Graph Theory

Lecture 5.

Q.1. $V = \mathbb{F}_2^{\cdot k}$,

$1 \leq i \leq k$.

$w + e_i$ & w are adjacent

$\forall \; e_i = (0, \cdots, 1, 0, \cdots 0)$
    ↳ i^{th} place.

regular, deg $= k$. Eulerian ✓ Hamiltonian.

$(1, w) \; (1, \mathbb{F}_2^{k-1})$

$(0, w) \; (0, \mathbb{F}_2^{k-1})$

$= \mathbb{F}_2^2$

$(0,1)$

$(0,0)$

$(1,1)$

$(1,0)$
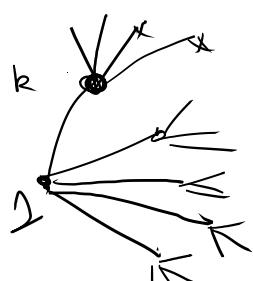
$i_1 \cdots i \overset{th}{(} \;$ coord.

$w \longrightarrow (0, 0, \cdots, 0)$ by dropping
    one coord at a time.

Connected ✓

✓ Bipartite — $\underline{\text{König}} \to G$ is bipartite iff it
    does not contain any
    odd cycle.

(Exercise)

**Q.2** Girth $= 5$, min deg k. $\Rightarrow |V(G)| \geq k^2 + 1$
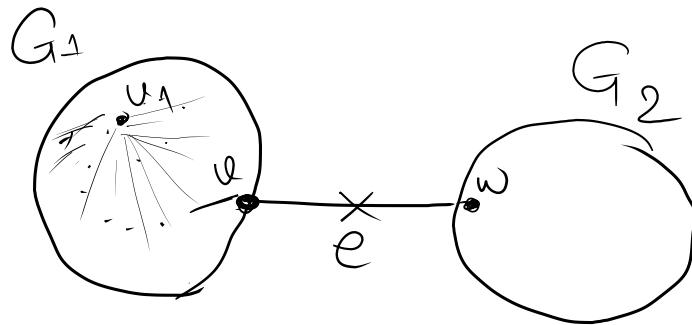
$1 + k + (k-1)k$, $k^2 + 1$

Q.3 $2k+1$ reg. graph. — G $\underline{k > 1}$ $k \geq 1$

$\exists\, e \in E(G)$ s.t. $G-e$ is disconnected.

$\Rightarrow$ $|V(G)| \geq 4k+6.$

G_1 u_1 v w G_2 e
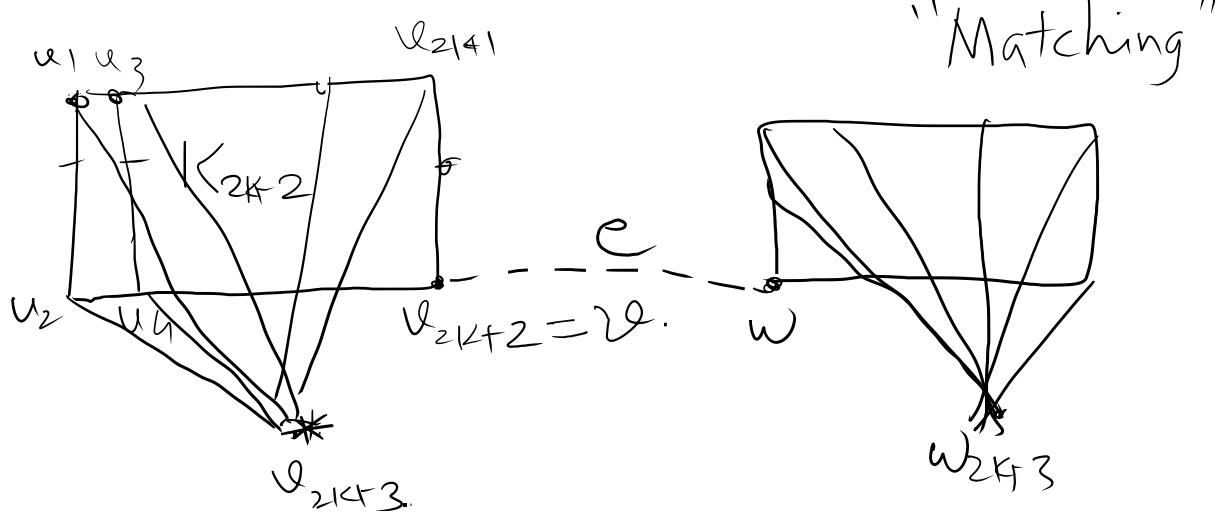
In G_i v, w has deg. $2k$,   all rest have deg. $\underline{2k+1}$.

$|V(G_i)| \geq \underline{2k+2}$ $i = 1,2$

$\longrightarrow \underset{v \in V(G)}{\underline{\sum \deg v}} = 2|E(G)|.$ ———— $\boxed{1}$

$\Downarrow$
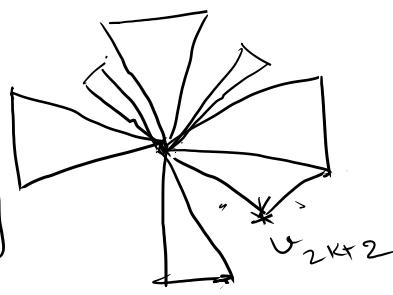
One can't have odd no. of odd deg. vertices in G.

$\Rightarrow$ $|V(G_i)| \geq 2k+3.$

"Matching"

$u_1\, u_3$ $v_2 | 4_1$

K_{2k+2}

u_2 u_4 $v_{2k+2} = 2v.$ w

$v_{2k+3}.$ w_{2k+3}

e
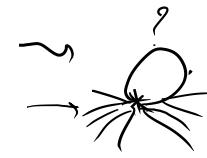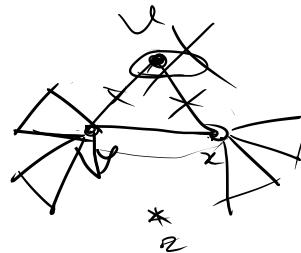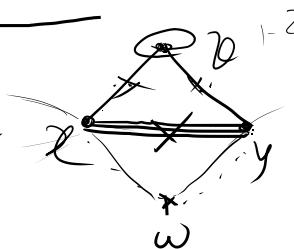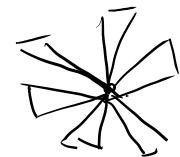
Q.4 (Erdős)

Erdős No. of a Mathematician.

2k+1 vertices.

v_{2k+2}

Proof is by induction

v

v_{1-2}

x y

w

z

z

→ idea of the proof.

$$|E(G)| \geq \frac{3(n-1)}{2}$$
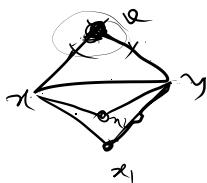
By induction.   Given such a G if ∃ a vertex which is part of only one △, remove it. & remove the two edges.

If the remaining edge of that △ is not a part of any other △, then identify the end pts of that edge & remove that edge too.

$$|V(G_1)| = |V(G)| - 2.$$
$$|E(G_1)| = |E(G)| - 3$$
} ind. hypo.

If → v x y x_1

$$\begin{cases} |V(G_1)| = |V(G)| - 1 \\ |E(G_1)| = |E(G)| - 2. \end{cases}$$

→ $2|E(G)| = \sum_{v} \deg(v) \geq 3n$.

$\Rightarrow |E(G)| \geq \frac{3n}{2} > \frac{3(n-1)}{2}$

(G. Dirac)(1952)   If in a conn. graph, the
min. deg $\geq \dfrac{|V(G)|}{2}$, then G is Hamiltonian.

Thm. $\sum \deg(v) = 2|E(G)|$.
(v vertices, G graph, E(G)-edges)

Pf. $S = \{ (v,e) / \; v$ is an end pt. of $e\}$
$e \in E(G)$
$v \in V(G)$

Two way counting.

① Form a set S of ordered tuples.
② Count $|S|$ by fixing first coord.
③ Count $|S|$ by fixing second coord.

result of ② = result ③.
‖   ‖

$\displaystyle\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

QED !!