

EXERCISES — I

1. Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Assume that f has a local minima at each point. Show that f is a constant.
2. What are the different metrics that you know on \mathbb{R} .
3. Show that every metric space is Hausdorff.
4. Compare the usual and the cofinite topology on \mathbb{R} .
5. Compare the usual and countable topology on \mathbb{R} .
6. ~~Describe~~ Describe as many bases as you can for the usual topology on \mathbb{R} .
7. Let X be the set of positive integers. Show that the collection of all arithmetic progressions of positive integers is a basis for a topology on X .
8. Show that if p is a prime, then the set $\{np : n \geq 1\}$ is closed in the topology on X in Problem 7. Conclude that there exist infinitely many primes.

9. Let Y be a subspace of X . Show that if U is open in Y and Y is open in X , then U is open in X .
10. Show that any finite set in a Hausdorff space is closed.
11. Show that arbitrary product of Hausdorff spaces is Hausdorff.
12. Show that X is Hausdorff if and only if $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.
13. Let $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function $i=1, \dots, k$
 Show that $Z = \left\{ (x_1, \dots, x_n) \mid f_i(x_1, \dots, x_n) = 0 \right\}_{i=1, \dots, k}$ is a closed subset of \mathbb{R}^n .
14. Suppose X is a space and that $X = \bigcup_{\alpha} A_{\alpha}$ where each A_{α} is closed in X . Assume that $f: X \rightarrow Y$ is such that $f_{\alpha} = f|_{A_{\alpha}}: A_{\alpha} \rightarrow Y$ is continuous for each α . Is f continuous? We had checked this when each A_{α} is open.

15. Show that \mathbb{R}^n is homeomorphic to \mathbb{R}
if and only if $n=1$. 3

16. Show that

$$X = \{(x_1, \dots, x_n) \mid \sum x_i^2 < 1\}$$

is homeomorphic to \mathbb{R}^n . Is every open convex subset of \mathbb{R}^n homeomorphic to \mathbb{R}^n ?

17. Let

$$S^n = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1\}.$$

Show that $S^n - (0, \dots, 0, 1)$ is homeomorphic to \mathbb{R}^n .

18. Let $f: X \rightarrow Y$ be a continuous function.
Show that X is homeomorphic to

$$\text{gr}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y.$$

$\text{gr}(f)$ is called the graph of f .

19. A function $f: [0,1] \rightarrow \mathbb{R}$ is said to be bounded if $\exists M$ such that

$$|f(x)| \leq M \quad \forall x \in [0,1].$$

Let $B[0,1]$ denote the set of bounded functions $f: [0,1] \rightarrow \mathbb{R}$. For $f, g \in B[0,1]$ show that

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

is a metric.

20. Let X be a space such that every

$$f: X \rightarrow \mathbb{R}$$

is continuous. What can you say about the topology of X ?

21. Let X be a connected space and $f: X \rightarrow \mathbb{R}$ continuous. Assume that for each $x \in X$ there exists a neighbourhood U of x restricted to which f is constant. Show that f is constant.

22. Let X be a space and $A, B \subseteq X$. Determine whether

$$\overline{A \cap B} = \overline{A} \cap \overline{B}$$

$$\overline{A - B} = \overline{A} - \overline{B}$$

23. If $A \subseteq X$, then the boundary, $\text{bd}(A)$, of A is defined by the expression

$$\text{bd}(A) = \overline{A} \cap \overline{(X - A)}$$

Show that $\text{Int}(A)$ and $\text{bd}(A)$ are disjoint and

that

$$\overline{A} = \text{Int}(A) \cup \text{bd}(A).$$

Show that $\text{bd}(A) = \emptyset$ if and only if A is both open and closed.