# Connected components :-

- **Discrete topology** : $(X, \tau)$

  $X$ is disconnected, $\{x\}_{x \in X}$ is maximal connected components of $X$.

- **Indiscrete topology** :-

  $X$ is connected, so only one component.

- **$\mathbb{R}$ with usual topology** :-

  $X$ is connected ; single component.

- **$\mathbb{R}_\ell$ Topology** :

  $\mathbb{R}_\ell = (-\infty, 0) \cup [0, \infty)$ = disconnected.

  singletons are components of $\mathbb{R}_\ell$ (why?)

  ie) $A \subseteq \mathbb{R}_\ell$, $|A| \geq 2$, then $A$ is disconnected.

- **Co finite topology** :-

  - If $X$ is finite, its components are singleton.
  - If $X$ is infinite, $X$ is connected (why?), so only one component.

- **Co countable topology** :-

  - If $X$ is countable, its components are singleton (why?)
  - If $X$ is uncountable, $X$ is connected. so only one component.

- **K-topology on $\mathbb{R}$ :** $(a,b)$ or $(a,b)\setminus K$.

$A_1 = (-\infty, 0)$ , $A_2 = (0, \infty)$

- $A_1$ & $A_2$ are connected in usual top.
- supspace top on $A_1$, $A_2$ are nothing but usual top.
- $\overline{A_1} = [-\infty, 0]$ , $\overline{A_2} = [0, \infty)$ connected.
- $\mathbb{R}_K = \overline{A_1} \cup \overline{A_2}$ is connected.
- Thefore only one connected component but there are two path components namely $(-\infty, 0]$ and $(0, \infty)$.

- **Region of convergence:**

$$A = \{ z \in \mathbb{C} : \Sigma a_n z^n \text{ converges} \} , \quad a_n\text{'s fixed.}$$

Is A connected ? Yes. (why?)

Hint: $\Sigma a_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$, $R =$ Radius of convergence.

$$\therefore \{ |z| < R \} \subseteq A \subseteq \{ |z| \le R \}$$

- Is $\mathbb{R}$ is homeomorphic to $\mathbb{R}_\ell$ ?

NO ; $\mathbb{R} =$ connected, $\mathbb{R}_\ell =$ disconnected.

- discrete topology $\Rightarrow$ totally disconnected. converse ?

No ; $\mathbb{R}_\ell$ is totally disconnected but not discrete topology.

$e_1 = (1, 0, 0, \dots)$ , $e_2 = (0, 1, 0, 0, \dots)$ , ....

$$c_{00} = \text{Span}\{e_1, e_2, \dots\}$$
$$= \{x = (x_n) : x_n \neq 0 \text{ for finitely many}\}.$$

$c_{00}$ is not complete metric for any norm!

$\ell^\infty$ = set of all bounded sequences. , $\|x\|_\infty = \sup |x_n|$

- $\mathbb{R}^\omega$ is connected (path) as product of (path) connected spaces.

**Uniform Topology:-**

$$d(x, y) = \sup_{n \in \mathbb{N}} \{\min\{1, |x_n - y_n|\}\}$$

$f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ , $a \in \mathbb{R}^\omega$.

$x \mapsto x + a$ $\qquad d(f(x), f(y)) = d(x, y) \quad \forall x, y.$

$f$ is a homeomorphism of $\mathbb{R}^\omega$.

$$\mathbb{R}^\omega = \ell^\infty \cup (\ell^\infty)^c = (\text{bdd seq}) \cup (\text{unbdd seq})$$
= separation.

- $\ell^\infty$ is open $\qquad B(x, 1/2) \subseteq \ell^\infty \quad \forall x \in \ell^\infty$

- $(\ell^\infty)^c$ is open $\qquad B(x, 1/2) \notin \ell^\infty$ if $x \notin \ell^\infty$.

$\mathbb{R}^\omega$ is not connected under uniform topology.
What are all its connected components?

$\because$ $f$ is a homeomorphism, it is enough to find $[0] = ?$

- $[a] = a + [0]$, $[0] =$ connected component of $0$.

<u>Claim</u>: $[0] = \ell^\infty$.

(*) $\ell^\infty$ is open, closed, $0 \in \ell^\infty$.

uniform top on $\ell^\infty$ is nothing but $(\ell^\infty, \|\cdot\|_\infty)$.

For $x \in \ell^\infty$

$$\alpha : [0,1] \to \ell^\infty$$
$$t \mapsto tx.$$

$$\|\alpha(t) - \alpha(s)\|_\infty = \|tx - sx\|_\infty = |t-s| \, \|x\|_\infty. \quad (\alpha \text{ is cts}).$$

$\alpha(0) = 0$, $\alpha(1) = x$.

$\alpha$ is a path between $0$ & $x$.

$\therefore$ $\ell^\infty$ is path connected.

Therefore $\ell^\infty$ is the (Path) connected component of $0$.

- For $x \notin \ell^\infty$, $[x] = x + [0] = x + \ell^\infty$.

$$= \{ y \in \mathbb{R}^\omega : x - y \in \ell^\infty \}$$

ie) $x$ & $y$ lie in the same component in $\mathbb{R}^\omega$ under uniform topology $\iff$ $x - y \in \ell^\infty$.

**<u>Box topology</u>:-**

- $x \in \ell^\infty$, $\prod_{n=1}^{\infty} (x_n - 1, x_n + 1) \subseteq \ell^\infty$ (open !)

- $x \notin \ell^\infty$, $\prod_{n=1}^{\infty} (x_n - 1, x_n + 1) \subseteq (\ell^\infty)^c$. (open !)

ie) $\mathbb{R}^\omega = \ell^\infty \cup (\ell^\infty)^c$ disconnected.

- $f: \mathbb{R}^\omega \to \mathbb{R}^\omega$ (box topology), $a \in \mathbb{R}^\omega$

  $x \mapsto x+a$

  $f$ is a homeomorphism. why?

- $[a] = a + [0]$, but $[0] = ?$

$C_{00} = \text{span}\{e_1, e_2, \ldots\}$

$C_n = \text{span}\{e_1, e_2, \ldots e_n\} \cong \mathbb{R}^n$ (path connected) $\forall n \in \mathbb{N}$.

$\bar{0} \in C_n \;\; \forall n \qquad \bar{0} = (0,0,0 \ldots)$

$\Rightarrow \quad C_{00} = \bigcup_{n=1}^{\infty} C_n$ is path connected, hence connected too.

- Let $x \notin C_{00}$.

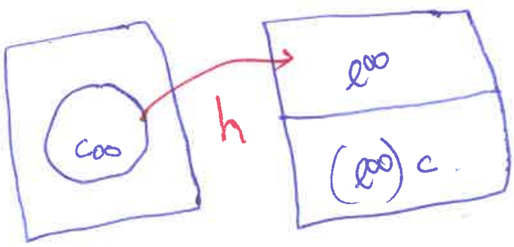
  $h: \mathbb{R}^\omega \to \mathbb{R}^\omega \qquad h(x) = (h_1(x), h_2(x), \ldots)$

$$h_n(u) = \begin{cases} u_n & , \text{ if } x_n = 0 \\ n\left(1 - \frac{u_n}{x_n}\right) & , \text{ if } x_n \neq 0. \end{cases}$$

- $h_n$ is linear map with non zero slope.
- $h$ is a homeomorphism (why?)

  $h(x) = 0, \qquad h(0) = $ ~~[illegible]~~ $=$ unbounded seq.

$\Rightarrow$ $[0]$ cannot contain elements from $(C_{00})^c$.




$\therefore \; [0] = C_{00}$

$[a] = \{ y \in \mathbb{R}^\omega : a - y \in C_{00} \}$.