One-parameter exponential family in natural form.

In the usual form, we have the density as: $f(x|\theta) = \exp(c(\theta)T(x) + d(\theta) + S(x))I_A(x)$. Define $\eta = c(\theta)$ for $\theta \in \Theta$. Let $\Gamma = \{\eta : \eta = c(\theta), \theta \in \Theta\}$. Then we get

$$f^*(x|\eta) = \exp(\eta T(x) + d_0(\eta) + S(x))I_A(x),$$

where $d_0(\eta) = d(c^{-1}(\eta))$ if c is one-one. Otherwise, since we must have

$$\begin{aligned} 1 &= \int_A f^*(x|\eta)\,dx = \int_A \exp(\eta T(x) + d_0(\eta) + S(x))\,dx \\ &= \exp(d_0(\eta)) \int_A \exp(\eta T(x) + S(x))\,dx, \end{aligned}$$

$d_0(\eta) = \log \left(\int_A \exp(\eta T(x) + S(x))\,dx\right)^{-1}$ or $\log \left(\sum_A \exp(\eta T(x) + S(x))\,dx\right)^{-1}$. Let $H = \{\eta : |d_0(\eta)| < \infty\}$. Whenever $\theta \in \Theta$, note that

$$\int_A \exp(c(\theta)T(x) + S(x))\,dx = \int_A \exp(\eta T(x) + S(x))\,dx < \infty$$

since

$$1 = \int_A f(x|\theta\,dx = \exp(d(\theta)) \int_A \exp(c(\theta)T(x) + S(x))\,dx.$$

Therefore, whenever $\theta \in \Theta$, $|d_0(\eta)| < \infty$ and $\eta \in H$.

$$f^*(x|\eta) = \exp(\eta T(x) + d_0(\eta) + S(x))I_A(x),$$

$\eta \in H$ is called exponential family in natural form. H can be shown to be an interval.

Theorem. If X has density

$$f(x|\eta) = \exp(\eta T(x) + d_0(\eta) + S(x))I_A(x),$$

and η is an interior point of H (i.e., $(\eta - \epsilon, \eta + \epsilon) \subset H$), the mgf of $T(X)$ exists and is

$$\psi(s) = E(\exp(sT(X)) = \exp(d_0(\eta) - d_0(s + \eta))$$

for s in some neighbourhood of 0. Also,

$$\begin{aligned} E\,[T(X)] &= -\frac{d}{d\eta} d_0(\eta), \\ Var\,[T(X)] &= -\frac{d^2}{d\eta^2} d_0(\eta). \end{aligned}$$

Proof. Note that

$$\begin{aligned} E(\exp(sT(X)) &= \int_A \exp(sT(x) + \eta T(x) + d_0(\eta) + S(x))\, dx \\ &= \exp(d_0(\eta)) \int_A \exp((s+\eta)T(x) + S(x))\, dx. \end{aligned}$$

Since η is an interior point of H, $s + \eta \in H$ if s is small enough. Therefore, $\int_A \exp((s+\eta)T(x) + S(x))\, dx < \infty$. But then $f(x|s+\eta)$ is also a density. Thus,

$$\exp(d_0(s+\eta)) \int_A \exp((s+\eta)T(x) + S(x))\, dx = 1$$

or

$$\int_A \exp((s+\eta)T(x) + S(x))\, dx = \exp(-d_0(s+\eta)).$$

Therefore,

$$\begin{aligned} E\left[T(X)\right] &= \frac{d}{ds} E\left[\exp(sT(X)\right]|_{s=0} \\ &= \exp(d_0(\eta) - d_0(s+\eta))\,(-d_0'(s+\eta))\,|_{s=0} = -d_0'(\eta), \end{aligned}$$

$$\begin{aligned} & E\left[T^2(X)\right] \\ = & \frac{d^2}{ds^2} E\left[\exp(sT(X)\right]|_{s=0} \\ = & \frac{d}{ds}\left\{\frac{d}{ds} E\left[\exp(sT(X)\right]\right\}|_{s=0} \\ = & \frac{d}{ds}\left\{-\exp(d_0(\eta) - d_0(s+\eta)) d_0'(s+\eta)\right\}|_{s=0} \\ = & \left\{\exp(d_0(\eta) - d_0(s+\eta))\,(d_0'(s+\eta))^2 - \exp(d_0(\eta) - d_0(s+\eta)) d_0''(s+\eta)\right\}|_{s=0} \\ = & (d_0'(\eta))^2 - d_0''(\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} Var\left[T(X)\right] &= E\left[T^2(X)\right] - E^2\left[T(X)\right] \\ &= -d_0''(\eta) + (d_0'(\eta))^2 - (-d_0'(\eta))^2 \\ &= -d_0''(\eta). \end{aligned}$$

Example. $X \sim$ Binomial(n,p), $0 < p < 1$, n fixed. Then $E(X) = np$, $Var(X) = np(1-p)$ and $E(\exp(sX)) = [p\exp(s) + (1-p)]^n$. Derive these using the result above. You will note that these formulas are not especially useful for such purposes. They are useful for deriving certain theoretical results instead.