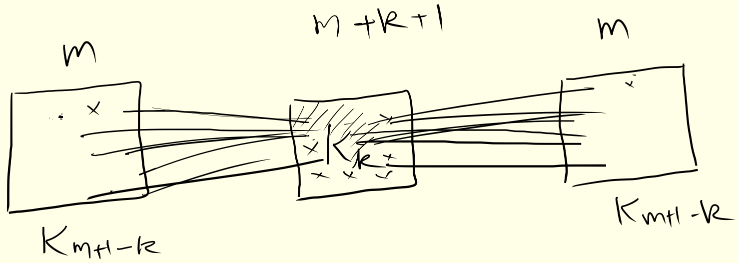
Graph Theory

Lecture 15

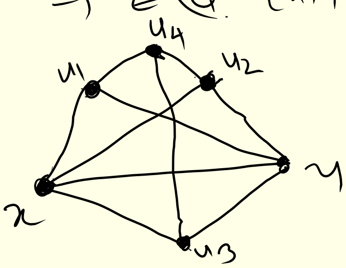
"Brooks Theorem" on vertex colouring

1. Solutions to assignment questions.

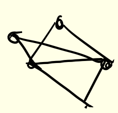
   1. $0 < k \le m$ find a k-connected G with $\delta(G) = m$.



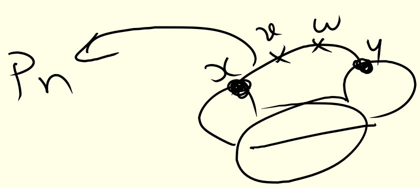
   2. Connectivity is 3, $G - \{v, w\}$ is connected $\forall\, v, w$.
   In particular $G - \{x, y\}$ is connected.
   $\Rightarrow \mathcal{E}(G - \{x, y\}) \geq 3$.



$*$ $E(G) \geq 10$.
this is the minimum.



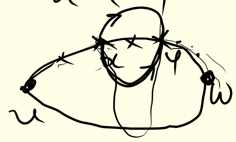
   3. minimally 2-connected. $\Leftrightarrow$ $G - e$ is not 2-conn. $\forall e$.
   Ear decomposition of G.



P_n be last ear.
If P_n is an edge then
$G - P_n$ is 2-connected.
$\therefore$ P_n is not an edge. $\therefore$ it contains
a vertex of deg. 2.

   4. $G - xy$ is 2-conn. iff x & y lie on a cycle in $G - xy$.

$\rightarrow G$ is 2-connected



Abel prize - Norwegian Government.

(born 1948) Laszlo Lavasz & Avi Wigderson

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Def^n_s:- • A k-colouring of a graph G is a map $f : V(G) \to S$ where $|S| = k$.

• A proper k-colouring of G is a k-colouring s.t. $f(v) \neq f(w)$ if $vw \in E(G)$

• G is called k-colourable if $\exists$ proper k-colouring of G.

• Chromatic number (denoted by $\chi(G)$) is the min. no. k s.t. $\exists$ a proper k-colouring of G.

Greedy algorithm.

A greedy colouring of G is ordering vertices in $V(G)$ as $v_1, \dots, v_n$ & given colours to the first i vertices, give the smallest available colour among $v_1, \dots, v_i, v_{i+1}$ to v_{i+1}.

i.e. give the smallest colour that is not given to $v_1, \dots, v_i$ to v_{i+1}

Theorem:- $\chi(G) \leq 1 + \Delta(G)$ $\Delta(G) = \max\limits_{v \in V} (\deg v)$

pf. Tweak the greedy algorithm by looking at only the neighbours of v_{i+1} in $v_1, \dots, v_i$.

v_{i+1} has at most $\Delta(G)$ neighbours in $v_1, \dots, v_i$

$\Rightarrow$ there always exist one colour free to give to v_{i+1} if we start with $1 + \Delta(G)$ colours.

Remark. If G = odd cycle or G = complete graph, then equality holds in above theorem.

Brooks Theorem (1941) If G is connected & if G is not an odd cycle or a complete graph, then $\chi(G) \le \Delta(G)$.

proof (Lovasz, 1975). (Theorem 5.1.22, Douglas West)

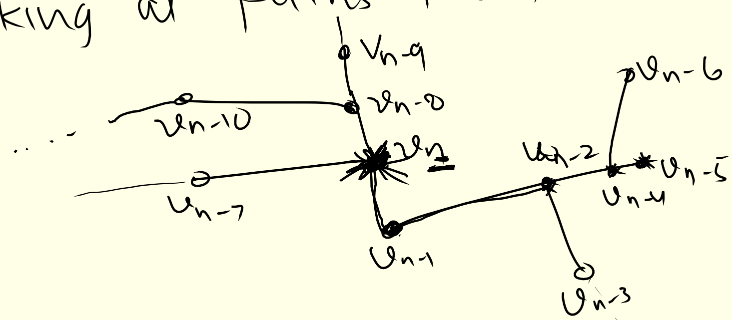
WLOG $k \ge 3$, where $k = \Delta(G)$.

Idea is to order vertices such that for any i, v_i has at most $k-1$ vertices among $v_1, \dots, v_{i-1}$ adjacent to it. Then use greedy alg. (better version)

Case 1 :- $\exists$ a vertex of deg. $< \Delta(G) = k$.

let $n = |V(G)|$ & let that vertex be v_n.

G is connected; $\exists$ spanning tree T. "Order vertices by looking at paths from v_n in T"



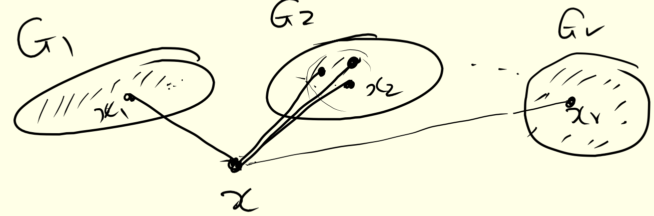
In this ordering every v_i is adj. to atleast one v_j with $j > i$ except v_n but $\deg(v_n) \le k-1$.

$\Rightarrow$ In this ordering $v_1, \dots, v_n$, $\forall i$ $\exists$ at most $k-1$ neighbours of v_i exist in $v_1, \dots, v_{i-1}$.

$\Rightarrow \exists$ k-colouring of G.

Case 2 G is k-regular.

Case 2a. G is not 2-connected. ie G has a cut-vertex say x.

$G =$ 

$r \le k$.

Each $G_1, G_2, \dots, G_r$ has max. deg $= k$. & $\exists$ vertices $x_i \in V(G_i)$ s.t. $\deg_{G_i}(x_i) = k-1$.

make this precise!

$\Rightarrow$ By the first case each G_i has a k-colouring. permute the colours of $G_1, \dots, G_r$ s.t. x can be given one of the k-colours to obtain a proper k-colouring of G.

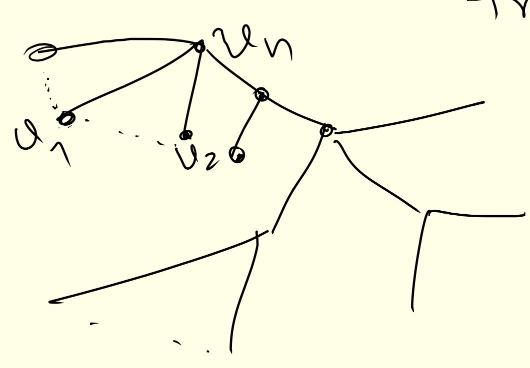
Remark. A proper colouring of G partitions $V(G)$ in to k-sets s.t. $\nexists$ any edge within any of the parts.

Case 2b G is 2-connected.

Best scenario! $\exists v_n \in V(G)$ s.t. $v_1, v_2 \in V(G)$ with $v_n v_1, v_n v_2 \in E(G)$; $v_1 v_2 \notin E(G)$ AND $G - \{v_1, v_2\}$ is connected.

$* \rightarrow$

If $*$ is true, then let T be a spanning tree of $G - \{v_1, v_2\}$.



order the vertices of T starting from v_n to v_3.

v_1, v_2 gets the same colour 1.

& $\forall 3 \le i < n$, v_i have atmost $k-1$ adj. vertices in $v_1, \dots, v_{i-1}$

$\therefore\ \exists$ proper k-colouring of $v_1, \dots, v_{n-1}$

but v_n has two neighbours of same colour (v_1, v_2)

$\therefore\ \exists$ at most $k-1$ colours in $N(v_n)$.

$\therefore\ G$ has k-colouring!

<u>Exercise</u> This $*$ is true for all 2-conn. k-regular graphs which are not complete graphs.