Quotient Topology :-

Quotient map :

$p : X \twoheadrightarrow Y$ surjective, $X, Y =$ Top. spaces

The map p is said to be quotient map if

$U \subseteq Y$ is open in $Y \iff p^{-1}(U)$ is open in X (or)

$U \subseteq Y$ is closed in $Y \iff p^{-1}(U)$ is closed in X.

- Quotient map $\Rightarrow$ Continuous ?
- Quotient map $\Rightarrow$ open map ?
- Quotient map $\Rightarrow$ Closed map ?

Saturated (w.r.to p) :

Let $p : X \to Y$ be a surjective map. ie.) $p : X \twoheadrightarrow Y$.

$A \subseteq X$ is called saturated if A is complete inverse image of some subset of Y.

ie) $A = p^{-1}(B)$ for some $B \subseteq Y$.

- $p : X \to Y$ is quotient map $\iff$
  1. (i) p is surjective
  2. (ii) p is continuous
  3. (iii) p maps open saturated sets to open sets

     (or)

     p maps closed saturated sets to closed sets.

Quotient topology:-

$$P : \underset{\text{Top. space}}{X} \twoheadrightarrow \underset{\text{set}}{A}$$

There exists exactly one topology τ on A for which p is a quotient map. It is called quotient top. induced by p.

Note: Define $U \subseteq A$ is open if $p^{-1}(U)$ is open in X.

Quotient space:-

$X^* =$ disjoint partition of X, $X =$ Top. space.

$$p : X \to X^*$$
$$x \mapsto [x] = \text{partition containing } x.$$

In the quotient topology induced by P, the space X^* is called a quotient space of X.

Theorem:

$$x \xmapsto{p} [x]$$

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, two heads, "g = \text{cts}"] & \\ \{g^{-1}(z) : z \in Z\} = X^* \arrow[r, dashed, hook, two heads, "f = \text{bijective}"'] & Z \end{tikzcd}$$

$$f([x]) := g(x)$$

$$f(g^{-1}(z)) = (z)$$

- f is homeomorphism $\Leftrightarrow$ g is quotient map
- Z is Hausdorff $\Rightarrow$ X^* is Hausdorff.

—— x ——

✱ Prove that continuous function with right inverse (cts) is quotient map.

Proof:

$P : X \xrightarrow{cts} Y$ & $f : Y \xrightarrow{cts} X$ s.t $P \circ f = Id$.

- P has right inverse $\Rightarrow$ P is onto (why?)
- Saturated open sets should maps to opens sets.

Let $A = \bar{p}^{1}(B)$, $B \subseteq Y$.

Suppose A is open. claim: $P(A) = B$ is open.

$$f^{-1}(A) = f^{-1}(\bar{p}^{1}(B)) = (P \cdot f)^{-1}(B) = B \text{ is open}$$

as f is cts.

Retraction:

$A \subseteq X$, $r : X \xrightarrow{cts} A$ is called retraction if $r(a) = a \ \forall \ a \in A$.

✱ Prove that every retraction is quotient map.

Proof:

$r : X \to A$, $f : A \to X$
$a \mapsto a$.

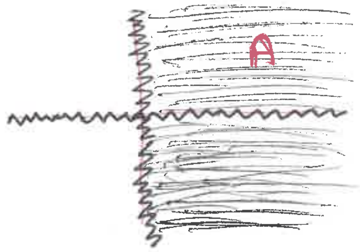
$(r \circ f)(a) = r(f(a)) = r(a) = a \quad \forall a \in A$.

$\therefore$ r has continuous right inverse.

Hence every retraction is quotient map.

Prove that

$q : A \to \mathbb{R}$ is quotient map.
$(x,y) \mapsto x$



$y=0$ (or) $x \geq 0$.

- q is onto, continuous. (clear)

- $f : \mathbb{R} \to A$ cts map.
  $x \mapsto (x,0)$

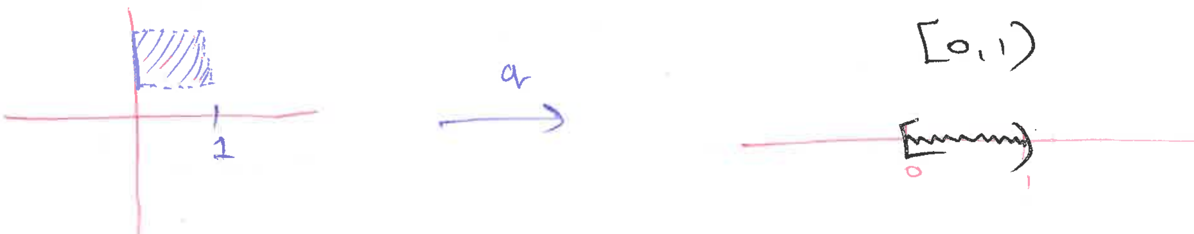
$(q \circ f)(x) = q(x,0) = x \quad \forall x \in \mathbb{R}.$

ie) q has cts right inverse $\therefore$ q is quotient.

- Is q closed map?

closed set in A — not closed in $\mathbb{R}$.

- Is q open map?



open in A — not open in $\mathbb{R}$.

$\therefore$ quotient map does not imply open map (or) closed map.

$X = \mathbb{R}^2$

⊛ Define an equivalence relation on $\mathbb{R}^2$ by

$(x_0, y_0) \sim (x_1, y_1)$ if $x_0 + y_0^2 = x_1 + y_1^2$.

$X^* =$ corresponding quotient space.

Prove that X^* is homeomorphic to $\mathbb{R}$.

Proof.

$$g : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x + y^2$$

- g is cts, g is onto (why?)
- $h : \mathbb{R} \to \mathbb{R}^2$
  $x \mapsto (x, 0)$

  $g(h(x)) = g(x, 0) = x \quad \forall x \in \mathbb{R}$.
- g has cts right inverse, $\therefore$ g is quotient map.

$$\begin{tikzcd} X = \mathbb{R}^2 \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, dashed, "f"'] & \mathbb{R} \end{tikzcd}$$

$X^* = \{ g^{-1}(x) : x \in \mathbb{R} \}$

Define $f : X^* \to \mathbb{R}$ by

$f([g^{-1}(x)]) = x$.

f is clearly a homeomorphism.

$= \times =$

$p : X \to Y$ $\quad A \subseteq X$.

$p|_A = q : A \to p(A)$.

Qn:

① $q^{-1}(V) = p^{-1}(V)$, $V \subseteq p(A)$?

② $p(U \cap A) = p(U) \cap p(A)$? $U \subseteq X$

- If A is saturated w.r.to p then

  i) $q^{-1}(V) = p^{-1}(V)$ & (ii) $p(U \cap A) = p(U) \cap p(A)$.

Proof:

- Note that $p^{-1}(V) \cap A = q^{-1}(V)$.

- $V \subseteq p(A)$ & A is saturated $\Rightarrow$ $p^{-1}(V) \subseteq A$.

  $\therefore$ $q^{-1}(V) = p^{-1}(V)$ $\forall$ $V \subseteq p(A)$.

- $p(U \cap A) \subseteq p(U)$ & $p(U \cap A) \subseteq p(A)$

  $\therefore$ $p(U \cap A) \subseteq p(U) \cap p(A)$.

Conversely, suppose $y \in p(U) \cap p(A)$.

$y = p(u) = p(a)$, $u \in U$, $a \in A$.

$u \in p^{-1}(p(a)) \subseteq A$ $\because$ A is saturated.

$\therefore$ $u \in A \cap U$ & $y = p(u) \in p(U \cap A)$.

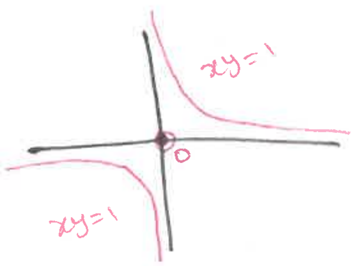
Hence $p(U) \cap p(A) = p(U \cap A)$.

$\times$

Restriction of quotient map:-

$\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ has cts right inverse
$(x,y) \mapsto x$ $\qquad$ $x \mapsto (x,0)$.

$\therefore$ π_1 is quotient map.



$A = \{(0,0)\} \cup \{(x,y) : xy = 1\}$

$\{(0,0)\}$ is open in A but

$\pi_1(0,0) = 0$ is not open in $\mathbb{R}$.

$\therefore$ Restriction of quotient map need not be quotient.

quotient map.

Theorem:

$p : X \to Y$ $A \subseteq X$ is saturated.

$q = p|_A : A \to p(A)$

we have the following:

1. If A is open (or closed) in X, then q is quotient map.

2. If p is open (or closed) map, then q is quotient map.

Proof-

$q : A \to p(A)$ is clearly cts & onto.

Let $V \subseteq p(A)$ such that $q^{-1}(V)$ is open in A.

claim: V is open in $p(A)$.

case-I A is open in X.

$\Downarrow$

$q^{-1}(V)$ is open in X $\Rightarrow$ $p^{-1}(V)$ is open in X

$\Rightarrow$ V is open in Y.

$\Rightarrow$ V is open in $p(A)$.

case-II p is open map

$q^{-1}(V)$ is open in A $\Rightarrow$ $p^{-1}(V) = q^{-1}(V) = U \cap A$, U = open in X

$\therefore$ $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$

$p(U)$ open in X.

$\therefore$ V is open in $p(A)$.

other case, similar proof will work.