Posterior inference

Model: $X|\theta$ has density $f(x|\theta)$
Prior: θ has density $\pi(\theta)$

Posterior density:

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{m(x)}$$

This is the probability density for θ after observing the data, $X = x$

It contains all the information on θ after observing the data

All inferences on θ must be based on this

Optimal estimators

$L(\theta, a) = (\theta - a)^2$ is the (squared error) loss when θ is estimated with a number a.

What is the best estimator for θ under this loss?

$\min_a E\left[(\theta - a)^2 | x\right] = ?$

$$
\begin{aligned}
E\left[(\theta - a)^2 | x\right] &= E\left[\{(\theta - E(\theta|x)) + (E(\theta|x) - a)\}^2 | x\right] \\
&= E\left[(\theta - E(\theta|x))^2\right] + (E(\theta|x) - a)^2 \\
&\geq E\left[(\theta - E(\theta|x))^2\right] = Var(\theta|x),
\end{aligned}
$$

with equality iff $a = \delta(x) = E(\theta|x)$

What is the optimal estimator if

$$L(\theta, a) = |\theta - a|?$$

Credible sets

$\pi(\theta|x)$ is the probability density for θ (after observing data, $X = x$)

Any subset $C = C(x) \subset \Theta$ which has probability

$$P(\theta \in C|x) = \int_C \pi(\theta|x)\, d\theta = 1 - \alpha$$

is a $100(1-\alpha)\%$ credible set for θ

Frequentist Confidence sets:

$$P_\theta(\underline{T}(X) \leq \theta \leq \bar{T}(X)) = 1 - \alpha$$

for all θ

Example.

$X_1, X_2, \cdots, X_n$ i.i.d. $N(\theta, \sigma^2)$, σ^2 is known. $\theta \sim N(\mu, \tau^2)$.

$$\theta | \mathbf{X} = \mathbf{x} \sim N\left(\frac{\tau^2}{\tau^2 + \sigma^2/n}\bar{x} + \frac{\sigma^2/n}{\tau^2 + \sigma^2/n}\mu, \frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n} \right).$$

Bayes estimate of θ is the posterior mean:

$$E(\theta|\mathbf{x}) = \frac{\tau^2}{\tau^2 + \sigma^2/n}\bar{x} + \frac{\sigma^2/n}{\tau^2 + \sigma^2/n}\mu,$$

Posterior variance:

$$Var(\theta|\mathbf{x}) = \frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n}.$$

$100(1-\alpha)\%$ HPD credible interval for θ is:

$$E(\theta|\mathbf{x}) \pm z_{1-\alpha/2}\, \mathrm{s.d.}(\theta|\mathbf{x})$$

$$\frac{\tau^2}{\tau^2 + \sigma^2/n}\bar{x} + \frac{\sigma^2/n}{\tau^2 + \sigma^2/n}\mu \pm z_{1-\alpha/2}\sqrt{\frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n}}$$