TEST - I

Answer all questions. All questions carry equal (= 5) marks. Write your names and Roll Numbers on the answer sheet. Upload your answers latest by 12:15 pm.

(1) Let (X, d) be a metric space. Given non-empty bounded subsets A, B of X define

$$\overline{d}(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}.$$

Show that $\overline{d}$ satisfies the triangle inequality, that is, if A, B, C are any three non-empty bounded subsets of X, then

$$\overline{d}(A, C) \leq \overline{d}(A, B) + \overline{d}(B, C).$$

Is $\overline{d}$ a metric on the set of all non-empty bounded subsets of X?

(2) For a path connected space X, let $P(X)$ denote the set of path components of X. Show that if two spaces X and Y are homeomorphic, then there exists a bijection $f : P(X) \longrightarrow P(Y)$.

(3) Suppose $f : X \longrightarrow Y$ is a function between two topological spaces with the property that for each $x \in X$, there exists a neighborhood U_x of x such that $f(y) = f(x)$ for all $y \in U_x$. Show that
  (a) f is continuous, and
  (b) if X is connected, then f is a constant.

(4) Let $\mathbb{C} = \mathbb{R}^2$ denote the set of complex numbers. Let G be a non-empty subset of $\mathbb{C}$ with the property that whenever $z, \omega \in G$, then $z^{-1} \in G$ and $z\omega \in G$. If G is compact as a subspace of $\mathbb{C} = \mathbb{R}^2$, then show that $|z| = 1$ for all $z \in G$.