How good is consistency as a measure of optimality of an estimator? We need a measure of accuracy. For large samples, we need a rate of convergence to the true parameter or parametric function. If we assume that our estimator is asymptotically unbiased (i.e. $E(\hat{\theta}-\theta) \underset{n\to\infty}{\longrightarrow} 0$) then we can use the asymptotic s.d. for this purpose. If we have an i.i.d sequence $X_1, X_2, \ldots$ with $E(X) = \mu$ and $Var(X) = \sigma^2$ then $\sqrt{n}(\bar{X}_n - \mu)$ is asymptotically normal. What about $g(\bar{X}_n)$ in such a situation for a smooth function g? We need the following result in this context.

Result. Suppose $\{a_n\} \uparrow \infty$ as $n \to \infty$, b fixed and

$$a_n(X_n - b) \underset{n\to\infty}{\xrightarrow{d}} X.$$

Let g be a continuous function which is differentiable, and let g' be continuous and $g'(b) \neq 0$. Then

$$a_n\left(g(X_n) - g(b)\right) \underset{n\to\infty}{\xrightarrow{d}} g'(b)X.$$

Proof. Note that

$$X_n - b = \frac{1}{a_n}\left[a_n(X_n - b)\right] \underset{n\to\infty}{\xrightarrow{d}} 0 \times X = 0.$$

Therefore $X_n \underset{n\to\infty}{\xrightarrow{P}} b$. Now, $a_n\left(g(X_n) - g(b)\right) = a_n\left(g'(X_n^*)(X_n - b)\right)$ where X_n^* lies between X_n and b. Therefore

$$|X_n^* - b| \leq |X_n - b| \underset{n\to\infty}{\xrightarrow{P}} 0.$$

Therefore, $X_n^* \underset{n\to\infty}{\xrightarrow{P}} b$ and hence $g'(X_n^*) \underset{n\to\infty}{\xrightarrow{P}} g'(b)$. It follows then that

$$g'(X_n^*)a_n\left(X_n - b\right) \underset{n\to\infty}{\xrightarrow{d}} g'(b)X.$$

Note, however, that if $g'(b) = 0$, then $a_n\left(g(X_n) - g(b)\right) \underset{n\to\infty}{\xrightarrow{P}} 0$.

Result. Suppose we have an i.i.d sequence $X_1, X_2, \ldots$ with $E(X) = \mu$ and $Var(X) = \sigma^2 < \infty$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Let h be differentiable, h' be continuous and $h'(\mu) \neq 0$. Then

$$\sqrt{n}\left(h(\bar{X}_n) - h(\mu)\right) \underset{n\to\infty}{\xrightarrow{d}} N\left(0, (h'(\mu))^2\sigma^2\right).$$

Proof. From CLT, $\sqrt{n}\left(\bar{X}_n-\mu\right) \xrightarrow[n\to\infty]{d} N\left(0,\sigma^2\right)$. Therefore, from the previous result,

$$\sqrt{n}\left(h(\bar{X}_n)-h(\mu)\right) \xrightarrow[n\to\infty]{d} h'(\mu)N\left(0,\sigma^2\right) = N\left(0,(h'(\mu))^2\sigma^2\right).$$

Example. $X_1,\ldots,X_n$ i.i.d Bernoulli(p). Let $S_n=\sum_{i=1}^n X_i$. Then $\hat{p}_n = S_n/n = \bar{X}_n$ satisfies $\sqrt{n}(\hat{p}_n-p)=\sqrt{n}(\bar{X}_n-p) \xrightarrow[n\to\infty]{d} N(0,p(1-p))$. Now consider estimating $q(p)=p(1-p)$ with $T_n=\hat{p}_n(1-\hat{p}_n)$. We have seen earlier that it is consistent. What can be said about its asymptotic distribution? Consider $h(x)=x(1-x)$ which is differentiable with $h'(x)=1-2x$. $h'(p)\neq 0$ if $p\neq 1/2$. Therefore, for $p\neq 1/2$,

$$\begin{aligned}\sqrt{n}\left(T_n-p(1-p)\right) = \sqrt{n}\left(h(\hat{p}_n)-h(p)\right) &\xrightarrow[n\to\infty]{d} N\left(0,(h'(p))^2p(1-p)\right)\\ &= N(0,p(1-p)(1-2p)^2).\end{aligned}$$

What happens when $p=1/2$? Recall how we proved the result:

$$a_n\left(g(X_n)-g(b)\right) \xrightarrow[n\to\infty]{d} g'(b)X$$

if $a_n(X_n-b) \xrightarrow[n\to\infty]{d} X$, g is differentiable, g' continuous and $g'(b)\neq 0$? We used Taylor series expansion: $g(X_n)=g(b)+g'(X_n^*)(X_n-b)$. If $g'(b)=0$, we need a further term:

$$\begin{aligned}g(X_n) &= g(b)+g'(b)(X_n-b)+\frac{1}{2}g''(X_n^*)(X_n-b)^2\\ &= g(b)+\frac{1}{2}g''(X_n^*)(X_n-b)^2,\end{aligned}$$

so that $g(X_n)-g(b)=\frac{1}{2}g''(X_n^*)(X_n-b)^2$. Assume that g'' is continuous at b and $g''(b)\neq 0$. Then $g''(X_n^*) \xrightarrow[n\to\infty]{P} g''(b)$ and $(a_n(X_n-b))^2 \xrightarrow[n\to\infty]{d} X^2$. Therefore,

$$\begin{aligned}a_n^2\left(g(X_n)-g(b)\right) &= \frac{1}{2}g''(X_n^*)\left\{a_n(X_n-b)\right\}^2\\ &\xrightarrow[n\to\infty]{d} \frac{1}{2}g''(b)X^2.\end{aligned}$$

Now consider the asymptotic distribution of $T_n=\hat{p}_n(1-\hat{p}_n)$ in the Bernoulli(p) example. Here $h(x)=x(1-x)$, $h'(x)=1-2x$ and $h''(x)=-2$. Also $\sqrt{n}(\hat{p}_n-\frac{1}{2}) \xrightarrow[n\to\infty]{d} N(0,\frac{1}{4})$. Therefore

$$n(\hat{p}_n(1-\hat{p}_n)-\frac{1}{4}) \xrightarrow[n\to\infty]{d} \frac{1}{2}(-2)\frac{1}{4}\chi_1^2 = -\frac{1}{4}\chi_1^2.$$

Example. Let $X_1, X_2, \ldots$ be an i.i.d sequence such that $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Then

$$\left(\sqrt{n}(\bar{X} - \mu), s^2\right) \xrightarrow[n\to\infty]{d} \left(N(0, \sigma^2), \sigma^2\right).$$

Therefore, by Slutsky,

$$\frac{\sqrt{n}(\bar{X} - \mu)}{s} \xrightarrow[n\to\infty]{d} N(0, 1).$$

Example. Let $X_1, X_2, \ldots$ be i.i.d $N(\mu, \sigma^2)$. Then

$$T_n = \frac{\sqrt{n}(\bar{X} - \mu)}{s} \sim t_{n-1}.$$

Thus note that as $n \to \infty$, $t_{n-1} \xrightarrow{d} N(0, 1)$ from the previous result. This can also be seen directly since the numerator is always $N(0, \sigma^2)$ whereas s in the denominator converges to σ in probability.

Asymptotic Normality.

Large sample normal approximation for estimators is desirable:
(i) to obtain estimation error;
(ii) to be able to compare efficiencies; and
(iii) for large sample tests.

Usually reasonable estimators $\hat{\theta}_n$ converge at the rate of $O_P(\frac{1}{\sqrt{n}})$. i.e., $\hat{\theta}_n \xrightarrow[n\to\infty]{P} \theta$ and $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{P}$ to some distribution.

Definition. Suppose $T_n(X_1, \ldots, X_n)$ is an estimator of $q(\theta)$. Then T_n is said to be asymptotically normal if

$$\sqrt{n}\left(T_n(X_1, \ldots, X_n) - q(\theta)\right) \xrightarrow[n\to\infty]{d} N(0, \sigma^2(\theta)).$$

Example. Let $X_1, X_2, \ldots$ be i.i.d such that $E(X) = \mu$ and $Var(X) = \sigma^2$. Consider $T_n(X_1, \ldots, X_n) = \bar{X}$. Then by CLT $\sqrt{n}(T_n - \mu) \xrightarrow[n\to\infty]{d} N(0, \sigma^2)$. We therefore say that $\bar{X}$ is asymptotically normal in this case.

Example. Let $X_1, X_2, \ldots$ be i.i.d Exp(θ). Then $E(X) = \frac{1}{\theta}$ and $Var(X) = \frac{1}{\theta^2}$. Consider $\hat{\theta} = \frac{1}{\bar{X}}$. Since, by WLLN, $\bar{X} \xrightarrow[n\to\infty]{P} \frac{1}{\theta}$, $\hat{\theta} = \frac{1}{\bar{X}} \xrightarrow[n\to\infty]{P} \theta$, hence it is consistent. Note that we have made use of the continuity of $h(x) = 1/x$ for $x > 0$. $h'(x) = -1/x^2$ is also continuous for $x > 0$. We have, by CLT,

$$\sqrt{n}\left(\bar{X} - \frac{1}{\theta}\right) \xrightarrow[n\to\infty]{d} N(0, \frac{1}{\theta^2}).$$

Therefore,

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow[n\to\infty]{d} h'(\frac{1}{\theta})N(0, \frac{1}{\theta^2}) = N\left(0, (\theta^2)^2\frac{1}{\theta^2} = \theta^2\right).$$