x} = (x_i)_{i\in I}$$

We say that x_i is the i-th coordinate of $\mathbf{x}$.

Suppose $\{A_i\}_{i\in I}$ is an indexed family of sets indexed by I. Assume that $A_i = A$ for all $i \in I$, When $I = \{1, 2, \ldots, n\}$ we use the notation

$$\Pi_i A_i = A^n$$

to denote the the cartesian product and the right hand side is called the n-fold product of A with itself. If $I = \{1, 2, \ldots\}$, then we use the notation

$$\Pi_n A_i = A^\omega$$

to denote the cartesian product and the right hand side is called the countable product of A with itself.

The above definitions of the cartesian product are simple and it is important that we understand them well. We shall be using the cartesian product construction often.

Remark 2.5. Set theory is fundamental in the study of mathematics. There are many models of set theory with differing basic axioms. It is therefore important to fix a model of set theory with which we work. Although we will not spell this out explicitly, we will assume that in our setup the cartesian product of an arbitrary family of *nonempty* sets is *nonempty.* This seemingly self evident fact is an important tool in many situations. This assumption is equivalent to any of the following : axiom of choice, Zorn's lemma, maximum principle, the well-ordering theorem and many others[2].

Having discussed the notion of cartesian products we move on to discuss ways in which two sets can be compared. Towards this we make some definitions. First recall that two sets A, B are said to be in *bijective correspondence* if the exists a bijective function $f : A \longrightarrow B$.

Definition 2.6. A set A is said to be *finite* if it is in bijective correspondence with the set $\{1, 2, \ldots, n\}$ for some non-negative integer n. In this case we say that the set X has *cardinality* n. By convention the set $\{1, 2, \ldots, n\}$ is empty if $n = 0$. The set A is said to be infinite if it is not finite.

That the cardinality of a finite set is well defined needs a proof. This and some other consequences will following from the seemingly self evident facts that we prove below.

Proposition 2.7. Let n be a positive integer. Let A be a set with $a \in A$. Then there exists a bijection

$$f : A \longrightarrow \{1, 2, \ldots, n + 1\}$$

if and only if there exists a bijection

$$g : A - \{a\} \longrightarrow \{1, 2, \ldots, n\}.$$

[2]Also look at https://plato.stanford.edu/entries/axiom-choice/ for an interesting discussion.

Proof. If g is a bijection, then clearly f can be defined to be a bijection. Conversely assume the existen of f. Then we define g as follows. Let $f(a) = k$ with $1 \leq k \leq n+1$. We set

$$g(x) = \begin{cases} f(x) & \text{if } f(x) < k \\ f(x) - 1 & \text{if } f(x) > k \end{cases}$$

It is easy to check that g is a bijection. □

Proposition 2.8. Let $f : A \longrightarrow \{1, 2, \ldots, n\}$ be a bijection for some positive integer n. If B is a proper subset of A, then there does not exist any bijection $g : B \longrightarrow \{1, 2, \ldots, n\}$. If B is nonempty, then there exists a bijection $B \longrightarrow \{1, 2, \ldots, m\}$ for some positive integer $m < n$.

Proof. The proposition is true if B is the empty set. So we assume B is nonempty and induct. The proposition is clearly true when $n = 1$. Assume the truth of the proposition for n and let

$$f : A \longrightarrow \{1, 2, \ldots, n+1\}$$

be a bijection and B a proper subset of A. Fix $b \in B$ (B is nonempty) and $a \in (A - B)$ (B is a proper subset). By Proposition 2.7 there exists a bijection

$$g : A - \{b\} \longrightarrow \{1, 2, \ldots, n\}.$$

As $B - \{b\} \subseteq A - \{b\}$ (and is a proper subset), the induction hypothesis now implies that

(1) there does not exist any bijection

$$B - \{b\} \longrightarrow \{1, 2, \ldots, n\}$$

and

(2) either $B - \{b\}$ is empty or there exists a bijection

$$B - \{b\} \longrightarrow \{1, 2, \ldots, p\}$$

for some $p < n$ is it is nonempty.

The point (1) along with Proposition 2.7 tells us that there does not exist any bijection

$$B \longrightarrow \{1, 2, \ldots, n\}.$$

To prove the last claim of the proposition we observe that if $B - \{b\} = \emptyset$, then there is a bijection $B \longrightarrow \{1\}$ and if it is nonempty, then again by Proposition 2.7 there is a bijection

$$B \longrightarrow \{1, 2, \ldots, p+1\}.$$

But as $p < n$, we have $p+1 < n+1$ completing the inductive step and the proof of the proposition. □

Here are some consequences of the above observations.

Corollary 2.9. If A is a finite set, then there does not exist any bijection of A with a proper subset of of itself.

Proof. Let $B \subsetneq A$ be a proper subset. Assume that there exists a bijection $f : B \longrightarrow A$ Since A is finite, there exists a bijection $g : A \longrightarrow \{1, 2, \ldots, n\}$ for some positive integer n. The composition

$$g^{-1} \circ f : B \longrightarrow \{1, 2, \ldots, n\}$$

is a bijection contradicting Proposition 2.8. □

Thus if a set A is in bijective correspondence with a proper subset of iteself, then it must be infinite. Using this fact it is easy to show that the integers $\mathbb{Z}$ is infinite.

Corollary 2.10. Cardinality of a finite set is well defined.

Proof. Exercise. □

Another self evident fact that needs a proof is the following.

Corollary 2.11. A subset of a finite set is finite.

Proof. This follows from Proposition 2.8. Notice that the cardinality of B is strictly less than that of A.

□

Corollary 2.12. Let B be a nonempty set. Then the following are equivalent.

(1) B is finite.
(2) There exists a surjective function $f : \{1, 2, \ldots, n\} \longrightarrow B$ for some positive integer n.
(3) There exists an injective function $g : B \longrightarrow \{1, 2, \ldots, n\}$ for some positove integer n.

Proof. Clearly, (1) implies (2). We now show that (2) implies (3). So assume that $f : \{1, 2, \ldots, n\} \longrightarrow B$ is a surjective function. Then we immediately notice that for any distinct $b, b' \in B$ we have

$$f^{-1}(b) \neq \emptyset; \quad f^{-1}(b) \cap f^{-1}(b') = \emptyset.$$

Now define $g : B \longrightarrow \{1, 2, \ldots, n\}$ by setting $g(b)$ to be the smallest element of $g^{-1}(b)$. This is evidently injective. Thus (2) implies (3). Finally we show that (3) implies (1) which will complete the proof. So assume that $g : B \longrightarrow \{1, 2, \ldots, n\}$ is injective. Thus B is in bijective correspondence with a subset of the finite set $\{1, 2, , \ldots, n\}$. By Corollary 2.11 B must be finite. Thus (3) implies (1). This completes the proof. □

Corollary 2.13. Finte union and products of finite sets is finite.

Proof. Exercise. □

Here are some examples.

Example 2.14. Let $X = \{0, 1\}$. We consider the countable cartesian product X^ω. By definition, X^ω the set of all functions

$$\mathbf{x} : I = \{1.2. \ldots\} \longrightarrow \{0, 1\}.$$

Recalling our notation, we have that

$$\{0, 1\}^\omega = \{\mathbf{x} = (x_i)_i \, : \, x_i = 0, 1\}.$$

We often write

$$\mathbf{x} = (x_1, x_2, x_3, \ldots)$$

and think of $\mathbf{x}$ as sequence in X. Let A be the subset of X^ω defined by

$$A = \{\mathbf{x} = (x_i) \, : \, x_1 = 0\}.$$

In other words A is the subset of all sequences with first term 0. Observe that A is a proper subset of X^ω. The function $f : X^\omega \longrightarrow A$ defined by

$$f(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

is clearly a bijection.

Example 2.15. Let $A = \{1, 2, \ldots, n\}$ and let $\mathcal{P}(A)$ be the set of all subsets of of A (also called the power set of A. Let $X = \{0, 1\}$. Define a function

$$f : \mathcal{P}(A) \longrightarrow X^n$$

as follows. Given $B \in \mathcal{P}(A)$ consider the function

$$f(B) : I \longrightarrow X$$

defined by

$$f(B)(i) = \begin{cases} 1 & \text{if } i \in B \\ 0 & \text{otherwise} \end{cases}$$

Then by definition $f(B) \in X^n$. Then it is easy to check that f is a bijection. It is an exercise to show that if A is finite, then so is $\mathcal{P}(A)$.

We now discuss infinite sets. We begin with a definition. We shall use the notation $\mathbb{Z}_+$ to denote the positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers.

Definition 2.16. A set A is said to be *countably infinite* if there exists a bijectiion

$$f : A \longrightarrow \mathbb{Z}_+$$

of A with the set of positive integers. A set A is said to be *countable* if it is either finite or countably infinite.

Here are some examples.

Example 2.17. The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}_+$ defined by

$$f(n) = \begin{cases} 2n & n > 0 \\ -2n + 1 & n \leq 0 \end{cases}$$

is a bijection showing that $\mathbb{Z}$ is countably infinite.

Example 2.18. The function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ defined by

$$f(m, n) = 2^{m-1}(2n - 1)$$

is a bijection showing that the cartesian product $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.

The following theorem tells us several interesting ways to decide when a set is countable.

Theorem 2.19. Let B be a nonempty set. Then the following are equivalent.

(1) B is countable.
(2) There is a surjective function $f : \mathbb{Z}_+ \longrightarrow B$.
(3) There exists an injective function $g : B \longrightarrow \mathbb{Z}_+$.

Proof. Assume that B is countable. If B is countably infinite, we are done. If B is finite then there exists a bijection $\{1, 2, \ldots, n\} \longrightarrow B$ for some positive integer n. This can be extended to a surjective function $\mathbb{Z}_+ \longrightarrow B$. Thus (1) implies (2).

Next assume that there exists a surjective function $f : \mathbb{Z}_+ \longrightarrow B$. Then an injective function $g : B \longrightarrow \mathbb{Z}_+$ can be constructed just as in Corollary 2.12. Thus (2) implies (3).

Finally assume that there is an injective function $g : B \longrightarrow \mathbb{Z}_+$. Thus B is in bijective correspondence with its image $g(B)$. If $g(B)$ is finite, then B is finite and hence countable. If $g(B)$ is infinite we proceed as follows.

We will use the following property of positive integers : Every nonempty subset of $\mathbb{Z}_+$ has a smallest element. This is called the well-ordering principle.

We now construct a bijection $h : \mathbb{Z}_+ \longrightarrow g(B)$ thereby completing the proof that $g(B)$ is countably infinite. First define $h(1)$ to be the smallest element of $g(B)$. Assume that we have defined $h(1), \ldots, h(n-1)$. We define $h(n)$ to be the smallest element of the set

$$g(B) - \{h(1), h(2), \ldots, h(n-1)\}.$$

The set $g(B) - \{h(1), h(2), \ldots, h(n-1)\}$ is nonempty. For if it were empty h would be a surjective map from

$$h : \{1, 2, \ldots, n-1\} \longrightarrow g(B)$$

which woul mean (by Corollary 2.12) that $g(B)$ is finite. A contradiction. Hence $h(n)$ is well defined. Thus the function

$$h : \mathbb{Z}_+ \longrightarrow g(B)$$

is well defined.

We first verify that h is one one. Suppose $m < n$. Then, by construction,

$$h(n) \in g(B) - \{h(1), \ldots, h(m), h(m+1), \ldots, h(n-1)\}$$

so in particular $h(m) \neq h(n)$.

Finally we verify that h is surjective. Let $p \in g(B)$. We shall show that there exists m with $h(m) = p$. Since h is injective, $h(\mathbb{Z}_+)$ is infinite and hence there exists $n \in \mathbb{Z}_+$ such that

$$h(n) > p$$

Thus the set

$$\{a \in \mathbb{Z}_+ \,:\, h(a) \geq p\}$$

is nonempty and therefore must have a smallest element say m. Then whenever $i < m$ we must have

$$h(i) < p.$$

Thus

$$p \notin \{h(1), \ldots, h(m-1)\}.$$

Since $h(m)$ is the smallest element of the set

$$g(B) - \{h(1), \ldots, h(m-1)\}$$

we must have $h(m) \leq p$. Whence $h(m) = p$. This completes the proof. $\square$

The proof actually shows that every infinite subset of $\mathbb{Z}_+$ is countably infinite.