

## Countability and Separation Axioms:

### First countable (Fc):

A space that has a countable basis at each of its points is called first countable space.

Eg.: metrizable spaces are F.c.

### Second countable (Sc):

If a space  $X$  has countable basis for its topology.

### Lindelof Space:

A space for which every open cover contains a countable sub cover is called Lindelof.

### Separable:

A space having a countable dense subset is called separable.

Eg.: compact metrizable space is second countable.

• S.c  $\Rightarrow$  lindelof

② S.c  $\Rightarrow$  separable

In a metrizable space, S.c  $\Leftrightarrow$  lindelof  $\Leftrightarrow$  separable.

For a dense set  $A$ ,  $\bigcup_{x \in A} \left( \bigcup_{n=1}^{\infty} B(x, r_n) \right)$  is a basis for its topology.

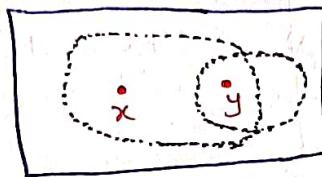
Eg.: The space  $\mathbb{R}_L$  satisfying all the countability axioms but the second countable.

Cor.:  $\mathbb{R}_l$  is not metrizable. (2)

### Separation Axioms!

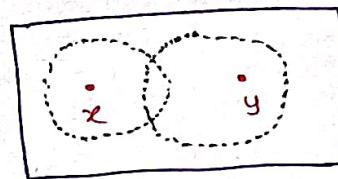
Trennungs axiom = separation axiom (German word)

Kolmogorov ( $T_0$ )



indiscrete top is  
not  $T_0$ .

Frechet ( $T_1$ )



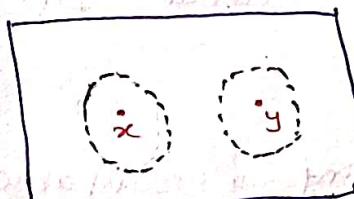
~~•~~ Sierpinski space:

$$X = \{\emptyset, \{x\}, \{x, y\}\}$$

$$T = \{\emptyset, X, \{\{x\}\}\}.$$

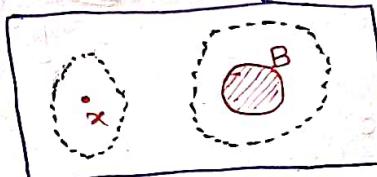
It is  $T_0$  but not  $T_1$ .

Hausdorff ( $T_2$ )



$\mathbb{R}$  with co-finite top is  $T_1$  but not  $T_2$ .

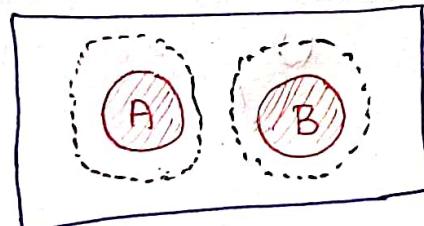
Regular ( $T_3$ )



$\mathbb{R}^K$  is  $T_2$  but not  $T_3$ .

Hint: Take  $\{\emptyset\} \times K$ .

Normal ( $T_4$ )

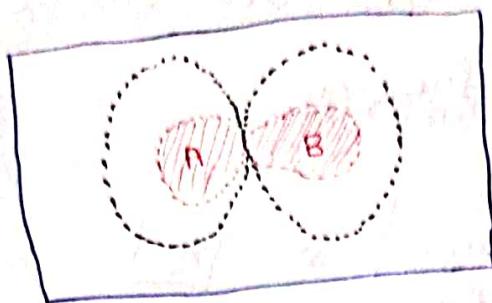


$\mathbb{R}_l$  is  $T_3$  but not  $T_4$ .

○ Subspace of  $T_4$  space need not be normal.

○ product of  $T_4$  spaces need not be normal.

Completely normal ( $T_5$ )



$$\bar{A} \cap B = \emptyset = A \cap \bar{B}$$

$$\underline{\underline{A}} = (0, \frac{1}{2})$$

$$B = (\frac{1}{2}, 1)$$

Eg:  $[0,1]$  is normal but not completely normal.

$$R^J \approx (0,1)^J \subseteq [0,1]^J = \text{compl} + T_2.$$

Urysohn ( $T_{2\frac{1}{2}}$ )

A space in which any two distinct points can be separated by closed neighborhoods.

completely  $T_2$ :

A space in which any two distinct points can be separated by continuous function.

$$(e) f: X \rightarrow [0,1]$$

$$f(x) = 0$$

$$f(y) = 1$$

Tychonoff ( $T_{3\frac{1}{2}}$ )

A closed set and any point outside it can be separated by continuous function.

$$f: X \rightarrow [0,1]$$

$$f(y) = 0 \quad \forall y \in \bar{A} = A$$

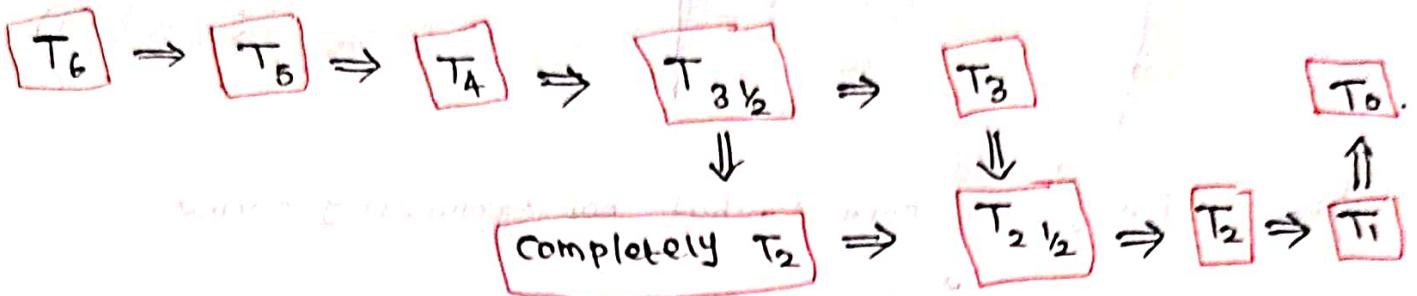
$$f(x) = 1$$

Perfectly normal ( $T_6$ )

$$f: X \rightarrow [0,1]$$

$$f^{-1}[0] = A, f^{-1}[1] = B.$$

Two disjoint closed sets are precisely separated by a continuous function.



Ex: Find counterexamples showing that none of these implications reverse.

Exercise:

Property:

Subspace      finite product      ctable product      uncountable product

$T_2$

✓

✓

✓

✓

$T_3$

✓

✓

✗

$T_4$

X  
(closed  
subspace)

X

(unsolved)

RJ

$T_5$

✓

✓

✓

✓

$T_6$

✓

✓

✓

✓

F.C

✓

✓

✓

✓

S.C

✓

✓

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✓

lin.

✓

✓

✓

✓

separable

✓

✓

✓

✓

## Properties:-

- $T_3 + \text{second countable} \Rightarrow T_4$  (Thm 32.1)
- $T_2 + \text{compact} \Rightarrow T_4$  (Thm 32.3)
- $T_2 + \text{loc. compact} \Rightarrow T_{3\frac{1}{2}}$
- $T_3 + \text{lindelof} \Rightarrow T_4$
- closed subspace of normal space is normal.

## Urysohn Lemma:-

- Let  $X$  be a normal space.  $A, B$  are disjoint closed sets of  $X$ . Then there exist a continuous map

$$f: X \rightarrow [0,1] \text{ s.t}$$

$$f(x) = 0 \quad \forall x \in A$$

$$f(x) = 1 \quad \forall x \in B.$$

Eg:- Let  $(X, d)$  be a <sup>bouned</sup> metric space.

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

$$\overline{A} = A, \overline{B} = B$$

$$A \cap B = \emptyset$$

$$\circ f: X \rightarrow [0,1]$$

$f$  iscts

$$\circ f(x) = 0 \Leftrightarrow x \in A, \quad f(x) = 1 \Leftrightarrow x \in B.$$

## Urysohn metrization thm:-

Every regular space with a countable basis is metrizable.

Rmk:- Every metrizable space is  $T_6$ .

Note:  $d(x, y) = \min \{1, d(x, y)\}$  or  $\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$

# Tietze extension theorem

Let  $X$  be a normal space and  $A$  be a closed subspace of  $X$ . Then

- ④  $f: A \xrightarrow{\text{cts}} [a, b]$  has extension  $\tilde{f}: X \xrightarrow{\text{cts}} [a, b]$
  - ⑤  $g: A \xrightarrow{\text{cts}} \mathbb{R}$  has extension  $\tilde{g}: X \xrightarrow{\text{cts}} \mathbb{R}$ .

# Hahn-Banach Extension!

Let  $Y$  be a subspace of a normed linear space  $X$  and  $f: Y \rightarrow \mathbb{C}$  be a continuous linear map. Then there exist a continuous linear map

$f: X \rightarrow C$  such that

$$0 \quad f(x) = f(x) \quad \forall x \in Y.$$

$$\textcircled{5} \quad \sup_{\substack{x \in Y \\ \|x\| \leq 1}} |f(x)| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |f(x)|$$