TOPOLOGY - LECTURE 4

1. Introduction

In the previous section we defined the notion of a topology τ on a set X. This is a collection of subsets of X satisfying certain conditions, namely,

(1) $\emptyset, X \in \tau$,
(2) τ is closed with respect to finite intersections, and
(3) τ is closed with respect to arbitrary unions.

We then defined the notion of a basis for a topology on a set X. A collection $\mathcal{B}$ of subsets of X is a basis for a topology on X if

(1) $X = \bigcup_{B \in \mathcal{B}} B$, and
(2) whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ with
$$x \in B \subseteq B_1 \cap B_2.$$

If $\mathcal{B}$ is a basis for a topology on X, then the topology τ determined (or generated) by $\mathcal{B}$ is
$$\tau = \{U \subseteq X \,:\, \text{for each } x \in U, \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq U\}.$$
Clearly, we have the following,

(1) $\mathcal{B} \subseteq \tau$. Thus every $B \in \mathcal{B}$ is open (in the τ topology) in X, and
(2) τ is precisely the collection of subsets of X that are union of elements of $\mathcal{B}$.

In the sequel we discuss some examples and methods of constructing new topological spaces.

2. Examples and constructions

One way of constructing a topology on a set X is in the presence of an order relation on the set. Let us quickly recall some definitions.

A *relation* on a set X is a subset $R \subseteq X \times X$ of the cartesian product. If $(x, y) \in R$ we write xRy and say that x is *related* (or *equivalent*) to y. It may so happen that for some $x, y \in X$ we have that x is not related to y, that is, $(x, y) \notin R$.

Definition 2.1. A relation R on a set X is called an equivalence relation if

(1) (reflexive) For every $x \in X$, xRx holds, that is, $(x, x) \in R$.
(2) (symmetric) If xRy holds, then yRx holds.

(3) (transitive) If xRy and yRz hold, the xRz holds.

It is traditional to use the symbold $\sim$ for an equivalence relation R. Given an equivalence relation $\sim$ on X and $x \in X$, the set $[x]$ is defined to be

$$[x] = \{y \in X \,:\, y \sim x\}.$$

Thus $[x]$ is the set of all elements in X that are equivalent to x and is called the *equivalence class* of x. We know that if $x, y \in X$, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. Thus the equivalence classes partition the set X into disjount equivalence classes.

Definition 2.2. A relation R is called an *order relation* provided

(1) (comparability) If $x, y \in X$ and $x \neq y$, then either xRy or yRx holds.
(2) (nonreflexive) If $x \in X$, then xRx does not hold, that is, $(x, x) \notin R$.
(3) (transitive) If xRy and yRz, then xRz holds.

Traditionally $<$ is the symbol used to denote an order relation. An ordered set is a pair $(X, <)$ with $<$ an order relation on the set X.

Definition 2.3. Let X be an ordered set with order $<$. Given $a, b \in X$ the set

$$(a, b) = \{x \in X \,:\, a < x < b\}$$

is called an open interval in X. If the above set is empty, then a is called the immediate predecessor of b and b the immediate successor of a.

We are now in a position to define a topology on an ordered set. Start with an ordered set $(X, <)$. Let $\mathcal{B}$ be the collection of sets of the following type : open intervals (a, b) in X, intervals of the form $[m, b)$ in X where m is the smallest element (which may not exist) in X, and intervals of the form $(a, M]$ in X where M is the largest element in X.

Definition 2.4. Let $(X, <)$ be an ordered set. Then the collection $\mathcal{B}$ defined above is a basis for a topology on X called the *order topology* on X.

That $\mathcal{B}$ is a basis is a straightforward verification and is left as an exercise. Note that the standard topology on $\mathbb{R}$ is the same as the order topology on $\mathbb{R}$ with the natural order.

Given two ordered sets $(X, <')$ and $(Y, <'')$, there is an order $<$ defined on the product $X \times Y$ by setting

$$(x, y) < (x', y')$$

if and only if either $x <' x'$ or $x = x'$ and $y <'' y'$. The order $<$ is called the *dictionary order* on $X \times Y$

We shall have occassion to consider the order topology on an ordered set, often to construct counter examples. We now turn our attention to construct new topological spaces. Our first effort will be to understand how one can define a topology on the product of two topological spaces.

Definition 2.5. Let X and Y be topological spaces with topologies τ' and τ'' respectively. Let $\mathcal{B}$ be the collection

$$\mathcal{B} = \{U \times V \,:\, U \in \tau', V \in \tau''\}.$$

Then $\mathcal{B}$ is abasis for a topology τ on $X \times Y$ called the *product topology* on $X \times Y$.

Let us first check that $\mathcal{B}$ is indeed a basis for a topology on $X \times Y$. It is clear that $X \times Y \in \mathcal{B}$ so that

$$X \times Y = \bigcup_{U \times V \in \mathcal{B}} U \times Y.$$

Now suppose that

$$(x, y) \in (U \times V) \cap (U' \times V')$$

so that $x \in U \cap U'$ and $y \in V \cap V'$. But $U \cap U'$ is open in X so that $U \cap U' \in \tau'$. Similarly, $V \times V' \in \tau''$. Thus

$$(x, y) \in (U \cap U') \times (V \cap V') \subseteq (U \times V) \cap (U' \times V').$$

Note that the collection $\mathcal{B}$ is in general not a topology on X. Before looking at examples let us look at the following fact.

Proposition 2.6. Suppose that $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ a basis for the topology on Y. Then the collection

$$\mathcal{D} = \{B \times C \,:\, B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. The proof is an exercise in unravelling the definitions. First we convince ourselves that $\mathcal{D}$ is a basis for a topology on $X \times Y$. Given $(x, y) \in X \times Y$ we can find $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B$ and $y \in C$. Thus

$$X \times Y = \bigcup_{B \times C \in \mathcal{D}} B \times C.$$

Next assume that $(x, y) \in (B_1 \times C_1) \cap (B_2 \times C_2)$ with $B_i \in \mathcal{B}$ and $C_i \in \mathcal{C}$, $i = 1, 2$. Then $x \in B_1 \cap B_2$ and therefore there exists $B \in \mathcal{B}$ (since $B_1 \cap B_2$ is open in X and $\mathcal{B}$ is a basis for the topology on X) with

$$x \in B \subseteq B_1 \cap B_2.$$

Similarly there exists $C \in \mathcal{C}$ with $y \in C \subseteq C_1 \cap C_2$. Then $B \times C \in \mathcal{D}$ and

$$(x, y) \in B \times C \subseteq (B_1 \times C_1) \cap (B_2 \times C_2).$$

Thus $\mathcal{D}$ is indeed a basis for a topology on $X \times Y$. Recall that the topology τ generated by $\mathcal{D}$ consists of precisely those sets that can be written as (arbitrary) union of sets in $\mathcal{D}$. Since every set in $\mathcal{D}$ is already open in the product topology we have

$$\tau \subseteq \text{the product topology on } X \times Y.$$

To prove the reverse inclusion let $U \subseteq X \times Y$ be an open set in the product topology. We claim that $U \in \tau$. To prove this it is enough to show that U can be written as the union of elements in $\mathcal{D}$. Now given $(x, y) \in U$ there exists a basic open set $V_1 \times V_2$ (open in the product topology on $X \times Y$) such that

$$(x, y) \in V_1 \times V_2 \subseteq V.$$

Thus V_1 is open in X and V_2 is open in Y. Since $\mathcal{B}$ is a basis for the topology on X we can find $B_x \mathcal{B}$ such that

$$x \in B \subseteq V_1.$$

Similarly there exists $C_y \in \mathcal{C}$ with

$$y \in C \subseteq V_2.$$

Thus

$$(x, y) \in B_x \times C_y \subseteq V_1 \times V_2 \subseteq V$$

and hence

$$V \bigcup_{(x,y) \in V} B_x \times C_y.$$

This show that every open set in the product topology is also open in the τ topology. This completes the proof. □

Example 2.7. Recall that the usual topology on $\mathbb{R}$ has as a basis the collection

$$\{(a, b) \, : \, a < b\}$$

of all open intervals. Thus the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has as a basis the collection

$$\{(a, b) \times (c, d) \, : \, a < b, c < d\}$$

of all open rectangles.

We will soon try to understand which is the right topology to work with when dealing with arbitrary products of topological spaces. It turns out that when dealing with arbitrary products it is convenient to specify a subbasis rather than a basis. So we turn our attention to the definition of subbasis for a short while.

Recall that given a set X, a collection $\mathcal{S}$ of subsets of X is said to be a subbasis for a topology on X provided the union of elements in $\mathcal{S}$ is the whole of X. Starting with such a subbasis $\mathcal{S}$ we look at the collection

$$\mathcal{B} = \{B \subseteq X \, : \, B = S_1 \cap \cdots \cap S_r, S_i \in \mathcal{S}\}.$$

Thus $\mathcal{B}$ equals precisely those sets that are equal to intersetion of finitely many sets in $\mathcal{S}$. We claim that $\mathcal{B}$ is a basis for a topology on X. First note that as $\mathcal{S} \subseteq \mathcal{B}$ we have that

$$X = \bigcup_{B \in \mathcal{B}} B.$$

Next suppose that $x \in B \cap B'$ for some $B, B' \in \mathcal{B}$. We need to find $B \in \mathcal{B}$ with

$$x \in B \subseteq B \cap B'.$$

We let $B = B \cap B'$. This shows that $\mathcal{B}$ is a basis for a topology on X. Thus $\mathcal{B}$ generates a topology, say τ, on X. We say that τ is the topology generated by the subbasis $\mathcal{S}$. Thus the topology generated by a subbasis consists of all arbitrary unions of finite intersections of elements of $\mathcal{S}$.

We now come back to the product topology on $X \times Y$ where X, Y are topological spaces. We let $\pi_1 : X \times Y \longrightarrow X$ be the projection to the first factor and π_2 the projection to the second factor. Now if $U \subseteq X$ is an open set then as

$$\pi_1^{-1}(U) = U \times Y$$

we have that $\pi_1^{-1}(U)$ is an open set in the product topology (because it is a basic open set by definition). Similarly for every open set $V \subseteq Y$, the set

$$\pi_2^{-1}(V) = X \times V$$

is a (basic) open set in the product topology on $X \times Y$.

Proposition 2.8. With the above notations, the collection

$$\mathcal{S} = \{\pi_1^{-1}(U) \,:\, U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \,:\, V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Suppose U, U', are open in X and V is open in Y. Then $U \times V$ is a basic open set in $X \times Y$ and

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Thus every basic open set in $X \times Y$ is a finite intersection of elements of $\mathcal{S}$. Further, every finite intersection of elements of $\mathcal{S}$ is clearly a basic open set (in the product topology). Thus $\mathcal{S}$ must be a subbasis for the product topology on $X \times Y$. $\square$

Let X be a topological space with a topology τ and Y a subset of X. Then τ induces in a natural way a topology on Y.

Definition 2.9. Let (X, τ) be a topological space and Y a subset of X. Then the collection

$$\tau_Y = \{U \cap Y \,:\, U \in \tau\}$$

is a topology on Y called the *subspace topology* on Y. Y with the subspace topology is called a subspace of X.

The verification that τ_Y is a topology on Y is an exercise. Thus in the subspace topology on Y a subset V of Y is open in Y if and only if V is the intersection of an open set in X with Y.

Here is an example.

Definition 2.10. In the subspace topology on $\mathbb{Z} \subseteq \mathbb{R}$, every subset of $\mathbb{Z}$ is open. Thus the subspace topology on $\mathbb{Z}$ is discrete.

This example shows that if Y is a subspace of X, then if a subset $U \subseteq Y$ is open in Y, then it need not be open in X.

Proposition 2.11. Suppose $\mathcal{B}$ is a basis for the topology τ on X. Let $Y \subseteq X$ be a subset. Then the collection

$$\mathcal{B}_Y = \{B' = B \cap Y \,:\, B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let us first check that $\mathcal{B}_Y$ is a basis for a topology on Y. Since the union of elements in $\mathcal{B}$ equals X we have that

$$\bigcup_{B' \in \mathcal{B}_Y} B' = Y.$$

Let $B_1', B_2' \in \mathcal{B}_Y$ and $y \in B_1' \cap B_2'$. Then by definition

$$y \in (B_1 \cap Y) \cap (B_2 \cap Y) = (B_1 \cap B_2) \cap Y$$

for some $B_1, B_2 \in \mathcal{B}$. We may thus find $B \in \mathcal{B}$ (since $\mathcal{B}$ is a basis) with

$$y \in B \subseteq B_1 \cap B_2.$$

Thus

$$y \in B' = B \cap Y \subseteq (B_1 \cap B_2) \cap Y = (B_1 \cap Y) \cap (B_2 \cap Y) = B_1' \cap B_2'$$

which shows that $\mathcal{B}_Y$ is a basis for a topology on Y. To check that $\mathcal{B}_Y$ is a basis for the subspace topology on Y it is enough to check that every open set in Y is the union of elements of $\mathcal{B}_Y$. $\square$

Here are some exercises.

Exercise 2.12. Define two points (x_0, y_0), (x_1, y_1) in the plane $\mathbb{R}^2$ to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. show that this is an equivalence relation. Describe the equivalence classes.

Exercise 2.13. Let R be a relation on a set X. Let Y be a subset of X. Then

$$R' = R \cap (Y \times Y)$$

is a relation on Y called the restriction of R to Y. Show that if R is an equivalence relation, then so is R'.

Exercise 2.14. Given a surjective function $f : X \longrightarrow Y$ define $x \sim x'$ if and only if $f(x) = f(x')$. Show that $\sim$ is an equivalence relation on X. Let X^* denote the set of $\sim$ equivalence classes. Construct a bijection $g : X^* \longrightarrow Y$

For example consider the following map

$$f : X = [0, 1] \longrightarrow \mathbb{R}^2$$

given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. This map is injective on $(0, 1)$. Thus under the equivalence relation in above exercise the equivalence class $[t] = \{t\}$ consists of a single point if $0 < t < 1$ and $[0] = \{0, 1\} = [1]$ has two points. The above exercise guarantees a bijection between X^* and the image of f which is the circle of radius 1.

Exercise 2.15. Let

$$R = \{(x, y) \,:\, y = x + 1 \text{ and } 0 < x < 2\}$$
$$R' = \{(x, y) \,:\, y - x \text{ is an integer}\}.$$

(1) Show that R, R' are equivalence relations on $\mathbb{R}$ and that $R \subseteq R'$. Describe the equivalence classes of R'.
(2) Show that given any collection of equivalence relationson a set X, their intersection is again an equivalence relation on X.

Exercise 2.16. Define a relation on the plane by setting

$$(x_0, y_0) < (x_1, y_1)$$

if either $y_0 - x_0^2 < y_1 - x_1^2$ or $y_0 - x_0^2 == y_1 - x_1^2$ and $x_0 < x_1$. Show that this is an order relation on the plane. Describe it geometrically.

Exercise 2.17. Show that an element in an ordered set has at most immediate successor and at most one predecessor. Show that a subset of an ordered set has at most one smallest element and at most one largest element.

Exercise 2.18. Consider the following oeder relations on $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(1) Dictionary order.
(2) $(x_0, y_0) < (x_1, y_1$ if either $x_0 - y_0 < x_1 - y_1$, or $x_0 - y_0 = x_1 - y_1$ and $y_0 < y_1$.
(3) $(x_0, y_0) < (x_1, y_1$ if either $x_0 + y_0 < x_1 + y_1$, or $x_0 + y_0 = x_1 + y_1$ and $y_0 < y_1$.

In these orders which elements have immediate predecessors? Which orderes have a smallest element. Show that all three orders are different.

Let X be an ordered set and Y a subset. An element $x \in X$ is called an *upper bound* for Y if for every $y \in Y$ we have $y \leq x$. The element x is said to the least upper bound of Y if it is an upper bound and for any other upper bound x' of Y we have $x \leq x'$. One can similarly define lower bounds and greatest lower bound. An ordered set X is said to have the *least upper bound property* if every subset that is bounded above has a least upper bound. We know that $\mathbb{R}$ has the least upper bound property in the usual order.

Exercise 2.19. Show that an ordered set has the least upper bound property then it has the greatest lower bound property.

Exercise 2.20. Let Y be a subspace of X and let A be a subset of Y. Show that the topology that A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Exercise 2.21. Let $Y = [-1, 1]$ with the subspaace topology from $\mathbb{R}$. Which of the following subsets are open wither in Y or $\mathbb{R}$. (1) $A = \{x : (1/2) < |x| < 1\}$, (2) $B = \{x : (1/2) < |x| \leq 1\}$, (3)$C = \{x : (1/2) \leq |x| < 1\}$, (4) $D = \{x : 0 < |x| < 1, (1/x) \notin \mathbb{Z}_+\}$.

Exercise 2.22. A map $f : X \longrightarrow Y$ is said to be an *open map* if $f(U)$ is open whenever U is open. Show that the two projection maps on $X \times Y$ are open maps.

Exercise 2.23. Suppose that L is a line in the plane. Describe the subspace topology on L in the cases that it is a subspace of $\mathbb{R}_\ell \times \mathbb{R}$, $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Exercise 2.24. Recognize the dictionary order topology on $\mathbb{R}^2$ as a product topology.

Exercise 2.25. Let $I = [0, 1]$. There are three topologies on $I \times I$ that we consider. The product topology, the dictionary order topology, and the subspace topology it inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ with the dictionary order topology. Compare these three topologies.