Example. Now consider the unknown variance case. Then as discussed previously, with the Jeffreys' prior, we have

$$\frac{\sqrt{n}(\theta - \bar{x})}{s}|\mathbf{x} \sim t_{n-1}.$$

Then, since

$$P(|\frac{\sqrt{n}(\theta - \bar{x})}{s}| \leq t_{n-1}(1-\alpha/2)|\mathbf{x}) = 1 - \alpha,$$

for $n \geq 2$, the HPD 100(1-α)% credible interval for θ is

$$\bar{x} \pm t_{n-1}(1-\alpha/2)\frac{s}{\sqrt{n}}.$$

Credible intervals are very easy to calculate unlike confidence intervals, the construction of which requires pivotal quantities or inversion of a family of tests. Consider the following example.

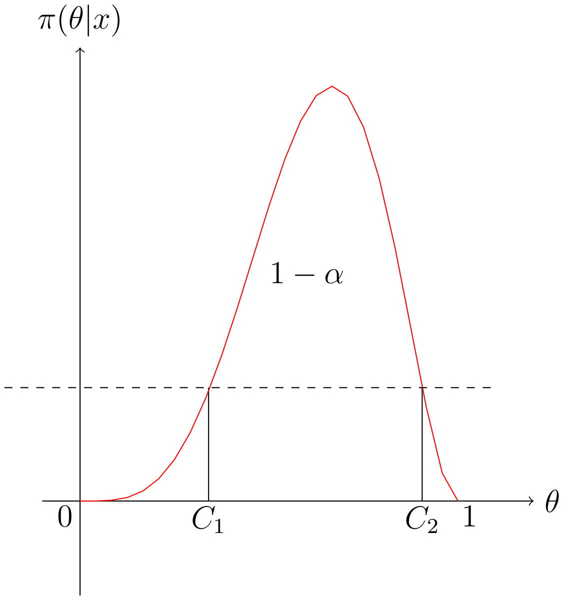
Example. $X|\theta \sim$ Binomial(n, θ) and $\theta \sim$ Beta(a, b). Then $\theta|X = x \sim$ Beta$(a+x, b+n-x)$. Therefore, the 100(1-α)% HPD credible set is (C_1, C_2) where C_1 and C_2 satisfy

$$\begin{aligned}
1 - \alpha &= \int_{C_1}^{C_2} \pi(\theta|x)\, d\theta \\
&= \int_{C_1}^{C_2} \frac{\Gamma(a+b+n)}{\Gamma(a+x)\Gamma(b+n-x)} \theta^{a+x-1}(1-\theta)^{b+n-x-1}\, d\theta, \text{ and} \\
\pi(C_1|x) &= \pi(C_2|x), \text{ or}
\end{aligned}$$

$$C_1^{a+x-1}(1-C_1)^{b+n-x-1} = C_2^{a+x-1}(1-C_2)^{b+n-x-1}$$

.

Solve for C_1 and C_2 numerically.



Example. $X_1, X_2, \ldots, X_n$ i.i.d $N(\theta, \sigma^2)$, σ^2 known. $\pi(\theta) \equiv C$. Then, from previous discussion,

$$\theta|x_1, x_2, \ldots, x_n \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right).$$

Therefore, the 100(1-α)% HPD credible set for θ is

$$\bar{x} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}},$$

which is the same as the corresponding confidence interval. Then what is the difference between the Bayesian and Frequentist intervals? It is in the interpretation.

When viewed as a credible set, $\bar{x} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$ has (posterior) probability $1-\alpha$ of containing θ. But when it is viewed as a confidence interval, this fixed set has no such meaning. The random interval $\bar{X} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$ has probability $1-\alpha$ of containing θ. In otherwords, if the procedure is employed over and over again, then the resulting intervals have long-run relative frequency of $1-\alpha$ of capturing θ inside. If $\alpha = 0.05$, the random interval has 19 out of 20 chance of containing θ, so we can have the confidence that the confidence interval from any data set has a good chance of capturing θ.

Prediction of a Future Observation

We have already done this informally earlier. Suppose the data are $x_1, \cdots, x_n$, where $X_1, \ldots, X_n$ are i.i.d. with density $f(x|\theta)$, e.g., $N(\mu, \sigma^2)$ with σ^2 known.

We want to predict the unobserved X_{n+1} or set up a predictive credible interval for X_{n+1}.

Prediction by a single number $t(x_1, \cdots, x_n)$ based on $x_1, \cdots, x_n$ with squared error loss amounts to considering prediction loss

$$\begin{aligned} E\left\{(X_{n+1}-t)^2|\boldsymbol{x}\right\} &= E\left[\left\{(X_{n+1}-E(X_{n+1}|\boldsymbol{x}))-(t-E(X_{n+1}|\boldsymbol{x}))\right\}^2|\boldsymbol{x}\right] \\ &= E\left\{(X_{n+1}-E(X_{n+1}|\boldsymbol{x}))^2|\boldsymbol{x}\right\}+(t-E(X_{n+1}|\boldsymbol{x}))^2 \end{aligned}$$

which is minimum at

$$t = E(X_{n+1}|\boldsymbol{x}).$$

To calculate the predictor we need to calculate the predictive distribution

$$\begin{aligned} \pi(x_{n+1}|\boldsymbol{x}) &= \int_\Theta \pi(x_{n+1}|\boldsymbol{x},\theta)\pi(\theta|\boldsymbol{x})\,d\theta \\ &= \int_\Theta f(x_{n+1}|\theta)\pi(\theta|\boldsymbol{x})\,d\theta. \end{aligned}$$

Let $\mu(\theta) = \int_{-\infty}^{\infty} xf(x|\theta)\,dx$. It can be shown that

$$E(X_{n+1}|\boldsymbol{x}) = E(\mu(\theta)|\boldsymbol{x}) = \int_\Theta \mu(\theta)\pi(\theta|\boldsymbol{x})\,d\theta$$

and hence for the normal problem the predictor is $\int_{-\infty}^{\infty} \mu\pi(\mu|\boldsymbol{x})\,d\mu$ = posterior mean of μ.

Similarly in the Binomial Example, the predictive probability that the next ball is red is

$$E(X_{n+1}|\boldsymbol{x}) = E(p|\boldsymbol{x}) = \frac{\alpha+r}{\alpha+\beta+n}$$

where $r = \sum_1^n x_i$.

A predictive credible interval for X_{n+1} is (c,d) where c and d are $100\alpha_1\%$ and $100(1-\alpha_2)\%$ quantiles of the predictive distribution of X_{n+1} given $\boldsymbol{x}$. Usually, one takes $\alpha_1 = \alpha_2 = \alpha/2$ as for credible intervals.

Testing of hypotheses: Model choice/criticism

$X \sim P_\theta$, $\theta \in \Theta$, with density or mass function $f(x|\theta)$. We want to test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$, $\Theta_0 \cup \Theta_1 = \Theta$. In principle this is just another Bayesian inference problem. Simply obtain $\pi(\theta|x)$ and compute

$$P(\Theta_0|x) = \int_{\Theta_0} \pi(\theta|x)\,d\theta \text{ and } P(\Theta_1|x) = \int_{\Theta_1} \pi(\theta|x)\,d\theta = 1 - P(\Theta_0|x).$$

If $P(\Theta_0|x) > 1/2$ (or a suitable threshold), or the posterior odds ratio (of H_0 relative to H_1), $P(\Theta_0|x)/P(\Theta_1|x) > 1$, accept H_0.

Example. Consider a blood test conducted for determining the sugar level of a person with diabetes two hours after he had his breakfast. It is of interest to see if his medication has controlled his blood sugar levels. Assume that the test result X is $N(\theta, 100)$, where θ is the true level. In the appropriate population (diabetic but under this treatment), θ is distributed according to a $N(100, 900)$. Then, marginally X is $N(100, 1000)$, and the posterior distribution of θ given $X = x$ is normal with
mean $= \frac{900}{1000}x + \frac{100}{1000}100 = 0.9x + 10$ and variance $= \frac{100\times900}{1000} = 90$.
Suppose we want to test $H_0 : \theta \leq 130$ versus $H_1 : \theta > 130$. If the blood test shows a sugar level of 130, what can be concluded? Note that, given this test result, the true mean blood sugar level (θ) may be assumed to be $N(127, 90)$, which is the posterior distribution. Consequently, we obtain,

$$\begin{aligned} P(\theta \leq 130|X = 130) &= \Phi\left(\frac{130 - 127}{\sqrt{90}}\right) = \Phi(.316) = 0.624, \text{ and hence} \\ P(\theta > 130|X = 130) &= 0.376. \end{aligned}$$

Therefore the Posterior odds ratio of H_0 relative to H_1 is $0.624/0.376 = 1.66$.