# QUIZ

(1) Consider the two statements: (P) $x^4 - 2$ is irreducible over $\mathbb{Q}(\sqrt{2})$, (Q) $x^4 - 2$ is irreducible over $\mathbb{Q}(\sqrt{3})$. Then,
(a) Both statements are true,
(b) (P) is true and (Q) is false,
(c) (P) is false and (Q) is true,
(d) none of the above

Answer: (c). We have that $x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2})$ is reducible over $\mathbb{Q}(\sqrt{2})$, so (P) is true. But (Q) is false, and we argue as follows. Suppose $x^4 - 2$ is reducible over $\mathbb{Q}(\sqrt{3})$, then it can either factor as the product of two degree 2 polynomials over $\mathbb{Q}(\sqrt{3})$ or as the product of a degree 3 and a degree 1 polynomial over $\mathbb{Q}(\sqrt{3})$ (and hence in this case, a root of $x^4 - 2$ belongs to $\mathbb{Q}(\sqrt{3})$). Let $\alpha = \sqrt[4]{2}$ be the real 4th root of 2. Then the roots of $x^4 - 2$ are $\alpha$, $-\alpha$. $i\alpha$ and $-i\alpha$. Attaching any of them to $\mathbb{Q}$ produces a degree 4 extension of $\mathbb{Q}$ (as $x^4 - 2$ is irreducible over $\mathbb{Q}$) and such an extension cannot be contained $\mathbb{Q}(\sqrt{3})$ which has degree 2 over $\mathbb{Q}$. Suppose $x^4 - 2$ factorizes as the product of two degree two polynomials over $\mathbb{Q}(\sqrt{3})$. Then consider one of these degree 2 factors and attach one (and hence both) of the roots of this factor to $\mathbb{Q}(\sqrt{3})$. We get a degree 2 extension of $\mathbb{Q}(\sqrt{3})$. It is clear that this degree 4 extension of $\mathbb{Q}$ contains the degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and hence is equal to it, and hence must be a normal extension. Hence this degree 4 extension must contain all the roots of $x^4 - 2$ (since it contains one of them), hence it must contain $\mathbb{Q}(\alpha, i)$ which has degree 8 over $\mathbb{Q}$, and this is a contradiction. You can do every step of this problem in several other ways and so your solution need not match with this one.

(2) Let $L$ be a splitting field of $x^5 - 1$ over $\mathbb{Q}$. Then $[L : \mathbb{Q}]$ is
(a) 1,
(b) 2,
(c) 4,
(d) 5

Answer: (c) is true. Let $\zeta = e^{2i\pi/5}$, then the roots of $x^5 - 1$ are $1, \zeta, \zeta^2, \zeta^3, \zeta^4$. The minimal polynomial of $\zeta$ over $\mathbb{Q}$ is $x^4 + x^3 + x^2 + x + 1 = 0$. Hence, the splitting field of $x^5 - 1$ over $\mathbb{Q}$ is $\mathbb{Q}(\zeta)$ which has degree 4 over $\mathbb{Q}$.

(3) Let $K$ be a splitting field of $x^6 - 1$ over $\mathbb{Q}$. Then $[K : \mathbb{Q}]$ is
(a) 1,
(b) 2,
(c) 3,
(d) 6.

Answer: (b) is true. The roots of $x^6 - 1$ are $1, \beta, \beta^2, \beta^3 = -1, \beta^4, \beta^5$ where $\beta = e^{2i\pi/6} = \cos(2\pi/6) + i\sin(2\pi/6) = \frac{1+\sqrt{3}i}{2}$. Hence a splitting field of $x^6 - 1$ is $\mathbb{Q}(\beta) = \mathbb{Q}(i\sqrt{3})$. The irreducible polynomial of $\beta$ over $\mathbb{Q}$ is $x^2 - x + 1$. Note $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$ over $\mathbb{Q}$.

(4) Let $E$ be a splitting field of $x^6+1$ over the finite field $F_2$ with two elements. Then $[E:F_2]$ is
   (a) 1,
   (b) 2,
   (c) 3,
   (d) 6.

   Answer: (b). Over $F_2$, $x^6+1 = x^6-1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1) = (x+1)^2(x^2+x+1)^2$. Note $x^2+x+1$ is irreducible over $F_2$.

(5) Let $F \subset L \subset K$ be three fields, and let $K/F$ be a normal extension. Consider the two following statements: (P) $K/L$ is a normal extension and (Q) $L/F$ is a normal extension. Then,
   (a) Both statements are true,
   (b) (P) is true and (Q) is false,
   (c) (P) is false and (Q) is true,
   (d) none of the above

   Answer: (b) is true, why?

(6) Let $t$ be a variable. Consider the two following statements: (P) $\mathbb{C}(t)/\mathbb{C}(t^3)$ is a normal extension and (Q)$F_2(t)/F_2(t^3)$ is a normal extension (here $F_2$ is a field with 2 elements). Then,
   (a) Both statements are true,
   (b) (P) is true and (Q) is false,
   (c) (P) is false and (Q) is true,
   (d) none of the above

   Answer: (b). The roots of $x^3 - t^3 = 0$ over $\mathbb{C}(t^3)$ are $t, \omega t, \omega^2 t$ where $\omega$ is a primitive cube root of unity. Now note $\omega, \omega^2$ belong to $\mathbb{C}$. The same reasoning gives that (Q) is false, as the equation $x^2+x+1$ is irreducible over $F_2$ and if $\alpha$ is a root of this equation then $\alpha$ does not belong to $F_2(t)$ (why?). Note that the three roots of $x^3-t^3$ over $F_2$ are $t, t\alpha, t\alpha^2$.