ON THE TOPOLOGY OF CERTAIN MATRIX GROUPS

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ABSTRACT. We illustrate how some basic algebraic properties of certain real and complex classical matrix groups have a significant say in the analysis of their topological structures.

1. Introduction

Let $\mathbb{K}$ be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. The groups of invertible square matrices over $\mathbb{K}$, also known as the *general linear groups*, play an important role in different branches of mathematics like linear algebra, field theory, Lie groups, differential geometry, representation theory, harmonic analysis, operator algebras and non commutative geometry, to name a few. Various subgroups of the general linear groups, *e.g.*, the *special linear groups, the orthogonal and special orthogonal groups, the unitary and special unitary groups* and *symplectic groups* have attracted the attention of some of the best minds in the world of mathematics for decades, and have also found significance in the world of physics. These subgroups are usually referred to as the *classical linear groups*. Operator algebraists have in fact developed a whole world of quantum versions of most of these classical groups.

Quite interestingly, apart from their applications to different areas, studying the topological properties of matrix groups is itself quite significant and occupies prominent space in mathematics literature. In this short article, we make an attempt to show how the linear algebraic results that we learn at undergraduate level turn out to provide deep implications towards the analysis of the topological structures of these classical groups.

2. Preliminaries

Throughout this article, $\mathbb{K}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers with the usual metric $d : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ given by $d(x, y) := |x - y|$. And, $M_n(\mathbb{K})$ will denote the space of $n \times n$ matrices with entries from $\mathbb{K}$.

2.1. **Topology on the matrix groups.** Recall that there is a natural identification between $M_n(\mathbb{K})$ and $\mathbb{K}^{n^2}$ via the canonical map

$$M_n(\mathbb{K}) \ni [a_{ij}] \mapsto (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots a_{nn}) \in \mathbb{K}^{n^2}.$$

Via this identification, $M_n(\mathbb{K})$ becomes a metric space with the usual metric. There are some natural continuous maps from and into $M_n(\mathbb{K})$. For instance, if $p_{rs} \in \mathbb{K}[x_{11}, x_{12}, \ldots, x_{mm}]$, $1 \leq r, s \leq n$, is a collection of n^2 polynomials in m^2 variables, then the map $M_m(\mathbb{K}) \ni [a_{ij}] \mapsto [p_{rs}(a_{11}, a_{12}, \ldots, a_{mm})] \in M_n(\mathbb{K})$ is continuous. And, if X is a metric space and $\varphi_{rs} : X \to \mathbb{K}$, $1 \leq r, s \leq n$ is a collection of n^2 continuous functions, then the map $X \ni x \mapsto [\varphi_{rs}(x)] \in M_n(\mathbb{K})$ is continuous. In particular, the determant function, being a polynomial function of the entries of a matrix is continuous.

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We aim to study the compactness and connectedness of the multiplicative group of invertible real and complex square matrices

$$GL(n,\mathbb{K}) := \{A \in M_n(\mathbb{K}) : A \text{ is invertible}\} \qquad \text{(general linear group)},$$

and its subgroups

$$\begin{array}{rcll} SL(n,\mathbb{K}) & := & \{A \in GL(n,\mathbb{K}) : \det(A) = 1\} & \text{(special linear group)} \\ O(n) & := & \{A \in GL(n,\mathbb{R}) : AA^T = I_n = A^TA\} & \text{(orthogonal group)} \\ SO(n) & := & \{A \in O(n) : \det(A) = 1\} & \text{(special orthogonal group)} \\ U(n) & := & \{U \in GL(n,\mathbb{C}) : UU^* = I_n = U^*U\} & \text{(unitary group)} \\ SU(n) & := & \{U \in U(n) : \det(U) = 1\} & \text{(special unitary group)} \\ Sp(n,\mathbb{K}) & := & \{A \in GL(2n,\mathbb{K}) : A^TJ_nA = J_n\} & \text{(symplectic group)}, \end{array}$$

where A^T and U^* denote the transpose of A and conjugate transpose of U, respectively, and $J_n = \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}$. It follows from definitions that $O(n)$ and $U(n)$ are closed under multiplication as well as inversion; so, they form multiplicative groups. Further, for a matrix A with determinant 1, multiplicativity of the determinant function yields $\det(A^{-1}) = 1$. In particular, $SL(n,\mathbb{K}), SO(n)$ and $SU(n)$ are all multiplicative groups. That symplectic matrices form a multiplicative group will be shown in Section 5.

2.2. **Some gems from the world of linear algebra.** We now recall some algebraic properties of the matrix algebra and its subsets (mostly without proof) which will be used in analyzing the topoligical structures of multiplicative groups of invertible matrices.

The Euclidean spaces admit natural inner products given by $\langle v, w\rangle = w^Tv$ for all $v, w \in \mathbb{R}^n$ and $\langle v, w\rangle = w^*v$ for all $v, w \in \mathbb{C}^n$, where we have treated the vectors v and w as column vectors. We would require the following definition of a positive semidefinite matrix and the subsequent results, the details of which may be found in any standard text of linear algebra, see [1, 3] for instance.

Definition 2.1. *A square matrix $P \in M_n(\mathbb{K})$ is said to be positive semidefinite if $\langle Px, x\rangle \geq 0$ for all $x \in \mathbb{K}^n$.*

Remark. For any $A \in M_n(\mathbb{C})$, A^*A is positive semidefinite and likewise for any $A \in M_n(\mathbb{R})$, A^TA is positive semidefinite and it is a fact that these are the only positive semidefinite matrices possible.

Proposition 2.2. *Let $P \in M_n(\mathbb{R})$ be a positive semidefinite matrix. Then P is symmetric, $\det(P) \geq 0$ and there exists a unique postive semidefinite matrix $P^{1/2} \in M_n(\mathbb{R})$ such that $(P^{1/2})^2 = P$ and $P^{1/2}$ is invertible if and only if P is so.*

We now prove the following important result also known as the *polar decomposition* of determinant one real matrices

Theorem 2.3. *Let $A \in SL(n,\mathbb{R})$. Then there exists a rotation matrix $R \in SO(n)$ and a real, symmetric and positive semidefinite matrix $P \in SL(n,\mathbb{R})$ such that $A = RP$.*

Proof. There is an obvious candidate for P, namely $P = (A^TA)^{1/2}$ and this forces R to be defined as $R = AP^{-1}$. Clearly, P is real, symmetric and positive semidefinite matrix. Further,

$$RR^T = AP^{-1}P^{-1}A^T = AP^{-2}A^T = A(A^TA)^{-1}A^T = I_n$$

and similarly $R^TR = I_n$ implying that R is orthogonal. In particular, $1 = \det(R^TR) = \det(R^T)\det(R) = \det(R)^2$ so that $\det(R) = 1$ or -1. Now, $1 = \det(A) = \det(R)\det(P)$ and, by Proposition 2.2 and the fact that P is invertible, we have $\det(P) > 0$. Therefore, we must have $\det(R) = 1$, i.e., $R \in SO(n)$. $\square$

We now state a result that justifies the name *rotation matrices* for the elements of $SO(n)$ and a proof can be found in [1, Theorem 6.39].

Proposition 2.4. *Any matrix in $SO(n)$ is orthogonally similar to a block diagonal of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_r$, where each A_i is $[1]$ or a 2×2 rotation matrix of the type $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in \mathbb{R}$.*

3. Real Classical groups

In this section, we discuss the topological properties of the real general linear group $GL(n, \mathbb{R})$ and its subgroups $SL(n, \mathbb{R}), O(n)$ and $SO(n)$.

Observe that $GL(1, \mathbb{R}) \cong (-\infty, 0) \cup (0, \infty)$ is open, non-compact and disconnected. Interestingly, the same properties hold in higher dimensions as well.

Proposition 3.1. *(1) $GL(n, \mathbb{R})$ is open and unbounded.*
(2) $GL(n, \mathbb{R})$ is not connected.

Proof. (1) The complement of $GL(n, \mathbb{R})$ in $M_n(\mathbb{R})$ is the set $\{A \in M_n(\mathbb{R}) : \det A = 0\}$. Since determinant is continuous and $\{0\}$ is closed in $\mathbb{R}$, $M_n(\mathbb{R}) \setminus GL(n, \mathbb{R})$ is closed. Therefore, $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R})$. Also, $kI_n \in GL(n, \mathbb{R})$ for all $k > 0$. So, $GL(n, \mathbb{R})$ is unbounded. (In particular, $GL(n, \mathbb{R})$ is not compact.) (2) Note that $\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is a surjective continuous map and $\mathbb{R} \setminus \{0\}$ is not connected. Since a continuous map sends a connected set onto a connected set, $GL(n, \mathbb{R})$ cannot be connected. $\square$

We shall, in fact, show that $GL(n, \mathbb{R})$ has precisely two (path) components, namely, $GL_+(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : \det(A) > 0\}$ and $GL_-(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : \det(A) < 0\}$. However, in order to achieve this, we will first have to analyze the topological properties of some of its subgroups.

Proposition 3.2. *The groups $O(n)$ and $SO(n)$ are compact.*

Proof. Write any matrix $A \in O(n)$ as $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ where each v_i is a row matrix. Then from the identity $AA^T = I_n$, we get $v_i v_i^T = 1$ for all $1 \leq i \leq n$. This implies that A is inside the unit ball of $\mathbb{R}^{n^2}$. Therefore $O(n)$ is a bounded subset of the Euclidean space $\mathbb{R}^{n^2}$. Let $\{A_k\}$ be any sequence in $O(n)$ and let $A_k \to A$ in $M_n(R)$. Taking limit as $k \to \infty$ in the relation $A_k A_k^T = A_k^T A_k = I_n$, by continuity of multiplication, we get $AA^T = A^T A = I_n$ proving that $A \in O(n)$. Thus $O(n)$ is closed too. Hence, by the Heine-Borel theorem, $O(n)$ is compact. If, in addition, each of the matrices A_k above have determinant 1 then by continuity of the determinant, we also see that $\det A = 1$, which shows that $SO(n)$ is closed in $O(n)$ and hence compact. $\square$

Theorem 3.3. *$O(n)$ is not connected whereas $SO(n)$ is path connected.*

Proof. Let $M \in O(n)$. Then, as seen in Theorem 2.3, $\det(M) \in \{1, -1\}$. Let

$$O_\pm(n) := \{M \in O(n) : \det(M) = \pm 1\} = GL_\pm(n, \mathbb{R}) \cap O(n).$$

Then $O_+(n)$, which is the same as the subgroup $SO(n)$, and $O_-(n)$ are open in the subspace topology and they form a disconnection of $O(n)$, implying that $O(n)$ is not connected.

Now, we show that any matrix in $SO(n)$ is joined to I_n by a path. Since $SO(1) = \{[1]\}$, let us assume that $n \geq 2$. Let $R \in SO(n)$. Then, by Proposition 2.4, there exists an orthogonal matrix $M \in O(n)$ such that

$$MRM^T = A_1 \oplus A_2 \oplus \cdots \oplus A_r,$$

where each A_i is $[1]$ or a 2×2 rotation matrix of the type $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in \mathbb{R}$. Without loss of generality, assume that, for some $k \leq r$, for $1 \leq i \leq k$, $A_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$ for some $\theta_i \in \mathbb{R}$, and that $A_i = [1]$ for $k \leq i \leq r$.

We can now look for an appropriate path. For each $1 \leq i \leq k$, consider the map $\varphi_i : [0,1] \to SO(2)$ given by $\varphi_i(t) = \begin{bmatrix} \cos(t\theta_i) & \sin(t\theta_i) \\ -\sin(t\theta_i) & \cos(t\theta_i) \end{bmatrix}$. Then each φ_i is a path in $SO(2)$ with end points I_2 and $\begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$. Therefore, the map $\varphi : [0,1] \to SO(n)$ given by

$$\varphi(t) = M^T \Big(\varphi_1(t) \oplus \varphi_2(t) \oplus \cdots \oplus \varphi_k(t) \oplus I_{n-2k} \Big) M$$

is a path in $SO(n)$ with end points $\varphi(0) = I_n$ and $\varphi(1) = R$, thereby, establishing that $SO(n)$ is path connected. □

Corollary 3.4. *$O(n)$ has precisely two path components, namely, $O_+(n)$ and $O_-(n)$.*

Proof. By Theorem 3.3, $O_+(n) = SO(n)$ is path connected. Now, let $A, B \in O_-(n)$ and fix a $C \in O_-(n)$. Then $AC, BC \in O_+(n)$ and, therefore, there exists a path φ in $O_+(n)$ joining AC and BC. Consider the map $\tilde{\varphi} : [0,1] \to O_-(n)$ given by $\tilde{\varphi}(t) = \varphi(t)C^{-1}$. Then $\tilde{\varphi}$ is a path in $O_-(n)$ joining A and B.

Also, we know that $O(n)$ is a disjoint union of $O_+(n)$ and $O_-(n)$, so these are the only two path components of $O(n)$. □

Note that $SL(1,\mathbb{R}) = \{[1]\}$ is clearly path connected and compact. However, in higher dimensions compactness takes a back seat.

Corollary 3.5. *$SL(n,\mathbb{R})$ is closed, path connected and is not compact for $n \geq 2$.*

Proof. Since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous and $SL(n,\mathbb{R}) = \det^{-1}(\{1\})$, it is closed. It is not bounded as it contains $SL(2,\mathbb{R})$ which contains the matrices $\begin{bmatrix} r & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$ for all $r > 0$.

For path connectedness, it is again enough to show that I_n is connected by a path to any other matrix in $SL(n,\mathbb{R})$. Let $A \in SL(n,\mathbb{R})$. Then, by Theorem 2.3, there exists an $R \in SO(n)$ and a real, symmetric and positive semidefinite matrix $P \in SL(n,\mathbb{R})$ such that $A = RP$. By Theorem 3.3, $SO(n)$ is path connected, so there exists a path $\varphi : [0,1] \to SO(n) \subset SL(n,\mathbb{R})$ with end points $\varphi(0) = I_n$ and $\varphi(1) = R$. Then the map $\tilde{\varphi} : [0,1] \to SL(n,\mathbb{R})$ given by $\tilde{\varphi}(t) = \varphi(t)P$ is a path in $SL(n,\mathbb{R})$ with end points $\tilde{\varphi}(0) = P$ and $\tilde{\varphi}(1) = RP = A$.

It now suffices to show that there exists a path in $SL(n,\mathbb{R})$ with end points I_n and P for then a path from I_n to A would be obtained by concatenating the paths from I_n to P and from P to A.

Since P is a symmetric matrix, there exists an orthogonal matrix Q such that QPQ^{-1} equals the diagonal matrix $D := \mathrm{diag}(r_1, r_2, \ldots, r_n)$, where $r_1, r_2, \ldots, r_n$ are the eigenvalues of P. Since P is positive semidefinite and invertible, $r_i > 0$, so that $1 + t(r_i - 1) > 0$, for all $1 \leq i \leq n$ and $0 \leq t \leq 1$. The map $\psi : [0,1] \to GL_+(n,\mathbb{R})$ given by

$$\psi(t) = \mathrm{diag}\Big(1 + t(r_1 - 1), 1 + t(r_2 - 1), \ldots, 1 + t(r_n - 1)\Big)$$

is a path with end points $\psi(0) = I_n$ and $\psi(1) = D$. Then, $\frac{1}{\sqrt[n]{\det(\psi(t))}} Q\psi(t)Q^{-1} \in SL(n,\mathbb{R})$ for all $0 \leq t \leq 1$, so that the map

$$[0,1] \ni t \mapsto \frac{1}{\sqrt[n]{\det(\psi(t))}} Q\psi(t)Q^{-1} \in SL(n,\mathbb{R})$$

is a path in $SL(n,\mathbb{R})$ joining I_n and P. □

We now have the required tools to show that $GL(n,\mathbb{R})$ has precisely two components, namely, $GL_+(n,\mathbb{R})$ and $GL_-(n,R)$.

Corollary 3.6. *$GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ are path connected and these are the only two path components of $GL(n,\mathbb{R})$.*

Proof. Let $A \in GL_+(n,R)$. Then $\tilde{A} := \frac{1}{\sqrt[n]{\det(A)}} A \in SL(n,\mathbb{R})$. So, by path connectedness of $SL(n,\mathbb{R})$ there exists a path φ in $SL(n,\mathbb{R}) \subset GL_+(n,\mathbb{R})$ with end points I_n and $\tilde{A}$. Also, there is an obvious path in $GL_+(n,\mathbb{R})$ with end points $\tilde{A}$ and A, namely, $[0,1] \ni t \to \left(\frac{1-t}{\sqrt[n]{\det(A)}} + t\right) A \in GL_+(n,\mathbb{R})$. Therefore, $GL_+(n,\mathbb{R})$ is path connected.

The fact that $GL_-(n,\mathbb{R})$ is also path connected follows on the lines of the proof of path connectedness of $O_-(n)$ as in Corollary 3.4.

Also, we know that $GL(n,\mathbb{R})$ is a disjoint union of $GL_+(n,\mathbb{R})$ and $GL_-(n,\mathbb{R})$ so these are the only two path components of $GL(n,\mathbb{R})$. □

4. Complex Classical groups

In this section, we take up the complex general linear group $GL(n,\mathbb{C})$ and it's subgroups $SL(n,\mathbb{C})$, $U(n)$ and $SU(n)$.

The elements of $U(n)$ are called unitary matrices and satisfy the equivalent angle preserving property:

$$\langle Uv, Uw\rangle = (Uw)^*(Uv) = w^*U^*Uv = w^*v = \langle v, w\rangle, \text{ for all } v, w \in \mathbb{C}^n$$

Proposition 4.1. *(1) $GL(n,\mathbb{C})$ is open and unbounded.*
(2) $GL(n,\mathbb{C})$ is path connected.

Proof. (1) Since $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is continuous and $A \in GL(n,\mathbb{C})$ if and only if $\det(A) \neq 0$, we see that $GL(n,\mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$ is open. Since $GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$, $GL(n,\mathbb{C})$ is unbounded.

(2) Let $A \in GL(n,\mathbb{C})$ with distinct (non-zero) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ and multiplicites $m_1, m_2, \ldots, m_r$, respectively. Then, $m_1 + m_2 + \cdots + m_r = n$. Since $\mathbb{C}$ is alebraically closed, A possesses a Jordan canonical form ([1, Corollary 2, p. 291]), that is, there exists a $P \in GL(n,\mathbb{C})$ such that $PAP^{-1} = A_1 \oplus A_2 \oplus \cdots A_r$, where each A_i is a block diagonal matrix of the form

$$A_i = J_{m_{1,i}}(\lambda_i) \oplus J_{m_{2,i}}(\lambda_i) \oplus \cdots \oplus J_{m_{k_i,i}}(\lambda_i),$$

with $m_{1,i} + m_{2,i} + \cdots + m_{k_i,i} = m_i$, where for each j, $J_{m_{j,i}}(\lambda_i)$ is the $m_j \times m_j$ Jordan block

$$J_{m_{j,i}}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}.$$

Note that for each $0 \neq \lambda \in \mathbb{C}$, we can easily find a path $\psi_\lambda : [0,1] \to \mathbb{C}$ with end points $\psi_\lambda(0) = 1$ and $\psi(1) = \lambda$ such that ψ_λ does not pass through the origin of $\mathbb{C}$. As a consequence, for each Jordan block $J_m(\lambda)$ with $\lambda \neq 0$, the map $\varphi_{m,\lambda} : [0,1] \to M_m(\mathbb{C})$ given by

$$\varphi_{m,\lambda}(t) = \begin{bmatrix} \psi_\lambda(t) & t & & & \\ & \psi_\lambda(t) & t & & \\ & & \ddots & \ddots & \\ & & & \psi_\lambda(t) & t \\ & & & & \psi_\lambda(t) \end{bmatrix}$$

is a path because each component of $\varphi_{m,\lambda}$ is continous, and has end points $\varphi_{m,\lambda}(0) = I_m$ and $\varphi_{m,\lambda}(1) = J_m(\lambda)$. Also, since 0 does not lie in the range of ψ_λ, we see that $\varphi_{m,\lambda}(t) \in GL(m,\mathbb{C})$ for all $t \in [0,1]$. Therefore, the map $\varphi : [0,1] \to GL(n,\mathbb{C})$ given by

$$\begin{aligned}\varphi(t) = \quad & P^{-1}\Big(\varphi_{m_{1,1},\lambda_1}(t) \oplus \varphi_{m_{2,1},\lambda_1}(t) \oplus \cdots \oplus \varphi_{m_{k_1,1},\lambda_1}(t) \oplus \varphi_{m_{1,2},\lambda_2}(t) \oplus \varphi_{m_{2,2},\lambda_2}(t) \oplus \\ & \cdots \oplus \varphi_{m_{k_2,2},\lambda_2}(t) \oplus \cdots \varphi_{m_{1,r},\lambda_r}(t) \oplus \varphi_{m_{2,r},\lambda_r}(t) \oplus \cdots \oplus \varphi_{m_{k_r,r},\lambda_r}(t)\Big)P\end{aligned}$$

is a path in $GL(n,\mathbb{C})$ with end points $\varphi(0) = I_n$ and $\varphi(1) = A$ and we are done. □

Note that $SL(1,\mathbb{C}) = \{1\}$ is compact and path connected. However, there is slight difference in compactness property in higher dimensions.

Corollary 4.2. *$SL(n,\mathbb{C})$ is closed, path connected and is not compact for $n \geq 2$.*

Proof. Let $A \in SL(n,\mathbb{C})$. Then by path connectedness of $GL(n,\mathbb{C})$ there exists a path $\varphi : [0,1] \to GL(n,\mathbb{C})$ such that $\varphi(0) = I_n$ and $\varphi(1) = A$. Note that components of the path φ are all paths in $\mathbb{C}$ and suppose they are given by $\varphi(t) = [\varphi_{ij}(t)]$. Then, if $\theta : [0,1] \to \mathbb{C}\backslash\{0\}$ is given by $\theta(t) = \det(\varphi(t))^{-1}$, by n-linearity of the determinant function, we see that the matrix $\begin{bmatrix} \theta(t)\varphi_{11}(t) & \dots & \theta(t)\varphi_{1n}(t) \\ \varphi_{21}(t) & \dots & \varphi_{2n}(t) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(t) & \dots & \varphi_{nn}(t) \end{bmatrix}$ has determinant 1 for all $0 \leq t \leq 1$. This suggests us to consider the map $\tilde{\varphi} : [0,1] \to SL(n,\mathbb{C})$ given by $\tilde{\varphi}(t) = \begin{bmatrix} \theta(t)\varphi_{11}(t) & \dots & \theta(t)\varphi_{1n}(t) \\ \varphi_{21}(t) & \dots & \varphi_{2n}(t) \\ \vdots & \ddots & \vdots \\ \varphi_{n1}(t) & \dots & \varphi_{nn}(t) \end{bmatrix}$. Since θ is continuous, it is easily seen that $\tilde{\varphi}$ is a path in $SL(n,\mathbb{C})$ with end points $\tilde{\varphi}(0) = I_n$ and $\tilde{\varphi}(1) = A$. Hence $SL(n,\mathbb{C})$ is path connected.

Since $SL(n,\mathbb{C}) = \det^{-1}(\{1\})$ and det is continuous, $SL(n,\mathbb{C})$ is a closed subset of $M_n(\mathbb{C})$. However, $SL(n,\mathbb{C})$ is not bounded as it contains $SL(n,\mathbb{R})$ which is unbounded, as seen in Corollary 3.5. □

Theorem 4.3. *$U(n)$ is compact and path connected.*

Proof. Since the map $M_n(\mathbb{C}) \ni A \mapsto A^*A \in M_n(\mathbb{C})$ is continuous and $U(n)$ is the inverse image of the singleton closed set $\{I_n\}$ under this map, we see that $U(n)$ is a closed subset of $M_n(\mathbb{C})$. Also, it is easily seen that that $U(n)$ lies in the closed unit ball of $\mathbb{C}^{n^2}$ under its usual metric. Hence, by the Heine-Borel theorem, $U(n)$ is compact.

Again, it is enough to show that any unitary matrix is connected to I_n by a path in $U(n)$. If $A \in U(n)$, then A is normal and therefore unitarily diagonalizable ([3, Corollary to Theorem 21, Chapter 8]), i.e., there exists a $U \in U(n)$ such that U^*AU is diagonal. Note that, for each $U \in U(n)$, the operation $Ad(U) : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $Ad(U)(X) = U^*XU$ is a homeomorphism; so, it is enough to prove that every diagonal unitary matrix is connected to I_n by a path in $U(n)$. Indeed, if $D = U^*AU$ is a diagonal unitary matrix, and φ is a path in $U(n)$ connecting I_n and D, then the map $\Phi : [0,1] \to U(n)$ given by $\Phi(t) = U\varphi(t)U^*$ is a path with $\Phi(0) = I_n$ and $\Phi(1) = U\varphi(1)U^* = A$.

Let $D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ be a diagonal unitary matrix, where $\theta_i \in \mathbb{R}$ for all $1 \leq i \leq n$. Consider the path $\varphi : [0,1] \to U(n)$ given by $\varphi(t) = \operatorname{diag}(e^{it\theta_1}, e^{it\theta_2}, \ldots, e^{it\theta_n})$. Clearly, φ is a path in $U(n)$ with $\varphi(0) = I_n$ and $\varphi(1) = D$. This proves our assertion. $\square$

Note that $U(1)$ has the obvious metric space structure, namely, it equals the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Also, $GL(1, \mathbb{C})$ is just the punctured complex plane at the origin. It is usually not possible to visualize the metric space structure of higher matrix groups, except for the following beautiful metric space realization of $SU(2)$.

Proposition 4.4. *The group $SU(2)$ is homeomorphic to the real Euclidean sphere*

$$S^3 := \{(x, y, w, z) \in \mathbb{R}^4 : x^2 + y^2 + w^2 + z^2 = 1\}.$$

Proof. Note that for every $A = [a_{ij}] \in SU(2)$, its inverse is given by $A^{-1} = A^*$. Direct calculation gives us that $A^{-1} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ and on the other hand, by definition, we have $A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{bmatrix}$. Comparing entry wise, we obtain $A = \begin{bmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & \overline{a_{11}} \end{bmatrix}$ and if $a_{rs} = x_{rs} + \iota y_{rs}$ for $x_{rs}, y_{rs} \in \mathbb{R}$, we obtain

$$1 = \det(A) = |a_{11}|^2 + |a_{12}|^2 = x_{11}^2 + y_{11}^2 + x_{12}^2 + y_{12}^2.$$

In particular, this induces a map

$$SU(2) \ni \begin{bmatrix} a_{11} & a_{12} \\ -\overline{a_{12}} & \overline{a_{11}} \end{bmatrix} \mapsto (x_{11}, y_{11}, x_{12}, y_{12}) \in S^3.$$

$SU(2)$, being a closed subset of $U(2)$, is compact; so, this map, being a continuous bijection from $SU(2)$ onto a Hausdorff space S^3 is a homeomorphism ([5, Theorem 26.6]). $\square$

This obviously tells us that $SU(2)$ is path connected. Motivated by this, we now show that compactness and path connectedness holds in higher dimensions as well.

Theorem 4.5. *$SU(n)$ is compact and path connected.*

Proof. By definition, $SU(n)$ is a closed subset of the compact space $U(n)$ and hence compact.

One is tempted to think that connectedness of $SU(n)$ can be deduced from that of $U(n)$ on the lines of Corollary 4.2. However, that trick does not provide us with a path in $SU(n)$. We actually try to imitate the proof of Theorem 4.3. Indeed, if $A \in SU(n)$, then there exists a $U \in U(n)$ such that $D := U^*AU$ is diagonal.

If $D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ for some $\theta_i \in \mathbb{R}$, we get $1 = \det(A) = \det(D) = e^{i\sum_{i=1}^{n}\theta_i}$, so that $e^{-i\sum_{i=1}^{n-1}\theta_i} = e^{i\theta_n}$. Consider the map $\varphi : [0,1] \to SU(n)$ given by

$$\varphi(t) = U \operatorname{diag}(e^{it\theta_1}, e^{it\theta_2}, \ldots, e^{it\theta_{n-1}}, e^{-it\sum_{i=1}^{n-1}\theta_i})\, U^*.$$

Clearly, φ is a path in $SU(n)$ with end points $\varphi(0) = I_n$ and $\varphi(1) = UDU^* = A$ because $e^{-i\sum_{i=1}^{n-1}\theta_i} = e^{i\theta_n}$. The above trick was motivated from the proof of [2, Proposition 1.10]. $\square$

5. Symplectic groups

Before discussing the topological properties of symplectic groups, let us quickly get a short overview of some of their algebraic properties.

The matrix $J_n = \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}$ belongs to $Sp(n, \mathbb{K})$, it satisfies the equalities $J_n^T = -J_n = J_n^{-1}$, and has determinant 1. Also, for any $A \in Sp(n, \mathbb{K})$, the defining equation $A^T J_n A = J_n$ yields $\det(A)^2 = 1$ so that $\det(A) = \pm 1$. We shall, in fact, see below that $Sp(n, \mathbb{K}) \subseteq SL(2n, \mathbb{K})$.

Notice that J_n induces a bilinear form $B : \mathbb{K}^{2n} \times \mathbb{K}^{2n} \to \mathbb{K}$ given by

$$B(x, y) = x^T J_n y = \sum_{i=1}^{k} (x_i y_{n+i} - x_{n+i} y_i), \tag{5.1}$$

which is easily seen to be non-degerate, i.e., $B(x, y) = 0$ for all $y \in \mathbb{K}^{2n}$ implies $x = 0$, skew-symmetric, i.e., $B(x, y) = -B(y, x)$, and we observe that

$$Sp(n, \mathbb{K}) = \{A \in M_{2n}(\mathbb{K}) : B(Ax, Ay) = B(x, y) \text{ for all } x, y \in \mathbb{K}^{2n}\}. \tag{5.2}$$

Lemma 5.1. *$Sp(n, \mathbb{K})$ is a group that is closed under transposition. Also, for $A \in Sp(n, \mathbb{K})$, we have $A^T = -J_n A^{-1} J_n$ and $A^{-1} = -J_n A^T J_n$.*

Proof. From (5.2), we see that $Sp(n, \mathbb{K})$ is multiplicatively closed, i.e., $AB \in Sp(n, \mathbb{K})$ for all $A, B \in Sp(n, \mathbb{K})$. Also, for $A \in Sp(n, \mathbb{K})$, we have

$$B(A^{-1}x, A^{-1}y) = B(AA^{-1}x, AA^{-1}y) = B(x, y)$$

for all $x, y \in \mathbb{K}^{2n}$. Thus, $Sp(n, \mathbb{K})$ is a subgroup of $GL(2n, \mathbb{K})$.

Then, for $A \in Sp(n, \mathbb{K})$, its defining condition yields $A^T = J_n A^{-1} J_n^{-1}$, which implies that $Sp(n, \mathbb{K})$ is closed under transposition and it also provides the desired expressions for A^T and A^{-1}. □

Lemma 5.2. *Let $A \in Sp(n, \mathbb{K})$ and $p(\lambda)$ be its characteristic polynomial. Then, the following hold:*

(1) $p(\lambda) = \pm \lambda^{2n} p(1/\lambda)$.
(2) If λ is an eigenvalue of A, then so is $1/\lambda$.
(3) A and A^{-1} have same eigenvalues.

Moreover, if $\mathbb{K} = \mathbb{R}$ and λ is an eigenvalue of A, then so are $\bar{\lambda}$ and $\big(\bar{\lambda}\big)^{-1}$.
In particular, $Sp(n, \mathbb{K}) \subseteq SL(2n, \mathbb{K})$.

Proof. (2) and (3) follow from (1), which holds as follows:

$$\begin{aligned}
p(\lambda) &= \det\big(A - \lambda I_{2n}\big) \\
&= \det\big(A^T - \lambda I_{2n}\big) \\
&= \det\big(-J_n A^{-1} J_n - \lambda I_{2n}\big) \\
&= \det\big(-J_n A^{-1} J_n + \lambda J_n J_n\big) \\
&= \det\big(-A^{-1} + \lambda I_{2n}\big) \\
&= \det(A^{-1}) \det\big(-I_{2n} + \lambda A\big) \\
&= \det(A^{-1}) \lambda^{2n} \det\Big(-\frac{1}{\lambda} I_{2n} + A\Big) \\
&= \pm \lambda^{2n} p(1/\lambda).
\end{aligned}$$

Since a complex root of a real polynomial always appears in conjugates we are done. □

Lemma 5.3. $Sp(1, \mathbb{K}) = SL(2, \mathbb{K})$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{K})$. Then,

$$\begin{aligned} A \in Sp(1, \mathbb{K}) &\iff A^T J_1 A = J_1 \\ &\iff \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &\iff \begin{bmatrix} 0 & -bc + ad \\ -ad + bc & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &\iff \det(A) = 1. \end{aligned}$$

□

However, for $n \geq 2$, it easily seen that $Sp(n, \mathbb{K}) \subsetneq SL(2n, \mathbb{K})$. For instance,

$$D_r := \operatorname{diag}\big(r, r, ..., r, \frac{1}{r^{2n-1}}\big) \in SL(2n, \mathbb{K}) \setminus Sp(n, \mathbb{K})$$

for every $r > 1$ because, unlike r, $1/r$ is not an eigenvalue of D_r.

We now move towards analyzing the topological properties of symplectic groups with the help of linear algebra. For that, we recall an important class of symplectic matrices called *symplectic transvections.* For each nonzero $u \in \mathbb{K}^{2n}$ and $\lambda \in \mathbb{K}$, consider the linear map $\tau = \tau_{u,\lambda} : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$ given by $\tau_{u,\lambda}(v) = v + \lambda B(v, u)u$, $v \in \mathbb{K}^{2n}$. Let $W = \{v \in \mathbb{K}^{2n} : B(u, v) = 0\}$. Then, it is easily seen that W is a hyperplane, i.e., $\dim\big(\mathbb{K}^{2n}/W\big) = 1$, $\tau_{|W} = Id_W$ and $\tau(v) - v \in W$ for all $v \in V$. A linear map of the form $\tau_{u,\lambda}$ is called a *symplectic transvection.*

Note that $B(\tau_{u,\lambda}(x), \tau_{u,\lambda}(y)) = B(x, y)$ for all $x, y \in \mathbb{K}^{2n}$; so that $\tau_{u,\lambda} \in Sp(n, \mathbb{K})$ for all $u \in \mathbb{K}^{2n}$ and $\lambda \in \mathbb{K}$. Also, $\tau_{u,0} = I_{2n}$ for all $u \in \mathbb{K}^{2n}$. Interestingly, the symplectic transvections generate the symplectic groups; a proof of this fact can be found, for instance, in [4, § 6.9].

Theorem 5.4. *$Sp(n, \mathbb{K})$ is generated by the symplectic transvections.*

The symplectic groups share some topological properties with the special linear groups.

Proposition 5.5. *$Sp(n, \mathbb{K})$ is closed and is not compact for all $n \in \mathbb{N}$.*

Proof. If $\{X_m\}$ is a sequence in $Sp(n, \mathbb{K})$ converging to some X in $M_{2n}(\mathbb{K})$, then $J_n = (X_m)^T J_n X_m \to X^T J_n X$ as $m \to \infty$, and hence $X^T J_n X = J_n$ implying that $X \in Sp(n, \mathbb{K})$. So, $Sp(n, \mathbb{K})$ is closed in $M_{2n}(\mathbb{K})$.

For each $r > 0$, consider the block diagonal matrix $A_r = B_r \oplus B_{\frac{1}{r}}$, where $B_r = \operatorname{diag}(r, \frac{1}{r}, 1, 1, \ldots, 1) \in SL(n, \mathbb{K})$. It is easily seen that $(A_r)^T J_n A_r = J_n$, i.e., $A_r \in Sp(n, \mathbb{K})$ for all $r > 0$ and $\{A_r : r > 0\}$ is not bounded in $M_{2n}(\mathbb{K})$. Hence, $Sp(n, \mathbb{K})$ is not compact. □

Exercise. Let G be a subgroup of $GL(n, \mathbb{K})$ generated by a set S. If each element of S can be joined by a path in G to the identity matrix I_n, then G is path connected.

Proposition 5.6. *$Sp(n, \mathbb{K})$ is path connected for all $n \in \mathbb{N}$.*

Proof. By Theorem 5.4, every symplectic matrix is a (finite) product of symplectic transvections. So, it is enough to show that every symplectic transvection can be connected to the identity matrix by a path in $Sp(n, \mathbb{K})$. Consider a symplectic transvection $\tau_{u,\lambda}$ and define $\gamma : [0, 1] \to Sp(n, \mathbb{K})$ by

$$\gamma(t) = \tau_{u,(1-t)\lambda}, t \in [0, 1].$$

It is an easy exercise to show that γ is a path in $Sp(n, \mathbb{K})$ with end points $\gamma(0) = \tau_{u,\lambda}$ and $\gamma(1) = \tau_{u,0} = I_{2n}$. □

Although both real and complex symplectic groups have similar basic topological groups, they are topologically different as they are known to have different “fundamental groups”, a notion studied in “algebraic topology”.

There is also a compact version of (complex) symplectic groups given by

$$Sp(n) = Sp(n, \mathbb{C}) \cap U(2n).$$

It is clearly compact and is known to be path-connected. However, a proof of it requires some advanced mathematics (“Lie group theory”) which is out of the reach of this discussion.

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