Graph Theory

Lecture 2

On MOODLE, New ref. book has bee added.

AMS Open Notes Series

→ Benny Sudakov – Graph Theory.

+ D. West – Introduction to G.T.

Connectedness in Graphs.

Path ; If any two vertices can be "joined by a path" then G - (graph) is connected.

Actually the relation $\sim$ on $V(G)$ is an equivalence relation where $v \sim w$ iff $\exists$ a v-w path.

Equivalence classes are called <u>connected components</u> of a graph.

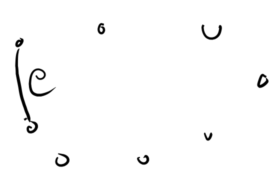
<u>Thm</u>: If a graph G has n vertices & k edges, then it must hat <u>at least</u> $n-k$ connected components.

<u>Pf</u>:- Note that this theorem needs to be proved only for $k \leq n-1$.

Prove this by induction. ① $k=0$.

Then $G =$  → n - points.

& G has $n = n-k$ conn. comp.

② $k=1$. $G =$ 

Thm. is true in this case too.

(3) Induction hypothesis : If G has n-vertices & $k-1$ - edges then G has $\geq n+1-k$ conn. comp.

(4) If $|V(G)| = n$ & $|E(G)| = k$

Then remove one of the edges e, use induction hypothesis & then observe that at most two conn. comp. of $G \setminus e$ can become into one compo. with rest being undisturbed.

$\exists$ # comp of $G \geq$ (# comp of $G \setminus e$) $- 1$

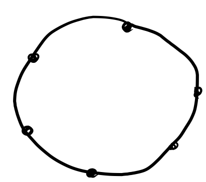
$\geq (n+1-k) - 1 = n-k$

QED.

Tree :- A Tree is a graph with n vertices $n-1$ edges & is connected.

(Extremal case in the above theorem!)

Def^n :- A Cycle of length n is a closed path with n edges. (v - v path)



= cycle of length 6.

Exercise: (1) Prove that a tree can not a cycle as a subgraph.

(2) Prove that given any two vertices v & w in a tree, $\exists$ a unique path bet^n v & w.

(3) A connected graph without cycle is a tree.

Forest :- Any graph without cycle is called a forest.

Exercise :- Let G be a graph, that is not connected. If $|V(G)| = 10$, then prove that $|E(G)| \leq 36$.

Does equality hold?

Sol^n :- If G_1 & G_2 are two conn. comp. of G, then G can not have any edge with one end pt in $V(G_1)$ & other in $V(G_2)$.

If $|V(G_1)| = m$ & $|V(G_2)| = n$ then from the possible 45 edges $({}^{10}C_2 = 45)$, mn edges must be missing.

To find upper bound on $|E(G)|$, we need consider only the case of 2 conn. comp.

$\Rightarrow$ $m+n = 10$. $\qquad 45 - mn \geq |E(G)|$

$\min\{mn \mid m+n = 10\}$.

1+9	9
2+8	16
3+7	21
4+6	24
5+5	25

$\Rightarrow$ $|E(G)| \leq 36$.

$\Rightarrow$ $G = K_9 \cup \{\cdot\}$ $\Leftrightarrow$ equality is achieved!

Exercise :- Let $A_1, \ldots, A_n$ be distinct subsets of $\{1, 2, \ldots, n\}$. P.T. there exist a number $1 \leq x \leq n$ s.t. $A_i \setminus \{x\}$ are also distinct.

Pf :- When $A_i \neq A_j$ becomes $A_i \setminus \{x\} = A_j - \{x\}$?

iff $A_i = A_j \cup \{x\}$ OR $A_j = A_i \cup \{x\}$.

iff $|A_i \Delta A_j| = 1$.

Aside : S, T two subsets of a set X, then $S \Delta T = S \cup T \setminus S \cap T$
$= (S \setminus T) \cup (T \setminus S)$.

on $\mathcal{P}(X)$, Δ is a binary relation & ϕ is identity

$A^2 = id$.

$(\mathcal{P}(X), \Delta)$ is an abelian group.

We form G whose vertices are labelled $A_1, \dots, A_n$ & $\{A_i, A_j\} \in E(G)$ iff $|A_i \Delta A_j| = 1$. Let the edge A_iA_j has colour x if $A_i \Delta A_j = \{x\}$

→ We need to prove that there is one colour missing from these edges. If G was a tree, $\exists$ only $n-1$ edges & we are done. If G was a forest, then it has k trees & each one have $n_i - 1$ edges $\sum n_i = n$. $k \geq 2$.

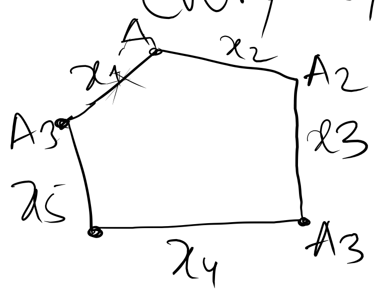
$$\Rightarrow \# = \sum n_i - k < n.$$

Hence we are done.

What happens if G has a cycle?

In the graph G (as described above),

<u>Claim</u> :- every cycle must have a repeated colour.

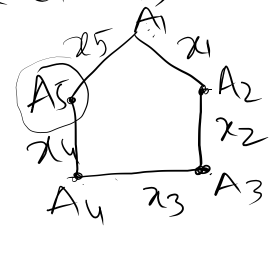


• Assuming this claim, we can drop an edge from G s.t. that a given cycle is broken & no. of colours are still same!

Repeating this argument (if needed), we arrive at a graph $\tilde{G} = G \setminus \{e_1, \dots, e_k\}$ s.t. $\tilde{G}$ is a forest and colours in $\tilde{G}$ = colours in G.

<u>QED.</u>

(proof of the claim)



Assume WLOG that $|A_1| = \max_i \{|A_i| \mid 1 \leq i \leq 5\}$

$\Rightarrow A_2 = A_1 \setminus \{x_1\}$; $A_5 = A_1 \setminus \{x_5\}$

$\{x_2\} = A_2 \triangle A_3$. $|A_3| = |A_1|$

or $|A_3| = |A_1| - 2$ -- & $|A_4| = |A_3| - 1$

we know that $|A_5| = |A_1| - 1$. $|A_4| = |A_1| - 3$.

$\Rightarrow$ $\exists$ i s.t. $|A_i| = |A_{i-1}| - 1$

& $|A_{i+1}| = |A_i| + 1$.

Also $\underline{A_5} \subset A_1 \supset A_2$

$\Rightarrow$ No new element remains by the time we reach A_5

$\Rightarrow$ If new element is added then it has to be removed also!

$\Rightarrow$ One colour is repeated.

$\Rightarrow$ only elements from A_1 are added or subtracted (one at a time).

A removed element must be added.

$\Rightarrow$ A colour is repeated. Thereby proving the claim!

$\rightarrow$ QED

<u>Exercise</u> :- Complete the proof of the claim by considering a cycle of length m (instead of 5).

——— x ——— x ——— x ———

Generate more exercises out of this.

$\rightarrow$ What happens if we take $A_1, \ldots, A_m$ distinct subsets of $\{A_1, \ldots, A_n\}$

$m \leq n$ $\longrightarrow$ add $B_{m+1}, \ldots, B_n$

$\rightarrow$ What happens if you remove x & y ?