Graph Theory

Lecture 21

Maxflow - Mincut Theorem.

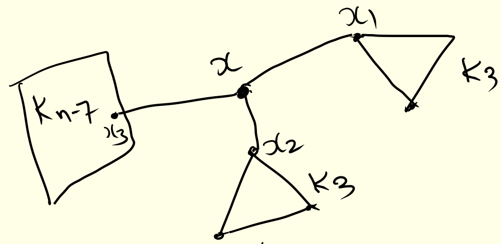
1. Solutions for Assignment 3 questions.

Q.1 $\underbrace{|N(S)| \geq |S| \quad \forall S \subseteq X}_{\text{Hall's thm.}} \Leftrightarrow G = X \sqcup Y$

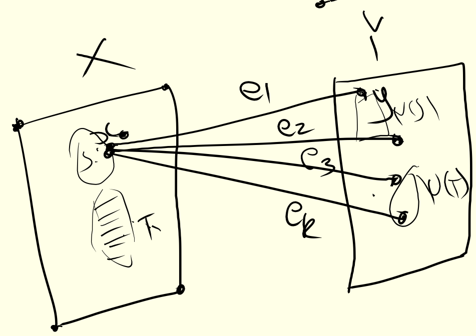
has a matching of size $|X|$.

$\rightarrow$ G is not bipartite & $\forall S \subset V(G)$ we must have $|N(S)| \geq |S|$. & G not having a 1-factor.

$\boxed{n > 10}$ $\forall n$, you just take



Q.2 :-



Case 2 — $\exists S \neq \phi$ s.t. $|N(S)| = |S|$

Case 1 — $\forall S \neq \phi \subset X$ we have $|N(S)| > |S|$.

In this case take any $x \in X$ & e be an edge xy. Then $\langle G - \{x, y\} \rangle$ also satisfies Hall's cond. $\therefore$ has a matching of size $X \setminus \{x\}$. That together with e ~~does the job~~! is the required matching.

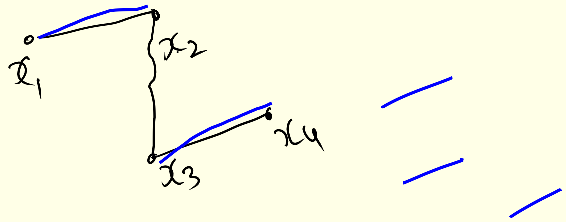
take $x \in S$. Let $N(x) \subseteq N(S)$.

do the same thing to complete the argument.

Q.3 : | G

A, B play a game of finding paths.

If G has 1-factor then player B chooses the vertex that occurs in the 1-factor. (That's the winning strategy)



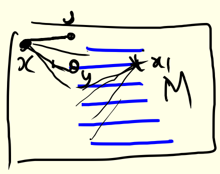
If G does not have 1-factor then $\exists$ a winning strategy A. Let M be a maximum matching.

i.e. M covers largest no. of vertices

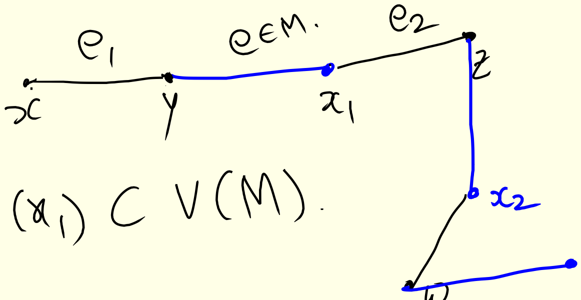
M has largest no. of edges.

$\therefore$ $\exists$ $x \notin V(M)$. A chooses that vertex x.

Now note $N(x) \subset V(M)$. $\therefore$ whichever y player B chooses A chooses the one that is adjacent to y in M.



Now if $\exists$ $z \in N(x_1)$ s.t. $z \notin V(M)$.



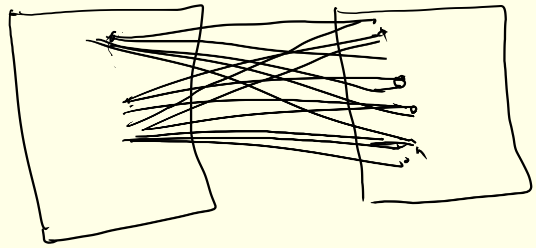
then $\{e_1, e_2\} \cup M - e$ is larger matching!

$\Rightarrow$ $N(x_1) \subset V(M)$.

Proceeding this way, player A will always have a choice for the next vertex. $\Rightarrow$ player B loses!

<u>Q.4</u> : # edges in a bipartite graph with max. matching of size $k-1$ is at most $\leq (k-1)n$ where n is the size of each part.

(König's theorem)



$|E(K_{k-1,n})| = (k-1)n.$

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2. Max flow - Mincut thm. (maxflow-mincut)

Transportation Networks. Maximize the flow given "feasibility" & "conservation of flow" constraints.

<u>Thm1</u>: $\max\limits_{f\text{-flow}} |f| = \min\limits_{x,y\text{-cut}} C(x,y)$

<u>Thm2</u>: If all capacities were integers, then $\exists$ a flow with max. strength having integral values.

<u>Application of these theorems.</u>

<u>Birkhoff's theorem</u>: If A is $n \times n$ integer mx. with non-negative entries with constant row sum & col^n sum (say) l. Then $A = P_1 + \cdots + P_l$ where P_i's are permutation matrices (const row sum = col^n sum = 1)

<u>Cor.</u> $\forall\, r < l$. one can <u>reduce few</u> of the entries of A to get B with non-negative integral entries with col^n sum = row sum = r.

<u>pf</u>. $B = P_{i_1} + \cdots + P_{i_r}$ for $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, l\}$.

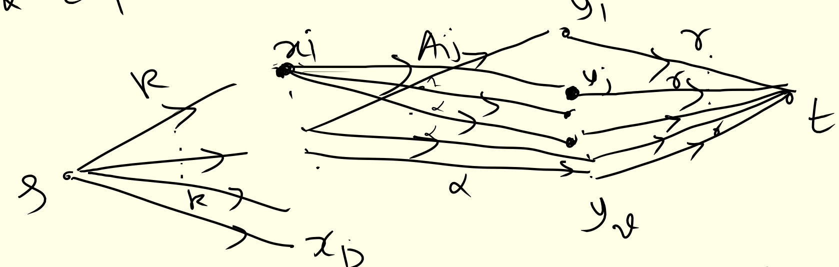
<u>Theorem</u>:- Let A be a $b \times v$ $(0,1)$-matrix with k ones per row & r ones per column. (This means $bk = vr$ = sum of all entries of A)

[Relates to the incidence matrix of a design].

Let k' & r' be integer s.t. $\exists$ $0<\alpha<1$ with
$k' = \alpha k$ & $r' = \alpha r$. (Note $\alpha \in \mathbb{Q}$)

Then, $\exists$ $(0,1)$ matrix A' whose row sums are k', column sums are r' & $A - A'$ is also a $(0,1)$ matrix. (ie A' is obtained by removing some of the 1's from A)

<u>Pf.</u> We construct a transportation network with vertices s, t & $x_1, \ldots, x_b$ & $y_1, \ldots, y_v$ with edges & capacities as follows:



ie $E(T) = \left\{ (s, x_i), (x_i, y_j), (y_j, t) \;\middle|\; \begin{matrix} 1 \le i \le b \\ 1 \le j \le v \end{matrix} \right\}$.

$c((s, x_i)) = k \quad \forall i \qquad c((x_i, y_j)) = A_{ij}$
$c(y_j, t) = r \quad \forall j$

Clearly, $\exists$ <u>flow</u> of strength bk where we give
$f(e) = c(e) \quad \forall e.$

Now we change capacities of this network as follows:
$c((s, x_i)) = k' \quad \forall i$
$c((y_j, t)) = r' \quad \forall i$
$c((x_i, y_j)) = A_{ij} \quad \forall \begin{matrix} 1 \le i \le b \\ 1 \le j \le v \end{matrix}$

max. possible flow can be $k'b \; (= r'v)$. We construct one by:
$f((s, x_i)) = k' \quad \forall i \qquad f((x_i, y_j)) = \alpha A_{ij} \quad \forall i, j.$
$f((y_j, t)) = r' \quad \forall j \qquad 0<\alpha<1.$

Since $\alpha < 1$, f is feasible.

Also $k' = \alpha k$ & $\exists$ k ones in each row
conservation law holds at x_i $\forall i$

Ily conservation law holds at all y_j's as each col^n has r non-zero entries

$\Rightarrow$ $\exists$ a max. flow g with integral entries !!!!

$A' = [A'_{ij}]$ where $A'_{ij} = g((x_i, y_j))$.

$\Rightarrow$ A' is the required $(0,1)$ matrix !

QED.

Ref. → Wilson + van Lint : A course in Combinatorics.
Chapter 7 - flows in networks.