

## EXERCISES — I

1. Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Assume that  $f$  has a local minima at each point. Show that  $f$  is a constant.
2. What are the different metrics that you know on  $\mathbb{R}$ .
3. Show that every metric space is Hausdorff.
4. Compare the usual and the cofinite topology on  $\mathbb{R}$ .
5. Compare the usual and countable topology on  $\mathbb{R}$ .
6. ~~Describe~~ Describe ~~as~~ as many bases as you can for the usual topology on  $\mathbb{R}$ .
7. Let  $X$  be the set of positive integers. Show that the collection of all arithmetic progressions of positive integers is a basis for a topology on  $X$ .
8. Show that if  $p$  is a prime, then the set  
$$\{np : n \geq 1\}$$
  
is closed in the topology on  $X$  in Problem 7. Conclude that there exist infinitely many primes.

9. Let  $Y$  be a subspace of  $X$ . Show that if  $U$  is open in  $Y$  and  $Y$  is open in  $X$ , then  $U$  is open in  $X$ .

10. Show that any finite set in a Hausdorff space is closed.

11. Show that arbitrary product of Hausdorff spaces is Hausdorff.

12. Show that  $X$  is Hausdorff if and only if

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in  $X \times X$ .

13. Let  $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function  $i=1, \dots, k$ . Show that

$$Z = \left\{ (x_1, \dots, x_n) \mid \left. \begin{array}{l} f_i(x_1, \dots, x_n) = 0 \\ i=1, \dots, k \end{array} \right\} \right\}$$

is a closed subset of  $\mathbb{R}^n$ .

14. Suppose  $X$  is a space and that  $X = \bigcup_{\alpha} A_{\alpha}$  where each  $A_{\alpha}$  is closed in  $X$ . Assume that  $f: X \rightarrow Y$  is such that

$$f_{\alpha} = f|_{A_{\alpha}}: A_{\alpha} \rightarrow Y$$

is continuous for each  $\alpha$ . Is  $f$  continuous? We had checked this when each  $A_{\alpha}$  is open.

15. Show that  $\mathbb{R}^n$  is homeomorphic to  $\mathbb{R}$  if and only if  $n=1$ . 3

16. Show that

$$X = \{(x_1, \dots, x_n) \mid \sum x_i^2 < 1\}$$

is homeomorphic to  $\mathbb{R}^n$ . Is every open convex subset of  $\mathbb{R}^n$  homeomorphic to  $\mathbb{R}^n$ ?

17. Let

$$S^n = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1\}.$$

Show that  $S^n - (0, \dots, 0, 1)$  is homeomorphic to  $\mathbb{R}^n$ .

18. Let  $f: X \rightarrow Y$  be a continuous function. Show that  $X$  is homeomorphic to

$$\text{gr}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y.$$

$\text{gr}(f)$  is called the graph of  $f$ .

19. A function  $f: [0,1] \rightarrow \mathbb{R}$  is said to be bounded if  $\exists M$  such that

$$|f(x)| \leq M \quad \forall x \in [0,1].$$

Let  $B[0,1]$  denote the set of bounded functions  $f: [0,1] \rightarrow \mathbb{R}$ . For  $f, g \in B[0,1]$  show that

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

is a metric.

20. Let  $X$  be a space such that every

$$f: X \rightarrow \mathbb{R}$$

is continuous. What can you say about the topology of  $X$ ?

21. Let  $X$  be a connected space and  $f: X \rightarrow \mathbb{R}$  continuous. Assume that for each  $x \in X$  there exists a neighborhood  $U$  of  $x$  restricted to which  $f$  is constant. Show that  $f$  is constant.

22. Let  $X$  be a space and  $A, B \subseteq X$ . Determine whether

$$\overline{A \cap B} = \overline{A} \cap \overline{B}$$

$$\overline{A - B} = \overline{A} - \overline{B}$$

23. If  $A \subseteq X$ , then the boundary,  $\text{bd}(A)$ , of  $A$  is defined by the expression

$$\text{bd}(A) = \overline{A} \cap \overline{(X - A)}$$

Show that  $\text{Int}(A)$  and  $\text{bd}(A)$  are disjoint and that

$$\overline{A} = \text{Int}(A) \cup \text{bd}(A).$$

Show that  $\text{bd}(A) = \emptyset$  if and only if  $A$  is both open and closed.