QUIZ

(1) Consider the finite Galois extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ with Galois group S_3 (here $\omega = exp(2i\pi/3)$). Then the subgroup of S_3 corresponding to the subextension $\mathbb{Q}(\omega)$ is
   (a) S_3
   (b) A_3 (the alternating group of order 3),
   (c) the subgroup $< Id, \sigma >$ of order 2 where $\sigma(\omega) = \omega^2$ and $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}$,
   (d) the trivial subgroup $< Id >$.
   Answer: (b), there are many ways to see this, for instance $|A_3| = 3 = [\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\omega)]$.

(2) Consider the finite Galois extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ with Galois group S_3 (here $\omega = exp(2i\pi/3)$). Then the subfield corresponding to the subgroup $< Id, \tau >$ of S_3, where $\tau(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and $\tau(\omega) = \omega^2$, is
   (a) $\mathbb{Q}(\omega)$,
   (b) $\mathbb{Q}(\sqrt[3]{2})$,
   (c) $\mathbb{Q}(\omega\sqrt[3]{2})$,
   (d) $\mathbb{Q}(\omega^2\sqrt[3]{2})$
   Answer: (d) is true. We have $\tau(\omega^2\sqrt[3]{2}) = (\tau(\omega))^2\tau(\sqrt[3]{2}) = (\omega^2)^2\omega\sqrt[3]{2} = \omega^2\sqrt[3]{2}$, hence $\mathbb{Q}(\omega^2\sqrt[3]{2})$ is contained in the fixed field of τ. Now both fields, the fixed field of τ and $\mathbb{Q}(\omega^2\sqrt[3]{2})$, have the same degree over $\mathbb{Q}$ (namely 3), hence they are equal. The fixed field of τ has degree 3 over $\mathbb{Q}$ because $\mathbb{Q}(\sqrt[3]{2}, \omega)$ has degree 2 (equal to the cardinality of the subgroup $< Id, \tau >$) over the fixed field of τ. It is clear that $\mathbb{Q}(\omega^2\sqrt[3]{2})$ has degree 3 over $\mathbb{Q}$.

(3) Let K be a field with 4 elements. Consider the extension of fields $K(t^{1/2})/\mathbb{F}_2(t)$. The purely inseparable closure and the separable closure of this extension are (respectively)
   (a) $\mathbb{F}_2(t)$ and $K(t)$,
   (b) $\mathbb{F}_2(t^{1/2})$ and $K(t)$,
   (c) $\mathbb{F}_2(t^{1/2})$ and $K(t^{1/2})$,
   (d) none of the above.
   Answer: (b) is true, since $\mathbb{F}_2(t^{1/2})$ is purely inseparable over $\mathbb{F}_2(t)$ and $K(t^{1/2})$ is separable over $\mathbb{F}_2(t^{1/2})$, while $K(t)$ is separable over $\mathbb{F}_2(t)$ and $K(t^{1/2})$ is purely inseparable over $K(t)$.