There is a simple interpretation for the posterior odds ratio, $P(\Theta_0|x)/P(\Theta_1|x)$ mentioned above. Consider the simple versus simple testing: $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. We can assume $\Theta = \{\theta_0, \theta_1\}$. Let $\pi_0 = P^\pi(\theta = \theta_0) = 1 - P^\pi(\theta = \theta_1)$. Then

$$\begin{aligned}
\pi(\theta|x) &= \frac{\pi(\theta)f(x|\theta)}{m(x)} = \frac{\pi(\theta)f(x|\theta)}{\pi_0 f(x|\theta_0) + (1-\pi_0)f(x|\theta_1)} \\
&= \left\{ \begin{array}{cc} \frac{\pi_0 f(x|\theta_0)}{m(x)} & \text{if } \theta = \theta_0; \\ \frac{(1-\pi_0)f(x|\theta_1)}{m(x)} & \text{if } \theta = \theta_1. \end{array} \right\}
\end{aligned}$$

Therefore, the posterior odds ratio of H_1 relative to H_0 is

$$\frac{(1-\pi_0)f(x|\theta_1)}{\pi_0 f(x|\theta_0)} = \frac{f(x|\theta_1)}{f(x|\theta_0)},$$

if $\pi_0 = 1/2$. This is nothing but the likelihood ratio used in the classical MP test. However the Bayesian use of it is simply to use it for expressing evidence against H_0 directly, without having to look for a reference distribution.

Testing a Point Null Hypothesis The problem is to test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$

Conceptually testing a point null is not a different problem, but there are complications. First of all, it is not possible to use a continuous prior density because any such prior will necessarily assign prior probability zero to the null hypothesis. Consequently, the posterior probability of the null hypothesis will also be zero. Intuitively, this is clear: if the null hypothesis is *a priori* impossible, it will remain so *a posteriori* also. Therefore, a prior probability of $\pi_0 > 0$ needs to be assigned to the point θ_0 and the remaining probability of $\pi_1 = 1 - \pi_0$ will be spread over $\{\theta \neq \theta_0\}$ using a density g_1. The complication now is that the prior π is of the form

$$\pi(\theta) = \pi_0 I\{\theta = \theta_0\} + (1-\pi_0)g_1(\theta)I\{\theta \neq \theta_0\}$$

and hence has both discrete and continuous parts. This complication appears whenever Θ_0 and Θ_1 have different dimensions when we test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$. Therefore we shall discuss this more general problem below.

Let π_0 and $1 - \pi_0$ be the prior probabilities of Θ_0 and Θ_1. Let $g_i(\theta)$ be the prior p.d.f. of θ on Θ_i (conditional on H_i being true), so that

$$\int_{\Theta_i} g_i(\theta)\, d\theta = 1.$$

Thus the prior $\pi(\theta)$ is specified by

$$\pi(\theta) = \pi_0 g_0(\theta) I\{\theta \in \Theta_0\} + (1-\pi_0) g_1(\theta) I\{\theta \in \Theta_1\}.$$

We do not require any longer that Θ_0 and Θ_1 be of the same dimension. We can now proceed as before and compute posterior probabilities or posterior odds. To obtain these posterior quantities, note that the marginal density of X under the prior π can be expressed as

$$\begin{aligned} m_\pi(x) &= \int_\Theta f(x|\theta)\pi(\theta)\, d\theta \\ &= \pi_0 \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta + (1-\pi_0) \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta, \end{aligned}$$

and hence the posterior density of θ given the data $X = x$ as

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{m_\pi(x)} = \begin{cases} \pi_0 f(x|\theta) g_0(\theta)/m_\pi(x) & \text{if } \theta \in \Theta_0; \\ (1-\pi_0) f(x|\theta) g_1(\theta)/m_\pi(x) & \text{if } \theta \in \Theta_1. \end{cases}$$

It follows then that

$$\begin{aligned} P^\pi(H_0|x) &= P^\pi(\Theta_0|x) = \frac{\pi_0}{m_\pi(x)} \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta \\ &= \frac{\pi_0 \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta}{\pi_0 \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta + (1-\pi_0) \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta} \quad \text{and} \\ P^\pi(H_1|x) &= P^\pi(\Theta_1|x) = \frac{(1-\pi_0)}{m_\pi(x)} \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta \\ &= \frac{(1-\pi_0) \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta}{\pi_0 \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta + (1-\pi_0) \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta}. \end{aligned}$$

Therefore, the posterior odds ratio of H_0 to H_1 is

$$\begin{aligned} \frac{P^\pi(\Theta_0|x)}{P^\pi(\Theta_1|x)} &= \frac{\pi_0 \int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta}{(1-\pi_0) \int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta} \\ &= \frac{\pi_0}{1-\pi_0} \frac{\int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta}{\int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta} \\ &= \frac{\pi_0}{1-\pi_0} \times \ \mathrm{BF}_{01}(x), \end{aligned}$$

where

$$\begin{aligned} \mathrm{BF}_{01}(x) &= \frac{P^\pi(\Theta_0|x)}{P^\pi(\Theta_1|x)} \Big/ \frac{P^\pi(\Theta_0)}{P^\pi(\Theta_1)} \\ &= \frac{\text{Posterior odds ratio}}{\text{Prior odds ratio}}. \end{aligned}$$

Thus, one may also report the *Bayes factor*, which does not depend on π_0. Note that the Bayes factor may be defined without reference to the prior odds ratio also:

$$\mathrm{BF}_{01} = \frac{\int_{\Theta_0} f(x|\theta) g_0(\theta)\, d\theta}{\int_{\Theta_1} f(x|\theta) g_1(\theta)\, d\theta} = \frac{m_0(x)}{m_1(x)},$$

where $m_i(x)$ is the marginal or predictive distribution of X under H_i. Clearly, $\mathrm{BF}_{10} = 1/\mathrm{BF}_{01}$. Also, the posterior odds ratio of H_0 relative to H_1 is

$$\left(\frac{\pi_0}{1-\pi_0}\right) \mathrm{BF}_{01},$$

which reduces to BF_{01} if $\pi_0 = \frac{1}{2}$. Thus, BF_{01} is an important evidential measure that is free of π_0. The smaller the value of BF_{01}, the stronger the evidence against H_0. As noted previously, the Bayes factor is the likelihood ratio in the simple verus simple case, a weighted likelihood ratio in the general case.

Example. In the blood sugar example, $\pi_0 = P^{\pi}(\theta \leq 130) = \Phi(\frac{130-100}{30}) = \Phi(1)$, so the prior odds ratio is $\pi_0/(1-\pi_0) = \Phi(1)/(1-\Phi(1)) = .8413/.1587 = 5.3$, and thus the Bayes factor turns out to be BF_{01} = posterior odds ratio/prior odds ratio = 1.66/5.3 = .313.

Consider an example on testing a point null hypothesis.

Example. Suppose $X \sim \text{Binomial}(n, \theta)$ and we want to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, a problem similar to checking whether a given coin is biased based on n independent tosses (where θ_0 will be taken to be 0.5). Under the alternative hypothesis, suppose θ is distributed as $\text{Beta}(\alpha, \beta)$. Then $m_1(x)$ is given by

$$m_1(x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)},$$

so that

$$\begin{aligned} BF_{01}(x) &= \binom{n}{x} \theta_0^x (1-\theta_0)^{n-x} \Big/ \left(\binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)} \right) \\ &= \theta_0^x (1-\theta_0)^{n-x} \Big/ \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)} \right) \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n-x)} \theta_0^x (1-\theta_0)^{n-x}. \end{aligned}$$

Hence, we obtain,

$$\begin{aligned}
\pi(\theta_0|x) &= \left\{1 + \frac{1-\pi_0}{\pi_0}\, BF_{01}^{-1}(x)\right\}^{-1} \\
&= \left\{1 + \frac{1-\pi_0}{\pi_0}\frac{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)}}{\theta_0^x(1-\theta_0)^{n-x}}\right\}^{-1}.
\end{aligned}$$