Corollary. The power of the MP level α test for testing $H_0 : P_\theta = P_{\theta_0}$ versus $H_1 : P_\theta = P_{\theta_1}$ is strictly larger than α unless $P_{\theta_1} = P_{\theta_0}$.

Proof. Consider $\phi(x) \equiv \alpha$. This is a level α test since $E_{\theta_0}\phi(X) = \alpha$. Since $E_{\theta_1}\phi(X) = \alpha$ also, the MP test has power at least $E_{\theta_1}\phi(X) = \alpha$. If the power of the MP test is exactly α, then $\phi(x) \equiv \alpha$ is also MP. Then from the earlier theorem, part (c), ϕ should satisfy:

$$\phi(x) = \begin{cases} 1 & \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} > k; \\ 0 & \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} < k. \end{cases}$$

Therefore $p_{\theta_1}(x) = p_{\theta_0}(x)$ w.p. 1. (Why is $k = 1$?) i.e., $P_{\theta_1} = P_{\theta_0}$.

Example. $X \sim \text{Binomial}(2, p)$. Test $H_0 : p = 1/2$ versus $H_1 : p = 3/4$ at level $\alpha = 0.01$. MP test ϕ has the form:

$$\phi(x) = \begin{cases} 1 & \frac{p_{3/4}(x)}{p_{1/2}(x)} > k; \\ 0 & \frac{p_{3/4}(x)}{p_{1/2}(x)} < k. \end{cases}$$

Note that

$$\begin{aligned} \frac{p_{3/4}(x)}{p_{1/2}(x)} &= \frac{\binom{2}{x}(3/4)^x(1/4)^{2-x}}{\binom{2}{x}(1/2)^x(1/2)^{2-x}} > k \text{ iff} \\ & 3^x 2^2 4^{-2} > k \text{ iff} \\ & 3^x > 4k \text{ iff} \\ & x > \frac{\log(4k)}{\log(3)} = k_1. \end{aligned}$$

i.e.,

$$\phi(x) = \begin{cases} 1 & \text{if } x > k_1; \\ 0 & \text{if } x < k_1; \\ \gamma & \text{if } x = k_1. \end{cases},$$

where k_1 and γ are chosen to satisfy the level condition. Now note that

$$p_{1/2}(x) = \begin{cases} 1/4 & \text{if } x = 0; \\ 1/2 & \text{if } x = 1; \\ 1/4 & \text{if } x = 2. \end{cases}$$

Since $p_{1/2}(2) = 1/4 > 0.01$, $k_1 = 2$. Since

$$\alpha = 0.01 = E_{1/2}\phi(X) = \gamma p_{1/2}(2) = \gamma/4,$$

the MP test rejects H_0 with probability 0.04 if $x = 2$; accepts otherwise.

Example. $X_1, \ldots, X_n$ i.i.d $N(\mu, \sigma^2)$, σ^2 known. Test $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$ where $\mu_1 > \mu_0$. $\bar{X}$ is sufficient for μ and $\bar{X} \sim N(\mu, \sigma^2/n)$. We switch notation now from $p_\theta(x)$ to $f(x|\theta)$. MP level α test is of the form:

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{f(\bar{x}|\mu_1)}{f(\bar{x}|\mu_0)} > k; \\ \gamma & \text{if } \frac{f(\bar{x}|\mu_1)}{f(\bar{x}|\mu_0)} = k; \\ 0 & \text{if } \frac{f(\bar{x}|\mu_1)}{f(\bar{x}|\mu_0)} < k. \end{cases}$$

Since $f(\bar{x}|\mu) = (2\pi)^{-1/2}\frac{\sqrt{n}}{\sigma}\exp\left(-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right)$,

$$\begin{aligned} \frac{f(\bar{x}|\mu_1)}{f(\bar{x}|\mu_0)} &= \exp\left(-\frac{n}{2\sigma^2}\left\{\bar{x}^2 + \mu_1^2 - 2\mu_1\bar{x} - \bar{x}^2 - \mu_0^2 + 2\mu_0\bar{x}\right\}\right) \\ &= \exp\left(-\frac{n}{2\sigma^2}\left\{\mu_1^2 - \mu_0^2 - 2\bar{x}(\mu_1 - \mu_0)\right\}\right) \\ &= \exp\left(\frac{n(\mu_1 - \mu_0)}{\sigma^2}\bar{x} - \frac{n}{2\sigma^2}(\mu_1^2 - \mu_0^2)\right), \end{aligned}$$

which is a monotone increasing function of $\bar{x}$. Therefore,

$$\frac{f(\bar{x}|\mu_1)}{f(\bar{x}|\mu_0)} > k \text{ iff } \bar{x} > c \text{ for some } c,$$

and hence

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \bar{x} > c; \\ \gamma & \text{if } \bar{x} = c; \\ 0 & \text{if } \bar{x} < c. \end{cases}$$

Choice of γ is not needed since $P_\mu(\bar{X} = c) = 0$. We need

$$\alpha = P_{\mu_0}\left(\bar{X} > c\right) = P_{\mu_0}\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{c - \mu_0}{\sigma/\sqrt{n}}\right),$$

so that $\frac{c-\mu_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$. Equivalently,

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}\ ; \\ 0 & \text{if } \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_{1-\alpha}. \end{cases}$$

This is simply the standard Z-test.

Uniformly Most Powerful (UMP) Tests.

For testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$, ϕ is level α UMP test if $\sup_{\theta \in \Theta_0} E_\theta \phi(X) \leq \alpha$ and

$E_\theta \phi(X) \geq E_\theta \phi^*(X)$ for all $\theta \in \Theta_1$, for any other test ϕ^* for which $\sup_{\theta \in \Theta_0} E_\theta \phi^*(X) \leq \alpha$. This extension of the MP theory is subject to rather strong conditions on the density $f(x|\theta)$:

I. $\Theta \subset \mathcal{R}^1$ (single parameter)

II. Monotone Likelihood Ratio (MLR). $P_\theta, \theta \in \Theta \subset \mathcal{R}^1$ with density $f(x|\theta)$ is said to have m.l.r if there exists a real valued function $T(x)$ such that for any $\theta < \theta'$, $P_\theta \neq P_{\theta'}$ and the likelihood ratio $\frac{f(x|\theta')}{f(x|\theta)}$ is a nondecreasing function of $T(x)$. i.e.,

$$\frac{f(x|\theta')}{f(x|\theta)} = h_{\theta,\theta'}(T(x)), \text{ where } h_{\theta,\theta'}(y) \nearrow y \text{ for fixed } \theta' > \theta.$$

Example. $X \sim \text{Binomial}(n, \theta)$. Then

$$\begin{aligned} f(x|\theta) &= \binom{n}{x} \theta^x (1-\theta)^{n-x}, \text{ and hence} \\ \frac{f(x|\theta')}{f(x|\theta)} &= \left(\frac{\theta'}{\theta}\right)^x \left(\frac{1-\theta'}{1-\theta}\right)^{n-x} \\ &= \left(\frac{1-\theta'}{1-\theta}\right)^n \left(\frac{\theta'/(1-\theta')}{\theta/(1-\theta)}\right)^x. \end{aligned}$$

For fixed $\theta' > \theta$, $\left(\frac{1-\theta'}{1-\theta}\right)^n$ is fixed and $\frac{\theta'}{1-\theta'} > \frac{\theta}{1-\theta}$, so that $\left(\frac{\theta'/(1-\theta')}{\theta/(1-\theta)}\right)^x$ is increasing in $T(x) = x$.

Theorem. If $P_\theta, \theta \in \Theta \subset \mathcal{R}^1$ belongs to a one-parameter exponential family having density

$$f(x|\theta) = \exp\left(c(\theta)T(x) + d(\theta) + S(x)\right) I_A(x)$$

with $c(.)$ strictly monotone in θ (strictly increasing or strictly decreasing) then $\{P_\theta\}$ has m.l.r. in $T(x)$ or $-T(x)$.

Proof. Note that

$$\frac{f(x|\theta')}{f(x|\theta)} = \exp\left(T(x)\left[c(\theta') - c(\theta)\right]\right) \exp\left(d(\theta') - d(\theta)\right)$$

is increasing in $T(x)$ if $c(.)$ is increasing, otherwise increasing in $-T(x)$.

Example. $X \sim U(0, \theta]$. Then $f(x|\theta) = \frac{1}{\theta} I(0 < x \leq \theta)$, so that for $\theta' > \theta > 0$,

$$\frac{f(x|\theta')}{f(x|\theta)} = \begin{cases} \frac{\theta}{\theta'} & \text{if } 0 < x \leq \theta; \\ \infty & \text{if } \theta < x \leq \theta'. \end{cases}$$

This shows that $U(0, \theta)$, $\theta > 0$ has monotone likelihood ratio (in $T(x) = x$) even though it is not an exponential family.