Topological Groups:

Topology + Group = Topological group.

Defn:

A topological group G is a group with an underlying topology such that

- the product map $m : G \times G \to G$ is continuous
- the inverse map $i : G \to G$ is continuous.

Eg:

Lie groups, Banach spaces, Top. vector spaces.

Thm: on a topological group G, the following maps are homeomorphisms:

- Left multiplication: For $g \in G$, $l_g : G \to G$
  $x \mapsto gx$.
- Right multiplication: For $g \in G$, $r_g : G \to G$
  $x \mapsto x.g$
- Inverse: $i : G \to G$
  $x \mapsto x^{-1}$
- conjugation: For $g \in G$, $c_g : G \to G$
  $x \mapsto gxg^{-1}$.

⊛ $A, B \subseteq G$. If A is open in G then AB, BA are also open in G. $AB = \{xy : x \in A, y \in B\}$

⊛ C = Closed, K = Compact subspace of G. Then CK and KC are closed in G.

* The following are equivalent:

1) G satisfies T_3
2) G satisfies T_2
3) G satisfies T_1
4) $\{e\}$ is closed.

T_0 = Kolmogorov

T_1 = Frechet

T_2 = Hausdorff

$T_{2\frac{1}{2}}$ = Urysohn

T_3 = Regular

$T_{3\frac{1}{2}}$ = Tychonoff

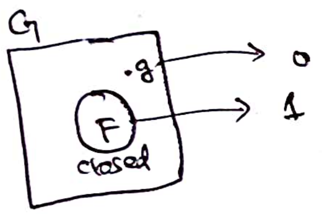
T_4 = normal

T_5 = completely normal

T_6 = perfectly normal.

Indeed.

G is $T_0 \iff G$ is $T_{3\frac{1}{2}}$.



* If H is a subgp $(H \leq G)$ then $\overline{H} \leq G$.
* If H is normal subgp $(H \trianglelefteq G)$ then $\overline{H} \trianglelefteq G$.

normal Top. space, normal subgp are different notions.

* The (path) connected component containing e is a closed normal subgp of G.

* open subgp of G is also closed.

closed subgp of finite index is open.

* $f : G \longrightarrow H$ cts map. G, H = Top. gps.

1) K = closed subgp $\Rightarrow f^{-1}(K)$ = closed subgp.

2) U = open subgp $\Rightarrow f^{-1}(U)$ = open subgp.

3) $K \trianglelefteq H \Rightarrow f^{-1}(K) \trianglelefteq G$.

Cor: $\ker f = f^{-1}\{e\}$ is closed normal subg of G (when H is T_1 space) cts +

- $f : G \longrightarrow H$
  gp homomorphism

1) G is connected $\Rightarrow$ $f(G)$ = connected.
2) G is compact $\Rightarrow$ $f(G)$ = compact.

Eg $f : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{R}$ is cts & homomorphism
$(a, b) \longmapsto a + b\sqrt{2}$

but $\mathrm{Im}(f)$ is not closed.

- A topological gp G is metrizable $\Leftrightarrow$ G is T_0 & first countable.

- $N_g = g \cdot N_e$, N_g = set of open sets containing g.

- A group homomorphism $\phi : G \to H$ is cts on G $\Leftrightarrow$ ϕ is continuous at one point of G.

- $\frac{G}{N}$ is Hausdorff $\Leftrightarrow$ N is a closed subgroup of G.

- Isomorphism theorems fails: (first & second).

$id : \mathbb{R}_d \to \mathbb{R}$ cts, gp homomorphism.
$x \mapsto x$

but $\frac{\mathbb{R}_d}{\ker(id)} = \mathbb{R}_d \not\cong \mathbb{R}$ as top. gp.

- $\frac{H+N}{N} \not\cong \frac{H}{H\cap N}$ (discrete)
(dense set)

$G = \mathbb{R}$, $H = \alpha\mathbb{Z}$, $\alpha \notin \mathbb{Q}$, $N = \mathbb{Z}$

- strangely enough, the Third Isomorphism Theorem holds.

$=\!=\!= \times =\!=\!=$

Topological Groups:

(G,τ) be T_1-space & $(G,*)$ is a group. G is called topological group if

(i) $*: G\times G \to G$
$(x,y) \mapsto x*y$ and

(ii) $\text{In}: G \to G$
$x \mapsto x^{-1}$ are continuous maps.

Hints for exercise from munkers:

① ◦ $(x,y) \mapsto xy^{-1}$ cts $\Rightarrow$ $G \to G\times G \to G$ is cts
$y \mapsto (e,y) \mapsto y^{-1}$

◦ $(x,y) \mapsto (x,y^{-1}) \mapsto x\cdot y$ is cts.

Note: $(x,y) \mapsto xy^{-1}$ is cts $\Leftrightarrow$ $(x,y) \mapsto x\cdot y$ & $x \mapsto x^{-1}$ are cts.

② $(\mathbb{Z},+)$, $(\mathbb{R},+)$, $(\mathbb{R}_+,\cdot)$, $(S^1,\cdot)$, $(GL_n(\mathbb{C}),\cdot)$ are top. gps.

$\mathcal{H}(\mathbb{D}) = \{ f:\mathbb{D} \to \mathbb{C} \;:\; f \text{ is analytic}\}$

~~Take~~ Take compact open topology

$f_n \to f \Leftrightarrow f_n \xrightarrow{u} f$ on every compact subsets of $\mathbb{D}$.

③ If $H \le G$, then H & $\bar{H}$ are top. gps.

$f(x,y) = xy^{-1}$, $x,y \in G$.

◦ $f(\bar{H}\times\bar{H}) = f(\overline{H\times H}) \subseteq \overline{f(H\times H)} \subseteq \bar{H}$ $\therefore \bar{H} \le G$

Fix $\alpha \in G$.

④ $f_\alpha : G \to G$ $\qquad\qquad$ $g_\alpha : G \to G$

$x \mapsto \alpha . x$ $\qquad\qquad$ $x \mapsto x . \alpha$.

- $\alpha x = \alpha y \iff \alpha^{-1} \alpha x = \alpha^{-1} \alpha y \iff x = y$ ie) f_α is 1-1
- $\alpha^{-1} y \mapsto \alpha . \alpha^{-1} y = y$ $\quad \forall y \in G$ ie) f_α is onto.
- $(y, x) \mapsto (\alpha, x) \mapsto \alpha x$ is cts.

$$\boxed{(f_\alpha)^{-1} = f_{\alpha^{-1}}} \qquad f_{\alpha^{-1}} \cdot f_\alpha (x) = \alpha^{-1} . \alpha . x = x$$

- f_α, g_α are homeomorphisms of G.
- Fix $\alpha, \beta \in G$.

$$f_{\beta\alpha^{-1}} (\alpha) = (\beta . \alpha^{-1}) \alpha = \beta$$

$\therefore$ $f_{\beta\alpha^{-1}}$ maps α to β.

<u>Rmk</u> $\quad \mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta} . \dfrac{a - z}{1 - \bar{a} z} : a \in \mathbb{D}, \theta \in \mathbb{R} \right\}$

$B_a = \dfrac{a - z}{1 - \bar{a} z}$ interchanges α & 0. Also $(B_a)^{-1} = B_a$.

$$a \overset{B_a}{\longmapsto} 0 \overset{B_b}{\longmapsto} b.$$

$\therefore$ $\forall a, b \in \mathbb{D}$, $\exists f \in \mathrm{Aut}(\mathbb{D})$ s.t $f(a) = b$.

5) $G/H = \{ [x] : x \in G \}$, $\quad [x] = \{ x \cdot h : h \in H \} = x \cdot H$.

$f_\alpha : G/H \rightarrow G/H$

$[x] \longmapsto [\alpha x]$ $\quad$ ie) $xH \longmapsto \alpha x H$, $\forall x \in G$.

- $p : G \rightarrow G/H$ is quotient map. i.e) Topology on G/H taken as " $p^{-1}(U)$ is open $\Leftrightarrow$ U is open."

- $xH = yH \Leftrightarrow y = x \cdot z$ for some $z \in H$.

- $[\alpha x] = [\alpha y] \Leftrightarrow \alpha y = \alpha x \cdot z \Leftrightarrow y = x \cdot z$ for some $z \in H$

$\Leftrightarrow [x] = [y]$.

- H is closed $\Rightarrow f_x(H) = xH$ is closed in G.

$\Rightarrow [x]$ is closed in G/H. $\quad \forall x$

$\Rightarrow G/H$ is T_1.

- $p : G \rightarrow G/H$

$x \longmapsto [x] = xH$

Let U be open in G; $\quad p(U) = \bigcup_{x \in U} xH$

$= \{ x \cdot h ; x \in U, h \in H \}$

$= \bigcup_{h \in H} (U \cdot h) =$ open.

$\therefore$ p is open map.

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$$\begin{tikzcd} G \arrow[d, "p"'] \arrow[dr, "p_\alpha = p \circ f_\alpha"] & \\ G/H \arrow[r, dashed, "h_\alpha"'] & G/H \end{tikzcd}$$

$p : x \longmapsto [x]$

$p_\alpha : x \longmapsto [\alpha x]$

$\therefore$ $h_\alpha : \frac{G}{H} \rightarrow \frac{G}{H}$, $[x] \longmapsto [\alpha x]$ is homeomorphism.

- $\bar{H} = H$, $H \trianglelefteq G$ $\Rightarrow$ G/H is a Top. group.

Proof:

- H is closed $\Rightarrow$ G/H is T_1 space.

- $[x].[y] := [xy]$

  Is it well defined?

Suppose $[x] = [x']$, $[y] = [y']$

$x = x'z$, $y = y'w$, for some $z, w \in G$.

$xy = x'z . y'w = x'(y'z)w$ $\quad (\because zy' = y'z)$

$= x'y' . zw$

ie) $[xy] = [x'y']$

- $[e]$ is the identity element of G/H.

  $[x]^{-1} = [x^{-1}]$.

- G/H is a group.

- $h([x],[y]) := [x][y]^{-1}$ $\qquad h : G/H \times G/H \to G/H$

  $h(xH, yH) = f(x,y)H$, $\quad f(x,y) = xy^{-1}$.

$$\begin{tikzcd}
G \times G \arrow[d, "p \times p"'] \arrow[dr, "p \circ f"] & \\
G/H \times G/H \arrow[r, dashed, "h"'] & G/H
\end{tikzcd}$$

$(x,y) \mapsto [xy^{-1}]$

$p\circ f$ is cts $\rightarrow$ h is cts.

$\therefore$ G/H is a topological group.

(6) (5)

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"'] \arrow[dr, "f(x) = e^{i2\pi x}"] & \\ \mathbb{R}/\mathbb{Z} \arrow[r, dashed, "h"'] & S^1 \end{tikzcd}$$

$[x] \longmapsto e^{i2\pi x}$

- f is quotient map
- $f(x) = f(y) \iff x\mathbb{Z} = y\mathbb{Z}$
- h is homeomorphism.

$\mathbb{R}/\mathbb{Z}$ is homeomorphic to $S^1 = \{z \in \mathbb{C} : |z| = 1\}$