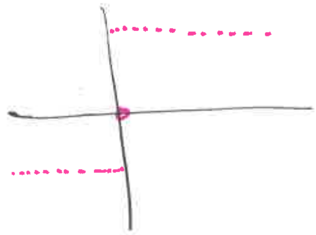
Product and box topologies:-

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0. \end{cases}$



Is f continuous function ?

Qn: Give a topology τ on $\mathbb{R}$ so that $f: \mathbb{R}_\tau \to \mathbb{R}$ is continuous ?

$$\mathcal{B} = \{ (-\infty, 0), \{0\}, [0, \infty) \}$$

$\tau_{\mathcal{B}}$ will work !

- $f_i : Y \to X_i$, $i = 1, 2 \cdots n$, X_i = Top. space.

$$\mathcal{B} = \bigcup_{i=1}^{n} f_i^{-1}(U_i),$$

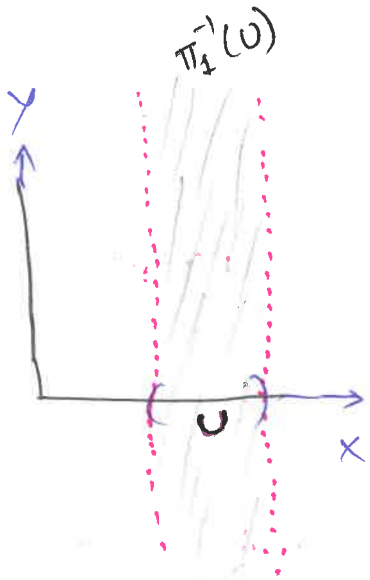
U_i = open in X_i

Then $\tau_{\mathcal{B}}$ is the smallest topology on Y such that $f_i : Y \to X_i$ is continuous for every $i = 1, 2, \cdots n$.

- $\pi_3 : X_1 \times X_2 \times X_3 \times \cdots \times X_n \to X_3$

$$(x_1, x_2, \cdots x_n) \mapsto x_3$$

$$\pi_3^{-1}(U) = X_1 \times X_2 \times U \times X_4 \times X_5 \cdots \times X_n.$$



- $X = \prod_{\alpha \in J} X_\alpha$, $x = (x_\alpha)$

$$\pi_\beta : \prod X_\alpha \to X_\beta$$

$$(x_\alpha)_{\alpha \in J} \mapsto x_\beta$$

(β^{th} projection).

$\pi_\beta^{-1}(U) = \prod_{\alpha \in J} V_\alpha$,

$V_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \neq \beta \\ U & \text{if } \alpha = \beta \end{cases}$

$S_\beta = \{ \pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta \}$

$S = \bigcup_{\beta \in J} S_\beta$, then $\mathcal{T}_S :=$ Product topology on $\prod_{\alpha \in J} X_\alpha$.

* Product topology on $\prod_{\alpha \in J} X_\alpha$ is nothing but the smallest topology on $\prod_{\alpha \in J} X_\alpha$ such that each projection $\pi_\beta : \prod X_\alpha \to X_\beta$ is continuous.

Basis : $B = \prod_{\alpha \in J} U_\alpha$, $U_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \neq \beta_1, \beta_2, \ldots, \beta_n \text{ for some finite elements.} \end{cases}$

Box topology:

$\mathcal{B} = \{ B = \prod_{\alpha \in J} U_\alpha , U_\alpha = \text{open in } X_\alpha \}$ generates box topology.

Theorem:

Let $\mathcal{B}_\alpha$ be a basis for X_α. Then

$\mathcal{B} = \{ B = \prod_{\alpha \in J} B_\alpha : B_\alpha \in \mathcal{B}_\alpha \}$ is a basis for box topology on $\prod X_\alpha$.

Proof:

* $\forall x \in X = \prod X_\alpha$, $\exists$ $B \in \mathcal{B}$ $\ni$ $x \in B$
* $x \in B_1 \cap B_2 \Rightarrow \exists$ $B_3 \in \mathcal{B}$ $\ni$ $x \in B_3 \subseteq B_1 \cap B_2$.

Qn: Can you give a basis for product topology on $\prod X_\alpha$?

$B = \prod_{\alpha \in J} B_\alpha$, $B_\alpha = X_\alpha$ except finitely many. $B_\alpha \in \mathcal{B}_\alpha$.

Theorem:-

Let A_α be a subspace of X_α $\forall \alpha \in J$. Then

$\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given box topology or product topology.

Proof:

$$\left(\prod A_\alpha\right) \cap \left(\prod U_\alpha\right) = \prod \left(A_\alpha \cap U_\alpha\right)$$

Theorem:-

If X_α is T_2-space $\forall \alpha \in J$ then $X = \prod_{\alpha \in J} X_\alpha$ is also T_2 space.

Proof:-

$x = (x_\alpha) \neq y = (y_\alpha) \Rightarrow \exists\, \beta \in J$ s.t $x_\beta \neq y_\beta$.

$\because X_\beta$ is T_2, $\exists\ U_\beta, V_\beta \in \tau_{X_\beta}$ s.t $x_\beta \in U_\beta$, $y_\beta \in V_\beta$, $U_\beta \cap V_\beta = \phi$

$U = \prod U_\alpha$, $V = \prod V_\alpha$, $U_\alpha = X_\alpha$ if $\alpha \neq \beta$, $V_\alpha = X_\alpha$ if $\alpha \neq \beta$

$$U \cap V = \prod_{\alpha \in J} \left(U_\alpha \cap V_\alpha\right) = \phi.$$

$\therefore X = \prod X_\alpha$ is T_2 space in both box & product top.

- $f_\alpha : A \to X_\alpha$, $f = (f_\alpha)$

$$f : A \to \prod_{\alpha \in J} X_\alpha = X$$

Theorem:- f is continuous (product top on X) $\Leftrightarrow$ each f_α is continuous.

Result:- $f : A \to X$ (box top) is continuous $\Rightarrow$ each f_α is cts.

- $f : A \to X$ (box top) is cts $\Rightarrow$ $f : A \to X$ (prod top) is cts. $\Leftrightarrow f_\alpha$ is cts $\forall \alpha \in J$

converse : fails !

$f : \mathbb{R} \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots$ $\quad f = (f_n)$

$x \mapsto (x, x, x, x, \cdots)$ $\quad f_n(x) = x$. is cts $\forall n$.

f is not continuous ! (why ?)

$B = (-1,1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$

$f^{-1}(B) = ?$

$x \in f^{-1}(B) \iff x \in (-1/n, 1/n) \ \forall n \iff x = 0$.

$f^{-1}(B) = \{0\}$ is not open.

usual top here !

Thm : $\{x_n\}$ be a seq in $X = \prod X_\alpha$ (prod . top).

$x_n \to x \iff \pi_\alpha(x_n) \to \pi_\alpha(x) \ \forall \alpha \in J$.

Proof : suppose $x_n \to x$. Fix $\beta \in J$.

claim : $\pi_\beta(x_n) \to \pi_\beta(x)$.

Let U_β is a open set in X_β containing $\pi_\beta(x)$.

Take $U = \prod_{\alpha\in J} U_\alpha$, $\quad U_\alpha = X_\alpha$ if $\alpha \neq \beta$.

$x \in U$! (why ?)

$x_n \in U \quad \forall n \geq N$ for some $N \in \mathbb{N}$.

$\Rightarrow \pi_\beta(x_n) \in U_\beta \quad \forall n \geq N$

ie) $\pi_\beta(x_n) \to \pi_\beta(x)$.

same proof works box top also !

Converse :

suppose $\pi_\alpha(x_n) \to \pi_\alpha(x) = x_\alpha \quad \forall \alpha \in J.$

claim: $x_n \to x$

Let U be an open set containing x.

$$x \in \prod_{\alpha \in J} U_\alpha \subseteq U, \quad U_\alpha = X_\alpha \text{ if } \alpha \neq \beta_1, \beta_2, \dots \beta_k.$$

$\Rightarrow x_\alpha \in U_\alpha \quad \forall \alpha.$

$\Rightarrow x_{\beta_i} \in U_{\beta_i} \quad \forall i = 1, 2 \dots k.$

$\Rightarrow \pi_{\beta_i}(x_n) \in U_{\beta_i} \quad \forall n \geq N_i,$

$N = \max\{N_1, N_2, \dots N_k\}$

$\Rightarrow \pi_{\beta_i}(x_n) \in U_{\beta_i} \quad \forall i = 1, 2 \dots k, \quad \forall n \geq N.$

Thus $x_n \in \prod_{\alpha \in J} U_\alpha \quad \forall n \geq N.$ ie) $x_n \to x.$

what about box top case ?

$x_1 = (1, 0, 0, \dots)$

$x_2 = (0, 1/2, 0, 0 \dots)$

$x_n = (0, 0, \dots, 0, 1/n, 0, 0, \dots)$

$\odot \; \pi_k(x_n) \to 0 \quad \forall k$

But $x_n \not\to 0 = (0, 0, 0 \dots).$

$B = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots \times (-1/n, 1/n) \times \dots$

$\odot \; x_n \notin B \quad \forall n \quad \because 1/n \notin (-1/n, 1/n)$

$\therefore x_n \not\to 0 = (0, 0 \dots)$ in box topology !

Question:

$\mathbb{R}^\infty = \{x = (x_1, x_2, \dots) : x_i \neq 0$ for atmost finitely many values of $i\}$

$= \text{span}\{e_1, e_2, \dots\}, \quad e_n = (0, 0, \dots 0, 1, 0, 0, \dots)$

what is the closure of $\mathbb{R}^{\infty}$ in box top? (in prod. top?)

Soln:- $\overline{\mathbb{R}^{\infty}} = \mathbb{R}^{\omega}$ (prod. top)

Let $x = (a_1, a_2, \dots) \in \mathbb{R}^{\omega}$ be an arbitrary point.

Take $x_n = (a_1, a_2, \dots a_n, 0, 0, 0 \dots)$ for each n.

- $x_n \in \mathbb{R}^{\infty}$ $\forall$ n.
- Let U be an open set containing x.

$x \in \prod_{n=1}^{\infty} U_n \subseteq U$, $U_\alpha = X_\alpha$ if $\alpha \neq \beta_1, \beta_2, \dots \beta_k$
$= \mathbb{R}$

$N = \max\{\beta_1, \beta_2, \dots \beta_k\}$, $U_n = \mathbb{R}$ $\forall n \geq N$.

- $x_n \in U$ $\forall$ $n \geq N$ $\Rightarrow$ $x_n \to x$.

ie) $\boxed{\overline{\mathbb{R}_\infty} = \mathbb{R}^{\omega}.}$

- $\overline{\mathbb{R}_\infty} = \mathbb{R}_\infty$ in box top

claim:- $\mathbb{R}_\infty^{c}$ is open set. Let $x \in \mathbb{R}_\infty^{c}$.

ie) $x_n \neq 0$ for infinitely many values of n.

ie) $x_{n_k} \neq 0$ $\forall$ $k = 1, 2, \dots$

Choose U_{n_k} (open set in $\mathbb{R}$) s.t $0 \notin U_{n_k}$, $x_{n_k} \in U_{n_k}$.

Take $U = \prod_{n=1}^{\infty} U_n$, $U_n = \mathbb{R}$ if $n \neq n_k$, $k = 1, 2 \dots$

$x \in U$, $U \cap \mathbb{R}_\infty = \phi$ $\therefore$ $\mathbb{R}_\infty^{c}$ is open set.

Hence $\boxed{\overline{\mathbb{R}_\infty} = \mathbb{R}_\infty.}$

Qn:- $a_1, a_2, \ldots$ & $b_1, b_2, \ldots$ are real numbers,

$a_i > 0 \ \forall i$ $\quad$ $h_n : \mathbb{R} \to \mathbb{R}$ by

$h_n(x) = \cancel{\ldots}\ a_n x + b_n$

$h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$h(x_1, x_2, \ldots) \longmapsto (h_1(x_1), h_2(x_2), \ldots)$$

- h_n is bijective map $(\mathbb{R} \to \mathbb{R})$
- $h^{-1}(y) = \dfrac{y - b_n}{a_n}$, $h^{-1} : \mathbb{R} \to \mathbb{R}$ $\quad h^{-1}(y) = \dfrac{y}{a_n} + \left(\dfrac{-b_n}{a_n}\right)$.
- each h_n is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.
- clearly h is bijective

$$h^{-1}(y_1, y_2, \ldots) = (h_1^{-1}(y_1), h_2^{-1}(y_2), \ldots)$$

- $h^{-1}\left(\prod_{n=1}^{\infty} U_n\right) = \prod_{n=1}^{\infty} \underbrace{h_n^{-1}(U_n)}_{\text{open}} = \text{open}.$

$\therefore$ h is a homeomorphism from $\mathbb{R}^\omega$ to $\mathbb{R}^\omega$ (in box & product top both).

Qn:- $f_\alpha : A \to X_\alpha$, $\alpha \in J$.

- show that there is a unique coarsest top on A to which each f_α is cts.

- $S_\alpha = \{ f_\alpha^{-1}(U_\alpha) : U_\alpha \text{ is open in } X_\alpha \}$

$\bigcup_{\alpha \in J} S_\alpha$ generates the required topology.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend right, "f_\alpha \circ g"'] & A \arrow[r, "f_\alpha = \text{cts}"] & X_\alpha \end{tikzcd}$$

Qn: g is cts $\Longleftrightarrow$ $f_\alpha \circ g$ is cts $\forall\, \alpha \in J$.

Proof If g is cts then $f_\alpha \circ g$ is cts ($\because$ f_α is cts $\forall\, \alpha \in J$)

suppose $f_\alpha \circ g$ is cts $\forall\, \alpha \in J$.

claim: $g : Y \to A$ is continuous.

$\Leftrightarrow$ $g^{-1}(U)$ is open $\forall\, U =$ open in A.

$\Leftrightarrow$ $g^{-1}(U)$ is open $\forall\, U =$ sub basis elt in A.

$\Leftrightarrow$ $g^{-1}\left(f_\beta^{-1}(U_\beta)\right)$ is open $\forall\, U_\beta =$ open in X_β, $\forall\, \beta \in J$.

$\Leftrightarrow$ $(f_\beta \circ g)^{-1}(U_\beta)$ is open $\forall\, U_\beta =$ open in X_β, $\forall\, \beta \in J$.

$\Leftrightarrow$ $f_\beta \circ g$ is cts $\forall\, \beta \in J$.

⊙ $f_\alpha : A \to X_\alpha$, $f = A \to \prod X_\alpha$, $f = (f_\alpha)_{\alpha \in J}$.

Then prove that f is open map. $[f : A \to f(A)]$

sol: Let $U \in \tau$, claim: $f(U)$ is open. Let $x \in f(U)$

$a = f(x)$, $a \in U$, $a \in V \subseteq U$, $V = \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$

Take $U_\alpha = X_\alpha$ if $\alpha \neq \alpha_1, \alpha_2, \ldots \alpha_n$.

$$V = \bigcap_{\alpha \in J} f_\alpha^{-1}(U_\alpha)$$

$$= f^{-1}\Big(\prod_{\alpha \in J} U_\alpha\Big).$$

$$f(V) = f(A) \cap \prod_{\alpha \in J} U_\alpha$$

$x \in f(V) \subseteq f(U)$

$\therefore$ $f(U)$ is open in the subspace topology.

=== x ===