

Countability and Separation Axioms:

①

First countable (FC):

A space that has a countable basis at each of its points is called first countable space.

Eg: metrizable spaces are F.C.

Second countable (SC):

If a space X has countable basis for its topology.

Lindelof space:-

A space for which every open cover contains a countable sub cover is called Lindelof.

separable:-

A space having a countable dense subset is called separable.

Eg: compact metrizable space is second countable.

• $S.C \Rightarrow \text{Lindelof}$

• $S.C \Rightarrow \text{separable}$

In a metrizable space, $S.C \Leftrightarrow \text{Lindelof} \Leftrightarrow \text{separable}$.

For a dense set A , $\bigcup_{x \in A} \left\{ \bigcup_{n=1}^{\infty} B(x, \frac{1}{n}) \right\}$ is a basis for its topology.

Eg: The space \mathbb{R}_l satisfies all the countability axioms but the second countable.

cor:

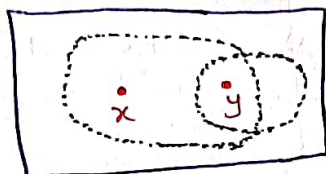
\mathbb{R}_ℓ is not metrizable.

(2)

Separation Axioms:

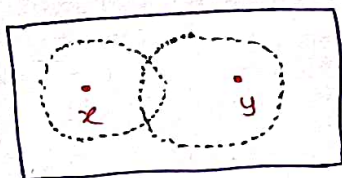
Trennungs axiom = separation axiom
(German word)

Kolmogorov (T_0)



indiscrete top is
not T_0 .

Frechet (T_1)



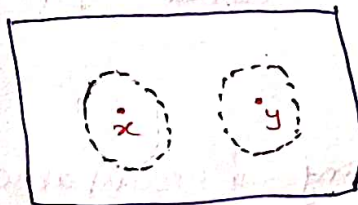
~~Se~~ Sierpinski space:

$$X = \{0, 1\}$$

$$\tau = \{\emptyset, X, \{0\}\}$$

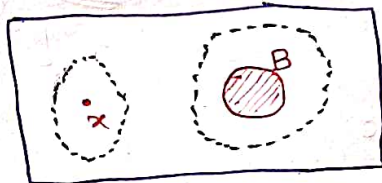
It is T_0 but not T_1 .

Hausdorff (T_2)



\mathbb{R} with co-finite
top is T_1 but
not T_2 .

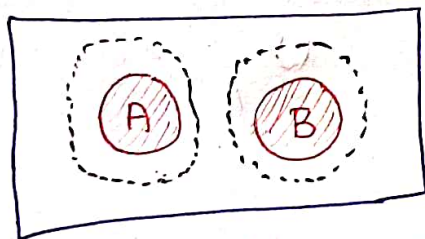
Regular (T_3)



\mathbb{R}_K is T_2 but
not T_3 .

Hint: Take $\{0\}$ & K .

Normal (T_4)

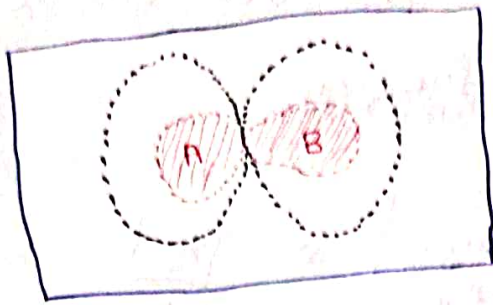


\mathbb{R}_ℓ^2 is T_3 but
not T_4 .

○ subspace of T_4 space need not be normal.

○ product of T_4 spaces need not be normal.

Completely normal (T_5)



$$\bar{A} \cap B = \phi = A \cap \bar{B}$$

Eg $A = (0, 1/2)$
 $B = (1/2, 1)$

Eg: $[0, 1]^J$ is normal but not completely normal.

$$\mathbb{R}^J \approx (0, 1)^J \subseteq [0, 1]^J = \text{cmpl} + T_2.$$

Urysohn ($T_{2 1/2}$)

A space in which any two distinct points can be separated by closed neighborhoods.

Completely T_2

A space in which any two distinct points can be separated by continuous function.

(e) $f: X \rightarrow [0, 1]$ $f(x) = 0$
 $f(y) = 1$

Tychonoff ($T_{3 1/2}$)

A closed set and any point outside it can be separated by continuous function.

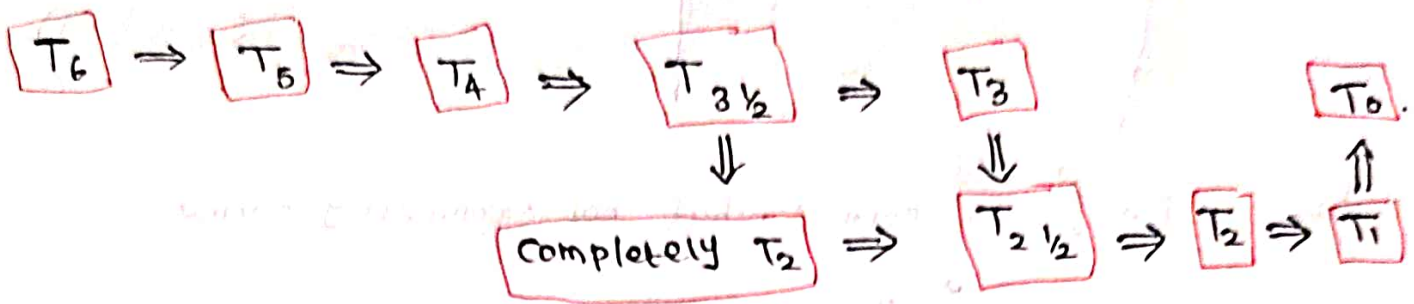
$f: X \rightarrow [0, 1]$ $f(y) = 0 \quad \forall y \in \bar{A} = A$
 $f(x) = 1.$

Perfectly normal (T_6)

$$f: X \rightarrow [0,1]$$

$$f^{-1}\{0\} = A, \quad f^{-1}\{1\} = B.$$

Two disjoint closed sets are precisely separated by a continuous function.



Ex: Find counterexamples showing that none of these implications reverse.

Exercise:

Property:	subspace	finite product	ctble product	uncountable product
T_2	✓	✓	✓	✓
T_3	✓	✓	✓	✓
T_4	✗ (closed subspace)	✗	✗ (unsolved)	✗ \mathbb{R}^J
T_5				
T_6				
F.C				
S.C				
lin.				
separable				

Properties:

- $T_3 + \text{second countable} \Rightarrow T_4$ (Thm 32.1)
- $T_2 + \text{compact} \Rightarrow T_4$ (Thm 32.3)
- $T_2 + \text{loc. compact} \Rightarrow T_{3\frac{1}{2}}$
- $T_3 + \text{lindehof} \Rightarrow T_4$
- closed subspace of normal space is normal.

Urysohn Lemma:

Let X be a normal space. A, B are disjoint closed sets of X . Then there exist a continuous map

$$f: X \rightarrow [0, 1] \text{ s.t.}$$

$$f(x) = 0 \quad \forall x \in A$$

$$f(x) = 1 \quad \forall x \in B.$$

Eg: Let (X, d) be a ^{bounded} metric space.

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

$$\bar{A} = A, \quad \bar{B} = B$$

$$A \cap B = \emptyset$$

$$\circ f: X \rightarrow [0, 1]$$

$$\circ f \text{ is cts}$$

$$\circ f(x) = 0 \Leftrightarrow x \in A, \quad f(x) = 1 \Leftrightarrow x \in B.$$

Urysohn metrization thm:

Every regular space with a countable basis is metrizable.

Rmk: Every metrizable space is T_6 .

Note: $\bar{d}(x, y) = \min \{1, d(x, y)\}$ or $\bar{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$

Tietze extension theorem

Let X be a normal space and A be a closed subspace of X . Then

① $f: A \xrightarrow{\text{cts}} [a, b]$ has extension $\tilde{f}: X \xrightarrow{\text{cts}} [a, b]$

② $g: A \xrightarrow{\text{cts}} \mathbb{R}$ has extension $\tilde{g}: X \xrightarrow{\text{cts}} \mathbb{R}$.

Hahn-Banach Extension!

Let Y be a subspace of a normed linear space X and $f: Y \rightarrow \mathbb{C}$ be a continuous linear map. Then there exist a continuous linear map

$\tilde{f}: X \rightarrow \mathbb{C}$ such that

$$\textcircled{1} \quad f(x) = \tilde{f}(x) \quad \forall x \in Y.$$

$$\textcircled{2} \quad \sup_{\substack{x \in Y \\ \|x\| \leq 1}} |f(x)| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\tilde{f}(x)|.$$

$$\underline{\underline{\quad \quad \quad}} \times \underline{\underline{\quad \quad \quad}}$$